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Dynamic Control Lyapunov Functions

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Abstract

This thesis studies the stabilization problem for nonlinear control systems. While for linear systems it has been successfully solved, a complete and satisfactory theory for nonlinear systems is not yet available. The aim of this work is to propose a new solution to the stabilization problem for the input affine control systems, the linearization of which around the origin is stabilizable.

The proposed procedure is strictly related to one of the main tools in stabilization theory: the notion of *Control Lyapunov function* (CLF). This concept was firstly introduced by Artstein in [3] and its importance is related to the fact that from the knowledge of a CLF it is possible to derive a stabilizing control law. The main shortcoming of this procedure however is that to construct a CLF a constrained differential inequality has to be solved.

The goal of the thesis is to introduce the new notion of *Dynamic Control Lyapunov function*. This is a CLF for an extended system, the additional dynamics of which are driven by a new controller. The main advantage of this approach is that the additional controller can be exploited to enforce the negativity of the time derivative of the Dynamic CLF along the trajectories of the closed-loop system, hence the origin is dynamically asymptotically stabilizated.

The motivation for introducing this new concept is that, as stated before, to construct a standard CLF a constrained differential inequality has to be solved. On the other hand, in this thesis it is shown that, for the class of nonlinear systems considered, it is possible to construct a Dynamic CLF in a simple way. Moreover it is shown that from the proposed Dynamic CLF it is possible to obtain not only a dynamic stabilizing feedback law but also a static one. Finally, the problem of deriving a standard CLF starting from a Dynamic CLF is also considered. This requires the solution of a differential inequality which, in general, is simpler than the constrained differential inequality characterizing the CLF.

The thesis is organised as follows. In **Chapter 1**, after a brief survey of the main concepts of stability and stabilization theory, the new concepts of *Dynamic Control Lyapunov functions* and *Algebraic $\bar{P}$ solution* are introduced. Moreover a class of candidate Dynamic CLFs, parametrized by the matrix $R = R^\top > 0$, is proposed. The aim of **Chapter 2** is to prove that there exist values of R such that the proposed functions are Dynamic CLFs. This result is obtained in a constructive way, i.e. deriving a nonlinear control law that statically asymptotically stabilizes the origin of the extended system. **Chapter 3** addresses the same problem from a geometric perspective yielding a sufficient condition on

the minimum singular value of R guaranteeing that the proposed functions are Dynamic CLFs. In **Chapter 4** the problem of deriving a standard CLF from the knowledge of a Dynamic CLF is studied. **Chapter 5** is devoted to applications to a few 2−dimensional and 3−dimensional examples. Finally the new concept of *weak algebraic* $\bar{P}$ *solution* is introduced and it is argued that this concept is useful to study input affine systems with non-stabilizable linearization.

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Contents

List of Symbols

$A^{\perp}$	Ortogonal matrix of A as defined in Definition 3.1.1
$A^{\perp\!\!\!\!-}$	Transposed of $A^{\perp}$
A^{*}	Complex conjugate transposed of A
$A(x)^{\perp}$	Ortogonal matrix of $A(x)$ as defined in Definition 3.2.1
$A(x)^{\perp\!\!\!\!-}$	Transposed of $A(x)^{\perp}$
$\bar{k}$	Positive scalar value as defined in Theorem 2.1.2 for linear systems and Theorem 2.2.3 for nonlinear systems
$\bar{l}$	Index of an algebraic $\bar{P}$ solution as defined in Definition 1.3.3
$\mathcal{O}(x)$	Continuous matrix-valued function such that $\mathcal{O}(0) = 0$
$\mathcal{O}(\|x\|^n)$	Continuous matrix-valued function such that $\lim_{x\to 0} \frac{\|\mathcal{O}(\|x\|^n)\|}{\|x\|^n} < \infty$
$\bar{P}$	Positive definite matrix
$p(x)$	Algebraic $\bar{P}$ solution as defined in Definition 1.3.2
Q	Positive definite matrix
R	Positive definite matrix
$V(x)$	(Candidate) Control Lyapunov function
$V(x,\xi)$	(Candidate) Dynamic Control Lyapunov function
$V_x(x)$	$\frac{\partial V(x)}{\partial x}$, also indicated with V_x
$V_x(x,\xi)$	$\frac{\partial V(x,\xi)}{\partial x}$, also indicated with V_x
$V_\xi(x,\xi)$	$\frac{\partial V(x,\xi)}{\partial \xi}$, also indicated with V_ξ
$\dot{V}(x,u)$	Time derivative of $V(x)$ along the trajectories of the control system $\dot{x} = f(x) + g(x)u$

$\dot{V}(x,\xi,u,w)$	Time derivative of $V(x,\xi)$ along the trajectories of the control system $\dot{x} = f(x) + g(x)u$, $\dot{\xi} = w$
$W(x)$	(Candidate) Lyapunov function
$\dot{W}(x)$	Time derivative of $W(x)$ along the trajectories of the autonomous system $\dot{x} = f(x)$
$\Gamma(x)$	Matrix-valued function that is a positive definite matrix for each x in a neighborhood of the origin (included)
Ω	Open set $\subset \mathbb{R}^{2n}$ and $0 \in \Omega$, unless otherwise stated
Ω_x	Open set $\subset \mathbb{R}^n$ and $0 \in \Omega_x$
$\lambda_i(A)$	i-th eigenvalue of A
$\bar{\lambda}(A)$	Maximum real eigenvalue of a symmetric matrix A
$\underline{\lambda}(A)$	Minimum real eigenvalue of a symmetric matrix A
$\sigma_i(A)$	i-th singular value of A
$\bar{\sigma}(A)$	Maximum singular value of A
$\underline{\sigma}(A)$	Minimum singular value of A

Chapter 1

Introduction

The objective of this chapter is to briefly survey the main concepts of stability and stabilizability theory used in this thesis. The chapter begins with some background definitions concerning autonomous nonlinear systems. In particular, the concepts of stability of equilibrium points and some basic notions of Lyapunov stability theory are provided. To extend these concepts to control system, in Section 1.2, the notion of Control Lyapunov function (CLF) is introduced and the main theorems about stabilizability theory are reviewed. Finally in Section 1.3 the new concept of Dynamic Control Lyapunov function is introduced and the connection between Dynamic CLF and standard CLF is analyzed. In Chapter 4, it is shown how, from the knowledge of a Dynamic CLF, in some particular cases, it is possible to obtain a standard CLF.

Note that all the main definitions and theorems mentioned in this chapter are taken from the books *Nonlinear Systems* by Khalil [10] and *Local Stabilizability of Nonlinear Control Systems* by Bacciotti [4] for autonomous systems and control systems, respectively. These books can be consulted for an in-depth exposition of stability and stabilizability theory.

1.1 Lyapunov Functions

Consider the autonomous nonlinear system

$$\dot{x} = f(x) \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Stability theory concerns the points x^* such that, if the initial state is $x(0) = x^*$, then $x(t) = x^*$ for all $t > 0$. These points are called equilibrium points.

Definition 1.1.1 (Equilibrium point). *A state $x^* \in \mathbb{R}^n$ is an equilibrium state or equilibrium point for system (1.1) if $x(0) = x^*$ implies $x(t) = x^*$ for all $t > 0$.*

Note that for system (1.1) the equilibrium points are the real roots of the equation $f(x) = 0$. The equilibrium points can be characterized in terms of their stability properties, as stated in the following section.

1.1.1 Stability properties

The simplest definition of stability of an equilibrium point is the Lyapunov stability.

Definition 1.1.2 (Lyapunov stability). *An equilibrium point x^* is Lyapunov stable if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\|x(0) - x^*\| < \delta \;\Rightarrow\; \|x(t) - x^*\| < \varepsilon, \quad \forall t \geq 0.$$

This means that an equilibrium point is Lyapunov stable when the trajectory $x(t)$ remains within a specified distance ε of x^*, whenever the initial point $x(0)$ is sufficiently near to x^*. An equilibrium, which is not stable, is said to be unstable. Another definition of stability is the following.

Definition 1.1.3 (Local asymptotic stability). *An equilibrium state x^* is locally asymptotically stable if it is stable and if δ can be chosen such that*

$$\|x(0) - x^*\| < \delta \;\Rightarrow\; \lim_{t\to\infty} x(t) = x^*.$$

When an equilibrium point is asymptotically stable, it is interesting to determine how large the constant δ can be taken in Definition 1.1.3. This leads to the definition of domain of attraction.

Definition 1.1.4 (Domain of attraction). *Suppose that x^* is an asymptotically stable equilibrium point. Then the set*

$$D(x^*) = \left\{ x(0) \in \mathbb{R}^n \mid \lim_{t\to\infty} x(t) = x^* \right\}$$

is called the domain of attraction of the equilibrium point x^*.

Note that determining analytically the exact domain of attraction might be difficult or even impossible. If the domain of attraction is all $\mathbb{R}^n$, then the equilibrium point is said to be globally asymptotically stable.

The next definition specifies an additional features of an asymptotically stable equilibrium.

Definition 1.1.5 (Exponential stability). *If x^* is a locally (globally) asymptotically stable equilibrium point and there exist two positive constants k and α such that*

$$\|x(t) - x^*\| < k\|x(0) - x^*\| e^{-\alpha t},$$

for each $x(0)$ in a neighborhood of x^* (for all $x(0) \in \mathbb{R}^n$) and each $t > 0$, then x^* is a locally (globally) exponentially stable equilibrium point.

1.1.2 Lyapunov functions

Lyapunov functions are a very powerful tool to investigate the stability properties of an equilibrium point.

Definition 1.1.6 (Lyapunov function). *Let x^* be an equilibrium point of system (1.1). The function $W(x) : \mathbb{R}^n \to \mathbb{R}$ is a local (global) Lyapunov function (LF) for system (1.1) at x^* if it is at least of class $\mathcal{C}^1$, (radially unbounded), and there exists Ω_x, neighborhood of x^*, ($\Omega_x = \mathbb{R}^n$), such that*

1. *$W(x)$ is positive definite in Ω_x and $W(x^*) = 0$;*

2. *$\dot{W}(x) = W_x(x)f(x)$ is negative definite in Ω_x.*

The importance of Lyapunov functions is clarified in the following theorem.

Theorem 1.1.1. *Let x^* be an equilibrium point of (1.1) and let $f(x)$ be continuous. If there is a Lyapunov function for system (1.1) at x^* then x^* is locally asymptotically stable.*

The proof of this theorem was given by Lyapunov in [14]. In other words Theorem 1.1.1 says that the presence of a Lyapunov function is a sufficient condition to guarantee the local asymptotic stability of an equilibrium point. It is immediate to wonder if this is also a necessary condition. The affirmative answer to this question is due to a class of theorems, known as Converse Lyapunov Theorems, [13, 15]. For brevity, it is hereby reported only the version due to J. Kurzweil, [13], which is one of the most general.

Theorem 1.1.2. *Consider system (1.1). Suppose that $f(x)$ is continuous and that x^* is a locally asymptotically stable equilibrium point. Then, there exists a local Lyapunov function $W(x)$ for system (1.1) at x^*, which is $\mathcal{C}^\infty$ in a neighborhood of x^*.*

1.2 Control Lyapunov Functions

The concept of Lyapunov function has been developed for autonomous systems, however it can be useful also for control systems design. In fact, candidate Lyapunov functions can be used in the design of feedback laws by choosing the control law to make the Lyapunov derivative negative on the trajectories of the closed-loop system, [9, 11, 8]. This idea leads to the definition of "Control Lyapunov function" (CLF), [3, 18]. Before describing this concept, however, it is important to recall the notion of stabilizability.

1.2.1 Stabilizability properties

Consider a nonlinear system described by an equations of the form

$$\dot{x} = f(x, u), \tag{1.2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $f : \mathbb{R}^{n \times m} \to \mathbb{R}^n$ is at least continuously differentiable and $f(0, 0) = 0$. The local stabilizability problem can be stated, in a very intuitive way, as the problem of finding a static feedback control law $u = u(x)$, defined in a neighborhood of the origin, such that the origin of the closed-loop system

$$\dot{x} = f(x, u(x))$$

is a locally asymptotically stable equilibrium point. If such a function $u(x)$ exists, the origin of system (1.2) is said to be *statically asymptotically stabilizable*. This formulation of the stabilization problem is very general and intuitive, but it is not precise. Indeed the regularity properties requested on the control law u have to be specified. To this end it is useful to introduce the following terminology [4].

Definition 1.2.1 (Local almost continuous stabilizability). *The origin of system (1.2) is locally almost continuously stabilizable if there exists a feedback control law $u = u(x)$ which is continuous in a punctured neighborhood of the origin (i.e. everywhere in the neighborhood except at $x = 0$) and such that the closed-loop system has a locally asymptotically stable equilibrium at the origin.*

Definition 1.2.2 (Local continuous stabilizability). *The origin of system (1.2) is locally continuously stabilizable if there exists a feedback control law $u = u(x)$ which is continuous in a neighborhood of the origin and such that the closed-loop system has a locally asymptotically stable equilibrium at the origin.*

Definition 1.2.3 (Local exponential stabilizability). *The origin of system (1.2) is locally exponentially stabilizable if there exists a feedback control law $u = u(x)$ which is continuous in a neighborhood of the origin and such that the closed-loop system has a locally exponentially stable equilibrium at the origin.*

Definition 1.2.4 (Local linear stabilizability). *The origin of system (1.2) is locally linearly stabilizable if the origin can be locally asymptotically stabilized by means of a linear control law $u = Kx$, where $K \in \mathbb{R}^{m\times n}$.*

Note that all the above definitions are equivalent if system (1.2) is linear. Therefore, in the linear case, the general term stabilizable can be used without further specification.

To study the stabilizability properties of a nonlinear system at the origin, it is usually convenient to resort to its linearization around the origin, i.e. the linear system

$$\dot{x} = Ax + Bu, \tag{1.3}$$

where $A = \frac{\partial f}{\partial x}(0,0)$ and $B = \frac{\partial f}{\partial u}(0,0)$. In fact a very important result on the stabilizability of a nonlinear system at the origin is the following [4].

Theorem 1.2.1. *If the linearized system (1.3) is stabilizable, then any feedback law $u = Kx$ which stabilizes the origin of (1.3) locally linearly exponentially stabilizes the origin of system (1.2).*

Finally another notion that it is important to recall is an extension of the stabilizability concept to the dynamic context.

Definition 1.2.5 (Dynamical stabilizability). *The system (1.2) is dynamically stabilizable at $x = 0$ if there exist an integer η and a function $w : \Omega_x \times \mathbb{R}^\eta \to \mathbb{R}^\eta$, $w(0,0) = 0$, such that the extended control system*

$$\begin{aligned} \dot{x} &= f(x,u), \\ \dot{\xi} &= w(x,\xi), \end{aligned} \tag{1.4}$$

is stabilizable at the origin by means of a feedback of the form $u = u(x,\xi)$.

Also for this concept a classification similar to the one given in Definitions 1.2.1, 1.2.2 and 1.2.3, based on the regularity of the control laws u and w, can be given.

Remark 1.1. In the following, if not specified, all the stability and stabilizability properties are local, i.e. the term asymptotically stabilizable means locally asymptotically stabilizable.

1.2.2 Control Lyapunov functions

To recall the concept of control Lyapunov functions consider system (1.2) and assume that the origin is continuously stabilizable. In other words suppose that there exists a continuous control law $\bar{u}(x)$ such that the equilibrium $x = 0$ of the closed-loop system

$$\dot{x} = f(x, \bar{u}(x)) \tag{1.5}$$

is asymptotically stable. According to the converse Lyapunov Theorem[1], system (1.5) possesses a local Lyapunov function, $V(x)$, i.e. a positive definite function (on Ω) whose time derivative along the trajectories of the closed-loop system satisfies

$$\dot{V}(x) = V_x(x) f(x, \bar{u}(x)) < 0 \text{ for all } x \in \Omega \backslash \{0\}.$$

Note that this condition, in particular, guarantees that

$$\inf_{u \in \mathbb{R}^m} [V_x(x) f(x, u)] < 0 \text{ for all } x \in \Omega \backslash \{0\}. \tag{1.6}$$

A positive definite function $V(x)$ that satisfies property (1.6) is said to be a control Lyapunov function for system (1.2).

Definition 1.2.6 (Control Lyapunov Function). *A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is a local (global) Control Lyapunov Function (CLF) for the system $\dot{x} = f(x, u)$, where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, if it is at least of class $\mathcal{C}^1$, (radially unbounded) and there exists an open set $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, ($\Omega = \mathbb{R}^n$), such that*

- ***Property 1:*** *$V(x) > 0$ for all $x \in \Omega \backslash \{0\}$, $V(0) = 0$;*
- ***Property 2:*** *$\inf_u [V_x(x) f(x, u)] < 0$ for all $x \in \Omega \backslash \{0\}$.*

With this terminology, the previous discussion can be restated by saying that if a system is continuously stabilizable at the origin then there exists a CLF. A more interesting question is if the converse statement holds. In other words, does the existence of a CLF imply the existence of some kind of stabilizing feedback law?

To give an answer to this question, we restrict our attention to systems affine in the control

$$\dot{x} = f(x) + g(x)u, \tag{1.7}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are at least continuously differentiable and $f(0) = 0$. For these systems the definition of CLF can be restated as follows.

[1] See Theorem 1.1.2.

Definition 1.2.7 (Control Lyapunov Function). *A function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is a local (global) Control Lyapunov Function (CLF) for the affine system (1.7), if it is at least of class $\mathcal{C}^1$, (radially unbounded) and there exists an open set $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, ($\Omega = \mathbb{R}^n$) such that:*

- ***Property 1:*** $V(x) > 0$ *for all* $x \in \Omega \backslash \{0\}$, $V(0) = 0$;
- ***Property 2:*** $\inf_u [V_x f(x) + V_x g(x) u] < 0$ *for all* $x \in \Omega \backslash \{0\}$.

As a matter of fact, for an affine system the second condition can be rewritten in a simpler way

Lemma 1.2.1. *Consider the affine system (1.7) the following properties are equivalent*

- ***Property 2:*** $\inf_u [V_x f(x) + V_x g(x) u] < 0$ *for all* $x \in \Omega \backslash \{0\}$;
- ***Property 2':*** $V_x g(x) = 0 \Rightarrow V_x f(x) < 0$ *for all* $x \in \Omega \backslash \{0\}$.

Proof. $2 \Rightarrow 2'$ The proof is straightforward indeed, for all $x \in \Omega \backslash \{0\}$

$$\inf_u \dot{V}(x,u) < 0 \;\Rightarrow\; \inf_u [V_x f(x) + V_x g(x) u] < 0 \tag{1.8}$$

In the particular case when $V_x g(x) = 0$ then (1.8) becomes

$$\inf_u [V_x f(x)] < 0 \;\Rightarrow\; V_x f(x) < 0$$

hence Property 2' holds.

$2' \Rightarrow 2$ Consider the scalar case, i.e. $m = 1$. For all $x \in \Omega \backslash \{0\}$

- if $V_x g(x) = 0$ then, by 2', $V_x f(x) < 0$ and

$$\inf_u \dot{V}(x,u) = \inf_u [V_x f(x)] = V_x f(x) < 0$$

- if $V_x g(x) \neq 0$ then define

$$\bar{u}(x) = -\frac{V_x f(x)}{V_x g(x)} - V_x g(x).$$

For each fixed value of x this is a well-defined scalar quantity, since $V_x g(x) \neq 0$, and

$$\dot{V}(x, \bar{u}(x)) = V_x f(x) - V_x g(x) \left[\frac{V_x f(x)}{V_x g(x)} + V_x g(x)\right] = -(V_x g(x))^2 < 0$$

hence

$$\inf_u \dot{V}(x,u) \leqslant \dot{V}(x, \bar{u}(x)) < 0.$$

The general case, $m > 1$, is similar to the scalar case.

- If $V_x g(x) = 0$ then, by 2', $V_x f(x) < 0$ and

$$\inf_u \dot{V}(x,u) = \inf_u [V_x f(x)] = V_x f(x) < 0.$$

- If $V_x g(x) \neq 0$ then there exists at least one index i, $1 \leqslant i \leqslant m$, such that $V_x g_i(x) \neq 0$. Then let

$$\begin{array}{rcll} \bar{u}_k &=& 0, & \text{for } k \neq i, \\ \bar{u}_i &=& -\frac{V_x f(x)}{V_x g_i(x)} - V_x g_i(x), & \text{otherwise.} \end{array}$$

Note that, for each fixed value of x, $\bar{u}(x)$ is a well-defined vector, since $V_x g_i(x) \neq 0$, and

$$\dot{V}(x,\bar{u}(x)) = V_x f(x) + \sum_k V_x g_k(x)\bar{u}_k = V_x f(x) - V_x g_i(x)\left[\frac{V_x f(x)}{V_x g_i(x)} + V_x g_i(x)\right] =$$
$$= -(V_x g_i(x))^2 < 0$$

hence

$$\inf_u \dot{V}(x,u) \leqslant \dot{V}(x,\bar{u}(x)) < 0.$$

□

In the previous proof, for each value of x such that $V_x g(x) \neq 0$, a control value $\bar{u}(x)$ has been generated to guarantee the negativity of $\dot{V}(x,\bar{u}(x))$. Note that if, on the other hand, $V_x g(x) = 0$, then $\dot{V}(x,\bar{u}(x)) = V_x f(x)$ does not depend on the value of the control law u, therefore the simplest choice is to define $\bar{u}(x) = 0$. The function $\bar{u}(x)$ stabilizes the origin in Ω, however in general it is not a smooth function, indeed it could be not even continuous. One can wonder if this is the best stabilizer that can be achieved or if the existence of a CLF implies, at least for an affine system, the existence of a smooth stabilizing feedback law.

The first result in this direction is due to Artstein. In [3] he showed that the existence of a CLF is a sufficient condition for local almost continuous stabilizability of the origin.

Theorem 1.2.2. *Consider the affine control system (1.7). If the CLF Property 2' holds, then the origin of system (1.7) is almost continuously stabilizable.*

On the other hand Sontag showed that if also continuity at the origin is required, then the CLF must satisfy an additional property, known as Small Control Property.

Definition 1.2.8 (Small Control Property)**.** *A CLF satisfies the Small Control Property (SCP) if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $x \in \Omega\backslash\{0\}$ satisfies $\|x\| < \delta$, then there exists some u, with $\|u\| < \varepsilon$, such that*

$$V_x f(x) + V_x(x) g(x) u < 0.$$

Theorem 1.2.3. *The existence of a CLF that satisfies the SCP is a necessary and sufficient condition for the existence of a control law, $u(x)$, smooth on $\Omega\backslash\{0\}$ and continuous at $x = 0$, that locally continuously stabilizes the origin.*

While the necessary part is obvious, the proof of the sufficiency is more complicated and can be found in [19]. Therein Sontag provided an explicit formula to construct a continuous control law starting from a CLF, $V(x)$, that satisfies the SCP and the dynamic of the system. Let $q(b) : \mathbb{R} \to \mathbb{R}$ be a function such that $q(0) = 0$ and $bq(b) > 0$, if $b \neq 0$, and consider the function

$$\Phi(a,b) \triangleq \begin{cases} 0, & \text{if } b = 0 \text{ and } a < 0, \\ \dfrac{a + \sqrt{a^2 + bq(b)}}{b}, & \text{otherwise.} \end{cases}$$

Moreover define

$$a(x) \triangleq V_x(x) f(x), \qquad B(x) \triangleq V_x(x) g(x), \qquad \beta(x) \triangleq \|B(x)\|^2. \tag{1.9}$$

Then

$$u(x) = \begin{cases} -B(x)\Phi(a(x), \beta(x)), & \text{for } x \neq 0, \\ 0, & \text{for } x = 0, \end{cases} \tag{1.10}$$

continuously stabilizes the origin. Equation (1.10) is known as Sontag's formula.

1.3 Dynamic Control Lyapunov Functions

In the previous section it has been shown how the knowledge of a CLF can be exploited to build a control law that stabilizes the origin, for example via the Sontag's formula. Unfortunately, to construct a CLF, the constrained differential inequality

$$V_x(x) g(x) = 0 \Rightarrow V_x(x) f(x) < 0, \quad x \neq 0, \tag{1.11}$$

has to be solved with respect to $V(x)$. In other words one needs to find a function $V(x)$ whose partial derivative $V_x(x)$ solves (1.11). Moreover, if the control law has to be continuous, than $V(x)$ must also satisfy the Small Control Property. This is clearly a difficult task to undertake, therefore usually a CLF is not easy to construct.

The aim of this work is to propose a method to construct a CLF, for an extended system, without solving any partial differential equation. This CLF can then be used to construct a control law that dynamically stabilizes the system. The main advantage of this approach is that the additional dynamics, of the extended system, can be exploited to enforce the negativity of the time derivative of the CLF along the trajectories of the closed-loop system. This idea was firstly introduced in [17], in the Lyapunov functions context, the following definition extends the concept therein to CLFs.

Definition 1.3.1 (Dynamic control Lyapunov function). *A function $V(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a Dynamic Control Lyapunov function, for the nonlinear system (1.2), if it is at least of class $\mathcal{C}^1$ and there exists an open set $\Omega \subset \mathbb{R}^{2n}$, $0 \in \Omega$, such that, if we consider the extended system*

$$\begin{aligned} \dot{x} &= f(x, u), \\ \dot{\xi} &= w, \end{aligned}$$

where $w(t) \in \mathbb{R}^n$, then

- ***Property 1:*** *$V(x,\xi) > 0$ for all $(x,\xi) \in \Omega \backslash (0,0)$, $V(0,0) = 0$;*
- ***Property 2:*** *$\inf_{(u,w)} \dot{V}(x,\xi,u,w) < 0$ for all $(x,\xi) \in \Omega\backslash(0,0)$.*

Note that $w(t) \in \mathbb{R}^n$ can be thought of as an additional control. Similarly to the CLF case, if the nonlinear system is affine in the control then the Property 2 is equivalent to the following property.

- **Property 2':**

$$\begin{cases} V_x(x,\xi)g(x) & = & 0 \\ V_\xi(x,\xi) & = & 0 \end{cases} \quad \Rightarrow \quad V_x(x,\xi)f(x) < 0 \quad \text{for all } (x,\xi) \in \Omega\backslash(0,0).$$

In other words, a Dynamic Control Lyapunov function is a CLF for the extended system

$$\begin{aligned} \dot{x} &= f(x,u), \\ \dot{\xi} &= w. \end{aligned}$$

Therefore the following statement is an immediate consequence of Theorems 1.2.2 and 1.2.3.

Theorem 1.3.1. *Consider the nonlinear system (1.2) and suppose that it admits a Dynamic CLF, $V(x,\xi)$. Then*

1. *the origin of (1.2) is dynamically almost continuously stabilizable;*
2. *if $V(x,\xi)$ satisfies the SCP, the origin of (1.2) is dynamically continuously stabilizable.*

Proof. The Dynamic CLF $V(x,\xi)$ is a CLF for the extended system

$$\begin{aligned} \dot{x} &= f(x,u), \\ \dot{\xi} &= w. \end{aligned} \tag{1.12}$$

Therefore, by Theorem 1.2.2, there exist two control laws $u(x,\xi)$ and $w(x,\xi)$ that almost continuously stabilize the origin of the extended system (1.12). By Definition 1.2.5, $u(x,\xi)$ and $w(x,\xi)$ are thus two dynamic almost continuous stabilizers for the origin of system (1.2), with $\eta = n$. Moreover if $V(x,\xi)$ satisfies the SCP, by Theorem 1.2.3, $u(x,\xi)$ and $w(x,\xi)$ can be chosen continuous at the origin. □

Note that if the control law u, derived from the Dynamic CLF, depends only on the state variable x, then it is a static stabilizer for the initial system (1.2).

Theorem 1.3.2. *Consider the nonlinear system (1.2) and suppose that there exist $u = u(x)$ and $w = w(x,\xi)$ such that the origin of the extended closed-loop system*

$$\begin{aligned} \dot{x} &= f(x,u(x)), \\ \dot{\xi} &= w(x,\xi), \end{aligned} \tag{1.13}$$

is asymptotically stable. Then $x = 0$ is an asymptotically stable equilibrium point of the closed-loop system

$$\dot{x} = f(x, u(x)). \tag{1.14}$$

Therefore $u = u(x)$ is a static stabilizer for system (1.2).

Proof. The idea of the proof is the same as that of Theorem 1 in [17] and it is hereby reported for completeness. By Lemma 4.5 of [10], the fact that the origin of system (1.13) is asymptotically stable is equivalent to the existence of a class $\mathcal{KL}$ function[2] β such that $\|(x(t), \xi(t))\| \leqslant \beta(\|(x(0), \xi(0))\|, t)$ for all $t \geq 0$ and for every $(x(0), \xi(0)) \in \Omega$. Suppose now that $x_0 \in \Omega_x$, where $\Omega_x = \{x \mid (x, 0) \in \Omega\}$. Moreover suppose that $(x(0), \xi(0)) = (x_0, 0)$ and define the corresponding trajectory of system (1.13) as $(x_0(t), \xi_0(t))$. Then $\|x_0(t)\| \leqslant \|(x_0(t), \xi_0(t))\| \leqslant \beta(\|(x_0, 0)\|, t) \triangleq \bar{\beta}(\|x_0\|, t)$. Consider now system (1.14) with the same initial condition x_0 and let $x(t)$ be the corresponding trajectory. Note that, since the dynamic of x in system (1.13) does not depend on ξ, it is immediate to prove that $x(t) \equiv x_0(t)$. Therefore $\|x(t)\| \leqslant \bar{\beta}(\|x_0\|, t)$ for all $t > 0$ and for all $x_0 \in \Omega_x$, thus proving the asymptotic stability of the origin of system (1.14).

□

As stated before, the main advantage of the proposed procedure is that a CLF for the extended system can be constructed without solving the partial differential inequality

$$V_x(x)g(x) = 0 \Rightarrow V_x(x)f(x) < 0, \quad x \neq 0. \tag{1.15}$$

Since we are not able to solve (1.15), we would like to solve the simpler problem

$$p(x)g(x) = 0 \Rightarrow p(x)f(x) < 0, \quad x \neq 0, \tag{1.16}$$

where $p(x)$ does not need to be a gradient vector, and then use this solution $p(x)$ to construct a Dynamic CLF. Unfortunately condition (1.16) is not sufficient to carry out the outlined procedure, therefore a slightly stronger constraint must be imposed. To this end, in the following it is required the existence of a scalar $\bar{l} > 0$ and of a matrix-valued function $\Gamma(x)$ such that, for all $l > \bar{l}$,

$$p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top \leqslant -x^\top \Gamma(x) x \tag{1.17}$$

where $\Gamma(x) = \Gamma(x)^\top > 0$. Note that condition (1.17) implies (1.16), as demonstrated by the following series of implications:

$$\exists\ l \text{ s.t. } p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top \leqslant -x^\top \Gamma(x) x$$

$$\Downarrow$$

$$\exists\ l \text{ s.t. } p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top < 0$$

$$\Downarrow$$

$$p(x)g(x) = 0 \Rightarrow p(x)f(x) < 0.$$

These considerations lead to the formal definition of algebraic $\bar{P}$ solution.

[2]See [10] for the definition of a $\mathcal{KL}$ function.

Definition 1.3.2 (Local algebraic $\bar{P}$ solution). *Let $\bar{P} = \bar{P}^\top > 0$ be a symmetric positive definite matrix, and let $\Omega \subset \mathbb{R}^n$ be an open set, $0 \in \Omega$. A continuously differentiable mapping $p(x) : \mathbb{R}^{n\times 1} \to \mathbb{R}^{1\times n}$ is said to be an algebraic $\bar{P}$ solution of (1.11) if*

(P1) $p(0) = 0$ and $p(x)$ is tangent in the origin to $\bar{P}$, namely $p_x(0) = \bar{P}$;

(P2) there exist a scalar $\bar{l} > 0$ and a matrix-valued function $\Gamma(x)$ such that, for all $l > \bar{l}$ and for all $x \in \Omega$,

$$p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top \leqslant -x^\top\Gamma(x)x, \tag{1.18}$$

where $\Gamma(x) = \Gamma(x)^\top > 0$, for all $x \in \Omega$.

Definition 1.3.3 (Index of an algebraic $\bar{P}$ solution). *The minimum value $\bar{l}$ such that Condition (P2), in Definition 1.3.2, is satisfied is called index of the algebraic $\bar{P}$ solution.*

Remark 1.2. If there exists a value $\hat{l}$ such that

$$p(x)f(x) - \hat{l}p(x)g(x)g(x)^\top p(x)^\top \leqslant -x^\top\Gamma(x)x,$$

for all $x \in \Omega$, then the same relation holds for all $l > \hat{l}$. In fact

$$\begin{aligned} p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top &= p(x)f(x) - \hat{l}p(x)g(x)g(x)^\top p(x)^\top - \\ &\quad - (l - \hat{l})p(x)g(x)g(x)^\top p(x)^\top \\ &\leqslant p(x)f(x) - \hat{l}p(x)g(x)g(x)^\top p(x)^\top \\ &\leqslant -x^\top\Gamma(x)x. \end{aligned}$$

Therefore in Definition 1.3.2 it is sufficient to check that there exists a value of $\hat{l}$ such that Condition (P2) is satisfied.

Remark 1.3. Condition (P2) can be difficult to check. However let[3] $p(x) = x^\top\tilde{P}(x)$ and $f(x) = F(x)x$, then a sufficient condition for (1.18) to hold is

$$x^\top\tilde{P}(0)g(0) = 0 \;\Rightarrow\; x^\top\tilde{P}(0)F(0)x < 0. \tag{1.19}$$

In fact if (1.19) is satisfied then, by Lemma A.2.1, there exists a value $\bar{l} > 0$ such that for all $l > \bar{l}$ the matrix

$$\frac{1}{2}(\tilde{P}(0)F(0) + F(0)^\top\tilde{P}(0)^\top) - l\tilde{P}(0)g(0)g(0)^\top\tilde{P}(0)^\top$$

is negative definite. Therefore, by Lemma A.2.3, there exists a neighborhood, $\Omega \subset \mathbb{R}^n$, of the origin such that the matrix

$$\begin{aligned} \Gamma(x) &\triangleq \frac{1}{2}(\tilde{P}(x)F(x) + F(x)^\top\tilde{P}(x)^\top) - l\tilde{P}(x)g(x)g(x)^\top\tilde{P}(x)^\top = \\ &= \frac{1}{2}(\tilde{P}(0)F(0) + F(0)^\top\tilde{P}(0)^\top) - l\tilde{P}(0)g(0)g(0)^\top\tilde{P}(0)^\top + \mathcal{O}(x) \end{aligned}$$

is negative definite, where we used $\tilde{P}(x) = \tilde{P}(0) + \mathcal{O}_P(x)$, $F(x) = F(0) + \mathcal{O}_F(x)$, $g(x) = g(0) + \mathcal{O}_g(x)$ and $\mathcal{O}(x)$ is the overall term due to sum and product of the previous ones. Therefore condition (1.18) holds with the equality sign for all $x \in \Omega$.

[3]See Lemma A.3.2 and Lemma A.3.1 in the appendix.

Note that the procedure described in Remark 1.3 relies only on the fact that $p(x)$ in tangent to $\tilde{P}(0)$ at the origin. This means in particular that for every function $p(x)$ tangent to $\tilde{P}(0)$ there exists a neighborhood of the origin Ω such that Condition (P2) holds but the set Ω can be very small. Therefore this procedure is useful to verify if a function is an algebraic $\bar{P}$ solution rather than to construct one. In fact we would like to find a function $p(x)$ that satisfies Condition (P2) in a set as large as possible, the ideal case being when $\Omega = \mathbb{R}^n$.

Definition 1.3.4 (Global algebraic $\bar{P}$ solution)**.** *If a local algebraic $\bar{P}$ solution satisfies Condition (P2) for all $x \in \mathbb{R}^n$ then it is said to be a global algebraic $\bar{P}$ solution.*

In the following the expression "algebraic $\bar{P}$ solution" is always used to mean a local algebraic $\bar{P}$ solution.

Finally, as stated before, the reason why we are interested in defining an algebraic $\bar{P}$ solution is that it can be used to construct a Dynamic CLF for the affine system (1.7). To this end consider the family of functions

$$V(x,\xi) = p(\xi)x + \frac{1}{2}\|x - \xi\|_R^2$$

where $R = R^\top$ is a positive definite matrix to be determined. In Chapters 2 and 3 it will be shown that, for some values of R, $V(x,\xi)$ is a CLF for the extended system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ \dot{\xi} &= w. \end{aligned} \tag{1.20}$$

As a result, for such values of R, $V(x,\xi)$ is a Dynamic CLF for system (1.7), with $\eta = n$.

Chapter 2

Construction of Dynamic CLFs and stabilizing feedback laws

In the previous chapter the concept of Dynamic Control Lyapunov Function has been introduced. Moreover a particular structure for the extended system and a class of candidate Control Lyapunov Functions has been proposed. The aim of this chapter is to prove that, for a suitable choice of R, the function $V(x, \xi) = p(\xi)x + \frac{1}{2}\|x - \xi\|_R^2$ is a CLF for the extended system (1.20). To simplify the exposition, the linear case is firstly analyzed and then the proposed approach is generalized to the nonlinear case.

Note that in the linear context, if the origin is stabilizable, it is possible to find a global CLF in a very simple way. Consider the system

$$\dot{x} = Ax + Bu, \tag{2.1}$$

and assume it is stabilizable. Then using, for example, Heymann Lemma a linear control law, $\bar{u} = Kx$, that stabilizes the equilibrium at the origin can be derived. Note that the origin of the closed-loop system

$$\dot{x} = (A + BK)x, \tag{2.2}$$

is asymptotically stable and hence it is possible to find a (global) Lyapunov Function. To this end let $\bar{P} = \bar{P}^\top > 0$ be the unique[1] solution of the Lyapunov equation

$$\bar{P}(A + BK) + (A + BK)^\top \bar{P} = -Q$$

where $Q = Q^\top > 0$ is an arbitrary symmetric positive definite matrix. Then the quadratic function

$$W(x) = \frac{1}{2} x^\top \bar{P} x \tag{2.3}$$

is a (global) Lyapunov function for the closed-loop system (2.2). In fact

1. $W(x)$ is positive definite and radially unbounded;

2. the time derivative of W along the trajectories of the closed-loop system is

$$\begin{aligned}\dot{W}(x) &\triangleq W_x(x)(A + BK)x = x^\top \bar{P}(A + BK)x = \\ &= x^\top \frac{\bar{P}(A + BK) + (A + BK)^\top \bar{P}}{2} x = -\frac{1}{2} x^\top Q x < 0\end{aligned} \tag{2.4}$$

[1]Since $(A + BK)$ is a Hurwitz matrix the Lyapunov equation has a unique positive definite solution.

for all $x \neq 0$.

Note now that (2.3) is also a (global) CLF for system (2.1). In fact

1. $W(x)$ is positive definite and radially unbounded;

2. (2.4) implies that $\dot{W}(x, \bar{u} = Kx) < 0$ for all $x \neq 0$, hence

$$\inf_u \dot{W}(x,u) \leqslant \dot{W}(x, \bar{u} = Kx) < 0 \text{ for all } x \neq 0.$$

Therefore in the linear case it is possible to construct a (global) CLF in a very simple way. The previous discussion proves the following statement.

Theorem 2.0.3. *Consider a linear control system $\dot{x} = Ax + Bu$, with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^n$. If this system is stabilizable then there exists a quadratic global CLF, $W(x) = \frac{1}{2}x^\top \bar{P}x$, i.e. there exists a matrix $\bar{P} = \bar{P}^\top > 0$ such that*

$$x^\top \bar{P}B = 0 \quad \Rightarrow \quad x^\top \bar{P}Ax < 0, \quad \textit{for all} \quad x \neq 0.$$

Note that the key point in the illustrated procedure is the knowledge of a feedback control law that stabilizes the system. In fact in this case a CLF can be obtained constructing a Lyapunov function for the closed-loop system.

Even if, in the linear case, it is possible to find a global CLF in such a simple way, in Section 2.1 a global algebraic $\bar{P}$ solution, as defined in Definition 1.3.4, is constructed. This algebraic $\bar{P}$ solution is then used to prove that $V(x,\xi) = \xi^\top \bar{P}x + \frac{1}{2}\|x - \xi\|_R^2$ is a global CLF for the extended linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ \dot{\xi} &= w. \end{aligned} \tag{2.5}$$

It is important to remark that, since in the linear case it is possible to derive a global CLF as illustrated above, the use of a Dynamic CLF does not give any additional benefit. However the discussion presented in Section 2.1 is useful to illustrate the main ideas of the proposed approach, in the simpler linear context. In Section 2.2 this procedure is then generalized to the nonlinear case.

For both cases the structure of the proof is the same as above.

1. First of all it is proved that $V(x,\xi)$ is positive definite.

2. Then Property 2 of Definition 1.3.1 is proved by imposing a particular feedback law and proving that $V(x,\xi)$ is a Lyapunov function for the closed-loop system.

Note that the proof of Property 2 yields a constructive procedure, since a particular control law is derived. For this reason in the following it is referred to as "*constructive proof*". Note also that if the feedback law used in the second part of the proof is continuous at the origin then $V(x,\xi)$ automatically satisfies the SCP.

2.1 Linear Systems

Consider a linear, time-invariant, system described by the equation

$$\dot{x} = Ax + Bu, \tag{2.6}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. Suppose that the origin is stabilizable. Then by Theorem 2.0.3 it possesses a quadratic CLF $W(x) = \frac{1}{2}x^\top \bar{P}x$, i.e. there exists a symmetric positive definite matrix $\bar{P}$ such that

$$x^\top \bar{P}B = 0 \quad \Rightarrow \quad x^\top \bar{P}Ax < 0\ , \quad \text{for all} \quad x \neq 0. \tag{2.7}$$

Note that $p(x) = x^\top \bar{P}$ is a global algebraic $\bar{P}$ solution. In fact

1. $p(0) = 0$ and the tangent at the origin is

$$\frac{\partial(x^\top \bar{P})}{\partial x} = \bar{P}.$$

2. In the linear case, the second condition of the algebraic $\bar{P}$ solution becomes

$$p(x)Ax - lp(x)BB^\top p(x)^\top \leqslant -x^\top \Gamma(x)x.$$

   Using $p(x) = x^\top \bar{P}$ yields

$$x^\top \bar{P}Ax - lx^\top \bar{P}BB^\top \bar{P}x \leqslant -x^\top \Gamma(x)x. \tag{2.8}$$

   Using Lemma A.2.1 it is immediate to show that (2.7) implies that there exists a value $\bar{l}$ such that for all $l > \bar{l}$ the matrix

$$S \triangleq \frac{\bar{P}A + A^\top \bar{P}}{2} - l\bar{P}BB^\top \bar{P}$$

   is negative definite. Therefore equation (2.8) is satisfied for all x, for $l > \bar{l}$ and $\Gamma(x) = -S$.

The mapping $x^\top \bar{P}$ is an exact differential, however suppose that instead of integrating $x^\top \bar{P}$, thus obtaining the quadratic CLF $W(x)$, the mapping $p(x) = x^\top \bar{P}$ is exploited to construct the auxiliary function

$$V(x,\xi) = \xi^\top \bar{P}x + \frac{1}{2}\|x - \xi\|_R^2, \tag{2.9}$$

with $\xi(t) \in \mathbb{R}^n$ and $R = R^\top > 0$ to be determined. The aim of the following section is to prove that, for a suitable choice of R, the function $V(x,\xi)$ is a global CLF for the extended linear system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ \dot{\xi} &= w. \end{aligned} \tag{2.10}$$

More in detail, Section 2.1.1 discusses Property 1 of a CLF, i.e. the positive definiteness, while in Section 2.1.2 a "*constructive proof* " of the validity of Property 2 is given. Note that (2.9) defines a family of functions parametrized in R. It is important to remark that not all the functions of this family are CLFs for system (2.10): in the following sections a condition on R is imposed to guarantee that $V(x,\xi)$ satisfies Property 1 and 2 of Definition 1.3.1.

2.1.1 Property 1: positive definiteness

First of all, to be a global CLF, $V(x,\xi)$ must be globally positive definite. The following theorem proves that this can be guaranteed by imposing a constraint on the value of R.

Theorem 2.1.1. *Consider the linear, time-invariant, system (2.6) and suppose that the origin is stabilizable. Let* $\bar{P} = \bar{P}^\top > 0$ *be a solution of (2.7). Then the function* $V(x,\xi) = \xi^\top \bar{P} x + \frac{1}{2}\|x-\xi\|_R^2$ *is globally positive definite if and only if* $R > \dfrac{\bar{P}}{2}$.

Proof. The proof of the theorem has been given in [17]: it is hereby reported for completeness. The change of variable $e = x - \xi$ yields

$$\tilde{V}(x,e) \triangleq V(x, x-e) = \frac{1}{2}\begin{bmatrix} x^\top & e^\top \end{bmatrix}\begin{bmatrix} 2\bar{P} & -\bar{P} \\ -\bar{P} & R \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} \triangleq \frac{1}{2}\begin{bmatrix} x^\top & e^\top \end{bmatrix} Q \begin{bmatrix} x \\ e \end{bmatrix}.$$

Thus by the Shur complement formula, see Theorem A.2.1, the matrix Q is positive definite if and only if both $A = 2\bar{P}$ and $S = R - \bar{P}(2\bar{P})^{-1}\bar{P} = R - \frac{\bar{P}}{2}$ are positive definite. Since $\bar{P} > 0$ by hypothesis, Q is positive definite if and only if $R - \dfrac{\bar{P}}{2} > 0 \Leftrightarrow R > \dfrac{\bar{P}}{2}$. □

2.1.2 Property 2: a "*constructive proof*" with $R^{-1} = \alpha \bar{P}^{-1}$

As stated before, to prove that $V(x,\xi) = \xi^\top \bar{P} x + \frac{1}{2}\|x-\xi\|_R^2$ satisfies the property

$$\inf_{u,w} \dot{V}(x,\xi,u,w) < 0 \quad \text{for all} \quad (x,\xi) \neq (0,0),$$

it is sufficient to find a pair of control laws $(\bar{u},\bar{w})$ such that

$$\dot{V}(x,\xi,\bar{u},\bar{w}) < 0 \text{ for all } \ (x,\xi) \neq (0,0).$$

Therefore the purpose of the following theorem is twofold. On the one hand it is a proof that, for certain values of R, $V(x,\xi)$ satisfies Property 2, while on the other hand a feedback control law that stabilizes the origin is derived. To do that, the following Lemma is needed.

Lemma 2.1.1. *Let* S *be an* $n \times n$ *matrix and* C *an* $m \times n$ *matrix of rank* m*, where* $m < n$*. If* $x^\top S x > 0$ *for all* $x \neq 0$ *such that* $Cx = 0$*, there exists a finite number* $\bar{\rho} \geqslant 0$ *such that, for all* $\rho > \bar{\rho}$*,* $x^\top(S + \rho C^\top C)x > 0$ *for all* $x \neq 0$.

Proof. Note that

$$x^\top S x = x^\top \frac{(S + S^\top)}{2} x = x^\top H x$$
$$x^\top(S + \rho C^\top C)x = x^\top \left(\frac{S + S^\top}{2} + \rho C^\top C\right) x = x^\top (H + \rho C^\top C)x.$$

Therefore it is equivalent to prove that if $x^\top H x > 0$ for all x such that $Cx = 0$, then there exists a finite $\bar{\rho} \geqslant 0$ such that, for all $\rho > \bar{\rho}$, condition $x^\top(H + \rho C^\top C)x > 0$ holds for all $x \neq 0$. The difference is that H is now an $n \times n$ symmetric matrix. Let Z denote a basis for the right null space of C, then condition $Cx = 0$ implies that $x \in \text{span}(Z)$, or $x = Zy$ for some vector y. Therefore the first constraint, $x^\top Hx > 0$ for all $x \neq 0$ such that $Cx = 0$, is equivalent to $y^\top Z^\top HZy > 0$ for all $y \neq 0$, i.e $Z^\top HZ > 0$. Finally Lemma A.2.1 guarantees that there exists a finite $\bar{\rho} \geqslant 0$ such that, for all $\rho > \bar{\rho}$, $H + \rho C^\top C > 0$ is positive definite, hence $x^\top(H + \rho C^\top C)x > 0$ for all $x \neq 0$. □

Theorem 2.1.2. *Consider the linear, time-invariant, system (2.6) and suppose that the origin is stabilizable. Let $\bar{P} = \bar{P}^\top > 0$ be a solution of (2.7). Consider the function $V(x,\xi) = \xi^\top \bar{P}x + \frac{1}{2}\|x - \xi\|_R^2$. Let $R^{-1} = \alpha \bar{P}^{-1}$, with $0 < \alpha < 2$, and let*

$$\bar{w} = -kV_\xi^\top = -k\bar{P}x + kR(x - \xi),$$

$$\bar{u} = Lx + M(x - \xi),$$

where k is a positive scalar and $L \in \mathbb{R}^{m\times n}, M \in \mathbb{R}^{m\times n}$. Then

1. *the function $V(x,\xi)$ is positive definite;*
2. *there exist $\bar{k} \geqslant 0$ and a choice of the matrices L and M such that, for all $k > \bar{k}$, the time derivative of $V(x,\xi)$ along the trajectories of the closed-loop system is globally negative definite.*

Therefore $V(x,\xi)$ is a (global) CLF for the extended system (2.10) and satisfies the SCP.

Proof. Note that $\alpha > 0$ implies $R > 0$, while $\alpha < 2$ implies

$$R = \frac{\bar{P}}{\alpha} > \frac{\bar{P}}{2},$$

hence Theorem 2.1.1 guarantees that $V(x,\xi)$ is positive definite. Moreover, note that $p(x) = x^\top \bar{P}$ is a global algebraic $\bar{P}$ solution[2]. Let $\bar{l}$ be the index of $p(x)$, as stated in Definition 1.3.3. Consider now the time derivative of $V(x,\xi)$ along the trajectories of the closed-loop system, i.e.

$$\begin{aligned}
\dot{V} &= V_x\dot{x} - kV_\xi V_\xi^\top \\
&= \begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} \begin{bmatrix} \bar{P}(A+BL) & \bar{P}BM \\ (R-\bar{P})(A+BL) & (R-\bar{P})BM \end{bmatrix} \begin{bmatrix} x \\ x-\xi \end{bmatrix} - \\
&\quad - k\begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} \begin{bmatrix} \bar{P} \\ -R \end{bmatrix} \begin{bmatrix} \bar{P} & -R \end{bmatrix} \begin{bmatrix} x \\ x-\xi \end{bmatrix} \\
&= -\begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} [S + kC^\top C] \begin{bmatrix} x \\ x-\xi \end{bmatrix},
\end{aligned}$$

where

$$C \triangleq \begin{bmatrix} \bar{P} & -R \end{bmatrix} \qquad S \triangleq \begin{bmatrix} \bar{P}(A+BL) & \bar{P}BM \\ (R-\bar{P})(A+BL) & (R-\bar{P})BM \end{bmatrix}$$

[2]See the introduction of Section 2.1.

Let N_C be the right null space of C, i.e.

$$\begin{aligned} N_C &\triangleq \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \ \text{ s.t. } \ C\left[x^\top \ \ (x-\xi)^\top\right]^\top = V_\xi = 0\} \\ &= \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \ \text{ s.t. } \ (x-\xi)^\top = x^\top \bar{P} R^{-1}\}. \end{aligned}$$

By Lemma 2.1.1, to guarantee that there exists $\bar{k} \geqslant 0$ such that $\dot{V} < 0$ for all $k > \bar{k}$, it is sufficient to demonstrate that $\dot{V}$ takes negative values for each value of $(x,\xi) \in N_C$. To this end, note that

$$\dot{V}\big|_{N_C} = V_x \dot{x}\big|_{N_C} = \left[\ x^\top \quad (x-\xi)^\top\ \right] \begin{bmatrix} \bar{P}(A+BL) & \bar{P}BM \\ (R-\bar{P})(A+BL) & (R-\bar{P})BM \end{bmatrix} \begin{bmatrix} x \\ x-\xi \end{bmatrix}\Bigg|_{N_C}. \tag{2.11}$$

Imposing $(x,\xi) \in N_C \Rightarrow (x-\xi)^\top = x^\top \bar{P} R^{-1}$, (2.11) becomes

$$\begin{aligned} V_x \dot{x}\big|_{N_C} &= x^\top \left[\ I \quad \bar{P}R^{-1}\ \right] \begin{bmatrix} \bar{P}(A+BL) & \bar{P}BM \\ (R-\bar{P})(A+BL) & (R-\bar{P})BM \end{bmatrix} \begin{bmatrix} I \\ R^{-1}\bar{P} \end{bmatrix} x = \\ &= x^\top \left[\bar{P}(A+BL) + \bar{P}R^{-1}(R-\bar{P})(A+BL) + \bar{P}BMR^{-1}\bar{P} + \bar{P}R^{-1}(R-\bar{P})BMR^{-1}\bar{P}\right] x. \end{aligned} \tag{2.12}$$

The choice $R^{-1} = \alpha \bar{P}^{-1}$ yields

$$\begin{aligned} V_x \dot{x}\big|_{N_C} &= x^\top \left[(2-\alpha)\bar{P}(A+BL) + \alpha(2-\alpha)\bar{P}BM\right] x \\ &= (2-\alpha)\ x^\top \left[\bar{P}A + \bar{P}B(L+\alpha M)\right] x \\ &= (2-\alpha)\ x^\top \left[\bar{P}A + \bar{P}BH\right] x, \end{aligned}$$

where $H \triangleq L + \alpha M$. Finally, imposing $H = -lB^\top \bar{P}$, with $l > \bar{l}$, yields

$$V_x \dot{x}\big|_{N_C} = (2-\alpha)\ x^\top \left[\bar{P}A - l(\bar{P}B)(B^\top \bar{P})\right] x \leq -(2-\alpha) x^\top \Gamma(x) x < 0, \tag{2.13}$$

where Condition (P2) of the algebraic $\bar{P}$ solution has been used. Therefore, since the restriction of $\dot{V}$ to the set N_C is a negative definite function, by Lemma 2.1.1, there exists a value $\bar{k} > 0$ such that $\dot{V}(x,\xi,\bar{u},\bar{w}) < 0$ for all $(x,\xi) \neq (0,0)$. Finally, since

$$\inf_{u,w} \dot{V}(x,\xi,u,w) \leq \dot{V}(x,\xi,\bar{u},\bar{w}) < 0 \quad \text{for all} \quad (x,\xi) \neq (0,0),$$

$V(x,\xi)$ is a global CLF for the extended system (2.10) and since $\bar{u}$ and $\bar{w}$ are continuous at the origin, the CLF satisfies the SCP.

□

Corollary 2.1.1. *Consider the linear, time-invariant, system (2.6) and suppose that the origin is stabilizable. Let $\bar{P} = \bar{P}^\top > 0$ be a solution of (2.7) and let $0 < \alpha < 2$, then there exists a value $\bar{l} > 0$ such that the following holds.*

1. *The control laws*

$$u(x,\xi) = Lx + M\xi,$$

$$w(x,\xi) = -k\bar{P}x + kR(x-\xi)$$

with L and M such that $L + \alpha M = -lB^\top \bar{P}$, for $l > \bar{l}$, and $k > \bar{k}$ dynamically stabilize the origin of (2.6).

2. *The control law*

$$u(x) = -lB^\top \bar{P}x$$

stabilizes the origin of (2.6), for each $l > \bar{l}$.

Proof. Consider the extended system

$$\dot{x} = Ax + Bu,$$
$$\dot{\xi} = w.$$

1. It has been shown in Theorem 2.1.2 that, if $R^{-1} = \alpha\bar{P}^{-1}$, with $0 < \alpha < 2$ and k is sufficiently large, the control law $u(x,\xi)$ and $w(x,\xi)$ stabilize the origin of the extended system. Therefore, according to Definition 1.2.5, they continuously dynamically stabilize the origin of (2.6).

2. Choosing $M = 0$, the previous dynamic control law becomes $u(x) = -lB^\top\bar{P}x$, which depends only on x. Therefore Theorem 1.3.2 can be applied and thus $u(x)$ is a static stabilizer for the origin of (2.6).

□

2.2 Nonlinear Systems

Consider now a nonlinear, time-invariant, system described by the equation

$$\dot{x} = f(x) + g(x)u, \tag{2.14}$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be at least continuously differentiable and $f(0) = 0$. Then, by Lemma A.3.1, there exists a non-unique continuous matrix-valued function $F(x)$ such that $f(x) = F(x)x$. Suppose also that the linearized system around the origin is stabilizable, hence the following theorem can be stated.

Theorem 2.2.1. *Consider a nonlinear, time-invariant, system described by the equation (2.14). If the linearized system around the origin is stabilizable then there exists a positive definite matrix* $\bar{P} = \bar{P}^\top > 0$*, such that*

$$x^\top\bar{P}B = 0 \;\Rightarrow\; x^\top\bar{P}Ax < 0, \quad \textit{for all} \;\; x \neq 0, \tag{2.15}$$

where $A = \frac{\partial f}{\partial x}|_{x=0} = F(0)$ *and* $B = g(0)$.

Proof. By Theorem 2.0.3, the linearized system admits a quadratic CLF, $W(x) = \frac{1}{2}x^\top\bar{P}x$, hence

$$x^\top\bar{P}B = 0 \;\Rightarrow\; x^\top\bar{P}Ax < 0 \quad \text{for all} \;\; x \neq 0.$$

□

Note that the simplest idea to construct a CLF for system (2.14) is to use the CLF of the linearized system, i.e.

$$W(x) = \frac{1}{2}x^\top\bar{P}x. \tag{2.16}$$

This is a local CLF for the nonlinear system. In fact

1. it is trivially positive definite;

2. $x^\top \bar{P} g(x) = 0 \;\Rightarrow\; x^\top \bar{P} f(x) < 0 \quad \text{for all} \;\; x \in \Omega\backslash\{0\}$.[3]

The main shortcoming in using (2.16) as a CLF for the nonlinear system is that the basin of attraction of the zero equilibrium may be very small. For this reason, in the following, the knowledge of $\bar{P}$ will be exploited to construct an algebraic $\bar{P}$ solution, as defined in Definition 1.3.2, and to prove that the function

$$V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2 \tag{2.17}$$

is a local CLF for the extended system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ \dot{\xi} &= w, \end{aligned} \tag{2.18}$$

where $\xi(t) \in \mathbb{R}^n$ and $R = R^\top > 0$ is an $n \times n$ matrix to be defined.

The main reason why this approach is under investigation is that the dynamic control law derived from (2.17) may be a better choice than the control law derived from (2.16). In fact

1. the basin of attraction of the zero equilibrium of system (2.14), controlled with a feedback law derived from (2.17), may be larger than the basin obtained with the control law derived from (2.16);

2. the rate of convergence towards the origin may be faster with the control law derived from (2.17) than the one derived from (2.16).

It is important to remark that these points have not been formally proved in the general context, but they have been observed in several examples, some of which are presented in Chapter 5.

The aim of the following sections is to prove that, for a suitable choice of R, the function $V(x,\xi)$ is indeed a local CLF for the extended system (2.18). More in detail, Section 2.2.1 deals with Property 1 of a CLF, i.e. positive definiteness, while in Section 2.2.2 a "*constructive proof*" of Property 2 is given.

2.2.1 Property 1: positive definiteness

To be a local CLF, $V(x,\xi)$ must be locally positive definite. The following theorem shows that this can be guaranteed by imposing the same constraint on the value of R given in the linear case. This is due to the fact that $V(x,\xi)$ can be locally approximated by the quadratic function used in the linear case.

[3] The proof of this point is not straightforward and it is given in the Appendix, see Theorem A.4.1.

Theorem 2.2.2. *Consider the nonlinear, time-invariant, system (2.14) and suppose that the linearized system around the origin is stabilizable. Let* $\bar{P} = \bar{P}^\top > 0$ *be a solution of (2.15) and* $p(x)$ *an algebraic* $\bar{P}$ *solution. Then if* $R > \dfrac{\bar{P}}{2}$ *the function* $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$ *is locally positive definite.*

Proof. Note that $p(\xi) = \xi^\top \bar{P} + \xi^\top \mathcal{O}(\xi)$, see Lemma A.3.2. Substituting in $V(x,\xi)$ yields

$$V(x,\xi) = \left[\xi^\top \bar{P} + \xi^\top \mathcal{O}(\xi)\right] x + \frac{1}{2}\|x-\xi\|_R^2 = V_2(x,\xi) + \xi^\top \mathcal{O}(\xi)x,$$

where $V_2(x,\xi) \triangleq \xi^\top \bar{P}x + \frac{1}{2}\|x-\xi\|_R^2$ is the same function used in Theorem 2.1.1 for the linear case. Therefore, if $R > \dfrac{\bar{P}}{2}$, $V_2(x,\xi)$ is positive definite and can be rewritten as

$$V_2(x,\xi) = \begin{bmatrix} x^\top & \xi^\top \end{bmatrix} Q \begin{bmatrix} x \\ \xi \end{bmatrix},$$

for a suitable matrix $Q = Q^\top > 0$. As a result

$$V(x,\xi) = \begin{bmatrix} x^\top & \xi^\top \end{bmatrix} [Q + S(x,\xi)] \begin{bmatrix} x \\ \xi \end{bmatrix},$$

where

$$S(x,\xi) \triangleq \frac{1}{2}\begin{bmatrix} 0 & \mathcal{O}(\xi)^\top \\ \mathcal{O}(\xi) & 0 \end{bmatrix}.$$

Note that $S(x,\xi) = S(x,\xi)^\top$, $S(0,0) = 0$ and that $V(x,\xi)$ is a continuous function. Therefore, by Lemma A.2.3, there exists a neighborhood $\Omega \subset \mathbb{R}^{2n}$ of $(0,0)$ such that $Q+S(x)$ is positive definite for all $(x,\xi) \in \Omega$, hence $V(x,\xi) > 0$ for all $(x,\xi) \in \Omega\backslash(0,0)$. □

2.2.2 Property 2: a "*constructive proof*" with $R^{-1} = \alpha \bar{P}^{-1}$

Similarly to the linear case, Property 2 of a CLF is demonstrated in a constructive way, i.e. by providing a pair of control laws $(\bar{u},\bar{w})$ such that

$$\dot{V}(x,\xi,\bar{u},\bar{w}) < 0 \text{ for all } (x,\xi) \in \Omega\backslash(0,0).$$

To this end the following lemma, which is a generalization of Lemma A.2.1 to the nonlinear case, is used.

Lemma 2.2.1. *Let* $H(x)$ *be an* $n \times n$ *symmetric matrix and* $C(x)$ *an* $m \times n$ *matrix of constant rank* m*, where* $m < n$*. Let* $Z(x)$ *denote a basis for the right null space of* $C(x)$ *and let* Ω *be a bounded set. Then if* $Z(x)^\top H(x) Z(x)$ *is positive definite for all* $x \in \Omega$ *there exists a finite* $\bar{\rho} \geqslant 0$ *such that, for all* $\rho > \bar{\rho}$, $[H(x) + \rho C(x)^\top C(x)] > 0$ *for all* $x \in \Omega$.

Proof. By Lemma A.2.1, for each fixed value $\bar{x} \in \Omega$, there exists a finite value $\bar{\rho}(\bar{x})$ such that for all $\rho > \bar{\rho}(\bar{x})$ the matrix $[H(\bar{x}) + \rho C(\bar{x})^\top C(\bar{x})]$ is positive definite. Note now that, since for each fixed value $\bar{x}$ the quantity $\rho(\bar{x})$ is finite and Ω is a bounded set, there exists a value $\bar{\rho} > 0$ such that

$$\sup_{\bar{x}\in\Omega} \bar{\rho}(\bar{x}) < \bar{\rho} < +\infty.$$

This concludes the proof, i.e. it shows that, for each $\bar{x} \in \Omega$,

$$\rho > \bar{\rho} \;\Rightarrow\; \rho > \bar{\rho}(\bar{x}) \;\Rightarrow\; [H(\bar{x}) + \rho C(\bar{x})^\top C(\bar{x})] > 0,$$

therefore $\rho > \bar{\rho}$ implies $[H(x) + \rho C(x)^\top C(x)] > 0$ for all $x \in \Omega$. □

Note now that in the nonlinear case the partial derivatives of $V(x,\xi)$ are

$$\begin{aligned} V_x &= p(\xi) + (x-\xi)^\top R = p(x) + (x-\xi)^\top (R - \Phi(x,\xi))^\top, \\ V_\xi &= x^\top p_\xi(\xi) - (x-\xi)^\top R, \end{aligned}$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a continuos matrix-valued function such that[4] $p(x) - p(\xi) = (x-\xi)^\top \Phi(x,\xi)^\top$.

To streamline the presentation of the following result, define the continuous matrix-valued function

$$\Delta(x,\xi) \triangleq p_\xi(\xi) R^{-1} (R - \Phi(x,\xi))^\top$$

and recall[5] that $f(x) = F(x)x$ and $p(x) = x^\top \tilde{P}(x)$.

Theorem 2.2.3. *Consider the nonlinear, time-invariant, system described by the equation (2.14). Suppose that there exists a local algebraic $\bar{P}$ solution, $p(x)$, where $\bar{P} = \bar{P}^\top > 0$ is a positive definite matrix, with index $\bar{l}$. Consider the extended system (2.18) and the function $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$. Let*

$$\begin{aligned} \bar{w} &= -kV_\xi^\top = -kp_\xi(\xi)^\top x + kR(x-\xi), \\ \bar{u} &= -lg(x)^\top p(x)^\top, \end{aligned}$$

where k is a positive scalar to be determined and $l > \bar{l}$. Moreover suppose that there exists a neighborhood of the origin, $\Omega \subset \mathbb{R}^{2n}$, such that

$$\frac{1}{2}\left[F(x)^\top - l\tilde{P}(x)g(x)g(x)^\top\right]\Delta(x,\xi)^\top + \frac{1}{2}\Delta(x,\xi)\left[F(x) - lg(x)g(x)^\top \tilde{P}(x)^\top\right] < \Gamma(x), \tag{2.19}$$

for all $(x,\xi) \in \Omega$. Then there exists $\bar{k} \geqslant 0$ such that, for all $k > \bar{k}$, the time derivative of $V(x,\xi)$ along the trajectories of the closed-loop system is negative definite in Ω.

Proof. The time derivative of $V(x,\xi)$ along the trajectories of the closed-loop system is

$$\begin{aligned} \dot{V} &= V_x\dot{x} + V_\xi\dot{\xi} = V_x[f(x) + g(x)\bar{u}] + V_\xi\bar{w} = V_x[f(x) - lg(x)g(x)^\top p(x)^\top] - kV_\xi V_\xi^\top \\ &= [p(x) + (x-\xi)^\top(R-\Phi(x,\xi))^\top][f(x) - lg(x)g(x)^\top p(x)^\top] - kV_\xi V_\xi^\top \\ &= [p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top] + \\ &\quad + (x-\xi)^\top(R-\Phi(x,\xi))^\top[f(x) - lg(x)g(x)^\top p(x)^\top] - kV_\xi V_\xi^\top \\ &\leqslant -x^\top\Gamma(x)x + (x-\xi)^\top(R-\Phi(x,\xi))^\top[f(x) - lg(x)g(x)^\top p(x)^\top] - kV_\xi V_\xi^\top. \end{aligned} \tag{2.20}$$

[4]See Lemma A.3.2.

[5]See Theorems A.3.1 and A.3.2.

Note that (2.20) can be rewritten as a quadratic form. To this end note that

$$f(x) - lg(x)g(x)^\top p(x)^\top = \left[F(x) - lg(x)g(x)^\top \tilde{P}(x)^\top\right] x \triangleq \bar{F}_l(x)x$$

and that

$$V_\xi = \begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} \begin{bmatrix} p_\xi(\xi) \\ -R \end{bmatrix}.$$

Therefore (2.20) becomes

$$\begin{aligned}
\dot{V} \leqslant & \begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} \begin{bmatrix} -\Gamma(x) & \frac{1}{2}\bar{F}_l(x)^\top(R - \Phi(x,\xi)) \\ \frac{1}{2}(R - \Phi(x,\xi))^\top \bar{F}_l(x) & 0 \end{bmatrix} \begin{bmatrix} x \\ x-\xi \end{bmatrix} - \\
& - k \begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} \begin{bmatrix} p_\xi(\xi) \\ -R \end{bmatrix} \begin{bmatrix} p_\xi(\xi)^\top & -R \end{bmatrix} \begin{bmatrix} x \\ x-\xi \end{bmatrix} \\
= & \begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} \left[S(x,\xi) - kC(x,\xi)^\top C(x,\xi)\right] \begin{bmatrix} x \\ x-\xi \end{bmatrix}, \qquad (2.21)
\end{aligned}$$

where

$$C(x,\xi) := \begin{bmatrix} p_\xi(\xi)^\top & -R \end{bmatrix}$$

and

$$S(x,\xi) := \begin{bmatrix} -\Gamma(x) & \frac{1}{2}\bar{F}_l(x)^\top(R - \Phi(x,\xi)) \\ \frac{1}{2}(R - \Phi(x,\xi))^\top \bar{F}_l(x) & 0 \end{bmatrix}.$$

Note that the columns of the matrix

$$Z(\xi) := \begin{bmatrix} I \\ R^{-1} p_\xi(\xi)^\top \end{bmatrix},$$

which has constant rank n, span the right kernel of the matrix $C(x,\xi)$. Consider now the restriction of the matrix $S(x,\xi)$ to the set $N_C = \{(x,\xi) : C(x,\xi)\left[x^\top \;\; (x-\xi)^\top\right]^\top\} = 0$, namely

$$\begin{aligned}
& Z(\xi)^\top S(x,\xi) Z(\xi) = \\
& = \begin{bmatrix} I & p_\xi(\xi) R^{-1} \end{bmatrix} \begin{bmatrix} -\Gamma(x) & \frac{1}{2}\bar{F}_l(x)^\top(R - \Phi(x,\xi)) \\ \frac{1}{2}(R - \Phi(x,\xi))^\top \bar{F}_l(x) & 0 \end{bmatrix} \begin{bmatrix} I \\ R^{-1} p_\xi(\xi)^\top \end{bmatrix} = \\
& = -\Gamma(x) + \frac{1}{2}\bar{F}_l(x)^\top (R - \Phi(x,\xi)) R^{-1} p_\xi(\xi)^\top + \frac{1}{2} p_\xi(\xi) R^{-1} (R - \Phi(x,\xi))^\top \bar{F}_l(x) = \\
& = -\Gamma(x) + \frac{1}{2}\bar{F}_l(x)^\top \Delta(x,\xi)^\top + \frac{1}{2}\Delta(x,\xi)\bar{F}_l(x). \qquad (2.22)
\end{aligned}$$

Condition (2.19) implies that $Z(\xi)^\top S(x,\xi) Z(\xi)$ is negative definite for all $(x,\xi) \in \Omega$. To summarize, we have shown that

$$\dot{V} \leqslant \begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} \left[S(x,\xi) - kC(x,\xi)^\top C(x,\xi)\right] \begin{bmatrix} x \\ x-\xi \end{bmatrix}$$

and that the restriction of $S(x,\xi)$ to $Z(\xi)$, i.e. the kernel of $C(x,\xi)$, is a negative definite matrix for all $(x,\xi) \in \Omega$. Therefore, by Lemma 2.2.1 , there exists a constant $\bar{k} > 0$ such that, for all $k > \bar{k}$, $\dot{V} < 0$ for all $(x,\xi) \in \Omega$ and $(x,\xi) \neq (0,0)$. □

Note that condition (2.19) can be difficult to check. The following corollaries give some sufficient conditions guaranteeing that condition (2.19) holds.

Corollary 2.2.1. *Consider the nonlinear, time-invariant, system described by equation (2.14). Suppose that there exists a local algebraic $\bar{P}$ solution, $p(x)$, where $\bar{P} = \bar{P}^\top > 0$ is a positive definite matrix, with index $\bar{l}$. Consider the extended system (2.18) and the function $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$. Let*

$$\bar{w} = -kV_\xi^\top = -kp_\xi(\xi)^\top x + kR(x-\xi),$$

$$\bar{u} = -lg(x)^\top p(x)^\top,$$

where k is a positive scalar to be determined and $l > \bar{l}$. If $R = \bar{P}$ then

1. *the function $V(x,\xi)$ is locally positive definite;*

2. *there exist $\bar{k} \geqslant 0$ such that, for all $k > \bar{k}$, the time derivative of $V(x,\xi)$ along the trajectories of the closed-loop system is locally negative definite.*

Therefore $V(x,\xi)$ is a local CLF for the extended system.

Proof. The first point is straightforward, since $R = \bar{P} > \frac{\bar{P}}{2}$, hence Theorem 2.2.2 applies. To prove the second point it is sufficient to show that if $R = \bar{P}$ then condition (2.19) is satisfied. Note now that if $R = \bar{P}$ then $\Delta(0,0) = 0$. Therefore the left hand side of the inequality (2.19) is zero at the origin, whereas the right hand side, i.e. $\Gamma(0)$, is positive definite. Hence by continuity there exists a non-empty set Ω where condition (2.19) holds. Therefore Theorem 2.2.3 can be applied. □

Corollary 2.2.2. *Consider the nonlinear, time-invariant, system described by equation (2.14) and suppose that the linearized system around the origin is stabilizable. Let $\bar{P} = \bar{P}^\top > 0$ be a solution of (2.15) and $p(x)$ an algebraic $\bar{P}$ solution, with index $\bar{l}$. Consider the extended system (2.18) and the function $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$. Let $R^{-1} = \alpha\bar{P}^{-1}$, with $0 < \alpha < 2$, and let*

$$\bar{w} = -kV_\xi^\top = -kp_\xi(\xi)^\top x + kR(x-\xi),$$

$$\bar{u} = -lg(x)^\top p(x)^\top,$$

where k is a positive scalar and $l > \bar{l}$. Then

1. *the function $V(x,\xi)$ is locally positive definite;*

2. *there exists $\bar{k} \geqslant 0$ such that, for all $k > \bar{k}$, the time derivative of $V(x,\xi)$ along the trajectories of the closed-loop system is locally negative definite.*

Therefore $V(x,\xi)$ is a local CLF for the extended system.

Proof. Firstly note that since the linearized system around the origin is stabilizable[6] there exists a matrix $\bar{P}$ solution of (2.15), i.e

$$x^\top \bar{P}B = 0 \;\Rightarrow\; x^\top \bar{P}Ax < 0, \quad \text{for all} \;\; x \neq 0, \tag{2.23}$$

[6]See Theorem 2.2.1.

where $A = \frac{\partial f}{\partial x}|_{x=0} = F(0)$ and $B = g(0)$. This condition guarantees the existence of an algebraic $\bar{P}$ solution. In fact consider an arbitrary continuously differentiable mapping $p(x) : \mathbb{R}^{n\times 1} \to \mathbb{R}^{1\times n}$ such that $p(0) = 0$ and $p_x(0) = \bar{P}$. Then

$$
\begin{aligned}
&p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top = \\
&= x^\top \frac{\tilde{P}(x)F(x) + F(x)^\top p(x)}{2} x - lx^\top \tilde{P}(x)g(x)g(x)^\top \tilde{P}(x)^\top x \\
&= x^\top \frac{\tilde{P}(0)F(0) + F(0)^\top \tilde{P}(0)}{2} x - lx^\top \tilde{P}(0)g(0)g(0)^\top \tilde{P}(0)^\top x + x^\top \mathcal{O}(x)x \\
&= x^\top \left[\frac{\bar{P}A + A^\top \bar{P}}{2} - lx^\top \bar{P}BB^\top \bar{P}x\right] x + x^\top \mathcal{O}(x)x \\
&= x^\top \left[S_l + \mathcal{O}(x)\right] x, \qquad (2.24)
\end{aligned}
$$

where

$$
S_l \triangleq \frac{\bar{P}A + A^\top \bar{P}}{2} - lx^\top \bar{P}BB^\top \bar{P}x.
$$

Using Lemma A.2.1 it is immediate to show that condition (2.23) implies that there exists a value $\bar{l}$ such that for all $l > \bar{l}$ the matrix S_l is negative definite. Therefore by Lemma A.2.3 there exists a neighborhood of the origin $\Omega_x \in \mathbb{R}^n$ such that $\Gamma(x) \triangleq -\left[S_l + \mathcal{O}(x)\right]$ is positive definite for all $x \in \Omega_x$. Substituting in (2.24) yields

$$
p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top = -x^\top \Gamma(x)x,
$$

where $\Gamma(x) = \Gamma(x)^\top > 0$ for all $x \in \Omega_x$. Therefore Condition (P2) of a local algebraic $\bar{P}$ solution is satisfied in Ω_x. Moreover, since it is satisfied with the equality sign, it follows that

$$
\Gamma(x) = -\left[\frac{\tilde{P}(x)F(x) + F(x)^\top p(x)}{2} - l\tilde{P}(x)g(x)g(x)^\top \tilde{P}(x)^\top\right]. \qquad (2.25)
$$

We are now ready to prove the two claims.

1) Condition $\alpha > 0$ implies $R > 0$, while $\alpha < 2$ implies

$$
R = \frac{\bar{P}}{\alpha} > \frac{\bar{P}}{2},
$$

hence Theorem 2.2.2 guarantees that $V(x,\xi)$ is locally positive definite.

2) Note that, using the condition $R^{-1} = \alpha \bar{P}^{-1}$, $\Delta(x,\xi)$ becomes

$$
\begin{aligned}
\Delta(x,\xi) = p_\xi(\xi)R^{-1}(R - \Phi(x,\xi))^\top &= p_\xi(\xi)\alpha\bar{P}^{-1}\left(\frac{1}{\alpha}\bar{P} - \Phi(x,\xi)\right)^\top \\
&= (\bar{P} + O(\xi))\alpha\bar{P}^{-1}\left(\frac{1}{\alpha}\bar{P} - \bar{P} + \mathcal{O}(x,\xi)\right) \\
&= (1-\alpha)\bar{P} + \mathcal{O}(x,\xi) = (1-\alpha)\tilde{P}(x) + \mathcal{O}(x,\xi), \qquad (2.26)
\end{aligned}
$$

hence

$$\frac{1}{2}\Delta(x,\xi)\left[F(x) - lg(x)g(x)^\top \tilde{P}(x)^\top\right] = \left[\frac{(1-\alpha)}{2}\tilde{P}(x) + \mathcal{O}(x,\xi)\right]\left[F(x) - lg(x)g(x)^\top \tilde{P}(x)^\top\right] =$$
$$= \frac{(1-\alpha)}{2}\left[\tilde{P}(x)F(x) - l\tilde{P}(x)g(x)g(x)^\top \tilde{P}(x)^\top\right] + \mathcal{O}(x,\xi).$$

Therefore the left hand side of condition (2.19) becomes

$$\begin{aligned}
&\frac{1}{2}\left[F(x)^\top - l\tilde{P}(x)g(x)g(x)^\top\right]\Delta^\top(x,\xi) + \frac{1}{2}\Delta(x,\xi)\left[F(x) - lg(x)g(x)^\top \tilde{P}(x)^\top\right] = \\
&= (1-\alpha)\left[\frac{\tilde{P}(x)F(x) + F(x)^\top \tilde{P}(x)^\top}{2} - l\tilde{P}(x)g(x)g(x)^\top \tilde{P}(x)^\top\right] + \mathcal{O}(x,\xi) = \\
&= -(1-\alpha)\Gamma(x) + \mathcal{O}(x,\xi)
\end{aligned}$$

where we have used (2.25). Hence condition (2.19) is satisfied if and only if

$$-(1-\alpha)\Gamma(x) + \mathcal{O}(x,\xi) < \Gamma(x)$$

or equivalently

$$-(2-\alpha)\Gamma(x) + \mathcal{O}(x,\xi) < 0$$

Recall that $\Gamma(x) = \Gamma(x)^\top > 0$ for all $x \in \Omega_x$, therefore, by Lemma A.2.3, there exists a neighborhood of the origin $\Omega_1 \subset \Omega_x$ such that $-(2-\alpha)\Gamma(x) + \mathcal{O}(x,\xi)$ is negative definite for all $x \in \Omega_1$. This concludes the proof, since condition (2.19) is locally satisfied and hence Theorem 2.2.3 can be applied. □

An immediate consequence of this result is the following statement.

Corollary 2.2.3. *Consider the nonlinear, time-invariant, system described by equation (2.14) and suppose that the linearized system around the origin is stabilizable. Let $\bar{P} = \bar{P}^\top > 0$ be a solution of (2.15) and $p(x)$ an algebraic $\bar{P}$ solution with index $\bar{l}$. Then the control law*

$$\bar{u} = -lg(x)^\top p(x)^\top,$$

with $l > \bar{l}$, locally exponentially stabilizes the origin.

Proof. In Corollary 2.2.2 it has been proved that the control law $u(x) = -lg(x)^\top p(x)^\top$ stabilizes the origin of the extended system

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u, \\
\dot{\xi} &= -kp_\xi(\xi)^\top x + kR(x-\xi),
\end{aligned}$$

where $R^{-1} = \alpha\bar{P}^{-1}$, with $0 < \alpha < 2$, and k sufficiently large. Therefore $u(x)$ is a dynamic stabilizer for system (2.14). Moreover, since u depends only on x, Theorem 1.3.2 can be applied, thus $u(x)$ is also a static stabilizer for the origin of (2.14). □

2.3 Conclusions

Let $p(x)$ be an algebraic $\bar{P}$ solution of

$$V_x g(x) = 0 \quad \Rightarrow V_x f(x) < 0 \ \text{ for all } \ x \neq 0,$$

with index $\bar{l}$. In this chapter it has been proved that there exist values of R such that the function

$$V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$$

is a CLF for the extended system (2.18). In particular the following results have been established.

Linear Systems

1. If $R^{-1} = \alpha \bar{P}^{-1}$, with $0 < \alpha < 2$, then $V(x,\xi) = \xi^\top \bar{P} x + \frac{1}{2}\|x-\xi\|_R^2$ is a global CLF.
2. The control laws $u = Lx + M\xi$, with $L + \alpha M = -lB^\top \bar{P}$ and $l > \bar{l}$, and $w = -k\bar{P}x + kR(x-\xi)$ with $k > \bar{k}$ dynamically stabilize the origin of system (2.6).
3. The control law $u = -lB^\top \bar{P}$ stabilizes the origin of system (2.6) for each value of $l > \bar{l}$.

Nonlinear Systems

1. If $R > \frac{\bar{P}}{2}$ and condition (2.19) holds, then $V(x,\xi)$ is a local CLF.
2. If $R = \bar{P}$ then condition (2.19) holds, therefore $V(x,\xi)$ is a local CLF.
3. If the linearized system around the origin is stabilizable and $R^{-1} = \alpha\bar{P}^{-1}$, with $0 < \alpha < 2$, then condition (2.19) holds, therefore $V(x,\xi)$ is a local CLF.
4. The control law $u = -lg(x)^\top p(x)^\top$ locally exponentially stabilizes the origin of system (2.14) for each value of $l > \bar{l}$.

Chapter 3

A geometric interpretation of the problem

In the previous section a constructive proof for Property 2 of a CLF has been given, by imposing $R^{-1} = \alpha \bar{P}^{-1}$ and a particular structure on the feedback law. The aim of this section is to give a illustration of the same property from a different perspective and without selecting any control law. In particular, under the hypothesis that the linearized system around the origin is stabilizable, a sufficient condition on the minimum singular value of R, to guarantee that $V(x,\xi)$ is a local CLF, is given.

To begin with we recall that the second property of a local Dynamic CLF for an affine system can be reformulated as follows.

- **Property 2':**

$$\begin{cases} V_\xi(x,\xi) &= 0 \\ V_x(x,\xi)g(x) &= 0 \end{cases} \quad \Rightarrow \quad V_x(x,\xi)f(x) < 0 \text{ for all } (x,\xi) \in \Omega \backslash (0,0).$$

Note that if $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$ then

$$V_\xi(x,\xi) = x^\top p_\xi(\xi) - (x^\top - \xi^\top)R,$$

hence $V_\xi(0,0) = 0$ and the partial derivative with respect to ξ at the origin is

$$\frac{\partial V_\xi(x,\xi)}{\partial \xi}\Big|_{(0,0)} = R > 0.$$

Therefore by the "implicit function Theorem"[1], there exists a neighborhood of the origin, $\Omega_x \subset \mathbb{R}^n$, such that, from the condition $V_\xi = 0$, it is possible to write ξ as a function of x, i.e. there exists a function $\xi = \xi(x)$ such that $V_\xi(x,\xi(x)) = 0$ for all $x \in \Omega_x$. Note that without loss of generality we can assume that $x \in \Omega_x \Rightarrow (x,\xi(x)) \in \Omega$. Substituting in Property 2' yields the equivalent condition

$$V_x(x,\xi(x))g(x) = 0 \quad \Rightarrow \quad V_x(x,\xi(x))f(x) < 0 \quad \text{for all } x \in \Omega_x \backslash \{0\}. \tag{3.1}$$

[1] See Theorem A.3.1.

Therefore the original $2n$–dimensional condition is now reduced to an n-dimensional condition. Moreover note that (3.1) is similar to the condition

$$p(x)g(x) = 0 \quad \Rightarrow p(x)f(x) < 0 \quad \text{for all } x \in \Omega\backslash\{0\}. \tag{3.2}$$

Note that if $p(x)$ is a local algebraic $\bar{P}$ solution then there exists a scalar $\bar{l} > 0$ such that, for all $l > \bar{l}$, $p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top \leqslant -x^\top \Gamma(x)x$ for all $x \in \Omega$. Moreover it has already been proved, see Section 1.3, that this in particular implies condition (3.2). Therefore, to demonstrate that $V(x,\xi)$ satisfies Property 2' it is sufficient to show that (3.2) implies (3.1).

This implication can be analyzed from a geometric perspective, since both (3.1) and (3.2) have a similar structure. Set

$$\begin{aligned} I^0_{Pg} &= \{x \neq 0 \ s.t. \ p(x)g(x) = 0\}, \\ I^-_{Pf} &= \{x \neq 0 \ s.t. \ p(x)f(x) < 0\}, \\ I^0_{Vg} &= \{x \neq 0 \ s.t. \ V_x(x,\xi(x))g(x) = 0\}, \\ I^-_{Vf} &= \{x \neq 0 \ s.t. \ V_x(x,\xi(x))f(x) < 0\}, \end{aligned}$$

where I^0_{Pg} and I^0_{Vg} are $(n-m)$-dimensional sets while I^-_{Pf} and I^-_{Vf} are n-dimensional sets. Note that condition (3.2) is satisfied if $(I^0_{Pg} \cap \Omega\backslash\{0\})$ is a subset of $(I^-_{Pf} \cap \Omega\backslash\{0\})$ while condition (3.1) is satisfied if $(I^0_{Vg} \cap \Omega_x\backslash\{0\})$ is a subset of $(I^-_{Vf} \cap \Omega_x\backslash\{0\})$.

To provide an illustration of Property 2' of a CLF, it is convenient to represent the sets above for some particular cases. The simplest one is when $n = 2$, so that the sets I^0_{Pg} and I^0_{Vg} are one dimensional (they are colored in blue in the following figures), while I^-_{Pf} and I^-_{Vf} are 2-dimensional (light-blue colored). Conditions (3.2) and (3.1) are therefore met if the blue line is inside the light-blue region, at least in a neighborhood of the origin.

A double integrator

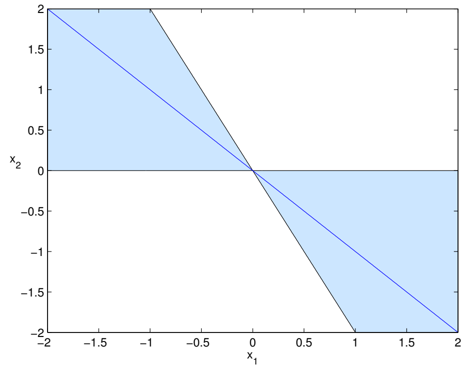
Consider a double integrator, namely

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= u, \end{aligned} \tag{3.3}$$

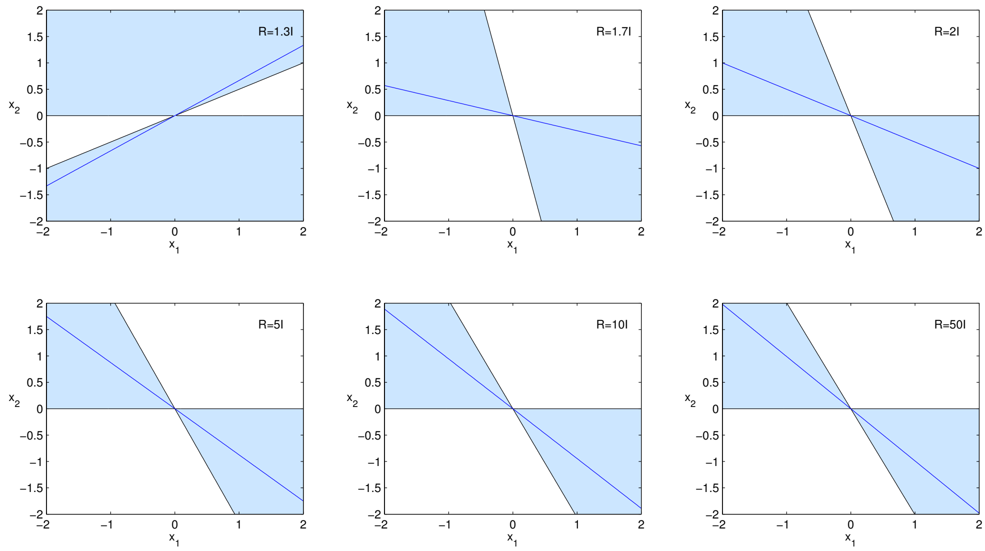
and introduce the matrix

$$\bar{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Figure 3.1 (a) gives a graphical visualization of condition (3.2), while Figure (b) gives a graphical visualization of condition (3.1) for different values of R. Note that the sets in Figure (b) are a rotated version of the sets in Figure (a) and that increasing R they are more and more similar to those in (a). For example for $R = 50I$ they are practically indistinguishable.



(a) The sets I^0_{Pg} and I^-_{Pf} for system (3.3)



(b) The sets I^0_{Vg} and I^-_{Vf} for different values of R

Figure 3.1: The sets I^0_{Pg}, I^-_{Pf}, I^0_{Vg} and I^-_{Vf} for system (3.3). Note that condition (3.2) is satisfied since the set I^0_{Pg} (blu line) is inside the set I^-_{Pf} (light blue region), in (a). On the other hand, for each of the plots in (b), condition (3.1) is satisfied only if the set I^0_{Vg} (blu line) is inside the set I^-_{Vf} (light blue region). Therefore, for this system, condition (3.1) is satisfied for each value of R used in the plots.

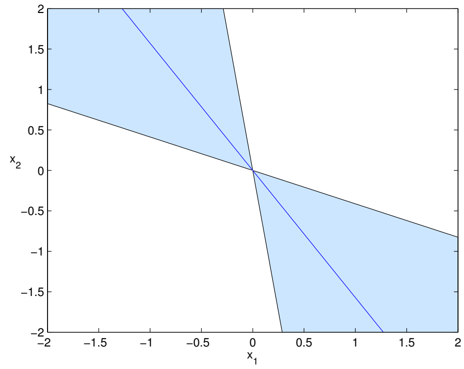
Consider now a different 2-dimensional linear system, namely

$$\begin{aligned}\dot{x}_1 &= x_1 + 2x_2,\\ \dot{x}_2 &= 3x_1 + u,\end{aligned} \tag{3.4}$$

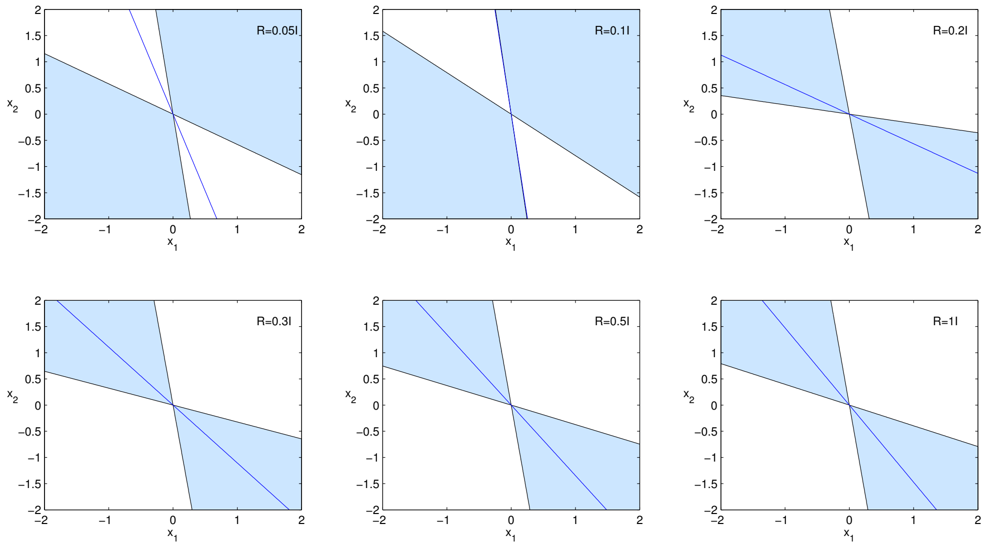
and the matrix

$$\bar{P} = \begin{bmatrix} 0.3281 & 0.0156 \\ 0.0156 & 0.2031 \end{bmatrix}.$$

Figure 3.2 gives a graphical visualization of conditions (3.2) and (3.1) for different values of R. Similarly to the previous example, the sets in Figure (b) are a rotated version of the sets in Figure (a), and when increasing R they become more and more similar to those in (a). Note however that, on the contrary to the previous example, condition (3.1) is not always satisfied, i.e. if $R < 0.1I$, the set I^0_{Vg} is not contained in the set I^-_{Vf}. Therefore one can deduce from these plots that, to satisfy condition (3.1), R must be sufficiently large.



(a) The sets I^0_{Pg} and I^-_{Pf} for system (3.4)



(b) The sets I^0_{Vg} and I^-_{Vf} for different values of R

Figure 3.2: The sets I^0_{Pg}, I^-_{Pf}, I^0_{Vg} and I^-_{Vf} for system (3.4). Note that condition (3.2) is satisfied since the set I^0_{Pg} (blu line) is inside the set I^-_{Pf} (light blue region), in (a). On the other hand, for each of the plots of (b), condition (3.1) is satisfied if the set I^0_{Vg} (blu line) is inside the set I^-_{Vf} (light blue region). Therefore, for this system, condition (3.1) is not satisfied for $R = 0.05I$ and $R = 0.1I$.

Planar nonlinear systems

We have carried out the same geometric analysis described in the previous section for two nonlinear planar (2-dimensional) systems. The first one is described by the equations

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2, \\ \dot{x}_2 &= u. \end{aligned} \tag{3.5}$$

Note that the linearized system around the origin is $\dot{x} = Ax + Bu$ where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3.6}$$

Therefore the matrix

$$\bar{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is such that

$$x^\top \bar{P} B = 0 \Rightarrow x^\top \bar{P} A x < 0, \quad \text{for all} \quad x \neq 0.$$

Select the row vector $p(x) = [2x_1 + x_2 - x_1^2 \;\; x_1 + x_2]$, then $p(x)$ is a local algebraic $\bar{P}$ solution for system (3.5), as detailed hereafter.

1. $p(0) = 0$ and the Jacobian of $p(x)$ is

$$p_x(x) = \begin{bmatrix} 2 - 2x_1 & 1 \\ 1 & 1 \end{bmatrix}$$

   hence $p_x(0) = \bar{P}$.

2. $p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top = -x^\top \begin{bmatrix} l + x_1(x_1 - 2) & l - 1 \\ l - 1 & l - 1 \end{bmatrix} x \triangleq x^\top \Gamma(x) x.$

   The matrix $\Gamma(x)$ is positive definite if and only if:

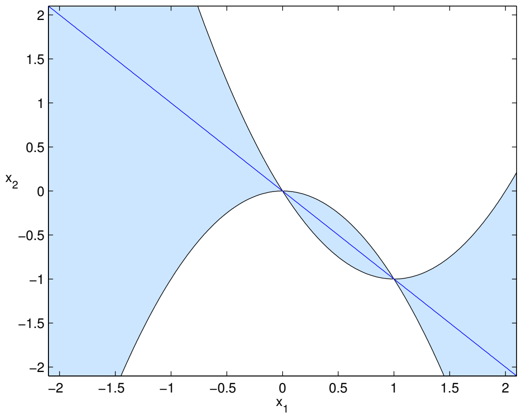
   (a) $l + x_1(x_1 - 2) > 0 \;\Rightarrow\; l > -x_1(x_1 - 2)$. Note that the term on the right hand side is upper bounded by 1 hence this condition is globally satisfied for $l > 1$;

   (b) $\det \Gamma(x) > 0 \;\Rightarrow\; (x_1 - 1)^2(l - 1) > 0 \;\Rightarrow l > 1$ and $x_1 \neq 1$.

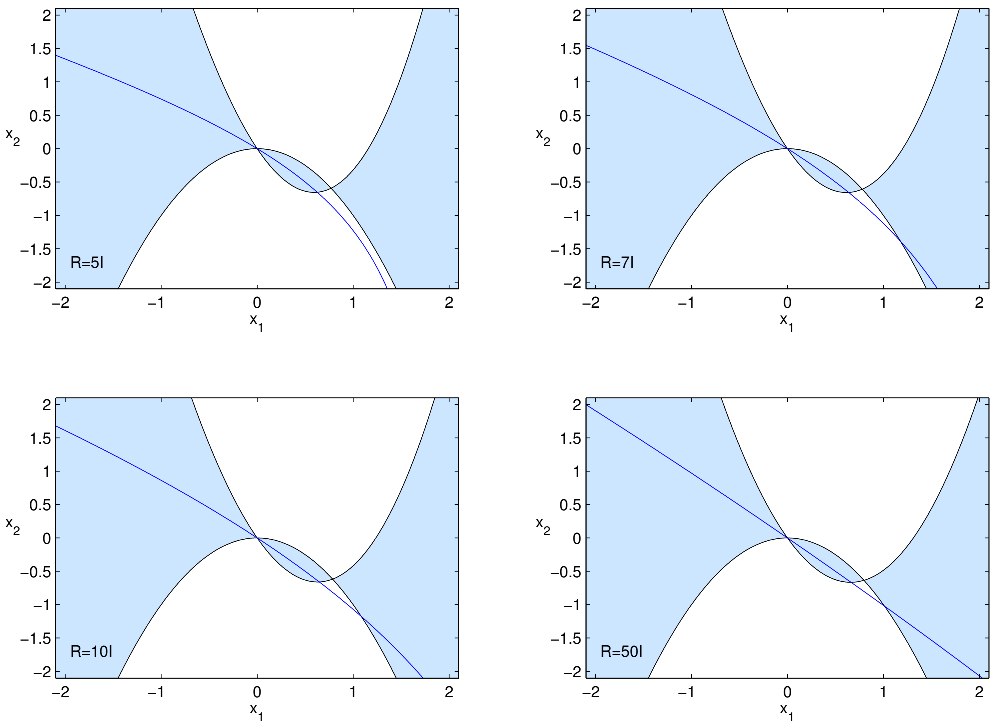
   Note that the determinant of $\Gamma(x)$ for $l > 1$ is always positive except for $x_1 = 1$, when it is zero independently of l. Therefore $\Gamma(x)$ is a positive definite matrix for all $x \neq (1, x_2)$ and $l > 1$.

Hence $p(x)$ is a local algebraic $\bar{P}$ solution in the set $\Omega = \{x \mid x_1 \neq 1\}$ and its index is $\bar{l} = 1$.

Figure 3.3 provides a graphical illustration of conditions (3.2) and (3.1) for this system. Note that, locally around the origin, the sets I^0_{Pg}, I^-_{Pf} in (a) are similar to the sets I^0_{Vg}, I^-_{Vf} in (b). Moreover also in this example the similarity between the sets increases as R increases.



(a) The sets I^0_{Pg} and I^-_{Pf} for system (3.5)



(b) The sets I^0_{Vg} and I^-_{Vf} for different values of R

Figure 3.3: The sets I^0_{Pg}, I^-_{Pf}, I^0_{Vg} and I^-_{Vf} for system (3.5). Note that condition (3.2) is satisfied since the set I^0_{Pg} (blu line) is inside the set I^-_{Pf} (light blue region) in (a). On the other hand, for each of the plots of (b), condition (3.1) is satisfied if the set I^0_{Vg} (blu line) is inside the set I^-_{Vf} (light blue region). Therefore, for this system, condition (3.1) is locally satisfied for each value of R used in the plots.

We now consider a second nonlinear example, namely the system

$$\dot{x}_1 = x_2 + x_1^3,$$
$$\dot{x}_2 = u. \tag{3.7}$$

Note that the linearized system is the same of the previous example therefore we can select

$$\bar{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Let $p(x) = [2x_1 + x_2 + x_1^3 \;\; x_1 + x_2 + x_1^3]$. This is an algebraic $\bar{P}$ solution for system (3.5) as detailed hereafter.

1. The Jacobian of $p(x)$ is

$$p_x(x) = \begin{bmatrix} 2 + 3x_1^2 & 1 \\ 1 + 3x_1^2 & 1 \end{bmatrix} \qquad \text{and} \qquad \bar{P} = p_x(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

   is such that

$$x^\top \bar{P} B = 0 \;\Rightarrow x^\top \bar{P} A x < 0, \quad \text{for all} \quad x \neq 0,$$

   with A and B as in (3.6).

2. $p(x)f(x) - lp(x)g(x)g(x)^\top p(x)^\top = -x^\top \begin{bmatrix} l(x_1^2+1)^2 - x_1^2(x_1^2+2) & l(x_1^2+1) - x_1^2 - 1 \\ l(x_1^2+1) - x_1^2 - 1 & l-1 \end{bmatrix} x$
   $\triangleq x^\top \Gamma(x) x$, where the matrix $\Gamma(x)$ is positive definite if and only if

   (a) $l(x_1^2+1)^2 - x_1^2(x_1^2+2) > 0 \;\Rightarrow\; l > \dfrac{x_1^2(x_1^2+2)}{(x_1^2+1)^2}$.

   Note that the term on the right hand side is upper bounded by one hence this condition is globally satisfied for $l > 1$;

   (b) $\det \Gamma(x) = l - 1 > 0 \;\Rightarrow l > 1$.

   Therefore $\Gamma(x)$ is a positive definite matrix for all $l > 1$.

$p(x)$ is therefore a global algebraic $\bar{P}$ solution with index $\bar{l} = 1$. Figure 3.4 and Figure 3.5 illustrate conditions (3.2) and (3.1) for this system, respectively.

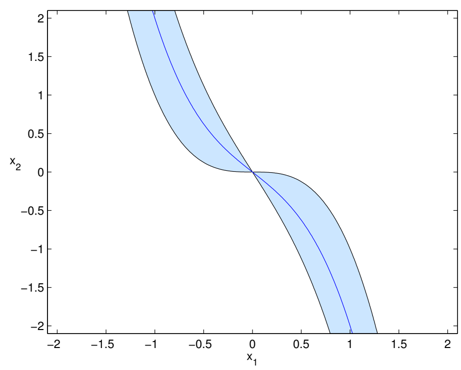


Figure 3.4: The sets I_{Pg}^0, I_{Pf}^- for system (3.5). Note that condition (3.2) is satisfied since the set I_{Pg}^0 (blu line) is inside the set I_{Pf}^- (light blue region).

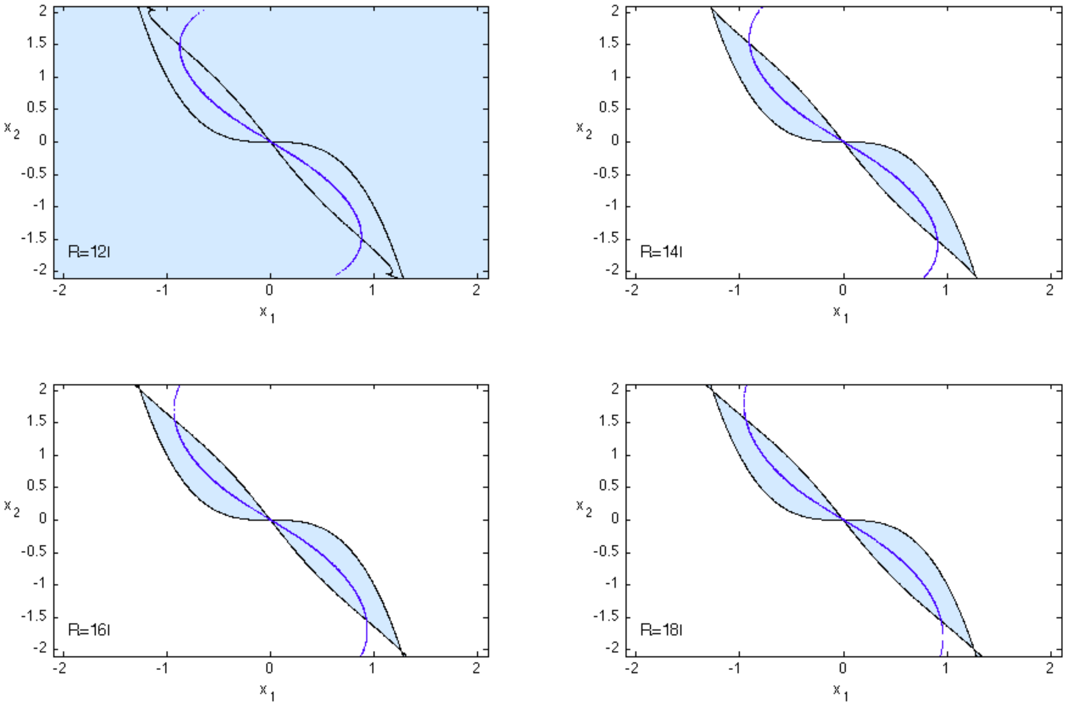


Figure 3.5: The sets I^0_{Vg}, I^-_{Vf} for system (3.5) and different values of R. Note that for each plot, condition (3.1) is satisfied if the set I^0_{Vg} (blu line) is inside the set I^-_{Vf} (light blue region). Therefore, for this system, condition (3.1) is locally satisfied for each value of R used in the plots.

A 3-dimentional linear system

It is interesting to verify if the previous considerations are valid also in the 3-dimensional case. In fact if this is true then it is very likely that these are general properties for any values of n. Of course the graphical representation is more difficult in this case since I^0_{Pg} and I^0_{Vg} are 2-dimensional sets, colored in red, while the sets I^-_{Pf} and I^-_{Vf} are 3-dimensional. Therefore only the boundaries of the sets I^-_{Pf} and I^-_{Vf} are represented in the following figures, i.e. the 2-dimensional sets $\partial I^t_P = \{x \neq 0 \ s.t. \ p(x)f(x) = 0\}$ and $\partial I^t_V = \{x \neq 0 \ s.t. \ V_x(x, \xi(x))f(x) = 0\}$.

Consider the 3-dimensional system

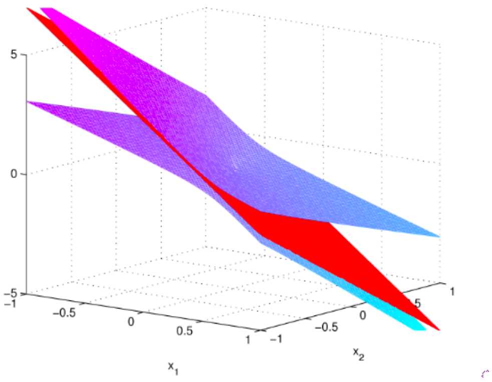
$$\begin{aligned} \dot{x}_1 &= x_2 + x_1, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= u, \end{aligned} \tag{3.8}$$

and let

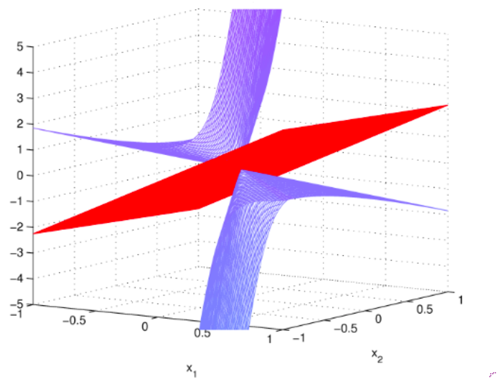
$$\bar{P} = \begin{bmatrix} 21 & 14 & 4 \\ 14 & 10 & 3 \\ 4 & 3 & 1 \end{bmatrix}.$$

Figure 3.6 illustrates conditions (3.2) and (3.1) for different values of R. Note that condition (3.2) does not depend on R, therefore the sets I^0_{Pg} and I^-_{Pf} are the same in each figure, but they are reported from different points of view to facilitate the comparison with I^-_{Vf} and I^0_{Vg}.

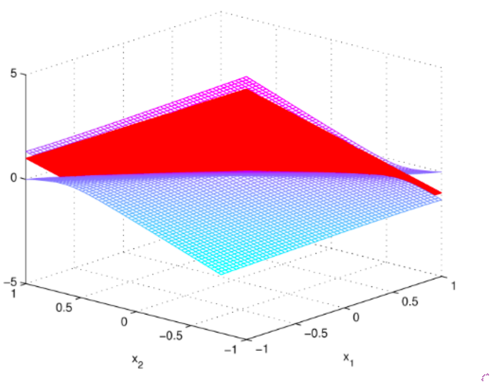
Note that for $R = 15I$ condition (3.1) is satisfied, since the set I^0_{Vg} is inside I^-_{Vf}, but I^0_{Vg} and I^-_{Vf} are very different from I^0_{Pg} and I^-_{Pf}. Moreover for $R = 20I$ condition (3.1) is not satisfied since the set I^0_{Vg} is not inside I^-_{Vf}. On the other hand, for larger values of R the sets I^0_{Vg} and I^-_{Vf} become more and more similar to I^0_{Pg} and I^-_{Pf} thus ensuring the validity of condition (3.1). For example for $R = 50I$ they are pratically undistinguishable.



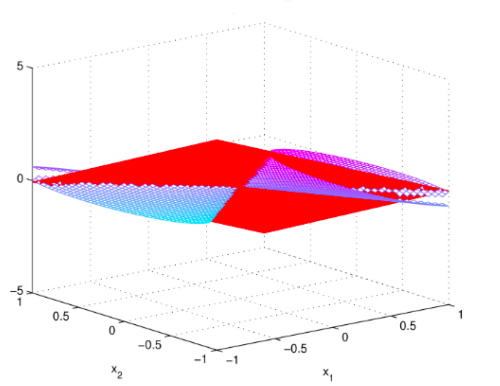
(a) The sets I^0_{Pg}, I^-_{Pf}



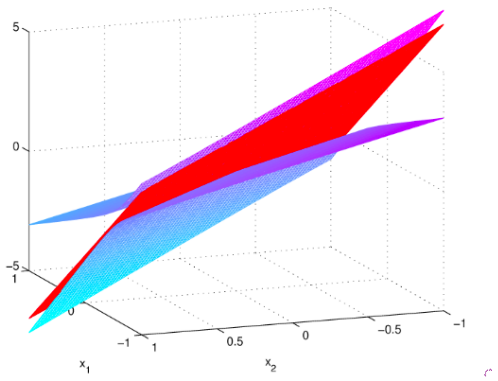
(b) The sets I^0_{Vg}, I^-_{Vf} for $R = 15I$



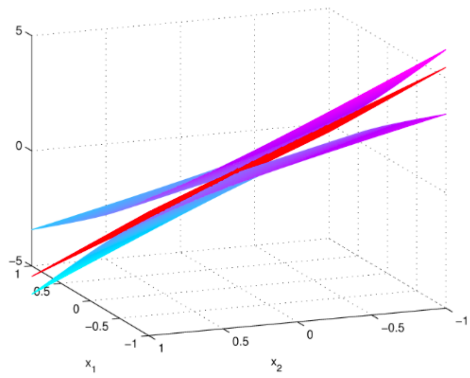
(c) The sets I^0_{Pg}, I^-_{Pf}



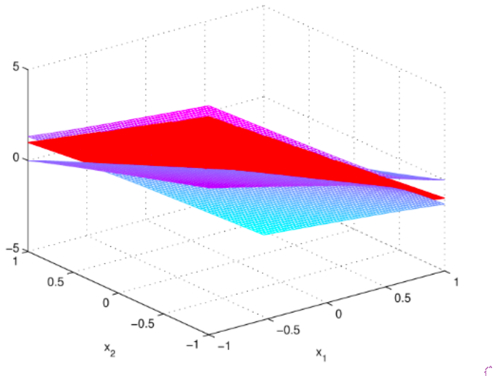
(d) The sets I^0_{Vg}, I^-_{Vf} for $R = 20I$



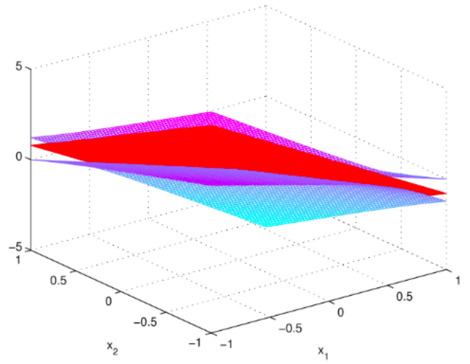
(e) The sets I^0_{Pg}, I^-_{Pf}



(f) The sets I^0_{Vg}, I^-_{Vf} for $R = 25I$



(g) The sets I^0_{Pg}, I^-_{Pf}



(h) The sets I^0_{Vg}, I^-_{Vf} for $R = 50I$

Figure 3.6: The sets I^0_{Pg}, I^-_{Pf}, I^0_{Vg} and I^-_{Vf} for system (3.8) and different values of R. Note that condition (3.2) is satisfied since the set I^0_{Pg}, red plane, is inside the boundary of set I^-_{Pf}, light blue region. These sets are reported in figures (a), (c), (e) and (g) from different perspective. On the other hand condition (3.1) is satisfied only if the set I^0_{Vg}, red plane, is inside the boundary of set I^-_{Vf}, light blue region. These sets are reported in figures (b), (d), (f) and (h) for different values of R. Note that condition (3.1) is not satisfied for $R = 20I$, while for $R = 50I$ the sets I^0_{Vg} and I^-_{Vf} are identical to the sets I^0_{Pg}, I^-_{Pf}.

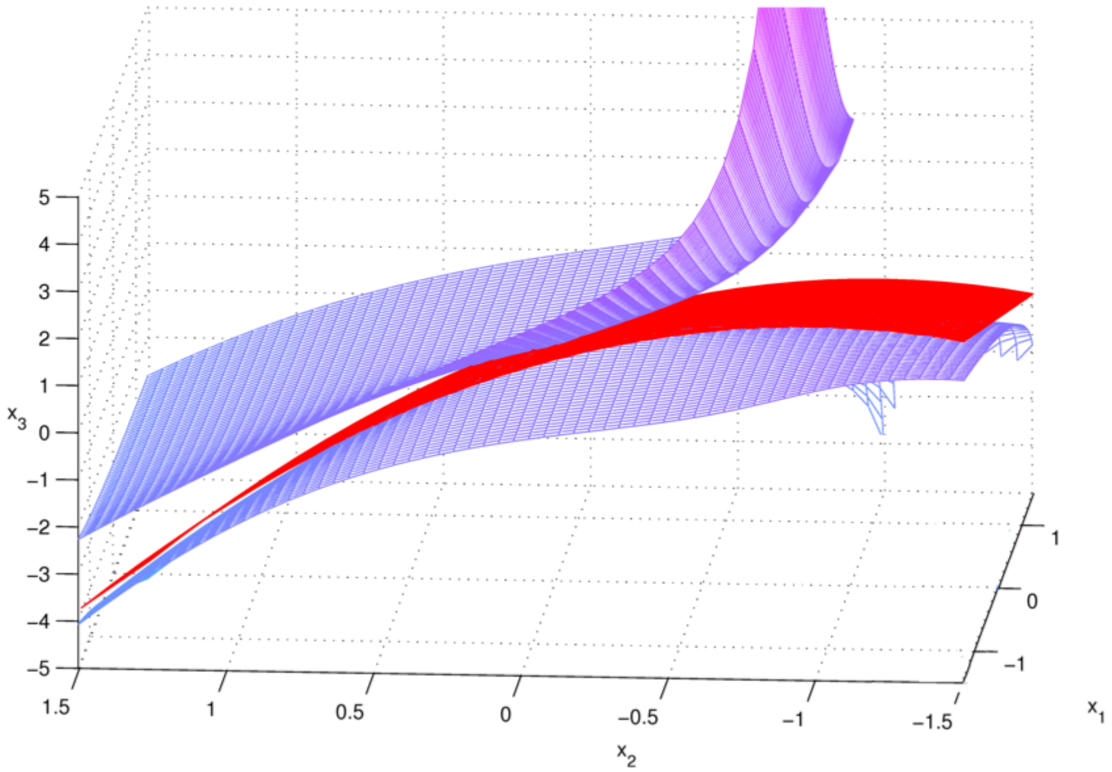
A 3-dimentional nonlinear system

Consider the 3-dimensional system

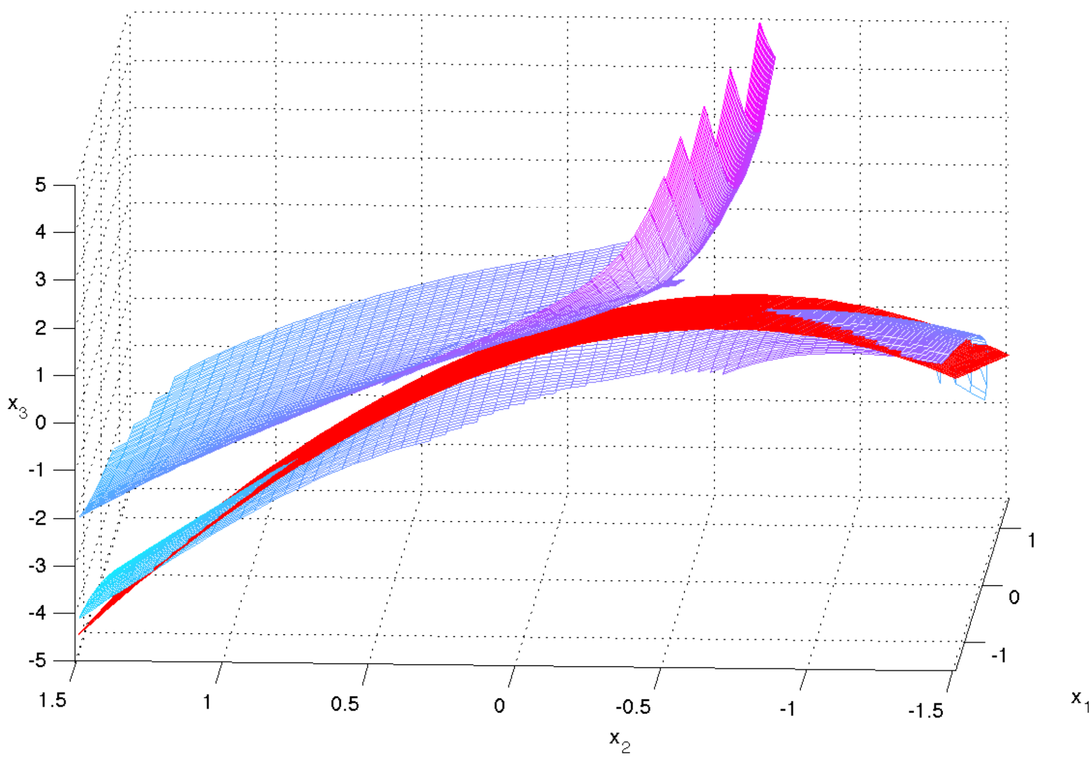
$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3 + x_2^2, \\ \dot{x}_3 &= u, \end{aligned} \tag{3.9}$$

and let

$$\bar{P} = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 5 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$



(a) The sets I^0_{Pg}, I^-_{Pf}



(b) The sets I^0_{Vg}, I^-_{Vf} for $R = 25I$

Figure 3.7: Graphical illustration of conditions (3.2) and (3.1) for $R = 25I$

Then the vector

$$p(x) = \left[\ x_2^2 + 3x_2 + 3x_1 + x_3, \quad x_1 + x_2 + (2x_2 + 2)(x_2^2 + 2x_2 + x_1 + x_3), \quad x_2^2 + 2x_2 + x_1 + x_3\ \right],$$

is a local algebraic $\bar{P}$ solution[2] for system (3.9).

Figures 3.7 gives a graphical illustration of conditions (3.2) and (3.1), for $R = 25I$. Note that condition (3.2) is satisfied since the set I^0_{Pg} (red) is inside the boundary of set I^-_{Pf} (light blue region), see Figure (a). On the other hand condition (3.1) is satisfied if the set I^0_{Vg} (red) is inside the boundary of set I^-_{Vf} (light blue region), see Figure (b). Hence condition (3.1) is locally satisfied for $R = 25I$.

All previous examples suggest that, since the sets I^0_{Vg} and I^-_{Vf} are similar to the sets I^0_{Pg} and I^-_{Pf}, there must be a relation between condition (3.1) and (3.2). Moreover it seems that the similarity between these sets increases when R increases. The aim of this chapter is to formalize and prove this intuition. In the following section the linear case is analyzed, while in Section 3.2 the nonlinear case is studied.

3.1 Linear Systems

Consider a linear stabilizable system, described by the equation

$$\dot{x} = Ax + Bu. \tag{3.10}$$

Theorem 2.0.3, guarantees that there exists a matrix $\bar{P}$ such that

$$x^\top \bar{P} B = 0 \quad \Rightarrow \quad x^\top \bar{P} A x < 0\ , \quad \text{for all} \quad x \neq 0. \tag{3.11}$$

The aim of this section is to find a sufficient condition on R such that (3.11) implies that $V(x,\xi) = \xi^\top \bar{P} x + \frac{1}{2}\|x - \xi\|^2_R$ is a (global) Dynamic CLF for system (3.10), or equivalently that

$$\left\{\begin{array}{rcl} V_\xi(x,\xi) & = & 0 \\ V_x(x,\xi)B & = & 0 \end{array}\right. \quad \Rightarrow \quad V_x(x,\xi)Ax < 0 \text{ for all } (x,\xi) \neq (0,0). \tag{3.12}$$

Following the same procedure used for the examples of the previous section, condition $V_\xi = 0$ can be used to express ξ as a function of x, i.e. $\xi = \xi(x)$. Therefore condition (3.12) rewrites as

$$V_x(x,\xi(x))B = 0 \quad \Rightarrow V_x(x,\xi(x))Ax < 0, \quad \text{for all } x \neq 0. \tag{3.13}$$

Note now that, in the linear case, the partial derivatives of $V(x,\xi)$ are

$$\begin{aligned} V_x &= \xi^\top \bar{P} + (x - \xi)^\top R = x^\top \bar{P} + (x - \xi)^\top (R - \bar{P}), \\ V_\xi &= x^\top \bar{P} - (x - \xi)^\top R, \end{aligned}$$

therefore, imposing condition $V_\xi = 0$ yields $(x - \xi)^\top = x^\top \bar{P} R^{-1}$. Set

$$\begin{aligned} N &= \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R} \text{ s.t. } \ (x - \xi)^\top = x^\top \bar{P} R^{-1}\} \\ &= \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R} \text{ s.t. } \ \xi = (I - \bar{P} R^{-1})^\top x \triangleq \xi(x)\}, \end{aligned}$$

[2]See Remark 1.3.

then the partial derivative $V_x(x,\xi)$, restricted to $V_\xi = 0$, is

$$\begin{aligned} V_x(x,\xi(x)) = V_x(x,\xi)\big|_N &= \left[x^\top \bar{P} + (x-\xi)^\top (R-\bar{P})\right]\big|_N = \left[x^\top \bar{P} + x^\top \bar{P}R^{-1}(R-\bar{P})\right] \\ &= \left[x^\top \bar{P} + x^\top \bar{P}(I - R^{-1}\bar{P})\right] = x^\top \left[\bar{P}(2I - R^{-1}\bar{P})\right] \\ &= x^\top (2I - \bar{P}R^{-1})\bar{P} = x^\top T\bar{P} \end{aligned}$$

where $T \triangleq (2I - \bar{P}R^{-1})$. As a result condition (3.13) is equivalent to

$$x^\top T\bar{P}B = 0 \quad \Rightarrow \quad x^\top T\bar{P}Ax < 0, \quad \text{for all} \quad x \neq 0. \tag{3.14}$$

It is important to remark a few facts.

1. As suggested by the graphical analysis, condition (3.14) is similar to condition (3.11), the only difference being the presence of the matrix T. As a consequence of the presence of this matrix, the sets I^0_{Vg} and I^-_{Vf} are a perturbation of I^0_{Pg} and I^-_{Pf}. The entity of the perturbation depends on the distance of T from a multiple of the identity matrix.

2. Imposing $R^{-1} = \alpha\bar{P}^{-1}$, as in the previous section, yields $T = (2-\alpha)I$. Therefore, if $2 - \alpha > 0$, the two conditions are equivalent. This is another proof that $V(x,\xi)$ is a CLF if $R = \dfrac{\bar{P}}{\alpha}$ and $0 < \alpha < 2$.

3. Condition (3.14) is satisfied provided that T is close enough to a multiple of the identity matrix. For instance, let $T = (\beta I + \gamma D)^{-1}$ with $D = D^\top$, $\beta > 0$ and $\gamma > 0$ arbitrarily small. If $z^\top = x^\top T$, then

$$x^\top T\bar{P}B = 0 \quad \Rightarrow \quad z^\top \bar{P}B = 0 \quad \Rightarrow \quad z^\top \bar{P}Az = -z^\top Qz,$$

   with $Q = Q^\top > 0$, where we used condition (3.11). Then

$$x^\top T\bar{P}Ax = z^\top \bar{P}AT^{-1}z = z^\top \bar{P}A(\beta I + \gamma D)z = \beta z^\top \bar{P}Az + \gamma z^\top \bar{P}ADz = \gamma z^\top (-\frac{\beta}{\gamma}Q + \bar{P}AD)z$$

   which is negative for arbitrary $x \neq 0$ and for sufficiently large[3] β/γ.

4. The fact that $V(x,\xi)$ can be a CLF also for values of T different from the identity matrix is important since, in the nonlinear case, we will find a similar scenario in which however T depends on x and ξ. Therefore we will not be able to guarantee that $T(x,\xi) = \alpha I$ but we will require that $T(x,\xi) \cong \alpha I$, at least locally around the origin.

We now characterize more precisely the values of R for which T is sufficiently close to a multiple of the identity matrix, i.e. we want to find a sufficient condition on R such that condition (3.11) implies condition (3.14). To do that it is convenient to rewrite both conditions (3.11) and (3.14) as conditions of negative definiteness of certain matrices. To this end it is useful to introduce the definition of orthogonal matrix and recall some of its properties.

[3]This fact is an immediate consequence of Lemma A.2.2.

Definition 3.1.1. *Given a matrix $A \in \mathbb{R}^{n\times m}$, with full column rank, we denote by the symbol $A^{\perp} \in \mathbb{R}^{(n-m)\times n}$ a matrix satisfying $A^{\perp}A = 0_{(n-m)\times m}$ and the rows of which are a basis for the left null subspace of A.*

Note that the matrix $A^{\perp}$ is not unique, as for example any scalar multiple of $A^{\perp}$ has the same properties. This implies that without loss of generality we can always impose $\|A^{\perp}\| = 1$. In the following we use the symbol $A^{\perp\!\!\!-}$ to indicate the transpose of $A^{\perp}$.

Lemma 3.1.1. *Suppose $A \in \mathbb{R}^{n\times m}$ and let $T \in \mathbb{R}^{n\times n}$ be an invertible matrix then $(TA)^{\perp} = A^{\perp}T^{-1}$, i.e.* $\mathrm{span}(TA)^{\perp} = \mathrm{span}(A^{\perp}T^{-1})$.

Proof. Let $x^{\top} \in \mathrm{span}(TA)^{\perp}$ then $x^{\top}TA = 0 \Rightarrow x^{\top}T \in \mathrm{span}(A^{\perp}) \Rightarrow x^{\top}T = y^{\top}A^{\perp} \Rightarrow x^{\top} = y^{\top}A^{\perp}T^{-1} \Rightarrow x^{\top} \in \mathrm{span}(A^{\perp}T^{-1})$. Viceversa let $x^{\top} \in \mathrm{span}(A^{\perp}T^{-1}) \Rightarrow x^{\top} = y^{\top}A^{\perp}T^{-1} \Rightarrow x^{\top}T = y^{\top}A^{\perp} \Rightarrow x^{\top}TA = 0 \Rightarrow x^{\top} \in \mathrm{span}((TA)^{\perp})$. □

We are now ready to prove the main theorem of this section.

Theorem 3.1.1. *Let $A \in \mathbb{R}^{n\times n}, \bar{P} \in \mathbb{R}^{n\times n}, R \in \mathbb{R}^{n\times n}$, with $\bar{P}$ and R symmetric and positive definite matrices, let $B \in \mathbb{R}^{n\times m}$ and $T = (2I - \bar{P}R^{-1})$. Suppose that T is invertible and let $Q = -(\bar{P}B)^{\perp}(A^{\top}\bar{P} + \bar{P}A)(\bar{P}B)^{\perp\!\!\!-}$, with $\|(\bar{P}B)^{\perp}\| = 1$. Then if*

$$\underline{\sigma}(R) > \frac{\|\bar{P}\|\left(2\|\bar{P}A\| + \underline{\sigma}(Q)\right)}{2\underline{\sigma}(Q)}, \tag{3.15}$$

condition

$$x^{\top}\bar{P}B = 0 \Rightarrow x^{\top}(\bar{P}A + A^{\top}\bar{P})x < 0 \quad , \ x \neq 0 \tag{3.16}$$

implies condition

$$x^{\top}T\bar{P}B = 0 \quad \Rightarrow \quad x^{\top}T\bar{P}Ax < 0 \quad , \ x \neq 0. \tag{3.17}$$

Proof. With the notation introduced above it is possible to rewrite[4] condition $x^{\top}\bar{P}B = 0$ as $x^{\top} = v^{\top}(\bar{P}B)^{\perp}$, with $v \in \mathbb{R}^{n-m}$. Substituting in the inequality (3.16) yields

$$x^{\top}\bar{P}B = 0,\ x \neq 0 \Rightarrow x^{\top}(\bar{P}A + A^{\top}\bar{P})x < 0$$

$$\Updownarrow$$

$$v^{\top}(\bar{P}B)^{\perp}(A^{\top}\bar{P} + \bar{P}A)(\bar{P}B)^{\perp\!\!\!-}v < 0 \text{ for all } v \neq 0.$$

Hence (3.16) is equivalent to

$$(\bar{P}B)^{\perp}(A^{\top}\bar{P} + \bar{P}A)(\bar{P}B)^{\perp\!\!\!-} = -Q < 0, \tag{3.18}$$

where $Q \in \mathbb{R}^{(n-m)\times(n-m)}$, $Q = Q^{\top} > 0$.

By repeating the same procedure for (3.17) the equivalence

[4] Note that $\bar{P}B \in \mathbb{R}^{n\times m}$ has full column rank since without loss of generality we can always assume that $B \in \mathbb{R}^{n\times m}$ is of full column rank, while $\bar{P} \in \mathbb{R}^{n\times n}$ is a square invertible matrix.

$$x^\top T\bar{P}B = 0 \;\;\Rightarrow\;\; x^\top T\bar{P}Ax < 0,\;\; x \neq 0$$
$$\Updownarrow$$
$$(T\bar{P}B)^\perp(A^\top\bar{P}T^\top + T\bar{P}A)(T\bar{P}B)^{\perp\top} < 0, \tag{3.19}$$

is obtained. Consider the diagram

$$\begin{array}{ccc} (3.16) & & (3.17) \\ \Updownarrow & & \Updownarrow \\ (3.18) & & (3.19) \end{array}$$

To prove that (3.16) $\Rightarrow$ (3.17), it is sufficient to prove that (3.18) $\Rightarrow$ (3.19). Note now that, since T is invertible, $(T\bar{P}B)^\perp = (\bar{P}B)^\perp T^{-1}$, i.e. $\text{span}(T\bar{P}B)^\perp = \text{span}((\bar{P}B)^\perp T^{-1})$, as stated in Lemma 3.1.1. This property yields

$$\begin{aligned}(T\bar{P}B)^\perp(A^\top\bar{P}T^\top + T\bar{P}A)(T\bar{P}B)^{\perp\top} &= (\bar{P}B)^\perp T^{-1}(A^\top\bar{P}T^\top + T\bar{P}A)T^{-\top}(\bar{P}B)^{\perp\top}\\ &= (\bar{P}B)^\perp(T^{-1}A^\top\bar{P} + \bar{P}AT^{-\top})(\bar{P}B)^{\perp\top}\\ &= (\bar{P}B)^\perp((\lambda I + M)A^\top\bar{P} + \bar{P}A(\lambda I + M^\top))(\bar{P}B)^{\perp\top}\\ &= \lambda(\bar{P}B)^\perp(A^\top\bar{P} + \bar{P}A)(\bar{P}B)^{\perp\top} +\\ &\quad + (\bar{P}B)^\perp(MA^\top\bar{P} + \bar{P}AM^\top)(\bar{P}B)^{\perp\top}\end{aligned}$$

where $M \triangleq T^{-1} - \lambda I$, and hence $T^{-1} = \lambda I + T^{-1} - \lambda I = \lambda I + M$, for an arbitrary $\lambda > 0$. Condition (3.18) implies that $(\bar{P}B)^\perp(A^\top\bar{P} + \bar{P}A)(\bar{P}B)^{\perp\top} = -Q$, and hence

$$\begin{aligned}(T\bar{P}B)^\perp(A^\top\bar{P}T^\top + T\bar{P}A)(T\bar{P}B)^{\perp\top} &= -\lambda Q + (\bar{P}B)^\perp(MA^\top\bar{P} + \bar{P}AM^\top)(\bar{P}B)^{\perp\top}\\ &= \lambda\left[-Q + (\bar{P}B)^\perp(\tilde{M}A^\top\bar{P} + \bar{P}A\tilde{M}^\top)(\bar{P}B)^{\perp\top}\right]\end{aligned}$$

where

$$\tilde{M} \triangleq \frac{M}{\lambda} = \frac{T^{-1} - \lambda I}{\lambda} = \frac{T^{-1}}{\lambda}(I - \lambda T) = \frac{T^{-1}}{\lambda}((1 - 2\lambda)I + \lambda\bar{P}R^{-1}). \tag{3.20}$$

The original problem is then reduced to find a sufficient condition that guarantees that

$$\left[-Q + (\bar{P}B)^\perp(\tilde{M}A^\top\bar{P} + \bar{P}A\tilde{M}^\top)(\bar{P}B)^{\perp\top}\right] < 0. \tag{3.21}$$

Using the fact that[5] $Q \geqslant \underline{\sigma}(Q)I$ and by the standard matrix norm properties, the left hand side in (3.21) can be bounded from above as

$$-Q + (\bar{P}B)^\perp(\tilde{M}A^\top\bar{P} + \bar{P}A\tilde{M}^\top)(\bar{P}B)^{\perp\top} \leqslant \left[-\underline{\sigma}(Q) + 2\|(\bar{P}B)^\perp\|^2\|\bar{P}A\|\|\tilde{M}\|\right]I.$$

Therefore, to satisfy (3.21), it is sufficient to have

$$-\underline{\sigma}(Q) + 2\|(\bar{P}B)^\perp\|^2\|\bar{P}A\|\|\tilde{M}\| < 0,$$

or equivalently

$$\|\tilde{M}\| < \frac{\underline{\sigma}(Q)}{2\|(\bar{P}B)^\perp\|^2\|\bar{P}A\|} = \frac{\underline{\sigma}(Q)}{2\|\bar{P}A\|} \tag{3.22}$$

[5] See the Appendix.

where, without loss of generality, it has been assumed that $\|(\bar{P}B)^{\perp}\| = 1$.

Equation (3.20) yields

$$\begin{aligned}\|\tilde{M}\| &\leqslant \frac{\|T^{-1}\|}{\lambda}\|(1-2\lambda)I + \lambda\bar{P}R^{-1}\| \\ &\leqslant \frac{\|T^{-1}\|}{\lambda}\left[|1-2\lambda| + \lambda\|\bar{P}\|\|R^{-1}\|\right] = \frac{1}{\lambda\underline{\sigma}(T)}\left[|1-2\lambda| + \lambda\|\bar{P}\|\|R^{-1}\|\right].\end{aligned}$$

Hence, to satisfy (3.22), it is sufficient to have

$$\frac{1}{\lambda\underline{\sigma}(T)}\left[|1-2\lambda| + \lambda\|\bar{P}\|\|R^{-1}\|\right] < \frac{\underline{\sigma}(Q)}{2\|\bar{P}A\|},$$

or equivalently

$$\left[|1-2\lambda| + \lambda\|\bar{P}\|\|R^{-1}\|\right] < \lambda\frac{\underline{\sigma}(Q)\underline{\sigma}(T)}{2\|\bar{P}A\|}. \tag{3.23}$$

Finally, using $\underline{\sigma}(A \pm B) \geqslant \underline{\sigma}(A) - \bar{\sigma}(B)$[6], yields

$$\underline{\sigma}(T) = \underline{\sigma}(2I - \bar{P}R^{-1}) \geqslant \underline{\sigma}(2I) - \bar{\sigma}(\bar{P}R^{-1}) = 2 - \|\bar{P}R^{-1}\| \geqslant 2 - \|\bar{P}\|\|R^{-1}\|.$$

Hence, to satisfy (3.23), it is sufficient to have

$$\left[|1-2\lambda| + \lambda\|\bar{P}\|\|R^{-1}\|\right] < \frac{\underline{\sigma}(Q)}{2\|\bar{P}A\|}\lambda(2 - \|\bar{P}\|\|R^{-1}\|).$$

Dividing by $\lambda > 0$ leads to

$$\left[\left|\frac{1}{\lambda} - 2\right| + \|\bar{P}\|\|R^{-1}\|\right] < \frac{\underline{\sigma}(Q)}{2\|\bar{P}A\|}(2 - \|\bar{P}\|\|R^{-1}\|),$$

which can be rewritten as

$$\|\bar{P}\|\|R^{-1}\|\left(1 + \frac{\underline{\sigma}(Q)}{2\|\bar{P}A\|}\right) < \frac{\underline{\sigma}(Q)}{\|\bar{P}A\|} - \left|2 - \frac{1}{\lambda}\right|. \tag{3.24}$$

Since λ was an arbitrary positive parameter, it is convenient to maximize the right hand side of inequality (3.24) by choosing $\lambda = 0.5$, so that it becomes

$$\|\bar{P}\|\|R^{-1}\|\left(1 + \frac{\underline{\sigma}(Q)}{2\|\bar{P}A\|}\right) < \frac{\underline{\sigma}(Q)}{\|\bar{P}A\|}. \tag{3.25}$$

Finally (3.25) can be rewritten as

$$\|R^{-1}\| < \frac{2\underline{\sigma}(Q)}{\|\bar{P}\|\left(2\|\bar{P}A\| + \underline{\sigma}(Q)\right)},$$

or equivalently

$$\underline{\sigma}(R) > \frac{\|\bar{P}\|\left(2\|\bar{P}A\| + \underline{\sigma}(Q)\right)}{2\underline{\sigma}(Q)}.$$

[6]see Section A.1.

Therefore, if condition (3.15) is satisfied, then (3.18) $\Rightarrow$ (3.19). Hence the following diagram

$$\begin{array}{ccc} (3.16) & & (3.17) \\ \Updownarrow & & \Updownarrow \\ (3.18) & \Rightarrow & (3.19) \end{array}$$

implies the relation: (3.16) $\Rightarrow$ (3.17). $\square$

Note that condition (3.15) can be satisfied by choosing a sufficiently large R. This result is in line with the fact that, for very large R

$$T = 2I - \bar{P}R^{-1} \cong 2I \quad \Rightarrow \quad \tilde{M} \cong \left(\frac{1}{2\lambda} - 1\right) I,$$

hence (3.21) becomes

$$-Q + (\bar{P}B)^{\perp}(\tilde{M}A^{\top}\bar{P} + \bar{P}A\tilde{M}^{\top})(\bar{P}B)^{\perp} \cong -\left(1 + \frac{1}{2\lambda} - 1\right) Q = -\frac{Q}{2\lambda} < 0.$$

The results discussed in this section can be summarized in the following statement.

Theorem 3.1.2. *Consider the linear, time-invariant, system (2.6) and suppose that the origin is stabilizable. Let* $\bar{P} = \bar{P}^{\top} > 0$ *be a solution of (2.7). Consider the extended system (2.10) and the function* $V(x,\xi) = \xi^{\top}\bar{P}x + \frac{1}{2}\|x - \xi\|_R^2$. *Let* $Q = -(\bar{P}B)^{\perp}(A^{\top}\bar{P} + \bar{P}A)(\bar{P}B)^{\perp}$, *assume* $\|(\bar{P}B)^{\perp}\| = 1$ *and suppose that* $R = R^{\top} > 0$ *satisfies*

$$\underline{\sigma}(R) > \frac{\|\bar{P}\|\left(2\|\bar{P}A\| + \underline{\sigma}(Q)\right)}{2\underline{\sigma}(Q)}. \tag{3.26}$$

Then the function $V(x,\xi)$ *is a global CLF for the extended system.*

Proof. To verify that $V(x,\xi)$ is positive definite it is sufficient to prove that (3.26) implies $R > \frac{\bar{P}}{2}$ and than use Theorem 2.1.1. To this end note that $Q = -(\bar{P}B)^{\perp}(A^{\top}\bar{P}+\bar{P}A)(\bar{P}B)^{\perp}$, with $\|(\bar{P}B)^{\perp}\| = 1$, implies

$$\underline{\sigma}(Q) \leqslant \bar{\sigma}(Q) = \|Q\| \leqslant 2\|(\bar{P}B)^{\perp}\|^2\|\bar{P}A\| = 2\|\bar{P}A\|.$$

Hence

$$\underline{\sigma}(R) > \frac{\|\bar{P}\|\left(2\|\bar{P}A\| + \underline{\sigma}(Q)\right)}{2\underline{\sigma}(Q)} \geqslant \frac{\|\bar{P}\|\left(\underline{\sigma}(Q) + \underline{\sigma}(Q)\right)}{2\underline{\sigma}(Q)} = \|\bar{P}\| = \bar{\sigma}(\bar{P}) > \frac{\bar{\sigma}(\bar{P})}{2}$$

which implies $R > \frac{\bar{P}}{2}$. Note that this condition implies also the invertibility of T. In fact

$$\det(T) = \det(2I - \bar{P}R^{-1}) = \det(2R - \bar{P})\det(R^{-1}) = 2^n \det\left(R - \frac{\bar{P}}{2}\right)\det(R^{-1}),$$

and, since $R - \frac{\bar{P}}{2} > 0$, $\det(R - \frac{\bar{P}}{2}) > 0$ and $\det(T) \neq 0$. The proof is then straightforward since, from Theorem 3.1.1, $V(x,\xi)$ satisfies also the Property 2 of a CLF. $\square$

3.2 Nonlinear Systems

Consider now a nonlinear affine system described by the equation

$$\dot{x} = f(x) + g(x)u, \tag{3.27}$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be at least continuously differentiable and $f(0) = 0$. Moreover suppose that the linearized system around the origin is stabilizable, i.e. there exists a positive definite matrix $\bar{P}$ such that:

$$x^\top \bar{P}B = 0 \;\Rightarrow\; x^\top \bar{P}Ax < 0 \quad \text{for all} \;\; x \neq 0, \tag{3.28}$$

where $A = \frac{\partial f}{\partial x}|_{x=0}$ and $B = g(0)$.

The aim of this section is to find a sufficient condition on R such that (3.28) implies that $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$ is a local Dynamic CLF, or equivalently that

$$\left\{ \begin{array}{rcl} V_\xi(x,\xi) & = & 0 \\ V_x(x,\xi)g(x) & = & 0 \end{array} \right. \;\Rightarrow\; V_x(x,\xi)f(x) < 0 \text{ for all } (x,\xi) \in \Omega\backslash(0,0), \tag{3.29}$$

where $\Omega \subset \mathbb{R}^{2n}$ is an open set, $0 \in \Omega$. In the introduction of this chapter it has already been proved that there exists a neighborhood of the origin $\Omega_x \subset \mathbb{R}^n$ such that, condition $V_\xi = 0$ implies the existence of a function $\xi(x)$ such that $V_\xi(x,\xi(x)) = 0$ for all $x \in \Omega_x$. Therefore, in the set Ω_x, (3.29) is equivalent to:

$$V_x(x,\xi(x))g(x) = 0 \;\;\Rightarrow\;\; V_x(x,\xi(x))f(x) < 0 \;\text{ for all } x \in \Omega_x\backslash\{0\}. \tag{3.30}$$

In the nonlinear case the partial derivatives of $V(x,\xi)$ are

$$\begin{aligned} V_x &= p(\xi) + (x-\xi)^\top R = p(x) + (x-\xi)^\top (R - \Phi(x,\xi))^\top \\ V_\xi &= x^\top p_\xi(\xi) - (x-\xi)^\top R \end{aligned}$$

where $\Phi(x,\xi)$ is defined as in Lemma A.3.2. Hence, the equation $V_\xi = 0$ cannot be explicitly solved for ξ. However note that $(x-\xi)^\top = x^\top p_\xi(\xi)R^{-1}$ implies $V_\xi = 0$. Substituting $(x-\xi)^\top$ with $x^\top p_\xi(\xi)R^{-1}$ in

$$V_x(x,\xi) = x^\top \tilde{P}(x) + (x-\xi)^\top (R - \Phi(x,\xi))^\top$$

yields

$$\begin{aligned} x^\top \left[\tilde{P}(x) + p_\xi(\xi)R^{-1}(R-\Phi(x,\xi))^\top\right] &= x^\top \left[I + p_\xi(\xi)R^{-1}(R-\Phi(x,\xi))^\top \tilde{P}(x)^{-1}\right]\tilde{P}(x) \\ &= x^\top T(x,\xi)\tilde{P}(x), \end{aligned} \tag{3.31}$$

where $T(x,\xi) \triangleq I + p_\xi(\xi)R^{-1}(R-\Phi(x,\xi))^\top \tilde{P}(x)^{-1}$. Note that (3.31) alone is not equivalent to V_x restricted to $V_\xi = 0$, since we do not totally use the condition $V_\xi = 0$ to substitute the variable ξ with the function $\xi(x)$. In other words V_x restricted to $V_\xi = 0$ is equivalent to equation (3.31) restricted to $V_\xi = 0$. Note, in fact, that equation (3.31) still depends both on x and ξ. However if the implication

$$x^\top T(x,\xi)\tilde{P}(x)g(x) = 0 \;\;\Rightarrow x^\top T(x,\xi)\tilde{P}(x)F(x)x < 0 \;\text{ for all }\; (x,\xi) \in \Omega\backslash(0,0) \tag{3.32}$$

holds then it also holds for the values of $(x, \xi) \in \Omega \backslash (0,0)$ such that $V_\xi = 0$. Therefore (3.32) is a sufficient condition for (3.30).

Similarly to the linear case we now want to derive a sufficient condition on R such that (3.28) implies (3.32). To do that it is useful to extend the definition of orthogonal matrix to a matrix-valued function.

Definition 3.2.1. *Let $A(x) \in \mathbb{R}(x)^{n \times m}$ be a continuous matrix-valued function defined in Ω. Suppose that $A(x)$ is a full column rank matrix for each $x \in \Omega$. The matrix $A(x)^\perp$ is a matrix-valued function such that, for each $\bar{x} \in \Omega$, $A(\bar{x})^\perp$ is an orthogonal matrix of $A(\bar{x})$, as defined in Definition 3.1.1.*

Note that $A(x)^\perp$ satisfies the following properties.

Lemma 3.2.1. *Let $A(x)^\perp \in \mathbb{R}(x)^{(n-m) \times n}$ be an orthogonal matrix to the continuous matrix-valued function $A(x) \in \mathbb{R}(x)^{n \times m}$ in Ω. Then, also the continuous matrix-valued function*

$$A_N(x) = \frac{A(x)^\perp}{\|A(x)^\perp\|}$$

is an orthogonal matrix to $A(x)$. Therefore it is always possible to choose $A(x)^\perp$ such that $\|A(x)^\perp\| = 1$.

Proof. Note that $A(x)^\perp$ has constant rank $n - m$ in Ω therefore $\|A(x)^\perp\| = \bar{\sigma}(A(x)^\perp) \neq 0$ for all $x \in \Omega$. Hence, since $A(x)^\perp$ is a continuous matrix-valued function, also $A_N(x)$ is continuous. Moreover for each $\bar{x} \in \Omega$

1. $A(\bar{x})^\perp A(\bar{x}) = 0 \ \Rightarrow A_N(\bar{x}) A(\bar{x}) = 0$;
2. if the rows of $A(\bar{x})^\perp$ are a basis for the left null subspace of $A(\bar{x})$ then the rows of $A_N(\bar{x})$ are a basis too.

Therefore, for each $\bar{x} \in \Omega$, $A_N(\bar{x})$ is an orthogonal matrix of $A(\bar{x})$, as defined in Definition 3.1.1. Hence $A_N(x)$ is an orthogonal matrix of $A(x)$ as defined in Definition 3.2.1. □

Lemma 3.2.2. *Let $A(x) \in \mathbb{R}^{n \times m}$, $m < n$, be a continuous matrix-valued function and suppose that $A(0)$ has rank m. Then there exists a neighborhood of the origin Ω_x such that $A(x)^\perp = A(0)^\perp + \mathcal{O}(x)$ for all $x \in \Omega_x$. Moreover the function $\mathcal{O}(x)$ is continuous.*

Proof. Note that an arbitrary rectangular matrix $M \in \mathbb{R}^{n \times m}$, $m < n$, has rank m if and only if $\det(M^\top M) \neq 0$.[7] Therefore the hypothesis that $A(0)$ has rank m is equivalent to $\det(A(0)^\top A(0)) \neq 0$. Now since the determinant of a matrix is a continuous function of its elements and $A(x)$ is continuous, the function $d(x) \triangleq \det(A(x)^\top A(x))$ is continuous. Therefore since $d(0) \neq 0$ there exists a neighborhood of the origin Ω_x^1, such that $d(x) \neq 0$ for all $x \in \Omega_x^1$. This fact in particular implies that $A(x)^\top A(x) \in \mathbb{R}^{m \times m}$ is an invertible matrix for all $x \in \Omega_x^1$. Therefore, in Ω_x^1, an explicit formula for an $n \times n$ orthogonal matrix is

$$A_n(x)^\perp = \left(I_n - A(x)\left[A(x)^\top A(x)\right]^{-1} A(x)^\top\right) \tag{3.33}$$

[7] This is due to the fact that $rank(M) = rank(M^\top M)$ and that, since $M^\top M \in \mathbb{R}^{m \times m}$, it has rank m if and only if $\det(M^\top M) \neq 0$. The first condition can be shown by proving the equality of the null spaces of M and $M^\top M$. Indeed, $Mx = 0$ if and only if $M^\top M x = 0$.

In fact

$$A_n(x)^\perp A(x) = (I_n - A(x)\left[A(x)^\top A(x)\right]^{-1} A(x)^\top)A(x) =$$
$$= A(x) - A(x)\left[A(x)^\top A(x)\right]^{-1}\left[A(x)^\top A(x)\right] = 0$$

Note now that (3.33) is a continuous matrix-valued function of x in Ω_x^1, since it is sum and product of continuous functions

$$A_n(x)^\perp = I_n - A(x)\frac{\mathrm{adj}\left[A(x)^\top A(x)\right]}{d(x)}A(x)^\top,$$

and $d(x) \neq 0$ for all $x \in \Omega_x^1$. Therefore

$$A_n(x)^\perp = (I_n - A(0)\left[A(0)^\top A(0)\right]^{-1} A(0)^\top) + \mathcal{O}(x)$$

where $\mathcal{O}(x)$ is continuous, moreover by definition the first term of the previous identity is $A_n(0)^\perp$. Note that $A_n(x)^\perp$ is not an orthogonal matrix as defined in 3.1.1, since it is not of full row rank. However consider the matrix $A_n(0)^\perp$, its row rank is of course $n-m$, therefore we can select $n-m$ independent rows. Define $A(x)^\perp$ the submatrix of $A_n(x)^\perp$ composed by the corresponding rows, then:

1. $A(x)^\perp$ inherits the continuity property from $A_n(x)^\perp$. Therefore $A(x)^\perp = A(0)^\perp + \mathcal{O}(x)$;
2. by the same argument used above, since $A(x)^\perp$ is continuous and $A(0)^\perp$ has full row rank $n-m$, there exists a neighborhood $\Omega_x \subset \Omega_x^1$ such that $A(x)^\perp$ is of full row rank for all $x \in \Omega_x$;
3. $A(x)^\perp A(x) = 0$ since $A(x)^\perp$ is a submatrix of $A_n(x)^\perp$ and $A_n(x)^\perp A(x) = 0$.

This ends the proof since $A(x)^\perp$ is now an orthogonal matrix with full row rank. □

Lemma 3.2.3. *Let $A(x) \in \mathbb{R}(x)^{n\times m}$ be a continuous matrix-valued function with constant rank m and let $T(x) \in \mathbb{R}(x)^{n\times n}$ be an invertible matrix for each value of $x \in \Omega$, where Ω is a neighborhood of the origin. Then $(T(x)A(x))^\perp = A(x)^\perp T(x)^{-1}$ for each value of $x \in \Omega$.*

Proof. The proof is straightforward since, by Lemma 3.1.1, for each value of $\bar{x} \in \Omega$ the following equivalence holds: $(T(\bar{x})A(\bar{x}))^\perp = A(\bar{x})^\perp T(\bar{x})^{-1}$. □

To prove that (3.28) implies (3.32) the following results are needed.

Lemma 3.2.4. *Let $T(x) \in \mathbb{R}(x)^{n\times n}$ be a continuous matrix-valued function then, if $T(0)$ is invertible, there exists a neighborhood of the origin, Ω, such that for all $x \in \Omega$ the matrix $T(x)$ is invertible.*

Proof. The claim is a direct consequence of the fact that the determinant of a matrix is a continuous function of its elements and $T(x)$ is a continuous matrix-valued function. Therefore, by composition, the function $\det\left[T(x)\right] : \mathbb{R}^n \to \mathbb{R}$ is a continuous function of x. Hence, if $\det\left[T(0)\right] \neq 0$ there is a neighborhood of the origin, Ω, such that $\det\left[T(x)\right] \neq 0$ for all $x \in \Omega$. □

Lemma 3.2.5. *Condition:*

$$x^\top \bar{P}B = 0 \;\Rightarrow x^\top A\bar{P}x < 0 \;\text{ for all }\; x \neq 0 \tag{3.34}$$

implies

$$\left[(\tilde{P}(x)g(x))^\perp\right]\left[\tilde{P}(x)F(x) + F(x)^\top \tilde{P}(x)^\top\right]\left[(\tilde{P}(x)g(x))^{\perp}\right] \triangleq -Q(x) < 0, \tag{3.35}$$

locally around $x = 0$.

Proof. Firstly note that the condition (3.34) is equivalent to the fact that

$$\bar{Q} \triangleq -(\bar{P}B)^\perp\left[\bar{P}A + A^\top\bar{P}\right](\bar{P}B)^{\perp} \tag{3.36}$$

is positive definite. Moreover $\tilde{P}(x)g(x)$ is a continuous matrix-valued function and $\tilde{P}(0)g(0) = \bar{P}B$. Without loss of generality assume that B has rank m, hence also $\bar{P}B$ has rank m. Therefore Lemma 3.2.2 can be applied and there exists a neighborhood of the origin Ω_x^1 such that

$$\left[\tilde{P}(x)g(x)\right]^\perp = (\bar{P}B)^\perp + \mathcal{O}(x).$$

Therefore in Ω_x^1 equation (3.35) becomes

$$\begin{aligned}-Q(x) &= \left[(\bar{P}B)^\perp + \mathcal{O}(x)\right]\left[\bar{P}A + A^\top\bar{P} + \mathcal{O}(x)\right]\left[(\bar{P}B)^{\perp} + \mathcal{O}(x)\right] = \\ &= (\bar{P}B)^\perp\left[\bar{P}A + A^\top\bar{P}\right](\bar{P}B)^{\perp} + \mathcal{O}(x) = \\ &= -\bar{Q} + \mathcal{O}(x),\end{aligned} \tag{3.37}$$

for all $x \in \Omega_x^1$, where we used (3.36), $F(0) = A$ and $\tilde{P}(0) = \bar{P}$. Finally, by Lemma A.2.3, there exists a neighborhood of the origin $\Omega_x^2 \subset \Omega_x^1$ such that $-Q(x) \triangleq -\bar{Q} + \mathcal{O}(x)$ is negative definite for all $x \in \Omega_x^2$. □

Lemma 3.2.6. *Assume that*

$$S(x,\xi) \triangleq \left[(T(x,\xi)\tilde{P}(x)g(x))^\perp\right]\left[T(x,\xi)\tilde{P}(x)F(x) + F(x)^\top\tilde{P}(x)^\top T(x,\xi)^\top\right]\left[(T(x,\xi)\tilde{P}(x)g(x))^{\perp}\right]$$

is negative definite for all $(x,\xi) \in \Omega$. *Then:*

$$x^\top T(x,\xi)\tilde{P}(x)g(x) = 0 \quad\Rightarrow\quad x^\top T(x,\xi)\tilde{P}(x)F(x)x < 0 \;\; \textit{for all} \;\; (x,\xi) \in \Omega\backslash(0,0). \tag{3.38}$$

Proof. $x^\top T(x,\xi)\tilde{P}(x)g(x) = 0$ implies $x^\top \in \text{span}(T(x,\xi)\tilde{P}(x)g(x))^\perp$, and hence there exists a vector $v \in \mathbb{R}^{(n-m)}$ such that $x^\top = v^\top (T(x,\xi)\tilde{P}(x)g(x))^\perp$. Note that $x \neq 0$ implies $v \neq 0$. Substituting in the second term of (3.38) yields

$$\begin{aligned}x^\top T(x,\xi)\tilde{P}(x)F(x)x &= x^\top\left[T(x,\xi)\tilde{P}(x)F(x) + F(x)^\top\tilde{P}(x)^\top T(x,\xi)^\top\right]x = \\ &= v^\top S(x,\xi)v.\end{aligned}$$

Note that since $S(x,\xi)$ is negative definite for all $(x,\xi) \in \Omega$ and $v \neq 0$ then

$$v^\top S(x,\xi)v < 0 \;\text{ for all }\; (x,\xi) \in \Omega\backslash(0,0),$$

therefore (3.38) holds. □

We are now ready to prove that (3.28) implies (3.32).

Theorem 3.2.1. *Consider the nonlinear, time-invariant, system (3.27) and suppose that the linearized system around the origin is stabilizable.*

1. *Let $\bar{P} = \bar{P}^\top > 0$ be a solution of (3.28) and $p(x)$ an algebraic $\bar{P}$ solution.*
2. *Consider the extended system (2.18) and the function $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$.*
3. *Let $A = \frac{\partial f}{\partial x}(0)$, $B = g(0)$ and $Q = -(\bar{P}B)^\perp(A^\top\bar{P}+\bar{P}A)(\bar{P}B)^{\perp}$, and assume $\|(\bar{P}B)^\perp\| = 1$.*
4. *Suppose that $R = R^\top > 0$ satisfies*

$$\underline{\sigma}(R) > \frac{\|\bar{P}\|\left(2\|\bar{P}A\| + \underline{\sigma}(Q)\right)}{2\underline{\sigma}(Q)}. \tag{3.39}$$

Then:

1. *$V(x,\xi)$ is locally positive definite;*
2. *there exists a neighborhood of the origin $\Omega \subset \mathbb{R}^{2n}$ such that (3.28) implies (3.32), hence:*

$$\left\{\begin{array}{rcl} V_\xi(x,\xi) & = & 0 \\ V_x(x,\xi)g(x) & = & 0 \end{array}\right. \quad \Rightarrow \quad V_x(x,\xi)f(x) < 0 \text{ for all } (x,\xi) \in \Omega\backslash(0,0).$$

Therefore $V(x,\xi)$ is a local CLF for the extended system (3.27).

Proof. Firstly note that, as shown in Theorem 3.1.2, condition (3.39) implies $R > \frac{\bar{P}}{2}$, thus Theorem 2.2.2 guarantees that $V(x,\xi)$ is locally positive definite.
To prove that (3.28) implies (3.32) we use the following chain of implications

$$(x^\top\bar{P}B = 0 \ \Rightarrow x^\top A\bar{P}x < 0) \text{ for all } \ x \neq 0$$

$$\Downarrow \text{ (by Lemma 3.2.5)}$$

$$\left[(\tilde{P}(x)g(x))^\perp\right]\left[\tilde{P}(x)F(x) + F(x)^\top\tilde{P}(x)^\top\right]\left[(\tilde{P}(x)g(x))^{\perp}\right] \triangleq -Q(x) < 0$$

$$\text{for all } \ x \in \Omega_x$$

$$\Downarrow \text{ (A)}$$

$$\left[(T(x,\xi)\tilde{P}(x)g(x))^\perp\right]\left[T(x,\xi)\tilde{P}(x)F(x) + F(x)^\top\tilde{P}(x)^\top T(x,\xi)^\top\right]\left[(T(x,\xi)\tilde{P}(x)g(x))^{\perp}\right] < 0$$

$$\text{for all } \ (x,\xi) \in \Omega\backslash(0,0)$$

$$\Downarrow \text{ (by Lemma 3.2.6)}$$

$$(x^\top T(x,\xi)\tilde{P}(x)g(x) = 0 \ \ \Rightarrow x^\top T(x,\xi)\tilde{P}(x)F(x)x < 0) \text{ for all } \ (x,\xi) \in \Omega\backslash(0,0)$$

The only fact to prove is that there exists a neighborhood of the origin $\Omega \subset \mathbb{R}^{2n}$ such that implication (A) is valid. Notice that $T(0,0) = I + \bar{P}R^{-1}(R-\bar{P})^\top\bar{P}^{-1} = 2I - \bar{P}R^{-1}$ hence, as proved in Theorem 3.1.2 and due to the fact that $R > \frac{\bar{P}}{2}$, the matrix $T(0,0)$ is invertible.

Therefore, by Lemma 3.2.4, there exists a neighborhood of the origin $\Omega_1 \subset \mathbb{R}^{2n}$ such that $T(x,\xi)$ is an invertible matrix for all $(x,\xi) \in \Omega_1$. Without loss of generality suppose that $(x,\xi) \in \Omega_1 \Rightarrow x \in \Omega_x$. Then, by Lemma 3.2.3, for all $(x,\xi) \in \Omega_1$:

$$\begin{aligned}
&\left[(T(x,\xi)\tilde{P}(x)g(x))^{\perp}\right]\left[T(x,\xi)\tilde{P}(x)F(x) + F(x)^{\top}\tilde{P}(x)^{\top}T(x,\xi)^{\top}\right]\left[(T(x,\xi)\tilde{P}(x)g(x))^{\pm}\right] = \\
&= (\tilde{P}(x)g(x))^{\perp}T(x,\xi)^{-1}\left[T(x,\xi)\tilde{P}(x)F(x) + F(x)^{\top}\tilde{P}(x)^{\top}T(x,\xi)^{\top}\right]T(x,\xi)^{-\top}(\tilde{P}(x)g(x))^{\pm} \\
&= \left[(\tilde{P}(x)g(x))^{\perp}\right]\left[\tilde{P}(x)F(x)T(x,\xi)^{-\top} + T(x,\xi)^{-1}F(x)^{\top}\tilde{P}(x)^{\top}\right]\left[(\tilde{P}(x)g(x))^{\pm}\right] \\
&= -\lambda Q(x) + \left[(\tilde{P}(x)g(x))^{\perp}\right]\left[\tilde{P}(x)F(x)M(x,\xi)^{\top} + M(x,\xi)^{-1}F(x)^{\top}\tilde{P}(x)^{\top}\right]\left[(\tilde{P}(x)g(x))^{\pm}\right],
\end{aligned} \tag{3.40}$$

where $M(x,\xi) \triangleq T(x,\xi)^{-1} - \lambda I$, with $\lambda > 0$ to be determined.

Therefore to prove that implication (A) is valid it is sufficient to find a neighborhood of the origin, $\Omega_2 \subset \Omega_1$, such that the matrix (3.40) is negative definite for all $(x,\xi) \in \Omega_2$. Let

$$\tilde{M}(x,\xi) \triangleq \frac{M(x,\xi)}{\lambda},$$

then, following the same procedure used in the linear case, it is sufficient to prove that

$$-Q(x) + \left[(\tilde{P}(x)g(x))^{\perp}\right]\left[\tilde{P}(x)F(x)\tilde{M}(x,\xi)^{\top} + \tilde{M}(x,\xi)^{-1}F(x)^{\top}\tilde{P}(x)^{\top}\right]\left[(\tilde{P}(x)g(x))^{\pm}\right] < 0, \tag{3.41}$$

is negative definite in a neighborhood of the origin, Ω_2. Using the fact that[8] $Q(x) \geqslant \underline{\sigma}(Q(x))I$ and the standard matrix norm properties, the left hand side in (3.41) can be bounded from above as

$$-Q(x) + \left[(\tilde{P}(x)g(x))^{\perp}\right]\left[\tilde{P}(x)F(x)\tilde{M}(x,\xi)^{\top} + \tilde{M}(x,\xi)^{-1}F(x)^{\top}\tilde{P}(x)^{\top}\right]\left[(\tilde{P}(x)g(x))^{\pm}\right] \leqslant$$

$$\leqslant \left[-\underline{\sigma}(Q(x)) + 2\|\tilde{P}(x)F(x)\|\|\tilde{M}(x,\xi)\|\right]I,$$

where, without loss of generality[9], it has been assumed that $\|(\tilde{P}(x)g(x))^{\perp}\| = 1$. Therefore, to satisfy (3.41), it is sufficient to have

$$-\underline{\sigma}(Q(x)) + 2\|\tilde{P}(x)F(x)\|\|\tilde{M}(x,\xi)\| < 0,$$

or equivalently

$$\|\tilde{M}(x,\xi)\| < \frac{\underline{\sigma}(Q(x))}{2\|\tilde{P}(x)F(x)\|} \qquad \text{for all } (x,\xi) \in \Omega_2. \tag{3.42}$$

Condition (3.42) can be locally satisfied by imposing the same constraint on R used in

[8] See the Appendix.
[9] See Lemma 3.2.1.

the linear case. To prove this statement, note that[10], by Lemma 3.2.2:

$$
\begin{aligned}
Q(x) &= -\left[(\tilde{P}(x)g(x))^{\perp}\right]\left[\tilde{P}(x)F(x) + F(x)^{\top}\tilde{P}(x)^{\top}\right]\left[(\tilde{P}(x)g(x))^{\pm}\right] \\
&= -\left[(\tilde{P}(0)g(0))^{\perp}\right]\left[\tilde{P}(0)F(0) + F(0)^{\top}\tilde{P}(0)^{\top}\right]\left[(\tilde{P}(0)g(0))^{\pm}\right] + \mathcal{O}(x) \\
&= -\left[(\bar{P}B)^{\perp}\right]\left[\bar{P}A + A^{\top}\bar{P}\right]\left[(\bar{P}B)^{\pm}\right] + \mathcal{O}(x) = Q + \mathcal{O}(x)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{M}(x,\xi) &= \frac{M(x,\xi)}{\lambda} = \frac{T(x,\xi)^{-1} - \lambda I}{\lambda} = \frac{T(0,0)^{-1} + \mathcal{O}(x,\xi) - \lambda I}{\lambda} \\
&= \frac{T^{-1} - \lambda I}{\lambda} + \mathcal{O}(x,\xi) = \tilde{M} + \mathcal{O}(x,\xi)
\end{aligned}
$$

where $\tilde{M}$, Q and T are defined as in Theorem 3.1.1 and all the functions labelled by $\mathcal{O}$ are continuous and vanish at the origin. Therefore condition (3.42) can be rewritten as

$$
||\tilde{M} + \mathcal{O}(x,\xi)|| = ||\tilde{M}|| + \mathcal{O}(x,\xi) < \frac{\underline{\sigma}(Q + \mathcal{O}(x))}{2||\bar{P}A + \mathcal{O}(x)||} = \frac{\underline{\sigma}(Q) + \mathcal{O}(x)}{2||\bar{P}A|| + \mathcal{O}(x)}, \tag{3.43}
$$

where we have used the fact that the norm operator and the singular values are continuous functions. Note now that, in a neighborhood of the origin, the denominator of the last term in equation (3.43) is always positive therefore equation (3.43) is equivalent to

$$
\begin{aligned}
\left[||\tilde{M}|| + \mathcal{O}(x,\xi)\right]\left[2||\bar{P}A|| + \mathcal{O}(x)\right] &< \underline{\sigma}(Q) + \mathcal{O}(x), \\
2||\tilde{M}||||\bar{P}A|| + 2\mathcal{O}(x,\xi)||\bar{P}A|| + ||\tilde{M}||\mathcal{O}(x) + \mathcal{O}(x,\xi)\mathcal{O}(x) &< \underline{\sigma}(Q) + \mathcal{O}(x), \\
2||\bar{P}A||\mathcal{O}(x,\xi) + ||\tilde{M}||\mathcal{O}(x) + \mathcal{O}(x,\xi)\mathcal{O}(x) - \mathcal{O}(x) &< \underline{\sigma}(Q) - 2||\tilde{M}||||\bar{P}A||,
\end{aligned} \tag{3.44}
$$

where the first term is a continuous function of (x,ξ) that vanishes at the origin. Therefore for each value of $\varepsilon > 0$ there exists a neighborhood, $\Omega_2 \subset \mathbb{R}^{2n}$, of the origin such that the first term is smaller than ε for all $(x,\xi) \in \Omega_2$. Note now that if

$$
||\tilde{M}|| < \frac{\underline{\sigma}(Q)}{2||\bar{P}A||} \tag{3.45}
$$

the right hand side of equation (3.44) is positive, therefore we can choose $\varepsilon = \underline{\sigma}(Q) - 2||\tilde{M}||||\bar{P}A||$ and find a set Ω_2 such that (3.44) holds for all $(x,\xi) \in \Omega_1 \cap \Omega_2$. Finally, as proved in Theorem 3.1.1, a sufficient condition on R to imply (3.45) is

$$
\underline{\sigma}(R) > \frac{||\bar{P}||\left(2||\bar{P}A|| + \underline{\sigma}(Q)\right)}{2\underline{\sigma}(Q)}.
$$

□

[10] $T(x,\xi)^{-1}$ is a continuous function because $T(x,\xi)$ is continuous and $T(x,\xi)^{-1} = \frac{adj(T(x,\xi))}{\det(T(x,\xi))}$ with $\det(T(x,\xi)) \neq 0$ for all $(x,\xi) \in \Omega_1$.

3.3 Conclusions

In this chapter the problem of constructing a Dynamic CLF has been analyzed from a geometric perspective. In the first part some illustrative examples have been considered for 2-dimensional and 3−dimensional linear and nonlinear systems. In Sections 3.1 and 3.2, using the intuition gained from the examples, the proof that $V(x,\xi)$ is a CLF has been given for the linear and nonlinear cases.

More in details a sufficient condition on the minimum singular value of R has been derived guaranteeing that the functions $V(x,\xi) = \xi^\top \bar{P} x + \frac{1}{2}\|x-\xi\|_R^2$ and $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$ are a global CLF for the extended system (2.10) and a local CLF for the extended system (2.18), respectively.

Chapter 4

From Dynamic Control Lyapunov Functions to Control Lyapunov Functions

Consider the nonlinear system

$$\dot{x} = f(x) + g(x)u, \tag{4.1}$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be at least continuously differentiable and $f(0) = 0$. Moreover suppose that the linearized system around the origin is stabilizable. The problem of constructing a Dynamic CLF for system (4.1) has been solved in the previous chapters. Moreover in Corollaries 2.1.1 and 2.2.3, starting from the proposed Dynamic CLF, an explicit, static, control law that stabilizes the origin of system (4.1) has been derived. From the knowledge of a Dynamic CLF, it is also possible to derive different control laws that dynamically stabilize the origin. For example, if the Dynamic CLF satisfies the small control property (SCP), Sontag's formula[1] can be applied, as stated in the following Theorem.

Theorem 4.0.1. *Consider the nonlinear system (4.1) and suppose that $V(x,\xi) : \mathbb{R}^{2n} \to \mathbb{R}$ is a Dynamic CLF for the system and satisfies the SCP. Let $q(b) : \mathbb{R} \to \mathbb{R}$ be a function such that $q(0) = 0$ and $bq(b) > 0$, if $b \neq 0$, and consider the function*

$$\Phi(a,b) \triangleq \begin{cases} 0 & \text{if } b = 0 \text{ and } a < 0, \\ \dfrac{a + \sqrt{a^2 + bq(b)}}{b} & \text{otherwise.} \end{cases}$$

Let

$$a(x,\xi) \triangleq V_x(x,\xi)f(x),\ B_u(x,\xi) \triangleq V_x(x,\xi)g(x),\ B_w(x,\xi) \triangleq V_\xi(x,\xi),\ \beta(x,\xi) \triangleq \|(B_u, B_w)\|^2.$$

Then the control laws

$$u(x,\xi) = \begin{cases} -B_u(x,\xi)^\top \Phi(a(x,\xi), \beta(x,\xi)), & \text{if } (x,\xi) \neq (0,0), \\ 0, & \text{if } (x,\xi) = (0,0), \end{cases} \tag{4.2}$$

[1] See Section 1.2.

$$w(x,\xi) = \begin{cases} -B_w(x,\xi)^\top \Phi(a(x,\xi),\beta(x,\xi)), & \text{if } (x,\xi) \neq (0,0), \\ 0, & \text{if } (x,\xi) = (0,0), \end{cases} \tag{4.3}$$

continuously dynamically stabilize the origin.

Proof. Consider the extended system

$$\dot{x} = f(x) + g(x)u,$$
$$\dot{\xi} = w.$$

By assumption, $V(x,\xi)$ is a CLF for the system and satisfies the SCP. Therefore Sontag's formula can be applied to derive a control law that statically continuously stabilizes the origin of the extended system. Note that, if we define the vector $z^\top = [x^\top \quad \xi^\top]$, then the functions $a(z)$ and $B(z)$, as defined in (1.9), become

$$a(z) = V_z(z)\begin{bmatrix} f(x) \\ 0 \end{bmatrix} = \begin{bmatrix} V_x(z) & V_\xi(z) \end{bmatrix}\begin{bmatrix} f(x) \\ 0 \end{bmatrix} = V_x(x,\xi)f(x)$$

$$B(z) = V_z(z)\begin{bmatrix} g(x) & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} V_x(z) & V_\xi(z) \end{bmatrix}\begin{bmatrix} g(x) & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} V_x(z)g(x) & V_\xi(z) \end{bmatrix}$$
$$= \begin{bmatrix} B_u(z) & B_w(z) \end{bmatrix}.$$

Finally the control vector $[u^\top \quad w^\top]^\top$ is obtained by applying formula (1.10). □

An interesting question is if, from the knowledge of a Dynamic CLF and a dynamic control law, it is possible to derive a static control law, or a "static" CLF for system (4.1). The aim of this chapter is to give an answer to this latter question. The general problem is addressed in the following section, while in Section 4.2, we focus on Dynamic CLFs with the structure given in Chapters 2 and 3, and the control laws of Corollaries 2.1.1 and 2.2.3.

4.1 General problem

The general problem of obtaining a CLF from the knowledge of a Dynamic CLF and a dynamic control law can be solved using the following theorems.

Theorem 4.1.1. *Consider the nonlinear system (4.1) and suppose that $V(x,\xi) : \mathbb{R}^{2n} \to \mathbb{R}$ is a Dynamic CLF for the system and satisfies the small control property. Let $u(x,\xi)$ and $w(x,\xi)$ be two continuous functions that stabilize the origin of the extended system*[2]

$$\dot{x} = f(x) + g(x)u,$$
$$\dot{\xi} = w. \tag{4.4}$$

If there exists a continuously differentiable mapping $h : \mathbb{R}^n \to \mathbb{R}^n$, $h(0) = 0$, such that

$$h_x(x)\left[f(x) + g(x)u(x,h(x))\right] = w(x,h(x)), \tag{4.5}$$

then $V_\mathcal{M} \triangleq V(x,h(x))$ is a CLF for system (4.1) and satisfies the small control property.

[2]Note that the existence of such $u(x,\xi)$ and $w(x,\xi)$ is guaranteed by the fact that $V(x,\xi)$ is a CLF for the extended system (4.4) and satisfies the SCP.

Proof. Condition (4.5) implies that the set $\mathcal{M} \triangleq \{(x,\xi) \text{ s.t. } \xi = h(x)\}$ is invariant along the trajectories of the closed-loop system. In fact

$$\dot{\xi}\big|_{\xi=h(x)} = w(x,h(x)) = h_x(x)\left[f(x) + g(x)u(x,h(x))\right] = h_x(x)\dot{x} = \dot{h}(x).$$

Therefore the restriction of system (4.4) to the invariant set is a copy of the dynamics of system (4.1), with the control law $\bar{u}(x) = u(x,h(x))$. Note now that

1. $V(x,\xi) > 0$ and $\dot{V}(x,\xi,u(x,\xi),w(x,\xi)) < 0$ for all $(x,\xi) \in \Omega\backslash(0,0)$ imply that there exists an open set $\Omega_x \subset \mathbb{R}^n$, $0 \in \Omega_x$, such that:

$$V_{\mathcal{M}}(x) \triangleq V(x,h(x)) > 0 \quad \text{for all} \quad x \in \Omega_x\backslash\{0\},$$

$$\dot{V}(x,h(x),u(x,h(x)),w(x,h(x))) < 0 \quad \text{for all} \quad x \in \Omega_x\backslash\{0\}.$$

2. The time derivative of $V_{\mathcal{M}}$ along the trajectories of system (4.1), controlled with $\bar{u}(x) = u(x,h(x))$, satisfies

$$\begin{aligned}
\dot{V}_{\mathcal{M}}(x,\bar{u}(x)) &= V_x(x,\lambda)\big|_{\lambda=h(x)}\dot{x}_{\bar{u}} + V_\lambda(x,\lambda)\big|_{\lambda=h(x)}\dot{h}_{\bar{u}}(x) \\
&= V_x(x,\lambda)\big|_{\lambda=h(x)}[f(x)+g(x)\bar{u}(x)] + V_\lambda(x,\lambda)\big|_{\lambda=h(x)}h_x(x)[f(x)+g(x)\bar{u}(x)] \\
&= V_x(x,\lambda)\big|_{\lambda=h(x)}\left[f(x) + g(x)u(x,h(x))\right] + V_\lambda(x,\lambda)\big|_{\lambda=h(x)}w(x,h(x)) \\
&= \dot{V}(x,\lambda,u(x,\lambda),w(x,\lambda))|_{\lambda=h(x)} < 0 \quad \text{for all} \quad x \in \Omega_x\backslash\{0\}.
\end{aligned}$$

Therefore the function $V_{\mathcal{M}}$ depends only on x, is positive definite in Ω_x and

$$\inf_u \dot{V}_{\mathcal{M}}(x,u) \le \dot{V}_{\mathcal{M}}(x,\bar{u}(x)) < 0 \quad \text{for all} \quad x \in \Omega_x\backslash\{0\}.$$

Hence $V_{\mathcal{M}}$ is a CLF for system (4.1). Moreover, since $\bar{u}(x)$ is continuous at the origin, $V_{\mathcal{M}}$ satisfies the small control property. □

In the following section this theorem is applied to the class of Dynamic CLFs derived in Chapters 2 and 3. However it is important to remark that, to use this result, equation (4.5) has to be solved. Note that this is a differential equation, which is simpler than the original constrained differential inequality (1.11), but can be still hard or impossible to solve. In such a case, the closed-form $h(x)$ of Theorem 4.1.1 can be replaced by an approximation, as clarified in the following statement.

Theorem 4.1.2. *Consider the nonlinear system (4.1) and suppose that $V(x,\xi) : \mathbb{R}^{2n} \to \mathbb{R}$ is a Dynamic CLF for this system that satisfies the small control property. Moreover suppose that $u(x,\xi)$ and $w(x,\xi)$ continuously stabilize the extended system (4.4). If there exists a continuous mapping $\hat{h} : \mathbb{R}^n \to \mathbb{R}^n$, $\hat{h}(0) = 0$, such that:*

$$\|\hat{h}_x(x)\left[f(x) + g(x)u(x,\hat{h}(x))\right] - w(x,\hat{h}(x))\| < -\frac{\dot{V}(x,\lambda,u(x,\lambda),w(x,\lambda))}{\|V_\lambda(x,\lambda)\|}\Big|_{\lambda=\hat{h}(x)} \tag{4.6}$$

for all $x \in \Omega_x\backslash\{0\}$, then $V_{\mathcal{M}} \triangleq V(x,\hat{h}(x))$ is a local CLF for the system (4.1) and satisfies the small control property.

Proof. The proof is similar to that of Theorem 4.1.1. Define Ω_x as in the proof of Theorem 4.1.1 and $f_g(x) \triangleq f(x) + g(x)u(x, \hat{h}(x))$. Then

1. $V_{\mathcal{M}}(x) \triangleq V(x, \hat{h}(x)) > 0$ for all $x \in \Omega_x \setminus \{0\}$;
2. the time derivative of $V_{\mathcal{M}}$ along the trajectories of system (4.1), controlled with $\bar{u}(x) = u(x, \hat{h}(x))$, satisfies

$$\begin{aligned}
\dot{V}_{\mathcal{M}}(x, \bar{u}(x)) &= V_x(x, \lambda)\big|_{\lambda=\hat{h}(x)} \dot{x}_{\bar{u}} + V_\lambda(x, \lambda)\big|_{\lambda=\hat{h}(x)} \dot{\hat{h}}_{\bar{u}}(x) \\
&= V_x(x, \lambda)\big|_{\lambda=\hat{h}(x)} f_g(x) + V_\lambda(x, \lambda)\big|_{\lambda=\hat{h}(x)} \hat{h}_x(x) f_g(x) \\
&= V_x(x, \lambda)\big|_{\lambda=\hat{h}(x)} f_g(x) + V_\lambda(x, \lambda)\big|_{\lambda=\hat{h}(x)} w(x, \hat{h}(x)) + \\
&\qquad + V_\lambda(x, \lambda)\big|_{\lambda=\hat{h}(x)} \left[\hat{h}_x(x) f_g(x) - w(x, \hat{h}(x))\right] \\
&\leq \dot{V}(x, \lambda, u(x, \lambda), w(x, \lambda))|_{\lambda=\hat{h}(x)} + \\
&\qquad + \|V_\lambda(x, \lambda)\|\big|_{\lambda=\hat{h}(x)} \|\hat{h}_x(x) f_g(x) - w(x, \hat{h}(x))\| < 0,
\end{aligned}$$

   where the last inequality holds in $\Omega_x \setminus \{0\}$ by condition (4.6).

Therefore the function $V_{\mathcal{M}}$ depends only on x, is positive definite in Ω_x and

$$\inf_u \dot{V}_{\mathcal{M}}(x, u) \leq \dot{V}_{\mathcal{M}}(x, \bar{u}(x)) < 0 \quad \text{for all} \quad x \in \Omega_x \setminus \{0\}.$$

Hence $V_{\mathcal{M}}$ is a CLF for system (4.1). Moreover, since $\bar{u}(x)$ is continuous at the origin, $V_{\mathcal{M}}$ satisfies the small control property. □

4.2 Construction of a CLF from the proposed Dynamic CLF

The aim of this section is to derive a "static" CLF for system (4.1) starting from a Dynamic CLF with the structure given in Chapters 2 and 3. To this end the control laws of Corollaries 2.1.1 and 2.2.3 are used. As usual the linear case is firstly analyzed, while the nonlinear problem is discussed in Section 4.2.2.

4.2.1 Linear Systems

Consider the linear time-invariant system

$$\dot{x} = Ax + Bu. \tag{4.7}$$

Theorem 4.2.1. *Consider system (4.7) and suppose that the origin is stabilizable. Let $\bar{P} = \bar{P}^\top > 0$ be a solution of (2.7) and suppose that R is such that $V(x, \xi) = \xi^\top \bar{P} x + \frac{1}{2}\|x - \xi\|_R^2$ is a global Dynamic CLF. Fix $k > \bar{k}$ and $l > \bar{l}$ as defined in Theorem 2.1.2 and Corollary 2.1.1. Moreover let $Y \in \mathbb{R}^{n \times n}$ be a solution of the Sylvester equation*[3]

$$kRY + Y(A - lBB^\top \bar{P}^\top) = k(R - \bar{P}). \tag{4.8}$$

Then $V_{\mathcal{M}}(x) = V(x, Yx)$ is a CLF for system (4.7).

[3] See [5].

Proof. This statement is a consequence of Theorem 4.1.1, with the choice $h(x) = Yx$. Note that if $l > \bar{l}$ and $k > \bar{k}$, the control laws

$$u(x) = -lB^\top \bar{P}^\top x,$$
$$w(x,\xi) = -k(\bar{P}x - R(x-\xi)),$$

stabilize the origin of the extended system. Moreover with this choice of control laws condition (4.5) becomes

$$Y\left[A - lBB^\top \bar{P}^\top\right]x = -k(\bar{P}x - R(x - Yx)) = -k(\bar{P} - R + RY)x.$$

Therefore condition (4.5) is (globally) satisfied if (4.8) holds. □

Note that it is always possible to find two values $k > \bar{k}$ and $l > \bar{l}$ such that (4.8) admits a solution. In fact the Sylvester equation

$$AX + XB = C$$

has a unique solution if and only if $\sigma(A) \cap \sigma(B) = \varnothing$, see [5]. Therefore (4.8) admits a unique solution if and only if $\sigma(kR) \cap \sigma(A - lBB^\top \bar{P}^\top) = \varnothing$ and this is true for all $(l,k) > (\bar{l},\bar{k})$.[4]

4.2.2 Nonlinear Systems

Theorem 4.2.2. *Consider the nonlinear, time-invariant, system (4.1) and suppose that the linearized system around the origin is stabilizable. Let $\bar{P} = \bar{P}^\top > 0$ be a solution of (2.15) and $p(x)$ an algebraic $\bar{P}$ solution, with index $\bar{l}$. Moreover suppose that R is such that the function $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$ is a local Dynamic CLF for (4.1) and define*

$$f_g(x) \triangleq f(x) - lg(x)g(x)^\top p(x)^\top,$$

with $l > \bar{l}$. Suppose that there exists a continuous mapping $h : \mathbb{R}^n \to \mathbb{R}^n$, $h(0) = 0$, such that

$$h_x(x)f_g(x) = -k\left[p_\xi(h(x))^\top x - R(x - h(x))\right] \tag{4.9}$$

where $k > \bar{k}$. Then $V_{\mathcal{M}} \triangleq V(x,h(x))$ is a CLF for the system (4.1) and satisfies the small control property.

Proof. Similarly to the linear case, this is a direct consequence of Theorem 4.1.1. $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$ is a CLF for the extended system

$$\dot{x} = f(x) + g(x)u,$$
$$\dot{\xi} = w, \tag{4.10}$$

[4]This is due to the fact that, since $R > 0$, the eigenvalues of kR are all positive. On the other hand, if $l > \bar{l}$, then the eigenvalues of $M = A - lBB^\top \bar{P}^\top$ are all negative because the Lyapunov equation $XM + M^\top X = -Q$ has a positive definite solution $X = \bar{P}$. In fact

$$\bar{P}M + M^\top \bar{P} = 2\left(\frac{\bar{P}A + A^\top \bar{P}}{2} - l\bar{P}BB^\top \bar{P}^\top\right)$$

is negative definite for each $l > \bar{l}$.

and in Corollary 2.2.3 it has been proved that the control laws $u(x) = -lg(x)^\top p(x)^\top$ and $w(x,\xi) = -k(p_\xi(\xi)^\top x - R(x-\xi))$, with $l > \bar{l}$ and $k > \bar{k}$, stabilize the origin of (4.10). Therefore, to use Theorem 4.1.1, it is sufficient to verify that $h(x)$ satisfies condition (4.5) and this is guaranteed by (4.9). □

Chapter 5

Examples and future directions

In this chapter three examples are presented to illustrate the theoretical results. The first example describes a simple 2-dimensional nonlinear system. The aim of this example is to compare the results obtained using, as algebraic $\bar{P}$ solutions, two mappings $p_1(x)$ and $p_2(x)$, where only the former is a gradient vector. Contrary to what one could expect, the latter yields better performance.

The second example is relative to the roll dynamic of an airplane. For this system it is shown that the control law derived using the method presented in Chapters 2 and 3 leads to better performance than the control law that can be derived from the CLF of the linearized system.

In the third case a model for the angular velocity of a rigid body is considered. For this system it is not possible to directly apply the theory developed in the previous chapters, since the linearized system around the origin is not stabilizable. To overcome this problem the notion of weak algebraic $\bar{P}$ solution is introduced and a procedure similar to the one described in Corollary 2.2.3 is given. Finally the performance of the resulting control law are compared with those given by the linear control law proposed in [1].

5.1 A planar nonlinear system

Consider the 2-dimensional, nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2, \\ \dot{x}_2 &= u. \end{aligned} \tag{5.1}$$

The linearized system around the origin is

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

which is reachable and thus stabilizable. Therefore, as stated in Theorem 2.0.3, there exists a matrix $\bar{P} = \bar{P}^\top > 0$ such that

$$x^\top \bar{P} B = 0 \Rightarrow x^\top \bar{P} A x < 0, \quad x \neq 0.$$

Since this is a very simple example it is possible to characterize all such matrices. Let $\bar{P}$ be described as

$$\bar{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}.$$

where the terms p_1 and $p_1 p_3 - p_2^2$ must be positive for $\bar{P}$ to be positive definite. Note that $p_2 = 0$ is not acceptable since in this case $x^\top \bar{P} B = 0$ implies $x_2 = 0$ and $x^\top \bar{P} A x = p_1 x_1 x_2$ would be only semi-definite. Since $x^\top \bar{P} B = 0$ implies $x_1 = (p_3/p_2) x_2$, substituting in $x^\top \bar{P} A x < 0$ yields

$$x^\top \bar{P} A x = \left(-\frac{p_1 p_3}{p_2} + p_2 \right) x_2^2 = -\frac{1}{p_2}(p_1 p_3 - p_2^2) x_2^2,$$

which, since $(p_1 p_3 - p_2^2)$ is positive for hypothesis, is negative if and only if $p_2 > 0$.
In the following we select

$$\bar{P} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

Algebraic $\bar{P}$ solution

To construct $V(x, \xi)$ an algebraic $\bar{P}$ solution is needed. Consider an arbitrary matrix-valued $\mathcal{C}^1$ function $p(x)$ such that $p(0) = 0$ and $p_x(0) = \bar{P}$, i.e. $p(x)$ satisfies Condition (P1), then locally $p(x)$ satisfies also Condition (P2), as stated in Remark 1.3. However, to extend the region in which Condition (P2) holds, we additionally impose the necessary condition

$$p(x) g(x) = 0 \ \Rightarrow p(x) f(x) < 0. \tag{5.2}$$

The region in which Condition (P2) holds will be evaluated a posteriori. As already stated, the solution of this problem is not unique; for example

- $p_1(x) = \begin{bmatrix} 2x_1 + x_2 - x_1^2 & x_1 + x_2 \end{bmatrix}$,
- $p_2(x) = \begin{bmatrix} 2x_1 + x_2 + x_1^2 & x_1 + x_2 + x_1^2 \end{bmatrix}$,

both satisfy (5.2). It is interesting to underline that, while the former is a gradient vector, the second one is not. Moreover note that Condition (P2) becomes

- $p_1(x) f(x) - l p_1(x) g(x) g(x)^\top p_1(x)^\top = -x^\top \begin{bmatrix} l + x_1(x_1 - 2) & l - 1 \\ l - 1 & l - 1 \end{bmatrix} x \triangleq x^\top \Gamma_1(x) x$, where the matrix $\Gamma_1(x)$ is positive definite if and only if:

  1. $l + x_1(x_1 - 2) > 0 \ \Rightarrow \ l > -x_1(x_1 - 2)$. Note that the term on the right hand side is upper bounded by one hence this condition is globally satisfied for $l > 1$;
  2. $\det \Gamma_1(x) > 0 \ \Rightarrow \ (x_1 - 1)^2 (l - 1) > 0 \ \Rightarrow l > 1$.

  Note that the determinant of $\Gamma_1(x)$ is always positive except for $x_1 = 1$, when it is zero independently of l. Therefore $\Gamma_1(x)$ is a positive definite matrix for all $x \neq (1, x_2)$. Hence $p_1(x)$ is a local algebraic $\bar{P}$ solution in the set $\Omega = \{x \mid x_1 \neq 1\}$ and its index is $\bar{l} = 1$.

- $p_2(x)f(x) - lp_2(x)g(x)g(x)^\top p_2(x)^\top = -x^\top \begin{bmatrix} l(x_1+1)^2 - x_1(x_1+2) & (l-1)(x_1+1) \\ (l-1)(x_1+1) & l-1 \end{bmatrix} x$ $\triangleq x^\top \Gamma_2(x) x$, where the matrix $\Gamma_2(x)$ is positive definite if and only if:

  1. $l(x_1+1)^2 - x_1(x_1+2) > 0 \;\Rightarrow\; l > \dfrac{x_1(x_1+2)}{(x_1+1)^2}$, however since the term on the right hand side is upper bounded by one this condition is globally satisfied for $l > 1$;
  2. $\det \Gamma_1(x) = l - 1 > 0 \;\Rightarrow l > 1$.

  Therefore $p_2(x)$ is a global algebraic $\bar{P}$ solution with index $\bar{l} = 1$.

Selection of R

As stated in Theorems 2.2.2, 3.2.1 and Corollary 2.2.2, two sufficient conditions for $V(x,\xi)$ to be a local CLF are

$$\text{Property 1:}\quad R > \frac{\bar{P}}{2},$$
$$\text{Property 2:}\quad R^{-1} = \alpha \bar{P}^{-1} \quad \text{or} \quad \underline{\sigma}(R) > \frac{\|\bar{P}\|\left(2\|\bar{P}A\| + \underline{\sigma}(Q)\right)}{2\underline{\sigma}(Q)},$$

where $Q = -(\bar{P}B)^\perp (A^\top \bar{P} + \bar{P}A)(\bar{P}B)^{\perp}$ and $\|(\bar{P}B)^\perp\| = 1$. Note that, since $m = 1$, the matrix

$$M = (PB)^\top (PB) I - (PB)(PB)^\top$$

is orthogonal to PB, but it is not a suitable orthogonal matrix, as defined in Section 3.1, since it is not of full row rank. In fact the null subspace of PB is one dimensional. A suitable orthogonal matrix can be obtained from M selecting, for example, the first row and normalizing it. With this choice, substituting the value of $\bar{P}$ imposed before, yields

$$(\bar{P}B)^\perp = [0.7071 \quad -0.7071], \qquad Q = 0.5, \qquad \underline{\sigma}(R) > 15.1.$$

5.1.1 The geometry of the problem

To better understand the problem, the sets I^0_{Pg}, I^-_{Pf}, I^0_{Vg} and I^-_{Vf}, as defined in Chapter 3, are hereby reported, both for $p_1(x)$ and $p_2(x)$, with different choices of R.

1) Gradient vector: $p_1(x) = \begin{bmatrix} 2x_1 + x_2 - x_1^2 & x_1 + x_2 \end{bmatrix}$

Figure 5.1 displays the sets I^0_{Pg}, I^-_{Pf} for this system, in particular the subset $I^0_{Pg} = \{x \mid p_1(x)g(x) = 0\}$ is reported in blue while $I^-_{Pf} = \{x \text{ t.c } p_1(x)f(x) < 0\}$ is light blue colored. Condition (5.2) is therefore met if the blue set is inside the light blue one. Note that this condition is not satisfied if $x_1 = 1$, as stated in the previous section.

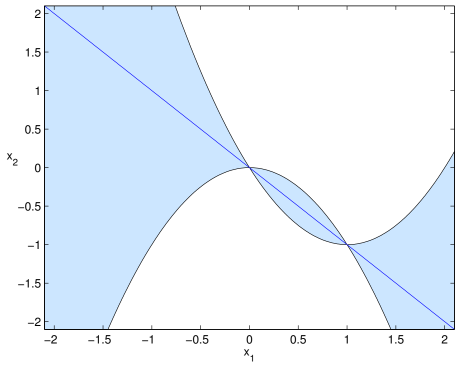
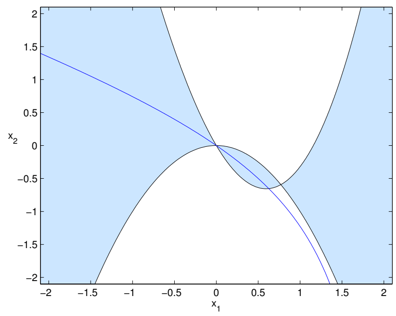
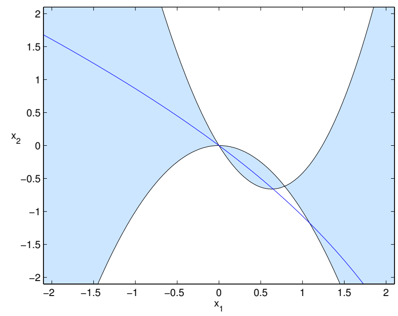


Figure 5.1: Sets I^0_{Pg}, I^-_{Pf} for system (5.1), $V(x,\xi)$ has been obtained using $p_1(x)$

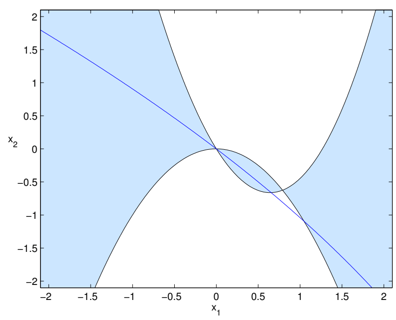
Figures 5.2 and 5.3 display the sets I^0_{Vg} and I^-_{Vf}, for different values of R. As in the previous figure, the subset $I^0_{Vg} = \{x \mid V_x(x,\xi(x))g(x) = 0\}$ is reported in blue while $I^-_{Vf} = \{x \mid V_x(x,\xi(x))f(x) < 0\}$ is light blue colored. Condition $V_x(x,\xi(x))g(x) = 0 \Rightarrow V_x(x,\xi(x))g(x) < 0$ is therefore met if the blue set is inside the light blue one. Of course this is the case only locally around the origin. Moreover note that the bound on $\underline{\sigma}(R)$ is, in this case, conservative. In fact $V(x,\xi)$ satisfies Property 2 also when R is much smaller.



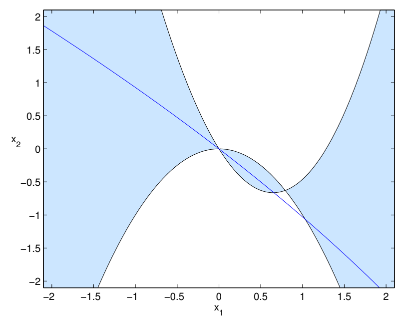
(a) $R = 5I$



(b) $R = 10I$

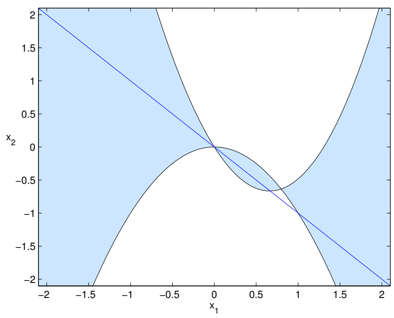


(c) $R = 15I$

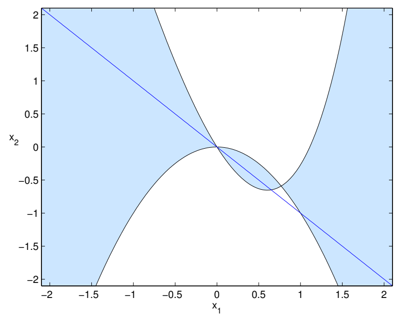


(d) $R = 20I$

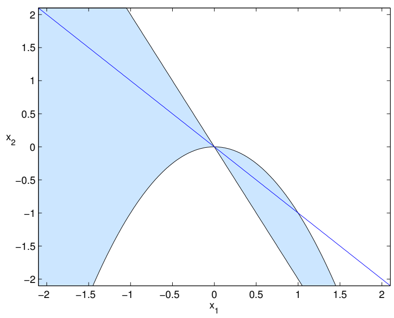
Figure 5.2: The sets I^0_{Vg} and I^-_{Vf} for different values of $R = cI$. $V(x,\xi)$ has been obtained using $p_1(x)$.



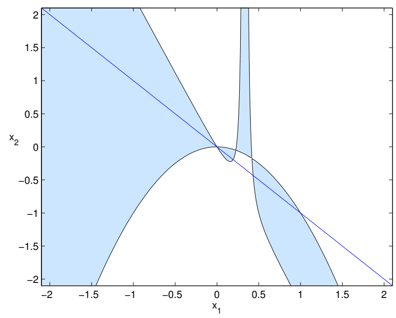
(a) $R^{-1} = 0.01P^{-1}$



(b) $R^{-1} = 0.5P^{-1}$



(c) $R^{-1} = P^{-1}$



(d) $R^{-1} = 1.5P^{-1}$

Figure 5.3: The sets I^0_{Vg} and I^-_{Vf} for different values of $R^{-1} = \alpha P^{-1}$ and $0 < \alpha < 2$. $V(x,\xi)$ has been obtained using $p_1(x)$.

2) Non-gradient vector: $p_2(x) = \left[\begin{array}{cc} 2x_1 + x_2 + x_1^2 & x_1 + x_2 + x_1^2 \end{array} \right]$

The same discussion can be repeated using $p_2(x)$ instead of $p_1(x)$. In Figure 5.4 the condition $p_2(x)g(x) = 0 \Rightarrow p_2(x)g(x) < 0$ is illustrated, while the sets I^0_{Vg} and I^-_{Vf} are reported in Figures 5.5 and 5.6.

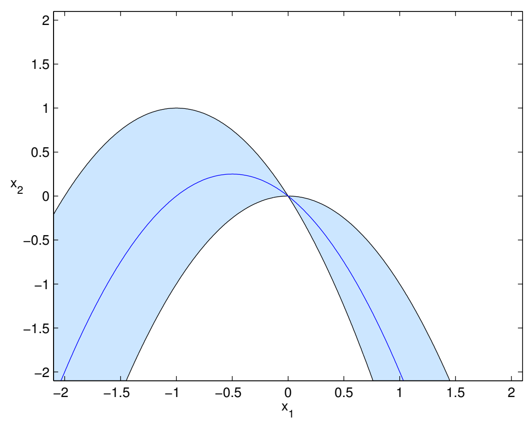
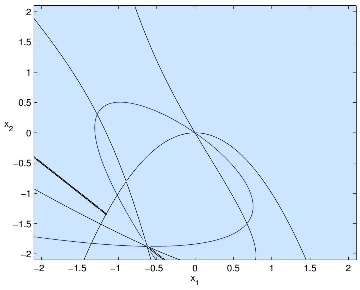
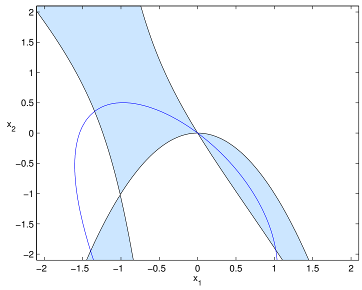


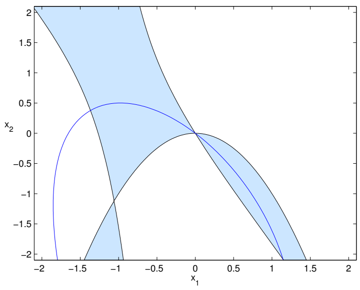
Figure 5.4: Sets I^0_{Pg}, I^-_{Pf} for system (5.1), $V(x,\xi)$ has been obtained using $p_2(x)$



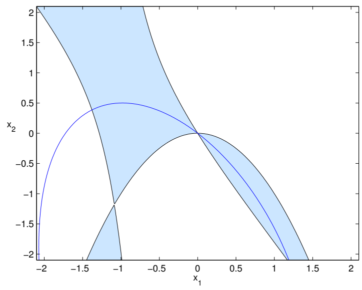
(a) $R = 5I$



(b) $R = 10I$

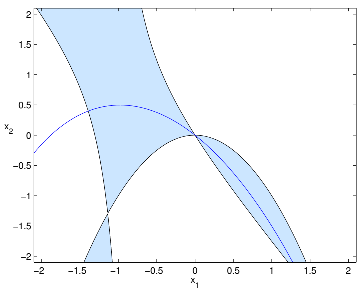


(c) $R = 15I$

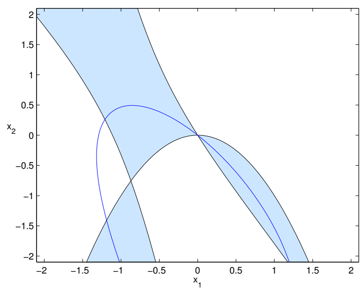


(d) $R = 20I$

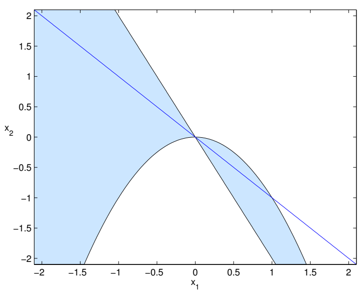
Figure 5.5: The sets I^0_{Vg} and I^-_{Vf} for different values of $R = cI$. $V(x,\xi)$ has been obtained using $p_2(x)$.



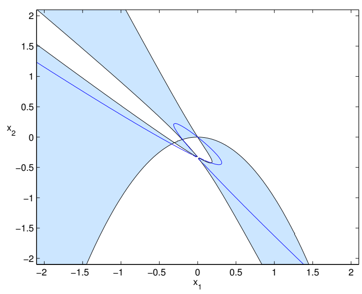
(a) $R^{-1} = 0.01P^{-1}$



(b) $R^{-1} = 0.5P^{-1}$



(c) $R^{-1} = P^{-1}$



(d) $R^{-1} = 1.5P^{-1}$

Figure 5.6: The sets I^0_{Vg} and I^-_{Vf} for different values of $R^{-1} = \alpha P^{-1}$ and $0 < \alpha < 2$. $V(x,\xi)$ has been obtained using $p_2(x)$.

5.1.2 Simulations

As discussed in Corollary 2.2.3, the algebraic $\bar{P}$ solution can be used to design the stabilizing control law

$$u_{nl}(x) = -l g(x)^{\top} p(x)^{\top}.$$

In the following sections this nonlinear control law is compared with the control law

$$u_{lin}(x) = -l B^{\top} \bar{P} x,$$

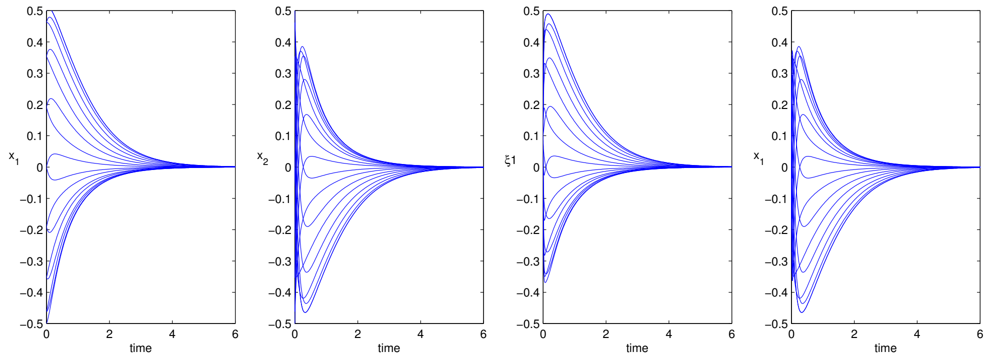
which can be derived from the linearized system.

1) Simulations with $p_1(x)$: gradient vector

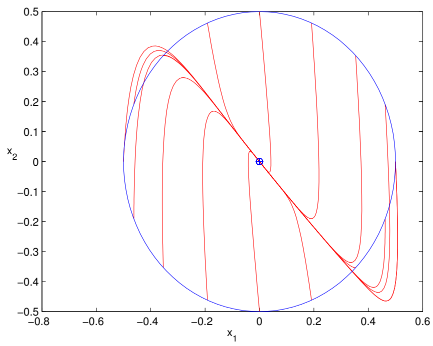
Using the Algebraic $\bar{P}$ solution $p_1(x)$ yields

$$u_{lin}(x) = -l B^{\top} \bar{P} x = u_{nl}(x) = -l g(x)^{\top} p_1(x)^{\top},$$

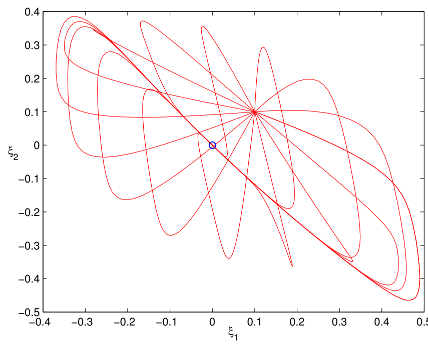
hence the trajectories of the closed-loop system, shown in Figure (5.7) and (5.8), coincide for both control laws.



(a) Time histories of the components of the state vector x



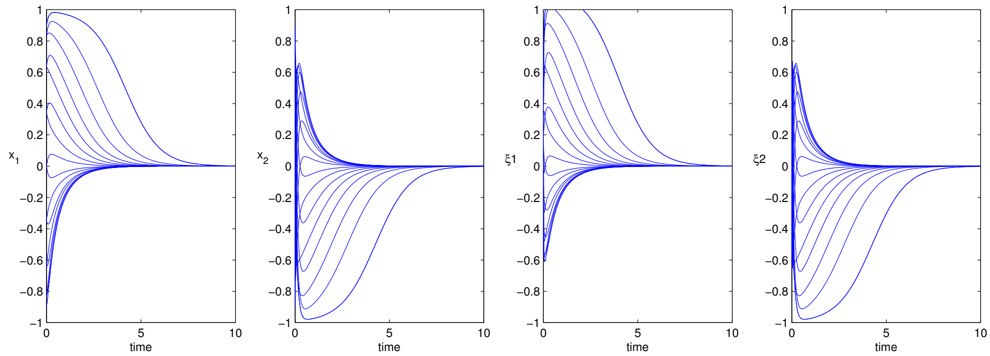
(b) Phase portrait on the x-plane



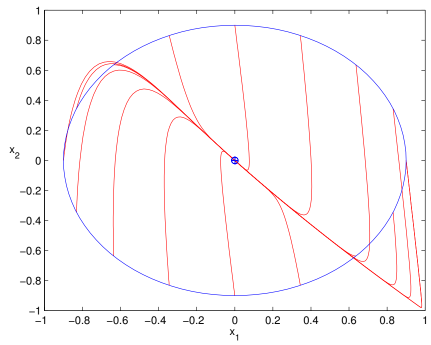
(c) Phase portrait on the ξ-plane

Figure 5.7: Trajectories of system (5.1), using the control law $u_{lin} = u_{nl} = -10 g(x)^{\top} p_1(x)^{\top}$, when the initial conditions (x_1, x_2) are on a circumference of radius $r = 0.5$ and $(\xi_1(0), \xi_2(0)) = (0.1, 0.1)$.

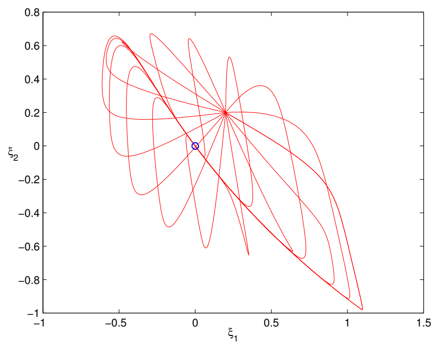
Therefore using the gradient vector $p_1(x)$, as algebraic $\bar{P}$ solution, is not a good choice since it leads to the same control law that can be derived simply from the linearized system around the origin. On the other hand it will be shown in the following that the algebraic $\bar{P}$ solution $p_2(x)$, which is not a gradient vector, improves the performance of the closed-loop system.



(a) Time histories of the components of the state vector (x, ξ)



(b) Phase portrait on the x-plane



(c) Phase portrait on the ξ-plane

Figure 5.8: Trajectories of system (5.1), using the control law $u_{lin} = u_{nl} = -10g(x)^{\top}p_1(x)^{\top}$, when the initial conditions (x_1, x_2) are on a circumference of radius $r = 0.9$ and $(\xi_1(0), \xi_2(0)) = (0.2, 0.2)$.

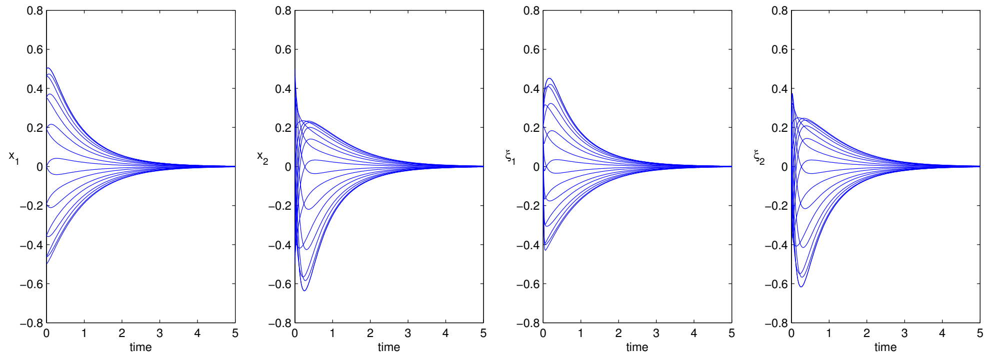
2) Simulations with $p_2(x)$: non gradient vector

Contrary to the previous case, if the algebraic $\bar{P}$ solution $p_2(x)$ is used, the two control laws

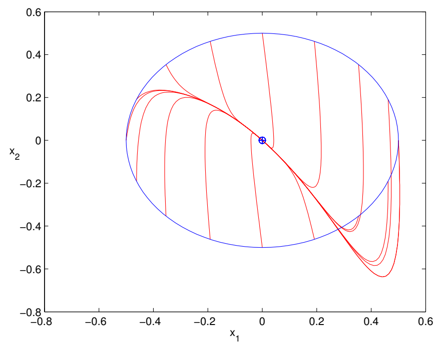
$$u_{lin}(x) = -lB^{\top}\bar{P}x,$$
$$u_{nl}(x) = -lg(x)^{\top}p_2(x)^{\top},$$

are different. Therefore it is interesting to compare the performances of the resulting closed-loop systems. Figures 5.9 and 5.10 show that, if the initial condition is on a circumference of radius $r = 0.5$, both the control laws drive the state to the origin.

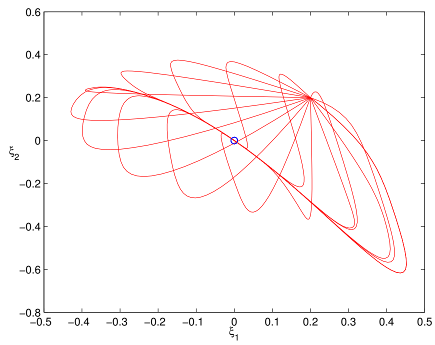
On the other hand, setting the initial condition on a circumference of radius $r = 0.95$, the trajectories of the system controlled with $u_{lin}(x)$ diverge, while the nonlinear controller still yields converging trajectories. This result is illustrated in Figures 5.11 and 5.12.



(a) Time histories of the components of the state vector (x, ξ)

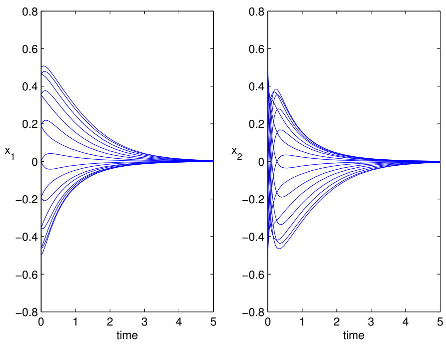


(b) Phase portrait on the x-plane

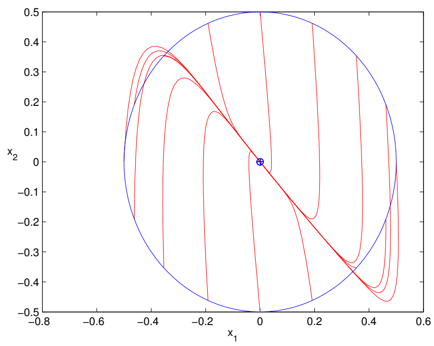


(c) Phase portrait on the ξ-plane

Figure 5.9: Trajectories of system (5.1), using the control law $u_{nl} = -10g(x)^\top p_2(x)^\top$, when the initial conditions (x_1, x_2) are on a circumference of radius $r = 0.5$ and $(\xi_1(0), \xi_2(0)) = (0.2, 0.2)$

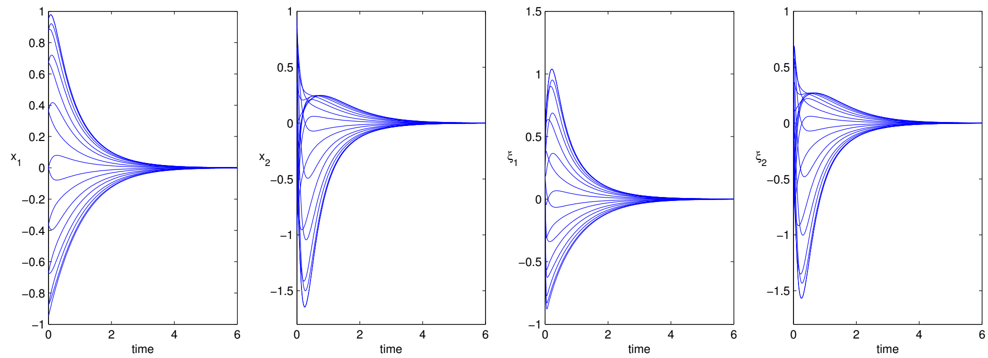


(a) Time histories of the components of the state vector (x, ξ)

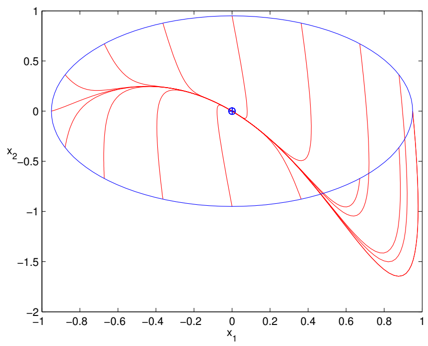


(b) Phase portrait on the x-plane

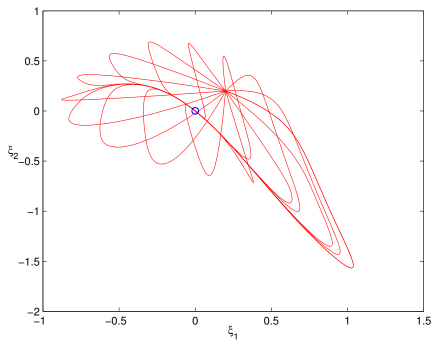
Figure 5.10: Trajectories of system (5.1), using the linear control law $u_{lin} = -10B^\top \bar{P}x$, when the initial conditions (x_1, x_2) are on a circumference of radius $r = 0.5$



(a) Time histories of the components of the state vector (x, ξ)

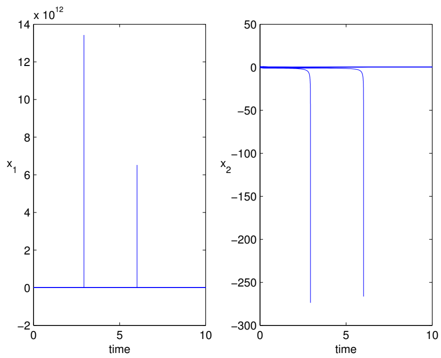


(b) Phase portrait on the x-plane

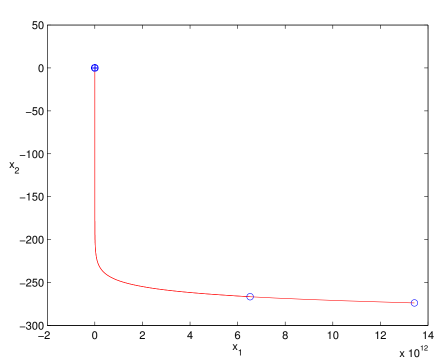


(c) Phase portrait on the ξ-plane

Figure 5.11: Trajectories of system (5.1), using the control law $u_{nl} = -10g(x)^{\top}p_2(x)^{\top}$, when the initial conditions (x_1, x_2) are on a circumference of radius $r = 0.95$ and $(\xi_1(0), \xi_2(0)) = (0.2, 0.2)$



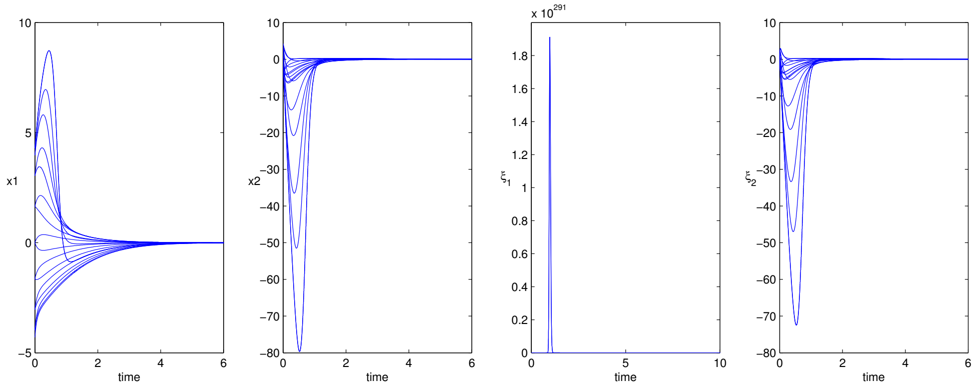
(a) Time histories of the components of the state vector x



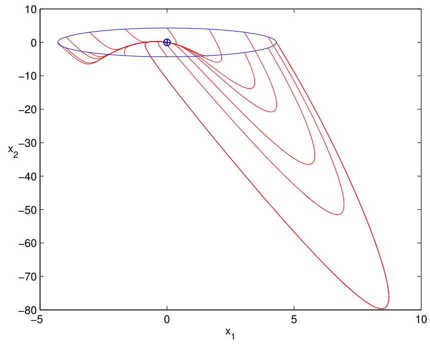
(b) Phase portrait on the x-plane

Figure 5.12: Trajectories of system (5.1), using the linear control law $u_{lin} = -10B^{\top}\bar{P}x$, when the initial conditions (x_1, x_2) are on a circumference of radius $r = 0.95$

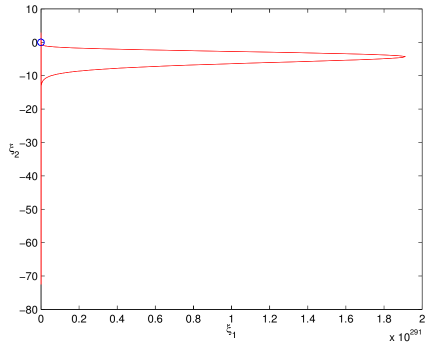
Therefore the control law $u_{nl}(x)$, constructed with the theory developed in Chapters 2 and 3, yields better performance than the performance resulting from the use of the control law $u_{lin}(x)$, that can be derived from the linearized system. Finally, Figure 5.13 shows that the basin of attraction associated to $u_{nl}(x)$ is sensitively larger than the one associated to $u_{lin}(x)$.



(a) Time histories of the components of the state vector (x, ξ)



(b) Phase portrait on the x-plane



(c) Phase portrait on the ξ-plane

Figure 5.13: Trajectories of system (5.1), using the control law $u_{nl} = -10g(x)^\top p_2(x)^\top$, when the initial conditions (x_1, x_2) are on a circumference of radius $r = 4.3$ and $(\xi_1(0), \xi_2(0)) = (0.2, 0.2)$

5.2 Aircraft: reduced-order model of roll mode

There are three primary ways for an aircraft to change its orientation. Pitch (movement of the nose up or down), roll (rotation around the longitudinal axis, that is, the axis which runs along the length of the aircraft) and yaw (movement of the nose to the left or right). Hereby a reduced model that considers only the roll mode is presented. The main aim of the controller, in this context, is to avoid an effect called wing rock, i.e. a limit cycle oscillation in the roll angle ϕ which can occur in high-performance aircrafts when flying at high angle-of-attack. The model that we consider is

$$\ddot{\phi} = \theta_2\phi + \theta_3\dot{\phi} + \theta_4|\phi|\dot{\phi} + \theta_5|\dot{\phi}|\dot{\phi}, \tag{5.3}$$

which is based on a wind tunnel test [16] at NASA Langley Research Center. The parameters θ_i depends on angle-of-attack, dynamic pressure, wing reference area, wing span, roll moment of inertia, and flight velocity. The main controller of the roll mode is the angle of deflection of the ailerons[1] and its effect can be modeled with first-order actuator dynamics

$$\tau\dot{\delta}_A = -\delta_A + u, \tag{5.4}$$

where δ_A is the aileron deflection angle, u is the control input and τ is the aileron time constant. The ultimate goal of this controller is to avoid the wing rock effect, stabilizing the roll angle ϕ to a constant value. Let $x_1 = \phi$, $x_2 = \dot{\phi}$ and $x_3 = \delta_A$ then the model for the controlled roll mode can be rewritten as

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= \theta_2 x_1 + \theta_3 x_2 + \theta_4|x_1|x_2 + \theta_5|x_2|x_2 + x_3,\\
\dot{x}_3 &= -x_3 + u,
\end{aligned} \tag{5.5}$$

where, for the sake of simplicity, we imposed $\tau = 1$. The goal of the controller is to stabilize the origin of system (5.5).

5.2.1 Linearized model

The linearized model around the origin is described by the equation

$$\dot{x} = Ax + Bu, \tag{5.6}$$

where

$$A = \frac{\partial f(x)}{\partial x}\Big|_{x=0} = \begin{bmatrix} 0 & 1 & 0 \\ \theta_2 & \theta_3 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Note that

$$\mathcal{R} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & \theta_3 - 1 \\ 1 & -1 & 1 \end{bmatrix}$$

has full rank for each value of the parameter vector θ, therefore the linearized system is reachable, and hence stabilizable, for each θ.

[1] Roll is controlled by movable sections on the trailing edge of the wings called ailerons.

So, by Theorem 2.0.3, there exist a, b, c, d, e, f such that

$$\bar{P} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = \bar{P}^\top,$$

is positive definite and $x^\top \bar{P} B = 0$ implies $x^\top \bar{P} A x < 0$. In addition note that the first condition implies

$$x^\top \bar{P} B = 0 \;\Rightarrow\; \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} c \\ e \\ f \end{bmatrix} = 0 \;\Rightarrow\; x_3 = -\frac{c x_1 + e x_2}{f} \triangleq x_{lr},$$

which substituted in $x^\top \bar{P} A x$ yields

$$Q(x) \triangleq x^\top \bar{P} A x\Big|_{x_3 = x_{lr}} = (a x_1 + b x_2 + c x_{lr}) x_2 + (b x_1 + d x_2 + e x_{lr})(\theta_2 + x_1 \theta_3 x_2 + x_{lr}) < 0, \quad (5.7)$$

which is a quadratic negative definite function.

Note now that, as stated at the beginning of Section 2.1, $p_{lin}(x) = x^\top \bar{P}$ is an algebraic $\bar{P}$ solution and let $\bar{l}$ be its index. Then, by Corollary 2.1.1, the control law $u_{lin} = -l B^\top \bar{P} x$, with $l > \bar{l}$, is a global stabilizer for the origin of the linearized system. Hence, by Theorem 1.2.1, u_{lin} is a local stabilizer for the origin of the nonlinear system (5.5).

It is important to remark that u_{lin} is the control law that can be derived using the linearization around the origin and, in most of the cases, it is the only known feedback law. The aim of the following section is to improve the performance of this control law using the theory developed in Chapters 2 and 3.

5.2.2 Nonlinear control

In the previous section we have proved that the linear control law $u_{lin} = -l B^\top \bar{P} x$ is a local stabilizer for system (5.5). To do that we have used the fact that, if $V(x) = \frac{1}{2} x^\top \bar{P} x$ is a CLF for the linearized system, then $p_{lin}(x) = x^\top \bar{P}$ is an algebraic $\bar{P}$ solution. In this section, we improve the performance of the closed-loop system using the nonlinear control law proposed in Corollary 2.2.3. To this end, we need to find a nonlinear algebraic $\bar{P}$ solution. The key point is that, if this new algebraic $\bar{P}$ solution has a larger set of definition than $p_{lin}(x) = x^\top \bar{P}$, then it is likely that also the region of attraction of the zero equilibrium of the closed-loop system, with the new control law, will be larger. To compare the two control laws, suppose to use the same matrix $\bar{P}$ found in the linearized case. Then, locally, each function $p(x)$ such that $p(0) = 0$ and $p_x(0) = \bar{P}$ is an algebraic $\bar{P}$ solution since condition

$$x^\top \tilde{P}(0) g(0) = 0 \;\Rightarrow x^\top \tilde{P}(0) F(0) x < 0$$

given in Remark 1.3 , becomes

$$x^\top \bar{P} B = 0 \;\Rightarrow x^\top \bar{P} A x < 0,$$

which is satisfied by construction. To extend the region of attraction of the zero equilibrium, since Condition (P2) for an algebraic $\bar{P}$ solution is very difficult to check, we limit our attention to the necessary condition

$$p(x)g(x) = 0 \ \Rightarrow p(x)f(x) < 0.$$

Note that the constraint $p(0) = 0$ and $p_x(0) = \bar{P}$ impose that

$$p(x) \triangleq \begin{bmatrix} p_1(x) & p_2(x) & p_3(x) \end{bmatrix} = x^\top \bar{P} + \begin{bmatrix} \tilde{p}_1(x) & \tilde{p}_2(x) & \tilde{p}_3(x) \end{bmatrix}$$

where $\tilde{p}_i(x)$ are at least $\mathcal{O}(\|x\|^2)$. Moreover note that, since the final goal is to implement the control law

$$u_{nl} = -lg(x)p(x)^\top = -lp_3(x)^\top,$$

that does not depend of the first two components of the vector $p(x)$, we can impose $\tilde{p}_1(x) = \tilde{p}_2(x) = 0$. Note now that $p(x)g(x) = 0$ implies $p_3(x) = 0$ and hence

$$x_3 = -\frac{cx_1 + ex_2 + \tilde{p}_3(x)}{f} \triangleq x_{3r}.$$

Substituting in $p(x)f(x)$ yields

$$\begin{aligned}
&\left(ax_1 + bx_2 - c\frac{cx_1 + ex_2 + \tilde{p}_3}{f}\right)x_2 + \\
&+\left(bx_1 + dx_2 - e\frac{cx_1 + ex_2 + \tilde{p}_3}{f}\right)(\theta_2 x_1 + \theta_3 x_2 + \theta_4|x_1|x_2 + \theta_5|x_2|x_2 + x_{3r}) = \\
&= \overbrace{(ax_1 + bx_2 + cx_{lr})x_2 + (bx_1 + dx_2 + ex_{lr})(\theta_2 x_1 + \theta_3 x_2 + x_{lr})}^{T_1} + \\
&\overbrace{+(bx_1 + dx_2 - \tfrac{ec}{f}x_1 + \tfrac{ec}{f}x_2)(\theta_4|x_1|x_2 + \theta_5|x_2|x_2)}^{T_2} - \\
&\overbrace{-\tfrac{\tilde{p}_3}{f}(cx_2 + e(\theta_2 x_1 + \theta_3 x_2 + \theta_4|x_1|x_2 + \theta_5|x_2|x_2 + x_{3r}) + bx_1 + dx_2 - \tfrac{e}{f}(cx_1 + ex_2))}^{T_3}
\end{aligned}$$

Note now that

1. the term T_1 is quadratic and it coincides with the function $Q(x)$ in (5.7), therefore it is globally negative definite;
2. the term T_2 has exactly order three;
3. if we suppose that $\tilde{p}_3(x)$ has order at least 3 then the term T_3 has order at least four.

Therefore if x is small the dominant term is T_1, which is negative definite, while if x is large the dominant term is T_3, that is negative definite if we impose

$$\begin{aligned}
\tilde{p}_3(x) &= (k_p(cx_2 + e(\theta_2 x_1 + \theta_3 x_2 + \theta_4|x_1|x_2 + \theta_5|x_2|x_2 + x_3) + bx_1 + dx_2 - \frac{e}{f}(cx_1 + ex_2)))^3 \\
&= k_p^3\left[(e\theta_2 + b - \frac{ec}{f})x_1 + (c + e\theta_3 + d - \frac{e^2}{f})x_2 + e\theta_4|x_1|x_2 + e\theta_5|x_2|x_2 + ex_3\right]^3
\end{aligned}$$

where $k_p > 0$ is an arbitrary scalar value. In fact this choice yields

$$T_3 = -\frac{k_p^3}{f}(cx_2 + e(\theta_2 x_1 + \theta_3 x_2 + \theta_4 |x_1| x_2 + \theta_5 |x_2| x_2 + x_{3r}) + bx_1 + dx_2 - \frac{e}{f}(cx_1 + ex_2))^4,$$

which is negative definite since $k_p > 0$ and $f > 0$. Therefore it is reasonable to suppose that the set Ω, in which Condition (P2) holds, is larger for $p(x) = x^\top \bar{P} + \begin{bmatrix} \tilde{p}_1(x) & \tilde{p}_2(x) & \tilde{p}_3(x) \end{bmatrix}$ than the set associated to the algebraic $\bar{P}$ solution derived from the linearized system, $p_{lin}(x)$. Finally, by Corollary 2.2.3, the control law $u_{nl}(x) = -lg(x)p(x)^\top$ is at least a local stabilizer for the origin of system (5.5).

5.2.3 Simulations

To test the behavior of the designed control laws, namely

$$u_{lin}(x) = -lB^\top \bar{P} x,$$
$$u_{nl}(x) = -lg(x)^\top p(x)^\top,$$

the values

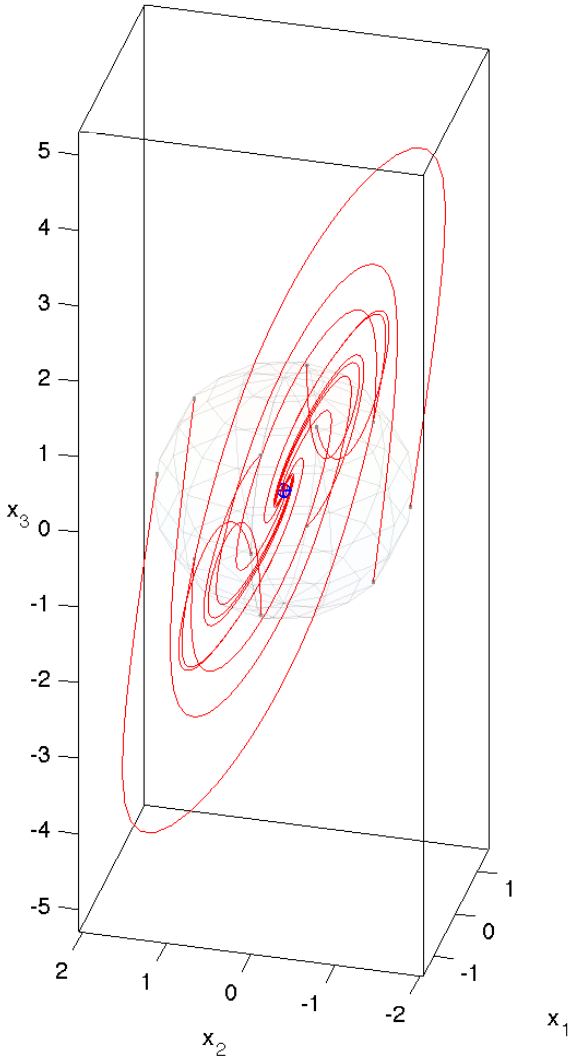
$$\begin{bmatrix} \theta_2 & \theta_3 & \theta_4 & \theta_5 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix} \tag{5.8}$$

have been used. It is important to remark that these values have been chosen to illustrate the difference between the two controls and do not necessarily refer to a realistic model. As a matter of fact, the aim of this example is to illustrate that, for a system with a structure similar to (5.5), the nonlinear control produces better performance than the classical control derived from the linearized system, more than solving the wing rock problem. For the same reason, in the following analysis, constraints on the state variables or on the control value are not taken in consideration (whereas in a real case these must of course be limited).

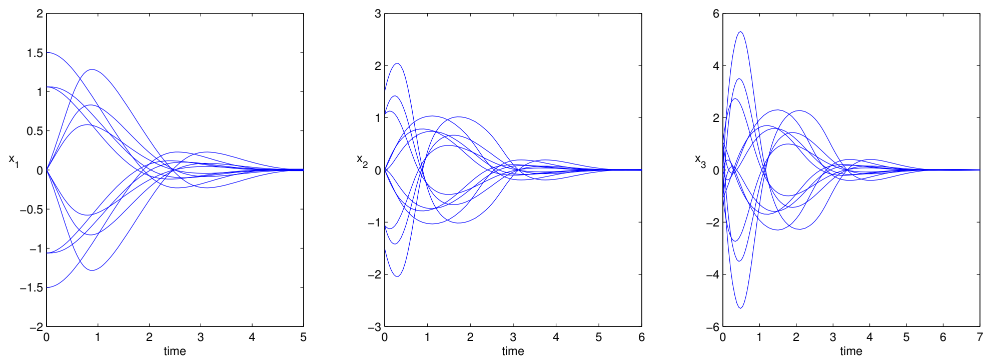
In all the simulations the value $l = 10$ is used both for $u_{lin}(x)$ and $u_{nl}(x)$, hence the linear part of the two control laws is exactly the same and the comparison is meaningful. Finally we fix $k_p = 1/10$ for the nonlinear control law.

Basin of attraction

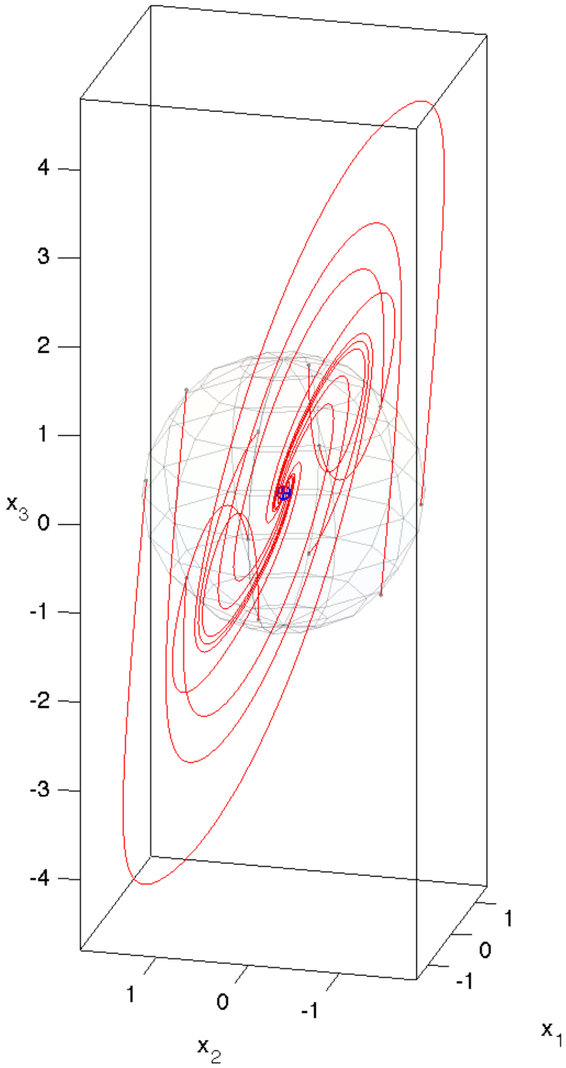
Figure 5.14 shows the trajectories of the two closed-loop systems, with initial conditions on a sphere of radius $r = 1.5$, in the $3D$ plane. Moreover the comparison between the norm of the state vector x and the value of the control laws is shown. To better understand the dynamics of the systems, in Figure 5.15, the projections of the trajectories on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, when the initial conditions are on a circumference of radius r on the (x_1, x_3)-plane and $x_2(0) = 0$, are also reported. This choice is due to the fact that the 3-dimentional graphs reveal that the most important dynamics are on the (x_1, x_3)-plane.
It is interesting to note that, while for $r = 1.5$, both control laws stabilize the origin, increasing r yields to the divergence of the trajectories of the system controlled with $u_{lin}(x)$, while the system controlled with $u_{nl}(x)$ has still converging trajectories. Figure 5.16 illustrates this situation for $r = 3$.



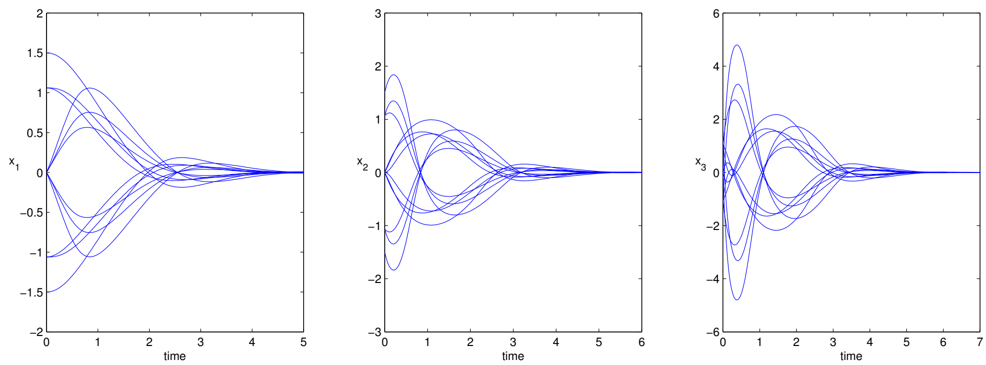
(a) Phase portrait with the control law $u_{lin}(x)$



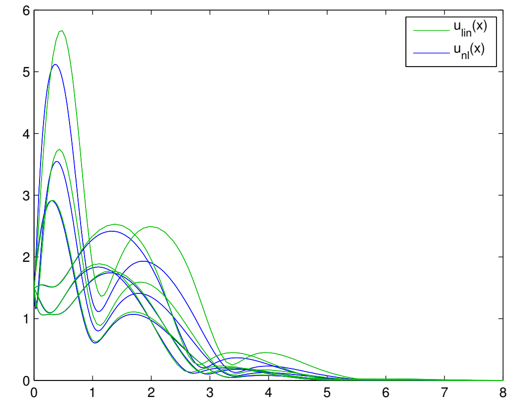
(b) Time histories of the components of the state vector x, with the control law $u_{lin}(x)$



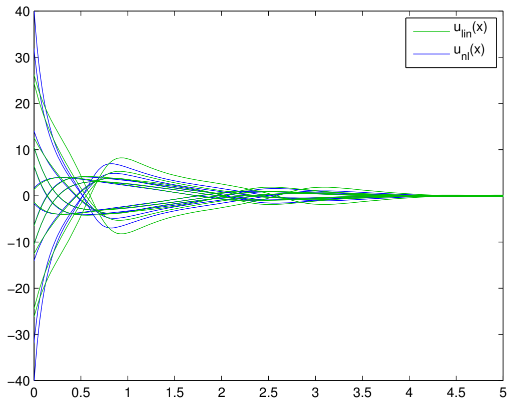
(c) Phase portrait with the control law $u_{nl}(x)$



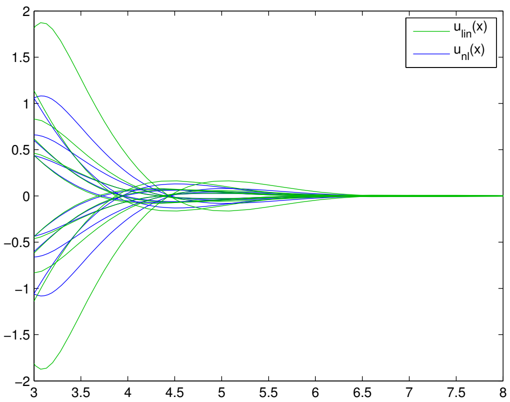
(d) Time histories of the components of the state vector x, with the control law $u_{nl}(x)$



(e) Time histories of the norm of the x state vector

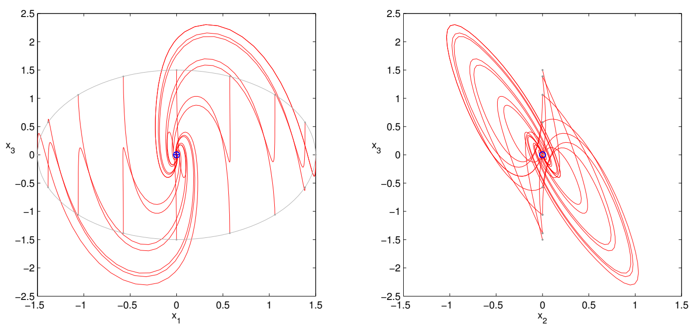


(f) Time histories of the control values respectively of $u_{lin}(x)$ and $u_{nl}(x)$

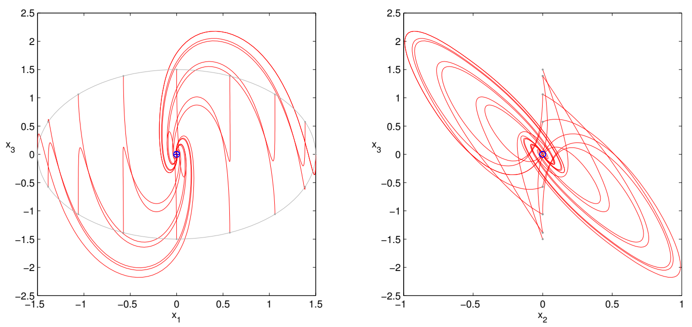


(g) Zoom of plot (f)

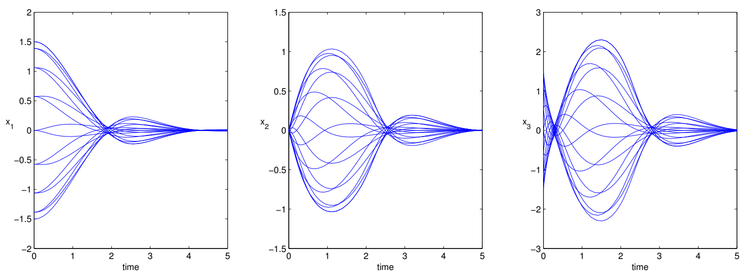
Figure 5.14: Simulation results for system (5.5) with initial conditions on a sphere of radius $r = 1.5$. Plots (a) and (b) are relative to the control law $u_{lin}(x)$, while plots (c) and (d) to the control law $u_{nl}(x)$. Plots (e), (f) and (g) show a comparison between the norm of the state vector and the values of the control in the two cases. In these plots and in the following ones the initial point, of each trajectory, is grey colored, while the final point is a blu circle.



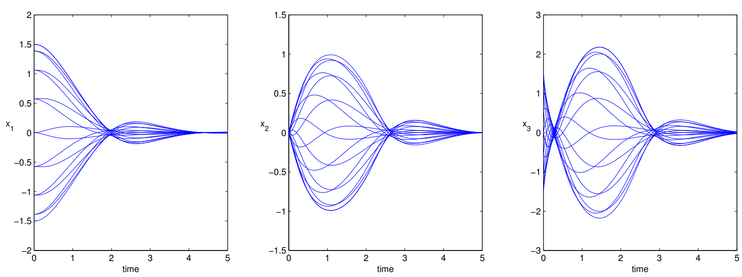
(a) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{lin}(x)$



(b) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{nl}(x)$

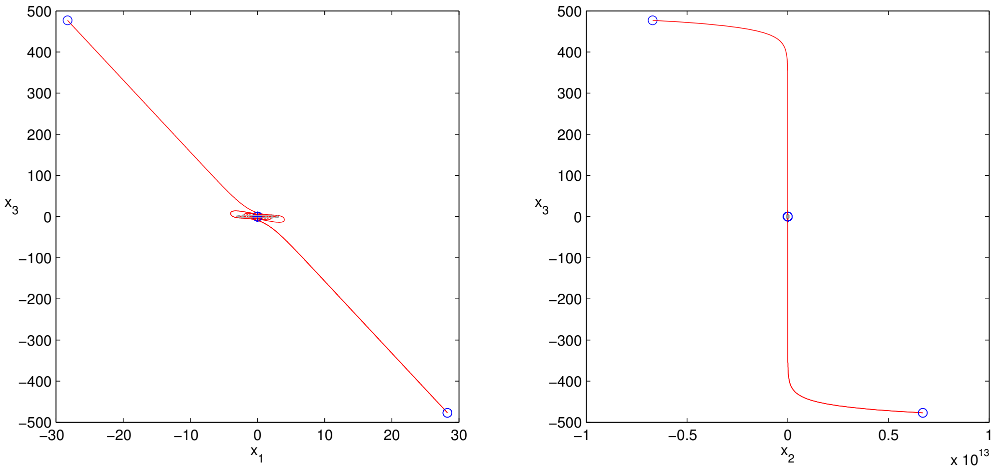


(c) Time histories of the components of the state vector x, with the control law $u_{lin}(x)$

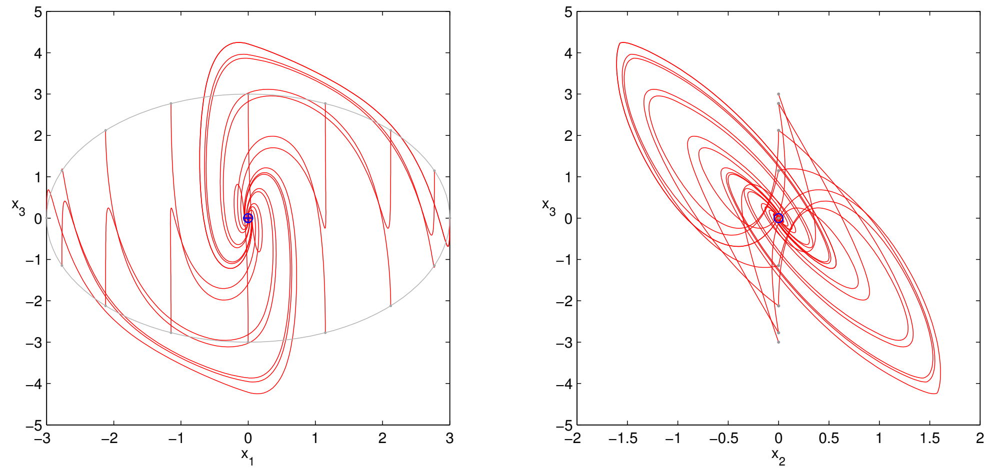


(d) Time histories of the components of the state vector x, with the control law $u_{nl}(x)$

Figure 5.15: Simulation results for system (5.5) with initial conditions on a circumference of radius $r = 1.5$ on the (x_1, x_3)-plane and $x_2(0) = 0$. Plots (a) and (c) are relative to the control law $u_{lin}(x)$, while plots (b) and (d) to the control law $u_{nl}(x)$.



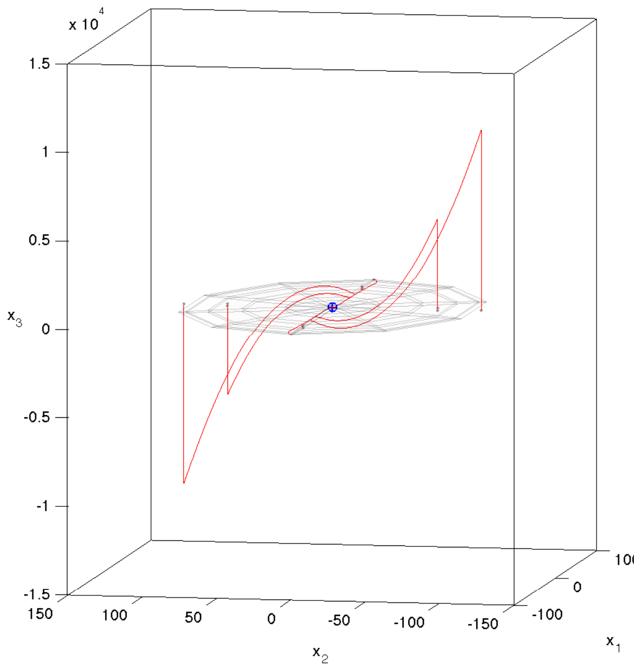
(a) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{lin}(x)$



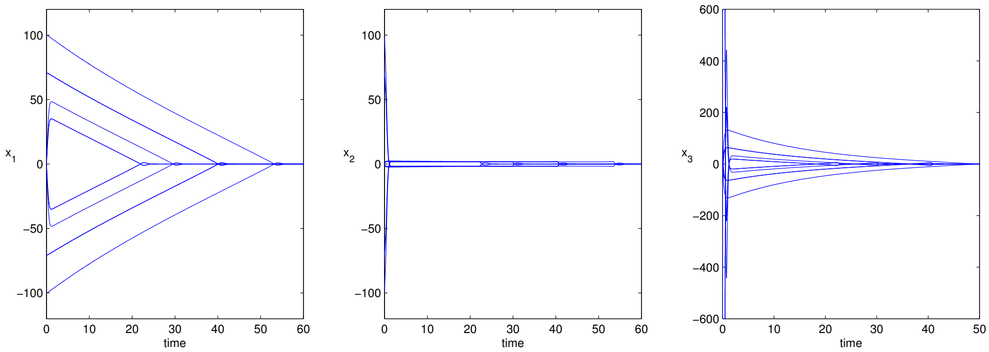
(b) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{nl}(x)$

Figure 5.16: Simulation results for system (5.5) with initial conditions on a circumference of radius $r = 3$ on the (x_1, x_3)-plane and $x_2(0) = 0$. Plots (a) are relative to the control law $u_{lin}(x)$, while plots (b) to the control law $u_{nl}(x)$.

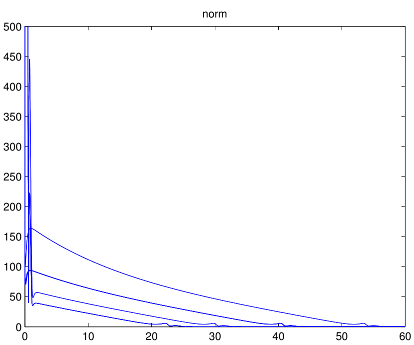
Finally Figures 5.17 and 5.18 show that the nonlinear control derived with the theory developed in Chapters 2 and 3 not only has a larger basin of attraction but it is a global stabilizer for the origin of system (5.5). This is a remarkable result: it proves that the proposed method not only can give better performance than a linear design but (in some cases) leads to globally stabilizing feedback law.



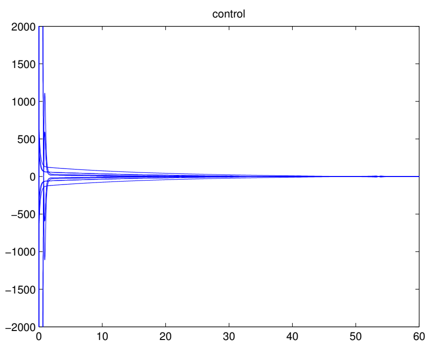
(a) Phase portrait for the control law $u_{nl}(x)$



(b) Time histories of the components of the state vector x, with the control law $u_{nl}(x)$

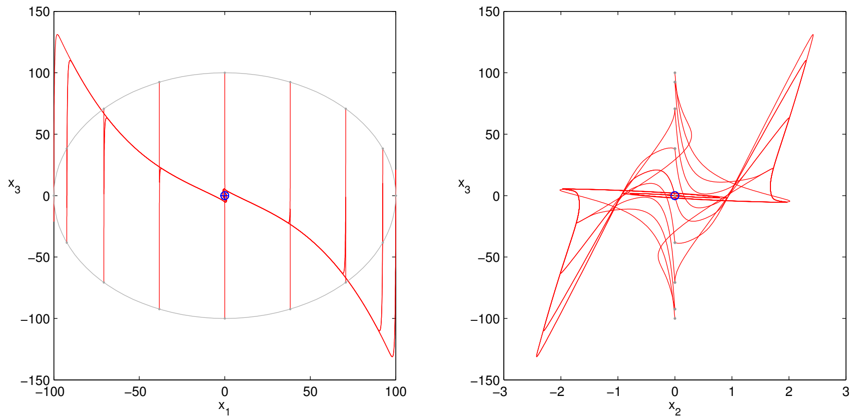


(c) Time histories of the norm of the x state vector

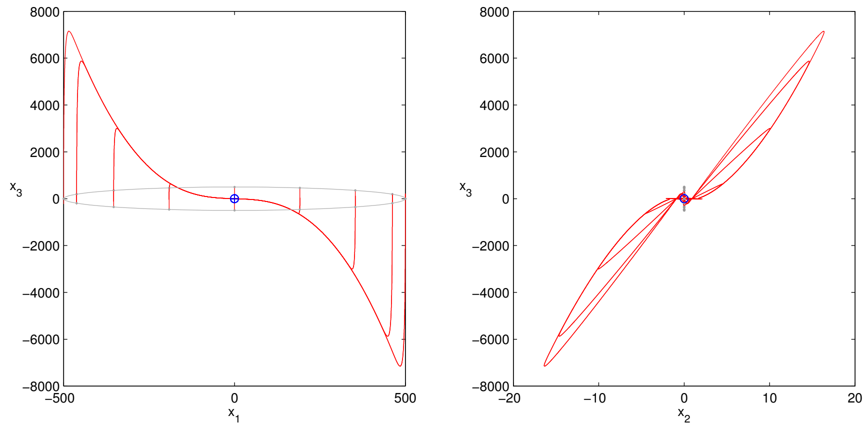


(d) Time histories of the control values $u_{nl}(x)$

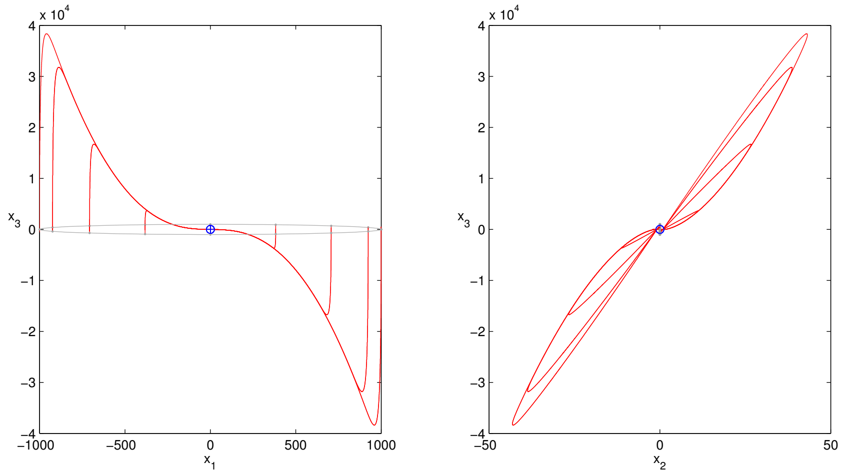
Figure 5.17: Simulation results for system (5.5) with initial conditions on a sphere of radius $r = 100$, with control law $u_{nl}(x)$.



(a) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{nl}(x)$ and $r = 100$



(b) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{nl}(x)$ and $r = 500$



(c) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{nl}(x)$ and $r = 1000$

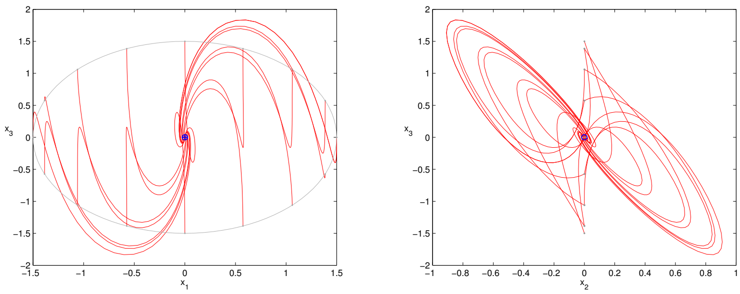
Figure 5.18: Simulation results for system (5.5) with initial conditions on a circumference of radius r on the plane (x_1, x_3)-plane, $x_2(0) = 0$ and control law $u_{nl}(x)$.

Parameter sensitivity

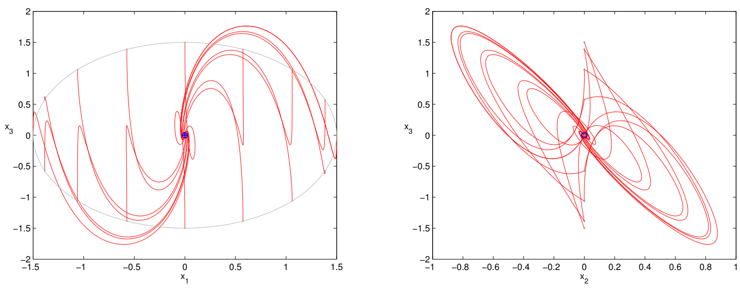
Another important issue to address is the robustness of the proposed method with respect to parameter's variations. Figure 5.19 shows the behavior of the two controlled systems when the same control law derived in the previous section, i.e. the one relative to the parameter vector (5.9), is used, but the real parameter vector is scaled by the scalar $d > 0$, i.e.

$$\begin{bmatrix} \theta_2 & \theta_3 & \theta_4 & \theta_5 \end{bmatrix} = d \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}. \tag{5.9}$$

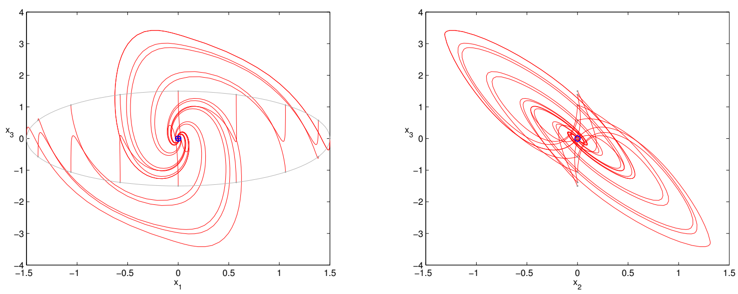
Note that with $d = 0.9$ or $d = 1.1$, corresponding to a 10% change on θ, both the control laws $u_{lin}(x)$ and $u_{nl}(x)$ still stabilize the origin. Whereas for $d = 1.2$, corresponding to a 20% change on θ, only the nonlinear controller yields converging trajectories. Therefore the proposed nonlinear controller not only is global, instead of local, but it is also more robust to parameters perturbations.



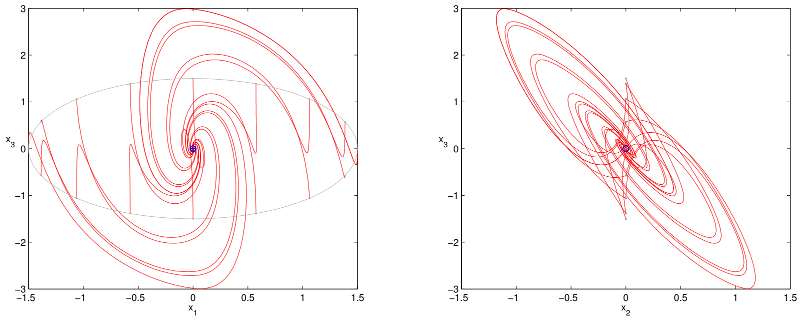
(a) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{lin}(x)$ and $d = 0.9$



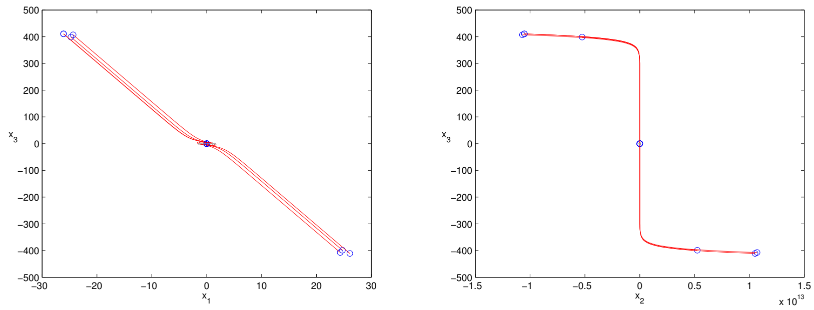
(b) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{nl}(x)$ and $d = 0.9$



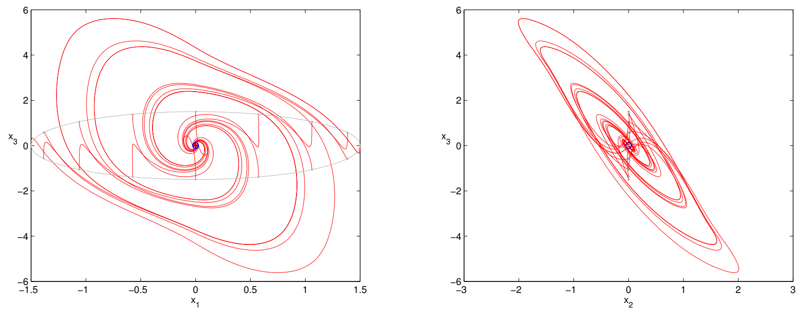
(c) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{lin}(x)$ and $d = 1.1$



(d) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{nl}(x)$ and $d = 1.1$



(e) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{lin}(x)$ and $d = 1.2$



(f) Projection of the dynamics on the (x_1, x_3)-plane and on the (x_2, x_3)-plane, with the control law $u_{nl}(x)$ and $d = 1.2$

Figure 5.19: Dynamics of system (5.5) with initial conditions on a circumference of radius $r = 1.5$ on the (x_1, x_3)-plane, $x_2(0) = 0$ and error on the parameters.

5.3 Stabilization of the angular velocity of a rigid body

The theoretical discussion developed in the previous chapters relies upon the fact that the linearized system around the origin is stabilizable. This is, in fact, a necessary condition for the existence of an algebraic $\bar{P}$ solution, as stated in the following theorem.

Theorem 5.3.1. *If the nonlinear system (1.7) possesses an algebraic $\bar{P}$ solution then the origin of the linearized system*

$$\dot{x} = Ax + Bu, \tag{5.10}$$

where $A = \frac{\partial f}{\partial x}(0)$ and $B = g(0)$, is stabilizable.

Proof. Since (1.7) possesses an algebraic $\bar{P}$ solution, there exists a matrix $\bar{P} = \bar{P}^\top > 0$ such that $p_x(0) = \bar{P}$ and a scalar $\bar{l} > 0$ such that, for all $l > \bar{l}$,

$$p(x)f(x) - lp(x)g(x)g(x)^\top p(x) \leq -x^\top \Gamma(x)x, \tag{5.11}$$

where $\Gamma(x) = \Gamma(x)^\top > 0$ for all $x \in \Omega$. In particular note that $\Gamma(0) = \Gamma(0)^\top > 0$. Let[2], as usual, $p(x) = x^\top \tilde{P}(x)$ and $f(x) = F(x)x$. Then $\tilde{P}(0) = \bar{P}$ and $F(0) = A$. Therefore (5.11) can be rewritten as

$$\begin{aligned} x^\top(\bar{P}A + \mathcal{O}(x))x - lx^\top(\bar{P}BB^\top\bar{P} + \mathcal{O}(x))x &\leq -x^\top(\Gamma(0) + \mathcal{O}(x))x \\ x^\top(\bar{P}A - l\bar{P}BB^\top\bar{P})x &\leq -x^\top\Gamma(0)x + \mathcal{O}(\|x\|^3). \end{aligned} \tag{5.12}$$

Consider now a neighborhood of the origin in which $\mathcal{O}(\|x\|^3) < x^\top \dfrac{\Gamma(0)}{2} x$. In such neighborhood (5.12) implies

$$x^\top(\bar{P}A - l\bar{P}BB^\top\bar{P})x \leq -x^\top\Gamma(0)x + \mathcal{O}(\|x\|^3) < -x^\top\frac{\Gamma(0)}{2}x, \tag{5.13}$$

and hence

$$-Q \triangleq \frac{\bar{P}A + A^\top\bar{P}}{2} - l\bar{P}BB^\top\bar{P} < -\frac{\Gamma(0)}{2} < 0.$$

Consider now the linearized system (5.10) and the function $V(x) = \frac{1}{2}x^\top\bar{P}x$, which is positive definite. If $\bar{u} = -lB^\top\bar{P}x$, then the time derivative of $V(x)$ along the trajectories of the closed-loop system is

$$\dot{V}(x) = V_x\dot{x} = x^\top\bar{P}(Ax + B\bar{u}) = x^\top\left[\frac{\bar{P}A + A^\top\bar{P}}{2} - l\bar{P}BB^\top\bar{P}\right]x = -x^\top Qx.$$

Therefore $V(x)$ is a Lyapunov function for the closed-loop linearized system and system (5.10) is stabilizable. □

In the previous example, since we have not been able to check Condition (P2), we have constructed $p(x)$ imposing the necessary condition[3]

$$p(x)g(x) = 0 \quad \Rightarrow p(x)f(x) < 0, \quad x \neq 0. \tag{5.14}$$

[2] See Lemmas A.3.1 and A.3.2.

[3] It has been proved in Section 1.3 that Condition (P2) implies (5.14).

This procedure is legitimated by the fact that, since $p(x)$ is tangent to $\bar{P}$, solution of the linearized problem, Remark 1.3 guarantees that it locally satisfies Condition (P2). Therefore the only reason for imposing (5.14) was to enlarge the region in which Condition (P2) is satisfied. Note, however that in the previous example, (5.14) is sufficient to guarantee that $p(x)$ was a global algebraic $\bar{P}$ solution.

Given the positive result of the previous example, one can wonder if this discussion can be extended to systems with non-stabilizable linearization, i.e. by imposing (5.14) instead of Condition (P2) in the definition of an algebraic $\bar{P}$ solution. To this end the following definition can be given.

Definition 5.3.1 (Local weak algebraic $\bar{P}$ solution)**.** *Let $\bar{P} = \bar{P}^\top > 0$ be a symmetric positive definite matrix, and let $\Omega \subset \mathbb{R}^n$ be an open set, $0 \in \Omega$. A continuously differentiable mapping $p(x) : \mathbb{R}^{n\times 1} \to \mathbb{R}^{1\times n}$ is said to be a local weak algebraic $\bar{P}$ solution of (1.11) if*

(P1) $p(0) = 0$ and $p(x)$ is tangent at the origin to $\bar{P}$, namely $p_x(0) = \bar{P}$;

(P3) $p(x)g(x) = 0 \;\Rightarrow\; p(x)f(x) < 0$ for all $x \in \Omega\backslash\{0\}$.

Note that Theorem 2.2.2 can still be applied. Therefore, if $R > \frac{\bar{P}}{2}$, the candidate CLF $V(x,\xi) = p(\xi)x + \frac{1}{2}\|x-\xi\|_R^2$ is locally positive definite. Unfortunately the proof of Theorem 2.2.3, on the other hand, relies upon Condition (P2) of the algebraic $\bar{P}$ solution. Therefore it is not guaranteed that, using a weak algebraic $\bar{P}$ solution, $V(x,\xi)$ satisfies Property 2 of a CLF. Hence it is not even proved whether the nonlinear control law $u_{nl} = -lg(x)^\top p(x)^\top$ stabilizes the origin or not. However the previous example suggests that, if $p(x)$ satisfies Condition (P3), this can be the case for particular systems.

In the following, a specific example is analyzed. In particular it is proved that, for this particular example, Condition (P3) is sufficient to guarantee the local stabilizability of the origin, using the nonlinear control law $u_{nl} = -lg(x)^\top p(x)^\top$. The theoretical discussion of this issue, in particular if Condition (P2) can be always replaced by Condition (P3) or if some additional property is needed, is not within the scope of this work and it is left as an open question.

5.3.1 Model description and linear feedback law

Consider a rigid body in an inertial reference frame. Let ω_1, ω_2 and ω_3 be the angular velocity components along a body fixed reference frame, having the origin at the center of gravity and consisting of the three principal axes. The Euler equations for the rigid body, with one control, are given by the equation

$$I\dot{\omega} = S(\omega)I\omega + Gu, \tag{5.15}$$

where $I = diag(I_1, I_2, I_3)$ is the inertial matrix, $\omega(t) \in \mathbb{R}^3$, $u(t) \in \mathbb{R}$, $G \in \mathbb{R}^{3\times 1}$ and

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}.$$

Note that system (5.15) can be rewritten as

$$\dot{x} = I^{-1}S(x)Ix + I^{-1}Gu \triangleq I^{-1}S(x)Ix + Bu, \tag{5.16}$$

where we have used the variable x instead of ω, to distinguish the two formulations. Hence, if we define $B^\top = [b_1\ b_2\ b_3]^\top$, system (5.16) rewrites as

$$\begin{aligned}
\dot{x}_1 &= \frac{I_2 - I_3}{I_1} x_2 x_3 + b_1 u \triangleq \alpha_1 x_2 x_3 + b_1 u, \\
\dot{x}_2 &= \frac{I_3 - I_1}{I_2} x_3 x_1 + b_2 u \triangleq \alpha_2 x_3 x_1 + b_2 u, \\
\dot{x}_3 &= \frac{I_1 - I_2}{I_3} x_1 x_2 + b_3 u \triangleq \alpha_3 x_1 x_2 + b_3 u.
\end{aligned} \tag{5.17}$$

Note that $A = \frac{\partial f}{\partial x}(0) = 0_{3\times 3}$, therefore the linearized system around the origin is not stabilizable. So, it is not possible to use the theory developed in Chapters 2 and 3. However the following theorem shows that it is possible to stabilize the origin of (5.17) by means of a linear control law of the same form as the one proposed in Corollary 2.1.1, with $\bar{P} = I$, the inertial matrix.

Theorem 5.3.2. *The feedback law $u = -lB^\top Ix$ globally asymptotically stabilizes the origin of system (5.17) if and only if b_1, b_2 and b_3 are all different from zero.*

Proof. The proof of this theorem, with $l = 1$, is a consequence of the Main Theorem in [1]. The argument used therein can be easily extended to the case $l \neq 1$, $l > 0$. Note that [1] analyzes system (5.15), therefore therein the control law assumes the form $u = -lG^\top\omega$. This is exactly the same control law proposed here since, substituting $\omega = x$ and $B = I^{-1}G$, yields

$$u = -lG^\top\omega = -l(IB)^\top x = -lB^\top Ix.$$

□

As already stated we now investigate if the control law $u_{nl} = -lg(x)^\top p(x)^\top$, which has the same form as the control law proposed in Corollary 2.2.3, stabilizes the origin. By Theorem 5.3.1, since the linearized system is not stabilizable, it is not possible to find an algebraic $\bar{P}$ solution, therefore we impose that $p(x)$ is a weak algebraic $\bar{P}$ solution.

5.3.2 Construction of a weak algebraic $\bar{P}$ solution

For the sake of simplicity, in the following we select $B^\top = [1\ 1\ 1]^\top$. Therefore system (5.17) becomes

$$\begin{aligned}
\dot{x}_1 &= \alpha_1 x_2 x_3 + u, \\
\dot{x}_2 &= \alpha_2 x_3 x_1 + u, \\
\dot{x}_3 &= \alpha_3 x_1 x_2 + u.
\end{aligned} \tag{5.18}$$

In agreement with Theorem 5.3.2 select

$$\bar{P} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix},$$

and define $p_{lin}(x) = x^\top \bar{P}$. Then

$$p_{lin}(x)f(x) = \begin{bmatrix} I_1 x_1 & I_2 x_2 & I_3 x_3 \end{bmatrix} \begin{bmatrix} \alpha_1 x_2 x_3 \\ \alpha_2 x_3 x_1 \\ \alpha_3 x_1 x_2 \end{bmatrix} = \left[(I_2 - I_3) + (I_3 - I_1) + (I_1 - I_3) \right] x_1 x_2 x_3 = 0.$$

Note that the conditions $p(0) = 0$ and $p_x(0) = \bar{P}$ imply

$$p(x) = x^\top \bar{P} + \begin{bmatrix} \tilde{p}_1(x) & \tilde{p}_2(x) & \tilde{p}_3(x) \end{bmatrix} = p_{lin}(x) + \begin{bmatrix} \tilde{p}_1(x) & \tilde{p}_2(x) & \tilde{p}_3(x) \end{bmatrix},$$

where $\tilde{p}_i(x)$ are at least $\mathcal{O}(\|x\|^2)$. We can assume, for instance

$$\begin{aligned} \tilde{p}_1(x) &= -(\alpha_1 x_3 - \alpha_3 x_1) x_2, \\ \tilde{p}_2(x) &= -(\alpha_2 x_3 - \alpha_3 x_2) x_1, \\ \tilde{p}_3(x) &= -\tilde{p}_1(x) - \tilde{p}_2(x) - \alpha_3 x_1 x_2. \end{aligned}$$

With this choice $p(x)f(x)$ becomes

$$\begin{aligned} p(x)f(x) &= \left[p_{lin}(x) + \begin{bmatrix} \tilde{p}_1(x) & \tilde{p}_2(x) & \tilde{p}_3(x) \end{bmatrix} \right] f(x) \\ &= p_{lin}(x)f(x) + \tilde{p}_1(x)\alpha_1 x_2 x_3 + \tilde{p}_2(x)\alpha_2 x_3 x_1 + \tilde{p}_3(x)\alpha_3 x_1 x_2 \\ &= 0 + \tilde{p}_1(x)\alpha_1 x_2 x_3 + \tilde{p}_2(x)\alpha_2 x_3 x_1 + (-\tilde{p}_1(x) - \tilde{p}_2(x) - \alpha_3 x_1 x_2)\alpha_3 x_1 x_2 \\ &= \tilde{p}_1(x)\left[\alpha_1 x_2 x_3 - \alpha_3 x_1 x_2\right] + \tilde{p}_2(x)\left[\alpha_2 x_3 x_1 - \alpha_3 x_1 x_2\right] - (\alpha_3 x_1 x_2)^2 \\ &= \tilde{p}_1(x) x_2 \left[\alpha_1 x_3 - \alpha_3 x_1\right] + \tilde{p}_2(x) x_1 \left[\alpha_2 x_3 - \alpha_3 x_2\right] - (\alpha_3 x_1 x_2)^2 \\ &= -x_2^2 \left[\alpha_1 x_3 - \alpha_3 x_1\right]^2 - x_1^2 \left[\alpha_2 x_3 - \alpha_3 x_2\right]^2 - (\alpha_3 x_1 x_2)^2. \end{aligned} \tag{5.19}$$

Therefore $p(x)f(x)$ is the sum of three non positive terms. We want to show that, under the constraint $p(x)g(x) = 0$, it is not possible that all the three terms are simultaneously zero. To this end, note that $p(x)g(x) = 0$ implies

$$\begin{aligned} p(x)g(x) = 0 \;&\Rightarrow\; p_1(x) + p_2(x) + p_3(x) = 0 \\ &\Rightarrow\; I_1 x_1 + \tilde{p}_1(x) + I_2 x_2 + \tilde{p}_2(x) + I_3 x_3 - \tilde{p}_1(x) - \tilde{p}_2(x) - \alpha_3 x_1 x_2 = 0, \end{aligned}$$

therefore a condition equivalent to $p(x)g(x) = 0$ is

$$I_1 x_1 + I_2 x_2 + I_3 x_3 - \alpha_3 x_1 x_2 = 0. \tag{5.20}$$

Note now that the third term in the right hand side of (5.19) is equal to zero if and only if $x_1 = 0$ or $x_2 = 0$ and

- if $x_1 = 0$ then (5.19) becomes $-x_2^2 \left[\alpha_1 x_3\right]^2$, which is zero if and only if x_2 or x_3 are zero;
- if $x_2 = 0$ then (5.19) becomes $-x_1^2 \left[\alpha_2 x_3\right]^2$, which is zero if and only if x_1 or x_3 are zero.

Finally, by (5.20), if two of the three variables x_1, x_2, x_3 are zero also the third one is zero. Therefore, under the constraint $p(x)g(x) = 0$, $p(x)f(x)$ assumes negative values for each $x \neq 0$. Thus, $p(x)$ is a global weak algebraic $\bar{P}$ solution.

5.3.3 Simulations

The aim of the simulations is to compare the performances of the control laws

$$u_{lin}(x) = -lB^\top \bar{P}x,$$
$$u_{nl}(x) = -lg(x)^\top p(x)^\top,$$

where $u_{lin}(x)$ is the control law given in Theorem 5.3.2, while $u_{nl}(x)$ has the same form as the control law derived in Corollary 2.2.3. It is important to remark that, since $p(x)$ is only a weak algebraic $\bar{P}$ solution, there is no guarante that $u_{nl}(x)$ stabilizes the origin.

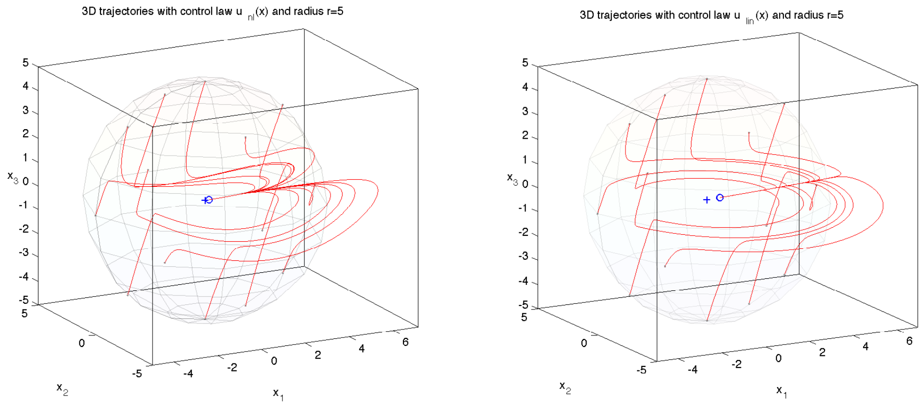
The following plots refer to the case $I_1 = 1$, $I_2 = 2$ and $I_3 = 3$ and $l = 10$. Figure 5.20 shows the phase portrait of the closed-loop systems, for some initial conditions on a sphere of radius $r = 5$. It appears from these figures that both control laws locally asymptotically stabilize the origin. To better understand the dynamics of the systems, Figures 5.21 displays the projections of the trajectories on the (x_1, x_2)-plane and on the (x_2, x_3)-plane, when the initial conditions are on a circumference of radius $r = 5$ on the (x_1, x_2)-plane and $x_3(0) = 0$. Moreover Figures 5.22 and 5.23 show the comparison between the time histories of the norm of the state vector x and the value of the control laws, for the two closed-loop systems, respectively. Note that $u_{nl}(x)$ provides better performance. In fact the norm of the state vector converges to zero at a higher rate even though the control amplitude is comparable. This is an important result: it demonstrate that, for this system, the nonlinear control, built with the weak algebraic $\bar{P}$ solution, not only is a local stabilizer but it guarantees better performance than $u_{lin}(x)$.

However, if we look at the basin of attraction of the two closed-loop system, the linear control law has to be preferred. This is due to the fact that $u_{lin}(x)$ is a global stabilizer for the origin of system (5.18), while increasing r yields to the instability of the trajectories of the system controlled with $u_{nl}(x)$.

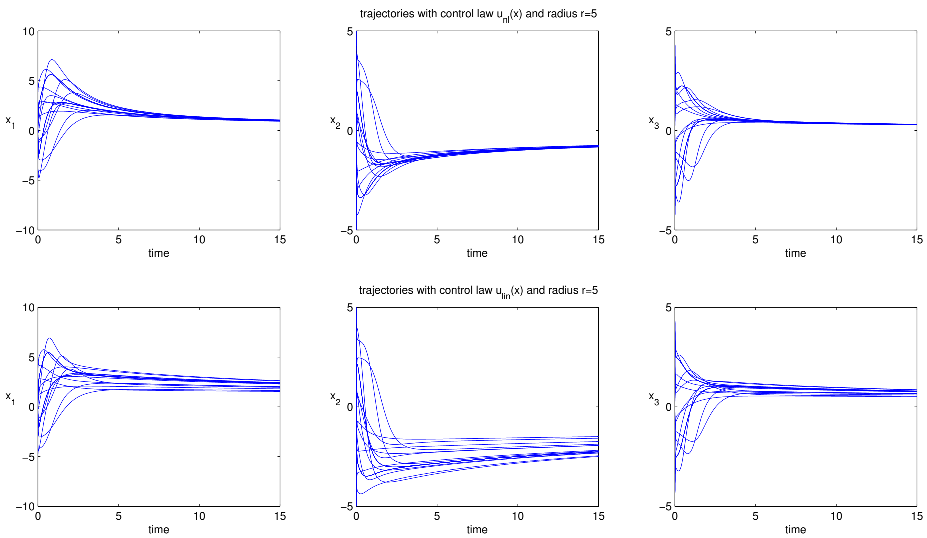
To summarize we have considered a system with non-stabilizable linearization around the origin and we have derived a global weak algebraic $\bar{P}$ solution. Mimicking Corollary 2.2.3 the weak algebraic $\bar{P}$ solution has been used to design the control law $u_{nl}(x) = -g(x)^\top p(x)^\top$. From the comparison between this control law and the one proposed in [1] it appears that:

1. $u_{nl}(x)$ is a local stabilizer while $u_{lin}(x)$ is a global one;
2. $u_{nl}(x)$ yields better local behavior than $u_{lin}(x)$.

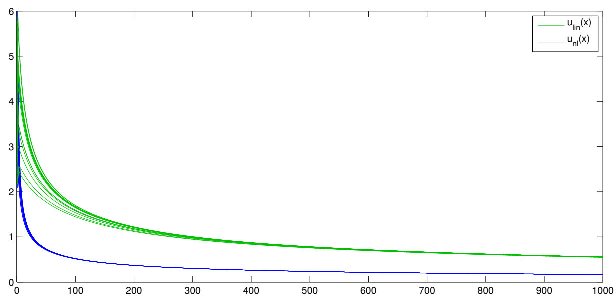
This result can be used, for example, to implement a control law that switches between $u_{lin}(x)$ and $u_{nl}(x)$, when x is small, to obtain a global stabilizer with overall good performances.



(a) Phase portrait of the closed-loop systems. ($u_{nl}(x)$ left graphs, $u_{lin}(x)$ right graphs)

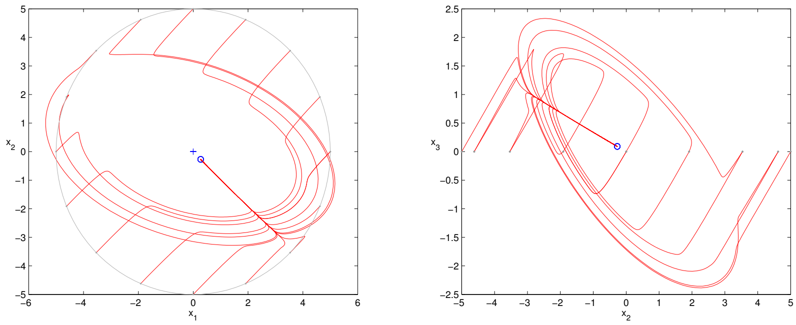


(b) Time histories of the state vector. ($u_{nl}(x)$ top graphs, $u_{lin}(x)$ bottom graphs)

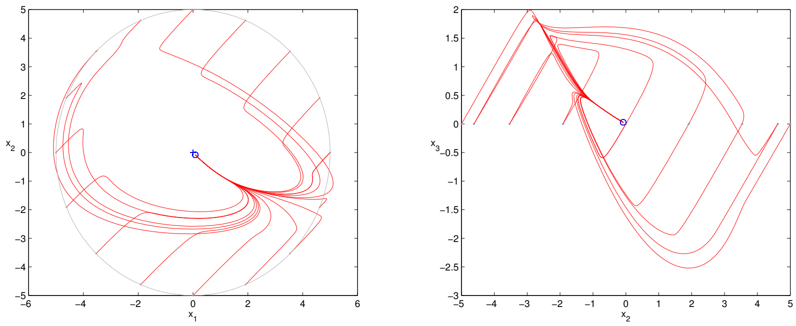


(c) Time histories of the norm of the state vector

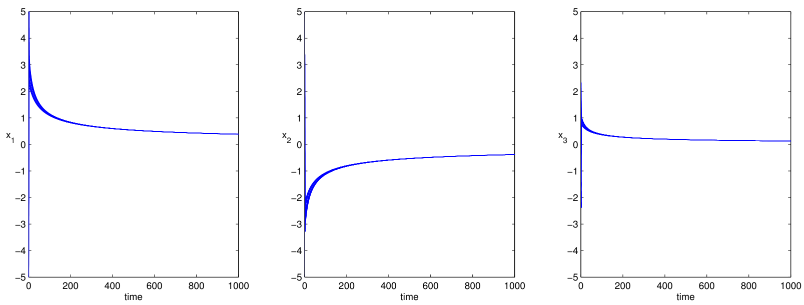
Figure 5.20: Simulation results for system (5.18) with initial conditions on a sphere of radius $r = 5$ and $l = 10$, controlled with $u_{lin}(x) = -10B^{\top}\bar{P}x$ and $u_{nl}(x) = -10g(x)^{\top}p(x)^{\top}$, respectively.



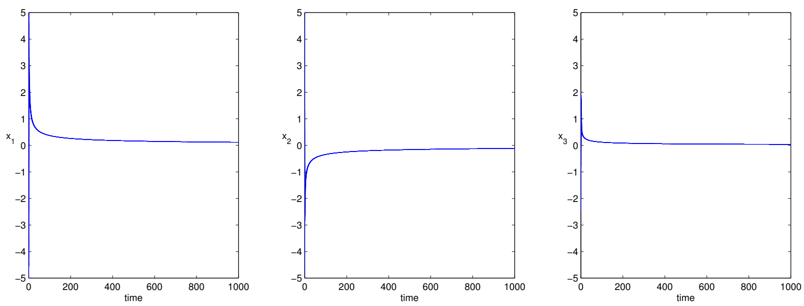
(a) Projection of the dynamics on the (x_1, x_2)-plane and on the (x_2, x_3)-plane, with control law $u_{lin}(x)$



(b) Projection of the dynamics on the (x_1, x_2)-plane and on the (x_2, x_3)-plane, with control law $u_{nl}(x)$



(c) Time histories of the state vector, with control law $u_{lin}(x)$



(d) Time histories of the state vector, with control law $u_{nl}(x)$

Figure 5.21: Simulation results for system (5.18) with initial conditions on a circumference of radius $r = 5$ on the (x_1, x_2)-plane and $x_3(0) = 0$, controlled with $u_{lin}(x) = -10B^\top \bar{P}x$ and $u_{nl}(x) = -10g(x)^\top p(x)^\top$, respectively.

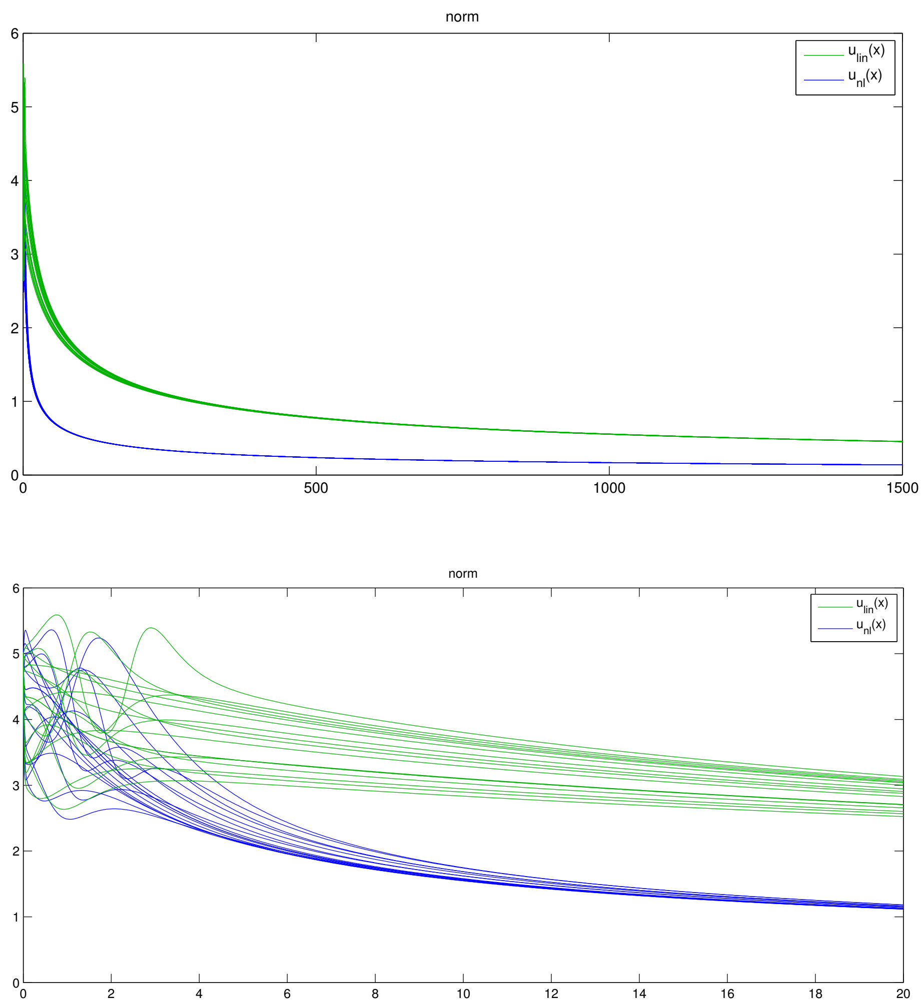


Figure 5.22: Time histories of norm of the state vector for the initial states used in Figure 5.21.

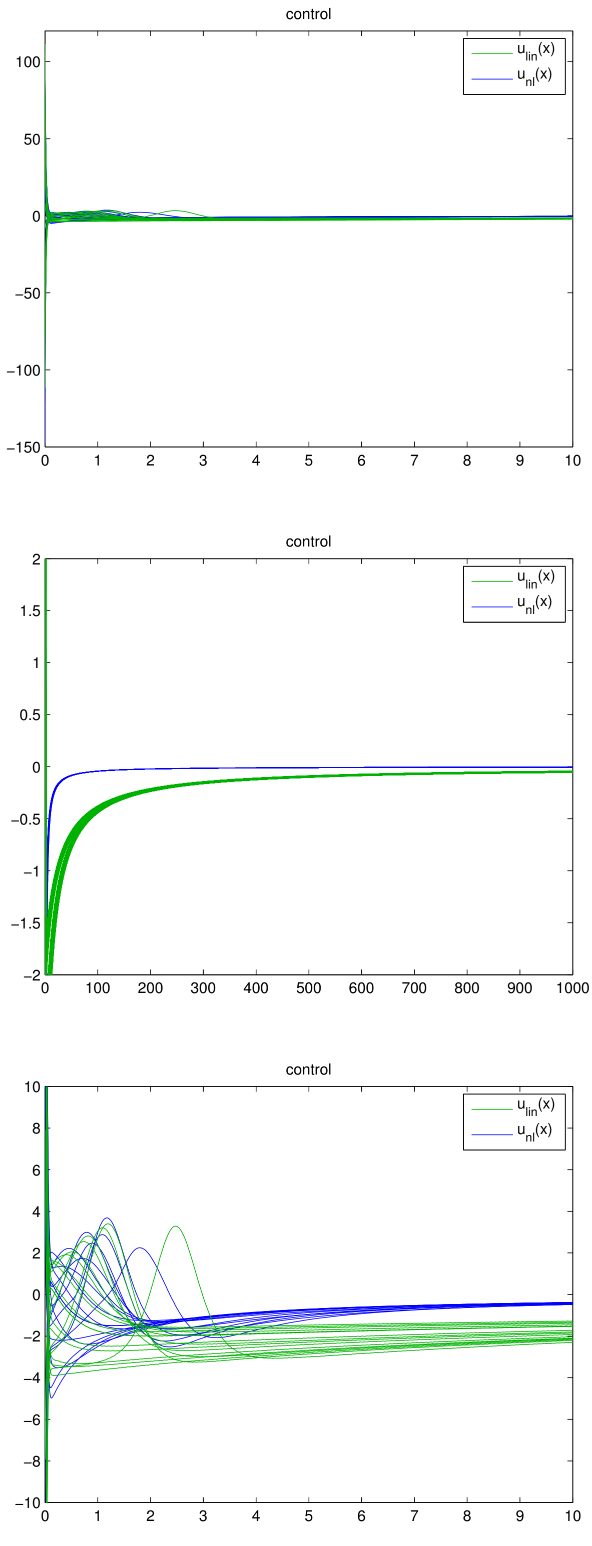


Figure 5.23: Time histories of the control signal $u_{lin}(x) = -10B^\top \bar{P}x$ and $u_{nl}(x) = -10g(x)^\top p(x)^\top$, for the initial states used in Figure 5.21

Conclusions

The goal of this thesis was to develop the new notion of *Dynamic Control Lyapunov function* and to illustrate its use in nonlinear control design. To this end, the main concepts of stability and stabilization theory have been surveyed in the first part of **Chapter 1**. In particular the notions of Lyapunov function (LF) for autonomous systems and Control Lyapunov function (CLF) for control systems have been reviewed. Starting from these concepts, in the second part of the chapter, the notion of Dynamic CLF has been introduced as an instrument to study dynamic asymptotic stabilizability of the origin of a nonlinear system. In particular, it has been proved that the existence of a Dynamic CLF, which satisfies the SCP, is a sufficient condition to guarantee continuous dynamic stabilizability of the origin. Finally the new definition of *Algebraic* $\bar{P}$ *solution* has been introduced to construct a class of candidate Dynamic CLFs $V(x,\xi)$ parametrized by a matrix $R = R^\top > 0$.

In **Chapter 2** it has been proved that there exist values of the matrix R such that the function $V(x,\xi)$ is indeed a Dynamic CLF. In particular, it has been shown, in the linear and nonlinear cases, that the choice $R^{-1} = \alpha\bar{P}^{-1}$, with $0 < \alpha < 2$, guarantees that $V(x,\xi)$ satisfies the properties of a Dynamic CLF. Moreover, in Corollaries 2.1.1 and 2.2.3 a control law that globally asymptotically stabilizes the origin of a linear system and a control law that locally asymptotically stabilizes the origin of an affine nonlinear system have been derived.

In **Chapter 3** a geometric interpretation of the problem has been given. Firstly some explanatory examples have been presented then, the intuition gained from these examples has been used to derive a sufficient condition on R to guarantee that the functions proposed in Chapter 1 are Dynamic CLFs.

To summarize, in the first three chapters the problem of constructing a Dynamic CLF for affine nonlinear systems, the linearization of which around the origin is stabilizable, has been solved and a nonlinear feedback control law has been explicitly derived.

In **Chapter 4** the problem of deriving a standard CLF from a Dynamic CLF has been addressed. In particular, Theorem 4.1.1 deals with the general problem of deriving a standard CLF from a Dynamic CLF, while Theorems 4.2.1 and 4.2.2 specify this result to a Dynamic CLF with the structure proposed above, in the linear and nonlinear cases, respectively.

In **Chapter 5** some applications have been presented and the control law proposed in Chapter 2 has been compared with the control law that can be derived exploiting the linearized system. Simulations show that, in these cases, the former performs better since the basin of attraction of the origin of the closed-loop system with this new control law is larger than the one obtained using the latter. In one example it is also shown that the nonlinear control law yields global asymptotical stabilizability of the origin. Finally an example of a system with non-stabilizable linear approximation has been considered. Since, as stated in Theorem 5.3.1, the stabilizability of the linearized system around the origin is a necessary condition for the existence of an algebraic $\bar{P}$ solution, the new notion of *weak algebraic* $\bar{P}$ *solution* has been introduced. From the simulations it appears that, also in this case, a nonlinear control law with the structure proposed in Corollary 2.2.3 stabilizes the origin and provides, at least locally, better performances than the standard control law proposed in [1].

Future Research

The research carried out in this thesis naturally leads to several open questions and suggests the following areas for future developments.

1. The procedure described in Chapters 2 and 3 relies upon the knowledge of an algebraic $\bar{P}$ solution. Remark 1.3 states that, if the linearized system around the origin is stabilizable, then it is always possible to construct a local algebraic $\bar{P}$ solution, but the region in which Condition (P2) is satisfied can be very small. An interesting open question is therefore if it possible to derive a procedure to construct an algebraic $\bar{P}$ solution that satisfies Condition (P2) in a region as large as possible, ideally for all $x \in \mathbb{R}^n$, for some classes of nonlinear systems.

2. In Chapter 2 the knowledge of a Dynamic Control Lyapunov function has been exploited to construct the stabilizing feedback law proposed in Corollary 2.2.3. This is not the only control law that can be derived from $V(x, \xi)$. For example, it has been shown in Theorem 4.0.1 how to use Sontag's formula to derive a control law that dynamically stabilizes the origin of the nonlinear system. The possibility of deriving also other control laws and the comparison between their performances requires further research.

3. In Chapter 4 a procedure to derive a Control Lyapunov function from the knowledge of a Dynamic Control Lyapunov function has been proposed. To this end the differential equation (4.5) has to be solved. The possibility of solving this problem using an approximate solution of (4.5) needs to be investigated.

4. Finally, the theory developed in this thesis requires stabilizability of the linearized system, however in Section 5.3 an example where this hypothesis was violated has been analyzed. To overcome this problem the concept of weak algebraic $\bar{P}$ solution has been introduced and it has been shown, through simulations, that the corresponding nonlinear control law yields better performance than the control law proposed in [1]. The theoretical discussion of this issue, in particular if the function $V(x, \xi)$ constructed using a weak algebraic $\bar{P}$ solution is always a Dynamic CLF or

if the nonlinear system must satisfies some additional properties, is left as an open question.

Appendix A

A.1 Singular Values

Consider the matrices $A \in \mathbb{C}^{m\times n}$ and $AA^* \in \mathbb{C}^{m\times m}$, where A^* denote the complex conjugate transpose of A. Let λ_i, $i = 1,..,m$, denote the eigenvalues of AA^*, and note that these are all real and nonnegative numbers. Moreover assume $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r \cdots \geq \lambda_m$. Note that if $r = rank(A) = rank(AA^*)$, then $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_r > 0$ and $\lambda_{r+1} = \cdots = \lambda_m = 0$.

Definition A.1.1 (Singular values)**.** *The positive square roots of λ_i, $i = 1,..,\min(m,n)$ are called singular values σ_i of the matrix A, i.e.*

$$\sigma_i = \sqrt{\lambda_i}.$$

Note that $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_r > 0$ and $\sigma_i = 0$ for all $i > r$.

There is an important relation between the singular values of A and its induced Hilbert or 2−norm, $\|A\|_2$. In fact

$$\|A\|_2 \triangleq \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_i(\sigma_i(A))$$

Important properties of singular values

Given a matrix $A \in \mathbb{C}^{m\times n}$ define $\sigma_i(A)$ the generic i^{th} singular value and

$$\bar{\sigma}(A) \triangleq \sigma_1(A), \qquad \underline{\sigma}(A) \triangleq \sigma_k(A) \tag{A.1}$$

where $k = \min(m,n)$. Recall that $\sigma_i(A)^2 = \lambda_i(AA^*)$ and that $\sigma_i(A) \geq 0$. In the following $B \in \mathbb{R}^{m\times n}$ while $C \in \mathbb{R}^{n\times p}$.

P1. $\sigma_i(\alpha A) = |\alpha|\sigma_i(A), \ \forall \alpha \in \mathbb{C}$.

P2. If A is square $\underline{\sigma}(A) \leq |\lambda_i(A)| \leq \bar{\sigma}(A)$.

P3. If A is square and invertible, $\bar{\sigma}(A) = \dfrac{1}{\underline{\sigma}(A^{-1})}$.

P4. $\bar{\sigma}(A+B) \leq \bar{\sigma}(A) + \bar{\sigma}(B)$.

P5. $\underline{\sigma}(A) - \bar{\sigma}(B) \leq \underline{\sigma}(A \pm B) \leq \underline{\sigma}(A) + \bar{\sigma}(B)$.

P6. $\underline{\sigma}(AC) \geq \underline{\sigma}(A)\underline{\sigma}(C)$.

P7. $\bar{\sigma}(AC) \leq \bar{\sigma}(A)\bar{\sigma}(C)$.

A.2 Properties of symmetric matrices

Theorem A.2.1 (Shur complement). *Let X be a symmetric matrix given by*

$$X = \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$$

with A and C square matrices, A invertible. Let S be the Schur complement of A in X, that is $S = C - B^\top A^{-1} B$, then X is positive definite if and only if A and S are both positive definite.

Proof. Can be found in [6]. □

Lemma A.2.1. *Let H be an $n \times n$ symmetric matrix and C an $m \times n$ matrix of rank m, where $m < n$. Let Z denote a basis for the right null space of C. Then $Z^\top H Z$ is positive definite if and only if there exists $\bar{\rho} \geqslant 0$ such that, for all $\rho > \bar{\rho}$, $H + \rho C^\top C$ is positive definite.*

Proof. The proof can be found in [2]. □

Theorem A.2.2. *Consider a symmetric matrix $H = H^\top$, then*

$$-\underline{\sigma}(H) I \leqslant H \leqslant \bar{\sigma}(H) I \tag{A.2}$$

Moreover if $H = H^\top > 0$ then

$$\underline{\sigma}(H) I \leqslant H \leqslant \bar{\sigma}(H) I \tag{A.3}$$

Proof. The claim is a consequence of the following property of a symmetric matrix

$$\lambda_{min}(H) I \leqslant H \leqslant \lambda_{max}(H) I$$

This property can be proved recalling that, since H is symmetric, all the eigenvalues are real and there exists an orthogonal matrix U, i.e. $U^\top = U^{-1}$, such that

$$U^\top H U = \begin{bmatrix} \lambda_1(H) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n(H) \end{bmatrix}$$

Then

$$U^\top \left[\lambda_{max}(H) I - H\right] U = \lambda_{max}(H) I - U^\top H U =$$

$$= \begin{bmatrix} \lambda_{max}(H) - \lambda_1(H) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{max}(H) - \lambda_n(H) \end{bmatrix} \geqslant 0$$

and

$$U^\top \left[H - \lambda_{min}(H) I\right] U = U^\top H U - \lambda_{min}(H) I =$$

$$= \begin{bmatrix} \lambda_1(H) - \lambda_{min}(H) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n(H) - \lambda_{min}(H) \end{bmatrix} \geqslant 0$$

Finally, since premultipling by $U^\top$ and postmultipling by U is equivalent to a change of coordinates

$$U^\top [\lambda_{max}(H)I - H]U \geqslant 0 \;\Leftrightarrow\; \lambda_{max}(H)I - H \geqslant 0 \;\Leftrightarrow\; H \leqslant \lambda_{max}(H)I$$

$$U^\top [H - \lambda_{min}(H)I]U \geqslant 0 \;\Leftrightarrow\; H - \lambda_{min}(H)I \geqslant 0 \;\Leftrightarrow\; H \geqslant \lambda_{min}(H)I$$

The original statement is, finally, a direct consequence of the fact that, since H is a symmetric matrix

$$-\underline{\sigma}(H) \leqslant \lambda_{min}(H) \leqslant \lambda_{max}(H) \leqslant \bar{\sigma}(H).$$

Moreover, if H is positive definite, all the eigenvalues are positive hence

$$\underline{\sigma}(H) \leqslant \lambda_{min}(H) \leqslant \lambda_{max}(H) \leqslant \bar{\sigma}(H).$$

□

Lemma A.2.2. *Let $Q = Q^\top > 0$ be a symmetric positive definite $n \times n$ matrix and $S = S^\top$ a symmetric $n \times n$ matrix. Then there exists a scalar value $\bar{\lambda} > 0$ such that $\lambda Q + S$ is positive definite for all $\lambda > \bar{\lambda}$. Moreover*

$$\bar{\lambda} \leqslant \frac{\bar{\sigma}(S)}{\underline{\sigma}(Q)} \triangleq \tilde{\lambda}$$

Proof. It is sufficient to demonstrate that $\lambda Q + S$ is positive definite for all $\lambda > \tilde{\lambda}$. The proof is then straightforward, indeed if $\lambda > \tilde{\lambda}$ then

$$\lambda \underline{\sigma}(Q) > \bar{\sigma}(S) = \bar{\sigma}(-S)$$

and using property (A.2) for $-S$ and (A.3) for Q yields

$$\lambda Q \geqslant \lambda \underline{\sigma}(Q)I > \bar{\sigma}(-S)I \geqslant -S$$

that implies $\lambda Q > -S$ hence $\lambda Q + S > 0$. □

Lemma A.2.3. *Consider the matrix $Q + S(x)$, where $Q = Q^\top > 0$ and $S(x) = S(x)^\top$ is a symmetric continuous matrix such that $S(0) = 0$. Then there exists a neighborhood of the origin Ω_x such that $Q + S(x) > 0$ for all $x \in \Omega_x$.*

Proof. Since $S(x)$ is a continuous matrix and $S(0) = 0$ there exists a neighborhood of the origin Ω_x such that $\bar{\sigma}(S(x)) = \|S(x)\| < \underline{\sigma}(Q)$ for all $x \in \Omega_x$. Therefore[1]

$$Q \geqslant \underline{\sigma}(Q)I > \bar{\sigma}(S(x))I = \bar{\sigma}(-S(x))I \geqslant -S(x)$$

and hence $Q + S(x) > 0$ for all $x \in \Omega_x$. □

[1] See Theorem A.2.2 in the Appendix.

A.3 Properties of $\mathcal{C}^1$ functions

Lemma A.3.1. *Consider a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}^n$ and suppose that $f(0) = 0$. Then there exists a non unique continuous matrix $F(x)$ such that $f(x) = F(x)x$.*

Proof. Note that

$$f(x) = f(x) - f(0) = \int_0^1 \frac{\partial f(sx)}{\partial s} ds = \int_0^1 \frac{\partial f(\zeta)}{\partial \zeta}\Big|_{\zeta=sx} x ds = \int_0^1 \frac{\partial f(\zeta)}{\partial \zeta}\Big|_{\zeta=sx} ds\ x \triangleq F(x)x.$$

Moreover, being the integral of a continuous function, $F(x)$ is a continuous matrix-valued function. □

Lemma A.3.2. *Consider a continuously differentiable function $p : \mathbb{R}^n \to \mathbb{R}^{1\times n}$ and suppose that $p(0) = 0$ and $p_x(0) = \bar{P}$. Then:*

1. *$p(x) = x^\top \bar{P} + x^\top \mathcal{O}(x)$;*
2. *there exists a non unique continuous matrix-valued function $\tilde{P}(x)$ such that $p(x) = x^\top \tilde{P}(x)$ and $\tilde{P}(x) = \bar{P} + \mathcal{O}(x)$;*
3. *there exists a continuos matrix-valued function $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $p(x) - p(\xi) = (x-\xi)^\top \Phi(x,\xi)^\top$.*

Proof. 1. This is an immediate consequence of Taylor's Theorem for multivariable functions. In fact the truncated series of $p(x)$ around zero, using the Lagrange form of the remainder, is

$$p(x) = p(0) + x^\top p_x(0) + \frac{1}{2} x^\top \left[H_1(\bar{x})x, \cdots, H_n(\bar{x})x\right]$$

where $H_i(x)$ is the Hessian of $p_i(x)$, i.e. the i-th component of $p(x)$, and $\bar{x}$ is a fixed point of $\mathbb{R}^n$. Upon setting $[H_1(\bar{x})x, \cdots, H_n(\bar{x})x] = \mathcal{O}(x)$ the previous equation becomes

$$p(x) = 0 + x^\top \bar{P} + x^\top \mathcal{O}(x)$$

2. Note that

$$p(x) = p(x) - p(0) = \int_0^1 \frac{\partial p(sx)}{\partial s} ds = \int_0^1 x^\top \frac{\partial p(\zeta)}{\partial \zeta}\Big|_{\zeta=sx} ds = x^\top \int_0^1 \frac{\partial p(\zeta)}{\partial \zeta}\Big|_{\zeta=sx} ds \triangleq x^\top \tilde{P}(x).$$

Moreover, being the integral of a continuous function, $\tilde{P}(x)$ is a continuous matrix-valued function. Finally, using the relation proved in the previous point,

$$x^\top \bar{P} + x^\top \mathcal{O}(x) = p(x) = x^\top \tilde{P}(x),$$

for every x, therefore $\tilde{P}(x) = \bar{P} + \mathcal{O}(x)$.

3. Set $z = x - \xi$, then

$$p(x) - p(\xi) = p(x) - p(x - z) = H(x, z).$$

Note that $H(x, 0) = 0$ and therefore

$$\begin{aligned} H(x,z) = H(x,z) - H(x,0) &= \int_0^1 \frac{\partial H(x, sz)}{\partial s} ds = \int_0^1 z^\top \frac{\partial H(x,\zeta)}{\partial \zeta}\Big|_{\zeta = sz} ds \\ &= z^\top \int_0^1 \frac{\partial H(x,\zeta)}{\partial \zeta}\Big|_{\zeta = sz} ds \triangleq z^\top \Phi(x,z)^\top. \end{aligned}$$

Moreover, since $H(x, z)$ is $\mathcal{C}^1$, $\Phi(x, z)$ is a continuous matrix valued function. Finally

$$p(x) - p(\xi) = H(x, z) = z^\top \Phi(x, z)^\top = (x - \xi)^\top \Phi(x, \xi)^\top.$$

□

Theorem A.3.1 (Implicit function theorem). *Let $f(x, y) : E \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a $\mathcal{C}^1$ function. Suppose that $(a, b) \in E$ and that $f(a, b) = 0$. If the jacobian matrix $\frac{\partial f}{\partial x}\big|_{(a,b)}$ is invertible then there exist two open sets $U \subset \mathbb{R}^n \times \mathbb{R}^m$ and $V \subset \mathbb{R}^m$ such that:*

1. *$(a, b) \in U$ and $b \in V$;*
2. *for every $y \in V$ there exists a unique value x such that $(x, y) \in U$ and $f(x, y) = 0$;*
3. *the function g such that $x = g(y)$ is of class $\mathcal{C}^1$ and $g(b) = a$;*
4. *$f(g(y), y) = 0$ for all $y \in V$.*

A.4 Local CLF obtained from the linearized system

Theorem A.4.1. *Consider a nonlinear, time-invariant, system described by the equation*

$$\dot{x} = f(x) + g(x)u, \tag{A.4}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ be at least continuously differentiable and $f(0) = 0$. Suppose also that the linearized system around the origin is stabilizable and hence there exists a positive definite matrix $\bar{P} = \bar{P}^\top > 0$ such that

$$x^\top \bar{P} B = 0 \;\Rightarrow\; x^\top \bar{P} A x < 0 \quad \text{for all} \quad x \neq 0, \tag{A.5}$$

where $A = \frac{\partial f}{\partial x}|_{x=0}$ and $B = g(0)$. Then

$$x^\top \bar{P} g(x) = 0 \;\Rightarrow\; x^\top \bar{P} f(x) < 0 \quad \text{for all} \quad x \in \Omega \backslash \{0\}. \tag{A.6}$$

Proof. Note that, since (A.4) is an affine system, condition (A.6) is equivalent to

$$\inf_u \left[x^\top \bar{P} f(x) + x^\top \bar{P} g(x) u \right] < 0 \quad \text{for all} \quad x \in \Omega \backslash \{0\}. \tag{A.7}$$

Consider now $\bar{u}(x) = -l B^\top \bar{P} x$, where $l > 0$ is a scalar value to be determined. Then

$$\begin{aligned} x^\top \bar{P} f(x) + x^\top \bar{P} g(x) \bar{u}(x) &= x^\top \bar{P} F(x) x - l x^\top \bar{P} g(x) B^\top \bar{P} x \\ &= x^\top \bar{P} (A + \mathcal{O}(x)) x - l x^\top \bar{P} (B + \mathcal{O}(x)) B^\top \bar{P} x \\ &= x^\top \left[\frac{\bar{P} A + A^\top \bar{P}}{2} - l \bar{P} B B^\top \bar{P} + \mathcal{O}(x) \right] x. \end{aligned} \tag{A.8}$$

Let Z be the right kernel of $(PB)^\top$ and note that condition (A.5) implies

$$Z^\top \left[\frac{\bar{P} A + A^\top \bar{P}}{2} \right] Z < 0. \tag{A.9}$$

Then, by Lemma A.2.1, there exist a value $\bar{l} > 0$ such that for all $l > \bar{l}$

$$\frac{\bar{P} A + A^\top \bar{P}}{2} - l \bar{P} B B^\top \bar{P} \triangleq -Q_l, \tag{A.10}$$

where $Q_l = Q_l^\top > 0$. Substituting (A.10) in equation (A.8) yields

$$x^\top \bar{P} f(x) + x^\top \bar{P} g(x) \bar{u} = x^\top (-Q_l + \mathcal{O}(x)) x$$

and finally, by Lemma A.2.3, there exists a neighborhood of the origin Ω such that

$$x^\top (-Q_l + \mathcal{O}(x)) x < 0 \quad \text{for all} \quad x \in \Omega \backslash \{0\}.$$

This conclude the proof since

$$\inf_u \left[x^\top \bar{P} f(x) + x^\top \bar{P} g(x) u \right] \leqslant x^\top \bar{P} f(x) + x^\top \bar{P} g(x) \bar{u} < 0 \quad \text{for all} \quad x \in \Omega \backslash \{0\}.$$

□

Bibliography

[1] D. Aeyels and M. Szafranski. Comments on the stabilizability of the angular velocity of a rigid body. *Systems & Control Letters*, 10(1):35–39, 1988.

[2] K.M. Anstreicher and M.H. Wright. A note on the augmented Hessian when the reduced Hessian is semidefinite. *SIAM Journal of Optimization*, 11(1):243–253, 2000.

[3] Z. Artstein. Stabilization with relaxed controls. *Nonlinear Ann. Theory Methods Applic.*, 7(11):1163–1173, 1983.

[4] A. Bacciotti. *Local stabilizability of nonlinear control systems*, volume 8. World Scientific Pub Co Inc, 1992.

[5] R.H. Bartels and G.W. Stewart. Solution of the matrix equation $AX + XB = C$. *Comm. ACM*, 15(9):820–826, 1972.

[6] S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, 2004.

[7] A. Isidori. *Nonlinear control systems*, volume 1. Springer Verlag, 1995.

[8] V. Jurdjevic and J.P. Quinn. Controllability and stability. *Differential equations*, 28:381–389, 1978.

[9] R. Kalman and J. Bertram. Control system analysis and design via the second method of Lyapunov:(i) continuous-time systems (ii) discrete time systems. *IRE Transactions on Automatic Control*, 4(3):112–112, 1959.

[10] H.K. Khalil and JW Grizzle. *Nonlinear systems.* Macmillan Publishing Company New York, 1992.

[11] AA Krasovsky. A new solution to the problem of a control system analytical design. *Automatica*, 7(1):45–50, 1971.

[12] M. Krstic, I. Kanellakopoulos, and P. Kokotovic. *Nonlinear & Adaptive Control Design.* Wiley-Interscience, 1995.

[13] J. Kurzweil. On the inversion of Lyapunov's second theorem on stability of motion. *Ann. Math. Soc. Transl. Ser.*, 2(24):19–77, 1956.

[14] A.M. Lyapunov. Problème general de la stabilite du mouvement. *Ann. Fac. Sci. Toulouse*, 9:203–474, 1907.

[15] J.L. Massera. Contributions to stability theory. *The Annals of Mathematics*, 64(1):182–206, 1956.

[16] L.T. Nguyen, R.D. Whipple, and J.M. Brandon. Recent experiences of unsteady aerodynamic effects on aircraft flight dynamics at high angle of attack. Technical report, DTIC, 1985.

[17] M. Sassano and A. Astolfi. Dynamic Lyapunov Functions. In *2011 IFAC World Congress*, volume 18, pages 3409–3414, 2011.

[18] E.D. Sontag. Lyapunov-like characterization of asymptotic controllability. *SIAM J. Contr. Optimiz.*, 21(3):462–471, 1983.

[19] E.D. Sontag. A universal construction of Artstein's theorem on nonlinear stabilization. *Systems & Control letters*, 13(2):117–123, 1989.

[20] E.D. Sontag. *Mathematical control theory: deterministic finite dimensional systems*, volume 6. Springer Verlag, 1998.