UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Astrophysics and Cosmology

Final Dissertation

Probing parity violation in the Early Universe

Thesis supervisor

Prof. Nicola Bartolo

Thesis co-supervisor

Dr. Alessandro Greco

Candidate

Alberto Salvarese

Academic Year 2021/2022

Abstract

In this thesis we briefly expose the shortcomings of the Hot Big-Bang model, thereby explaining the inflationary solution. Then, we describe a scalar single-field slow-roll model of Inflation, and introduce the topic of cosmological perturbation theory. After having summarized the standard ways to deal with cosmological perturbations, we introduce the gauge invariant scalar curvature perturbation field ζ, and explain why it is necessary to seek for its correlation functions of order higher than two in order to discriminate between inflationary models. We thus expose two important formalisms that allow us to treat cosmological correlation functions. The first one is the in-in formalism, which enables one to compute correlation functions in a similar way to what its done in the S-matrix approach. The second one is the ADM formalism, with which it turns out easier to take into account perturbations of the involved fields, in the context of General Relativity. We then apply these formalisms to two-, three-, and four-points correlation functions in the framework of single-field slow-roll Inflation, and of the $I^2(\phi)\tilde{F}F$ model, where the kinetic term of a gauge field is coupled to the inflaton. Finally, we introduce the $f(\phi)\tilde{W}W$ model, which induces parity-violating signatures from the gravity sector. This model goes under the name of gravitional Chern-Simons term, coupling the inflaton with the Weyl tensor W. We then compute several two-, and three-points functions to highlight possible sources of parity violation. In the end, we compute the $\langle\zeta\zeta\zeta\zeta\rangle$ trispectrum mediated by tensor fields in the Chern-Simons model. This is indeed one of the main goals of this thesis: to see whether such a graviton-mediated trispectrum carries a parity-breaking signature. We investigate this possibility in details under several assumptions, discussing under which conditions $\langle\zeta\zeta\zeta\zeta\rangle$ turns out to be parity-violating.

Contents

Introduction

During the past century, the Hot Big-Bang (HBB) model was proposed to explain the expansion of our Universe. It described how the Universe evolved from its initial conditions to what we observe today, the presence of light-elements and their abundances, and the Cosmic Microwave Background. This model relies on two main ideas: General Relativity and the Cosmological Principle. Therefore, since the former gives us a law of gravity that connects it to the amount of matter we have, and the latter provides us a guide to study (approximately) the distribution of matter in the Universe, we can compute its dynamics, revealing how it evolved in time. Without entering into the details regarding how differently the Universe evolves considering different types of matter, which are specified by their equation of state ω, the widely accepted picture shows a Universe that shrinks going back in time, until it reaches a singularity, with density and temperature that tend to infinity.
Although several predictions of the HBB model were confirmed by observations, a lot of other issues remain unsolved. The most important ones go under the name of the Hot Big-Bang model shortcomings. Specifically, these can be summarize in three different "problems": the *horizon problem*, the *flatness problem*, and the *unwanted relics* problem. The first one rests on the fact that although the Hubble horizon delimiting the causally connected regions shrinks going to earlier times, the largest scales we observe today are in perfect agreement with the cosmological principle. Thus, how can it be that very large scales, which were not in causal connection between one another, now show similar properties? The second problem has to do with the fact that the Universe should have been originated with a very precise (fine-tuned) value of the parameter density, namely extremely close to unity, in order to be as we observe it today. So a natural question is: why such a special value? The last problem, which was actually the first to be observed, regards the spontaneous symmetry breakings (SBB) that could have happened in the very first moments of the Universe. As the Universe expanded, it led to lower and lower temperatures, possibly undergoing towards a series of SBBs. It is possible to show that from these breakings several topological defects would have arisen, yielding to extremely high values of the density parameter, in conflict with what we observe today. All these issues can be solved by admitting an early phase of the Universe's evolution during which the Universe underwent an extremely rapid accelerated expansion, called Cosmological Inflation [1]. In this scenario, in a very short period of time, the Universe increased its dimensions by a factor of $\sim 10^{25-30}$. By doing that, it is possible to show that the comoving Hubble radius defining our horizon increases rapidly, thereby including all the largest scales we observe today in a physically connected region of space. Then, it pushes the parameter density at the end of Inflation to assume values extremely closed to one (at least if Inflation lasts for enough time), thereby solving the fine tuning problem. Finally, Inflation would dilute away all the unwanted relics created through SBBs, hence decreasing their densities to an acceptable level.
A simple model of Inflation is given by supposing the Universe to be dominated by the energy density of the vacuum associated to a scalar field's potential. A very important success of such an inflationary model regards the prediction of the CMB temperature fluctuations. By studying the power-spectrum, i.e. the Fourier transform of the two-points correlation function, of the inflaton's field, it is possible to connect it to the density and temperature perturbations we have in the CMB, obtaining an estimate which is pretty close to what is observed. Being Inflation such an important feature of our Universe, a great effort has been made to properly describe its mechanism. A lot of models were proposed since its introduction, exploring possibilities with one or more quantum fields. However, if on the one hand the analysis of the power spectrum gives us several important observables to test Inflation, on the other one it does not tell everything about the mechanism underlying this theory. In fact, different models can lead to similar predictions for their power spectra. By exploring the extremely high energies at which Inflation is

supposed to take place, we would be able not only to properly describe the mechanism that created the Universe we observe, but also, we could get access to energies we won't ever be able to reproduce in a particle accelerator. Therefore, these early stages could provide us a way to test also high energy theories beyond the current Standard Model of particle physics and of gravity. For example, it could be possible to test theories that include corrections to General Relativity, or that predict the presence of other heavy particles, or new fundamental forces. For instance, we could get information about forces that break parity. Since the Sakharov conditions must be fulfilled to have a matter-antimatter asymmetry, we need a force that violate the CP symmetry, as Weak Interaction is not able to provide the observed amount of asymmetry. This new force may as well violate parity, so that by studying the very first moments of Inflation, we could find the right answer also for this modern cosmological issue.
A way to do that is to look at higher order correlation functions of the field(s) responsible for Inflation. By admitting the possibility of having non-zero higher statistics than the power spectrum, i.e. n-points correlation functions $(n > 2)$, we could robustly discriminate between different models of Inflation. Since a purely Gaussian random field is completely described by its power spectrum, contributions coming from (connected) n-points correlation functions are called non-Gaussianities. In this thesis, we explore the possibility of having signatures of parity violations from the very Early Universe. There are in fact different ways to obtain parity-odd signals from an inflationary background, also from the scalar sector. For example, one can look at the complex terms that can arise from the scalar curvature trispectrum [2]. Also, it would be possible to search for such a signal by following the procedure proposed by Liu et al. [3], where the presence of an additional scalar field introduces a dihedral angle between the momenta in the Fourier space that eventually breaks parity. In this work, we adopt the first approach, by including a modification of Einstein's General Relativity. In this theory, a parity violating Chern-Simons term appears in the gravitational part of the theory's Lagrangian [4–7]. This correction arises naturally in an Effective Field Theory approach to Inflation, where General Relativity is thought as an effective theory at energies lower than a certain cut-off energy scale. By allowing other higher order corrections to be non-vanishing, the Chern-Simons and the Gauss-Bonnet terms show up. Since in this thesis a particular emphasis is given to parity-violation, the Gauss-Bonnet term is neglected. Specifically, the Gauss-Bonnet term affects the evolution of the inflaton's field without producing any parity-odd signature. On the contrary, the Chern-Simons term implies a different evolution for right- and left-handed gravitons, which affects the primordial correlation functions that from the end of Inflation imprint themselves in the CMB. Several parity-violating tensor and scalar-tensor mixed bispectra have already been computed [8–10]. Thus, we explore here the possibility of having the same induced parity-odd signal in a purely scalar primordial trispectrum.
The thesis is organized as follows. In Chapter 1, we recall how Inflation was introduced to solves the HBB shortcomings presented above. We also describe a simple single-field slow-roll inflationary model, and introduce the topic of cosmological perturbation theory. In Chapter 2 we introduce the formalisms we used: the in-in formalism, and the ADM formalism. For the former, we derive the perturbative formula we will use intensively during this work, and the diagrammatic rules that can be inferred from it. For the latter, we briefly explain the reasons that led to its introduction. Then, we compute known correlation functions for the Gauge invariant scalar curvature perturbation ζ, two different mixed bispectra we will encounter again in the fourth chapter. In Chapter 3 we replicate some computations for the $I^2(\phi)\tilde{F}F$ model of Inflation, in which the (small) anisotropy of the Gauge field produces a non-vanishing signature of parity violation [2]. In Chapter 4 we finally introduce the $f(\phi)\tilde{W}W$ model, on which we mainly focus to derive some original results for this thesis. We show how the introduced Chern-Simons term breaks parity, and compute several non-Gaussian parity-breaking contributions. In the last section of this chapter, the original calculation of the graviton-mediated scalar curvature trispectrum is shown. The computation is done firstly by using standard de Sitter gravitons, and then by considering Chern-Simons modified ones. In the last chapter (5), we explain in a better way the topic of non-Gaussianities, and their importance in discriminating between inflationary models. Also, we provide a brief introduction to the techniques that are used to constrain the amount of non-Gaussianities from the observations of CMB and Large Scale Structures (LSS), and a brief summary of their actual constraints obtained from the Planck 2018 [11], and eBOSS [12] missions.
In this thesis, where not explicitly specified, the following conventions are used: $c = \hbar = M_{Pl} = 1$

Chapter 1

Inflation

We start this thesis by reporting the basic elements which are going to be frequently used in what follows. After having introduced them, we will briefly discuss the motivations that led to the introduction of the Inflationary paradigm. In the end of the chapter, we will finally introduced the single field slow-roll model of Inflation.

1.1 Basic tools

The cosmological principle of modern Cosmology states that the Universe appears to be homogeneous and isotropic. This means that whenever we are and wherever we are, we see the same properties in any direction at large scales ($\sim$ 100Mpc). To describe such a kind of Universe, we use the well known Friedmann-Lemaitre-Robertson-Walker (FLRW) metric, whose line element is given by (Appendix A.1)

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \tag{1.1}$$

where t is the cosmic time, $a(t)$ is the scale-factor, which defines the uniform expansion of the Universe as a function of time, and r, θ, and ϕ are spherical comoving coordinates. These quantities are said to be comoving since they do not represent physical coordinates, which enlarge due to the scale-factor. On the contrary, they remain fixed and do not change as the Universe expands. In order to connect these coordinates $\lambda_{comoving}$ to physical coordinates $\lambda_{physical}$ we have $\lambda_{physical} = a(t)\lambda_{comoving}$. Now, with the aim of understanding how the Universe evolves, we need its equations of motion (EoM). We can calculate them through the variational principle, but we do not know the exact action of all the possible fields present in the Universe. On the other hand, since the Einstein's field equations (EFE) contain also the EoM, we could just solve them for the metric we have. In order to do so, we also need the energy-momentum tensor of the matter content. Instead of build the exact form of such an object, we can think of it as a tensor that respects the symmetries of the metric we observe, and neglect the microscopic contribution of the fundamental fields. Thus, we just consider the matter content in a macroscopic way. Namely, it has to be homogeneous and isotropic. The energy-momentum tensor of a perfect fluid preserves these characteristics. By labelling with u_μ the four-velocity of the cosmic fluid's particle and defining $h_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu}$, with $g_{\mu\nu}$ being the metric, we have

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu}, \qquad T_{\mu\nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{1.2}$$

where ρ is the fluid's density and p its isotropic pressure. Then, by applying Eqs. 1.1 and 1.2 to the EFE

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{1.3}$$

we get the well known Friedmann equations

$$H^2 = \frac{8}{3}\pi G\rho - \frac{k}{a^2} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right), \tag{1.4}$$

where H is the Hubble parameter $H(t) = \frac{\dot{a}(t)}{a(t)}$, which tells us at which rate the Universe is expanding, and k is the curvature of the Universe (Appendix A.1). From the Bianchi identity, i.e. from $T^{\mu\nu}_{;\nu} = 0$, it follows also the continuity equation

$$\dot{\rho} + 3H\left(\rho + p\right) = 0. \tag{1.5}$$

Being the latter dependent on the other two Friedmann equations, we end up with a system of two equations in three unknowns ($a(t)$, $\rho(t)$ and $p(t)$). Thus, as customary in these scenarios, we have to impose an equation of state (EOS) $p = w\rho$. By solving Eq. 1.5, we get

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+w)\rho = 0 \quad \Longrightarrow \quad \rho(t) \propto a(t)^{-3(1+w)}, \tag{1.6}$$

and hence from the first of Eq. 1.4, by putting $k = 0$,

$$a(t) \propto t^{\frac{2}{3(1+w)}}. \tag{1.7}$$

We can now focus on three main cases: (i) the Universe is matter dominated, so that we get $w = 0$, $p = 0$, which leads to $\rho \propto a^{-3}$; (ii) the Universe is radiation dominated, from which it follows $w = \frac{1}{3}$, $\rho \propto a^{-4}$; (iii) the Universe is dominated by the vacuum energy, implying $w = -1$, and a constant ρ[1]. In the second case, we observe an additional a^{-1} with respect to the first one, since the radiation suffers also from the cosmological redshift, that further decreases its energy-density.
In order to proceed in describing the three flows of the Hot Big-Bang model, we need to introduce the concept of cosmological Horizons. For our purposes, the most important kind of horizon is defined by the so called Hubble Radius

$$R_c = \frac{c}{H(t)}. \tag{1.8}$$

By defining the Hubble Time $\tau_H = H^{-1}$ as the time during which the scale factor doubles itself, then Eq. 1.8 represents the physical distance travelled by light in a Hubble time. We also define the Comoving Hubble Radius by simply dividing R_c by $a(t)$. Restoring natural units we then obtain $r_H(t) = \frac{1}{\dot{a}}$. Once we calculate the Hubble Radius, all the points outside this sphere are not causally connected to us, since the information has not had enough time yet to travel from that point to us. This means that regions greater than a "Hubble sphere" at a time t cannot be physically connected in their whole extension.

[1]It is easy to show that by considering the Eistein's field equations and adding a constant $-\Lambda g_{\mu\nu}$ to their left hand side. Then, by considering in those equations an effective stress-energy tensor $T^{eff}_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$, and assuming it to be diagonal (at first approximation this is supported by the cosmological principle), we find $\rho_\Lambda = -p_\Lambda$

1.2 Shortcomings of the Hot Big-Bang model

In the Hot Big Bang (HBB) model of Stardard Cosmology, started to arise some issues in the '60s. Specifically, the three main concerns were the Horizon problem, the Flatness problem, and the Monopole problem. We start by discussing the Horizon problem.

From the previous section, we know that the Hubble radius, $r_H \propto t^{\frac{1+3w}{3(1+w)}}$, increases during both the radiation dominated and the matter dominated epoch. We also know that for scales larger than that radius, we cannot have casual connection. Nevertheless, we observe a highly homogeneous and isotropic Universe at any scale. For scales that stayed disconnected for most part of their history, we see the same macroscopic homogeneity, isotropy, temperature and polarisation of the CMB. How could it be that regions that could not communicate between themselves turn out to be so similar? The Inflation paradigm solves this problem by admitting a period during which the Universe underwent a tremendous accelerated expansion. It implies indeed $\ddot{a}(t) > 0$, and hence $\dot{r}_H = -\frac{\ddot{a}(t)}{a^2(t)} < 0$. Thus, it follows that during this period, the Hubble radius actually increased itself going backwards in time, at a rate proportional to the acceleration of the scale factor. In principle, this solves the Horizon problem by revealing a portion of the Universe's history in which all the scales were in causal connection. To completely solve the problem, we need Inflation to have lasted for a sufficient amount of time to guarantee a causal connection within the largest scales we are observing today. Therefore, we also have to quantify how much the Universe expanded. In order to do so, we introduce the so called Number of e-folds

$$N = \log\left(\frac{a_f}{a_i}\right), \tag{1.9}$$

where a_f and a_i are respectively the scale factor at the end and at the beginning of Inflation. This quantity describes how much a certain portion of the Universe expanded in logarithmic scale, or for which amount of time Inflation lasted. To get an estimate of N, we simply start with the condition $r_H(t_i) \geq r_H(t_0)$, i.e. by requiring that the (comoving) Hubble radius at the beginning of Inflation was greater than the Hubble radius now. Then, it follows

$$\frac{1}{a_i H_i} \geq \frac{1}{a_0 H_0} \quad\implies\quad \frac{a_f}{a_i} \geq \frac{H_i}{H_0}\frac{a_f}{a_0}. \tag{1.10}$$

We want now to rewrite the right-hand side of the last equation as a function of quantities related to post-inflationary epochs. Firstly, we use the first Friedmann equation to rewrite H_i. We neglect the term proportional to the curvature assuming it to be null and we are left with

$$H^2 \propto \rho \propto a^{-3(1+w)} \quad\implies\quad \frac{H_i}{H_0} = \left(\frac{a_i}{a_f}\right)^{-\frac{3}{2}(1+w_{inf})}\frac{H_f}{H_0}. \tag{1.11}$$

Thus,

$$\left(\frac{a_f}{a_i}\right)^{-\frac{(1+3_{inf})}{2}} \geq \frac{T_0}{T_f}\frac{H_f}{H_0}, \tag{1.12}$$

where we used the approximation $T \propto a^{-1}$ with T being the temperature of the Universe. We notice now that in order to get $\ddot{a} > 0$, from Eq. 1.7 we must have $w_{inf} < -\frac{1}{3}$. We then have a positive exponent and, by using the definition of the number of e-folds, we get

$$N \geq \frac{2}{|1+3w_{inf}|}\left[\log\left(\frac{T_0}{H_0}\right) + \log\left(\frac{H_f}{T_f}\right)\right]. \tag{1.13}$$

Here, we have $T_0 \approx 2.7\text{K} \sim 10^{-13}\text{GeV}$, $H_0 \approx 10^{-42}\text{GeV}$ [13,14], from which $\log{(T_0/H_0)} \approx 67$. T_f is the temperature of the Universe at the end of Inflation, when it started to be dominated by radiation, in the so called Re-heating phase. The value of the second logarithm we find in the right-hand side of the equation is model dependent and it is one of the reasons behind the uncertainties in the number of e-folds. A rough estimate of T_f can be made by considering again the first Friedmann equation and the expression for the radiation's energy density

$$\rho_\gamma = \frac{\pi^2}{30} g_* T^4, \tag{1.14}$$

where g_* is the relativistic degrees of freedom, which is definitely model dependent. Hence we get

$$H_f^2 = \frac{8\pi G}{3}\frac{\pi^2}{30} g_* T_f^4 \sim \frac{T_f^4}{M_{Pl}^2}, \tag{1.15}$$

with $\frac{T_f}{M_{Pl}} \leq 10^{-14}$ [15]. Also the choice of w_{inf} is largely model dependent. By choosing a cosmological constant value, i.e. $W_{inf} \sim -1$, at the very end we find

$$N \gtrsim 60 - 70. \tag{1.16}$$

This number translates in an expansion of the order of $\frac{a_f}{a_i} = e^N \sim 10^{23}$. Hence, in order to solve the Horizon problem, the Universe should have undergone a huge expansion.
Once stated the basic paradigm and computed a first constraint for the Inflation's duration, we pass to discuss the Flatness problem [16]. We firstly define the so called critical density as $\rho_c = \frac{3H^2}{8\pi G}$, which is the density the Universe should have if it were flat. From this quantity, we also define the density parameter $\Omega(t) = \frac{\rho(t)}{\rho_c(t)}$. Then, if Ω were greater, smaller, or equal to 1, we would have respectively a closed, open, or a flat Universe. It is now trivial to find

$$\Omega(t) - 1 = k r_H^2(t). \tag{1.17}$$

Since in a standard FLRW Universe the comoving Hubble radius increases with time, then if there were even a tiny deviation of Ω from 1 at the beginning of the Universe, we would expect it to increase. Therefore, we can think of a close (open) Universe becoming more and more close (open) as the cosmic time passes. Nevertheless, the observations today show $|\Omega_0 - 1| < 0.4\%$ (at 95% confidence level), which is completely consistent with a spatial flatness. For what we said above, trying to extrapolate the parameter density backwards in time, we should find even smaller deviations. At the Planck time, it results indeed $|\Omega_{Pl} - 1| < 10^{-62}$. Also, it is possible to recast this problem in terms of the life-time of the Universe. It turns out that a Universe can live up to $\sim 10^{10}$years only if Ω assume very special value at the Planck time [17]. Therefore, we see that the Flatness puzzle is actually a fine-tuning problem. Again, the inflationary paradigm solves this issue. In fact, by admitting a period of a rapidly decreasing r_H, any reasonable value of Ω different from 1 would be pushed towards this "special" value anyway. Even in this case, we can constraint the duration of this accelerated expansion by means of the number of e-folds. Starting from Eq. 1.17 we have $|\Omega(t_f) - 1| \approx |\Omega(t_{Pl}) - 1| \leq 10^{-64}$, hence, by requiring $|\Omega_{T_i} - 1| \sim O(1)$, we find

$$\frac{|\Omega - 1|_{t_f}}{|\Omega - 1|_{t_i}} \sim \left(\frac{a_i}{a_f}\right)^2 = e^{-2N} \qquad \Longrightarrow \qquad N \gtrsim 60 - 70. \tag{1.18}$$

This scenario also allows us to neglect the curvature term in the Friedmann equations, as we done above. That is because, within a certain range of values for k, which is dictated by Eq. 1.18, thanks to Inflation we will always end up with a flat Universe.
Finally, we briefly introduce the Unwanted relics problem, then providing a solution to it. We start to do so by summarize the concepts of a Spontaneous Symmetry Breaking (SSB). In the Early Universe, a (spontaneous)

symmetry breaking occurs when, for example, a phase transition takes place. At this point, the Lagrangian of the theory remains symmetric for all the states but the vacuum, for instance, under a certain transformation. We can imagine of having a potential of the type

$$V(\phi, T) = V_0(\phi) + \frac{1}{2}\alpha\lambda\phi^2 T^2, \tag{1.19}$$

with $V_0 = \frac{\lambda}{4}\left(\phi^2 - \sigma^2\right)^2$. We see that at high temperature the potential resembles a characteristic parabolic-shape $\propto T^2$, with a global minimum at $\phi = 0$. But as the Universe evolves expanding, it cools down. After having reached a certain critical temperature $T_c = \frac{\sigma}{\alpha^{1/2}}$, at which the global minimum becomes a saddle point, the potential changes its shape towards a Gizburg-Landau one $\propto \left(\phi^2 - \sigma^2\right)^2$. Now, the previous global minimum changes to an unstable maximum, thereby causing the scalar field to roll down to one of the two new minima ($\pm\sigma$) where, in this case, parity is broken. These phase transitions do not occur perfectly anyway, and there could be regions where this passage does not happen. Then, the scalar field remains stacked in its original false vacuum. These imperfections give rise to topological defects as domain walls (spontaneous breaking of a discrete symmetry), cosmic strings (SSB of a $U(1)$ gauge group), and magnetic monopoles (SSB of GUT theories) [18, 19]. The mechanism responsible for the production of cosmological defects as a consequence of phase transition is known as the Kibble mechanism. By considering domain walls as an example, it can be show that these are very massive and stable. In particular, they imply a density parameter of the order of $\Omega_x \sim \frac{1}{\sigma_x} \sim 10^{14} \gg 1$, which is in conflict with what we observe. Inflation solves this puzzle, since its accelerated expansion would completely wash away the number density of such relics ($n_x \propto a^{-3Ht}$). Thus, although a large number of, e.g., domain walls could have been produced, they would be diluted during the expansion of the Universe.

1.3 The paradigm

We already have pointed out that Inflation is capable of solving the shortcomings exposed above, and that in order to achieve this goal it has to admit a period of accelerated expansion. We have also seen that this feature corresponds to ask for an equation of state $w_{inf} < -1/3$. There are now several different models aiming to describe Inflation [20]. For our purpose, we start with the simplest possible scenario: the single field slow-roll Inflation.
We can start by considering a model on Inflation driven by cosmological constant, for which $w = -1 < -1/3$. Then, we add a constant term $\Lambda g_{\mu\nu}$ to the right hand side of EFE. We have the right to do that, as it satisfies both the mathematical consistency and the Bianchi identity. If on the one hand this term might figure as a mere geometric term, on the other one we can describe it as the energetic contribution given by continuous creations and annihilations of virtual particles in a curved space-time. In this view, Λ represents the vacuum energy-density of a certain combination of quantum fields. By remembering the EoS associated to the vacuum energy, if we consider the vacuum expectation value (vev) of the stress-energy tensor associated to that field, we get

$$\langle T^{\Lambda}_{\mu\nu}\rangle = \langle \rho + p\rangle u_\mu u_\nu + \langle p\rangle g_{\mu\nu} = -\langle\rho_\Lambda\rangle g_{\mu\nu}. \tag{1.20}$$

Then, by taking the vev of the EFE, we obtain

$$\langle G_{\mu\nu}\rangle = 8\pi G\langle T^{eff}_{\mu\nu}\rangle = 8\pi G\langle T_{\mu\nu}\rangle + 8\pi G\langle T^{\Lambda}_{\mu\nu}\rangle = -8\pi G\langle\rho_\Lambda\rangle g_{\mu\nu}, \tag{1.21}$$

where we considered the classical part of the effective stress-energy tensor as vanishing in the vacuum, since no particles are present. This shows us that also the vacuum fluctuations of quantum fields contribute to the curvature of the space-time, and hence they cannot be neglected. By considering now a scalar field Lagrangian $\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$ in a flat spacetime, for its stress-energy tensor we obtain

$$\begin{aligned} T^{\mu\nu} =& -\frac{\partial\mathcal{L}}{\partial\partial_\nu\phi}\partial^\mu\phi + g^{\mu\nu}\mathcal{L} \\ =&\frac{1}{2}\partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}V(\phi). \end{aligned} \tag{1.22}$$

By imposing a constant value of the vev of this field, in order to resemble the cosmological constant's case, we have $\langle\partial_\mu\phi\rangle = 0$, and hence

$$\langle T_{\mu\nu}\rangle = -\langle V(\phi)\rangle g_{\mu\nu}, \tag{1.23}$$

which is exactly what we found above. We notice that we can impose $\langle\phi\rangle$ to be constant only because ϕ is a scalar field, whose value does not change if we perform either a rotation or a Lorentz boost. This encourages us to continue with a model of Inflation driven by a scalar field. A clarification that is worth to be mentioned is the fact that we cannot actually require a perfectly constant value of the vev. Although this is useful to show the starting point of our simple model, eventually Inflation must end. Therefore, in full generality, we should instead require $\langle\phi\rangle = f(t)$. We thus proceed in analyzing the scalar field scenario. The potential's form is dictated by the precise model, and for the moment we leave it as general as possible. In curved spacetime, in order to find the corresponding stress-energy tensor, we have to use its general definition $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$. In spite of the fact that the total action of the system is given by the gravity's, the matter's and the inflaton's contributions, during Inflation, the dominant energy is that of the latter. Hence, we remain with $S \approx S_\phi = \int d^4x\sqrt{-g}\mathcal{L}_\phi$. We get

$$\begin{aligned} T_{\mu\nu} =& -\frac{2}{\sqrt{-g}}\left[\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}\mathcal{L}_\phi + \sqrt{-g}\frac{\partial\mathcal{L}_\phi}{\partial g^{\mu\nu}}\right] \\ =&\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V(\phi)\right], \end{aligned} \tag{1.24}$$

where the equation $\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-g}}{2}g_{\mu\nu}$ was used [2]. Being a quantum field, we expect the inflaton to have quantum fluctuations $\delta\phi$, which could hopefully explain also the density perturbations we observe in the CMB. Hence, at first order we introduce these fluctuations in the inflaton's field, which we rewrite as

$$\phi(\vec{x},t) = \phi_0(t) + \delta\phi(\vec{x},t). \tag{1.25}$$

Here, $\phi_0 = \phi(t)$ represents the average classical background motion of our field, which has to respect the cosmological principle, whereas $\delta\phi$ are its quantum fluctuations, which can carry spatial dependence. To perform this splitting, we have to require $\langle\delta\phi\rangle^2 \ll \phi_0^2$ ($\langle\delta\phi\rangle = 0$ by definition). Once these first tools are set, we start by focusing on ϕ_0. From its stree-energy it follows

$$\begin{aligned} T_0^{0(\phi_0)} &= -\frac{1}{2}\dot{\phi}_0^2 - V(\phi) \\ T_0^{i(\phi_0)} &= 0 = T_i^{0(\phi_0)} \\ T_j^{i(\phi_0)} &= \left(\frac{1}{2}\dot{\phi}_0^2 - V(\phi)\right)\delta_j^i. \end{aligned} \tag{1.26}$$

[2] Given a matrix A, starting from $\det e^A = e^{TrA}$, we have $\log\left(\det e^A\right) = TrA$. Hence, once called $g_{\mu\nu} = \left(e^A\right)_{\mu\nu}$, by differentiating both sides we are left with $\frac{d\det(g_{\mu\nu})}{\det(g_{\mu\nu})} = Tr\left(\frac{dg_{\mu\nu}}{g_{\mu\nu}}\right) = Tr\left(g^{\mu\nu}dg_{\mu\nu}\right)$. Then, the results follows by substituting $g = -\sqrt{-g}\sqrt{-g}$

T^μ_ν is therefore diagonal, and by defining $T_0^{0(\phi_0)} = \rho_{\phi(t)}$ and $T_j^{i(\phi_0)} = p_\phi(t)\delta^i_j$, we note two important things. Firstly, ρ_ϕ is just given by a kinetic term summed to a potential one, so that we can see ϕ_0 as a particle propagating along its potential curves. Secondly, there is a sign mismatch in the potential's sign between ρ_ϕ and p_ϕ. The latter observation is crucial to realize Inflation. In fact, if we could assume $\frac{\dot\phi_0^2}{2} \ll V(\phi_0)$, we would find an equation of state $w \approx -1$. This is what is called the *first slow-roll condition*, and allows us to produce a quasi de-Sitter expansion of the Universe, in complete analogy with the effect of a cosmological constant. However, we would eventually obtain this condition to be fulfilled anyway, as from Eq. 1.6 the kinetic energy density contribution would rapidly decrease as the Universe expands.
In order to properly study the dynamics of our field $\phi(\vec{x}, t)$, we need its equation of motion, which turns out to be (Appendix A.2)

$$\ddot\phi + 3H\dot\phi - \frac{\nabla^2}{a^2}\phi = -\frac{\partial V}{\partial \phi}. \tag{1.27}$$

This resembles the usual Klein-Gordon equation when we consider $H \to 0$. Thus, the expansion of the Universe emerges through $3H\dot\phi$, that represents a friction term in the field's dynamics. Now, we can couple this equation with the first Friedmann's equation, obtaining a system of two equations that allows us to describe the evolution of the inflaton

$$\begin{cases} H^2 = \frac{8\pi G}{3}\left(\frac{\dot\phi^2}{2} + V(\phi)\right) \\ \ddot\phi_0 + 3H\dot\phi_0 = -\frac{\partial V}{\partial\phi}, \end{cases} \tag{1.28}$$

where we restricted ourselves to the background field, and we considered the inflaton's energy density as dominant with respect to all the other possible contributions. By applying the first slow-roll condition, we get the same situation that we face considering a particle that moves under a constant potential and a viscous friction term. The particle's acceleration is exponentially decreased by the action of the viscous force, that in turn drives the system towards a constant velocity solution. By remembering that $V(\phi)$ is nearly constant for our hypothesis, from Eq. 1.28 we get exactly the same scenario. After a brief initial period, the *second slow-roll condition* $\ddot\phi_0 \ll 3H\dot\phi_0$ takes place, and the inflaton's dynamics tends towards a phase of slow-roll $\dot\phi_0 \approx -\frac{V_{,\phi_0}}{3H}$. During this phase, $V(\phi)$ mimics an effective cosmological constant and Inflation occurs. Once the full model is specified, i.e. once $V(\phi)$ is stated, through the equation above we can determine the dynamics of the inflaton and the subsequent expansion of the Universe.
It is useful to specify now some quantities that quantify this "Slow-Roll" dynamics. These are the so called *first* and *second slow-roll parameters* and in their most general form read respectively

$$\epsilon = -\frac{\dot H}{H^2} \qquad \text{and} \qquad \eta = -\frac{\ddot\phi}{H\dot\phi}. \tag{1.29}$$

By exploiting the first Friedmann equation shown above and the slow-roll conditions, we get from the first parameter

$$\epsilon \approx 4\pi G\frac{\dot\phi^2}{H^2} = \frac{3}{2}\frac{\dot\phi^2}{V(\phi)}. \tag{1.30}$$

It thus follows that in our model $\dot H = -4\pi G\dot\phi^2 \leq 0$, which gives us either a constant or a decreasing expansion in time, and that ϵ is directly proportional to the ratio between the kinetic and potential energy of the inflaton. Hence, thanks to the first slow roll condition, we need $\epsilon \ll 1$ to have an inflationary Universe. Also, we can rewrite ϵ as $\epsilon \approx \frac{1}{16\pi G}\frac{V'^2}{V^2} \ll 1$, with $' = \partial/\partial\phi$. We then see that ϵ tells us how much the potential energy is greater than the kinetic one, or, in other words, when the potential is "flat enough" to make Inflation starts. From the second slow-roll condition it immediately follows that also the second parameter has to be $\eta < 1$. It is also possible to write it in another way, by introducing η_V. We have in fact $\eta = \eta_V - \epsilon$ with $\eta_V = \frac{1}{3}\frac{V''}{H^2}$. Although ϵ seems to tell us all what we need to have Inflation [3], the second slow-roll parameter constrains the potential to be flat for "enough time", thereby allowing a slow-roll phase for the sufficient amount of time depicted by the number of e-folds. We have indeed that $\eta,\epsilon \ll 1$ implies $\eta_V \ll 1$ too, which in turn gives us $V''/H^2 \sim V''/V \ll 1$. Not only the potential turns out to be flat enough, but also this behaviour is forced to last for a sufficiently long amount of time.

1.4 Inflaton's perturbations

One of the primary goals of the inflationary paradigm was to explain the density perturbations $\frac{\delta\rho}{\rho}$ that we observe on large scales ($\gg H^{-1}$). In order to do so we have now to consider the quantum contribution to the inflaton field, which is the one that carries the space-dependence. This will be very useful also because in this work we would like to consider cosmological quantum correlators. Hence, we need to quantize the inflaton field and to study its dynamics. From Eq. 1.27, we can decouple two independent equations

$$\begin{cases} \ddot{\phi}_0 + 3H\dot{\phi}_0 = -V'(\phi_0) \\ \delta\ddot{\phi} + 3H\delta\dot{\phi} - \frac{\nabla^2}{a^2}\delta\phi = -V''(\phi_0)\delta\phi \end{cases} \tag{1.31}$$

where we expanded $V(\phi)$ at first order in perturbations. By going now in Fourier space, we have $\frac{\nabla^2}{a^2} \to -\frac{k^2}{a^2}$, where k is the wavenumber of considered mode function. In this way, by considering only superhorizon scales, we have $\lambda_{suph} \gg H^{-1}$, $k \ll aH$ and thus $\frac{\nabla^2}{a^2} \ll H^2$. Then, since $H \sim \partial/\partial t$ indicates the characteristic time-scale of the Universe's expansion, we can roughly approximate $3H\delta\dot{\phi} \sim 3H^2\delta\phi$. Therefore, the second equation of the system becomes $\delta\ddot{\phi} + 3H^2\delta\phi = -V''\delta\phi$. In other words, we are considering the inflaton's fluctuations as spatially constant on small scales, a process known as *coarse-graining*. By considering a de Sitter expansion for simplicity, and differentiating with respect to time the first line of Eq. 1.31 we obtain two identical equations for $\dot{\phi}_0$ and $\delta\phi$. This tells us that these equalities are not completely independent. In fact, one finally finds $\delta\phi(\vec{x},t) = -\delta t(\vec{x})\dot{\phi}_0(t)$[4], and hence

$$\phi(\vec{x},t) = \phi_0(t - \delta t(\vec{x})). \tag{1.32}$$

In fact, by expanding it to first order in δt we find the same equation. This is a quite interesting result, that shows us that ϕ will assume the same value of ϕ_0 in every spacetime's points, with the same time-evolution but at slightly different times. The inflaton's fluctuations translate themselves into little time-deviations on the history of each point of the Universe.

We proceed now by writing our fluctuations as a series of wave-functions. In order to do that, we should solve the Helmholtz equation $\nabla^2 Q_\kappa + k^2 Q_\kappa = 0$, where κ takes care of the curvature of the Universe. Since at the end of Inflation we saw that κ is almost perfectly null, no matter what initial Universe's curvature we choose, we can consider simply a spatially flat spacetime ($\kappa = 0$). Then, we simply have $Q_\kappa = e^{i\vec{k}\vec{x}}$ and our fluctuations' Fourier transform reads $\delta\phi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}}\delta\phi(\vec{k},t)$ with $\delta\phi(\vec{k},t) = \delta\phi^*(-\vec{k},t)$ being the wave-function of wave-mode k. Thanks to the previous result, we know that, at first order, wave-functions of different modes evolve independently in time, so that for a given k-mode the equation of motion exactly resembles the one for the real fluctuations. Namely,

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k = -V''\delta\phi_k. \tag{1.33}$$

[3] From requiring $\ddot{a} = \frac{d}{dt}(aH) = aH^2(1-\epsilon) > 0$ we see that the condition $\epsilon \ll 1$ fulfills the Inflation's requirements by itself

[4] In order to show that a certain level of dependence exists between the two lines of Eq. 1.31, we use the Wronskian. We have $W(\dot{\phi}_0, \delta\phi) = \det\begin{pmatrix} \dot{\phi}_0 & \delta\phi \\ \ddot{\phi}_0 & \delta\dot{\phi} \end{pmatrix} = \dot{\phi}_0\delta\dot{\phi} - \ddot{\phi}_0\delta\phi$. By considering the time derivative of the Wronskian, we see clearly that it eventually goes to zero, as $\dot{W} = -3HW$, i.e. $W \propto e^{-3Ht}$. The solution of $W = 0$ is given by [21] $\delta\phi(\vec{x},t) = -\delta t(\vec{x})\dot{\phi}_0(t)$

Passing now to the conformal time, the equation becomes

$$\delta\phi_k'' + 2\mathcal{H}\delta\phi_k' + k^2\delta\phi_k = -a^2\frac{\partial^2 V}{\partial\phi^2}\delta\phi_k, \tag{1.34}$$

where, here and after, $' = \partial/\partial\tau$ and $\mathcal{H}$ is the Hubble parameter expressed as a function of conformal time. By rewriting the field as $\tilde{\delta\phi_k} \equiv \delta\phi_k/a(t)$, we finally obtain

$$\delta\phi_k'' + \left[k^2 - \frac{a''}{a} + a^2\frac{\partial^2 V}{\partial\phi^2}\right]\delta\phi_k = 0. \tag{1.35}$$

To properly quantize these inflaton's perturbatinos, we proceed canonically. First of all, we need to define the fluctuations' conjugate momenta $\delta\pi(\tau,\vec{x}) = \frac{\partial\mathcal{L}}{\partial\partial_0\delta\phi} = a^2\partial_\tau\delta\phi(\tau,\vec{x})$. Then, we proceed by computing their Poisson brackets, hence we promote these to commutators (multiplied by i since we are in natural units) and the fields to operators. Firstly, we get $\{\delta\phi(\tau,\vec{x}),\delta\phi'(\tau,\vec{y})\} = a^{-2}\delta^{(3)}(\vec{x}-\vec{y})$ [22], while all the other brackets between these fields are vanishing. Then, we promote $\delta\phi(\tau,\vec{x})$ and $\delta\pi(\tau,\vec{x})$ to operators in Fourier space

$$\begin{aligned}\hat{\delta\phi}(\tau,\vec{x}) &= \int\frac{d^3k}{(2\pi)^3}e^{i\vec{x}\vec{k}}\left(u_k(\tau)\hat{a}(\vec{k}) + u_k^*(\tau)a^\dagger(-\vec{k})\right)\\ \hat{\delta\pi}(\tau,\vec{x}) &= a^2\int\frac{d^3k}{(2\pi)^3}e^{i\vec{x}\vec{k}}\left(u_k'(\tau)\hat{a}(\vec{k}) + u_k^*(\tau)a^\dagger(-\vec{k})\right).\end{aligned} \tag{1.36}$$

Here, as usual, $a_{\vec{k}}$, $a^\dagger_{-\vec{k}}$ are the annihilation and creation operators respectively, where $a_{\vec{k}}|0\rangle = 0$ ($\langle 0|a^\dagger_{-\vec{k}} = 0$), with $|0\rangle$ the vacuum state. By inverting the last equations, one could define the commutation relations between the ladder operators, which reads

$$\left[a_{\vec{k}}, a^\dagger_{-\vec{k}'}\right] = (2\pi)^3\delta^{(3)}\left(\vec{k}+\vec{k'}\right) \quad \text{and} \quad \left[a_{\vec{k}}, a_{\vec{k}'}\right] = 0 = \left[a^\dagger_{-\vec{k}}, a^\dagger_{-\vec{k}'}\right]. \tag{1.37}$$

By promoting now our Poisson brackets, we find

$$\begin{aligned}\left[\hat{\delta\phi}(\tau,\vec{x}), \hat{\delta\phi}'(\tau,\vec{y})\right] =& \int\frac{d^3kd^3k'}{a^2(2\pi)^6}e^{i\vec{x}\vec{k}}e^{i\vec{y}\vec{k'}}\left[(u_k a_{\vec{k}} + u_k^* a^\dagger_{-\vec{k}})(u'_{k'}a_{\vec{k}'} + u'^*_{k'}a^\dagger_{-\vec{k}'}) - (u'_{k'}a_{\vec{k}'} + u'^*_{k'}a^\dagger_{-\vec{k}'})(u_k a_{\vec{k}} + u_k^* a^\dagger_{-\vec{k}})\right]\\ =& a^{-2}\int\frac{d^3kd^3k'}{(2\pi)^6}e^{i\vec{x}\vec{k}}e^{i\vec{y}\vec{k'}}\left[u_k u'^*_{k'}\left[a_{\vec{k}}, a^\dagger_{-\vec{k}'}\right] - u'_{k'}u_k^*\left[a_{\vec{k}'}, a^\dagger_{-\vec{k}}\right]\right]\\ =& a^{-2}i\delta^{(3)}(\vec{x}-\vec{y}),\end{aligned} \tag{1.38}$$

where the last passage is valid only if our mode-functions obey the normalization condition

$$u_k^*(\tau)u'_\tau - u_k(\tau)u'^*_k(\tau) = -i. \tag{1.39}$$

Once the operators are defined, we have to specify what they act on. Besides, the physical meaning of an operator is itself unclear until the states' space in which it can act is not defined. This topic is one of the main caveat of quantum field theory in curved spacetime. The first road one usually follows to define a space of all the possible states is to define the vacuum state, thereby building up the entire space by letting the creation operator to act on it. However, whereas it is quite an easy task in Minkowski space, in curved spacetime it is not. Normally, one could find the vacuum state as the one with the lowest energy. But in curved space-time, the Hamiltonian is in general time dependent. Also, as long as $\hat{\delta\phi}(\tau,\vec{x})$ remains the same, we could in principle chose the mode-functions to have different forms. This of course would modify $a^{(\dagger)}$ and hence the vacuum's definition. In spite of all these difficulties, we can overcome them by requiring that at very small scales the space-time approaches the Minkowski one. As a consequence of the equivalence principle, we can in fact require that, locally, every point of any space-time we chose resembles the Minkowski space-time. This of course would imply an uniqueness of the vacuum state, just as in ordinary quantum field theories. The so obtained vacuum is called Bunch-Davies vacuum state [22, 23], and it implies a plane-wave solution for out mode-function

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}. \tag{1.40}$$

Now, we would like to properly solve the equation of motion for our field's fluctuations. We start from Eq.1.35 which represents nothing but the equation of motion for a harmonic oscillator with a frequency that changes in time due to the expansion of the Universe. Before proceeding in finding the most general solution, we consider for simplicity the case of a massless field in de Sitter (dS). In fact, this first approximation allows us to see how inflaton's fluctuations can freeze-out becoming classical perturbations of the spacetime. By choosing a massless field in dS, $\partial^2/\partial\phi^2 = m_\phi^2 = 0$ and we have a constant H. From the latter it follows $a \propto e^{Ht}$, and hence, since $d\tau = \frac{dt}{a}$, one finds $\tau = -\frac{1}{aH}$ (with a proper choice of integration's constants). Thus, it is possible to write $\frac{a''}{a} = \frac{2}{r_H^2}$[5]. The equation of motion then reads

$$\delta\phi_k'' + \left[k^2 - \frac{2}{r_H^2}\right]\delta\phi_k = 0, \tag{1.41}$$

where we can put $|\delta\phi_k| = |u_k|$. To show how quantum fluctuations inside the horizon freeze-out becoming classical, we just have to restrict ourselves to two limits. In the so called *subhorizon regime* we expect scales of dimensions $\lambda_{sub} \ll r_H$, which implies $k^2 \gg \frac{a''}{a}$ and hence, the above equation reduces to $\delta\phi_k''(\tau) + k^2\delta\phi_k(\tau) = 0$. On the contrary, by considering the opposite scenario, in the *superhorizon regime* we have scales $\lambda \gg r_H$, so that $k^2 \ll \frac{a''}{a}$. Therefore, well outside the horizon we will have $\delta\phi_k''(\tau) - \frac{a''}{a}\delta\phi_k(\tau) = 0$. The first equation is now easy to solve with the initial condition dictated by our previous vacuum choice. Being a harmonic oscillator's equation with a constant frequency, its solution is a superposition of two plane-waves. By requiring it to resemble the Bunch-Davies solution, we find $u_k^{sub} = \frac{1}{\sqrt{2k}}e^{-ik\tau}$, and hence $\delta\phi_k^{sub} = \frac{aH}{\sqrt{2k^3}}e^{-ik\tau}$, where we inverted the relation $\tilde{\delta\phi}_k = \delta\phi_k/a$. On the other hand, on superhorizon scales, we have a general solution $u_k(\tau) = B(k)a(\tau) + A(k)a(\tau)^{-2} \approx B(k)a(\tau)$ as the second, decaying term gets washed out by Inflation. In terms of the field's fluctuations then $|\delta\phi_k(\tau)|^{sup} = B(k)$. In the end, requiring the continuity of the solution, we force $B(k) = \frac{aH}{\sqrt{2k^3}}$. Inside the horizon, by looking at macroscopic spacetime regions, the fluctuations are averaged out due to their oscillatory behaviour. On superhorizon scales, microphysics does not work anymore and hence the perturbations cannot annihilate between one another. In other words, if inside a definite Hubble volume the continuous creation and annihilation of $\delta\phi_k$ guarantees a vev $\langle\delta\phi_k^{sub}\rangle = 0$, as the Inflation takes place this Hubble radius shrinks, enclosing less and less physical space. Therefore, the created fluctuations outside the shrinking horizon are no more in causal connection between themselves, and get frozen-out. This is the mechanism that consents the creation of classical perturbations from quantum fluctuations. Proceeding now in finding the exact (massless) solution in dS, we start again with

$$u_k''(\tau) + \left[k^2 - \frac{2}{\tau^2}\right]u_k(\tau) = 0. \tag{1.42}$$

By rewriting $u_k(\tau) = \tau U_k(\tau)$, we have

[5] In perfect dS, H is constant. We have: $\frac{a''}{a} = \frac{d}{dt}(a\dot{a}) = \frac{d}{dt}(a^2H) = 2a^2H^2 = \frac{2}{r_H^2} = \frac{2}{\tau^2}$, where also the last equality holds only in a perfect dS Universe

$$U_k''(\tau) + \frac{2U_k(\tau)}{\tau} + U(\tau)_k \left[k^2 - \frac{2}{\tau^2}\right] = 0. \tag{1.43}$$

If we then change the variable over which we are derivating putting $y = \tau K$, we find the first-order ($l = 1$) spherical Bessel's function

$$k^2 \frac{d^2}{dy^2} U_k(y) + 2k \frac{d}{dy} U_k(y) + U_k(y) \left[1 - \frac{2}{y^2}\right] = 0, \tag{1.44}$$

which is resolved by a superposition of the two Hankel's functions

$$J_l(y) = (-y)^l \left(\frac{1}{y}\frac{d}{dy}\right)^l \frac{\sin y}{y} \qquad \text{and} \qquad J_l(y) = -(-y)^l \left(\frac{1}{y}\frac{d}{dy}\right)^l \frac{\cos y}{y}. \tag{1.45}$$

Hence, we can write our solution as a linear combination of these functions: $U_k(y) = c_1 J_l(y) + c_2 J_l(y)$. In our case, we have $l = 1$, from which it follows

$$U_k(y) = c_1 \left(-\frac{\cos y}{y} + \frac{\sin y}{y^2}\right) - c_2 \left(\frac{\sin y}{y} + \frac{\cos y}{y^2}\right). \tag{1.46}$$

Thus, by using the exponential form of sinusoidal functions

$$u_k(\tau) = \frac{1}{2k^2\tau^2} \left[e^{ik\tau} \left(-c_1 k\tau - ic_1 + ic_2 k\tau - c_2\right) + e^{-ik\tau} \left(-c_1 k\tau + ic_1 - ic_2 k\tau - c_2\right)\right]. \tag{1.47}$$

In general, to solve a nth order differential equation, one needs to specify all the n integration's constants. Thus, in order to do that, we need n-independent equation. In this case, the second constraint we need can be found in the initial conditions. In fact, we saw that at $t \to -\infty$, the solution has to resembles the usual wave-plane solution of canonical quantum field theories in Minkowski space. Hence, by looking at $u_k(\tau)^{sub}$, we see that for asymptotically small wavelengths ($|k\tau| >> 1$), only the second term of Eq.1.47 must survive. This gives us $c_1 = ic_2$ from the first term. Then, by imposing our Bunch-Davies initial condition to the second term

$$\lim_{|k\tau| \to +\infty} \frac{1}{2k^2\tau^2} [-c_1 k\tau + ic_1 - c_1 k\tau + ic_1] = \frac{1}{\sqrt{2k}}, \tag{1.48}$$

we get $c_1 = -\sqrt{\frac{k}{2}}\tau$, $c_2 = i\sqrt{\frac{k}{2}}\tau$. In the end, for the re-scaled mode-function $\tilde{u_k}(\tau) = u_k(\tau)/a$ we find

$$\begin{aligned} \tilde{u_k}(\tau) =& \frac{e^{-ik\tau}}{a\sqrt{2k}} \left(1 - \frac{i}{k\tau}\right)\Big|_{dS} = \frac{aH}{a\sqrt{2k^3}} (i - k\tau) e^{-ik\tau} \\ =& \frac{iH}{\sqrt{2k^3}} (1 + ik\tau) e^{-ik\tau}. \end{aligned} \tag{1.49}$$

From now on, we will use only the mode-function we have just computed. Therefore, we rename it as $\tilde{u_k}(\tau) = u_k(\tau)$. We see that the last equation resembles both the limits we briefly sketched above. On superhorizon scales, it reduces to the constant value $|u_k| \approx H/\sqrt{2k^3}$, while for subhorizon wavelengths it gives us the usual plane-wave solution we wanted. Now that we have found the exact solution in a perfect dS Universe, one could extend this result to the case of massive fields in a quasi-dS spacetime. The most general result, accounting for a non-vanishing inflaton's mass, is given by solving

$$u_k(\tau) + \left[k^2 - \frac{2}{\tau^2}\left(1 + \frac{3}{2}\epsilon\right) + \frac{m_\phi^2}{H^2\tau^2}\right] = 0, \tag{1.50}$$

where in this case the conformal time has been approximated as $\tau \approx -\frac{1}{aH(1-\epsilon)}$ and the inflaton's mass comes as an effective term from the slow-roll parameter $\eta_V = \frac{1}{3}\frac{m_\phi^2}{H^2} \ll 1$. Once solved, the solution shows a typical dependence on $k^{-\epsilon+\eta_V}$ (Appendix A.3). This unique feature of inflationary models is quite important, as it reflects itself in the dependence on the momentum k of the (scalar) power spectrum.
However, up to now we considered only scalar perturbations in our model, owing to the fact that no matter what metric we start with, the Universe will be described by a flat FLRW space-time. Even by neglecting possible quantum fluctuations of the metric, according to Eistein's equations, the inflaton's perturbations produce perturbations in the spacetime's geometry by warping the stress-energy tensor. In turn, these geometric fluctuations back-react affecting the inflaton's dynamics, and so on and so forth. For this reason, we cannot think of inflaton's perturbations without looking also at the metric's ones. In order to account for these geometrical contributions, we need to perturb the equation of motion we found for the inflaton's field (Eq.1.27). Before doing that, however, we need to introduce the topic of cosmological perturbations, and the approaches one can use to study them.

1.5 Cosmological perturbations

As we saw, the inflaton's field produces fluctuations while following its dynamics. These produce a back-reaction from the gravitational sector, which affect their own evolution. Thus, in order to build a consistent model to explain, for instance, primordial perturbations, we need to take into account also the geometrical contributions that can arise. Besides, since in this thesis a particular regard will be given to gravitons, which are nothing else than metric perturbations, we must consider this topic before proceeding. Before passing to analyze the cosmological perturbations' contribution to our inflationary model, we first need to solve aa puzzle. Treating perturbations, one usually deals with two different spacetimes. One is given by the unperturbed background, which in our case will be the FLRW one, while the other represents the physical and perturbed Universe. In fact, if we want to study a given quantity f, the perturbations of that will be $\Delta f = f - f_0$, where f_0 is our reference's value, expressed in a Universe that we consider as known. We call these two spacetimes $\mathcal{M}$ and $\mathcal{M}_0$. We need to provide a relation between these two "worlds", since the comparison between, e.g., two tensors or vectors can be done only if they are expressed in the same spacetime. A *gauge choice* provides a one-to-one map between the x_0^μ and the x^μ points, where the subscript 0 belongs to unperturbed quantities. A change between two gauge choices is called a *gauge transformation*. This correspondence should not be misguided with an usual coordinates' transformation. In fact, besides inducing of course a coordinate transformation in $\mathcal{M}$, a gauge transformation changes also the point in $\mathcal{M}_0$, corresponding to a certain point in the physical space. Thus, under a gauge transformation, the perturbation of a quantity that undergoes a change of coordinates as a scalar might not be invariant [24]. Since in principle we do not have a way to discriminate a gauge's choice over another, the so called *gauge issue*, arises from to the fact that this freedom we have in choosing the $\mathcal{M}_0$-to-$\mathcal{M}$ map implies an ambiguity in the definition of the perturbation. For example, by choosing two different gauges ψ and ϕ, for the same quantity f we might get $\Delta f = f - f_0$ ($f_0 \xrightarrow{\psi}$f) and $\tilde{\Delta f} = \tilde{f} - f_0$ ($f_0 \xrightarrow{\phi} \tilde{f}$) with $\Delta f \neq \tilde{\Delta f}$. Therefore, we need to know how to pass from a gauge to another, in order to relate two differently-transformed quantities.
We start by considering a general change of coordinates $x^\mu \to x'^\mu = x^\mu - \xi^\mu$. According to that, scalar, vector and tensor functions will then transform as $\phi'(x') \equiv \phi(x)$, $V'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu}V_\nu(x)$ ($V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}V^\nu$), and $T'^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}(x') = \frac{\partial x'^{\alpha_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\alpha_n}}{\partial x^{\nu_n}} \frac{\partial x^{\mu_1}}{\partial x'^{\beta_1}} \cdots \frac{\partial x^{\mu_m}}{\partial x'^{\beta_n}} T^{\nu_1 \ldots \nu_n}_{\mu_1 \ldots \mu_m}(x)$ respectively. Now, we parametrize our spacetimes by introducing a parameter κ, such that, for $\kappa = 0$, $\mathcal{M}_\kappa = \mathcal{M}_0$ is our idealized FLRW Universe. By identifying then a point $P \in \mathcal{M}_0$, we choose a gauge ψ which maps P into $O = \psi_\kappa(P) \in \mathcal{M}_\kappa$. However, we can also choose another gauge ϕ such that $O = \phi_\kappa(Q)$ ($Q \in \mathcal{M}_0$). Being a gauge transformation invertible, we can write $Q = \phi_\kappa^{-1}(O) = \phi_\kappa^{-1}(\psi_\kappa(P)) = \Psi_\kappa(P)$. Now, we fix a coordinate reference on the ideal spacetime x^μ, and introduce a vector ξ^μ which defines κ as the parameter describing the path along which we pass from $x^\mu(P)$ to $x^\mu(Q)$. Namely, $\xi^\mu = \frac{dx^\mu}{d\kappa}$, so that we can write, at first order in κ, $x^\mu(Q) = x^\mu(P) + \kappa\xi^\mu(x^\mu(P))$. Then, our Ψ is such

that $x^\mu(P) \xrightarrow{\Psi_\kappa} x^\mu(P) + \kappa\xi^\mu(\kappa = 0)$. We see therefore that a gauge transformation is carried by the vector ξ^μ. However, since we are still in the same geometry, we could have find the relation between the points P and$Q$ simply by introducing a change of coordinate $x^\mu \to y^\mu$ in a way that $y^\mu(Q) \equiv x^\mu(P)$. For what we saw above, a change of coordinate, again at first order, can be written as $y^\mu(Q) = x^\mu(Q) - \kappa\xi^\mu(x(P))$. Thus, by using the expression of our active coordinate transformation Ψ and Taylor-expanding the whole result up to first order in κ, we have

$$y^\mu(Q) = x^\mu(Q) - \kappa\xi^\mu\left(x^\mu(Q) - \kappa\xi^\mu(\kappa = 0)\right) \approx x^\mu - \kappa\xi^\mu(x(Q)). \tag{1.51}$$

As we see, this represents a simple coordinate transformation. Aiming to provide how tensors transform under a gauge transformation, we consider now two tensor fields Z and $\tilde{Z}$ whose contravariant components are respectively Z^μ and $\tilde{Z}^\mu$ in the x-coordinate system. By imposing now that the x-components of the second field in P are equal to the y-components of the first one in Q we find $\tilde{Z}^\mu(x(P)) = Z'^\mu(y(Q)) = \frac{\partial y^\mu}{\partial x^\nu}\big|_Q Z^\nu(x(Q))$. From Eq.1.51 it thus follows

$$\begin{aligned}
\tilde{Z}^\mu(x(P)) =& \frac{\partial}{\partial x^{nu}}\left[x^\nu(Q) - \kappa\xi^\mu(Q)\right]\Big|_Q Z^\nu(x(Q)) = \delta^\mu_\nu Z^\nu(x(Q)) - \kappa\frac{\partial\xi^\mu}{\partial x^\nu}\Big|_Q Z^\nu(x(Q)) \\
=& Z^\mu(x(P)) + \kappa\frac{\partial Z^\mu}{\partial x^\nu}\Big|_P \xi^\nu(P) - \kappa\frac{\partial\xi^\mu}{\partial x^\nu}\Big|_P\left(Z^\nu + \frac{\partial Z^\nu}{\partial x^\rho}\Big|_P \kappa\xi^\rho(P)\right) \\
\approx& Z^\mu(x(P)) + \kappa\frac{\partial Z^\mu}{\partial x^\nu}\Big|_P \xi^\nu(P) - \kappa\frac{\partial\xi^\mu}{\partial x^\nu}\Big|_P Z^\nu \\
=& Z^\mu(x(P)) + \kappa\mathcal{L}_\xi Z^\mu(P),
\end{aligned} \tag{1.52}$$

where we applied the active transformation $x^\mu(Q) = X^\mu(P) + \kappa\xi^\mu(P)$ to express $\xi^\mu(Q)$, and expanded $Z^\mu(Q)$ at first order in κ. Finally, we see that the last equality compares two different vector fields in the same point $P \in \mathcal{M}_0$. Here $\mathcal{L}_\xi Z^\mu = \frac{\partial Z^\mu}{\partial x^\nu}\xi^\nu - \frac{\partial\xi^\mu}{\partial x^\nu}Z^\nu$ is the Lie derivative of Z^μ along ξ^μ[6]. Therefore, our gauge transformation acts in adding a Lie-derivative term to our initial vector field Z^μ. By following the same line of reasoning, we find that scalar, and tensor fields are gauge-transformed in the same way as vector fields, with a Lie derivative respectively given by [25]

$$\begin{aligned}
\mathcal{L}_\xi f = f_{,\mu}\,\xi^\mu \quad \text{and} \quad \mathcal{L}_\xi T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_m} =& T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_m\;,\mu}\xi^\mu + \\
& - T^{\mu\ldots\alpha_n}_{\beta_1\ldots\beta_m}\xi^{\alpha_1}{}_{,\mu} - \cdots - T^{\alpha_1\ldots\mu}_{\beta_1\ldots\beta_m}\xi^{\alpha_n}{}_{,\mu} + \\
& + T^{\alpha_1\ldots\alpha_n}_{\mu\ldots\beta_m}\xi^{,\mu}_{\beta_1} + \cdots + T^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\mu}\xi^{,\mu}_{\beta_m}.
\end{aligned} \tag{1.53}$$

In this way, we solved the gauge issue, since now we are able to connect a (scalar, vector, or tensor) field expressed through a gauge choice to the same field underwent a different gauge map. In fact, we have $\tilde{\Delta f} = \tilde{f} - f_0$, $\Delta f = f - f_0$ where $\tilde{f} = f_0 + \tilde{\Delta f} = f + \mathcal{L}_\xi f = f_0 + \Delta f + \mathcal{L}_\xi f$. Hence, in the end, $\tilde{\Delta f} = \Delta f + \mathcal{L}_\xi f_0$, where we restrict ourselves to linear order in ξ.

1.6 Perturbations and gauges

Once understood how we can relate quantities expressed in different gauges, we can proceed to perturb our fictitious FLRW metric. Also, we want to explore some gauge choice's possibilities, and introduce a very useful alternative that we are going to use vastly in the following sections. To introduce small perturbations to our usual metric, we have to discriminate between scalar, vector, and tensor perturbations. In doing so, we decompose the metric's perturbations in these three categories, obtaining

[6] In curved spacetime, the Lie derivative of a function f is again its directional derivative along a certain vector field. However, in this case the usual derivative has to be changed with the covariant one. Considering again vector fields, one has $\mathcal{L}_u w^\alpha = u^\beta\nabla_\beta w^\alpha - w^\beta\nabla_\beta u^\alpha$ [25]

$$
\begin{aligned}
g_{00} =& - a^2(\tau)\left[1 + 2\sum_{r=1}^{+\infty}\frac{1}{r!}\psi^{(r)}(\vec{x},\tau)\right] \\
g_{0i} =& a^2(\tau)\sum_{r=1}^{+\infty}\frac{1}{r!}\omega_i^{(r)}(\vec{x},\tau) \\
g_{ij} =& a^2(\tau)\left\{\left[1 - 2\sum_{r=1}^{+\infty}\frac{1}{r!}\varphi^{(r)}(\vec{x},\tau)\right]\delta_{ij} + \sum_{r=1}^{+\infty}\frac{1}{r!}\chi_{ij}^{(r)}(\vec{x},\tau)\right\}.
\end{aligned}
\tag{1.54}
$$

Here, r represents the perturbation's order, ψ and ω_i are the Lapse and Shift functions, while the two distinct terms in the spatial part of the metric are respectively the non-vanishing and vanishing-trace tensor perturbations. Specifically, the traceless $\chi_{ij}(\vec{x},\tau)$ term contains also gravitational waves. We note that here the metric is written as $g_{\mu\nu} = g_{\mu\nu}^{FLRW} + \delta g_{\mu\nu}$. Now, thanks to the Helmholtz theorem we can decompose a vector in its divergenceless and irrotational components as $\omega_i = \partial_i\omega^{\parallel} + \omega_i^{\perp}$ ($\partial^i\omega_i^{\perp} = 0$). Also tensor perturbations can be decomposed as $\chi_{ij} = D_{ij}\chi^{\parallel} + \chi_{i,j}^{\perp} + \chi_{j,i}^{\perp} + \chi_{ij}^{T}$. Analogously, the first term acts as the scalar (traceless $D_{ij} = \partial_i\partial_j - (1/3)\delta_{ij}\nabla^2$) part, the two in the middle shows the divergenless vector part, and the latter is given by the transverse and traceless tensor contribution. This indeed represents gravitational waves propagating in a FLRW Universe.
Since our aim is to perturb EFEs, thereby obtaining a more consistent model of Inflation that takes into account the interplay between geometry and evolution of the inflaton's field, we have to perturb the stress-energy tensor too. To do so, we can approximate our tensor as a perfect fluid's one: $T_{\mu\nu} = (\rho+p)u_\mu u_\nu + p g_{\mu\nu}$, with $T_{\mu\nu} = T_{\mu\nu}^{(0)} + \delta T_{\mu\nu}$, where the former is the usual stress-energy tensor for a perfect fluid at rest. So we proceed perturbing ρ, p, and u:

$$
\begin{aligned}
\rho(\vec{x},\tau) =& \rho^{(0)}(\tau) + \sum_{r=1}^{+\infty}\frac{1}{r!}\delta^{(r)}\rho(\vec{x},\tau) \\
p(\vec{x},\tau) =& p^{(0)}(\tau) + \sum_{r=1}^{\infty}\frac{1}{r!}\delta^{(r)}p(\vec{x},\tau) \\
u^\mu(\vec{x},\tau) =& \frac{1}{a}\left(\delta_0^\mu + \sum_{r=1}^{\infty}\frac{1}{r!}V^\mu(\vec{x},\tau)\right).
\end{aligned}
\tag{1.55}
$$

Again, we note that the first term for each decomposition represents the analogous quantities in a FLRW metric. In fact, isotropy and homogeneity guarantee the unperturbed density and pressure to depend upon time only. In the latter equality, δ_0^μ/a represents the velocity of the cosmic fluid at rest, while the perturbation's term expresses the (generally) non-vanishing velocities of fluid's elements. For the latter, it is possible to write $V^0 = -\psi$, $V^i = \partial^i V^{\parallel} + V^{i\perp}$[7]. There are two ways that can be followed to deal with the gauge issue. The first one is to choose a gauge, make there certain computations, and eventually go back to the original spacetime through the gauge-transformation laws we found. The second approach is to find a basis of gauge-invariant quantities. We start by exploring the first possibility.
By applying now a general linear gauge transformation $\xi_\mu(\vec{x},\tau) = (\alpha, \partial_i\beta + d_i)$, we have $\tilde{g}_{\mu\nu} = g_{\mu\nu}g_{\mu\nu;\alpha}\xi^\alpha + g_{\alpha\nu}\xi_\mu^{;\alpha} + g_{\mu\alpha}\xi_\nu^{;\alpha} = g_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu}$, as the covariant derivative of the metric tensor vanishes and we can consider a flat FLRW metric. However, we can use a simpler trick to transform the several perturbations we introduced in the metric tensor [26]. We can apply an infinitesimal gauge coordinate transformation $\tilde{x}^\mu = x^\mu + \xi^\mu$. Through that, the line element is left unchanged $\tilde{ds}^2 = ds^2$. Also, by considering first order perturbations only, the different perturbations are decoupled from each other[8], and can be studied separately. In this context, looking at the 00-component we find

[7] It can be shown that at linear order this is the case by computing $u_\mu u^\mu = -1$

[8] This can be easily seen by looking at how the metric tensor transforms. At linear order we can in fact approximate $g_{\mu\nu} = g_{\mu\nu}^{(0)}$

$$\tilde{a}^2(\tilde{\tau})\left(1+2\tilde{\psi}\right)d\tilde{\tau}^2 = a^2(\tau)\left(1+2\psi\right)d\tau^2, \tag{1.56}$$

with $\tilde{a}^2(\tilde{\tau}) = \tilde{a}^2(\tau+\xi^0) \approx a^2(\tau) + 2a\frac{\partial a}{\partial \tau}\xi^0$ and $\tilde{d\tau} = \frac{\partial(\tau+\xi^0)}{\partial x^\mu}dx^\mu = (1+\xi'^0)d\tau$. Therefore

$$\begin{aligned} a^2(\tau)d\tau^2 + 2a^2(\tau)d\tau^2\psi =& (a^2(\tau) + 2aa'\xi^0)\left(1+2\tilde{\psi}\right)(d\tau^2 + 2d\tau^2\xi'^0) \\ \approx& a^2(\tau)d\tau^2 + 2a^2(\tau)d\tau^2\tilde{\psi} + 2a^2(\tau)d\tau^2\frac{a'}{a}\xi^0 + 2\xi'^0a^2(\tau)d\tau^2, \end{aligned} \tag{1.57}$$

which implies $\tilde{\psi} = \psi - \frac{a'}{a}\xi^0 - \xi'^0$.
Now, by following the same approach, it is possible to find the expressions for all the other introduced perturbations, which read

$$\begin{aligned} &\tilde{\psi} = \psi - \alpha' - \frac{a'}{a}\alpha && \tilde{\omega}_i = \omega_i - \alpha_{,i} + \beta'_{,i} + d'_i \\ &\tilde{\varphi} = \varphi - \frac{1}{3}\nabla^2\beta - \frac{a'}{a}\alpha && \tilde{\chi}_{ij} = \chi_{ij} + 2D_{ij}\beta + d_{i,j} + d_{j,i} \\ &\tilde{\delta\rho} = \delta\rho + \rho'^{(0)}\alpha && \tilde{V}^i = V^i - \beta'^{,i} - d^i \\ &\tilde{V}_\parallel = V_\parallel - \beta' && \tilde{V}^{i\perp} = V^{i\perp} - d^i, \end{aligned} \tag{1.58}$$

where we decomposed v^i by means of Helmholtz theorem. Once α, β and d^i are specified, the gauge is chosen. Just from these results, one can see that a proper gauge's choice might be particularly useful to solve a specific problem. In fact, by changing the gauge-variables we introduced, we can get rid of several terms in the perturbed EFEs, thereby getting a simplification of the equations one is using. Some example of notable gauges are the *Poisson*, *Synchronous*, and the *Uniform curvature gauge.*
The Poisson gauge is specified by the constraints $\tilde{\omega^\parallel} = 0, \tilde{\chi}^\parallel = 0$, and $\tilde{\chi_i^\perp} = 0$. It corresponds to a gauge that closely resembles a Newtonian treatment of gravity. In order to see this, one has to perturb the 00 and the $0i$ EFEs. Then, by applying this gauge, it follows $\nabla^2\Phi_H = -4\pi Ga^2\epsilon_m$, which is very similar to the Poisson equation in Newtonian gravity. Here, $\Phi_H = -\varphi - \frac{1}{6}\nabla^2\chi^\parallel + \frac{a'}{a}\omega^\parallel - \frac{a'}{2a}\chi'^\parallel$ is one of the two *gauge invariant Bardeen gravitational potentials* [24] we will introduce in the next section, and $\epsilon_m = \delta\rho + \rho'_0(V^\parallel + \omega^\parallel)$ is the perturbations' energy density. In the Synchronous gauge, one has $\tilde{\psi} = 0$, $\tilde{\omega}^\parallel = 0 = \tilde{\omega}_i^\perp$, from which it simply follows $\tilde{\delta g}_{00} = \tilde{\delta g}_{0i} = 0$. Namely, in this gauge only the purely spatial part of the metric is perturbed, and hence all the observers share the same proper time. The uniform curvature gauge is specified by requiring $\tilde{\varphi} = 0$, $\tilde{\chi}^\parallel = 0 = \tilde{\chi}_i^\perp$. It is defined by selecting τ-constant hypersurfaces where the spatial metric tensor is perturbed only by tensors, i.e. only by χ_{ij}^T. This last gauge will be useful to define a very important quantity that we are going to use intensively in this thesis: the scalar curvature perturbation on uniform density hypersurfaces.

1.7 Gauge-invariant quantities

As we pointed out, the covariance of General Relativity implies a complete freedom in the gauge's choice, until we can find a gauge transformation that brings us from our initial background to the chosen gauge. This means that only gauge-invariant quantities have an intrinsic physical meaning. Of course, this implies that the other gauge-dependent quantities acquire a physical significance only when, under a certain gauge choice, they can be related to some gauge-independent quantity [24]. We start the gauge-independent approach by focusing on scalar

in all the $\sim\xi$ terms. In this way, computing how the 00-component transforms, for instance, only the corresponding 00-terms are considered. Hence, there is no mixing between different perturbations. This is not true anymore when the procedure is generalized to greater orders.

quantities. It is possible to obtain some gauge-independent variables from both the metric and the stress-energy tensors. Let us first consider those arising from the geometric part of EFEs. From Eq.1.54, we see that through Helmholtz's theorem four scalar perturbations can be identified. That noticed, by specifying a gauge we can fix two scalar functions. Thus, it follows that only two gauge-independent scalar quantities can be constructed from the metric tensor. These are the so called *Bardeen gravitational potentials*, which read

$$\begin{aligned} 2\Psi_A =& 2\psi + 2\omega'^{\|} + 2\frac{a'}{a}\omega^{\|} - \left(\chi''^{\|} + \frac{a'}{a}\chi'^{\|}\right) \\ 2\Phi_H =& -2\varphi - \frac{1}{3}\nabla^2\chi^{\|} + 2\frac{a'}{a}\omega^{\|} - \frac{a'}{a}\chi'^{\|}. \end{aligned} \tag{1.59}$$

In the zero-shear gauge, the shear scalar $\sigma = -\omega^{\|} + \frac{1}{2}\chi^{\|}$ vanishes. In these so defined hypersurfaces, we have $\Psi_A = \psi$, and $\Phi_H = -\hat{\varphi} = -\varphi - \frac{1}{6}\nabla^2\chi^{\|}$. From these relations, we can see that the Bardeen's gravitational potentials identify the amplitude of the perturbation in the lapse function and the amount of deformation of the perturbed hypersurfaces respectively. We pass now to the energy-matter sector (Eq.1.55), where we find [24]

$$2V_s = 2v^{\|} + \chi'^{\|}, \tag{1.60}$$

$$\epsilon_m = \delta\rho + \rho_0'(v^{\|} + \omega^{\|}), \tag{1.61}$$

$$2E_g = 2\delta\rho + \rho_0'\left(2\omega^{\|} - \chi'^{\|}\right). \tag{1.62}$$

As done above, to understand the physical meaning of these quantities, we chose a useful gauge. From the former, by choosing the zero-shear gauge, we get $V_s = v^{\|} + \omega^{\|}$ where ω represents the velocity of an observer moving normally with respect to τ-constant hypersurfaces, and $v^{\|}$ is the scalar perturbation to the matter's velocity. Thus, in the Poisson slicing, V_s corresponds to the magnitude of the matter's proper velocity. By looking at the second equation, and going in the comoving gauge ($v^{\|} + \omega^{\|} = 0$), we find $\epsilon_m = \delta\rho$. Therefore, it follows that ϵ_m can be seen as the most natural definition of the energy-density perturbation of the matter. Another way to see that, is to consider how we defined the comoving gauge. We saw that the condition $v^{\|} + \omega^{\|} = 0$ corresponds indeed to require a vanishing matter's momentum density, which is the case in its rest frame. The last gauge invariant quantity defines another expression for the energy density perturbation. Going in fact to the zero-shear gauge, we find again $E_g = \delta\rho$.

Let us now discuss the vector gauge invariant quantities. By looking only to geometrical contributions at first, we have two vectors $\omega_i^{\perp}$,$\chi_i^{\perp}$ and a vector-condition given by a gauge choice ($d_i = \xi_i - \partial_i\beta$). Thus, we are left with a single possible gauge invariant vector

$$\psi_i = \omega_i^{\perp} - \chi_i'^{\perp}, \tag{1.63}$$

which is related to the amplitude of the vector part of the geometric shear tensor $\sigma_{\mu\nu}$[9], and describes effect related to the frame-dragging effect in GR. As in the case of scalar quantities from the matter sector, we can identify two different forms for the matter's velocity perturbation

$$V_s^i = v^{i\perp} + \chi'^{i\perp} \qquad \text{and} \qquad V_c^i = v^{i\perp} + \omega^{i\perp}, \tag{1.64}$$

[9]The shear tensor $\sigma_{\mu\nu}$ describes the shear viscosity that can form between two subsequent layers of matter's fluid. It is given by $\sigma_{\mu\nu} = \frac{1}{2}h_\mu^\alpha h_\nu^\beta\left(u_{\alpha;\beta} + u_{\beta;\alpha}\right) - \frac{1}{2}\Theta h_{\mu\nu}$, where Θ represents the isotropic expansion of the hypersurface we are considering (in FLRW, $\Theta = 3H$), and $h_{\mu\nu} = g_{\mu\nu} + u_{mu}u_\nu$ is the projection tensor on spatial hypersurfaces orthogonal to u_μ

which represent respectively the shear's amplitude and the vorticity of the matter velocity field. In particular, as we will briefly explain in what follows, V_{ic} is related to the angular velocity of a fluid element.
Finally, the tensor perturbations at linear order are all gauge invariant. This can be see by considering the gauge transformation we constructed at the beginning of the section. In fact, it does not contribute with any tensor contribution.

1.8 Perturbed Einstein's Field Equations

In order to study the evolution of both the metric and all the other gauge invariant quantities, we need to consider the perturbed EFEs. Thus, we have to perturb them, together with the Bianchi identity, which encodes the equations of motion for the matter described by $T^{\mu\nu}$. We start by perturbing at linear order only the geometrical part of the EFEs, considering a null source of gravity. Since the entire computation is quite long, and lies outside the purposes of this thesis, we just report here the main steps and final results. The first elements we need to perturb are the Chrystoffel's symbols. By considering only linear order contributions in the perturbations, we find

$$\delta\Gamma^{\alpha(1)}_{\beta\gamma} = \frac{1}{2}\delta g^{\alpha\tau(1)}\left(g_{\beta\tau,\gamma} + g_{\gamma\tau,\beta} - g_{\beta\gamma,\tau}\right) + \frac{1}{2}g^{\alpha\tau}\left(\delta g^{(1)}_{\beta\tau,\gamma} + \delta g^{(1)}_{\gamma\tau,\beta} - \delta g^{(1)}_{\beta\gamma,\tau}\right). \tag{1.65}$$

Then, by using Eq.1.54, we get the following components for our (fully) linearly perturbed Chrystoffel's symbols[10]

$$\begin{aligned}
&\delta\Gamma^{0(1)}_{00} = \psi' \qquad \delta\Gamma^{0(1)}_{0i} = \partial_i\psi + \frac{a'}{a}\omega_i \qquad \delta\Gamma^{i(1)}_{00} = \frac{a'}{a}\omega^i + \omega'^i + \partial^i\psi\\
&\delta\Gamma^{0(1)}_{ij} = -2\frac{a'}{a}\psi\delta_{ij} - \frac{1}{2}\left(\partial_i\omega_j + \partial_j\omega_i\right) - 2\frac{a'}{a}\varphi\delta_{ij} - \varphi'\delta_{ij} + \frac{a'}{a}\chi_{ij} + \frac{1}{2}\chi'_{ij}\\
&\delta\Gamma^{i(1)}_{0j} = -\varphi'\partial^i_j + \frac{1}{2}\partial_j\omega^i - \frac{1}{2}\partial^i\omega_j\frac{1}{2}\chi'^i_j\\
&\delta\Gamma^{i(1)}_{jk} = -\partial_j\varphi\delta^i_k - \partial_k\varphi\delta^i_j + \partial^i\varphi\delta_{jk} - \frac{a'}{a}\omega^i\delta_{jk} + \frac{1}{2}\partial_j\chi^i_k + \frac{1}{2}\partial_k\chi^i_j - \frac{1}{2}\partial^i\chi_{jk}.
\end{aligned} \tag{1.66}$$

Thus, through these expressions we can find the perturbed Ricci tensor and hence the Ricci curvature. For the first one we have

$$\delta R^{(1)}_{\alpha\beta} = \delta\Gamma^{\gamma(1)}_{\alpha\beta,\gamma} - \delta\Gamma^{\gamma(1)}_{\gamma\alpha,\beta} + \delta\Gamma^{\gamma(1)}_{\gamma\tau}\Gamma^{\tau}_{\alpha\beta} + \Gamma^{\gamma}_{\gamma\tau}\delta\Gamma^{\tau^{(1)}}_{\alpha\beta} - \delta\Gamma^{\gamma(1)}_{\beta\tau}\Gamma^{\tau}_{\gamma\alpha} - \Gamma^{\gamma}_{\beta\tau}\delta\Gamma^{\tau(1)}_{\gamma\alpha}, \tag{1.67}$$

whose components, restricted to only scalar contributions, read [27]

$$\begin{aligned}
\delta R^{(1)}_{00} &= \frac{a'}{a}\partial_i\partial^i\omega^{\parallel} + \partial_i\partial^i\omega'^{\parallel} + \partial_i\partial^i\psi + 3\varphi'' + 3\frac{a'}{a}\varphi' + 3\frac{a'}{a}\psi'\\
\delta R^{(1)}_{oi} &= \frac{a''}{a}\partial_i\omega^{\parallel} + \frac{a'^2}{a^2}\partial_i\omega^{\parallel} + 2\partial_i\varphi' + 2\frac{a'}{a}\partial_i\psi + \frac{1}{2}\partial_k D^k_i\chi^{\parallel}\\
\delta R^{(1)}_{ij} &= \left[-\frac{a'}{a}\psi' - 5\frac{a'}{a}\varphi' - 2\frac{a''}{a}\psi - 2\left(\frac{a'}{a}\right)^2\psi - 2\frac{a''}{a}\varphi - 2\left(\frac{a'}{a}\right)^2 - \varphi'' + \partial_k\partial^k\varphi - \frac{a'}{a}\partial_k\partial^k\omega^{\parallel}\right]\delta_{ij} +\\
&\quad - \partial_i\partial_j\omega'^{\parallel} + \frac{a'}{a}D_{ij}\chi'^{\parallel} + \frac{a''}{a}D_{ij}\chi^{\parallel} + \left(\frac{a'}{a}\right)^2 D_{ij}\chi^{\parallel} + \frac{1}{2}D_{ij}\chi''^{\parallel} + \partial_i\partial_j\varphi - \partial_i\partial_j\psi - 2\frac{a'}{a}\partial_i\partial_j\omega^{\parallel} +\\
&\quad + \frac{1}{2}\partial_k\partial_i D^k_j\chi^{\parallel} + \frac{1}{2}\partial_k\partial_j D^k_i\chi^{\parallel} - \frac{1}{2}\partial_k\partial^k D_{ij}\chi^{\parallel}.
\end{aligned} \tag{1.68}$$

[10] By considering only scalar contributions one would find exactly the connections reported in [27]. For example, we see that by Helmholtz expanding the vector modes in the second line, the two ω-contributions cancel each other out by inverting the order of the derivatives.

Finally, for the Ricci curvature $\delta R^{(1)} = \delta g^{\alpha\beta(1)} R_{\alpha\beta} + g^{\alpha\beta}\delta R^{(1)}_{\alpha\beta}$, we have

$$\delta R^{(1)} = \frac{1}{a^2}\left(-6\frac{a'}{a}\partial_i\partial^i\omega^{\|} - 2\partial_i\partial^i\omega'^{\|} - 2\partial_i\partial^i\psi - 6\varphi'' - 6\frac{a'}{a}\psi' - 18\frac{a'}{a}\varphi' - 12\frac{a''}{a}\psi + 4\partial_i\partial^i\psi + \partial_k\partial^i D^k_i\chi^{\|}\right). \quad (1.69)$$

Now that all the elements of the Einstein's tensor $G_{\mu\nu}$ have been perturbed at linear order, one can proceed in perturbing that too. As always, it will be given by $\delta G^{(1)}_{\alpha\beta} = R^{(1)}_{\alpha\beta} - \frac{1}{2}\left(g^{(1)}_{\alpha\beta}R + g_{\alpha\beta}R^{(1)}\right)$ Specifically, its components read

$$\begin{aligned}
\delta G^{0(1)}_0 &= \frac{1}{a^2}\left[6\left(\frac{a'}{a}\right)^2\psi + 6\frac{a'}{a}\varphi' + 2\frac{a'}{a}\nabla^2\omega^{\|} - 2\nabla^2\varphi - \frac{1}{2}\partial_k\partial^i D^k_i\chi^{\|}\right] \\
\delta G^{0(1)}_i &= \frac{1}{a^2}\left[-2\frac{a'}{a}\partial_i\psi - 2\partial_i\varphi' - \frac{1}{2}\partial_k D^k_i\chi'^{\|}\right] \\
\delta G^{i(1)}_j &= \frac{1}{a^2}\Bigg[\Bigg(2\frac{a'}{a}\psi' + 4\frac{a''}{a}\psi - 2\left(\frac{a'}{a}\right)^2\psi + \nabla^2\psi + 4\frac{a'}{a}\varphi + 2\varphi'' - \nabla^2\varphi + 2\frac{a'}{a}\nabla^2\omega^{\|} + \nabla^2\omega'^{\|} + \\
&\quad + \frac{1}{2}\partial_k\partial^m D^k_m\chi^{\|}\Bigg)\delta^i_j - \partial^i\partial_j\psi + \partial^i\partial_j\varphi - 2\frac{a'}{a}\omega^{\|} - \partial^i\partial_j\omega'^{\|} + \frac{a'}{a}D^i_j\chi'^{\|} + \frac{1}{2}D^i_j\chi''^{\|} + \\
&\quad + \frac{1}{2}\partial_k\partial^i D^k_j\chi^{\|} + \frac{1}{2}\partial_k\partial_j D^{ik}\chi^{\|} - \frac{1}{2}\partial_k\partial^k D^i_j\chi^{\|}\Bigg].
\end{aligned} \quad (1.70)$$

We note that up to now these expressions are completely general. In fact, we did not apply any gauge to write them, from which it follows how convenient a proper gauge choice can be in treating cosmological perturbations. We then proceed in find the perturbed stress-energy tensor. However, instead of choosing a perfect fluid's one, we want to take into account possible anisotropies. In doing that our stress-energy tensor reads

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + \Pi_{\mu\nu}, \quad (1.71)$$

with $\Pi_{\mu\nu}$ transverse and traceless. We can of course rewrite it owing to Helmholtz theorem as all the other tensor quantities we introduced above, getting $\Pi_{ij} = D_{ij}\Pi^{\|} + \Pi^{\perp}_{i,j} + \Pi^{\perp}_{j,i} + \Pi^T_{ij}$. At the end we find

$$\begin{aligned}
\delta T^0_0 &= \rho_o + \delta\rho(\tau,\vec{x}) & \delta T^0_i &= (\rho_o + p_0)\left(v^i + \omega^i\right) \\
\delta T^i_0 &= -(\rho_0 + p_0)\, v^i & \delta T^i_j &= p_0\left[\left(1 + \frac{\delta\rho}{\rho_0}\right) + \frac{\Pi^i_j}{p_0}\right].
\end{aligned} \quad (1.72)$$

Here, we can see that the perturbed spatial stress-energy components, are given by the usual (perturbed) isotropic pressure plus some anisotropic deviations. Also, we see that the $i0$ and $0i$ components, express the energy flux in the i-th direction. As we pointed out above, at linear order we can decouple scalars, vectors and tensors. Hence, for scalar perturbations, the perturbed Einstein's Field Equations read

$$3\frac{a'}{a}\left(\hat{\varphi} + \frac{a'}{a}\psi\right) - \nabla^2\left(\hat{\varphi} + \frac{a'}{a}\sigma\right) = -4\pi G a^2\delta\rho, \quad (1.73)$$

$$\hat{\varphi}' + \frac{a'}{a}\psi = -4\pi G a^2\left(\rho_0 + p_0\right)\left(v^{\|} + \omega^{\|}\right), \quad (1.74)$$

for the 00 and the $0i$ components respectively, and

$$\hat{\varphi}'' + 2\frac{a'}{a}\hat{\varphi}' + \frac{a'}{a}\psi + \left[2\left(\frac{a'}{a} + \left(\frac{a'}{a}\right)^2\right)\right]\psi = 4\pi G a^2 \left(\frac{\delta\rho}{\rho_0} + \frac{2}{3}\nabla^2\frac{\Pi^{\parallel}}{p_0}\right)\rho_0, \tag{1.75}$$

$$\sigma' + 2\frac{a'}{a}\sigma + \hat{\varphi} - \psi = 8\pi G a^2 \Pi^{\parallel}, \tag{1.76}$$

for the trace and traceless part of the ij components, where the former will correspond to pressure perturbations and contributions from Π^i_j. Here, $\hat{\varphi}$ is the so called curvature perturbation defined above. To get a closer look to these equations, we put ourselves in the Poisson gauge. Here, we have $\omega^{\parallel} = \xi^{\perp}_i = \chi'^{\parallel} = 0$. Therefore, from the definition of the Bardeen's potential and the curvature perturbation, it follows $\Phi_H = -\hat{\varphi} = -\varphi$, $\Psi_A = \psi$, $\sigma = 0$. Hence, from Eq.1.76, we have [24]

$$\Phi_H + \Psi_A = -8\pi G a^2 \Pi^{\parallel}. \tag{1.77}$$

Although the right hand side of the latter can be not completely negligible due to the presence of propagating light particles in the Universe, we will stick ourselves to the $\Pi^{\parallel} = 0$ scenario. By doing so, we can rewrite Eq.1.75 as an evolution equation for Φ_H, which reads

$$\Phi''_H + 3\left(1 + c_s^2\right)\frac{a'}{a}\Phi'_H + \left[2\frac{a'}{a} + \left(1 + 3c_s^2\right)\left(\frac{a'}{a}\right)^2 - c_s^2\nabla^2\right]\Phi_H = 0, \tag{1.78}$$

where we also assumed adiabatic perturbations. In such a way, we could write $\delta p = c_s^2\delta\rho + \delta p_{non-adiabatic} \approx c_s^2\delta\rho$. We then used Eq.1.73 to relate $\delta\rho$ to the gravitational potentials. The above equation corresponds to a propagation equation for Φ_H, which is seen as a scalar perturbation propagating with the adiabatic speed of sound c_s^2, and undergoing a frictional term $\frac{a'}{a}\Phi'_H$ given by the expansion of the Universe.
For vector perturbations, we find, from the $0i$ perturbed EFEs,

$$\nabla^2\psi_i = 16\pi G a^2 \left(\rho_0 + p_0\right) V_{ic}, \tag{1.79}$$

which represents a constraint equation for the vector ψ_i. From the ij Einstein's equations then, it follows an evolution equation for ψ_i. In particular, it shows that the vector components of the anisotropic stress-energy tensor, act as a source term for ψ_i. However, given a null anisotropic stress-energy tensor, this vector perturbation is rapidly vanishing as the Universe expands, since its amplitude turns out to be proportional to a^{-3} [28]. Finally, from the continuity equation given by the Bianchi identity, for vector modes we find

$$\left[\left(\rho_0 + p_0\right) V_{ic}\right]' + 4\frac{a'}{a}\left(\rho_0 p_0\right) V_{ic} = -\nabla_k\left(\Pi^{\perp k}_{,i} + \Pi^{\perp,k}_{i}\right), \tag{1.80}$$

which corresponds to the Kelvin's circulation theorem. In fact, in absence of vorticity sources (right hand side equal to zero), we find

$$\left(\rho_0 + p_0\right) V_{ic} a^3 a = constant. \tag{1.81}$$

Being now $\left(\rho_0 + p_0\right) a^3 V_i c$ an energy times a velocity, from the latter we have that the angular momentum of matter is conserved in time along the fluid's trajectories.
In the end, for tensors perturbations, from the purely spatial and traceless perturbed EFEs, we find

$$\chi_{ij}^{\prime\prime T}+2\frac{a'}{a}\chi_{ij}^{\prime T}-\nabla^2\chi_{ij}^T=16\pi Ga^2p_0\Pi_{ij}^T. \tag{1.82}$$

As for the vector case, the right hand side encodes all the possible source terms given by the anisotropic stress-energy tensor. Except for possible contributions arising from this term, Eq.1.82 is identical in its form to the equation of motion we found for the inflaton field. This is a pretty important and useful result, which allows us to describe the dynamics of a graviton with a rescaling of the inflaton's mode-function, as we are going to see in the next chapter.

1.9 The curvature perturbation

The intrinsic spatial curvature ${}^{(3)}R$ reads in general

$${}^{(3)}R=\frac{6k}{a^2}+\frac{12k}{a^2}\hat{\varphi}+\frac{4}{a^2}\nabla^2\hat{\varphi}, \tag{1.83}$$

where $\hat{\varphi}$ is the curvature perturbation we introduced above. By considering a flat spacetime ($k=0$) we find

$${}^{(3)}R=\frac{4}{a^2}\nabla^2\hat{\varphi}. \tag{1.84}$$

Now, by going to the uniform curvature gauge $\varphi=0=\chi^{\parallel}$, it follows $\hat{\varphi}=0$ and hence ${}^{(3)}R=0$, it is now clear why it is called the uniform curvature gauge. For the same reason, usually $\hat{\varphi}$ is called curvature perturbation, as a non-vanishing value of that can lead to perturbations in the intrinsic curvature. However, this quantity is not gauge invariant, but it is defined only in this particular gauge. In fact, given a generic gauge transformation, we have $\tilde{\hat{\varphi}}=-\frac{a'}{a}\xi^0+\varphi+\frac{\nabla^2\chi}{6}=-\frac{a'}{a}\xi^0+\hat{\varphi}$. Instead, we would like to obtain a gauge invariant quantity for curvature perturbations.
By looking at Eq.1.58, we see that $\delta\rho$ transforms as $\tilde{\delta\rho}=\delta\rho+\frac{\dot{\rho}}{a}\xi^0$. Therefore, if we considered $\frac{\delta\rho}{\dot{\rho}^{(0)}}$, we would have

$$\frac{\tilde{\delta\rho}}{\dot{\rho}^{(0)}}=\frac{\delta\rho}{\dot{\rho}^{(0)}}+\frac{1}{a}\alpha. \tag{1.85}$$

By observing then that $H=\frac{\dot{a}}{a}=a'$, we could define

$$\zeta=-\hat{\Phi}-H\frac{\delta\rho}{\dot{\rho}}. \tag{1.86}$$

This is the so called *curvature perturbation on uniform energy density gauge*, since, by passing on the *Uniform density gauge* ($\delta\rho=0$), it resembles exactly the scalar curvature perturbation $\hat{\varphi}$. This is gauge invariant by construction, and represents a very important quantity since it remains nearly constant outside the horizon. This characteristic allows us to study (and eventually observe) its value directly at horizon crossing, thereby letting us to probe the inflationary epoch. We note that it is not a case that the interplay between curvature perturbations and energy-density fluctuations produce a gauge invariant quantity. Now, in order explicitly show tha ζ is constant outside the horizon, we take into account the energy-density continuity equation for scalar perturbations (it comes from the perturbed 0-Bianchi identity $T^{0\mu}_{,\mu}$), which reads

$$\delta\rho'+3\frac{a'}{a}\left(\delta\rho+\delta p\right)-3\left(\rho_0+p_0\right)\hat{\varphi}'+\left(\rho_0+p_0\right)\nabla^2(v^{\parallel}+\omega^{\parallel}+\sigma)=0. \tag{1.87}$$

By exploiting the gauge invariance of ζ, we impose the uniform energy-density gauge, getting

$$+3\left(\rho_0+p_0\right)\zeta'+3\frac{a'}{a}\delta p+\left(\rho_0+p_0\right)\nabla^2\left(v^{\parallel}+\omega^{\parallel}+\sigma\right)=0, \tag{1.88}$$

where we can neglect the derivative term, as on superhorizon scales k are quite small and passing to Fourier space $\nabla^2 \to -k^2$, becoming negligible. Hence, we are left with

$$\zeta'=-\frac{a'}{a}\frac{\delta p}{\rho_0+p_0}\approx-\frac{a'}{a}\frac{\delta p_{non-ad}}{\rho_0+p_0}, \tag{1.89}$$

where we decomposed δp into its adiabatic and non-adiabatic contributions and applied the chosen gauge. For single field models of slow-roll inflation, however, we have $\delta p_{non-ad}\propto\frac{k^2}{a^2}\Phi_H$, which tends to zero on superhorizon scales, giving us a constant scalar curvature perturbation.

1.10 Correlation functions

Now that we have (breafly) introduced the topic of cosmological perturbations and how to treat them, we can proceed in perturbing the inflaton's field equation of motion. From Appendix A.2, we have $\Box\phi=\frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{\mu\nu}\phi_{,\mu}\right)_{,\nu}=\frac{\partial V}{\partial(\phi)}$, and hence

$$\begin{aligned}\delta\left(\Box\phi\right)=&\frac{1}{2(-g^{(0)})^{3/2}}\delta\sqrt{-g}\left(\sqrt{-g^{(0)}}g^{(0)\mu\nu}\phi_{,\mu}\right)_{,\nu}+\frac{1}{\sqrt{-g^{(0)}}}\left(\delta\sqrt{-g}g^{(0)\mu\nu}\phi_{,\mu}\right)_{,\nu}+\\&+\frac{1}{\sqrt{-g^{(0)}}}\left(\sqrt{-g}\delta g^{\mu\nu}\phi_{,\mu}\right)_{,\nu}+\frac{1}{\sqrt{-g^{(}0)}}\left(\sqrt{-g^{(0)}}g^{(0)\mu\nu}\delta\phi_{,\mu}\right)_{,\nu}\end{aligned} \tag{1.90}$$

and

$$\delta\frac{\partial V}{\partial\phi}=\delta\phi\frac{\partial^2V}{\partial\phi^2}. \tag{1.91}$$

By using the fact that $\delta\sqrt{-g}=-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}$, and considering only scalar perturbations at first order (which are decoupled from all the other as we stressed before), we find [29]

$$\delta\phi''+2\frac{a'}{a}\delta\phi'-\nabla^2\delta\phi-\psi'\phi'-3\varphi'\phi'-\nabla^2\omega^{\parallel}\phi'=-\delta\phi\frac{\partial^2V}{\partial\phi^2}a^2-2\psi\frac{\partial V}{\partial\phi}. \tag{1.92}$$

To find this equation, the slows-roll conditions ($\phi''\psi$,$\phi''\varphi\ll1$), and the first order approximation of the inflaton's field ($\phi(\tau,\vec{x})=\phi(\tau)+\delta\phi(\tau,\vec{x})$) were used. Also, from the first term of Eq.1.90 contributions $\propto a^{-8}$ arise, which can be easily neglect since during inflation they would be rapidly washed out. In the spatially flat gauge ($\chi^{\parallel}=0=\varphi$) then

$$\delta\phi''+2\frac{a'}{a}\delta\phi'-\nabla^2\delta\phi-\psi'\phi'-\nabla^2\omega^{\parallel}\phi'=-\delta\phi\frac{\partial^2V}{\partial\phi^2}a^2-2\psi\frac{\partial V}{\partial\phi}. \tag{1.93}$$

This can be solved by introducing a new gauge-invariant variable. This is the so called Sasaki-Mukhanov variable [30]

$$Q_\phi=\delta\phi+a\frac{\phi'}{a'}\hat{\varphi}, \tag{1.94}$$

which relates the perturbations of the inflaton's field with the perturbations of the metric through the scalar curvature perturbation $\hat{\varphi}$. This is the analogous of the curvature perturbation ζ we introduced above. In the same way, this new invariant takes into account the interplay between the fluctuations of our slow-rolling field with the back-reaction of the metric. By defining now, as done for $\delta\phi$ the rescaled variable $\tilde{Q_\phi} = aQ_\phi$, the linearly perturbed evolution equation we obtained reduces to

$$\tilde{Q_\phi}'' + \left(k^2 - \frac{a''}{a} + \mathcal{M}_\phi^2 a^2\right)\tilde{Q_\phi} = 0, \tag{1.95}$$

where we expressed our fields in the Fourier space. Here $\mathcal{M}_\phi^2 = \frac{\partial^2 V}{\partial \phi^2} - \frac{8\pi G}{a^2}\left(\frac{a^5}{a'}\phi'^2\right)'$ plays the role of an effective mass. It can be expressed at first order in the slow-roll parameters becoming $\mathcal{M}_\phi \approx H^2\left(3\eta - 6\epsilon\right)$[11]. We see that this equation has the exact form of Eq.1.35. Thus, it can be solved in the same way, observing that here we have, by using $a^2H^2 \approx \tau^{-1}$

$$\tilde{Q}_\phi'' + \left(k^2 - \frac{\nu_\phi^2 - \frac{1}{4}}{\tau^2}\right)\tilde{Q}_\phi = 0, \tag{1.96}$$

with $\nu_\phi^2 = \frac{9}{4} - 3\eta_V + 9\epsilon$ ($\nu_\phi \approx \frac{3}{2} - \eta_V + 3\epsilon$, at $\mathcal{O}(\epsilon, \eta)$).
Having found a way to properly describe how the inflaton's field's fluctuations evolve realistically, namely taking into account also the back-reaction of gravity, we need to compare the theoretical predictions with observations. In order to do that, we need to define some observables. We saw that the inflaton's fluctuations are stochastic, and can be described by a quantum field $\delta\phi(\tau, \vec{x})$ which describes at each point of the space-time the amplitude of the scalar perturbations of ϕ. These fluctuations, although having a null vacuum expectation value when considered alone, can be correlated through different points of the space-time. We thus define the so called *two-points correlation function*, which indicates the statistical dependence between two variables, to be

$$\xi(r) = \langle \delta(\vec{x} + \vec{r}, \tau)\delta(\vec{x}, \tau)\rangle\,, \tag{1.97}$$

where it depends only on the distance between the two δ-functions in a homogeneous and isotropic Universe. By considering the above two-points correlation function and taking its Fourier transform, we can define the so called *power spectrum*

$$\begin{aligned}\xi = \langle \delta(\vec{x} + \vec{r}, \tau)\delta(\vec{x}, \tau)\rangle &= \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} e^{i\vec{k_1}(\vec{x}+\vec{r})} e^{\vec{k_2}\vec{x}} \left\langle \delta(\vec{k_1}, \tau)\delta(\vec{k_2}, \tau)\right\rangle \\ &= \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} e^{i\vec{k_1}(\vec{x}+\vec{r})} e^{\vec{k_2}\vec{x}} (2\pi)^3 \delta^{(3)}\left(\vec{k_1} + \vec{k_2}\right) P(|\vec{k_1}|) \\ &= \int \frac{d^3k_1}{(2\pi)^3} e^{i\vec{k_1}\vec{r}} P(|\vec{k_1}|).\end{aligned} \tag{1.98}$$

Here $\left\langle \delta(\vec{k_1})\delta(\vec{k_2})\right\rangle = \left\langle \delta^+(\vec{k_1})\delta^-(\vec{k_2})\right\rangle = u^\delta(k_1)u^{*\delta}(k_2)(2\pi)^3\delta^{(3)}(\vec{k_1} + \vec{k_2})$ is simply a contraction between the two fields of which we want to calculate the power spectrum. Therefore, we have $P_\delta(k_1) = u^\delta(k_1)u^{*\delta}(k_2)$. It is then possible to define the *adimensional power spectrum* $\Delta(k)$ by computing the Fourier transform of the variance, i.e. the correlation function between the same field in the same spacetime point. Now that we have briefly introduced a way to forecast a certain level of correlation between our stochastic variables, we can proceed in applying these to our Inflation's model. We had $|\delta\phi_k| = \frac{|u_k|}{a}$ in the Fourier space. Therefore, it follows

[11] From its definition, computing the time derivative and expressing all in cosmic time, the terms not suppressed by second-order slow-roll parameters are $V_{\phi\phi} = 3\eta H^2 = 3\eta_V H^2 - 3\epsilon H^2$ and $-3\frac{8\pi G}{\phi^2} = -6\epsilon H^2$

$$P_{\delta\phi}(k_1) = |\delta\phi_k|^2 = \frac{|u_k|^2}{a^2}. \tag{1.99}$$

In a perfect dS Universe, we found (Eq.1.49) that on superhorizon scales we have $|\delta\phi_k| = \frac{H}{\sqrt{2k^3}}$ from which it follows $P_{\delta\phi,dS}(k) = \frac{H^2}{2k^3}$. To have a more realistic scenario, we can consider the Sasaki-Mukhanov gauge invariant variable we defined above. By observing that its equation of motion Eq.1.96 is identical in form to the inflaton's one, we can proceed in the same way to find its mode function outside the horizon. In the end we find

$$|Q_\phi(\tau)| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{3\epsilon-\eta_V} \qquad \Longrightarrow \qquad P_\phi(k) = \frac{H^2}{2k^3}\left(\frac{k}{aH}\right)^{-3\epsilon+\eta_V}. \tag{1.100}$$

By considering now its adimensional power spectrum $\Delta_\phi(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{2\eta_V-6\epsilon}$, we can define the so called *spectral index* $n(k)$

$$n(k) - 1 = \frac{d\log\Delta_\phi(k)}{d\log k}, \tag{1.101}$$

which describes the shape of the power spectrum. A spectral index $n = 1$ (Harrison Zel'dovich spectrum [31, 32]) would imply a scale invariant power spectrum, while a constant n would show a power-law evolving power spectrum. A remarkable example of spectral index is given by the *scalar spectral index of primordial density perturbations*. It is possible to show that the adimensional power spectrum for primordial energy perturbations $\Delta_{\delta\rho/\rho}(k)$ can be related to the inflaton's power spectrum[12]. This provides a very important link between Inflation and its forecasts on the fluctuations we observe in the CMB. Specifically, we have $\delta_{\delta\rho/\rho}(k) = \frac{H^2}{\dot{\phi}^2}\Delta_{\delta\phi}(k)$ and hence

$$n_s - 1 = \frac{d\log\Delta_{\delta\rho/\rho}(k)}{d\log k} = 2\eta_V - 6\epsilon. \tag{1.102}$$

This shows us that inflationary models predict a scalar spectral index that deviates from a Harrison-Zel'dovich one by $\mathcal{O}(\epsilon,\eta)$. From the observations, it was found [33]

$$n_s = 0.9649 \pm 0.0042, \tag{1.103}$$

which allows non negligible deviation from a scale-constant power spectrum. This can be understood by observing that during the inflationary period, both H, and $\dot{\phi}$ change very slowly (slow-roll conditions) but do not remain perfectly equal. Since then different scales cross the horizon at different time, each fixed-scale power spectrum will carry a slightly different value of $H, \dot{\phi}$. We note that the spectral index and the power spectrum are extremely useful observable quantities, that can be used to have direct information about the inflationary epoch. For instance [27], through the ζ's power spectrum, which we will formally introduce in the next chapter, and its observed value $P_\zeta(k) \approx 2.95 \times 10^{-9}A$ (with $A = 0.6 - 1$ depending upon the model), it is possible to constrain the value of the Hubble parameter during Inflation. This turned out to be $H \approx (0.9-1.2) \times 10^{15}\sqrt{\epsilon}GeV$ [34], which in turn allowed to get an estimate for the energy scale of inflation $V^{1/4} \approx (6.3 - 7.1) \times 10^{16}\epsilon^{1/4}GeV$. Unfortunately, a (nearly) scale-invariant power spectrum, can be expected even without admitting the existence of Inflation [35]. A strong prediction of the inflationary model would be instead given by the relation between the scalar and the tensor spectral index. By repeating the whole procedure sketched above, one can find the power spectrum for tensor perturbations and compute the *tensor to scalar amplitude ratio*

[12] Through the δN formalism, it is possible to show that the shift in the number of e-folds produced by inflaton's fluctuations from point to point δN can be related to ζ, which is constant out of the horizon. We have in fact $\delta N \approx -H\frac{\delta\rho_\phi}{\dot{\rho_\phi}} = \zeta$. Moreover, we can see that at linear order $\delta N \approx -H\frac{\delta\phi}{\dot{\phi}}$, which establishes a direct link between $\delta\phi$ and $\delta\rho$

$$r = 16\epsilon. \tag{1.104}$$

In particular, this implies $r = -8n_T$, with n_T being the tensor spectral index. Observing it would be a clear proof of Inflation, as no other Early Universe's models forecast the same relation.
However, another useful observable could be defined. Up to now, we introduced and talked about only two-points correlation functions. These give us power spectra which are determined by the expansion rate during Inflation and its time dependence, but they do not tell anything about possible interactions of the inflaton's field with itself or other quantum fields. Therefore, given several different models of Inflation with the same expansion rates and time evolution, we will obtain very similar power spectra, which then result to be degenerate with respect to the chosen model [36]. This third observable is given by the deviation from a pure Gaussian statistic in the CMB anisotropies; namely, the presence of higher-order correlation functions. These deviations from Gaussianity are called non-Gaussianities, and represent a sensitive probe of, for instance, the interactions undergone by the inflaton. In doing so, they allow us to explore properties of the inflationary paradigm that are not accessible through the two-points functions only. Non-Gaussian contributions are expected to be negligible in models of inflation where all the following features are fulfilled: slow-roll, single field, canonical kinetic term (the speed of propagation of inflaton's fluctuations equals the speed of light), adiabatic Bunch-Davies vacuum choices. By imposing one (or more) of these condition to be unsatisfied, large non-Gaussianities can arise. In order to see how non-Gaussianities can discriminate between different models, one can consider for instance the connected three-points function. it correlates, e.g., density perturbations at three points in space. Thus, when Fourier transformed, the bispectrum correlates fluctuations with three wave-vectors through the Dirac's delta that arise from the transform. Therefore, these three vectors are forced to form a triangle in the Fourier space. A remarkable observation is that violations of each of the conditions reported above result in a different signal, which is translated in a different triangular shape [36]. The three main triangular configurations are given by the *squeezed limit* ($k_3 \ll k_2 \approx k_1$), the *equilateral* ($k_1 \approx k_2 \approx k_3$), and the *folded* ($k_3 \approx k_2 \approx 2k_1$) triangle. These three possibilities are given respectively by multi-field models, models with non-canonical terms, and models with non Bunch-Davies vacuum choices, and represent a sort of basis for all the other possibilities that one has to build an inflationary mechanism. Specifically, when more than one condition sketched above is violated, a linear combination of these shapes can arise. Phenomenologically, the level of non-Gaussianity for the squeezed limit in the Fourier decomposition can be parametrized by the non linearity parameter f_{NL} [37] through the equality

$$\Phi_H(\vec{x}) = \Phi_L(\vec{x}) + f_{NL}\left[\Phi_L^2(\vec{x}) - \left\langle \Phi_L^2(\vec{x}) \right\rangle\right], \tag{1.105}$$

with Φ_L are Gaussian linear perturbations and Φ_H the Bardeen's gravitational potential.
Another important example of non-Gaussianity is given by the four-points correlation function, or its corresponding Trispectrum in the Fourier space. An important characteristic of this function is that it represents the minimum n-point correlator ($n > 2$) that can explicitly show parity-violating signatures [2]. A theory is said to be symmetric under parity if, under a space inversion $\vec{x} \to -\vec{x}$, its dynamics does not change. This inversion, in Fourier space, corresponds to ask for $\vec{k} \to -\vec{k}$[13]. Therefore, a n-points correlator which is symmetric under parity transformations has to be such that

$$\left\langle \prod_{i=1}^{n} \zeta_{\vec{k}_i} \right\rangle = \left\langle \prod_{i=1}^{n} \zeta_{-\vec{k}_i} \right\rangle. \tag{1.106}$$

As we are going to see, the curvature fluctuation ζ shares the same mode function of the inflaton's field, besides a (real) rescaling. Thus, this equation, together with the fact that $\zeta^*_{\vec{k}} = \zeta_{-\vec{k}}$, guarantees the whole n-point correlator to be real. Hence, if Eq.1.106 is not satisfied, i.e. if parity is violated, then the correlator includes imaginary contributions. For a rotational-invariant scenario, this signal is null for $n \leq 3$, implying that the trispectrum $n = 4$ is the lowest-order correlator able to show parity-violating contributions.

[13]By computing the Fourier transform of $\zeta_{-\vec{x}}$ it is in fact sufficient to flip the sign of the momentum to get this relation

In this thesis, a great effort will be given in the computation of the trispectrum for two kind of Inflationary models, and their complex (parity-violating) contributions. In the next chapter we will introduce a formalism that allows us to compute n-points correlation functions and their corresponding in the Fourier space.

Chapter 2

Adopted Formalism

2.1 In-In formalism

2.1.1 Exact formula

Although we want to calculate correlation functions, this problem is quite different from what is done in QFT [38,39]. Firstly, we are interested in expectation values of products of fields at the same fixed time, rather than in calculating S-matrix elements. Secondly, we do not impose conditions on the fields at very early and very late times, as in QFT for the states at the beginning and at the end. We would rather require conditions only at very early times, when $k\tau \gg 1$. Finally, in inflationary theories, the fields we want to consider are, for instance, fluctuations of the inflaton field, the metric, and other possible fields which can show up in different inflation's models. In this scenario, the time dependence of those fluctuations is governed by a fluctuations Hamiltonian. Therefore, in order to calculate correlation functions, we use the so called "in-in formalism", which was firstly introduced by Schwinger and Keldysh [40, 41]. In order to describe this formalism, we consider our operator Q to be the product of generic fields $\delta\phi_a$ and their conjugate momenta $\delta\pi_a$, with a denoting different fields. In the Heisenberg picture, we want to calculate

$$\langle Q \rangle \equiv \langle \Omega | Q(t_e) | \Omega \rangle. \tag{2.1}$$

Here, t_e denotes the time at which inflation ends, and $|\Omega\rangle$ represents the vacuum state for the interacting theory at the far past t_0. We start by considering a common quantized system, whose canonical variables $\phi_a(\vec{x},t)$ and $\pi_a(\vec{x},t)$ satisfy the usual algebra

$$[\phi_a(\vec{x},t), \pi_b(\vec{x},t)] = i\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \qquad [\phi_a(\vec{x},t), \phi_b(\vec{y},t)] = 0 = [\pi_a(\vec{x},t), \pi_b(\vec{y},t)], \tag{2.2}$$

and the following equations of motion

$$\dot{\phi}(\vec{x},t) = i[H[\phi(t),\pi(t)], \phi_a(\vec{x},t)] \qquad \dot{\pi_a}(\vec{x},t) = i[H[\phi(t),\pi(t)], \pi_a(\vec{x},t)], \tag{2.3}$$

where H is the Hamiltonian of the system. We consider now each field as decomposed in its background and fluctuations, namely

$$\phi_a(\vec{x},t) = \bar{\phi}_a(\vec{x},t) + \delta\phi_a(\vec{x},t) \qquad \pi_a(\vec{x},t) = \bar{\pi}_a(\vec{x},t) + \delta\pi_a(\vec{x},t), \tag{2.4}$$

where the background contributions are just (real or complex) numbers. In this way, Eq.2.2 becomes

$$\begin{aligned}
&[\delta\phi_a(\vec{x},t),\delta\pi_b(\vec{y},t)] = i\delta_{ab}\delta^{(3)}(\vec{x}-\vec{y}) \\
&[\delta\phi_a(\vec{x},t),\delta\phi_b(\vec{y},t)] = 0 = [\delta\pi_a(\vec{x},t),\delta\pi_b(\vec{y},t)].
\end{aligned} \tag{2.5}$$

We now expand the Hamiltonian as

$$\begin{aligned}
H[\phi(t),\pi(t)] &= \bar{H} + \delta H + \delta^2 H + ... \\
&= H[\bar{\phi}(t),\bar{\pi}(t)] + \sum_a \int d^3x \frac{\partial \mathcal{H}}{\partial \bar{\phi}_a(\vec{x},t)}\delta\phi_a(\vec{x},t) + \sum_a \int d^3x \frac{\partial \mathcal{H}}{\partial \bar{\pi}_a(\vec{x},t)}\delta\pi_a(\vec{x},t) + \\
&\quad + \tilde{H}[\delta\phi(t),\delta\pi(t);t],
\end{aligned} \tag{2.6}$$

where $\tilde{H}$ includes second and higher orders in perturbations. By using this last expansion, Eq.2.5, and noticing that Eq.2.3 implies $\dot{\bar{\phi}}_a(\vec{x},t) = \frac{\partial \mathcal{H}}{\partial \bar{\pi}_a}$, $\dot{\bar{\pi}}_a(\vec{x},t) = -\frac{\partial \mathcal{H}}{\partial \bar{\phi}_a}$, we find

$$\begin{aligned}
\dot{\phi}_a(\vec{x},t) = \dot{\bar{\phi}}_a + \dot{\delta\phi}_a &= i\left[H[\bar{\phi},\bar{\pi}],\bar{\phi}_a + \delta\phi_a\right] + i\left[\delta H[\phi,\pi],\bar{\phi}_a + \delta\phi_a\right] + i\left[\tilde{H}[\delta\phi,\delta\pi;t],\bar{\phi}_a + \delta\phi_a\right] \\
&= i\left[\delta H[\phi,\pi],\delta\phi_a\right] + i\left[\tilde{H}[\delta\phi,\delta\pi;t],\delta\phi_a\right] \\
&= i\sum_b \left(\int d^3y \frac{\partial \mathcal{H}}{\partial \bar{\pi}_b(\vec{y},t)}[\delta\pi_b(\vec{y},t),\delta\phi(\vec{x},t)_a]\right) + i\left[\tilde{H}[\delta\phi,\delta\pi;t],\delta\phi_a(\vec{x},t)\right] \\
&= \frac{\partial \mathcal{H}}{\partial \bar{\pi}_a(\vec{x},t)} + i\left[\tilde{H}[\delta\phi,\delta\pi;t],\delta\phi_a(\vec{x},t)\right].
\end{aligned} \tag{2.7}$$

Analogously, we obtain $\dot{\pi}_a(\vec{x},t) = -\frac{\partial \mathcal{H}}{\partial \bar{\phi}_a(\vec{x},t)} + i\left[\tilde{H}[\delta\phi,\delta\pi],\delta\pi_a(\vec{x},t)\right]$. From these two equations, we finally find the following equation of motion for the perturbations

$$\dot{\delta\phi}_a(\vec{x},t) = i\left[\tilde{H}[\delta\phi,\delta\pi;t],\delta\phi_a(\vec{x},t)\right] \qquad \dot{\delta\pi}_a(\vec{x},t) = i\left[\tilde{H}[\delta\phi,\delta\pi;t],\delta\pi_a(\vec{x},t)\right]. \tag{2.8}$$

This shows us that although H generates the time dependence of ϕ_a,π_a, the time dependence of their perturbations is actually carried out by $\tilde{H}$. This is a pretty important result, which tells us that in order to study the evolution of fields' fluctuations, we only need the perturbed Hamiltonian up to a desired order.
Since our final goal is to compute correlation functions, wee need an operator formalism. In order to construct it, we proceed as in QFT, by introducing an unitary time-evolution operator $U(t,t_0)$ ($U^\dagger U = \mathbb{I}$), such that

$$\delta\phi_a(\vec{x},t) = U^{-1}(t,t_0)\delta\phi_a(\vec{x},t_0)U(t,t_0) \qquad \delta\pi_a(\vec{x},t) = U^{-1}(t,t_0)\delta\pi_a(\vec{x},t_0)U(t,t_0). \tag{2.9}$$

From these equations, by differentiating both sides with respect to time and using Eq.2.8, it is easy to see that U satisfies

$$\frac{d}{dt}U(t,t_0) = -i\tilde{H}[\delta\phi(t_0),\delta\pi(t_0);t]U(t,t_0) \;\;\text{with}\;\; U(t_0,t_0) = 1. \tag{2.10}$$

In the scenarios we are interested in, we can take $t_0 \to -\infty$, by which we consider time early enough so that the wavelengths we want to study are deep inside the horizon [38]. To explicitly calculate U, we can use again perturbation theory, by further splitting $\tilde{H}$ into two different parts,

$$\tilde{H}[\delta\phi(\vec{x},t),\delta\pi(\vec{x},t)] = H_0^I[\delta\phi(\vec{x},t),\delta\pi(\vec{x},t)] + H_{int}^I[\delta\phi(\vec{x},t),\delta\pi(\vec{x},t)], \tag{2.11}$$

Here, H_0 is the kinematic term, which describes the leading evolution of the perturbed fields, whereas H_I is the interaction term. In this view, we want now to calculate U as a power series of the perturbation H_I. In order to do that, we continue to follow what is done in canonical QFT, by defining an interaction picture where fluctuation operators $\delta\phi_a^I(\vec{x},t)$ and $\delta\pi_a^I(\vec{x},t)$ evolve as stated by H_0:

$$\dot{\delta\phi}_a^I = i\left[H_0^I[\delta\phi^I(t),\delta\pi^I(t);t],\delta\phi^I(\vec{x},t)\right] \qquad \dot{\delta\pi}_a^I = i\left[H_0^I[\delta\phi^I(t),\delta\pi^I(t);t],\delta\pi^I(\vec{x},t)\right], \tag{2.12}$$

with the initial conditions

$$\delta\phi_a^I(t_0) = \delta\phi_a(t_0) \qquad \delta\pi_a^I(t_0) = \delta\pi_a(t_0). \tag{2.13}$$

Before proceeding to find an expression for U, we note that the explicit dependence on time of H_0 allows us to consider the time argument of the fields $\delta\phi^I$ and $\delta\pi^I$ to have any value. In particular we can choose it to be t_0, so that we have

$$H_0^I[\delta\phi^I(t),\delta\pi^I(t);t] = H_0^I[\delta\phi(t_0),\delta\pi(t_0);t]. \tag{2.14}$$

Also, we observe that since H_0 is quadratic in the fields, the operators of our interaction picture are free fields and satisfy linear wave equations [38].
Now, by introducing a time-evolution unitary operator U_0 as we did above, from Eq. 2.12 it follows that

$$\frac{d}{dt}U_0(t,t_0) = -iH_0^I[\delta\phi(t_0),\delta\pi(t_0);t]U_0(t,t_0) \;\text{ with }\; U_0(t_0,t_0)=1. \tag{2.15}$$

In order to calculate the effects of the interaction H_I in our correlation function, we need a way to express our operator Q in this newly defined interaction picture. To do that, we rewrite Eq.2.1 in terms of the interaction fields by making use of the unitarity of both the time-evolution operators

$$\begin{aligned}
\langle\Omega|Q[\delta\phi_a(\vec{x},t),\delta\pi_a(\vec{x},t)]|\Omega\rangle &= \langle\Omega|U^{-1}(t,t_0)Q[\delta\phi_a(\vec{x},t_0),\delta\pi_a(\vec{x},t_0)]U(t,t_0)|\Omega\rangle \\
&= \langle\Omega|U^{-1}(t,t_0)U_0(t,t_0)U_0^{-1}(t,t_0)Q[\delta\phi_a(\vec{x},t_0),\delta\pi_a(\vec{x},t_0)]U_0(t,t_0)U_0^{-1}(t,t_0)U(t,t_0)|\Omega\rangle \\
&= \langle\Omega|F^{-1}(t,t_0)U_0^{-1}(t,t_0)Q[\delta\phi_a(\vec{x},t_0),\delta\pi_a(\vec{x},t_0)]U_0(t,t_0)F(t,t_0)|\Omega\rangle \\
&= \langle\Omega|F^{-1}(t,t_0)Q[\delta\phi_a^I(\vec{x},t),\delta\pi_a^I(\vec{x},t)]F(t,t_0)|\Omega\rangle,
\end{aligned} \tag{2.16}$$

where

$$F(t,t_0) = U_0^{-1}(t,t_0)U(t,t_0). \tag{2.17}$$

By differentiating this latter equation and by employing Eqs.2.10, 2.15 and 2.11, we find

$$\frac{d}{dt}F(t,t_0) = -iH_{int}^I(t)F(t,t_0) \;\text{ with }\; F(t_0,t_0)=1, \tag{2.18}$$

and hence

$$F(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H^I_{int}(t)dt\right), \tag{2.19}$$

where T indicates the fact that all the time variables must be time-ordered. Finally, by calling $\bar{T}$ the anti time-ordering operator, our correlation function becomes [38, 39]

$$\langle Q\rangle = \langle\Omega|\left[\bar{T}\exp\left(i\int_{t_0}^{t} H^I_{int}(t)dt\right)\right] Q[\delta\phi^I(t),\delta\pi^I(t)]\left[T\exp\left(-i\int_{t_0}^{t} H^I_{int}(t)dt\right)\right]|\Omega\rangle. \tag{2.20}$$

2.1.2 Vacuum choice

We would like to express this last equation in terms of the vacuum of the free theory, instead of using the interacting theory's one. In this case, thanks to the unitarity of the time-evolution operator $F(t,t_0)$, we can simply substitute $|\Omega\rangle$ with the Bunch-Davis vacuum state. To better show this feature, we consider the Bunch-Davis vacuum as our initial ground state for H_0 and we evolve it from t_0 to t:

$$\begin{aligned} e^{-i\tilde{H}(t-t_0)}|0\rangle &= e^{-i\tilde{H}(t-t_0)}\sum_n |n\rangle\langle n||0\rangle = \sum_n e^{-iE_n(t-t_0)}|n\rangle\langle n|0\rangle = \\ &= e^{-iE_\Omega}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq\Omega} e^{-iE_n(t-t_0)}|n\rangle\langle n|0\rangle, \end{aligned} \tag{2.21}$$

where we introduced a complete set of interacting states $|n\rangle$ making use of the completeness relation $\mathbb{I} = \sum_n |n\rangle\langle n|$. Here, E_n, E_Ω are respectively the eigenvalues of $\tilde{H}$ for its eigenstates $|n\rangle$ and $|\Omega\rangle$. Now, since we would like to calculate correlation functions by means of perturbation theory, we require H_I to be small enough to expand our operator F up to a certain order. By making this assumption, we are implying $\langle\Omega|0\rangle \neq 0$. In fact, by considering a quite small perturbation, our initial state does not get changed completely. By returning to the last expression, we notice that since $|\Omega\rangle$ represents the interacting vacuum, we have $E_n > E_\Omega$, as by definition the ground state is the state with minimum energy. Hence, by performing a time-shift ϵ in the complex plane $t_0 \to -\infty(1+i\epsilon)$, the terms proportional to $|n\rangle$ get suppressed. The only contribution we are left with is the first one in Eq. 2.21, which is suppressed more slowly that the others. Thus, we get

$$\begin{aligned} |\Omega\rangle &= \lim_{t_0\to i\infty} \frac{1}{e^{-iE_\Omega(t-t_0)}\langle\Omega|0\rangle} e^{-i\tilde{H}(t-t_0)}|0\rangle \\ &= \lim_{t_0\to i\infty} \frac{1}{e^{-iE_\Omega(t-t_0)}\langle\Omega|0\rangle} e^{-i\tilde{H}(t-t_0)}e^{+iH^I_0(t-t_0)}|0\rangle \\ &= \lim_{t_0\to i\infty} \frac{1}{e^{-iE_\Omega(t-t_0)}\langle\Omega|0\rangle} F(t,t_0)|0\rangle, \end{aligned} \tag{2.22}$$

since $e^{+iH^I_0(t-t_0)}|0\rangle = \sum_{n=0}^{+\infty}\frac{i^n}{n!}(t-t_0)^n H^{In}_0|0\rangle = |0\rangle$ and $F(t,t_0) = U_0^{-1}(t,t_0)U(t,t_0)$. Analogously, for the bra of the state we obtain

$$\langle\Omega| = \lim_{t_0\to i\infty}\langle 0|F^{-1}(t,t_0)\frac{1}{e^{+iE_\Omega(t-t_0)}\langle\Omega|0\rangle}. \tag{2.23}$$

By considering now Eq. 2.20, we find [42]

$$\langle\Omega|Q(t)|\Omega\rangle = \frac{1}{|\langle\Omega|0\rangle|^2}\langle 0|\left[\bar{T}\exp\left(i\int_{t_0(1+i\epsilon)}^{t} H^I_{int}(t)dt\right)\right] Q^I(t)\left[T\exp\left(-i\int_{t_0(1+i\epsilon)}^{t} H^I_{int}(t)dt\right)\right]|0\rangle, \tag{2.24}$$

where $|\langle\Omega|0\rangle|^2$ are the usual "vacuum bubbles" that accounts for the projection of $|0\rangle$ onto $|\Omega\rangle$ [43]. Since now the overall scalar factor does not depend upon the choice of the operator Q, we can put $Q = \mathbb{I}$, obtaining $1 = \frac{1}{|\langle\Omega|0\rangle|^2}\langle 0|F^{-1}F|0\rangle = \langle\Omega|0\rangle|^{-2}$, from which it follows that we can replace the Bunch-Davis vacuum to the interacting one in our computation. A deeper and more detailed discussion about the exchange of the two vacua and the used time-shift is reported in [42, 43]. By reading Eq. 2.24 from right to left, we see that the system starts in the interacting vacuum $|\Omega\rangle$ and evolves forward in time from t_0 to t where the operator $Q(t)$ occurs. Then, it evolves back to t_0 thanks to F^{-1}. This is another difference with respect to the usual S-matrix approach. Here, the initial states evolves in time, "interacts" with the operator and then goes back to itself, rather than evolving further to a final state. This is the reason why the formalism is called "in-in".

2.1.3 Perturbative expansion

For our purposes, it is easier to deal with cosmological correlation functions by expanding Eq.2.24 in the perturbation H_I. There are two equivalent ways to expand the above expression [39]. However, for what concerns the work done in this thesis, we will focus only on the following one

$$\langle Q(t)\rangle = \sum_{n=0}^{\infty}(-i)^n\int_{t_0}^{t}dt_n\int_{t_0}^{t_n}dt_{n-1}\cdots\int_{t_0}^{t_2}dt_1\langle 0|\Big[...\Big[\Big[Q^I(t), H^I_{int}(t_n)\Big], H^I_{int}(t_{n-1})\Big], ..., H^I_{int}(t_1)\Big]|0\rangle. \tag{2.25}$$

In order to prove the equivalence between the latter expression and Eq. 2.24, we proceed by induction. For the zeroth order the equivalence is straightforward: $H^{I(0)}_{int} = 0$, therefore Eq. 2.24, as well as Eq.2.25, reduces to $\langle Q(t)\rangle = \langle 0|Q^I(t)|0\rangle$. We now assume the two expressions are equivalent for the $(N-1)$th order. Then, we differentiate Eq. 2.24 with respect to time t, obtaining

$$\frac{d}{d\tilde{t}}\langle Q(t)\rangle = \frac{d}{d\tilde{t}}\langle F^{-1}(\tilde{t}, t_0)Q^I(t)F(\tilde{t}, t_0)\rangle = -i\langle F^{-1}\Big[Q^I(t), H^I_{int}(\tilde{t})\Big]F\rangle. \tag{2.26}$$

Now, we can rewrite the right hand side of this expression up to order $(N-1)$ using Eq.2.25 because of our assumption:

$$\langle\Big[Q^I(t), H^I_{int}(\tilde{t})\Big]\rangle\Big|_{N-1} = (-i)^{N-1}\int_{t_0}^{\tilde{t}}dt_{N-1}\cdots\int_{t_0}^{t_2}dt_1\langle\Big[...\Big[\Big[Q^I(t), H^I_{int}(\tilde{t})\Big], H^I_{int}(t_{N-1})\Big]..., H^I_{int}(t_1)\Big]\rangle. \tag{2.27}$$

Once inserted into Eq. 2.26, this is nothing but the derivative of Eq. 2.25 with respect to $\tilde{t}$ at Nth order, which in fact reads

$$\frac{d}{d\tilde{t}}\langle Q(t)\rangle\Big|_N = \frac{d}{d\tilde{t}}(-i)^N\int_{t_0}^{\tilde{t}}dt_N\cdots\int_{t_0}^{t_2}dt_1\langle[...]\rangle = (-i)^N(1)\int_{t_0}^{\tilde{t}}dt_{N-1}\cdots\int_{t_0}^{t_2}dt_1\langle[...]\rangle, \tag{2.28}$$

where we changed $t \to \tilde{t}$. Hence, the time-derivatives of the two expressions are equivalent up to order N. Since the two equations give the same result for $t \to -\infty$ to all orders, they are also equal for any arbitrary $\tilde{t}$ at Nth order [38, 42].

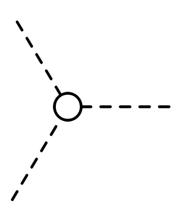
At this point, for each term arising from our expansion, we have to deal with (anti-) time-ordered products of different fields, which can be not trivial at all. Nevertheless, there is another type of ordering product, that is a lot simpler to deal with: the normal-ordering product $\mathcal{N}$. In fact, this acts in changing the position of its operators in a way in which all the annihilation operators are shifted to the right, and all the creation ones to the left. The order of momenta on the left or on the right does not matter, due to the commutativity between the same operators. In this way, we have for instance $\mathcal{N}[a(\vec{k})a(\vec{p})a^\dagger(\vec{q})a^\dagger(\vec{r})] = a^\dagger(\vec{q})a^\dagger(\vec{r})a(\vec{p})a(\vec{k})$. It is clear that if one wanted to calculate the vacuum expectation value of a certain operator, the normal-ordering simplifies a lot the calculation, since $\langle 0|a^\dagger = 0 = a|0\rangle$. It exists a powerful theorem that connects these two types of products and allows us to pass from the former to the latter. This is the so called Wick's theorem whose proof and implications are provided in [44–46]. Here, we limit ourself to discuss how this theorem is applied in the in-in formalism. Formally, we have

$$
\begin{aligned}
T[A_1A_2...A_n] =& \mathcal{N}[A_1A_2...A_n] + \sum_{i,j} \overbracket{A_iA_j}\mathcal{N}[A_1...\check{A}_i...\check{A}_j...A_n] + \sum_{i,j,k,l} \overbracket{A_iA_j}\overbracket{A_kA_l} N[A_1...\check{A}_i...\check{A}_j...\check{A}_k...\check{A}_l...A_n]... \\
&+ \sum_{\text{all contractions}} \overbracket{A_1A_2}...\overbracket{A_iA_j}...\overbracket{A_{n-1}A_n},
\end{aligned}
\tag{2.29}
$$

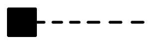
where $\check{A}_i$ means that the field is removed from the normal ordering, $A, B \sim a_{A/B} + a^\dagger_{A/B}$, and $\overbracket{AB} = \langle 0|T[A,B]|0\rangle$ is a scalar function, which can thus be put outside the $\mathcal{N}$-product. We can observe now, that in an inflationary background one does not have a single Feynman propagator taking care of the time-ordering, as in Minkowski spacetime. Thus, in this scenario, the time-ordering can be considered only in the final integration [39]. Furthermore, we notice that once the T-product is decomposed through Wick's theorem, only the term where all the fields are contracted survives between two vacuum states. In fact, being a contraction a scalar, we would get $\langle 0|[\overbracket{A_1...A_i}\overbracket{A_jA_k}...A_n]|0\rangle = 0$. Of course, the same is true if we leave two or more fields uncontracted.

2.1.4 Brief introduction to diagrammatic rules

The procedure to compute correlation functions by means of in-in formalism, is now quite similar to what is done in usual QFT. In Eq. 2.25, the Nth order term corresponds to a diagram with N vertices. Once the perturbed Hamiltonians are expressed in Fourier space, each vertex can be labelled with space and time coordinates. Also, a vertex has a number of attached lines depending upon the number of perturbed fields the corresponding hamiltonian has. For instance, an interacting Hamiltonian $H(\tau) \propto \int \frac{d^3pd^3qd^3k}{(2\pi)^9)}\delta\phi(\vec{p},\tau)\delta\phi(\vec{q},\tau)\delta\phi(\vec{k},\tau)$ will correspond to a vertex with three external legs:

$H(\tau) \propto$ 

There are then the external lines one could add to a diagram. These are represented by each field present in the $Q^I(t)$ operator. These, are labelled by the different space-time coordinates that appear in the expectation value we want to compute. For instance, the two fields $\zeta_{\vec{k}_1}(\tau_1)$ and $\gamma^s_{\vec{k}_1}(\tau_1)$ are depicted as

$\zeta_{\vec{k}_1}(\tau_1) \propto$  and $\gamma_{\vec{k}_1}(\tau_1) \propto$ 

These represent external fields, and as we are going to see in the next sessions, they will be always considered out of the horizon. Of course, all the external lines are connected to either a vertex or other external legs, just as in Feynman diagrams. There is however a difference with respect to the usual Feynman rules. In the in-in formalism, in fact, we do not have only one time-ordered product. We also have an anti-time-ordering operator, which takes us back to the initial state. Therefore, we need to distinguish between "positive" and "negative" vertices, arising respectively from the time-ordering and the anti-time-ordering [38]. As a consequence, a diagram with N vertices will give us 2^N contributions, carried by the 2^N ways of choosing a vertex to be "positive" or "negative". Each "positive" ("negative") or "right" ("left") vertex will contribute a factor $+i$ $(-i)$. For the same reason, a line connecting two "right" ("left") vertices will correspond to the usual (opposite) Wick's contraction. This property of the in-in formalism, under certain circumstances, allows one to write a correlation function as a product of commutators of the involved fields. In fact, as we are going to point out when we will study the $f(\phi)\tilde{F}F$ model, if we have an equal number of vertices and attached fields, we can proceed as mentioned earlier. This is because if to one vertex is attached only one field, it contributes with both the usual and the opposite Wick's contraction (with a minus sign), which is exactly the definition of the commutator between this field and the external one which is attached to. In the model we are going to apply this observation, we can in fact remove the gauge-field from the whole expectation value, leaving only the scalar curvature perturbation ζ attached to each vertex. Of course, this is just a trick that could simplify the procedure to get the final expectation value. Having only one field attached to each vertex has not a physical meaning in terms of connected diagrams. We will give an example of this new set of rules also in the section dedicated to the mixed bispectra.
Here, each vertex is represented as an empty circle, while the external fields as a full square. Both gravitons and electromagnetic fields are depicted as wiggly lines. However, since we are going to study separately inflationary models with vector and tensor fields, the correct interpretation will be always clear. In this thesis, we are going to consider tree-level diagram only, thus we will never encounter any loop. For more details about the in-in formalism's diagrammatic rules and their derivation, one can look in [38].

2.2 The ADM formalism

The Arnowitt-Deser-Misner (ADM) formalism enables us to rewrite the Einstein's field equations in a way that allows a Hamiltonian formulation of General Relativity [47]. In doing that, as we will very briefly mention, the ADM metric has the advantage of describing the Einstein's theory of gravity reducing the number of degrees of freedom to the minimum, by separating the physical from the fictitious ones [48]. The formalism is based on the idea that the covariance of General Relativity under a coordinate change introduces some redundant variables, which make it difficult to analyse the dynamics of the gravitational field, and define of its Hamilton's equations.
Consider a classical mechanical system with a number n of degrees of freedom. Its action, can be written as

$$S = \int_{t_i}^{t_f} d^t L = \int_{t_i}^{t_f} dt \left(\sum_{l=1}^{n} p_l \dot{q}_l - H(p,q) \right), \tag{2.30}$$

where here the dot indicates the usual time derivative. We see that L is expressed here linearly in time derivatives. p_i and q_i can be treated as independent variables, and since by varying them we obtain the so called first-order equations of motion, the above Lagrangian could be referred to as the first-order or canonical form. However, it is possible to rewrite this action in a more compact way. By considering the time t as a further degree of freedom q_{n+1}, which is treated as a function of a new parameter τ, we can write

$$S = \int_{\tau_i}^{\tau_f} d\tau L_\tau = \int_{\tau_i}^{\tau_f} \left(\sum_{l=1}^{n+1} p_l q_l' \right), \tag{2.31}$$

where here $' = d/d\tau$. In order to make the two Lagrangians perfectly equivalent, we need to impose the constraint equation $p_{n+1} + H(p,q) = 0$. In fact, by starting from Eq.2.31, we have

$$\begin{aligned} S &= \int_{\tau_i}^{\tau_f} d\tau \left(p_1 q_1' + p_2 q_2' + ... + p_n q_n'\right) = \int_{\tau_i}^{\tau_f} d\tau \left(p_1 q_1' + ... + p_n q_n' - H(p,q) q_{n+1}'\right) \\ &= \int_{\tau_i}^{\tau_f} d\tau \frac{dq_{n+1}}{d\tau} \left(\sum_{l=1}^{n} p_l \frac{dq_l}{dq_{n+1}} - H \right) = \int_{q_{n+1}^i}^{q_{n+1}^f} dq_{n+1} \left(\sum_{l=1}^{n} p_l \frac{dq_l}{dq_{n+1}} - H \right), \end{aligned} \tag{2.32}$$

which perfectly resembles Eq.2.30 if we put $t = q_{n+1}$. Thus, we see that we can rewrite the usual action for a classical system in a form that includes the time t as a further degree of freedom. The price to pay in doing so is that we introduced a new parameter τ, and we had to specify a constraint equation for p_{n+1}. It is possible to write the parametrized action in a more general way. Instead of specify the constraint equation, one could replace it by adding the additional term $-NR(p_{n+1}, p, q)$ to Eq.2.31. Here, $N(\tau)$ plays the role of a Lagrange multiplier, and by varying it, we get the constraint equation $R = 0$. This in principle can be any equation with solution our condition $p_{n+1} = -H$ [48]. The so written action

$$S = \int_{\tau_i}^{\tau_f} d\tau \left(\sum_{l=1}^{n+1} p_l q_l' - NR \right) \tag{2.33}$$

is now covariant under a "coordinate" transformation $\tau \to \tilde{\tau}$. If on the one hand the above action gained covariance, on the other one in order to reach this result we had to introduced another degree of freedom, carried by the Lagrange multiplier N. However, we know that N, rather than being physical, arises from the performed parametrization. In fact, by computing the equation of motion for q_{n+1}, it follows $q_{n+1}' = N \frac{\partial R}{\partial q_{n+1}}$. Since no other equations determine N, this shows that our multiplier is fully specified by q_{n+1} and viceversa. Therefore, these two interdependent degrees of freedom are left arbitrary by the dynamics, so that we can choose whichever function q_{n+1} of τ and use it as a new independent variable. This whole procedure has shown that one can go from a parametrized form of the action to the first-order or canonical one by inserting the solution of the constraint equation and imposing coordinate conditions, which are given by varying the action with respect to the newly-introduced variables. This in turn leads to the introduction of an intrinsic coordinate q_{n+1}, whose specification fixes N and hence the dynamics. Eq.2.30 is then determined by $q_{n+1} = t$ (thus $N = 0$) and the imposition of the constraint equation $p_{n+1} = -H$.
In a parametrized form then, a theory is invariant under a re-parametrization, just as General Relativity is invariant (in its expression) under a general change of coordinates. Following this line of reasoning, General Relativity can be seen as in an "already parametrized" form [48]. For this reason, it could be useful to rewrite the theory in its canonical expression. It would allows to precisely determine its dynamical modes, as in the first-order form one has the minimum number of physical fields. Moreover, a theory's canonical form leads directly to simple Poisson brakets relations, which are essential to quantize it. By looking at the first order action we wrote (Eq.2.30), we can see that it immediately yields to the well known Hamilton's equations $\dot{q}_i = \partial H/\partial p$, $\dot{p}_i = -\partial H/\partial q$. These show two important features of canonical form: firstly, the fields' equations involve first-order time derivative; secondly, the time has been single out, thereby recasting the theory in a $3+1$ form [48]. The ADM formalism is based exactly on these two goals. The first one can be achieved by rewriting the Hilbert-Einstein action in its Palatini's form. This was constructed by considering the metric $g_{\mu\nu}$ and the connection $\Gamma^\alpha_{\mu\nu}$ as two independent quantities [49]. From what we saw above, by varying the action with respect to these two variables we gain the Einstein's field equations and the constraints which define the two fields with respect to one another. The Palatini action is given by

$$S^P = \int d^4x \mathfrak{g}^{\mu\nu} R_{\mu\nu}(\Gamma), \tag{2.34}$$

where $\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ and $R_{\mu\nu}$ is the usual Ricci tensor, which is now dependent only on $\Gamma^\alpha_{\mu\nu}$. By varying the action with respect to the metric field, it is straightforward to find the EFEs, while the constraint equation between $g_{\mu\nu}$ and $\Gamma^\alpha_{\mu\nu}$ is found through the equations of motion for the connection [48]. In fact, we have

$$\frac{\delta S}{\delta g^{\mu\nu}} = \int d^4x \left(-\frac{1}{2} g_{\mu\nu} R + R_{\mu\nu} \right) \sqrt{-g} = 0, \tag{2.35}$$

where we used $\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-g}}{2} g_{\mu\nu}$, and, from $R_{\mu\nu}(\Gamma) = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} - \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu}$,

$$\mathfrak{g}^{\mu\nu} \frac{\partial R_{\mu\nu}}{\partial \Gamma^\tau_{\omega\rho}} - \partial_\xi \frac{\partial \mathfrak{g}^{\mu\nu} R_{\mu\nu}}{\partial \partial_\xi \Gamma^\tau_{\omega\rho}} = \Gamma^\alpha_{\alpha\tau}\mathfrak{g}^{\omega\rho} - \Gamma^\rho_{\nu\tau}\mathfrak{g}^{\omega\nu} - \Gamma^\omega_{\tau\mu}\mathfrak{g}^{\rho\mu} - \mathfrak{g}^{\omega\rho}_{\tau} + \left(\Gamma^\omega_{\mu\nu}\mathfrak{g}^{\mu\nu} + \mathfrak{g}^{\omega\nu}_{,\nu} \right) \delta^\rho_\tau = 0. \tag{2.36}$$

Now, for $\tau = \rho$ we just find an equivalence between two contracted Christoffel's symbols. However, since in principle $\tau \neq \rho$, we obtain

$$\mathfrak{g}^{\omega\rho}{}_{,\tau} + \Gamma^\rho_{\nu\tau}\mathfrak{g}^{\omega\nu} + \Gamma^\omega_{\tau\mu}\mathfrak{g}^{\rho\mu} - \mathfrak{g}^{\omega\rho}\Gamma^\alpha_{\tau\alpha} = 0, \tag{2.37}$$

which can be solved for the Christoffel's symbols getting the usual $\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha} \right)$.
For the second purpose, that is to split the space-time in $3+1$ we can imagine to foliate the four-dimensional spacetime into a one parameter family of three-hypersurfaces. We do not enter into the mathematical details of that operation. We would rather briefly explain the physical interpretation of this procedure and its final result, following the treatment done in [50]. As parameter, we choose the embedding time t, so that our spacetime can be seen as $\mathcal{M} = \left\{ \Sigma_t \middle| t \in \mathbb{R} \right\}$, and described by coordinates $x^\mu = (t, x^i)$. Each three-hypersurface is then specified by a three-geometry ${}^3g_{ij}(t, x, y, z)$. We consider now two constant-time three-slices. The upper would be the one at $t + dt$, whereas the lower is given at t. In order to describe the embedding spacetime, we need to known the three-geometries of each hypersurface, given respectively by ${}^3g_{ij}(t + dt, x, y, z)$ and ${}^3g_{ij}(t, x, y, z)$, and a way to describe how one could go from the lower to the upper slice. To do so, since they are separated only in time, their separating proper "distance" is simply the amount (lapse) of time between them, which in general can be written as $N(t, x, y, z)dt$. In the same way, we need to fix a coordinate system in both slices. We put $x^i_{low} = x^i$ and $x^i_{upper} = x^i - N^i(t, x, y, z)dt$. Here N^i describe the coordinates' shift that can occur moving through the foliation. The embedding four-geometry is then described by the usual line-element $ds^2 = -dt^2 + dl^2$ between $x^\mu = (t, x^i)$ and $x^\mu + dx^\mu = (t + dt, x^i + dx^i)$. Here $x^\mu(t)$ is chosen to belong in the lower slice (the corresponding upper-point given by the previous relations), and $x^\mu(t + dt)$ in the upper one. For the time interval dt^2 we already found Ndt. For the spatial contribution, we stay fixed to the upper hypersurface, getting $dl^i(t + dt) = (x^i + dx^i)(t + dt) - x^i_{up}(t + dt) = dx^i + N^i dt$. Therefore, in the end we find [50]

$$ds^2 = -(N^2 - N_i N^i)dt^2 +^3 g_{ij} dx^i dx^j + N_i dx^i dt + N_j dx^j dt, \tag{2.38}$$

the corresponding metric and its inverse respectively being

$$g_{\mu\nu} = \begin{pmatrix} -(N^2 - N_i N^i) & N_j \\ N_i & {}^3g_{ij} \end{pmatrix} \qquad \text{and} \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & \left({}^3g^{ij} - \frac{N^i N^j}{N^2} \right) \end{pmatrix}, \tag{2.39}$$

where one can easily derive the latter from the former by imposing $g_{\mu\alpha} g^{\alpha\nu} = \delta^\nu_\mu$. Now, by inserting these metric components inside the Christoffel's symbols' definition, one can find them expressed in the ADM formalism. Then, also the Riemann tensor, the Ricci tensor and scalar can be found in the same way, thereby giving a final expression for the Hilbert-Einstein action in this newly defined metric. We report only the final results in the following. A pretty useful result concerns instead the determinant of the metric, which reads[1]

$$\sqrt{-g} = N\sqrt{\det({}^3g_{ij})} = N\sqrt{{}^3g}. \tag{2.40}$$

Before proceeding in showing the aforementioned results, we need to introduce a couple of variables which are essential in the ADM formalism. We introduced this new way of expressing General Relativity with a parallelism with classical mechanics, in which the two independent variables describing the dynamics of a system are the position q and its conjugate momentum p. Here, we should do the same for each slice of the three-spacetime. However, while the first variable we need is given by the three-metric ${}^3g_{ij}$, the quantity which plays the role of a momentum is more difficult to find, as it refers to a motion in time, which can happen only by exiting the three-surface. This quantity, of fundamental relevance in this formalism, is the so called *extrinsic curvature* K_{ij} [47, 48]. Geometrically, it gives the radii of curvature of a three-slice as observed from the embedding spacetime. Another way to define it, would be given by imagining a series of normal vectors to a surface. These normal vectors are given by $n_\mu = (-N, \vec{0})$ (hence, $n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right)$). The extrinsic curvature provides the curvature's radius of a slice by describing how these normal vectors converge or diverge with respect to each other. In this sence, K_{ij} represents the covariant spatial part of the four-dimensional covariant derivative of the unit normal vector n_μ [48][2]. Thus, we have

$$\begin{aligned} K_{ij} = -\nabla_i n_j &= -\,\partial_i n_j + \Gamma^\mu_{ij} n_\mu = -\Gamma^0 ij N \\ &= \frac{1}{2N}\left({}^3\nabla_i N_j +^3 \nabla_j N_i - {}^3\dot{g}_{ij}\right), \end{aligned} \tag{2.41}$$

where ${}^3\nabla_i$ is the spatial covariant derivative referred to the three metric ${}^3g_{ij}$, which reads ${}^3\nabla_i N_j = \partial_i N_j -^3 \Gamma_{ijk} N^k$, with ${}^3\Gamma_{ijk} \equiv \Gamma_{ijk} = \frac{1}{2}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right)$. Since we want to rewrite the action, we need to compute $\sqrt{-g}R = N\sqrt{{}^3g}R$. The Ricci scalar reads

$$R = {}^3R + K_{ij}K^{ij} + K^i_i K^j_j - \frac{2}{N}\dot{K}^i_i + 2\frac{N^j}{N}\,{}^3\nabla_j K^i_i - \frac{2}{N}\,{}^3g^{ij}\,{}^3\nabla^3_i\nabla_j N, \tag{2.42}$$

which gives us [51]

$$\sqrt{-g}R = \sqrt{{}^3g}N\left({}^3R + K_{ij}K^{ij} - K^i_i K^j_j\right) - 2\sqrt{{}^3g}N\nabla_\mu\left(K^i_i n^\mu\right) - 2\sqrt{{}^3g}\,{}^3\Delta N. \tag{2.43}$$

By rewriting this last result in way that allows the appearance of total derivative terms, which can hereby integrated out in the action, in the end we find

$$S = \int dt d^3x \sqrt{{}^3g}N\left({}^3R + K^{ij}K_{ij} - K^i_i K^j_j\right). \tag{2.44}$$

[1] Given a block matrix $G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ if D is invertible, then $\det(G) = \det(A - BD^{-1}C)\det(G)$. In our case, we have $\det(g_{\mu\nu}) = \det(-(N^2 - N_i N_j\,{}^3g^{ij}))\det({}^3g_{ij})$

[2] To show the unitarity of this vector, it is sufficient verify that $n_\mu n^\mu = -1$

2.3 ζ correlation functions

We learnt that in order to calculate cosmological correlation functions by means of the in-in formalism, we need the interacting Hamiltonian. Also, we saw that to calculate the correlator between n fields, we need the Hamiltonian to be perturbed at n-th order. An exercise that can be done to get used to the in-in formalism and the ADM metric is given by the computation of the bispectrum of the scalar curvature field ζ. The first step is to compute the second-order action, which allows us to get the equation of motion for that field.
We consider the usual Hilbert-Einstein action referred to a slow-roll model of inflation (Sec.1.3). Since we have to consider perturbations of the metric as well as of the field, it is more convenient to work within the ADM formalism. By considering $M_{Pl} = 1$, and defining $E_{ij} = NK_{ij}$, the aforementioned action reads

$$S = \frac{1}{2} \int dtd^3x\sqrt{h}N \left[^{(3)}R + 2X - 2V + N^{-2}\left(E^{ij}E_{ij} - E^2\right)\right], \tag{2.45}$$

where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. Hence, by using the newly defined ADM metric (Eq.2.39), it follows

$$X = \frac{1}{2}\left[N^{-2}\left(\dot{\phi}^2 - 2N^i\dot{\phi}\partial_i\phi + N^iN^j\partial_i\phi\partial_j\phi\right) - h^{ij}\partial_i\phi\partial_j\phi\right]. \tag{2.46}$$

The action now reads

$$S = \frac{1}{2}\int d^4x\sqrt{h}N\left[^3R + N^{-2}\left(E_{ij}E^{ij} - E^2\right) + N^{-2}\left(\dot{\phi} - N^j\partial_j\phi\right)^2 - h^{ij}\partial_i\phi\partial_j\phi - 2V(\phi)\right]. \tag{2.47}$$

As we mentioned in the previous section, the ADM formalism allows us to recognise the physical fields from the "parametric" ones. The lapse and shift functions N, and N_i are our Lagrange multiplier, that tell us how we can pass from a three-slice at time t to another at time $t + dt$ [48]. Since their are not dynamical, we can find their expression and putting them back into the action. In this way, we reduce the number of degrees of freedom by two and we can study the dynamics of the remaining ones with fixed N and N_i. To do so, we can calculate their equation of motion through the Euler-Lagrange equations, which now are given by

$$\frac{\partial\mathcal{L}}{\partial N} -^3\nabla_k\frac{\partial\mathcal{L}}{\partial\partial_kN} = 0 \qquad \text{and} \qquad \frac{\partial\mathcal{L}}{\partial N^i} -^3\nabla_k\frac{\partial\mathcal{L}}{\partial\, ^3\nabla_kN^i} = 0. \tag{2.48}$$

From the former, we see that only the first term is different from zero, as there are no contributions proportional to $N_{,i}$. Therefore we find the so called *Hamiltonian constraint* to be

$$^3R - \frac{1}{N^2}\left(E_{ij}E^{ij} - E^2\right) - \frac{1}{N^2}\left(\dot{\phi} - N^i\partial_i\phi\right)^2 - h^{ij}\partial_i\phi\partial_j\phi - 2V(\phi) = 0. \tag{2.49}$$

For the latter, we have $E_{ij} = E_{ij}(N^l)$. From Eq.2.41, one finds

$$\begin{aligned} \frac{\partial E_{ij}}{\partial N^l} &= 0 \\ \frac{\partial E_{ij}}{\partial\, ^3\nabla_kN^l} &= \frac{1}{2}\left(\delta_i^k\delta_{jl} + \delta_j^k\delta_{il}\right) \\ \frac{\partial E_i^i}{\partial\, ^3\nabla_kN^l} &= h^{in}\frac{\partial E_{in}}{\partial\, ^3\nabla_kN^l} = h_l^k. \end{aligned} \tag{2.50}$$

Now, only term in the Lagrangian that depend upon the three-covariant derivative of the shift functions is $\left(E_{ij}E^{ij} - E^2\right)$. We have

$$\frac{\partial \mathcal{L}}{\partial\, {}^3\nabla_k N^l} = \frac{\partial}{\partial\, {}^3\nabla_k N^l}\left(E_{ij}E^{ij} - E^i_i E^j_j\right) = \left(2E^{ij}\frac{\partial E^{ij}}{\partial\, {}^3\nabla_k N^l} - 2E^j_j h^{mi}\frac{\partial E_{im}}{\partial^3\, \nabla_k N^l}\right) \\ = N^{-1}\left(2E^k_l - 2E^n_n h^k_l\right), \tag{2.51}$$

and

$$\frac{\partial \mathcal{L}}{\partial N^l} = 2N^{-1}\left(\dot{\phi} - N^j\partial_j\phi\right)\partial_l\phi. \tag{2.52}$$

Hence, in the end the *momentum constraint* reads

$${}^3\nabla_j\left[N^{-1}\left(E^j_i - E\delta^j_i\right)\right] = N^{-1}\left(\dot{\phi} - N^l\partial_l\phi\right)\partial_i\phi. \tag{2.53}$$

Before proceeding in the treatment, it is useful to choose a gauge, in order to simplify the computation. Since we want to calculate the power spectrum and the bispectrum for the scalar curvature field ζ, it is convenient to pass to the comoving gauge

$$\delta\phi = 0 \qquad \text{and} \qquad h_{ij} = e^{2\rho}\left[e^{2\zeta}\delta_{ij} + \chi^T_{ij}\right], \tag{2.54}$$

where χ^T_{ij} represents gravitational waves, and $e^\rho = a(t)$ is the scale factor, from which it follows $H = \frac{\dot{a}}{a} = \dot{\rho}$. By aplying it, since the only spatial-dependent contribution of the inflaton comes from its perturbations, we are left with [38]

$${}^3R - N^{-2}(E_{ij}E^{ij} - E^2) - N^{-2}\dot{\phi}^2 - 2V = 0, \tag{2.55}$$

$${}^3\nabla_j[N^{-1}(E^j_i - \delta^j_i E)] = 0. \tag{2.56}$$

We note that no conditions have been expressed for the potential. The only thing we require is for it to satisfy the slow-roll conditions, which implies $V = 3H^2 - \frac{1}{2}\dot{\phi}^2$. To proceed in finding the expression for the lapse and shift functions, we can approximate each of them at first order in scalar perturbations: $N \approx 1 + 2\Phi(t, \vec{x})$, $N^i = \tilde{N}^i + \partial^i B \approx \partial^i B$. This is because, to calculate the quadratic and cubic action for ζ it is not necessary to compute N and N^i to orders higher than one [52]. Besides, since we are interested only in the scalar part perturbations of the metric, we can forget about the tensor contributions ($\chi^T_{ij} = 0$). We start with the Hamiltonian constraint. The first step to do so is to calculate the Chrystoffel symbol, for which we have.

$$\Gamma^k_{ij} = \frac{1}{2}h^{kl}(h_{il,j} + h_{jl,i} - h_{ij,l}) = \partial_j\zeta\delta^k_i + \partial_i\zeta\delta^k_j - \partial^k\zeta\delta_{ij}. \tag{2.57}$$

Hence, for the spatial Ricci curvature we obtain

$$\begin{aligned} {}^3R &= h^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^l_{ij}\Gamma^k_{kl} - \Gamma^l_{ik}\Gamma^k_{jl}\right) \\ &= a^{-2}e^{-2\zeta}\delta^{ij}\Big[2\partial_i\partial_j\zeta - \partial_k\partial^k\zeta\delta_{ij} - 3\partial_i\partial_j\zeta + \left(\partial_j\zeta\delta^l_i + \partial_i\zeta\delta^l_j - \partial^l\zeta\delta_{ij}\right)3\partial_l\zeta \\ &\quad - \left(\partial_k\zeta\delta^l_i + \partial_i\zeta\delta^l_k - \partial_l\zeta\delta_{ik}\right)\left(\partial_j\zeta\delta^k_l + \partial_l\zeta\delta^k_j - \partial^k\zeta\delta_{jl}\right)\Big] \\ &= -4a^{-2}\partial_k\partial^k\zeta + \mathcal{O}(\zeta^2), \end{aligned} \tag{2.58}$$

where the Taylor expansion at first order $e^{-2\zeta} \approx 1 - 2\zeta$ was used. In the same way, by observing that $N^k = \delta^{kl}\partial_l B \sim \mathcal{O}(\zeta)$, for the extrinsic curvature E_{ij} we have

$$\begin{aligned} E_{ij} &= -\frac{1}{2}\left[\dot{h^{ij}} - h_{ik}\partial_j N^k - h_{jk}\partial_i N^k - N^k\partial_k h_{ij}\right] \\ &= -\frac{1}{2}a^2 e^{2\zeta}\left[2(H+\dot{\zeta})\delta_{ij} - \delta_{ik}\partial_j N^k - \delta_{jk}\partial_i N^k - N^k\delta_{ij}\partial_k\zeta\right] \\ &= -\frac{1}{2}a^2\left[(H + H\zeta + \dot{\zeta})\delta_{ij} - \partial_i\partial_j B\right] + \mathcal{O}(\zeta^2). \end{aligned} \tag{2.59}$$

Then, by considering $E^i_i = E_{ij}h^{ji}$ it is easy to show that

$$E^2 = \left(E^i_i\right)^2 = 9H^2 + 18H\dot{\zeta} - 6H(\partial_k\partial^k B) + \mathcal{O}(\zeta^2), \tag{2.60}$$

whence it follows that

$$E_{ij}E^{ij} - E^2 = -6H^2 - 12H\dot{\zeta} + 4H(\partial^2 B) + \mathcal{O}(\zeta^2). \tag{2.61}$$

Now we have all the ingredients needed to compute the Hamiltonian constraints. We just have to sum up all the terms and truncate them at first order in scalar fluctuations. We get

$$-\partial_k\partial^k\left[a^{-2}\zeta + HB\right] - 3H\left[2\Phi H - \dot{\zeta}\right] + \Phi\dot{\phi}^2 = 0. \tag{2.62}$$

For the momentum constraint we have to evaluate only E^j_i, since the other element we need has already been computed above. By using again the metric to raise indices we have

$$E^j_i = E_{il}h^{lj} = -(H+\dot{\zeta})\delta^j_i + \partial_i\partial^j B + \mathcal{O}(\zeta^2). \tag{2.63}$$

Then, by considering $N^{-1} \approx 1 - 2\Phi(t, \vec{x})$, we obtain

$$\begin{aligned} &{}^3\nabla_j\left[(1-2\Phi)\left((H+\dot{\zeta})\delta^j_i - \partial_i\partial^j B - \delta^j_i(3H + 3\dot{\zeta} - \partial_l\partial^l B)\right)\right] = 0 \\ &{}^3\nabla_j\left[-2(H+\dot{\zeta} - 2H\Phi)\delta^j_i - \partial_i\partial^j B + \delta^j_i\partial_l\partial^l B\right] = 0. \end{aligned} \tag{2.64}$$

As we saw above, the Crystoffel's symbols are of order one in scalar fluctuations. Hence, when we apply the covariant derivative on the left hand side, they can be forgotten. In fact, they act only on the tensors present in the equation, which are already of order one in scalar perturbations. By focusing for a moment only on these tensor terms we have $-\partial_j\partial_i\partial^j B + \Gamma^j_{jl}\partial_i\partial^l B - \Gamma^l_{ji}\partial_l\partial^j B + \partial_i\partial_l\partial^l B + \Gamma^k_{il}\partial_k\partial^l B - \Gamma^l_{ik}\partial_l\partial^k B \approx -\partial_i\partial_j\partial^j B + \partial_i\partial_l\partial^l B = 0$. Therefore our momentum constraint turns out to be simply

$$2\partial_i(2H\Phi - \dot{\zeta}) = 0, \tag{2.65}$$

whence we can easily obtain the expression for Φ at first order in scalar perturbations:

$$\Phi^{(1)} = \frac{\dot{\zeta}}{H}. \tag{2.66}$$

Putting now this form of N inside Eq.2.62 leads to

$$\begin{aligned}\partial_k\partial^k\left[a^{-2}\zeta+HB\right]-\dot{\phi}^2\frac{1}{2}\frac{\dot{\zeta}}{H}=0\\ \partial_k\partial^k B=-\frac{a^{-2}}{H}\partial_k\partial^k\zeta+\frac{\dot{\phi}^2}{2}\frac{\dot{\zeta}}{H}.\end{aligned}\tag{2.67}$$

At the very end, we can rewrite the lapse and shift functions at first order in scalar perturbations as [38]

$$N^{(1)}=1+\frac{\dot{\zeta}}{H},\quad N_i^{(1)}=\partial_i B,\quad B=-\frac{\zeta}{a^2H}+\frac{\dot{\phi}^2}{2H^2}\partial^{-2}\dot{\zeta},\tag{2.68}$$

where $\frac{\dot{\phi}^2}{2H^2}=\epsilon$, as $M_{Pl}=1$.

2.3.1 Quadratic action and ζ power spectrum

As a first example for the application of the in-in formalism, we calculate the power spectrum for the scalar curvature field ζ. It corresponds to the two-point correlation function, therefore we need the corresponding action to be perturbed up to second-order in ζ. The usual action of the inflaton's field minimally coupled with gravity and expressed in the ADM formalism reads

$$S=\frac{1}{2}\int d^4x\sqrt{h}N\left({}^3R-N^{-2}(E_{ij}E^{ij}-E^2)+(1+N^{-2})\dot{\phi}^2-6\dot{\rho}^2\right),\tag{2.69}$$

which can be rewrite as

$$S=\int d^4x\left\{\sqrt{h}N\,{}^3R-\frac{\sqrt{h}}{N}\left(E_{ij}E^{ij}-E^2\right)+\left(\sqrt{h}N+\frac{\sqrt{h}}{N}\right)\dot{\phi}^2-6\sqrt{h}NH^2\right\}.\tag{2.70}$$

Now, we proceed by perturbing each piece up to $\mathcal{O}(\zeta^2)$. Instead of expanding each term up to the desired order thereby simplifying the expression, we will firstly derive all the exact forms of the different factors. In this way calculations remain more restricted in the number of terms and we can make some simplifications more easily. The exact expressions for the spatial Ricci's curvature,the square root of the determinant of the metric, and the extrinsic curvature tensor E_{ij} are given respectively by

$$\hat{R}=a^{-2}e^{-2\zeta}\left(-4\partial_i\partial^i\zeta-2\partial_i\zeta\partial^i\zeta\right),\tag{2.71}$$

$$\sqrt{h}=\sqrt{\det\left(a^2e^{2\zeta}\delta_{ij}\right)}=a^3e^{3\zeta},\tag{2.72}$$

$$E_{ij}=a^2e^{2\zeta}\left[\left(\dot{\rho}+\dot{\zeta}-\partial_k\zeta\partial^k B\right)\delta_{ij}-\partial_i\partial_j B\right].\tag{2.73}$$

Therefore, we have, by remembering that $N,N^i\sim\mathcal{O}(\zeta)$ (Eq.2.68)

$$\left(\sqrt{h}N\right)^{(2)}=-2a\left[\partial_i\zeta\partial^i\zeta+2\zeta\partial_i\partial^i\zeta+\frac{\dot{\zeta}}{H}\partial_i\partial^i\zeta\right],\tag{2.74}$$

$$\left(\frac{\sqrt{h}}{N}\right)^{(2)} = a^3\left[\left(\frac{\dot{\zeta}}{H}\right)^2 - 3\frac{\zeta\dot{\zeta}}{H} + 9\zeta^2\right], \tag{2.75}$$

$$\begin{aligned}\left(\frac{\sqrt{h}}{N}(E_{ij}E^{ij} - E^2)\right)^{(2)} =& a^3\left(12H\partial_i\zeta\partial^i B - 6\dot{\zeta}^2 + 4\dot{\zeta}\partial_i\partial^i B + (\partial_i\partial_j B)^2 - (\partial_i\partial^i B)^2\right)\\ &+a^3\left(3\zeta - \frac{\dot{\zeta}}{H}\right)\left(4H\partial_i\partial^i B - 12H\dot{\zeta}\right) + 6a^3 H\left(\left(\frac{\dot{\zeta}}{H}\right)^2 - 3\frac{\zeta\dot{\zeta}}{H} + \frac{9}{2}\zeta^2\right).\end{aligned} \tag{2.76}$$

By inserting all these blocks into the action, we find

$$\begin{aligned}S = \int d^4x\Bigg\{& -2a\partial_i\partial^i\zeta - 2a\partial_i\partial^i\zeta\left(2\zeta + \frac{\dot{\zeta}}{H}\right) - 12a^3H\partial_i\zeta\partial^i B - 12a^3H\zeta\partial_i\partial^i B +\\ &- a^3\left[(\partial_i\partial_j B)^2 - (\partial_i\partial^i B)^2\right] - 9a^3\zeta^2H^2 + 27a^3H^2\Bigg\},\end{aligned} \tag{2.77}$$

where we used $\epsilon = -\frac{\dot{\phi}^2}{2H}$. Now, by integrating by parts to get rid of some terms, which can be rewritten as total derivatives (i.e. non-dynamical terms), we can simplify the action a lot. For instance, we focus on the last two terms of the first line. Specifically, the second one can be rewritten by using the Leibniz rule as $-12a^3H\zeta\partial_i\partial^i B = \partial_i[-12a^3H\zeta\partial^i B] + 12a^3H\partial_i\zeta\partial^i B$. Thus, the last term simplifies to zero with the third piece of the action, while the total derivative term can be neglected. In this way, in the end one finds [52]

$$S^{(2)} = \int dt d^3x\, a\epsilon\left[a^2\dot{\zeta}^2 - \partial_i\zeta\partial^i\zeta\right]. \tag{2.78}$$

Now that we have the second-order action of the scalar curvature field, we can use it to find its equation of motion. By varying the action with respect to our field we obtain [52]

$$\frac{d}{d\tau}\left(a^2\epsilon\zeta'\right) - a^2\epsilon\partial_i\partial^i\zeta = 0. \tag{2.79}$$

Since it describes the motion of a free-field, we can decompose it in Fourier modes getting

$$\zeta(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}\zeta_{\vec{k}}(\tau). \tag{2.80}$$

Before continuing to find the ζ's power spectrum, we need to find its mode function. To do that, as we done for the inflaton's field, we have to solve its equation of motion. First of all, we start to quantize the field by rewriting it as an operator

$$\zeta_{\vec{k}}(\tau) = u^\zeta_k(\tau)a_{\vec{k}} + u^{\zeta*}_k(\tau)a^\dagger_{-\vec{k}} = \zeta^+_{\vec{k}}(\tau) + \zeta^-_{\vec{k}}(\tau). \tag{2.81}$$

Thus, by passing in the Fourier space and neglecting $\mathcal{O}(\epsilon^2)$ terms, we find

$$u_k''^{\zeta} + 2\frac{a'}{a}u_k'^{\zeta} + k^2 u_k^{\zeta} = 0, \tag{2.82}$$

which is exactly the equation we found for the inflaton's perturbations. To properly quantize ζ, we need now to specify its conjugate momenta and then its commutation relations. For the former, we find $\pi^{\zeta} = 2a^2\epsilon u_k'^{\zeta}$. Therefore, as Poisson brackets we obtain $\{\zeta(\vec{x},\tau), \zeta'(\vec{y},\tau)\} = (2a^2\epsilon)^{-1}\delta^{(3)}(\vec{x}-\vec{y})$. By promoting these to commutators we find the same normalization condition we obtained for the inflaton's perturbations (Eq.1.39). Hence, at the very end, we see that the scalar curvature field resembles the dynamics of a free scalar massless field, once the rescaling $\sqrt{2\epsilon}$ is taken into account. We thus have

$$u^{\zeta}(k,\tau) = \frac{iH}{\sqrt{4\epsilon k^3}}(1+ik\tau)e^{-ik\tau}. \tag{2.83}$$

We are now ready to find the ζ's two-points function, which reads

$$\begin{aligned}\left\langle \zeta_{\vec{k_1}}(\tau)\zeta_{k_2}(\tau)\right\rangle =& \langle 0|\zeta_{\vec{k_1}}(\tau)\zeta_{\vec{k_2}}(\tau)|0\rangle = \langle 0|\zeta^+_{\vec{k_1}}(\tau)\zeta^-_{\vec{k_2}}(\tau)|0\rangle \\ =&(2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2})\frac{H^2}{4\epsilon k^3}(1+k^2\tau^2).\end{aligned} \tag{2.84}$$

2.3.2 Cubic action and ζ bispectrum

In the same way we computed the second-order action for the scalar curvature field, we can now derive its cubic action. Since the procedure is much more long that the one for the quadratic action, we highlight the main passages of the calculation. The same procedure could be followed to compute the cubic action for tensor and tensor-scalar perturbations. We start from Eq.2.70 and we proceed in perturbing these terms up to third order in ζ. To do that, again we start from the exact solution of Christoffel's symbols, three-dimensional Ricci scalar, and extrinsic curvature (Eqs.2.42,2.72,2.73). Then, the first term in the action reads

$$\begin{aligned}\left(\sqrt{h}N\,{}^3R\right)^{(3)} &= a^3e^{3\zeta}\left(1+\frac{\dot\zeta}{H}\right)a^{-2}e^{-2\zeta}\left(-4\partial_i\partial^i\zeta - 2\partial_i\partial^i\zeta\right)\Big|_{(3)} \\ &= -2a\left(\zeta^2\partial_i\partial^i\zeta + \zeta\partial_i\zeta\partial^i\zeta + \frac{2}{H}\dot\zeta\zeta\partial_i\partial^i\zeta + \frac{1}{H}\dot\zeta\partial_i\zeta\partial^i\zeta\right),\end{aligned} \tag{2.85}$$

where the Taylor expansion $e^{\zeta} \approx 1+\zeta+\frac{\zeta^2}{2}+\frac{\zeta^3}{6}$ was used and only third order terms were considered. In the same way, for the remaining pieces of the action we find

$$\left(\sqrt{h}N\right)^{(3)} = a^3\left(\frac{9}{2}\zeta^3 + \frac{9}{2}\frac{1}{H}\dot\zeta\zeta^2\right), \tag{2.86}$$

$$\left(\frac{\sqrt{h}}{N}\right)^{(3)} = a^3\left(\frac{9}{2}\zeta^3 - \frac{9}{2}\frac{1}{H}\zeta^2\dot\zeta + \frac{3}{H}\zeta\dot\zeta^2 - \left(\frac{\dot\zeta}{H}\right)^3\right), \tag{2.87}$$

$$\begin{aligned}\left(\frac{\sqrt{h}}{N}\left(E_{ij}E^{ij}-E^2\right)\right)^{(3)} &= a^3e^{3\zeta}\left[1-\frac{\dot\zeta}{H}+\left(\frac{\dot\zeta}{H}\right)^2-\left(\frac{\dot\zeta}{H}\right)^3\right]\Big[-6\left(H+\dot\zeta-\partial_k\zeta\partial^kB\right)^2\\ &\quad+4\left(H+\dot\zeta-\partial_k\zeta\partial^kB\right)\partial_i\partial^iB+(\partial_i\partial_jB)(\partial^i\partial^jB)-(\partial_i\partial^iB)^2\Big]\Big|_{(3)}\\ &= a^3\Big[-27H^2\zeta^3-27H\zeta^2\dot\zeta+36H\zeta\partial_k\zeta\partial^kB+18H\zeta^2(\partial_k\partial^kB)\\ &\quad-4\partial_k\zeta\partial^kB\partial_i\partial^iB+\left(3\zeta-\frac{\dot\zeta}{H}\right)\left(\partial_i\partial_jB\partial^i\partial^jB-(\partial_i\partial^iB)^2\right)\Big],\end{aligned} \tag{2.88}$$

where the expansions $N^{-1}\approx 1-\frac{\dot\zeta}{H}+\left(\frac{\dot\zeta}{H}\right)^2-\left(\frac{\dot\zeta}{H}\right)^3$ and $e^{3\zeta}\approx 1+3\zeta+\frac{9}{2}\zeta^2+\frac{9}{2}\zeta^3$ were used. We proceed now in writing the action in a form in which many terms are re-written as total derivatives. In this way they do not contribute to the action, being non-dynamical. We start to do so with the first term we expanded. From the last two contributions of $\left(\sqrt{h}N\,{}^3R\right)^{(3)}$, by using the Leibniz rule of derivation in order to remove time derivatives from ζ, we have

$$\begin{aligned}-4\frac{a}{H}\dot\zeta\zeta\partial_i\partial^i\zeta-2\frac{a}{H}\dot\zeta\partial_i\zeta\partial^i\zeta = &-2\partial_0\left\{\frac{a}{H}\zeta^2\partial_i\partial^i\zeta+\frac{a}{H}\zeta\partial_i\zeta\partial^i\zeta\right\}+2a\zeta^2\partial_i\partial^i\zeta+2\frac{a}{H}\zeta^2\partial_i\partial^i\dot\zeta-\frac{2\dot H}{H^2}a\zeta^2\partial_i\partial^i\zeta+\\ &+2a\zeta\partial_i\zeta\partial^i\zeta+4\frac{a}{H}\zeta\partial_i\dot\zeta\partial^i\zeta-\frac{2\dot H}{H^2}a\zeta\partial_i\zeta\partial^i\zeta.\end{aligned} \tag{2.89}$$

We see that the first and the fourth terms of this equation and the first two ones of Eq.2.85 simplify away, thereby giving us

$$\begin{aligned}\left(\sqrt{h}N\,{}^3R\right)^{(3)} &= -2a\frac{\dot H}{H}\zeta\partial_i\zeta\partial^i\zeta-2a\frac{\dot H}{H}\zeta^2\partial_i\partial^i\zeta-2\partial_0\left\{\frac{a}{H}\left(\zeta^2\partial_i\partial^i\zeta+\zeta\partial_i\zeta\partial^i\zeta\right)\right\}+\frac{2a}{H}\zeta^2\partial_i\partial^i\dot\zeta+\frac{4a}{H}\zeta\partial_i\zeta\partial^i\dot\zeta\\ &= -2\partial_0\left\{\frac{a}{H}\left(\zeta^2\partial_i\partial^i\zeta+\zeta\partial_i\zeta\partial^i\zeta\right)\right\}+2\partial_i\left\{\frac{a}{H}(\zeta^2\partial^i\dot\zeta)\right\}-\frac{2a}{H^2}\dot H\zeta^2\partial_i\partial^i\zeta-\frac{2a}{H^2}\dot H\zeta\partial_i\zeta\partial^i\zeta\\ &= -2\partial_0\left\{\frac{a}{H}\left(\zeta^2\partial_i\partial^i\zeta+\zeta\partial_i\zeta\partial^i\zeta\right)\right\}+2\partial_i\left\{\frac{a}{H}(\zeta^2\partial^i\dot\zeta)\right\}-2\partial_i\left\{\frac{a}{H^2}\dot H\zeta^2\partial^i\zeta\right\}+\frac{2a}{H^2}\dot H\zeta\partial_i\zeta\partial^i\zeta\\ &= -2\partial_0\left\{\frac{a}{H}\left(\zeta^2\partial_i\partial^i\zeta+\zeta\partial_i\zeta\partial^i\zeta\right)\right\}+2\partial_i\left\{\frac{a}{H^2}\zeta^2\left[H\partial^i\dot\zeta-\dot H\partial^i\zeta\right]\right\}-2a\left(\frac{\dot\phi}{H}\right)^2\zeta\partial_i\zeta\partial^i\zeta.\end{aligned} \tag{2.90}$$

We used the Leibniz rule to unify the last two terms of the first line and again to expand the term $\propto \dot H\zeta^2$ in the second line. At the end we used the slow-roll condition $\dot\phi\approx-\frac{V_{,\phi}}{3H}$, which implies $\dot H=-M_{Pl}^{-2}\frac{\dot\phi^2}{2}$. For the other terms we have:

$$\begin{aligned}
\frac{\sqrt{h}}{N}\left(E_{ij}E^{ij}-E^2\right)^{(3)} =& a^3\Big[-27H\zeta^3-27H\zeta^2\dot{\zeta}+18H\partial_k\zeta^2\partial^k B+18H\zeta^2\left(\partial_k\partial^k B\right)+\\
&-4\partial_k\zeta\partial^k B\partial_i\partial^i B+\left(3\zeta-\frac{\dot{\zeta}}{H}\right)\left(\partial_i\partial_j B\partial^i\partial^j B-(\partial_i\partial^i B)^2\right)\Big]\\
=& a^3\left[-4\partial_k\zeta\partial^k B\partial_i\partial^i B+\left(3\zeta-\frac{\dot{\zeta}}{H}\right)\left(\partial_i\partial_j B\partial^i\partial^j B-(\partial_i\partial^i B)^2\right)\right]+\\
&+\partial_k\left[18a^3H\zeta^2\partial^k B\right]-9\partial_0\left[Ha^3\zeta^3\right]+a^3\left[9\dot{H}\zeta^3\right],
\end{aligned} \tag{2.91}$$

where, in the first line, we integrated the first and the third term and we used the Leibniz rule on the second one;

$$\left(-6\sqrt{h}NH^2\right)^{(3)}=-6a^3\left[\frac{9}{2}\zeta^3+\frac{9}{2}\frac{1}{H}\dot{\zeta}\zeta^2\right]H^2, \tag{2.92}$$

and

$$\left(N+\frac{1}{N}\sqrt{h}\dot{\phi}^2\right)^{(3)}=\left[9\zeta^3\dot{\phi}^2-\dot{\phi}^2\frac{\dot{\zeta}^3}{H^3}+3\zeta\frac{\dot{\zeta}^2}{H^2}\dot{\phi}^2\right]a^3. \tag{2.93}$$

By combining them we find

$$\begin{aligned}
\sqrt{h}\left\{\frac{1}{N}\left(E_{ij}E^{ij}-E^2\right)-6NH^2+\left(N+\frac{1}{N}\right)\dot{\phi}^2\right\}=& a^3\frac{\dot{\phi}^2}{H^2}\left(3\zeta-\frac{\dot{\zeta}}{H}\right)\dot{\zeta}^2+9a^3\left[2\dot{H}+\dot{\phi}^2\right]\zeta^3-18\partial_0\left[Ha^3\zeta^3\right]+\\
&+\partial_k\left[18a^3H\zeta^2\partial^k B\right]+a^3\left[-4\partial_k\zeta\partial^k B\partial_i\partial^i B+\left(3\zeta-\frac{\dot{\zeta}}{H}\right)\left(\partial_i\partial_j B\partial^i\partial^j B-(\partial_i\partial^i B)^2\right)\right].
\end{aligned} \tag{2.94}$$

Thus, the action reads

$$\begin{aligned}
S=&\frac{1}{2}\int d^4x 2\epsilon a^3\left(3\zeta-\frac{\dot{\zeta}}{H}\right)\dot{\zeta}^2-2\epsilon a^3\zeta\partial_k\zeta\partial^k\zeta+\\
&+a^3\left[-4\partial_k\zeta\partial^k B\partial_i\partial^i B+\left(3\zeta-\frac{\dot{\zeta}}{H}\right)\left(\partial_i\partial_j B\partial^i\partial^j B-(\partial_i\partial^i B)^2\right)\right]+D_0,
\end{aligned} \tag{2.95}$$

where

$$\begin{aligned}
D_0=&-\partial_0\left[18a^3H\zeta^3+2\frac{a}{H}\zeta\partial_k(\zeta\partial^k B)\right]+\partial_k\left[18a^3H\zeta^2\partial^k B+2\frac{a}{H^2}\zeta^2(H\partial^k\zeta-\ddot{\rho}\partial^k\zeta)\right]+\\
&+9a^3[2\dot{H}+\dot{\phi}^2]\zeta^3+a\frac{1}{H^2}[2\dot{H}+\dot{\phi}^2]\zeta\partial_k\zeta\partial^k\zeta.
\end{aligned} \tag{2.96}$$

We see that the D_0 contribution contains all the non-dynamical terms we inserted so far to rewrite the action. Now, one can realize that while the first line is suppressed by ϵ, the second line does not show this trend at all. In order to make clear the true dependences on ϵ, we replace $B = -\frac{1}{a^2 H}\zeta + \chi$ with $\partial_i \partial^i \chi = \epsilon\zeta$. By following the same procedure sketched above, i.e. by trying to make total derivative terms manifest (Appendix B.1), at the very end we find

$$\begin{aligned} S =& \frac{1}{2}\int d^4x \Big\{ 2\epsilon^2 a^3 \dot{\zeta}^2 \zeta + 2\epsilon^2 a \zeta \partial_k \zeta \partial^k \zeta - 4\epsilon a^3 \dot{\zeta} \partial_k \zeta \partial^k \chi + \\ &+ 2\epsilon a^3 \dot{\zeta} \zeta^2 \partial_0 \left[\frac{\ddot{\phi}}{\dot{\phi} H} + \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \right] - \epsilon^3 a^3 \dot{\zeta}^2 \zeta + \epsilon a^3 \zeta \partial_k \partial_l \chi \partial^k \partial^l \chi + \mathcal{F} + \mathcal{D} \Big\}, \end{aligned} \tag{2.97}$$

where the last two terms represent respectively contributions proportional to the EOM of ζ, namely $\frac{d}{dt}\left[2\epsilon a^3 \dot{\zeta}\right] - 2\epsilon a \partial_k \partial^k \zeta = 0$, and total derivatives. Since several terms are proportional to the ζ's equation of motion [52], we can perform a trick that allows us to write the action in a further simplified expression. In fact, we can rewrite Eq.2.97 as

$$\begin{aligned} S^{(3)} =& M_{Pl}^2 \int d^4x \Big\{ \epsilon^2 a^3 \dot{\zeta}^2 \zeta + \epsilon^2 a \zeta \partial_i \zeta \partial^i \zeta - 2\epsilon^2 a^3 \dot{\zeta} \partial_i \zeta \partial^i \zeta \left(\partial^{-2}\dot{\zeta}\right) + \\ &+ \epsilon a^3 (\dot{\epsilon} - \dot{\eta}) \dot{\zeta} \zeta^2 - \frac{1}{2} a^3 \epsilon^3 \left[\dot{\zeta}\zeta^2 - \zeta \partial_i \partial_j \left(\partial^{-2}\dot{\zeta}\right) - \partial^i \partial^j \left(\partial^{-2}\dot{\zeta}\right) \right] + \\ &+ 2\epsilon \left\{ \frac{d}{dt}\left(a^3 \dot{\zeta}\right) - a \partial_i \partial^i \zeta \right\} f(\zeta) \Big\}, \end{aligned} \tag{2.98}$$

with

$$\begin{aligned} f(\zeta) =& \frac{1}{2}\left[\epsilon - \eta\right]\zeta^2 + \frac{\zeta\dot{\zeta}}{H} - \frac{1}{4a^2 H^2}\left[\partial_i \zeta \partial^i \zeta - \partial^{-2}\partial_i \partial_j \left(\partial^i \zeta \partial^j \zeta\right)\right] + \\ &+ \frac{1}{2}\epsilon \left[\partial_i \zeta \partial^i \left(\partial^{-2}\dot{\zeta}\right) - \partial^{-2}\partial_i \partial_j \left(\partial^i \zeta \partial^j \left(\partial^{-2}\dot{\zeta}\right)\right) \right], \end{aligned} \tag{2.99}$$

where $\eta = -\frac{\ddot{\phi}}{\dot{\phi} H}$ is the second slow-roll parameter. Since now $f(\zeta)$ is quadratic in the scalar field, we can perform a shift $\zeta \to \zeta_n + f(\zeta_n)$. This shift introduces cubic and quartic contributions to the expression we found for the action, which can be neglected. Moreover, because of the shift, we gain a third-order contribution also from the second order action for ζ. In fact, we have

$$\begin{aligned} S^{(2\to3)}_{\zeta_n+f} &= \int d^4x a^3 \epsilon \left(\dot{\zeta}^2 - \frac{1}{a^2}\partial_i \zeta \partial^i \zeta\right) \\ &\approx \int d^4x 2\epsilon a^3 \left(\dot{\zeta}_n \dot{f} - \frac{1}{a^2}\partial_i \zeta_n \partial^i f\right) \\ &\approx -\int d^4x f(\zeta_n) 2\epsilon \left\{ \frac{d}{dt}\left(a^3 \dot{\zeta}_n\right) - a \partial_i \partial^i \zeta_n \right\}, \end{aligned} \tag{2.100}$$

where in the penultimate passage we neglected higher order terms in the shifted field, and in the last passage we used the Leibniz rule to remove the time derivative on the function f. Furthermore, we neglected total derivatives terms that arose from the manipulation we done on $\dot{f}$. Hence, when we shift the field, we gain another contribution from the second order action which cancels away with the last line of Eq.2.98. Moreover, we note that the second line of that equation is suppressed by the slow-roll parameters with respect to the first line ($\mathcal{O}\left(\epsilon^3,\eta^3\right)$). Hence, in the end we can write

$$S_n^{(3)} = \int d^4x\epsilon^2 \left\{a^3\dot{\zeta_n}^2\zeta_n + a\zeta_n\partial_k\zeta_n\partial^k\zeta_n - 2a^3\dot{\zeta_n}\partial_k\zeta_n\partial^k(\partial^{-2}\dot{\zeta_n}) + ...\right\}, \tag{2.101}$$

where the dots indicate the $\mathcal{O}(\epsilon^2)$ terms.
Now that we have the third-order action, we can compute the bispectrum for scalar curvature fluctuations. To do so, we exploit the in-in formalism we introduced above through Eq.2.25. For the interaction Hamiltonian we find

$$H_{int}^{\zeta\zeta\zeta} = -\epsilon\int d^3x\int\frac{d^3p_1d^3p_2d^3p_3}{(2\pi)^9}e^{i(\vec{p_1}+\vec{p_2}+\vec{p_3})\vec{x}}a\left(\zeta'_{n\vec{p_1}}\zeta'_{n\vec{p_2}}\zeta_{n\vec{p_3}} - p_{2k}p_3^k\zeta_{n\vec{p_1}}\zeta_{n\vec{p_2}}\zeta_{n\vec{p_3}} - 2\frac{p_{2k}p_3^k}{p_3^2}\zeta'_{n\vec{p_1}}\zeta_{n\vec{p_2}}\zeta'_{n\vec{p_3}}\right). \tag{2.102}$$

However, before diving into the computation, we remember that we want to find the bispectrum of ζ, rather than the one of the shifted field ζ_n, which reads

$$\left\langle\zeta_{\vec{k_1}}(\tau)\zeta_{\vec{k_2}}(\tau)\zeta_{\vec{k_3}}(\tau)\right\rangle = \left\langle\zeta_{n\vec{k_1}}\zeta_{n\vec{k_2}}\zeta_{n\vec{k_3}}\right\rangle + \left\langle\zeta_{n\vec{k_1}}\zeta_{n\vec{k_2}}f\left(\zeta_{n\vec{k_3}}\right)\right\rangle + \left\langle\zeta_{n\vec{k_1}}f\left(\zeta_{n\vec{k_2}}\right)\zeta_{n\vec{k_3}}\right\rangle + \left\langle f\left(\zeta_{n\vec{k_1}}\right)\zeta_{n\vec{k_2}}\zeta_{n\vec{k_3}}\right\rangle. \tag{2.103}$$

Therefore, one should in principle compute the first term of the latter by means of in-in formalism, and the remaining three fourth-order terms just by contracting the different fields between themselves. We follow [53] in the procedure. Since the entire computation is quite long, we report here only the computation of the cubic part for two reasons. Firstly, only this term has to be solved through the in-in formalism; secondly, a similar computation for the $\langle\zeta\zeta f(\zeta)\rangle$ contribution will be explicitly done in the next session studying mixed correlators.
By applying Eq.2.25 we find

$$\begin{aligned}\left\langle\prod_{i=1}^3\zeta_{n\vec{k_i}}\right\rangle =& i\epsilon^2\int_{-\infty}^{\tau}d\tau_1a^2\int\frac{d^3p_1d^3p_2}{(2\pi)^6}\Big\langle\left[\zeta_{n\vec{k_1}}\zeta_{n\vec{k_2}}\zeta_{n\vec{k_3}},\zeta'_{n\vec{p_1}}\zeta'_{n\vec{p_2}}\zeta_{n-\vec{p_1}-\vec{p_2}}\right]+\\ &- p_{2k}p_3^k\left[\zeta_{n\vec{k_1}}\zeta_{n\vec{k_2}}\zeta_{n\vec{k_3}},\zeta_{n\vec{p_1}}\zeta_{n\vec{p_2}}\zeta_{n-\vec{p_1}-\vec{p_2}}\right] - 2\frac{p_{2k}p_3^k}{p_3^2}\left[\zeta_{n\vec{k_1}}\zeta_{n\vec{k_2}}\zeta_{n\vec{k_3}},\zeta'_{n\vec{p_1}}\zeta_{n\vec{p_2}}\zeta'_{n-\vec{p_1}-\vec{p_2}}\right]\Big\rangle\\ =& i\epsilon^2\int_{-\infty}^{\tau}d\tau_1a^2\int\frac{d^3p_1d^3p_2}{(2\pi)^6}\langle C_1 - p_{2k}p_3^kC_2 - 2\frac{p_{2k}p_3^k}{p_3^2}C_3\rangle,\end{aligned} \tag{2.104}$$

where we can send $\tau\to0$, as we are interested in the bispectrum when it is out of the horizon. Now, we proceed by applying the Wick's theorem to solve each term arising from the commutators. In order to do that, the following results are useful

$$\overbrace{\zeta_{n\vec{k_i}}(\tau\to0)\zeta_{n\vec{p_j}}}(\tau_1) = (2\pi)^3\delta^{(3)}(\vec{ki}+\vec{p_j})\frac{H^2}{4\epsilon k_i^3}(1-ik_i\tau_1)e^{ik_i\tau_1}, \tag{2.105}$$

$$\overbrace{\zeta_{n\vec{k_i}}(\tau\to0)\zeta'_{n\vec{p_j}}}(\tau_1) = (2\pi)^3\delta^{(3)}(\vec{ki}+\vec{p_j})\frac{H^2}{4\epsilon k_i^3}k_i^2\tau_1e^{ik_i\tau_1}. \tag{2.106}$$

In the end, by remembering that a Wick's contraction is nothing else than the v.e.v. $\langle\zeta^+\zeta^-\rangle$ of our field, we get that the opposite contraction $\zeta_{n\vec{p_j}}(\tau_1)\zeta'_{n\vec{k_i}}(\tau)$ is just the complex conjugate of Eq.2.105. Then from Eq.2.104 one obtains

$$\begin{aligned}C_1 =&(2\pi)^9\delta^{(3)}(\vec{k_1}+\vec{p_1})\delta^{(3)}(\vec{k_2}+\vec{p_2})\delta^{(3)}(\vec{k_3}-\vec{p_1}-\vec{p_2})\frac{H^6}{2^6\epsilon^3\prod_i k_i^3}\\ &\left[k_1^2k_2^2\tau_1^2(1-k_3\tau_1)e^{i\tau_1(k_1+k_2+k_3)}-\text{c.c.}\right]+5 \text{ perms in } \vec{k_i},\end{aligned} \tag{2.107}$$

$$\begin{aligned}C_2 =&(2\pi)^9\delta^{(3)}(\vec{k_1}+\vec{p_1})\delta^{(3)}(\vec{k_2}+\vec{p_2})\delta^{(3)}(\vec{k_3}-\vec{p_1}-\vec{p_2})\frac{H^6}{2^6\epsilon^3\prod_i k_i^3}\\ &\left[(1-ik_1\tau_1)(1-ik_2\tau_1)(1-ik_3\tau_1)e^{i\tau_1(k_1+k_2+k_3)}-\text{c.c.}\right]+5 \text{ perms in } \vec{k_i},\end{aligned} \tag{2.108}$$

and

$$\begin{aligned}C_3 =&(2\pi)^9\delta^{(3)}(\vec{k_1}+\vec{p_1})\delta^{(3)}(\vec{k_2}+\vec{p_2})\delta^{(3)}(\vec{k_3}-\vec{p_1}-\vec{p_2})\frac{H^6}{2^6\epsilon^3\prod_i k_i^3}\\ &\left[k_1^2k_3^2\tau_1^2(1-ik_2\tau_1)e^{i\tau_1(k_1+k_2+k_3)}-\text{c.c.}\right]+5 \text{ perms in } \vec{k_i}.\end{aligned} \tag{2.109}$$

We observe that it is now possible to find immediately also all the terms that follow from the different permutations. Besides the Dirac's deltas in fact, the multiplicative factors in front of the square brackets are symmetric in the momenta. Once integrated over the Fourier space, it is easy to switch the momenta and find all the other combinations. For C_1, since two of the internal fields are equal (two time-derivatives), the possible permutations are reduced to three and a factor $\times 2$ shows up. This is because, after having contracted the non-derivative external field, the result is symmetric under the exchange of the other external legs. The same reasoning would be true also for the other two terms, but a slightly different procedure has to be considered for those, due to the fact that they are multiplied by internal momenta. Therefore we have

$$\begin{aligned}I_1 = \int\frac{d^3p_1d^3p_2}{(2\pi)^6}C_1 = (2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\frac{H^6}{2^6\epsilon^3\prod_i k_i^3}2\tau_1^2e^{ik_t\tau_1}\left[k_1^2k_2^2(1-ik_3\tau_1)+\right.\\ \left.+k_2^2k_3^2(1-ik_1\tau_1)+k_1^2k_3^2(1-ik_2\tau_1)\right]-\text{c.c.},\end{aligned} \tag{2.110}$$

$$\begin{aligned}I_2 = \int\frac{d^3p_1d^3p_2}{(2\pi)^6}(-p_{2k}p_3^kC_2) = -(2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\frac{H^6}{2^6\epsilon^3\prod_i k_i^3}2\big(\vec{k_2}\cdot\vec{k_3}+\vec{k_1}\cdot\vec{k_3}+\vec{k_1}\cdot\vec{k_2}\big)\\ \left[(1-ik_1\tau_1)(1-ik_2\tau_1)(1-ik_3\tau_1)e^{ik_t\tau_1}\right]-\text{c.c.},\end{aligned} \tag{2.111}$$

and

$$\begin{aligned}I_3 = \int\frac{d^3p_1d^3p_2}{(2\pi)^6}-2\frac{p_{2k}p_3^k}{p_3^2}C_3 = -(2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\frac{H^6}{2^6\epsilon^3\prod_i k_i^3}2\tau_1^2e^{ik_t\tau_1}\Big\{k_1^2(\vec{k_2}\cdot\vec{k_3})\big[2-i\tau_1(k_2+k_3)\big]\\ k_2^2(\vec{k_1}\cdot\vec{k_3})\big[2-i\tau_1(k_1+k_3)\big]+k_3^2(\vec{k_1}\cdot\vec{k_2})\big[2-i\tau_1(k_1+k_2)\big]\Big\}-\text{c.c.},\end{aligned} \tag{2.112}$$

where $k_t = k_1 + k_2 + k_3$. Now, to find the bispectrum for the shifted field we have to integrate over conformal time. We have, by putting $D = (2\pi)^3\delta^(3)(\vec{k_1} + \vec{k_2} + \vec{k_3})\frac{H^6}{2^5\epsilon^3\prod_i k_i^3}$, and approximating $a(\tau) \approx -\frac{1}{\tau H}$,

$$\begin{aligned}\int_{-\infty}^{0}\frac{d\tau_1}{\tau_1^2 H^2}I_1 =& D\int_{-\infty}^{0}\frac{d\tau_1}{H^2}\left(k_1^2k_2^2 + k_1^2k_3^2 + k_2^2k_3^2\right)e^{ik_t\tau_1} + \\ & - i\tau_1\left(k_1k_2^2k_3^2 + k_2k_1^2k_3^2 + k_3k_1^2k_2^2\right)e^{ik_t\tau_1} - \text{c.c.}.\end{aligned} \tag{2.113}$$

Before proceeding it is worth to mention a technique which comes in hand when we have to integrate complex exponential functions. The Minkowski spacetime is described by the well known line-element $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. The analogue four-dimensional Euclidean space, instead, is given by $ds^2 = dt^2 + dx^2 + dy^2 + dz^2$. By observing that the Minkowsi's metric resembles the four dimensional Euclidean one if $t \to it'$, a Wick rotation allows us to pass from the former spacetime to the latter, where the problem one is facing might be easier to solve. Once the problem is solved, by applying the inverse transformation it is possible to express the solution in the original spacetime. It is possible to see the Wick's rotation in terms of an analytical continuation of the signature parameter ω, by writing $\eta_{\alpha\beta} = diag(w, 1, ..., 1)$ with $w \in \mathbb{C}$ [54]. [3] By Wick rotating the whole expression, thereby integrating by parts the second line, it follows

$$\int_{-\infty}^{0}\frac{d\tau_1}{\tau_1^2 H^2}I_1 = 2i\frac{D}{H^2}\left(\frac{\sum_{i<j}k_i^2k_j^2}{k_t} + \frac{\sum_{i;j,k\neq i}k_ik_j^2k_k^2}{k_t^2}\right). \tag{2.114}$$

The exact same treatment can be followed to find the integrated I_3. Which reads

$$\begin{aligned}\int_{-\infty}^{0}\frac{d\tau_1}{\tau_1^2 H^2}I_3 =& 2i\frac{D}{H^2}\left(2\frac{\sum_{i;j,k\neq i}k_i^2(\vec{k_j}\cdot\vec{k_k})}{k_t} + \frac{\sum_{i;j,k\neq i}k_i^2(\vec{k_j}\cdot\vec{k_k})(k_j + k_k)}{k_t^2}\right) \\ =& i\frac{D}{H^2}\left(-\sum_i k_i^3 + \frac{6}{k_t}\sum_{i<j}k_i^2k_j^2 - \frac{1}{k_t}\sum_i k_i^4 + 2\frac{k_1k_2k_3}{k_t^2}\sum_i k_i^2\right),\end{aligned} \tag{2.115}$$

where we used the fact that thanks to the Dirac's deltas $\vec{k_i}\cdot\vec{k_j} = \frac{1}{2}[k_k^2 - k_i^2 - k_j^2]$ $(i, j, k = 1, 2, 3)$. For the second integrand, one more step is necessary, since a τ_1^{-1} dependence comes out from the scale factor's approximation we have done. In fact, one has

$$\begin{aligned}\int_{-\infty}^{0}\frac{d\tau_1}{\tau_1^2 H^2}I_2 =& -\frac{D}{H^2}\left(\textstyle\sum_{i>j}\vec{k_i}\cdot\vec{k_j}\right)\int_{-\infty}^{0}\frac{d\tau_1}{\tau_1^2}\left[1 - i\tau_1k_t - \tau_1^2(k_1k_2 + k_1k_3 + k_2k_3) + ik_1k_2k_3\tau_1^3\right]e^{ik_t\tau_1} - \text{c.c.} \\ =& -\frac{D}{H^2}\left(\textstyle\sum_{i>j}\vec{k_i}\cdot\vec{k_j}\right)\int_{-\infty}^{0}d\tau_1\,\frac{e^{ik_t\tau_1}}{\tau_1^2} - ik_t\frac{e^{ik_t\tau_1}}{\tau_1} - \sum_{i<j}k_ik_je^{ik_t\tau_1} + ik_1k_2k_3\tau_1e^{ik_t\tau_1} - \text{c.c.}.\end{aligned} \tag{2.116}$$

Now, the first term of the integrand can be integrated by part, giving us $\int_{-\infty}^{0}d\tau_1e^{ik_t\tau_1}\tau_1^{-2} = -\tau_1^{-1}e^{ik_t\tau_1} + ik_t\int_{-\infty}^{0}d\tau_1\tau_1^{-1}e^{ik_t\tau_1}$. This allows us to get rid of the second term of Eq.2.116. Now we are left with

[3] We note that this is mathematically consistent only if the transition to the Euclidean space starts from a Minkowski one. In fact, if one considered a curved time-dependent spacetime, or a non-diagonal metric with non vanishing $dtdx$-coefficients, by Wick rotating the temporal coordinate the Euclideanization would not be fulfilled. Therefore, in such cases, the Wick rotation has not a clear mathematical explanation. It is possible though to redefine it in a more formal way, which allows one to operate this trick also in non-Minkowskian space-times [55]

$$
\begin{aligned}
\int_{-\infty}^{0}\frac{d\tau_1}{\tau_1^2 H^2}I_2 =& -\frac{D}{H^2}\left(\sum\nolimits_{i>j}\vec{k_i}\cdot\vec{k_j}\right)\left(-\frac{1}{\tau_1}e^{ik_t\tau}\Big|_{-\infty}^{0}+\int_{-\infty}^{0}d\tau_1\, ik_1k_2k_3\tau_1 e^{ik_t\tau_1}-\sum_{i>j}k_ik_je^{ik_t\tau_1}\right)-\text{c.c.}\\
=& -\frac{D}{H^2}\left(\sum\nolimits_{i>j}\vec{k_i}\cdot\vec{k_j}\right)\left(-\frac{1}{\tau_1}\left(\cos\left(k_t\tau_1\right)+i\sin\left(k_t\tau_1\right)\right)\Big|_{-\infty}^{0}+i\frac{\sum_{i<j}k_ik_j}{k_t}+i\frac{k_1k_2k_3}{k_t^2}\right)-\text{c.c.}\\
=& -2i\frac{D}{H^2}\left(\sum\nolimits_{i>j}\vec{k_i}\cdot\vec{k_j}\right)\left(k_t+\frac{\sum_{i<j}k_1k_j}{k_t}+\frac{\prod_i k_i}{k_t^2}\right),
\end{aligned}
\tag{2.117}
$$

where in the end we considered the complex conjugate, which simplified all the real contributions, and took the limits for $\tau_1 \to -\infty$ and $\tau_1 \to 0$. From the former the first term reduces to zero, while from the latter we get the well known result $\lim_{(k_t\tau_1)\to 0} ik_t\frac{\sin(k_t\tau_1)}{k_t\tau_1} = ik_t$.
At the very end, by summing together all the terms we have found so far (once multiplied by $i\epsilon^2$) and the contribution that comes from the field's shifting, we obtain [52]

$$
\left\langle \zeta_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\right\rangle = (2\pi)^3\delta^{(3)}\left(\sum_i^3\vec{k_i}\right)\frac{H^4}{2^5\epsilon^2\prod_i k_i^3}\left[2(\epsilon-\eta)\sum_i^3 k_i^3+\epsilon\sum_{ij}^3 k_i^2k_j+\epsilon\frac{8}{k_t}\sum_{i<j}^3 k_i^2k_j^2\right]. \tag{2.118}
$$

Here the first contribution inside the square brackets is given by the fourth-order term carried by the field-shifting. In particular, only the first term of the function $f(\zeta)$ will contribute to the bispectrum, as the other ones are vanishing if the total three-points function is considered well outside the horizon. In fact, by looking at Eq.2.99, we see that all terms but the former include derivative of ζ and hence of its mode-function. The contributions proportional to ζ' vanish simply because the temporal derivative introduces a τ-proportionality, while the ones that contain only spatial derivatives vanish because of the scale-factor they have in front (in dS $a \approx -\frac{1}{\tau H}$).

2.4 Mixed bispectra

As we are going to talk about an inflationary model in which gravity is modified at high energies, it is useful to start working with gravitons and their correlation functions. Hence, we will focus here on two-points functions between scalars and tensors. In particular, we will calculate the bispectra $\langle\gamma\zeta\zeta\rangle$, which will be the analogous vertex we are going to consider for the final result of this thesis, and $\langle\gamma\gamma\zeta\rangle$, which will be encountered again while studying the $f(\phi)\tilde{W}W$ model.
Before jumping into the calculation, we need to find the mode functions of tensor fields, as we want to calculate these two-points functions in Fourier space. We could in principle repeat the whole reasoning we did for the ζ's case, starting from the tensors' equation of motion and hence quantizing the field. However, here we follow a different approach [52]. For our fields, we would like to resemble the usual harmonic oscillator's mode function, for which we already know the solution. To do so we exploit the action of a canonically normalized scalar field in de Sitter space, thereby imposing it to describe our fields. In order to find the perturbations' equation of motion at first order, we need the action perturbed up to second order in the specific perturbation. The kinetic part of the action of a free scalar field f is

$$
S = \frac{1}{2}\int d^3x d\tau a^2\left[f'^2-(\partial_l f)^2\right], \tag{2.119}
$$

where again $' = \frac{\partial}{\partial\tau}$. For our curvature perturbations, we have instead

$$
S = \frac{1}{2}\int d^3x d\tau a^2 2\epsilon\left[\zeta'^2-(\partial_l \zeta)^2\right]. \tag{2.120}
$$

Thus, to easily describe ζ as f, and hence quantize it with the usual mode functions for such a field, we consider $\zeta = \frac{f}{\sqrt{2\epsilon}}$. For the tensor field one more step is required. The second order action for the gravity component is given by

$$S = \frac{1}{16\pi G}\int d^3x d\tau a^2 \frac{1}{4}\left[\gamma'_{ij}\gamma'_{ij} - \partial_l\gamma_{ij}\partial_l\gamma_{ij}\right]. \tag{2.121}$$

By opening now the two fields in Fourier space, we notice that each tensor field contributes with two scalar fields. By remembering the property $\epsilon_{ij}(\vec{k})^s\epsilon_{ij}(\vec{k})^{s'} = 2\delta^{ss'}$, we find the following action for each real, massless, scalar field

$$S^{(s)} = \frac{1}{16\pi G}\int d^3x d\tau a^2 \frac{1}{2}\left[(\gamma'^s_{\vec{k}})^2 - (\partial_i\gamma^s_{\vec{k}})^2\right]. \tag{2.122}$$

Therefore, by considering $M_{pl}^2 = 1$, we get $\gamma = \sqrt{16\pi G}f = \sqrt{2}f$, where $u_k^f(\tau) = \frac{iH}{\sqrt{2k^3}}(1+ik\tau)e^{-ik\tau}$. Finally, we can quantize the two fields as

$$\zeta_{\vec{p}}(\tau) = u_p^\zeta(\tau)a_{\vec{p}} + u_p^{\zeta *}(\tau)a^\dagger_{-\vec{p}} \qquad \gamma^s_{\vec{p}}(\tau) = u_p^\gamma(\tau)a^s_{\vec{p}} + u_p^{\gamma *}(\tau)a^{s\dagger}_{-\vec{p}}, \tag{2.123}$$

with

$$u_p^\zeta(\tau) = \frac{iH}{\sqrt{4\epsilon p^3}}(1+ip\tau)e^{-ip\tau}, \quad u_p^\gamma(\tau) = \frac{iH}{\sqrt{p^3}}(1+ip\tau)e^{-ip\tau}, \tag{2.124}$$

and where the usual commutation relations hold

$$[a_{\vec{p}}, a^\dagger_{-\vec{q}}] = \delta^{(3)}(\vec{p}+\vec{q})(2\pi)^3, \quad [a^r_{\vec{p}}, a^{s\dagger}_{-\vec{q}}] = \delta^{(3)}(\vec{p}+\vec{q})\delta^{rs}(2\pi)^3. \tag{2.125}$$

The former is referred to the scalar field, whereas the latter to the tensor one. All the other scalar-scalar and tensor-tensor commutators vanish. We are now ready to start our computations.

$\langle\gamma\gamma\zeta\rangle$ bispectrum

From the famous paper of J. Maldacena [52], the action for a single field slow-roll inflation expressed in terms of the so called shifted field ζ^c reads

$$S = \int d^4x \frac{1}{2}\epsilon H a^5 \dot{\gamma_{ij}}\dot{\gamma_{ij}}\partial^{-2}\dot{\zeta}^c + ... \tag{2.126}$$

where

$$\zeta = \zeta^c + f(\zeta^c), \quad f(\zeta^c) = -\frac{1}{32}\gamma_{ij}\gamma^{ij} + \frac{1}{16}\partial^{-2}(\gamma_{ij}\partial^2\gamma^{ij}) + ... \tag{2.127}$$

Here, the dots indicate terms that are higher order in the slow roll approximation. In order to evaluate the two-points function between a scalar and two tensors, we thus have to take into account this shift. Specifically, we have to evaluate two different terms:

$$\langle\zeta(\vec{x},\tau)\gamma_{ij}(\vec{y},\tau)\gamma_{ij}(\vec{z},\tau)\rangle = \underbrace{\langle\zeta^c(\vec{x},\tau)\gamma_{ij}(\vec{y},\tau)\gamma_{ij}(\vec{z},\tau)\rangle}_{(1)} + \underbrace{\langle f(\zeta^c(\vec{x},\tau))\gamma_{ij}(\vec{y},\tau)\gamma_{ij}(\vec{z},\tau)\rangle}_{(2)}. \tag{2.128}$$

In Fourier space we get

$$(1) = \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} e^{i\vec{x}\vec{k}_1} e^{i\vec{y}\vec{k}_2} e^{i\vec{z}\vec{k}_3} \sum_{ss'} \epsilon^s_{ij}\epsilon^{s'}_{ij} \langle \zeta^c_{k_1}\gamma^s_{k_2}\gamma^{s'}_{k_3}\rangle, \tag{2.129}$$

where

$$\zeta(\vec{x},\tau) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \zeta_{\vec{p}}(\tau), \quad \gamma_{ij}(\vec{x},\tau) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \sum_s \epsilon^s_{ij}(\vec{p})\gamma^s_{\vec{p}}(\tau). \tag{2.130}$$

To evaluate the first term of Eq. 2.128, we proceed using the in-in formalism we introduced above:

$$\langle \zeta^c_{k_1}\gamma^s_{k_2}\gamma^{s'}_{k_3}\rangle = -i\int d^3w \int_{-\infty}^{\tau\to 0} d\tau' a \int \frac{d^3p d^3q d^3k}{(2\pi)^9} e^{i\vec{w}(\vec{p}+\vec{q}+\vec{k})} \sum_{r,r'} \epsilon^r_{lm}(\vec{p})\epsilon^{r'}_{lm}(\vec{q}) \frac{\epsilon H a^5}{2}\frac{1}{k^2a^3} \langle T[\zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}, \gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}]\rangle, \tag{2.131}$$

where $\partial^{-2}\zeta'^c_{\vec{k}} = -\frac{1}{k^2}\zeta'^c_{\vec{k}}$, $H^I_{int} = -\int d^3w\mathcal{L}_{int}$ and $\tau \to 0$ since we are looking for the bispectrum well outside the horizon. By expanding now the time-ordered product inside the integral through the Wick theorem [38,39] we have

$$\begin{aligned} \langle T[\zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}, \gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}]\rangle &= \langle T\left(\zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}\gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}\right)\rangle - c.c. \\ &= \langle \zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}\gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}\rangle + \langle \zeta^c_{\vec{k}_1}\gamma^s_{\vec{k}_2}\gamma^{s'}_{\vec{k}_3}\gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}\rangle - c.c.. \end{aligned} \tag{2.132}$$

A contraction between two fields is nothing but vacuum expectation value of the time-ordered product of the two fields [39]. As we pointer out in Sec. 2.1, in an inflationary background we can evaluate fields' contractions forgetting about the time-ordering and leave this complication to the final integration in time. By remembering that $\tau \to 0$, since we want to calculate the bispectra well outside the horizon, and decomposing each Fourier mode in its ladder operators, one has

$$\begin{aligned} \langle 0|\gamma^s_{\vec{p}}(\tau)\gamma'^r_{\vec{q}}(\tau')|0\rangle &= \langle \left[\gamma^{s+}_{\vec{p}}(\tau), \gamma'^{r-}_{\vec{q}}(\tau')\right]\rangle \\ &= (16\pi G)(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\delta^{rs}\left(\frac{iH}{\sqrt{2p^3}}(1+ip\tau)e^{-ip\tau}\right)\left(\frac{-iH}{\sqrt{2q^3}}(1-iq\tau')e^{+iq\tau'}\right)' \\ &= (16\pi G)(2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\delta^{rs}\frac{H^2}{2\sqrt{p^3q^3}}q^2\tau' e^{+iq\tau'}, \end{aligned} \tag{2.133}$$

$$\begin{aligned} \langle 0|\zeta^c_{\vec{p}}(\tau)\zeta'^c_{\vec{q}}(\tau')|0\rangle &= \langle \left[\zeta^{c+}_{\vec{p}}(\tau), \zeta'^{c-}_{\vec{q}}(\tau')\right]\rangle \\ &= (2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\left(\frac{iH}{\sqrt{4\epsilon p^3}}(1+ip\tau)e^{-ip\tau}\right)\left(\frac{-iH}{\sqrt{4\epsilon q^3}}(1-iq\tau')e^{+iq\tau'}\right)' \\ &= (2\pi)^3\delta^{(3)}(\vec{p}+\vec{q})\frac{H^2}{4\epsilon\sqrt{p^3q^3}}q^2\tau' e^{-iq\tau'}. \end{aligned} \tag{2.134}$$

where τ' is the conformal time of the internal fields, i.e. those carried by the interacting Hamiltonian, so in this case $'$ does not refer to a time-derivation. Hence, by taking all the contractions we finally get

$$\begin{aligned}\langle T[\zeta^c_{\vec{k_1}}\gamma^s_{\vec{k_2}}\gamma^{s'}_{\vec{k_3}},\gamma'^r_{\vec{p}}\gamma'^{r'}_{\vec{q}}\zeta'^c_{\vec{k}}]\rangle &= (16\pi G)^2(2\pi)^9\delta^{(3)}(\vec{k_1}+\vec{k})\frac{H^6}{16\epsilon\sqrt{k_1^3k_2^3k_3^3p^3q^3k^3}}p^2q^2k^2\tau'^3e^{i\tau'(p+q+k)}\\ &\left(\delta^{(3)}(\vec{k_2}+\vec{p})\delta^{(3)}(\vec{k_3}+\vec{q})\delta^{rs}\delta^{r's'}+\delta^{(3)}(\vec{k_2}+\vec{q})\delta^{(3)}(\vec{k_3}+\vec{p})\delta r's\delta^{rs'}\right)-c.c..\end{aligned} \tag{2.135}$$

Now, by putting it inside the integral, we note that the different deltas act in the same way on the momentum-dependent factors. In fact, the permutation $k_2 \leftrightarrow k_3$ would just produce a switch between two momenta, which can be rearranged in a single order since all the remain terms commute. We thus obtain

$$\begin{aligned}\langle\zeta^c_{k_1}\gamma^s_{k_2}\gamma^{s'}_{k_3}\rangle &= -i\frac{H^7}{4\prod_i(k_i^3)}\epsilon_{lm}(-\vec{k_2})^s\epsilon_{lm}(-\vec{k_3})^{s'}(2\pi)^3k_2^2k_3^2\int\frac{d^3w}{(2\pi)^3}e^{i\vec{w}(\vec{k_1}+\vec{k_2}+\vec{k_3})}\left(\int_{-\infty}^{\tau\to 0}d\tau'a^3e^{+i\tau'k_t}\tau'^3-c.c\right).\\ &= -i\frac{H^7}{4\prod_i(k_i^3)}\epsilon^s_{lm}\epsilon^{s'}_{lm}(2\pi)^3k_2^2k_3^2\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\left(\int_{-\infty}^{\tau\to 0}d\tau'\left(-\frac{1}{H^3\tau'^3}\right)e^{+i\tau'k_t}\tau'^3-c.c.\right)\\ &= i\frac{H^4}{4\prod_i(k_i^3)}\epsilon^s_{lm}\epsilon^{s'}_{lm}(2\pi)^3k_2^2k_3^2\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\left(\int_{-\infty}^{\tau\to 0}d\tau'e^{+i\tau'k_t}-c.c.\right),\end{aligned} \tag{2.136}$$

where $k_t=\sum_i k_i$. By applying now a Wick rotation $\tau'\to -i\tau''$ we have a simple integral of an exponential factor multiplied by $-i$, which simplifies to 1 with the i in front. Finally, summing also the complex conjugate that gives us a factor $\times 2$, we obtain

$$\langle\zeta^c_{\vec{k_1}}\gamma^s_{\vec{k_2}}\gamma^{s'}_{\vec{k_3}}\rangle=(2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3})\frac{H^4}{\prod_i(2k_i^3)}4\frac{k_2^2k_3^2}{k_t}\epsilon^s_{lm}\epsilon^{s'}_{lm}. \tag{2.137}$$

For the second term we have to express the shifting function in Fourier space, just as the other tensor fields. Doing that we obtain

$$(2)=\int\frac{d^3pd^3qd^3k_2d^3k_3}{(2\pi)^{12}}e^{i\vec{x}(\vec{p}+\vec{q})}e^{i\vec{y}\vec{k_2}}e^{i\vec{z}\vec{k_3}}\sum_{ss'}\epsilon^s_{ij}\epsilon^{s'}_{ij}\sum_{rr'}\epsilon^r_{lm}\epsilon^{r'}_{lm}\left(-\frac{1}{32}\langle\gamma^r_{\vec{p}}\gamma^{r'}_{\vec{q}}\gamma^s_{\vec{k_2}}\gamma^{s'}_{\vec{k_3}}\rangle+\frac{1}{16}\frac{q^2}{|\vec{p}+\vec{q}|^2}\langle\gamma^r_{\vec{p}}\gamma^{r'}_{\vec{q}}\gamma^s_{\vec{k_2}}\gamma^{s'}_{\vec{k_3}}\rangle\right). \tag{2.138}$$

Here the factor $\frac{q^2}{|\vec{p}+\vec{q}|^2}$ shows up from the term $\partial^{-2}(\gamma_{ij}\partial^2\gamma_{ij})$ in the shifted action. To evaluate now the correlation functions inside the parenthesis, we must use the Wick's theorem again. However, in order to obtain only connected diagrams, we avoid contractions between $\vec{x}$ and itself. So that the only contractions will be

$$\langle\gamma^r_{\vec{p}}\gamma^{r'}_{\vec{q}}\gamma^s_{\vec{k_2}}\gamma^{s'}_{\vec{k_3}}\rangle=\langle\gamma^r_{\vec{p}}\gamma^{r'}_{\vec{q}}\gamma^s_{\vec{k_2}}\gamma^{s'}_{\vec{k_3}}\rangle+\langle\gamma^r_{\vec{p}}\gamma^{r'}_{\vec{q}}\gamma^s_{\vec{k_2}}\gamma^{s'}_{\vec{k_3}}\rangle, \tag{2.139}$$

where

$$\langle\gamma^r_{\vec{p}}(\tau)\gamma^s_{\vec{k_2}}(\tau)\rangle=(16\pi G)(2\pi)^3\delta^{(3)}(\vec{p}+\vec{k_2})\delta^{rs}\left(\frac{iH}{\sqrt{2p^3}}(1+ip\tau)e^{-ip\tau}\right)\left(\frac{-iH}{\sqrt{2k_2^3}}(1-ik_2\tau)e^{ik_2\tau}\right). \tag{2.140}$$

Considering again the limit $\tau \to 0$, we have, by putting $\int D = \int \frac{d^3p d^3q d^3k_2 d^3k_3}{(2\pi)^{12}} e^{i\vec{p}\vec{x}} e^{i\vec{q}\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}}$,

$$
\begin{aligned}
(2) = \int D \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2})\epsilon^{s'}_{ij}(\vec{k_3}) \sum_{r,r'} \epsilon^r_{lm}(\vec{p})\epsilon^{r'}_{lm}(\vec{q}) \left[-\frac{1}{32} \langle \gamma^r_{\vec{p}} \gamma^{r'}_{\vec{q}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle + \frac{1}{16} \frac{q^2}{|\vec{p}+\vec{q}|^2} \langle \gamma^r_{\vec{p}} \gamma^{r'}_{\vec{q}} \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle \right] \\
= (16\pi G)^2 (2\pi)^6 \int D \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2})\epsilon^{s'}_{ij}(\vec{k_3}) \sum_{r,r'} \epsilon^r_{lm}(\vec{p})\epsilon^{r'}_{lm}(\vec{q}) \frac{H^4 (2\pi)^6}{4\sqrt{p^3 q^3 k_2^3 k_3^3}} \left(-\frac{1}{32} + \frac{1}{16} \frac{q^2}{|\vec{p}+\vec{q}|} \right) \times \\
\times \left(\delta^{rs}\delta^{r's'} \delta^{(3)}(\vec{k_2}+\vec{p})\delta^{(3)}(\vec{k_3}+\vec{q}) + \delta^{rs'}\delta^{r's} \delta^{(3)}(\vec{k_2}+\vec{q})\delta^{(3)}(\vec{k_3}+\vec{p}) \right) \\
= (16\pi G)^2 \int \frac{d^3k_2 d^3k_3}{(2\pi)^{12}} e^{-i(\vec{k_2}+\vec{k_3})\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}} \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2})\epsilon^{s'}_{ij}(\vec{k_3}) \frac{H^4 (2\pi)^6}{4k_2^3 k_3^3} \left(-\frac{2}{32} + \frac{1}{16} \frac{k_2^2 + k_3^2}{|\vec{k_2}+\vec{k_3}|^2} \right) \epsilon^s_{lm}(-\vec{k_2})\epsilon^{s'}_{lm}(-\vec{k_3}).
\end{aligned}
\tag{2.141}
$$

Considering now that $(16\pi G)^2 = (2M_P^{-2})^2 = 4$, we obtain

$$
(2) = \int \frac{d^3k_2 d^3k_3}{(2\pi)^6} e^{-i(\vec{k_2}+\vec{k_3})\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}} \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2})\epsilon^{s'}_{ij}(\vec{k_3}) \frac{H^4}{16 k_2^3 k_3^3} \left(-1 + \frac{k_2^2 + k_3^2}{|\vec{k_2}+\vec{k_3}|^2} \right) \epsilon^s_{lm}(-\vec{k_2})\epsilon^{s'}_{lm}(-\vec{k_3}). \tag{2.142}
$$

Now, in order to make the expression more symmetric in the momentum space, we can multiply and divide by k_1^3, and integrate over $\vec{k_1}$ under the condition that $\vec{k_1} + \vec{k_2} + \vec{k_3} = 0$

$$
\begin{aligned}
(2) &= \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} e^{i\vec{k_1}\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}} \sum_{s,s'} \epsilon^s_{ij}\epsilon^{s'}_{ij} (2\pi)^3 \delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3}) \frac{H^4}{\prod_i (2k_i^3)} \left(-\frac{k_1^3}{2} + \frac{k_1}{2}(k_2^2 + k_3^2) \right) \epsilon^s_{lm}\epsilon^{s'}_{lm} \\
&= \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} e^{i\vec{k_1}\vec{x}} e^{i\vec{k_2}\vec{y}} e^{i\vec{k_3}\vec{z}} \sum_{s,s'} \epsilon^s_{ij}(\vec{k_2})\epsilon^{s'}_{ij}(\vec{k_3}) \langle f_{\vec{k_1}}(\zeta^c) \gamma^s_{\vec{k_2}} \gamma^{s'}_{\vec{k_3}} \rangle.
\end{aligned}
\tag{2.143}
$$

In the end, we can rewrite the full two-points function in the momentum space by summing together the two contributions getting [52]

$$
\langle \zeta_{k_1} \gamma^s_{k_2} \gamma^{s'}_{k_3} \rangle = (2\pi)^3 \delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3}) \frac{H^4}{\prod_i (2k_i^3)} \left(-\frac{k_1^3}{2} + \frac{1}{2} k_1 (k_2^2 + k_3^2) + 4 \frac{k_2^2 k_3^2}{k_t} \right) \epsilon^s_{lm}\epsilon^{s'}_{lm}. \tag{2.144}
$$

$\langle \gamma\zeta\zeta \rangle$ bispectrum

For the $\gamma\zeta\zeta$-bispectrum, we have instead [52]

$$
S = \int d^4x \epsilon a \gamma_{ij} \partial^i \zeta \partial^j \zeta, \tag{2.145}
$$

from which the interacting Hamiltonian reads

$$
\begin{aligned}
H_{int}(t) &= -\int d^3\vec{x}\epsilon a\gamma_{ij}\partial^i\zeta\partial^j\zeta \\
&= -\int d^3x\epsilon a\int\frac{d^3p_1d^3p_2d^3p_3}{(2\pi)^9}\sum_{r=\pm}\epsilon^r_{ij}(\hat{p}_1)i^2p_2^ip_3^j\gamma^r_{\vec{p_1}}(t)\zeta_{\vec{p_2}}(t)\zeta_{\vec{p_3}}(t)e^{i\vec{x}(\vec{p_1}+\vec{p_2}+\vec{p_3})} \\
&= \epsilon a(t)\int\frac{d^3p_1d^3p_2d^3p_3}{(2\pi)^6}\delta^{(3)}(\vec{p_1}+\vec{p_2}+\vec{p_3})\sum_r\epsilon^r_{ij}(\hat{p}_1)p_2^ip_3^j\gamma^r_{\vec{p_1}}(t)\zeta_{\vec{p_2}}(t)\zeta_{\vec{p_3}}(t),
\end{aligned}
\tag{2.146}
$$

where we expanded each field in its Fourier modes. Thus, we simply have

$$
\begin{aligned}
\langle\gamma_{ij}\zeta\zeta\rangle &= i\int_{-\infty}^t dt'\langle T\left[H_{int}(t'),\gamma_{ij}(\vec{x},t)\zeta(\vec{y},t)\zeta(\vec{z},t)\right]\rangle \\
&= i\int\frac{d^3k_1d^3k_2d^3k_3}{(2\pi)^9}e^{i(\vec{k_1}\cdot\vec{x}+\vec{k_2}\cdot\vec{y}+\vec{k_3}\cdot\vec{z})}\sum_{s=\pm}\epsilon^s_{ij}(\hat{k_1})\int_{-\infty}^t dt'\langle T\left[H_{int},\gamma^s_{\vec{k_1}}(t)\zeta_{\vec{k_2}}(t)\zeta_{\vec{k_3}}(t)\right]\rangle \\
&= \int\frac{d^3k_1d^3k_2d^3k_3}{(2\pi)^9}e^{i(\vec{k_1}\cdot\vec{x}+\vec{k_2}\cdot\vec{y}+\vec{k_3}\cdot\vec{z})}\sum_{s=\pm}\epsilon^s_{ij}(\hat{k_1})\langle\gamma^s_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\rangle.
\end{aligned}
\tag{2.147}
$$

Again, we want to calculate our bispectrum when it is well outside the horizon. Thus, by considering the mode functions given by Eq. 2.124 one has

$$
\gamma^r_{\vec{k_1}}(\tau')\gamma^s_{\vec{k_2}}(\tau) = (2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2})\delta^{rs}\frac{H^2}{\sqrt{k_1^3k_2^3}}(1+ik_1\tau')e^{-ik_1\tau}
\tag{2.148}
$$

for a connection between two gravitons, and

$$
\zeta_{\vec{k_1}}(\tau')\zeta_{\vec{k_2}}(\tau) = (2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2})\frac{H^2}{4\epsilon\sqrt{k_1^3k_2^3}}\left(1+ik_1\tau'\right)e^{-ik_1\tau'}
\tag{2.149}
$$

for a scalar-scalar one.

Hence, by means of Wick's theorem, we find

$$
\begin{aligned}
\langle\left[\gamma_{\vec{p_1}}\zeta_{\vec{p_2}}\zeta_{\vec{p_3}},\gamma_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\right]\rangle = &\left(\gamma^r_{\vec{p_1}}\zeta_{\vec{p_2}}\zeta_{\vec{p_3}}\gamma^s_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}+\gamma^r_{\vec{p_1}}\zeta_{\vec{p_2}}\zeta_{\vec{p_3}}\gamma^s_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}-c.c.\right) \\
=&(2\pi)^9\delta^{(3)}\left(\vec{k_1}+\vec{p_1}\right)\delta^{rs}\frac{H^6}{M_{Pl}^216\epsilon^2}\frac{1}{\sqrt{\prod_i k_i^3p_i^3}}(1+ip_1\tau')(1+ip_2\tau')(1+ip_3\tau')\times \\
&\times e^{-i\tau'(p_1+p_2+p_3)}\left(\delta^{(3)}(\vec{p_2}+\vec{k_2})\delta^{(3)}(\vec{p_3}+\vec{k_3})+\delta^{(3)}(\vec{p_2}+\vec{k_3})\delta^{(3)}(\vec{p_3}+\vec{k_2})\right)-c.c.,
\end{aligned}
\tag{2.150}
$$

where we considered $8\pi G = M_{Pl}^{-2}$. Now, by considering the conformal time rather than the cosmic one, and approximating $a(\tau)\approx\frac{1}{\tau H}$, we obtain

$$
\begin{aligned}
\langle\gamma^s_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\rangle =& i\epsilon\delta^{rs}\int d\eta' a^2\int\frac{d^3p_1d^3p_2d^3p_3}{(2\pi)^6}(2\pi)^9\delta^{(3)}(\vec{p_1}+\vec{p_2}+\vec{p_3})\frac{H^6}{16\epsilon^2}\delta^{(3)}(\vec{k_1}+\vec{p_1})\frac{p_2^ip_3^j}{\sqrt{\prod_l k_l^3p_l^3}}\sum_r\epsilon^r_{ij}(\hat{p_1})\times\\
&\times\left(1+i\tau'(p_1+p_2+p_3)-\tau'^2(p_1p_2+p_1p_3+p_2p_3)-i\tau'^3p_1p_2p_3\right)e^{-i\tau' K}\times\\
&\times\left(\delta^{(3)}(\vec{p_2}+\vec{k_2})\delta^{(3)}(\vec{p_3}+\vec{k_3})+\delta^{(3)}(\vec{p_2}+\vec{k_3})\delta^{(3)}(\vec{p_3}+\vec{k_2})\right)-c.c.\\
=& i(2\pi)^3\delta^{(3)}\left(\sum_i\vec{k_i}\right)\frac{H^4}{16\epsilon}\epsilon^s_{ij}(\hat{k}_1)\frac{k_2^ik_3^j}{\prod_l k_l^3}\times\\
&\times 2\int d\tau'\frac{1}{\tau'^2}\left(1+i\tau'(k_1+k_2+k_3)-\tau'^2(k_1k_2+k_1k_3+k_2k_3)-i\tau'^3k_1k_2k_3\right)e^{-i\tau' K}-c.c.,
\end{aligned}
\tag{2.151}
$$

since the Dirac's deltas, when applied on the second last line, give us the same term, which is taken into account in the 2 factor in front of the time-integral. We named also K as the sum of the three external momenta k_i. By focusing only on the time integral $I(\tau)$, we have

$$
\begin{aligned}
I=&\int_{-\infty}^{\tau\to 0}d\tau'\left[\frac{e^{-i\tau' K}}{\tau'^2}+\frac{e^{-i\tau' K}}{\tau'}iK-e^{-i\tau' K}(k_1k_2+k_1k_3+k_2k_3)-ik_1k_2k_3e^{-i\tau' K}\tau'-c.c.\right]\\
=&-\frac{1}{\tau'}e^{-i\tau' K}\Big|_{-\infty}^{0}-iK\int_{-\infty}^{0}d\tau'\frac{e^{-i\tau' K}}{\tau'}+iK\int_{-\infty}^{0}d\tau'\frac{e^{-i\tau' K}}{\tau'}-i\sum_{i>j}k_ik_j\frac{1}{K}e^{\tau' K}\Big|_{-\infty}^{0}\\
&-ik_1k_2k_3i^2\left(\frac{1}{K}\tau''e^{K\tau''}\Big|_{-\infty}^{0}-\frac{1}{K}\int_{-\infty}^{0}d\tau''e^{K\tau''}\right)-c.c..
\end{aligned}
\tag{2.152}
$$

Here, we integrated by part the first term, so that the second piece that we obtain cancels out with its opposite. Also, we Wick-rotated the last two terms. Finally, by rewriting the first term in terms of sinusoidal functions, we get

$$
I=2i\left[K-\frac{\sum_{i>j}k_ik_j}{K}-\frac{k_1k_2k_3}{K^2}\right],
\tag{2.153}
$$

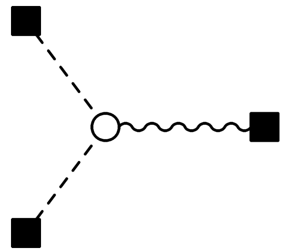
since all the real terms sum themselves to zero due to the complex conjugate. At the very end, we find [52]

$$
\langle\gamma^s_{\vec{k_1}}\zeta_{\vec{k_2}}\zeta_{\vec{k_3}}\rangle=(2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}+\vec{k_3}\right)\frac{2H^4}{\epsilon\prod_i(2k_i^3)}\epsilon^s_{ij}(\hat{k_1})k_2^ik_3^j\left[-K+\frac{\sum_{i>j}k_ik_j}{K}+\frac{k_1k_2k_3}{K^2}\right].
\tag{2.154}
$$

Diagrammatic rules: application

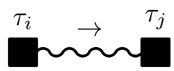
Just to directly apply the diagrammatic rules we mentioned above, we try to reproduce the last result by starting from the diagram of the interaction. We have

$$\left\langle \gamma^s_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\right\rangle =$$

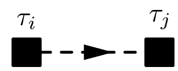


where

$$\gamma^s_{\vec{k}_i}(\tau_i)\gamma^r_{\vec{k}_j}(\tau_j) = \langle\gamma^{s+}\gamma^{r-}\rangle =$$

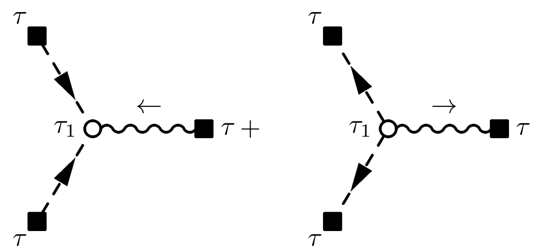


$$\zeta_{\vec{k}_i}(\tau_i)\zeta_{\vec{k}_j}(\tau_j) = \langle\zeta^{+}\zeta^{-}\rangle =$$



are the "right" Wick's contraction (in our case in fact $\tau_i = \tau \to 0$, $\tau_j < \tau_i$). For the "left" ones, it is sufficient to take the opposite contraction. Thus, our bispectrum will have 2 contributions, which in the end, as we saw, will be one the complex conjugate of the other. We have

$$\left\langle \gamma^s_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\right\rangle =$$



from which, by applying the diagrammatic rules and integrating over time, it follows

$$\begin{aligned}\left\langle \gamma^s_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\right\rangle =& -\int_{-\infty}^{\tau\to 0} d\tau_1 a\left[\gamma^s\zeta\zeta\left(iH^{\text{"}right\text{"}}\right) + \left(-iH^{\text{"}left\text{"}}\right)\gamma^s\zeta\zeta\right] \\ =& 2i\int_{-\infty}^{\tau\to 0}\frac{d\tau_1}{\tau_1 H}\int\frac{d^3p_1 d^3p_2 d^3p_3}{(2\pi)^6}\delta^{(3)}(\vec{p_1}+\vec{p_2}+\vec{p_3})\left(\gamma^s_{\vec{k}_1}\gamma^r_{\vec{p}_1}\zeta_{\vec{k}_2}\zeta_{\vec{p}_2}\zeta_{\vec{k}_3}\zeta_{\vec{p}_3} - \gamma^r_{\vec{p}_1}\gamma^s_{\vec{k}_1}\zeta_{\vec{p}_2}\zeta_{\vec{k}_2}\zeta_{\vec{p}_3}\zeta_{\vec{k}_3}\right),\end{aligned} \tag{2.155}$$

where the factor $\times 2$ arises from the symmetry of this specific problem. Being indeed tensor and scalar fields considered as uncorrelated, they evolve independently, and we can contract an external momentum k_i with only two internal ones getting the same result.

Chapter 3

$I^2(\phi)F\tilde{F}$ parity violating trispectrum

We present here a first example of parity violating trispectrum. Firstly, we briefly present the $I^2(\phi)\tilde{F}F$ model. Secondly, we calculate the power-spectra needed for the computation of the bispectrum and the trispecturm. Finally, we will show the parity violation led by this model. We will refer our entire discussion to [2,56,57]. We start studying the model by introducing its action [57, 58]

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + I^2(\phi)\left(-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{\gamma}{4}\tilde{F}^{\mu\nu}F_{\mu\nu}\right)\right], \tag{3.1}$$

where γ is a constant parameter, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the field strength of A^μ and $\tilde{F}^{\mu\nu} = \frac{\epsilon^{\mu\nu\alpha\beta}}{2\sqrt{-g}}F_{\alpha\beta}$ is its dual tensor. While all the coupling terms affect the inflaton's dynamics, only the last one can produce signatures of parity violation, as it is a pseudo-scalar quantity [57]. Since the deep study of this model is not the primary goal of this work, we report here only the most important steps that are needed to compute the correlation functions we want. We perturb the field, allowing for the presence of fluctuations, getting $\phi(\tau,\vec{x}) = \phi_0(\tau) + \delta\phi(\tau,\vec{x})$ and hence $I(\phi) = I_0(\phi) + \delta I(\tau,\vec{x})$. It is shown that the second order term in the field fluctuations is actually subdominant with respect to the other ones, and thus we can neglect it in our calculations. Another useful assumption is to put $I_0(\phi) \propto a^n(\tau)$ during Inflation, and that it becomes a constant when Inflation ends. We also introduce now the electromagnetic convention $\vec{E} = -\frac{I_0(\tau)}{a^2}\vec{A}'$, $\vec{B} = \frac{I_0(\tau)}{a^2}\vec{\nabla}\times\vec{A}$, and put ourselves in the Lorentz gauge, where $A_0 = \vec{\nabla}\cdot\vec{A} = 0$.

As we wish to reproduce some parity violation, we need to set $\gamma \neq 0$. But as we stated in the first chapter, a non vanishing vev for a vector field is not symmetric under rotations leading to an anisotropic expansion of the Universe. However, for $\gamma = 0$, it was shawn that the vector field's energy density must be much smaller than the inflaton's one, for the model to be consistent with the CMB results. The situation gets even more delicate with $\gamma \neq 0$, and hence we can neglect the geometric contributions than can arise from the vector field spoiling the FLRW metric [59]. By introducing now the canonical vector field $\vec{V} = I_0(\tau)\vec{A}$ and expanding it at first order, we see that the gauge field's vev must satisfies $\vec{V}_0'' - \frac{I_0''}{I}\vec{V}_0 = 0$, which leads to a vanishing and a constant vev for $\vec{B}_0$ and $\vec{E}_0$ respectively. Specifically, the latter depends upon the I_0's dependence on the scale factor. For $n = -2$, $\vec{E}_0(\tau) = \vec{E}^{vev}$. We skip now the construction of the mode-functions of our fields, whose procedure can be found in [57,58]. The results that are important for our purposes are

$$\delta E_+(\tau,k) = -\frac{e^{\pi\xi}}{\xi^{3/2}}\frac{3H^2}{2\sqrt{\pi k^3}} = \frac{3}{k\tau}\delta B_+(\tau,k), \tag{3.2}$$

and

$$\langle \delta E_i(\vec{k_1},\tau_1)\delta E_j(\vec{k_2},\tau_2)\rangle \approx (2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}\right)\frac{9H^4}{4\pi}\frac{e^{2\pi\xi}}{\xi^3}k_1^{-3}\epsilon_i^{(+)}(\hat{k_1})\epsilon_j^{(+)}(\hat{k_2}), \tag{3.3}$$

where $\xi \equiv -n\gamma$ We reported only the one-point correlation function between two electric fields since, as the previous equation shows, on superhorizon scales the magnetic components are much smaller than the other ones. In this model, the power spectra of curvature perturbations depend upon the corrections given by the interaction with the vector field [60]. For example, the direction dependence of the vector field's vev is straightly shown in the curvature power spectra, as we will see. We set ourselves in the spatially flat gauge, where the only contributions to the spatial part of the metric are given by gravitational waves $\delta g_{ij} = a^2 h_{ij}$. We use the scalar curvature perturbations defined as $\zeta = -H\frac{\delta\rho}{\dot{\rho}} \approx -\frac{H}{\dot{\phi}}\delta\phi$. In principle, the vector contributions enter in perturbing the g_{00} and g_{0i} components of the metric, but since the subsequent couplings are suppressed, with respect to the topological $\delta\phi\delta A^2$-couplings, it is possible to set $\delta g_{00} = \delta g_{0i} = 0$ [57,58,60]. Then, since we are interested in studying the power spectra given by the gauge field-curvature interactions, we can just look at the interaction Hamiltonian that carries these information. In other words, by writing $\langle\zeta\zeta\rangle \approx \langle\zeta\zeta\rangle_0 + \langle\zeta\zeta\rangle_1$, we focus ourselves on the correlators at first order in the gauge field. By treating the vector field perturbatively, we expand the different terms in the I^2-coupling searching for contributions up to $\delta\phi\delta A^2$. We get

$$\delta\left(I^2(\phi)\right) = 2I_0(\phi)\frac{\partial I_0(\phi)}{\partial\phi}\delta\phi = -2I_0\frac{I_0'}{a}\left(-\frac{\delta\phi}{\dot{\phi}}\right) \approx -2\frac{I_0 I_0'}{aH}\zeta, \tag{3.4}$$

$$\begin{aligned} F^{\mu\nu}F_{\mu\nu} =& -a^{-2}\partial_\tau A^i\partial_\tau A_i a^{-2}2 + \left(\partial_i A_j - \partial_j A_i\right)\left(\partial_k A_l - \partial_l A_k\right)a^{-4}\delta^{ik}\delta^{jl} \\ =& -2\frac{|\vec{E}|^2}{I_0^2} + 2\frac{|\vec{B}|^2}{I_0^2} \approx -2E_i^{(0)}\delta E^i + 2B_i^{(0)}\delta B^i - \delta E_i\delta E^i + \delta B_i\delta B^i, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \frac{\gamma}{8}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}F_{\mu\nu} =& \frac{\gamma}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\mu A_\nu\partial_\alpha A_\beta = \frac{\gamma}{2}\left[-2\left(-\frac{A_i'}{a^2}\right)\left(\frac{\vec{\nabla}\times\vec{A}}{a^2}\right)^i\frac{a^3}{I_0^2}I_0^2\right] \\ \approx& -\frac{\gamma a^3}{I_0^2}\left[\delta E_i B^{(0)i} + E_i^{(0)}\delta B^i + \delta E_i\delta B^i\right]. \end{aligned} \tag{3.6}$$

Thus, we can separate the so perturbed action in the $A\delta\phi\delta A$ and $\delta\phi\delta A^2$ contributions

$$S^{(\zeta\delta A)} = \int d\tau d^3x\left(-\frac{2a^3}{H}\right)\frac{I_0'}{I_0}\zeta\left[\left(E_i^{(0)}\delta E^i - B_i^{(0)}\delta B^i\right) - \gamma\left(E_i^{(0)}\delta B^i + B_i^{(0)}\delta E^i\right)\right], \tag{3.7}$$

$$S^{(\zeta\delta A^2)} = \int d\tau d^3x\left(-\frac{2a^3}{H}\right)\frac{I_0'}{I_0}\zeta\left[\frac{1}{2}\left(\delta E_i\delta E^i - \delta B_i\delta B^i\right) - \gamma\delta E_i\delta B^i\right]. \tag{3.8}$$

.

For studying the perturbed power spectra for gravitational modes, one has to follow the same procedure perturbing $\sqrt{-\delta g} \approx \sqrt{-g} + \delta g_{\mu\nu}\sqrt{-g}g^{\mu\nu}$ getting

$$\begin{aligned} S^{(h\delta A^2)} =& \int d\tau d^3x a I_0^2\delta g_{\mu\nu}\sqrt{-g}g^{\mu\nu}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \\ =& \int d\tau d^3x a^4 I_0^2 a^2 h_{ij}g^{ij}\left[-\frac{1}{2I_0^2}\left(2B_i^{(0)}\delta B^i + 2E_i^{(0)}\delta E^i + \delta B_i\delta B^i + \delta E_i\delta E^i\right)\right]. \end{aligned} \tag{3.9}$$

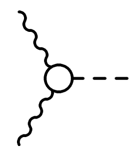
By focusing now only on scalar correlation functions, we can neglect the magnetic modes, as we saw they contribute much less than the electric ones. Hence, we can find the two interaction Hamiltonians that follow from Eqs. 3.7, 3.8. By simply expanding each field in the Fourier space and substituting them inside the corresponding action, knowing that $S = \int dt d^3x \mathcal{L} = -\int d\tau a \mathcal{H}$, we get

$$H^{(\zeta\delta A)} \approx -\frac{4E_i^{vev}}{H^4\tau^4}\int \frac{d^3p}{(2\pi)^3}\delta E^i(\vec{p},\tau)\zeta_{-\vec{p}}(\tau), \tag{3.10}$$

and

$$H^{\zeta\delta A^2} \approx -\frac{2\eta^{ij}}{H^4\tau^4}\int \frac{d^3p d^3p'}{(2\pi)^6}\delta E_i(\vec{p},\tau)\delta E_j(\vec{p},\tau)\zeta_{-\vec{p}'}(\tau)\zeta_{-\vec{p}-\vec{p}'}(\tau). \tag{3.11}$$

We used also the relation $\frac{I_0'}{I_0} = \frac{\left[(-H\tau)^2\right]'}{(-H\tau)} = 2\frac{H}{H\tau}$. It is now useful to diagrammatically represent these Hamiltonians, that are depicted by

$H^{(\zeta\delta A)} =$  and $H^{(\zeta\delta A^2)} =$ 

As, in the computation of the bispectra we did in the previous chapter, we will disassemble all the following correlation functions in several contractions. However, as we are going to see in what follows, we will decompose our expectation values in products of the our fields' commutators at different times. For the scalar field we have

$$\begin{aligned}\langle\left[\zeta_{k_1}(\tau_1),\zeta_{\vec{k}_2}(\tau_2)\right]\rangle =&\langle 0|\left\{u_{k_1}(\tau_1)u^*_{k_2}(\tau_2)\left[a(\vec{k_1}),a^\dagger(-\vec{k_2})\right]+u^*_{k_1}(\tau_1)u_{k_2}(\tau_2)\left[a^\dagger(-\vec{k_1}),a(\vec{k_2}))\right]\right\}|0\rangle\\ =&u_{k_1}(\tau_1)u^*_{k_2}(\tau_2)\langle\left[a(\vec{k_1}),a^\dagger(-\vec{k_2})\right]\rangle - u^*_{k_1}(\tau_1)u_{k_2}(\tau_2)\langle\left[a(\vec{k_2}),a^\dagger(-\vec{k_1})\right]\rangle\\ =&(2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}\right)\frac{H^2}{4\epsilon k_1^3}\left((1+ik_1\tau_1)(1-ik_1\tau_2)e^{-ik_1(\tau_1+\tau_2)}-c.c.\right).\end{aligned} \tag{3.12}$$

By expanding now the exponentials up to third order in $-k_i\tau_i \ll 1$ (i.e. when the modes are well outside the horizon), all the real term cancel between themselves, and we are left with

$$\langle\left[\zeta_{k_1}(\tau_1),\zeta_{\vec{k}_2}(\tau_2)\right]\rangle = -(2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2})\frac{iH^2}{6\epsilon}(\tau_1^3-\tau_2^3). \tag{3.13}$$

For the tensorial modes the same procedure holds, but the mode functions, as we saw, are given by $u_k^{h_{ij}}(\tau) = u_k^{\zeta}(\tau)\sqrt{2\epsilon}$. We are now ready to calculate the power spectra.

Power spectrum

At tree-level the only two-points contribution that can arise from the scalar-vector interaction is given by

$\langle\zeta_{\vec{k_1}}(\tau)\zeta_{\vec{k_2}}(\tau)\rangle_1 =$ 

By means of the in-in formalism, we thus have

$$\left\langle \prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau) \right\rangle_1 = -\int^{\tau} d\tau_1 \int^{\tau_1} d\tau_2 \left\langle \left[\left[\prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau), H^{(\zeta\delta A)}(\tau_1) \right], H^{(\zeta\delta A)}(\tau_2) \right] \right\rangle . \tag{3.14}$$

Now, at first order in perturbations scalars and vectors evolve independently between each other, so that we will have only contractions between the same kind of fields. Moreover, in the integration over time, only the long-wavelength contributions survive, as the small-wavelength cancel each other out due to their rapid oscillation [2]. This means that the dominant terms are given by considering the time interval $\left[\tau_{in} = -\frac{1}{k_{in}} \ll 1, \tau \to 0\right]$, so that we can consider the gauge field as classical and hence commutating. Thanks to this fact, and to Wick's theorem, the expectation value can be divided in two separate expectation values [2, 57][1]. One then obtains

$$\begin{aligned}
\left\langle \prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau) \right\rangle_1 =& -\left(\frac{4}{H^2}\right)^2 E_i^{vev} E_j^{vev} \int_{-\frac{1}{k_1}}^{\tau} \frac{d\tau_1}{\tau_1^4} \int_{-\frac{1}{k_2}}^{\tau_1} \frac{d\tau_2}{\tau_2^4} \\
& \int \frac{d^3p_1 d^3p_2}{(2\pi)^6} \langle \delta E^i(\vec{p_1},\tau_1) \delta E^j(\vec{p_2},\tau_2) \rangle \left\langle \left[\zeta_{\vec{k_1}}(\tau), \zeta_{-\vec{p_1}}(\tau_1) \right] \right\rangle \left\langle \left[\zeta_{\vec{k_2}}(\tau), \zeta_{-\vec{p_2}}(\tau_2) \right] \right\rangle \\
=& -\left(\frac{4}{H^2}\right)^2 E_i^{vev} E_j^{vev} \int_{-\frac{1}{k_1}}^{\tau} \frac{d\tau_1}{\tau_1^4} \int_{-\frac{1}{k_2}}^{\tau_1} \frac{d\tau_2}{\tau_2^4} \\
& \int \frac{d_1^3 d^3p_2}{(2\pi)^6} \epsilon^{(+)i}(\hat{p_1}) \epsilon^{(+)j}(\hat{p_2}) \left(-\frac{e^{\pi\xi}}{\xi^{3/2}}\right)^2 \left(\frac{3H^2}{2\sqrt{\pi}}\right)^2 \frac{(2\pi)^3 \delta^{(3)}(\vec{p_1}+\vec{p_2})}{(p_1p_2)^{3/2}} \times \\
& \times (2\pi)^6 \left[\left(-\frac{iH^2}{6\epsilon}\right)^2 (\tau^3-\tau_1^3)(\tau^3-\tau_2^3) \delta^{(3)}(\vec{k_1}-\vec{p_1}) \delta^{(3)}(\vec{k_2}-\vec{p_2}) + (\vec{k_1} \leftrightarrow \vec{k_2}) \right] \\
=& (2\pi)^3 \delta^{(3)}\left(\vec{k_1}+\vec{k_2}\right) \left(\frac{4}{H^4}\right)^2 \frac{E_i^{vev} E_j^{vev}}{k_1^3} \int_{-\frac{1}{k_1}}^{\tau} \frac{d\tau_1}{\tau_1^4} \int_{-\frac{1}{k_2}}^{\tau_1} \frac{d\tau_2}{\tau_2^4} (\tau^3-\tau_1^3)(\tau^3-\tau_2^3) \\
& \left[\epsilon^{(+)i}(\hat{k_1}) \epsilon^{(+)j}(\hat{k_2}) + \epsilon^{(+)i}(\hat{k_2}) \epsilon^{(+)j}(\hat{k_1}) \right] \frac{e^{2\pi\xi}}{\xi^3} \frac{H^8}{16\pi\epsilon^2},
\end{aligned} \tag{3.15}$$

where we simply substitute the corresponding contractions' and commutators' expressions to the various expectation values. By sending now $\tau \to 0$, as we are intersted in the power-spectrum when it is well outside the horizon, the time integrals reduce to

$$\left\langle \prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau) \right\rangle_1 = (2\pi)^3 \delta^{(3)}(\vec{k_1}+\vec{k_2}) \left(\frac{4}{H^4}\right)^2 \frac{H^8 e^{2\pi\xi}}{16\pi\epsilon^2\xi^3} \sum_{s=+,-} \frac{\epsilon^{(s)i}(\hat{k_1}) \epsilon^{(s)j*}(\hat{k_1}) E_i^{vev} E_j^{vev}}{k_1^3} \int_{-k_1^{-1}}^{0} \frac{d\tau_1}{\tau_1} \int_{-k_2^{-1}}^{\tau_1} \frac{d\tau_2}{\tau_2}, \tag{3.16}$$

where we used the property $\epsilon_i^s(-\hat{k}) = \epsilon_j^{s*}(\hat{k})$, with $s = +, -$. We note that if we had the same upper bound for the τ_2-integral, we would resemble the exact definition of the number of e-folds N_{k_2} calculated from k-mode's horizon exit to the end of inflation, just as done by the τ_1-integration. There is however a well known result that relates nested time integrals to a simpler time integral in which the boundaries are the same for all the integrated variables. It is well known in fact that $\int^{\tau} d\tau_1 \int^{\tau_1} d\tau_2 \cdots \int^{\tau_{n-1}} d\tau_n f(\tau_1, ..., \tau_n) = \frac{1}{n!} \int^{\tau} f(\tau_1, ...\tau_n)$. Therefore, from the whole integration in time, we get a factor $\frac{1}{2} N_{k_1}^2$ as the Dirac's delta forces the two momenta to be the same in modulus. For the directional part we have

[1]We note that in this model we can reduce the entire scalar expectation value to the product of scalar commutators because there is an equal number of vertices and scalar correlation functions. This is true for all the following correlation functions

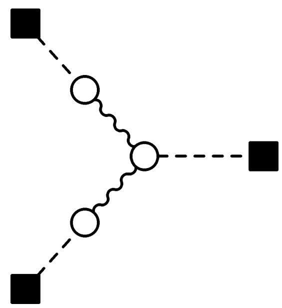
$$\sum_{s=+,-} \epsilon^{(s)i}(\hat{k_1})\epsilon^{(s)*j}(\hat{k_1})E_i^0 E_j^0 \delta^{(3)}(\vec{k_1}+\vec{k_2}) = \left[1-\left(\hat{k_1}\cdot\hat{E}^{vev}\right)^2\right](E^{vev})^2\,\delta^{(3)}(\vec{k_1}+\vec{k_2}), \tag{3.17}$$

where we applied the well known equality $\sum_s \epsilon_i^s(\hat{k})\epsilon_j^{s*}(\hat{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}$[2]
Thus, in the end it follows[3]

$$\left\langle \prod_{i=1}^{2} \zeta_{\vec{k_i}}(\tau) \right\rangle_1 = (2\pi)^3 \frac{E_{vev}^2}{2\pi\epsilon^2 M_{Pl}^4}\frac{e^{2\pi\xi}}{\xi^3}\frac{N_{k_1}^2}{k_1^3}\left[1-\left(\hat{k_1}\cdot\hat{E}^{vev}\right)^2\right]\delta^{(3)}\left(\vec{k_1}+\vec{k_2}\right). \tag{3.18}$$

Bispectrum

Analogously, we want now to calculate the 1-st order perturbation of the three-point scalar correlation function. The tree-level contribution to this bispectrum is given by

$\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1 =$ 

where we have that each vertex identifies a different interaction-time. By the definition of ensemble average, from which we built our discussion regarding correlation functions, we need to take into account all the possible scenarios. Therefore, we have to split our bispectrum in three contributions, accounting for the three different possible series of interactions we have. Namely, we can have the first $H^{(\zeta\delta A)}$ acting at τ_1, the second one at τ_2, and $H^{\zeta\delta A^2}$ at τ_3, or the exact opposite, or yet another combination. Hence, we write our bispectrum as

$$\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1 = \left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} + \left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{121} + \left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{112}, \tag{3.19}$$

with $\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{abc} = i\int^{\tau} d\tau_a \int^{\tau_a} d\tau_b \int^{\tau_b} d\tau_c \left\langle \left[\left[\left[\prod_{l=1}^{3}\zeta_{\vec{k_l}}(\tau), H(\tau_a)\right], H(\tau_b)\right], H(\tau_c)\right]\right\rangle$. For each vertex we will use the corresponding Hamiltonian calculated above. We have to use here both types of Hamiltonian, since we introduced a vertex that connects two internal vector fields. By following the same line of reasoning exposed above, for each piece of Eq. 3.19 we can decompose the whole expectation value in two separate ones, respectively for the scalar and vector part. Considering as an example $\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211}$, we have

[2]We expect $\sum_{s=+,-} \epsilon^{(s)i}(\hat{k_1})\epsilon^{(s)*j}(\hat{k_1})$ to be equal to a certain tensor D_{ij}. Now, we exploit two properties of polarization vectors to identify D_{ij}. Since $\epsilon_i^s(\hat{k})\epsilon_i^r(\hat{k}) = \delta^{rs}$, then we have $\epsilon^{(r)i}(\hat{k})\sum_{s=+,-}\epsilon^{(s)i}(\hat{k_1})\epsilon^{(s)*j}(\hat{k_1}) = \epsilon_j^{(r)}(\hat{k})$, and the analog would be true if we multiplied by $\epsilon^{(r)j}$. Therefore, D_{ij} should have a form which reduces to a Kronecker delta when contracted with one polarization vector. The general form is therefore $D_{ij} = \delta_{ij} + c_{ij}$, with $c_{ij}\epsilon^{(r)i}(\hat{k}) = 0$. Another property of polarization vectors is that $\hat{k}_i\epsilon^{(r)i}(\hat{k}) = 0$. Hence, in the end we can write the equality used above

[3]In [57], the authors do not have a factor $(2\pi)^3$ since they adopted a different convention for the Fourier tranforms and hence the mode functions.

$$
\begin{aligned}
\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} =& - i \int^{\tau} d\tau_1 \int^{\tau_1} d\tau_2 \int^{\tau_2} d\tau_3 \int \frac{\prod_{n=1}^{4} d^3 p_n}{(2\pi)^{12}} \frac{2}{H^4 \tau_1^4} \langle \delta E_i(\vec{p_1},\tau_1) \delta E^j(\vec{p_2},\tau_1) \delta E^k(\vec{p_3},\tau_2) \delta E^l(\vec{p_4},\tau_3) \rangle \times \\
& \times \frac{E_k^{vev} E_l^{vev}}{H^8 \tau_2^4 \tau_3^4} 16 \left[\left\langle \left[\zeta_{\vec{k_1}}(\tau), \zeta_{-\vec{p_1}-\vec{p_2}}(\tau_1) \right] \right\rangle \left\langle \left[\zeta_{\vec{k_2}}(\tau), \zeta_{-\vec{p_3}}(\tau_2) \right] \right\rangle \left\langle \left[\zeta_{\vec{k_3}}(\tau), \zeta_{-\vec{p_4}}(\tau_3) \right] \right\rangle + 5 \text{ perms in } \vec{k_l} \right],
\end{aligned}
\tag{3.20}
$$

where have already dissembled the scalar fields' expectation value in their commutators, by means of Wick's theorem. Just for simplicity, we redefine $\frac{32 E_k^{vev} E_l^{vev}}{(H2\pi)^{12}} = B_{kl}$. Then, from Eq. 3.13, it follows

$$
\begin{aligned}
\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} =& - i B_{kl} (2\pi)^9 \left(-\frac{iH^2}{6\epsilon} \right)^3 \int^{\tau} d\tau_1 \int^{\tau_1} d\tau_2 \int^{\tau_2} d\tau_3 \prod_{m=1}^{3} \frac{(\tau^3 - \tau_m^3)}{\tau_m^4} \\
& \int d^3 p_1 \delta^{(3)}(\vec{k_1} - \vec{p_1} - \vec{p_2}) \langle \delta E_i(\vec{p_1},\tau_1) \delta E^i(\vec{k_1} - \vec{p_1},\tau_1) \delta E^k(\vec{k_2},\tau_2) \delta E^l(\vec{k_3},\tau_3) \rangle \\
& + 2 \text{ perms in } \vec{k_l}.
\end{aligned}
\tag{3.21}
$$

We focus now on the expectation value between vector fields. Again, owing to Wick's theorem, and by looking only to connected diagrams, we have

$$
\begin{aligned}
\langle \delta E_i \delta E^i \delta E^k \delta E^l \rangle =& \delta E_i \delta E^i \delta E^k \delta E^l + \delta E_i \delta E^i \delta E^k \delta E^l \\
=& (2\pi)^6 \epsilon^{(+)i}(\hat{p_1}) \epsilon^{(+)k}(\hat{k_2}) \epsilon_i^{(+)}(\widehat{k_1 - p_1}) \epsilon^{(+)l}(\hat{k_3}) \frac{e^{4\pi\xi}}{\xi^6} \left(\frac{3H^2}{2\sqrt{\pi}} \right)^4 \frac{1}{k_2^3 k_3^3} \\
& \left(\delta^{(3)}(\vec{p_1} + \vec{k_2}) \delta^{(3)}(\vec{k_1} - \vec{p_1} + \vec{k_3}) + \delta^{(3)}(\vec{p_1} + \vec{k_3}) \delta^{(3)}(\vec{k_1} - \vec{p_1} + \vec{k_2}) \right).
\end{aligned}
\tag{3.22}
$$

Integrating out the different Dirac's deltas, we end up with two equal contributions. Namely, the last line of Eq. 3.21 reads

$$
\int d^3 p_1 d^3 p_2 \langle \dots \rangle = \frac{e^{4\pi\xi}}{\xi^6} \left(\frac{3H^2}{2\sqrt{\pi}} \right)^4 \frac{2}{k_2^3 k_3^3} \delta^{(3)}(\vec{k_1} + \vec{k_2} + \vec{k_3}) \epsilon^{(-)i}(\hat{k_2}) \epsilon^{(+)k}(\hat{k_2}) \epsilon_i^{(-)}(\hat{k_3}) \epsilon^{(+)l}(\hat{k_3}). \tag{3.23}
$$

This shows explicitly a symmetry of the system. That is, if we exchange the two external legs connected to the $H^{\zeta\delta A}$-vertex, we obtain the same result. Once the external field connected through $H^{\zeta\delta A}$ is fixed, the choice of the other two gives us the same factor. Therefore, since we have only three momenta, the total number of independent diagram is three and a factor $\times 2$ shows up. Hence, at the end we have

$$
\begin{aligned}
\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} =& (2\pi)^3 \delta^{(3)}(\vec{k_1} + \vec{k_2} + \vec{k_3}) \frac{3H^2 E_k^{vev} E_l^{vev}}{2\pi^2 \epsilon^3} \frac{e^{4\pi\xi}}{\xi^6} \frac{1}{k_2^3 k_3^3} \epsilon^{(-)i}(\hat{k_2}) \epsilon^{(+)k}(\hat{k_2}) \epsilon_i^{(-)}(\hat{k_3}) \epsilon^{(+)l}(\hat{k_3}) \\
& \int^{\tau} d\tau_1 \frac{\tau^3 - \tau_1^3}{\tau_1^4} \int^{\tau_1} d\tau_2 \frac{\tau^3 - \tau_2^3}{\tau_2^4} \int^{\tau_2} d\tau_3 \frac{\tau^3 - \tau_3^3}{\tau_3^4} \\
& + 2 \text{ perms in } \vec{k_l}.
\end{aligned}
\tag{3.24}
$$

Again, we can send $\tau \to 0$, and apply the same trick as above to reduce the nested integral in a single one. However, while we have two time integrals that resemble exactly the definition of the number of e-folds, respectively for the modes that exit the horizon at $-\frac{1}{k_2}$ and $-\frac{1}{k_3}$, the integral over τ_1 depends upon the three moment, which are constrained by the Dirac's delta. Since we want to be sure to consider only long-wavelength contributions, and as $k_1 = |\vec{k_2} + \vec{k_3}|$, we require $\tau_1 > \text{Max}\left[-\frac{1}{k_1}, -\frac{1}{k_2}, -\frac{1}{k_3}\right]$. Namely, since a $H^{\zeta\delta A^2}$-vertex connects three momenta, we want to pick the higher time of horizon-crossing and set it to be the lower bound of our final integration in time.

$$\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1^{211} = (2\pi)^3 \delta^{(3)}(\vec{k_1} + \vec{k_2} + \vec{k_3}) \frac{H^2 E_{vev}^2}{4\pi^2 \epsilon^3} \frac{e^{4\pi\xi}}{\xi^6} \frac{N_{k_3} N_{k_2} \text{Min}\left[N_{k_1}, N_{k_2}, N_{k_3}\right]}{k_2^3 k_3^3} \epsilon^{(-)i}(\hat{k_2}) \epsilon^{(+)k}(\hat{k_2}) \epsilon_i^{(-)}(\hat{k_3}) \epsilon^{(+)l}(\hat{k_3}) + 2 \text{ perms in } \vec{k_l}. \tag{3.25}$$

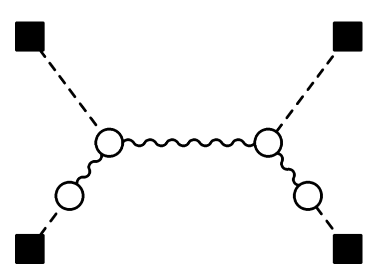
In principle, we would have to calculate all the other terms of Eq. 3.19. However, we notice that for each term we would get exactly the same three contributions depicted in the latter equation. Therefore, we can simply skip the whole calculation and multiply the result above by 3. We just have to follow the same reasoning for the nested time integral and relabel its variables in order to end up with an identical result for each term [57]. At the very end, we find

$$\left\langle \prod_{l=1}^{3} \zeta_{\vec{k_l}}(\tau) \right\rangle_1 = (2\pi)^3 \delta^{(3)}(\vec{k_1} + \vec{k_2} + \vec{k_3}) \frac{3H^2 E_{vev}^2}{4\pi^2 \epsilon^3} \frac{e^{4\pi\xi}}{\xi^6} \frac{N_{k_3} N_{k_2} \text{Min}\left[N_{k_1}, N_{k_2}, N_{k_3}\right]}{k_2^3 k_3^3} \mathcal{C}_{\vec{k_2}, \vec{k_3}} + 2 \text{ perms in } \vec{k_l}, \tag{3.26}$$

with $\mathcal{C}_{\vec{k_2}, \vec{k_3}} = \hat{E}_k^{vev} \hat{E}_l^{vev} \epsilon^{(-)i}(\hat{k_2}) \epsilon^{(+)k}(\hat{k_2}) \epsilon_i^{(-)}(\hat{k_3}) \epsilon^{(+)l}(\hat{k_3})$.
Since we just want to show how this type of calculation is done, we will not go through the procedure to rewrite the directional part in a more compact way. Besides, we are going to do that explicitly in the next subsection. The final equation, in the two opposite limits $\gamma = 0$,$\gamma \neq 0$ is computed in [57].
We note that the two terms arising from a different way to contract the scalar fields can be immediately written as the one above. Namely, by just substituting the old $H^{\zeta\delta A}$- and $H^{\zeta\delta A^2}$-connected fields' momenta with the new ones. While the latter will appear only in $\text{Min}[N_{k_1}, N_{k_2}, N_{k_3}]$, the former will define $\mathcal{C}_{\vec{k}, \vec{k'}}$, the other two number of e-folds and the $\sim 1/k^3 k'^3$ factor.

Trispectrum

Now, we study the four-points correlation function for the scalar-curvature field. Again, we restrict ourselves to the single tree-level contribution we can draw, which reads

$\left\langle \prod_{l=1}^{4} \zeta_{\vec{k_l}}(\tau) \right\rangle_1 =$ 

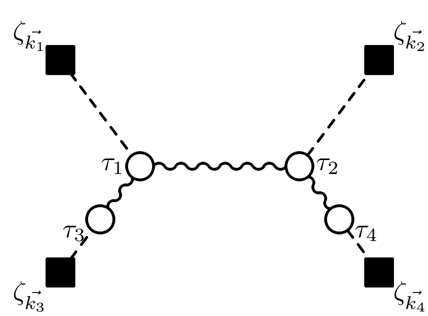
Again, the full trispectrum is given by

$$\begin{aligned}\left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1 =& \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{1122} + \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{1212} + \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{1221} + \\ &+ \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{2121} + \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{2112} + \left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{2211} .\end{aligned} \tag{3.27}$$

In complete analogy with what we have done so far, we proceed in writing explicitly the trispectrum. For the same reason explained above, we can separate the whole expectation value in two simpler ones. Then, we proceed in applying Wick's theorem, thereby simplifying each of them in several scalar commutators. We have

$$\begin{aligned}\left\langle\prod_{l=1}^{4}\zeta_{\vec{k_l}}(\tau)\right\rangle_1^{2211} =& \frac{64E_k^{vev}E_l^{vev}}{H^{16}}\int^{\tau}\frac{d\tau_1}{\tau_1^4}\int^{\tau_1}\frac{d\tau_2}{\tau_2^4}\int^{\tau_2}\frac{d\tau_3}{\tau_3^4}\int^{\tau_3}\frac{d\tau_4}{\tau_4^4}\int\frac{d^3p_1d^3q_1d^3p_2d^3q_2d^3p_3d^3p_4}{(2\pi)^{18}}(2\pi)^{12}\left(\frac{-iH^2}{6\epsilon M_{Pl}^2}\right)^4 \\ & \left\langle\delta E^i(\vec{p_1},\tau_1)\delta E_i(\vec{q_1},\tau_1)\delta E^j(\vec{p_2},\tau_2)\delta E_j(\vec{q_2},\tau_2)\delta E^k(\vec{p_3},\tau_3)\delta E^l(\vec{p_4},\tau_4)\right\rangle(\tau^3-\tau_1^3)(\tau^3-\tau_2^3) \\ & (\tau^3-\tau_3^3)(\tau^3-\tau_4^3)\delta^{(3)}\left(\vec{k_1}-\vec{p_1}-\vec{q_1}\right)\delta^{(3)}\left(\vec{k_2}-\vec{p_2}-\vec{q_2}\right)\delta^{(3)}\left(\vec{k_3}-\vec{p_3}\right)\delta^{(3)}\left(\vec{k_4}-\vec{p_4}\right) \\ & + 23 \text{ perms in } \vec{k_l} \\ =& \frac{64E_k^{vev}E_l^{vev}}{H^{16}}\left(\prod_{i=1}^{4}\int^{\tau_{i-1}}\frac{d\tau_i}{\tau_i^4}(\tau^3-\tau_i^3)\right)\int\frac{d^3p_1d^3p_2}{(2\pi)^6}\left(\frac{-iH^2}{6\epsilon M_{Pl}^2}\right)^4 \\ & \left\langle\delta E^i(\vec{p_1},\tau_1)\delta E_i(\vec{k_1}-\vec{p_1},\tau_1)\delta E^j(\vec{p_2},\tau_2)\delta E_j(\vec{k_2}-\vec{p_2},\tau_2)\delta E^k(\vec{k_3},\tau_3)\delta E^l(\vec{k_4},\tau_4)\right\rangle \\ & + 23 \text{ perms in } \vec{k_l},\end{aligned} \tag{3.28}$$

where we expanded the scalar expectation value by using Eq. 3.13 and put $\tau_0 = \tau$. Since the complete calculation is quite long, we proceed reporting the most important passages. For simplicity, we forget about the 23 permutations for the moment, and depict only the contribution explicitly written above. This corresponds to the diagram



By expanding the expectation value of the vector fields we find, neglecting the non-connected diagrams

$$
\begin{aligned}
\langle \dots \rangle =& (2\pi)^9 \left(\frac{9H^4 e^{2\pi\xi}}{4\pi\xi^3}\right)^3 \frac{\epsilon_i^{(+)}(\hat{p_1})\epsilon^{(+)i}(k_1 \hat{-} p_1)\epsilon_j^{(+)}(\hat{p_2})\epsilon^{(+)j}(k_2 \hat{-} p_2)\epsilon^{(+)k}(\hat{k_3})\epsilon^{(+)l}(\hat{k_4})}{(p_1|\vec{k_1}-\vec{p_1}|p_2|\vec{k_2}-\vec{p_2}|k_3k_4)^{3/2}} \times \\
& \times \Big\{ \delta^{(3)}(\vec{p_1}+\vec{p_2}) \Big[\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_3})\delta^{(3)}(\vec{k_2}-\vec{p_2}+\vec{k_4}) + \delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_4})\delta^{(3)}(\vec{k_2}-\vec{p_2}+\vec{k_3})\Big] \\
& \delta^{(3)}(\vec{p_1}+\vec{k_2}-\vec{p_2}) \Big[\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_3})\delta^{(3)}(\vec{p_2}+\vec{k_4}) + \delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_4})\delta^{(3)}(\vec{p_2}+\vec{k_3})\Big] \\
& \delta^{(3)}(\vec{p_1}+\vec{k_3}) \Big[\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_2}-\vec{p_2})\delta^{(3)}(\vec{p_2}+\vec{k_4}) + \delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{p_2})\delta^{(3)}(\vec{k_2}-\vec{p_2}+\vec{k_4})\Big] \\
& \delta^{(3)}(\vec{p_1}+\vec{k_4}) \Big[\delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{k_2}-\vec{p_2})\delta^{(3)}(\vec{p_2}+\vec{k_3}) + \delta^{(3)}(\vec{k_1}-\vec{p_1}+\vec{p_2})\delta^{(3)}(\vec{k_2}-\vec{p_2}+\vec{k_3})\Big] \Big\} \\
=& (2\pi)^9 \delta^{(3)}\left(\sum_{i=1}^4 \vec{k_i}\right)\left(\frac{9H^4 e^{2\pi\xi}}{4\pi\xi^3}\right)^3 4 \frac{\epsilon^{(-)i}(\hat{k_3})\epsilon^{(+)k}(\hat{k_3})\epsilon^{(-)j}(\hat{k_4})\epsilon^{(+)l}(\hat{k_4})}{k_3^3 k_4^3} \times \\
& \times \left[\frac{\epsilon_i^{(+)}(\hat{k_{24}})\epsilon_j^{(-)}(\hat{k_{24}})}{k_{24}^3} + \frac{\epsilon_j^{(+)}(\hat{k_{23}})\epsilon_i^{(-)}(\hat{k_{23}})}{k_{23}^3}\right].
\end{aligned}
\tag{3.29}
$$

From the last line, we see that if we exchange the two external legs connected to the simplest vertex, the result remains the same. This is because all the indices are actually dummy and can be relabelled. This reduces the number of independent diagrams to 12, and multiplies the whole result by a factor 2. By calling now the directional part $\mathcal{C}^{k_i,k_j}(k_k)$ we see that for every diagram we have

$$
\mathcal{C}^{k_i,k_j}(k_k) = 4E_k^{\hat{v}ev}E_l^{\hat{v}ev}\frac{\epsilon^{(-)i}(\hat{k_i})\epsilon^{(+)k}(\hat{k_i})\epsilon^{(-)j}(\hat{k_j})\epsilon^{(+)l}(\hat{k_j})}{k_i^3 k_j^3}\left[\frac{\epsilon_i^{(+)}(\hat{k_{ki}})\epsilon_j^{(-)}(\hat{k_{ki}})}{k_{ki}^3} + \frac{\epsilon_j^{(+)}(\hat{k_{kj}})\epsilon_i^{(-)}(\hat{k_{kj}})}{k_{kj}^3}\right], \tag{3.30}
$$

where k_i,k_j are the momenta connected to the simplest vertex, while k_k is one of the other two. Thanks to this observation, we can now write all the other 11 permutations easily. However, since k_k can be referred either to k_1 or k_2 without having any differences in the final result, we get that by exchanging the two $H^{\zeta\delta A^2}$-external lines we obtain the same form of $\mathcal{C}$. Therefore, the number of independent diagrams further decrease to 6, while another factor $\times 2$ shows up. In the end, we thus have

$$
\left\langle \prod_{l=1}^4 \zeta_{\vec{k_l}}(\tau)\right\rangle_1^{2211} = (2\pi)^3\delta^{(3)}\left(\sum_{i=1}^4 \vec{k_i}\right)\frac{3^5 H^4 E_{vev}^2}{\pi^3 M_{Pl}^8 \epsilon^4}\frac{e^{6\pi\xi}}{\xi^9}\left(\prod_{i=1}^4\int^{\tau_{i-1}}\frac{d\tau_i}{\tau_i^4}(\tau^3-\tau_i^3)\right)\mathcal{C}^{k_3k_4}(k_2) + 5 \text{ perms in } \vec{k_l}. \tag{3.31}
$$

Now, for the nested time-integrals, we proceed as above. We extend each integral's upper bound to the same value $\tau \to 0$ while dividing by $4! = 24$. Then, we use the same approximation applied in the previous section, which constraints our integral to $\tau_i > -\frac{1}{k_i}$. Just to briefly recap, we neglect the short-wavelength contributions since due to their highly oscillatory behaviour they cancel each other [2]. Therefore, we have

$$
\begin{aligned}
\prod_i \int^{\tau} d\tau_i \frac{1}{\tau_i} =& \frac{1}{24}\int_{\mathrm{Max}\left[-\frac{1}{k_1},-\frac{1}{k_3},-\frac{1}{k_{24}}\right]}^0 \frac{d\tau_1}{\tau_1}\int_{\mathrm{Max}\left[-\frac{1}{k_2},-\frac{1}{k_4},-\frac{1}{k_{13}}\right]}^0 \frac{d\tau_2}{\tau_2}\int_{-\frac{1}{k_3}}^0 \frac{d\tau_3}{\tau_3}\int_{-\frac{1}{k_3}}^0 \frac{d\tau_4}{\tau_4} \\
=& \frac{1}{24}\mathrm{Min}\left[N_{k_1},N_{k_3},N_{k_{24}}\right]\mathrm{Min}\left[N_{k_2},N_{k_4},N_{k_{13}}\right]N_{k_3}N_{k_4}.
\end{aligned}
\tag{3.32}
$$

In the very end, by calculating all the other terms of Eq. 3.27 the total trispectrum $t^{\vec{k_1},\vec{k_2}}_{\vec{k_3},\vec{k_4}}$ depicted in the diagram above reads [2][4]

$$\begin{aligned}\mathbf{t}^{\vec{k_1},\vec{k_2}}_{\vec{k_3},\vec{k_4}} =&(2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}+\vec{k_3}+\vec{k_4}\right)\frac{3^5}{4\pi^3}\frac{H^4E^2_{vev}}{M^8_{Pl}\epsilon^8}\frac{e^{6\pi\xi}}{\xi^9}\mathcal{C}^{\vec{k_3},\vec{k_4}}(\vec{k_2})\\ &\mathrm{Min}\left[N_{k_1},N_{k_3},N_{k_{24}}\right]\mathrm{Min}\left[N_{k_2},N_{k_4},N_{k_{13}}\right]N_{k_3}N_{k_4}.\end{aligned} \tag{3.33}$$

Here the parity violation can be explicitly seen by rewriting the $\mathcal{C}$ function using

$$\epsilon^-_i(\hat{k})\epsilon^+_j(\hat{k}) = \frac{1}{2}\left[\delta_{ij}-\hat{k}_i\hat{k}_j+i\epsilon_{ijk}\hat{k}^k\right]. \tag{3.34}$$

By considering for instance $\mathcal{C}^{\vec{k_3},\vec{k_4}}(\vec{k_2})$ of the result above, and restricting ourselves to just the first term arising from Eq.3.30, we find

$$\begin{aligned}\mathrm{Im}\left(\mathcal{C}^{\vec{k_3},\vec{k_4}}(\hat{k_2})\right) = \frac{1}{8k_3^3k_4^3k_{23}^3}\Big\{&-\vec{k_{23}}\left[\hat{E}\times\hat{k_4}\right]\left(\hat{E}\cdot\hat{k_3}\right)-\hat{k_{23}}\left[\hat{E}\times\hat{k_3}\right]\left(\hat{E}\cdot\hat{k_4}\right)-\hat{k_3}\left[\hat{E}\times\hat{k_4}\right]\left(\hat{E}\cdot\hat{k_3}\right)+\\ &-\hat{k_4}\left[\hat{E}\times\hat{k_3}\right]\left(\hat{E}\cdot\hat{k_4}\right)-\hat{k_4}\left[\hat{E}\times\hat{k_{23}}\right]\left(\hat{E}\cdot\hat{k_4}\right)+\hat{k_3}\left[\hat{E}\times\hat{k_{23}}\right]\left(\hat{E}\cdot\hat{k_3}\right)+\\ &+\hat{k_2}\left[\hat{E}\times\hat{k_4}\right]\left(\hat{k_{23}}\cdot\hat{k_3}\right)\left(\hat{E}\cdot\hat{k_3}\right)+\hat{k_2}\left[\hat{E}\times\hat{k_3}\right]\left(\hat{k_{23}}\cdot\hat{k_4}\right)\left(\hat{E}\cdot\hat{k_4}\right)+\\ &+\hat{k_2}\left[\hat{k_4}\times\hat{k_3}\right]\left(\hat{E}\cdot\hat{k_3}\right)\left(\hat{E}\cdot\hat{k_4}\right)+\hat{k_{23}}\left[\left(\hat{E}\times\hat{k_3}\right)\times\left(\hat{E}\times\hat{k_4}\right)\right]\Big\}.\end{aligned} \tag{3.35}$$

This result can be then translated into the induced CMB trispectrum, which, thanks to the presence of the vacuum expectation value $\hat{E}^{vev}$, breaks the rotational invariance. This was explicitly done in [2]. Now that we have reproduced a result analogous to what we have to treat in this work, we can proceed in studying our specific model.

[4]The numerical factors between the original result differ from the one reported here. In [2], a factor $1/2^{15}$ arises from the three vector-vector contractions, since there it was used $\xi = 2|\gamma|$.

Chapter 4

The $f(\phi)\tilde{W}W$ model

In order to describe a quantum field theory, a typical way to proceed is to specify a Lagrangian for a particular field, thus constructing its action and use it to find the field's equation of motion through the action principle. The Lagrangian of a theory has to be always an invariant under any spacetime and Lorentz transformations. This is because we want the same Lagrangian to describe the same physical theory everywhere and after any Lorentz's boost, which implies that it has to be a scalar. For example, in the familiar Lagrangian for a point-form body of mass m in classical mechanics one finds only scalar terms. In fact, the only vector that shows up is easily contracted with itself giving us its squared modulus. This is the velocity vector, which is given by the first time derivative of the position $\vec{x}$. Also by considering quantum electrodynamics we find a Lagrangian which contains first derivative terms of the type $\partial_\mu A^\mu$, which are contracted with one another by the Minkowski metric. In general relativity, the theory's main feature is the metric $g_{\mu\nu}$ itself. However, it is not possible to find scalar invariants out of first derivatives of the metric. To understand why, it is sufficient to notice that to be an invariant $\mathcal{L}$ has to have a value that is independent upon the coordinate system's choice. Thanks to the equivalence principle, however, nearby a point it is always possible to find a coordinate system where all the first derivatives of the metric vanish[1]. Hence, by building up a Lagrangian proportional to $\partial_\alpha g_{\mu\nu}$, one obtains a theory with an action that can be either null or not depending upon the chosen coordinate. The Hilbert-Einstein action is instead built by considering only covariant terms proportional to second-derivatives of the metric tensor. The only covariant term that respects these requests is the Ricci's scalar $R = R_{\mu\nu}g^{\mu\nu}$. The action then reads

$$S^{HE} = \frac{M_{Pl}^2}{2}\int d^4x\sqrt{-g}R, \tag{4.1}$$

where $\sqrt{-g} = \sqrt{-det\,(g_{\mu\nu})}$ makes the integration's measure covariant. A usual way to modify the Einstein's theory of gravity is to go to higher order terms in the derivatives.
When we introduced the theory of Inflation, we started from the following action

$$S = \int d^4x\sqrt{-g}\left[-\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{4.2}$$

This theory allowed us to depict a first model for explaining the observed fluctuations in the CMB matter density. These entail the Hubble parameter and the physical wave-number k/a at horizon exit to have a value $H = k/a \approx \sqrt{\epsilon} \times 2 \times 10^{14}$GeV [61], with ϵ the first slow-roll parameter. Being this value much less than the Planck's mass one ($M_{Pl} \approx 2.4 \times 10^{18}$GeV), we are allow to consider General Relativity as our underlying theory for gravity. However,

[1]For example, Riemann normal coordinates constitute a local inertial frame, where the neighborhood of any point q of a spacetime resembles Minkowski: $g_{\mu\nu}(x_q) = \eta_{\mu\nu}$, $\partial_\alpha g_{\mu\nu}\big|_{x_q} = 0$

at horizon crossing H (and hence k/a) is not negligible compared to whichever energy scale characterizing the theory underlying Inflation [6], and at early times the physical wave-number of perturbations can be exponentially larger than at horizon crossing. Therefore, it might be useful to go beyond the Einstein's theory of relativity and consider some correction to its action. In order to do that, we can consider Eq. 4.2 as the first term of an Effective Field Theory expansion. Then, whatever fundamental theory this will be, it will contain some higher derivatives terms that are proportional to negative powers of some large cut-off energy scale M. From the first slow-roll parameter $\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2M_{Pl}^2 H^2}\dot{\phi}$, it follows that the background field ϕ, described by the above Lagrangian, evolves in time as $\dot{\phi} = \sqrt{2\epsilon} M_{Pl} H$. This means that in a Hubble time, it undergoes a change $\frac{\dot{\phi}}{H} = \sqrt{2\epsilon} M_{Pl}$. We expect that M cannot be much smaller than $\sqrt{2\epsilon} M_{Pl}$. In fact, by following the above evolution of ϕ (i.e. $\delta\phi$), we find an order of magnitude of the inflaton's energy scale at a Hubble time before the horizon crossing. As we started our discussion by saying that at early times we expect the Inflation's energy scale to be not entirely negligible compared with other possible underlying theories, we impose this limit to be reasonable for our corrections. Therefore, it follows that the expansion parameter $\frac{H}{M}$ has to be not much greater than $\frac{H}{\sqrt{2\epsilon} M_{Pl}} \approx 6 \times 10^{-5}$ [6]. We will assume $M \sim \sqrt{2\epsilon} M_{Pl}$. Our Lagrangian now reads

$$\mathcal{L}_0 = \sqrt{-g}\left[\frac{M_{Pl}^2}{2} R - \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{4.3}$$

The leading correction to the so-written theory, will be given by a sum of all possible covariant terms with four spacetimes derivative and unitary coefficients. We thus have [6]

$$\begin{aligned}\Delta\mathcal{L} =& \sqrt{-g}\Big[f_1(\phi)\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)^2 + f_2(\phi) g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\Box\phi + f_3(\phi)\left(\Box\phi\right)^2 + f_4(\phi) R^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \\ &+ f_5(\phi) R g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + f_6(\phi) R\Box\phi + f_7(\phi) R^2 + f_8(\phi) R^{\mu\nu} R_{\mu\nu} + f_9(\phi) W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma}\Big] + \\ &+ f_{10}(\phi)\epsilon^{\mu\nu\rho\sigma} W_{\mu\nu}^{\alpha\beta} W_{\rho\sigma\alpha\beta}.\end{aligned} \tag{4.4}$$

In the last two terms, instead of Riemann tensors, we used the Weyl tensor, which is defined as

$$W_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta} - \frac{1}{2}\left(g_{\mu\alpha}R_{\nu\beta} - g_{\mu\beta}R_{\nu\alpha} - g_{\nu\alpha}R_{\mu\beta} + g_{\nu\beta}R_{\mu\alpha}\right) + \frac{R}{6}\left(g_{\mu\alpha}g_{\nu\beta} - g_{\nu\alpha}g_{\mu\beta}\right). \tag{4.5}$$

The last term of our Lagrangian's correction does not change if it is written using the Weyl tensor [6, 8], while for the second last term it is sufficient to redefine f_7 and f_8. The Weyl tensor is nothing but the traceless part of the Riemann tensor, and it turns out to be more convenient than $R_{\mu\nu\alpha\beta}$ in Eq.4.4 to perform future computations.
Before proceeding in analyze the theory in its entirety, we would like to get rid of some additional terms. In fact, by not doing that, we would find more than the usual two adiabatic modes for single-field inflation with strange commutation relations (e.g. ϕ would commute with $\dot{\phi}$), or we would ended up with dynamical auxiliary fields, as their time derivatives, although not in $\mathcal{L}_0$, enter in the correction $\Delta\mathcal{L}_0$. We can actually neglect some terms since, from the point of view of effective field theory, Eq. 4.4 represents just the lowest additional term in a M^{-a} expansion ($a < 0$), and we have to get rid of all the contributions that cannot be expanded in this way [62]. We thus eliminate all the time derivative of the auxiliary ADM-fields by using

$$\Box\phi = V'(\phi) \qquad \text{and} \qquad R_{\mu\nu} = -\left(\frac{1}{M_{Pl}}\right)^2 \partial_\mu\phi\partial_\nu\phi - V(\phi) g_{\mu\nu}, \tag{4.6}$$

which are the field equations derived from the leading terms in the action. This procedure, somehow, corresponds to find the physical dynamics of the fields we are interested in (through the Euler-Lagrange equations), and then impose it to all the other possible corrections. The latter, come into play as higher order terms to a dynamics that has to be satisfied. Hence, considering again the example of auxiliary fields, if the physics we observe at leading order does not see them as dynamical, all its correction cannot making them so. In this way, one obtains a consistent series expansion of the (unknown) underlying theory. By using the dynamics expressed by Eq.4.6, we can rule out all the terms in Eq.4.4 with the exception of the first one and the last two. For a detailed analysis of all the neglected terms, one can look at [6]. Therefore, in the end, with proper redefinitions of ϕ, $V(\phi)$, and f_1, the final Lagrangian will be

$$\mathcal{L} = \mathcal{L}_0 + \sqrt{-g}\left[f_1\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right)^2 + f_9 W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma}\right] + f_{10}\epsilon^{\mu\nu\rho\sigma}W^{\alpha\beta}_{\mu\nu}W_{\rho\sigma\alpha\beta}. \tag{4.7}$$

Here, the last second term represents a Gauss-Bonnet contribution, which modifies the dynamics of the inflaton[2], and the last one is a Chern-Simons term, which breaks parity. These two terms, as well as the third one, are coupled to the inflaton's field through the f_n coupling functions, which in principle must be fixed by quantum gravity. However, since it does not exist yet a consistent theory for quantum gravity, it is impossible to determine these couplings. Therefore, we will deal with these functions just by leaving them free [7]. The first term of the above equation represents a counterterm to ultraviolet divergenless, which has to be considered when the leading terms of Eq.4.3 are expanded beyond tree-level. However, in this work we are not interested in those limits, hence we can forget about it. Thus, our Lagrangian reads

$$\mathcal{L} = \sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{g^{\mu\nu}}{2}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + f_9(\phi)W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma} + f_{10}(\phi)\epsilon^{\mu\nu\rho\sigma}W^{\alpha\beta}_{\mu\nu}W_{\rho\sigma\alpha\beta}. \tag{4.8}$$

In particular, we will pay attention to the Chern-Simons term, since it is the one that breaks parity. In addition to that, however, it has other useful properties [8]. Once computed in a flat FLRW space-time, it is automatically zero, since the metric is conformally invariant [3]. This implies that the Chern-Simons term does not modify the background dynamics of the inflaton. Hence, in absence of the Gauss-Bonnet contribution, we will be able to use the usual mode function for ϕ. Moreover, the Chern-Simons term is invariant under Weyl transformation of the metric, which will be useful in the following computations. In the end, we note that this term is a total derivative one, hence the coupling f_{10} is extremely important to have signatures of the theory in the studied dynamics [64]. Thus, the Lagrangian we will use is given by the usual one, with the addition of the Chern-Simons term

$$\mathcal{L} = \sqrt{-g}\left[\frac{M_{Pl}^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + f(\phi)\epsilon^{\mu\nu\rho\sigma}W^{\alpha\beta}_{\mu\nu}W_{\rho\sigma\alpha\beta}. \tag{4.9}$$

In the following, we will adopt the ADM formalism we introduced in Sec.2.2, and fix ourselves in the spatially-flat gauge, where the three dimensional metric is given by

$$g_{ij} = a^2\left[\delta_{ij} + \gamma_{ij}\right] \qquad \text{with} \qquad \gamma^i_i = 0 = \partial^i\gamma_{ij}. \tag{4.10}$$

The advantage of this slicing is that the terms dominant in the slow-roll hypothesis are clear in studying primordial non-Gaussianities. On the other hand, the scalar curvature on uniform energy-density hypersurfaces ζ does not appear explicitly. In fact, all the scalar perturbations as set to zero. In details, we can pass to describe non-Gaussianities in that variable through a non-linear relation [52]

[2]In fact, it can be rewritten as $f_9(\phi)R^2_{GB} = f_9(\phi)\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2\right)$. By expressing each term in the flat FLRW metric, thereby solving the Euler-Lagrange equation for ϕ, a new term $\frac{3}{2}H^2(\dot{H} + H^2)f_{9,\phi}$ arises, which in principle is not vanishing and hence directly contributes to define the evolution of the inflaton's field [7]

[3]Actually, it is possible to show that also by considering a curvature $k = \pm 1$, we would get a vanishing Weyl tensor [63]

$$
\begin{aligned}
\zeta =&\zeta_1 + \left(\frac{1}{2}\frac{\ddot{\phi}}{\dot{\phi}H} + \frac{1}{4}\frac{\dot{\phi}^2}{H^2}\right)\zeta_1^2 + \\
&+ \frac{1}{H}\dot{\zeta}_1\zeta_1 - \frac{1}{4}\frac{a^{-2}}{H^2}\partial^{-2}\partial_i\partial_j\left(\partial^i\zeta_1\partial^j\zeta_1\right) + \frac{1}{2H}\partial_i B\partial^i\zeta_1 - \frac{1}{2H}\partial^{-2}\partial_i\partial_j\left(\partial^i B\partial^j\zeta_1\right) - \frac{1}{4H}\dot{\gamma_{ij}}\partial^i\partial^j\zeta_1,
\end{aligned}
\tag{4.11}
$$

where ϕ is the classical inflaton's field, H is the usual Hubble's parameter and B is one of the auxiliary fields introduced in Sec.2.2. Also, here

$$
\zeta_1 = -\frac{H}{\dot{\phi}}\delta\phi. \tag{4.12}
$$

From the equation above, we can remove the whole second line, as after horizon crossing it is vanishing [52]. Moreover, the ζ_1^2 contribution of the first line represents a disconnected term [8]. Therefore, in the following calculation we will consider $\zeta \approx -\frac{H}{\dot{\phi}}\delta\phi$.
As done in studying the ζ's dynamics in the previous sections, we should now constrain the two auxiliary fields introduced by the ADM formalism. We notice that now another gravity term is present in the action, and that in principle this can modify our previously defined N and N_i. However, before jumping into the full computation, let us discuss if the newly introduced Chern-Simons term is really going to modify the first order values of N, N_i. Certainly, as N is scalar, it will not be touched by the parity-breaking Chern-Simons contribution. Even N_i, through the Helmholtz theorem can be written as a parity-even scalar B and a vector part $\tilde{N}_i$. We remind that in Sec.2.2 we found

$$
N_i^{(1)} = \partial_i B, \qquad \text{with} \qquad B = -\frac{\zeta}{a^2 H} + \frac{\dot{\phi}^2}{2H^2}\partial^{-2}\dot{\zeta}, \tag{4.13}
$$

so that only the scalar part of N_i contributed. Now, either $\tilde{N}_i$ remain an auxiliary field, or it becomes dynamical thanks to the action of the Chern-Simons term. For the former, at first order in perturbations (δ, γ_{ij}) we cannot construct any non-trivial transverse vector, and hence we can put $\tilde{N}_i = 0$. For the latter, instead, we would have an additional evolving vector perturbation, which in principle could interact with both $\delta\phi$, γ_{ij}. However, the goal of the thesis does not concern the study of other degrees of freedom. Therefore, we will stick to the standard gravity ($\tilde{N}_i = 0$) case [8].

4.1 Parity-violating correlation functions

Our purpose is to find if there are some parity-violating signatures in the ζ trispectrum given by two connected vertices of the type scalar-scalar-graviton. In order to do that, we need to perturb the interaction Hamiltonian up to second order in scalar and at first order in tensor perturbations. Before doing that, however, it is interesting to study other possibilities of parity breaking in this modified gravity theory. Also, since we are going to consider a tensor-tensor contraction, it is useful to start by studying the power-spectra of these modified gravitons. As we said above, since we are not going to consider the Gauss-Bonnet term in our action, the inflaton's field behaves as reported in Chapter 1. For the full derivation of the scalar mode-function in a theory described by Eq.4.8, one can look in [7].
If on the one hand the Chern-Simons term does not affect the quadratic scalar part of $\mathcal{L}$, on the other hand it does so in the quadratic tensor section. We thus need to find the second-order perturbed action for tensor perturbations. A very useful observation is that since our choice for the metric reproduces a vanishing Weyl tensor, we have to compute only a first order perturbed $W_{\mu\nu\alpha\beta}$. Therefore, the only term which we have to compute is $f(\phi)\epsilon^{\mu\nu\rho\sigma}W^{\alpha\beta}_{\mu\nu}\big|^{(1)}_T W_{\alpha\beta\rho\sigma}\big|^{(1)}_T$. Furthermore, we can put $N = 1, N_i = 0$ as we do not want to consider scalar perturbations. Thanks to that and by using conformal time, we find $g_{\mu\nu} = a^2\gamma_{\mu\nu}$ (with $\gamma_{00} = -1$). We can actually

compute the Weyl tensors only on $\gamma_{\mu\nu}$. By noting that since we wrote the Chern-Simons term in a Weyl invariant way, through a Weyl transformation $g_{\mu\nu} \to e^{-2\omega} g_{\mu\nu}$, and by choosing $\omega = \log a$ we have indeed $g_{\mu\nu} \to g'_{\mu\nu} = \gamma_{\mu\nu}$. The resulting second-order action reads [8]

$$S^{\gamma\gamma} = \int d\tau \frac{d^3k}{(2\pi)^3} \sum_{s=L,R} A^2_{T,s} \left[\left|\gamma'_s(k,\tau)\right|^2 - k^2 \left|\gamma_s(k,\tau)\right|^2 \right], \tag{4.14}$$

with

$$A^2_{T,s} = \frac{M^2_{Pl}}{2} a^2 \left(1 - 8\lambda_s \frac{k}{a} \frac{\dot{f(\phi)}}{M^2_{Pl}} \right) = \frac{M^2_{Pl}}{2} a^2 \left(1 - \lambda_s \frac{k_{phys}}{M_{CS}} \right), \tag{4.15}$$

and where we expanded each gravitational wave in its Fourier modes through

$$\gamma_{ij}(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \sum_{s=L,R} \epsilon^s_{ij}(\hat{k}) \gamma_s(k,\tau). \tag{4.16}$$

Here, we expressed the gravitational waves as decomposed in their left ($s = -2$) and right ($s = +2$) polarizations [4]. The polarization tensors are defined as $\epsilon^{R/L}_{ij} = \frac{1}{\sqrt{2}} \left(\epsilon^+_{ij} \pm i\epsilon^\times_{ij} \right)$ and the following properties hold

$$\begin{aligned}
\epsilon^L_{ij}(\hat{k})\epsilon^{ij}_L(\hat{k}) =& \epsilon^R_{ij}(\hat{k})\epsilon^{ij}_R(\hat{k}) = 0 \\
\epsilon^L_{ij}(\hat{k})\epsilon^{ij}_R(\hat{k}) =& 2 \\
\epsilon^L_{ij}(-\hat{k}) =& \epsilon^R_{ij}(\hat{k}) \\
k_l \epsilon^{mlj} \epsilon^i_{(s)j}(\hat{k}) =& -i\lambda_s k \epsilon^{im}_{(s)}(\hat{k}) \\
\gamma^*_s(\hat{k}) =& \gamma_s(\hat{k}),
\end{aligned} \tag{4.17}$$

with $\lambda_s = \pm 1$ if $s = \pm 2$ ($s = R/L$) respectively. In Eq.4.15, we introduced the $M_{CS} = \frac{M^2_{Pl}}{8\dot{f(\phi)}}$ term. This is the so-called Chern-Simons mass and it represents the energy cut-off of this effective model [7]. We also notice that from the mentioned equation, we can already put some constraints on the gravitons' equation of motion. In particular, for physical momenta k/a higher than our cut-off M_{CS}, we see that if $\lambda_s = +1$, $A^2_{T,s}$ becomes negative. This would imply the appearance of modes with a negative kinetic energy, i.e. ghost fields. In principle, ghosts might imply an ill-defined quantum theory at high energies and create severe problems in finding a consistent UV completion (i.e. the generalization of an effective field theory above its energy threshold). Also, ghosts can create serious instabilities in the evolution of perturbations [65]. To avoid the problems that might arise from possible ghosts, we consider gravitons with momenta below our cut-off energy $k_{phys} < M_{CS}$ at the beginning of inflation. Now, we need to put also $k_{phys} \gg H$ when Inflation starts, in order to have λ_{phys} well inside the horizon, and hence be able to consider perturbations arising from a Bunch-Davies vacuum. Therefore, by requiring these two conditions, we end up with $\frac{H}{M_{CS}} \ll 1$ at the beginning of Inflation.

As done in Sec.2.4, we can find the mode-function for our gravitons by rescaling the field γ_s. In fact, by putting [8]

$$\mu_s = A_{T,s}\gamma_s, \tag{4.18}$$

[4] It is straightforward to see that the decompositions obtained through the usual $+,\times$ and through L,R polarizations give the same final form for the Fourier transform of γ_{ij}. In order to see this, one can use the definition of left- and right- polarization tensors and the fact that $\gamma^R = \frac{1}{\sqrt{2}} \left(\gamma^+ - i\gamma^- \right)$ and $\gamma^L = \frac{1}{\sqrt{2}} \left(\gamma^+ + i\gamma^- \right)$ [5]

we find

$$\mathcal{L}^{\gamma\gamma} = A_{T,s}^2 \gamma_s'^2 - k^2 A_{T,s}^2 \gamma_s^2 = \mu_s'^2 + \frac{A_{T,s}'^2}{A_{T,s}^2}\mu_s^2 - 2\mu_s'\mu_s \frac{A_{T,s}'}{A_{T,s}} - k^2\mu_s^2, \tag{4.19}$$

whose correspondent Euler-Lagrange equation read

$$\partial_\tau \frac{\partial \mathcal{L}^{\mu\mu}}{\partial \mu'} - \frac{\partial \mathcal{L}^{\mu\mu}}{\partial \mu} = \mu_k''^s + \left(k^2 - \frac{A_{T,s}''}{A_{T,s}}\right)\mu_k^s = 0, \tag{4.20}$$

where the new $\frac{A''_{T,s}}{A_{T,s}}$ term plays the role of an effective potential. Now we have

$$\begin{aligned}\frac{A_{T,s}'}{A_{T,s}} = \frac{a}{2}\frac{(\dot{A_{T,s}})^2}{A_{T,s}^2} =& \frac{a}{2}\frac{1}{A_{T,s}^2}\left(a\dot{a}M_{Pl}^2\left(1-\lambda_s\frac{k}{aM_{CS}}\right) + \frac{a^2M_{Pl}^2}{2}\lambda_s\frac{k}{a}\frac{H}{M_{CS}}\right)\\ =& aH + \lambda_s\frac{kH}{2M_{CS}},\end{aligned} \tag{4.21}$$

where the first order approximation in $\frac{H}{M_{CS}} \ll 1$ has been used, and M_{CS} has been considered constant in time. Then, by considering a (quasi) dS Universe [7], we have $\tau = -\frac{1}{aH(1-\epsilon)}$ and hence

$$\frac{A_{T,s}'}{A_{T,s}} \approx -\frac{1+\epsilon}{\tau} + \lambda_s\frac{kH}{2M_{CS}}, \tag{4.22}$$

$$\begin{aligned}\frac{A_{T,s}''}{A_{T,s}} =& \frac{1+\epsilon}{\tau^2} + \lambda_s\frac{k\dot{H}a}{M_{CS}} + \frac{1}{\tau^2}(1+\epsilon)^2 + \lambda_s^2\left(\frac{kH}{M_{CS}}\right)^2 - 2\lambda_s\frac{kH}{M_{CS}}\frac{(1+\epsilon)}{\tau}\\ \approx& \frac{2+3\epsilon}{\tau^2} - \frac{kH}{\tau M_{CS}}.\end{aligned} \tag{4.23}$$

Therefore, in the end, we find

$$\mu_k''^s + \left(k^2 - \frac{\nu_T^2 - \frac{1}{4}}{\tau^2} + \lambda_s\frac{kH}{M_{CS}\tau}\right)\mu_k^s = 0, \tag{4.24}$$

with $\nu_T \approx \frac{3}{2} + \epsilon$, and ϵ being the usual slow-roll parameter. The equation above differs from what we found for tensor perturbations (Sec.2.4) just by the effective term $\lambda_s \frac{k}{\tau}\frac{H}{M_{CS}}$. It is indeed this term that explicitly breaks the parity, since it introduces a difference in the evolution for right- and left-polarized modes. Now, we proceed by quantizing the field, as done several times up to now. We have

$$\mu^s(k,\tau) = u^s(k,\tau)a^s(\vec{k}) + u^{s*}(k,\tau)a^{s\dagger}(-\vec{k}), \tag{4.25}$$

with

$$\left[a^s(\vec{k}), a^{r\dagger}(-\vec{k}')\right] = (2\pi)^3\delta^{(3)}\left(\vec{k}+\vec{k}'\right)\delta^{sr} \qquad \text{and} \qquad \left[[a^s(\vec{k}), a^r(\vec{k}')\right] = 0 = \left[a^{s\dagger}(-\vec{k}), a^{r\dagger}(-\vec{k}')\right]. \tag{4.26}$$

Then, we can rewrite Eq.4.24 in terms of the mode-functions we just defined, thereby solving it for those. Once imposed a Bunch-Davies vacuum, the exact solution reads [7,8]

$$u^s(k,\tau) = 2\sqrt{\frac{(-k\tau)^3}{k}}e^{-i(\frac{\pi}{4}-\frac{\pi\nu_T}{2})}e^{-ik\tau}U\left(\frac{1}{2}+\nu_T-\lambda_s\frac{H}{M_{CS}},1+2\nu_T,2ik\tau\right)e^{\frac{\pi}{4}\lambda_s\frac{H}{M_{CS}}}, \tag{4.27}$$

where $U\left(\frac{1}{2}+\nu_T-\lambda_s\frac{H}{M_{CS}},1+2\nu_T,2ik\tau\right)$ is the confluent hypergeometric function [66]. By expanding the solution at zero-th order in $k\tau$, i.e. going on superhorizon regimes, the solution reads

$$u^s(k,\tau)^{sup} = \sqrt{\frac{1}{2k^2\tau^2k}}e^{-i\frac{\pi}{4}(1-2\nu_T)}\frac{\Gamma(\nu_T)}{\Gamma(\frac{3}{2})}\left(\frac{-k\tau}{2}\right)^{3-2\nu_T}e^{\frac{\pi}{4}\lambda_s\frac{H}{M_{CS}}}. \tag{4.28}$$

Now that we have the mode-functions both inside and outside the horizon, we can start considering some observables. As done in the previous chapter, we begin by considering the power spectra given by our parity-violating gravitons. One has

$$\begin{aligned}\langle\gamma^s_{ij}(\vec{k})\gamma^{ijr}(\vec{k'})\rangle =&\sum_{s,r}\epsilon^s_{ij}(\hat{k})\epsilon^{ijr}(\hat{k}')\frac{\langle\mu^s_k\mu^r_{k'}\rangle}{A^2_{T,s}} = (2\pi)^3\delta^{rs}\delta^{(3)}\left(\vec{k}+\vec{k'}\right)\sum_{s,r}\epsilon^s_{ij}(\hat{k})\epsilon^{ijr}(\hat{k}')\frac{|u^s_k|^2}{A^2_{T,s}}\\ =&(2\pi)^3\delta^{sr}\delta^{(3)}\left(\vec{k}+\vec{k'}\right)\left[2\frac{|u^R_k|^2}{A^2_{T,R}}+2\frac{|u^L_k|^2}{A^2_{T,L}}\right] = (2\pi)^3\delta^{sr}\delta^{(3)}\left(\vec{k}+\vec{k'}\right)2\Delta^{tot}_T,\end{aligned} \tag{4.29}$$

where we used the properties given by Eqs.4.17. From here, we see that, as happens in the usual slow-roll scenario, the total tensor power spectrum is considered as the sum of its two polarized power spectra, which read

$$\Delta^R_T = 2\frac{|u^R_k|^2}{A^2_{T,R}} \qquad \text{and} \qquad \Delta^L_T = 2\frac{|u^L_k|^2}{A^2_{T,L}}. \tag{4.30}$$

Hence, by going well outside the horizon they become

$$\Delta^R_T = 2\frac{\Delta_T}{2}e^{\frac{\pi}{4}\frac{H}{M_{CS}}} \qquad \text{and} \qquad \Delta^L_T = 2\frac{\Delta_T}{2}e^{-\frac{\pi}{4}\frac{H}{M_{CS}}}. \tag{4.31}$$

respectively, with $\Delta_T = \frac{4}{k^3}\frac{H^2}{M^2_{Pl}}\left(\frac{-k\tau}{2}\right)^{3-2\nu_T}$ being the total power spectrum of tensor perturbations in the standard slow-roll model, i.e. without the Chern-Simons term. Since we have imposed $\frac{H}{M_{CS}} \ll 1$, we can expand the exponentials above, thereby defining a relative difference between right- and left-polarized power-spectra [7,8] (and Refs. therein)

$$\Theta = \frac{\Delta^R_T-\Delta^L_T}{\Delta^R_T+\Delta^L_T} \approx \frac{\pi}{2}\left(\frac{H}{M_{CS}}\right), \tag{4.32}$$

as all the other components of the tensor mode-functions do not depend upon the chosen polarization. Since we constrained the parameter H/M_{CS} to be very small during inflation, we expect a tiny parity-violating signatures coming from the parameter just defined. Motivated by this reason, it could be interesting to investigate the quantity of parity violation carried by higher order correlation functions.

4.2 Parity-violating $f(\phi)\tilde{W}W$ bispectra

The only parity-violating bispectra that can arise from the Chern-Simons term are given by $\langle\gamma\gamma\gamma\rangle$,$\langle\gamma\gamma\zeta\rangle$, and $\langle\gamma\zeta\zeta\rangle$. This is because, as reported in Chapter 1, for purely scalar correlation functions the lowest order statistics carrying parity violation is the trispectrum, whose computation is shown in the next session. As above, we will adopt the in-in formalism to deal with these cosmological correlation functions, sending $\tau \to 0$, since we are interested in the observables when they are well outside the horizon. In the following, we briefly report the three correlators, which was studied in details in [8]. We will limit ourselves to report the important steps, underlying the ones which are important for our final computation of the $\langle\zeta\zeta\zeta\zeta\rangle$ trispectrum.

4.2.1 $\langle\gamma\gamma\gamma\rangle$ bispectrum

As done several time up to now, we start by searching for the interacting Hamiltonian. This comes from the interacting Lagrangian expanded up to third order in tensor fields. Since the Chern-Simons term we have vanishes when evaluated in a FLRW background, we do not have to compute the Weyl tensor perturbed up to third order. Hence we have

$$\mathcal{L}^{\gamma\gamma\gamma} = f(\phi)\epsilon^{\mu\nu\rho\sigma}\left[W^{\alpha\beta}_{\mu\nu}\Big|^{(2)}_{T} W_{\rho\sigma\alpha\beta}\Big|^{(1)}_{T} + W^{\alpha\beta}_{\mu\nu}\Big|^{(1)}_{T} W_{\rho\sigma\alpha\beta}\Big|^{(2)}_{T} \right]. \tag{4.33}$$

Now, instead of explicitly compute the Weyl tensor up to second order in tensor fields, we can proceed getting an idea of the strength of such an interaction. By using the properties of the Chern-Simons term, we can in fact see that it can be rewritten $f(\phi)$ multiplied by a total derivative term: $\mathcal{L}^{\gamma\gamma\gamma} = f(\phi)\nabla_\alpha A^\alpha$. Here A^α is the Chern-Simons topological current, and it is given by [67, 68]

$$A^\alpha = \epsilon^{\alpha\beta\gamma\delta}\left(\Gamma^\sigma_{\beta\rho}\partial_\gamma\Gamma^\rho_{\delta\sigma} + \frac{2}{3}\Gamma^\sigma_{\beta\rho}\Gamma^\rho_{\gamma\xi}\Gamma^\xi_{\delta\sigma} \right). \tag{4.34}$$

Moreover, we can also rewrite $\mathcal{L}^{\gamma\gamma\gamma}$ as $\nabla_\alpha\left(f(\phi)A^\alpha\right) - \left(\partial_\alpha f(\phi)\right)A^\alpha$, where the former term is non-dynamical and the latter can be computed considering the background inflaton's field, so that we can consider $\alpha = 0$. Therefore, we have $\mathcal{L}^{\gamma\gamma\gamma} = -f'(\phi)A^0 + \nabla_\alpha\left(f(\phi)A^\alpha\right)$, with only the first term that directly contributes to the dynamics of our tensor fields. Now, for what concerns the explicit form of A^0, we can perform a Weyl's transformation making A^α to be dependent only on γ_{ij} and its derivatives. Hence, by considering $\mathcal{H}^{\gamma\gamma\gamma} = -\mathcal{L}^{\gamma\gamma\gamma}$, by means of the in-in formalism we find

$$\langle\gamma^s(\vec{k_1})\gamma^r(\vec{k_2})\gamma^t(\vec{k_3})\rangle = i\int_{-\infty}^{\tau\to 0} d\tau_1 f'(\phi)\left\langle\left[\gamma^s_{\vec{k_1}}(\tau)\gamma^r_{\vec{k_2}}(\tau)\gamma^t_{\vec{k_3}}(\tau), A^0(\tau_1)\right]\right\rangle. \tag{4.35}$$

Now, in order to solve the entire braket, we saw that we have to proceed by contracting all the fields through Wick's contractions. As always, a contraction between two tensor fields is given by

$$\langle\gamma^s_{\vec{k_1}}(\tau)\gamma^r_{\vec{k_2}}(\tau)\rangle = (2\pi)^3\delta^{sr}\delta^{(3)}\left(\vec{k_1} + \vec{k_2}\right)u^{s\gamma}(k_1,\tau)u^{r*\gamma}(k_2,\tau), \tag{4.36}$$

where now $u^{s\gamma}(k,\tau) = \frac{u^s(k,\tau)}{A_{T,s}}$. By taking the slow-roll parameters and $\frac{H}{M_{CS}}$ to be vanishing, we have $\frac{1}{A_{T,S}} \approx \frac{\sqrt{2}}{M_{Pl}a}$, and the solution becomes [8]

$$u^{s\gamma}(k,\tau) = \frac{iH}{M_{Pl}\sqrt{k^3}}(1 + ik\tau)e^{-ik\tau}, \tag{4.37}$$

which perfectly resembles the solution in a dS space. Thus, by using this graviton's mode function in the three-points function above, it follows [8]

$$\langle \gamma^s_{\vec{k}_1} \gamma^r_{\vec{k}_2} \gamma^t_{\vec{k}_3} \rangle = -i \int_{-\infty}^{\tau \to 0} d\tau_1 \left(\frac{1}{H\tau_1} \right) \dot{f}(\phi) \left(\prod_{i=1}^{3} \frac{H^2}{M_{Pl}^3 k_i^3} \right) f(k_i, \tau_1) e^{i k_T \tau_1}, \tag{4.38}$$

where $k_T = k_1 + k_2 + k_3$, and $f(k_1, \tau_1) = (1 + i\tau_1(k_1 + k_2 + k_3) - \tau_1^2(k_1k_2 + k_2k_3 + k_1k_3) - ik_1k_2k_3\tau_1^3)$. We observe that $a \approx -\frac{1}{\tau H}$ has been used since we are considering slow-roll parameters at lowest order. Furthermore, due to this approximation, we can put H, and $\dot{f}$ to be nearly constant. From that it also follows that M_{CS} is constant too. We will come back to this assumption later on. By solving the time integral performing a Wick's rotation, we can rewrite the three gravitons correlation function as

$$\langle \gamma^s_{\vec{k}_1} \gamma^r_{\vec{k}_2} \gamma^t_{\vec{k}_3} \rangle \sim \frac{H}{M_{CS}} \left(\sum_{i \neq j} \Delta_T(k_i) \Delta_T(k_j) \right) M(k_i), \tag{4.39}$$

where $M(k_i)$ represents a function of momenta and Δ_T is the dS tensor power-spectrum we introduced before. We note, that $M(k_i)$ comes from the integration over time of the polynomial function f, and hence it turns out to be complex. The parity violation is then encoded in this function. However, the whole correlator is multiplied by $\frac{H}{M_{CS}} \ll 1$, giving us the same suppression suffered by the power spectra.

4.2.2 $\langle \zeta\gamma\gamma \rangle$ bispectrum

The interaction Hamiltonian we have to consider for this type of bispectrum has to contain perturbations up to first order in scalar and second order in tensor fields. Starting from the Lagrangian we have [8]

$$\begin{aligned} \mathcal{L}^{\delta\phi\gamma\gamma} = & \epsilon^{\mu\nu\rho\sigma} \Bigg[\left(\frac{\partial f(\phi)}{\partial \phi} \right) \delta\phi W^{\alpha\beta}_{\mu\nu}\Big|^{(1)}_T W_{\rho\sigma\alpha\beta}\Big|^{(1)}_T + f(\phi) W^{\alpha\beta}_{\mu\nu}\Big|^{(1)}_S W_{\rho\sigma\alpha\beta}\Big|^{(2)}_T + f(\phi) W^{\alpha\beta}_{\mu\nu}\Big|^{(2)}_T W_{\rho\sigma\alpha\beta}\Big|^{(1)}_S \\ & + f(\phi) W^{\alpha\beta}_{\mu\nu}\Big|^{(1)}_T W_{\rho\sigma\alpha\beta}\Big|^{(2)}_{ST} + f(\phi) W^{\alpha\beta}_{\mu\nu}\Big|^{(2)}_{ST} W_{\rho\sigma\alpha\beta}\Big|^{(1)}_T \Bigg], \end{aligned} \tag{4.40}$$

where the subscripts indicate the nature of a certain perturbation, and the superscript its order. In the gauge we chose, scalar perturbations appear in the metric only through first order approximation of N, and N_i [8]. By expressing these by means of the slow-roll parameters, it follows that $N, N_i \sim \sqrt{\epsilon}\delta\phi$. Therefore, all the terms coming from scalar metric perturbations are suppressed with respect to $\delta\phi$ thanks to slow-roll. The dominant term in $\mathcal{L}^{\delta\phi\gamma\gamma}$ is then the first one. It reads, once expressed in the Fourier space and integrated over d^3x,

$$\begin{aligned} L^{\delta\phi\gamma\gamma} = & -\sum_s \lambda_s \int \frac{d^3k d^3p d^3q}{(2\pi)^6} \delta^{(3)} \left(\vec{k} + \vec{p} + \vec{q} \right) \Bigg\{ \left(\frac{\partial f(\phi)}{\partial \phi} \right) p \delta\phi'_{\vec{k}} \left[\gamma'^{s}_{ij\,\vec{p}} \gamma'^{2ij}_{\vec{q}} + (\vec{p} \cdot \vec{q})\, \gamma^s_{ij}(\vec{p}) \gamma^{sij}(\vec{q}) \right] + \\ & + a \left(\dot{\phi} \frac{\partial^2 f(\phi)}{\partial \phi^2} \right) p \delta\phi(\vec{k}) \left[\gamma'^{s}_{ij}(\vec{p}) \gamma'^{sij}(\vec{q}) + \left(\vec{p} \cdot \vec{q} \gamma^s_{ij}(\vec{p}) \gamma^{sij}(\vec{q}) \right) \right] \Bigg\}. \end{aligned} \tag{4.41}$$

In this vertex, we have terms that are proportional to the second derivative of the coupling function f with respect to the inflaton's field. This is not suppressed by $\frac{H}{M_{CS}}$ as the other results, and hence, in principle, we can expect a non-negligible amount of parity-violation. Also, we observe that since we got rid of the Gauss-Bonnet term, the scalar mode-function resembles the usual one in dS (Eq.1.49). Thus, by approximating H and $f(\phi)$ to be nearly constant during Inflation, and putting $a \approx -\frac{1}{\tau H}$ as before, we have

$$\begin{aligned}\langle\gamma_{R\vec{k_1}}\gamma_{R\vec{k_2}}\delta\phi_{\vec{k_3}}\rangle =& -i(2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}+\vec{k_3}\right)\mathrm{Im}\left\{\left[k_1(I_1+I_2)+k_1(\vec{k_1}\cdot\vec{k_2})(I_3+I_4)\right]\epsilon_{ij}^{R*}(\hat{k_1})\epsilon_R^{ij*}(\hat{k_2})-\text{c.c.}\right\}+\\ &+\left(\vec{k_1}\leftrightarrow\vec{k_2}\right),\end{aligned} \tag{4.42}$$

where the integrals I_1, I_2, and I_3 are time integrals of polynomial functions in τ_1 multiplied by a complex exponential $e^{iK_T}\tau_1$, since each of them contains a time derivative of the mode functions that simplifies the $\frac{1}{\tau_1}$ factor. Thus, to solve them it is sufficient to perform a Wick's rotation. For the I_4 integral, instead, another step is necessary, since in this case we are not able to get rid of the τ_1 at the denominator. In particular, we have

$$\begin{aligned}I_4 =& a\frac{M_{Pl}^2}{H}\left(\prod_{i=1}^{3}\frac{H^2}{M_{Pl}^2 2k_i^3}\right)\left(\dot{\phi}\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)\left\{\int_{-\infty}^{0}\frac{d\tau_1}{\tau_1}e^{iK_T\tau_1}+iK_T\int_{-\infty}^{0}d\tau_1 e^{iK_T\tau_1}+\right.\\ &\left.-\prod_{i\neq j}k_ik_j\int_{-\infty}^{0}d\tau_1\tau_1 e^{iK_T\tau_1}-ik_1k_2k_3\int_{-\infty}^{0}d\tau_1\tau_1^2 e^{iK_T\tau_1}\right\}.\end{aligned} \tag{4.43}$$

Here, while for the last three integrals one can proceed as explained above, for the first one it is possible to use the Exponential Integral $Ei(z)$. We do not solve it explicitly now, as it is going to be necessary later on, when we will consider an explicitly parity-breaking mode-function. The only thing we stress now, is that all these contributions but the first give us a real contribution, which is cancelled thanks to the c.c. term in Eq.4.42. For the same reason, we can restrict the result of $Ei(z)$ to be purely imaginary, obtaining $Ei(iK_T\tau) = -i\frac{\pi}{2}$. At the very end one finds

$$\langle\gamma^R_{\vec{k_1}}\gamma^R_{\vec{k_2}}\delta\phi_{\vec{k_3}}\rangle = (2\pi)^3\delta^{(3)}\left(\sum_n\vec{k_n}\right)4\pi\frac{H^6}{M_{Pl}^4\prod_n(2k_n^3)}\left(\dot{\phi}\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)(k_1+k_2)(\vec{k_1}\cdot\vec{k_2})\epsilon_{ij}^R(\hat{k_1})\epsilon_R^{ij}(\hat{k_2}), \tag{4.44}$$

and

$$\langle\gamma^L_{\vec{k_1}}\gamma^L_{\vec{k_2}}\delta\phi_{\vec{k_3}}\rangle = -(2\pi)^3\delta^{(3)}\left(\sum_n\vec{k_n}\right)4\pi\frac{H^6}{M_{Pl}^4\prod_n(2k_n^3)}\left(\dot{\phi}\frac{\partial^2 f(\phi)}{\partial\phi^2}\right)(k_1+k_2)(\vec{k_1}\cdot\vec{k_2})\epsilon_{ij}^L(\hat{k_1})\epsilon_L^{ij}(\hat{k_2}), \tag{4.45}$$

because of the λ_s that multiplies the whole result. It is then possible to show that the mixed correlators between left- and right- polarized gravitons vanish [8]. As we see, this result is not suppressed by the parameter $\frac{H}{M_{CS}}$, and the parity breaking shows itself by a different evolution of the correlator with left- and right-gravitons. By using Eq.4.12, we can express the above correlators as functions of the scalar curvature perrturbation. Then, as done for the power-spectra, we can define a relative parity-violation coefficient

$$\Pi = \frac{\langle\gamma_R\gamma_R\zeta\rangle-\langle\gamma_L\gamma_L\zeta\rangle}{\langle\gamma_R\gamma_R\zeta\rangle+\langle\gamma_L\gamma_L\zeta\rangle} = \frac{96\pi}{25}H^2\frac{\partial^2 f(\phi)}{\partial\phi^2}, \tag{4.46}$$

from which we see that by requiring the second derivative of the coupling function to be large enough, we might obtain a non negligible amount of parity violation. A complete analysis of the possible constraints that can be inferred from this result is reported in [8].

4.2.3 $\langle\zeta\zeta\gamma\rangle$ bispectrum

For the purpose of this thesis, this is the most important three-points correlation function, since the trispectrum we are going to compute is given by the connection of two vertices of this type. The interaction Lagrangian reads [8]

$$
\begin{aligned}
\mathcal{L}^{\delta\phi\delta\phi\gamma} =&\epsilon^{\mu\nu\rho\sigma}\left[\frac{\partial f(\phi)}{\partial\phi}\delta\phi W^{\alpha\beta}_{\mu\nu}\Big|^{(1)}_S W_{\rho\sigma\alpha\beta}\Big|^{(1)}_T + \frac{\partial f(\phi)}{\partial\phi}\delta\phi W^{\alpha\beta}_{\mu\nu}\Big|^{(1)}_T W_{\rho\sigma\alpha\beta}\Big|^{(1)}_S + f(\phi)W^{\alpha\beta}_{\mu\nu}\Big|^{(2)}_S W_{\rho\sigma\alpha\beta}\Big|^{(1)}_T +\right.\\
&\left.+ f(\phi)W^{\alpha\beta}_{\mu\nu}\Big|^{(1)}_T W_{\rho\sigma\alpha\beta}\Big|^{(2)}_S + f(\phi)W^{\alpha\beta}_{\mu\nu}\Big|^{(2)}_{ST} W_{\rho\sigma\alpha\beta}\Big|^{(1)}_S + f(\phi)W^{\alpha\beta}_{\mu\nu}\Big|^{(1)}_S W_{\rho\sigma\alpha\beta}\Big|^{(2)}_{ST}\right],
\end{aligned}
\tag{4.47}
$$

where, as in the previous case, the subscripts and the superscripts identify the kind and the order of perturbation. Again, for the same reasons expressed above, the dominant contributions are carried by the first two terms, which depend only on linearly perturbed Weyl's tensors and therefore can be easily computed. One obtains

$$
\mathcal{L}^{\delta\phi\delta\phi\gamma} = -8\frac{\partial f(\phi)}{\partial\phi}\sqrt{\epsilon}\left(\partial^l\delta\phi\right)\epsilon^{ijk}\left[\left(\partial_k\delta\phi\right)\partial_i\gamma'_{lj}\right], \tag{4.48}
$$

where we can substitute $\frac{\partial f(\phi)}{\partial\phi} = \frac{\dot{f(\phi)}}{\dot{\phi}} = \frac{M^2_{Pl}}{8M_{CS}\dot{\phi}}$. Thanks to the in-in formalism the correlator reads

$$
\begin{aligned}
\langle\delta\phi_{\vec{k_1}}\delta\phi_{\vec{k_2}}\gamma^s_{\vec{k_3}\,mb}\rangle =&(2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}+\vec{k_3}\right)\sqrt{\epsilon}\frac{H^5}{4M_{CS}\dot{\phi}\prod_n k_n^3}\epsilon^{ijk}\sum_r\epsilon^{(-r)}_{lj}(\hat{k_3})\epsilon^{(s)}_{mb}(\hat{k_3})\delta^{sr}k_1^l k_{2k}k_{3i}k_3^2\\
&\int_{-\infty}^{\tau\to 0} d\tau_1\left(1+i\tau_1(k_1+k_2)-k_1k_2\tau_1^2\right)e^{i\tau_1 K_T},
\end{aligned}
\tag{4.49}
$$

where again $K_T = k_1+k_2+k_3$. One can then solve the time integral through a Wick's rotation, obtaining a complex function that depends only on the modulus of the external momenta, namely

$$
\mathcal{G} = -i\left(\frac{1}{K_T}-\frac{k_1+k_2}{K_T^2}+2\frac{k_1k_2}{K_T^3}\right). \tag{4.50}
$$

By caling $\mathcal{F} = \epsilon^{ijk}\sum_r\epsilon^{(-r)}_{ij}\epsilon^{(s)}_{mb}k_1^l k_{2k}k_{3i}$, in the end we find

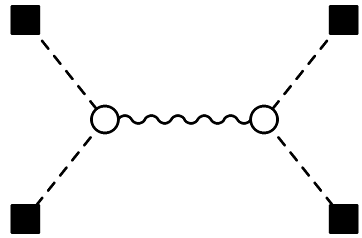
$$
\langle\zeta_{\vec{k_1}}\zeta_{\vec{k_2}}\gamma^s_{\vec{k_3}\,mb}\rangle = (2\pi)^3\delta^{(3)}\left(\sum_n\vec{k_n}\right)\frac{H^4}{\sqrt{2}8M_{CS}M^3_{Pl}\epsilon\prod_n k_n^3}\mathcal{F}\mathcal{G}, \tag{4.51}
$$

from which it is clear that the amount of parity violation is suppressed by the $\frac{H}{M_{CS}}$ factor in front.

4.3 Parity-violating $f(\phi)\tilde{W}W$ trispectrum

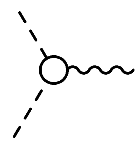
In order to compute the trispectrum $\langle\prod_{i=1}^4\zeta_{\vec{k}_i}(\tau)\rangle$ for the model $f(\phi)\tilde{W}W$, we proceed by applying the in-in formalism to the following tree-level contribution

$$\left\langle \prod_{i=1}^4 \zeta_{\vec{k_i}}(\tau) \right\rangle_1 =$$



where the subscript 1 indicates the leading order contribution to the scalar curvature's trispectrum given by the interaction Hamiltonian

$$H^{\delta\phi\delta\phi\gamma} = -L_{int}^{\delta\phi\delta\phi\gamma} =$$



.

Since we want to compute the trispectrum for scalar curvature perturbations, we rewrite the interaction Lagrangian expressed in Eq.4.48 by using $\zeta \approx -\frac{H}{\dot{\phi}}\delta\phi$, $\frac{\partial f(\phi)}{\partial \phi} = \frac{M_{Pl}^2}{8M_{CS}\dot{\phi}}$ and $\epsilon \simeq \frac{1}{2}\frac{\dot{\phi}^2}{H^2}M_{Pl}^{-2}$. We find

$$\mathcal{L}_{int}^{\zeta\zeta\gamma} = -\frac{\sqrt{2}M_{Pl}^3}{M_{CS}H}\epsilon(\partial^l\zeta)\epsilon^{ijk}\left[(\partial_k\zeta)\partial_i\gamma'_{lj}\right]. \tag{4.52}$$

By expanding now all fields in Fourier space and integrating over d^3x, we find

$$H_{int}^{\zeta\zeta\gamma}(\tau) = -i\frac{\sqrt{2}M_{Pl}^3}{M_{CS}H}\epsilon\int\frac{d^3kd^3pd^3q}{(2\pi)^6}\sum_s k^l\epsilon^{ijk}p_kq_i\epsilon_{lj}^s(\hat{q})\zeta_{\vec{k}}(\tau)\zeta_{\vec{p}}(\tau)\gamma_{\vec{q}}^{\prime s}(\tau)\delta^{(3)}(\vec{k}+\vec{p}+\vec{q}). \tag{4.53}$$

In order to proceed in the calculation, we cannot rewrite the trispectrum as in [2] as a multiplication between the v.e.v. of the gauge fields and the scalar commutators. This is because of two things. The first one is that we are not interested (only) in the long-wavelengths approximation. Hence, we do not have a classical commuting tensor field, and hereby we cannot put it outside the whole expectation value of the in-in formula. The second one is due to the fact that we do not have an equal number of vertices and scalar contractions. As explained above, this does not allow us to write the entire scalar expectation value as a multiplication between commutators of scalar at different times. In details, we have two vertices and four scalar fields' contractions, which gives us just four terms (two for each vertex, 2.1), out of which we are not able to write four commutators (sixteen terms). Therefore, we will apply directly the in-in formalism equation, getting

$$\begin{aligned}\left\langle \prod_{n=1}^4 \zeta_{\vec{k}_n}(\tau)\right\rangle =& \frac{2M_{Pl}^6\epsilon^2}{M_{CS}^2H^4}\int^\tau\frac{d\tau_1}{\tau_1}\int^{\tau_1}\frac{d\tau_2}{\tau_2}\int\frac{d^3q_1d^3p_1d^3n_1d^3q_2d^3p_2d^3n_2}{(2\pi)^{12}}\delta^{(3)}\left(\vec{q_1}+\vec{p_1}+\vec{n_1}\right)\\ &\delta^{(3)}\left(\vec{q_2}+\vec{p_2}+\vec{n_2}\right)\sum_{s,r}\epsilon_{lj}^s(\hat{n_1})\epsilon_{mb}^r(\hat{n_2})\epsilon^{ijk}q_1^lp_{1k}n_{1i}\epsilon^{abc}q_2^mp_{2c}n_{2a}\\ &\left\langle\left(\prod_{n=1}^4\zeta_{\vec{k_n}}\right)\zeta_{\vec{q_1}}\zeta_{\vec{p_1}}\gamma_{\vec{n_1}}^{'s}\zeta_{\vec{q_2}}\zeta_{\vec{p_2}}\gamma_{\vec{n_2}}^{'r} - \zeta_{\vec{q_1}}\zeta_{\vec{p_1}}\gamma_{\vec{n_1}}^{'s}\left(\prod_{n=1}^4\zeta_{\vec{k_n}}\right)\zeta_{\vec{q_2}}\zeta_{\vec{p_2}}\gamma_{\vec{n_2}}^{'r}+\right.\\ &\left.-\zeta_{\vec{q_2}}\zeta_{\vec{p_2}}\gamma_{\vec{n_2}}^{'r}\left(\prod_{n=1}^4\zeta_{\vec{k_n}}\right)\zeta_{\vec{q_1}}\zeta_{\vec{p_1}}\gamma_{\vec{n_1}}^{'s}+\zeta_{\vec{q_2}}\zeta_{\vec{p_2}}\gamma_{\vec{n_2}}^{'r}\zeta_{\vec{q_1}}\zeta_{\vec{p_1}}\gamma_{\vec{n_1}}^{'s}\left(\prod_{n=1}^4\zeta_{\vec{k_n}}\right)\right\rangle,\end{aligned} \tag{4.54}$$

where we did not show the times explicitly for simplicity of notation. However, it should be clear that the external momenta $\vec{k_n}$ are referred to the time τ, while the internal one to τ_1 and τ_2, as highlighted by their subscripts.

Now, we split our computation in two parts. Firstly, we will evaluate the temporal contribution, i.e. the one arising from the time integration. Secondly, we will calculate the directional term, from which we wish to see some parity-violating signatures.

4.3.1 Temporal contribution

We start to study the temporal integration without any approximation. Hence, instead of following the example of [2, 57, 59] as in the previous chapter, we will consider the contributions coming from all wavelengths.
Since we are interested in the trispectrum when it is well outside the horizon, we can send $\tau \to 0$, getting the simplified expression. By using the usual dS mode-functions for both scalar and tensor fields, i.e. by considering $\frac{H}{M_{CS}}$ at zero-th order as done above, for the scalar-scalar and tensor-tensor contractions we have respectively

$$\begin{aligned}\overbrace{\zeta^{+}_{\vec{k_1}}(\tau_1)\zeta^{-}_{\vec{k_2}}}(\tau_2) =&(2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2})\frac{H^2}{4M_{Pl}^2\epsilon k_1^3}\left(1+ik_1(\tau_1-\tau_2)+k_1^2\tau_1\tau_2\right)e^{-ik_1(\tau_1-\tau_2)}\\ =&(2\pi)^3\delta^{(3)}(\vec{k_1},\vec{k_2})\frac{H^2}{4M_{Pl}^2\epsilon k_1^3}a(k_1,\tau_1,\tau_2),\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}\overbrace{\gamma'^{s}_{\vec{k_1}}(\tau_1)\gamma'^{r}_{\vec{k_2}}}(\tau_2) =&(2\pi)^3\delta^{sr}\delta^{(3)}(\vec{k_1}+\vec{k_2})\frac{H^2}{M_{Pl}^2}k_1\tau_1\tau_2 e^{-ik_1(\tau_1-\tau_2)}\\ =&(2\pi)^3\delta^{sr}\delta^{(3)}(\vec{k_1}+\vec{k_2})\frac{H^2}{M_{Pl}^2}a^{\gamma}(k_1,\tau_1,\tau_2),\end{aligned} \tag{4.56}$$

where we restored the M_{Pl} dependence. Now, we can substitute these expression into Eq. 4.54, obtaining

$$\begin{aligned}\left\langle\prod_n^4\zeta_{\vec{k}_n}(\tau)\right\rangle =&-(2\pi)^3\frac{H^6}{M_{CS}^2M_{Pl}^4 2^7\epsilon^2}\int^{\tau\to 0}\frac{d\tau_1}{\tau_1}\int^{\tau_1}\frac{d\tau_2}{\tau_2}\int d^3q_1d^3p_1d^3q_2d^3p_2\frac{\mathcal{F}(\vec{q_1},\vec{p_1},\vec{q_2},\vec{p_2})}{\prod_n k_n^3}\\ &\Big\{a(k_1,\tau_1)a(k_2,\tau_1)a(k_3,\tau_2)a(k_4,\tau_2)a^{\gamma}(|-\vec{q_1}-\vec{p_1}|,\tau_1,\tau_2)+\\ &-a^*(k_1,\tau_1)a^*(k_2,\tau_1)a(k_3,q_2)a(k_4,p_2)a^{\gamma}(|-\vec{q_1}-\vec{p_1}|,\tau_1,\tau_2)+\text{c.c.}\Big\}\delta^{(3)}(\vec{k_1}+\vec{q_1})\delta^{(3)}(\vec{k_2}+\vec{p_1})\\ &\delta^{(3)}(\vec{k_3}+\vec{q_2})\delta^{(3)}(\vec{k_4}+\vec{p_2})\delta^{(3)}(-\vec{q_1}-\vec{p_1}-\vec{q_2}-\vec{p_2})+23\text{ perms in }\vec{k_n}\\ =&-(2\pi)^3\delta^{(3)}(\vec{k_1}+\vec{k_2}+\vec{k_3}+\vec{k_4})\frac{H^6}{M_{CS}^2M_{Pl}^4 2^7\epsilon^2}\frac{\mathcal{F}(\vec{k_1},\vec{k_2},\vec{k_3},\vec{k_4})}{\prod_n k_n^3}\\ &\int^{\tau\to 0}\frac{d\tau_1}{\tau_1}\int^{\tau_1}\frac{d\tau_2}{\tau_2}\Big\{A(k_1,k_2,k_3,k_4,k_{12})-A(-k_1,-k_2,k_3,k_4,k_{12})\\ &-A(k_1,k_2,-k_3,-k_4,-k_{12})+A(-k_1,-k_2,-k_3,-k_4,-k_{12})\Big\}+23\text{ perms in }\vec{k_n},\end{aligned} \tag{4.57}$$

where now

$$A(k_1,k_2,k_3,k_4,k_{12})=(1-ik_1\tau_1)(1-ik_2\tau_1)(1-ik_3\tau_2)(1-ik_4\tau_2)k_{12}\tau_1\tau_2 e^{i\tau_1(k_1+k_2-k_{12})}e^{i\tau_2(k_3+k_4+k_{12})}, \tag{4.58}$$

as we sent τ to zero, and $|\vec{q_1}+\vec{p_1}| = |\vec{k_1}+\vec{k_2}| = k_{12}$. Also, we notice that the complex conjugate of a certain a function is given by exchanging the momentum with its opposite. By applying this line of reasoning, we just have to solve the first term of the integration in time, thereby extending the result by just flipping the sign of the correspondent momenta. Furthermore, we can constrain k_{12} to be strictly less than $\vec{k_1}+\vec{k_2}$ because of the Levi-Civita symbol contained in $\mathcal{F}$. We also notice that by flipping two external legs connected to the same vertex, the result does not change. This implies an overall factor of $\times 4$ and a decrease of the permutations from 24 to 6. We will be able to do that for the entire trispectrum because also the directional part exhibits the same symmetry, as we are going to see. By extending the time-integration domain to $\tau \to 0$ for each integral, hence dividing by $2! = 2$, the first integrated element reads

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\tau\to 0} d\tau_1 d\tau_2 A_1 = I_1 =& \frac{1}{2}\int_{-\infty}^{\tau\to 0} d\tau_1 d\tau_2 \big(1 - i\tau_1(k_1+k_2) - i\tau_2(k_3+k_4) - \tau_1\tau_2(k_1+k_2)(k_3+k_4) - k_1k_2\tau_1^2 - k_3k_4\tau_2^2 \\ &+ i\tau_1\tau_2^2 k_3k_4(k_1+k_2) + i\tau_1^2\tau_2 k_1k_2(k_3+k_4) + k_1k_2k_3k_4\tau_1^2\tau_2^2\big)e^{i\tau_1(k_1+k_2-k_{12})}e^{i\tau_2(k_3+k_4+k_{12})}.\end{aligned} \tag{4.59}$$

By Wick-rotating now each time variables (as done in Eq. 5.42 of [8]), we have

$$\begin{aligned}I_1 =& -\frac{1}{2}\Bigg[\frac{1}{(k_1+k_2-k_{12})(k_3+k_4+k_{12})}\bigg(1 + \frac{k_1+k_2}{k_1+k_2-k_{12}} + \frac{k_3+k_4}{k_3+k_4+k_{12}} + 2\frac{k_1k_2}{(k_1+k_2-k_{12})^2} + \\ &+ 2\frac{k_3k_4}{(k_3+k_4+k_{12})^2}\bigg) + \frac{2}{(k_1+k_2-k_{12})^2(k_3+k_4+k_{12})^2}\bigg(\frac{k_1k_2(k_3+k_4)}{k_1+k_2-k_{12}} + \frac{k_3k_4(k_1+k_2)}{k_3+k_4+k_{12}} + \\ &+ 2\frac{k_1k_2k_3k_4}{(k_1+k_2-k_{12})(k_3+k_4+k_{12})} + \frac{(k_1+k_2)(k_3+k_4)}{2}\bigg)\Bigg].\end{aligned} \tag{4.60}$$

We see that $I_1 = I(k_1,k_2,k_3,k_4,k_{12})$ is completely real, and for what we said above, we can infer that the entire nested time-integral will be so. Also, by remembering the complex conjugate contribution we got before, we see that even if there were a complex contribution, it would sum itself to zero. The final result will then be

$$\mathcal{G}^{All} = I(k_1,k_2,k_3,k_4,k_{12}) - I(-k_1,-k_2,k_3,k_4,k_{12}) - I(k_1,k_2,-k_3,-k;4,-k_{12}) + I(-k_1,-k_2,-k_3,-k_4,-k_{12}). \tag{4.61}$$

4.3.2 Directional contribution

No matter what approximation we follow for the different fields' mode functions, from the two tensors' contraction a Kronecker delta for the gravitons' polarizations will appear. That understood, we can set our summation over the s, r polarization to be restricted to the s one only. Then we are left with

$$\mathcal{F} = \sum_s \epsilon_{lj}^{(s)}(\hat{k_{12}})\epsilon_{mb}^{(-s)}(\hat{k_{12}})\epsilon^{ijk}k_1^l k_{2k}(k_{12})_i \epsilon^{abc}k_3^m k_{4c}(k_{34})_a, \tag{4.62}$$

where we applied the Dirac's deltas coming from Wick's contractions (i.e. $\vec{q_1},\vec{p_1},\vec{q_2},\vec{p_2} \to \vec{k_1},\vec{k_2},\vec{k_3},\vec{k_4}$). It is possible to observe now, that if we exchange the order of two external legs attached to the same vertex, the trispectrum remains the same. In fact, by using the relation

$$k_i\epsilon^{jik}\epsilon_{lj}^s(\hat{k}) = -i\lambda_s k\epsilon_l^{sk}(\hat{k}), \tag{4.63}$$

with $\lambda_s = \pm 1$, $\mathcal{F}$ reads

$$\epsilon^s_{lj}(\hat{k}_{12})\epsilon^r_{mb}(\hat{k}_{34})\epsilon k_1^l k_{2k}(k_{12})_i \epsilon^{abc} k_3^m k_{4c}(k_{34})_a = K^2 \epsilon^s_{lk}(\hat{K})\epsilon^{-s}_{mc}(\hat{K}) k_1^l k_2^k k_3^m k_4^c. \tag{4.64}$$

Therefore, the 23 permutations reduce to 6 "independent" diagrams, leaving us with a factor 4 multiplying the whole trispectrum. Now, in order to search for parity violation signatures, i.e. for complex contributions of the trispectrum [2], we can rewrite the polarization tensors as polarization vectors, thanks to [69]

$$\epsilon^{(\pm 2)}_{ij}(\hat{k}) = \sqrt{2}\epsilon^{(\pm 1)}_i(\hat{k})\epsilon^{(\pm 1)}_j(\hat{k}). \tag{4.65}$$

Here, $R = +2$ and $L = -2$. Thus, we have, by calling $k_{12} = K$ here and after,

$$\begin{aligned}
\mathcal{F} &= -\epsilon^{ijk} k_1^l k_{2k} K_i \epsilon^{abc} k_3^m k_{4c} K_a \left[\epsilon^{(+2)}_{lj}(\hat{K})\epsilon^{(-2)}_{mb}(\hat{K}) + \epsilon^{(-2)}_{lj}(\hat{K})\epsilon^{(+2)}_{mb}(\hat{K})\right] \\
&= -\epsilon^{ijk} k_1^l k_{2k} K_i \epsilon^{abc} k_3^m k_{4c} K_a 2\left[\epsilon^{(+1)}_l(\hat{K})\epsilon^{(-1)}_m(\hat{K})\epsilon^{(+1)}_j(\hat{K})\epsilon^{(-1)}_b(\hat{K}) + \epsilon^{(-1)}_l(\hat{K})\epsilon^{(+1)}_m(\hat{K})\epsilon^{(-1)}_j(\hat{K})\epsilon^{(+1)}_b(\hat{K})\right].
\end{aligned} \tag{4.66}$$

For simplicity, we concentrate now only on the term between square brackets. By using the equation

$$\epsilon^{(-1)}_i(\hat{K})\epsilon^{(+1)}_j(\hat{K}) = \frac{1}{2}\left[\delta_{ij} - \hat{K}_i\hat{K}_j + i\epsilon_{ijk}\hat{K}^k\right], \tag{4.67}$$

it reduces to

$$\begin{aligned}
\left[\epsilon^{(+2)}_{lj}(\hat{K})\epsilon^{(-2)}_{mb}(\hat{K}) + \epsilon^{(-2)}_{lj}(\hat{K})\epsilon^{(+2)}_{mb}(\hat{K})\right] =& \frac{1}{4}\left[\delta_{ml} - \hat{K}_m\hat{K}_l + i\epsilon_{mlp}\hat{K}^p\right]\left[\delta_{bj} - \hat{K}_b\hat{K}_j + i\epsilon_{bjq}\hat{K}^q\right] + \\
&+\frac{1}{4}\left[\delta_{lm} - \hat{K}_l\hat{K}_m + i\epsilon_{lmp}\hat{K}^p\right]\left[\delta_{jb} - \hat{K}_j\hat{K}_b + i\epsilon_{jbq}\hat{K}^q\right].
\end{aligned} \tag{4.68}$$

From this relation, we can immediately see that all the imaginary contributions cancel between each other, since a single permutation in the Levi-Civita symbol on the second line gives rise to the same complex term of the first line but with a minus sign. In the same way, all the real terms get multiplied by a factor 2. In the end, by doing all the multiplications thereby contracting with the different momenta in $\mathcal{F}$, we obtain

$$\begin{aligned}
\mathcal{F} = -\frac{1}{2}\Big[& K^2\left[(\vec{k}_2\cdot\vec{k}_4)(\vec{k}_1\cdot\vec{k}_3) - (\hat{K}\cdot\vec{k}_1)(\hat{K}\cdot\vec{k}_3)(\vec{k}_2\cdot\vec{k}_4)\right] - (\vec{k}_1\cdot\vec{k}_3)(\vec{K}\cdot\vec{k}_4)(\vec{K}\cdot\vec{k}_2) + \frac{1}{K^2}\prod_i(\vec{K}\cdot\vec{k}_i) + \\
&- \left[\vec{K}\cdot(\hat{K}\times\vec{k}_2)\right]\left[\vec{K}\cdot(\hat{K}\times\vec{k}_4)\right](\vec{k}_1\cdot\vec{k}_3) - \left[\vec{K}\cdot(\vec{K}\times\vec{k}_4)\right]\left[\vec{k}_1\cdot(\vec{k}_3\times\hat{K})\right]\hat{K}\cdot\vec{k}_2 + \\
&- \left[\vec{K}\cdot(\vec{k}_2\times\vec{k}_4)\right]\left[\hat{K}\cdot(\vec{k}_1\times\vec{k}_3)\right](\hat{K}\cdot\vec{K})\Big].
\end{aligned} \tag{4.69}$$

The contributions given by the second line are null, since we can perform a cyclic permutation of the momenta in each term getting a cross product between two parallel vectors. Hence, we simply get

$$\begin{aligned}\mathcal{F} = &-\frac{1}{2}\Big[K^2\left[(\vec{k}_1\cdot\vec{k}_3)(\vec{k}_2\cdot\vec{k}_4)-(\hat{K}\cdot\vec{k}_1)(\hat{K}\cdot\vec{k}_3)(\vec{k}_2\cdot\vec{k}_4)\right]+\frac{1}{K^2}\prod_i(\vec{K}\cdot\vec{k}_i)+\\ &-(\vec{k}_1\cdot\vec{k}_3)(\vec{K}\cdot\vec{k}_2)(\vec{K}\cdot\vec{k}_4)+\left[\vec{K}\cdot(\vec{k}_1\times\vec{k}_3)\right]\left[\vec{K}\cdot(\vec{k}_2\times\vec{k}_4)\right]\Big].\end{aligned} \tag{4.70}$$

In the end, by combining the temporal and the directional contributions, we find

$$\begin{aligned}\left\langle\prod_n^4\zeta_{\vec{k}_n}(\tau\to 0)\right\rangle = &-(2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\frac{H^6}{2M_{CS}^2M_{Pl}^4\epsilon^2}\frac{K}{\prod_i(2k_i^3)}\mathcal{F}(\vec{k_n})\mathcal{G}^{All}\\ &+5\text{ perms in }\vec{k_n}.\end{aligned} \tag{4.71}$$

We see that there are no traces of parity violation. This is because having two connected vertices induces a cancellation of the imaginary part that a single vertex can produce. In details, a single vertex can exhibit a imaginary contribution which is proportional to the graviton's momentum. By connecting two indices, the momentum of the second graviton is forced to be equal and opposite to the first one's, hence giving us a null complex contribution.

4.3.3 Parity-violating mode-function: constant M_{CS}

A way to get a parity violation signature from our trispectrum would be considering the entire mode-function for Chern-Simons gravitons. This shows in fact a phase that is explicitly parity-violating. To study this case, we can proceed by considering two different scenarios: namely a constant Chern-Simons mass M_{CS} in time, and an evolving one. For the directional dependence we expect an important difference with respect to the previous case. The new parity-violating wave-function contains indeed a term which is explicitly fixed by the graviton's polarization. To proceed in the computation for a constant Chern-Simons mass, we need to make a few steps back, and remember that at the beginning we found the graviton's mode-function by shifting the field as $\gamma^s = \mu^s/A_{T,s}$. While in the previous cases we put $H/M_{CS}\sim 0$, now we relax this condition, obtaining

$$\frac{1}{A_{T,s}} = \frac{1}{\sqrt{\frac{M_{Pl}^2a^2}{2}\left(1-\lambda_s\frac{k}{aM_{CS}}\right)}} \approx \frac{\sqrt{2}}{M_{Pl}a}\left(1+\lambda_s\frac{k}{2aM_{CS}}\right), \tag{4.72}$$

where we expanded the expression of $A_{T,s}$ up to first order in $\frac{k_{phys}}{M_{CS}}$. This is justified by the fact that this parameter is very small since the beginning of Inflation, but also because the second order term would not carry any parity violation. Also, not considering $\frac{H}{M_{CS}}$ as vanishing implies a modification of the mode-function expressed in Eq.4.27. Specifically, we do not have $U(2,4,2ik\tau)$ anymore, but $U\left(2-\lambda_s\frac{H}{M_{CS}},4,2ik\tau\right)$, where we chose a perfectly dS Universe (namely $\epsilon\sim 0$ as done before). For a very small parameter $r_s=\lambda_s\frac{H}{M_{CS}}$, the confluent hypergeometric function we have to solve can be Taylor expanded through Mathematica, reading

$$U(2-r_s,4,2ik\tau) = U(2,4,2ik\tau) - r_sU^{(1,0,0)}(2,4,2ik\tau)+\mathcal{O}(r_s^2), \tag{4.73}$$

where $U^{(1,0,0)}$ is the first-derivative of U with respect to r_s. However, the computation of this function is very long and complicated, therefore, since r_s is still considered very small, we restrict ourselves to $U(2,4,2ik\tau)$ for simplicity. Thus, following the same procedure done in [7, 8], our new graviton's mode function reads [5]

[5] In [7, 8], the authors used $A_{T,s}$ expanded up to zeroth order in slow-roll and H/M_{CS}. Also, they assumed these factors to be vanishing in the confluent hypergeometric function. By doing the same while considering H/M_{CS} up to first order, we have equal contributions simply multiplied by a factor $[1+\lambda_s(k/2aM_{CS})]$. Thus, since the mode-function $u^{\gamma\ s} = u^s/A_{T,s}$ here differs only by this factor, we have the result expressed above

$$
\begin{aligned}
u^{\gamma s}(k,\tau) =& \frac{iH}{M_{Pl}\sqrt{k^3}}(1+ik\tau)e^{-ik\tau}\left(1+\lambda_s\frac{k_{phys}}{2M_{CS}}\right)e^{\lambda_s\frac{\pi H}{4M_{CS}}} \\
\approx& \frac{iH}{M_{Pl}\sqrt{k^3}}(1+ik\tau)e^{-ik\tau}\left[1+\lambda_s\left(\frac{\pi}{4}\frac{H}{M_{CS}}+\frac{k_{phys}}{2M_{CS}}\right)\right],
\end{aligned}
\tag{4.74}
$$

with $\frac{H}{M_{CS}} \ll 1$, $k_{phys}/M_{CS} \ll 1$, and hence

$$
u'^{\gamma\, s}(k\tau) = \frac{iH}{M_{Pl}\sqrt{k^3}}k^2\tau e^{-ik\tau}\left[1+\lambda_s\left(\frac{\pi H}{4M_{CS}}+\frac{k_{phys}}{2M_{CS}}\right)\right] - \frac{iH}{M_{Pl}\sqrt{k^3}}(1+ik\tau)e^{-ik\tau}\lambda_s k\frac{H}{2M_{CS}}. \tag{4.75}
$$

Our tensor-tensor contraction then reads, up to first order in $\frac{H}{M_{CS}}$

$$
\begin{aligned}
\left\langle \gamma'^s(\vec{k_1},\tau_1)\gamma'^r(\vec{k_2},\tau_2)\right\rangle \approx& (2\pi)^3\delta^{sr}\delta^{(3)}\left(\vec{k_1}+\vec{k_2}\right)\frac{H^2}{M_{Pl}^2k_1^3}\left[k_1^4\tau_1\tau_2 + k^4\tau_1\tau_2\lambda_s\frac{\pi H}{2M_{CS}}+\right.\\
&\left. + k_1^5\tau_1\tau_2\lambda_s\frac{H}{2M_{CS}}(\tau_1+\tau_2) - k_1^3\lambda_s\frac{H}{2M_{CS}}(\tau_1+\tau_2)\right]\\
=& \langle\gamma'^s\gamma'^r\rangle_1 + \langle\gamma'^s\gamma'^r\rangle_2 + \langle\gamma'^s\gamma'^r\rangle_3 + \langle\gamma'^s\gamma'^r\rangle_4\,,
\end{aligned}
\tag{4.76}
$$

where we used $a(\tau_i) \approx -(\tau_i H)^{-1}$. As shown above, once inserted the new mode-function into the in-in formula, we end up with four different terms, each of which can be written as a constant factor in front, multiplied by a directional and a temporal contribution. Here, the first term contributes with the usual parity-even trispectrum we found earlier, so we can neglect it in the following computation. Before solving the several integrals in time, we take a look at the directional contributions that arise from other three terms in the new power spectrum.

4.3.4 Directional contribution

By considering the K's dependence as part of the temporal contribution, we have

$$
\mathcal{F}_{odd} = -\sum_s \epsilon_{lj}^{(s)}(\hat{K})\epsilon_{mb}^{(-s)}(\hat{K})\epsilon^{ijk}k_1^l k_{2k}K_i\epsilon^{abc}k_3^m k_{4c}K_a\lambda_s\frac{\pi H}{2M_{CS}}, \tag{4.77}
$$

for the second term of Eq.4.76, and

$$
\mathcal{F}_{odd} = -\sum_s \epsilon_{lj}^{(s)}(\hat{K})\epsilon_{mb}^{(-s)}(\hat{K})\epsilon^{ijk}k_1^l k_{2k}K_i\epsilon^{abc}k_3^m k_{4c}K_a\lambda_s\frac{H}{2M_{CS}}, \tag{4.78}
$$

for the other two. We see that the only difference is given by a factor π. So, from now on we refer to $\mathcal{F}_{odd}$ as the parity-violating directional contribution in general, multiplying it by π when it will be necessary.
As before, we expand the sum over the polarization tensors, but this time we get a minus sign between them carried by $\lambda_s = \lambda_{L/R} = \mp 1$. In details, we find

$$
\mathcal{F}_{odd} = -\frac{H}{M_{CS}}\epsilon^{ijk}k_1^l k_{2k}K_i\epsilon^{abc}k_3^m k_{4c}K_a\left[\epsilon_{lj}^{(+2)}(\hat{K})\epsilon_{mb}^{(-2)}(\hat{K}) - \epsilon_{lj}^{(-2)}(\hat{K})\epsilon_{mb}^{(+2)}(\hat{K})\right]. \tag{4.79}
$$

Thus, we apply again Eq.4.67, obtaining the same result above, but with a minus sign between the two terms

$$\mathcal{F}_{odd} = -\frac{H}{2M_{CS}}\epsilon^{ijk}k_1^l k_{2k}K_i\epsilon^{abc}k_3^m k_{4c}K_a\frac{1}{4}\Big\{\left[\delta_{ml} - \hat{K}_m\hat{K}_l + i\epsilon_{mlp}\hat{K}^p\right]\left[\delta_{bj} - \hat{K}_b\hat{K}_j + i\epsilon_{bjq}\hat{K}^q\right] + \\ - \left[\delta_{lm} - \hat{K}_l\hat{K}_m + i\epsilon_{lmp}\hat{K}^p\right]\left[\delta_{jb} - \hat{K}_j\hat{K}_b + i\epsilon_{jbq}\hat{K}^q\right]\Big\}. \tag{4.80}$$

This flip in sign allows all the imaginary contributions to get a factor $\times 2$ while simplifying away the real ones. We have

$$\mathcal{F}_{odd} = -i\frac{H}{2M_{CS}}\epsilon^{ijk}k_1^l k_{2k}K_i\epsilon^{abc}k_3^m k_{4c}K_a\left[\delta_{ml}\epsilon_{bjq}\hat{K}^q + \delta_{bj}\epsilon_{mlp}\hat{K}^p - \hat{K}_m\hat{K}_l\epsilon_{bjq}\hat{K}^q - \hat{K}_b\hat{K}_j\epsilon_{mlp}\hat{K}^p\right], \tag{4.81}$$

which reads, once made the proper contractions

$$\mathcal{F}_{odd} = i\frac{H}{2M_{CS}}\Big\{K\left(\vec{k_1}\cdot\vec{k_3}\right)\left[\vec{K}\cdot\left(\vec{k_2}\times\vec{k_4}\right)\right] + K^2\left(\vec{k_2}\cdot\vec{k_4}\right)\left[\vec{K}\cdot\left(\vec{k_1}\times\vec{k_3}\right)\right] + \\ + \left(\vec{k_1}\cdot\vec{K}\right)\left(\vec{k_3}\cdot\vec{K}\right)\left[\hat{K}\cdot\left(\vec{k_2}\times\vec{k_4}\right)\right]\Big\}. \tag{4.82}$$

4.3.5 Temporal contribution

As far as the temporal contribution is concerned, by remembering that we are interested in trispectrum's complex terms only, we need to have real functions that multiply $\mathcal{F}_{odd}$. This allows us to immediately get rid of the third contribution in Eq.4.76. In fact, this reads

$$\bar{\mathcal{G}_3} = \frac{1}{2}K^5\int_{-\infty}^{\tau\to 0}\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}\left(\bar{A}_3(k_1,k_2,k_3,k_4,K) - \bar{A}_3(-k_1,-k_2,k_3,k_4,K) + \text{c.c.}\right), \tag{4.83}$$

with

$$\bar{A}_3(k_1,k_2,k_3,k_4,K) = \tau_1\tau_2(\tau_1+\tau_2)(1-ik_1\tau_1)(1-ik_2\tau_1)(1-ik_3\tau_2)(1-ik_4\tau_2)e^{i\tau_1(k_1+k_2-K)}e^{i\tau_2(k_3+k_4+K)}, \tag{4.84}$$

and after a Wick's rotation it gives us a complex function of the momenta. Moreover, because of the complex conjugate, the whole contribution simplifies to zero.
For the second term, we have the same temporal contribution found in the $\frac{H}{M_{CS}}\sim 0$ case ($\bar{\mathcal{G}_2} = \bar{\mathcal{G}_1} = \mathcal{G}^{All}$).
Lastly, for the last term we have a slightly more complicated integral to solve. In fact, if for the other three terms a factor $\tau_1\tau_2$ simplified the $(\tau_1\tau_2)^{-1}$ dependence carried by the in-in formula in conformal time, here it does not happen. We have

$$\bar{\mathcal{G}_4} = -\frac{1}{2}K^3\int_{-\infty}^{\tau\to 0}\frac{d\tau_1}{\tau_1}\frac{d\tau_2}{\tau_2}\left(\bar{A}_4(k_1,k_2,k_3,k_4,K) - \bar{A}_4(-k_1,-k_2,k_3,k_4,K) + \text{c.c.}\right), \tag{4.85}$$

with

$$\bar{A}_4(k_1,k_2,k_3,k_4,K) = (\tau_1+\tau_2)(1-ik_1\tau_1)(1-ik_2\tau_1)(1-ik_3\tau_2)(1-ik_4\tau_2)e^{i\tau_1(k_1+k_2-K)}e^{i\tau_2(k_3+k_4+K)}. \tag{4.86}$$

Thanks to the presence of the complex conjugate, we can write $\bar{\mathcal{G}}_4 = 2\text{Re}\left[\bar{I}_4(k_1,k_2,k_3,k_4,K) - \bar{I}_4(-k_1,-k_2,k_3,k_4,K)\right]$ with

$$\begin{aligned}\bar{I}_4(k_1,k_2,k_3,k_4,K) = -\frac{K^3}{2}\Bigg\{ \int_{-\infty}^{\tau\to 0} d\tau_1 \int_{-\infty}^{\tau\to 0} d\tau_2 e^{i\tau_1(k_1+k_2-K)} e^{i\tau_2(k_3+k_4+K)} \left(\frac{1}{\tau_2}+\frac{1}{\tau_1}\right) \\ \Big[1 - i\tau_1(k_1+k_2) - i\tau_2(k_3+k_4) - \tau_1\tau_2(k_1+k_2)(k_3+k_4) - k_1k_2\tau_1^2 - k_3k_4\tau_2^2 + \\ + i\tau_1\tau_2^2 k_3k_4(k_1+k_2) + i\tau_1^2\tau_2 k_1k_2(k_3+k_4) + k_1k_2k_3k_4\tau_1^2\tau_2^2\Big]\Bigg\}.\end{aligned} \tag{4.87}$$

Then, we rewrite the integral as follows

$$\begin{aligned}\bar{I}_4 = -\frac{K^3}{2}\Bigg[\left(\int_{-\infty}^{\tau\to 0} d\tau_1 \frac{e^{i\tau_1 K_T}}{\tau_1}\right)\left(\int_{-\infty}^{\tau\to 0} d\tau_2 e^{i\tau_2 K_T'}\left(1 - i\tau_2(k_3+k_4) - k_3k_4\tau_2^2\right)\right) + \\ + \left(\int_{-\infty}^{\tau\to 0} d\tau_2 \frac{e^{i\tau_2 K_T'}}{\tau_2}\right)\left(\int_{-\infty}^{\tau\to 0} d\tau_1 e^{i\tau_1 K_T}\left(1 - i\tau_1(k_3+k_4) - k_3k_4\tau_1^2\right)\right) + \\ + \int_{-\infty}^{\tau\to 0} d\tau_1 d\tau_2 e^{i\tau_1 K_T} e^{i\tau_2 K_T'}\Bigg(- i\sum_i k_i - (\tau_1+\tau_2)(k_1+k_2)(k_3+k_4) - k_1k_2\tau_1 - k_3k_4\tau_2 + \\ + i\tau_1\tau_2\left(k_1k_2(k_3+k_4) + k_3k_4(k_1+k_2)\right) + i\tau_1^2 k_1k_2(k_3+k_4) + i\tau_2^2 k_3k_4(k_1+k_2) + \prod_i k_i\tau_1\tau_2(\tau_1+\tau_2)\Bigg)\Bigg],\end{aligned} \tag{4.88}$$

where $K_T = k_1+k_2-K$, $K_T' = k_3+k_4+K$. We focus now only on the last two lines. We apply as usual a Wick's rotation, and send $\tau \to 0$. In this way, we get a purely imaginary contribution, which can be neglected thanks to the complex conjugate in Eq.4.85. In details, we would get a factor i overall as each term contains an odd number of time variables. We are thus left with the first two lines, which have the same form. Thereby, we can solve the first one and generalize the result to the second. The second parenthesis gives us a purely complex result. One can indeed Wick rotate the time variable getting, as before, an overall i factor. Hence, we have to multiply it by a complex term, in order to end up with a real contribution. Specifically, we have

$$\int_{-\infty}^{\tau\to 0} d\tau_2 e^{i\tau_2 K_T'}\left(1 - i\tau_2(k_3+k_4) - k_3k_4\tau_2^2\right) = -i\left(\frac{1}{K_T'} + \frac{k_3+k_4}{K_T'^2} + 2\frac{k_3k_4}{K_T'^3}\right). \tag{4.89}$$

The first parenthesis can be rewritten by promoting the τ_1 variable to a complex one $\tau_1 = i\tau_1'$, and Wick rotating the integration contour [8]. Thus, we have

$$\lim_{\tau\to 0^-}\int_{-i\infty}^{i\tau} d\tau_1' \frac{e^{-K_T\tau_1'}}{\tau_1'} = \lim_{K_T\tau\to 0}\int_{-i\infty}^{iK_T\tau} dx \frac{e^{-x}}{x}, \tag{4.90}$$

where we also changed the variable $x = K_T\tau_1'$, $dx = K_T d\tau_1'$. This is the so called *exponential integral in the complex plane $Ei(K_T\tau)$* [70], which is solved by

$$Ei(K_T\tau) = \lim_{K_T\tau\to 0}\left(-\gamma - \ln|iK_T\tau| - \sum_{n=1}^{+\infty}\frac{(-iK_T\tau)^n}{n!n}\right), \tag{4.91}$$

with $\gamma \approx 0.5772156649$ being the Euler-Mascheroni constant. Now, since we are interested in the real part of the integral, we can restrict ourselves to the imaginary contribution of $Ei(K_T\tau) = -\gamma - \lim_{K_T\tau\to 0}\ln|K_T\tau| + i\frac{\pi}{2}$[6], namely $\mathrm{Im}Ei(K_T\tau) = +i\frac{\pi}{2}$. Thus, in the end we have

$$\begin{aligned}\bar{\mathcal{G}} =& 2\mathrm{Re}\left[\bar{I}_1 - \bar{I}_2\right] \\ =& -\frac{\pi K^3}{2}\left[\frac{k_1+k_2}{(k_1+k_2)^2-K^2} + (k_1+k_2)\frac{(k_1+k_2)^2+K^2}{(k_1+k_2+K)^2(k_1+k_2-K)^2}\left(1+2\frac{k_1+k_2}{(k_1+k_2)^2-K^2}\right)\right].\end{aligned} \tag{4.92}$$

By collecting all the pieces, we find

$$\left\langle \prod_n^4 \zeta_{\vec{k}_n}(\tau\to 0)\right\rangle = \left\langle \prod_n^4 \zeta_{\vec{k}_n}(\tau\to 0)\right\rangle_{even} + i\left\langle \prod_n^4 \zeta_{\vec{k}_n}(\tau\to 0)\right\rangle_{odd}, \tag{4.93}$$

with

$$\left\langle \prod_n^4 \zeta_{\vec{k}_n}(\tau\to 0)\right\rangle_{odd} = (2\pi)^3\delta^{(3)}\left(\sum_n \vec{k_n}\right)\frac{\pi H^7}{2M_{Pl}^4 M_{CS}^3\prod_n(2k_n^3)}\frac{\mathrm{Im}\left(\mathcal{F}_{odd}\right)}{K^3}\left[\bar{\mathcal{G}_2}(k_n) + \bar{\mathcal{G}_4}(k_n)\right] + 5 \text{ perms in } \vec{k_n}, \tag{4.94}$$

where we also took into account the $\times 4$ factor coming from the symmetry of the problem[7], and we rescaled the $\bar{\mathcal{G}_4}$ function in order to have an overall π factor. We note that now $\left\langle \prod_n \zeta_{\vec{k_n}}\right\rangle_{even}$ contains another term given by the newly defined power-spectrum. We see that by using a modified graviton's mode-function, we can obtain a non vanishing parity-violating contribution. However, the price we pay for finding that, is a higher suppression of our parity-odd trispectrum. The Chern-Simons factor encoded in the so defined gravitons results indeed in an overall factor $\frac{H^3}{M_{CS}}^3$, which strongly suppresses any imaginary contribution, making the parity-odd signature completely negligible.

4.3.6 Parity-violating mode-function: increasing M_{CS}

Let us now consider a time dependent Chern-Simons term. As reported in [10], a decreasing M_{CS} in time could yield to high values of H/M_{CS}, that eventually would cause the appearance of ghosts during Inflation. Therefore, we must consider only an increasing mass in time. In full generality, also the parameter $\xi = \frac{\dot{M}_{CS}}{M_{CS}H}$ and $\omega = \frac{\ddot{M}_{CS}}{M_{CS}H^2}$ could be much greater than one and time dependent. In fact, it was shown that large values of ξ and ω, would cause an enhancement of the parity odd contribution given by the $\langle\gamma\gamma\zeta\rangle$ bispectrum, that can therefore be seen as a useful tool to test Chern-Simons gravity [10]. However, even if we do not consider too large derivative terms for our Chern-Simons mass, namely $\xi \sim 1$, we end up with a parameter H/M_{CS} that becomes smaller and smaller during inflation. Through the definitions made just above we find [10]

$$\frac{A''_{T,s}}{A_{T,s}} = \frac{2}{\tau^2}\left(1 - \frac{\lambda_s}{2}k\tau\frac{H}{M_{CS}}\mathcal{A}\right), \tag{4.95}$$

[6]The complex logarithm can be written as $\ln z = \ln|z| + i\mathrm{Arg}(z)$, with $\mathrm{Arg}(z)$ is the argument of the complex number z, i.e. the angle between the positive real axis and the line that identifies z in the complex plane. In this case, being $z = iK_T\tau$, it is a purely imaginary number, which therefore lies in the negative ($\tau < 0$, $K_T = k_1 + k_2 - K > 0$ for what said above) complex axis ($\mathrm{Arg}(z) = -\pi/2$)

[7]The directional contribution exhibits the same symmetry of the one computed for a dS tensor correlator. It is easy to see that also all the new $\bar{\mathcal{G}}$ functions are the same if we flip two left (right) momenta between themselves. Then, as before, this reduces the number of independent diagram to 6.

with

$$\mathcal{A} = \frac{1}{(1-\lambda_s k_{phys}/M_{CS})^2}\left\{\left[1-\xi+\frac{\omega}{2}-\frac{\xi}{2H\tau}\right]\left(1-\lambda_s\frac{k_{phys}}{M_{CS}}\right)-\lambda_s\frac{k_{phys}}{2M_{CS}}\left[\frac{1}{2}+\xi+\frac{\xi^2}{2}\right]\right\}. \tag{4.96}$$

In order to encode an evolution for M_{CS} in our trispectrum, we should solve the (shifted) graviton's equation of motion with the effective potential given by this new $A''_{T,s}/A_{T,s}$. However, if initially we require k_{phys}/M_{CS},$H/M_{CS} \ll 1$ to avoid ghosts, then, regardless how much ξ and ω are large, H/M_{CS} washes away any parity-violating contribution in the effective potential. For that reason, one should also expect that very soon the equation of motion for gravitons resemble the usual ones in standard gravity. For this reason, we cannot consider the gravitons we used above to find a non-vanishing parity breaking. The only contributions that would be able to carry a non-vanishing, yet negligible, amount of parity violation outside the horizon are the ones that exit it at τ close enough to the beginning of Inflation. These would have a momentum given by $k=-H/\tau$, that tends to zero if the conformal time is chosen to be very large in modulus (as it should to guarantee some parity-breaking signatures). For all the other modes with higher momenta, they do not exit the horizon immediately after Inflation starts. Thus, we would have an amount of time $\tau \in (-\infty, -1/k]$ during which they can be neglected in the nested time-integral due to their highly oscillatory behaviour (as reported in [2, 57, 59]), and a period $\tau \in [-1/k, 0]$ during which they are out of the horizon. But with a large evolution of M_{CS}, the so-carried parity-breaking would be negligible.
In [10], it was possible to encode an evolution of the Chern-Simons mass since from the $\langle\gamma\gamma\zeta\rangle$ bispectrum one could define a parameter $\Pi = \frac{96}{25}H^2\frac{\partial^2 f(\phi)}{\partial\phi^2}$. Hence, by using the definition of ξ, and M_{CS}, we have

$$\xi = -\frac{1}{HM_{CS}}\left(\frac{1}{\dot{f}^2}\right)\sqrt{2\epsilon}\frac{HM_{Pl}^3}{8}\dot{\phi}\frac{\partial^2 f(\phi)}{\partial\phi^2} = -\sqrt{2\epsilon}M_{Pl}\frac{\partial^2 f(\phi)/\partial\phi^2}{\partial f(\phi)/\partial\phi}. \tag{4.97}$$

Therefore, it is possible to directly connect Π to ξ, through [10]

$$\Pi = -\frac{12}{25}\frac{\Theta}{\epsilon}\xi, \tag{4.98}$$

with Θ being the relative power spectrum's parity violation. From that important constraints on M_{CS} follow. In fact, by leaving it to be constant, one is forced to have $|\Pi| < \frac{12}{25}\frac{\Theta}{\epsilon}$, but if on the other hand ξ is treated as a free parameter, also the relative parity-violation carried by $\langle\gamma\gamma\zeta\rangle$ becomes free, and through the observation of Π one could constrain and test Chern-Simons modifications to General Relativity.
In our vertex, however, a similar term does not appear, and hence we are left with a more and more suppressed trispectrum.

Chapter 5

Observational constraints

We introduced non-Gaussianities (NG) and their importance in the first chapter of this thesis. In particular, we stressed that while a power spectrum statistic, which is the higher correlation function we can hope for a perfectly Gaussian field, is nearly degenerate with respect to different models of Inflation that can be proposed, NG can instead underline differences between them, thereby helping us to discrimate the favourable ones. As we already said, the three-points function (or its Fourier transform, the bispectrum) is the lowest-order statistics we can use to search for NG, as well as the four-points function (or the trispectrum) is the lowest correlator that allows us to search for a certain amount of parity-violation. However, up to now we considered only primordial NG, i.e. the ones that emerge at the end of Inflation ($\tau \to 0$). Of course, we are not able to observe at such distances, and therefore we need a way to connect these observables to what we can measure. In this last chapter, we briefly present the different techniques that are used to constrain primordial NG and what are the actual limits set by the observations. Also, although we have already introduces the concept of shapes, here we generalize a little bit this topic, including also the shapes given by the trispectrum. In particular, this will allows us to see the importance of writing correlation functions using shapes. Thus, we will briefly discuss the shapes of the bispectrum, while expose those of the trispectrum in more details.

5.1 Shapes

We saw that the CMB temperature fluctuations can be seen as seeded by the quantum fluctuations of the inflaton's field, which therefore identify the CMB as a random field. Also, in the section dedicated to cosmological perturbations, we understood how to link these inflaton's fluctuations to a gauge invariant quantity we adopted for all the subsequent discussion: the scalar curvature perturbations on uniform energy density gauge ζ. Therefore, by going to the $\delta\rho = 0$ slicing, we can correlate the Bardeen's gravitational potential Φ_H to the CMB anisotropies through ζ. Of course, the whole reasoning can be applied to CMB polarization perturbations as well. The primordial power spectrum and bispectrum of the primordial gravitational potential are given respectively by

$$\left\langle \Phi_H(\vec{k_1})\Phi_H(\vec{k_2}) \right\rangle = (2\pi)^3\delta^{(3)}\left(\vec{k_1} + \vec{k_2}\right) P_\Phi(k_1), \tag{5.1}$$

and

$$\left\langle \Phi_H(\vec{k_1})\Phi_H(\vec{k_2})\Phi_H(\vec{k_3}) \right\rangle = (2\pi)^3\delta^{(3)}\left(\vec{k_1} + \vec{k_2} + \vec{k_3}\right) B_\Phi(k_1, k_2, k_3). \tag{5.2}$$

We see that while in the former the Dirac's delta enforces a momentum conservation, in the latter equation we have also the triangular condition $\vec{k_1}+\vec{k_2}+\vec{k_3} = 0$. We already pointed out that each inflationary model predicts a different shape of this triangle in Fourier space, and that all the possible triangles can be depicted as a linear combination of three most important shapes: equilateral, squeezed, and folded. The mpst studied of these configurations is the second one, in which one could express the gravitational potential as splitted in a linear and a non-linear (NL) contribution, as done in Eq.1.105. By expressing the non-linear term $f_{NL}\left[\Phi_L^2(\vec{x})-\left\langle\Phi_L^2(\vec{x})\right\rangle\right]$ in Fourier space, it is possible to write $\left\langle\Phi_{H\vec{k_1}}\Phi_{H\vec{k_2}}\Phi_{\vec{k_3}}\right\rangle$ as a function of Φ_H power spectra [71]

$$\left\langle\Phi_{H\vec{k_1}}\Phi_{H\vec{k_2}}\Phi_{H\vec{k_3}}\right\rangle=(2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}+\vec{k_3}\right)\left[P_{\Phi_H}(k_1)P_{\Phi_H}(k_2)+P_{\Phi_H}(k_2)P_{\Phi_H}(k_3)+P_{\Phi_H}(k_1)P_{\Phi_H}(k_3)\right]. \quad (5.3)$$

Then, by considering the different momenta's permutations, one can define the bispectrum as

$$B_{\Phi_H}(k_1,k_2,k_3)=2f_{NL}\frac{\Delta_\Phi^2}{\prod_i k_i^2}\left[\sum_i\frac{k_1^2}{k_2k_3}+\frac{k_2^2}{k_1k_3}+\frac{k_3^2}{k_1k_2}\right]. \quad (5.4)$$

We define the so called shape, the function $S^{squeezed}$ given by

$$S^{squeezed}(k_1,k_2,k_3)=\frac{1}{N}\left(k_1k_2k_3\right)^2 B_{\Phi_H}, \quad (5.5)$$

with N being a normalization factor. It is useful to write the bispectrum as a function of the shape, because it is adimensional and invariant under a certain rescaling of all the momenta, in the case of scale-invariant bispectra. As anticipated, this squeezed shape forms a basis for all the other, together with other two important ones. We have the equilateral shape [39]

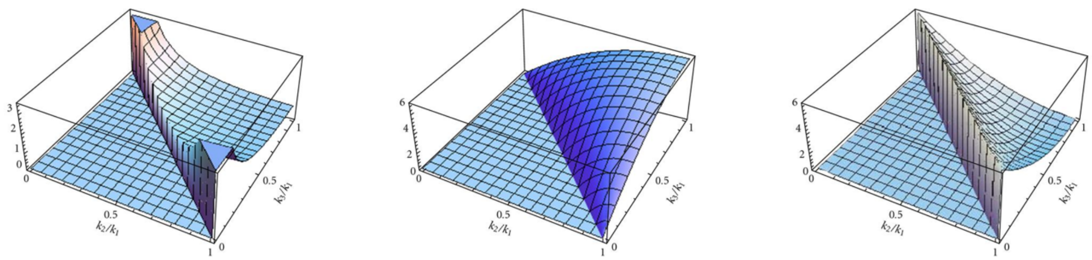
$$S^{equilateral}(k_1,k_2,k_3)=-6\left(\frac{k_1^2}{k_2k_3}+\frac{k_2^2}{k_1k_3}+\frac{k_3^2}{k_1k_2}\right)+6\left(\frac{k_1}{k_2}+\frac{k_1}{k_3}+\frac{k_2}{k_1}+\frac{k_2}{k_3}+\frac{k_3}{k_1}+\frac{k_3}{k_2}\right)-12, \quad (5.6)$$

and the folded one

$$S^{folded}=6\left(\frac{k_1^2}{k_2k_3}+\frac{k_2^2}{k_1k_3}+\frac{k_3}{k_1k_2}\right)-6\left(\frac{k_1}{k_2}+\frac{k_1}{k_3}+\frac{k_2}{k_1}+\frac{k_2}{k_3}+\frac{k_3}{k_1}+\frac{k_3}{k_2}\right)+18. \quad (5.7)$$

Even though the equilateral one is not directly derived by a particular model, it represents a phenomenologically motivated ansatz for data analysis in higher derivatives models or Dirac-Born-Infeld Inflation. The folded one, as already specified, is typical for inflationary scenarios with a non-Bunch-Davies vacuum choice [39,71]. Fig.5.1 shows the shapes just introduced. It is then possible to identify several other shapes that lie between the three exposed above. These, are called *intermediate shapes*. Also, it is possible to have different non-Gaussianities coming from different mechanism in a single model. In such a case, one could expect a certain level of mixing between two or more shapes.

As we focused mostly on the trispectrum, it is now useful to introduce the concept of shapes in the context of four-points correlation functions. First of all, we expect the shapes to be far more complicated than the ones for the bispectrum. This is because, as done by the three-points functions, the four-points function constrains together four momenta, which can therefore escape from the planarity condition forced by the Dirac's delta in the bispectrum. Thus, we expect that in principle the four momenta might be also non-planar. However, in what follows we restrict ourselves to the planar scenario, since the CMB experiments performed to measure trispectra's contribution contribute mainly from planar quadrangles [72].
We start by an inflationary model given by $S=\frac{1}{2}\int d^4x\sqrt{-g}\left[M_{Pl}^2R+2P(X,\phi)\right]$ where $X\equiv-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the kinetic term of the inflaton's Lagrangian $P(X,\phi)$. In this way, we are not fixing the kinetic term, thereby leaving it to be non-canonical. In fact, as we saw, a non-canonical kinetic term can produce higher NG contributions. The scalar curvature trispectrum is given, at leading order, by two contributions: the first one arises from a fourth-order



(a) (b) (c)

Figure 5.1: Figure 1: the figure shows an example of a plot for the (a) local, (b) equilateral, and (c) folded shape [39]. The first shapes was obtained by computing the scalar curvature perturbation bispectrum of the curvaton model, while the other two are respectively given by Eqs. 5.6, and 5.7.

scalar vertex, while the other one comes from two connected third-order scalar vertices, as the one we encountered studying the $f(\phi)\tilde{W}W$ model. In this case, however, the field that connects the two vertices is ζ. We start from the second contribution. The cubic Lagrangian is given by [72]

$$\begin{aligned}\mathcal{L}^{(3)} &= \left(\frac{1}{2}P_{,XX}\dot{\phi} + \frac{1}{6}P_{,XXX}\dot{\phi}^3\right) a^3 \dot{\delta\phi}^3 - \frac{1}{2}P_{,XX}\dot{\phi}a\dot{\delta\phi}\left(\nabla\delta\phi\right)^2 \\ &= -2a^3\frac{\lambda}{H^3}\dot{\zeta}^3 + a\frac{\sigma}{H^3}\left(1-c_s^2\right)\dot{\zeta}\partial_i\zeta\partial^i\zeta,\end{aligned} \tag{5.8}$$

with $c_s^2 = \frac{P_{,X}}{P_{,X}+2XP_{,xx}}$ being the sound speed, $\lambda = X^2P_{,XX} + \frac{2}{3}X^3P_{,XXX}$, and $\sigma = XP_{,X} + 2X^2P_{,XX} = \frac{H^2\epsilon}{c_s^2}$. We thus have two different interaction Hamiltonians we need to use in the in-in formalism. Before proceeding in doing that, however, we have to compute the mode-function for our model. This turns out to be [72]

$$u^\zeta = \frac{iH}{4\epsilon c_s k^3}\left(1 + ikc_s\tau\right)e^{ikc_s\tau}. \tag{5.9}$$

We see that the fact that the kinetic term is not canonical introduces a speed of propagation $c_s < 1$ in the field's mode-function, as anticipated earlier. The commutation relations do not change, so that we can write $\left\langle \zeta_{\vec{k_1}}\zeta_{\vec{k_2}}\right\rangle = (2\pi)^3\delta^{(3)}\left(\vec{k_1}+\vec{k_2}\right)u(\vec{k_1})u^*(\vec{k_1})$. After having applied the in-in formula, thereby contracting all the free-fields, for the first interaction Hamiltonian

$$H_1(\tau_1) = 2a^3\frac{\lambda}{H}\int\prod_{n=1}^{3}\frac{d^3p_n}{(2\pi)^3}\dot{\zeta}(\vec{p_1},\tau_1)\dot{\zeta}(\vec{p_2},\tau_1)\dot{\zeta}(\vec{p_3},\tau_1)(2\pi)^3\delta^{(3)}\left(\sum_{n=1}^{3}\vec{p_n}\right), \tag{5.10}$$

we have

$$\begin{aligned}\langle\zeta\zeta\zeta\zeta\rangle^{H_1} =& (2\pi)^9\delta^{(3)}\left(\sum_n \vec{k_n}\right)P_\zeta^3\frac{9}{8}\left(\frac{\lambda}{\sigma}\right)^2\frac{k_{12}}{k_1k_2k_3k_4}\left[\frac{1}{(k_1+k_2+k_{12})^3}\frac{1}{(k_3+k_4+k_{12})^3}+\right. \\ &\left. +2\frac{1}{(k_1+k_2+k_{12})^3}\left(6\frac{(k_1+k_2+k_{12})^2}{K^5}+3\frac{k_1+k_2+k_{12}}{K^4}+\frac{1}{K^3}\right)\right] + 23 \text{ perms in } \vec{k_n},\end{aligned} \tag{5.11}$$

where $k_{12} = |\vec{k_1}+\vec{k_2}|$ and $K = k_1+k_2+k_3+k_4$. For the second interacting Hamiltonian

$$H_2(\tau_1) = a\frac{\sigma}{H^3}(1-c_s^2)\int\prod_{n=1}^{3}\frac{d^3p_n}{(2\pi)^3}\left(\vec{p_2}\cdot\vec{p_3}\right)\dot{\zeta}(\vec{p_1},\tau_1)\zeta(\vec{p_2,\tau_1})\zeta(\vec{p_3},\tau_1)(2\pi)^3\delta^{(3)}\left(\sum_{n=1}^{3}\vec{p_n}\right), \tag{5.12}$$

and for all the other mixed trispectra $\langle\zeta\zeta\zeta\zeta\rangle^{ab}$ $(a,b=1,2)$, the computation is the same. For more details one should look in [72].
Now, we have to consider the fourth-order vertex interaction. By defining the parameter

$$\mu = \frac{1}{2}X^2P_{,XX} + 2X^3P_{,XXX} + \frac{2}{3}X^4P_{,XXX}, \tag{5.13}$$

the fourth-order perturbed Lagrangian reads [73]

$$\begin{aligned}\mathcal{L}^{(4)} =& a^3\frac{\mu}{H^4}\dot{\zeta}^4 - \frac{a}{H}(3\lambda-\sigma(1-c_s^2))\partial_i\zeta\partial^i\zeta\dot{\zeta}^2 + \frac{1}{4aH}\sigma(1-c_s^2)\partial_i\zeta\partial^i\zeta\partial_j\zeta\partial^j\zeta \\ =& \mathcal{L}_1^{(4)} + \mathcal{L}_2^{(4)} + \mathcal{L}_3^{(4)}.\end{aligned} \tag{5.14}$$

Then, by proceeding again applying the in-in formalism, we get [72]

$$\begin{aligned}\langle\zeta\zeta\zeta\zeta\rangle\Big|_{\mathcal{L}_1^{(4)}} =& 36\left(\frac{\mu}{\sigma}-9\frac{\lambda^2}{\sigma^2}\right)\frac{\prod_{n=1}^{4}k_n^2}{K^5} \\ \langle\zeta\zeta\zeta\zeta\rangle\Big|_{\mathcal{L}_2^{(4)}+\mathcal{L}_3^{(4)}} =& -\frac{1}{8}\left(3\frac{\lambda}{\sigma}-\frac{1}{c_s^2}+1\right)\frac{k_1^2k_2^2\left(\vec{k_3}\cdot\vec{k_4}\right)}{K^3}\left[1+3\frac{(k_3+k_4)}{K}+12\frac{k_3k_4}{K^2}\right]+ \\ &+\frac{1}{32}\left(\frac{1}{c_s^2-1}\right)\frac{\left(\vec{k_1}\cdot\vec{k_2}\right)\left(\vec{k_3}\cdot\vec{k_4}\right)}{K}\left[1+\frac{\sum_{i<j}k_1k_j}{K^2}+3\frac{k_1k_2k_3k_4}{K^3}\sum_n\frac{1}{k_n}+12\frac{k_1k_2k_3k_4}{K^4}\right] \\ &+ 23 \text{ perms in } \vec{k_n}.\end{aligned} \tag{5.15}$$

In the end, by recollecting all the pieces, and passing to the proper gauge, we find

$$\langle\zeta^4\rangle = (2\pi)^9P_\zeta^3\delta^{(3)}\left(\sum_n\vec{k_n}\right)\prod_n\frac{1}{k_n^3}\mathcal{T}(k_1,k_2,k_3,k_4,k_{12},k_{14}), \tag{5.16}$$

with

$$\mathcal{T} = \left(\frac{\lambda}{\sigma}\right)^2T_{s1} + \frac{\lambda}{\sigma}\left(\frac{1}{c_s^2}-1\right)T_{s2} + \left(\frac{1}{c_s^2}-1\right)^2T_{s3} + \left(\frac{\mu}{\sigma}-9\frac{\lambda^2}{\sigma^2}\right)T_{c1} + \left(3\frac{\lambda}{\sigma}-\frac{1}{c_s^2}+1\right)T_{c2} + \left(\frac{1}{c_s^2}-1\right)T_{c3}, \tag{5.17}$$

where the subscripts s, and c indicate the contributions coming from the third-order connected vertices and the fourth-order vertex respectively. An interesting observation, is that by requiring the parameter $\frac{\lambda}{\sigma},\frac{1}{c_s^2},\frac{\mu}{\sigma}\gg1$, the first four terms dominate, as they are of order $\mathcal{O}\left((\lambda/\sigma)^2,(1/c_s^2)^2,(\mu/\sigma)^2\right)$. This tells us that we have four possible shapes which are determined by three parameters [72]. Before proceeding to study them, we want to quantize the magnitude of the NG coming from this trispectrum. In order to do so, as done for the parameter f_{NL}, we define t_{NL} for each shape as

$$\left\langle \zeta^4 \right\rangle^{RT} = (2\pi)^9 P_\zeta^3 \delta^{(3)} \left(\sum_n \vec{k_n} \right) \frac{1}{k^9} t_{NL}, \tag{5.18}$$

where RT means that the trispectrum has been computed in the *regular tetrahedron limit* ($k_1 = k_2 = k_3 = k_4 = k_{12} = k_{14} = k$). The only difference between t_{NL} and f_{NL} is that while the latter is defined once the local limit is chosen for the bispectrum, the former is not, as even in the local configuration, as we are going to see, there is not a single shape [72]. In details, the size of the different shapes now read

$$\begin{aligned} t_{NL}^{s1} =& 0.250 \left(\frac{\lambda}{\sigma} \right)^2 & t_{NL}^{s2} &= 0.420 \frac{\lambda}{\sigma} \left(\frac{1}{c_s^2} - 1 \right) & t_{NL}^{s3} &= 0.305 \left(\frac{1}{c_s^2} - 1 \right)^2 \\ t_{NL}^{c1} =& 0.0352 \left(\frac{\lambda}{\sigma} - 9 \frac{\lambda^2}{\sigma^2} \right) & t_{NL}^{c2} &= 0.0508 \left(3 \frac{\lambda}{\sigma} - \frac{1}{c_s^2} + 1 \right) & t_{NL}^{c3} &= 0.0503 \left(\frac{1}{c_s^2} - 1 \right)^2 . \end{aligned} \tag{5.19}$$

Finally, before going to analyze different shapes, it is useful to introduce the local limit for the trispectrum, as done for the bispectrum. This is given by the following ansatz [74, 75]:

$$\zeta(\vec{x}) = \zeta_L + \frac{3}{5} f_{NL} \left(\zeta_L^2 - \left\langle \zeta_L^2 \right\rangle \right) + \frac{9}{25} g_{NL} \left(\zeta_L^3 - 3 \left\langle \zeta_L^2 \right\rangle \zeta_L \right). \tag{5.20}$$

and the resulting trispectrum becomes $\mathcal{T} = f_{NL}^2 T_{loc1} + g_{NL} T_{loc2}$, whence we see that even in the local limit we have more than one shape. The corresponding sizes are $t_{NL}^{loc1} = 2.16 f_{NL}^2$, and $t_{NL}^{loc2} = 1.08 g_{NL}$. We are now ready to study the trispectrum's shapes in different limits.

First of all, we have to constrain the domain if the shape functions by means of the consistency relations. There are two of them for the trispectrum, one for the sqeezed limit and one for the folded limit. A detailed analysis of the consistency relations applied to the current model has been done in [72]. After having checked them, we can consider the shapes under certain limits, setting them to zero when the corresponding momenta do not form a closed tetrahedron.

The first limit we apply is the *equilateral limit* ($k_1 = k_2 = k_3 = k_4$). The corresponding plot are shown in Fig.C.1. The external momenta are chosen to be $k_1 = k_2 = k_3 = k_4$, and the dominant T_{s1}, T_{s2}, T_{s3}, T_{c1}, and local T_{loc1}, T_{loc2} shapes are considered as functions of k_{12}/k_1 and k_{14}/k_1. One can see that T_{loc1} blows up at the boundaries. This is important since it helps to distinguish these shapes from T_{loc1} originated by f_{NL}.

The second limit we choose is the *folded limit* (Fig.C.2), where the condition $k_{12} = 0$ is imposed. In this limit one has $k_3 = k_4$ and $k_1 = k_2$. Again, the non-local dominant shapes and T_{loc2} are plotted as functions of k_4/k_1 and k_{14}/k_1. We observe that T_{loc2} is the only one that does not vanish as $k_4 \to 0$. This feature can thus be used to disriminate between the plotted shapes.

The third limit we consider is the so called *specialized planar limit* (Fig.C.3). Here, one puts $k_1 = k_3 = k_{14}$, while imposing the tetrahedron to be a planar quadrangle. Once defined our tetrahedron to have angles at one vertex $\cos\theta = \frac{k_1^2 + k_{14}^2 - k_4^2}{2k_1 k_{14}}$, $\cos\beta = \frac{k_2^2 + k_{14}^2 - k_3^2}{2k_2 k_{14}}$, and $\cos\alpha = \frac{k_1^2 + k_2^2 - k_{12}^2}{2k_1 k_2}$, by imposing these conditions, and the inequality $1 - \cos^2\theta - \cos^2\beta - \cos^2\alpha + 2\cos\theta\cos\beta\cos\alpha \geq 0$, it is possible to find the following constraint for k_{12} [72]:

$$k_{12} = \sqrt{k_1^2 + \frac{k_2 k_4}{2k_1^2} \left(k_2 k_3 \pm \sqrt{(4k_1^2 - k_2^2)(4k_1^2 - k_4^2)} \right)}. \tag{5.21}$$

Only the plus sign solution is considered, since it is possible to relate the minus sign one to another plus sign by changing the limit to $k_1 = k_2 = k_{14}$. The shapes are then plotted as functions of k_2/k_1, and k_4/k_1. We note immediately that as $k_2 \to k_4$, k_{13} vanishes, causing T_{loc1} to increases rapidly along the diagonal of the plot. All the other shapes remain finite. Also, towards the origin ($k_2, k_4 \to 0$), all the non-local functions vanishes rapidly ($\mathcal{O}(k_2^2, k_4^2)$), while the local ones do not.

Finally, we consider a limit close to the *double-squeezed limit* (Fig.C.4), by imposing $k_3 = k_4 = k_{12} \to 0$, and the tetrahedron to be planar as before. The same shapes as above are shown as functions of k_{12}/k_1 and k_{14}/k_1. Also, the shapes are divided by $\prod_{n=1}^{4} k_n$, in order to sharpen the differences between local and non-local ones. The latter, do not show any divergence, while the former do exhibit such a behaviour both in the limits $k_4 \to 0$, and $k_4, k_{14} \to 1$.

With the exception of the first considered limit, the configurations we studied forces the tetrahedron to be planar. As already pointed out, this is an important scenario as the main contributions from the CMB signals come from planar quadrangles [39]. We also note that while important differences arose between local and non-local shapes, the latter were often pretty similar between themselves. However, some differences can be pointed out. First of all, in the double squeezed limit (Fig.C.4), we can notice that while the shape arising from the fourth-order vertex, T_{c1} goes to zero as $\mathcal{O}(k_3^4)$, the other third-order vertices ones vanish as $\mathcal{O}(k_3^2)$ [72]. Moreover, it is possible to note immediately a difference in the behaviour at $k_4 \to 1$ and at $k_4 \to 0$. In the latter limit in fact, T_{c1} vanishes, while T_{s1}, T_{s2}, T_{s3} do not. Furthermore, as usually done in data analysis, the parameters that define the shapes can be tuned in order to enlarge the differences between the considered shapes.

Just from this little summary, we can see how important the shapes can be in discriminate between different contributions to the trispectrum. By admitting strong four-points NG for the scalar curvature perturbations, we can in fact study its shapes and hence understand which parameters (λ/σ, $1/c_s^2$, μ/σ) mainly contribute to that. Then, this would clearly indicate specific properties of the action we chose for our inflationary model. In order to make a concrete example, we consider the following Lagrangian, typical for models of DBI Inflation [76, 77]:

$$\mathcal{L} = -f^{-1}(\phi)\sqrt{1 - 2Xf(\phi)} + f^{-1}(\phi) - V(\phi), \tag{5.22}$$

with X the kinetic term defined as above. Here, $c_s \ll 1$, $\frac{\lambda}{\sigma} = \frac{1}{2}\left(1/c_s^2 - 1\right)$, and $\frac{\mu}{\sigma} = \frac{1}{4}\left(\frac{5}{c_s^2} - 4\right)\left(\frac{1}{c_s^2} - 1\right)$, from which we expect large trispectra because of the non-kinetic term. From Eqs.5.19, we see that the dominant terms in the trispectrum are T_{s1},T_{s2},T_{s3}, and T_{c1} as they are $\sim \mathcal{O}(c_s^{-4})$. The estimator t_{NL} then reads $t_{NL}^{DBI} \approx 0.542/c_s^4$. As a comparison, we consider a k-inflation model [78], with

$$\mathcal{L} \sim \frac{(-X + X^2)}{\phi^2}, \tag{5.23}$$

with again $c_s \ll 1$, $\frac{\lambda}{\sigma} = \frac{1-c_s^2}{2}$, and $\frac{\mu}{\sigma} = \frac{1-c_s^2}{4}$. Here the dominant contribution is given by T_{s3}, with $t_{NL}^{k} \approx 0.305/c_s^4$. Even though we saw that the differences between T_{si} are quite small in all the limits we considered, one can see that through T_{c1} it is possible to discriminate between the two models, as in the latter this contribution is subdominant with respect to T_{s3}.

5.2 Observations

Since we talked about how to recognize and discriminate models through their own shapes, it is now worth it to mention how we can directly observe NGs and their properties. Two main observables are used to constraint primordial NG and parity violation: the Cosmic Microwave Background (CMB) and Large Scale Structures (LSS).

5.2.1 The Cosmic Microwave Background

The first way to test primordial NG is to look at the fluctuations of the CMB. It is possible to consider both temperature and polarization fluctuations. We will focus on the former, but an analogous analysis can be done for the latter. The CMB temperature fluctuations' field can be seen as a two-dimensional field on a sphere, and defined as

$$\frac{\Delta T}{T}(\theta, \phi) = \sum_{l=0}^{+\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi). \tag{5.24}$$

This corresponds to decompose the entire field in a multiplication between a spherical harmonics Y_{lm} and a corresponding amplitude a_{lm} along each direction (θ, ϕ) in the sky. Here, l, indicates the n-pole expansion, namely how the fluctuation distributes itself in the sphere. $l = 1$ (monopole) corresponds to have a maximum of ΔT at one pole and a minimum at the opposite one, $l = 2$ to two opposite maxima and so on. In other words, it is possible to link l to the angular scale variation of the fluctuations. A big l means that temperature fluctuations changes on small angular scales, while tiny l indicates that a certain fluctuation varies slowly as the angular direction changes. m expresses the orientation of this distribution. From the equation above, we see that all the information for each direction is encoded in the harmonic modes a_{lm}. As for any other random field, it is possible to describe the CMB temperature fluctuations through correlation functions. The most studied one is given by the CMB power spectrum

$$\left\langle a_{l_1 m_1} a^*_{l_2 m_2} \right\rangle = C_{l_1} \delta_{l_1 l_2} \delta_{m_1 m_2}, \tag{5.25}$$

where the Kronecker's deltas ensure that multipoles with different wave-numbers are non correlated, and C_l encodes the amplitude of the power spectrum. Then, by observing $\frac{\Delta T}{T}$ at different scales, thereby computing C_l, it is possible to reconstruct the entire power-spectrum of the CMB, which turns out to be strongly dependent upon several cosmological parameters. This of course helps us to get important estimations and constraints on them. Also primordial NG are imprinted in the CMB, and can therefore be studied through its observations. In order to provide a link between these observables, one can convolute $\Phi(\vec{k})$ with a transfer function $\Delta_l(k)$, getting [71]

$$a_{lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \Delta_l(k) Y_{lm}(\hat{k}) \Phi(\hat{k}), \tag{5.26}$$

where the transfer function encodes all the effects that can be observed in the CMB. Then, the CMB bispectrum is given by

$$\begin{aligned} B^{l_1 l_2 l_3}_{m_1 m_2 m_3} = \langle a_{l_1 m_1} a_{l_2 m_3} a_{l_3 m_3} \rangle = (4\pi)^3 (-i)^{l_1 + l_2 + l_3} \int & \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} \Delta_{l_1}(k_1) \Delta_{l_2}(k_2) \Delta_{l_3}(k_3) \\ & \left\langle \Phi_H(\vec{k_1}) \Phi_H(\vec{k_2}) \Phi_H(\vec{k_3}) \right\rangle Y_{l_1 m_1}(\hat{k_1}) Y_{l_2 m_2}(\hat{k_2}) Y_{l_3 m_3}(\hat{k_3}), \end{aligned} \tag{5.27}$$

which can be rewrite by means of Bessel's functions $j_{l_n}(k_n x)$ as

$$\begin{aligned} B^{l_1 l_2 l_3}_{m_1 m_2 m_3} &= \left(\frac{2}{\pi}\right)^3 \int x^2 dx \int \prod_{i=1} \left(d^3k_i \Delta_{l_i}(k_i) j_{l_i}(k_i x)\right) B_\Phi(k_1, k_2, k_3) \int d\Omega_{\hat{x}} Y_{l_1 m_1}(\hat{x}) Y_{l_2 m_2}(\hat{x}) Y_{l_3 m_3}(\hat{x}) \\ &= G^{l_1 l_2 l_3}_{m_1 m_2 m_3} b_{l_1 l_2 l_3}, \end{aligned} \tag{5.28}$$

where $b_{l_1 l_2 l_3}$ is the so called reduced bispectrum, and $G^{l_1 l_2 l_3}_{m_1 m_2 m_3}$ is the Gaunt integral, whose explicit expression reads

$$G^{l_1 l_2 l_3}_{m_1 m_2 m_3} \equiv \int d\Omega_{\hat{x}} Y_{l_1 m_1}(\hat{x}) Y_{l_2 m_2}(\hat{x}) Y_{l_3 m_3}(\hat{x}) = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \tag{5.29}$$

Here, the matrices denote the Wigner-$3j$ symbols. The reduced bispectrum is more convenient than the initial one, since all the geometric factors associated to the Gaunt integral can be factorized out thanks to homogeneity and isotropy of the cosmological principle. However, inside the integral, two important physical constraints are present, that cannot be seen anymore in the reduced bispectrum. Firstly, it enforces the rotational-invariance forcing $|l_1 - l_j| \leq l_k \leq l_1 + l_j$. Secondly, the sum of multipoles l_i has to be even to guarantee invariance under parity transformation; namely it enforces $l_1 + l_2 + l_3$ to be even. In the end one can write

$$b_{l_1 l_2 l_3} = \left(\frac{2}{\pi}\right)^3 \int x^2 dx \int dk_1 dk_2 dk_3 (k_1 k_2 k_3)^2 B_\Phi(k_1 k_2 k_3) \Delta_{l_1}(k_1) \Delta_{l_2}(k_2) \Delta_{l_3}(k_3) j_{l_1}(k_1 x) j_{l_2}(k_2 x) j_{l_3}(k_3 x), \tag{5.30}$$

which is the key equation for the CMB analysis of primordial NG. In fact, it relates the primordial bispectrum to the reduced CMB's one. Then, this can be rewritten in terms of the shape function S introduced above for the bispectrum, thereby providing the relation between CMB's analysis and primordial bispectrum's features we need to discriminate between inflationary models. All the analysis can be generalized to the trispectrum case. For a detailed analysis of the whole procedure and the explicit representation of different inflationary models, one can look at [71]

5.2.2 Large Scale Structures

Another way to test primordial NG is to analyse the matter distribution. It is customary to define a density parameter $\delta(\vec{x}) = \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}}$. It is set to be $\delta = -1$ in voids at late times and $\delta \gg 1$ in collapsed objects, as dark matter halos. Primordial NG induces signatures in the statistics of the matter density field. In fact, through the Bardeen's gravitational potential, they can be related to the matter density perturbations thanks to the Poisson equation. At linear order in perturbation theory we have [79]

$$\delta(\vec{k}, z) = \alpha(k, z) \Phi_H(\vec{k}, z), \tag{5.31}$$

where $\alpha(k, z) = \frac{2c^2 k^2 T(k) D(z)}{3\Omega_m H_0^2} \frac{g(0)}{g(z_*)}$, with z being the redshift, $D(z)$ the linear growth function, $T(k)$ the transfer function and $g(z) \propto (1 + z)D(z)$ the potential growth function. Therefore, through Eq.5.31, we can relate the primordial statistics to the matter density's one, getting, for example

$$P_m(k, z) = \alpha^2 P_{\Phi_H}(k, z_*), \tag{5.32}$$

where z_* indicates the redshift corresponding to primordial times during which primordial NG left the horizon.
Primordial NG can therefore imprints observable signatures in the large scale structures (LSS) [80]. However, there are two main contributions that makes these possible constraints difficult to be set. First of all, the matter density distribution is highly non-Gaussian itself. Because of the gravitational instability, it indeed departs from a Gaussian distribution even if the initial conditions are Gaussian [71]. Gravitational instability produces in fact a non-zero bispectrum, which, in the next-to-leading order in cosmological perturbation theory, reads [80]

$$B(\vec{k_1}, \vec{k_2}, \vec{k_3}) = 2\left[P(k_1)P(k_2)J(\vec{k_1}, \vec{k_2}) + P(k_2)P(k_3)J(\vec{k_2}, \vec{k_3}) + P(k_1)P(k_3)J(\vec{k_1}, \vec{k_3})\right], \tag{5.33}$$

with, in a dS Universe,

$$J(\vec{k_1}, \vec{k_2}) = \frac{5}{7} + \frac{\vec{k_1} \cdot \vec{k_2}}{2k_1 k_2}\left(\frac{k_1}{k_2} + \frac{k_2}{k_1}\right) + \frac{2}{7}\left(\frac{\vec{k_1} \cdot \vec{k_2}}{k_1 k_2}\right)^2 \tag{5.34}$$

being the gravitational instability kernel. Unfortunately, this contribution turns out to be larger than the one given by primordial NG.
The second issue is related to the fact that in principle primordial NG act on the late-time dark matter density bispectrum. However, since we do not observe directly the dark matter's distribution, we have another non-Gaussian term coming from bias that contributes against the signal we want to detect.
Another probe of primordial NG can be inferred by looking at the so called rare events: namely large voids or galaxy clusters. These are objects that form from the tail of the nearly Gaussian initial matter density distribution and hence are largely sensitive even to small deviations from Gaussianity [80, 81]. In particular, it was shown that the abundance of voids is sensitive to non-Gaussianity. In fact, this abundance turns out to be dependent upon the skewness S_3 (i.e. the three-point function) of primordial NGs [82]:

$$\frac{dn}{dR} = \frac{9}{2\pi^2}\sqrt{\frac{\pi}{2}}\frac{1}{R^4}e^{-\delta_v^2/2\sigma_M^2}\left\{\left|\frac{d\ln\sigma_M}{d\ln M}\right|\left[\frac{|\delta_v|}{\sigma_M} - \frac{S_3\sigma_M}{6}\left(\frac{\delta_v^4}{\sigma_M^4} - 2\frac{\delta_v^2}{\sigma_M^2} - 1\right)\right] + \frac{1}{6}\frac{dS_3}{dM}\sigma_M\left(\frac{\delta_v^2}{\sigma_M^2} - 1\right)\right\}, \tag{5.35}$$

where δ_v is the critical underdensity responsible for the production of a void, σ_M is the rms mass function, and R is the radius of a sphere whence the mass M has been removed creating a void. We see that in principle S_3 could be dependent on the mass scale M. If it were not true, then the second term of the above equation would simply vanish. The disadvantage of this probe is that, since the three-point function is given by the integral over the bispectrum, this the effects of primordial NGs are less sensitive to the shapes.
A different possibility of obtaining NG's constraints is given by the clustering of dark matter halos. Primordial NG in fact seems to affect the process of clustering of these objects by inducing a scale-dependent bias for halos on large-scales [83]. Moreover, this probe is also reactive to the primordial NG's shape. Equilateral types of NG indeed produce effects which are strongly suppressed with respect to local types [80].

5.3 Parity-violation probes

We would like to end this last chapter by highlighting how parity violations can be seen through the two main observables we depicted earlier. Starting from the CMB observations, we see that by applying a parity transformation to the scalar curvature perturbation, we get $\zeta_{\vec{k}} \to \zeta_{-\vec{k}}$ as pointed out before. This transforms the harmonic coefficients we used to decompose the CMB fluctuations as $a_{lm} \to (-1)^l a_{lm}$ [2]. Hence, after a parity inversion we will have, for the CMB trispectrum,

$$\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a_{l_4m_4}\rangle \to (-1)^{l_1+l_2+l_3+l_4}\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a_{l_4m_4}\rangle\,. \tag{5.36}$$

Therefore, we can set the two condition $\sum_n l_n = even$ if the CMB induced trispectrum is invariant under parity transformation, and $\sum_n l_n = odd$ if it violates parity. In terms of CMB polarizations, these can be then decomposed in even and odd contributions, making it possible to define parity-violating observables, for instance, from the product of opposite-parity polarizations [84]. Also the temperature bispectrum can be seen as a possible observable of parity-violation. In fact, if a scalene triangle projected on the observed CMB sphere is observed more often than its mirror image, them we would have a sign of parity-breaking, since these two triangles cannot be superimposed in a 2D space.
By following this last approach, a new idea has been proposed for searching for parity-violating signals also in LSS. In the model we studied in this thesis, parity violation led to a difference in the evolution of two chirality states of GWs, somehow indicating a preferred one. This then induced a parity violation in the correlation functions of scalar curvature perturbations, which can be related to density fluctuations as we seen above. As done for ΔT^{CMB}, the idea for LSS is to use four-points correlation functions in a 3D space, since, as for the three-point functions (triangles) in a spherical 2D space, it is the lowest order correlator that cannot be rotated in its mirror image. Therefore, by using the 3D distribution of galaxies, one can search for parity-violations in terms of a preferred configuration for the induced tetrahedra (with sides of the order $\sim 1 - 100$Mpc [85]). Since these large structures

are thought to be seeded by density fluctuations, which in turn are strictly related by primordial perturbations, we can in principle relate the (possibly) observed "3D parity violation" to the primordial one.

5.4 Actual constraints

The last released results regarding primordial non-Gaussianity are given by the Planck collaboration's analysis of Planck full mission. In particular, it analysed in details temperature and $E-$mode polarization of the CMB in order to obtain constraints on primordial NG (PNG). The entire analysis can be found in [11]. In this work, also lower-l multipoles ($4 \leq l < 40$) were considered with respect to the Planck mission's alaysis of 2015. The following values for the local, equilateral, and orthogonal bispectrum amplitudes was found: $f_{NL}^{local} = -0.9 \pm 5.1$, $f_{NL}^{equilateral} = -26 \pm 47$, and $f_{NL}^{orthogonal} = -38 \pm 24$ at 68% confidence level, which consistently resemble the constraints from the previous analysis. Despite the consideration of lower multipoles in polarization, these results do not show improvements in the errors because of two main reasons. Firstly, even though the local shape is expected to be more sensitive to low-l, it is actually less dependent upon the polarization, implying that none of the three shapes are sensitive enough to low multipoles. Secondly, the most realistic simulations of this analysis introduced higher noise levels. However, the polarization bispectra have been well constrained and are now reliable.
Also the constraints on the trispectrum have been refined. We saw that in the local form the trispectrum can be written in terms of two parameters f_{NL}^2, often related to the parameter τ_{NL}, and g_{NL}. The results following from the data analysis of Planck 2013 and 2018 find the flowing apmplitudes: $\tau_{NL} < 2800$ (95% CL) [86], and $g_{NL}^{local} = (-5.8 \pm 6.5) \times 10^4$ (68% CL) [11]. The analysis of the Planck 2018 mission reached a better level of sensitivity for what concerns the amplitude of higher order correlation functions' amplitudes, and produced the first (only) polarization-based CMB bispectrum constraints. The obtained results are consistent with the expectations from the standard single-field slow-roll model, and strongly constrain the parameter space of other models, since usually they produce higher values of NG. However, these goals are not enough to rule out or confirm scenarios alternative to Inflation. It was suggested by several studies that future experiments should seek for $f_{NL}^{local} \sim 1$, which would allow us to confirm or not many multi-field models of Inflation. By requiring a similar precision for non-local amplitudes, on the other hand, it would enable us to distinguish slow-roll single-field models from the more complex ones we briefly sketched in the previous chapters.
The latest constraint on f_{NL}^{local} comes from the extended Baryon Oscillation Spectroscopic Survey (eBOSS) quasar samples, reporting an estimation $-51 < f_{NL}^{local} < 21$ at 95% CL. Later, J.P. Dai and J.Q. Xia [87] used the multitracer technique in a simulated catalog of halos to probe the local-type amplitude of PNG. They reached a reduction of 73% of the errors of f_{NL}^{local}, which makes the next generation of large field galaxy surveys a powerful tool for constraining PNGs.
The actual constraints on parity violation in the Universe have instead been obtained by measuring the rotation of the polarization's plane of photons. Thus, parity-violation has been tested as coming from the electromagnetic sector [88]. Considering for instance a Maxwell's Lagrangian coupled with a Chern-Simons term, as studied briefly in Chapter 3, we expect right- and left- handed photons to evolve asymmetrically. This difference in the evolution, denoted with α mixes the modes of CMB polarization, generating correlations between them that would be zero otherwise. The estimates provided by the data analysis of Planck 2015 mission are $\alpha \approx (0^\circ.31 - 0^\circ.35) \pm 0^\circ.05$ (68% CL). These results show a very small rotation angle, which is consistent with no rotation, i.e. no parity violation. However, in [89] new estimates have been found from Planck 2018, which rule out the possibility of having a null birefringence angle at 99.2% CL. The new value for the angle results to be $\alpha \approx 0^\circ.35 \pm 0^\circ.14$. Another way to constraint Parity violation, arising from through the CMB analysis is to look at the so called tensor nonlinearity parameter f_{NL}^{tens}, defined as

$$f_{NL}^{tens} \equiv \lim_{k_i \to k} \frac{B_h^{+++}(\vec{k_i})}{F_\zeta^{equil}(k_i)}, \tag{5.37}$$

with $F_\zeta^{equil} = \left(\frac{5}{3}\right)^3 \frac{B_{\Phi_H}^{equil}}{f_{NL}^{equil}}$. This parameter can be decomposed in its P-even and P-odd contributions. Its analysis is reported in [11], and gives us a result which is consistent with no parity violation in the primordial Universe.

Chapter 6

Conclusions

In this thesis we introduced the topic of cosmological Inflation, thus describing a single-field slow-roll model. Then, we addressed the cosmological perturbation theory, explaining the two ways one can follow to deal with the gauge issue in General Relativity. We introduced the very important gauge invariant quantity ζ, stressing how it can be related directly to the scalar curvature perturbations, and hence to the Bardeen's gravitational potential, which can be directly linked to CMB fluctuations. In the second part (Chapters 2 and 3) we explained in details the formalism we used, providing detailed calculations of several known results, both from the single-field slow-roll model of Chapter 1 [52], and from a more complex model, where the inflaton's field interacts with a primordial gauge field, which can be seen as the electromagnetic one [2, 56, 57, 59, 60]. We explicitly showed how parity violation can be carried out from the calculation of the ζ's trispectrum, i.e. the Fourier transform of the four-points correlation function, following [2]. In the original part of this work, we wanted to search for parity violation signatures from the gravitational sector in the Early Universe. In fact, this possibility could give us important hints about corrections to General Relativity at high energies. We concentrated in computing the scalar curvature perturbation's trispectrum mediated by gravitons, since the four-points correlation function is the lowest order scalar statistics that allows us to get parity violation. Thus, we introduced an Effective Field Theory approach [6] showing up some corrections to General Relativity at the high energies characterizing Inflation. Hence, we proceeded by building up the tools we needed for the final computation of $\langle\zeta\zeta\zeta\zeta\rangle$ in this beyond-Einstein's gravity model. Then, we computed different two-points and three-points correlation functions in this framework [8], highlighting how for just one bispectrum the complex contribution arising from parity violation was not highly suppressed by the Chern-Simons mass. Finally, we proceeded with the central computation of the thesis. We started by considering simply de Sitter gravitons, as the interaction vertex already carried a parity-violating term: $\epsilon^{ijk}k_i\epsilon^{(s)}_{lj}(\hat{k}) = i\lambda_s k\epsilon^{(s)k}_{l}(\hat{k})$. However, this trispectrum did not show any parity-odd signature, i.e. any complex term. This was given by the fact that the correlation between the two gravitons forced one of them to have the opposite momentum with respect to the other, which once reflected to the polarization tensor compensated the parity-violation carried by the other vertex. This fact could have be seen a priori, since by using the equation written above, it could be seen than, besides some other real terms, each vertex would have corresponded to a de Sitter one. We computed it in details anyway. Then, we considered Chern-Simons gravitons instead. In order to do so, we divided the problem in two cases: we considered a constant Chern-Simons mass, and then allowed it to be time dependent. In the first case scenario we just proceeded as before, but with another mode-function for the tensor fields. This explicitly showed two corrections that violate parity. Then, we performed again a detailed computation of the trispectrum, obtaining a clear signature of parity-violation. This was because in the summation of the gravitons' polarizations, the parity-breaking corrections flipped the sign of the polarization tensors giving us a complex contribution from the trispectrum's directional part. The disadvantage of this procedure was that, by using a Chern-Simons modified mode-function for gravitons, we added a further M_{CS}^{-1} factor to the final result. Being $\langle\zeta\zeta\zeta\zeta\rangle \propto \left(\frac{H}{M_{CS}}\right)^3$, our trispectrum resulted to be strongly suppressed, as $\frac{H}{M_{CS}} \ll 1$. In the second case scenario, things got even worse. In fact, it was shown that in order to avoid ghost fields, an evolving Chern-Simons mass can only be increasing in time. However, since at the beginning of Inflation we already had to impose $\frac{H}{M_{CS}} \ll 1$ [7,8], this evolution results in a complete suppression of all the parity-violating signatures of our three-vertex. Besides, as pointed out in [10], all the Chern-Simons related terms in the tensor fields' equation of motion would get rapidly washed out, leaving to very small deviation from General Relativity, which can be consider as the underlying theory. Hence, also the gravitons' mode function would be given by the usual de Sitter one. The only contribution that not only does not undergo a strong suppression, but also can enjoy an increasing Chern-Simons mass, is the $\langle\zeta\gamma\gamma\rangle$ bispectrum, which can then be seen as an important tool to investigate Chern-Simons corrections to the current theory of gravity [10]. Another way to search for parity-violation would be given by following the procedure proposed by Liu et al. [3]. They added another scalar field in their trispectrum computation, in order to enforce the presence of a dihedral angle in the momentum space. Under a parity transformation, this angle transforms as $\phi \rightarrow -\phi$, implying $\sin(\phi) \rightarrow -\sin(\phi)$, which breaks parity. It would be possible to encode this idea in our model by adding another (scalar) field to our initial Lagrangian, thereby computing the same scalar curvature trispectrum but with different vertices. Namely, we would still get an internal graviton correlator, but, by naming the new scalar field σ, we would get vertices of the type $\zeta - \sigma - \gamma$, where σ would be then connected to another external ζ. Another possibility, we cannot concretely apply yet, would be given by a successful procedure to deal with ghosts [7]. In this way, we could consider also a (slowly) decreasing Chern-Simons mass, which can therefore enhance our parity-odd signatures. A final observation it is worth to make is that the exact same procedure we adopted can be generalized to all other types of fields. Perhaps it could be possible to analyze gravitational waves from Inflation, and see what we should expect from their correlators in different inflationary models. In fact, as reported in [90], the gravitational waves from Inflation will be potentially detectable from the future LISA mission, making it complementary to CMB analyses to probe the very Early Universe.

Appendices

Appendix A

A.1 Friedmann-Lemaitre-Robertson-Walker metric

The cosmology principle states that wherever we are and whenever we are, the Universe at large scales (~ 100 Mpc) appears to be homogeneous and isotropic. The observations of the CMB highlight a relative variation of temperature of the order $\frac{\delta T}{T} \sim 10^{-5}$. Thanks to this principle, at each cosmic time t, all the imaginary observers see the same macroscopic Universe. Hence, by starting from the usual four-dimensional line element $ds^2 = -dt^2 + dl^2$, we can focus only on the space coordinates of the Universe, the time ones implying only a possible rescaling. Moreover, because of the isotropy hypothesis, we can think the metric to be symmetric. Now, since the Universe is homogeneous, we are able to find a constant Ricci scalar to be the same in every point. In order to see this, one can think that being the density of the Universe distributed in a highly homogeneous way, also the curvature will be more or less constant everywhere. This Ricci scalar can be greater, smaller or equal to zero. The latter case would produce a familiar four-dimensional Euclidean space, with line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2. \tag{A.1}$$

We consider instead $R > 0$. The opposite case is analogous. An example of a space with a positive curvature is the sphere. Hence, we simply want to generalize a spherical surface in three dimensions to a three-sphere. In order to do that, we just start with the usual three-dimensional sphere's line element $dl^2 = d\vec{x}^2$ (where in Cartesian coordinates $\vec{x} = (x_1, x_2, x_3)$) and we add a fourth generic dimension u:

$$dl^2 = d\vec{x}^2 + du^2. \tag{A.2}$$

Now, we want to constrain this new dimension in order to get a three-sphere. Thus, we simply introduce $\vec{x}^2 + u^2 = a^2$, with a constant in space, which is the usual equation for a spherical surface. In such a way, by differentiating, we get

$$2\vec{x} \cdot d\vec{x} + 2u du = 0 \quad \Longrightarrow \quad du = -\frac{\vec{x} \cdot d\vec{x}}{u}. \tag{A.3}$$

We then re-scale these quantities in order to have physical quantities, getting $\vec{x} = a\vec{x}_{phys}$ and $u = au_{phys}$. Therefore, our constraint now becomes $\vec{x}^2_{phys} + u^2_{phys} = 1$. By inserting the newly constrained u_{phys} in Eq. A.2, and forgetting the new relabelling for simplicity, we obtain

$$dl^2_{phys} = a^2 \left(d\vec{x}^2_{phys} + du^2_{phys} \right) = a^2 \left(d\vec{x}^2 + \frac{\vec{x}^2 d\vec{x}^2}{1 - \vec{x}^2} \right). \tag{A.4}$$

Finally, by changing our Cartesian coordinates to spherical ones ($d\vec{x}^2 = d\vec{r}^2 + r^2 d\Omega^2$, $\vec{x} d\vec{x} = \vec{r} d\vec{r}$), we find

$$dl^2 = a^2\left(dr^2 + r^2 d\Omega^2 + \frac{r^2 dr^2}{1-r^2}\right) = a^2\left(\frac{dr^2}{1-r^2} + r^2 d\Omega^2\right). \tag{A.5}$$

For $R < 0$, one finds the same result with a plus sign instead of a minus sign in the denominator of the first term on the right hand side. An example of surface which have a negative curvature is the hyperboloid. In this case we would add the fourth dimension as constrained by $\vec{x}^2 - u^2 = 1$. Finally, by introducing also the time dimension, we obtain the famous FLRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right], \tag{A.6}$$

where $k = -1, 0, +1$ if the curvature is negative (open Universe), null (flat Universe), or positive (closed Universe) respectively.

A.2 Inflaton's equation of motion

From the variational principle we can easily find the Euler-Lagrange equation for the inflaton field, which in a curved space reads

$$\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0. \tag{A.7}$$

From that, considering a scalar field's Lagrangian, it follows

$$-\frac{\partial V(\phi)}{\partial \phi} + \nabla_\mu \left(g^{\mu\beta}\partial_\beta \phi\right) = 0, \tag{A.8}$$

where $\nabla_\mu \partial^\mu \phi = \partial_\mu \partial^\mu \phi + \Gamma^\mu_{\mu\beta}\partial^\beta \phi$ with $\Gamma^\mu_{\mu\beta} = \frac{1}{2} g^{\mu\tau} g_{\mu\tau,\beta}$. We consider now the equality $\partial_\beta \sqrt{-g} = -\frac{1}{2\sqrt{-g}}\partial_\beta g$. We note that it is possible to write $\partial_\beta g \approx \frac{\det|g_{\mu\tau} + \partial_\nu g_{\mu\tau}\delta x^\nu| - \det|g_{\mu\tau}|}{\delta x^\beta} = \frac{g \det|\mathbb{I} + \delta^\mu_\alpha \delta^\tau_\beta g^{\alpha\beta}\partial_\nu g_{\mu\tau}\delta x^\nu| - g}{\delta x^\beta}$. Hence, since we can make the first approximation under the limit of $\delta x^\mu \to 0$, we have $\det(\mathbb{I} + \epsilon \mathbb{X}) = 1 + Tr(\mathbb{X})\epsilon + \mathcal{O}(\epsilon^2) \approx 1 + Tr(\mathbb{X})\epsilon$. By calling $\epsilon = \delta x^\nu \ll 1$, in the end one gets

$$\partial_\beta \sqrt{-g} = \frac{\sqrt{-g}}{2} g^{\mu\tau} g_{\mu\tau,\beta} \quad \Longrightarrow \quad \Gamma^\mu_{\mu\beta} = \frac{\partial_\beta \sqrt{-g}}{\sqrt{-g}}. \tag{A.9}$$

Therefore, the inflaton field's equation of motion finally reads

$$\frac{1}{\sqrt{-g}}\left(\sqrt{-g}\phi^{,\mu}\right)_{,\mu} = \frac{\partial V}{\partial \phi}. \tag{A.10}$$

Then, if we consider a FLRW metric with $k = 0$, we simply find

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2}{a^2}\phi = -\frac{\partial V}{\partial \phi}. \tag{A.11}$$

If we started with a non-flat FLRW Universe, we would find the same equation of motion, the only difference being the expression for the Laplacian, that can however be expressed as above by using coordinates of a non-flat space.

A.3 Quasi de Sitter Universe

In a quasi dS Universe, we cannot consider H as a constant anymore. Therefore, we need to look back at the approximation we made for the conformal time $\tau \approx -\frac{1}{aH}$. This time, by considering a slowly evolving Hubble's parameter, we have $H(t) \approx H(t_0) + \dot{H}(t_0)t$. Hence

$$\begin{aligned} a(t) =& exp\left(\int H dt\right) = exp\left(\int H(t_0) + \dot{H}(t_0) t dt\right) = exp\left(\int H(t_0) - \epsilon H^2(t_0) t dt\right) \\ =& exp\left(H_0 t - \epsilon H_0^2 \frac{t^2}{2}\right), \end{aligned} \tag{A.12}$$

where we used $\epsilon = -\dot{H}/H^2$, which vanishes in dS. Now, by considering the last equation and $d\tau = dt/a(t)$, we can Taylor-expand the conformal time as $\tau \approx \tau(\epsilon = 0) + \frac{d\tau}{d\epsilon}\Big|_{\epsilon=0} \epsilon$. Here $\tau_0 = -\frac{1}{H_0 a(\tau)}$ and

$$\begin{aligned} \frac{d\tau}{d\epsilon}\Big|_{\epsilon=0} =& \frac{d}{d\epsilon} \int exp\left(-H_0 t + \epsilon H_0^2 \frac{t^2}{2}\right) dt\Big|_{\epsilon=0} = \frac{1}{H_0} \int exp\left(-H_0 t + \epsilon H_0^2 \frac{t^2}{2}\right) \frac{(H_0 t)^2}{2}\Big|_{\epsilon=0} d(H_0 t) \\ =& \frac{1}{H_0} \int exp\left(-H_0 t\right) \frac{(H_0 t)^2}{2} d(H_0 t) \approx \frac{1}{H_0} \int \exp -H_0 t d(H_0 t), \end{aligned} \tag{A.13}$$

where in the last passage we neglected the term $\propto (H_0 t)^2$ as it if subdominant in the time-integral's domain ($t_{in} \approx 10^{-36}$s, $t_{fin} \approx 10^{-33}$s). Therefore, in the end, we find

$$\tau \approx -\frac{1}{aH(1-\epsilon)}. \tag{A.14}$$

The equation of motion for a massive inflaton field in quasi dS is given by

$$u_k''(\tau) + \left[k^2 - \frac{a''}{a} + m_\phi^2 a^2\right] u_k(\tau) = 0. \tag{A.15}$$

As the mass comes from an already small slow-roll parameter, in the last term we can consider an usual dS conformal time. We have then

$$\tau^2 u_k''(\tau) + \left[k^2\tau^2 - \left(\nu^2 - \frac{1}{4}\right)\right] u_k(\tau) = 0, \tag{A.16}$$

where $\nu^2 = \frac{9}{4} + 3\epsilon - 3\eta_V$ ($\nu \approx \frac{3}{2} + \epsilon - \eta_V$). This is again a Bessel's type equation. By solving it, on superhorizon scales we find

$$|\delta\phi_k|^{sup} = \frac{H}{\sqrt{2k}} \left(\frac{k}{aH}\right)^{-\epsilon+\eta_V}. \tag{A.17}$$

Appendix B

B.1 Third-order action simplification

We report here an example of simplification for the several terms that arise from the first form of the third-order action for the scalar curvature field ζ. By rewriting all the B fields in terms of ζ and χ, three sets of terms arise; namely, with a a^3, a, and a^{-1} dependence. We have

$$\begin{aligned} S = \int d^4x a^3 \epsilon \left(3\zeta - \frac{\dot{\zeta}}{H}\right)\dot{\zeta}^2 - a^3\epsilon^2\left(3\zeta - \frac{\dot{\zeta}}{H}\right)\dot{\zeta}^2 - 2a^3\epsilon\dot{\zeta}\partial_i\zeta\partial^i\partial^i\chi + a^3\left(3\zeta - \frac{\dot{\zeta}}{H}\right)\partial_i\partial_j\chi\partial^i\partial^j\chi + \\ - a\epsilon\left(\zeta - 2\frac{\dot{\zeta}}{H}\right)\partial_i\zeta\partial^i\zeta + a\epsilon\left(3\zeta - \frac{\dot{\zeta}}{H}\right)\dot{\zeta}\partial_i\partial^i\zeta - 2\frac{a}{H}\left(3\zeta - \frac{\dot{\zeta}}{H}\right)\partial_i\partial_j\zeta\partial^i\partial^j\zeta + \\ + 4\frac{a}{H}\partial_i\zeta\partial^i\chi\partial_j\partial^j\zeta + \frac{a^{-1}}{H^2}\left(3\zeta - \frac{\dot{\zeta}}{H}\right)\left[\partial_i\partial_j\zeta\partial^i\partial^j\zeta - \left(\partial_i\partial^i\zeta\right)^2\right] - 4\frac{a^{-1}}{H^2}\partial_i\zeta\partial^i\zeta\partial_j\partial^j\zeta + D_0. \end{aligned} \tag{B.1}$$

The whole procedure is very long, so we will just sketch the example for the $\propto a^{-1}$ terms. The entire computation can be found in [53]. We have

$$F = \frac{a^{-1}}{H^2}3\zeta\partial_i\partial_j\zeta\partial^i\partial^j\zeta - \frac{a^{-1}}{H^2}3\zeta\left(\partial_i\partial^i\zeta\right)^2 - 4\frac{a^{-1}}{H^2}\partial_i\zeta\partial^i\zeta\partial_j\partial^j\zeta - \frac{a^{-1}}{H^3}\dot{\zeta}\partial_i\partial_j\zeta\partial^i\partial^j\zeta + \frac{a^{-1}}{H^3}\dot{\zeta}\left(\partial_i\partial^i\zeta\right)^2. \tag{B.2}$$

The last two terms can be rewritten thanks to the Leibniz rule. These then reads

$$\frac{a^{-1}}{H^3}\dot{\zeta}\partial_i\partial_j\zeta\partial^i\partial^j\zeta = \partial_0\left[-\frac{a^{-1}}{H^3}\zeta\partial_i\partial_j\zeta\partial^i\partial^j\zeta\right] - \frac{a^{-1}}{H^2}\zeta\partial_i\partial_j\zeta\partial^i\partial^j\zeta - 3\frac{a^{-1}}{H^4}\dot{H}\zeta\partial_i\partial_j\zeta\partial^i\partial^j\zeta + 2\frac{a^{-1}}{H^3}\zeta\partial_i\partial_j\dot{\zeta}\partial^i\partial^j\zeta, \tag{B.3}$$

and

$$\frac{a^{-1}}{H^3}\dot{\zeta}\left(\partial_i\partial^i\zeta\right)^2 = \partial_0\left[\frac{a^{-1}}{H^3}\zeta\left(\partial_i\partial^i\zeta\right)^2\right] + \frac{a^{-1}}{H^2}\zeta\left(\partial_i\partial^i\zeta\right)^2 + 3\frac{a^{-1}}{H^4}\dot{H}\zeta\left(\partial_i\partial^i\zeta\right)^2 - 2\frac{a^{-1}}{H^3}\zeta\left(\partial_i\partial^i\zeta\right)\left(\partial_j\partial^j\dot{\zeta}\right). \tag{B.4}$$

By calling A their sum, we have

$$
\begin{aligned}
A = & -\frac{a^{-1}}{H^2}\zeta\left[\partial_i\partial_j\zeta\partial^i\partial^j\zeta-\left(\partial_i\partial^i\zeta\right)^2\right]+3\frac{a^{-1}}{H^2}\epsilon\zeta\left[\partial_i\partial_j\zeta\partial^i\partial^j\zeta-\left(\partial_i\partial^i\zeta\right)^2\right]+\\
&+\partial_0\left\{\frac{a^{-1}}{H^3}\zeta\left[\left(\partial_i\partial^i\zeta\right)^2-\partial_i\partial_j\zeta\partial^i\partial^j\zeta\right]\right\}+\partial_i\left\{\dot{\zeta}\left[\partial^i\dot{\zeta}\partial_j\partial^j\zeta-\partial^j\dot{\zeta}\partial_j\partial^i\zeta\right]\right\}+C,
\end{aligned}
\tag{B.5}
$$

with

$$
C=-2\frac{a^{-1}}{H^3}\left(\partial_i\dot{\zeta}\right)\left[\partial_j\partial^i\zeta\partial^j\zeta-\partial_j\partial^j\zeta\partial^i\zeta\right].
\tag{B.6}
$$

After some basic algebraic simplifications, A can be written as

$$
\begin{aligned}
A = & -\frac{a^{-1}}{H^3}\dot{\zeta}\partial_i\partial_j\zeta\partial^i\partial^j\zeta+\frac{a^{-1}}{H^3}\dot{\zeta}\left(\partial_i\partial^i\zeta\right)^2=-\frac{1}{3}\frac{a^{-1}}{H^2}\zeta\left[\partial_i\partial_j\zeta\partial^i\partial^j\zeta-\left(\partial_i\partial^i\zeta\right)^2\right]+\frac{a^{-1}}{H^2}\epsilon\zeta\left[\partial_i\partial_j\zeta\partial^i\partial^j\zeta-\left(\partial_i\partial^i\zeta\right)^2\right]+\\
&+\frac{1}{3}\partial_0\left\{\frac{a^{-1}}{H^3}\zeta\left[\left(\partial_i\partial^i\zeta\right)^2-\partial_i\partial_j\zeta\partial^i\partial^j\zeta\right]\right\}+\frac{2}{3}\partial_i\left\{\frac{a^{-1}}{H^3}\dot{\zeta}\left[\partial^i\zeta\partial_j\partial^j\zeta-\partial^j\zeta\partial_j\partial^i\zeta\right]\right\}+\\
&+\frac{a^{-1}}{H^2}\left[\dot{H}-\frac{\dot{\phi}^2}{2}\right]\zeta\left[\left(\partial_i\partial^i\zeta\right)^2-\partial_i\partial_j\zeta\partial^i\partial^j\zeta\right],
\end{aligned}
\tag{B.7}
$$

where, by restoring the Planck mass $M_{Pl}^{-2}=1$ inside the first brackets of the last line, we have the slow-roll condition $\dot{H}=M_{Pl}^{-2}\frac{\dot{\phi}}{2}$. The first term of the latter, when summed to all the other a^{-1}-terms, gives us a new set of total derivatives, which can be found, again, by applying the Leibniz rule. At the very end, the part of the action proportional to a^{-1} can be written as

$$
S=\int d^4x\frac{a^{-1}}{H^2}\epsilon\left[\partial_i\zeta\partial^i\zeta\partial_j\partial^j\zeta-\zeta\partial_i\partial_j\left(\partial^i\zeta\partial^j\zeta\right)\right]+N.
\tag{B.8}
$$

where N includes all the non-dynamical terms. For all the other contributions the procedure is identical to the one sketched above, with just a few differences for the $\propto a$ set of terms.

Appendix C

Figures

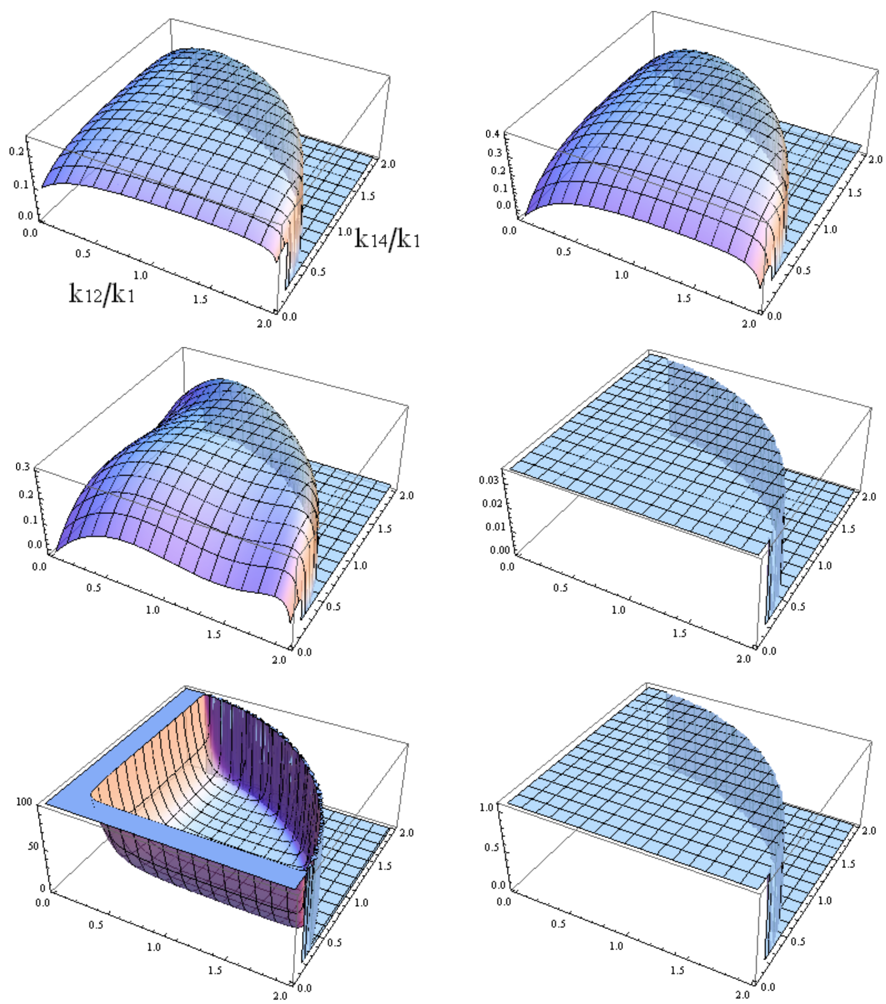


Figure C.1: Equilateral limit: the four external momenta are considered to be equal. The dominant T_{s1},T_{s2},T_{s3},T_{c1} and the local T_{loc1}, T_{loc2} shapes are plotted as functions of k_{12}/k_1, k_{14}/k_1

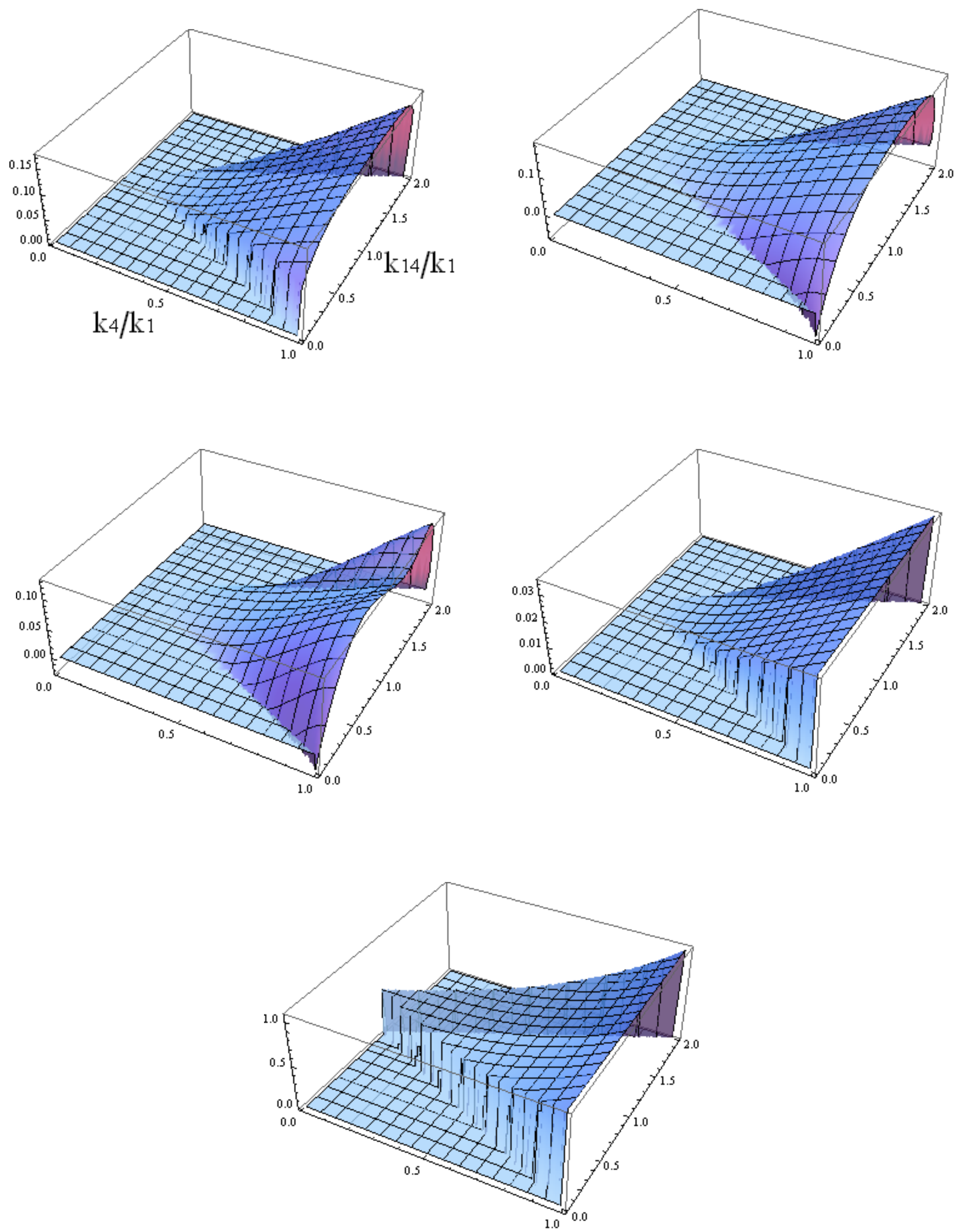


Figure C.2: Folded limit: k_{12} is considered to be vanishing. The dominant T_{s1},T_{s2},T_{s3},T_{c1} and the local T_{loc2} shapes are plotted as functions of k_{14}/k_1, k_4/k_1

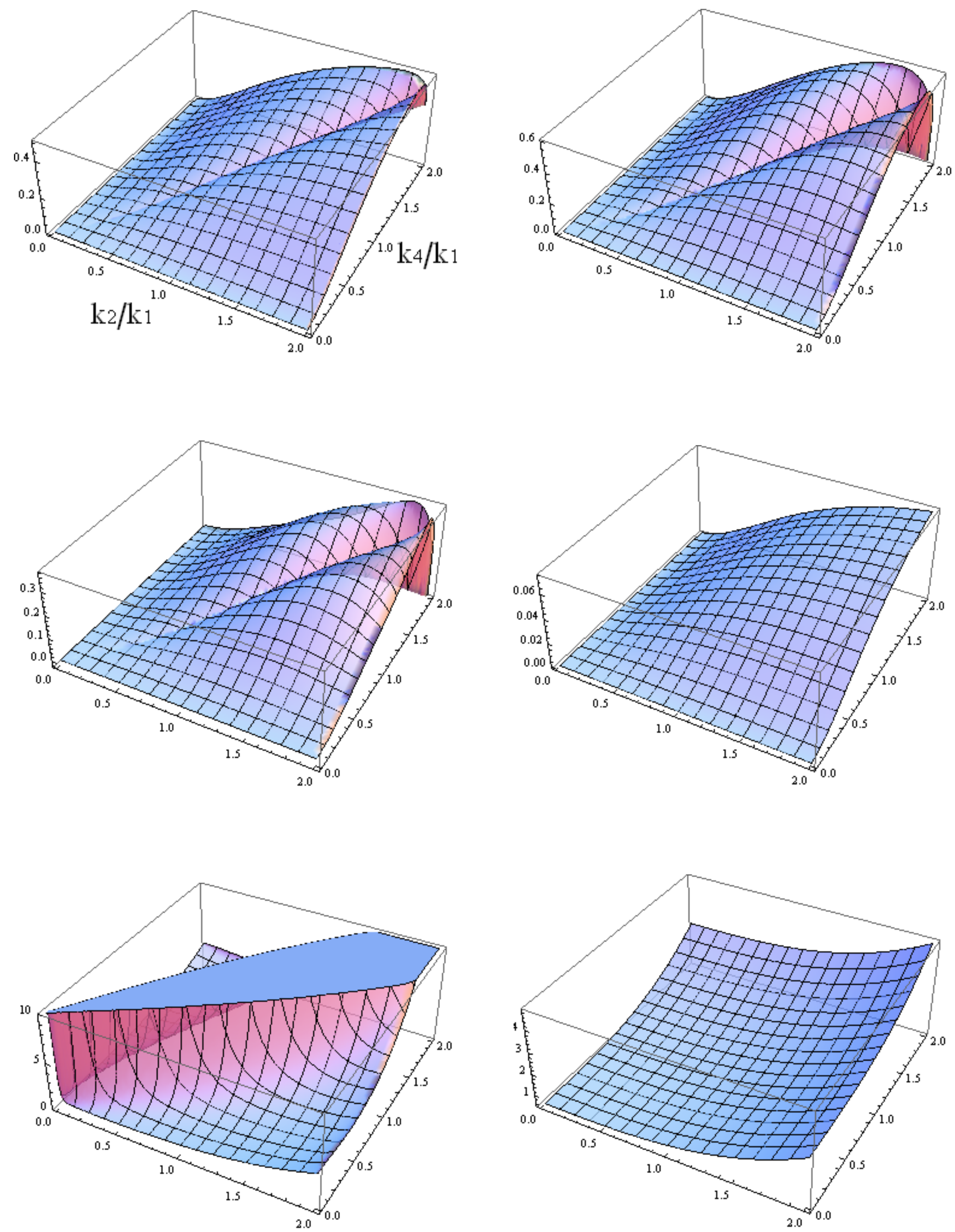


Figure C.3: Specialized planar limit: the external momenta are considered to be $k_1 = k_3 = k_{14}$. The dominant T_{s1},T_{s2},T_{s3},T_{c1} and the local T_{loc1}, T_{loc2} shapes are plotted as functions of k_2/k_1, k_4/k_1

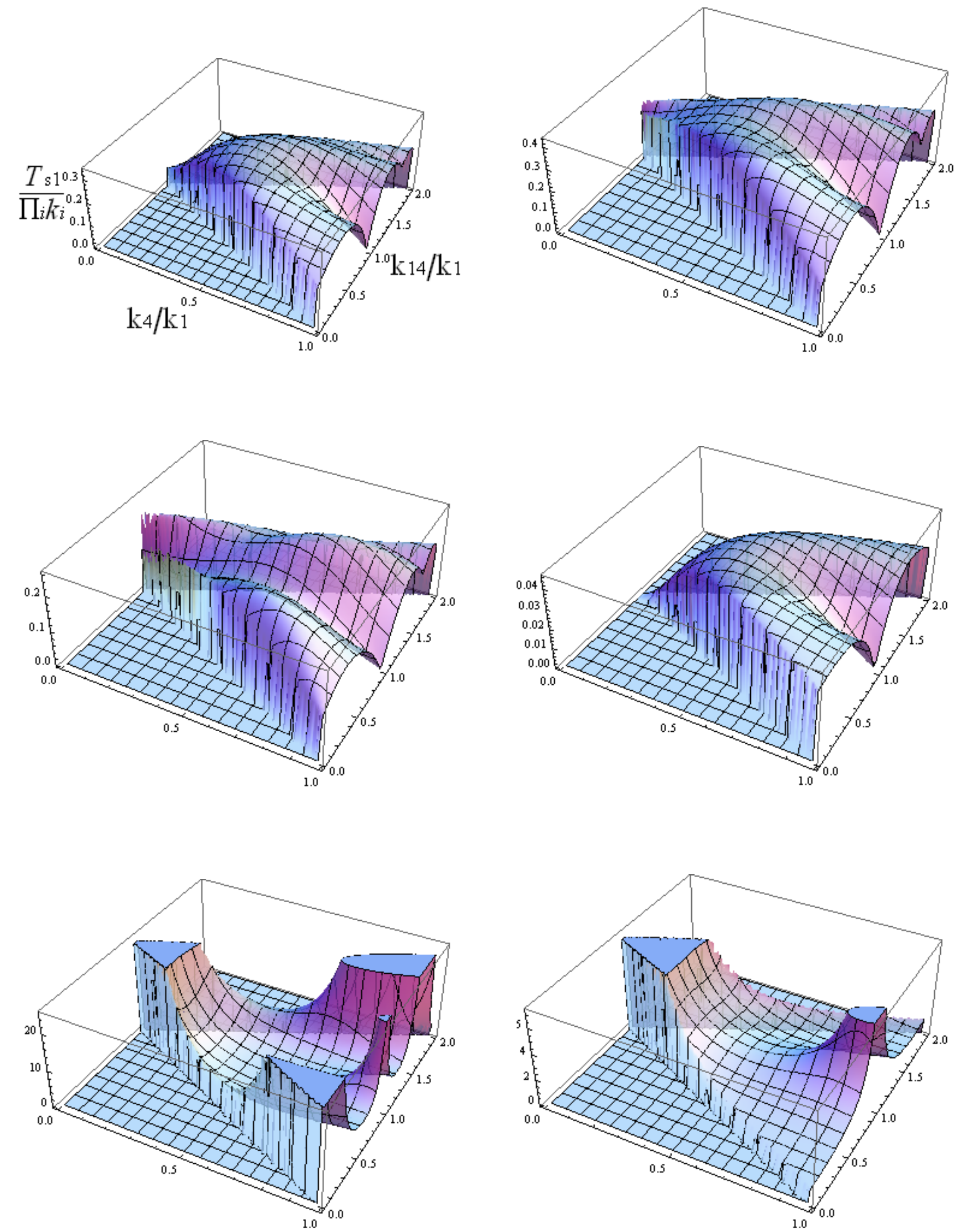


Figure C.4: Almost double squeezed limit: the external momenta are considered to be $k_3 = k_4 = k_{12}$. The dominant T_{s1},T_{s2},T_{s3},T_{c1} and the local T_{loc1}, T_{loc2} shapes, once divided by $\prod_{n=1}^{4} k_n$, are plotted as functions of k_2/k_1, k_4/k_1

Acknowledgments

I want to thank Professor Nicola Bartolo and Dr. Alessandro Greco for the availability they showed to me during this work. Their help was crucial for me to understand the technical details of the adopted formalism. Also, different useful discussions with them helped me to better comprehend several topics of modern Cosmology.
Then, firstly I want to thank my parents, for both the precious possibility they gave me to study, and their strong support over these years. Secondly, I have to thank my closest friends Filippo, Alberto, and Luca, for all the days we spent studying and discussing. I want to thank also my girlfriend Vittoria for the great support, and Bianca, Francesco, and Davide for the days off spent hanging on rocks. Also, I thank the University mates I met during this part of my life. However, a special thanks goes to each person that somehow helped me during my University path, a part of this goal belongs also to them.

Bibliography

[1] A. H. Guth, "Inflationary universe: A possible solution to the horizon and flatness problems," *Phys. Rev. D*, vol. 23, pp. 347–356, Jan 1981.

[2] M. Shiraishi, "Parity violation in the cmb trispectrum from the scalar sector," *Physical Review D*, vol. 94, Oct 2016.

[3] T. Liu, X. Tong, Y. Wang, and Z.-Z. Xianyu, "Probing p and cp violations on the cosmological collider," *Journal of High Energy Physics*, vol. 2020, Apr 2020.

[4] A. Lue, L. Wang, and M. Kamionkowski, "Cosmological signature of new parity-violating interactions," *Physical Review Letters*, vol. 83, pp. 1506–1509, aug 1999.

[5] S. Alexander and J. Martin, "Birefringent gravitational waves and the consistency check of inflation," *Physical Review D*, vol. 71, Mar 2005.

[6] S. Weinberg, "Effective field theory for inflation," *Physical Review D*, vol. 77, Jun 2008.

[7] M. Satoh, "Slow-roll inflation with the gauss-bonnet and chern-simons corrections," *Journal of Cosmology and Astroparticle Physics*, vol. 2010, p. 024–024, Nov 2010.

[8] N. Bartolo and G. Orlando, "Parity breaking signatures from a chern-simons coupling during inflation: the case of non-gaussian gravitational waves," *Journal of Cosmology and Astroparticle Physics*, vol. 2017, p. 034–034, Jul 2017.

[9] N. Bartolo, L. Caloni, G. Orlando, and A. Ricciardone, "Tensor non-Gaussianity in chiral scalar-tensor theories of gravity," *JCAP*, vol. 03, p. 073, 2021.

[10] N. Bartolo, G. Orlando, and M. Shiraishi, "Measuring chiral gravitational waves in chern-simons gravity with cmb bispectra," *Journal of Cosmology and Astroparticle Physics*, vol. 2019, p. 050–050, Jan 2019.

[11] Y. Akrami *et al.*, "Planck 2018 results. IX. Constraints on primordial non-Gaussianity," *Astron. Astrophys.*, vol. 641, p. A9, 2020.

[12] E. Castorina, N. Hand, U. Seljak, F. Beutler, C.-H. Chuang, C. Zhao, H. Gil-Marín, W. J. Percival, A. J. Ross, P. D. Choi, K. Dawson, A. de la Macorra, G. Rossi, R. Ruggeri, D. Schneider, and G.-B. Zhao, "Redshift-weighted constraints on primordial non-gaussianity from the clustering of the eBOSS DR14 quasars in fourier space," *Journal of Cosmology and Astroparticle Physics*, vol. 2019, pp. 010–010, sep 2019.

[13] D. J. Fixsen, "The Temperature of the Cosmic Microwave Background," *The Astrophysical Journal*, vol. 707, pp. 916–920, nov 2009.

[14] N. Aghanim, Y. Akrami, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini, A. J. Banday, R. B. Barreiro, N. Bartolo, and et al., "Planck 2018 results," *Astronomy & Astrophysics*, vol. 641, p. A6, Sep 2020.

[15] R. Goswami and U. A. Yajnik, "Reheating constraints to modulus mass for single field inflationary models," *Nuclear Physics B*, vol. 960, p. 115211, 2020.

[16] S. W. Hawking and W. Israel, *General relativity: an Einstein centenary survey.* Cambridge, reprint ed., 1979.

[17] A. H. Guth, "Inflationary universe: A possible solution to the horizon and flatness problems," *Phys. Rev. D*, vol. 23, pp. 347–356, Jan 1981.

[18] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects.* Cambridge Monographs on Mathematical Physics, 1994.

[19] S. Weinberg, "Gauge and global symmetries at high temperature," *Phys. Rev. D*, vol. 9, pp. 3357–3378, Jun 1974.

[20] V. Vennin, J. Martin, and C. Ringeval, "Cosmic inflation and model comparison," *Comptes Rendus Physique*, vol. 16, no. 10, pp. 960–968, 2015. Cosmic inflation / Inflation cosmique.

[21] A. H. Guth and S.-Y. Pi, "Fluctuations in the new inflationary universe," *Phys. Rev. Lett.*, vol. 49, pp. 1110–1113, Oct 1982.

[22] S. Weinberg, *Cosmology.* Cosmology, OUP Oxford, 2008.

[23] T. S. Bunch and P. C. W. Davies, "Quantum field theory in de Sitter space - Renormalization by point-splitting," *Proceedings of the Royal Society of London Series A*, vol. 360, pp. 117–134, Mar. 1978.

[24] J. M. Bardeen, "Gauge-invariant cosmological perturbations," *Phys. Rev. D*, vol. 22, pp. 1882–1905, Oct 1980.

[25] J. L. Friedman and N. Stergioulas, *Lie derivatives, forms, densities, and integration.* Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2013.

[26] A. Riotto, "Inflation and the theory of cosmological perturbations," 2017.

[27] N. Bartolo, E. Komatsu, S. Matarrese, and A. Riotto, "Non-gaussianity from inflation: theory and observations," *Physics Reports*, vol. 402, no. 3, pp. 103–266, 2004.

[28] H. Kodama and M. Sasaki, "Cosmological Perturbation Theory," *Progress of Theoretical Physics Supplement*, vol. 78, pp. 1–166, 01 1984.

[29] V. Acquaviva, N. Bartolo, S. Matarrese, and A. Riotto, "Gauge-invariant second-order perturbations and non-gaussianity from inflation," *Nuclear Physics B*, vol. 667, no. 1, pp. 119–148, 2003.

[30] V. F. Mukhanov, "Quantum theory of gauge-invariant cosmological perturbations," *Zh. Eksp. Teor. Fiz*, vol. 94, no. 1, 1988.

[31] Y. B. Zeldovich, "A Hypothesis, Unifying the Structure and the Entropy of the Universe," *Monthly Notices of the Royal Astronomical Society*, vol. 160, pp. 1P–3P, 10 1972.

[32] E. R. Harrison, "Fluctuations at the threshold of classical cosmology," *Phys. Rev. D*, vol. 1, pp. 2726–2730, May 1970.

[33] Planck Collaboration, "Planck 2018 results - x. constraints on inflation," *A&A*, vol. 641, p. A10, 2020.

[34] C. L. Bennett and et al, "First-year wilkinson microwave anisotropy probe (WMAP) observations: Preliminary maps and basic results," *The Astrophysical Journal Supplement Series*, vol. 148, pp. 1–27, sep 2003.

[35] S. Dodelson, *Modern cosmology.* San Diego, CA: Academic Press, 2003.

[36] E. Komatsu, N. Afshordi, N. Bartolo, and et al., "Non-gaussianity as a probe of the physics of the primordial universe and the astrophysics of the low redshift universe," 2009.

[37] D. S. Salopek and J. R. Bond, "Nonlinear evolution of long-wavelength metric fluctuations in inflationary models," *Phys. Rev. D*, vol. 42, pp. 3936–3962, Dec 1990.

[38] S. Weinberg, "Quantum contributions to cosmological correlations," *Phys. Rev. D*, vol. 72, p. 043514, Aug 2005.

[39] X. Chen, "Primordial non-gaussianities from inflation models," *Advances in Astronomy*, vol. 2010, p. 1–43, 2010.

[40] J. Schwinger, "Brownian motion of a quantum oscillator," *Journal of Mathematical Physics*, vol. 2, no. 3, pp. 407–432, 1961.

[41] L. V. Keldysh, "Diagram technique for nonequilibrium processes," *Sov. Phys. JETP*, vol. 20, p. 1018, 1965. [Zh. Eksp. Theor. Fiz. **47**, 1515 (1964)].

[42] M. Baumgart and R. Sundrum, "De Sitter Diagrammar and the Resummation of Time," *JHEP*, vol. 07, p. 119, 2020.

[43] M. Baumgart and R. Sundrum, "Manifestly Causal In-In Perturbation Theory about the Interacting Vacuum," *JHEP*, vol. 03, p. 080, 2021.

[44] G. C. Wick, "The evaluation of the collision matrix," *Phys. Rev.*, vol. 80, pp. 268–272, Oct 1950.

[45] F. Mandl and G. Shaw, *Quantum Field Theory.* A Wiley-Interscience publication, Wiley, 2010.

[46] M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory.* Boulder, CO: Westview, 1995. Includes exercises.

[47] R. Arnowitt, S. Deser, and C. W. Misner, "Dynamical Structure and Definition of Energy in General Relativity," *Physical Review*, vol. 116, pp. 1322–1330, Dec. 1959.

[48] R. Arnowitt, S. Deser, and C. W. Misner, "Republication of: The dynamics of general relativity," *General Relativity and Gravitation*, vol. 40, p. 1997–2027, Aug 2008.

[49] A. Palatini, "Deduzione invariantiva delle equazioni gravitazionali dal primo principio di hamilton," *Rendiconti del Circolo Matematico di Palermo (1884-1940)*, 1919.

[50] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation.* San Francisco: W. H. Freeman, 1973.

[51] A. Golovnev, "Adm analysis and massive gravity," 2013.

[52] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," *JHEP*, vol. 05, p. 013, 2003.

[53] H. Collins, "Primordial non-gaussianities from inflation," 2014.

[54] V. D. Ivashchuk, "Wick rotation, regularization of propagators by a complex metric and multidimensional cosmology," 1997.

[55] M. Visser, "How to wick rotate generic curved spacetime," 2017.

[56] N. Barnaby, R. Namba, and M. Peloso, "Phenomenology of a pseudo-scalar inflaton: naturally large nongaussianity," *Journal of Cosmology and Astroparticle Physics*, vol. 2011, p. 009–009, Apr 2011.

[57] N. Bartolo, S. Matarrese, M. Peloso, and M. Shiraishi, "Parity-violating cmb correlators with non-decaying statistical anisotropy," *Journal of Cosmology and Astroparticle Physics*, vol. 2015, p. 039–039, Jul 2015.

[58] N. Barnaby, R. Namba, and M. Peloso, "Observable non-gaussianity from gauge field production in slow roll inflation, and a challenging connection with magnetogenesis," *Physical Review D*, vol. 85, Jun 2012.

[59] N. Bartolo, S. Matarrese, M. Peloso, and A. Ricciardone, "Anisotropic power spectrum and bispectrum in the f(phi)f2 mechanism," *Physical Review D*, vol. 87, Jan 2013.

[60] N. Bartolo, S. Matarrese, M. Peloso, and M. Shiraishi, "Parity-violating and anisotropic correlations in pseudoscalar inflation," *Journal of Cosmology and Astroparticle Physics*, vol. 2015, p. 027–027, Jan 2015.

[61] D. N. Spergel, R. Bean, O. Doré, M. R. Nolta, C. L. Bennett, J. Dunkley, G. Hinshaw, N. Jarosik, E. Komatsu, L. Page, H. V. Peiris, L. Verde, M. Halpern, R. S. Hill, A. Kogut, M. Limon, S. S. Meyer, N. Odegard, G. S.

Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, "Three-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Implications for Cosmology," *apjs*, vol. 170, pp. 377–408, June 2007.

[62] J. Z. Simon, "Higher-derivative lagrangians, nonlocality, problems, and solutions," *Phys. Rev. D*, vol. 41, pp. 3720–3733, Jun 1990.

[63] M. Iihoshi, S. V. Ketov, and A. Morishita, "Conformally flat FRW metrics," *Progress of Theoretical Physics*, vol. 118, pp. 475–489, sep 2007.

[64] R. Jackiw, "S. s. chern and chern-simons terms," 2005.

[65] B. Himmetoglu, C. R. Contaldi, and M. Peloso, "Ghost instabilities of cosmological models with vector fields nonminimally coupled to the curvature," *Phys. Rev. D*, vol. 80, p. 123530, 2009.

[66] "*NIST Digital Library of Mathematical Functions*." http://dlmf.nist.gov/, Release 1.1.4 of 2022-01-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[67] R. Jackiw and S.-Y. Pi, "Chern-simons modification of general relativity," *Physical Review D*, vol. 68, Nov 2003.

[68] D. Grumiller and N. Yunes, "How do black holes spin in chern-simons modified gravity?," *Physical Review D*, vol. 77, Feb 2008.

[69] M. Shiraishi, "Probing the early universe with the cmb scalar, vector and tensor bispectrum," *Springer Theses*, 2013.

[70] E. Masina, "Useful review on the exponential-integral special function," 2019.

[71] M. Liguori, E. Sefusatti, J. R. Fergusson, and E. Shellard, "Primordial non-gaussianity and bispectrum measurements in the cosmic microwave background and large-scale structure," *Advances in Astronomy*, vol. 2010, 2010.

[72] X. Chen, B. Hu, M.-x. Huang, G. Shiu, and Y. Wang, "Large primordial trispectra in general single field inflation," *Journal of Cosmology and Astroparticle Physics*, vol. 2009, p. 008–008, Aug 2009.

[73] M.-x. Huang and G. Shiu, "Inflationary trispectrum for models with large non-gaussianities," *Physical Review D*, vol. 74, Dec 2006.

[74] N. Kogo and E. Komatsu, "Angular trispectrum of cmb temperature anisotropy from primordial non-gaussianity with the full radiation transfer function," *Physical Review D*, vol. 73, Apr 2006.

[75] T. Okamoto and W. Hu, "Angular trispectra of cmb temperature and polarization," *Physical Review D*, vol. 66, Sep 2002.

[76] E. Silverstein and D. Tong, "Scalar speed limits and cosmology: Acceleration from d-cceleration," *Physical Review D*, vol. 70, Nov 2004.

[77] M. Alishahiha, E. Silverstein, and D. Tong, "Dbi in the sky: Non-gaussianity from inflation with a speed limit," *Physical Review D*, vol. 70, Dec 2004.

[78] C. Armendáriz-Picón, T. Damour, and V. Mukhanov, "k-inflation," *Physics Letters B*, vol. 458, p. 209–218, Jul 1999.

[79] A. Slosar, C. Hirata, U. Seljak, S. Ho, and N. Padmanabhan, "Constraints on local primordial non-gaussianity from large scale structure," *Journal of Cosmology and Astroparticle Physics*, vol. 2008, p. 031, Aug 2008.

[80] L. Verde, "Non-gaussianity from large-scale structure surveys," *Advances in Astronomy*, vol. 2010, p. 1–15, 2010.

[81] C. Wagner and L. Verde, "N-body simulations with generic non-gaussian initial conditions II: halo bias," *Journal of Cosmology and Astroparticle Physics*, vol. 2012, pp. 002–002, mar 2012.

[82] M. Kamionkowski, L. Verde, and R. Jimenez, "The void abundance with non-gaussian primordial perturbations," *Journal of Cosmology and Astroparticle Physics*, vol. 2009, pp. 010–010, jan 2009.

[83] N. Dalal, O. Doré, D. Huterer, and A. Shirokov, "Imprints of primordial non-gaussianities on large-scale structure: Scale-dependent bias and abundance of virialized objects," *Physical Review D*, vol. 77, Jun 2008.

[84] A. Lue, L. Wang, and M. Kamionkowski, "Cosmological signature of new parity-violating interactions," *Phys. Rev. Lett.*, vol. 83, pp. 1506–1509, Aug 1999.

[85] R. N. Cahn, Z. Slepian, and J. Hou, "A Test for Cosmological Parity Violation Using the 3D Distribution of Galaxies," 10 2021.

[86] P. A. R. Ade, N. Aghanim, C. Armitage-Caplan, M. Arnaud, M. Ashdown, F. Atrio-Barandela, J. Aumont, C. Baccigalupi, A. J. Banday, R. B. Barreiro, J. G. Bartlett, N. Bartolo, and et al, "Planck 2013 results. XXIV. constraints on primordial non-gaussianity," *Astronomy & Astrophysics*, vol. 571, p. A24, oct 2014.

[87] J.-P. Dai and J.-Q. Xia, "Constraints on primordial non-gaussianity using the multitracer technique for skew spectra," *The Astrophysical Journal*, vol. 905, p. 127, Dec 2020.

[88] N. Aghanim, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini, A. J. Banday, R. B. Barreiro, N. Bartolo, and et al, "Planck intermediate results," *Astronomy & Astrophysics*, vol. 596, p. A110, dec 2016.

[89] Y. Minami and E. Komatsu, "New extraction of the cosmic birefringence from the planck 2018 polarization data," *Physical Review Letters*, vol. 125, nov 2020.

[90] N. Bartolo, C. Caprini, V. Domcke, D. G. Figueroa, J. Garcia-Bellido, M. C. Guzzetti, M. Liguori, S. Matarrese, M. Peloso, A. Petiteau, A. Ricciardone, M. Sakellariadou, L. Sorbo, and G. Tasinato, "Science with the space-based interferometer LISA. IV: probing inflation with gravitational waves," *Journal of Cosmology and Astroparticle Physics*, vol. 2016, pp. 026–026, dec 2016.