Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

Corso di Laurea Magistrale in Matematica

Torsion obstructions to the existence of manifold bundles

Relatori:

Ulrich Bunke
Remke Kloosterman
Christoph Winges

Laureando:

Francesco Roana
Matricola: 2053892

Anno Accademico 2022/2023

Contents

Introduction

Given a map $f\colon M \to B$ of closed topological manifolds, is f homotopic to the projection map of a fiber bundle of closed manifolds? The fibering problem has a long tradition in geometric topology: since the 1960s it has been intensively studied and has accompanied the development of the subject, from surgery theory and the s-cobordism theorem applied in the fibering theorems of Browder and Levine [BL65] and Farrell [Far71] to algebraic K-theory used by Steimle in [Ste12]. In this thesis we discuss this problem starting from the work of Farrell, Lück and Steimle [FLS09], which provides two obstructions whose vanishing is a necessary condition for a map of closed manifold to be homotopic to a fiber bundle. Moreover, we compare what they achieved with Farrell's fibering theorem [Far71] over a circle and Steimle's stable fibering theorems [Ste12], which both provide a complete set of obstructions for their cases.

Let us briefly describe what is presented. The main idea to investigate the fibering problem is to take advantage of the notion of fibration. Fibrations are in fact so similar to fiber bundles that is quite easy to see if they are actually fiber bundles, but at the same time they are so general that any map $f\colon M \to B$ can be converted into a fibration $\widehat{f}\colon \mathrm{FIB}(f) \to B$ such that its total space $\mathrm{FIB}(f)$ is homotopy equivalent to M. The strategy is first to reduce the problem to the fibration $\widehat{f}$ and then to check what happens during the "conversion" of the map f into $\widehat{f}$. Note that if f is homotopic to a fiber bundle, then the fiber of $\widehat{f}$ has necessarily the homotopy type of a finite CW-complex. Therefore, this will always be assumed during the discussion.

Let us focus our study to fibrations. A big difference between fibrations and fiber bundles is that the fiber transport of a fiber bundle along a path is a homeomorphism, while that of a fibration in general is not. The first obstruction $\theta(f)$ is therefore meant to measure how "simple" is the fiber transport of the fibration $\widehat{f}$ along a loop in B. Since by construction this is homotopy invariant and fiber bundles have "simple" fiber transport, then if f is homotopic to a fiber bundle, the obstruction $\theta(f)$ necessarily vanishes.

At this point, we go back to the general problem and we focus on the conversion of the map f into the fibration $\widehat{f}$. The idea is to check whether we lose any "fiber bundle information" during this operation. The second obstruction $\tau_{\mathrm{fib}}(f)$, therefore, aims to measure how "simple" is the homotopy equivalence $M \xrightarrow{\simeq} \mathrm{FIB}(f)$. By construction this has obviously to vanish for fiber bundles. Therefore, by homotopy invariance, if f is homotopic to a fiber bundle, then $\tau_{\mathrm{fib}}(f)$ vanishes.

The tool we want to use to measure the "simplicity" of a map is the Whitehead torsion: a map is said "simple", or rather, a simple homotopy equivalence, if its Whitehead torsion vanishes. However, this is defined in general in the category of finite CW-complexes, while we mainly work with spaces that only have the homotopy type of a finite CW-complex, for example with closed manifolds. Therefore, we have to extend the definition of Whitehead torsion to this kind of spaces. The idea is to choose for any such space a finite CW-model up to simple homotopy equivalence and to use it to compute the torsion. We call this a simple structure. Equipping a space with a simple structure is particularly easy for closed manifolds because they have a canonical choice of finite CW-model, but it is quite long for the total space of a fibration, for example for $\mathrm{FIB}(f)$, since this in general is not a manifold. The construction of a simple structure for the total space of a fibration will be of great importance for us and it will be studied extensively.

At the end of this investigation we obtain the following theorem.

Theorem 0.1. *Let $f\colon M \to B$ be a map of closed manifolds with path-connected B. Suppose that for some $b \in B$ the homotopy fiber of f has the homotopy type of a finite* CW*-complex. Then if f is homotopic to a map $g\colon M \to B$ which is the projection of a locally trivial fiber bundle with a closed manifold as fiber, both $\theta(f)$ and $\tau_{fib}(f)$ vanish.*

Once this has been proved, it is natural to ask whether, or rather, when, the converse implication also holds. In second half of this thesis we give an answer to this question in the following two different ways.

First of all, we focus on the particular case of smooth maps $f\colon M \to S^1$ from a smooth manifold M to a circle. The idea is to compare the obstructions $\theta(f)$ and $\tau_{\text{fib}}(f)$ with Farrell's obstructions of [Far71]. It turns out that these invariants do not coincide, but the vanishing of $\theta(f)$ and $\tau_{\text{fib}}(f)$ implies the vanishing of Farrell's obstructions. Therefore, by Farrell's fibering theorem, we conclude that in this case the converse implication of Theorem 0.1 also holds.

Secondly, we study the fibering problem in the more general context of algebraic K-theory. The idea is to apply the same strategy described above, but taking advantage of the more powerful tools of algebraic K-theory. In this way we actually obtain a complete set of obstructions, but for the more general stable fibering problem presented in [Ste12]. More precisely, the obstructions $\theta(f)$ and $\tau_{\text{fib}}(f)$ generalize naturally in algebraic K-theory to Steimle's obstructions and their vanishing is both a necessary and sufficient condition for a map $f\colon M \to B$ of manifolds to stably fiber, where stabilization is given by crossing M with disks of sufficiently high dimension.

The work is structured as follows. In Chapter 1, we present the tools that we need to apply our strategy. In particular, first we introduce the category of fibrations, then we define fiber transports and fiber trivialization and finally we explain how to convert any map into a fibration.

In Chapter 2, we focus on the Whitehead torsion theory and we extend it to spaces of the homotopy type of a finite CW-complex. More precisely, first we recall the classical Whitehead torsion, then we introduce simple structures to get the new notion of Whitehead torsion and we describe the simple structure on the total space of a fibration and finally we present as an example the case of closed manifolds by equipping them with a preferred canonical choice of simple structure.

In Chapter 3, we give a first answer to the fibering problem. Namely, we describe the obstructions $\theta(f)$ and $\tau_{\text{fib}}(f)$ and we prove Theorem 0.1.

In Chapter 4, we study the particular case of smooth maps $f\colon M \to S^1$ from a smooth manifold M to a circle. In particular, first we explain how in this case the two obstructions $\theta(f)$ and $\tau_{\text{fib}}(f)$ can be summarized in one single invariant ${\tau_{\text{fib}}}'(f)$ and we investigate the case of mapping tori and h-cobordisms, then we describe a Bass-Heller-Swan decomposition for the Whitehead group of $\pi_1(M)$ and finally we present Farrell's work [Far71] and we compare its obstruction with ${\tau_{\text{fib}}}'(f)$. We obtain that for this kind of maps the vanishing of $\theta(f)$ and ${\tau_{\text{fib}}}'(f)$ is both a necessary and sufficient condition for f being homotopic to a fiber bundle.

In Chapter 5, we give an overview to Steimle's work [Ste12]. In detail, we introduce the parametrized and excisive A-theory characteristics, we extend the notion of Whitehead torsion to algebraic K-theory and we prove that Steimle's obstructions form a complete obstruction theory in algebraic K-theory for existence and uniqueness of the stable fibering problem. Moreover, we show that this is actually the generalization of the fibering problem.

Finally, I would like to thank Christoph Winges for the interesting thesis topic he gave me and the excellent support during this work.

Chapter 1

The category of fibrations

In this chapter we present the category of fibrations and its properties. The notion of fibration is fundamental to solve the fibering problem. In fact, it generalizes the concept of fiber bundles in such a way that it no longer requires the local structure of the map, but maintains axiomatically the fiber transport property. Moreover, it is so general that any map can be converted into a fibration. In other words, it allows us to easily reduce the fibering problem to this category and, thus, simplify it.

The work is structured as follows. In Section 1.1, we present the category of fibrations and we define fiber transports and fiber trivializations. Moreover, given a fibration $p\colon E \to B$, we use them to describe the fiber transport functor

$$T\colon \pi B \to \mathrm{ho}(\mathbf{Top})$$

from the grupoid of B to the homotopy category of **Top**, the category of topological spaces. This will be useful in the following to work with fibers in a functorial way. Finally, in Section 1.2, we recall how to convert a map into a fibration.

The main references are [Whi78, Chapter I], [Die08, Chapter 5] and [Swi75, pp. 341–345].

Notation 1.1. Throughout this thesis, we denote $I = [0, 1]$.

1.1 Fiber homotopy equivalences and fiber trivializations

In this section we describe the category of fibrations. In particular, we define it and we study fiber homotopy equivalences, fiber transports and fiber trivializations. Moreover, we prove some of their particular properties that will be useful in the following chapters.

The category of fibrations

Let us start with the definition of fibrations between topological spaces.

Definition 1.2. (i) A map $p\colon E \to B$ in **Top**, the category of topological spaces, is a *fibration* if it has the *homotopy lifting property*, that is, if given a map $f\colon Z \to E$ for a topological space Z and a homotopy $H\colon Z \times I \to B$ such that $p \circ f = H(-, 0)$, there exists a homotopy $H'\colon Z \times I \to E$ with $H'(-, 0) = f$ and $p \circ H' = H$:

$$\begin{tikzcd} Z \times \{0\} \cong Z \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ Z \times I \arrow[r, "H"] \arrow[ur, dotted, "H'"] & B \end{tikzcd}$$

(ii) Let $p\colon E \to B$ and $p'\colon E' \to B'$ be fibrations. A *map of fibrations* $(\overline{f}, f)\colon p \to p'$ consists of a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\overline{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

We denote by **Fib** the category whose objects are fibrations and morphisms are maps of fibrations.

Let us consider the subcategory $\mathbf{Fib}(B)$ of **Fib** whose objects are fibrations $p\colon E \to B$ with fixed base space B and whose morphisms $p \to p'$ from $p\colon E \to B$ to $p'\colon E' \to B$ are maps of fibrations of the kind $(\overline{f}, \mathrm{id}_B)\colon p \to p'$, that is, maps $f\colon E \to E'$ such that $p' \circ f = p$. Then $\mathbf{Fib}(B)$ has the following natural notion of homotopy equivalence, called fiber homotopy equivalence, which is based on the notion of fiber homotopy.

Definition 1.3. Let $p\colon E \to B$ and $p'\colon E' \to B$ be objects of $\mathbf{Fib}(B)$ and X be in **Top**. Then:

(i) A homotopy $H\colon X \times I \to E$ is called a *fiber homotopy* if $p \circ H$ is stationary, that is, $p \circ H(x,t) = p \circ H(x,0)$ for all $(x,t) \in X \times I$.

(ii) Two maps $f_0, f_1\colon X \to E$ with $p \circ f_0 = p \circ f_1$ are called *fiber homotopic* $f_0 \simeq_p f_1$ if there is a fiber homotopy $H\colon X \times I \to E$ with $H(-,0) = f_0$ and $H(-,1) = f_1$.

(iii) A *fiber homotopy equivalence* $\overline{f}\colon E \to E'$ from p to p' is a morphism $(\overline{f}, \mathrm{id}_B)\colon p \to p'$ in $\mathbf{Fib}(B)$ such that there exists another morphism $(\overline{g}, \mathrm{id})\colon p' \to p$ with $\overline{g} \circ \overline{f} \simeq_p \mathrm{id}_E$ and $\overline{f} \circ \overline{g} \simeq_{p'} \mathrm{id}_{E'}$.

$$\begin{tikzcd} E \arrow[rr, bend right, "\overline{f}"'] \arrow[dr, "p"'] & & E' \arrow[ll, bend right, "\overline{g}"'] \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

Fiber homotopies respect the homotopy lifting property, as explained in the following lemma.

Lemma 1.4. *Let $p\colon E \to B$ be in* **Fib**. *Consider two fiber homotopic maps $f, g\colon X \to E$ over B. Let F and G be solutions of the two following homotopy lifting problems, respectively.*

$$\begin{tikzcd} X \times \{0\} \arrow[r, "f"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dotted, "F"] & B \end{tikzcd} \qquad \begin{tikzcd} X \times \{0\} \arrow[r, "g"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times I \arrow[r, "h"'] \arrow[ur, dotted, "G"] & B \end{tikzcd}$$

Then F and G are fiber homotopic over B.

Proof. At first, note that the two homotopy liftings problems are over the same homotopy h: this is well-defined because f and g are fiber homotopic over B. Now, let $H\colon X \times I \to E$ be a fiber homotopy between f and g and let $h'\colon X \times I \times I \to B$ be the homotopy which is constantly h, that is, such that $h'(x,t,s) = h(x,s)$. Denote by A the subset $\{0\} \times I \cup I \times \{0,1\}$ and define $l\colon X \times A \to E$ by

$$l|_{X \times I \times \{0\}} = F, \quad l|_{X \times I \times \{1\}} = G, \quad l|_{X \times \{0\} \times I} = H$$

Note that this is well-defined because $F(x,0) = f(x) = H(x,0)$ and $G(x,0) = g(x) = H(x,1)$. Now, by using the standard homeomorphism $I \times I \to I \times I$ which carries A to $\{0\} \times I$, we may interpret the homotopy lifting property for p as saying that there is a lifting $L\colon X \times I \times I \to E$ of h' which is l on $X \times A$. The map L is then a fiber homotopy between F and G. $\square$

The fiber transport functor

Let us now construct the functor $T\colon \pi B \to \mathrm{ho}(\mathbf{Top})$. Recall that, given a space B, we denote by πB the grupoid of B, that is, the category whose objects are the points of B and whose morphisms are the homotopy classes of paths in B. Moreover, we denote by $\mathrm{ho}(\mathbf{Top})$ the homotopy category of **Top**. As the name "fiber transport functor" says, the idea is that, given a fibration $p\colon E \to B$, the functor T maps any point b of B to the fiber of p over b and any homotopy class of paths of B to its associated fiber transport. Therefore, to describe it rigorously, we have first to define the fiber transport of a fibration p along a path. We do it by considering the general situation of a pullback of p along any map $f\colon X \to B$ and then by applying it to the case $X = \{*\}$ and $f\colon \{*\} \to B$, $* \mapsto b$.

Notation 1.5. Let $p\colon E \to B$ be in **Fib**. We denote the pullback of p along a map $f\colon X \to B$ by $p_f\colon f^*E \to X$.

$$\begin{tikzcd} f^*E \arrow[r, "\overline{f}"] \arrow[d, "p_f"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

Note that p_f is in **Fib** as well. Indeed, recall that $f^*E = \big\{(x, e) \in x \times E | f(x) = p(e)\big\}$ with topology induced by product topology and consider the following commutative diagram.

$$\begin{tikzcd} Z \times \{0\} \cong Z \arrow[r, "g"] \arrow[d, hook] & f^*E \arrow[d, "p_f"] \\ Z \times I \arrow[r, "G"'] & X \end{tikzcd}$$

A lifting $G'\colon Z \times I \to f^*E$ of the homotopy G is given by the same G in the first coordinate and by a lifting to E of $f \circ G\colon Z \times I \to B$ in the second one.

Lemma 1.6. *Let $p\colon E \to B$ be in **Fib**. Then:*

(i) Let $H\colon X \times I \to B$ be a homotopy $f_0 \simeq f_1\colon X \to B$. Let $H'\colon f_0^*E \times I \to E$ be a solution of the homotopy lifting problem for $H \circ (p_{f_0} \times \mathrm{id}_I)\colon f_0^*E \times I \to B$ and $\overline{f_0}\colon f_0^*E \to E$.

$$\begin{tikzcd} f_0^*E \arrow[rr, "\overline{f_0}"] \arrow[d, hook] & & E \arrow[d, "p"] \\ f_0^*E \times I \arrow[r, "p_{f_0} \times \mathrm{id}_I"'] \arrow[urr, dashed, "H'"] & X \times I \arrow[r, "H"'] & B \end{tikzcd}$$

Define $g_H\colon f_0^*E \to f_1^*E$ by $H'(-,1)$ and p_{f_0} using the pullback property of f_1^*E. Then $(g_H, \mathrm{id})\colon f_0^*E \to f_1^*E$ is a fiber map and H' is a homotopy $\overline{f_1} \circ g_H \simeq \overline{f_0}$.

(ii) Let $K\colon X \times I \to B$ be a second homotopy such that $f_0 \simeq f_1$ and $M\colon X \times I \times I \to B$ be a homotopy relative $X \times \{0,1\}$ between H and K. Then M induces a fiber homotopy $L\colon f_0^*E \times I \to f_1^*E$ from g_H to g_K.

Proof. Throughout this proof, we denote by $\widetilde{k}\colon Z \to E$ a lifting to E of any map $k\colon Z \to B$.

(i) By construction, we have $H'(-,1) = \widetilde{f_1 \circ p_{f_0}}$ and thus $p \circ H'(-,1) = p \circ (\widetilde{f_1 \circ p_{f_0}}) = f_1 \circ p_{f_0}$. Therefore, the outer square of the following diagram commutes and $g_H\colon f_0^*E \to f_1^*E$ exists and is well-defined by pullback property.

$$\begin{tikzcd}
f_0^*E \arrow[dr, dashed, "g_H"] \arrow[drr, bend left, "H'(-,1)"] \arrow[ddr, bend right, "p_{f_0}"'] & & \\
& f_1^*E \arrow[r, "\overline{f_1}"] \arrow[d, "p_{f_1}"'] & E \arrow[d, "p"] \\
& X \arrow[r, "f_1"'] & B
\end{tikzcd}$$

Moreover, again by pullback property, we have that g_H is a fiber map since $p_{f_1} \circ g_H = p_{f_0}$ and that H' is a homotopy $\overline{f_1} \circ g_H \simeq \overline{f_0}$ since $\overline{f_1} \circ g_H = H'(-,1) \simeq H'(-,0) = \overline{f_0}$.

(ii) As with H, let $K'\colon f_0^*E \times I \to E$ be a solution of the homotopy lifting problem for the homotopy $K \circ (p_{f_0} \times \mathrm{id}_I)\colon f_0^*E \times I \to B$ and the map $\overline{f_0}\colon f_0^*E \to E$ and let g_K be defined by $K'(-,1)$ and p_{f_0} using the pullback property of f_1^*E. As in the proof of Lemma 1.4, denote by A the subset $\{0\} \times I \cup I \times \{0,1\}$ and define $M'\colon f_0^*E \times A \to E$ by

$$M'|_{f_0^*E \times I \times \{0\}} = H', \quad M'|_{f_0^*E \times I \times \{1\}} = K', \quad M'|_{f_0^*E \times \{0\} \times I} = \overline{f_0}$$

Note that M' is well-defined, since $H'(e',0) = \overline{f_0}(e') = K'(e',0)$ for any $e' \in f_0^*E$. Moreover, by definition of M, we have $p \circ M' = M \circ (p_{f_0} \times \mathrm{id}_A)|_{f_0^*E \times A}$. By using the standard homeomorphism $I \times I \to I \times I$ which carries A to $\{0\} \times I$ and the homotopy lifting property for p, there is a lifting $M'\colon f_0^*E \times I \times I \to E$ of $M \circ (p_{f_0} \times \mathrm{id}_I \times \mathrm{id}_I)$ which extends the definition already given on $f_0^*E \times A$. Consider following the map.

$$L\colon f_0^*E \times I \to f_1^*E, \qquad (e',t) \mapsto \big(p_{f_0}(e'), M'(e',1,t)\big)$$

Note that it is well-defined because

$$p \circ M'(e',1,t) = M \circ (p_{f_0} \times \mathrm{id}_I \times \mathrm{id}_I)(e',1,t) = M\big(p_{f_0}(e'),1,t\big) = f_1\big(p_{f_0}(e')\big)$$

Now, we have $L(e',0) = \big(p_{f_0}(e'), H'(e',1)\big) = g_H(e')$ and similarly $L(e',1) = g_K(e')$. Furthermore, $p_{f_1} \circ L(e',t) = p_{f_0}(e') = p_{f_1} \circ L(e',0)$. Hence, L is a fiber homotopy from g_H to g_K. □

Remark 1.7. Note that g_H is not uniquely defined in general, because it depends on the choice of a lifting H'. However, since different liftings are homotopic, it is uniquely defined up to homotopy.

The map $g_H\colon f_0^*E \to f_1^*E$ plays the role of transport between pullbacks and so, in the particular case where $X = \{*\}$, of fiber transport. Therefore, we are interested in investigating its properties.

Lemma 1.8. *In the situation of the previous Lemma 1.6, the map $g_H\colon f_0^*E \to f_1^*E$ is a fiber homotopy equivalence.*

The proof of this lemma follows by the following more general property of the map g_H.

Proposition 1.9. *Let $p\colon E \to B$ be in* **Fib**. *Consider three maps $f_0, f_1, f_2\colon X \to B$ in* **Top** *and let $H, K\colon X \times I \to B$ be homotopies such that $f_0 \simeq f_1$ and $f_1 \simeq f_2$, respectively. Denote by $H * K$ the homotopy which is the "concatenation" of H and K, that is, the homotopy $H * K\colon X \times I \to B$ defined by*

$$H * K(x,t) = \begin{cases} H(x,2t) & \text{if } \ 0 \le t \le \frac{1}{2} \\ K(x,2t-1) & \text{if } \ \frac{1}{2} \le t \le 1 \end{cases}$$

Then, using the notation of Lemma 1.6, there is a fiber homotopy $g_K \circ g_H \simeq_{p_{f_2}} g_{H*K}$.

Proof. Let $H' \colon f_0^* E \times I \to E$ and $K' \colon f_1^* E \times I \to E$ be liftings of $H \circ (p_{f_0} \times \mathrm{id}_I)$ and $K \circ (p_{f_1} \times \mathrm{id}_I)$, respectively, with $H'(-,0) = \overline{f_0}$ and $K'(-,0) = \overline{f_1}$. Then g_H and g_K fit in the following commutative diagram.

$$
\begin{tikzcd}
f_0^*E \arrow[dr, dashed, "g_H"] \arrow[drrr, bend left, "{H'(-,1)}"] \arrow[ddrr, bend right, "p_{f_0}"'] & & & \\
& f_1^*E \arrow[dr, dashed, "g_K"] \arrow[drr, bend left, "{K'(-,1)}"] \arrow[ddr, bend right, "p_{f_1}"'] & & \\
& & f_2^*E \arrow[r, "\overline{f_2}"] \arrow[d, "p_{f_2}"'] & E \arrow[d, "p"] \\
& & X \arrow[r, "f_2"] & B
\end{tikzcd}
\tag{1.1}
$$

Similarly, let $(H * K)' \colon f_0^* E \times I \to E$ be a lifting of $(H * K) \circ (p_{f_0} \times \mathrm{id}_I)$ with $(H * K)'(-,0) = \overline{f_0}$. We have

$$
\begin{tikzcd}
f_0^*E \arrow[dr, dashed, "g_{H*K}"] \arrow[drr, bend left, "{(H*K)'(-,1)}"] \arrow[ddr, bend right, "p_{f_0}"'] & & \\
& f_2^*E \arrow[r, "\overline{f_2}"] \arrow[d, "p_{f_2}"'] & E \arrow[d, "p"] \\
& X \arrow[r, "f_2"'] & B
\end{tikzcd}
\tag{1.2}
$$

Consider the homotopy $M \colon f_0^* E \times I \times I \to B$ defined by $M(e', t, s) = H * K\big(p_{f(0)}(e'), t\big)$ for $e' \in f_0^* E$ and $t, s \in I$. As in the proof of Lemma 1.4, let $A = \{0\} \times I \cup I \times \{0,1\}$ and define the map $M' \colon f_0^* E \times A \to E$ by

$$
M'|_{f_0^* E \times I \times \{0\}} = (H * K)', \quad M'|_{f_0^* E \times I \times \{1\}} = H' * \big(K' \circ (g_H \times \mathrm{id}_I)\big), \quad M'|_{f_0^* E \times \{0\} \times I} = \overline{f_0}
$$

where $H' * \big(K' \circ (g_H \times \mathrm{id}_I)\big)$ is the "concatenation" of the homotopies H' and $K' \circ (g_H \times \mathrm{id}_I)$. Note that M' is well-defined, since $(H * K)'(e', 0) = \overline{f_0}(e') = H' * (K' \circ \big(g_H \times \mathrm{id}_I)\big)(e', 0)$. Moreover, we have $p \circ M' = M|_{f_0^* E \times A}$. Indeed, we have

$$
\begin{aligned}
&p \circ (H * K)' = (H * K) \circ (p_{f_0} \times \mathrm{id}_I) \\
&p \circ \overline{f_0} = f_0 \circ p_{f_0}|_{\{0\} \times I} = (H * K) \circ (p_{f_0} \times \mathrm{id}_I) \\
&p \circ \Big(H' * \big(K' \circ (g_H \times \mathrm{id}_I)\big)\Big) = \begin{cases} H \circ (p_{f_0} \times \mathrm{id}_I) & \text{if} \quad 0 \le t \le \frac{1}{2} \\ K \circ \big((p_{f_1} \circ g_H) \times \mathrm{id}_I\big) = K \circ (p_{f_0} \times \mathrm{id}_I) & \text{if} \quad \frac{1}{2} \le t \le 1 \end{cases} \\
&\qquad\qquad\qquad\qquad\qquad\qquad = (H * K) \circ (p_{f_0} \times \mathrm{id}_I)
\end{aligned}
$$

Hence, by using the standard homeomorphism $I \times I \to I \times I$ which carries A to $\{0\} \times I$ and by the homotopy lifting property, there exists a lifting $M' \colon f_0^* E \times I \times I \to E$ of M which extends the definition given on $f_0^* E \times A$. Consider, now, the following map.

$$
L \colon f_0^* E \times I \to f_2^* E, \qquad (e', t) \mapsto \big(p_{f_0}(e'), M'(e', 1, t)\big)
$$

Note that it is well-defined since $p \circ M'(e', 1, t) = M(e', 1, t) = (H * K)\big(p_{f_0}(e'), 1\big) = f_2\big(p_{f_0}(e')\big)$.

Moreover, it is a homotopy such that $g_K \circ g_H \simeq g_{H*K}$. Indeed, we have

$$\begin{aligned} L(e',0) &= \big(p_{f_0}(e'), M'(e',1,0)\big) \\ &= \Big(p_{f_0}(e'), (H*K)'(e',1)\Big) \\ &= \Big(p_{f_2}\big(g_{H*K}(e')\big), \overline{f_2}\big(g_{H*K}(e')\big)\Big) \\ &= g_{H*K}(e') \end{aligned}$$

where we have used the commutativity of diagram (1.2), and

$$\begin{aligned} L(e',1) &= \big(p_{f_0}(e'), M'(e',1,1)\big) \\ &= \Big(p_{f_0}(e'), H' * \big(K' \circ (g_H \times \mathrm{id}_I)\big)(e',1)\Big) \\ &= \Big(p_{f_0}(e'), K' \circ (g_H \times \mathrm{id}_I)(e',1)\Big) \\ &= \Big(p_{f_0}(e'), K'\big(g_H(e'),1\big)\Big) \\ &= \Big(p_{f_2}\big(g_K \circ g_H(e')\big), \overline{f_2}\big(g_K \circ g_H(e')\big)\Big) \\ &= g_K \circ g_H(e') \end{aligned}$$

where we have used the commutativity of diagram (1.1). Finally, L is a fiber homotopy since $p_{f_2} \circ L(e',t) = p_{f_0}(e') = p_{f_2} \circ L(e',0)$. Thus, L is a fiber homotopy from $g_K \circ g_H$ to g_{H*K}. □

The proof of Lemma 1.8 now follows easily by Proposition 1.9.

Proof of Lemma 1.8. Define $H^-: X \times I \to B$ to be the homotopy H "reversed", that is, the homotopy such that $H^-(x,t) = H(x,1-t)$. Then, by Proposition 1.9, we have $g_{H^-} \circ g_H \simeq g_{H*H^-}$. Moreover, since there is an obvious homotopy $M: X \times I \times I \to B$ from $H * H^-$ to the constant homotopy $f_0: X \times I \to B$, $(x,t) \mapsto f_0(x)$, by Lemma 1.6(ii) we obtain $g_{H^-} \circ g_H \simeq g_{f_0}$. Now, by definition, we have that g_{f_0} is (homotopic to) the identity $\mathrm{id}_{f_0^*E}$. Hence, we conclude that $g_{H^-} \circ g_H \simeq \mathrm{id}_{f_0^*E}$. Similarly, also $g_H \circ g_{H^-} \simeq \mathrm{id}_{f_1^*E}$. Therefore, the map g_H is a homotopy equivalence. Finally, since all the homotopies that we have used are fiber homotopies, g_H is a fiber homotopy equivalence. □

Before considering the particular case $X = \{*\}$, let us study how the map g_H behaves in case of restrictions and fiber homotopy equivalences. These properties will be particulary useful in the following chapters.

Lemma 1.10. *Consider the situation of Lemma 1.6 and let Y be a subspace of X. Denote by $K: Y \times I \to B$ the homotopy $H|_{Y\times I}$ between the restrictions $f_0|_Y$ and $f_1|_Y$. Then, under the identification $(f_0|_Y)^*E = (f_0^*E)|_Y$ and $(f_1|_Y)^*E = (f_1^*E)|_Y$, we have that g_K is fiber homotopic to $g_H|_Y$.*

Proof. Consider the lifting $H': f_0^*E \times I \to E$ of H of Lemma 1.6. Then it is clear that the restriction of H' to $(f_0|_Y)^*E \times I$ is a solution of the homotopy lifting problem of Lemma 1.6 for the homotopy K. Therefore, we can conclude by Lemma 1.6(ii). □

Lemma 1.11. *Let $p: E \to B$ and $p': E' \to B$ be in* **Fib** *and $\varphi: E \to E'$ be a fiber homotopy equivalence. Consider a homotopy $H: X \times I \to B$ between two maps $f_0, f_1: X \to B$ and let*

$g_H\colon f_0^*E \to f_1^*E$ and $g'_H\colon f_0^*E' \to f_1^*E'$ be defined as in Lemma 1.6. Then the following diagram commutes up to fiber homotopy.

$$\begin{tikzcd} f_0^*E \arrow[r, "f_0^*\varphi"] \arrow[d, "g_H"'] & f_0^*E' \arrow[d, "g'_H"] \\ f_1^*E \arrow[r, "f_1^*\varphi"] & f_1^*E' \end{tikzcd}$$

where $f_i^*\varphi$ is the map on pullbacks induced by the following diagram for $i = 0, 1$.

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[d, equal] & B \arrow[d, equal] & E \arrow[l, "p"'] \arrow[d, "\varphi"] \\ X \arrow[r, "f_i"] & B & E' \arrow[l, "p'"'] \end{tikzcd}$$

Proof. Let $\psi\colon E' \to E$ be a fiber homotopy inverse of φ. Consider the following diagram

$$\begin{tikzcd} f_0^*E' \times \{0\} \arrow[r, "f_0^*\psi"] \arrow[d, hook] & f_0^*E \times \{0\} \arrow[rr, "\overline{f_0}"] \arrow[d, hook] & & E \arrow[r, "\varphi"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ f_0^*E' \times I \arrow[r, "f_0^*\psi \times \mathrm{id}_I"] & f_0^*E \times I \arrow[r] \arrow[urr, dotted, "H'"] & X \times I \arrow[r, "H"] & B \arrow[r, equal] & B \end{tikzcd}$$

where the map $f_0^*\psi$ is defined in the same way as $f_0^*\varphi$ and the middle rectangle is a homotopy lifting problem with solution H'. Obviously, also the outer rectangle defines a homotopy lifting problem and its solution is given by the map

$$H'' = \varphi \circ H' \circ (f_0^*\psi \times \mathrm{id}_I)\colon f_0^*E' \times I \to E'$$

Now, the maps $\overline{f_0}\colon f_0^*E' \times \{0\} \to E'$ and $\varphi \circ \overline{f_0} \circ f_0^*\psi$ are fiber homotopic. Hence, by Lemma 1.4, the map H'' is fiber homotopic to the solution H''' of the following homotopy lifting problem.

$$\begin{tikzcd} f_0^*E' \times \{0\} \arrow[rr, "\overline{f_0}"] \arrow[d, hook] & & E' \arrow[d, "p'"] \\ f_0^*E' \times I \arrow[r] \arrow[urr, dotted, "H'''"] & X \times I \arrow[r, "H"'] & B \end{tikzcd}$$

In particular, the maps $H''(-,1) = \varphi \circ H'(-,1) \circ f_0^*\psi$ and $H'''(-,1)$ from f_0^*E' to E' are fiber homotopic. Therefore, using the definition of pullback, the maps $p_{f_0}\colon f_0^*E \to X$ and $p'_{f_0}\colon f_0^*E' \to X$ and the definition of g_H and g'_H, we obtain that $f_1^*\varphi \circ g_H \circ f_0^*\psi$ and g'_H are fiber homotopic and the lemma holds. □

We reduce now to the case of fibers. In particular, we apply Lemma 1.6 with $X = \{*\}$ and $f\colon X \to B$, $* \mapsto b$ for some $b \in B$, so that the pullback is the fiber $F_b = p^{-1}(b)$ and $\overline{f}$ is the inclusion $F_b \hookrightarrow E$.

Definition 1.12. Let $w\colon I \to B$ be a path in B and consider its homotopy class $[w]$. Apply Lemma 1.6 to $X = \{*\}$ and the homotopy $H = w$ between the maps $f_0, f_1\colon X \to B$ such that $f_i(*) = w(i)$ for $i = 0, 1$. We define $t[w]\colon F_{w(0)} \to F_{w(1)}$ to be the homotopy class of the maps g_H and we call it the *fiber transport* along w.

Remark 1.13. (i) The fiber transport has the following properties:

- if v and w are path with $v(1) = w(0)$, then $t[w] \circ t[v] = t[v * w]$;

- the constant path c_b induces the identity on F_b.

Indeed, the first one follows by Proposition 1.9 and the second one is obtained by defining H to be the constant homotopy.

(ii) Using the notation of Lemma 1.6, for each $x \in X$ the map $F_{f_0(x)} \to F_{f_1(x)}$ induced by g_H represents the fiber transport along the path $H(x, -)$.

We can now finally define the fiber transport functor T.

Definition 1.14. Let $p\colon E \to B$ be in **Fib**. We define the *fiber transport functor*

$$T\colon \pi B \to \mathrm{ho}(\mathbf{Top})$$

from the grupoid πB of B to the homotopy category of **Top** to be the functor which maps any $b \in B$ to its fiber F_b and any homotopy class $[w]$ of paths in B to its associated fiber transport $t[w]$.

Remark 1.15. This functor is well-defined by Remark 1.13.

Fiber trivializations

To conclude this section, let us introduce the notion of fiber trivialization and study its properties.

Definition 1.16. Let $p\colon E \to B$ be in **Fib** and $f\colon X \to B$ be a map in **Top**. Consider $x \in X$ and $b \in B$. Let $w\colon I \to B$ be a path from b to $f(x)$. A *fiber trivialization* of f^*E with respect to (b, x, w) is a fiber homotopy equivalence $T\colon F_b \times X \to f^*E$ over X such that the map $F_b \to F_{f(x)}$ induced by T represents the fiber transport $t[w]$ for p along w.

$$\begin{tikzcd} F_b \times X \arrow[r, "T"] \arrow[dr, bend right, "\mathrm{pr}"'] & f^*E \arrow[r, "\overline{f}"] \arrow[d, "p_f"'] & E \arrow[d, "p"] \\ & X \arrow[r, "f"'] & B \end{tikzcd}$$

Remark 1.17. The notion of fiber trivialization is well-defined. More precisely, any fiber homotopy equivalence $T\colon F_b \times X \to f^*E$ induces a map $F_b \to F_{f(x)}$. Indeed, if $(e, x) \in F_b \times X$ with x fixed, then we have

$$p \circ \overline{f} \circ T(e, x) = f \circ p_f \circ T(e, x) = f\big(\mathrm{pr}(e, x)\big) = f(x)$$

Therefore, $\overline{f} \circ T(-, x)$ is a map $F_b \to F_{f(x)}$.

Lemma 1.18. *Consider the situation of Definition 1.16 and suppose that X is contractible. Then:*

(i) There exists a fiber trivialization with respect to (b, x, w).

(ii) Any two fiber trivializations with respect to (b, x, w) are fiber homotopic.

(iii) Let $T_i\colon F_{b_i} \times X \to f^*E$ be a fiber trivialization with respect to (b_i, x_i, w_i) for $i = 0, 1$. Choose a path $v\colon I \to X$ from x_0 to x_1. Let $t\colon F_{b_0} \to F_{b_1}$ be a representative of the fiber transport of p along $w_0 * f(v) * w_1^-$. Then we get a fiber homotopy such that $T_1 \circ (t \times \mathrm{id}_X) \simeq_{p_f} T_0$.

$$\begin{tikzcd} F_{b_0} \times X \arrow[rr, "t\times \mathrm{id}_X"] \arrow[dr, "T_0"'] & \arrow[d, phantom, "\Rightarrow"] & F_{b_1} \times X \arrow[dl, "T_1"] \\ & f^*E & \end{tikzcd}$$

(iv) *Let $H\colon X \times I \to B$ be a homotopy such that $f_0 \simeq f_1$. Let v be the path in B from $f_0(x)$ to $f_1(x)$ given by $H(x,-)$ and let w_0 be a path from b to $f_0(x)$. Put $w_1 = w_0 * v$. Let $T_i\colon F_b \times X \to f_i^*E$ be a fiber trivialization of f_i^*E with respect to the triple (b, x, w_i) for $i = 0, 1$ and let $g_H\colon f_0^*E \to f_1^*E$ be the fiber homotopy equivalence of Lemma 1.6(i). Then we get a fiber homotopy over X such that $g_H \circ T_0 \simeq_{p_{f_1}} T_1$.*

$$\begin{tikzcd} & F_b \times X \arrow[dl, "T_0"'] \arrow[dr, "T_1"] \arrow[d, phantom, "\Nearrow"] & \\ f_0^*E \arrow[rr, "g_H"'] & & f_1^*E \end{tikzcd}$$

Proof. (i) Consider the situation of Definition 1.16. By contractibility of X, there exists a homotopy $G\colon X \times I \to X$ relative $\{x\}$ from the constant map $\mathrm{const}_x\colon X \to \{x\} \hookrightarrow X$ to the identity id_X. Define $H\colon X \times I \to B$ to be the homotopy

$$(x,t) \mapsto \begin{cases} w(t) & \text{if } 0 \le t \le \frac{1}{2} \\ f \circ G(x,t) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

from the constant map $\mathrm{const}_b\colon X \to \{b\} \hookrightarrow B$ to f. This is well-defined by definition of the path w. Then, by Lemma 1.6(i), since the pullback of p along the map const_b is $F_b \times X$, there exists a fiber homotopy equivalence $T = g_H\colon F_b \times X \to f^*E$. Now, the path $H(x,-)\colon I \to B$ is (homotopic to) the path w. Therefore, by Remark 1.13(ii), the map $\overline{f} \circ T(-,x)\colon F_b \to F_{f(x)}$ induced by T represents the fiber transport $t[w]$. Hence, T is a fiber trivialization with respect to (b, x, w).

(ii) Let $T_0, T_1\colon F_b \times X \to f^*E$ be two fiber trivializations of f^*E with respect to (b, x, w). Consider the following homotopy lifting problem

$$\begin{tikzcd} F_b \times X \times \{0,1\} \cup F_b \times \{x\} \times I \arrow[r, "\lambda"] \arrow[d, hook] & f^*E \arrow[d, "p_f"] \\ F_b \times X \times I \arrow[r, "\mathrm{pr}_X"'] \arrow[ur, dotted, "H"] & X \end{tikzcd} \tag{1.3}$$

where $\mathrm{pr}_X\colon F_b \times X \times I \to X$ is the projection map $(e, x, s) \mapsto x$ onto X and the map $\lambda\colon F_b \times X \times \{0,1\} \cup F_b \times \{x\} \times I \to f^*E$ is defined by

$$(e,y,s) \mapsto \begin{cases} T_0(e,y) & \text{if } s = 0 \\ T_1(e,y) & \text{if } s = 1 \\ \big(t[w](e), x\big) & \text{if } y = x \end{cases}$$

Note that λ is well-defined since both T_0 and T_1 are fiber trivializations. Moreover, the left vertical map of diagram (1.3) defines a strong deformation retract by [Whi78, (5.2)]. Indeed, we have

$$F_b \times X \times \{0,1\} \cup F_b \times \{x\} \times I = (F_b, \emptyset) \times \big(X, \{x\}\big) \times \big(I, \{0,1\}\big)$$

and $\{0,1\}$ and $\emptyset$ are neighborhood deformation retracts of I and F_b, respectively, and $\{x\}$ is a strong deformation retract of X. Therefore, we can conclude by [Whi78, Lemma (7.15)] that there exists a solution $H\colon F_b \times X \times I \to f^*E$ of the previous lifting problem (1.3). Such H is by construction exactly the required fiber homotopy $T_0 \simeq T_1$.

(iii) By construction, using the notation of Lemma 1.6, we have that $t \times \mathrm{id}_X = g_K$ for the homotopy $K\colon X \times I \to B$ defined by $K(x,t) = w_0 * f(v) * w_1^-(t)$ for $x \in X$ and $t \in I$. Moreover, by part (i) and (ii) of this lemma, we may assume without loss of generality that $T_1 = g_{H_1}$ where $H_1\colon X \times I \to B$ is a homotopy from the constant map const_{b_1} to f such that $H_1(x_1,t) = w_1(t)$. Therefore, by Proposition 1.9, there exists a fiber homotopy such that

$$T_1 \circ (t \times \mathrm{id}_X) = g_{H_1} \circ g_K \simeq g_{K*H_1}$$

Now, the map $K * H_1$ is a homotopy from the constant map const_{b_0} to the map f such that $K * H_1(x_1,t) = \big(w_0 * f(v) * w_1^-\big) * w_1(t)$. Moreover, we obviously have that

$$w_0 * f(v) * w_1^- * w_1 \simeq w_0 * f(v)$$

Therefore, the map $T_1 \circ (t \times \mathrm{id}_X)$ is up to fiber homotopy a fiber trivialization with respect to $\big(b_0, x_1, w_0 * f(v)\big)$. Let $H\colon X \times I \to B$ to be the homotopy of part (i) from const_{b_0} to f that realizes $T_1 \circ (t \times \mathrm{id}_X)$ as such fiber trivialization. Then, H may be obviously written up to homotopy as

$$H(x,t) = \begin{cases} w_0(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \widetilde{H}(x, 2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

where $\widetilde{H}$ is a homotopy relative $\{x_0\}$ from $\mathrm{const}_{f(x_0)}$ to f such that $\widetilde{H}(x_1,t) = f(v)$. Therefore, H realizes $T_1 \circ (t \times \mathrm{id}_X)$ also as fiber trivialization with respect to the triple (b_0, x_0, w_0). Hence, we can conclude by part (ii).

(iv) Assume without loss of generality by part (i) and (ii) that for $i = 0,1$ we have $T_i = g_{H_i}$ where $H_i\colon X \times I \to B$ is a homotopy from the constant map const_b to f such that $H_i(x,t) = w_i(t)$. Then, by Proposition 1.9, we obtain that $g_H \circ g_{H_0}$ is fiber homotopic to $g_{H_0 * H}$, which is a fiber trivialization with respect to $w_0 * v = w_1$. Therefore, we can conclude by part (ii). □

1.2 Turning map into a fibration

In this section we recall how to convert any map into a fibration. This will be of great importance in the following chapters to reduce the fibering problem to the category of fibrations.

Proposition 1.19. *Let $f\colon X \to B$ be a map in* **Top**. *Define the space* $\mathrm{FIB}(f)$ *to be the set*

$$\mathrm{FIB}(f) = \{(x,w) \in X \times B^I | w(0) = f(x)\}$$

with the product topology, where B^I is the space of maps $I \to B$ with the compact-open topology. Consider the following maps:

$$\begin{aligned} \widehat{f}\colon X \to B, &\qquad (x,w) \mapsto w(1) \\ \lambda_f\colon X \to \mathrm{FIB}(f), &\qquad x \mapsto (x, \mathrm{const}_{f(x)}) \\ \mu_f\colon \mathrm{FIB}(f) \to X, &\qquad (x,w) \mapsto x \end{aligned}$$

Then, $\widehat{f}\colon X \to B$ is in **Fib**, *the map λ_f is a homotopy equivalence and μ_f is a homotopy inverse of λ_f. Moreover $\widehat{f} \circ \lambda_f = f$ and $f \circ \mu_f \simeq \widehat{f}$.*

Proof. We prove first that λ_f is a homotopy equivalence with homotopy inverse μ_f. Obviously, we have that $\mu_f \circ \lambda_f = \mathrm{id}_X$. On the other hand, the composition $\lambda_f \circ \mu_f(x,w) = (x, \mathrm{const}_{f(x)})$

is homotopic to $\mathrm{id}_{\mathrm{FIB}(f)}$ via a homotopy $H\colon \mathrm{FIB}(f)\times I \to \mathrm{FIB}(f)$ defined by $(x,w,t)\mapsto(x,w_t)$ where $w_t\colon I\to B$ is the path $w_t(s)=w(t\cdot s)$. Therefore, we can conclude.

Now, the equation $\widehat{f}\circ\lambda_f = f$ follows by a direct computation. Moreover, by pre-composition with μ_f, it also holds that $f\circ\mu_f\simeq\widehat{f}$. Hence, it remains only to show that $\widehat{f}$ is in **Fib**. For this, consider the following homotopy lifting problem.

$$\begin{tikzcd} Z \arrow[r, "g_0"] \arrow[d, hook] & \mathrm{FIB}(f) \arrow[d, "\widehat{f}"] \\ Z\times I \arrow[r, "G"'] \arrow[ur, dashed, "\widetilde{G}"] & B \end{tikzcd}$$

Write the map g_0 as $g_0(z)=\big(h(z),w_z\big)$ for $h\colon Z\to X$ and $w_z\colon I\to B$. Define $\widetilde{G}\colon Z\times I\to\mathrm{FIB}(f)$ by

$$\widetilde{G}(z,t)=\big(h(z),w_z * G(z,-\cdot t)\big)$$

where the second coordinate is the path w_z followed by the path $G(z,-\cdot t)\colon I\to B$, $s\mapsto G(z,s\cdot t)$. This is well-defined because $G(z,0)=\widehat{f}\circ g_0(z)=w_z(1)$. We claim that $\widetilde{G}$ is a solution of the previous homotopy lifting problem. Indeed, it is a continuous map, at level 0 it is (homotopic to) $g_0(z)=(h(z),w_z)$ and $\widehat{f}\circ\widetilde{G}(z,t)=w_z*G(z,-\cdot t)(1)=G(z,t)$. Hence, $\widehat{f}$ is in **Fib** . □

Definition 1.20. The map $\widehat{f}\colon \mathrm{FIB}(f)\to B$ of the previous proposition is called the *fibration associated* to f. Its fiber over b is called the *homotopy fiber* of f over b and is denoted by $\mathrm{hofib}(f)_b$.

To conclude this chapter, we study how this construction fits with fibrations and homotopies.

Lemma 1.21. *(i) If $f\colon E\to B$ is already in* **Fib**, *then $\lambda_f\colon E\to\mathrm{FIB}(f)$ is a fiber homotopy equivalence. In particular, the homotopy fibers of f are homotopy equivalent to the actual fibers.*

(ii) If $H\colon X\times I\to B$ is a homotopy such that $f\simeq g\colon X\to B$, then it induces a fiber homotopy equivalence $\widehat{H}\colon \mathrm{FIB}(f)\to\mathrm{FIB}(g)$.

Proof. (i) Consider the following homotopy lifting problem

$$\begin{tikzcd} \mathrm{FIB}(f) \arrow[r, "\mu_f"] \arrow[d, hook] & E \arrow[d, "f"] \\ \mathrm{FIB}(f)\times I \arrow[r, "G"'] \arrow[ur, dashed, "\widetilde{G}"] & B \end{tikzcd}$$

where $G\colon \mathrm{FIB}(f)\times I\to B$ is a homotopy defined by $(e,w,t)\mapsto w(t)$. Denote by η the map $\widetilde{G}(-,1)\colon \mathrm{FIB}(f)\to E$. Then, we claim that η is a fiber homotopy inverse of λ_f. Indeed, let us first show that $\lambda_f\circ\eta$ is homotopic to the identity. Consider the homotopy $H\colon \mathrm{FIB}(f)\times I\to\mathrm{FIB}(f)$ defined by $H(e,w,t)=\big(\widetilde{G}(e,w,t),w'\big)$ where $w'\colon I\to B$ is the restriction of the path w to $[t,1]$. We have that H is such that $\mathrm{id}_{\mathrm{FIB}(f)}\simeq\lambda_f\circ\eta$. Indeed, at level 0 we have

$$H(e,w,0)=\big(\widetilde{G}(e,w,0),w\big)=\big(\mu_f(e,w),w\big)=(e,w)$$

and at level 1 it holds

$$H(e,w,1)=\big(\eta(e,w),\mathrm{const}_{w(1)}\big)=\lambda_f\big(\eta(e,w)\big)$$

Moreover, H is a fiber homotopy, since

$$\widehat{f} \circ H(e, w, t) = w'(1) = w(1) = \widehat{f}(e, w) = \widehat{f} \circ H(e, w, 0)$$

Therefore, $\lambda_f \circ \eta$ is fiber homotopic to the identity $\mathrm{id}_{\mathrm{FIB}(f)}$.

It remains to prove that $\eta \circ \lambda_f$ is fiber homotopic to id_E. For this, let $K\colon E \times I \to E$ be the homotopy defined by $K(e, t) = \widetilde{G}\big(\lambda_f(e), t\big)$. Then, K is a fiber homotopy, since for any $t \in I$ it holds

$$f \circ K(e, t) = f \circ \widetilde{G}\big(\lambda_f(e), t\big) = G\big(\lambda_f(e), t\big) = G\big(e, \mathrm{const}_{f_e}, t\big) = f(e)$$

Moreover, we have that

$$K(e, 0) = \widetilde{G}\big(\lambda_f(e), 0\big) = \mu_f\big(\lambda_f(e)\big) = e$$

and

$$K(e, 1) = \widetilde{G}\big(\lambda_f(e), 1\big) = \eta \circ \lambda_f(e)$$

Therefore, K is a fiber homotopy such that $\mathrm{id}_E \simeq \eta \circ \lambda_f$.

(ii) Let $H\colon X \times I \to B$ be a homotopy such that $f \simeq g\colon X \to B$. Let $H^-\colon X \times I \to B$ be the homotopy H "reversed". Define $\widehat{H}\colon \mathrm{FIB}(f) \to \mathrm{FIB}(g)$ by $\widehat{H}(x, w) = (x, v)$ where $v\colon I \to B$ is the path given by

$$v(t) = \begin{cases} H^-(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ w(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

The path v is well-defined because $H^-(x, 1) = H(x, 0) = f(x) = w(0)$. We claim that $\widehat{H}$ is a fiber homotopy equivalence. Indeed, it is obviously a fiber map, because we have $\widehat{g}\big(\widehat{H}(x, w)\big) = w(1) = \widehat{f}(x, w)$. Moreover, if we apply the same construction to the homotopy H^-, then we get a fiber homotopy inverse $\widehat{H^-}$ of $\widehat{H}$. Let us prove that $\widehat{H^-} \circ \widehat{H} \simeq_{\widehat{f}} \mathrm{id}_{\mathrm{FIB}(f)}$. The other composition is completely dual, since $H = (H^-)^-$. Consider the map $\widehat{H^-} \circ \widehat{H}$. It sends the element $(x, w) \in \mathrm{FIB}(f)$ to an element $(x, u) \in \mathrm{FIB}(f)$ where $u\colon I \to B$ is the following path:

$$u(t) = \begin{cases} H(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ H^-(x, 4t-2) & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\ w(4t-3) & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases}$$

Define the homotopy $K\colon \mathrm{FIB}(f) \times I \to \mathrm{FIB}(f)$ by $K(x, w, s) = (x, u_s)$ where $u_s\colon I \to B$ is the following path:

$$u_s(t) = \begin{cases} H(x, 2ts) & \text{if } 0 \leq t \leq \frac{1}{2} \\ H^-\big(x, 1 - s + (4t-2)s\big) & \text{if } \frac{1}{2} \leq t \leq \frac{3}{4} \\ w(4t-3) & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases}$$

Then K is well-defined because $u_s(0) = H(x, 0) = f(x)$. Moreover, at level 0 it is (homotopic to) $\mathrm{id}_{\mathrm{FIB}(f)}$ and at level 1 it is $\widehat{H^-} \circ \widehat{H}$. Finally, it is a fiber homotopy, since

$$\widehat{f} \circ K(x, w, s) = u_s(1) = w(1) = \widehat{f}(x, w) = \widehat{f} \circ K(x, w, 0)$$

Therefore, K is a fiber homotopy such that $\widehat{H^-} \circ \widehat{H} \simeq_{\widehat{f}} \mathrm{id}_{\mathrm{FIB}(f)}$ and we can conclude. $\square$

Chapter 2

Whitehead torsion and simple structures

The obstructions to fibering a manifold measure the "simplicity" of a certain homotopy equivalence. This chapter is devoted to the development of the tools to make this kind of measurements. In particular, we focus on the notion of Whitehead torsion of a homotopy equivalence. The goal is to extend it from the category **FCW** of finite CW-complexes, which is the natural environment of this tool, to the category **Man** of closed topological manifolds, where the fibering problem is defined. The natural idea to do this is to choose some preferred finite CW-model on closed manifolds and compute the torsion using it. More generally, since our strategy consists turning maps into fibrations and the total space of the fibration associated to a map of closed manifolds is not in general a closed manifold, we consider the category **TFCW** of spaces of the homotopy type of a finite CW-complex, of which **Man** is a subcategory, and we equip its object with a new structure, called simple structure, which consists on the choice of a CW-model up to simple homotopy equivalence. Then, we can define the new Whitehead torsion simply as the classical one computed on the level of these structures.

The work is structured as follows. In Section 2.1, we review the classical Whitehead torsion in **FCW**. In Section 2.2, we introduce the notion of simple structure, we extend the Whitehead torsion theory to **TFCW** and we explain how simple structures fit with all the canonical construction of this category, in particular with fibrations. Finally, in Section 2.3, we present the particular case of **Man** as an example by simply defining the preferred simple structure on a closed manifold.

2.1 The Whitehead torsion

The aim of this section is to present briefly the notion of Whitehead torsion and to summarize its more important properties. A more detailed description may be found for example in [Coh73] or in [LM23].

Let G be a group and consider the group $K_1(\mathbb{Z}G)$ as defined in [LM23, Section 3.2]. We define the *Whitehead group* $\mathrm{Wh}(G)$ of G to be the quotient of $K_1(\mathbb{Z}G)$ with the subgroup generated by the elements $[\pm g]$ with $g \in G$, where $[\pm g]$ is the equivalence class in $K_1(\mathbb{Z}G)$ of the (1×1)-matrix with entry $\pm g \in \mathbb{Z}G$. Moreover, if X is a space, we define the *Whitehead group* $\mathrm{Wh}(\pi X)$ of X to be

$$\mathrm{Wh}(\pi X) = \bigoplus_{C \in \pi_0(X)} \mathrm{Wh}\left(\pi_1(C)\right)$$

Note that the rule $X \mapsto \mathrm{Wh}(\pi X)$ defines a functor $\mathrm{Wh}\colon \mathbf{Top} \to \mathbf{Ab}$ from the category of topological spaces to the category of abelian groups. We denote by $f_*\colon \mathrm{Wh}(\pi X) \to \mathrm{Wh}(\pi Y)$ the group homomorphism given by $\mathrm{Wh}(f)$ for any morphism $f\colon X \to Y$ in **Top**.

The *Whitehead torsion* is now a map which assigns to any homotopy equivalence $f\colon X \to Y$ in **FCW** an element $\tau(f) \in \mathrm{Wh}(\pi Y)$. An explicit algebraic construction may be found in [LM23, Section 3.3].

The most important properties of the Whitehead torsion are listed in the following lemma.

Lemma 2.1 ([LM23, Theorem 3.1]). *The Whitehead torsion has the following properties:*

(i) (Homotopy invariance) Let $f \simeq g\colon X \to Y$ be homotopic maps in **FCW**. *Then the homomorphism $f_*, g_*\colon \mathrm{Wh}(\pi X) \to \mathrm{Wh}(\pi Y)$ agree. In addition, if f and g are homotopy equivalences, then*

$$\tau(f) = \tau(g)$$

(ii) (Composition formula) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two homotopy equivalences in **FCW**. *Then*

$$\tau(g \circ f) = \tau(g) + g_*\tau(f)$$

(iii) (Sum formula) Consider the following diagram in **FCW**

$$\begin{tikzcd}
X_0 \arrow[rr, hook, "a_1"] \arrow[dr, "f_0"] \arrow[dd, "a_2"'] & & X_1 \arrow[dr, "f_1"] \arrow[dd, "b_1" near start] & \\
& Y_0 \arrow[rr, hook, crossing over, "i_1" near start] & & Y_1 \arrow[dd, "j_1"] \\
X_2 \arrow[rr, "b_2" near start] \arrow[dr, "f_2"'] & & X \arrow[dr, "f"] & \\
& Y_2 \arrow[rr, "j_2"'] \arrow[from=uu, crossing over, "i_2" near end] & & Y
\end{tikzcd}$$

where the front and back squares are pushouts. Assume that f_i is a homotopy equivalence for any $i = 1, 2, 3$ and that f is the map induced by f_0, f_1 and f_2 and the pushout property. Let $j_0\colon Y_0 \to Y$ be the map $j_1 \circ i_1 = j_2 \circ i_2$. Then f is a homotopy equivalence and

$$\tau(f) = (j_1)_*\tau(f_1) + (j_2)_*\tau(f_2) - (j_0)_*\tau(f_0)$$

(iv) (Product formula) Let $f_1\colon X_1 \to Y_1$ and $f_2\colon X_2 \to Y_2$ be two homotopy equivalences between connected objects of **FCW**. *Define $i_1\colon Y_1 \to Y_1 \times Y_2$ and $i_2\colon Y_2 \to Y_1 \times Y_2$ to be the inclusions $y \mapsto (y, y_2)$ and $y \mapsto (y_1, y)$ for some base point $y_2 \in Y_2$ and $y_1 \in Y_1$, respectively. Then*

$$\tau(f_1 \times f_2) = \chi(Y_1) \cdot (i_2)_*\tau(f_2) + \chi(Y_2) \cdot (i_1)_*\tau(f_1)$$

where the integer $\chi(Y_j)$ denotes the Euler characteristic for $j = 1, 2$.

(v) (Topological invariance) Let $f\colon X \to Y$ be a homeomorphism in **FCW**. *Then*

$$\tau(f) = 0$$

Simple homotopy theory

In the last part of this section, we show briefly the geometric interpretation of Whitehead group and Whitehead torsion: the simple homotopy theory. References for this subsection are [LM23, Section 3.4] and [Coh73, Chapter 2].

Notation 2.2. For any CW-complex X, we denote by X_n the n-skeleton of X.

Consider the upper hemisphere S^{n-1}_+ of S^{n-1}. Note that the pair (D^n, S^{n-1}_+) has a natural relative CW-structure, given by an $(n-1)$-cell to obtain S^{n-1} and an n-cell to get D^n. Denote by $S^{n-2} \subset S^{n-1}_+$ the equator. Let X be in **FCW** and $\varphi\colon S^{n-1}_+ \to X$ be a map such that $\varphi(S^{n-2}) \subset X_{n-2}$ and $\varphi(S^{n-1}_+) \subset X_{n-1}$. Let Y be the space defined by the following pushout square

$$\begin{tikzcd} S^{n-1}_+ \arrow[r, "\varphi"] \arrow[d, hook, "i"'] & X \arrow[d, "j"] \\ D^n \arrow[r, "\psi"'] & Y \end{tikzcd}$$

where i is the canonical inclusion. Then Y has the following canonical CW-structure:

$$Y_k = \begin{cases} j(X_k) & \text{if } k \leq n-2 \\ j(X_{n-1}) \cup \psi(S^{n-1}) & \text{if } k = n-1 \\ j(X_k) \cup \psi(D^n) & \text{if } k \geq n \end{cases}$$

In other words, Y is obtained from X by attaching one $(n-1)$-cell and one n-cell. Moreover, the map j is a homotopy equivalence. We call such a $j\colon X \to Y$ an *elementary expansion* and any homotopy inverse of j an *elementary collapse*.

Definition 2.3. We say that a map $f\colon X \to Y$ in **FCW** is a *simple homotopy equivalence* if there is a sequence of maps $X = X[0] \xrightarrow{f_0} X[1] \xrightarrow{f_1} X[2] \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} X[n] = Y$ in **FCW** such that f_i is either an elementary expansion or an elementary collapse for any $i = 0, \dots, n-1$ and f is homotopic to the composition $f_{n-1} \circ \dots \circ f_0$.

Now, fix a space X and let $\mathcal{C}(X)$ be the set of finite CW-complexes Y relative X such that the inclusion $X \hookrightarrow Y$ is a homotopy equivalence. Define $\mathrm{Wh}^{\mathrm{geo}}(X)$ to be the set $\mathcal{C}(X)$ modulo the equivalence relation generated by cellular isomorphisms and elementary expansions. By [Coh73, (6.1)], the addition $[Y]+[Z] = [Y \cup_X Z]$ makes $\mathrm{Wh}^{\mathrm{geo}}(X)$ an abelian group with the zero element given by $[X]$. This group is called the *geometric Whitehead group* of X and is isomorphic to the Whitehead group $\mathrm{Wh}(\pi X)$ by [LM23, Theorem 3.37]. In other words, $\mathrm{Wh}^{\mathrm{geo}}(X)$ is the geometric interpretation of $\mathrm{Wh}(\pi X)$. As with $\mathrm{Wh}(-)$, note that also $\mathrm{Wh}^{\mathrm{geo}}(-)$ defines a functor. We denote again by f_* the group homomorphism $\mathrm{Wh}^{\mathrm{geo}}(f)\colon \mathrm{Wh}^{\mathrm{geo}}(X) \to \mathrm{Wh}^{\mathrm{geo}}(Z)$ for a map $f\colon X \to Z$ in **Top**.

Consider now a homotopy equivalence $f\colon X \to Z$ in **FCW**. Then, by [Coh73, Section 22], the geometric interpretation of the Whitehead torsion $\tau(f)$ of f is given by the element $f_*\big[\,\mathrm{cyl}(f)\big]$ in $\mathrm{Wh}^{\mathrm{geo}}(Z)$, where $\mathrm{cyl}(f)$ is the mapping cylinder of f seen as a finite CW-complex relative X and f_* is defined as above.

In this geometric framework, it is not difficult to understand that the Whitehead torsion measures the "simplicity" of a homotopy equivalence. More precisely, it is the obstruction for a homotopy equivalence to be a simple homotopy equivalence, as stated in the following lemma.

Lemma 2.4 (Obstruction property, [Coh73, (22.2)])**.** *A homotopy equivalence $f\colon X \to Z$ in* **FCW** *is a simple homotopy equivalence if and only if $\tau(f) \in \mathrm{Wh}(Z)$ vanishes.*

2.2 Simple structures

In this section we extend the notion of Whitehead torsion of a homotopy equivalence from the category **FCW** of finite CW-complexes to the category **TFCW** of spaces of the homotopy type of a finite CW-complex. The idea is to add a new structure to these spaces, called simple structure, which is well-constructed as it is compatible with the canonical operations of this category. In particular, this structure fits well with fibrations and therefore it is particularly well-suited to study the fibering problem.

Simple structures and Whitehead torsion

Let us start by defining the following equivalence relation. Fix a space Y in **TFCW** and consider the category $\mathbf{FCW}_{/Y}$ whose objects are homotopy equivalences $f\colon X \to Y$ in **TFCW** with X in **FCW** and whose morphisms $g\colon f_1 \to f_2$ with $f_i\colon X_i \to Y$ for $i = 1,2$ are maps $g\colon X_1 \to X_2$ in **FCW** such that $f_2 \circ g = f_1$.

$$\begin{tikzcd} X_1 \arrow[rr, "g"] \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}$$

We say that two homotopy equivalences $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ in $\mathbf{FCW}_{/Y}$ are *simply equivalent* if there exists a morphism $g\colon f_1 \to f_2$ such that $\tau(g) = 0$. Note that being simply equivalent is well-defined, since any morphism $g\colon f_1 \to f_2$ is by construction a homotopy equivalence. Moreover, it is an equivalence relation. Indeed, if we denote by f_2^{-1} a homotopy inverse of f_2, then, by homotopy invariance of the Whitehead torsion, we have $\tau(g) = \tau(f_2^{-1} \circ f_1)$. Therefore, by obstruction property, f_1 is simply equivalent to f_2 if and only if $f_2^{-1} \circ f_1$ is a simple homotopy equivalence. Using this and Lemma 2.1, it is easy to conclude that this is an equivalence relation.

Definition 2.5. (i) A *simple structure* ξ on a space Y in **TFCW** is a choice of a simple equivalence class of an object $u\colon Z \to Y$ of $\mathbf{FCW}_{/Y}$. Moreover, if Y is in **FCW**, we call the simple structure represented by id_Y the *canonical simple structure* $\xi_{\mathrm{can}}(Y)$ on Y. We denote by **SStruct** the category whose objects are pairs (Y, ξ) with Y in **TFCW** and ξ a simple structure on Y and whose morphism $f\colon (X, \xi) \to (Y\eta)$ are the morphisms $f\colon X \to Y$ in **TFCW**.

(ii) Let $f\colon (X, \xi) \to (Y, \eta)$ be a homotopy equivalence in **SStruct**. We define its Whitehead torsion by

$$\tau(f) = v_*\tau(v^{-1} \circ f \circ u) \in \mathrm{Wh}(\pi Y) \tag{2.1}$$

where $u\colon X' \to X$ and $v\colon Y' \to Y$ are representatives of the simple structures ξ and η, respectively, and $\tau(v^{-1} \circ f \circ u) \in \mathrm{Wh}(\pi Y')$ is the classical Whitehead torsion recalled in the previous section.

$$\begin{tikzcd} X' \arrow[d, "u"', "\simeq"] & Y' \arrow[d, "v", "\simeq"'] \\ X \arrow[r, "\simeq", "f"'] & Y \end{tikzcd}$$

Remark 2.6. • Note that the Whitehead torsion in **SStruct** is well-defined. Indeed, let $u_1\colon X_1' \to X$ and $v_1\colon Y_1' \to Y$ be other two representatives of ξ and η, respectively. Then, by Lemma 2.1 and the definition of simple structure, we have

$$v_*\tau(v^{-1} \circ f \circ u) = v_*\tau(v^{-1} \circ v_1 \circ v_1^{-1} \circ f \circ u_1 \circ u_1^{-1} \circ u) = (v_1)_*\tau(v_1^{-1} \circ f \circ u_1)$$

- By abuse of notation, we denote the Whitehead torsion in **SStruct** as the classical one: in the following, it will be clear from the context which definition we are using.

Before investigating the properties of the new definition of Whitehead torsion, we need to understand how simple structures fit with the operations of the category **TFCW**. In particular, we are interested in its behavior with homotopy equivalences, disjoint unions, pushouts and products. Let us start with the two easiest cases.

Example 2.7. Let $f\colon X \to Y$ be a homotopy equivalence in **TFCW** and let ξ be a simple structure on X. Then Y has a canonical choice of simple structure such that $\tau(f) = 0$ denoted by $f_*\xi$ and defined as follows. Consider a representative $u\colon X' \to X$ of ξ. Then we define $f_*\xi$ to be the simple structure represented by $f \circ u\colon X' \to Y$. Note that it is obviously well-defined.

Example 2.8. Let X and Y be in **SStruct** with simple structures ξ and η, respectively. Then the disjoint union $X \amalg Y$ has canonical choice of simple structure denoted by $\xi \amalg \eta$ and defined as follows. Let $u\colon Z \to X$ and $v\colon W \to Y$ be representatives of ξ and η, respectively. We define $\xi \amalg \eta$ to be the simple structure represented by $u \amalg v\colon Z \amalg W \to X \amalg Y$. Note it is obviously well-defined by the sum formula of Lemma 2.1.

Let us construct now with the pushout simple structure.

Notation 2.9. From now on, in this section, we assume without loss of generality that all the maps in **FCW** are cellular. This can be done by using the cellular approximation theorem [Hat02, Theorem 4.8]. Note that by homotopy invariance this does not change the torsion of the maps involved.

Construction/Proposition 2.10. Let (X_i, ξ_i) be in **SStruct** for $i = 0, 1, 2$ and consider the following pushout square in **Top**

$$\begin{tikzcd} Y_0 \arrow[r, hook, "i_1"] \arrow[d, "i_2"'] & Y_1 \arrow[d, "j_1"] \\ Y_2 \arrow[r, "j_2"'] & Y \end{tikzcd}$$

where i_1 a cofibration. Then Y is in **TFCW** and there is a canonical simple structure ξ on Y which we call the *pushout simple structure* and which is constructed as follows. Let $u_i\colon X_i \to Y_i$ be homotopy equivalences representing ξ_i for $i = 0, 1, 2$. Choose the following pushout diagram in **FCW**

$$\begin{tikzcd} X_0 \arrow[r, hook, "a_1"] \arrow[d, "a_2"'] & X_1 \arrow[d, "b_1"] \\ X_2 \arrow[r, hook, "b_2"] & X \end{tikzcd} \tag{2.2}$$

where a_1 and b_2 are inclusions of CW-subcomplexes and the n-skeleton X_n of X is the subspace $b_1\big((X_1)_n\big) \cup b_2\big((X_2)_n\big)$ for every $n \geq -1$. The pushout property yields a map $u\colon X \to Y$ which is a homotopy equivalence by [Die08, Proposition (5.3.4)]. Define the pushout simple structure ξ on Y to be the one represented by u.

Remark 2.11. The fact that Y is in **TFCW** follows immediately by construction, since X is obviously in **FCW** and $u\colon X \to Y$ is a homotopy equivalence.

To show that this construction is well-defined, we need the following particular property of the Whitehead torsion.

Lemma 2.12. *Consider the following commutative diagram in* **Top**

$$\begin{tikzcd} X_1 \arrow[d, "u_1"] & X_0 \arrow[l, hook'] \arrow[r, "a_2"] \arrow[d, "u_0"] & X_2 \arrow[d, "u_2"] \\ Y_1 & Y_0 \arrow[l, hook', "i_1"'] \arrow[r, "i_2"] & Y_2 \\ Z_1 \arrow[u, "v_1"] & Z_0 \arrow[l, hook'] \arrow[r, "c_2"] \arrow[u, "v_0"] & Z_2 \arrow[u, "v_2"] \end{tikzcd} \tag{2.3}$$

where the top and bottom rows are cellular pushout diagrams in **FCW**, *i_1 is a cofibration and all the vertical maps are homotopy equivalences. Let X, Y and Z be the pushouts of the three rows, respectively, and $u\colon X \to Y$ and $v\colon Z \to Y$ be the maps induced by the u_i and v_i, respectively. Then, the following formula holds in* $\mathrm{Wh}(\pi Y)$

$$v_*\tau(v^{-1} \circ u) = (j_1)_*(v_1)_*\tau(v_1^{-1} \circ u_1) + (j_2)_*(v_2)_*\tau(v_2^{-1} \circ u_2) - (j_0)_*(v_0)_*\tau(v_0^{-1} \circ u_0) \tag{2.4}$$

where $j_i\colon Y_i \to Y$ for $i = 0, 1, 2$ are the structure maps of the pushout.

Proof. Let us assume first that i_2 and c_2 are cofibrations. In this particular case, the homotopy inverses of the maps v_i induce a homotopy inverse of the map v by [Die08, Proposition (5.2.8)]. Hence, we can conclude by applying the sum formula of the Whitehead torsion.

Let us consider now the general case. By [Die08, Section 5.3], diagram (2.3) induces a diagram where (X_2, a_2), (Y_2, j_2) and (Z_2, c_2) are replaced by the inclusions in the respective mapping cylinder $\mathrm{cyl}(a_2)$, $\mathrm{cyl}(j_2)$ and $\mathrm{cyl}(c_2)$. Moreover, such inclusions are cofibrations. Therefore, since $\mathrm{cyl}(a_2)$ and $\mathrm{cyl}(c_2)$ are in **FCW**, if we prove that the replacement does non change the Whitehead torsions involved, we can reduce to the previous case and conclude. But such replacement is done through the projection maps $p\colon \mathrm{cyl}(a_2) \to X_2$ and $q\colon \mathrm{cyl}(c_2) \to Z_2$, which are simple homotopy equivalence by [Coh73, (5.1A)]. Therefore, the claim follows easily by composition formula and obstruction property of the Whitehead torsion. □

Proof of Construction/Proposition 2.10. Let us start by proving that diagram (2.2) exists. In particular, we show that, given the representative (X_0, u_0) of ξ_0, there exists a representative (X_1, u_1) of ξ_1 such that X_0 is a subcomplex of X_1 and the following diagram commutes.

$$\begin{tikzcd} X_0 \arrow[r, hook, "a_1"] \arrow[d, "u_0"'] & X_1 \arrow[d, "u_1"] \\ Y_0 \arrow[r, "i_1"'] & Y_1 \end{tikzcd}$$

Consider a representative $(\widetilde{X_1}, \widetilde{u_1})$ of ξ_1 and let $\widetilde{a_1}\colon X_0 \to \widetilde{X_1}$ be a map such that $\widetilde{u_1} \circ \widetilde{a_1} \simeq i_1 \circ u_0$. Then, by [Die08, Section 5.3], the map $\widetilde{a_1}$ factors as $\widetilde{a_1} = p \circ a_1$ where $a_1\colon X_0 \to \mathrm{cyl}(\widetilde{a_1})$ is the cofibration given by the inclusion into the mapping cylinder and $p\colon \mathrm{cyl}(\widetilde{a_1}) \to \widetilde{X_1}$ is a homotopy equivalence. Define $X_1 = \mathrm{cyl}(\widetilde{a_1})$ in **FCW**. We obtain the following homotopy commutative diagram.

$$\begin{tikzcd} X_0 \arrow[r, hook, "a_1"] \arrow[d, "u_0"'] & X_1 \arrow[d, "\widetilde{u_1} \circ p"] \arrow[dl, phantom, "H"] \\ Y_0 \arrow[r, "i_1"'] & Y_1 \end{tikzcd}$$

Let u_1 be the map obtained by the cofibration property of a_1 applied to the homotopy H with initial condition $\widetilde{u_1} \circ p$. Then (X_1, u_1) is the wanted representative of ξ_1. Indeed, it is a representative of ξ_1 because we have $\tau(\widetilde{u_1}^{-1} \circ u_1) = \tau(p) = 0$, since $u_1 \simeq \widetilde{u_1} \circ p$ and p is a simple homotopy equivalence by [Coh73, (5.1A)]. Moreover, by construction, a_1 is an inclusion and $u_1 \circ a_1 = i_1 \circ u_0$.

We show now that the construction does not depend on the choice of representatives u_i of the simple structures ξ_i for $i = 0, 1, 2$. Let $v_i \colon Z_i \to Y_i$ be other representatives of the simple structures ξ_i for $i = 0, 1, 2$. Then, by definition, $\tau(u_i^{-1} \circ v_i) = 0$ for $i = 0, 1, 2$. Therefore, by Lemma 2.12, since the right hand side of formula (2.4) vanishes, we have that $\tau(v^{-1} \circ u) = 0$. Hence, the induced simple structure is the same. □

Example 2.13. Let X be in **FCW**. Consider the following pushout describing the n-skeleton of X.

$$\begin{tikzcd}
\coprod_{I_n} S^{n-1} \arrow[r, "\coprod_{i\in I_n} q_i"] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{I_n} D^n \arrow[r, "\coprod_{i\in I_n} Q_i"'] & X_n
\end{tikzcd}$$

Equip $\coprod_{I_n} S^{n-1}$, $\coprod_{I_n} D^n$ and X_{n-1} with the canonical simple structure with respect to any CW-structure. Then the pushout simple structure on X_n agrees with the canonical simple structure with respect to any CW-structure. Indeed, by the cellular approximation theorem [Hat02, Theorem 4.8] and by [Coh73, (7.1)], we can assume without loss of generality that the characteristic maps are cellular. Moreover, by topological invariance of the Whitehead torsion, we can choose our preferred CW-structures on the spaces involved. Therefore, we can reduce to the case where S^{n-1} has some finite CW-structure, D^n has the CW-structure obtained from the one of S^{n-1} by attaching one n-cell, X_{n-1} and X_n has the given CW-structures and the characteristic maps are cellular. But in this case, the claim is obviously true, since the CW-structure on X_n is exactly the pushout of the other structures.

Finally, we present how to construct a canonical simple structure on the product of two spaces in **SStruct**.

Construction/Proposition 2.14. Let (X, ξ) and (Y, η) be in **SStruct**. Let $u \colon Z \to X$ and $v \colon W \to Y$ be representatives of ξ and η, respectively. We define the *product simple structure* $\xi \times \eta$ on $X \times Y$ to be the one represented by $u \times v \colon Z \times W \to X \times Y$.

Proof. We prove that the product simple structure $\xi \times \eta$ on $X \times Y$ does not depend on the choice of representatives of ξ and η. If X and Y are connected, this follows directly by the product formula of Lemma 2.1. Hence, let us assume that X or Y are not connected. Let $u' \colon Z' \to X$ and $v' \colon W' \to Y$ be other two representatives of ξ and η, respectively. Any space is the disjoint union of its connected components. Moreover, the connected components of a finite CW-complex are again finite CW-complexes. Therefore, we can conclude by the sum formula and the previous case as follows

$$\tau\big((u \times v)^{-1} \circ (u' \times v')\big) = \coprod_{i \in \pi_0(X),\ j \in \pi_0(Y)} \tau\big((u_i \times v_j)^{-1} \circ (u_i' \times v_j')\big) = 0$$

where u_i, u_i', v_j and v_j' are the homotopy equivalences induced by u, u', v and v' on connected components, respectively. □

Thanks to these constructions, we can finally study the properties of the Whitehead torsion in **SStruct**. Both Definition 2.5 and the simple structures defined above are very clear and natural. Therefore, it is easy to understand that this new definition has the same behavior of the classical one, as stated in the following lemma.

Lemma 2.15. *The Whitehead torsion in* **SStruct** *satisfies the* homotopy invariance *and the* composition formula *as stated in Lemma 2.1. Moreover, if we equip pushouts with the pushout simple structure and products with the product simple structure, it satisfies also the* sum formula *and the* product formula.

Proof. This lemma follows by direct computation using Definition 2.5 and Lemma 2.1. □

The simple structure on a total space of a fibration

A natural question at this point is whether simple structures are suitable for our case. Given a map $f\colon M \to B$ in **Man**, our strategy to solve the fibering problem is to convert f into a fibration and to compute some torsion to understand what happens if f is homotopic to a fiber bundle. As we have just studied, to do this, we need to equip the total space $\mathrm{FIB}(f)$, which is not in general in **Man**, with a simple structure. Therefore, a more precise question is whether there is a canonical way to equip the total space E of a fibration $p\colon E \to B$ with a simple structure, given those of the base space B and the fiber F. The remainder of this section is devoted to answer to this question in the particular case where B is in **FCW**. Once we know this and the first obstruction, under some particular assumption on the fibration, the construction in the case where B is in **TFCW** can be obtained quite easily by taking the pullback over a representative of the simple structure of B: we will present this in the next chapter.

The program is as follows: first we construct the simple structure on E and we check that it is well-defined, then we study how this depends on its initial data. The result will be a very canonical and explicit construction, although at first glance it may seem quite artificial. In particular, it makes very explicit the role of the fiber and its simple structure, which is of great importance for us, being interested in fiber bundles.

Let us start with the construction.

Definition 2.16. Let B be a connected CW-complex with base point $b \in B$. Denote by $I(B)$ the set of open cells of B and by $\dim(c)$ the dimension of a cell $c \in I(B)$. A *spider* at b for B is a collection of path w_c indexed by $c \in I(B)$ such that $w_c(0) = b$ and $w_c(1) \in c$.

Construction/Proposition 2.17. Consider $p\colon E \to B$ in **Fib** such that B is a path-connected object of **FCW** and the fiber is in **TFCW**. Given a base point $b \in B$, a spider s at b and a simple structure ζ on F_b, we construct a preferred simple structure $\xi(b, s, \zeta)$ on E as follows. Let B_n be the n-skeleton of B and define $E_n = p^{-1}(B_n)$. We construct a preferred simple structure ξ_n on E_n inductively for $n = -1, 0, 1, \ldots$, so that $\xi(b, s, \zeta) = \xi_N$ for $N \in \mathbb{N}$ such that $E_N = E$.

(i) The case $n = -1$ is trivial.

(ii) As for $n = 0$, let $B_0 = \coprod_{i \in I_0} \{b_i\}$. Then we have $E_0 = \coprod_{i \in I_0} F_{b_i}$. Let w_i be the path of the spider s from b to b_i for $i \in I_0$ and consider the induced fiber transports $t[w_i]\colon F_b \to F_{b_i}$. We equip F_{b_i} with the simple structure $\zeta_i = t[w_i]_* \zeta$ of Example 2.7. We define ξ_0 as simple structure of the disjoint union of the ζ_i presented in Example 2.8.

(iii) Assume now that we have constructed ξ_{n-1} on E_{n-1}. Choose the following pushout for the n-skeleton B_n of B.

$$\begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i} & B_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod_{i \in I_n} Q_i} & B_n \end{array}$$

Let $x_i \in D^n \setminus S^{n-1}$ for $i \in I_n$ be a point such that $Q_i(x_i) = w_i(1)$, where w_i is the path of the spider s associated to the cell indexed by $i \in I_n$. We get by Lemma 1.18(i) a fiber trivialization

$$T_i\colon F_b \times D^n \to Q_i^*E$$

of Q_i^*E with respect to (b, x_i, w_i) which induces by restriction a homotopy equivalence

$$t_i\colon F_b \times S^{n-1} \to q_i^*E$$

Equip Q_i^*E and q_i^*E with the simple structures $(T_i)_*\big(\zeta\times\xi_{\mathrm{can}}(D^n)\big)$ and $(t_i)_*\big(\zeta\times\xi_{\mathrm{can}}(S^{n-1})\big)$ and consider the following pushout diagram with a cofibration as left vertical map.

$$\begin{tikzcd} \coprod_{i\in I_n} q_i^*E \arrow[r] \arrow[d, hook, "\iota"'] & E_{n-1} \arrow[d, hook, "i_{E_{n-1}}"] \\ \coprod_{i\in I_n} Q_i^*E \arrow[r] & E_n \end{tikzcd} \tag{2.5}$$

We define ξ_n to be the pushout simple structure on E_n.

To prove that the construction is well-defined, we need the following preliminary lemma.

Lemma 2.18 ([Lüc89, Lemma 1.26]). *Let $p\colon X \to Y$ be in* **Fib**. *Let*

$$\begin{tikzcd} Y_0 \arrow[r, "j_1"] \arrow[d, hook, "j_2"'] \arrow[dr, "k"] & Y_1 \arrow[d, hook, "k_1"] \\ Y_2 \arrow[r, "k_2"'] & Y \end{tikzcd}$$

be a commutative pushout square such that the vertical maps are cofibrations. Then the pullback construction yields a pushout

$$\begin{tikzcd} k^*X \arrow[r, "\overline{j_1}"] \arrow[d, hook, "\overline{j_2}"'] & k_1^*X \arrow[d, hook, "\overline{k_1}"] \\ k_2^*X \arrow[r, "\overline{k_2}"'] & X \end{tikzcd}$$

where the vertical maps are still cofibrations.

Proof of Construction/Proposition 2.17. We have to show that the construction is well-defined. In particular, we have to check that diagram (2.5) is a pushout, that the restriction t_i of T_i is a well-defined homotopy equivalence and that the construction does not depend on the choices of the fiber trivialization T_i of Q_i^*E and of the characteristic maps Q_i and q_i.

- Let us start by proving that diagram (2.5) is a pushout. Consider the following diagram where the top and the bottom squares are pullbacks by construction.

$$\begin{tikzcd} \coprod_{i\in I_n} q_i^*E \arrow[rr, "\coprod \overline{q_i}"] \arrow[dr, "\coprod p_{q_i}"] \arrow[dd, "\iota"'] & & E_{n-1} \arrow[dr, "p"] \arrow[dd, hook, "i_{E_{n-1}}"' near start] & \\ & \coprod_{i\in I_n} S^{n-1} \arrow[rr, "\coprod q_i" near start] \arrow[dd, hook, "j" near start] & & B_{n-1} \arrow[dd, hook, "i_{B_{n-1}}"] \\ \coprod_{i\in I_n} Q_i^*E \arrow[rr, "\coprod \overline{Q_i}" near start] \arrow[dr, "\coprod p_{Q_i}"'] & & E_n \arrow[dr, "p"] & \\ & \coprod_{i\in I_n} D^n \arrow[rr, "\coprod Q_i"] & & B_n \end{tikzcd} \tag{2.6}$$

The right square is obviosly a pullback and so, by pasting law [AHS90, Proposition 11.10], also the diagonal square

$$\begin{tikzcd} \coprod_{i\in I_n} q_i^*E \arrow[r] \arrow[d, "\coprod p_{q_i}"'] & E_n \arrow[d, "p"] \\ \coprod_{i\in I_n} S^{n-1} \arrow[r] & B_n \end{tikzcd}$$

is a pullback. Therefore, since j and $i_{B_{n-1}}$ of diagram (2.6) are cofibrations, we can apply Lemma 2.18 and conclude that diagram (2.5) is a pushout with cofibrations as vertical maps.

- We prove now that the restriction t_i of T_i is a well-defined homotopy equivalence. Consider the following diagram.

$$\begin{tikzcd} F_b \times S^{n-1} \arrow[rr] \arrow[dr, "t_i"] \arrow[dd] & & S^{n-1} \arrow[dr, equal] \arrow[dd, hook] & \\ & q_i^*E \arrow[rr] \arrow[dd] & & S^{n-1} \arrow[dd, hook] \\ F_b \times D^n \arrow[rr] \arrow[dr, "T_i"] & & D^n \arrow[dr, equal] & \\ & Q_i^*E \arrow[rr, "p_{Q_i}"'] & & D^n \end{tikzcd} \tag{2.7}$$

  We claim first that the front and back square are pullback squares. This is obvious for the back one. For the front one consider diagram (2.6). We have already showed that the diagonal square and the bottom square are pullbacks. Hence, again by pasting law [AHS90, Proposition 11.10], also the left square is so. Therefore, since all these claims hold also if we take a single $i \in I_n$ instead of taking the coproduct, the front square of the previous diagram (2.7) is a pullback as well.

  Now, the map induced by pullback property is by commutativity the restriction of the map T_i, that is, it is exactly the map t_i. Therefore, $t_i\colon F_b \times S^{n-1} \to q_i^*E$ is a well-defined map. It remains to prove that this is a homotopy equivalence. This follows by coglueing theorem [FP90, Theorem A.4.19] using that $F_b \times D^n \to D^n$ and $p_{Q_i}\colon Q_i^*E \to D^n$ are fibrations. Indeed, the first map is a projection and any projection is fibration by [Whi78, Theorem 7.7] and the second map is by construction the pullback of the fibration p and thus it is a fibration as well.

- It remains to show that the construction does not depend on the choices of the fiber trivialization T_i and of the characteristic maps. As for characteristic maps, this follows directly by the Lemma 2.21 below by taking $(b', s', \zeta') = (b, s, \zeta)$, but with different characteristic maps. Thus, let us consider the case of T_i. Since D^n is contractible, we obtain from $T_i\colon F_b \times D^n \to Q_i^*E$ a homotopy equivalence $S_i\colon F_b \to Q_i^*E$. This induces a simple structure $(S_i)_*\zeta$ on Q_i^*E which agrees with the simple structure $(T_i)_*\big(\zeta \times \xi_{\mathrm{can}}(D^n)\big)$. Indeed, the transition map $S_i^{-1} \circ T_i$ is exactly the contraction $F_b \times D^n \to F_b$, which is obviously a simple homotopy equivalence. Moreover, if we replace the map T_i with S_i in diagram (2.7), we get a diagram where the front and back square are homotopy pullbacks: the first one because it is a pullback over a fibration, the second one by construction. Hence, we can apply the homotopy version of the coglueing theorem [Mat76, Corollary 7] to obtain again the map $t_i\colon F_b \times S^{n-1} \to q_i^*E$ up to homotopy. To sum up, the simple structure on Q_i^*E

and the map t_i can also be obtained using $S_i\colon F_b \to Q_i^* E$ instead of T_i and therefore the fiber trivialization T_i is only used to get the homotopy equivalence S_i. Hence, we can conclude by Lemma 1.18(ii) and the homotopy invariance of the Whitehead torsion that the construction does not depend on the choice of T_i. $\square$

Remark 2.19. In the literature, there are other ways to get the same simple structure of Construction 2.17 on the total space of a fibration with fiber and base space in **TFCW**, for example see [FP90, Theorem 5.4.2]. Nevertheless, Construction 2.17 is the most suitable for us. Indeed, although the alternative construction may seem more natural, they are certainly more implicit. In particular, they do not make explicit the initial data as well as our construction, especially the role of the fiber and its simple structure.

Example 2.20. Let $p\colon B \times F \to B$ be a trivial bundle such that B and F are in **FCW**. Then for any spider s the simple structure $\xi\big(b, s, \xi_{\mathrm{can}}(F)\big)$ on $B \times F$ agrees with the product simple structure. Indeed, if B_n is the n-skeleton of B, then $E_n = p^{-1}(B_n) = B_n \times F$. Moreover, diagram (2.6) appears as follows.

$$\begin{tikzcd}
\coprod_{i\in I_n} S^{n-1} \times F \arrow[rr] \arrow[dr] \arrow[dd] & & B_{n-1} \times F \arrow[dr, "p"] \arrow[dd, hook] & \\
& \coprod_{i\in I_n} S^{n-1} \arrow[rr] \arrow[dd, hook] & & B_{n-1} \arrow[dd, hook] \\
\coprod_{i\in I_n} D^n \times F \arrow[rr] \arrow[dr] & & B_n \times F \arrow[dr, "p"] & \\
& \coprod_{i\in I_n} D^n \arrow[rr] & & B_n
\end{tikzcd}$$

Therefore, since by Example 2.13 the pushout simple structure on the skeleton of a CW-complex coincide with the canonical simple structure on it, by definition of product simple structure we have $\xi_n = \xi_{\mathrm{can}}(B_n) \times \xi_{\mathrm{can}}(F)$ and $\xi(b, s, \xi_{\mathrm{can}}(F)) = \xi_{\mathrm{can}}(B) \times \xi_{\mathrm{can}}(F)$.

Dependence of the simple structure on the total space of a fibration

We conclude this section by showing how the simple structure on the total space of a fibration depends on the choice of (b, s, ζ).

Lemma 2.21. *Consider the situation of Construction 2.17 and suppose that another choice (b', s', ζ') of the triple has been made. Given a cell $c \in I(B)$, let u_c be any path in the interior of c from $w_c(1)$ to $w'_c(1)$, where w_c and w'_c are given by the spiders s and s'. Let $v_c = w_c * u_c * (w'_c)^-$. Then the homotopy class relative endpoints $[v_c]$ is independent of u_c. Let $(i_{b'})_*\colon \mathrm{Wh}(\pi F_{b'}) \to \mathrm{Wh}(\pi E)$ be the homomorphism induced by the inclusion $i_{b'}\colon F_{b'} \to E$. Then the following holds in $\mathrm{Wh}(\pi E)$.*

$$\tau\Big(\big(E, \xi(b, s, \zeta)\big) \xrightarrow{\mathrm{id}} \big(E, \xi(b', s', \zeta')\big)\Big) = \sum_{c \in I(B)} (-1)^{\dim(c)} \cdot (i_{b'})_* \tau\big((F_b, \zeta) \xrightarrow{t[v_c]} (F_{b'}, \zeta')\big) \tag{2.8}$$

Remark 2.22. The fact that the homotopy class $[v_c]$ is independent of u_c is trivial, since any cell of a CW-complex is contractible.

Proof. As in Construction 2.17, let us denote by B_n the n-skeleton of B and let $E_n = p^{-1}(B_n)$. Let ξ_n and ξ'_n be the simple structures on E_n induced by $\xi(b, s, \zeta)$ and $\xi(b', s', \zeta')$ respectively, that is, the simple structures at the n-th inductive step of Construction 2.17. We prove inductively on $n = -1, 0, 1, \ldots$ that

$$\tau\big((E_n, \xi_n) \xrightarrow{\mathrm{id}} (E_n, \xi_n')\big) = \sum_{c \in I(B_n)} (-1)^{\dim(c)} \cdot (i_{b'})_* \tau\big((F_b, \zeta) \xrightarrow{t[v_c]} (F_{b'}, \zeta')\big) \tag{2.9}$$

The case $n = -1$ is trivial. Consider the case $n = 0$. Let $B_0 = \coprod_{i \in I_0} \{b_i\}$. Denote by w_i, w_i' and v_i the paths w_{b_i}, w_{b_i}' and v_{b_i} for any $i \in I_0$. We have that $E_0 = \coprod_{i \in I_0} F_{b_i}$. Therefore, we can conclude by the sum formula as follows

$$\begin{aligned}
&\tau\big((E_0, \xi_0) \xrightarrow{\mathrm{id}} (E_0, \xi_0')\big) \\
=& \sum_{i \in I_0} \tau\Big(\big(F_{b_i}, t[w_i]_* \zeta\big) \xrightarrow{\mathrm{id}} \big(F_{b_i}, t[w_i']_* \zeta'\big)\Big) \\
=& \sum_{i \in I_0} \tau\big((F_b, \zeta) \xrightarrow{t[v_i]} (F_{b'}, \zeta')\big)
\end{aligned}$$

where the simple structures on the second line are those of Example 2.7.

Let us now assume that (2.9) holds for E_{n-1}. Suppose that the following push-out diagrams defining B_n induce the simple structures ξ_n and ξ_n' on E_n as in the inductive step of Construction 2.17, respectively.

$$\begin{tikzcd}
\coprod_{i \in I_n} S^{n-1} \arrow[r, "\coprod_{i \in I_n} q_i"] \arrow[d, hook] & B_{n-1} \arrow[d, hook] \\
\coprod_{i \in I_n} D^n \arrow[r, "\coprod_{i \in I_n} Q_i"] & B_n
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod_{i \in I_n'} S^{n-1} \arrow[r, "\coprod_{i \in I_n'} q_i'"] \arrow[d, hook] & B_{n-1} \arrow[d, hook] \\
\coprod_{i \in I_n'} D^n \arrow[r, "\coprod_{i \in I_n'} Q_i'"] & B_n
\end{tikzcd} \tag{2.10}$$

We can assume without loss of generality that $I_n = I_n'$ with $Q_i(D^n) = Q_i'(D^n)$ for all $i \in I_n$, since I_n and I_n' are in bijection with the connected components of $B_n \setminus B_{n-1}$ by the following homeomorphisms.

$$\coprod_{i \in I_n} (D^n \setminus S^{n-1}) \xrightarrow{\coprod_{i \in I_n} Q_i} B_n \setminus B_{n-1} \xleftarrow{\coprod_{i \in I_n'} Q_i'} \coprod_{i \in I_n'} (D^n \setminus S^{n-1})$$

Moreover, again without loss of generality, we can assume $Q_i(0) = Q_i'(0)$ for all $i \in I_n$ by pre-composing with a homeomorphism $(D^n, S^{n-1}) \to (D^n, S^{n-1})$.

Now, in order to compare the different characteristic maps, we thicken B_{n-1} into B_n and we glue the remaining part using correspondent modified characteristic maps. More precisely, consider for $t \in I$ the map $\mu_t \colon D^{n+1} \to D^{n+1}$ given by the multiplication with t and let

$$\begin{aligned}
D^n(t) &= \mu_t(D^n), & S^{n-1}(t) &= \mu_t(S^{n-1}), \\
Q_i(t) &= Q_i \circ \mu_t, & q_i(t) &= Q_i(t)\big|_{S^{n-1}}
\end{aligned}$$

Note that for $t = 1$ we have $\mu_t = \mathrm{id}_{D^n}$. Moreover, if $t < 1$, the maps $Q_i(t)$, and therefore also their restriction $q_i(t)$, are topological embeddings. Consider the following two spaces.

$$B_{n-1}(t) = B_{n-1} \cup Q_i\big(\overline{D^n \setminus D^n(t)}\big), \qquad E_{n-1}(t) = p^{-1}\big(B_{n-1}(t)\big)$$

For $t > 0$, we get the following new pushout diagram for B_n, where the vertical maps are cofibrations.

$$\begin{tikzcd}
\coprod_{i \in I_n} S^{n-1} \arrow[r, "\coprod_{i \in I_n} q_i(t)"] \arrow[d, hook] & B_{n-1}(t) \arrow[d, hook] \\
\coprod_{i \in I_n} D^n \arrow[r, "\coprod_{i \in I_n} Q_i(t)"] & B_n
\end{tikzcd}$$

Thus, we can apply the pullback construction to get, by Lemma 2.18, the following pushout square with cofibrations as vertical maps.

$$\begin{tikzcd} \coprod_{i\in I_n} q_i(t)^*E \arrow[r, "\coprod_{i\in I_n} \overline{q_i(t)}"] \arrow[d, hook, "\iota(t)"'] & E_{n-1}(t) \arrow[d, hook, "i_{E_{n-1}(t)}"] \\ \coprod_{i\in I_n} Q_i(t)^*E \arrow[r, "\coprod_{i\in I_n} \overline{Q_i(t)}"] & E_n \end{tikzcd} \tag{2.11}$$

Note that for $0 < t < 1$ the horizontal maps are still topological embeddings. Now, we equip the spaces of the previous diagram with the following simple structures.

- As for $E_{n-1}(t)$, since the inclusion $B_{n-1} \hookrightarrow B_{n-1}(t)$ is an homotopy equivalence, so is the inclusion $j\colon E_{n-1} \hookrightarrow E_{n-1}(t)$. Hence, we equip $E_{n-1}(t)$ with the simple structure $j_*\,\xi_{n-1}$ induced by the simple structure ξ_{n-1} on E_{n-1}.

- Consider the two spaces on the left of (2.11). Since μ_t is trivially homotopic to the identity map id_{D^n}, there is a homotopy $H_i\colon D^n\times I \to B_n$ such that $Q_i(t) \simeq Q_i$ for all $i \in I_n$. Denote by $h_i\colon S^{n-1} \times I \to B_{n-1}(t)$ the homotopy such that $q_i(t) \simeq q_i$ which is the restriction of the homotopy H_i to $S^{n-1}\times I$. By Lemma 1.6, there exist the following two fiber homotopy equivalences.
$$g_{H_i}\colon Q_i(t)^*E \to Q_i^*E, \qquad g_{h_i}\colon q_i(t)^*E \to q_i^*E$$
Moreover, by Lemma 1.10, we can assume that g_{h_i} is the restriction of g_{H_i} for all $i \in I_n$. We equip $Q_i(t)^*E$ and $q_i(t)^*E$ with the simple structures induced by those of Q_i^*E and q_i^*E through g_{H_i} and g_{h_i}, respectively.

- We equip E_n with the simple structure ξ_n.

Claim 1. With these choices of simple structures, diagram (2.11) is simple pushout, that is, the pushout simple structure on E_n agrees with ξ_n.

Proof of Claim 1. Let η be the pushout simple structure on E_n given by diagram (2.11). We show that $\tau\big((E_n, \eta) \xrightarrow{\mathrm{id}} (E_n, \xi_n)\big) = 0$. Consider the following natural transformation of pushout diagrams.

$$\begin{tikzcd} \coprod_{i\in I_n} Q_i(t)^*E \arrow[d, hook] & \coprod_{i\in I_n} q_i(t)^*E \arrow[l, hook'] \arrow[r, "\coprod_{i\in I_n} \overline{q_i(t)}"] \arrow[d, hook, "j_0"] & E_{n-1}(t) \arrow[d, equal] \\ \coprod_{i\in I_n} Q_i^*E \arrow[d, equal] & \coprod_{i\in I_n} h_i^*E \arrow[l, hook'] \arrow[r, "\coprod_{i\in I_n} \overline{h_i}"] & E_{n-1}(t) \\ \coprod_{i\in I_n} Q_i^*E & \coprod_{i\in I_n} q_i^*E \arrow[l, hook'] \arrow[r, "\coprod_{i\in I_n} \overline{q_i}"] \arrow[u, hook, "j_1"'] & E_{n-1} \arrow[u, hook] \end{tikzcd} \tag{2.12}$$

Note that all the inclusions are well-defined. Indeed, by definition of the map $Q_i(t)$, the space $Q_i(t)^*E$ can be viewed as a subspace of Q_i^*E. Moreover, by seeing h_i as the restriction of Q_i to $D^n \setminus \mu_t(D^n)$, the space h_i^*E can be viewed as a subspace of Q_i^*E. Now, by construction, all the vertical maps of diagram (2.12) are homotopy equivalences and the induced maps on pushouts are the identity of E_n. Furthermore, ξ_n is by construction the pushout simple structure on E_n given by the bottom row, while η is the one given by top row. Therefore, fixed $i \in I_n$, if we prove that all the i-th vertical compositions have vanishing torsion, we can conclude by the sum formula of Lemma 2.15.

This follows easily for the right column because the torsion of inclusion $E_{n-1} \hookrightarrow E_{n-1}(t)$ vanishes by construction.

Consider the middle vertical composition $j_1^{-1} \circ j_0$. We claim that this is component-wise homotopic to g_{h_i}. Indeed, apply Lemma 1.6 to the fibration $h_i^*E \to S^{n-1} \times I$ with f_0 and f_1 given by the inclusions $i_k\colon S^{n-1} = S^{n-1} \times \{k\} \hookrightarrow S^{n-1} \times I$ for $k = 0, 1$ and H given by the obvious homotopy $\mathrm{id}_{S^{n-1}\times I}$. By definition the homotopy h_i is such that $q_i(t) \simeq q_i$. Therefore we have $i_0^* h_i^* E = q_i(t)^*E$ and $i_1^* h_i^* E = q_i^* E$. Moreover, by the following diagram, using that $p_{i_0} = p_{q_i(t)}$, we obtain that a lifting G of the homotopy $\mathrm{id}_{S^{n-1}\times I} \circ (p_{q_i(t)} \times I)$ induce a lifting of the homotopy $h_i \circ (p_{q_i(t)} \times I)$.

$$\begin{tikzcd}
q_i(t)^*E \arrow[r, "\overline{i_0}"] \arrow[d, hook] & h_i^*E \arrow[r] \arrow[d] & E_{n-1}(t) \arrow[d, "p"] \\
q_i(t)^*E \times I \arrow[r, "p_{q_i(t)}\times \mathrm{id}_I"'] \arrow[ur, dashed, "G"] & S^{n-1} \times I \arrow[r, "h_i"'] & B_{n-1}(t)
\end{tikzcd}$$

Hence, we have that $g_{\mathrm{id}_{S^{n-1}\times I}}\colon q_i(t)^*E \to q_i^*E$ is homotopic to g_{h_i}. Now, by construction, using the notation of Notation 1.5, we have that j_0 and j_1 are component-wise $\overline{i_0}$ and $\overline{i_1}$. Moreover, again by Lemma 1.6 we have $\overline{i_1} \circ g_{\mathrm{id}_{S^{n-1}\times I}} \simeq \overline{i_0}$, which implies that $\overline{i_1}^{-1} \circ \overline{i_0} \simeq g_{\mathrm{id}_{S^{n-1}\times I}} \simeq g_{h_i}$. Therefore, the map $j_1^{-1} \circ j_0$ is component-wise homotopic to g_{h_i} as wanted. It follows now directly by the construction of the simple structure on $q_i(t)^*E$ that this map has vanishing torsion.

It remains to show that inclusion $Q_i(t)^*E \hookrightarrow Q_i^*E$ on the left of diagram (2.12) has vanishing torsion. Apply Lemma 1.6 to the fibration $Q_i^*E \to D^n$ and the homotopy $K\colon D^n \times I \to D^n$ such that $\mathrm{id}_{D^n} \simeq \mu_t$. Since $\overline{\mathrm{id}_{D^n}} = \mathrm{id}_{Q_i^*E}$, we obtain that $\overline{\mu_t} \simeq \overline{\mathrm{id}_{D^n}} \circ g_K = g_K$. Now, by definition of $Q_i(t)$, we have that $Q_i(t)^*E = \mu_t^* Q_i^* E$. In particular, the map $\overline{\mu_t}$ is exactly the map $Q_i(t)^*E \hookrightarrow Q_i^*E$ of diagram (2.12). Therefore, by the following diagram, using that $p_{\mu(t)} = p_{Q_i(t)}$, we get that a lifting K' of the homotopy $K \circ (p_{Q_i(t)} \times \mathrm{id}_I)$ induces a lifting $\overline{Q_i} \circ K'$ of the homotopy $Q_i \circ K \circ (p_{Q_i(t)} \times \mathrm{id}_I)$, which is exactly the homotopy $H_i \circ (p_{Q_i(t)} \times \mathrm{id}_I)$ by construction.

$$\begin{tikzcd}
Q_i(t)^*E \arrow[rr] \arrow[d, hook] & & Q_i^*E \arrow[r, "\overline{Q_i}"] \arrow[d] & E_n \arrow[d, "p"] \\
Q_i(t)^*E \times I \arrow[r, "p_{Q_i(t)}\times \mathrm{id}_I"'] \arrow[urr, dotted, "K'"] & D^n \times I \arrow[r, "K"'] & D^n \arrow[r, "Q_i"'] & B_n
\end{tikzcd}$$

In particular, we obtain that g_K, and so $\overline{\mu_t}$, is homotopic to g_{H_i}. Hence, we can conclude again by the construction of the simple structure on $Q_i(t)^*E$ and by homotopy invariance. $\square$

We can now apply all the previous construction to Q_i' to get $B_{n-1}(t)'$ and $E_{n-1}(t)'$ together with maps $Q_i'(t)$ and $q_i'(t)$. Obviously, Claim 1 holds also in this case.

The next step is to compute the torsion of the identity map $(E_n, \xi_n) \to (E_n, \xi_n')$ by using the sum formula applied to some pushout diagram of the shape of (2.11) and the torsion of $\mathrm{id}\colon (E_{n-1}, \xi_{n-1}) \to (E_{n-1}, \xi_{n-1}')$. Let us start with the following claim.

Claim 2. There are two continuous and strictly monotonically increasing maps $\varepsilon, \delta\colon I \to I$ such that for $t \in I$ it holds

$$B_{n-1}(t)' \subset B_{n-1}(\varepsilon(t)) \subset B_{n-1}(\delta(t))'$$

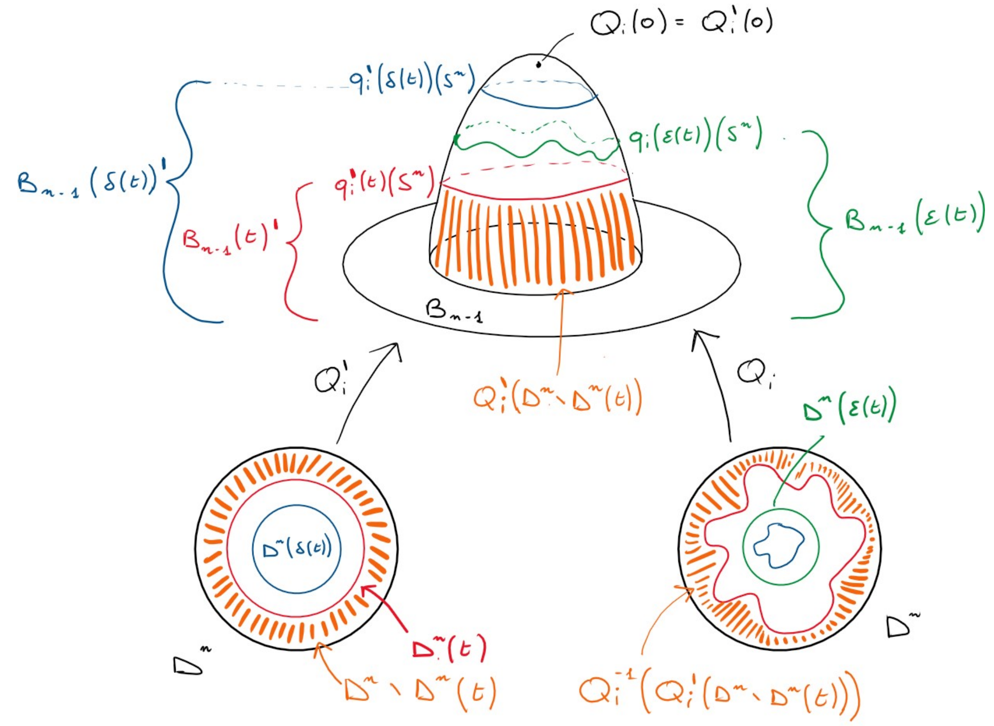
Proof of Claim 2. We construct the map ε, the map δ is defined analogously. Consider the map

$$\overline{\varepsilon}\colon [0,1) \to [0,1), \qquad t \mapsto \min\left\{ \|x\| : x \in \bigcup_{i \in I_n} Q_i^{-1}\Big(Q_i'\big(\overline{D^n \setminus D^n(t)}\big)\Big) \right\}$$

It is obviously continuous and monotonically increasing. Moreover, it is also strictly increasing. Indeed, we have $\overline{\varepsilon}(t) = 0$ for $t = 0$ and $\overline{\varepsilon}(t) > 0$ for $t > 0$ and the minimum always occurs on the boundary of the set considered. Therefore, $\overline{\varepsilon}$ extends to a continuous and strictly increasing function $\varepsilon\colon [0,1] \to [0,1]$. Now, this is the wanted function ε. Indeed, by definition, we have

$$Q_i'\big(\overline{D^n \setminus D^n(t)}\big) \subset Q_i\Big(\overline{D^n \setminus D^n\big(\varepsilon(t)\big)}\Big)$$

for any $i \in I_n$. Therefore, the inclusion $B_{n-1}(t)' \subset B_{n-1}\big(\varepsilon(t)\big)$ holds. $\square$



Now, using the maps ε and δ of Claim 2, we define a homotopy $k\colon B_n \times I \to B_n$ such that:

(i) $k(-,t)|_{B_{n-1}(t)'} = \mathrm{id}_{B_{n-1}(t)'}$ for all $t \in I$, in particular $k(-,0) = \mathrm{id}_{B_n}$;

(ii) $k\big(Q_i'(s),t\big) = Q_i'\big(\frac{ts}{\|s\|}\big)$ for all $t \in I$, $i \in I_n$ and $\delta(t) \le \|s\| < t$;

(iii) $k\big(Q_i'(s),t\big) = Q_i'\big(\frac{ts}{\delta(t)}\big)$ for all $t \in I$, $i \in I_n$ and $0 < \|s\| \le \delta(t)$;

(iv) $k\big(Q_i'(0),t\big) = Q_i'(0)$ for all $i \in I_n$.

For a geometric idea of what the homotopy k does, look at the picture above. The idea is that at time $t \in I$ we squeeze the annulus $Q_i'\Big(D^n(t) \setminus D^n\big(\delta(t)\big)\Big)$ into the circle $q_i'(t)(S^n)$ and we broaden the disk $Q_i'\big(\delta(t)\big)(D^n)$ to the disk $Q_i'(t)(D^n)$ for any $i \in I_n$.

Claim 3. The map $k\colon B_n \times I \to B_n$ is well-defined and continuous.

Proof of Claim 3. Consider first the region $B_{n-1} \times I$. Here the homotopy k is obviously well-defined. To prove that it is continuous, by condition (i) it suffices to show $k\big(Q_i'(s),t\big)$ converges to $Q_i'(s)$ for $(\|s\|,t) \to (1,1)$. This is clear in the region of (ii) and also in the one of (iii) if $\delta(t) \to 1$. Therefore, k is continuous on $B_{n-1} \times I$.

We are left to prove the claim in any connected component of $(B_n \setminus B_{n-1}) \times I$. Consider the following homotopy.

$$(D^n \setminus S^{n-1}) \times I \to D^n \setminus S^{n-1}, \qquad (s,t) \mapsto \begin{cases} s & \text{if } \|s\| \geq t \\ \frac{ts}{\|s\|} & \text{if } \delta(t) \leq \|s\| \leq t \\ \frac{ts}{\delta(t)} & \text{if } 0 < \|s\| \leq \delta(t) \\ 0 & \text{if } s = 0 \end{cases}$$

Note that its geometric idea is similar to the one explained before the statement of the claim. It is easy to check that this function is continuous and well-defined. Therefore, by applying the homeomorphism $Q'_i \colon D^n \setminus S^{n-1} \to B_n \setminus B_{n-1}$, also the homotopy k is continuous and well-defined. $\square$

Let now $0 < t < 1$ and define $f = k(-,t) \colon B_n \to B_n$. This map is homotopic to the identity $\mathrm{id}_{B_n} \colon B_n \to B_n$ through the obvious following homotopy

$$k' \colon B_n \times I \to B_n, \qquad (x,t') \mapsto k(x, t \cdot t') \tag{2.13}$$

Moreover, it sends the space $B_{n-1}\big(\varepsilon(t)\big)$ to $B_{n-1}(t)'$, the space $q_i\big(\varepsilon(t)\big)(S^{n-1})$ to $q'_i(t)(S^{n-1})$ and the space $Q_i\big(\varepsilon(t)\big)(D^n)$ to $Q'_i(t)(D^n)$ for any $i \in I_n$: this can be seen geometrically in the picture after Claim 2 using the geometric idea of k. Therefore, f factors as the following natural transformation of pushout diagram

$$\begin{tikzcd} \coprod_{i\in I_n} D^n \arrow[d, "\coprod_{i\in I_n} V_i"'] & \coprod_{i\in I_n} S^{n-1} \arrow[l, hook'] \arrow[r, hook, "\coprod_{i\in I_n} q_i(\varepsilon(t))"] \arrow[d, "\coprod_{i\in I_n} v_i"'] & B_{n-1}\big(\varepsilon(t)\big) \arrow[d, "r"] \\ \coprod_{i\in I_n} D^n & \coprod_{i\in I_n} S^{n-1} \arrow[l, hook'] \arrow[r, hook, "\coprod_{i\in I_n} q'_i(t)"] & B_{n-1}(t)' \end{tikzcd} \tag{2.14}$$

where v_i and V_i are homeomorphisms for any $i \in I_n$ and r is the retraction of $B_{n-1}\big(\varepsilon(t)\big)$ into $B_{n-1}(t)'$ obtained by restriction of f.

Claim 4. Denote by $(\sigma_{D^n}, \sigma_{S^{n-1}}) \colon (D^n, S^{n-1}) \to (D^n, S^{n-1})$ the reflection at the equator $x_n = 0$. Then there exists a homotopy

$$(L, l) \colon (D^n, S^{n-1}) \times I \to (D^n, S^{n-1})$$

such that $(V_i, v_i) \simeq (\mathrm{id}_{D^n}, \mathrm{id}_{S^{n-1}})$ or $(V_i, v_i) \simeq (\sigma_{D^n}, \sigma_{S^{n-1}})$ for all $i \in I_n$.

Proof of Claim 4. Consider the following diagram, which is commutative by construction of v_i.

$$\begin{tikzcd} S^{n-1} \arrow[r, hook, "\mu_{\varepsilon(t)}"] \arrow[d, "v_i"'] & D^n \setminus \big(S^{n-1} \cup \{0\}\big) \arrow[r, "Q_i"] & Q_i(D^n) \setminus \Big(Q_i(S^{n-1}) \cup \big\{Q_i(0)\big\}\Big) \arrow[d, "f"] \\ S^{n-1} \arrow[r, hook, "\mu_t"] & D^n \setminus \big(S^{n-1} \cup \{0\}\big) \arrow[r, "Q'_i"] & Q_i(D^n) \setminus \Big(Q_i(S^{n-1}) \cup \big\{Q_i(0)\big\}\Big) \end{tikzcd}$$

We have that Q_i and Q'_i are homeomorphisms, the maps $\mu_{\varepsilon(t)}$ and μ_t are homotopy equivalences and the map f on the right, which is actually a restriction of the map f defined above, is also a homotopy equivalence, being homotopic to the identity map. Therefore, also v_i is a homotopy equivalence. In particular it has mapping degree 1 or -1. Now, define α to be id_{D^n} if the mapping degree of v_i is 1 and σ_{D^n} if it is -1. Then, by construction, there exists in both cases a homotopy $H\colon S^{n-1}\times I\to S^{n-1}$ such that v_i is homotopic to $\alpha|_{S^{n-1}}$. Extend H by cofibration property of the inclusion $S^{n-1}\hookrightarrow D^n$ to a homotopy $L'\colon D^n\times I\to D^n$ between V_i and a map $L'_1\colon D^n\to D^n$ such that its restriction to S^{n-1} is $\alpha|_{S^{n-1}}$. Define L to be the concatenation of the homotopy L' with the homotopy $L''\colon D^n\times I\to D^n$ relative S^{n-1} from L'_1 to α given by $(x,t)\mapsto t\cdot\alpha(x)+(1-t)\cdot L'_1(x)$ and let $l\colon S^{n-1}\times I\to S^{n-1}$ be the restriction of L. Then (L,l) is by construction the wanted homotopy between (V_i,v_i) and $(\alpha,\alpha|_{S^{n-1}})$. □

From now on, let us assume without loss of generality that (V_i,v_i) is homotopic to the identity: if this is not the case, it suffices to apply $(\sigma_{D^n},\sigma_{S^{n-1}})$ to the image of (Q_i,q_i). By Lemma 1.6, the homotopy k' defined in (2.13) induces a fiber homotopy equivalence $g_{k'}\colon E_n\to f^*E$ and a map $g=\overline{f}\circ g_{k'}\colon E_n\to E_n$ over f which is homotopic to the identity of E_n. Moreover, since k' is a homotopy relative B_{n-1}, the restriction of g to a map $E_{n-1}\to E_{n-1}$ is still homotopic to the identity. Now, g factors as the following natural transformation of pushout squares, which lies over the previous diagram (2.14).

$$\begin{tikzcd}
\coprod_{i\in I_n} Q_i\big(\varepsilon(t)\big)^*E \arrow[d, "\coprod_{i\in I_n}\Phi_i"'] & \coprod_{i\in I_n} q_i\big(\varepsilon(t)\big)^*E \arrow[l, hook'] \arrow[r, hook, "\coprod_{i\in I_n}\overline{q_i\left(\varepsilon(t)\right)}"] \arrow[d, "\coprod_{i\in I_n}\varphi_i"'] & E_{n-1}\big(\varepsilon(t)\big) \arrow[d, "\psi"] \\
\coprod_{i\in I_n} Q'_i(t)^*E & \coprod_{i\in I_n} q'_i(t)^*E \arrow[l, hook'] \arrow[r, hook, "\coprod_{i\in I_n}\overline{q'_i(t)}"] & E_{n-1}(t)'
\end{tikzcd} \tag{2.15}$$

All the vertical maps are homotopy equivalences because they are induced by the homotopy equivalences of diagram (2.14). Hence, if we denote by $j\colon E_{n-1}(t)\hookrightarrow E_n$ the canonical inclusion, by Lemma 2.15 we obtain:

$$\begin{aligned}
&\tau\big((E_n,\xi_n)\xrightarrow{\mathrm{id}}(E_n,\xi'_n)\big)=\\
=&\tau\big((E_n,\xi_n)\xrightarrow{g}(E_n,\xi'_n)\big)=\\
=&\sum_{i\in I_n}\overline{Q'_i(t)}_*\tau(\Phi_i)-\sum_{i\in I_n}j_*\,\overline{q'_i(t)}_*\tau(\varphi_i)+j_*\tau(\psi)
\end{aligned}$$

Now, $E_{n-1}\big(\varepsilon(t)\big)$ and $E_{n-1}(t)'$ have the simple structures induced by the inclusion of E_{n-1}. Moreover, the restriction of ψ to E_{n-1} is by construction exactly the restriction of g to E_{n-1} and so it is homotopic to the identity. Therefore, we obtain $\tau(\psi)=\tau\big((E_{n-1},\xi_{n-1})\xrightarrow{\mathrm{id}}(E_{n-1},\xi'_{n-1})\big)$ and

$$\begin{aligned}
&\tau\big((E_n,\xi_n)\xrightarrow{\mathrm{id}}(E_n,\xi'_n)\big)=\\
=&\sum_{i\in I_n}\overline{Q'_i(t)}_*\tau(\Phi_i)-\sum_{i\in I_n}j_*\,\overline{q'_i(t)}_*\tau(\varphi_i)+(j_{n-1})_*\tau\big((E_{n-1},\xi_{n-1})\xrightarrow{\mathrm{id}}(E_{n-1},\xi'_{n-1})\big)
\end{aligned} \tag{2.16}$$

where $j_{n-1}\colon E_{n-1}\hookrightarrow E_n$ is the inclusion.

At this point, we have reached the goal of calculating how the simple structure on E_n changes using the case of E_{n-1}, which is known by inductive assumption. To conclude the proof, it remains only to compute the torsion of the maps Φ_i and φ_i. The idea is to modify them through the homotopies L_i of the base space to get fiber homotopy equivalences $\overline{\Phi}_i$ and $\overline{\varphi}_i$ whose torsion is easily computable. More precisely, recall that by construction of f there is a homotopy k' such that $f\simeq\mathrm{id}_{B^n}$. On the other hand, by Claim 4, there exists a homotopy (L_i,l_i) for $i\in I_n$ such that $(V_i,v_i)\simeq(\mathrm{id}_{D^n},\mathrm{id}_{S^{n-1}})$. Therefore, if we consider the fiber homotopy equivalences $g_{k'}$ and g_{L_i} induced by these homotopies by Lemma 1.6, we get for $i\in I_n$ the following composition

$$\overline{\Phi_i}\colon Q_i\big(\varepsilon(t)\big)^*E\xrightarrow{Q_i\left(\varepsilon(t)\right)^*g_{k'}}Q_i\big(\varepsilon(t)\big)^*f^*E=V_i^*Q'_i(t)^*E\xrightarrow{g_{L_i}}Q'_i(t)^*E$$

where the equality holds because by construction of f and by diagram (2.14) we have that $f \circ Q_i\big(\varepsilon(t)\big) = Q_i'(t) \circ V_i$ and the map $Q_i\big(\varepsilon(t)\big)^* g_{k'}$ is obtained as pullback of the following commutative diagram.

$$\begin{tikzcd}
D^n \arrow[r, "Q_i(\varepsilon(t))"] \arrow[d, equal] & B_n \arrow[d, equal] & E_n \arrow[l, "p"'] \arrow[d, "g_{k'}"] \\
D^n \arrow[r, "Q_i(\varepsilon(t))"] & B_n & f^*E \arrow[l, "p_f"']
\end{tikzcd}$$

Define $\overline{\varphi_i}\colon q_i\big(\varepsilon(t)\big)^* E \to q_i'(t)^*E$ to be the restriction of $\overline{\Phi}_i$.

Claim 5. (i) There exists a homotopy H between (Φ_i, φ_i) and $(\overline{\Phi_i}, \overline{\varphi_i})$.

(ii) With the notation of equation (2.16), it holds:

$$\overline{Q_i'(t)}_*\tau(\Phi_i) - j_*\,\overline{q_i'(t)}_*\tau(\varphi_i) = (-1)^n \cdot (i_{b'})_*\tau\big((F_b,\zeta) \xrightarrow{t[v_i]} (F_{b'},\zeta')\big) \tag{2.17}$$

where v_i is the path of the statement of Lemma 2.21 associated to the i-th n-cell.

Proof of Claim 5. (i) Let us start by noting that the composition

$$Q_i\big(\varepsilon(t)\big)^*E \xrightarrow{Q_i(\varepsilon(t))^* g_{k'}} Q_i\big(\varepsilon(t)\big)^* f^*E \xrightarrow{\overline{Q_i(\varepsilon(t))}} f^*E \xrightarrow{\overline{f}} E_n$$

is equal to the map $\overline{Q_i'(t)} \circ \Phi_i$. Indeed, we have $\overline{Q_i\big(\varepsilon(t)\big)} \circ Q_i\big(\varepsilon(t)\big)^* g_{k'} = g_{k'} \circ \overline{Q_i\big(\varepsilon(t)\big)}$ by construction of $Q_i\big(\varepsilon(t)\big)^* g_{k'}$ and $\overline{f} \circ g_{k'} \circ \overline{Q_i\big(\varepsilon(t)\big)} = g \circ \overline{Q_i\big(\varepsilon(t)\big)} = \overline{Q_i'(t)} \circ \Phi_i$ by definition of Φ_i in diagram (2.15). It follows that the composition

$$Q_i\big(\varepsilon(t)\big)^*E \xrightarrow{Q_i(\varepsilon(t))^* g_{k'}} Q_i\big(\varepsilon(t)\big)^* f^*E = V_i^* Q_i'(t)^*E \xrightarrow{\overline{V_i}} Q_i'(t)^*E$$

is equal to Φ_i because their compositions with the injective map $\overline{Q_i'(t)}$ coincide, since by definition of f we have $f \circ Q_i\big(\varepsilon(t)\big) = Q_i'(t) \circ V_i$. Therefore, to get $\overline{\Phi}_i$ from Φ_i it suffices to change $\overline{V_i}$ into g_{L_i} through some homotopy. This follows easily by Lemma 1.6 applied to the fibration $p_{Q_i'(t)}\colon Q_i'(t)^*E \to D_n$ and the homotopy L_i. In fact, we get $g_{L_i} = \mathrm{id} \circ g_{L_i} = \overline{\mathrm{id}} \circ g_{L_i} \simeq \overline{V_i}$. To conclude, note that this construction is compatible with the restriction to $q_i\big(\varepsilon(t)\big)E$. Therefore, part (i) holds.

(ii) By part (i), it suffices to prove equation (2.17) using $\overline{\Phi_i}$ and $\overline{\varphi_i}$ instead of Φ_i and φ_i. Let w_i and w_i' be the paths in the spiders s and s' associated to the i-th n-cell. We assume without loss of generality that $w_i(1)$ and $w_i'(1)$ are contained in the image of the modified characteristic maps $Q_i\big(\varepsilon(t)\big)$ and $Q_i'(t)$, respectively: if this is not the case, it suffices to take a homotopic path with this property instead of w_i and w_i'. The simple structures on $Q_i\big(\varepsilon(t)\big)^*E$ and $Q_i'(t)^*E$ are defined by the respective fiber trivializations

$$T_i\colon F_b \times D^n \to Q_i\big(\varepsilon(t)\big)^*E \qquad \text{and} \qquad T_i'\colon F_{b'} \times D^n \to Q_i'(t)^*E$$

with respect to the triples $\big(b, w_i(1), w_i\big)$ and $\big(b', w_i'(1), w_i'\big)$, where, by definition of characteristic maps, we have used points of B_n instead of D^n in the notation. In particular, the torsion of the maps T_i and T_i' vanishes. Now, without loss of generality, again changing the path w_i' up to homotopy, we can assume that $\overline{\Phi}_i \circ T_i$ is a fiber trivialization with respect to the triple $\big(b, w_i'(1), w_i * u_i\big)$, for some path u_i in the i-th n-cell of B from $w_i(1)$ to

$w_i'(1)$. Therefore, by Lemma 1.18, there exists the following fiber homotopy commutative diagram.

$$\begin{tikzcd}
F_b \times D^n \arrow[r, "T_i"] \arrow[d, "t[v_i]\times \mathrm{id}_{D^n}"'] & Q_i\big(\varepsilon(t)\big)^*E \arrow[d, "\overline{\Phi_i}"] \\
F_{b'} \times D^n \arrow[r, "T_i'"] & Q_i'(t)^*E
\end{tikzcd}$$

Note that everything we have done so far is compatible with the restriction to S^n. we denote by $t_i' \colon F_{b'} \times S^{n-1} \to q_i'(t)^*E$ the restriction of the fiber trivialization T_i' to $F_{b'} \times S^{n-1}$. Then, by Lemma 2.15 and the vanishing of $\tau(T_i)$ and $\tau(T_i')$, we have

$$\begin{aligned}
&\overline{Q_i'(t)}_*\tau(\overline{\Phi_i}) - j_*\,\overline{q_i'(t)}_*\tau(\overline{\varphi_i}) \\
=&\overline{Q_i'(t)}_*(T_i')_*\tau\big(t[v_i]\times \mathrm{id}_{D^n}\big) - j_*\,\overline{q_i'(t)}_*(t_i')_*\tau\big(t[v_i]\times \mathrm{id}_{S^{n-1}}\big) \\
=&\chi(D^n)\cdot (i_{b'})_*\tau\big(t[v_i]\big) - \chi(S^{n-1})\cdot (i_{b'})_*\tau\big(t[v_i]\big) \\
=&(-1)^n\cdot (i_{b'})_*\tau\big(t[v_i]\big)
\end{aligned}$$

Therefore, equation (2.17) holds. $\square$

We can now finally conclude the proof of Lemma 2.21. Indeed, by (2.16), (2.17) and the inductive hypothesis, we have that

$$\begin{aligned}
&\tau\big((E_n,\xi_n) \xrightarrow{\mathrm{id}} (E_n,\xi_n')\big) \\
=&\sum_{i\in I_n}(-1)^n\cdot (i_b)_*\tau\big((F_b,\zeta) \xrightarrow{t[v_i]} (F_{b'},\zeta')\big) + \sum_{c\in I(B_{n-1})}(-1)^{\dim(c)}\cdot (i_{b'})_*\tau\big((F_b,\zeta) \xrightarrow{t[v_c]} (F_{b'},\zeta')\big) \\
=&\sum_{c\in I(B_n)}(-1)^{\dim(c)}\cdot (i_{b'})_*\tau\big((F_b,\zeta) \xrightarrow{t[v_c]} (F_{b'},\zeta')\big)
\end{aligned}$$

$\square$

2.3 The preferred simple structure on a manifold I

In this last section of this chapter, we present the case of **Man**, the category of closed topological manifolds, as an example of the previous section. In particular, we define a preferred simple structure on any closed manifold, so that the Whitehead torsion theory can be applied to this category simply by using the theory developed in **SStruct**. There are three choices of a simple structure on a manifold M in **Man** that we can consider canonical: one defined using triangulations, one using handlebody decompositions and one using disk bundles. However, it turns out that only the last one always exists. Therefore, we will choose this as the preferred one. On the other hand, it is possible to prove that, whenever they exist, all these canonical choices coincide. Hence, we will actually think of each of them as the preferred simple structure on M.

Let us start with the first canonical choice of simple structure M, based on the notion of triangulation of a space.

Definition 2.23. If X is in **Top**, a pair (K, h) where K is a simplicial complex and $h\colon |K| \to X$ is a homeomorphism from the geometric realization of K to X is called a *triangulation* of X.

Assume that M admits a smooth triangulation $h\colon |K| \to M$, that is, a triangulation such that the restriction of h to a simplex is a smooth C^∞-embedding. For example, assume that M is a smooth manifold (see [Mun63, Theorem 10.6 on page 103]). Then, the map h gives M the type of a simplicial complex and, therefore, by [FP90, Theorem 5.2.1], of a CW-complex, which we may assume finite by compactness. We denote by ξ_1 the simple structure on M represented by the triangulation $h\colon |K| \to M$.

The second way of equipping M with a simple structure uses its handlebody decomposition.

Definition 2.24. A *handlebody decomposition* of a closed manifold M is a finite sequence of manifolds $\emptyset = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_r$ such that:

- M_i is obtained from M_{i-1} by attaching a q_i-handle, that is, $M_i = M_{i-1} + (\varphi_i^{q_i})$ for some $\varphi_i^{q_i}\colon S^{q_i-1} \times D^{n-q_i} \hookrightarrow \partial M_{i-1}$;
- M_r is diffeomorphic to M.

Assume that M possesses a finite handlebody decomposition. For example assume that M is smooth (this follows by Morse theory, see for example [Mil63, part I]) or that $\dim(M) \geq 6$ (see [KS77, Theorem 2.1, Essay III]). By [LM23, Section 2.2], we can assume without loss of generality that the handles are ordered by dimension. Therefore, we can redefine the filtration $\emptyset = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_r$ of the previous definition to be such that:

- $M_{-1} = \emptyset$;
- M_i obtained from M_{i-1} by attaching all the i-handles;
- M_n is diffeomorphic to M for $n = \dim(M)$.

At this point, it is not hard to construct by induction on $i = -1, 0, 1, \ldots$ a finite CW-complex X and a homotopy equivalence $f\colon M \to X$ which gives M the type of a finite CW-complex: the idea is to squeeze any handle to get a cell (see [LM23, Section 2.3]). We define ξ_2 to be the simple structure on M represented by f.

Finally, there is the last canonical choice of simple structure on M, which is constructed using disk bundles. Consider any closed disk bundle T over M and select a smooth triangulation h over it. The simple structure on T given by h induces a simple structure ξ_3 on M via retraction onto M. Now, by [KS77, Theorem 4.1, Essay III], every manifold M in **Man** admits a simple structure of the kind of ξ_3. Therefore, among others, this is the most suitable choice to be the *preferred simple structure* on M. We define $\xi^{\mathrm{Top}}(M)$ to be the simple structure ξ_3.

In conclusion of this chapter, we compare these three different canonical choices of simple structure on M. In particular, we assume that the simple structures ξ_1 and ξ_2 on M exist and we compare them with the preferred one $\xi^{\mathrm{Top}}(M)$.

Theorem 2.25 ([KS77, Theorem 5.10 and 5.11, Essay III])**.** *Whenever they exist, the three simple structures ξ_1, ξ_2 and $\xi^{Top}(M)$ on M defined above agree.*

Therefore, once we know that all three choices ξ_1, ξ_2 and $\xi^{\mathrm{Top}}(M)$ exist, we can actually think of each of them as the preferred simple structure $\xi^{\mathrm{Top}}(M)$ on M.

Chapter 3

The fibering problem

The goal of this chapter is to apply the results obtained so far in the context of the fibering problem. In particular, given a map $f\colon M \to B$ in **Man**, we define two torsion obstructions whose vanishing is a necessary condition for the existence of a fiber bundle homotopic to f: $\theta(f)$ and $\tau_{\text{fib}}(f)$. As presented in the previous chapters, the strategy is to convert the map f into the fibration $\widehat{f}\colon \operatorname{FIB}(f) \to B$ and to compute some Whitehead torsion in **SStruct** to understand what happens if such a bundle homotopic to f exists. Therefore, we start by reducing the question to the category **Fib** of fibrations. We define $\theta(f)$ to be the obstruction that measures how "simple" is the fiber transport in $\operatorname{FIB}(f)$ along any loop of B. More precisely, $\theta(f)$ is the element in $H^1\Big(B; \operatorname{Wh}\left(\pi \operatorname{FIB}(f)\right)\Big)$ represented by the map that sends any homotopy class $[w]$ of loops in B to the Whitehead torsion of its associated fiber transport $t[w]$. Since any fiber transport of a fiber bundle is a homeomorphism, it easy to conclude that the vanishing of this obstruction is a necessary condition for f to be homotopic to a fiber bundle. Once we know that $\theta(f) = 0$, we come back to the general problem. We find out that another necessary condition for f being homotopic to a fiber a bundle is that we do not lose any information during the conversion λ_f of f into $\widehat{f}$. In particular, the simple structure used on $\operatorname{FIB}(f)$ to get that $\theta(f) = 0$, which is the simple structure on the total space of a fibration defined in the previous chapter, has to agree with the one induced by the preferred structure $\xi^{\text{Top}}(M)$ on M through λ_f. Therefore, the second obstruction $\tau_{\text{fib}}(f)$ measures how "simple" is to convert f into $\widehat{f}$ by computing the Whitehead torsion of λ_f and its vanishing is again necessary for f being homotopic to a fiber bundle.

In conclusion, we obtain the following theorem.

Theorem. *Let $f\colon M \to B$ be a map of closed manifolds with path-connected B. Suppose that for some $b \in B$ the homotopy fiber of f has the homotopy type of a finite* CW*-complex. Then if f is homotopic to a map $g\colon M \to B$ which is the projection of a locally trivial fiber bundle with a closed manifold as fiber, both $\theta(f)$ and $\tau_{\mathit{fib}}(f)$ vanish.*

The work is structured as follows. In Section 3.1, we introduce the obstruction $\theta(p)$ for a fibration p and we focus on the case where $\theta(p) = 0$. In Section 3.2, we present some compatibility results related to the simple structure on the total space of a fibration and, assuming $\theta(p) = 0$, we generalize the construction of this simple structure to the case where the base space is in **TFCW** instead of **FCW**. In Section 3.3, we explain why the preferred simple structure on closed manifolds defined in Section 2.3 is well-suited for our case. Finally, in Section 3.4, we define the two obstructions in the general case and we prove the theorem stated above.

3.1 Simple fibrations

In this section we describe the obstruction $\theta(p)$ for a map $p\colon E \to B$ in **Fib** such that its fiber F is in **TFCW**. Moreover, we show that it is really an obstruction to the fibering problem and, since we are interested in fibrations p such that $\theta(p) = 0$, we study how the simple structure on the total space of a fibration behaves in this case.

Let us start with the construction of $\theta(p)$.

Definition 3.1. Let $p\colon E \to B$ be an object of **Fib** such that B is path-connected and the fiber F_b of p over a point $b \in B$ is in **TFCW**. Let ζ be a simple structure on F_b. Consider the fiber transport functor $T\colon \pi B \to \mathrm{ho}(\mathbf{Top})$ of Definition 1.14 and restrict it to the full subcategory $\pi_1(B, b)$ of πB. We get a group homomorphism $[w] \mapsto t[w]$ from the fundamental group $\pi_1(B, b)$ of B to the group of fiber transports $t[w]$ along loops $[w] \in \pi_1(B, b)$. We define $\theta^b(p)$ to be the composition of such group homomorphism with the Whitehead torsion map and the homomorphism $(i_b)_*\colon \mathrm{Wh}(\pi F_b) \to \mathrm{Wh}(\pi E)$ induced by the inclusion $i_b\colon F_b \hookrightarrow E$, that is,

$$\theta^b(p)\colon \pi_1(B, b) \to \mathrm{Wh}(\pi E), \qquad [w] \mapsto (i_b)_*\tau\big(t[w]\colon (F_b, \zeta) \to (F_b, \zeta)\big)$$

Remark 3.2. The map $\theta^b(p)$ is independent of the choice of ζ. Indeed, let η be another simple structure on F_b. Then, by Lemma 2.15, we have:

$$\begin{aligned}
&(i_b)_*\tau\big(t[w]\colon (F_b, \eta) \to (F_b, \eta)\big)\\
=&(i_b)_*\tau\big((F_b, \eta) \xrightarrow{\mathrm{id}} (F_b, \zeta) \xrightarrow{t[w]} (F_b, \zeta) \xrightarrow{\mathrm{id}} (F_b, \eta)\big)\\
=&(i_b)_*\tau\big((F_b, \zeta) \xrightarrow{\mathrm{id}} (F_b, \eta)\big) + (i_b)_*\tau\big((F_b, \zeta) \xrightarrow{t[w]} (F_b, \zeta)\big) + (i_b)_*t[w]_*\tau\big((F_b, \eta) \xrightarrow{\mathrm{id}} (F_b, \zeta)\big)\\
=&(i_b)_*\tau\big((F_b, \zeta) \xrightarrow{\mathrm{id}} (F_b, \eta)\big) + (i_b)_*\tau\big((F_b, \zeta) \xrightarrow{t[w]} (F_b, \zeta)\big) - (i_b)_*\tau\big((F_b, \zeta) \xrightarrow{\mathrm{id}} (F_b, \eta)\big)\\
=&(i_b)_*\tau\big(t[w]\colon (F_b, \zeta) \to (F_b, \zeta)\big)
\end{aligned}$$

where the second to last equality holds because $i_b \circ t[w]$ is homotopic to i_b by Lemma 1.6 and $\tau\big((F_b, \eta) \xrightarrow{\mathrm{id}} (F_b, \zeta)\big) = -\tau\big((F_b, \zeta) \xrightarrow{\mathrm{id}} (F_b, \eta)\big)$ by composition formula and Definition 2.5(ii).

Lemma 3.3. *Consider the situation of Definition 3.1. Then:*

(i) *$\theta^b(p)$ is a well-defined group homomorphism.*

(ii) *Let $b' \in B$ be another point and $\gamma\colon I \to B$ be a path in B from b' to b. Denote by $c_\gamma\colon \pi_1(B, b) \to \pi_1(B, b')$ the isomorphism given by conjugation with γ. Then the following diagram is commutative:*

$$\begin{tikzcd}
\pi_1(B, b) \arrow[rr, "c_\gamma"] \arrow[dr, "\theta^b(p)"'] & & \pi_1(B, b') \arrow[dl, "\theta^{b'}(p)"] \\
& \mathrm{Wh}(\pi E) &
\end{tikzcd}$$

Proof. (i) By the properties of the Whitehead torsion and of fiber transports, it suffices to prove that $\theta^b(p)$ respects the group operations of $\pi_1(B, b)$ and $\mathrm{Wh}(\pi E)$. Let $v, w\colon I \to B$ be loops at b in B. Then, by Lemma 2.15, we have:

$$\begin{aligned}
\theta^b(p)(v * w) &= (i_b)_*\tau\big(t[v * w]\big)\\
&= (i_b)_*\tau\big(t[v] \circ t[w]\big)\\
&= (i_b)_*\tau\big(t[v]\big) + (i_b)_*(t[v])_*\tau\big(t[w]\big)\\
&= (i_b)_*\tau\big(t[v]\big) + (i_b)_*\tau\big(t[w]\big)\\
&= \theta^b(p)(v) + \theta^b(p)(w)
\end{aligned}$$

where we have used that $i_b \circ t[v] \simeq i_b$ by Lemma 1.6. Therefore, $\theta^b(p)$ is a well-defined group homomorphism.

(ii) Consider the fiber transport $t[\gamma]\colon F_{b'} \to F_b$ associated to γ. By Remark 3.2, we can choose without loss of generality two simple structures on F_b and $F_{b'}$ such that $\tau\big(t[\gamma]\big) = 0$. Therefore, by Lemma 2.15, we obtain that

$$\tau\big(t[\gamma^- * w * \gamma]\big) = \tau\big(t[\gamma]^{-1} \circ t[w] \circ t[\gamma]\big) = \big(t[\gamma]^{-1}\big)_* \tau\big(t[w]\big)$$

Hence, since $i_{b'} \circ t[\gamma]^{-1} \simeq i_b$ by Lemma 1.6, we can conclude as follows.

$$\begin{aligned}\theta^{b'}(p)\big(c_\gamma(w)\big) &= (i_{b'})_* \tau\big(t[\gamma]^{-1} \circ t[w] \circ t[\gamma]\big)\\ &= (i_{b'})_* \big(t[\gamma]^{-1}\big)_* \tau\big(t[w]\big)\\ &= (i_b)_* \tau\big(t[w]\big)\\ &= \theta^b(p)(w)\end{aligned}$$ □

Using the map $\theta^b(p)$, we can now define the obstruction $\theta(p)$.

Definition 3.4. Consider the situation of Definition 3.1. Since any Whitehead group is an abelian group, then $\theta^b(p)\colon \pi_1(B, b) \to \mathrm{Wh}(\pi E)$ factors through a map $\theta^b(p)\colon H_1(B) \to \mathrm{Wh}(\pi E)$. We define

$$\theta(p) \in H^1\big(B; \mathrm{Wh}(\pi E)\big)$$

to be the element determined by $\theta^b(p)\colon H_1(B) \to \mathrm{Wh}(\pi E)$ by the Universal Coefficient Theorem [Hat02, Theorem 3.2].

Remark 3.5.

- The obstruction $\theta(p)$ is well-defined. Indeed, since B is path-connected by assumption, the group homomorphism $H^1\big(B; \mathrm{Wh}(\pi E)\big) \to \mathrm{Hom}\,\big(H_1(B), \mathrm{Wh}(\pi E)\big)$ of the Universal Coefficient Theorem is actually an isomorphism. Therefore, $\theta(p)$ is uniquely defined by the map $\theta^b(p)$.
- The obstruction $\theta(p)$ is independent of $b \in B$. Indeed, it follows easily by Lemma 3.3(ii) that the map $\theta^b(p)\colon H_1(B) \to \mathrm{Wh}(\pi E)$ is so.
- The obstruction $\theta(p)$ can be also defined in the following faster alternative way: given a loop $w\colon S^1 \to B$ at b in B, we define $\theta(p)$ to be the element of $H^1\big(B; \mathrm{Wh}(\pi E)\big)$ that under the restriction map

$$H^1\big(B; \mathrm{Wh}(\pi E)\big) \xrightarrow{w^*} H^1\big(S^1; \mathrm{Wh}(\pi E)\big) \cong \mathrm{Wh}(\pi E)$$

is sent to $(i_b)_* \tau\big(t[w]\colon (F_b, \zeta) \to (F_b, \zeta)\big)$ for an arbitrary simple structure ζ on F_b. This definition is obviously equivalent to Definition 3.4. However, it is definitely more implicit, since it is less clear what $\theta(p)$ represents geometrically.

Let us now give a name to the fibrations p whose obstruction $\theta(p)$ vanishes.

Definition 3.6. Let $p\colon E \to B$ be an object of **Fib** whose fiber F is in **TFCW**. We call p *simple* if $\theta(p|_C) = 0$ holds for any component $C \in \pi_0(B)$ with respect to the restriction $p|_C\colon E|_C \to C$.

Remark 3.7. By construction of $\theta(p|_C)$, an object p in **Fib** is simple if and only if we have $\tau\big(t[w]\colon (F_b, \zeta) \to (F_b, \zeta)\big) = 0$ for any point $b \in B$, any path $w\colon I \to B$ and any simple structure ζ on F_b.

By construction, it is very easy to show now that $\theta(p)$ really defines an obstruction to the fibering problem.

Lemma 3.8. *Let $p\colon E \to B$ be a locally trivial fiber bundle in* **Top** *with typical fiber in* **FCW** *and paracompact base space. Then p is a simple fibration.*

Proof. The map p is in **Fib** by [Whi78, p. 33]. Moreover, since the fiber transport of a locally trivial fiber bundle is a homeomorphism and the Whitehead torsion of a homeomorphism vanishes by Lemma 2.1(v), then p is also simple. □

To conclude this section, we apply Lemma 2.21 of Chapter 2 to the case of simple fibration. In this way, we study how the simple structure on the total space of a fibration p depends on its initial data if we have $\theta(p) = 0$. This will be useful in Section 3.4 to define the second obstruction to the fibering problem.

Corollary 3.9. *Let $p\colon E \to B$ be a simple fibration such that B is a path-connected object of* **FCW** *and the fiber is in* **TFCW**. *Consider two points $b, b' \in B$, two spiders s at b and s' at b' in B and two simple structures ζ on F_b and ζ' on $F_{b'}$ and let $\xi(b, s, \zeta)$ and $\xi(b', s', \zeta')$ be the corresponding preferred simple structures on E as defined in Construction 2.17. Let $(i_{b'})_*\colon \mathrm{Wh}(\pi F_{b'}) \to \mathrm{Wh}(\pi E)$ be the homomorphism induced by the inclusion $i_{b'}\colon F_{b'} \to E$. Define*

$$\tau_0 = (i_{b'})_* \tau\big(t\colon (F_b, \zeta) \to (F_{b'}, \zeta')\big)$$

where $t\colon F_b \to F_{b'}$ represents the fiber transport $t[w]$ for some path $w\colon I \to B$ from b to b'. Then τ_0 is independent of the choice of w and

$$\tau\Big(\mathrm{id}\colon \big(E, \xi(b, s, \zeta)\big) \to \big(E, \xi(b', s', \zeta')\big)\Big) = \chi(B) \cdot \tau_0 \tag{3.1}$$

Proof. We prove first that τ_0 does not depend on the path in B from b to b'. Let w and v be two such paths. Then, since $i_{b'} \circ t[v] \simeq i_b$ by Lemma 1.6, by Lemma 2.15 we have

$$\begin{aligned}
(i_{b'})_* \tau\big(t[w]\big) &= (i_{b'})_* \tau\big(t[v] \circ t[v]^{-1} \circ t[w]\big) \\
&= (i_{b'})_* \tau\big(t[v] \circ t[w * v^-]\big) \\
&= (i_{b'})_* \tau\big(t[v]\big) + (i_{b'})_* t[v]_* \tau\big(t[w * v^-]\big) \\
&= (i_{b'})_* \tau\big(t[v]\big) + (i_b)_* \tau\big(t[w * v^-]\big) \\
&= (i_{b'})_* \tau\big(t[v]\big)
\end{aligned}$$

where the last equality holds because $(i_b)_* \tau\big(t[w * v^-]\colon (F_b, \zeta) \to (F_b, \zeta)\big) = 0$ as p is simple.

It remains to show that (3.1) holds. This follows by Lemma 2.21. Indeed, if for any cell $c \in I(B)$ we define $v_c = w_c * u_c * (w'_c)^-$ where w_c and w'_c are the path given by the spiders s and s' and u_c is any path in the interior of c from $w_c(1)$ to $w'_c(1)$, then by Lemma 2.21 we have

$$\tau\Big(\big(E, \xi(b, s, \zeta)\big) \xrightarrow{\mathrm{id}} \big(E, \xi(b', s', \zeta')\big)\Big) = \sum_{c \in I(B)} (-1)^{\dim(c)} \cdot (i_{b'})_* \tau\big((F_b, \zeta) \xrightarrow{t[v_c]} (F_{b'}, \zeta')\big)$$

Moreover, by the independence of τ_0 of the path from b to b', we have for any $c \in I(B)$ that

$$(i_{b'})_* \tau\big((F_b, \zeta) \xrightarrow{t[v_c]} (F_{b'}, \zeta')\big) = \tau_0$$

Therefore, if we denote by I_n the set of n-cells of B for any $n \in \mathbb{N}$, we can conclude as follows.

$$\begin{aligned}
\tau\Big(\big(E, \xi(b,s,\zeta)\big) \xrightarrow{\mathrm{id}} \big(E, \xi(b',s',\zeta')\big)\Big) &= \sum_{c \in I(B)} (-1)^{\dim(c)} \cdot (i_{b'})_* \tau\big((F_b,\zeta) \xrightarrow{t[v_c]} (F_{b'},\zeta')\big) \\
&= \sum_{c \in I(B)} (-1)^{\dim(c)} \cdot \tau_0 \\
&= \left(\sum_{n \in \mathbb{N}} (-1)^n |I_n|\right) \cdot \tau_0 \\
&= \chi(B) \cdot \tau_0 \qquad \square
\end{aligned}$$

Notation 3.10. Let $p\colon E \to B$ be simple a fibration such that B is a path-connected object of **FCW** and the fiber is in **TFCW**. Then, by Corollary 3.9, it follows easily that the simple structure $\xi(b,s,\zeta)$ is independent of the spider s. In this case, we denote it briefly by $\xi(b,\zeta)$.

The previous result has the following immediate consequence.

Corollary 3.11. *Consider the situation of the Corollary 3.9. Assume in addition that $\chi(B) = 0$. Then E carries a preferred simple structure.*

3.2 Some compatibility results

In this section, we present some results related to the simple structure on the total space of a fibration of Construction 2.17. In particular, using the theory of the previous section, we show how this structure is compatible with fiber homotopy equivalences, pushouts and pullbacks by simple homotopy equivalences. Moreover, in case of simple fibrations, we generalize its construction to fibrations whose base space is in **TFCW** instead of **FCW**. All this will allow us in the following sections to work with maps in **Man** and to study the second obstruction $\tau_{\mathrm{fib}}(f)$. Note that what is presented in this section may appear technical, but instead it is easy to realize that everything is completely canonical.

Compatibility with fiber homotopy equivalences

Let us start by considering the case of fiber homotopy equivalences.

Lemma 3.12. *Let $p\colon E \to B$ and $p'\colon E' \to B$ be two objects of $\mathbf{Fib}(B)$ such that B is a path-connected object of* **FCW** *and their fibers are in* **TFCW**. *Consider a fiber homotopy equivalence $\overline{f}\colon E \to E'$ from p to p'. Fix a point $b \in B$ and a spider s at b in B and define ζ and ζ' to be simple structures on the fibers F_b and F'_b of p and p' over b, respectively. Denote by $\overline{f}_b\colon F_b \to F'_b$ the homotopy equivalence induced by $\overline{f}$ and by $\overline{f}_*\colon H^1\big(B, \mathrm{Wh}(\pi E)\big) \xrightarrow{\cong} H^1\big(B, \mathrm{Wh}(\pi E')\big)$ the isomorphism induced by $\overline{f}$. Then we have*

$$\tau\Big(\overline{f}\colon \big(E, \xi(b,s,\zeta)\big) \to \big(E', \xi(b,s,\zeta')\big)\Big) = \chi(B)(i_b)_*\tau\big(\overline{f}_b\colon (F_b,\zeta) \to (F'_b,\zeta')\big) \tag{3.2}$$

$$\theta(p') = \overline{f}_*\big(\theta(p)\big) \tag{3.3}$$

Proof. Let us start with the proof of (3.2). This formula follows by the following equality of simple structures on E, where we use the notation of Example 2.7.

$$\overline{f}_*^{-1}\xi(b,s,\zeta') = \xi\big(b,s,(\overline{f}_b^{-1})_*\zeta'\big) \tag{3.4}$$

Indeed, assume that this holds. Then, by Lemma 2.21 and by construction of $\overline{f}_*^{-1}\xi(b,s,\zeta')$ and $(\overline{f}_b^{-1})_*\zeta'$, we have

$$\begin{aligned}
\tau\Big(\big(E,\xi(b,s,\zeta)\big) \xrightarrow{\overline{f}} \big(E',\xi(b,s,\zeta')\big)\Big) &= \tau\Big(\big(E,\xi(b,s,\zeta)\big) \xrightarrow{\mathrm{id}} \big(E,\overline{f}_*^{-1}\xi(b,s,\zeta')\big) \xrightarrow{\overline{f}} \big(E',\xi(b,s,\zeta')\big)\Big) \\
&= \overline{f}_*\tau\Big(\big(E,\xi(b,s,\zeta)\big) \xrightarrow{\mathrm{id}} \big(E,\overline{f}_*^{-1}\xi(b,s,\zeta')\big)\Big) \\
&= \overline{f}_*\tau\bigg(\big(E,\xi(b,s,\zeta)\big) \xrightarrow{\mathrm{id}} \Big(E,\xi\big(b,s,(\overline{f}_b^{-1})_*\zeta'\big)\Big)\bigg) \\
&= \sum_{c\in I(B)} (-1)^{\dim(c)}\cdot \overline{f}_*(i_{b'})_*\tau\Big((F_b,\zeta) \xrightarrow{t[v_c]} \big(F_b,(\overline{f}_b^{-1})_*\zeta'\big)\Big) \\
&= \sum_{c\in I(B)} (-1)^{\dim(c)}\cdot (i_b)_*\tau\big((F_b,\zeta) \xrightarrow{\overline{f}_b\circ t[v_c]} (F_b',\zeta')\big)
\end{aligned}$$

where for the last equality we have used that $\overline{f}\circ i_b = i_b\circ \overline{f}_b$ and the composition formula. Moreover, we have $t[v_c]=\mathrm{id}_{F_b}$, because, by construction, in this case the path $v_c = w_c * (w_c)^{-1}$ is obviously homotopic to the constant path const_b. Therefore, if we denote by I_n the set of n-cells of B, we can conclude as follows.

$$\begin{aligned}
\tau\Big(\big(E,\xi(b,s,\zeta)\big) \xrightarrow{\overline{f}} \big(E',\xi(b,s,\zeta')\big)\Big) &= \sum_{c\in I(B)} (-1)^{\dim(c)}\cdot (i_b)_*\tau\big((F_b,\zeta) \xrightarrow{\overline{f}_b\circ t[v_c]} (F_b',\zeta')\big) \\
&= \sum_{c\in I(B)} (-1)^{\dim(c)}\cdot (i_b)_*\tau\big((F_b,\zeta) \xrightarrow{\overline{f}_b} (F_b',\zeta')\big) \\
&= \left(\sum_{n\in\mathbb{N}} (-1)^n |I_n|\right)\cdot (i_b)_*\tau\big((F_b,\zeta) \xrightarrow{\overline{f}_b} (F_b',\zeta')\big) \\
&= \chi(B)(i_b)_*\tau\big(\overline{f}_b\colon (F_b,\zeta)\to (F_b',\zeta')\big)
\end{aligned}$$

Hence, we are left to prove (3.4). For this, let B_n be the n-skeleton of B and define $E_n = p^{-1}(B_n)$ and $E_n' = (p')^{-1}(B_n)$. Denote by ξ_n and ξ_n' the simple structures on E_n induced by $\xi\big(b,s,(\overline{f}_b^{-1})_*\zeta'\big)$ and $\xi(b,s,\zeta')$, that is, the simple structures at the n-th inductive step of Construction 2.17. We show by induction that the map $\overline{f}_n\colon (E_n,\xi_n)\to(E_n',\xi_n')$ induced by $\overline{f}$ is simple, so that also

$$\overline{f}\colon \Big(E,\xi\big(b,s,(\overline{f}_b^{-1})_*\zeta'\big)\Big) \to \big(E',\xi(b,s,\zeta')\big)$$

is so and (3.4) holds.

For $n=-1$, this is trivial. Consider the case $n=0$. We have that

$$\overline{f}_0\colon E_0 = \coprod_{i\in I_0} F_{b_i} \to \coprod_{i\in I_0} F_{b_i}' = E_0'$$

and the simple structures ξ_0 and ξ_0' are given by the disjoint union of the simple structures $t[w_i]_*(\overline{f}_b^{-1})_*\zeta'$ on F_{b_i} and $t'[w_i]_*\zeta'$ on F_{b_i}', respectively, where w_i is the path in s associated to $\{b_i\}$ for any $i\in I_0$. Moreover, by Lemma 1.11, the following diagram commutes up to fiber homotopy.

$$\begin{tikzcd}
F_b \arrow[r, "\overline{f}_b"] \arrow[d, "{t[w_i]}"'] & F_b' \arrow[d, "{t'[w_i]}"] \\
F_{b_i} \arrow[r, "\overline{f}"] & F_{b_i}'
\end{tikzcd}$$

Therefore, by Lemma 2.15 and by definition of the simple structure $(\overline{f}_b^{-1})_*\zeta'$, we can conclude as follows.

$$\tau\Big(\big(F_{b_i}, t[w_i]_*(\overline{f}_b^{-1})_*\zeta'\big) \xrightarrow{\overline{f}} \big(F'_{b_i}, t'[w_i]_*\zeta'\big)\Big) = \tau\Big(\big(F_b, (\overline{f}_b^{-1})_*\zeta'\big) \xrightarrow{\overline{f}_b} \big(F'_b, \zeta'\big)\Big) = 0$$

Assume now that $\overline{f}\colon (E_{n-1}, \xi_{n-1}) \to (E'_{n-1}, \xi'_{n-1})$ is simple. Then the map $\overline{f}_n$ factors as the following natural transformation of pushout squares.

$$\begin{tikzcd}
\coprod_{i\in I_n} Q_i^*E \arrow[d, "\coprod_{i\in I_n} Q_i^*\overline{f}"'] & \coprod_{i\in I_n} q_i^*E \arrow[l, hook'] \arrow[r, hook, "\coprod_{i\in I_n} \overline{q_i}"] \arrow[d, "\coprod_{i\in I_n} q_i^*\overline{f}"'] & E_{n-1} \arrow[d, "\overline{f}_{n-1}"] \\
\coprod_{i\in I_n} Q_i^*E' & \coprod_{i\in I_n} q_i^*E' \arrow[l, hook'] \arrow[r, hook, "\coprod_{i\in I_n} \overline{q_i}"] & E'_{n-1}
\end{tikzcd} \tag{3.5}$$

Moreover, ξ_n and ξ'_n are the simple structure induced on E_n and E'_n by these pushout diagrams. Therefore, it suffices to show that the left and middle vertical maps are simple and we are done. We prove it for $Q_i^*\overline{f}$: the proof for $q_i^*\overline{f}$ is obtained analogously by restricting all occurring homotopy equivalences. Recall that the spaces Q_i^*E and Q_i^*E' have the simple structures induced by the fiber trivializations $T_i\colon F_b \times D^n \to Q_i^*E$ and $T'_i\colon F'_b \times D^n \to Q_i^*E$ with respect to the same triple $(b, w_i(1), w_i)$, where w_i is the path in s associated to the i-th n-cell. Note that here, as in the proof of Lemma 2.21, we have used without loss of generality by definition of characteristic maps a point of B instead of a point of D^n as second entry of the triple. Now, by Lemma 1.11, the following diagram commutes up to fiber homotopy

$$\begin{tikzcd}
F_b \times D^n \arrow[r, "\overline{f}_b \times \mathrm{id}_{D^n}"] \arrow[d, "T_i"'] & F'_b \times D^n \arrow[d, "T'_i"] \\
Q_i^*E \arrow[r, "Q_i^*\overline{f}"] & Q_i^*E'
\end{tikzcd}$$

and the upper map is simple by product formula and by definition of the simple structure $(\overline{f}_b^{-1})_*\zeta'$ on F_b. Hence, we can conclude that $Q_i^*\overline{f}$ is simple as well. We have therefore proved that $\overline{f}_n$ is simple and, by induction, that (3.4) holds.

Let us prove now the second formula (3.3). By Definition 3.4, it suffices to show that $\theta^b(p') = \overline{f}_*\big(\theta^b(p')\big)$ as maps $\pi_1(B,b) \to \mathrm{Wh}(\pi E')$. Since $\theta^b(p)$ and $\theta^b(p')$ are independent of the simple structures on the fibers, we can assume without loss of generality that $\zeta' = (\overline{f}_b)_*\zeta$ to obtain

$$\tau\big((F_b, \zeta) \xrightarrow{\overline{f}_b} (F'_b, \zeta')\big) = 0$$

Moreover, by Lemma 1.11, the following diagram commutes up to fiber homotopy for any loop $w\colon S^1 \to B$ in B.

$$\begin{tikzcd}
F_b \arrow[r, "\overline{f}_b"] \arrow[d, "{t[w]}"'] & F'_b \arrow[d, "{t'[w]}"] \\
F_b \arrow[r, "\overline{f}_b"] & F'_b
\end{tikzcd}$$

Therefore, by Lemma 2.15, for any $[w] \in \pi_1(B,b)$ we have

$$\begin{aligned}
\theta^b(p')[w] &= (i'_b)_*\tau\big((F'_b,\zeta') \xrightarrow{t'[w]} (F'_b,\zeta')\big)\\
&= (i'_b)_*\tau\big((F'_b,\zeta') \xrightarrow{\overline{f}_b^{-1}} (F_b,\zeta) \xrightarrow{t[w]} (F_b,\zeta) \xrightarrow{\overline{f}_b} (F'_b,\zeta')\big)\\
&= (i'_b)_*\overline{f}_{b*}\tau\big((F_b,\zeta) \xrightarrow{t[w]} (F_b,\zeta)\big)\\
&= \overline{f}_*(i_b)_*\tau\big((F_b,\zeta) \xrightarrow{t[w]} (F_b,\zeta)\big)\\
&= \overline{f}_*\theta^b(p)[w]
\end{aligned}$$

where we have used that $\overline{f} \circ i_b = i'_b \circ \overline{f}_b$. This completes the proof of Lemma 3.12. $\square$

Compatibility with pushouts

We study now the compatibility of the simple structure on the total space of a simple fibration with pushouts. For this, we need the following extension of the notion of spider from spaces to maps.

Definition 3.13. Let X be a CW-complex and B be a path-connected object of **Top**. Consider a map $f\colon X \to B$ in **Top** and fix $b \in B$. A *spider* at b for f is a collection of paths w_c in B indexed by $c \in I(X)$, the set of open cells of X, such that $w_c(0) = b$ and $w_c(1)$ is in the image under f of a point in the open cell c.

The choice of a spider is, in general, a necessary ingredient for the construction of the simple structure on the total space of a fibration. Thanks to the previous definition we can generalize this construction as follows.

Definition 3.14. Let $p\colon E \to B$ be a fibration over a path-connected object B in **Top** such that its fiber is in **TFCW**, let X be in **FCW** and let $f\colon X \to B$ be a map. Choose a point $b \in B$ a spider s for f at b and a simple structure ζ on the fiber F_b of p over b. Then, Construction 2.17 generalizes to the construction of a simple structure $\xi(f,b,s,\zeta)$ on the pullback f^*E simply by applying the same construction to the fibration $f^*E \to X$, but using the fiber transport over B instead of the fiber transport over X. Obviously, we have that $\xi(b,s,\zeta)$ of Construction 2.17 is exactly $\xi(\mathrm{id}_B,b,s,\zeta)$.

Consider, now, the following pushout diagram in **FCW**.

$$\begin{array}{ccc}
B_0 & \xrightarrow{\varphi} & B_1\\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j}\\
B_2 & \xrightarrow{\Phi} & B
\end{array}$$

where B is path connected. Assume without loss of generality that the pairs (B_2,B_0) and (B,B_1) are CW-pairs, the maps i and j are inclusions, B is obtained by B_1 by attaching the relative cells of (B_2,B_0) and all maps are cellular. This can be done using the same argument of the proof of Construction/Proposition 2.10. Consider a simple fibration $p\colon E \to B$. Let $b \in B$ and let ζ be a simple structure on the fiber F_b of p over b. Denote by $\xi = \xi(b,\zeta)$ the associated simple structure on E. Recall that it is independent of the choice of a spider by Notation 3.10. Choose any spider s_1 for j at b, any spider s_2 for Φ at b and any spider s_0 for $j \circ \varphi$ at b. We obtain simple structures $\xi_1 = \xi(j,b,s_1,\zeta)$ on $E|_{B_1} = j^*E = p^{-1}(B_1)$, $\xi_2 = \xi(\Phi,b,s_2,\zeta)$ on Φ^*E

and $\xi_0 = \xi(j \circ \varphi, b, s_0, \zeta)$ on $\varphi^*(E|_{B_1})$ and the following commutative diagram

$$\begin{array}{ccc} \big(\varphi^*(E|_{B_1}), \xi_0\big) & \xrightarrow{\overline{\varphi}} & (E|_{B_1}, \xi_1) \\ \Big\downarrow{\scriptstyle \bar{i}} & & \Big\downarrow{\scriptstyle \bar{j}} \\ (\Phi^* E, \xi_2) & \xrightarrow{\overline{\Phi}} & (E, \xi) \end{array} \tag{3.6}$$

which is a pushout by Lemma 2.18.

Lemma 3.15. *Under the assumptions above, the square* (3.6) *is a simple pushout, that is, the pushout simple structure on E agrees with ξ.*

Proof. Since p is simple, we construct a spider s for B at b out of s_1 and s_1 as follows and we assume without loss of generality to use it to get ξ. By assumptions, the set $I(B)$ of the open cells of B decomposes as a disjoint union

$$I(B) = I(B_1) \coprod I(B_2, B_0)$$

where $I(B_1)$ and $I(B_2, B_0)$ are the sets of open cells of B_1 and of the relative CW-complex (B_2, B_0), respectively. Thus, we can take s as the union of the two collection of paths given by s_1 and the subset of s_2 induced by the inclusion $I(B_2, B_0) \subset I(B_2)$. Now, it suffices to prove the following claim to conclude.

Claim. Even if p is non-simple, if we define $\xi = \xi(b, s, \zeta)$ with this particular choice of s, then the square (3.6) is a simple pushout.

We prove it by induction over the dimension k of the relative CW-complex (B_2, B_0). If $B_2 = B_0$, then the claim obviously holds, because in this case $B = B_1$ and $s = s_1$ and therefore $\xi = \xi_1$ is trivially the pushout simple structure. Let now assume that the claim holds for $k = n-1$ and suppose that (B_2, B_0) has dimension n. Let $B_2^{(n-1)}$ and $B^{(n-1)}$ be the relative $(n-1)$-skeleta. Denote by $i' \colon B_0 \hookrightarrow B_2^{(n-1)}$ and $i'' \colon B_2^{(n-1)} \hookrightarrow B_2$ the inclusions and let $j' \colon B_1 \hookrightarrow B^{(n-1)}$ and $j'' \colon B^{(n-1)} \hookrightarrow B$ be the corresponding inclusions of subcomplexes. Let $\Phi' \colon B_2^{(n-1)} \to B^{(n-1)}$ be the restriction of Φ. We obtain the following commutative diagram.

$$\begin{array}{ccc} \varphi^*(E|_{B_1}) & \xrightarrow{\overline{\varphi}} & E|_{B_1} \\ \Big\downarrow{\scriptstyle \overline{i'}} & & \Big\downarrow{\scriptstyle \overline{j'}} \\ (\Phi')^*(E|_{B^{(n-1)}}) & \xrightarrow{\overline{\Phi'}} & E|_{B^{(n-1)}} \\ \Big\downarrow{\scriptstyle \overline{i''}} & & \Big\downarrow{\scriptstyle \overline{j''}} \\ \Phi^* E & \xrightarrow{\overline{\Phi}} & E \end{array}$$

Restrict s and s_2 to get spiders on $B^{(n-1)}$ and $B_2^{(n-1)}$ for j'' and $j'' \circ \Phi'$ at b. Then, we can equip all the spaces of the diagram with simple structures. We have to prove that the outer square is a simple pushout. By construction of the pushout simple structure and by the pasting law [AHS90, Proposition 11.10], it suffices to check that the upper and lower square are simple pushouts. For the upper one, this is true by induction hypothesis. For the lower one, this is an easy consequence of the construction of the simple structure on E. Indeed, $\xi(b, s, \zeta)$ is constructed as the pushout simple structure of the square where the other corners are the $(n-1)$-skeleton $E|_{B^{(n-1)}}$ and the pullbacks of E along the characteristic and attaching maps. But since $(B_2, B_2^{(n-1)})$ as relative CW-complex is equivalent to $\coprod_{i \in I_n}(D^n, S^{n-1})$, then the two simple structures that they induce on E agree. Hence, the claim holds. $\square$

Compatibility with pullbacks by simple homotopy equivalences

Finally, we present the compatibility of the simple structure on the total space of a simple fibration with pullbacks by a simple homotopy equivalence. This can be used to generalize the construction of the simple structure on the total space of a simple fibration with base space in **FCW** to the case where the base space is in **TFCW**.

Lemma 3.16. *Let $f\colon B' \to B$ be a map in* **FCW**. *Let $p\colon E \to B$ be a simple fibration whose fiber is in* **TFCW**. *Suppose that f is a simple homotopy equivalence and consider the following pullback square.*

$$\begin{tikzcd} f^*E \arrow[r, "\overline{f}"] \arrow[d, "p_f"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

For every component $C \in \pi_0(B')$ choose a base point x_C and equip the fiber $(p_f)^{-1}(x_C)$ with a simple structure ζ'_C. Let ζ_C be a simple structure on the fiber $p^{-1}\big(f(x_C)\big)$ such that

$$\tau\Big(\overline{f}|_{(p_f)^{-1}(x_C)}\colon \big((p_f)^{-1}(x_C), \zeta'_C\big) \to \Big(p^{-1}\big(f(x_C)\big), \zeta_C\Big)\Big) = 0 \tag{3.7}$$

Equip f^*E and E with the simple structures $\xi' = \xi(x_C, \zeta'_C)$ and $\xi = \xi\big(f(x_C), \zeta_C\big)$ in the notation of Notation 3.10. Then

$$\tau\big(\overline{f}\colon (f^*E, \xi') \to (E, \xi)\big) = 0$$

Proof. Assume without loss of generality that B', and thus also B, is path-connected and let $x = x_C$. Denote also $(p_f)^{-1}(x)$ simply by F_x and ζ'_C and ζ_C simply by ζ' and ζ. Note that $\overline{f}|_{(p_f)^{-1}(x)}\colon (p_f)^{-1}(x) \to p^{-1}\big(f(x)\big)$ is a homotopy equivalence by [Die08, Proposition (5.5.10)]. Therefore, the simple structures ζ' and ζ are well-defined. We prove this lemma in three steps.

(i) Assume first that f is a elementary expansion. In this case, B is defined by the following pushout square

$$\begin{tikzcd} S^{n-1}_+ \arrow[r, "q"] \arrow[d, hook] & B' \arrow[d, hook, "f"] \\ D^n \arrow[r, "Q"] & B \end{tikzcd}$$

where S^{n-1}_+ is the upper hemisphere of S^{n-1} and and the pair (D^n, S^{n-1}_+) has a natural relative CW-structure, given by an $(n-1)$-cell and an n-cell (see Section 2.1). Hence, E and $\overline{f}$ fit in the following commutative diagram

$$\begin{tikzcd} q^*(E|_{B'}) \arrow[r, "\overline{q}"] \arrow[d, hook, "\widetilde{f}"'] & f^*E = E|_{B'} \arrow[d, hook, "\overline{f}"] \\ Q^*E \arrow[r, "\overline{Q}"] & E \end{tikzcd}$$

which is a simple pushout by Lemma 2.18 and Lemma 3.15. It follows that if we apply the sum formula to the following diagram, it suffices to show that $\tau(\widetilde{f}) = 0$ to conclude that also $\tau(\overline{f}) = 0$.

$$\begin{tikzcd} q^*E|_{B'} \arrow[r, equal] \arrow[d, hook, "\widetilde{f}"'] & q^*E|_{B'} \arrow[r, "\overline{q}"] \arrow[d, equal] & E|_{B'} \arrow[d, equal] \\ Q^*E & q^*E|_{B'} \arrow[l, hook'] \arrow[r, "\overline{q}"] & E|_{B'} \end{tikzcd}$$

Since $\overline{f}$ is an inclusion, we denote $\overline{f}(x)$ simply by x. By Lemma 1.18 and Lemma 1.10, there exist two fiber trivializations $t\colon F_x \times S^{n-1}_+ \to q^*E|_{B'}$ and $T\colon F_x \times D^n \to Q^*E|_{B'}$ with respect to triples of the kind (x, y, const_x) such that the following diagram commutes.

$$\begin{tikzcd} F_x \times S^{n-1}_+ \arrow[r, "t"] \arrow[d, hook, "\iota"'] & q^*E|_{B'} \arrow[d, hook, "\widetilde{f}"] \\ F_x \times D^n \arrow[r, "T"] & Q^*E \end{tikzcd}$$

Now, the torsion of the two horizontal maps vanish by Lemma 3.12. Indeed, $t|_{F_x}$ and $T|_{F_x}$ are the identities by construction and equation (3.7) holds. Moreover, by a similar argument of Example 2.20, we have that the simple structures on $F_x \times S^{n-1}_+$ and $F_x \times D^n$ as total spaces of a projection are exactly the product simple structures. Hence, we obtain that $\tau(\iota) = 0$ because the inclusion $S^{n-1}_+ \hookrightarrow D^n$ is a simple homotopy equivalence by construction. Therefore, we can conclude that $\widetilde{f}$, and thus $\overline{f}$, has vanishing torsion.

(ii) We consider now the case where f is an elementary collapse. Let $l\colon B \to B'$ be an elementary expansion which is a homotopy inverse of f and let $H\colon B \times I \to B$ a homotopy such that $f \circ l \simeq \mathrm{id}_B$. Then, by Lemma 1.6, the map $\overline{f}\colon f^*E \to E$ is homotopic to the composition

$$g_H \circ \bar{l}^{-1}\colon f^*E \to l^*f^*E \to E$$

Now, we have that $\tau(\bar{l}^{-1}) = 0$ because $\tau(\bar{l}\,) = 0$ by the previous case. Moreover, by Lemma 3.12 and by equation (3.7), we have also that $\tau(g_H) = 0$. Therefore, we conclude that $\overline{f}$ has vanishing torsion.

(iii) Finally, we consider the general case where f is a simple homotopy equivalence. Then f is, by definition, homotopic to a composition $l = l_1 \circ \cdots \circ l_n$ of elementary expansion and collapses. Let $H\colon B' \times I \to B$ be a homotopy such that $f \simeq l$. Then, by Lemma 1.6, there exists a fiber homotopy equivalence $g_H\colon f^*E \to l^*E$ such that $\bar{l} \circ g_H \simeq \overline{f}$. Now, $\tau(\bar{l}) = 0$ by the previous cases and $\tau(g_H) = 0$ by Lemma 3.12 and by equation (3.7). Therefore, we can conclude that $\overline{f}$ has vanishing torsion. □

To conclude this section, we use the previous result to construct a well-defined simple structure on the total space of a simple fibration whose base space is in **TFCW**. This allows us to use from now on manifolds and their preferred structure as base spaces, as the fibering problem requires.

Definition 3.17. Let $p\colon E \to B$ be a simple fibration such that B and the fiber F_b of p over a point $b \in B$ are in **TFCW**. Suppose in addition that B is path-connected and let η and ζ be a simple structures on B and F_b, respectively. Then, we can define a simple structure on a total space E as follows. Choose a representative $f\colon X \to B$ of η and consider the pullback map $\overline{f}\colon f^*E \to E$. It is a homotopy equivalence by [Die08, Proposition (5.5.10)]. We can arrange by possibly changing f up to homotopy that $b = f(x)$ for some $x \in X$. Give f^*E the simple structure $\xi(x, \zeta)$. We equip E with the simple structure $\overline{f}_*\xi(x, \zeta)$ such that

$$\tau(\overline{f}\colon f^*E \to E) = 0$$

and we denote it by $\xi(\eta, b, \zeta)$.

Remark 3.18.
- The previous construction is well-defined by Lemma 3.16.
- By construction, the structure $\xi(\eta, b, \zeta)$ has all the properties as $\xi(b, s, \zeta)$. In particular, Lemma 3.12 holds also for this kind of simple structure.

3.3 The preferred simple structure on a manifold II

In this section, we consider the preferred simple structure $\xi^{\mathrm{Top}}(M)$ on a closed manifold M defined in Section 2.3 and we use all the machinery of the previous two sections to study its property. It turns out that $\xi^{\mathrm{Top}}(M)$ is particularly well-suited for us, since it is compatible with locally trivial fiber bundles.

Let us state and prove the main lemma.

Lemma 3.19. *Let $F \to M \xrightarrow{p} B$ be a locally trivial fiber bundle in* **Man** *with path-connected B. Then we have:*

$$\theta(p) = 0$$
$$\xi^{Top}(M) = \xi\big(\,\xi^{Top}(B), b, \xi^{Top}(F)\big)$$

where $\xi\big(\,\xi^{Top}(B), b, \xi^{Top}(F)\big)$ has been defined in Definition 3.17.

Proof. We have already proved $\theta(p) = 0$ in Lemma 3.8. Thus, let us consider the second formula. We prove it first in case $p\colon B \times F \to F$ is a globally trivial fiber bundle. By construction, if $f\colon X \to B$ is a representative of $\xi^{\mathrm{Top}}(B)$, then the simple structure $\xi\big(\,\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big)$ is the simple structure induced through the homotopy equivalence $f \times \mathrm{id}_F\colon X \times F \to B \times F$ by the simple structure $\xi\big(x, \xi^{\mathrm{Top}}(F)\big)$ on $X \times F$, where $x \in X$ is such that $f(x) = b$. Namely, we have

$$\xi\big(\,\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big) = (f \times \mathrm{id}_F)_* \,\xi\big(x, \xi^{\mathrm{Top}}(F)\big)$$

Now, choose a representative $g\colon Y \to F$ of $\xi^{\mathrm{Top}}(F)$ and consider the projection $X \times Y \to X$. By Example 2.20, the simple structure on the total space $X \times Y$ is exactly the product simple structure $\xi_{\mathrm{can}}(X) \times \xi_{\mathrm{can}}(Y)$. Moreover, by Lemma 3.12, we have that

$$\tau\Bigg(\,\mathrm{id}_X \times g\colon \Big(X \times Y, \xi_{\mathrm{can}}(X) \times \xi_{\mathrm{can}}(Y)\Big) \to \Big(X \times F, \xi\big(x, \xi^{\mathrm{Top}}(F)\big)\Big)\Bigg) = 0$$

Therefore, we obtain that

$$\xi\big(\,\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big) = (f \times g)_* \left(\xi_{\mathrm{can}}(X) \times \xi_{\mathrm{can}}(Y)\right)$$

In particular, $\xi\big(\,\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big)$ is the product simple structure $\xi^{\mathrm{Top}}(B) \times \xi^{\mathrm{Top}}(F)$ on $B \times F$. Therefore, since obviously the product of preferred simple structures on manifolds is the preferred simple structure on the product manifold, then we get that $\xi\big(\,\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big)$ agrees with $\xi^{\mathrm{Top}}(B \times F)$.

Let us consider now the general case. Since every fiber transport, being a homeomorphism, has vanishing torsion, then, by Corollary 3.9, the simple structure $\xi\big(\,\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big)$ does not depend on the point b. Thus, we can overlook the point b. Suppose now that $\dim(B) \geq 6$. Then, by [KS77, Theorem 2.1, Essay III], there exists a handlebody decomposition

$$D^n = B_0 \subset B_1 \subset \cdots \subset B_k = B$$

We proceed by induction over k. If $k = 0$, then the bundle is trivial, as D^n is contractible. Hence, the claim follows by the previous case. Let us assume that the claim holds for B_{k-1}. The space B_k is obtained from B_{k-1} by attaching a handle H, that is, by the following pushout diagram.

$$\begin{tikzcd} B_{k-1} \cap H \arrow[r] \arrow[d] & B_{k-1} \arrow[d] \\ H \arrow[r] & B_k \end{tikzcd}$$

Therefore, if we apply Lemma 2.18 and Lemma 3.15, we get that the following diagram is a simple pushout

$$\begin{array}{ccc} \Big(E|_{B_{k-1}\cap H}, \xi\big(\xi^{\mathrm{Top}}(B_{k-1}\cap H), b, \xi^{\mathrm{Top}}(F)\big)\Big) & \longrightarrow & \Big(E|_{B_{k-1}}, \xi\big(\xi^{\mathrm{Top}}(B_{k-1}), b, \xi^{\mathrm{Top}}(F)\big)\Big) \\ \downarrow & & \downarrow \\ \Big(E|_{H}, \xi\big(\xi^{\mathrm{Top}}(H), b, \xi^{\mathrm{Top}}(F)\big)\Big) & \longrightarrow & \Big(E|_{B_k}, \xi\big(\xi^{\mathrm{Top}}(B_k), b, \xi^{\mathrm{Top}}(F)\big)\Big) \end{array}$$

where we denote by $E|_Y$ the subspace $p^{-1}(Y)$ of M for any subspace $Y \subset B$. Now, the simple structure $\xi\big(\xi^{\mathrm{Top}}(B_{k-1}), b, \xi^{\mathrm{Top}}(F)\big)\Big)$ on the upper-right corner agrees with the preferred one by induction hypothesis. Moreover, since by contractibility of H the bundle is trivial on $E|_H$ and on its subspace $E|_{B_{k-1}\cap H}$, also the simple structures on the left column agree with the preferred one. Therefore, by using the handlebody method to define the preferred simple structure on a manifold (see Theorem 2.25), it follows immediately that the pushout simple structure on $E|_{B_k}$ given by the previous diagram is $\xi^{\mathrm{Top}}(E|_{B_k})$. Hence, since the previous pushout is simple, the two structures $\xi\big(\xi^{\mathrm{Top}}(B_k), b, \xi^{\mathrm{Top}}(F)\big)\Big)$ and $\xi^{\mathrm{Top}}(E|_{B_k})$ on $E|_{B_k}$ agree.

It remains to consider the case $\dim(B) \leq 5$. Let N be a 1-connected object in **Man** such that $\dim(N) \geq 6$ and $\chi(N) = 1$. For example, let $N = (\mathbb{CP}^2 \times \mathbb{CP}^2)\#4(S^3 \times S^5)$. Apply what we have just proved to the fiber bundle $p \times \mathrm{id}_N \colon M \times N \to B \times N$. Then, for any $n \in N$, we have

$$\xi^{\mathrm{Top}}(M \times N) = \xi\big(\xi^{\mathrm{Top}}(B \times N), (b, n), \xi^{\mathrm{Top}}(F)\big) \tag{3.8}$$

By construction, the simple structure on the right hand side coincide with the simple structure $\xi\big(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big) \times \xi^{\mathrm{Top}}(N)$. Moreover, $\dim(N \times B) \geq 6$. Therefore, by Lemma 2.15 and by equation (3.8), we can conclude as follows.

$$\begin{aligned} &\tau\bigg(\big(M, \xi^{\mathrm{Top}}(M)\big) \xrightarrow{\mathrm{id}} \Big(M, \xi\big(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big)\Big)\bigg) \\ =&\tau\bigg(\big(M, \xi^{\mathrm{Top}}(M)\big) \xrightarrow{\mathrm{id}} \Big(M, \xi\big(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big)\Big)\bigg) \cdot \chi(N) \\ =&\tau\bigg(\big(M \times N, \xi^{\mathrm{Top}}(M) \times \xi^{\mathrm{Top}}(N)\big) \xrightarrow{\mathrm{id}} \Big(M \times N, \xi\big(\xi^{\mathrm{Top}}(B), b, \xi^{\mathrm{Top}}(F)\big) \times \xi^{\mathrm{Top}}(N)\Big)\bigg) \\ =&\tau\bigg(\big(M \times N, \xi^{\mathrm{Top}}(M) \times \xi^{\mathrm{Top}}(N)\big) \xrightarrow{\mathrm{id}} \Big(M \times N, \xi\big(\xi^{\mathrm{Top}}(B \times N), (b, n), \xi^{\mathrm{Top}}(F)\big)\Big)\bigg) \\ =&0 \end{aligned}$$

□

3.4 Fiber torsion obstructions

In this section, given a map $f \colon M \to B$ in **Man**, we define the two obstructions $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$ for the existence of a manifold bundles. Moreover, we finally apply our strategy to study the fibering problem by proving the main theorem of this thesis, which claims that the vanishing of such obstructions is a necessary condition for a map being homotopic to the projection of a locally trivial fiber bundle.

Notation 3.20. Throughout this section, we denote by abuse of notation the simple structures $\xi(\xi^{\mathrm{Top}}(B), b, \zeta)$ constructed in Definition 3.17 simply by $\xi(b, \zeta)$. This can be done lightly by Remark 3.18, since all the results that hold for $\xi(b, \zeta)$ still hold for $\xi(\xi^{\mathrm{Top}}(B), b, \zeta)$.

Let us start by introducing $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$. As expected, they are defined by turning the map f in question into a fibration as explained in Section 1.2.

Definition 3.21. Let $f\colon M \to B$ be a map in **Man** with path-connected B. Suppose that for some (and hence all) $b \in B$ the homotopy fiber $\mathrm{hofib}(f)_b$ is in **TFCW**.

(i) We define the element
$$\theta(f) \in H^1\big(B; \mathrm{Wh}(\pi M)\big)$$
to be the image of the obstruction $\theta(\widehat{f})$ of the fibration $\widehat{f}\colon \mathrm{FIB}(f) \to B$ defined in Definition 3.4 under the isomorphism $(\mu_f)_*\colon H^1\Big(B; \mathrm{Wh}\left(\pi\, \mathrm{FIB}(f)\right)\Big) \to H^1\big(B; \mathrm{Wh}(\pi M)\big)$ induced by the homotopy equivalence $\mu_f\colon \mathrm{FIB}(f) \to M$.

(ii) Suppose that $\theta(f)$ vanishes. Let $(\mu_f \circ i_b)_*\colon \mathrm{Wh}\left(\pi\, \mathrm{hofib}(f)_b\right) \to \mathrm{Wh}(\pi M)$ be the map induced by the composite $\mathrm{hofib}(f)_b \xrightarrow{i_b} \mathrm{FIB}(f) \xrightarrow{\mu_f} M$. We define the *fiber torsion obstruction*
$$\tau_{\mathrm{fib}}(f) \in \mathrm{coker}\Big(\chi(B)(\mu_f \circ i_b)_*\colon \mathrm{Wh}\left(\pi\, \mathrm{hofib}(f)_b\right) \to \mathrm{Wh}(\pi M)\Big)$$
to be the class for which a representative in $\mathrm{Wh}(\pi M)$ is the image of the Whitehead torsion
$$\tau\Big(\lambda_f\colon \big(M, \xi^{\mathrm{Top}}(M)\big) \to \big(\mathrm{FIB}(f), \xi(b,\zeta)\big)\Big)$$
under the isomorphism $(\mu_f)_*\colon \mathrm{Wh}\left(\pi\, \mathrm{FIB}(f)\right) \to \mathrm{Wh}(\pi M)$ for some choice of a base point $b \in B$ and a simple structure ζ on $\mathrm{hofib}(f)_b$.

Note that, as explained at the beginning of this section, the geometric intuition is that $\theta(f)$ measures how "simple" are fiber transports along loops, while $\tau_{\mathrm{fib}}(f)$ measures how "simple" is to convert f into the fibration $\widehat{f}$. Let us show now that these two obstructions are well-defined and they does not depend on the simple structure ζ on $\mathrm{hofib}(f)_b$ and on $b \in B$.

Remark 3.22.

- The image of the map $(\mu_f \circ i_b)_*\colon \mathrm{Wh}\left(\pi\, \mathrm{hofib}(f)_b\right) \to \mathrm{Wh}(\pi M)$ is independent of the choice of $b \in B$ and thus so is the cokernel
$$\mathrm{coker}\Big(\chi(B)(\mu_f \circ i_b)_*\colon \mathrm{Wh}\left(\pi\, \mathrm{hofib}(f)_b\right) \to \mathrm{Wh}(\pi M)\Big)$$
Indeed, let b' be another base point. Then, by Lemma 1.6, the fiber transport along some path w from b to b' defines a homotopy equivalence $t[w]\colon \mathrm{hofib}(f)_b \to \mathrm{hofib}(f)_{b'}$ such that $\mu_f \circ i_b$ and $\mu_f \circ i_{b'} \circ t[w]$ are homotopic. Hence, they induce the same map on Whitehead groups and $\mu_f \circ i_b$ and $\mu_f \circ i_{b'}$ have the same image.

- The obstruction $\tau_{\mathrm{fib}}(f)$ is a well-defined invariant of f. Indeed, let us assume that $\theta(f) = 0$. We already know that the spider in this case does not play a role (see Notation 3.10). We show that it is independent also of the base point $b \in B$ and the simple structure ζ on $\mathrm{hofib}(f)_b$. Suppose we have made a different choice of a base point $b' \in B$ and of a simple structure ζ' on $\mathrm{hofib}(f)_{b'}$. Then, by Lemma 2.15, we have
$$\begin{aligned}\tau\Big(\lambda_f\colon \big(M, \xi^{\mathrm{Top}}(M)\big) \to \big(\mathrm{FIB}(f), \xi(b',\zeta')\big)\Big) - \tau\Big(\lambda_f\colon \big(M, \xi^{\mathrm{Top}}(M)\big) \to \big(\mathrm{FIB}(f), \xi(b,\zeta)\big)\Big) \\ = -\tau\Big(\mathrm{id}\colon \big(\mathrm{FIB}(f), \xi(b',\zeta')\big) \to \big(\mathrm{FIB}(f), \xi(b,\zeta)\big)\Big)\end{aligned}$$
Therefore, by Corollary 3.9, we can conclude that the difference vanishes in the cokernel of $\chi(B)(\mu_f \circ i_b)_*$ and thus $\tau_{\mathrm{fib}}(f)$ is well-defined.

We state and prove now the main theorem of this thesis. The proof follows the strategy presented at the beginning of this chapter and it is very linear and natural. This emphasizes how the two vanishing condition are really intrinsic in the property of being homotopic to a locally trivial fiber bundle and, therefore, how the two invariants in question are really obstructions to the fibering problem.

Theorem 3.23. *Let $f\colon M \to B$ be a map in* **Man** *with path-connected B. Suppose that for some (and hence all) $b \in B$ the homotopy fiber* $\mathrm{hofib}(f)_b$ *is in* **TFCW**. *Then*

(i) The element $\theta(f)$ depends only on the homotopy class of f. Moreover, if $\theta(f)$ vanishes, then the same statement holds for the fiber torsion obstruction $\tau_{fib}(f)$.

(ii) If f is homotopic to a map $g\colon M \to B$ which is the projection of a locally trivial fiber bundle with fiber F in **Man**, *then both $\theta(f)$ and $\tau_{fib}(f)$ vanish.*

Proof. (i) Let $H\colon M \times I \to B$ be a homotopy between two maps $f, g\colon M \to B$ in **Man** such that for some (and hence all) $b \in B$ their homotopy fibers $\mathrm{hofib}(f)_b$ and $\mathrm{hofib}(g)_b$ are in **TFCW**. Consider the fiber homotopy equivalence $\widehat{H}\colon \mathrm{FIB}(f) \to \mathrm{FIB}(g)$ of Lemma 1.21, which is defined by $\widehat{H}(x, w) = (x, v)$ for the following path $v\colon I \to B$

$$v(t) = \begin{cases} H^{-}(x, 2t) & \text{if } 0 \le t \le \frac{1}{2} \\ w(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Recall that $\lambda_f\colon M \to \mathrm{FIB}(f)$ is defined by $x \mapsto (x, \mathrm{const}_{f(x)})$, where $\mathrm{const}_{f(x)}\colon I \to B$ is the constant path $t \mapsto f(x)$. Therefore, the map $\widehat{H}$ has the obvious property that $\widehat{H} \circ \lambda_f \simeq \lambda_g$, which implies that $\mu_g \circ \widehat{H} \simeq \mu_f$. Moreover, Lemma 3.12 implies that the isomorphism

$$\widehat{H}_*\colon H^1\Big(B; \mathrm{Wh}\left(\pi\,\mathrm{FIB}(f)\right)\Big) \to H^1\Big(B; \mathrm{Wh}\left(\pi\,\mathrm{FIB}(g)\right)\Big)$$

sends $\theta(\widehat{f})$ to $\theta(\widehat{g})$. Hence, by Definition 3.21, since homotopic maps induce the same map on Whitehead groups, we get

$$\theta(g) = (\mu_g)_*\theta(\widehat{g}) = (\mu_g)_*\widehat{H}_*\theta(\widehat{f}) = (\mu_f)_*\theta(\widehat{f}) = \theta(f)$$

Now, suppose $\theta(f) = 0$. Then, the fibrations $\widehat{f}\colon \mathrm{FIB}(f) \to B$ and $\widehat{g}\colon \mathrm{FIB}(g) \to B$ are simple by definition and by homotopy invariance. Fix a base point $b \in B$ and a simple structure ζ on $\mathrm{hofib}(f)_b$. Equip $\mathrm{hofib}(g)_b$ with a simple structure ζ' such that the homotopy equivalence $\widehat{H}_b\colon \left(\mathrm{hofib}(f)_b, \zeta\right) \to \left(\mathrm{hofib}(g)_b, \zeta'\right)$ induced by $\widehat{H}$ has vanishing torsion. Then, by Lemma 3.12, we have that

$$\tau\Big(\widehat{H}\colon \left(\mathrm{FIB}(f), \xi(b,\zeta)\right) \to \left(\mathrm{FIB}(g), \xi(b,\zeta')\right)\Big) = 0$$

Therefore, since $\widehat{H} \circ \lambda_f \simeq \lambda_g$, we obtain

$$\tau(\lambda_g) = \tau(\widehat{H} \circ \lambda_f) = \widehat{H}_*\tau(\lambda_f)$$

and

$$(\mu_g)_*\tau(\lambda_g) = (\mu_g)_*\widehat{H}_*\tau(\lambda_f) = (\mu_f)_*\tau(\lambda_f)$$

Hence, by definition, we conclude that $\tau_{\mathrm{fib}}(f) = \tau_{\mathrm{fib}}(g)$, since we have already shown that the definition of τ_{fib} is independent of the choice of the point b and of the simple structure on the homotopy fiber.

(ii) Let g be a fiber bundle homotopic to f. By (i), we have $\theta(f) = \theta(g)$. Moreover, by Lemma 1.21, the map $\lambda_g\colon M \to \mathrm{FIB}(g)$ is a fiber homotopy equivalence from $\widehat{g}$ to g. Therefore, by Lemma 3.12, we can compute $\theta(g)$ directly from the bundle g instead of $\widehat{g}$. Now, Lemma 3.8 implies that $\theta(g) = 0$. Hence, we conclude that also $\theta(f) = 0$.

Consider now $\tau_{\mathrm{fib}}(f)$. Since $\theta(f) = 0$, then we have $\tau_{\mathrm{fib}}(f) = \tau_{\mathrm{fib}}(g)$ by part (i). Therefore, it suffices to prove that $\tau_{\mathrm{fib}}(g) = 0$. As above, the map $\lambda_g\colon M \to \mathrm{FIB}(g)$ is a fiber homotopy equivalence from $\widehat{g}$ to g. Fix a base point $b \in B$ and equip $\mathrm{hofib}(g)_b$ with a simple structure ζ such that the homotopy equivalence $(\lambda_g)_b\colon \big(F, \xi^{\mathrm{Top}}(F)\big) \to \big(\mathrm{hofib}(g)_b, \zeta\big)$ induced by λ_g has vanishing torsion. Then, by Lemma 3.12 and Lemma 3.19, we have

$$\tau\Big(\lambda_g\colon \big(M, \xi^{\mathrm{Top}}(M)\big) \to \big(\mathrm{FIB}(g), \xi(b,\zeta)\big)\Big) = 0$$

Hence, we can conclude by definition that $\tau_{\mathrm{fib}}(g)$, and hence $\tau_{\mathrm{fib}}(f)$, vanishes, since τ_{fib} is independent of the choice of b and ζ. □

Remark 3.24. Once this theorem is proved, the following natural question arises: does the converse implication of Theorem 3.23(ii) hold? The aim the following chapters is to try to answer to this question. In particular, in the fourth chapter we prove that if $B = S^1$, then the vanishing of $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$ is also a sufficient condition for f being homotopic to a fiber bundle, while in the fifth chapter we present the more general stable fibering problem, which has a complete obstruction theory in algebraic K-theory for existence and uniqueness.

Let us conclude this chapter by studying some special case of the fiber torsion obstructions.

Example 3.25. Let $f\colon M \to B$ be a map in **Man** with path-connected B. Suppose that for some (and hence all) $b \in B$ the homotopy fiber $\mathrm{hofib}(f)_b$ is in **TFCW**. If $\chi(B) = 0$ and $\theta(f)$ vanishes, then the invariant $\tau_{\mathrm{fib}}(f)$ defined in Definition 3.21 is an element of $\mathrm{Wh}(\pi M)$. More precisely, if $\chi(B) = 0$, then by Corollary 3.11 the space $\mathrm{FIB}(f)$ carries a preferred simple structure ξ and the element $\tau_{\mathrm{fib}}(f)$ is the image of

$$\tau\Big(\lambda_f\colon \big(M, \xi^{\mathbf{Top}}(M)\big) \to \big(\mathrm{FIB}(f), \xi\big)\Big) \in \mathrm{Wh}\left(\pi\, \mathrm{FIB}(f)\right)$$

under the isomorphism $(\mu_f)_*\colon \mathrm{Wh}\left(\pi\, \mathrm{FIB}(f)\right) \to \mathrm{Wh}(\pi M)$.

Example 3.26. Let $f\colon M \to B$ be a map in **Man** with path-connected B and M. Suppose that for some (and hence all) $b \in B$ the homotopy fiber $\mathrm{hofib}(f)_b$ it in **TFCW**. Assume in addition that the Whitehead group of the kernel of $\pi_1(f)\colon \pi_1(M) \to \pi_1(B)$ is trivial. For example, assume that $\pi_1(f)$ is bijective. Then $\theta(f)$ vanishes. Indeed, fix $b \in B$. By Definition 3.21 and Definition 3.4, the obstruction $\theta(f)$ is represented in $H^1\big(B; \mathrm{Wh}(\pi M)\big)$ by the map

$$\pi_1(B,b) \to \mathrm{Wh}(\pi M), \qquad [w] \mapsto (\mu_f \circ i_b)_* \tau\Big(t[w]\colon \big(\mathrm{hofib}(f)_b, \zeta\big) \to \big(\mathrm{hofib}(f)_b, \zeta\big)\Big)$$

where ζ is any simple structure on $\mathrm{hofib}(f)_b$. Therefore, it suffices to show that the group homomorphism $(\mu_f \circ i_b)_*\colon \mathrm{Wh}\left(\pi\, \mathrm{hofib}(f)_b\right) \to \mathrm{Wh}(\pi M)$ is trivial. For this, fix $x \in \mathrm{hofib}(f)_b$ and consider the long exact sequence of homotopy groups for the fibration $\mathrm{FIB}(f) \to B$.

$$\cdots \to \pi_2(B,b) \to \pi_1\big(\mathrm{hofib}(f)_b, x\big) \xrightarrow{\pi_1(i_b)} \pi_1\big(\mathrm{FIB}(f), x\big) \xrightarrow{\pi_1(\widehat{f})} \pi_1(B,b) \to \dots$$

We obtain that $\pi_1(f) \circ \pi_1(\mu_f) \circ \pi_1(i_b) = \pi_1(\widehat{f}) \circ \pi_1(i_b) = 0$. In particular, we get that $\mathrm{im}\,\big(\pi_1(\mu_f) \circ \pi_1(i_b)\big) \subset \ker \pi_1(f)$. It suffices now to apply the Whitehead torsion functor to this inclusion to realize that the map $(\mu_f \circ i_b)_*\colon \mathrm{Wh}\left(\pi\, \mathrm{hofib}(f)_b\right) \to \mathrm{Wh}(\pi M)$ is trivial, since it factors through $\mathrm{Wh}\left(\ker \pi_1(f)\right)$, which is trivial by assumption.

Chapter 4

The fibering problem over a circle

In this chapter we present an example where the vanishing of the fiber torsion obstructions $\theta(f)$ and $\tau_{\text{fib}}(f)$ is both a sufficient and necessary condition for a map $f\colon M \to B$ in **Man** being homotopic to a fiber bundle. In particular, we show that this is the case if f is a smooth map with base space $B = S^1$. This points out that the two obstructions $\theta(f)$ and $\tau_{\text{fib}}(f)$ do not vanish for all maps in **Man** and, therefore, it emphasizes once again that they are actually obstructions for the fibering problem. Since we already know that the vanishing of $\theta(f)$ and $\tau_{\text{fib}}(f)$ is a necessary condition, we have only to focus on the "sufficiency" part. The idea is to prove it using the results obtained by Farrell in [Far71]. More precisely, first we show that for a smooth map $f\colon M \to S^1$ over a circle the obstructions $\theta(f)$ and $\tau_{\text{fib}}(f)$ combine in one single invariant ${\tau_{\text{fib}}}'(f)$ and then we investigate how ${\tau_{\text{fib}}}'(f)$ is related to Farrell's obstructions $c(f)$ and $\tau(f)$. It turns out that if we consider the twisted Bass-Heller-Swan decomposition of $\mathrm{Wh}(\pi M)$

$$\mathrm{Wh}(\pi M) \cong \mathrm{Wh}(G \rtimes \mathbb{Z}) \cong X_1(\mathbb{Z}G, \alpha) \oplus NK_1(\mathbb{Z}G, \alpha) \oplus NK_1(\mathbb{Z}G, \alpha^{-1})$$

given for example in [FH70], then the two Farrell's obstructions $c(f)$ and $\tau(f)$ are in some sense the projections of ${\tau_{\text{fib}}}'(f)$ on two different components of $\mathrm{Wh}(\pi M)$. This observation together with the main theorem of [Far71] implies that the vanishing of $\theta(f)$ and $\tau_{\text{fib}}(f)$ is also a sufficient condition for a smooth map $f\colon M \to S^1$ being homotopic to a fiber bundle. To sum up, we obtain the following theorem.

Theorem 4.1. *Let $f\colon M \to S^1$ be a map in the subcategory* **Diff** *of* **Man** *whose object are smooth manifolds and morphisms are smooth maps. Assume that M is connected of dimension $\dim(M) \geq 6$ and that the homotopy fiber of f is in* **TFCW**. *Suppose in addition that the homomorphism $\pi_1(f)\colon \pi_1(M) \to \pi_1(S^1)$ is surjective. Then the following are equivalent:*

(i) $\theta(f)$ and $\tau_{\text{fib}}(f)$ vanish;

(ii) ${\tau_{\text{fib}}}'(f)$ vanishes;

(iii) $c(f)$ and $\tau(f)$ vanish;

(iv) the map f is homotopic to a smooth fiber bundle.

The work is structured as follows. In Section 4.1 and Section 4.2 we define ${\tau_{\text{fib}}}'(f)$, we prove that assertion (i) of the previous theorem implies assertion (ii) and and we study deeply the relations between ${\tau_{\text{fib}}}'(f)$ and the obstructions $\theta(f)$ and $\tau_{\text{fib}}(f)$. In the Section 4.3 we present the algebraic background that we need for the other implications and, in particular, the previous decomposition of $\mathrm{Wh}(\pi M)$. In Section 4.4 we introduce Farrell's obstructions and we show that (iii) is equivalent to (iv). Finally, in Section 4.5, we complete the proof of Theorem 4.1 by showing that (ii) implies (iii).

4.1 The fiber torsion obstruction over S^1

The goal of this section is to investigate how the two fiber torsion obstructions $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$ behave in case of maps $f\colon M \to S^1$ in **Man**. In particular, we define a single invariant ${\tau_{\mathrm{fib}}}'(f)$ which summarizes both $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$ and we show that if $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$ vanish, then ${\tau_{\mathrm{fib}}}'(f)$ vanishes as well. Then, we present the example of mapping tori and we study how the obstructions can be described in this case. This will be very useful in the following sections.

Let us start with the definition of ${\tau_{\mathrm{fib}}}'(f)$.

Notation 4.2. In the following, we denote by $e\colon \mathbb{R} \to S^1$ the universal covering $t \mapsto \exp(2\pi i t)$ of S^1 and by F the homotopy fiber $\mathrm{hofib}(f)_{e(0)}$ of f over $e(0)$. Moreover, we denote simply by $i = i_{e(0)}\colon F \hookrightarrow \mathrm{FIB}(f)$ the canonical inclusion.

Definition 4.3. Let $f\colon M \to S^1$ be a map in **Man**. Consider the CW-structure on S^1 given by $\{e(0)\}$ as 0-skeleton and S^1 as 1-skeleton. Let s be the spider at $e(0)$ given by the constant path $\mathrm{const}_{e(0)}$ at $e(0)$ for the 0-cell and by the path $w\colon I \to S^1$, $t \mapsto \exp(\pi i t)$ for the 1-cell. Equip $\mathrm{FIB}(f)$ with the simple structure $\xi\big(e(0), s, \zeta\big)$ defined in Section 2.2 for any choice of simple structure ζ of F. We define the *fiber torsion obstruction* ${\tau_{\mathrm{fib}}}'(f) \in \mathrm{Wh}(\pi M)$ to be the Whitehead torsion

$$\tau\Big(\mu_f\colon \Big(\mathrm{FIB}(f), \xi\big(e(0), s, \zeta\big)\Big) \to \big(M, \xi^{\mathrm{Top}}(M)\big)\Big)$$

where μ_f is the canonical homotopy equivalence introduced in Section 1.2 and $\xi^{\mathrm{Top}}(M)$ is the preferred simple structure on M defined in Section 2.3.

Remark 4.4. The element ${\tau_{\mathrm{fib}}}'(f)$ is well-defined because the simple structure $\xi\big(e(0), s, \zeta\big)$ on $\mathrm{FIB}(f)$ is independent of the choice of the simple structure ζ on F. Indeed, by Lemma 2.21, given another choice ζ' of simple structure on F, we have that

$$\begin{aligned}
&\tau\Big(\Big(\mathrm{FIB}(f), \xi\big(e(0), s, \zeta\big) \xrightarrow{\mathrm{id}} \Big(\mathrm{FIB}(f), \xi\big(e(0), s, \zeta'\big)\Big)\Big) \\
&= i_*\tau\big((F,\zeta) \xrightarrow{\mathrm{id}} (F,\zeta')\big) - i_*\tau\big((F,\zeta) \xrightarrow{t[v_1]} (F,\zeta')\big)
\end{aligned}$$

where $v_1 = w * w^-$. But this is clearly homotopic to a constant path. Therefore, the above torsion vanishes and we can conclude.

Notation 4.5. We denote the simple structure $\xi\big(e(0), s, \zeta\big)$ on $\mathrm{FIB}(f)$ simply by $\xi(e(0), s)$.

Remark 4.6. In the following, we identify $H^1\big(S^1; \mathrm{Wh}(\pi M)\big) = \mathrm{Wh}(\pi M)$ using the standard generator of $\pi_1(S^1) \cong H_1(S^1) \cong \mathbb{Z}$ represented by the identity map id_{S^1}. In particular, we consider $\theta(f)$ as an element of $\mathrm{Wh}(\pi M)$. Recall that, by Example 3.25, when it exists, also $\tau_{\mathrm{fib}}(f)$ is in $\mathrm{Wh}(\pi M)$ since $\chi(S^1) = 0$.

We compare now ${\tau_{\mathrm{fib}}}'(f)$ with $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$. It turns out that the two fiber torsion obstructions can be obtained by ${\tau_{\mathrm{fib}}}'(f)$, which therefore sums them up into one single invariant. Moreover, we get that assertion (i) of Theorem 4.1 implies assertion (ii).

Lemma 4.7. *Consider the situation of Definition 4.3. Denote by*

$$-f\colon S^1 \to S^1$$

the map $\mathit{con} \circ f$ *where* $\mathit{con}\colon S^1 \to S^1$, $z \mapsto \overline{z}$ *is the orientation reversing diffeomorphism defined by complex conjugation. Then the following holds:*

(i) We have
$$\theta(f) = \tau_{\mathit{fib}}{}'(f) - \tau_{\mathit{fib}}{}'(-f)$$

(ii) If $\theta(f) = 0$, then
$$\tau_{\mathit{fib}}(f) = -\,\tau_{\mathit{fib}}{}'(f)$$

(iii) If $\theta(f)$ and $\tau_{\mathit{fib}}(f)$ vanish, then $\tau_{\mathit{fib}}{}'(f)$ vanishes as well.

Proof. (i) Let $\overline{s}$ be the spider at $e(0)$ for S^1 given by the constant path at $e(0)$ for the 0-cell and by the path $\overline{w}\colon I \to S^1$, $t \mapsto \exp(-\pi i t)$ for the 1-cell. Then we obviously have

$$\tau_{\mathrm{fib}}{}'(-f) = \tau\Big(\mu_f\colon \big(\mathrm{FIB}(f), \xi\big(e(0), \overline{s}\big)\big) \to \big(M, \xi^{\mathrm{Top}}(M)\big)\Big)$$

Therefore, by Lemma 2.15, we obtain

$$\tau_{\mathrm{fib}}{}'(-f) - \tau_{\mathrm{fib}}{}'(f) = \tau\Big(\big(\mathrm{FIB}(f), \xi\big(e(0), \overline{s}\big)\big) \xrightarrow{\mathrm{id}} \big(\mathrm{FIB}(f), \xi\big(e(0), s\big)\big)\Big)$$

Hence, we can conclude by Lemma 2.21, since for any simple structure ζ on F we get

$$\begin{aligned}
\tau_{\mathrm{fib}}{}'(-f) - \tau_{\mathrm{fib}}{}'(f) &= i_*\tau\big((F,\zeta) \xrightarrow{\mathrm{id}} (F,\zeta)\big) - i_*\tau\big((F,\zeta) \xrightarrow{t[v_1]} (F,\zeta)\big) \\
&= -i_*\tau\big((F,\zeta) \xrightarrow{t[\overline{w}*w^-]} (F,\zeta)\big) \\
&= i_*\tau\big((F,\zeta) \xrightarrow{t[w*\overline{w}^-]} (F,\zeta)\big)
\end{aligned}$$

which is by definition exactly the obstruction $\theta(f) \in \mathrm{Wh}(\pi M)$.

(ii) This follows immediately by Definition 3.21, Definition 4.3 and Lemma 2.15.

(iii) This follows immediately by the previous claims. □

Let us now focus on the example of mapping tori. The goal is to check how the obstructions $\theta(f)$ and $\tau_{\mathrm{fib}}{}'(f)$ behave in this case. This will be very useful in the following sections to conclude the proof of Theorem 4.1. Let us start by recalling the definition of mapping torus of a map.

Definition 4.8. Let Y be in **Top** and $v\colon Y \to Y$ be a self-map. We define the *mapping torus* T_v of v to be the following pushout

$$\begin{tikzcd}
Y \amalg Y \arrow[r, "\mathrm{id} \amalg \mathrm{id}"] \arrow[d, hook, "i"'] & Y \arrow[d, "\bar{i}"] \\
\mathrm{cyl}(v) \arrow[r, "p"] & T_v
\end{tikzcd} \tag{4.1}$$

where i is the inclusion of the front and the back into the mapping cylinder. In other words, T_v is given by
$$T_v = Z \times I \,/\, \big\langle (y,0) \sim \big(v(y),1\big) \mid y \in Y \big\rangle$$

Remark 4.9. If Y is in **TFCW**, then T_v is clearly in **TFCW** and has preferred simple structure. Indeed, choose a simple structure ξ on Y and give $\mathrm{cyl}(v)$ the simple structure $(i_1)_*\xi$ defined in Example 2.7, where $i_1\colon Y \cong Y \times \{1\} \hookrightarrow \mathrm{cyl}(v)$ is the homotopy equivalence given by the back inclusion. Then the pushout simple structure on T_v is independent of ξ by the sum formula of Lemma 2.15. Therefore, it defines a preferred simple structure on T_v. In the following, we will implicitly assume that we have equipped any mapping cylinder with this preferred choice of simple structure.

Example 4.10. Let Y_1 and Y_2 be homotopy equivalent spaces in **TFCW** and let $u\colon Y_1 \to Y_2$ be a homotopy equivalence between them. Consider two self-homotopy equivalences $v_1\colon Y_1 \to Y_1$ and $v_2\colon Y_2 \to Y_2$ such that $v_2 \circ u \simeq u \circ v_1$ via a homotopy $h\colon Y_1 \times I \to Y_2$.

$$\begin{tikzcd} Y_1 \arrow[r, "v_1"] \arrow[d, "u"'] & Y_1 \arrow[d, "u"] \\ Y_2 \arrow[r, "v_2"'] \arrow[ur, phantom, "\overset{h}{\Rightarrow}"] & Y_2 \end{tikzcd}$$

Define $t_{u,h}\colon \operatorname{cyl}(v_1) \to \operatorname{cyl}(v_2)$ and $T_{u,h}\colon T_{v_1} \to T_{v_2}$ as maps induced by h by pushout property, which are homotopy equivalences by [Mat76, Corollary 9]. Then, even though in general (the homotopy class of) the map $T_{u,h}$ depends on the choice of u and h, its torsion is independent of u and h and the following equation holds.

$$\tau(T_{u,h}\colon T_{v_1} \to T_{v_2}) = 0$$

Indeed, by Lemma 2.15, using the notation of the commutative diagram (4.1), we obtain that

$$\begin{aligned} \tau(T_{u,h}) &= p_*\tau(t_{u,v}) + \bar{i}_*\tau(u) - \big(\bar{i} \circ (\mathrm{id} \amalg \mathrm{id})\big)_*\tau(u \amalg u) \\ &= p_*\tau(t_{u,v}) + \bar{i}_*\tau(u) - (\bar{i} \circ \mathrm{id})_*\tau(u) - (\bar{i} \circ \mathrm{id})_*\tau(u) \\ &= p_*\tau(t_{u,v}) - (p \circ i_1)_*\tau(u) \end{aligned}$$

Therefore, if the mapping cylinder $\operatorname{cyl}(v_2)$ is defined by the following pushout

$$\begin{tikzcd} Y_2 \arrow[r, "v_2"] \arrow[d, "i_1"'] & Y_2 \arrow[d, "i_1"] \\ Y_2 \times I \arrow[r, "\overline{v_2}"] & \operatorname{cyl}(v_2) \end{tikzcd}$$

we can conclude by Lemma 2.15 as follows.

$$\begin{aligned} \tau(T_{u,h}) &= p_*\tau(t_{u,v}) - (p \circ i_1)_*\tau(u) \\ &= p_*(\overline{v_2})_*\tau(u \times \mathrm{id}_I) + p_*(i_1)_*\tau(u) - p_*(\overline{v_2} \circ i_1)_*\tau(u) - (p \circ i_1)_*\tau(u) \\ &= p_*(\overline{v_2})_*(i_1)_*\tau(u) - p_*(\overline{v_2} \circ i_1)\tau(u) \\ &= 0 \end{aligned}$$

The strategy now to describe the obstructions $\theta(f)$ and $\tau_{\mathrm{fib}}{}'(f)$ in terms of mapping tori is first to construct a particular homotopy equivalences using the previous example and then to use this to compute the obstructions.

Let $f\colon M \to S^1$ be a map in **Man**. Consider the following pullback diagram

$$\begin{tikzcd} \overline{M} \arrow[r, "\bar{e}"] \arrow[d, "\bar{f}"'] & M \arrow[d, "f"] \\ \mathbb{R} \arrow[r, "e"] & S^1 \end{tikzcd}$$

where e is the universal covering of Notation 4.2. Let $l_1\colon \overline{M} \to \overline{M}$ be the map induced by the action of the generator $1 \in \mathbb{Z} \cong \pi_1(S^1)$ by deck transformations. Denote by $t\colon F \to F$ the fiber transport induced by the same generator, that is, the map $F \to F$ which sends (x, w) to $\big(x, w * (e|_I)\big)$. Then l_1 and t are self-homotopy equivalences. Define the map

$$h\colon \overline{M} \to F, \qquad x \mapsto \big(\bar{e}(x), w\big)$$

where $w\colon I \to S^1$ is the path $w(s) = \exp\big(2\pi i\bar{f}(x)(1-s)\big)$.

Proposition 4.11. *The map $h\colon \overline{M} \to F$ is a well-defined homotopy equivalence such that $h \circ l_1 = t \circ h$. Therefore, by Example 4.10, it induces a homotopy equivalence $\varphi\colon T_{l_1} \to T_t$ such that $\tau(\varphi) = 0$.*

Proof. We prove first that h is well-defined map over F. This follows by a direct computation. Indeed, we have

$$w(0) = \exp\big(2\pi i \overline{f}(x)\big) = e \circ \overline{f}(x) = f \circ \overline{e}(x)$$

and

$$\widehat{f}\big(\overline{e}(x), w\big) = w(1) = \exp(0) = e(0)$$

where $\widehat{f}\colon \operatorname{FIB}(f) \to S^1$ is the fibration associated to f.

Let now $(y, v) \in F$. Since $v\colon I \to S^1$ is a path in S^1, there exists a $\overline{s} \in \mathbb{R}$ such that v is homotopic to the path $s \mapsto \exp\big(2\pi i \overline{s}(1-s)\big)$. Define $k\colon F \to \overline{M}$ to be the map which sends any element $(y, v) \in F$ to the element in $\overline{M}$ defined by the pair $(y, \overline{s})$. This is is well-defined because by construction $\overline{s}$ is such that $e(\overline{s}) = v(0) = f(y)$. We claim that k is a homotopy inverse of h. Indeed, we obviously have that $k \circ h = \mathrm{id}_{\overline{M}}$. On the other hand, we have that

$$h \circ k(y, v) = h(y, \overline{s}) = \big(y, s \mapsto \exp\big(2\pi i \overline{s}(1-s)\big)\big)$$

and thus $h \circ k$ is homotopic to id_F by construction. Therefore, h is a homotopy equivalence.

It remains to check that $h \circ l_1 = t \circ h$. For this, let $(x, \overline{s})$ be an element in $\overline{M}$. We have that

$$h \circ l_1(x, \overline{s}) = h(x, \overline{s} + 1) = (x, w)$$

where $w\colon I \to S^1$ is the path $s \mapsto \exp\big(2\pi i(\overline{s}+1)(1-s)\big) = \exp\big(2\pi i \overline{s}(1-s)\big) + \exp\big(2\pi i(1-s)\big)$. But this is $t \circ h(x, \overline{s})$ by construction and hence we can conclude. $\square$

Consider now the space $e^* \operatorname{FIB}(f)$ defined by the following pullback square

$$\begin{tikzcd}
e^* \operatorname{FIB}(f) \arrow[r] \arrow[d] & \operatorname{FIB}(f) \arrow[d, "\widehat{f}"] \\
I \arrow[r, "e|_I"] & S^1
\end{tikzcd}$$

Then, by the second cube theorem [Mat76, Theorem 25], this fits in the following homotopy pushout square.

$$\begin{tikzcd}
F \amalg F \arrow[r, "\mathrm{id} \amalg \mathrm{id}"] \arrow[d, hook] & F \arrow[d] \\
e^* \operatorname{FIB}(f) \arrow[r] & \operatorname{FIB}(f)
\end{tikzcd}$$

We define the homotopy equivalence

$$\psi\colon T_t \to \operatorname{FIB}(f) \tag{4.2}$$

to be the pushout of the following commutative diagram

$$\begin{tikzcd}
F \times I \arrow[d, "u"] & F \amalg F \arrow[l, hook'] \arrow[r, "\mathrm{id} \amalg t"] \arrow[d, "\mathrm{id} \amalg t"'] & F \arrow[d, "\mathrm{id}"] \\
e^* \operatorname{FIB}(f) & F \amalg F \arrow[l, hook'] \arrow[r, "\mathrm{id} \amalg \mathrm{id}"] & F
\end{tikzcd}$$

where the left vertical map is a fiber trivialization with respect to the triple $\big(e(0), 1, e\big)$.

Definition 4.12. In the previous situation, we define the homotopy equivalence

$$\widehat{e}\colon T_{l_1} \to M$$

to be the composition $T_{l_1} \xrightarrow{\varphi} T_t \xrightarrow{\psi} \mathrm{FIB}(f) \xrightarrow{\mu_f} M$ where φ is the homotopy equivalence of Proposition 4.11, ψ is the homotopy equivalence (4.2) and μ_f is the canonical homotopy equivalence defined in Section 1.2.

Remark 4.13. The homotopy equivalence $\widehat{e}$ may be obtained also in the following faster, but more implicit way. Since we have that $\overline{e} \circ l_1 = \overline{e}$, then the map $\overline{e}\colon \overline{M} \times I \to M$ which sends (x,t) to $\overline{e}(x)$ induces a map $\widehat{e}\colon T_{l_1} \to M$. This is a homotopy equivalence because the following diagram of homotopy fiber sequences commutes and T_{l_1} and M are in **TFCW**.

$$\begin{tikzcd}
\overline{M} \arrow[r] \arrow[d, "\simeq"'] & T_{l_1} \arrow[r] \arrow[d] & S^1 \arrow[d, equal] \\
F \arrow[r] & M \arrow[r, "f"] & S^1
\end{tikzcd}$$

The explicit construction of Definition 4.12, however, is more suitable for us, because it is useful to prove the following lemma.

We can finally compute the obstructions $\theta(f)$ and ${\tau_{\mathrm{fib}}}'(f)$ using mapping tori.

Lemma 4.14. *In the previous situation, we have*

$$\theta(f) = \overline{e}_* \, \tau(l_1\colon \overline{M} \to \overline{M}) \tag{4.3}$$

$$\tau_{\mathit{fib}}{}'(f) = \tau(\widehat{e}\colon T_{l_1} \to M) \tag{4.4}$$

where we equip T_{l_1} with the preferred simple structure defined in Remark 4.9, M with the preferred simple structure $\xi^{Top}(M)$ and $\overline{M}$ with any simple structure.

Proof. Let us start by proving (4.3). By definition, we have

$$\theta(f) = (\mu_f \circ i)_* \tau(t\colon F \to F)$$

for any choice of simple structure on F, where $i\colon F \to \mathrm{FIB}(f)$ is the inclusion. Moreover, by construction, we have $h \circ l_1 = t \circ h$ and $\overline{e} = \mu_f \circ i \circ h$. Therefore, we conclude by Lemma 2.15 as follows

$$\begin{aligned}
\theta(f) &= (\mu_f \circ i)_* \tau(t\colon F \to F) \\
&= (\mu_f \circ i)_* \tau(h \circ l_1 \circ h^{-1}\colon F \to F) \\
&= (\mu_f \circ i)_* \big(\tau(h) + h_* \tau(l_1) + h_*(l_1)_* \tau(h^{-1})\big) \\
&= (\mu_f \circ i)_* \big(\tau(h) + h_* \tau(l_1) + t_* h_* \tau(h^{-1})\big) \\
&= (\mu_f \circ i)_* \big(\tau(h) + h_* \tau(l_1) - \tau(h)\big) \\
&= (\mu_f \circ i \circ h)_* \tau(l_1\colon \overline{M} \to \overline{M}) \\
&= \overline{e}_* \tau(l_1\colon \overline{M} \to \overline{M})
\end{aligned}$$

We prove now (4.4). By definition we have

$$\tau_{\mathrm{fib}}{}'(f) = \tau\Big(\mu_f\colon \Big(\mathrm{FIB}(f), \xi\big(e(0), s\big)\Big) \to \Big(M, \xi^{\mathrm{Top}}(M)\Big)\Big)$$

By unraveling its construction, it is easy to check that the simple structure $\xi\big(e(0),s\big)$ on $\mathrm{FIB}(f)$ is the one induced by the homotopy equivalence ψ in (4.2), where we equip T_t with its preferred simple structure (see for example [Ste07, pp. 51–52]). Therefore, we obtain

$$\tau_{\mathrm{fib}}{}'(f) = \tau(\mu_f \circ \psi\colon T_t \to M)$$

Moreover, since by Proposition 4.11 the torsion of φ vanishes, we get

$$\tau_{\mathrm{fib}}{}'(f) = \tau(\mu_f \circ \psi \circ \varphi\colon T_{l_1} \to M)$$

Therefore, by definition of $\widehat{e}$, we can conclude that

$$\tau_{\mathrm{fib}}{}'(f) = \tau(\widehat{e}\colon T_{l_1} \to M)$$

□

4.2 Gluing h-cobordisms

In this section we describe the obstructions $\theta(f)$, $\tau_{\mathrm{fib}}(f)$ and $\tau_{\mathrm{fib}}{}'(f)$ in case of maps $f\colon M \to S^1$ obtained by gluing the two components of the boundary of an h-cobordism together. It turns out that in this case the vanishing of $\tau_{\mathrm{fib}}{}'(f)$ is actually equivalent to the vanishing of $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$, proving that the invariant $\tau_{\mathrm{fib}}{}'(f)$ actually summarizes the obstructions $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$. All this will be applied to smooth maps in the last section of this chapter using the Pontrjagin-Thom construction and will be useful to prove that for smooth maps over a circle the invariant $\tau_{\mathrm{fib}}{}'(f)$ is equivalent to the obstructions $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$. Moreover, we will use this description of the obstructions also to show that the vanishing of $\tau_{\mathrm{fib}}{}'(f)$ implies the vanishing of Farrell's obstructions.

Let us start by recalling the definition of h-cobordism.

Definition 4.15. (i) An *h-cobordism* $(W, \partial_0 W, \partial_1 W)$ consists of a manifold W in **Man** whose boundary ∂W has a disjoint decomposition $\partial W = \partial_0 W \coprod \partial_1 W$ such that the inclusions $i_k\colon \partial_k W \to W$ for $k = 0, 1$ are homotopy equivalences.

(ii) We define the Whitehead torsion of an h-cobordism $(W, \partial_0 W, \partial_1 W)$ as

$$\tau(W) = (i_0)_*^{-1}\tau(i_0\colon \partial_0 W \to W) \in \mathrm{Wh}(\pi\partial_0 W)$$

where ∂W_0 and W are equipped with the preferred simple structure of a closed manifold.

Remark 4.16. Let $w_1\colon \pi_1(\partial_0 W) \to \{\pm 1\}$ be the orientation homomorphism of $\partial_0 W$. The w_1-twisted involution $\iota\colon \mathbb{Z}\big[\pi_1(\partial_0 W)\big] \to \mathbb{Z}\big[\pi_1(\partial_0 W)\big]$ on the group ring $\mathbb{Z}\big[\pi_1(\partial_0 W)\big]$ is the homomorphism defined by

$$\overline{\sum_{g \in \pi_1(\partial_0 W)} \lambda_g \cdot g} = \sum_{g \in \pi_1(\partial_0 W)} w_1(g) \cdot \lambda_g \cdot g^{-1}$$

Let

$$\delta\colon \mathrm{Wh}(\pi\partial_0 W) \to \mathrm{Wh}(\pi\partial_0 W)$$

be the involution induced by ι on Whitehead groups. Then, by [Mil66, Duality Theorem], if we denote by W^* the h-cobordism where the roles of $\partial_0 W$ and $\partial_1 W$ are interchanged, we have

$$\tau(W^*) = (-1)^{\dim(\partial_0 W)} \cdot (i_1)_*^{-1}\,(i_0)_*\,\delta\,\tau(W) \tag{4.5}$$

In other words, the homomorphism δ corresponds geometrically to turn the h-cobordism W upside down.

We glue now the two components of the boundary an h-cobordism together and we study the obstructions in this situation. Consider an h-cobordism $(W; \partial_0 W, \partial_1 W)$ and let $g\colon \partial_1 W \to \partial_0 W$ be a homeomorphism. Define W_g to be the manifold in **Man** obtained from W by gluing $\partial_1 W$ and $\partial_0 W$ by g, that is, as the following pushout.

$$\begin{tikzcd} \partial_0 W \amalg \partial_1 W \arrow[r, "\mathrm{id} \amalg g"] \arrow[d, hook, "i_0 \amalg i_1"'] & \partial_0 W \arrow[d, hook, "l"] \\ W \arrow[r, "\mathrm{pr}"] & W_g \end{tikzcd} \tag{4.6}$$

Then for any map $f'\colon W \to I$ in **Top** such that $f'(\partial_0 W) = \{0\}$ and $f'(\partial_1 W) = \{1\}$, there exists a map $f_g\colon W_g \to S^1$ defined as pushout of the following diagram.

$$\begin{tikzcd} W \arrow[d, "f'"'] & \partial_0 W \amalg \partial_1 W \arrow[l, hook', "i_0 \amalg i_1"'] \arrow[r, "\mathrm{id} \amalg g"] \arrow[d, "f'"] & \partial_0 W \arrow[d, "f'"] \\ I & \{0\} \amalg \{1\} \arrow[l, hook'] \arrow[r] & \{0\} \end{tikzcd}$$

Remark 4.17. Since I is convex, then the f_g is unique up to homotopy.

Denote again by
$$\delta\colon \mathrm{Wh}(\pi W_g) \to \mathrm{Wh}(\pi W_g)$$
the $w_1(W_g)$-twisted involution, where $w_1(W_g)\colon \pi_1(W_g) \to \{\pm 1\}$ is the orientation homomorphism of W_g.

Lemma 4.18. *(i) We have*
$$\theta(f_g) = l_* \tau(g \circ i_1^{-1} \circ i_0) = \big((-1)^{\dim(W)} \cdot \delta + \mathrm{id}\big) l_* \tau(W)$$

(ii) We have
$$\tau_{fib}{}'(f_g) = (-1)^{\dim(\partial_0 W)} \cdot \delta\, l_* \tau(W) = l_* \tau(W) - \theta(f_g)$$

(iii) If $\theta(f_g) = 0$, then
$$\tau_{fib}(f_g) = -l_* \tau(W)$$

(iv) The following are equivalent:

(a) $l_* \tau(W)$ vanishes;

(b) $\tau_{fib}{}'(f_g)$ vanishes;

(c) $\theta(f_g)$ and $\tau_{fib}(f_g)$ vanish.

Proof. (i) Let us start by noting that the map $l_*\colon \mathrm{Wh}(\pi \partial_0 W) \to \mathrm{Wh}(\pi W_g)$ induced by the inclusion $l\colon \partial_0 W \hookrightarrow W$ is compatible with the involution δ. Namely, we have $l_*\, \delta = \delta\, l_*$. Indeed, by construction, since $\partial_0 W$ is in the boundary of W, the orientation covering of $\partial_0 W$ is the orientation covering of W restricted to $\partial_0 W$. Therefore, we have that $w_1(\partial_0 W) = w_1(W_g) \circ \pi_1(l)$. It follows immediately that $l_*\, \delta = \delta\, l_*$.

Consider now the following pullback diagram

$$\begin{tikzcd} \overline{W_g} \arrow[r, "\overline{e}"] \arrow[d, "\overline{f_g}"'] & W_g \arrow[d, "f_g"] \\ \mathbb{R} \arrow[r, "e"] & S^1 \end{tikzcd}$$

where e is the universal covering of Notation 4.2. If we restrict e to the unit interval I, the pullback is by construction exactly W. Therefore, the space $\overline{W_g}$ may be seen as the space obtained from $W \times \mathbb{Z}$ by identifying (x, n) and $\big(g(x), n+1\big)$ for any $x \in \partial_1 W$ and the map $l_1 \colon \overline{W_g} \to \overline{W_g}$ defined in the previous section may be seen as the map which sends (y, n) to $(y, n+1)$ for any $y \in W$. Define $\bar{l} \colon \partial_0 W \hookrightarrow \overline{W_g}$ to be the pullback of the following diagram.

$$
\begin{tikzcd}
\{0\} \arrow[r, "e"] \arrow[d, hook] & S^1 \arrow[d, equal] & \partial_0 W \arrow[l, "f_g"'] \arrow[d, hook, "l"] \\
\mathbb{R} \arrow[r, "e"] & S^1 & W_g \arrow[l, "f_g"']
\end{tikzcd}
$$

This is a homotopy equivalence by coglueing theorem [FP90, Theorem A.4.19] and, in the previous model, it corresponds to the map which sends x to $(x, 0)$ for any $x \in \partial_0 W$. Therefore, it is easy to see that by construction we have $\bar{l}^{-1} \circ l_1 \circ \bar{l} \simeq g \circ i_1^{-1} \circ i_0$.

Now, if we equip $\overline{W_g}$ with the simple structure such that $\bar{l}$ has vanishing torsion, then by Lemma 4.14 and Lemma 2.15, we obtain

$$
\begin{aligned}
\theta(f_g) &= \bar{e}_* \tau(l_1 \colon \overline{W_g} \to \overline{W_g}) \\
&= \bar{e}_* \bar{l}_* \tau(\bar{l}^{-1} \circ l_1 \circ \bar{l} \colon \partial_0 W \to \partial_0 W) \\
&= l_* \tau(g \circ i_1^{-1} \circ i_0 \colon \partial_0 W \to \partial_0 W)
\end{aligned}
$$

Moreover, since $g \circ i_1^{-1} \circ i_0$ corresponds to the fiber transport $\partial_0 W \to \partial_0 W$ induced by the action of the generator $1 \in \pi_1(S^1)$, we get that $g_*\,(i_1)_*^{-1}\,(i_0)_* = \mathrm{id}$. Hence, we can conclude by Lemma 2.15 and by Remark 4.16 as follows

$$
\begin{aligned}
\theta(f_g) &= l_* \tau(g \circ i_1^{-1} \circ i_0) \\
&= l_* \tau(g) + (l \circ g)_* \tau(i_1^{-1}) + (l \circ g \circ i_1^{-1})_* \tau(i_0) \\
&= 0 - l_* g_* (i_1)_*^{-1} \tau(i_1) + l_* (g \circ i_1^{-1})_* \tau(i_0) \\
&= l_* \big(- g_* \tau(W^*) + (g \circ i_1^{-1})_* \tau(i_0) \big) \\
&= l_* \big(- (i_0)_*^{-1} (i_1)_* \tau(W^*) + (i_0)_* \tau(i_0) \big) \\
&= l_* \big(- (-1)^{\dim(\partial_0 W)} \cdot \delta\, \tau(W) + \tau(W) \big) \\
&= l_* \big((-1)^{\dim(W)} \cdot \delta + \mathrm{id} \big) \tau(W) \\
&= \big((-1)^{\dim(W)} \cdot \delta + \mathrm{id} \big) l_* \tau(W)
\end{aligned}
$$

where in the last equality we have used that $l_*\, \delta = \delta\, l_*$.

(ii) Define the homotopy equivalence $\lambda \colon T_{g \circ i_1^{-1} \circ i_0} \to W_g$ as pushout of the following diagram

$$
\begin{tikzcd}
\partial_0 W \times I \arrow[d, "h \circ (i_0 \times \mathrm{id}_I)"'] & \partial_0 W \amalg \partial_0 W \arrow[l, hook'] \arrow[r, "\mathrm{id} \amalg g \circ i_1^{-1} \circ i_0"] \arrow[d, "\mathrm{id} \amalg i_1^{-1} \circ i_0"] & \partial_0 W \arrow[d, equal] \\
W & \partial_0 W \amalg \partial_1 W \arrow[l, hook', "i_0 \amalg i_1"'] \arrow[r, "\mathrm{id} \amalg g"] & \partial_0 W
\end{tikzcd}
$$

where $h \colon W \times I \to W$ is a homotopy between id_W and $i_1 \circ i_1^{-1}$. Then, since by construction we have $\bar{l} \circ (g \circ i_1^{-1} \circ i_0) \simeq l_1 \circ \bar{l}$, we obtain by Lemma 4.14 and by Example 4.10 the following equation.

$$
{\tau_{\mathrm{fib}}}'(f_g) = \tau(\widehat{e} \colon T_{l_1} \to W_g) = \tau(\lambda \colon T_{g \circ i_1^{-1} \circ i_0} \to W_g)
$$

Therefore, by equation (4.5), diagram (4.6) and Lemma 2.15, we can conclude as follows

$$\begin{aligned}
{\tau_{\mathrm{fib}}}'(f_g) &= \tau(\lambda) \\
&= -(\mathrm{pr}\circ i_1)_*\tau(i_1^{-1}\circ i_0) + \mathrm{pr}_*\,\tau\big(h\circ(i_0\times \mathrm{id}_I)\big) \\
&= -(\mathrm{pr}\circ i_1)_*\tau(i_1^{-1}) - (\mathrm{pr}\circ i_1\circ i_1^{-1})_*\tau(i_0) + \mathrm{pr}_*\,\tau(h) + (\mathrm{pr}\circ h)_*\tau(i_0\times \mathrm{id}_I) \\
&= (\mathrm{pr}\circ i_1\circ i_1^{-1})_*\tau(i_1) - \mathrm{pr}_*\,\tau(i_0) + \mathrm{pr}_*\,\tau(\mathrm{id}_W\times 0) + \mathrm{pr}_*\,\tau(i_0) \\
&= (\mathrm{pr}\circ i_1\circ i_1^{-1})_*\tau(i_1) \\
&= (\mathrm{pr}\circ i_0)_*\,(i_0^{-1}\circ i_1)_*\,(i_1)_*^{-1}\tau(i_1) \\
&= l_*\,(i_0^{-1}\circ i_1)_*\tau(W^*) \\
&= (-1)^{\dim(\partial_0 W)}\cdot l_*\,\delta\,\tau(W) \\
&= (-1)^{\dim(\partial_0 W)}\cdot \delta\, l_*\tau(W) \\
&= l_*\tau(W) - \theta(f_g)
\end{aligned}$$

where we have used that h is homotopic to the homotopy $\mathrm{id}_W\times 0\colon W\times I\to W\times\{0\}\cong W$ and that $l_*\,\delta = \delta\, l_*$.

(iii) This follows immediately by Lemma 4.7(ii) and claim (ii).

(iv) This follows immediately by the previous claims (i), (ii) and (iii).

□

Therefore, in case of maps $f\colon M\to S^1$ obtained by gluing the two components of the boundary of an h-cobordism together the invariant ${\tau_{\mathrm{fib}}}'(f)$ summarizes completely the obstructions $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$.

Remark 4.19. We will see that the homomorphism $l_*\colon\ \mathrm{Wh}(\pi\partial_0 W)\to \mathrm{Wh}(\pi W_g)$ is not injective in general. In particular, its behavior depends on how the fiber transport $g\circ i_1^{-1}\circ i_0\colon \partial_0 W\to\partial_0 W$. Therefore, it can happen that the h-cobordism W is non-trivial, that is, by the s-cobordism theorem [LM23, Theorem 2.1], that $\tau(W)$ is not zero, but both obstruction $\theta(f_g)$ and $\tau_{\mathrm{fib}}(f_g)$ vanish.

4.3 A Bass-Heller-Swan decomposition for Whitehead groups

In this section we present the algebraic background that is needed to define Farrell's obstructions $c(f)$ and $\tau(f)$. The goal is to introduce the groups where Farrell's obstructions are defined and to study the relation between them. It turns out that they fit together into a Bass-Heller-Swan decomposition of some Whitehead group $\mathrm{Wh}(G)$.

The work is structured as follows: first we introduce a twisted Bass-Heller-Swan decomposition in the more general context of algebraic K-theory and then we apply it to the case of polynomial rings, which is the one we are interested in. Main references for this part are [LS15] for the algebraic K-theory part and [Gra88], [FH70] and [Far71] for the polynomial ring part.

A twisted Bass-Heller-Swan decomposition

Let us start by presenting a twisted Bass-Heller-Swan decomposition in algebraic K-theory. Note that we will make use of the definition of K-theory given by Waldhausen in [Wal85] and we will give for known all the results about it. Let us only recall that this is given for Waldhausen categories $\mathscr{C}$, that is, for categories with a choice of a subcategory of cofibrations and a subcategory of weak equivalences satisfying the axioms of [Wal85, p. 9], and it is defined as the loop space $K(\mathscr{C}) = \Omega|wS_.|\mathscr{C}$. See [Wal85, Section 1.3] for more details.

Notation 4.20. Throughout this part, $\mathscr{A}$ denotes a (small) additive category and $\Phi\colon \mathscr{A} \xrightarrow{\cong} \mathscr{A}$ denotes an automorphism of additive categories.

We introduce first the two categories used in the decomposition: the Φ-twisted finite Laurent category and the Nil-category.

Definition 4.21. We define the Φ-*twisted finite Laurent category* $\mathscr{A}_\Phi[t,t^{-1}]$ as follows:

- Its objects are the objects of $\mathscr{A}$.
- Given two objects A and B in $\mathscr{A}_\Phi[t,t^{-1}]$, a morphism $f\colon A \to B$ in $\mathscr{A}_\Phi[t,t^{-1}]$ is a formal sum
$$f = \sum_{i\in\mathbb{Z}} f_i \cdot t^i$$
where $f_i\colon \Phi^i(A) \to B$ is a morphism in $\mathscr{A}$ and only finitely many f_i are non-trivial.
- Given two morphisms $f = \sum_{i\in\mathbb{Z}} f_i \cdot t^i\colon A \to B$ and $g = \sum_{j\in\mathbb{Z}} g_j \cdot t^j\colon B \to C$ in $\mathscr{A}_\Phi[t,t^{-1}]$, the composite $g \circ f\colon A \to C$ is defined by
$$g \circ f = \sum_{k\in\mathbb{Z}} \left(\sum_{\substack{i,j\in\mathbb{Z}\\ i+j=k}} g_j \circ \Phi^j(f_i) \right) \cdot t^k$$

Moreover, we denote by $i_0\colon \mathscr{A} \to \mathscr{A}_\Phi[t,t^{-1}]$ the inclusion functor which is the identity on objects and which sends a morphism $f\colon A \to B$ in $\mathscr{A}$ to $f \cdot t^0\colon A \to B$.

Remark 4.22.

- The Φ-twisted finite Laurent category $\mathscr{A}_\Phi[t,t^{-1}]$ is an additive category. Indeed, it has the obvious direct sum and the obvious structure of abelian group on sets of morphisms coming from the corresponding structures in $\mathscr{A}$.
- The fundamental relation for a morphism $f\colon A \to B$ in $\mathscr{A}_\Phi[t,t^{-1}]$ is the following.
$$(\mathrm{id}_{\Phi(B)} \cdot t) \circ (f \cdot t^0) = \Phi(f) \cdot t$$

Definition 4.23. We define $\mathscr{A}_\Phi[t]$ and $\mathscr{A}_\Phi[t^{-1}]$ as the additive subcategories of $\mathscr{A}_\Phi[t,t^{-1}]$ whose objects are the objects of $\mathscr{A}$ and whose morphisms $f\colon A \to B$ are the formal sums $\sum_{i\in\mathbb{Z}} f_i \cdot t^i$ with $f_i = 0$ for $i < 0$ and $i > 0$, respectively. In addition, we denote by $\mathrm{ev}_0^{\pm}\colon \mathscr{A}_\Phi[t^{\pm 1}] \to \mathscr{A}$ the functor given by the evaluation at t^0, that is, the functor which is the identity on objects and which sends a morphism $\sum_{i\in\mathbb{Z}} f_i \cdot t^i$ in $\mathscr{A}_\Phi[t]$ or $\mathscr{A}_\Phi[t^{-1}]$ respectively to f_0, and by $i_\pm\colon \mathscr{A} \to \mathscr{A}_\Phi[t^{\pm 1}]$ the restriction of the functor i_0 to the categories $\mathscr{A}_\Phi[t]$ and $\mathscr{A}_\Phi[t^{-1}]$, respectively.

Definition 4.24.

(i) A morphism $f\colon \Phi(A) \to A$ of $\mathscr{A}$ is called Φ-nilpotent if for some $n \geq 1$ the n-fold composite
$$f^{(n)} = f \circ \Phi(f) \circ \cdots \circ \Phi^{n-1}(f)\colon \Phi^n(A) \to A$$
is trivial.

(ii) We define $\mathrm{Nil}(\mathscr{A},\Phi)$ to be the category whose objects are pairs (A,φ) where A is an object of $\mathscr{A}$ and $\varphi\colon \Phi(A) \to A$ is a Φ-nilpotent morphism in $\mathscr{A}$ and whose morphisms $(A,\varphi) \to (B,\mu)$ are morphisms $u\colon A \to B$ in $\mathscr{A}$ such that the following diagram commutes.
$$\begin{tikzcd} \Phi(A) \arrow[r, "\varphi"] \arrow[d, "\Phi(u)"'] & A \arrow[d, "u"] \\ \Phi(B) \arrow[r, "\mu"] & B \end{tikzcd}$$

Remark 4.25. The category $\mathrm{Nil}(\mathscr{A},\Phi)$ has the obvious structure of exact category induced by $\mathscr{A}$: a sequence in $\mathrm{Nil}(\mathscr{A},\Phi)$ is exact if the underlying sequence in $\mathscr{A}$ is (split) exact.

The twisted Bass-Heller-Swan decomposition is based on the K-theory of the previous categories. In particular, it uses the NK-terms and the mapping torus of $K(\Phi^{-1})$, which are presented in the following definition. Recall that any additive (actually, any exact) category has a canonical Waldhausen category structure where the cofibrations are admissible monomorphisms and the weak equivalences are the isomorphisms. Therefore, the K-theory of all the categories above can actually be computed.

Definition 4.26.

- We define $NK\big(\mathscr{A}_\Phi[t^{\pm 1}]\big)$ to be the homotopy fiber of the map of spectra $K(\mathrm{ev}_0^{\pm})\colon K\big(\mathscr{A}_\Phi[t^{\pm 1}]\big) \to K(\mathscr{A})$. Moreover, we denote by
$$\mathrm{b}^{\pm}\colon NK\big(\mathscr{A}_\Phi[t^{\pm 1}]\big) \to K\big(\mathscr{A}_\Phi[t^{\pm 1}]\big)$$
the canonical map of spectra.
- We define the mapping torus $T_{K(\Phi^{-1})}$ of $K(\Phi^{-1})$ as the following pushout.
$$\begin{tikzcd} K(\mathscr{A}) \vee K(\mathscr{A}) = K(\mathscr{A}) \wedge \partial I_+ \arrow[r, hook] \arrow[d, "K(\Phi^{-1})\vee \mathrm{id}_{K_{\mathscr{A}}}"'] & K(\mathscr{A}) \wedge I_+ \arrow[d] \\ K(\mathscr{A}) \arrow[r] & T_{K(\Phi^{-1})} \end{tikzcd}$$
Consider the natural transformation $S\colon i_0 \circ \Phi^{-1} \to i_0$ of functors from $\mathscr{A}$ to $\mathscr{A}_\Phi[t,t^{-1}]$ of additive categories defined for any object A in $\mathscr{A}$ by the isomorphism $\mathrm{id}_A \cdot t\colon \Phi^{-1}(A) \to A$. This induces a homotopy $K(S)\colon K(\mathscr{A}) \wedge I_+ \to K\big(\mathscr{A}_\Phi[t,t^{-1}]\big)$ from $K(i_0) \circ K(\Phi^{-1})$ to $K(i_0)$. We define
$$\mathrm{a}\colon T_{K(\Phi^{-1})} \to K\big(\mathscr{A}_\Phi[t,t^{-1}]\big)$$
to be the map of spectra from the torus $T_{K(\Phi^{-1})}$ to $K\big(\mathscr{A}_\Phi[t,t^{-1}]\big)$ obtained by the homotopy $K(S)$.

We can now finally conclude this short dissertation in algebraic K-theory by stating the twisted Bass-Heller-Swan decomposition for K-theory of additive categories. A complete proof of this theorem and a more detailed description of all previously defined objects can be found in [LS15].

Theorem 4.27 ([LS15, Theorem 0.4])**.** *Let $\mathscr{A}$ be an additive category which is idempotent complete and let $\Phi\colon \mathscr{A} \to \mathscr{A}$ be an automorphism of additive categories. Then:*

(i) There is a weak homotopy equivalence of spectra, natural in $(\mathscr{A},\Phi)$,
$$a \vee b^+ \vee b^-\colon T_{K(\Phi^{-1})} \vee NK\big(\mathscr{A}_\Phi[t]\big) \vee NK\big(\mathscr{A}_\Phi[t^{-1}]\big) \xrightarrow{\simeq} K\big(\mathscr{A}_\Phi[t,t^{-1}]\big)$$

(ii) Denote by $(\mathbf{AddCat}_{ic})^{\mathbb{Z}}$ the category of functors from $\mathbb{Z}$ to the category $\mathbf{AddCat}_{ic}$ of idempotent complete additive categories and by **Spectra** *the category of spectra. Then there exists a functor $E\colon (\mathbf{AddCat}_{ic})^{\mathbb{Z}} \to$* **Spectra** *and weak homotopy equivalences of spectra, natural in $(\mathscr{A},\Phi)$,*
$$\Omega NK\big(\mathscr{A}_\Phi[t]\big) \xleftarrow{\simeq} E(\mathscr{A},\Phi)$$
$$K(\mathscr{A}) \vee E(\mathscr{A},\Phi) \xrightarrow{\simeq} K\big(\mathrm{Nil}(\mathscr{A},\Phi)\big)$$

Remark 4.28. By applying homotopy groups to the previous theorem, we obtain for all $n \geq 1$ a natural splitting

$$K_n\big(\mathscr{A}_\Phi[t,t^{-1}]\big) \xrightarrow{\cong} X_n\big(\mathscr{A}_\Phi[t,t^{-1}]\big) \oplus NK_n\big(\mathscr{A}_\Phi[t]\big) \oplus NK_n\big(\mathscr{A}_\Phi[t^{-1}]\big)$$

where $X_n(\mathscr{A}_\Phi[t,t^{-1}])$ is the cokernel of the split injective homomorphism

$$K_n(b^+) \oplus K_n(b^-) \colon NK_n\big(\mathscr{A}_\Phi[t]\big) \oplus NK_n\big(\mathscr{A}_\Phi[t^{-1}]\big) \to K_n\big(\mathscr{A}_\Phi[t,t^{-1}]\big)$$

Instead, if $n = 0$, we have that

$$\pi_0\Big(NK\big(\mathscr{A}_\Phi[t]\big)\Big) = \pi_0\Big(NK\big(\mathscr{A}_\Phi[t^{-1}]\big)\Big) = 0 \tag{4.7}$$

Indeed, recall that $K_0(\mathscr{A})$ is obtained as the Grothendieck construction of the abelian monoid of stable isomorphism classes of objects in $\mathscr{A}$ under direct sum, where two objects A and B in $\mathscr{A}$ are stable isomorphic if there exists an object C such that $A \oplus C$ and $B \oplus C$ are isomorphic. In particular, $K_0(\mathscr{A})$ is a relation on the objects on $\mathscr{A}$. Therefore, since $i_\pm \colon \mathscr{A} \to \mathscr{A}_\Phi[t,t^{-1}]$ is bijective on objects and by construction $\mathrm{ev}_0^\pm \circ i_\pm = \mathrm{id}_\mathscr{A}$, then the group homomorphism

$$K_0(\mathrm{ev}_0^\pm) \colon K_0\big(\mathscr{A}_\Phi[t^{\pm 1}]\big) \to K(\mathscr{A})$$

is bijective and (4.7) holds.

Finally, we obtain the following long exact sequence

$$\begin{array}{l} \ldots \xrightarrow{\partial_{n+1}} K_n(\mathscr{A}) \xrightarrow{K_n(\Phi)-\mathrm{id}} K_n(\mathscr{A}) \xrightarrow{K_n(i_0)} X_n(\mathscr{A}_\Phi[t,t^{-1}]) \xrightarrow{\partial_n} K_{n-1}(\mathscr{A}) \to \\ \ldots \longrightarrow K_0(\mathscr{A}) \xrightarrow{K_0(\Phi)-\mathrm{id}} K_0(\mathscr{A}) \xrightarrow{K_0(i_0)} K_0(\mathscr{A}_\Phi[t,t^{-1}]) \longrightarrow 0 \end{array} \tag{4.8}$$

A Bass-Heller-Swan decomposition for polynomial rings

We are interested now in applying the previous construction to obtain a decomposition for Whitehead groups. We do this in three steps: first we apply it to general polynomial rings, then we reduce the study to K_1 and finally we focus on group rings $\mathbb{Z}G$ to conclude.

Let us start with the case of polynomial rings. Let R be a ring with unit 1 and let $\alpha \colon R \to R$ be a ring automorphism. Define the α-twisted finite Laurent series ring $R_\alpha[t,t^{-1}]$ as follows.

- Additively, it is $R_\alpha[t,t^{-1}] = R[t,t^{-1}]$.
- Multiplicatively, for $f = at^n$ and $g = bt^m$ with $n, m \in \mathbb{Z}$, we have

$$f \cdot g = (at^n) \cdot (bt^m) = a\alpha^n(b)t^{n+m}$$

Let $\mathscr{R}$ be the category whose objects are the natural numbers and whose morphism from m to n are given by the abelian group of $(n \times m)$-matrices with entries in R with the composition given by matrix multiplication. Note that there is a natural direct sum in this category given by the usual sum of natural numbers on objects and by taking block matrices on morphisms. Moreover, the automorphism α induce an obvious automorphism $\Phi \colon \mathscr{R} \to \mathscr{R}$ of categories. Then we have that:

(i) $\mathscr{R}$ is a skeleton of the category of finitely generated free right R-modules.

(ii) $\mathscr{R}_\Phi[t,t^{-1}]$ is a skeleton for the category of finitely generated free right modules over the group ring $R_\alpha[t,t^{-1}]$.

(iii) The idempotent completion $\mathrm{Idem}(\mathscr{R})$ of $\mathscr{R}$ is a skeleton of the additive category of finitely generated projective right R-modules.

(iv) The subcategories $\mathscr{R}_\Phi[t]$ and $\mathscr{R}_\Phi[t^{-1}]$ of $\mathscr{R}_\Phi[t,t^{-1}]$ are a skeleton of the additive categories of finitely generated free right R-modules over the subrings $R_\alpha[t]$ and $R_\alpha[t^{-1}]$ of $R_\alpha[t,t^{-1}]$, respectively.

(v) $\mathrm{Nil}\big(\mathrm{Idem}(\mathscr{R}),\Phi\big)$ is a skeleton of the category $\mathrm{Nil}(R,\alpha)$ defined as follows:

- Its objects are pairs (P,f) where P is a finitely generated projective right R-module and f is a nilpotent α-semilinear endomorphism of P, that is, f is a nilpotent endomorphism such that $f(xr)=f(x)\alpha(r)$ for any $x\in P$ and any $r\in R$;
- Given two objects (P,f) and (Q,g) in $\mathrm{Nil}(R,\alpha)$, a morphism $\varphi\colon (P,f)\to(Q,g)$ in $\mathrm{Nil}(R,\alpha)$ is an R-linear homomorphism $\varphi\colon M\to N$ such that $g\circ\varphi=\varphi\circ f$.

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ Q & \xrightarrow{g} & Q \end{array}$$

(vi) $NK_n\big(\mathscr{R}_\Phi[t^{\pm 1}]\big)$ is isomorphic to the kernel $NK_n(R,\alpha^{\pm 1})$ of the group homomorphism $K_n(\mathrm{ev}_0^{\pm})\colon K_n\big(R_\alpha[t^{\pm 1}]\big)\to K_n(R)$ where $\mathrm{ev}_0^{\pm}\colon R_\alpha[t^{\pm 1}]\to R$ is the evaluation at zero.

We obtain that $\pi_n\big(K(\mathscr{R})\big)=K_n(R)$ for $n\geq 1$ and the map $\mathbb{Z}\to K_0(\mathscr{R})$ which sends n to $[R^n]$ is surjective and also bijective if $R^n\cong R^m$ implies $m=n$. Furthermore, passing to the idempotent completion $\mathrm{Idem}(\mathscr{R})$, we get that $\pi_n\Big(K\big(\mathrm{Idem}(\mathscr{R})\big)\Big)=K_n(R)$ for $n\geq 0$, where $K_0(R)$ is the group obtained applied the Grothendieck construction to the category of finitely generated projective right R-modules. Therefore, Theorem 4.27(i) reduces for $\mathscr{A}=\mathscr{R}$ and $n\geq 1$ to the following twisted Bass-Heller-Swan decomposition of $K_n\big(R_\alpha[t,t^{-1}]\big)$, given in [Gra88, Theorem 2.3] or, for $n=1$, in [FH70, Theorem 19]

$$K_n\big(R_\alpha[t,t^{-1}]\big)\cong X_n(R,\alpha)\oplus NK_n(R,\alpha)\oplus NK_n(R,\alpha^{-1}) \tag{4.9}$$

while Theorem 4.27(ii) reduces for $\mathscr{A}=\mathscr{R}$ and $n\geq 1$ to the following classical isomorphism, given in [Gra88, p. 361].

$$K_{n-1}\big(\mathrm{Nil}(R,\alpha)\big)\cong K_{n-1}(R)\oplus NK_n(R,\alpha)$$

Remark 4.29. The decomposition stated in [Gra88, Theorem 2.3] has $F_{n-1}(R,\alpha)$ instead of $X_n(R,\alpha)$, where $F_i(R,\alpha)$ is defined to be π_i of the homotopy fiber of the map

$$K(\alpha)-\mathrm{id}\colon K(R)\to K(R)$$

However, such decomposition is completely equivalent to (4.9). Indeed, by the long exact sequence (4.8), it is easy to see that $F_i(R,\alpha)$ plays exactly the same role as $X_{i+1}(R,\alpha)$.

Let us now focus on the case where $n=1$. In this case, the previous results reduce to the following two decompositions.

$$K_1(R_\alpha[t,t^{-1}])\cong X_1(R,\alpha)\oplus NK_1(R,\alpha)\oplus NK_1(R,\alpha^{-1}) \tag{4.10}$$
$$K_0(\mathrm{Nil}(R,\alpha))\cong K_0(R)\oplus NK_1(R,\alpha) \tag{4.11}$$

Moreover, by the long exact sequence (4.8), we obtain the following short exact sequence

$$0 \to K_1(R)/\operatorname{im}\big(K_1(\alpha) - \mathrm{id}\big) \xrightarrow{K_1(i_0)} X_1(R,\alpha) \to \ker\big(K_0(\alpha) - \mathrm{id}\big) \to 0 \tag{4.12}$$

where $K_1(i_0)$ is the map induced by the inclusion $i_0\colon R \hookrightarrow R_\alpha[t,t^{-1}]$.

Now, consider the following two groups.

Definition 4.30. Consider the cyclic subgroup $F'(R)$ of $K_0\big(\mathrm{Nil}(R,\alpha)\big)$ generated by the class $[R,0]$ of the pair $(R,0)$ in $\mathrm{Nil}(R,\alpha)$.

- We define $C(R,\alpha) = K_0\big(\mathrm{Nil}(R,\alpha)\big)/F'(R)$.
- We define $\widetilde{C}(R,\alpha)$ to be the subgroup of $C(R,\alpha)$ generated by $[R^n,f]$ for (R^n,f) in $\mathrm{Nil}(R,\alpha)$.

These groups will be of main importance for us in the next sections. In particular, $C(R,\alpha)$ is the group where the first obstruction lies, while $\widetilde{C}(R,\alpha)$ is an explicit description of the group $NK_1(R,\alpha)$, as the stated in the following proposition.

Proposition 4.31. *We have that $NK_1(R,\alpha) \cong \widetilde{C}(R,\alpha)$.*

Proof. By [FH70, Proposition 6] the following short exact sequence splits.

$$0 \to \widetilde{C}(R,\alpha) \to K_0\big(\mathrm{Nil}(R,\alpha)\big) \to K_0(R) \to 0$$

Therefore, by (4.11), we can conclude that $NK_1(R,\alpha) \cong \widetilde{C}(R,\alpha)$. □

Once we have these groups, it is not difficult to realize that decomposition (4.11) reduces to a decomposition for $C(R,\alpha)$. Indeed, define the reduced K_0-group of R to be the quotient $\widetilde{K}_0(R) = K_0(R)/F(R)$ where $F(R)$ is the cyclic subgroup of $K_0(R)$ generated by the class $[R]$. Then the split exact sequence of the previous proof reduces easily to the following one (see again [FH70, Proposition 6])

$$0 \to \widetilde{C}(R,\alpha) \to C(R,\alpha) \to \widetilde{K}_0(R) \to 0$$

Therefore, we obtain that

$$C(R,\alpha) \cong \widetilde{K}_0(R) \oplus NK_1(R,\alpha) \tag{4.13}$$

To conclude this part about $n = 1$, we introduce the following map over $C(R,\alpha^{-1})$.

Definition 4.32. We define the group homomorphism $p\colon K_1\big(R_\alpha[t,t^{-1}]\big) \to C(R,\alpha^{-1})$ as follows. Consider a representative

$$f\colon \big(R_\alpha[t,t^{-1}]\big)^n \to \big(R_\alpha[t,t^{-1}]\big)^n$$

of any element in $K_1\big(R_\alpha[t,t^{-1}]\big)$. Denote by r_{t^m} the right multiplication by t^m. Then, there exists a $m \in \mathbb{N}$ such that $r_{t^m} \circ f$ is a map

$$r_{t^m} \circ f\colon \big(R_\alpha[t]\big)^n \to \big(R_\alpha[t]\big)^n$$

We define p to be the group homomorphism which sends the element represented by f to the class $[\mathrm{coker}(r_{t^m} \circ f), r_t]$ in $C(R,\alpha^{-1})$.

Remark 4.33.
- The map p is well-defined by [Far71, pp. 320–321].
- By [Far71, Corollary 1.9], we have that $p(f) \in \ker\big(K_0(\alpha) - \mathrm{id}\big)$ for any f in $K_1\big(R_\alpha[t,t^{-1}]\big)$. Therefore, by (4.10) and (4.12), the map p can be seen as a sort of projection from $K_1\big(R_\alpha[t,t^{-1}]\big)$ to the "sum" of $NK_1(R,\alpha^{-1})$ with the "component" of $X_1(R,\alpha)$ given by $\ker\big(K_0(\alpha) - \mathrm{id}\big)$. Such "sum" is in $C(R,\alpha^{-1})$ by (4.13).

To conclude this section, let us apply all the previous construction to the group ring $\mathbb{Z}G$ for a group G and summarize it in a single lemma. This provides in particular a Bass-Heller-Swan decomposition of Whitehead groups, which will be important for us in the following sections.

Lemma 4.34. *Let G be a group and $\alpha\colon G \to G$ be an automorphism. By abuse of notation, we denote by $\alpha\colon \mathbb{Z}G \to \mathbb{Z}G$ the automorphism induced by α on $\mathbb{Z}G$. Let $G \rtimes_\alpha \mathbb{Z}$ be the semidirect product. We identify $\mathbb{Z}G_\alpha[t,t^{-1}] \cong \mathbb{Z}(G \rtimes_\alpha \mathbb{Z})$ through the standard isomorphism which is the identity on $\mathbb{Z}G$ and which sends $t \in \mathbb{Z}G_\alpha[t,t^{-1}]$ to $1_\mathbb{Z} \in \mathbb{Z}(G \rtimes_\alpha \mathbb{Z})$. Then the following holds:*

(i) We have a decomposition

$$\mathrm{Wh}(G \rtimes_\alpha \mathbb{Z}) \cong X_1(\mathbb{Z}G,\alpha) \oplus NK_1(\mathbb{Z}G,\alpha) \oplus NK_1(\mathbb{Z}G,\alpha^{-1})$$

(ii) We have a short exact sequence

$$0 \to \mathrm{Wh}(G)/\operatorname{im}\big(\alpha_* - \mathrm{id}\big) \xrightarrow{(i_0)_*} X_1(\mathbb{Z}G,\alpha) \to \ker\big(\widetilde{K}_0(\alpha) - \mathrm{id}\big) \to 0$$

where $\alpha_*\colon \mathrm{Wh}(G) \to \mathrm{Wh}(G)$ is the homomorphism induced on Whitehead groups by $K_1(\alpha)\colon K_1(\mathbb{Z}G) \to K_1(\mathbb{Z}G)$ and $(i_0)_*$ is the homomorphism induced by the inclusion $i_0\colon \mathbb{Z}G \hookrightarrow \mathbb{Z}G_\alpha[t,t^1]$.

(iii) The map p defined in Definition 4.32 factors through a map

$$p\colon\ \mathrm{Wh}(G \rtimes_\alpha \mathbb{Z}) \to C(\mathbb{Z}G,\alpha^{-1})$$

Proof. See [FH70, Theorem 21] and [Far71, pp. 321–322]. □

4.4 Farrell's obstructions over S^1

In this section we finally present the two Farrell's obstructions $c(f)$ and $\tau(f)$ and we prove the following theorem, which is the main result of [Far71] and which provides the equivalence of part (iii) and (iv) of Theorem 4.1.

Theorem 4.35. *Let $f\colon M \to S^1$ be a map in* **Diff**. *Assume that M is connected of dimension* $\dim(M) \geq 6$ *and that the homomorphism $\pi_1(f)\colon \pi_1(M) \to \pi_1(S^1)$ is surjective. Then there exists a smooth fiber bundle $\overline{f}\colon M \to S^1$ homotopic to f if and only if the following three conditions holds:*

(i) the covering space $\overline{M}$ of M corresponding to the subgroup $G = \ker \pi_1(f)$ of $\pi_1(M)$ is in **TFCW**;

(ii) $c(f)$ vanishes;

(iii) $\tau(f)$ vanishes.

Remark 4.36. The conditions we focus on are (ii) and (iii) because these are the not obvious ones. Indeed:

- We require that $\pi_1(f)$ is surjective because a necessary condition for f to be homotopic to a fiber bundle is that $\pi_1(f)$ is not the zero map. Indeed, assume by contradiction that f is (homotopic to) a fiber bundle in **Diff** with fiber F and $\pi_1(f) = 0$. Then by the homotopy exact sequence for the fiber sequence $F \to M \xrightarrow{f} S^1$ we obtain that $\pi_1(S^1) \cong \mathbb{Z} \hookrightarrow \pi_0(F)$. But since F is compact, this is a contradiction. Once we know that $\pi_1(f)$ is not the zero map, we can assume without loss of generality that this is surjective. Indeed, if this is not the case, then there exists an $n \in \mathbb{N}$ such that $\operatorname{im} \pi_1(f) \cong n\mathbb{Z}$. Therefore, there exists a lift g of f along the n-fold covering space of S^1.

$$\begin{tikzcd} & S^1 \arrow[d] & z \arrow[d, mapsto] \\ M \arrow[r, "f"'] \arrow[ur, dotted, "g"] & S^1 & z^n \end{tikzcd}$$

By construction, the map g is such that $\pi_1(g)$ is surjective and it is homotopic to a fiber bundle whenever f is so. Hence, it suffices to take g instead of f.

Note that, since M is connected, requiring that $\pi_1(f)$ is surjective corresponds geometrically to requiring that F is connected as well. This can be proved easily using the homotopy exact sequence of $\overline{f}$.

- Condition (i) of the previous theorem is obviously necessary for f being homotopic to a smooth fiber bundle $\overline{f}$. Indeed, by construction, $\overline{M}$ has the same homotopy type of the fiber F of $\overline{f}$ (see for example the homotopy equivalence $h\colon \overline{M} \to F$ of Proposition 4.11) and this is in **TFCW** as it is in **Diff**.

The strategy to prove this theorem is to use cobordisms and the s-cobordism theorem. In particular, first we provide a cobordism using the Pontrjagin-Thom construction. Then, we introduce the first obstruction $c(f)$ to measure whether such cobordism is an h-cobordism. Finally, if this is the case, that is, if $c(f) = 0$, we define $\tau(f)$ as the torsion obstruction of the s-cobordism theorem and we use it to check if the h-cobordism in question is trivial. If this happens, then there exists a smooth fiber bundle $\overline{f}\colon M \to S^1$ homotopic to f.

Geometric interpretation of the fibering problem

Let us start by giving the geometric interpretation of the fibering problem over the circle. This is based on the notion of splitting of the manifold M with respect to f and allows us to construct a cobordism starting from M.

Notation 4.37. In the following, M denotes a connected manifold in **Diff** and $f\colon M \to S^1$ denotes a continuous map in **Diff** such that $\pi_1(f)\colon \pi_1(M) \to \pi_1(S^1)$ is surjective.

Definition 4.38. A pair (N, ν) is said a *splitting* of M with respect to f if N is a closed submanifold of M of codimension 1 and ν is a framing of the normal bundle of N such that under the Pontrjagin-Thom construction [Ran02, pp. 127–128] the pair (N, ν) corresponds to f. We will also denote it by N.

Remark 4.39.

- A splitting (N, ν) of M with respect to f always exists. Indeed, such a splitting can be obtained for example in the following way. Change f in its homotopy class to a smooth map g which is transversal to $\{e(0)\} \subset S^1$ and define N to be the preimage of $e(0)$ under g. Then, by construction, N is a splitting of M with respect to f.
- If (N, ν) is a splitting of M with respect to f, then obviously $(N, -\nu)$ is a splitting of M with respect to $-f$.

Once we have a splitting (N, ν) of M with respect to f, it is not difficult in our situation to improve it to get a new splitting which has better properties. More precisely, let $G = \ker \pi_1(f)$.

Define $\overline{M}$ to be the covering space of M corresponding to G, that is, such that $\pi_1(\overline{M}) = G$. For example, let $\overline{M}$ defined as in Section 4.1 as the following pullback

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\overline{e}} & M \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{R} & \xrightarrow{e} & S^1 \end{array}$$

where e is the universal covering of Notation 4.2. Note that since G is normal, we do not care about base points. Suppose that $\overline{M}$ is in **TFCW**. Then we obtain the following:

(i) Since $\pi_1(f)$ is surjective, then, by [BL65, p. 157], we can pass by exchanging handles of dimension 1 to a connected splitting.

(ii) Since $\overline{M}$ is in **TFCW**, then G is finitely presented. Therefore, we can work with generators and exchange handles of dimension 1 and 2 to obtain a new splitting N which is such that the homomorphism $\pi_1(i)\colon \pi_1(N) \to \pi_1(M)$ induced by the inclusion $i\colon N \hookrightarrow M$ is a monomorphism with image G (see [Far71, p. 11] and [Bro65, Lemma 3.1]).

Notation 4.40. From now on, we assume without loss of generality that N has the two properties above.

Now, we want to see if we can further improve our splitting N of M and, if this is the case, we want to study its properties. For this, choose a lifting $(\widehat{N}, \widehat{\nu})$ of (N, ν) to $\overline{M}$. Then $\widehat{N}$ divides $\overline{M}$ into two connected components. Denote by B the component into which $\widehat{\nu}$ points and by A the other one. To simplify the notation, we will denote simply by (N, ν) the pair $(\widehat{N}, \widehat{\nu})$. Let $l_1\colon \overline{M} \to \overline{M}$ be the generator of the group of deck transformations such that $A \subset l_1(A)$. Denote by $\widetilde{M}$ the universal covering of $\overline{M}$ and by $p\colon \widetilde{M} \to \overline{M}$ the covering projection. Let $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{N}$ be $p^{-1}(A)$, $p^{-1}(B)$ and $p^{-1}(N)$, respectively. Since, by Notation 4.40, the inclusion map $N \hookrightarrow \overline{M}$ induces an isomorphism on fundamental groups, then $\widetilde{N}$ is connected and simply connected. Moreover, since we have $N \subset A \subset \overline{M}$ and $N \subset B \subset \overline{M}$, then the inclusions $A \hookrightarrow \overline{M}$ and $B \hookrightarrow \overline{M}$ induce epimorphisms on fundamental groups. Therefore, $\widetilde{A}$ and $\widetilde{B}$ are connected and, by Van-Kampen's theorem [Hat02, Theorem 1.20] applied to $\widetilde{N}$, $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{M}$, they are also simply connected. In other words, the inclusion maps $A \hookrightarrow \overline{M}$ and $B \hookrightarrow \overline{M}$ actually induce isomorphisms on fundamental groups.

Remark 4.41. Since $\overline{M}$ and N are in **TFCW**, the inclusion maps $A \hookrightarrow \overline{M}$ and $B \hookrightarrow \overline{M}$ induce isomorphisms on fundamental groups and A, B, N and $\overline{M}$ are connected, then by [Sie65, Complement 6.6] also A and B are in **TFCW**.

Now, consider the group $H_i(\widetilde{M}, \widetilde{A}; \mathbb{Z})$ for $i \in \mathbb{Z}$. Since we can identify G with the group of deck transformation of $\widetilde{M}$, then this is actually a $\mathbb{Z}G$-module. We denote it by $H_i(\overline{M}, A; \mathbb{Z}G)$.

Definition 4.42. We say that a splitting N is *s-connected* if the following conditions are satisfied:

- N is connected;
- the homomorphism $\pi_1(i)\colon \pi_1(N) \to \pi_1(M)$ induced by the inclusion $i\colon N \hookrightarrow M$ is a monomorphism with image G;
- $H_i(\overline{M}, A; \mathbb{Z}G) = 0$ for $i \leq s$.

Remark 4.43. By construction, we have that $H_0(\overline{M}, A; \mathbb{Z}G) = H_1(\overline{M}, A; \mathbb{Z}G) = 0$. Therefore, 1-connected splittings of M with respect to f always exist.

Connected splittings have the following properties.

Lemma 4.44. *If N is an s-connected splitting, then $H_{s+1}(\overline{M}, A; \mathbb{Z}G)$ is a finitely generated $\mathbb{Z}G$-module.*

Proof. By excision, we have that $H_{s+1}(\overline{M}, A; \mathbb{Z}G) \cong H_{s+1}(B, N; \mathbb{Z}G)$. Thus, it suffices to prove the claim for $H_{s+1}(B, N; \mathbb{Z}G)$. Let N_s be the s-skeleton of N in some triangulation of N. Consider the following homology long exact sequence for the triple $\widetilde{N}_s \subset \widetilde{N} \subset \widetilde{B}$.

$$\cdots \to H_i(N, N_s; \mathbb{Z}G) \to H_i(B, N_s; \mathbb{Z}G) \to H_i(B, N; \mathbb{Z}G) \to \dots$$

Since $H_i(N, N_s; \mathbb{Z}G) = 0$ for $i \leq s$, we have that $H_i(B, N_s; \mathbb{Z}G) = H_i(B, N; \mathbb{Z}G) = 0$ for $i \leq s$ and that $H_{s+1}(B, N; \mathbb{Z}G)$ is a quotient of $H_{s+1}(B, N_s; \mathbb{Z}G)$. Therefore, we can conclude by [Wal65, Theorem A]. Indeed, since B is in **TFCW**, then we have that $H_{s+1}(B, N_s; \mathbb{Z}G)$ is finitely generated and so also $H_{s+1}(B, N; \mathbb{Z}G)$ is so. □

Lemma 4.45. *Let N be a 1-connected splitting of M with respect to f. Then N is also 2-connected.*

Proof. Let W be the manifold $\overline{l_1(A) \setminus A}$. This is a connected cobordism with two boundary components $\partial_0 W = N$ and $\partial_1 W = l_1(N)$. With an argument similar to that for Notation 4.40, we can show that the inclusion maps $l_1(N) \hookrightarrow W$ and $N \hookrightarrow W$ induce isomorphisms on fundamental groups. Hence, we have $H_1(W, N; \mathbb{Z}G) = H_1\big(W, l_1(N); \mathbb{Z}G\big) = 0$. Consider the following homology exact sequence for the triple $\widetilde{A} \subset \widetilde{l_1(A)} \subset \widetilde{M}$.

$$\cdots \to H_2(\overline{M}, A; \mathbb{Z}G) \xrightarrow{j_*} H_2\big(\overline{M}, l_1(A); \mathbb{Z}G\big) \xrightarrow{\partial} H_1\big(l_1(A), A; \mathbb{Z}G\big) \to \dots$$

By excision, we have $H_1\big(l_1(A), A; \mathbb{Z}G\big) \cong H_1(W, N; \mathbb{Z}G) = 0$. Therefore j_* is surjective. Moreover, the collection of modules $\Big\{H_i\big(\overline{M}, l_1^m(A); \mathbb{Z}G\big)\Big\}_{m \in \mathbb{Z}}$ with i fixed form a direct system whose maps are induced by the inclusions $\big(\widetilde{M}, \widetilde{l_1^m(A)}\big) \hookrightarrow \big(\widetilde{M}, \widetilde{l_1^n(A)}\big)$ for $n \geq m$ and whose direct limit is $H_i(\overline{M}, \overline{M}; \mathbb{Z}G) = 0$. Hence, for $i = 2$, since $H_2(\overline{M}, A; \mathbb{Z}G)$ is finitely generated by Lemma 4.44, we obtain that $H_2(\overline{M}, A; \mathbb{Z}G) = 0$ and N is 2-connected. □

We introduce now an automorphism $\alpha \colon \mathbb{Z}G \to \mathbb{Z}G$ and a nilpotent α-semilinear endomorphism of $H_s(\overline{M}, A; \mathbb{Z}G)$ for s-connected splittings. Moreover, we use these to show that there exist also $(n-3)$-connected splittings.

Let $\widetilde{M}$ be the universal covering of M. Then we can identify the group $\pi_1(M)$ with the group of deck transformation of $\widetilde{M}$. Let $t \in \pi_1(M)$ be such that $\pi_1(f)(t)$ is the generator of $\pi_1(S^1) \cong \mathbb{Z}$ determined by the orientation of S^1 used for the Pontrjagin-Thom construction in Definition 4.38. Using the notations above, we have that $t \colon \widetilde{M} \to \widetilde{M}$ covers $l_1 \colon \overline{M} \to \overline{M}$.

Remark 4.46. The element t may not be uniquely defined. However, here and in the following, the choice of t will be considered fixed.

Since $A \subset \overline{M}$ is such that $A \subset l_1(A)$, then t_*^{-1} induces an endomorphism of $H_i(\widetilde{M}, \widetilde{A})$ for any i. Moreover, the conjugation by t defines an automorphism α of G which induces an automorphism α of $\mathbb{Z}G$. It is immediate to check that t_*^{-1} is an α-semilinear endomorphism of $H_i(\overline{M}, A; \mathbb{Z}G)$ for any i.

Lemma 4.47. *If N is an $(s-1)$-connected splitting of M with respect to f, then t_*^{-1} is a nilpotent α-semilinear endomorphism of $H_s(\overline{M}, A; \mathbb{Z}G)$.*

Proof. Let $j\colon (\widetilde{M},\widetilde{A}) \hookrightarrow \big(\widetilde{M},\widetilde{l_1^m(A)}\big)$ be the inclusion. We have that $H_s(\overline{M},A;\mathbb{Z}G)$ is finitely generated by Lemma 4.44. Moreover, we have $\varinjlim H_s(\overline{M},l_1^m(A);\mathbb{Z}G)=0$. Therefore, there exists an $m\in\mathbb{N}$ such that

$$j_*\colon H_s(\overline{M},A;\mathbb{Z}G)\to H_s\big(\overline{M},l_1^m(A);\mathbb{Z}G\big)$$

is zero. But $t^{-m}\colon(\widetilde{M},\widetilde{A})\to(\widetilde{M},\widetilde{A})$ is by construction the composite of $j\colon(\widetilde{M},\widetilde{A})\to\big(\widetilde{M},\widetilde{l_1^m(A)}\big)$ and $t^{-m}\colon\big(\widetilde{M},\widetilde{l_1^m(A)}\big)\to(\widetilde{M},\widetilde{A})$, where we have used that t covers l_1. Hence, we get that

$$(t_*^{-1})^m=t_*^{-m}=t_*^{-m}\circ j_*^m=0 \qquad \square$$

Remark 4.48. Using the automorphism $\alpha\colon G\to G$, we can also obtain a group isomorphism $G\rtimes_\alpha\mathbb{Z}\to\pi_1(M)$ which is the identity on G and maps $1\in\mathbb{Z}$ to $t\in\pi_1(M)$. In the following, we will use this to identify $G\rtimes_\alpha\mathbb{Z}=\pi_1(M)$.

We prove now that $(n-3)$-connected splittings exist.

Lemma 4.49. *There exist $(n-3)$-connected splittings of M with respect to f.*

Proof. We prove by induction for $s\le n-3$ that if there exists an $(s-1)$-connected splitting, then there exists also an s-connected splitting. We already know by Lemma 4.45 that 2-connected splittings exist. Hence, let N be an $(s-1)$-connected splitting of M with respect to f. Denote for simplicity by K the group $H_s(\overline{M},A;\mathbb{Z}G)$ and by φ the homomorphism $t_*^{-1}\colon K\to K$. By Lemma 4.47, there exists an $m\in\mathbb{N}$ such that $\varphi^m=0$. Moreover, by Lemma 4.44, the group K is finitely generated. Therefore, if we set $K_i=\operatorname{im}\varphi^{m-i}$, we obtain a filtration of K

$$0=K_0\subset K_1\subset\cdots\subset K_m=K$$

given by finitely generated submodules of K such that $\varphi(K_i)\subset K_{i-1}$.

Consider now the following homology exact sequence for the triple $\widetilde{A}\subset\widetilde{l_1(A)}\subset\widetilde{M}$.

$$\cdots\to H_i\big(l_1(A),A;\mathbb{Z}G\big)\xrightarrow{i_*}H_i(\overline{M},A;\mathbb{Z}G)\xrightarrow{j_*}H_i\big(\overline{M},l_1(A);\mathbb{Z}G\big)\to\dots \tag{4.14}$$

By construction, we have that $t^{-1}\colon(\widetilde{M},\widetilde{A})\to(\widetilde{M},\widetilde{A})$ is the composite of $j\colon(\widetilde{M},\widetilde{A})\to\big(\widetilde{M},\widetilde{l_1(A)}\big)$ and $t^{-1}\colon\big(\widetilde{M},\widetilde{l_1(A)}\big)\to(\widetilde{M},\widetilde{A})$ and the homomorphism $t_*^{-1}\colon H_s\big(\overline{M},l_1(A);\mathbb{Z}G\big)\to H_s(\overline{M},A;\mathbb{Z}G)$ is a monomorphism. Therefore, we obtain the following:

- Let $\widehat{x}$ be one of a fixed finite collection of generators for K_1. Since $\varphi(x)\in K_0=0$, by construction of t_*^{-1}, we obtain that $j_*(\widehat{x})=0$. Hence, by the homology exact sequence (4.14), there exists an $x\in H_s\big(l_1(A),A;\mathbb{Z}G\big)$ such that $i_*(x)=\widehat{x}$.

- Let W be the manifold $\overline{l_1(A)\setminus A}$. By excision, we have $H_i\big(l_1(A),A;\mathbb{Z}G\big)\cong H_i\big(W,N;\mathbb{Z}G\big)$ for any i. Moreover, by the homology sequence (4.14), we obtain $H_i\big(l_1(A),A;\mathbb{Z}G\big)=0$ for $i<s-1$. Indeed, since $H_i(\overline{M},A;\mathbb{Z}G)=0$ for $i\le s-1$, then, by construction of t_*^{-1} and l_1, also $H_i\big(\overline{M},l_1(A);\mathbb{Z}G\big)=0$ for $i\le s-1$ and hence $H_i\big(l_1(A),A;\mathbb{Z}G\big)=0$ for $i<s-1$. Therefore, there exists a handlebody decomposition

$$N\times I=W_0\subset W_1\subset W_2\subset\cdots\subset W_r\cong W$$

  such that the manifold W_i is obtained from W_{i-1} by attaching a handle of dimension $s-1\le q_i\le n-2$ (since there exist 2-connected splittings, it suffices $q_i\le n-2$). Moreover, by [LM23, Sections 2.2-2.3], if $x\in H_s(W,N;\mathbb{Z}G)$, since $s\le n-3$, we obtain that $\dim W_1=s$, that is, that W_1 is obtained by W_0 by adding an s-handle, and that there exists a generator $\overline{x}$ of $H_s(W_1,N;\mathbb{Z}G)$ such that $i'_*(\overline{x})=x$ where $i'\colon(\widetilde{W_1},\widetilde{N})\hookrightarrow(\widetilde{W},\widetilde{N})$ is the canonical inclusion.

Now, pick W_1 as above, that is, such that $\dim W_1 = s$. Denote by N' the component $\partial_1 W_1$ of the cobordism W_1 and by A' the manifold $A \cup W_1$. We can assume without loss of generality that N' is again a splitting of M with respect to f. Consider the following homology exact sequence for the triple $\widetilde{A} \subset \widetilde{A'} \subset \widetilde{M}$.

$$\cdots \to H_i(A', A; \mathbb{Z}G) \to H_i(\overline{M}, A; \mathbb{Z}G) \to H_i(\overline{M}, A'; \mathbb{Z}G) \to \dots \tag{4.15}$$

By construction we have $H_i(A', A; \mathbb{Z}G) = 0$ for $i \neq s$. Moreover, $H_i(\overline{M}, A; \mathbb{Z}G) = 0$ for $i \leq s-1$. Therefore, we obtain that $H_i(\overline{M}, A'; \mathbb{Z}G) = 0$ for $i \leq s-1$. In other words, N' is again an $(s-1)$-connected splitting. Denote by K' the module $H_s(\overline{M}, A'; \mathbb{Z}G)$. Then the sequence (4.15) in dimension s becomes

$$\cdots \to \mathbb{Z}G \xrightarrow{u_*} K \xrightarrow{v_*} K' \to 0 \to \dots$$

Denote by φ' the homomorphism $t_*^{-1} \colon K' \to K'$. It is easy to check that $\varphi' \circ v_* = v_* \circ \varphi$. Therefore, since v_* is surjective, we obtain easily that $(\varphi')^m = 0$. Let $K_i' = \operatorname{im}(\varphi')^{m-i}$. Then, since $\varphi' \circ v_* = v_* \circ \varphi$, we have that v_* induces surjective maps $K_i \to K_i'$ for any i. In particular, for $i = 1$, we obtain the following short exact sequence

$$0 \to \langle \widehat{x} \rangle \to K_1 \xrightarrow{v_*} K_1' \to 0$$

where $\langle \widehat{x} \rangle$ is the free submodule of K_1 generated by a generator $\widehat{x}$ of a fixed finite collection of generators for K_1. Indeed, given $\widehat{x}$, by the two observation above there exists a generator $\overline{x}$ of $H_s(W_1, N; \mathbb{Z}G) \cong H_s(A', A; \mathbb{Z}G) \cong \mathbb{Z}G$ such that $\overline{x}$ is sent to $\widehat{x}$. Therefore, $\langle \widehat{x} \rangle$ fits in the previous sequence. Then we get that K_1' is generated by one fewer element than K_1. After repeating this process a finite number of time, we obtain an s-connected splitting. This completes the proof of Lemma 4.49. □

The obstruction to pseudofibering a circle

Let us now focus on defining the obstruction $c(f)$. Let (N, ν) be a splitting of M with respect to f. Let $E(\nu) \hookrightarrow M$ be a tubular neighborhood of (N, ν). Define M_N to be the manifold $M \setminus E(\nu)^{\circ}$, where $E(\nu)^{\circ}$ denotes the interior of $E(\nu)$. This can be identified with the cobordism $W = \overline{l_1(A) \setminus A}$ of the proof of Lemma 4.45. In other words, $\big(M_N, N, l_1(N)\big)$ is a cobordism. Now, by Lemma 4.45 and Lemma 4.49, we can assume that $H_s(\overline{M}, N; \mathbb{Z}G) = 0$ for $s \neq 3$. If $H_3(\overline{M}, A; \mathbb{Z}G)$ happens to be zero, then it is easy to check that M_N is an h-cobordism. However, there is no evidence for which it has to vanish. Therefore, this homology group is an obstruction for M_N being an h-cobordism, that is, according with the following definition, for M pseudofibering a circle. This is exactly what the first obstruction $c(f)$ is. Nevertheless, we want to have $c(f) \in C(\mathbb{Z}G; \alpha)$. Therefore, we need to check whether $H_3(\overline{M}, A; \mathbb{Z}G)$ is a projective module. This is what we do in the first part of this subsection.

Definition 4.50. We say that the manifold M pseudofibers a circle with respect to f if there exists a splitting N of M with respect to f such that M_N is an h-cobordism.

Remark 4.51. It is easy to check that a manifold M pseudofibers a circle if and only if there exists a splitting N of M with respect to f such that the inclusion $i \colon N \to \overline{M}$ induces an isomorphism on fundamental groups and $H_s(\overline{M}, A; \mathbb{Z}G) = 0$ for any $s \in \mathbb{Z}$.

In order to simplify further the notation, we give also the following definition.

Definition 4.52. A splitting (N, ν) of M with respect to f is said *s-bi-connected* if (N, ν) is s-connected and $(N, -\nu)$ is $(n-s)-1$-connected, that is, if the following conditions hold:

- $H_i(\overline{M}, A; \mathbb{Z}G) = 0$ for any $i \leq s$;

- $H_i(\overline{M}, B; \mathbb{Z}G) = 0$ for any $i \leq (n-s)-1$.

By Lemma 4.45 and Lemma 4.49, we get the following lemma, which again points out that we need to study the group $H_3(\overline{M}, A; \mathbb{Z}G)$ to check if M pseudofibers.

Lemma 4.53. *There exist 2-bi-connected splitting.*

Moreover, s-bi-connected splittings have the following property.

Lemma 4.54. *A splitting (N, ν) is s-bi-connected if and only if there exists a handlebody decomposition of $\big(M_N, N, l_1(N)\big)$ consisting only of handles of dimension s and $s+1$.*

Proof. Assume first that (N, ν) is s-bi-connected. By Lemma 4.53, we can assume without loss of generality that $s \geq 2$. Then, the inclusion maps $N \hookrightarrow M_N$ and $l_1(N) \hookrightarrow M_N$ induce isomorphisms on fundamental groups. Consider the following homology exact sequence for the triple $\widetilde{N} \subset \widetilde{M_N} \subset \widetilde{B}$.

$$\cdots \to H_i(M_N, N; \mathbb{Z}G) \to H_i(B, N; \mathbb{Z}G) \xrightarrow{j_*} H_i(B, M_N; \mathbb{Z}G) \to \dots \tag{4.16}$$

By excision, we have $H_i(B, N; \mathbb{Z}G) \cong H_i(\overline{M}, A; \mathbb{Z}G)$ and $H_i(B, M_N; \mathbb{Z}G) \cong H_i\big(\overline{M}, l_1(A); \mathbb{Z}G\big)$. In particular, we have $H_i(B, N; \mathbb{Z}G) = 0$ for $i \leq s$ and, since by construction the homomorphism $t_*^{-1} \colon H_s\big(\overline{M}, l_1(A); \mathbb{Z}G\big) \to H_s(\overline{M}, A; \mathbb{Z}G)$ is a monomorphism, also $H_i(B, M_N; \mathbb{Z}G) = 0$ for $i \leq s$. Therefore, we obtain $H_i(M_N, N; \mathbb{Z}G) = 0$ for $i < s$. In other words, there exists a handlebody decomposition of M_N consisting only of handles of dimension greater than or equal to s.

Consider now the following homology sequence for the triple $\widetilde{l_1(N)} \subset \widetilde{M_N} \subset \widetilde{l_1(A)}$.

$$\cdots \to H_i\big(M_N, l_1(N); \mathbb{Z}G\big) \to H_i\big(l_1(A), l_1(N); \mathbb{Z}G\big) \to H_i\big(l_1(A), M_N; \mathbb{Z}G\big) \to \dots$$

By a similar argument as above, we have $H_i\big(l_1(A), M_N; \mathbb{Z}G\big) = H_i\big(l_1(A), l_1(N); \mathbb{Z}G\big) = 0$ for $i \leq n-s-1$. Therefore, we obtain that $H_i\big(M_N, l_1(N); \mathbb{Z}G\big) = 0$ for $i < n-s-1$. In other words, there exists a handlebody decomposition of M_N such that the dual decomposition consists only of handles of dimension greater than or equal to $n-s-1$. But this is equivalent to say that there exists a handlebody decomposition of M_N consisting only of handles of dimension less than or equal to $s+1$. Hence, we obtain a handlebody decomposition of M_N consisting only of handles of dimension s and $s+1$.

Conversely, assume that there exists a handlebody decomposition of $\big(M_N, N, l_1(N)\big)$ consisting only of handles of dimension s and $s+1$. Consider the homology exact sequence (4.16). By assumption we have that $H_i(M_N, N; \mathbb{Z}G) = 0$ for $i < s$. Therefore, the homomorphism j_* is an isomorphism for $i < s$ and an epimorphism for $i = s$. In particular, by excision, this holds also for the homomorphism

$$j_* \colon H_i(\overline{M}, A; \mathbb{Z}G) \to H_i\big(\overline{M}, l_1(A); \mathbb{Z}G\big)$$

Now, by Lemma 4.44, the module $H_i(\overline{M}, A; \mathbb{Z}G)$ is finitely generated for any $i \leq s$. Moreover, we have

$$\varinjlim_m H_i\big(\overline{M}, l_1^m(A); \mathbb{Z}G\big) = H_i\big(\overline{M}, \overline{M}; \mathbb{Z}G\big) = 0$$

for any i. Hence we obtain $H_i(\overline{M}, A; \mathbb{Z}G) = 0$ for $i \leq s$. Namely, (N, ν) is s-connected. By a similar argument, it follows also that $(N, -\nu)$ is $(n-s)-1$-connected. Therefore, we can conclude that (N, ν) is s-bi-connected. $\square$

We are now ready to prove that $H_3(\overline{M}, A; \mathbb{Z}G)$ is a projective module and then to define the first obstruction.

Lemma 4.55. *If (N,ν) is an s-bi-connected splitting of M, then $H_{s+1}(\overline{M},A;\mathbb{Z}G)$ is a projective $\mathbb{Z}G$-module.*

Proof. By Lemma 4.54, we have that the pair (B,N) has the homotopy type of a relative CW-complex (X,N) where X is obtained by N by attaching cells of dimension s and $s+1$ (see [LM23, Section 2.3]). In particular, we get that for $i \neq s, s+1$

$$H_i(B,N;\mathbb{Z}G) = H_i(X,N;\mathbb{Z}G) = 0$$

Denote by (X_s,N) the relative subcomplex of (X,N) obtained by attaching only the s-cells. Consider the following exact sequence, which comes from the homology exact sequence of the triple $\widetilde{N} \subset \widetilde{X_s} \subset \widetilde{X}$.

$$0 \to H_{s+1}(X,N;\mathbb{Z}G) \to H_{s+1}(X,X_s;\mathbb{Z}G) \xrightarrow{\partial} H_s(X_s,N;\mathbb{Z}G) \to H_s(X,N;\mathbb{Z}G) \to 0$$

By excision, we have $H_i(X,N;\mathbb{Z}G) \cong H_i(B,N;\mathbb{Z}G) \cong H_i(\overline{M},A;\mathbb{Z}G)$ for any i. Therefore, the above sequence becomes

$$0 \to H_{s+1}(\overline{M},A;\mathbb{Z}G) \to H_{s+1}(X,X_s;\mathbb{Z}G) \xrightarrow{\partial} H_s(X_s,N;\mathbb{Z}G) \to 0$$

Moreover, by construction, $H_s(X_s,N;\mathbb{Z}G)$ is a free $\mathbb{Z}G$-module. Hence, this sequence splits. Now, since also $H_{s+1}(X,X_s;\mathbb{Z}G)$ is a free $\mathbb{Z}G$-module, we can conclude that $H_{s+1}(\overline{M},A;\mathbb{Z}G)$ is a projective $\mathbb{Z}G$-module. □

Definition 4.56. (i) By Lemma 4.55, Lemma 4.44 and Lemma 4.47, if (N,ν) is an s-bi-connected splitting of M, then the pair $\left(H_{s+1}(\overline{M},A;\mathbb{Z}G), t_*^{-1}\right)$ is in the category $\mathrm{Nil}(\mathbb{Z}G,\alpha)$ presented in the previous section. We denote it by $c(N,\nu)$.

(ii) If (N,ν) is an s-bi-connected splitting of M with respect to f, we define

$$c(f) = (-1)^{s+1}\left[c(N,\nu)\right] \in C(\mathbb{Z}G,\alpha)$$

where $C(\mathbb{Z}G,\alpha)$ is the reduced K_0-group of $\mathrm{Nil}(\mathbb{Z}G,\alpha)$ defined in Definition 4.30.

Remark 4.57. The obstruction $c(f)$ does not depend on the choice of the splitting (N,ν). This follows by the fact that there exists a group homomorphism $\gamma\colon \mathrm{Wh}(\pi M) \to C(\mathbb{Z}G,\alpha)$ such that

$$\gamma(-\,{\tau_{\mathrm{fib}}}'(f)) = c(f)$$

where ${\tau_{\mathrm{fib}}}'(f)$ is the obstruction of Definition 4.3. Indeed, ${\tau_{\mathrm{fib}}}'(f)$ is independent of the splitting and hence so is $c(f)$. The existence of γ will be proved in the next section.

To conclude this part, we show that $c(f)$ is actually the obstruction for M to pseudofiber a circle. For this, we need the following technical result, whose proof can be found in [Far71, Chapter V].

Lemma 4.58 ([Far71, Lemma 5.1])**.** *Let (N,ν) be an s-bi-connected splitting of M with respect to f where $2 \leq s \leq n-4$ and let (P,φ) be an object of $\mathrm{Nil}(\mathbb{Z}G,\alpha)$ such that $[P,\varphi] = \left[c(N,\nu)\right]$. Then, there exists an s-bi-connected splitting (N',ν') such that $c(N',\nu') \cong (P,\varphi)$.*

Theorem 4.59. *The manifold M pseudofibers a circle with respect to f if and only if $c(f)$ vanishes.*

Proof. Suppose first that M pseudofibers a circle with respect to f. Let N be a splitting of M with respect to f such that M_N is an h-cobordism. Then by construction $c(f)=0$.

Conversely, assume that $c(f)$ vanishes. Then by Lemma 4.53, there exists a 2-bi-connected splitting (N,ν) such that $\left[c(N,\nu)\right]$ is the identity of $C(\mathbb{Z}G,\alpha)$. Obviously, we have that also $[0,0]$ is the identity of $C(\mathbb{Z}G,\alpha)$, where 0 denotes both the trivial group and the trivial homomorphism. Therefore, by Lemma 4.58, there exists a 2-bi-connected-splitting (N',ν') of M with respect to f such that $c(N',\nu') \cong (0,0)$. But this means that $H_3(\overline{M},A';\mathbb{Z}G)=0$. Hence, by Remark 4.51, M pseudofibers a circle. □

Farrell's torsion obstruction to fibering a circle

In this last part of this section, we introduce the second obstruction $\tau(f)$ for $f\colon M \to S^1$ being homotopic to a smooth fiber bundle. This is well-defined only when $c(f) = 0$, that is, when M pseudofibers a circle with respect to f and it is (related to) the torsion obstruction of the h-cobordism given by the s-cobordism theorem. More precisely, assume $c(f) = 0$. Then by Theorem 4.59 there exists a splitting (N, ν) of M with respect to f such that $M_N = \overline{l_1(A) \setminus A}$ is an h-cobordism $\big(M_N, N, l_1(N)\big)$. By the s-cobordism theorem [LM23, Theorem 2.1], M_N is diffeomorphic to the trivial cobordism $N \times I$ if and only if the Whitehead torsion $\tau(M_N)$ of the h-cobordism vanishes in $\mathrm{Wh}(\pi N) \cong \mathrm{Wh}(G)$. If this is the case, then there exists a smooth fiber bundle $\overline{f}\colon M \to S^1$ homotopic to f. Indeed, the projection $M_N \cong N \times I \to I$ induces a map $M \to S^1$ which is by construction a smooth fiber bundle homotopic to f. At this point, one could suggest to take $\tau(M_N)$ as obstruction. However, this is not well-defined because it depends in general on the splitting. Indeed, it is possible that $\tau(M_N) \neq 0$ even though there exists another splitting (N', ν') such that $\tau(M_{N'}) = 0$. We have therefore to measure the ambiguity in order to get a well-defined invariant.

For this, consider the group homomorphism $\alpha_*\colon \mathrm{Wh}(G) \to \mathrm{Wh}(G)$ of Lemma 4.34 induced by the homomorphism $K_1(\alpha)\colon K_1(\mathbb{Z}G) \to K_1(\mathbb{Z}G)$ and denote for simplicity by $\mathrm{Wh}_\alpha(G)$ the group $\mathrm{Wh}(G)/\operatorname{im}\big(\alpha_* - \mathrm{id}\big)$.

Definition 4.60. Assume that $c(f) = 0$ and let (N, ν) be a splitting of M with respect to f such that $M_N = \overline{l_1(A) \setminus A}$ is an h-cobordism. We define $\tau(f) \in \mathrm{Wh}_\alpha(G)$ to be the image of $\tau(M_N)$ under the projection map

$$q\colon \mathrm{Wh}(G) \to \mathrm{Wh}_\alpha(G)$$

Remark 4.61. The obstruction $\tau(f)$ is well-defined. Indeed:

(i) Consider the lifting $l_1(N)$ of N to $\overline{M}$. The map $l_1\colon (M_N, N) \to \big(l_1(M_N), l_1(N)\big)$ is a diffeomorphism. Moreover, the inclusion $j\colon l_1(N) \to l_1(M_N)$ is given by the composition

$$l_1(N) \xrightarrow{l_1^{-1}} N \xhookrightarrow{i} M_N \xrightarrow{l_1} l_1(M_N)$$

Therefore, since $(l_1)_*$ is by construction α_*^{-1}, by Definition 4.15 and Lemma 2.15, we obtain

$$\begin{aligned}\tau\big(l_1(M_N)\big) &= j_*^{-1}\tau(j)\\ &= \alpha_*^{-1}\, i_*\, \alpha_*\tau(l_1 \circ i \circ l_1^{-1})\\ &= \alpha_*^{-1}\, i_*\, \alpha_*\, \alpha_*^{-1}\tau(i)\\ &= \alpha_*^{-1}\, i_*\tau(i)\\ &= \alpha_*^{-1}\tau(M_N)\end{aligned}$$

It follows that $q\big(\tau(M_N)\big)$ is independent of the lifting of N to $\overline{M}$.

(ii) Let (N', ν') be another splitting such that $M_{N'}$ is an h-cobordism. By the previous point, we can assume without loss of generality that $l_1^{-1}(A') \subset l_1(A) \subset A'$. Define $W = \overline{A' \setminus l_1(A)}$.

Claim. The cobordism $\big(W, l_1(N), N'\big)$ is an h-cobordism.

Proof. Set $V = \overline{l_1(A) \setminus l_1^{-1}(A')}$. Consider the sequence of cobordism given by V, W and $l_1(V)$. We have that $V \cup W = l_1^{-1}(M_{N'})$ and $W \cup l_1(V) = l_1(M_N)$. Hence, they are

h-cobordism. Now, by the s-cobordism theorem [LM23, Theorem 2.1], any h-cobordism H has an "inverse" H^{-1}. Hence, we obtain the following two trivial h-cobordisms.

$$\big(l_1^{-1}(M_{N'})^{-1} \cup V\big) \cup W, \qquad W \cup \big(l_1(V) \cup l_1(M_N)^{-1}\big)$$

Moreover, we have

$$l_1^{-1}(M_{N'})^{-1} \cup V = l_1^{-1}(M_{N'})^{-1} \cup V \cup W \cup l_1(V) \cup l_1(M_N)^{-1} = l_1(V) \cup l_1(M_N)^{-1}$$

Therefore, W has the right inverse equal to the left inverse and so it is invertible. We obtain the following sequence

$$\begin{tikzcd} l_1(N) \arrow[r, hook] \arrow[rr, bend left, "\simeq"] & W \arrow[r, hook] \arrow[rr, bend right, "\simeq"'] & W \cup W^{-1} \arrow[r, hook] & W \cup W^{-1} \cup W = W \end{tikzcd}$$

where the composition of any two consecutive horizontal maps is a homotopy equivalence. Hence, by 2-out-of-6 property, we get that all the three horizontal individual maps, in particular the inclusion $l_1(N) \hookrightarrow W$, are homotopy equivalences. By a similar argument, also the inclusion $N' \hookrightarrow W$ is a homotopy equivalence. Therefore, we can conclude that $(W, l_1(N), N')$ is an h-cobordism. □

Now, set $W_1 = M_N \cup W$ and $W_2 = W \cup M_{N'}$. Since the composition of h-cobordisms is an h-cobordism, then these are h-cobordisms (W_1, N, N') and $\big(W_2, l_1(N), l_1(N')\big)$ and the following equations hold.

$$\tau(W_1) = \tau(M_N) + \tau(W)$$
$$\tau(W_2) = \tau(W) + \tau(M_{N'})$$

Moreover, since the map $l_1\colon (W_1, N) \to \big(W_2, l_1(N)\big)$ is a diffeomorphism, we obtain as above that $\tau(W_2) = \alpha_*^{-1}\tau(W_1)$. Therefore, we can conclude that $q\big(\tau(M_N)\big) = q\big(\tau(M_{N'})\big)$.

At this point, we can finally conclude the proof of Farrell's main theorem, that is, of Theorem 4.35.

Theorem 4.62. *Assume that $c(f) = 0$. Then there exists a smooth fiber bundle $\overline{f}\colon M \to S^1$ homotopic to f if and only if $\tau(f)$ vanishes.*

The proof is a consequence on the following result.

Lemma 4.63. *If $c(f) = 0$ and $x \in q^{-1}\big(\tau(f)\big)$, then there exists a splitting (N', ν') of M such that $M_{N'} = \overline{l_1(A') \setminus A'}$ is an h-cobordism and $\tau(M_{N'}) = x$.*

Proof of Lemma 4.63. Since $c(f) = 0$, by Theorem 4.59 there exists a splitting (N, ν) of M with respect to f such that M_N is an h-cobordism. Moreover, by definition, we have that $q\big(\tau(M_N)\big) = \tau(f)$. Hence, there exists $y \in \mathrm{Wh}(G)$ such that

$$x = \tau(M_N) + y - \alpha_*(y)$$

Now, by the s-cobordism theorem [LM23, Theorem 2.1], there exists an h-cobordims (W_1, N, N') such that $\tau(W_1) = \alpha_*(y)$. Denote by W_1^{-1} its inverse, which exists again by the s-cobordism theorem. Then $W_1 \cup W_1^{-1}$ is a trivial h-cobordism. In particular, we can identify it without loss of generality with half of a narrow tubular neighborhood of N. Therefore, N' is a splitting of M with respect to f. Define $A' = A \cup W_1$ and $M_{N'} = \overline{l_1(A') \setminus A'}$. Then, since $W_1 \cup M_{N'} = M_N \cup l_1(W_1)$, we have

$$\tau(W_1 \cup M_{N'}) = \tau\big(M_N \cup l_1(W_1)\big)$$

Moreover, the following two equations hold.

$$\begin{aligned} \tau(W_1 \cup M_{N'}) &= \tau(W_1) + \tau(M_{N'}) = \alpha_*(y) + \tau(M_{N'}) \\ \tau\big(M_N \cup l_1(W_1)\big) &= \tau(M_N) + \tau\big(l_1(W_1)\big) = \tau(M_N) + \alpha_*^{-1}\tau(W_1) = \tau(M_N) + y \end{aligned}$$

Therefore, we obtain

$$\tau(M_{N'}) = \tau(M_N) + y - \alpha_*(y) = x$$

This proves Lemma 4.63. □

Proof of Theorem 4.62. Assume first that there exists a smooth fiber bundle $\overline{f}\colon M \to S^1$ homotopic to f. Let N be the fiber of $\overline{f}$. Then M_N is clearly diffeomorphic to $N \times I$, that is, M_N is a trivial cobordism. Hence, by the s-cobordism theorem [LM23, Theorem 2.1], we have $\tau(M_N) = 0$ and $\tau(f)$ vanishes.

Conversely, assume that $\tau(f) = 0$. Then by the previous lemma applied to $x = \tau(f) = 0$ there exists a splitting (N, ν) of M with respect to f such that $\tau(M_N) = 0$. Hence, again by the s-cobordism theorem [LM23, Theorem 2.1], M_N is diffeomorphic to $N \times I$, which means, as explained before Definition 4.60, that there exists a smooth fiber bundle $\overline{f}\colon M \to S^1$ homotopic to f. □

4.5 Fibering a manifold over a circle

In the last section of this chapter we recap and we complete the proof of Theorem 4.1. In particular we obtain finally an example of a map $f\colon M \to B$ in **Man** where the vanishing of the obstructions $\theta(f)$ and $\tau_{\text{fib}}(f)$ is both a sufficient and necessary condition for f being homotopic to a fiber bundle.

Let us recall the statement of the theorem.

Theorem (Theorem 4.1)**.** *Let $f\colon M \to S^1$ be a map in* **Diff***. Assume that M is connected of dimension $\dim(M) \geq 6$ and that the homotopy fiber of f is in* **TFCW***. Suppose in addition that the homomorphism $\pi_1(f)\colon \pi_1(M) \to \pi_1(S^1)$ is surjective. Then the following are equivalent:*

(i) $\theta(f)$ and $\tau_{fib}(f)$ vanish;

(ii) ${\tau_{fib}}'(f)$ vanishes;

(iii) $c(f)$ and $\tau(f)$ vanish;

(iv) the map f is homotopic to a smooth fiber bundle.

Proof. Let us first summarize what we have achieved so far.

- By Lemma 4.7, we have that (i) implies (ii). Actually, they are also equivalent by Lemma 4.18(iv). Indeed, let N be a splitting of M with respect to f and consider the cobordism M_N of the previous section obtained from M by deleting a tubular neighborhood of N. The identity map $g = \mathrm{id}_N\colon N \to N$ is a diffeomorphism $\partial_1 M_N \to \partial_0 M_N$. Consider in the notation of Section 4.2 the manifold $(M_N)_g$ and a map $f'\colon M_N \to I$ well-defined up to homotopy. Then, there exists a diffeomorphism $\psi\colon (M_N)_g \to M$ such that $f \circ \psi = f_g$. In particular, we can use f_g and Lemma 4.18 to study f. Therefore, we can conclude that (i) is equivalent to (ii) by Lemma 4.18(iv).

- By Theorem 4.35, we have that (iii) is equivalent to (iv)
- By Theorem 3.23, we have that (iv) implies (i).

It remains only to show that (ii) implies (iii). Hence, assume that ${\tau_{\mathrm{fib}}}'(f) = 0$. Let us focus first on $c(f)$. Consider the situation of the previous chapter, that is, let (N, ν) be a splitting of M with respect to f. Consider the covering $\overline{M}$ of M with $\pi_1(\overline{M}) = \ker \pi_1(f) = G$. For example, let $\overline{M}$ be the pullback of f over the universal covering $e\colon \mathbb{R} \to S^1$ of S^1 of Notation 4.2, as in Section 4.1. Choose a lifting of (N, ν) to $\overline{M}$ and denote it again by (N, ν). Then N divides $\overline{M}$ into two connected components. Denote by B the component into which ν points and by A the other one. Let $l_1 \colon \overline{M} \to \overline{M}$ be the generator of the group of deck transformations such that $A \subset l_1(A)$. Using the notation of Section 4.1, we have by Lemma 4.14 that

$$ {\tau_{\mathrm{fib}}}'(f) = \tau(\widehat{e}\colon T_{l_1} \to M) $$

Consider now the map $p\colon \mathrm{Wh}(\pi M) = \mathrm{Wh}(G \rtimes_\alpha \mathbb{Z}) \to C(\mathbb{Z}G, \alpha^{-1})$ of Lemma 4.34. By [Far71, Theorem 4.1] there exists a duality isomorphism $\Delta\colon C(\mathbb{Z}G, \alpha) \to C(\mathbb{Z}G, \alpha^{-1})$ which maps $c(f)$ to $c(-f)$. Define the map

$$ \gamma = \Delta^{-1} \circ p\colon\ \mathrm{Wh}(\pi M) \to C(\mathbb{Z}G, \alpha) $$

We claim that $\gamma\big(-{\tau_{\mathrm{fib}}}'(f)\big) = c(f)$, so that, since ${\tau_{\mathrm{fib}}}'(f) = 0$ by assumption, then also $c(f) = 0$. To prove it, note that since we have the identification $G \rtimes_\alpha \mathbb{Z} = \pi_1(M)$ of Remark 4.48, we obtain that

$$ -{\tau_{\mathrm{fib}}}'(f) = -\tau(\,\widehat{e}\,) = \tau\big(u = (\,\widehat{e}\,)^{-1}\colon M \to T_{l_1}\big) $$

Therefore, it suffices to show that $p\big(\tau(u)\big) = c(-f)$. This follows by a "simple structure" version of [Far71, Lemma 3.8-3.9]. Indeed, equip T_{l_1} with the preferred simple structure on a mapping torus defined in Remark 4.9 and M with its preferred simple structure $\xi^{\mathrm{Top}}(M)$. Suppose that N is s-bi-connected with respect to f. This is equivalent to say that N is an $(n-s-1)$-bi-connected splitting with respect to $-f$. Define E to be the following pullback

$$ \begin{array}{ccc} E & \longrightarrow & T_{l_1} \\ \big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ \mathbb{R} & \xrightarrow{\ e\ } & S^1 \end{array} $$

where φ is a map coming from the projection $\overline{M} \times I \to I$. Then E is a covering of T_{l_1} and $\overline{M}$ divides E into two connected components A' and B'. Moreover, by construction, there exists a lift $u'\colon \overline{M} \to E$ of u such that $(u')^{-1}(A') = A$, $(u')^{-1}(B') = B$ and $(u')^{-1}(\overline{M}) = N$. Denote by $\widetilde{M}$ the universal covering of $\overline{M}$ (and hence of M), by $\widetilde{E}$ the universal covering of E (and hence of T_{l_1}) and by $\widetilde{u}\colon \widetilde{M} \to \widetilde{E}$ a lift of u to universal coverings. Consider the following commutative diagram, where the rows are the two long exact homology sequences for the inclusions $\widetilde{B} \subset \widetilde{M}$ and $\widetilde{B'} \subset \widetilde{E}$ and the vertical arrows are induced by $\widetilde{u}\colon (\widetilde{M}, \widetilde{B}) \to (\widetilde{E}, \widetilde{B'})$.

$$ \begin{array}{ccccccccc} \ldots \longrightarrow & H_k(B;\mathbb{Z}G) & \xrightarrow{i_*} & H_k(\overline{M};\mathbb{Z}G) & \xrightarrow{j_*} & H_k(\overline{M},B;\mathbb{Z}G) & \longrightarrow & H_{k-1}(B;\mathbb{Z}G) & \longrightarrow \ldots \\ & {\scriptstyle\widetilde{u}_*}\big\downarrow & & {\scriptstyle\cong}\big\downarrow & & \big\downarrow & & {\scriptstyle\widetilde{u}_*}\big\downarrow & \\ \ldots \longrightarrow & H_k(B';\mathbb{Z}G) & \xrightarrow{\cong} & H_k(E;\mathbb{Z}G) & \longrightarrow & 0 & \longrightarrow & H_{k-1}(B';\mathbb{Z}G) & \longrightarrow \ldots \end{array} $$

We have that $H_i(E, B'; \mathbb{Z}G) = \varinjlim H_i\big(l_1^{-m}(B'), B'; \mathbb{Z}G\big) = 0$. Moreover, since the splitting N is $(n-s)-1$-bi-connected, we obtain that $H_i(\overline{M}, B; \mathbb{Z}G) = 0$ for $i \neq (n-s)-1, n-s$. Therefore, the homorphism $\widetilde{u}_*\colon H_i(B;\mathbb{Z}G) \to H_i(B';\mathbb{Z}G)$ is an isomorphism for $i \neq (n-s)-1, n-s$. Now, for $i = (n-s)-1, n-s$ the diagram reduces to

$$\begin{tikzcd}
0 \arrow[r] & H_{n-s}(B;\mathbb{Z}G) \arrow[r, "i_*"] \arrow[d, "\widetilde{u}_*"'] & H_{n-s}(\overline{M};\mathbb{Z}G) \arrow[r, "j_*"] \arrow[d, "\cong"'] & H_{n-s}(\overline{M},B;\mathbb{Z}G) \arrow[r, equal] \arrow[d] & {} \\
0 \arrow[r] & H_{n-s}(B';\mathbb{Z}G) \arrow[r, "\cong"] & H_{n-s}(E;\mathbb{Z}G) \arrow[r] & 0 \arrow[r, equal] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, equal] & H_{n-s}(\overline{M},B;\mathbb{Z}G) \arrow[r] \arrow[d] & H_{n-s-1}(B;\mathbb{Z}G) \arrow[r] \arrow[d, "\widetilde{u}_*"'] & H_{n-s-1}(\overline{M};\mathbb{Z}G) \arrow[r] \arrow[d, "\cong"] & 0 \\
{} \arrow[r, equal] & 0 \arrow[r] & H_{n-s-1}(B';\mathbb{Z}G) \arrow[r, "\cong"] & H_{n-s-1}(E;\mathbb{Z}G) \arrow[r] & 0
\end{tikzcd}$$

We claim that the map j_* is the trivial map. Indeed, by Lemma 4.44, we have that the homology group $H_{n-s}(\overline{M},B;\mathbb{Z}G)$ is a finitely generated $\mathbb{Z}G$-module. Moreover, we have that $\varinjlim H_{n-s}\big(\overline{M}, l_1^{-m}(B);\mathbb{Z}G\big) = 0$. Therefore, there exists an integer r such that

$$j'_*\colon H_{n-s}(\overline{M},B;\mathbb{Z}G) \to H_{n-s}\big(\overline{M}, l_1^{-r}(B);\mathbb{Z}G\big)$$

is the zero map. In particular, the map $j'_*\colon H_{n-s}\big(\overline{M}, l_1^{r}(B);\mathbb{Z}G\big) \to H_{n-s}(\overline{M},B;\mathbb{Z}G)$ is the zero map. It suffices now to note that $j\colon \widetilde{M} \to (\widetilde{M},\widetilde{B})$ is the composition of the two inclusion maps $\widetilde{M} \hookrightarrow \big(\widetilde{M}, \widetilde{l_1^r(B)}\big)$ and $j'\colon \big(\widetilde{M}, \widetilde{l_1^r(B)}\big) \to (\widetilde{M},\widetilde{B})$ to conclude that $j_* = 0$.

It follows that the homomorphism i_* of the diagram above, and therefore the homomorphism $\widetilde{u}_*\colon H_{n-s}(B;\mathbb{Z}G) \to H_{n-s}(B';\mathbb{Z}G)$, is an isomorphism and that the homomorphism $\widetilde{u}_*\colon H_{n-s-1}(B;\mathbb{Z}G) \to H_{n-s-1}(B';\mathbb{Z}G)$ is an epimorphism with kernel isomorphic to the group $H_{n-s}(\overline{M},B;\mathbb{Z}G)$. To sum up, we have obtained the following:

- $\widetilde{u}_*\colon H_k(B;\mathbb{Z}G) \to H_k(B';\mathbb{Z}G)$ is epimorphic for any $k \in \mathbb{Z}$;
- $\widetilde{u}_*\colon H_k(B;\mathbb{Z}G) \to H_k(B';\mathbb{Z}G)$ is also monomorphic if $k \neq n-s-1$;
- $\ker \widetilde{u}_* = H_{n-s}(\overline{M},B;\mathbb{Z}G)$ is a projective $\mathbb{Z}G$-module by Lemma 4.55 if $k = n-s-1$;
- u may be supposed cellular;
- N, A and B may be seen as subcomplexes of $\overline{M}$.

Hence, we can conclude as follows by [Far71, Lemma 3.8].

$$p\big(-{\tau_{\mathrm{fib}}}'(f)\big) = p\big(\tau(u)\big) = (-1)^{n-s}\big[\ker \widetilde{u}_*, t_*\big] = (-1)^{n-s}\big[H_{n-s}(\overline{M},B;\mathbb{Z}G), t_*\big] = c(-f)$$

Therefore, we have proved that if ${\tau_{\mathrm{fib}}}'(f) = 0$, then also $c(f) = 0$.

Let us now show that if ${\tau_{\mathrm{fib}}}'(f) = 0$, then also $\tau(f) = 0$. Note that $\tau(f)$ is well-defined because we have just proved that $c(f) = 0$. Consider the inclusion map $l\colon N \hookrightarrow M$. As explained at the beginning of this proof, we can look at M as the manifold $(M_N)_g$ where $g = \mathrm{id}_N$. Therefore, the map l induces a map $\pi_1(l)\colon G = \pi_1(N) \to \pi_1(M) = G \rtimes_\alpha \mathbb{Z}$ which is the identity on G. We obtain then the following homomorphism of Whitehead groups

$$l_*\colon \mathrm{Wh}(G) = \mathrm{Wh}(\pi N) \to \mathrm{Wh}(\pi M) = \mathrm{Wh}(G \rtimes_\alpha \mathbb{Z})$$

Now, this homomorphism is by construction exactly the homomorphism induced also by the inclusion $i_0\colon \mathbb{Z}G \hookrightarrow \mathbb{Z}G_\alpha[t,t^{-1}] \cong \mathbb{Z}(G \rtimes_\alpha \mathbb{Z})$. Therefore, by Lemma 4.34(ii), it factors through the monomorphism

$$l'_*\colon \mathrm{Wh}_\alpha(G) \to \mathrm{Wh}(G \rtimes_\alpha \mathbb{Z})$$

We claim that l'_* sends the obstruction $\tau(f)$ to $(-1)^{\dim(N)} \cdot \delta\big(\, \tau_{\mathrm{fib}}{}'(f)\big)$, where δ is the twisted involution defined in Section 4.2. To prove this, it suffices by definition to show that given any splitting (N, ν) of M with respect to f such that M_N is an h-cobordism, then

$$l_*\big(\tau(M_N)\big) = (-1)^{\dim(N)} \cdot \delta\big(\, \tau_{\mathrm{fib}}{}'(f)\big) \tag{4.17}$$

But this follows immediately by Lemma 4.18(ii). Therefore, since l'_* is a monomorphism, we get that $\tau(f) = 0$ if $\tau_{\mathrm{fib}}{}'(f) = 0$ (actually by (4.17) this is an equivalence).

We have then concluded the proof of Theorem 4.1. □

Remark 4.64. The idea of the proof above is that in some sense $c(f)$ and $\tau(f)$ are "components" of $\tau_{\mathrm{fib}}{}'(f)$ with respect to the Bass-Heller-Swan decomposition of $\mathrm{Wh}(\pi M)$ given in Lemma 4.34.

$$\mathrm{Wh}(G \rtimes_\alpha \mathbb{Z}) \cong X_1(\mathbb{Z}G, \alpha) \oplus NK_1(\mathbb{Z}G, \alpha) \oplus NK_1(\mathbb{Z}G, \alpha^{-1})$$

Indeed, according to Remark 4.33 and using the following short exact sequence, again given in in Lemma 4.34,

$$0 \to \mathrm{Wh}(G)/\operatorname{im}\big(\alpha_* - \mathrm{id}\big) \xrightarrow{l'_*} X_1(\mathbb{Z}G, \alpha) \to \ker\big(\widetilde{K}_0(\alpha) - \mathrm{id}\big) \to 0$$

we have that $c(f)$ is a sort of projection of $\tau(f)$ into the "sum" of $NK_1(\mathbb{Z}G, \alpha)$ with the "component" of $X_1(\mathbb{Z}G, \alpha)$ given by $\ker\big(\widetilde{K}_0(\alpha) - \mathrm{id}\big)$, while $\tau(f)$ is the "component" of $\tau_{\mathrm{fib}}{}'(f)$ in the subgroup $\mathrm{Wh}_\alpha(G)$ of $X_1(\mathbb{Z}G, \alpha)$. From this point of view, it is easy to understand that assertion (ii) of the theorem implies assertion (iii).

Chapter 5

The stable fibering problem

The goal of this chapter is to use algebraic K-theory to investigate whether there exists a set of obstructions whose vanishing is both a necessary and sufficient condition for a general map $f\colon M \to B$ in **Man** being homotopic to projection map of a fiber bundle. The idea is to apply the strategy used in Chapter 3 for the fibering problem, but using algebraic K-theory. This leads to the definition of two invariants $\mathrm{Wall}(p)$ and $o(f)$ that generalize naturally the two obstructions $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$. However, it turns out that these are not obstructions for the fibering problem, but they form a complete set of obstructions for existence and uniqueness of the more general *stable* fibering problem, which is formulated as follows: is a map $f\colon M \to B$ between compact topological manifolds homotopic to a fiber bundle with compact manifolds as fibers, if we allow to stabilize M by crossing with disks of sufficiently high dimension? More precisely, let $f\colon M \to B$ be a map in the category **Cpt** of compact manifolds with boundary (which we call compact manifolds for short). We say that f *stably fibers* if there exists a $n \in \mathbb{N}$ such that the composite

$$f \circ \mathrm{Proj}\colon M \times D^n \to M \to B$$

is homotopic to the projection of a fiber bundle whose fibers are in **Cpt**. The stable fibering problem consists of investigating when and in how many different ways a map f in **Cpt** stably fibers and $\mathrm{Wall}(p)$ and $o(f)$ provide an answer to these questions. Note that the problem is formulated in the category **Cpt** because as soon as we cross the total space M with disks, this leaves the category **Man** of closed manifolds and becomes an object of **Cpt**.

The main references for this chapter are [Ste12] and its longer version [Ste10]. The dissertation requires a wide use of algebraic K-theory and consists largely of extending the whole Whitehead torsion theory in this context. In particular, it is based on the parametrized A-theory characteristic developed in [DWW03] and it uses spectra and spectral sequences. What we present in this chapter is just an overview of the whole argument. In particular, most results are only stated without proof and we give for known most of the prerequisites. However, a reference is provided for all missing parts.

The work is structured as follows. In Section 5.1, we briefly present the parametrized A-theory characteristic and the excisive A-theory characteristic. In Section 5.2, we generalize the Whitehead torsion theory by describing the Whitehead spectrum and the parametrized Whitehead torsion. In Section 5.3, we introduce the geometric assembly map and we state its most important properties. In Section 5.4, we define the two obstructions $\mathrm{Wall}(p)$ and $o(f)$ and we study the stable fibering problem by proving the two main theorems of this chapter on existence and uniqueness. Finally, in Section 5.5, we compare the obstructions $\mathrm{Wall}(p)$ and $o(f)$ with $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$.

5.1 Parametrized Euler characteristics

In this section we present briefly the parametrized A-theory characteristic and the excisive A-theory characteristic defined in [DWW03]. These tools will be of fundamental importance in the next section to define the parametrized Whitehead torsion.

Let us start with the general definition of characteristic of a functor.

Notation 5.1. Throughout this chapter, we denote by **Cat** the category of small categories.

Definition 5.2. Let $\mathscr{C}$ be in **Cat** and $\mathcal{F}\colon \mathscr{C} \to \mathbf{Cat}$ be a functor. For a fixed object C in $\mathscr{C}$, denote by $\mathscr{C}_{/C}$ the over category, that is, the category whose objects are morphisms $D \to C$ in $\mathscr{C}$ and whose morphisms are commutative triangles. Let $\mathscr{C}_{/?}\colon \mathscr{C} \to \mathbf{Cat}$ be the functor which sends an object C in $\mathscr{C}$ to the category $\mathscr{C}_{/C}$. A *characteristic* for $\mathcal{F}$ is a natural transformation

$$\chi\colon \mathscr{C}_{/?} \to \mathcal{F}$$

Remark 5.3. Unraveling the definition, we see that a characteristic χ is completely determined by the following data:

- For any object C in $\mathscr{C}$, a *characteristic object* $C^!$ in $\mathcal{F}(C)$, which corresponds to the image of the identity morphism id_C of C in $\mathscr{C}$ under the functor $\chi(C)$.
- For any morphism $\varphi\colon C \to D$ in $\mathscr{C}$, a morphism $\varphi^!\colon \varphi_*(C^!) \to D^!$ in $\mathcal{F}(D)$ such that $(\mathrm{id}_C)^! = \mathrm{id}_{C^!}$ and the 1-cocycle condition $(\psi\varphi)^! = \psi^! \circ \psi_*(\varphi^!)$ is satisfied for all morphisms $\varphi\colon C \to D$ and $\psi\colon D \to E$ in $\mathscr{C}$, where η_* denotes the functor $\mathcal{F}(\eta)$ for any morphism η in $\mathscr{C}$.

By taking geometric realizations, this construction suggests to define a characteristic also for functors over **Top**.

Definition 5.4. Let $\mathscr{C}$ be in **Cat** and $F\colon \mathscr{C} \to \mathbf{Top}$ be a functor. A *characteristic* for F is a natural transformation

$$\chi\colon \left|\mathscr{C}_{/?}\right| \to F$$

where $|-|\colon \mathbf{Cat} \to \mathbf{Top}$ is the geometric realization functor.

Remark 5.5. (i) The space of characteristics for F is $\operatorname{holim} F$, the homotopy limit of F.

(ii) For any functor $\mathcal{F}\colon \mathscr{C} \to \mathbf{Cat}$, if we define $F\colon \mathscr{C} \to \mathbf{Top}$ by $F(C) = |\mathcal{F}(C)|$, then a characteristic $\chi\colon \mathscr{C}_{/?} \to \mathcal{F}$ for $\mathcal{F}$ induces a characteristic $\chi\colon \left|\mathscr{C}_{/?}\right| \to F$ for F.

(iii) Let $*\colon \mathscr{C} \to \mathbf{Top}$ be the terminal functor and $\operatorname{hocolim} F \to \operatorname{hocolim} * = |\mathscr{C}|$ be the obvious canonical map from the homotopy colimit of F to the geometric realization of $\mathscr{C}$. A characteristic χ for F may be seen as a section of the canonical map $\operatorname{hocolim} F \to |\mathscr{C}|$. Indeed, the natural transformation χ induces a lift

$$\begin{tikzcd} & \operatorname{hocolim} F \arrow[d] \\ \operatorname{hocolim}\left|\mathscr{C}_{/?}\right| \arrow[ur, dotted, "\chi_*"] \arrow[r, "\alpha"] & |\mathscr{C}| \end{tikzcd}$$

of the canonical projection α. Moreover, since α is a homotopy equivalence, we get a homotopy equivalence

$$\alpha^*\colon\ \Gamma\begin{pmatrix} \operatorname{hocolim} F \\ \downarrow \\ |\mathscr{C}| \end{pmatrix} \xrightarrow{\simeq} \mathrm{Lift}\begin{pmatrix} & \operatorname{hocolim} F \\ & \downarrow \\ \operatorname{hocolim}\left|\mathscr{C}_{/?}\right| \longrightarrow & |\mathscr{C}| \end{pmatrix}$$

from the space of section of $\operatorname{hocolim} F \to |\mathscr{C}|$ to the space of lift of the diagram on the right hand side (see Notation below). Therefore, we get a canonical map

$$\operatorname{holim} F \longrightarrow \Gamma \left(\begin{array}{c} \operatorname{hocolim} F \\ \downarrow \\ |\mathscr{C}| \end{array} \right)$$

If F sends all morphisms of $\mathscr{C}$ to homotopy equivalences, then the previous map is a zigzag of weak homotopy equivalences by [Dwy96, Proposition 3.12] and, in particular, any characteristic χ of F may be seen, up to homotopy, as a section of the canonical map $\operatorname{hocolim} F \to |\mathscr{C}|$.

Notation 5.6. In the previous remark and in the following, when we refer to a space of lifts, we will always implicitly assume that the vertical map has been converted into a fibration: in a Kan fibration for simplicial sets or Hurewicz fibration for topological spaces.

The parametrized A-theory characteristic

We apply now all this construction to Waldhausen's A-theory to get the parametrized A-theory characteristic.

First of all, let us briefly review the definition of A-theory of a space. As in the previous chapter, we use of the definition of K-theory given by Waldhausen in [Wal85]. Denote by $\mathscr{R}^{\mathrm{fd}}(X)$ the category of homotopy finitely dominated retractive spaces over X, that is, the category whose objects are diagrams

$$Y \underset{s}{\overset{r}{\rightleftarrows}} X$$

such that $r \circ s = \mathrm{id}_X$, the map s is a cofibration and Y is a homotopy finitely dominated space over X and whose morphisms are maps over and relative X. This category can be equipped with the following Waldhausen category structure: a morphism in $\mathscr{R}^{\mathrm{fd}}(X)$ is a weak equivalence or a cofibration if its underlying map of spaces is a homotopy equivalence or, respectively, a cofibration.

Definition 5.7. We define the *A-theory* $A(X)$ of a space X as the K-theory $K\big(\mathscr{R}^{\mathrm{fd}}(X)\big)$ of $\mathscr{R}^{\mathrm{fd}}(X)$.

We construct now a characteristic using the functor $X \mapsto A(X)$. Let $p\colon E \to B$ be in **Fib**, that is, let $p\colon E \to B$ be a fibration. Assume that B is the geometric realization of a simplicial set $B_.$ and that the fibers of p are homotopy finitely dominated. Let $\mathscr{C} = \operatorname{simp} B_.$ be the category of simplices of $B_.$. Define the functor $f\colon \mathscr{C} \to \mathbf{Top}$ by $f(\sigma) = E_\sigma$ for any simplex $\sigma\colon \Delta^k \to B_.$, where E_σ is the following pullback.

$$\begin{array}{ccc} E_\sigma & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta^k & \xrightarrow{|\sigma|} & B \end{array}$$

Note that by construction f sends all simplices to homotopy finitely dominated spaces and all morphisms to homotopy equivalences. Then, we define a characteristic for $\mathcal{F} = \mathscr{R}^{\mathrm{fd}} \circ f$ using Remark 5.3 as follows:

- For any simplex $\sigma\colon \Delta^k \to B$ in $\mathscr{C}$, choose $\sigma^! = f(\sigma) \times S^0 = E_\sigma \times S^0$ in $\mathcal{F}(\sigma) = \mathscr{R}^{\mathrm{fd}}(E_\sigma)$.

- For any map $e\colon \sigma \to \sigma'$ in $\mathscr{C}$, we have $e_*(\sigma^!) = \big(E_\sigma \times \{-1\}\big) \cup \big(E_{\sigma'} \times \{1\}\big)$ in $\mathscr{R}^{\mathrm{fd}}(E_{\sigma'})$. Define $e^!\colon e_*(\sigma^!) \to (\sigma')^!$ to be the morphism

$$(x,a) \mapsto \begin{cases} (e(x),-1) & \text{if } a=-1 \\ (y,1) & \text{if } a=1 \end{cases}$$

in $\mathscr{R}^{\mathrm{fd}}(E_{\sigma'})$. Note that the 1-cocycle condition is satisfied.

By Remark 5.5(ii), this defines also a characteristic for $F = |\mathcal{F}| = \big|\mathscr{R}^{\mathrm{fd}} \circ f\big|$. Moreover, by [Wal85, p. 12], there is a natural map

$$\big|\mathscr{R}^{\mathrm{fd}}(E_\sigma)\big| \to K\big(\mathscr{R}^{\mathrm{fd}}(E_\sigma)\big) = A(E_\sigma)$$

reminiscent of the group completion. Therefore, by composition of natural transformation, we obtain a characteristic $\chi(p)$ in $\operatorname{holim}_{\sigma\in\operatorname{simp} B_\cdot} A(E_\sigma)$ for the functor

$$\sigma \mapsto A(E_\sigma)$$

which we call the *parametrized A-theory Euler characteristic* of p. By Remark 5.5(iii), if we denote by $A_B(E) \to B$ the fibration associated with the composite $\operatorname{hocolim} F \to |\mathscr{C}| \to B$, where the last map is the homotopy equivalence $|\mathscr{C}| \simeq B$ given by Kan's last vertex map, then $\chi(p)$ can be seen up to homotopy as a section of the fibration $A_B(E) \to B$ over B obtained by applying the functor A fiberwise.

The excisive A-theory characteristic

The A-theory functor $X \mapsto A(X)$ is a functor on the category of finitely dominated spaces which is not a homology theory. In particular, it does not satisfy the excision axiom. By [WW95], there exists a excisive functor $X \mapsto A^{\%}(X)$ which approximate $A(-)$. To conclude this section, we construct a characteristic for this functor and we call it the excisive A-theory characteristic.

Let us briefly recall how $A^{\%}(X)$ is defined. By [DWW03, Section 7], if X is a Euclidean neighborhood retract (ENR) as defined in [Hat02, p. 527], then $A^{\%}(X)$ can be explicitly constructed using Waldhausen categories in the following way. Given a ENR space X, denote by $\mathscr{R}^{\mathrm{ld}}(\mathbb{J}X)$ the category whose objects are diagrams

$$Y \underset{s}{\overset{r}{\rightleftarrows}} X \times [0,\infty)$$

such that $r \circ s = \mathrm{id}_{X\times[0,\infty)}$ and Y is a homotopy locally finitely dominated space over $X \times [0,\infty)$ as defined in [DWW03, p. 48] and whose morphisms are maps over and relative $X \times [0,\infty)$. This category has the following notion of homotopy, which leads to very natural notions of cofibration and weak equivalence. Consider two objects

$$Y \underset{s_i}{\overset{r_i}{\rightleftarrows}} X \times [0,\infty)$$

for $i = 0,1$ and two morphisms $f,g\colon Y_0 \to Y_1$ of $\mathscr{R}^{\mathrm{ld}}(\mathbb{J}X)$. We say that a map $H\colon Y_0 \times I \to Y_1$ is a *controlled homotopy* between f and g if H is a homotopy between f and g in the usual sense and it commutes with maps s_0 and s_1, but it commutes with retractions r_0 and r_1 only in a controlled way (see [DWW03, p. 47]). Then, the category $\mathscr{R}^{\mathrm{ld}}(\mathbb{J}X)$ has a Waldahausen category structure where weak equivalences are controlled homotopy equivalences and cofibrations are the maps with the controlled homotopy extension property. Denote by $A^{\mathbb{J}}(X)$ the K-theory of $\mathscr{R}^{\mathrm{ld}}(\mathbb{J}X)$. Consider the functor

$$I\colon \mathscr{R}^{\mathrm{fd}}(X) \to \mathscr{R}^{\mathrm{ld}}(\mathbb{J}X)$$

which sends an object $r\colon Y \rightleftarrows X : s$ of $\mathscr{R}^{\mathrm{fd}}(X)$ to the object $\overline{Y} \rightleftarrows X \times [0,\infty)$ of $\mathscr{R}^{\mathrm{ld}}(\mathbb{J}X)$ where $\overline{Y}$ is the following pushout

$$\begin{tikzcd}
X = X \times \{0\} \arrow[r, hook] \arrow[d, "s"] & X \times [0,\infty) \arrow[d] \\
Y \arrow[r] & \overline{Y}
\end{tikzcd}$$

Then, I is an embedding of categories and an exact functor of Waldhausen categories. Therefore, it induces a map of K-theory spaces

$$I_*\colon A(X) \to A^{\mathbb{J}}(X)$$

Let $\mathcal{V}(X)$ be the full subcategory of $\mathscr{R}^{\mathrm{ld}}(\mathbb{J}X)$ whose objects are proper retractive ENRs over $X \times [0,\infty)$, that is, retractive spaces $r\colon Y \to X \times [0,\infty) : s$ where Y is an ENR and r is a proper map. Denote by $J\colon \mathcal{V}(X) \to \mathscr{R}^{\mathrm{ld}}(\mathbb{J}X)$ the inclusion functor. By [DWW03, p. 7.8], the category $\mathcal{V}(X)$ has a Waldhausen category structure such that the functor J is exact and such that $V(X) = K\big(\mathcal{V}(X)\big)$ is contractible. In particular, J induces a map $J_*\colon V(X) \to A^{\mathbb{J}}(X)$.

Definition 5.8. For a compact ENR X, we define $A^{\%}(X)$ has the homotopy limit

$$A^{\%} = \operatorname{holim}\big(A(X) \xrightarrow{I_*} A^{\mathbb{J}}(X) \xleftarrow{J_*} V(X)\big) \simeq \operatorname{hofib}\big(A(X) \xrightarrow{I_*} A^{\mathbb{J}}(X)\big)$$

and we call it the *excisive A-theory* of X. The natural map $\alpha\colon A^{\%}(X) \to A(X)$ is called the *assembly map.*

We construct now a characteristic for the excisive A-theory. Let $p\colon E \to B$ be a bundle in **Cpt** such that B is the geometric realization of a simplicial set $B_.$. Define $tB_.$ as the simplicial set whose n-simplices are pairs (σ,θ) where:

- σ is an n-simplex of B;
- θ is an equivalence relation on E_σ with quotient space E_σ^θ such that the two projections make up a homeomorphism $E_\sigma \to \Delta^n \times E_\sigma^\theta$.

Then the functor $f\colon \operatorname{simp} tB_. \to \mathbf{Top}$ which sends $(\sigma,\theta) \mapsto E_\sigma^\theta$ maps all objects to compact ENRs and all morphisms to homeomorphisms, which are cell-like maps. For a compact ENR X, consider now the category $\mathscr{R}^{\%}(X)$ defined as the following pullback.

$$\begin{tikzcd}
\mathscr{R}^{\%}(X) \arrow[r] \arrow[d] & \mathcal{V}(X) \arrow[d, "J"] \\
\mathscr{R}^{\mathrm{fd}}(X) \arrow[r, "I"] & \mathscr{R}^{\mathrm{ld}}(\mathbb{J}X)
\end{tikzcd}$$

It has the obvious Waldhausen category structure such that the functors $\mathscr{R}^{\%}(X) \to \mathscr{R}^{\mathrm{fd}}(X)$ and $\mathscr{R}^{\%}(X) \to \mathcal{V}(X)$ are exact. It follows immediately that we have a commutative diagram

$$\begin{tikzcd}
K\big(\mathscr{R}^{\%}(X)\big) \arrow[r] \arrow[d] & V(X) \arrow[d] \\
A(X) \arrow[r] & A^{\mathbb{J}}(X)
\end{tikzcd}$$

and a natural map $K\big(\mathscr{R}^{\%}(X)\big) \to A^{\%}(X)$. We define a characteristic for the functor $\mathcal{F} = \mathscr{R}^{\%} \circ f$ using Remark 5.3 as follows:

- For any pair (σ, θ), choose $(\sigma, \theta)^v = E^\theta_\sigma \times \{0\} \amalg E^\theta_\sigma \times [0, \infty)$ as retractive spaces over $E^\theta_\sigma \times [0, \infty)$, which is in $\mathcal{V}(E^\theta_\sigma)$ by [DWW03, p. 53], and $(\sigma, \theta)^h = E^\theta_\sigma \times S^0$ in $\mathscr{R}^{\mathrm{fd}}(E^\theta_\sigma)$. Define $(\sigma, \theta)^!$ in $\mathscr{R}^{\%}(E^\theta_\sigma)$ as the element represented by $\big((\sigma, \theta)^v, (\sigma, \theta)^h\big)$.

- For any map $e\colon (\sigma, \theta) \to (\sigma', \theta')$, define $e^v\colon e_*\big((\sigma, \theta)^v\big) \to (\sigma', \theta')^v$ in $\mathcal{V}(E^\theta_\sigma)$ by

  $$e^v = f(e) \amalg \mathrm{id}\colon e_*\big((\sigma, \theta)^v\big) = E^\theta_\sigma \amalg E^{\theta'}_{\sigma'} \times [0, \infty) \to E^{\theta'}_{\sigma'} \amalg E^{\theta'}_{\sigma'} \times [0, \infty) = (\sigma', \theta')^v$$

  and $e^h\colon e_*\big((\sigma, \theta)^h\big) \to (\sigma', \theta')^h$ in $\mathscr{R}^{\mathrm{fd}}(E^\theta_\sigma)$ as the map $e^!$ of the parametrized A-theory case. By [BD07, Lemma 2.9], we have $J(e^v) = I(e^h)$. We set $e^! = (e^v, e^h)$. This satisfies the 1-cocylce condition again by [BD07, Lemma 2.9].

By Remark 5.5(ii), this defines also a characteristic for $F = |\mathcal{F}| = \big|\mathscr{R}^{\%} \circ f\big|$. Moreover, by [Wal85, p. 12] and by construction of $\mathscr{R}^{\%}(X)$, there is a natural map

$$\Big|\mathscr{R}^{\%}(E^\theta_\sigma)\Big| \to K\big(\mathscr{R}^{\%}(E^\theta_\sigma)\big) \to A^{\%}(E^\theta_\sigma)$$

where the first map is the map reminiscent of the group completion. Therefore, by composition, we obtain a characteristic $\chi_e(p)$ in $\mathrm{holim}_{(\sigma,\theta)} A^{\%}(E^\theta_\sigma)$ for the functor

$$(\sigma, \theta) \mapsto A^{\%}(E^\theta_\sigma)$$

Now, the projection $E_\sigma \to E^\theta_\sigma$ for a pair (σ, θ) in $\mathrm{simp}\, tB_.$ defines a natural transformation which induces by [DWW03, Corollary 2.7] a weak homotopy equivalence

$$\operatorname*{holim}_{\sigma} A^{\%}(E_\sigma) \to \operatorname*{holim}_{(\sigma,\theta)} A^{\%}(E^\theta_\sigma)$$

Hence, up to homotopy, the excisive characteristic $\chi_e(p)$ defines a section of a suitable fibration $A^{\%}_B(E) \to B$. Moreover, by [BD07, p. 10], the images of $\chi(p)$ and $\chi_e(p)$ in $\mathrm{holim}_{(\sigma,\theta)} A(E^\theta_\sigma)$ are connected by a canonical path. Therefore, we obtain an element

$$\begin{aligned}\chi^{\%}(p) \in \mathrm{holim}\,\big(\operatorname*{holim}_{\sigma} A(E_\sigma) \xrightarrow{\simeq} \operatorname*{holim}_{(\sigma,\theta)} A(E^\theta_\sigma) \leftarrow \operatorname*{holim}_{(\sigma,\theta)} A^{\%}(E^\theta_\sigma)\big)& \\ \simeq \mathrm{hofib}\,\big(\operatorname*{holim}_{\sigma} A^{\%}(E_\sigma) \xrightarrow{\alpha} \operatorname*{holim}_{\sigma} A(E_\sigma)\big)& \quad (5.1)\end{aligned}$$

which projects to $\chi(p)$. We call $\chi^{\%}(p)$ the *excisive A-theory Euler characteristic* of p or the *parametrized $A^{\%}$-theory Euler characteristic* of p.

Remark 5.9. The last homotopy equivalence of (5.1) holds by [BD07, Proposition 2.17]. This implies that, informally, $\chi^{\%}(p)$ can be understood as a refinement of $\chi(p)$ in the sense that it defines, up to homotopy, an element

$$\chi^{\%}(p) \in \mathrm{Lift}\begin{pmatrix} & A^{\%}_B(E) \\ & \downarrow \\ B \xrightarrow{\chi(p)} & A_B(E)\end{pmatrix}$$

In particular, it defines a section of the fibration $A^{\%}_B(E) \to B$ over B obtained by applying the functor $A^{\%}$ fiberwise, together with a path from $\alpha\chi^{\%}(p)$ to $\chi(p)$.

5.2 The parametrized Whitehead torsion

The goal of this section is to introduce and study the parametrized Whitehead torsion

$$\tau\colon \mathscr{S}_n(p) \longrightarrow \Gamma \left(\begin{array}{c} \Omega \operatorname{Wh}_B(E) \\ \downarrow \\ B \end{array} \right) \tag{5.2}$$

from the structure space on a bundle $p\colon E \to B$ to the space of sections of the fibration $\Omega \operatorname{Wh}_B(E) \to B$ obtained from p by applying the functor $\Omega \operatorname{Wh}$ fiberwise. Here, $\operatorname{Wh}(-)$ is the connective topological Whitehead functor as defined by Waldhausen in [Wal85, Section 3], that is, the functor which sends any space X to the spectrum $\operatorname{Wh}(X)$ defined by

$$\Omega \operatorname{Wh}(X) = \operatorname{hofib}\left(A^{\%}(x) \xrightarrow{\alpha} A(X)\right) \tag{5.3}$$

The strategy is as follows. First, we define structure spaces on fibration and we prove that they are naturally weak homotopy equivalent to some spaces of lifts. Then, we construct a parametrized excisive characteristic by composing such weak homotopy equivalence with a sort of "universal bundle". Finally, we define the parametrized Whitehead torsion as difference of such excisive characteristic. What we obtain is a very natural generalization of the classical Whitehead torsion in the context of algebraic K-theory. In particular, all the classical properties of Lemma 2.1 of Chapter 2 have a very natural generalization to the parametrized Whitehead torsion.

Structures spaces on fibrations

Let us start by studying structure spaces on fibrations.

Definition 5.10. Let $p\colon E \to B$ be in **Fib** over a paracompact space B. An *n-dimensional compact manifold structure* on p is a commutative diagram

$$\begin{tikzcd} E' \arrow[rr, "\varphi"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd} \tag{5.4}$$

such that

- the map $p'\colon E' \to B$ is a the projection map of a fiber bundle in **Cpt** with n-dimensional manifolds in **Cpt** as fibers;
- the map φ is a homotopy equivalence.

Denote by $\mathscr{S}_n(p)_0$ the set of all n-dimensional compact manifold structure on p.

Given a fibration $p\colon E \to B$ be in **Fib** over a paracompact space B, we want to construct a simplicial set $\mathscr{S}_n(p)_.$ such that $\mathscr{S}_n(p)_0$ is the set of 0-simplices. The idea is to cross E and B with the standard k-simplex Δ^k and to use $\mathscr{S}_n(p \times \mathrm{id}_{\Delta^k})_0$ as set of k-simplices. However, we need to define the simplicial operations. For this, consider the following pullback of p.

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d, "p_0"'] & E \arrow[d, "p"] \\ B_0 \arrow[r, "f"] & B \end{tikzcd} \tag{5.5}$$

A compact manifold structure on p induces a compact manifold structure on p_0 of the same dimension by restriction with f. Indeed, consider an n-dimensional compact manifold structure (5.4) on p. Denote by f^*E the pullback of p' by f and by $p_f\colon f^*E \to B$ the projection given by the pullback. Let φ' be the map defined as pullback of the following diagram.

$$\begin{tikzcd} B_0 \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, equal] & E' \arrow[l, "p'"'] \arrow[d, "\varphi"] \\ B_0 \arrow[r, "f"] & B & E \arrow[l, "p"'] \end{tikzcd}$$

Then, φ' is a homotopy equivalence by coglueing theorem [FP90, Theorem A.4.19] and p_f is by construction a bundle of n-dimensional compact manifolds. Therefore, p_0 has the following induced compact manifold structure.

$$\begin{tikzcd} f^*E \arrow[rr, "\varphi'"] \arrow[dr, "p_f"'] & & E_0 \arrow[dl, "p_0"] \\ & B_0 & \end{tikzcd}$$

Definition 5.11. Let $p\colon E \to B$ be in **Fib** over a paracompact space B. We define the simplicial set $\mathscr{S}_n(p)_\cdot$ by $\mathscr{S}_n(p)_k = \mathscr{S}_n(p \times \mathrm{id}_{\Delta^k})_0$ where the simplicial operations are induced by restriction on the level of standard simplices. The *space of n-dimensional compact manifold structures* $\mathscr{S}_n(p)$ on p is the geometric realization

$$\mathscr{S}_n(p) = |\mathscr{S}_n(p)_\cdot|$$

If B is a point, we simply write $\mathscr{S}_n(E)$ for $\mathscr{S}_n(p)$.

Remark 5.12. The construction of the space $\mathscr{S}_n(p)$ is functorial in the following two ways:

- Let $p\colon E \to B$ and $p'\colon E' \to B$ be in **Fib** and consider a fiber homotopy equivalence $\psi\colon p \to p'$. Then, ψ induces a simplicial map $\psi_*\colon \mathscr{S}_n(p)_\cdot \to \mathscr{S}_n(p')_\cdot$ by composition and, therefore, a map on structure spaces.

- Let $p\colon E \to B$ be a fibration and consider a pullback diagram as (5.5). Then the restriction operation induces a map $f^*\colon \mathscr{S}_n(p) \to \mathscr{S}_n(p_0)$.

Note that, by [Ste10, p. 15], these operations are homotopy invariant. More precisely, the first one sends fiber homotopy equivalences which are fiber homotopic to homotopic maps and the second one sends homotopy equivalences to homotopy equivalences.

As explained at the beginning of this section, the goal now is to "write" structure spaces as spaces of lifts. More precisely, we want to find a natural weak homotopy equivalence from $\mathscr{S}_n(p)$ to some space of lifts. Let us start with some notation.

Definition 5.13. Let F and B be in **Top** with B paracompact. We define:

- $\mathbf{Bun}_n(B;F)$ to be the category whose objects are bundles $E \to B$ with fibers compact n-dimensional topological manifolds homotopy equivalent to F and whose morphisms are isomorphisms of such bundles;

- $\mathbf{Fib}(B;F)$ to be the category whose objects are fibrations over B with fibers homotopy equivalent to F and whose morphisms are fiber homotopy equivalences.

Using them, we can construct two simplicial categories in the following way. Let **cpCW** be the category of compact CW-complexes with continuous maps. Define two functors

$$\mathbf{Bun}_n(B;F)\colon \mathbf{cpCW}^{\mathrm{op}} \to \mathbf{Cat}, \qquad \mathbf{Fib}(B;F)\colon \mathbf{cpCW}^{\mathrm{op}} \to \mathbf{Cat}$$

by the rules $X \mapsto \mathbf{Bun}_n(B \times X; F)$ and $X \mapsto \mathbf{Fib}(B \times X; F)$, respectively. Consider an explicit system of simplices in **cpCW**, that is, an embedding of categories $\Delta \to \mathbf{cpCW}$ which sends $[n]$ to an n-simplex and a morphism $[m] \to [n]$ to the corresponding face or degeneracy map. Then, by precomposition, we obtain two simplicial small categories

$$\mathbf{Bun}_n(B;F)_. \colon \Delta^{op} \to \mathbf{cpCW}^{op} \to \mathbf{Cat}, \qquad \mathbf{Fib}(B;F)_. \colon \Delta^{op} \to \mathbf{cpCW}^{op} \to \mathbf{Cat}$$

Note that the construction is well-defined up to isomorphism, since different choices of systems of simplices lead to naturally isomorphic simplicial small categories. Moreover, since B is paracompact, then any bundle over $B \times \Delta^n$ is a fibration by [Whi78, p. 33]. Therefore, there is a natural transformation $\mathbf{Bun}_n(B;F)_. \to \mathbf{Fib}(B;F)_.$.

Definition 5.14. We define $\mathrm{Bun}_n(B;F)_.$ and $\mathrm{Fib}(B;F)_.$ to be the simplicial sets given by the 0-nerves $N_0\,\mathbf{Fib}(B;F)_.$ and $N_0\,\mathbf{Bun}_n(B;F)_.$, respectively, and we denote by $\mathrm{Bun}_n(B;F)$ and $\mathrm{Fib}(B;F)$ their geometric realizations.

By [Ste10, pp. 20–21], the simplicial sets $\mathrm{Bun}_n(B;F)_.$ and $\mathrm{Fib}(B;F)_.$ have the following properties.

Lemma 5.15. *(i) For any fibration $p\colon E \to B$ in $\mathbf{Fib}(B;F)$ over a metrizable locally equiconnected base space B, there is a simplicial homotopy equivalence*

$$\mathscr{S}_n(p)_. \longrightarrow \mathrm{hofib}_p\left(\mathrm{Bun}_n(B;F)_. \to \mathrm{Fib}(B;F)_.\right)$$

which is natural in B.

(ii) For any locally finite simplicial set $X_.$, there are natural simplicial isomorphisms

$$\mathrm{Bun}_n\left(|X_.|;F\right)_. \cong \mathrm{map}_.\left(X_., \mathrm{Bun}_n(*;F)_.\right)$$
$$\mathrm{Fib}\left(|X_.|;F\right)_. \cong \mathrm{map}_.\left(X_., \mathrm{Fib}(*;F)_.\right)$$

(iii) For any fibration $p\colon E \to B$ over a base space B which is the geometric realization of a locally finite simplicial set $B_.$, there is a natural simplicial isomorphism

$$\mathrm{hofib}_p\left(\mathrm{Bun}_n(B;F)_. \to \mathrm{Fib}(B;F)_.\right) \longrightarrow \mathrm{Lift}\left(\begin{array}{ccc} & & \mathrm{Bun}_n(*;F)_. \\ & & \downarrow \\ B_. & \xrightarrow[p]{} & \mathrm{Fib}(*;F)_. \end{array}\right)$$

where $p\colon B_. \to \mathrm{Fib}(*;F)_.$ is given by $p\colon E \to B$ and the previous claim and where the vertical map on the right hand side is assumed to be converted into a fibration by Notation 5.6.

By the previous lemma, we obtain immediately the wanted weak homotopy equivalence.

Corollary 5.16 ([Ste10, Corollary 2.8])**.** *If B is a locally finite ordered simplicial complex, then there exists a natural weak homotopy equivalence*

$$\mathscr{S}_n(p) \longrightarrow \mathrm{Lift}\left(\begin{array}{ccc} & & \mathrm{Bun}_n(*;F) \\ & & \downarrow \\ B & \xrightarrow{p} & \mathrm{Fib}(*;F) \end{array}\right)$$

Remark 5.17. Since both domain and target of the previous weak homotopy equivalence are homotopy invariant, there exists still a weak homotopy equivalence, well-defined up to homotopy, also if B is homotopy equivalent to a locally finite ordered simplicial complex.

The parametrized excisive characteristic

Let us now go on with the second step to construct the parametrized Whitehead torsion: the definition a parametrized excisive characteristic. The idea is to compose the weak homotopy equivalence of Corollary 5.16 with the parametrized characteristic of some sort of "universal bundle". Let us start with the definition of such bundle.

Definition 5.18. Denote by $\widetilde{\mathscr{B}}$ the space $|\mathrm{Bun}_n(*;F)_{\cdot}|$ and by $\mathscr{B}$ the space $|\mathrm{Fib}(*;F)_{\cdot}|$. By Lemma 5.15, we can associate to the identity map $\mathrm{Bun}_n(*;F)_{\cdot} \to \mathrm{Bun}_n(*;F)_{\cdot}$ a map

$$\widetilde{\mathscr{P}} \colon \widetilde{\mathscr{E}}_n \to \widetilde{\mathscr{B}}$$

which is a bundle over every locally finite subcomplex of $\widetilde{\mathscr{B}}$. Similarly, we can associate to the identity map $\mathrm{Fib}(*;F)_{\cdot} \to \mathrm{Fib}(*;F)_{\cdot}$ a map

$$\mathscr{P} \colon \mathscr{E} \to \mathscr{B}$$

which is a fibration over every locally finite subcomplex of $\mathscr{B}$. We call these maps *universal bundles*.

Remark 5.19. Note that we do not know, in general, if $\widetilde{\mathscr{P}}$ and $\mathscr{P}$ are actually bundles or fibrations, but this does not matter. Indeed, the properties of being a bundle or a fibration over a locally finite subcomplex is good enough to define parametrized characteristics, since these only use the restrictions over simplices, which are locally finite subcomplexes.

Choose a representative

$$\chi(\mathscr{P}) \in \Gamma \begin{pmatrix} A_{\mathscr{B}}(\mathscr{E}) \\ \downarrow \\ \mathscr{B} \end{pmatrix}$$

of the parametrized A-theory characteristic of the universal bundle $\mathscr{P}$. It has the following important property.

Lemma 5.20 ([Ste10, Lemma 3.3])**.** *Let $p \colon E \to B$ be in* **Fib** *over a space which is the geometric realization of a simplicial set $B_{\cdot}$ such that the fibers are homotopy finitely dominated. Consider a map $f_{\cdot} \colon B_{\cdot} \to \mathscr{B}_{\cdot}$ and an element $\chi \in \mathrm{holim}_{\sigma \in \mathrm{simp}\, B_{\cdot}} A(E_\sigma)$. Then, there is a zigzag of weak homotopy equivalences*

$$\mathrm{hofib}_\chi \big(\operatorname*{holim}_{\sigma \in \mathrm{simp}\, B_{\cdot}} A^{\%}(E_\sigma) \to \operatorname*{holim}_{\sigma \in \mathrm{simp}\, B_{\cdot}} A(E_\sigma) \big) \simeq \mathrm{Lift} \begin{pmatrix} & A^{\%}_{\mathscr{B}}(\mathscr{E}) \\ & \downarrow \\ B \xrightarrow{\chi(\mathscr{P}) \circ f} & A_{\mathscr{B}}(\mathscr{E}) \end{pmatrix} \tag{5.6}$$

which is natural in B.

Remark 5.21. The idea of the previous lemma is contained in the following homotopy commutative diagram.

$$\begin{tikzcd}
B \arrow[r, dotted] \arrow[dr, "\chi"] \arrow[dd, "f"'] & A^{\%}_B(E) \arrow[r] \arrow[d, "\alpha"] & A^{\%}_{\mathscr{B}}(\mathscr{E}) \arrow[dd, "\alpha"] \\
 & A_B(E) \arrow[dr] & \\
\mathscr{B} \arrow[rr, "\chi(\mathscr{P})"] & & A_{\mathscr{B}}(\mathscr{E})
\end{tikzcd}$$

where the maps $A_B(E) \to A_{\mathscr{B}}(\mathscr{E})$ and $A^{\%}_B(E) \to A^{\%}_{\mathscr{B}}(\mathscr{E})$ are the maps given by naturality by [Ste10, Lemma 3.1]. Indeed, if there is an element in the left hand side of (5.6), then there exists a dotted arrow in the previous diagram and the outer square homotopy commutes. Conversely, if the outer square homotopy commutes, then, again by naturality, there exists a dotted arrow and, therefore, an element on the left hand side of (5.6).

We apply, now, the previous lemma to the forgetful map $f\colon \widetilde{\mathscr{B}} \to \mathscr{B}$ which considers a bundle as a fibration and to a representative $\chi(\widetilde{\mathscr{P}}) \in \operatorname{holim}_{\sigma\in \operatorname{simp}\widetilde{\mathscr{B}}_.} A(\widetilde{\mathscr{E}}_\sigma)$ of the parametrized A-theory charateristic of $\widetilde{\mathscr{P}}$. Since the excisive A-theory characteristic $\chi^{\%}(\widetilde{\mathscr{P}}) \in \operatorname{holim}_{\sigma\in \operatorname{simp}\widetilde{\mathscr{B}}_.} A^{\%}(\widetilde{\mathscr{E}}_\sigma)$ of the universal bundle $\widetilde{\mathscr{P}}$ is sent to $\chi(\widetilde{\mathscr{P}})$ by the assembly map α, we obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathscr{B}} & \longrightarrow & A^{\%}_{\mathscr{B}}(\mathscr{E}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathscr{B} & \xrightarrow{\chi(\mathscr{P})} & A_{\mathscr{B}}(\mathscr{E}) \end{array} \tag{5.7}$$

Composing it with the weak homotopy equivalence of Corollary 5.16, we get what we call the parametrized excisive characteristic.

Definition 5.22. Let $p\colon E \to B$ be in $\mathbf{Fib}(B;F)$ with a fiber F which is homotopy finitely dominated. Assume that B is homotopy equivalent to a locally finite simplicial complex $B_.$. Choose a representative $\chi(\mathscr{P})$ of the parametrized A-theory characteristic of $\mathscr{P}$. By abuse of notation, denote by $\chi(p)$ the map $\chi(\mathscr{P})\circ p$. The *parametrized excisive characteristic* of p is the composite

$$\chi^{\%}\colon \mathscr{S}_n(p) \xrightarrow{\simeq} \operatorname{Lift}\left(\begin{array}{ccc} & & \widetilde{\mathscr{B}} \\ & & \downarrow \\ B & \xrightarrow{p} & \mathscr{B} \end{array}\right) \longrightarrow \operatorname{Lift}\left(\begin{array}{ccc} & & A^{\%}_{\mathscr{B}}(\mathscr{E}) \\ & & \downarrow \\ B & \xrightarrow{\chi(p)} & A_{\mathscr{B}}(\mathscr{E}) \end{array}\right)$$

where the first map is the map of Corollary 5.16 and the second map is given by composition with diagram (5.7).

Remark 5.23. The parametrized excisive characteristic is well-defined up to homotopy.

The parametrized Whitehead torsion

At this point, we can finally define the parametrized Whitehead torsion. The idea is to introduce a sum operation on the target of the parametrized excisive characteristic and define the torsion as $\chi^{\%}(\,\cdot\,) - \chi^{\%}(\mathrm{id})$.

For this let us consider the range of the parametrized excisive characteristic

$$\operatorname{Lift}\left(\begin{array}{ccc} & & A^{\%}_{\mathscr{B}}(\mathscr{E}) \\ & & \downarrow \\ B & \xrightarrow{\chi(p)} & A_{\mathscr{B}}(\mathscr{E}) \end{array}\right) \tag{5.8}$$

This is, by Lemma 5.20, weak homotopy equivalent for some $\chi \in \operatorname{holim}_{\sigma\in\operatorname{simp} B_.} A(E_\sigma)$ to

$$\operatorname{hofib}_\chi\big(\operatorname*{holim}_{\sigma\in\operatorname{simp} B_.} A^{\%}(E_\sigma) \to \operatorname*{holim}_{\sigma\in\operatorname{simp} B_.} A(E_\sigma)\big)$$

which has a natural loop structure. Therefore, also (5.8) carries such a structure. In particular, we can define up to homotopy equivalence a sum operation on (5.8). We obtain a "difference map" given by

$$\mathrm{Lift}\left(\begin{array}{c} A^{\%}_{\mathscr{B}}(\mathscr{E}) \\ \downarrow \\ B \xrightarrow{\chi(p)} A_{\mathscr{B}}(\mathscr{E}) \end{array}\right) \times \mathrm{Lift}\left(\begin{array}{c} A^{\%}_{\mathscr{B}}(\mathscr{E}) \\ \downarrow \\ B \xrightarrow{\chi(p)} A_{\mathscr{B}}(\mathscr{E}) \end{array}\right) \xrightarrow{-} \mathrm{Lift}\left(\begin{array}{c} A^{\%}_{\mathscr{B}}(\mathscr{E}) \\ \downarrow \\ B \xrightarrow{0} A_{\mathscr{B}}(\mathscr{E}) \end{array}\right)$$

Now, by Lemma 5.20, the range of the previous map is homotopy equivalent to

$$\mathrm{hofib}_0\big(\operatorname*{holim}_{\sigma\in\mathrm{simp}\,B_.} A^{\%}(E_\sigma) \to \operatorname*{holim}_{\sigma\in\mathrm{simp}\,B_.} A(E_\sigma)\big)$$

Indeed, by looking at the diagram of Remark 5.21, it easy to see that if $\chi(\mathscr{P})\circ f = 0$, then $\chi = 0$ as $f \neq 0$. Therefore, by definition of Whitehead spectrum in (5.3), we obtain that the range of the difference map is

$$\operatorname*{holim}_{\sigma\in\mathrm{simp}\,B_.} \Omega\,\mathrm{Wh}(E_\sigma) \simeq \Gamma\left(\begin{array}{c} \Omega\,\mathrm{Wh}_B(E) \\ \downarrow \\ B \end{array}\right)$$

Definition 5.24. Let $p\colon E \to B$ be in $\mathbf{Fib}(B;F)$ with a fiber F which is homotopy finitely dominated. Assume that B is homotopy equivalent to a locally finite simplicial complex $B_.$.

(i) The *parametrized Whitehead torsion map*

$$\tau\colon \mathscr{S}_n(p) \times \mathscr{S}_n(p) \to \Gamma\left(\begin{array}{c} \Omega\,\mathrm{Wh}_B(E) \\ \downarrow \\ B \end{array}\right)$$

is given by $\tau(\,\cdot\,,\,?\,) = \chi^{\%}(\,\cdot\,) - \chi^{\%}(\,?\,)$ where $\chi^{\%}$ is the parametrized excisive characteristic of Definition 5.22.

(ii) If p is itself in $\mathrm{Bun}_n(B;F)$, then it defines a canonical element $\mathrm{id} \in \mathscr{S}_n(p)$. In this case, we define the *parametrized Whitehead torsion map* as

$$\tau = \tau(\,\cdot\,,\mathrm{id})\colon\ \mathscr{S}_n(p) \to \Gamma\left(\begin{array}{c} \Omega\,\mathrm{Wh}_B(E) \\ \downarrow \\ B \end{array}\right)$$

Remark 5.25. As the parametrized excisive characteristic, the parametrized Whitehead torsion map is well-defined up to homotopy.

To conclude this section, we state the properties of the parametrized Whitehead torsion.

Lemma 5.26. *Let $p\colon E \to B$ and $p'\colon E' \to B$ be in $\mathbf{Bun}_n(B;F)$ and assume that B is homotopy equivalent to a locally finite simplicial complex $B_.$.*

(i) (Naturality)

(a) The parametrized excisive characteristic $\chi^{\%}$, and hence also the parametrized Whitehead torsion τ, is natural with respect to fiber homotopy equivalences. More precisely, let $\varphi\colon p \to p'$ be a fiber homotopy equivalence. By Lemma 5.15(ii), we can look at p

and p' as maps $B \to \mathscr{B}$ and at $\varphi\colon p \to p'$ as a homotopy between p and p'. Therefore, φ induces a homotopy between $\chi(p) \simeq \chi(p')\colon B \to A_{\mathscr{B}}(\mathscr{E})$. Consider the map

$$\varphi_*\colon \operatorname{Lift}\begin{pmatrix} & A^{\%}_{\mathscr{B}}(\mathscr{E}) \\ & \downarrow \\ B \xrightarrow{\chi(p)} & A_{\mathscr{B}}(\mathscr{E}) \end{pmatrix} \longrightarrow \operatorname{Lift}\begin{pmatrix} & A^{\%}_{\mathscr{B}}(\mathscr{E}) \\ & \downarrow \\ B \xrightarrow{\chi(p')} & A_{\mathscr{B}}(\mathscr{E}) \end{pmatrix}$$

induced by standard fiber transport along the homotopy given by φ. Then we have $\varphi_* \circ \chi^{\%} \simeq \chi^{\%} \circ \varphi_*$.

(b) *The parametrized Whitehead torsion τ is compatible with pullbacks. More precisely, given a map $f\colon B_0 \to B$ and a pullback square as* (5.5)*, the following square with obvious map commutes up to homotopy.*

$$\begin{tikzcd} \mathscr{S}_n(p) \arrow[r, "f^*"] \arrow[d, "\tau"'] & \mathscr{S}_n(p_0) \arrow[d, "\tau"] \\ \Gamma\begin{pmatrix} \Omega \operatorname{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} \arrow[r, "f^*"] & \Gamma\begin{pmatrix} \Omega \operatorname{Wh}_{B_0}(E_0) \\ \downarrow \\ B_0 \end{pmatrix} \end{tikzcd}$$

(ii) *(Composition rule) Let $\varphi\colon p \to p'$ be a fiber homotopy equivalence. Then, the following diagram commutes up to homotopy*

$$\begin{tikzcd} \mathscr{S}_n(p) \arrow[r, "\varphi_*"] \arrow[d, "\tau"'] & \mathscr{S}_n(p') \arrow[d, "\tau-\tau(\varphi)"] \\ \Gamma\begin{pmatrix} \Omega \operatorname{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} \arrow[r, "\varphi_*"] & \Gamma\begin{pmatrix} \Omega \operatorname{Wh}_B(E') \\ \downarrow \\ B \end{pmatrix} \end{tikzcd}$$

(iii) *(Homotopy invariance) Let $\varphi\colon E' \to E$ be a fiber homeomorphism of bundles. Then we have $\tau(\varphi) = 0$.*

(iv) *(Stabilization) Consider the stabilization map $S\colon \mathscr{S}_n(p) \to \mathscr{S}_{n+1}(p)$ which on k-simplices is defined by the rule $(E' \to E) \mapsto (E' \times I \to E' \to E)$ where E' is a bundle of n-dimensional compact manifolds over $B \times \Delta^k$. Define $\mathscr{S}(p) = \operatorname{hocolim}_n \mathscr{S}_n(p)$. Then the parametrized Whitehead torsion τ of Definition 5.24 extends to a stabilized torsion*

$$\tau\colon \mathscr{S}(p) \to \Gamma\begin{pmatrix} \Omega \operatorname{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix}$$

(v) *(Product rule) Given a contractible k-dimensional compact manifold X, consider the map $- \times X\colon \mathscr{S}_n(p) \to \mathscr{S}_{n+k}(p)$ which on k-simplices is defined by the rule*

$$(E' \to E) \mapsto (E' \times X \to E' \to E)$$

where E' is a bundle of n-dimensional compact manifolds over $B \times \Delta^k$. Since we have that $S \circ (- \times X) \simeq (- \times X) \circ S$, there exists a stabilized version $- \times X\colon \mathscr{S}(p) \to \mathscr{S}(p)$. Then the following diagram commutes up to homotopy

$$\begin{tikzcd} \mathscr{S}(p) \arrow[rr, "- \times X"] \arrow[dr, "\tau"'] & & \mathscr{S}(p) \arrow[dl, "\tau"] \\ & \Gamma\begin{pmatrix} \Omega \operatorname{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} & \end{tikzcd}$$

(vi) *(Additivity) Consider the category whose objects are the objects of $\mathbf{Fib}(B;F)$ and whose morphisms $p_i \to p_j$ are fiberwise maps from p_i to p_j. Denote by $\Box$ the following commutative diagram in this category*

$$\begin{tikzcd} p_0 \arrow[r] \arrow[d] \arrow[dr, "j_0"] & p_1 \arrow[d, "j_1"] \\ p_2 \arrow[r, "j_2"'] & p_3 \end{tikzcd}$$

where we assume that all maps on the level of total spaces are cofibrations and that the total space $E(p_3)$ is the pushout of the total spaces $E(p_1)$ and $E(p_2)$ over $E(p_0)$. We say that a cube

$$\begin{tikzcd} q_0 \arrow[rr] \arrow[dr] \arrow[dd] & & q_1 \arrow[dr] \arrow[dd] & \\ & p_0 \arrow[rr] \arrow[dd] & & p_1 \arrow[dd] \\ q_2 \arrow[rr] \arrow[dr] & & q_3 \arrow[dr] & \\ & p_2 \arrow[rr] & & p_3 \end{tikzcd} \tag{5.9}$$

is an n-dimensional structure on $\Box$ if q_i is an object of $\mathbf{Bun}_n(B;F_i)$ for $i = 1,2,3$ respectively, q_0 is an object of $\mathbf{Bun}_{n-1}(B;F_0)$, all the maps $q_i \to p_i$ are fiber homotopy equivalences and the q-square is a codimension 1 splitting of q_3, that is, q_i is a locally flat subbundle of q_3 of codimension 0 if $i = 1,2$ and of codimension 1 if $i = 0$ and the total space of q_0 is the intersection of the total spaces of q_1 and q_2. Define a simplicial set $\mathscr{S}_n(\Box)_\cdot$ where the k-simplices are the n-dimensional structures (5.9) parametrized over Δ^k and denote by $\mathscr{S}_n(\Box)$ its geometric realization. Let $\alpha_i\colon \mathscr{S}_n(\Box) \to \mathscr{S}_n(p_i)$ for $i = 1,2,3$ and $\alpha_0\colon \mathscr{S}_n(\Box) \to \mathscr{S}_{n-1}(p_0)$ be the forgetful maps. Then the following diagram commutes up to homotopy

$$\begin{tikzcd} \mathscr{S}_n(\Box) \arrow[rr, "\alpha_3"] \arrow[d, "\prod_{i=0}^2 \tau\circ\alpha_i"'] & & \mathscr{S}_n(p_3) \arrow[d, "\tau"] \\ \prod_{i=0}^2 \Gamma\begin{pmatrix} \Omega \operatorname{Wh}_B(E_i) \\ \downarrow \\ B \end{pmatrix} \arrow[rr, "(j_1)_*+(j_2)_*-(j_0)_*"] & & \Gamma\begin{pmatrix} \Omega \operatorname{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} \end{tikzcd}$$

where the addition operation in the lower line is defined by Lemma 5.20 between spaces which are weakly homotopy equivalent to the given ones. In other words, we have

$$(j_1)_* \circ \tau \circ \alpha_1 + (j_2)_* \circ \tau \circ \alpha_2 - (j_0)_* \circ \tau \circ \alpha_0 \simeq \tau \circ \alpha_3$$

(vii) (Stabilized additivity) Using the notation of the previous claim, the following diagram commutes up to homotopy:

$$\begin{tikzcd} \mathscr{S}(\square) \arrow[rr, "\alpha_3"] \arrow[d, "\prod_{i=0}^2 \tau\circ\alpha_i"'] & & \mathscr{S}(p_3) \arrow[d, "\tau"] \\ \prod_{i=0}^2 \Gamma\begin{pmatrix} \Omega\,\mathrm{Wh}_B(E_i) \\ \downarrow \\ B \end{pmatrix} \arrow[rr, "(j_1)_*+(j_2)_*-(j_0)_*"] & & \Gamma\begin{pmatrix} \Omega\,\mathrm{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} \end{tikzcd}$$

(viii) (Comparison with the unparametrized case) Let M be in **Cpt** *of dimension n. Then the map*

$$\tau\colon \pi_0\,\mathscr{S}_n(M) \longrightarrow \pi_0\Omega\,\mathrm{Wh}(M) \cong \mathrm{Wh}(\pi M) \tag{5.10}$$

sends $f\colon N \to M$ to the Whitehead torsion of f

Idea of proof. Part (i), (ii) and (iii) follow directly by definition. Part (iv) and (v) are consequences of a lax naturality of the excisive characteristic (see [Ste10, Theorem 3.10]). The additivity properties (vi) and (vii) are proved by describing $\mathscr{S}_n(\square)$ as a suitable space of lifts and using an additivity result for the parametrized and excisive A-theory characteristics (see [Ste10, Theorem 3.15]). Finally, the comparison with the classical Whitehead torsion (viii) follows by using Waldhausen's notation from [Wal85, Chapter 3] and by noting that the codomain $\pi_0\Omega\,\mathrm{Wh}(M)$ of (5.10) is by construction the geometric Whitehead group $\mathrm{Wh}^{\mathrm{geo}}(M)$ of M as defined in Section 2.1. A complete proof of these properties can be found in [Ste10, Section 3]. □

Remark 5.27. The previous lemma shows that the parametrized Whitehead torsion is actually the generalization of the classical Whitehead torsion in algebraic K-theory. Indeed, it reduces to the classical Whitehead torsion by part (viii) and properties (ii), (iii), (v) and (vi) are the natural generalizations of Lemma 2.1.

5.3 The geometric assembly map

The goal of this section is to introduce the geometric assembly map on structure spaces. This is the last important tool that we need to study the stable fibering problem.

Let us start with its definition.

Definition 5.28. (i) Let $p\colon E \to B$ be in $\mathbf{Fib}(B;F)$. We define a product map

$$\beta\colon \mathscr{S}_k(B) \times \mathscr{S}_n(p) \longrightarrow \mathscr{S}_{n+k}(E)$$

which on m-simplices is defined as follows: let $x \in \mathscr{S}_k(B)_m$ and $y \in \mathscr{S}_n(p)_m$ be represented by

$$\begin{tikzcd} B' \arrow[rr, "\varphi"] \arrow[dr] & & B\times\Delta^m \arrow[dl] \\ & \Delta^m & \end{tikzcd} \qquad \begin{tikzcd} E' \arrow[rr, "\psi"] \arrow[dr] & & E\times\Delta^m \arrow[dl] \\ & B\times\Delta^m & \end{tikzcd}$$

Then the image of (x, y) is given by the following diagram

$$\begin{tikzcd} \varphi^*E' \arrow[rr, "\psi\circ\overline{\varphi}"] \arrow[dr] & & E\times\Delta^m \arrow[dl] \\ & \Delta^m & \end{tikzcd}$$

where $\varphi^* E$ is the following pullback.

$$\begin{tikzcd} \varphi^* E' \arrow[r, "\overline{\varphi}"] \arrow[d] & E' \arrow[d] \\ B' \arrow[r, "\varphi"'] & B \times \Delta^m \end{tikzcd}$$

(ii) If B is a k-dimensional compact manifold, then the identity id_B defines a point in $\mathscr{S}_k(B)$. We define the *geometric assembly map* α as

$$\alpha = \beta(\mathrm{id}_B, -)\colon\ \mathscr{S}_n(p) \longrightarrow \mathscr{S}_{n+k}(E)$$

Remark 5.29. (i) The map β is well-defined. Indeed, the map $\varphi^* E' \to \Delta^m$ is a bundle by construction and $\varphi^* E$ is a $(n+k)$-dimensional compact manifold, being the total space of the bundle $\varphi^* E' \to B'$ of n-dimensional compact manifold over the k-dimensional compact manifold B'.

(ii) The name "geometric assembly map" is due to the fact that geometrically α takes all the structure on the fibers of p and assembles them into one big structure.

(iii) If $B = \{*\}$, then the map $\alpha\colon \mathscr{S}_n(E) \to \mathscr{S}_n(E)$ is canonically homotopic to the identity map $\mathrm{id}_{\mathscr{S}_n(E)}$. Indeed, in this case $\alpha(x)$ and x are canonically homeomorphic for any $x \in \mathscr{S}_n(E)_m$ and this homeomorphism provides a homotopy from α to $\mathrm{id}_{\mathscr{S}_n(E)}$.

A natural question about the geometric assembly map is whether it can be translated into an "algebraic" version through the parametrized Whitehead torsion. More precisely, we ask whether in the following diagram a dotted arrow exists which makes the diagram commutative.

$$\begin{tikzcd} \mathscr{S}_n(p) \arrow[rr, "\alpha"] \arrow[d, "\tau"'] & & \mathscr{S}_{n+k}(E) \arrow[d, "\tau"] \\ \Gamma\begin{pmatrix} \Omega\,\mathrm{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} \arrow[rr, dotted] & & \Omega\,\mathrm{Wh}(E) \end{tikzcd}$$

Note that the codomain of the right-hand side torsion is $\Omega\,\mathrm{Wh}(E)$ because

$$\Gamma\begin{pmatrix} \Omega\,\mathrm{Wh}_*(E) \\ \downarrow \\ * \end{pmatrix} \simeq \operatorname*{holim}_{\sigma \in \mathrm{simp}\,*} \Omega\,\mathrm{Wh}(E_\sigma) \cong \Omega\,\mathrm{Wh}(E)$$

Such a dotted arrow actually exists. Let us construct it in the following definition.

Definition 5.30. Let $p\colon E \to B$ be in $\mathbf{Fib}(B; F)$. Assume for simplicity that B is path-connected and choose a base point $b \in B$. Denote by F_b the fiber of p over b. We define the map $\overline{\alpha}$ as the following composition

$$\overline{\alpha}\colon\ \Gamma\begin{pmatrix} \Omega\,\mathrm{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix} \longrightarrow \Omega\,\mathrm{Wh}(F_b) \xrightarrow{\chi(B)\cdot j_*} \Omega\,\mathrm{Wh}(E)$$

where the first map is the map

$$\Gamma\begin{pmatrix}\Omega\,\mathrm{Wh}_B(E)\\ \downarrow\\ B\end{pmatrix}\longrightarrow\Gamma\begin{pmatrix}\Omega\,\mathrm{Wh}_{\{b\}}(F_b)\\ \downarrow\\ \{b\}\end{pmatrix}\cong\Omega\,\mathrm{Wh}(F_b)$$

induced by restriction from B to $\{b\}$, the map j_* is the map induced by the inclusion $F_b \hookrightarrow E$ and $\chi(B)$ is the Euler characteristic of B.

Remark 5.31. The map $\overline{\alpha}$ is well-defined up to homotopy. Indeed, a different choice of base point $b' \in B$ leads to a map $\overline{\alpha}'$ which is homotopic to $\overline{\alpha}$.

Theorem 5.32 ([Ste10, Theorem 4.2]). *Let* $p\colon E \to B$ *be in* $\mathbf{Bun}_n(B;F)$ *and assume that* B *is a compact connected topological manifold. Then the following diagram commutes up to homotopy.*

$$\begin{tikzcd}
\mathscr{S}_n(p) \arrow[r, "\alpha"] \arrow[d, "\tau"'] & \mathscr{S}_{n+k}(E) \arrow[d, "\tau"] \\
\Gamma\begin{pmatrix}\Omega\,\mathrm{Wh}_B(E)\\ \downarrow\\ B\end{pmatrix} \arrow[r, "\overline{\alpha}"] & \Omega\,\mathrm{Wh}(E)
\end{tikzcd}$$

At this point, it is easy to realize that the geometric assembly map α commutes with stabilization up to canonical homotopy. In particular, it induces a "stable" geometric assembly map

$$\alpha\colon\ \mathscr{S}(p) \to \mathscr{S}(E)$$

Therefore, we can to obtain the following stable version of the previous theorem.

Theorem 5.33 ([Ste10, Theorem 4.3]). *Let* $p\colon E \to B$ *be in* $\mathbf{Bun}_n(B;F)$ *and assume that* B *is a compact connected topological manifold. Then the following diagram commutes up to homotopy.*

$$\begin{tikzcd}
\mathscr{S}(p) \arrow[r, "\alpha"] \arrow[d, "\tau"'] & \mathscr{S}(E) \arrow[d, "\tau"] \\
\Gamma\begin{pmatrix}\Omega\,\mathrm{Wh}_B(E)\\ \downarrow\\ B\end{pmatrix} \arrow[r, "\overline{\alpha}"] & \Omega\,\mathrm{Wh}(E)
\end{tikzcd} \tag{5.11}$$

We can say also more about diagram (5.11). Indeed, the following theorem holds.

Theorem 5.34 ([Ste10, Theorem 4.4]). *Consider the situation of Theorem 5.33. Then diagram* (5.11) *is a weak homotopy pullback.*

This theorem will be very important to study the stable fibering problem in the next section.

Remark 5.35. The previous theorem is proved by investigating how much information gets lost under the parametrized torsion map τ. In particular, the strategy is to show using some microbundle theory [Mil64] that all the ambiguity introduced by applying τ is given by the invariance under stabilization, which we already know, and the invariance under change of tangential structure on E. Using this, it follows that diagram (5.11) is a weak homotopy pullback.

5.4 Stable fibering obstructions

In this section we finally define the two obstructions $\mathrm{Wall}(p)$ and $o(f)$ for stably fibering a manifold and we prove that they form a complete obstruction theory in algebraic K-theory for existence and uniqueness of the stable fibering problem. This is a consequence of Theorem 5.34 and of the Riemann-Roch theorem with converse for topological manifolds [DWW03, Corollary 10.18].

Let us start by recalling the stable fibering problem.

Definition 5.36. Let $f\colon M \to B$ be a map in **Cpt**.

(i) We say that f *stably fibers* if there exists a $n \in \mathbb{N}$ such that the composite

$$f \circ \mathrm{Proj}\colon M \times D^n \to M \to B$$

is homotopic to the projection of a fiber bundle whose fibers are in **Cpt**.

(ii) Let $\mathcal{C}$ be the set of all bundles maps $g\colon M \times D^n \to B$ for some $n \in \mathbb{N}$ which are homotopic to $f \circ \mathrm{Proj}$. We say that two elements g and g' in $\mathcal{C}$ are *equivalent* and we denote $g \sim g'$ if, after possibly further stabilization, g and g' are isomorphic through a homeomorphism $i\colon M \times D^N \to M \times D^N$ such that i is homotopic to the identity map $\mathrm{id}_{M\times D^N}$, that is, if $g \circ i = g'$.

The stable fibering problem consists of the following two question:

(i) When does a map $f\colon M \to B$ in **Cpt** stably fiber?

(ii) How many different ways are there for f to stably fiber? Namely, how can $\mathcal{C}/\sim$ be described?

The idea is to apply the same strategy that we have used for the fibering problem in Chapter 3. More precisely, first we convert the map f into a fibration p and we reduce the study to p and then we check what information we lose during the conversion.

Notation 5.37. In the following, given a map $f\colon M \to B$ in **Cpt**, we denote simply by $p \circ \lambda$ with $p\colon E \to B$ a fibration and $\lambda\colon M \to E$ a homotopy equivalence the factorization of f given in Section 1.2. In other words, we denote by E the space $\mathrm{FIB}(f)$, by $\lambda\colon M \to E$ the homotopy equivalence λ_f and by $p\colon E \to B$ the fibration $\widehat{f}$.

Let us define the first obstruction.

Definition 5.38. Let $f\colon M \to B$ be a map in **Cpt** and assume that the fibers of p are homotopy finitely dominated. Fix a point $b \in B$ and denote by F_b the fiber of p over b. We define the *parametrized Wall obstruction* $\mathrm{Wall}(p) \in H^0\big(B; \mathrm{Wh}(F_b)\big)$ as the image of the parametrized A-theory characteristic

$$\chi(p) \in \Gamma \begin{pmatrix} A_B(E) \\ \downarrow \\ B \end{pmatrix}$$

under the map induced by the natural transformation $A(X) \to \mathrm{Wh}(X)$.

Remark 5.39. By $H^0\big(B; \mathrm{Wh}(F_b)\big)$ we mean the 0-th cohomology of B with twisted coefficients in the Whitehead spectrum of the fiber F_b. More precisely, by definition we have

$$\mathbb{H}^\bullet\left(B; \mathrm{Wh}(F_b)\right) = \Gamma \begin{pmatrix} \mathrm{Wh}_B(E) \\ \downarrow \\ B \end{pmatrix}$$

which is by construction weak homotopy equivalent to $\operatorname{holim}_{\sigma\in\operatorname{simp} B}\operatorname{Wh}(E_\sigma)$. We denote by $H^i\big(B;\operatorname{Wh}(F_b)\big) = \pi_{-i}\,\mathbb{H}^\bullet\left(B;\operatorname{Wh}(F_b)\right)$ the i-th cohomology of B with twisted coefficients in the Whitehead spectrum of the fiber F_b.

Remark 5.40. As its name says, the parametrized Wall obstruction $\operatorname{Wall}(p)\in H^0\big(B;\operatorname{Wh}(F_b)\big)$ may be understood as the parametrized version of the Wall finiteness obstruction of the fiber defined in [Wal65]. It will be clearer what this means in the next section.

Geometrically, $\operatorname{Wall}(p)$ is the obstruction for p being fiber homotopy equivalent to a fiber bundle in **Cpt**. More precisely, the following lemma holds.

Lemma 5.41. *Let $f\colon M\to B$ be a map in* **Cpt** *and assume that the fibers of p are homotopy finitely dominated. Then the parametrized Wall obstruction* $\operatorname{Wall}(p)\in H^0\big(B;\operatorname{Wh}(F_b)\big)$ *vanishes if and only if p is fiber homotopy equivalent to a fiber bundle in* **Cpt**.

Proof. Consider the following weak homotopy fibration sequence

$$\Gamma\left(\begin{array}{c} A_B^{\%}(E)\\ \downarrow\\ B\end{array}\right)\longrightarrow\Gamma\left(\begin{array}{c} A_B(E)\\ \downarrow\\ B\end{array}\right)\longrightarrow\Gamma\left(\begin{array}{c} \operatorname{Wh}_B(E)\\ \downarrow\\ B\end{array}\right)$$

By definition, the obstruction $\operatorname{Wall}(p)$ is zero if and only if $\chi(p)$ is in the image of the fiberwise assembly map up to homotopy, that is, if and only if $\chi(p)$ lifts over the fiberwise assembly map up to homotopy. But, by the Riemann-Roch theorem with converse for topological manifolds [DWW03, Corollary 10.18], this is equivalent to say that p is fiber homotopy equivalent to a fiber bundle in **Cpt**. $\square$

Remark 5.42. Lemma 5.41 is the solution of the stable fibering problem in case of fibration. Note that we have not used any stabilization. This tells us that the "stable" part of the problem is contained in the second obstruction.

Let us now go on with our strategy and define the second obstruction.

Definition 5.43. Let $f\colon M\to B$ be a map in **Cpt** and assume that the fibers F_b of p are homotopy finitely dominated. Suppose in addition that $\operatorname{Wall}(p)=0$. Then, by Lemma 5.41, we can factor f as $q\circ\lambda'$ where $q\colon E'\to B$ is a fiber bundle in **Cpt** and $\lambda'\colon M\to E'$ is a homotopy equivalence. We define the *parametrized torsion obstruction*

$$o(f)\in\operatorname{coker}\Big(\pi_0(\overline{\alpha})\colon H^0\big(B;\Omega\operatorname{Wh}(F_b)\big)\to\operatorname{Wh}(\pi M)\Big)$$

to be the class for which a representative in $\operatorname{Wh}(\pi M)$ is

$$(\lambda')_*^{-1}\tau(\lambda'\colon M\to E')\in\operatorname{Wh}(\pi M)$$

Remark 5.44. (i) The map $\overline{\alpha}$ is the map

$$\mathbb{H}^\bullet\left(B;\Omega\operatorname{Wh}(F_b)\right)\longrightarrow\mathbb{H}^\bullet\left(\{b\};\Omega\operatorname{Wh}(F_b)\right)\simeq\Omega\operatorname{Wh}(F_b)\xrightarrow{\chi(B)\cdot j_*}\Omega\operatorname{Wh}(E)\simeq\Omega\operatorname{Wh}(M)$$

given in Definition 5.30, where it is defined only for path-connected B. If B is not path-connected, then $\overline{\alpha}$ is defined component-wise.

(ii) The parametrized torsion obstruction $o(f)$ is well-defined. Indeed, choose another factorization $f = \overline{q} \circ \overline{\lambda'}$ where $\overline{q}\colon \overline{E'} \to B$ is a fiber bundle in **Cpt** and $\overline{\lambda'}\colon M \to \overline{E'}$ is a homotopy equivalence. Then, by Lemma 2.15, we have

$$(\overline{\lambda'})_*^{-1}\tau(\overline{\lambda'}) = (\lambda')_*^{-1}\tau(\lambda') - (\lambda')_*^{-1}\tau\Big(\lambda' \circ (\overline{\lambda'})^{-1}\Big)$$

Being a fiber homotopy equivalence, the map $\lambda' \circ (\overline{\lambda'})^{-1}\colon \overline{E'} \to E'$ is in the image of $\pi_0(\alpha')$ in the following diagram

$$\begin{tikzcd} \pi_0\,\mathscr{S}(q) \arrow[r, "\pi_0(\alpha')"] \arrow[d, "\tau"'] & \pi_0\,\mathscr{S}(E') \arrow[d, "\tau"] \\ H^0\big(B; \mathrm{Wh}(F'_b)\big) \arrow[r, "\pi_0(\overline{\alpha'})"] & \mathrm{Wh}(\pi E') \end{tikzcd}$$

which is π_0 of a weak homotopy pullback square by Theorem 5.34. By pullback property, this is equivalent to say that the corresponding element $\tau\Big(\lambda' \circ (\overline{\lambda'})^{-1}\Big) \in \mathrm{Wh}(\pi E')$ is in the image of $\pi_0(\overline{\alpha'})$ where $\overline{\alpha'}$ is the composition of Definition 5.30. Therefore, the class of $\tau\Big(\lambda' \circ (\overline{\lambda'})^{-1}\Big)$ is zero in $\operatorname{coker} \pi_0(\overline{\alpha'})$. It suffices now to note that $(\lambda')_*^{-1}$ induces a bijection from $\operatorname{coker} \pi_0(\overline{\alpha'})$ to $\operatorname{coker} \pi_0(\overline{\alpha})$ to conclude that $o(f)$ is not affected by $(\lambda')_*^{-1}\tau\big(\lambda' \circ (\overline{\lambda'})^{-1}\big)$ and so it is well-defined.

Now that we have defined the two obstructions $\mathrm{Wall}(p)$ and $o(f)$, we can finally state the existence and classification theorems that completely solve the stable fibering problem.

Theorem 5.45 (Existence). *A map $f\colon M \to B$ in* **Cpt** *stably fibers if and only if the following conditions hold:*

(i) the fibers of p are homotopy finitely dominated;

(ii) the parametrized Wall obstruction $\mathrm{Wall}(p) \in H^0\big(B; \mathrm{Wh}(F_b)\big)$ vanishes;

(iii) the parametrized torsion obstruction $o(f) \in \operatorname{coker} \pi_0(\overline{\alpha})$ vanishes.

Theorem 5.46 (Classification). *Given a map $f\colon M \to B$ in* **Cpt**, *there is a bijection*

$$\mathcal{C}/\!\sim\; \longrightarrow \ker \pi_0(\overline{\alpha})$$

The proof of these theorems is based on (the stabilized version of) the following lemma, which is the key result that connects the stable fibering problem with the geometric assembly map.

Lemma 5.47 ([Ste10, Lemma 5.3]). *(i) A fibration $p\colon E \to B$ is fiber homotopy equivalent to a bundle of k-dimensional compact manifolds if and only if $\mathscr{S}_k(p)$ is non empty.*

(ii) A map $f\colon M^{n+k} \to B^n$ is homotopic to a bundle of k-dimensional manifolds if and only if the element defined by $\lambda\colon M \to E$ is in the image of the map

$$\pi_0(\alpha)\colon \pi_0\,\mathscr{S}_k(p) \longrightarrow \pi_0\,\mathscr{S}_{n+k}(E)$$

(iii) There is a bijection from $\mathcal{C}/\!\sim$ to the preimage of $[\lambda]$ under the map $\pi_0(\alpha)$.

Proof. Statement (i) follows immediately by definition and (iii) implies (ii). Therefore, it suffices to prove (iii). This follows essentially by definition. See [Ste10, pp. 76–77] for the details. □

The stabilized version of this lemma, which we state now, follows by the unstable version above and by the fact that

$$\operatorname*{colim}_n \pi_0 \mathscr{S}_n(p) \xrightarrow{\cong} \pi_0 \operatorname*{hocolim}_n \mathscr{S}_n(p) = \pi_0 \mathscr{S}(p)$$

This is the version that we will use to prove Theorem 5.45 and Theorem 5.46.

Lemma 5.48. *(i) A fibration $p\colon E \to B$ is fiber homotopy equivalent to a bundle in* **Cpt** *if and only if $\mathscr{S}(p)$ is non empty.*

(ii) A map $f\colon M \to B$ stably fibers if and only if the element defined by $\lambda\colon M \to E$ is in the image of the map

$$\pi_0(\alpha)\colon \pi_0 \mathscr{S}(p) \longrightarrow \pi_0 \mathscr{S}(E)$$

(iii) There is a bijection from $\mathcal{C}/\sim$ to the preimage of $[\lambda]$ under the map $\pi_0(\alpha)$.

We can now finally prove the two theorems that solve the stable fibering problem.

Proof of Theorem 5.45. It is easy to see that conditions (i) and (ii) are necessary. Indeed, let us assume that f is homotopic to a bundle g in **Cpt**. Then assertion (i) clearly holds. Moreover, in this case the homotopy from f to g induces a fiber homotopy equivalence from p to g. Therefore, by Lemma 5.41, we have that $\mathrm{Wall}(p)$ vanishes and (ii) holds.

Suppose now that (i) and (ii) hold. We prove that under these assumptions the map $f\colon M \to B$ stably fibers if and only if also (iii) holds. By Lemma 5.41, we can factor f as $q \circ \lambda'$ where $q\colon E' \to B$ is a fiber bundle in **Cpt** and $\lambda'\colon M \to E'$ is a homotopy equivalence. Fix a point $b \in B$ and denote by F'_b the fiber of q over b. Consider the following commutative diagram

$$\begin{tikzcd} \pi_0 \mathscr{S}(q) \arrow[r, "\pi_0(\alpha')"] \arrow[d, "\tau"'] & \pi_0 \mathscr{S}(E') \arrow[d, "\tau"] \\ H^0\big(B; \mathrm{Wh}(F'_b)\big) \arrow[r, "\pi_0(\overline{\alpha'})"] & \mathrm{Wh}(\pi E') \end{tikzcd}$$

which is π_0 of the weak homotopy pullback square of Theorem 5.34. By Lemma 5.48, the map f stably fibers if and only if the element defined by λ' in the upper right-hand corner is in the image of the upper horizontal map $\pi_0(\alpha')$. By pullback property, this is equivalent to say that the corresponding element $\tau(\lambda')$ in $\mathrm{Wh}(\pi E')$ is in the image of the lower horizontal map $\pi_0(\overline{\alpha'})$. Therefore, f stably fibers if and only if the class of $\tau(\lambda')$ is zero in the coker $\pi_0(\overline{\alpha'})$. Now, note that $(\lambda')_*^{-1}$ induces a bijection from $\operatorname{coker}\pi_0(\overline{\alpha'})$ to $\operatorname{coker}\pi_0(\overline{\alpha})$ where $\overline{\alpha}$ is the composition of Remark 5.44(i) and this sends the class of $\tau(\lambda')$ in $\operatorname{coker}\pi_0(\overline{\alpha'})$ to the obstruction $o(f)$ in $\operatorname{coker}\pi_0(\overline{\alpha})$. Hence, we obtain that the map f stably fibers if and only if $o(f) = 0$. ☐

Proof of Theorem 5.46. By Lemma 5.48, we have that $\mathcal{C}/\sim$ is in bijection with $\pi_0(\alpha)^{-1}\big([\lambda]\big)$, which is in bijection with $\pi_0(\overline{\alpha})^{-1}\big(\tau(\lambda)\big)$ by diagram

$$\begin{tikzcd} \pi_0 \mathscr{S}(p) \arrow[r, "\pi_0(\alpha)"] \arrow[d, "\tau"'] & \pi_0 \mathscr{S}(E) \arrow[d, "\tau"] \\ H^0\big(B; \mathrm{Wh}(F_b)\big) \arrow[r, "\pi_0(\overline{\alpha})"] & \mathrm{Wh}(\pi E) \end{tikzcd}$$

Therefore, it is also in bijection to $\ker \pi_0(\overline{\alpha})$ since α is a infinite loop map. ☐

5.5 Comparison with the fiber torsion obstructions

As already pointed out, to define the obstructions $\mathrm{Wall}(p)$ and $o(f)$ we have used the same strategy used to define $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$ in Chapter 3. It is not surprising then that they have a very similar construction and they play exactly the same roles respectively. The goal of this last section is to show that $\mathrm{Wall}(p)$ and $o(f)$ are actually the generalization of $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$ in algebraic K-theory. More precisely, this section is devoted to present the following theorem.

Theorem 5.49. *Let $f\colon M \to B$ be a map in* **Man**. *Factor it as $f = p \circ \lambda$ according to Notation 5.37. Then:*

(i) The image of the parametrized Wall obstruction $\mathrm{Wall}(p)$ under the restriction

$$H^0\big(B; \mathrm{Wh}(F_b)\big) \to H^0\big(\{b\}; \mathrm{Wh}(F_b)\big) \cong \widetilde{K}_0\big(\mathbb{Z}[\pi F_b]\big)$$

is the Wall finiteness obstruction of the fiber (see [Wal65]).

(ii) Assume that the homotopy fiber F_b is homotopy finite. The image the parametrized Wall obstruction $\mathrm{Wall}(p)$ under the secondary homomorphism

$$\ker\Big(H^0\big(B; \mathrm{Wh}(F_b)\big) \to H^0\big(B; \pi_0 \mathrm{Wh}(F_b)\big)\Big) \to H^1\big(B; \mathrm{Wh}(\pi F_b)\big) \xrightarrow{j_*} H^1\big(B; \mathrm{Wh}(\pi M)\big)$$

is $\theta(f)$, where j_* is the homomorphism induced by the inclusion $j\colon F_b \hookrightarrow M$.

(iii) Assume that $\mathrm{Wall}(p)$ vanishes. Consider the map

$$\pi_0(\overline{\alpha})\colon H^0\big(B; \Omega \mathrm{Wh}(F_b)\big) \to \mathrm{Wh}(\pi F_b) \xrightarrow{\chi(B)\cdot j_*} \mathrm{Wh}(\pi M)$$

where $\overline{\alpha}$ is the composition of Remark 5.44(i). Then $\pi_0(\overline{\alpha})$ induces a map

$$\operatorname{coker}\big(\pi_0(\overline{\alpha})\big) \to \operatorname{coker}\big(\mathrm{Wh}(\pi F_b) \xrightarrow{\chi(B)\cdot j_*} \mathrm{Wh}(\pi M)\big)$$

which maps the parametrized torsion obstruction $o(f)$ to $\tau_{\mathit{fib}}(f)$. In particular, if $\chi(B) = 0$, then we have

$$o(f) = \tau_{\mathit{fib}}(f) \in \mathrm{Wh}(\pi M)$$

We give a complete proof only of part (iii) of this theorem.

Proof of part (iii). Since $\mathrm{Wall}(p) = 0$, we can assume that p is a bundle in **Cpt**. Moreover, by Lemma 3.19 and Remark 3.22, we can equip E with the canonical simple structure $\xi^{\mathrm{Top}}(E)$ defined in Section 2.3. Therefore, since both $o(f)$ and $\tau_{\mathrm{fib}}(f)$ are given by the respective class of $(\lambda)_*^{-1}\tau(\lambda)$, we can conclude. □

As for part (i) and (ii), they are immediate consequences of the following theorem, whose proof can be found in [Ste10, Section 5.6].

Theorem 5.50 ([Ste10, Theorem 5.19]). *(i) Let $p\colon E \to B$ be a fibration over a* CW*-complex with fiber F_b over $b \in B$. Then there is a 4-th quadrant spectral sequence*

$$E_2^{p,q} = H^p\big(B; \pi_{-q} \mathrm{Wh}(F_b)\big) \Rightarrow H^{p+q}\big(B; \mathrm{Wh}(F_b)\big)$$

where the E_2-term consists of ordinary cohomology with twisted coefficients in the system of abelian groups $\{b \mapsto \pi_{-q} \mathrm{Wh}(F_b)\}$.

(ii) If B is d-dimensional with $d < \infty$, then the corresponding filtration

$$\cdots \supset \mathscr{F}^{p,q} \supset \mathscr{F}^{p+1,q-1} \supset \ldots$$

of $H^{p+q}\big(B; \mathrm{Wh}(F_b)\big)$ is finite and the spectral sequence converges in the strongest possible sense. Namely, we have

$$\begin{aligned} \mathscr{F}^{0,n} &= H^n\big(B; \mathrm{Wh}(F_b)\big) && \text{for all } n \\ \mathscr{F}^{d+1,n-d-1} &= 0 && \text{for all } n \\ \mathscr{F}^{p,q}/\mathscr{F}^{p+1,q-1} &= E_\infty^{p,q} && \text{for all } p, q \end{aligned}$$

(iii) The image of the parametrized Wall obstruction $\mathrm{Wall}(p)$ *under the edge homomorphism*

$$H^0\big(B; \mathrm{Wh}(F_b)\big) \to H^0\big(B; \pi_0 \mathrm{Wh}(F_b)\big) \subset \coprod_{[b] \in \pi_0 B} \widetilde{K}_0\big(\mathbb{Z}[\pi_1 F_b]\big)$$

is the Wall finiteness obstruction of the fiber.

(iv) Suppose that all the fibers are in **TFCW** *and let $\gamma \colon S^1 \to B$ be a loop. Then the naturally defined secondary homomorphism*

$$\ker\Big(H^0\big(B; \mathrm{Wh}(F_b)\big) \to H^0\big(B; \pi_0 \mathrm{Wh}(F_b)\big)\Big) \to H^1\big(B; \pi_1 \mathrm{Wh}(F_b)\big)$$

followed by the restriction map

$$\gamma^* \colon H^1\big(B; \mathrm{Wh}(\pi F_b)\big) \to H^1\big(S^1; \pi_1 \mathrm{Wh}(F_b)\big) \cong \mathrm{Wh}(\pi F_b)$$

maps the parametrized Wall obstruction $\mathrm{Wall}(p)$ *to the element defined by the torsion of the fiber transport along γ.*

We conclude this section by explaining what are the edge and secondary homomorphisms of this theorem. In this way, we clarify also all the maps of Theorem 5.49.

Consider the spectral sequence of the previous theorem. By definition, we have exact sequences

$$0 \to \mathscr{F}^{p+1,q-1} \to \mathscr{F}^{p,q} \to E_\infty^{p,q}$$

for all p, q. Moreover, since the spectral sequence is limited to the 4-th quadrant, we obtain

$$\begin{gathered} E_2^{0,0} \supseteq E_3^{0,0} \supseteq \cdots \supseteq E_\infty^{0,0} \\ E_2^{1,-1} \supseteq E_3^{1,-1} \supseteq \cdots \supseteq E_\infty^{1,-1} \end{gathered}$$

Therefore, we get the following two exact sequences

$$\begin{gathered} 0 \to \mathscr{F}^{1,-1} \to \mathscr{F}^{0,0} \xrightarrow{\alpha_0} E_2^{0,0} \\ 0 \to \mathscr{F}^{2,-2} \to \mathscr{F}^{1,-1} \xrightarrow{\alpha_1} E_2^{1,-1} \end{gathered} \tag{5.12}$$

The edge and secondary homomorphism of Theorem 5.50, and hence of Theorem 5.49, are exactly the homomorphisms α_0 and α_1. Indeed, α_0 is by definition a map

$$\alpha_0 \colon H^0\big(B; \mathrm{Wh}(F_b)\big) \to H^0\big(B; \pi_0 \mathrm{Wh}(F_b)\big)$$

and since, by exactness of (5.12), $\mathscr{F}^{1,-1}$ is the kernel of α_0, then α_1 is a map

$$\alpha_1 \colon \ker\Big(H^0\big(B; \mathrm{Wh}(F_b)\big) \to H^0\big(B; \pi_0 \mathrm{Wh}(F_b)\big)\Big) \to H^1\big(B; \pi_1 \mathrm{Wh}(F_b)\big)$$

Remark 5.51. The previous maps α_0 and α_1, as for the map of Theorem 5.49(iii), are the most natural maps we can think of to compare our obstructions: they are very easy and nothing is artificial in their construction. Therefore, the obstructions $\mathrm{Wall}(p)$ and $o(f)$ that completely solve the stable fibering problem are the very natural generalization of the Wall finiteness obstruction, $\theta(f)$ and $\tau_{\mathrm{fib}}(f)$ in the context of algebraic K-theory.

Bibliography

[AHS90] Jiri Adamek, Horst Herrlich, and George E. Strecker. *Abstract and Concrete Categories: The Joy of Cats*. New York, NY: John Wiley and Sons, 1990. URL: http://katmat.math.uni-bremen.de/acc/.

[BD07] Bernard Badzioch and Wojciech Dorabiala. *Additivity for parametrized topological Euler characteristic and Reidemeister torsion*. arXiv:math/0506258. Nov. 2007. DOI: 10.48550/arXiv.math/0506258. URL: http://arxiv.org/abs/math/0506258 (visited on 05/11/2023).

[BL65] W. Browder and J. Levine. "Fibering manifolds over a circle". In: *Commentarii Mathematici Helvetici* 40.1 (1965), pp. 153–160. ISSN: 1420-8946. DOI: 10.1007/BF02564368.

[Bro65] W. Browder. "Structures on $M \times \mathbb{R}$". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 2 (1965), pp. 337–345. ISSN: 1469-8064, 0305-0041. DOI: 10.1017/S0305004100003947.

[Cha74] T. A. Chapman. "Topological Invariance of Whitehead Torsion". In: *American Journal of Mathematics* 96.3 (1974), pp. 488–497. ISSN: 0002-9327. DOI: 10.2307/2373556. URL: https://www.jstor.org/stable/2373556.

[Coh73] Marshall M. Cohen. *A Course in Simple-Homotopy Theory*. Ed. by P. R. Halmos. Vol. 10. Graduate Texts in Mathematics. New York, NY: Springer, 1973. ISBN: 978-0-387-90055-1. DOI: 10.1007/978-1-4684-9372-6. URL: http://link.springer.com/10.1007/978-1-4684-9372-6.

[Die08] Tammo tom Dieck. *Algebraic Topology*. EMS Textbooks in Mathematics. European Mathematical Society, 2008. ISBN: 978-3-03719-548-2. DOI: 10.4171/048. URL: https://ems.press/books/etb/57.

[DK01] James Davis and Paul Kirk. *Lecture Notes in Algebraic Topology*. Vol. 35. Graduate Studies in Mathematics. ISSN: 1065-7339. American Mathematical Society, 2001. ISBN: 978-1-4704-2088-8. DOI: 10.1090/gsm/035. URL: http://www.ams.org/gsm/035.

[DWW03] W. Dwyer, M. Weiss, and B. Williams. "A parametrized index theorem for the algebraic K-theory Euler class". In: *Acta Mathematica* 190.1 (2003), pp. 1–104. ISSN: 1871-2509. DOI: 10.1007/BF02393236.

[Dwy96] W. Dwyer. "The centralizer decomposition of BG". In: ed. by Carles Broto, Carles Casacuberta, and Guido Mislin. Basel: Birkhäuser Basel, 1996, pp. 167–184. ISBN: 978-3-0348-9018-2.

[Far71] F. T. Farrell. "The Obstruction to Fibering a Manifold over a Circle". In: *Indiana University Mathematics Journal* 21.4 (1971), pp. 315–346. ISSN: 0022-2518. URL: https://www.jstor.org/stable/24890336.

[FH70] F. T. Farrell and W.-C. Hsiang. "A formula for $K_1 R_\alpha[T]$". In: *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*. Ed. by Alex Heller. Amer. Math. Soc., Providence, R.I., 1970, pp. 192–218. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=0260836.

[FLS09] F. T. Farrell, Wolfgang Lück, and Wolfgang Steimle. *Obstructions to Fibering a Manifold*. Aug. 2009. DOI: 10.48550/arXiv.0901.1250. URL: http://arxiv.org/abs/0901.1250 (visited on 02/06/2023).

[FP90] Rudolf Fritsch and Renzo Piccinini. *Cellular Structures in Topology*. Cambridge Studies in Advanced Mathematics. Cambridge, NY: Cambridge University Press, 1990. ISBN: 978-0-521-32784-8. DOI: 10.1017/CBO9780511983948. URL: https://www.cambridge.org/core/product/identifier/9780511983948/type/book.

[Gra88] Daniel R. Grayson. "The K-theory of semilinear endomorphisms". In: *Journal of Algebra* 113.2 (1988), pp. 358–372. ISSN: 0021-8693. DOI: https://doi.org/10.1016/0021-8693(88)90165-2. URL: https://www.sciencedirect.com/science/article/pii/0021869388901652.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge, NY: Cambridge University Press, 2002. ISBN: 978-0-521-79540-1. URL: https://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[KS77] Robion C. Kirby and Laurence C. Siebenmann. *Foundational Essays on Topological Manifolds, Smoothings, and Triangulations*. Vol. 88. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 1977. ISBN: 978-0-691-08190-8. DOI: 10.1515/9781400881505.

[LM23] Wolfgang Lück and Tibor Macko. "Surgery Theory: Foundations". Ongoing book project. Jan. 2023. URL: http://www.him.uni-bonn.de/lueck/data/sb.pdf (visited on 03/01/2023).

[LS15] Wolfgang Lück and Wolfgang Steimle. *A twisted Bass-Heller-Swan decomposition for the algebraic K-theory of additive categories*. arXiv:1309.1353 [math]. June 2015. DOI: 10.48550/arXiv.1309.1353. URL: http://arxiv.org/abs/1309.1353 (visited on 03/22/2023).

[Lüc89] Wolfgang Lück. *Transformation Groups and Algebraic K-Theory*. Vol. 1408. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1989. ISBN: 978-3-540-46827-1. DOI: 10.1007/BFb0083681. URL: http://link.springer.com/10.1007/BFb0083681.

[Mat76] Michael Mather. "Pull-Backs in Homotopy Theory". In: *Canadian Journal of Mathematics* 28.2 (1976), pp. 225–263. ISSN: 1496-4279. DOI: 10.4153/CJM-1976-029-0.

[Mil63] J. Milnor. *Morse Theory*. Vol. 51. Annals of Mathematic Studies. Princeton, NJ: Princeton University Press, 1963. ISBN: 978-0-691-08008-6. DOI: 10.1515/9781400881802.

[Mil64] J. Milnor. "Microbundles: Part I". In: *Topology* 3 (1964), pp. 53–80. ISSN: 0040-9383. DOI: 10.1016/0040-9383(64)90005-9. URL: https://www.sciencedirect.com/science/article/pii/0040938364900059.

[Mil66] J. Milnor. "Whitehead torsion". In: *Bulletin of the American Mathematical Society* 72.3 (1966), pp. 358–426. ISSN: 0002-9904, 1936-881X. DOI: 10.1090/S0002-9904-1966-11484-2. URL: https://www.ams.org/bull/1966-72-03/S0002-9904-1966-11484-2/.

[Mun63] James R. Munkres. *Elementary Differential Topology*. Vol. 54. Annals of Mathematics Studies. Princeton, NJ: Princeton University Press, 1963. ISBN: 978-0-691-09093-1. DOI: 10.1515/9781400882656.

[Ran02] Andrew Ranicki. *Algebraic and Geometric Surgery*. Oxford Mathematical Monographs. Oxford: Oxford University Press, 2002. ISBN: 978-0-19-850924-0. DOI: 10.1093/acprof:oso/9780198509240.001.0001.

[Sie65] Laurence C. Siebenmann. "The obstruction to finding a boundary for an open manifold of dimension greater than five". Ph.D.Thesis. Princeton University, 1965. URL: http://www.math.uchicago.edu/~shmuel/tom-readings/Siebenmann%20thesis.pdf.

[Ste07] Wolfgang Steimle. *Whitehead-Torsion und Faserungen*. June 2007. DOI: 10.48550/arXiv.0706.3973. URL: http://arxiv.org/abs/0706.3973 (visited on 02/08/2023).

[Ste10] Wolfgang Steimle. "Obstructions to stably fibering manifolds". Ph.D.Thesis. Universität Münster, 2010. URL: https://miami.uni-muenster.de/Record/e132a01e-6a5c-42e2-8037-ee13b68d5e38.

[Ste12] Wolfgang Steimle. "Obstructions to stably fibering manifolds". In: *Geometry & Topology* 16.3 (Aug. 2012), pp. 1691–1724. ISSN: 1364-0380, 1465-3060. DOI: 10.2140/gt.2012.16.1691. URL: http://arxiv.org/abs/1105.4744 (visited on 03/30/2023).

[Swi75] Robert M. Switzer. *Algebraic Topology — Homotopy and Homology*. Berlin, Heidelberg: Springer, 1975. ISBN: 978-3-540-42750-6. DOI: 10.1007/978-3-642-61923-6. URL: http://link.springer.com/10.1007/978-3-642-61923-6.

[Wal65] C. T. C. Wall. "Finiteness Conditions for CW-Complexes". In: *Annals of Mathematics* 81.1 (1965), pp. 56–69. ISSN: 0003-486X. DOI: 10.2307/1970382. URL: https://www.jstor.org/stable/1970382.

[Wal85] Friedhelm Waldhausen. "Algebraic K-theory of spaces". In: *Algebraic and Geometric Topology*. Ed. by Andrew Ranicki, Norman Levitt, and Frank Quinn. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer, 1985, pp. 318–419. ISBN: 978-3-540-39413-6. DOI: 10.1007/BFb0074449.

[Whi78] George W. Whitehead. *Elements of Homotopy Theory*. Vol. 61. Graduate Texts in Mathematics. New York, NY: Springer, 1978. ISBN: 978-1-4612-6320-3. DOI: 10.1007/978-1-4612-6318-0. URL: http://link.springer.com/10.1007/978-1-4612-6318-0.

[WW95] M. Weiss and B. Williams. "Assembly". In: *Novikov Conjectures, Index Theorems, and Rigidity*. Ed. by Andrew Ranicki, Jonathan M. Rosenberg, and Steven C. Ferry. Vol. 2. London Mathematical Society Lecture Note Series. Cambridge, NY: Cambridge University Press, 1995, pp. 332–352. ISBN: 978-0-521-49795-4. DOI: 10.1017/CBO9780511629365.014.