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Infinite Gaussian Measures in String Theory

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Contents

1 Introduction **2**
- 1.1 Why do we need measures in QFT? . . . 4
- 1.2 Lagrangian for fields . . . 6
- 1.3 Application to String Theory and Polyakov's contribution . . . 7

2 Gaussian Measures in QFT **9**
- 2.1 Introduction . . . 9
- 2.2 Heuristic definition of the Feynman-Kač measure for the free field 10
- 2.3 Infinite product measures . . . 12
- 2.4 Feynman-Kač measure for the free field . . . 14
- 2.5 Concentration in L^2 of the Feynman-Kač measure for the free field 14
- 2.6 H^s spaces . . . 17
  - 2.6.1 Definition and first properties of the H^s-spaces . . . 18
  - 2.6.2 In and out ℓ^2 . . . 20
- 2.7 Definition of the measure (2.8) . . . 22
  - 2.7.1 Construction of the measure . . . 23
  - 2.7.2 Concentration of the measure in L^2 . . . 25
  - 2.7.3 H^s spaces . . . 25

3 Polyakov measure **26**
- 3.1 Nabu-Goto action vs Polyakov action . . . 26
  - 3.1.1 Nabu-Goto action . . . 26
- 3.2 Polyakov action . . . 29
  - 3.2.1 Polyakov Action advantages . . . 32
- 3.3 Polyakov Measure . . . 36
  - 3.3.1 Diagonalization of the Laplace-Beltrami operator . . . 38
  - 3.3.2 Definition of the measure . . . 39
  - 3.3.3 Concentration of the measure . . . 39
- 3.4 Complete Polyakov Measure . . . 40

A Weyl's asymptotic formula for $\mathbb{T}^2$ **42**

Bibliography **44**

Chapter 1

Introduction

One of the major revolutions of the 20^{th} century Physics was the introduction of **functional integration methods** in the study of quantum phenomena. Since the first experiments on particles was clear that the Physics of the atomic world is very different from Classical Physics. Even the concept of "particle" characterized by a position and a velocity in the space is out of sense, as well as the "natural" idea that the energy of a particle is a continuous quantity.
A non trivial probabilistic structure emerged by experiments, changing deeply the intuition on the dynamics. It was one of the biggest achievement of Functional Analysis to develop a striking framework capable to give a precise description and prediction of the QM[1]. With this strong achievement, since the middle of 20^{th} century the so called **second quantization** moved its first steps: the Quantum Field Theory was born. Despite the successes of the QM, the QFT[2] is a very hard mathematical problem not completely solved even today. There are basically two approaches to the problem. The first was based on an extention of the first quantization and involve algebraic methods in Functional Analysis. We do not mention this subject in this thesis.
The second method was actually introduced by the Nobel price R. Feynman and it is based on an idea of one of the fathers of QM: P. Dirac. In his book on QM, *Principles of Quantum Mechanics*, published yet in 1930, Dirac recognizes that the fundamental quantity of the QM, that is the **propagator**[3] (i.e. the integral kernel which allows the computation of transition probabilities for the quantum system) was expressed as an integral on the space of all possible paths of an exponential of type

$$e^{\frac{i}{\hbar}A(q)}$$

where q is the path and A is the classical action of the underlying system and $\hbar$ is the Planck constant. However it was R. Feynman, beginning with its PhD Thesis to develop the subject. Actually he defined a formal measure on the

[1] Quantum Mechanics
[2] Quantum Field Theory
[3] see formula (1.1).

path space (i.e. The space of all possible trajectories, even without any physical meaning) and started to develop a formal machinery to derive prediction on the system.
Once acquainted with QM, Feynman launched himself into the challenge of QFT. The basic idea was quite simple: just replace the classical action with the appropriate action describing the Physics of the classical fields (i.e. The Maxwell equations). In this case the formal measure works on the space of all possible fields. Working on this subject jontly with Tomonaga and Schwinger he was awarded by the Nobel Price for Physics in 1965 just for the theory of **quantum electrodynamics**.

Despite the straordinary succes of the functional integration methods introduced by Feynman, the main mathematical problem was that they weren't based on a rigorous ground. The Feynman measures were only formally defined and nobody up to now has been able to construct a rigorous theory, except for some exotic cases. However, yet in the fifties of 20^{th} century M. Kač proved that changing time in imaginary time (this corresponds to swich the Schrodinger equation into the heat equation) a completely rigorous construction of path space measures was possible, at least for the case corresponding to QM. Such measure is called now **Feynman-Kač measure**.
Since then many mathematicians studied the problem to extend the construction to QFT. The major success was achieved by J. Glimm and A. Jaffe in a series of works culminated in the book *Quantum Physics* (Springer 1984), where, however, the construction is done in full generality in the case of physical dimension of the space time equal to d=2 and in some particular case in the case d=3. The phisically interesting problem d=4 is yet open.
Even if an incomplete theory was developed, the positive results encouraged to believe that the functional integration methods are the right way in order to construct a rigorous framework for QFT. Indeed the next challenge was to include General Relativity into the description, taking account of the geometric structure of the space time and, hopefully (the big dream of every Physicist) the **unified theory**, that is theory able to include all the phenomena of Physics.
In this direction, the last decades of 20^{th} century were characterized by the introduction of a new theory: the **String Theory**. Actually, this theory is quite technical and seems a agglomerate of very complicate mathematical statements without a clear context.
In this jungle, in a couple of articles published on Physics Letters A.M. Polyakov proposed once again to describe strings toward integration on the space of 2-dim surfaces, using as action the geometrical area defined by a Riemaniann metric. Actually his idea was even more daring: he proposed to consider a measure in the product space {2-dim surfaces}×{riemanian metrics}. The literature on the mathematical solution of this problem is reduced to few attempts and it is basically completely open up to now.

The aim of this thesis is to introduce a bit of mathematical rigour in the

definition of the measures of type:

$$e^{-A(\phi)}d\phi,$$

where A is a particular action functional (the Nabu-Goto action), and ϕ is a field (mathematically spoken $\phi : \mathbb{R}^d \to \mathbb{R}^n$), in particular ϕ is a parameterization of a 2-dim surface with smooth boundary (it will be also a compact Riemannian manifold[4]) then $\phi : \mathbb{R}^2 \to \mathbb{R}^n$. Polyakov, introducing a functional with another variable, splits the problems into two parts, one of these is a kind of $e^{-\frac{1}{2}\int \langle \nabla\phi | \nabla\phi \rangle_g \, dx} d^\infty\phi$ where g is the metric associated to the Riemannian manifold. Therefore Chapter 2 is dedicated to the definition of the infinite product measure: $e^{-\frac{1}{2}\int |\nabla\phi|^2 \, dx} d^\infty\phi$, where $\phi : \mathbb{R}^d \to \mathbb{R}^n$. Finally in Chapter 3 will be presented in detail Polyakov idea and will be applied the results of the previous chapter to define, at least, the first part of the over-mentioned measure.

In this work there is no purpose to study physical phenomenons, or to motivate it. However follows a simple excursus useful to a better comprehension of the problem[5].

1.1 Why do we need measures in QFT?

In C.M.[6] the trajectories can be seen as stationary points of some functional (for example the statinary points of the action $A(q) = \int_{t_0}^{t_1} \frac{1}{2}\dot{q}(t)^2 - V(q(t))\, dt$ are the trajectories of a particle with mass $m = 1$, immersed in a forces field generated by a potential $-V$).
In QM we can't give the same interpretation, innovatively we can think that a particle can follow any path between two fixed positions. The interpretation which has more correspondences with experimental results, is to assign a probability density to each path. Then a particle's state is described by a probability distribution that has not to be meant as usual because the state space is $L^2(\mathbb{R}, \mathbb{C}) = \left\{\phi : \mathbb{R} \longrightarrow \mathbb{C} : \phi \text{ is measurable, } \|\phi\|_2^2 \equiv \int_{\mathbb{R}} |\phi(x)|^2 < +\infty\right\}$, and a state is said to be a probability distribution in the following sense: if $\|\phi\|_2 = 1$, the probability that a particle in the state ϕ is in $[a, b]$ is $\int_a^b |\phi|^2 \, dx$ (Analogously in Quantum Field Theory the state space will be $\mathcal{S} = L^2_{\mathbb{C}}\left(L^2_{\mathbb{R}}(\mathbb{R})\right)$).
The final purpose is then to construct a measure on the space of all paths connecting the initial and the final positions of the particle (which are fixed) and that evaluates the probabilities of a particular "trajectory" between the two observed positions. In QM we denote the the amplitude to propagate from a point q_I to a point q_F in time τ as

$$\langle q_F | e^{-iH\tau} | q_I \rangle, \tag{1.1}$$

[4]**Riemannian manifold**: is a couple (M, g) where M is a real differentiable manifold M, and g is a pointwise smooth inner product over the tangent space"

[5]The results that follows are not topics of this work, then they are presented without proof. Those are only useful to understand what is the context in which the definition of infinite gaussian measure originates.

[6]Classical Mechanics

where the symbol $|q\rangle$ stands for "the state in which the particle is at q, and H is the Hamiltonian[7]. In fact the equation that describes the evolution of a state is **Schrödinger equation**[8] $\dot{\phi}(t) = -\frac{i}{\hbar}H\phi(t)$, where $\phi(t) = \phi(t, \#)$, with formal solution $\phi(t) = e^{-\frac{i}{\hbar}tH}\phi(0)$. Does this solution have sense mathematically speaking?

We turn back to follow Dirac's formulation of the problem of defining the path integral representation (see [8]) and we have, in the free-particle case:

$$\langle q_F|e^{-iH\tau}|q_I\rangle = \int e^{i\int_0^\tau \frac{1}{2}m\dot{q}^2\,dt}\,Dq(t),$$

where $Dq(t)$ suggests an integration over all possible paths $q(t)$ such that $q(0) = q_I$ and $q(\tau) = q_F$. Analogously if the particle is in a potential the formula becomes $\langle q_F|e^{-iH\tau}|q_I\rangle = \int e^{i\int_0^\tau \left(\frac{1}{2}m\dot{q}^2 - V(q)\right)\,dt}\,Dq(t)$, then, in general, if $L(q, \dot{q})$ is the Lagrangian:

$$\langle q_F|e^{-iH\tau}|q_I\rangle = \int e^{i\int_0^\tau L(q,\dot{q})\,dt}\,Dq(t) \text{“} = \text{”} \int e^{iA(q)}\,Dq(t).$$

in fact in C.M. the quantity $A(q) = \int_0^\tau L(q, \dot{q})\,dt$ is called "action".
For simplicity it's better to introduce some useful notations: $\langle F|e^{-iH\tau}|I\rangle$ specify that a particle starts from a initial state I and ends in some final state F, and is:

$$\langle F|e^{-iH\tau}|I\rangle = \int \left(\int \langle F|q_F\rangle\langle q_F|e^{-iH\tau}|q_I\rangle\langle q_I|I\rangle dq_I\right) dq_F.$$

We will call $Z = \langle 0|e^{-iH\tau}|0\rangle$ where 0 is the ground state. After the rotation to the Euclidean time ($t \to -it$) we can write:

$$Z = \int e^{-\int_0^\tau L(q,\dot{q})\,dt}\,Dq(t), \tag{1.2}$$

therefore, it doesn't matter if the hamiltonian (and then $\int_0^\tau L(q, \dot{q})\,dt$) is not a bounded operator.

In Quantum Field Theory, instead of the particles q, we consider fields which are functions $\phi(t, \mathbf{x})$ where $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ [9]. The action becomes

$$A(\phi) = \int \left(\int L(\phi) d^n x\right) dt,$$

while the path integral defining a scalar field theory is:

$$Z = \int e^{iA(\phi)}\,D\phi.$$

[7] This is Dirac's formulation.

[8] This is the partial derivatives equation: $i\hbar\partial_t\phi(t, x) = -\frac{\hbar^2}{2m}\partial_{xx}\phi(t, x) + V(x)\phi(t, x)$.

[9] In particular we consider $n = 2$ then $\mathbf{x}$ represents the position, but notice that if $n = 1$ QFT is QM.

More generally we want to disturb the vacuum (particles could be created and annihilated) then we consider the path integral:

$$Z = \int e^{i\int(\int[L(\phi)+J(x)\phi(x)]d^n x)dt} D\phi \tag{1.3}$$

where $J(x) = J(t,x)$ can vanish except in some regions. This fuctional is impossible to do except when $L(\phi) = \frac{1}{2}\left((\partial\phi)^2 - m^2\phi^2\right)$. The corrisponding theory is called the **free** (or Gaussian) theory. The equation of motion is the Klein-Gordon equation, which is introduced in the following section.
Integrating by parts (1.3) we have

$$Z \to Z = \int e^{i\int\left(\int\left[-\frac{1}{2}\phi\left(\partial^2+m^2\phi^2\right)\phi+J(x)\phi(x)\right]d^n x\right)dt} D\phi. \tag{1.4}$$

The space of all ϕ is clearly not finite dimensional and it's not clear how this measure can be definined. We have already said that we can consider the imaginary time; this simplify considerably the problem because e^{-y} tends to zero if y is not limited, while e^{iy} do not behave in the same way.

1.2 Lagrangian for fields

Let talk about the quantum theory of the electromagnetic field and try to find a more explicit expression for the lagrangian used in formula (1.4). The analogue of the Newton Equations (in C.M.), is the Klein-Gordon equation:

$$(\Box + m^2)\phi(\xi) \equiv \partial_{tt}\phi(x,t) - \partial_{xx}\phi(x,t) + m^2\phi(x,t) = -f(\phi(x,t)).$$

where, for simplicity, $\phi = \phi(x,t) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, and $\Box$ is the D'Alembert operator. We now try to rewrite the equation as a Hamiltonian system: we look for a function $H(\phi,\psi)$, called Hamiltonian, such that

$$\begin{cases} \partial_t\phi = \psi \\ \partial_t\psi = -\left(-\partial_{xx}\phi + m^2\phi + f(\phi)\right) \end{cases} = \begin{cases} \partial_t\phi = \partial_\psi H \\ \partial_t\psi = -\partial_\phi H \end{cases}$$

The solution is the functional

$$H(\phi,\psi) = \frac{1}{2}\int_{\mathbb{R}} [\psi(x)^2 + \partial_x\phi(x)^2 + m^2\phi(x)^2]\,dx + \int_{\mathbb{R}} F(\phi(x))\,dx,$$

where $H : D \to \mathbb{R}$ and D could be $C_c^\infty(\mathbb{R})\times C_c^\infty(\mathbb{R}) \subset L^2_{\mathbb{R}}(\mathbb{R})\times L^2_{\mathbb{R}}(\mathbb{R})$, F it's a primitive for f.

However we need the lagrangian: we know that the lagrangian has to be such that $\partial_\phi A(\phi) = 0$ implies the Klein-Gordon equation, where

$$A(\phi) = \int_{\mathbb{R}^2} \mathcal{L}(\phi, \partial_t\phi, \partial_x\phi)\,dt\,dx.$$

This condition leads to the expression

$$\mathcal{L}(\phi, \partial_t\phi, \partial_x\phi) = \frac{1}{2}m^2\phi^2 + F(\phi) - \frac{1}{2}(\partial_t\phi^2 - \partial_x\phi^2);$$

if $f \equiv 0$ is called the lagrangian of the free field.
If we pass to the imaginary time ($t \to it$) we have

$$\frac{1}{2}m^2\phi^2 + F(\phi) - \frac{1}{2}(\partial_t\phi^2 - \partial_x\phi^2) \overset{t\to it}{\longrightarrow} \frac{1}{2}m^2\phi^2 + F(\phi) - \frac{1}{2}((i\partial_t\phi)^2 - \partial_x\phi^2) =$$
$$= \frac{1}{2}m^2\phi^2 + F(\phi) + \frac{1}{2}(\partial_t\phi^2 + \partial_x\phi^2) = \frac{1}{2}m^2\phi^2 + F(\phi) + \frac{1}{2}\|\nabla\phi\|_2^2.$$

Consequently the action becomes

$$iA(\phi) = i\int\left(\int L(\phi)dx\right)dt = i\left[\frac{1}{2}m^2\phi^2 + F(\phi) + \frac{1}{2}\|\nabla\phi\|_2^2\right]$$

and $e^{iA(\phi)}$ is replaced by $e^{-A(\phi)}$.
In the first section of the second chapter we will define the measure for the free field $\mu(d\phi) = e^{-A(\phi)} = e^{-\left[\frac{1}{2}m^2\phi^2 + F(\phi) + \frac{1}{2}\|\nabla\phi\|_2^2\right]}$, where we consider $F(\phi) \equiv 0$ (because $f = 0$ implies F is constant, then we can pick $F(\phi) \equiv 0$); this measure will be a infinite product of gaussian measure.

1.3 Application to String Theory and Polyakov's contribution

The main innovation of String Theory is to consider the particle as strings (1-dimensional), and we can imagine that a string, moving in space-time, describes a surface, as a particle describes paths. In analogy with what happens in QM we can't determine with precision what is the string position in a particular moment, or what is the surface it describes moving from a position to another, therefore we should contruct a probability density.

In *Quantum Geometry of Bosonic Strings*, Polyakov says that it's necessary "to develop an art of handling sums over random surfaces. These sums replace the old-fashioned sums over random paths." This is due to the heuristic description given just before, in fact, continues Polyakov, "all transition amplitude are given by the sums over all possible surfaces with fixed boundary". This is exactly what we want to be able to compute.

He considers the case of *free strings*, this means that there is no interaction if it cross each other; then he recall that the amplitudes of free particles are simbolically defined as follows: denote by $P_{p,q}$ a path from p to q and by L the lenght of the path, then we have the amplitude $G(p,q) = \sum_{(paths)} e^{-mL(P_{p,q})}$. By analogy, Polyakov writes the heuristic formula $G(C) = \sum_{(S_C)} e^{-m^2 A(S_C)}$, where C is a loop, the sum is over all possible surfaces S_C whose bound is the

loop C, and finally $A(S_C)$ is the area of S_C. He try to develop this formalism starting from the bosonic case[10].

In Chapter 3 we try to give a rigorous definition to these sums, or better to these integrals. Polyakov's idea is to replace the functional area with another functional, called Polyakov action, which does not bring different phisical results, but it's very helpful in the mathematical definition of such a probability distribution.

[10] In the article *Quantum Geometry of Fermionic Strings* he studies the fermionic case, as the title suggests.

Chapter 2

Gaussian Measures in QFT

2.1 Introduction

In this Chapter we study the problem of a rigorous definition of the Feynman-Kač measure for the free field case. This means to give a rigorous meaning to the measure:

$$e^{-\frac{1}{2}\int_D(\phi^2+|\nabla\phi|^2)\,dx}\,d^\infty\phi \tag{2.1}$$

on a suitable space of fields $\phi : D \to \mathbb{R}$, where D is a region with compact closure. This measure will be a sort of infinite gaussian measure with covariance $(\mathbb{I}-\Delta)^{-1}$. To simplify the presentation, we will assume that D is the 2-dimensional thorus. In particular in the second section we will notice that such a measure has a heuristic expression as infinite product of gaussian measure and then the third section summarize without proofs the relevant material on the construction of infinite product measures in order to apply the results in a rigorous definition of the Feynman-Kač measure. So in Section 4 the heuristic definition will gain sense.
Once we have a rigorous definition of (2.2), we study the problem of concentration of this measure. This is a relevant problem because it turns out that (2.2) is concentrated on distributional fields, i.e. on functional rather than functions. Section 5 is intended to motivate our investigation of the H^s spaces, which are defined in the subsequent section, in order to find the set of concentration of the measure.

The discussion on Feynman-Kač free field measure is extended to the definition of

$$e^{-\frac{1}{2}\int_D|\nabla\phi|^2\,dx}\,d^\infty\phi \qquad \phi : D \to \mathbb{R},$$

which is more close to the kind of measure we will treat in next chapter in the case of String Theory. Section 7 deals with the case of the Feynman-Kač with a quite simple extension of the results obtained in the previous sections: definition of the measure and its concentration.

2.2 Heuristic definition of the Feynman-Kač measure for the free field

Let D be a region with compact closure in $\mathbb{R}^2$ and let $\phi : D \to \mathbb{R}$, we can suppose ϕ to be a smooth map; in this section we look closely at the possible definitions of a measure of this kind:

$$e^{-\frac{1}{2}\int_D (\phi^2 + |\nabla\phi|^2)\, dx}\; d^\infty\phi. \tag{2.2}$$

A more complete theory may be obtained generalizing to the case $\phi : D \subset \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}^>$ then, when necessary, we will specify the general case, but for simplicity we discuss esplicitely the case $d = 2$.

Integrating by parts and supposing that $\phi : D \to \mathbb{R}$ is equal to zero in ∂D we have:

$$\begin{aligned}\int_D \nabla\phi \cdot \nabla\phi\, dx = \sum_i \int_D \partial_i\phi\partial_i\phi\, dx = -\sum_i \int_D \partial_i\left(\partial_i\phi\right)\phi\, dx = \\ = -\int_D \left(\sum_i \partial_i^2\phi\right)\; \phi\, dx = -\int_D \Delta\phi\, \phi\, dx,\end{aligned}$$

where Δ is the Laplace operator. Then

$$\begin{aligned}\int_D \phi^2 + |\nabla\phi|^2\, dx = \int_D \phi^2\, dx \;+\; \int_D |\nabla\phi|^2\, dx = \int_D \phi^2\, dx \;-\; \int_D \Delta\phi\, \phi\, dx = \\ = \int_D (\phi \;-\; \Delta\phi)\phi\, dx = \langle(\mathbb{I} \;-\; \Delta)\phi, \phi\rangle_{\mathrm{L}^2}\end{aligned}$$

So formally we have:

$$e^{-\frac{1}{2}\int_D \phi^2 + |\nabla\phi|^2\, dx}\; d^\infty\phi = e^{-\frac{1}{2}\langle(\mathbb{I}-\Delta)\phi,\phi\rangle_{\mathrm{L}^2}}\; d^\infty\phi. \tag{2.3}$$

The important point to note here is the form of the heuristic measure: there is an evident similarity with the gaussian measure finite dimensional. We recall that, if C is a positive-definite $n \times n$ matrix, a gaussian measure of null mean and variance C is defined as:

$$\mathcal{N}(0, C)(dx) = \frac{e^{-\frac{1}{2}(\mathrm{C}^{-1}x\cdot x)}}{\sqrt{(2\pi)^n \det \mathrm{C}}} dx.$$

By analogy it is natural to think to (2.3) (or better to (2.2)) as a infinite dimensional gaussian:

$$e^{-\frac{1}{2}\langle(\mathbb{I}-\Delta)\phi,\phi\rangle_{\mathrm{L}^2}}\; d\phi = \mathcal{N}(0, (\mathbb{I} \;-\; \Delta)^{-1})\,(d\phi), \tag{2.4}$$

of mean 0 and variance $C = (\mathbb{I} \;-\; \Delta)^{-1}$.

How to define the measure?

From now on we make the assumption that the domain is $D = [0, 2\pi]^2$; and let $e_k(x) = e^{i(k\cdot x)}$ where $k \in \mathbb{Z}^2$. $\{e_k\}_{k\in\mathbb{Z}^2}$ is an orthonormal base for $L^2(D)$. We notice that these e_k are also eigenfunction of the operator $\mathbb{I} - \Delta$ and

$$(\mathbb{I} - \Delta)e_k(x) = (1 + |k|^2)e_k(x) \qquad \forall k \in \mathbb{Z}^2, \quad x \in D,$$

so the eigenvalues are $\lambda_k = 1 + |k|^2$ where $\in \mathbb{Z}^2$. In fact if $k = (k_1, k_2)$ and $\mathbf{x} = (x_1, x_2) \in D$

$$\begin{aligned}(\mathbb{I} - \Delta)e_k(\mathbf{x}) &= e_k(\mathbf{x}) - \partial_x\partial_x e_k(\mathbf{x}) - \partial_y\partial_y e_k(\mathbf{x}) = \\ &= e_k(\mathbf{x}) - ik_1\partial_x e_k(\mathbf{x}) - ik_2\partial_y e_k(\mathbf{x}) = e_k(\mathbf{x}) + k_1^2\, e_k(\mathbf{x}) + k_2^2\, e_k(\mathbf{x}) = \\ &= e_k(\mathbf{x}) + (k_1^2 + k_2^2)e_k(\mathbf{x}) = (1 + |k|^2)e_k(\mathbf{x}).\end{aligned}$$

In particular

$$\langle(\mathbb{I} - \Delta)\phi, \phi\rangle = \sum_{k\in\mathbb{Z}^2}(1 + |k|^2)\phi_k^2.$$

Identifying ϕ with its "coordinates" (Fourier coefficients), $\phi \equiv (\phi_k)_{k\in\mathbb{Z}^2}$ so that $\phi = \sum_{k\in\mathbb{Z}^2}\phi_k e_k$, we have identified $L^2(D)$ and $\ell^2(\mathbb{Z}^2)$ because $\phi \in \ell^2(\mathbb{Z}^2) \Leftrightarrow \sum_{k\in\mathbb{Z}^2}\phi_k e_k$ converges in $L^2(D)$. However, thanks to this identification, we have:

$$(\mathbb{I} - \Delta)\phi = \sum_{k\in\mathbb{Z}^2}\phi_k(\mathbb{I} - \Delta)e_k = \sum_{k\in\mathbb{Z}^2}\phi_k(1 + |k|^2)e_k.$$

By this follows the useful formal equality

$$\langle(\mathbb{I} - \Delta)\phi, \phi\rangle = \sum_{k\in\mathbb{Z}^2}(1 + |k|^2)\phi_k^2.$$

The previous calculations make it legitimate to write formally:

$$\begin{aligned}e^{-\frac{1}{2}\langle(\mathbb{I}-\Delta)\phi,\phi\rangle}d^\infty\phi &= e^{-\frac{1}{2}\sum_{k\in\mathbb{Z}^2}(1+|k|^2)\phi_k^2}d\phi_1 d\phi_2 \dots d\phi_n \dots = \\ &\stackrel{*}{=} \bigotimes \frac{e^{-\frac{1}{2}(1+|k|^2)\phi_k^2}}{\sqrt{2\pi(1 + |k|^2)^{-1}}}d\phi_k = \\ &= \bigotimes_{k\in\mathbb{Z}^2}\mu_k,\end{aligned} \tag{2.5}$$

where we set:

$$\mu_k \equiv \frac{e^{-\frac{1}{2}(1+|k|^2)\phi_k^2}}{\sqrt{2\pi(1 + |k|^2)^{-1}}}d\phi_k.$$

The " $\stackrel{*}{=}$ " recall us that these formulas are heuristic.
Moreover notice that $\mu_k = \mathcal{N}_{\mathbb{R}}(0, (1 + |k|^2)^{-1})$, it's a probability (gaussian) measure over $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Remark 2.2.1. *It was necessary to diagonalize the Laplace operator, then to find eigenvalues and respective eigenfunctions. When we change the domain D from $[0, 2\pi]^2$ into a general region, with compact closure, of the plane (or in $\mathbb{R}^d$,) then we have to find eigenvalues and eigenfunction of the Laplace-Beltrami operator.*

In the following section we will see how to define infinite product of probability measure. We will touch only a few aspects of the thery and it is not our purpose to give proofs of the results that will be stated, for the proofs we refer the reader to [2].

2.3 Infinite product measures

Let $(X_i, \mathcal{M}_i, \mu_i)$ be probability spaces, that is X_i are totally finite measurable spaces (i.e. $\mu_i(X_i) < \infty$) such that the measure $\mu_i(X_i) = 1$, and $X = \times_{i=1}^{\infty} X_i$ the Cartesian product. We would like to define in X the concept of measurability and a measure, and we start with the definition of a σ-algebra on X. The first step in the

Definition 2.3.1 (Mesurable rectangle)**.** *In the above hypothesis we define a (measurable)* ***rectangle*** *as a set of the form*

$$\times_{i=1}^{\infty} A_i$$

where $A_i \subseteq X_i \quad \forall i$ (are measurable: $A_i \in \mathcal{M}_i$), and $A_i \neq X_i$ only for a finite number of i.

We construct a measurable space $(X, \mathcal{M})$:

Definition 2.3.2 (The σ-algebra $\mathcal{M}$)**.** *If $\mathcal{R}$ is the class of all measurable rectangles of X, $\mathcal{M} = \mathfrak{M}(\mathcal{R})$[1], i.e. $\mathcal{M}$ is the σ-algebra generated by the class of all measurable rectangle. We shall write $\mathcal{M} = \times_{i=1}^{\infty} \mathcal{M}_i$.*

Recall 2.3.3. *$\mathcal{M} = \mathfrak{M}(\mathcal{R})$ means that $\mathcal{M}$ is the (σ-algebra) intersection of all σ-algebras which contain $\mathcal{R}$.*

Definition 2.3.4 (J-cylinder)**.** *Let $J \subset \mathbb{N}^{>}$ finite, and two points of X $\mathbf{x} = (x_1, x_2, \ldots) \quad \mathbf{y} = (y_1, y_2, \ldots)$. We will say that $\mathbf{x}$ and $\mathbf{y}$* ***agree*** *on J and we will write*

$$\mathbf{x} \equiv \mathbf{y} \quad (J), \qquad \text{if } x_i = y_i \quad \forall i \in J.$$

A set E is a J-cylinder if, when $\mathbf{x} \equiv \mathbf{y}$ (J), then $\mathbf{x} \in E \Leftrightarrow \mathbf{y} \in E$.

Example 2.3.5 (useful example of a J-cylinder)**.** *If $J = \{1, 2, \ldots, n\}$ and A_j is an arbitrary subset of X_j, $j = 1, \ldots, n$ then the rectangle $A_1 \times A_2 \times \ldots \times A_n \times X_{n+1} \times X_{n+2} \times \ldots$ is a J-cylinder. In fact for each point $x \in X$ are not important the coordinates $x_i \quad i \geq n+1$, but only the belonging of $(x_1, x_2, \ldots, x_n)$ to $A_1 \times \ldots \times A_n$.*

[1] This notation for the generated σ-algebra is used in [4].

Notation 2.3.6. $X^{(n)} = \times_{i=n+1}^{\infty} X_i \qquad n = 0, 1, 2, \ldots$ *and* $I_n = \{1, 2, \ldots, n\}$

Definition 2.3.7 ($X^{(n)}$-section)**.** *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$ *and let* E *be a subset of* X*,* $E(x_1, \ldots, x_n)$ *is the* $X^{(n)}$*-section of* E*:*

$$E(\mathbf{x}) = \{\mathbf{y} \in X^{(n)} \mid (\mathbf{x}, \mathbf{y}) \in E\}$$

Lemma 2.3.8. *Every section of a rectangle in* X *is a rectangle in* $X^{(n)}$*;*
Every section of a measurable rectangle in X *is a measurable rectangle in* $X^{(n)}$*.*

Proof. Trivial □

Theorem 2.3.9. *Let* $E \subseteq X$*,*
$J = I_n$*, if* E *is a measurable* J*-cylinder, then* $E = A \times X^{(n)}$*,*
where A *is a measurable subset of* $X_1 \times X_2 \times \ldots \times X_n$*.*

Proof. See [2] 38.A. □

Let $\mathcal{F}$ be the class of all measurable sets which are I_n-cylinders for some value of n. This should be the domain of definition of a set function μ such that, for every measurable I_n-cylinder $A \times X^{(n)}$, $\mu(A \times X^{(n)}) = (\mu_1 \times \mu_2 \times \ldots \times \mu_n)(A)$.

Remark 2.3.10. *If* $m, n \geq 0$*, and we can suppose* $m < n$*. A* I_n *cylinder* E *can be also a* I_m *cylinder.*

Otherwise there is the following:

Proposition 2.3.11. *The function is well-defined and the domain is appropriate*

Proof. The previous theorem says that $E = A \times X^{(m)}$ and at the same time $E = B \times X(n)$, but we also can write $E = A \times X^{(m)} = (A \times X_{m+1} \times \ldots \times X_n \times X^{(n)}$. Finally the equality $B \times X^{(n)} = (A \times X_{m+1} \times \ldots \times X_n) \times X^{(n)}$ leads to $B = A \times X_{m+1} \times \ldots \times X_n$.
If E is measurable, both A and B are measurable and $(\mu_1 \times \ldots \times \mu_m)(A) = (\mu_1 \times \ldots \times \mu_n)(B)$.
It follows that μ is well-defined on $\mathcal{F}$ □

Proposition 2.3.12. *In the hypotesis of the preceeding definition the following conclusions hold*

- $\mathcal{F}$ *is an Algebra;*
- $\mathfrak{M}(\mathcal{F}) = \mathcal{M}$*;*
- *the set function* μ *on* $\mathcal{F}$ *is finite, non negative, and finitely additive.*

Proof. Easy. □

Notation 2.3.13. $X^{(n)}$ *may be seen as a product space, so we can costruct* $\mathcal{F}^{(n)}$ *and* $\mu^{(n)}$ *exactly as* $\mathcal{F}$ *and* μ*.*

Fact 2.3.14. *If $E \in \mathcal{F}$ then every section $E(x_1, x_2, \ldots, x_n) \in \mathcal{F}^{(n)}$ and*

$$\mu(E) = \int \cdots \int \mu^{(n)}(E(x_1, \ldots, x_n)) d\mu_1(x_1) \cdots d\mu_n(x_n)$$

Proof. It follows from the results in [2] for finite dimensional product spaces. □

Theorem 2.3.15. *In the opening Hypothesis ($\{(X_i, \mathcal{M}_i, \mu_i)\}$ totally finite spaces with measure with $\mu_i(X_i) = 1$) there exists a unique measure μ on the σ-algebra $\mathcal{M} = \times_{i=1}^{\infty} \mathcal{M}_i$, such that, for every measurable set $E = A \times X^{(n)}$,*

$$\mu(E) = (\mu_1 \times \cdots \times \mu_n)(A)$$

Proof. See [2], theorem 38.B □

2.4 Feynman-Kač measure for the free field

In our case there are the following:

Hypothesis 1. $(X_i, \mathcal{M}_i, \mu_i) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu_i) \qquad \forall i \in \mathbb{Z}^2$ *where μ_i are the gaussian measure $\mathcal{N}_{\mathbb{R}}(0, (1 + |i|^2)^{-1})$*

Clearly in Theorem (2.3.15) there is no dependency on the indexing (we can replace $\mathbb{Z}^2$ with $\mathbb{N}^>$) and the theorem says:

$$\exists! \mu \text{ su } \mathbb{R}^\infty = \{(\phi_k)_{k \in \mathbb{Z}^2}\}$$

such that $(\mathbb{R}^\infty, \mathcal{M}, \mu)$ is a space with measure where $\mathcal{M}$ is the σ-algebra generated by the I_n cylinders.
By definition $\mu = \bigotimes_{k \in \mathbb{Z}^2} \mu_k$

Definition 2.4.1.

$$\mathcal{N}(0, (\mathbb{I} - \Delta)^{-1}) \equiv \mu$$

and also, if we recall (2.3) and (2.5),

$$e^{-\frac{1}{2}\int_D \phi^2 + |\nabla\phi|^2 \, dx} \, d^\infty\phi \equiv \mu$$

Finally we have solved the problem (2.4).

2.5 Concentration in L^2 of the Feynman-Kač measure for the free field

The measure obtained in the previous section is defined on $\mathbb{R}^\infty$ but we would like to find the sets on which the measure is concentrated. It is to be expected that $L^2(D)$ is one of those sets; obviously in the same hypothesis of the previous sections ($D = [0, 2\pi]^2 = \mathbb{T}^2$, etc.). The crucial fact is that $L^2(D)$ is a negligible set with respect to the measure previously defined.

The proof fills all the section and to semplify it we need to recall the identification $L^2 \equiv \ell^2(\mathbb{Z}^2) = \{(\phi_k)_k : \sum_{k\in\mathbb{Z}^2} \phi_k^2 < \infty\}$, already done in (2.2) when we identified $\phi \in L^2(D)$ with its Fourier coefficients $\phi \equiv (\phi_k)_k$ because

$$\phi \in \ell^2(\mathbb{Z}^2) \Longleftrightarrow \sum_{k\in\mathbb{Z}^2} \phi_k e_k \text{ converges in } L^2(D).$$

Theorem 2.5.1.

$$\mu(\ell^2(\mathbb{Z}^2)) = 0 \tag{2.6}$$

In particular $\mu\{\|\phi\|_2 = \infty\} = 1$

Moreover might be proved that:

Proposition 2.5.2. *If* $d = dim(D(= \mathbb{T})) \geq 2$ *then* $\boxed{\mu(\ell^2(\mathbb{Z}^d)) = 0}$.

Proof. This is (2.14) that will be proved at page 25. □

Lemma 2.5.3. *We notice that* $\int_{\mathbb{R}^\infty} \|\phi\|_{\ell^2}^2 \, d\mu = \infty$. *(This don't implies that* $\mu\{\|\phi\|_{\ell^2}^2 = +\infty\} = 1$*)*

Proof. (Lemma)
Let us prove it assuming ϕ is real valued. Racall that $\mu_k = \mathcal{N}_{\mathbb{R}}(0, \frac{1}{1+|k|^2}x)$ is probability (gaussian) measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, and also we set $\sigma_k \equiv \frac{1}{1+|k|^2}$; then

$$\begin{aligned}
\int_{\mathbb{R}^\infty} \|\phi\|_{\ell^2}^2 \, d\mu &= \int_{\mathbb{R}^\infty} \sum_{k\in\mathbb{Z}^2} \phi_k^2 \, d\mu = \int_{\mathbb{R}^\infty} \lim_n \sum_{k:|k|\leq n} \phi_k^2 \, d\mu = \\
&\overset{\substack{monotone \\ conv.}}{=} \sum_{k\in\mathbb{Z}^2} \int_{\mathbb{R}^\infty} \phi_k^2 \, d\mu = \\
&\overset{Halmos}{=} \sum_{k\in\mathbb{Z}^2} \int_{\mathbb{R}} \phi_k^2 \, d\mu_k = \sum_{k\in\mathbb{Z}^2} \int_{\mathbb{R}} \phi_k^2 \frac{1}{\sqrt{2\pi\sigma_k}} e^{-\frac{\phi_k^2}{2\sigma_k}} \, d\phi_k \\
&\overset{variance}{=} \sum_{k\in\mathbb{Z}^2} \sigma_k = \sum_{k\in\mathbb{Z}^2} \frac{1}{1+|k|^2}.
\end{aligned}$$

Since

$$\sum_{k\in\mathbb{Z}^2} \frac{1}{1+|k|^2} \sim \int_{\mathbb{R}^2} \frac{1}{1+|k|^2} \, dk$$

we can compute this Lebesgue integral in $\mathbb{R}^2$ using polar coordinates:

$$\int_{\mathbb{R}^2} \frac{1}{1+|k|^2} \, dk = \int_0^\infty \int_0^{2\pi} \frac{\rho}{1+\rho^2} \, d\theta \, d\rho = 2\pi \int_0^\infty \frac{\rho}{1+\rho^2} \, d\rho = +\infty.$$

□

Proof. [2] Fix $\varepsilon > 0$, recall that $\sum_{k\in\mathbb{Z}^2} \sigma_k = +\infty$; the basic idea of the proof is to show that

$$\int_{\mathbb{R}^\infty} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi) = 0, \tag{2.7}$$

it follows the conclusion, in fact:

$$\begin{aligned}\int_{\mathbb{R}^\infty} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi) &= \int_{\{\phi\in\mathbb{R}^\infty:\|\phi\|_2<+\infty\}} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi) + \int_{\{\|\phi\|_2=+\infty\}} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi)\\ &= \int_{\{\|\phi\|_2<+\infty\}} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi) + 0 = \int_{\{\|\phi\|_2<+\infty\}} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi);\end{aligned}$$

where the sets $\{\|\phi\|_2 < +\infty\}$ and $\{\|\phi\|_2 = +\infty\}$ are respectively $\{\phi \in \mathbb{R}^\infty : \|\phi\|_2 < +\infty\}$ and $\{\phi \in \mathbb{R}^\infty : \|\phi\|_2 = +\infty\}$. Letting $\varepsilon \to 0$ we have

$$\int_{\{\phi\in\mathbb{R}^\infty:\|\phi\|_2<+\infty\}} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi) \to \int_{\{\phi\in\mathbb{R}^\infty:\|\phi\|_2<+\infty\}} 1\, d\mu(\phi) = \mu\left(\|\phi\|_2 < +\infty\right).$$

We have used the Monotone Convergence Theorem (let $\varepsilon = \frac{1}{n},\ n \in \mathbb{N}$ and the function $f_n = e^{-\frac{1}{n}\|\phi\|_2^2}$, then $\lim_{\varepsilon\to 0}$ can be replaced by $\lim_{n\to\infty}$). This means that $\mu\left(\|\phi\|_2 < +\infty\right) = 0$:

$$0 = \int_{\mathbb{R}^\infty} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi) = \int_{\{\phi\in\mathbb{R}^\infty:\|\phi\|_2<+\infty\}} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi) \overset{\varepsilon\to 0}{\longrightarrow} \mu\left(\|\phi\|_2 < +\infty\right).$$

We are reduced to proving (2.7):

$$\int_{\mathbb{R}^\infty} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi) = \int_{\mathbb{R}^\infty} \bigotimes_{k\in\mathbb{Z}^2} \frac{e^{-\varepsilon\|\phi\|_2^2} e^{-\frac{\phi_k^2}{2\sigma_k}}}{\sqrt{2\pi\sigma_k}}\, d\phi_k = \prod_{k\in\mathbb{Z}^2} \int_{\mathbb{R}} \frac{e^{-\left(\varepsilon\|\phi\|_2^2 + \frac{\phi_k^2}{2\sigma_k}\right)}}{\sqrt{2\pi\sigma_k}}\, d\phi_k;$$

Calculating separately the integrals[3]:

$$\int_{\mathbb{R}} \frac{e^{-\left(\varepsilon\|\phi\|_2^2 + \frac{\phi_k^2}{2\sigma_k}\right)}}{\sqrt{2\pi\sigma_k}}\, d\phi_k = \frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\varepsilon + \frac{1}{2\sigma_k}}\sqrt{2\pi\sigma_k}} = \frac{\sqrt{\pi}}{\sqrt{\pi(2\varepsilon\sigma_k + 1)}} = \frac{1}{\sqrt{2\varepsilon\sigma_k + 1}}.$$

Therefore $\int_{\mathbb{R}^\infty} e^{-\varepsilon\|\phi\|_2^2}\, d\mu(\phi) = \prod_{k\in\mathbb{Z}^2} (2\varepsilon\sigma_k + 1)^{-\frac{1}{2}}$, and we only need to show that $\prod_{k\in\mathbb{Z}^2} (2\varepsilon\sigma_k + 1)^{-\frac{1}{2}} = 0$. To do it we use a trick:

$$\prod_{k\in\mathbb{Z}^2} (2\varepsilon\sigma_k + 1)^{-\frac{1}{2}} = \prod_{k\in\mathbb{Z}^2} e^{-\frac{1}{2}\log(2\varepsilon\sigma_k+1)} = e^{-\frac{1}{2}\sum_{k\in\mathbb{Z}^2}(\log(2\varepsilon\sigma_k+1))},$$

but it's well known that $log(1 + x) \overset{x\to 0}{\sim} x$ (are infinitesimal of the same order), then

$$e^{-\frac{1}{2}\sum_{k\in\mathbb{Z}^2}(\log(2\varepsilon\sigma_k+1))} \sim e^{-\frac{1}{2}\sum_{k\in\mathbb{Z}^2}(2\varepsilon\sigma_k)} = e^{-\varepsilon\sum_{k\in\mathbb{Z}^2}\sigma_k} \overset{\sum_k \sigma_k=+\infty}{=} e^{-\infty} = 0$$

□

[2] see the appendix to have another idea for the solution.

[3] Is the integral number 69 in "Formulario" by G.De Marco

Resuming, μ which has been defined as a probability measure over $\mathbb{R}^\infty$, is such that $\mu(\ell^2) = 0$.

Is now our purpose to answer to the following questions:

1. Is there true that $\mu\left(L^2(\mathbb{T})\right) = 1$ (i.e. in the case $d = dimD = 1$)?
2. Where is the measure concentrated (we give a brief exposition for the case $d = 2$)?

Proposition 2.5.4. *The measure μ of the space $L^2(\mathbb{T})$ is equal to one. Identifying $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$, we only have to check that $\mu\left(\ell^2(\mathbb{Z})\right) = 1$.*

Proof. It suffices to prove

$$\int_{\mathbb{R}^\infty} \|\phi\|^2_{\ell^2} \, \mathrm{d}\mu < \infty$$

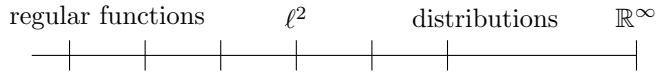
because this implies that $\mu\left\{\|\phi\|^2_{L^2} < \infty\right\} = 1$. By the same calculation done in the proof of Lemma 2.5.3 follows the conclusion:
$\int_{\mathbb{R}^\infty} \|\phi\|^2_{\ell^2} \, \mathrm{d}\mu = \sum_{n\in\mathbb{Z}} \frac{1}{1+|n|^2} = 1+2\sum_{k\in\mathbb{N}\setminus\{0\}} \frac{1}{1+n^2}$ and this sum converges. □

The sets of concentration of the measure are subsets of $\mathbb{R}^\infty$ such that the measure of these sets is equal to 1.
These sets have to contain ℓ^2, so it's necessary to look for the intemediate subsets.

2.6 H^s spaces

In the previous section has been shown that $\ell^2(\mathbb{Z}^d) \equiv L^2(\mathbb{T}^d) \subset \mathbb{R}^\infty$ $(d = 2)$ is a null set with respect to μ. In this section we will introduce an increasing family of $\mathbb{R}^\infty$ subspaces, such that ℓ^2 is a particular case. The advantage of defining these subspaces lies in the fact that some of those are the natural context where the concentration of the measure problem has to be studied.

In simple words in ℓ^2 there are spaces of more and more regular functions $(\mathcal{C}^n, \quad n \geq 2)$, while between ℓ^2 and $\mathbb{R}^\infty$ there are sets containing maps that are more and more irregular, or even not functions; and we shall call them distributions. Schematically: regular functions $\leftrightarrow \ell^2 \leftrightarrow$ irregular maps (distributions), as is shown in the picture below.



2.6.1 Definition and first properties of the H^s-spaces

Definition 2.6.1. *Let $s \in \mathbb{R}$, let us denote by H^s-space the set:*

$$H^s = \{(\phi_k)_{k\in\mathbb{Z}^2} \in \mathbb{R}^\infty \mid \sum (1+|k|^2)^s \phi_k^2 < \infty\}.$$

This definition provides a decreasing family of subsets of $\mathbb{R}^\infty$, and an element of the family is ℓ^2. If $s = 0$ we have $H^0 = \ell^2$ and, according to the well known identification, we can write $H^0 = L^2$, in fact:

$$H^0 = \{(\phi_k) \in \mathbb{R}^\infty \mid \sum (1+|k|^2)^0 \phi_k^2 < \infty\} = \{(\phi_k) \in \mathbb{R}^\infty \mid \sum \phi_k^2 < \infty\} = \ell^2.$$

Maybe H^s-spaces have many of the properties of ℓ^2. In fact, the next propositions will state that for all $s \in \mathbb{R}$ H^s are scalar product spaces, then normed spaces.

Proposition 2.6.2. *H^s is a scalar product space $\forall s, s \in \mathbb{R}$.*

Proof. Let fix $s \in \mathbb{R}$. We claim that

$$\langle (\phi_k), (\psi_k) \rangle_{H^s} = \sum (1+|k|^2)^s \phi_k \psi_k$$

defines a scalar product on H^s; plainly H^s is an $\mathbb{R}$-vector space. We only need to show that:

- $\langle (\phi_k + \eta_k), (\psi_k) \rangle_{H^s} = \langle (\phi_k), (\psi_k) \rangle_{H^s} + \langle (\eta_k), (\psi_k) \rangle_{H^s}$
- $\langle (\lambda\phi_k), (\psi_k) \rangle_{H^s} = \lambda \langle (\phi_k), (\psi_k) \rangle_{H^s}$
- $\langle (\phi_k), (\phi_k) \rangle_{H^s} \geq 0$ se $\phi \neq 0$

It's immediate that the three properties are satisfied. □

Proposition 2.6.3. *H^s, $s \in \mathbb{R}$, are also normed spaces as every scalar product space. Let $\|\#\|_{H^s}$ the norm defined by the scalar product, then $\|\#\|_{H^s} \geq \|\#\|_{\ell^2}$.*

Proof. The inner product defines the norm which satisfies the inequality:

$$\|(\phi_k)\|_{H^s}^2 \equiv \sum (1+|k|^2)^s \phi_k^2 \geq \sum \phi_k^2 = \|(\phi_k)\|_{\ell^2}^2.$$

And trivially $\|\#\|_{H^0} = \|\#\|_{\ell^2}$. □

Proposition 2.6.4. *Let $s > 0$ then H^s is isomorphic to H^{-s}: $H^s \cong H^{-s}$.*

Proof. The simple isomorphisms are: $(\phi_k)_k \in H^s \to \left(\phi_k (1+|k|^2)^s\right)_k \in H^{-s}$ and $(\phi_k)_k \in H^{-s} \to \left(\frac{\phi_k}{(1+|k|^2)^s}\right)_k \in H^s$ □

Proposition 2.6.5. *H^{-s} can be identify with the topological dual of H^s, therefore* $\boxed{H^s = \left(H^{-s}\right)^*}$.

Proof. Let $\phi \in H^{-s}$. Consider the map

$$\begin{aligned} F : H^s &\to \mathbb{R} \\ (\psi_k) &\to \sum \phi_k \psi_k \end{aligned}.$$

Playnly this is a linear map; we proceed to show that it is also continuous.

$$\sum \phi_k \psi_k = \sum (1+|k|^2)^{-s/2} \phi_k (1+|k|^2)^{s/2} \psi_k$$

which converges because, applying Hölder inequality we have:

$$\begin{aligned} |F(\psi)| =& | \sum (1+|k|^2)^{-s/2} \phi_k (1+|k|^2)^{s/2} \psi_k \, | \\ &\leq \left(\sum (1+|k|^2)^{-s} \phi_k^2 \right)^{1/2} \left(\sum (1+|k|^2)^{s} \psi_k^2 \right)^{1/2}. \\ &= \|\phi\|_{H^{-s}} \|\psi\|_{H^s} < \infty \end{aligned}$$

Therefore $|F(\psi)| \leq \|\phi\| \|\psi\| \forall \psi$ and this means that F is continuous. It remains to prove that the map $F : H^{-s} \to (H^s)^*$ is injective and surjective.

Let us suppose that ϕ goes to 0, i.e. that

$$\sum_k \phi_k \psi_k = 0$$

for every $\psi \in H^s$. In particular this holds for ψ such that $\psi_{\bar{k}} = 1$ and $\psi_h = 0$ for each $h \neq \bar{k}$. But this imply $\phi_{\bar{k}} = 0$ and, since $\bar{k}$ is arbitrary, $\phi = 0$.

Now take $F \in (H^s)^*$, we must find $\phi \in H^{-s}$ such that $F = F_\phi$.
Let consider the isometric immersion:

$$\begin{aligned} S : H^s &\longrightarrow \ell^2 \\ (\psi_k)_k &\longrightarrow \left(\psi_k (1+|k|^2)^{\frac{s}{2}} \right)_k \end{aligned}$$

this map clearly is linear and isometric ($\|\psi\|_s = \sum_k \psi_k^2 (1+|k|^2)^s = \|\psi_k^2 (1+|k|^2)^{s/2}\|_2$). Than we define a function:

$$\begin{aligned} Q : S(H^s) \subseteq \ell^2 &\longrightarrow \mathbb{R} \\ S(\psi) &\longrightarrow F(\psi), \end{aligned}$$

which is "$Q = F \circ S^{-1}$ well defined (because S is injective) and continuous: $\|F(\psi)\| \leq \|F\| \, \|\psi\|_s = \|F\| \, \|S(\psi)\|$. By Hahn-Banach theorem Q has an extension $\tilde{Q} : \ell^2 \to \mathbb{R}$ (more easily we can notice that $S(H^s)$ is dense in ℓ^2, then exists the extension for Q because of the theorem of extension of uniformely continuous functions), and by Rietz theorem $\exists! (\alpha_k)_k \in \ell^2$ such that $\tilde{Q}(\beta) = \sum_k \alpha_k \beta_k$. If $\psi \in H^s$ we have $\tilde{Q}(S(\psi)) = Q(S(\psi)) = F(\psi)$, but also, being $S(\psi) \in \ell^2$, also holds $\tilde{Q}(S(\psi)) = \sum_k \alpha_k \left(S(\psi)\right)_k$. $\left(S(\psi)\right)_k = \psi_k (1+|k|^2)^{s/2}$, therefore

$$F(\psi) = \sum_k \alpha_k \psi_k (1+|k|^2)^{\frac{1}{2}} = \sum_k \left(\alpha_k (1+|k|^2)^{\frac{1}{2}} \right) \psi_k.$$

Set $\phi_k = \alpha_k \left(1+|k|^2\right)^{s/2}$, if this is a function in H^{-s}, we have concluded: $\sum_k \frac{\phi^2}{(1+|k|^2)^s} = \sum_k \frac{\alpha_k^2 (1+|k|^2)^s}{(1+|k|^2)^s} = \sum_k \alpha_k^2 < +\infty$ because α is a function in ℓ^2. $\square$

In fact can be prooved (we accept it):

Theorem 2.6.6. *H^s are separable Hilbert spaces.*

2.6.2 In and out ℓ^2

$\boxed{s > 0}$
If $s > 0$ $H^s \subseteq \ell^2$

Proof. Let $s > 0$, the definition for the H^s spaces allow us to conclude:

$$H^s = \{(\phi_k) \in \mathbb{R}^\infty \mid \sum (1 + |k|^2)^s \phi_k^2 < \infty\}$$

but $(1 + |k|^2)^s \geq 1$, then we have

$$\phi_k^2 \leq (1 + |k|^2)^s \phi_k^2 \text{ and if } \sum (1 + |k|^2)^s \phi_k < \infty \Rightarrow \sum \phi_k^2 < \infty$$

In other words, if $\phi \in H^s$ then ϕ is also in ℓ^2. □

Example 2.6.7.

$$H^1 = \{\phi_k \mid \sum (1 + |k|^2)\phi_k^2 < \infty\}$$

If $(\phi_k) \in H^1$ then $(1 + |k|^2)\phi_k^2 \overset{k\to\infty}{\to} 0$; this happens for example if $\phi_k = o\left(\frac{1}{k}\right)$, and than if $\phi \in \mathcal{C}^1$. In fact $\phi_k = o\left(\frac{1}{k^l}\right)$ $\exists l > 0$ because ϕ_k, $k \in \mathbb{Z}^2$, are coefficients of the Fourier series of ϕ and the following result is valid ([6] 13.8.1):

Proposition 2.6.8. *Let $l \in \mathbb{N}$ and $f \in \mathcal{C}^l$ which is also τ-periodic, then the Fourier series coefficient $c_n(f)$ is $o\left(\frac{1}{n^l}\right)$ for $n \to \pm\infty$.*

Always for $(\phi_k) \in H^1 \subseteq \ell^2$, we define the function in L^2

$$\boxed{\Phi = \sum \phi_k e_k}.$$

Therefore H^1 can be written in terms of Φ:

$$H^1 = \{\phi_k \mid \sum (1 + |k|^2)\phi_k^2 < \infty = \{(\phi_k) \mid \|\Phi\|_{\ell^2}^2 + \|\nabla\Phi\|_{\ell^2}^2 < \infty\};$$

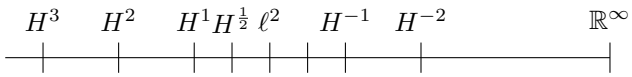
in fact $\sum (1 + |k|^2)\phi_k^2$ is exactly $\|\Phi\|_{\ell^2}^2 + \|\nabla\Phi\|_{\ell^2}^2$.

We notice that the derivative of Φ is in L^2: $\Phi \in L^2$ and also $\partial\Phi \in L^2$. This means that Φ is a function in L^2 as all its partial derivative. This calculation prove it:

$$\partial_{x_1} \sum \phi_k e_k = \sum \phi_k i k^{(1)} e_k$$

and this quantity is in L^2 because $(i\phi_k k^{(1)})_k \in \ell^2$ being $\sum |i\phi_k k^{(1)}|^2 = \sum (\phi_k k^{(1)})^2 \leq \sum \phi_k^2 |k|^2 \leq \infty$ by hypothesis.

The completion of the family



Being subsets of ℓ^2, H^s are negligible sets if $s > 0$. In the last part of the subsection our aim is to analyse the remaining case $s < 0$.

$\boxed{s \notin \mathbb{Z}}$

In order to give a complete description to these spaces we accept the following:

Fact 2.6.9. *If $s \notin \mathbb{Z}$, for example $s = \frac{1}{2}$, we have by the definition:*

$$H^{1/2} = \{\phi_k \mid \sum (1 + |k|^2)^{1/2} \phi_k^2 < \infty\}$$

$$\phi = \sum \phi_k e_k \in L^2(\mathbb{T}^2) \text{ and also is } 1/2 - \text{Hölder function}$$

with more precision are α-Hölder functions for all $\alpha < 1/2$.

$\boxed{s < 0}$

$$H^s = \{(\phi_k) \mid \sum (1 + |k|^2)^s \phi_k^2 < \infty\}$$

Example 2.6.10. *Let $d = 2$.*
If $s = -1$, $H^{-1} = \{(\phi_k) \mid \sum \frac{1}{1+|k|^2} \phi_k^2 < \infty\}$. It's easier to work with positive number; then, given $s > 0$, we consider

$$H^{-s} = \{\phi_k \mid \sum \frac{1}{(1 + |k|^2)^s} \phi_k^2 < \infty\}.$$

If $s > 1$ we have $\phi_k \equiv 1$ belongs to H^{-s} because $\sum \frac{1}{(1+|k|^2)^s}$ converges; but it doesn't converge if $s = 1$ ($\sum_{k \in \mathbb{Z}^d} \frac{1}{1+|k|^2} \sim \int_0^\infty \frac{1}{1+\rho^2} \rho^{d-1} d\rho = \infty$ if $d \geq 2$ and in our case $d = 2$), therefore $\phi_k \equiv 1$ doesn't belong to H^{-1}.

The point of the section is the looking for the sets of concentration of the measure μ and the following result will solve our problem.

Proposition 2.6.11. *$\exists s^*$, $s^* > 0$ such that if $s \leq s^*$ then $\mu(H^{-s}) = 1$.*

Proof. Surely s^* is the smaller positive number s such that

$$\int_{\mathbb{R}^\infty} \|\phi\|_{H^{-s}}^2 \, d\mu < \infty,$$

in fact $\int_{\mathbb{R}^\infty} \|\phi\|_{H^{-s}}^2 \, d\mu = \int_{\{\|\phi\|_{H^{-s}}^2 < \infty\}} \|\phi\|_{H^{-s}} \, d\mu + \int_{\{\|\phi\|_{H^{-s}} = \infty\}} \|\phi\|_{H^{-s}}^2 \, d\mu$, and

if it is finite this means that $\mu(\{\|\phi\|_{H^{-s}} = \infty\}) = 0$.

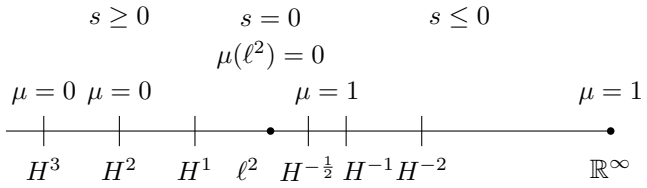
$$\begin{aligned}\int_{\mathbb{R}^\infty} \|\phi\|^2_{H^{-s}}\, d\mu &= \int \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-s}\phi_k^2\, d\mu_k \\ &= \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-s} \int \phi_k^2\, d\mu_k = \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-s}\sigma_k \\ &= \sum_{k\in\mathbb{Z}^2} \frac{(1+|k|^2)^{-s}}{1+|k|^2} = \sum_{k\in\mathbb{Z}^2} (1+|k|^2)^{-s-1}\end{aligned}$$

and $\sum_{k\in\mathbb{Z}^2}(1+|k|^2)^{-s-1} - 1 \sim \int_{\mathbb{R}^2\setminus(-1,1)^2}(1+|(x_1,x_2)|^2)^{-s-1}dx_1dx_2 =$
$= 2\pi\int_1^\infty \frac{\rho^{d-1}}{(1+\rho^2)^{1+s}}d\rho$ and also $\int_1^\infty \frac{\rho^{d-1}}{(1+\rho^2)^{1+s}}d\rho \le \int_1^\infty \frac{\rho^{d-1}}{\rho^{2(1+s)}}d\rho = \int_1^\infty \frac{1}{\rho^{2(1+s)-(d-1)}}d\rho$.
In this case $d = 2$ then $\int_1^\infty \frac{1}{\rho^{2(1+s)-(d-1)}}d\rho = \int_1^\infty \frac{1}{\rho^{1+2s}}d\rho$ and

$$\int_1^\infty \frac{1}{\rho^{1+2s}}d\rho < \infty \Leftrightarrow 1+2s > 1 \Leftrightarrow s > 0.$$

□

We would like to conclude the section with a resuming picture:



2.7 Definition of the measure (2.8)

Let $\phi : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^d$ region with compact closure (in particular we will consider the case of the torus $D = [0, 2\pi]^d = \mathbb{T}^d$). In this section we want to give sense to the expression

$$e^{-\frac{1}{2}\int_D |\nabla\phi|^2\, dx}\; d^\infty\phi. \tag{2.8}$$

We now indicate how the same tecniques and calculations that enabled us to give sense to (2.2), may be used to define the measure suggested by the formula (2.8). Let us follow the same order. Integrating by parts, ..., formally we have:

$$\int_D |\nabla\phi|^2\, dx = -\int_D \Delta\phi\, \phi\, dx = \langle -\Delta\phi, \phi\rangle_{\mathrm{L}^2},$$

and we obtain

$$e^{-\frac{1}{2}\int_D |\nabla\phi|^2\, dx}\; d^\infty\phi = e^{-\frac{1}{2}\langle -\Delta\phi,\phi\rangle_{\mathrm{L}^2}}\; d^\infty\phi \tag{2.9}$$

in the same way we have concluded (2.3) at page 10.

2.7.1 Construction of the measure

There still is an analogy with the gaussian measure of a finite gaussian measure of mean 0 and variance $C = (-\Delta)^{-1}$. As before, it is natural to think to (2.9) as an infinite dimensional gaussian:

$$e^{-\frac{1}{2}\langle -\Delta\phi,\phi\rangle_{L^2}}\, d\phi = \mathcal{N}(0,(-\Delta)^{-1})\, d\phi. \tag{2.10}$$

How to define the measure? Case $D = \mathbb{T}^d$

Let $D = [0,2\pi]^d = \mathbb{T}^d$ and $e_k(x) = e^{i(k\cdot\mathbf{x})}$ where $k \in \mathbb{Z}^d$. We notice that $\{e_k\}_{k\in\mathbb{Z}^d}$ is an orthonormal base for $L^2(D)$. We can identify $L^2(D)$ and $\ell^2(\mathbb{Z}^d)$ because

$$\phi \in \ell^2(\mathbb{Z}^d) \Leftrightarrow \sum_{k\in\mathbb{Z}^d} \phi_k e_k \text{ converges in } L^2(D).$$

Consequently each $\phi : D \to \mathbb{R}$ can be written as $\phi = \sum_{k\in\mathbb{Z}^d} \phi_k e_k$ because in "coordinates" $\phi \equiv (\phi_k)_{k\in\mathbb{Z}^d}$.

Proposition 2.7.1 (Diagonalization of the Laplace operator)**.** *$(e_k)_k$ is an eigenfunction with eigenvalue $|k|^d$ in fact:*

$$(-\Delta)e_k = |k|^d e_k.$$

Moreover

$$(-\Delta)\phi = \sum_{k\in\mathbb{Z}^d} \phi_k |k|^d e_k((x_1,x_2,\ldots,x_d))$$

then $\forall\phi\ \langle -\Delta\phi,\phi\rangle = \sum_{k\in\mathbb{Z}^d} |k|^d \phi_k^2$.
In particular, when $d = 2$ the statement of the proposition becomes: $(e_k)_k$ is an eigenfunction with eigenvalue $|k|^2$ in fact $(-\Delta)e_k = |k|^2 e_k$. Moreover $(-\Delta)\phi = \sum_{k\in\mathbb{Z}^2} \phi_k |k|^2 e_k((x,y))$ then $\forall\phi\ \langle -\Delta\phi,\phi\rangle = \sum_{k\in\mathbb{Z}^2} |k|^2 \phi_k^2$.

Proof. Trivial (see Subsection 2.2). □

Our proof is a direct computation but this proposition can have a confirmation of it's validity by Weyl's asymptotic formula:

Theorem 2.7.2 (Weyl's asymptotic formula for the torus $\mathbb{T}$)**.** *Let $N(\lambda)$ be the number of eigenvalues, counted with multiplicity, $\leq \lambda$ then, as $\lambda \to \infty$*

$$N(\lambda) \sim \omega_d \lambda^{\frac{d}{2}} \frac{vol(\mathbb{T})}{(2\pi)^d} \tag{2.11}$$

and also the m-th eigenvalue $(\lambda_m)^{\frac{d}{2}} \sim \frac{(2\pi)^d}{\omega_d}\frac{m}{vol(\mathbb{T})}$. In particular for bounded domains in $\mathbb{R}^d$ (as is the torus) G.Polya conjectured that $(\lambda_m)^{\frac{d}{2}} \geq \frac{(2\pi)^d}{\omega_d}\frac{m}{vol(\mathbb{T})}$ $\forall m \geq 1$.

Proof. See [7] (page 30 and 33). Here we show how this theorem gives the same eigenvalues asymptotic behaviour. Let us estimate the number of eigenvalues, counted with multiplicity, $\leq n^d$, as $n \to \infty$ with W.a.f.: $N(n^d) \sim \omega_d n^{\frac{d^2}{2}} \frac{vol(\mathbb{T})}{(2\pi)^d}$, then we can estimate what is the asymptotic behaviour of the $N(n^d)^{th}$ eigenvalue. Set $m = N(\lambda)$ then, noticing that letting $\lambda \to \infty$, we have $(\lambda_m)^{\frac{d}{2}} \sim \lambda$. In particular in fact, if $m = N(n^d)$, letting $n \to \infty$, $(\lambda_m)^{\frac{1}{2}} \sim \frac{(2\pi)}{\omega_d^{1/d}} \left(\frac{m}{vol(\mathbb{T})}\right)^{1/d} \sim \frac{(2\pi)}{\omega_d^{1/d}} \left(\frac{\omega_d n^{d^2/2} vol(\mathbb{T})}{(2\pi)^d vol(\mathbb{T})}\right)^{1/d} \sim n^{\frac{d}{2}}$. Polya's conjecture gives us $\lambda_m^{1/2} \geq \frac{(2\pi)}{\omega_d^{1/d}} \left(\frac{m}{vol(\mathbb{T})}\right)^{1/d} \overset{n\to\infty}{\sim} n^{\frac{d}{2}}$, then $\lambda_m \geq n^d$ if $n \to \infty$. □

The solution of the problem (2.10) is more delicate: it's necessary to put some restrictions on ϕ.

Theorem 2.7.3 (Definition of the measure)**.** *Let $\phi \in L^2(D)$ such that $\int_D \phi d\lambda = 0$; and ϕ is not constant (that is to say $\phi \not\equiv 0$, because of the previous assumption). We can now give sense to (2.10).*

Notation 2.7.4. *For simplicity we call $L_0^2(D) = \left\{\phi \in L^2 | \int_D \phi d\lambda = 0\right\}$, and*

$$\ell_0^2\left(\mathbb{Z}^d\right) = \{(\phi_k)_{k\in\mathbb{Z}^d} | \phi_0 = 0\} \cong \ell^2\left(\mathbb{Z}^d \setminus \{\mathbf{0}\}\right),$$

so that we can even talk about the identification between the two spaces.

Proof. Heuristically we have:

$$e^{-\frac{1}{2}\langle -\Delta\phi, \phi\rangle} d^\infty\phi = e^{-\frac{1}{2}\sum_{k\in\mathbb{Z}^d} |k|^d \phi_k^2} d\phi_1 d\phi_2 \dots d\phi_n \dots = \tag{2.12}$$

$$= \bigotimes_{k\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \frac{e^{-\frac{1}{2}|k|^d\phi_k^2}}{\sqrt{2\pi(|k|^d)^{-1}}} d\phi_k \tag{2.13}$$

And we set by definition:

$$\mu_k \equiv \frac{e^{-\frac{1}{2}|k|^d\phi_k^2}}{\sqrt{2\pi(|k|^d)^{-1}}} d\phi_k,$$

then $\mu_k = \mathcal{N}_{\mathbb{R}}(0, (|k|^d)^{-1})$ is a probability gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

A simple application of theorem 2.3.15 assures that:

$$\exists! \mu \text{ su } \mathbb{R}^\infty = \{(\phi_k)_{k\in\mathbb{Z}^d\setminus\{\mathbf{0}\}}\}$$

such that $\mu = \bigotimes_{k\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} \mu_k$. Finally we set $\mathcal{N}(0, (-\Delta)^{-1}) \equiv \mu$ and also: $e^{-\frac{1}{2}\int_D |\nabla\phi|^2 dx} d^\infty\phi \equiv \mu$ after having recalled (2.9). We have just given sense to (2.10). □

Remark 2.7.5 (Why the condition on ϕ is essential to the proof)**.** *Let $(\phi_k)_{k\in\mathbb{Z}^d} \equiv \phi \in \ell^2(\mathbb{Z}^d) \equiv L^2(D)$; notice that $\phi_0 = \langle \phi, e_0\rangle_{L^2} = \int_D \phi\, d\lambda$. ϕ_0 is a constant and $e^{-\frac{1}{2}|\mathbf{0}|^d\phi_0^2} d\phi_0 = d\phi_0$ is not a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. But*

$\phi_0 = 0 \Leftrightarrow \int_D \phi \, d\lambda = 0$ *and we are allowed to fix* ϕ_0 *as we prefer because* $(\phi_k)_{k\neq 0} \in \ell^2 \Leftrightarrow (\phi_k)_k \in \ell^2$, *then we fix* $\phi_0 = 0$.
Moreover $\langle -\Delta\phi, \phi\rangle = \int_D |\nabla\phi|^2 \, d^2\xi = 0 \Leftrightarrow \phi$ *is constant then we need this restriction on "*ϕ*-space" to define the mesure* $e^{\langle -\Delta\phi,\phi\rangle} d\phi$.

2.7.2 Concentration of the measure in L^2

We recall the Notation 2.7.4 and we restrict our attention to the measure of $L^2_0(\mathbb{T}^d)$.

Proposition 2.7.6.

$$\mu(\ell^2_0(\mathbb{Z}^d)) = 0 \tag{2.14}$$

Follows that $\mu\{\|\phi\|_2 = \infty\} = 1$

Proof. We can use exactly the proof of the formula (2.6) at page 15, because the only hypothesis we used is: $\sum_{k\in\mathbb{Z}^d} \sigma_k = +\infty$; in fact, being $\sigma_k = |k|^d$, we have $\sum_{k\in\mathbb{Z}^d} \sigma_k = \sum_{k\in\mathbb{Z}^d} \frac{1}{|k|^d} \sim \int_{\mathbb{R}^d} \frac{1}{|k|^d}\, dk = \int_0^{+\infty} \frac{\rho^{d-1}}{\rho^d}\, d\rho = \int_0^{+\infty} \frac{1}{\rho}\, d\rho = +\infty$. □

2.7.3 H^s spaces

In order to find the concentration sets of the Feynman-Kač measure, we try to estimate the measure of the H^s-spaces with $s < 0$. We recall the definition:

Definition 2.7.7. *Let* $s \in \mathbb{R}$, *let us denote by* H^s*-space the set:*

$$H^s = \{(\phi_k)_{k\in\mathbb{Z}^d\setminus\{0\}} \in \mathbb{R}^\infty \mid \sum_{k\in\mathbb{Z}^d} (1 + |k|^d)^s \phi_k^2 < \infty\}.$$

Proposition 2.7.8. $\mu(H^{-s}) = 1 \; \forall s > 0$.

Proof. The proof is the same: surely s^* is the smaller positive number s such that

$$\int_{\mathbb{R}^\infty} \|\phi\|^2_{H^{-s}}\, d\mu < \infty.$$

Imposing this condition we have:

$$\int_{\mathbb{R}^\infty} \|\phi\|^2_{H^{-s}}\, d\mu = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (1+|k|^d)^{-s}\sigma_k = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \frac{(1+|k|^d)^{-s}}{|k|^d} \leq \sum_{k\in\mathbb{Z}^d\setminus\{0\}} (|k|^d)^{-s-1}$$

and this quantity is $\sim \int_{\mathbb{R}^d\setminus(-1,1)^d} (|x|^d)^{-(s+1)} dx_1 dx_2 \dots dx_d =$
$= \omega_d \int_1^\infty \frac{\rho^{d-1}}{\rho^{d(1+s)}} d\rho = \omega_d \int_1^\infty \frac{1}{\rho^{(ds+1)}} d\rho$. And $\int_1^\infty \frac{1}{\rho^{1+ds}} d\rho < \infty \Leftrightarrow 1 + ds > 1 \Leftrightarrow$
$s > 0$. □

We wish to investigate the case $\phi : D \subset \mathbb{R}^d \to \mathbb{R}^n$ where D is a region with compact closure and piecewise $\mathcal{C}^\infty$ boundary, and not only $D = \mathbb{T}^d$ and $n = 1$. The following Chapter, in addition to the definition of Polyakov measure, gives as consequence the above-mentioned generalization.

Chapter 3

Polyakov measure

In his article *Quantum Geometry of Bosonic Strings* [1], Polyakov starts his analysis from the research of the minimum of the Nabu-Goto Action, then he notices that there is another functional, which gives the same results. This functional is the Polyakov functional and is essential in the construction of the measure.

The first section provides a detailed exposition with proofs of the preliminary statements, present in [1], necessary to the definition of the measure. Polyakov measure will be defined in Section 2. We will define the gaussian part of the measure, clearly applying the results of Chapter 1.

3.1 Nabu-Goto action vs Polyakov action

3.1.1 Nabu-Goto action

String Theory (and in particular, Polyakov Action [1]) studies the area action functional:

$$A = \int_D d^2\xi \, (\det\|h_{ab}\|)^{\frac{1}{2}} \qquad h_{ab} = \partial_a \phi^\mu \partial_b \phi^\mu, \tag{3.1}$$

where we use the Einstein notation, in other words $h_{ab} = \sum_{\mu=1}^{n} \partial_a \phi^\mu \partial_b \phi^\mu$. The computation is over the unitary circle D [1], and $\phi : D \to \mathbb{R}^n$ is the parameterization (we can suppose it to be in $\mathcal{C}^\infty$); $\Sigma = \phi(D)$ is the surface over which we compute the area with the formula (3.1). In String Theory A is called **Nabu-Goto action** [3].

[1] In some proof we will apply some theorems, for example the Divergence theorem or the Fundamental lemma in the calculus of variation, then we need some hypothesis to be fulfilled: D compact subset of $\mathbb{R}^d$ (in our case $d = 2$, but always remember that in any case $d \geq 2$) with piecewise $\mathcal{C}^\infty$ boundary (or even without boundary).
Moreover, later, D will be a compact 2-dim Riemannian manifold with asociated metric g, this means that D has to be a compact topological space that is locally euclidean. Being ϕ a diffeomorphism we can talk about D as it was $\phi(D)$.

The formula for the area in effect is:

$$A(\phi) = \int_D \sqrt{\det[J^t J]}\, d^2\xi,$$

and obviosly J is the jacobian: ϕ'.
We would like to find the stationary point of this functional.

Proposition 3.1.1 (Stationary point of A)**.** *Let $\phi, \delta\phi : D \to \mathbb{R}^n$ where D is a region in $\mathbb{R}^d$, $\phi \in \mathcal{C}^k$, $k \geq 2$, $\delta\phi \in \mathcal{C}^h$, $h \geq 1$ and $\delta\phi \mid_{\partial D} \equiv 0$,*

$$\delta A = \frac{1}{2} \int \sqrt{h} h^{ab} \delta h_{ab}$$

where $\delta h_{ab} = \partial_{(a}\phi^\mu \partial_{b)} \delta\phi^\mu$ denotes the "symmetrization of indexes":

$$\begin{aligned} \delta h_{ab} &= \partial_{(a}\phi^\mu \partial_{b)} \delta\phi^\mu = \\ &= \partial_a \phi^\mu \partial_b(\delta\phi^\mu) + \partial_b \phi^\mu \partial_a(\delta\phi^\mu) \end{aligned}$$

Notation 3.1.2. *$\phi^\mu(\xi_1, \xi_2)$ is the parameterization[2].*

Proof. Let $v : D \to \mathbb{R}^n$ with $v \mid_{\partial D} \equiv 0$, and recall that $\phi\prime^t \phi\prime = (h_{ab})_{ab}$ and also $h^{ab} = (h^{-1})_{ab}$. We introduce the notation $h = \det(\phi\prime^t \phi\prime)$.
A stationary point must satisfy the condition:

$$\frac{d}{dt} A(\phi + t \cdot v) \mid_{t=0} = 0.$$

Explicating calculations and remembering the differential of the funcion *det* we have:

$$\begin{aligned} &\frac{d}{ds} A(\phi + sv) \mid_{s=0} = \int_D \sqrt{\det[(\phi + s\cdot v)\prime^t (\phi + s\cdot v)\prime]}\, d^2\xi \mid_{s=0} = \\ &= \int_D \frac{1}{2} \frac{\frac{d}{ds}\det[(\phi + s\cdot v)\prime^t (\phi + s\cdot v)\prime].[\frac{d}{ds}((\phi + s\cdot v)\prime^t (\phi + s\cdot v)\prime)]}{\sqrt{\det[(\phi + s\cdot v)\prime^t (\phi + s\cdot v)\prime]}}\, d^2\xi \mid_{s=0} = \\ &= \int_D \frac{1}{2} \frac{\frac{d}{ds}\det[(\phi + s\cdot v)\prime^t (\phi + s\cdot v)\prime].[\frac{d}{ds}(\phi\prime t \phi\prime + s\cdot v\prime^t \phi\prime + s\phi\prime^t v\prime + s^2 v\prime^t v\prime)]}{\sqrt{\det[\phi\prime^t(\phi)\prime]}}\, d^2\xi \mid_{s=0} \\ &= \int_D \frac{1}{2} \frac{\frac{d}{ds}\det[(\phi + s\cdot v)\prime^t (\phi + s\cdot v)\prime].[v\prime^t \phi\prime + \phi\prime^t v\prime + 2s\cdot v\prime^t v\prime)]}{\sqrt{\det[\phi\prime^t(\phi)\prime]}}\, d^2\xi \mid_{s=0} \\ &= \int_D \frac{1}{2} \frac{\det[\phi\prime^t \phi\prime]}{\sqrt{\det[\phi\prime^t(\phi)\prime]}} \mathrm{tr}[(\phi\prime^t \phi\prime)^{-1}(v\prime^t \phi\prime + \phi\prime^t v\prime)]\, d^2\xi \end{aligned}$$

Now can be used the notation recalled above for $\det(\phi\prime^t \phi\prime)$ to write:

$$\begin{aligned} \frac{d}{ds} A(\phi + sv) \mid_{s=0} &= \int_D \frac{1}{2} \sqrt{\det[\phi\prime^t \phi\prime]}\ \mathrm{tr}[(\phi\prime^t \phi\prime)^{-1}(v\prime^t \phi\prime + \phi\prime^t v\prime)]\, d^2\xi \\ &= \frac{1}{2} \int_D \sqrt{h}\ \mathrm{tr}[(\{h_{ab}\}_{a,b=1,\ldots,d})^{-1}(v\prime^t \phi\prime + \phi\prime^t v\prime)]\, d^2\xi. \end{aligned}$$

[2]There is a typing error in the second line, second page, of the article [1]

We are reduced to study $\mathrm{tr}[(\{h_{ab}\}_{a,b=1,\dots,d})^{-1}(v\prime^t\phi\prime + \phi\prime^t v\prime)]$.

$$\mathrm{tr}[(\{h_{ab}\}_{a,b=1,\dots,d})^{-1}(v\prime^t\phi\prime + \phi\prime^t v\prime)] = \sum_{a=1}^{d}\left((\phi\prime^t\phi\prime)^{-1}(v\prime^t\phi\prime + \phi\prime^t v\prime)\right)_{aa} =$$

$$= \sum_{a=1}^{d}\left[\sum_{b=1}^{d}\left(\left((\phi\prime^t\phi\prime)^{-1}\right)_{ab}(v\prime^t\phi\prime + \phi\prime^t v\prime)_{ba}\right)\right] = \sum_{a,b=1}^{d}\left[h^{ab}(v\prime^t\phi\prime + \phi\prime^t v\prime)_{ba}\right].$$

In particular the quantity in parentheses can be written as:

$$(v\prime^t\phi\prime + \phi\prime^t v\prime)_{ba} = \sum_{\mu=1}^{n}(v\prime_{\mu b}\phi\prime_{\mu a} + \phi\prime_{\mu b}v\prime_{\mu a}) = \sum_{\mu=1}^{n}(\partial_b v^\mu\partial_a\phi^\mu + \partial_b\phi^\mu\partial_a v^\mu) =$$

$$= \sum_{\mu=1}^{n}(\partial_b(\delta\phi^\mu)\partial_a\phi^\mu + \partial_b\phi^\mu\partial_a(\delta\phi^\mu)),$$

where we noticed that $\delta\phi = \frac{d}{ds}(\phi + t\cdot v) = v$ e quindi $\delta\phi^\mu = v^\mu$.
Finally the minimal area is exactly the value in the unique critical point and this establishes the desired formula:

$$0 = \delta A = \frac{1}{2}\int_D \sqrt{h}\sum_{a,b=1}^{d}\left[h^{ab}\sum_{\mu=1}^{n}(\partial_b(\delta\phi^\mu)\partial_a\phi^\mu + \partial_b\phi^\mu\partial_a(\delta\phi^\mu))\right]. \qquad (3.2)$$

□

Theorem 3.1.3 (The Eulero-Lagrange equation)**.** *Now we suppose also that $\phi \in \mathcal{C}^k,\ k \geq 2$. Integrating by parts (3.2) follows the Eulero-Lagrange equation:*

$$\partial_a\left(\sqrt{h}\,h^{ab}\partial_b\phi^\mu\right) = 0 \qquad h_{ab} = \partial_a\phi^\mu\partial_b\phi^\mu,\ \ \mu = 1,\dots,n. \qquad (3.3)$$

In the proof will be used the following theorems: the Divergence theorem and the Fundamental lemma in the calculus of variations, in particular we use a more general formulation:

Lemma 3.1.4 (The du Bois-Reymond lemma)**.** *Suppose that f is a locally integrable function defined on an open set $D \in \mathbb{R}^d$. If*

$$\int_D f(x)h(x)dx = 0 \ \textit{ for all } h \in C_{00}^\infty(D),$$

then $f(x) = 0$ for almost all $x \in D$. $C_{00}^\infty(D)$ is the space of all infinitely differentiable functions defined on D whose support is a compact set contained in D.

In our theorem the functions are $\mathbb{R}^n-$valued but this is not a problem: if $f = (f^1, f^2, \dots, f^n)$ we can apply the theorem to each $f^\mu : D \to \mathbb{R}$.

Proof. Consider (3.2) which is $\delta A = 0$ then:

$$\begin{aligned} 0 &= \frac{1}{2}\int_D \sqrt{h} \sum_{a,b=1}^{d} \left[h^{ab} \sum_{\mu=1}^{n} (\partial_b(\delta\phi^\mu)\partial_a\phi^\mu + \partial_b\phi^\mu\partial_a(\delta\phi^\mu)) \right] \\ &= \frac{1}{2}\sum_{a,b=1}^{d}\sum_{\mu=1}^{n}\int_D \sqrt{h}\left[h^{ab}\left(\partial_b(\delta\phi^\mu)\partial_a\phi^\mu + \partial_b\phi^\mu\partial_a(\delta\phi^\mu)\right)\right]. \end{aligned} \tag{3.4}$$

Let us apply the Divergence Theorem: define $w = (w_i)_i$, $w_i = \sqrt{h}\, h^{ai}\partial_a\phi^\mu\, v^\mu$

$$\int_D \nabla\cdot w = \int_{\partial D} w\cdot\vec{n} \overset{by\ def}{\Leftrightarrow} \int_D \sum_{b=1}^{d}\partial_b w_b = \int_{\partial D}\sum_b w_b n_b$$

$$\int_D \sum_{b=1}^{d}\partial_b(\sqrt{h}\, h^{ab}\partial_a\phi^\mu\, v^\mu) = \int_{\partial D}\sum_b(\sqrt{h}\, h^{ab}\partial_a\phi^\mu\, v^\mu)n_b \overset{v|_{\partial D}\equiv 0}{=} 0$$

then we have:

$$\begin{aligned} 0 &= \int_D \sum_{b=1}^{d}\partial_b(\sqrt{h}\, h^{ab}\partial_a\phi^\mu\, v^\mu) \\ &= \int_D \sum_{b=1}^{d}\left[\partial_b\left(\sqrt{h}\, h^{ab}\partial_a\phi^\mu\right)\, v^\mu + (\sqrt{h}\, h^{ab}\partial_a\phi^\mu)\partial_b v^\mu\right] \Rightarrow \\ &\Rightarrow \int_D \sum_{b=1}^{d}\left(\sqrt{h}\, h^{ab}\partial_a\phi^\mu\partial_b v^\mu\right) = -\int_D \sum_{b=1}^{d}\left[\partial_b\left(\sqrt{h}\, h^{ab}\partial_a\phi^\mu\right)\, v^\mu\right] \overset{v|_{\partial D}\equiv 0}{=} 0 \\ &\Rightarrow \int_D \sum_{b=1}^{d}\left(\sqrt{h}\, h^{ab}\partial_a\phi^\mu\partial_b v^\mu\right) = 0. \end{aligned}$$

But v is arbitrary, then the Fundamental lemma in the calculus of variations completes the proof. □

3.2 Polyakov action

From now on we consider the functional

$$\begin{aligned} W\left((\phi^\mu)_\mu, (g_{ab})_{ab}\right) &= \frac{1}{2}\int \sqrt{g} g^{ab}\partial_a\phi^\mu\partial_b\phi^\mu d^2\xi \\ &= \frac{1}{2}\int\sqrt{g}\sum_{a,b=1}^{d}\sum_{\mu=1}^{n} g^{ab}\partial_a\phi^\mu\partial_b\phi^\mu d^2\xi \end{aligned}$$

where g is a metric. In String Theory W is called **Polyakov action**. When we introduce a metric, (D, g) becomes a compact Riemannian manifold, and from

now on, will be connected and without boundary (this happens even if $D = \mathbb{T}^2$ or D is the unitary circle: we can consider it so).

There are lots of reasons that allow us to introduce this new functional instead of the area (3.1) but this is the argument of the following subsection, now we only notice that, at least formally, it brings to the same conclusions, that is to say that preserves the physical features. For one thing we recover the Eulero-Lagrange equation and then that the stress-energy-momentum tensor is equal to zero.

Proposition 3.2.1 (Eulero-Lagrange equation)**.** *See the statement for the formula (3.3) at page 28.*

Remark 3.2.2. *Variating ϕ formally we have (3.3), however there is g instead of h. In fact formally the formulas are equal, but g doesn't depend on ϕ. The solution will be linear equation with respect to ϕ.*

Proof.

$$\begin{aligned}
0 &= \frac{d}{dt} W(g, \phi + t\, v) \mid_{t=0} \\
&= \frac{1}{2} \int \sqrt{g} \sum_{a,b,\mu} \frac{d}{dt} \left[g^{ab} \partial_a (\phi^\mu + t\, v^\mu) \partial_b (\phi^\mu + t\, v^\mu) \right] \mid_{t=0} d^2\xi \\
&= \frac{1}{2} \int \sqrt{g} \sum_{a,b,\mu} g^{ab} \left[\partial_a v^\mu \partial_b (\phi^\mu + t\, v^\mu) + \partial_a (\phi^\mu + t\, v^\mu) \partial_b v^\mu \right] \mid_{t=0} d^2\xi \\
&= \frac{1}{2} \int \sqrt{g} \sum_{a,b,\mu} g^{ab} \left[\partial_a v^\mu \partial_b \phi^\mu + \partial_a \phi^\mu \partial_b v^\mu \right] d^2\xi
\end{aligned}$$

It's now evident that (3.4), is the same formula but, instead of h (which is the metric induced by the euclidean metric), there is the metric g. Eulero-Lagrange equations follow by the same method as in the proof for the minimal area (3.1). □

Proposition 3.2.3 (The stress-energy-momentum tensor is equal to zero)**.** *Doing the variation with respect to the metric, we have:*

$$0 = \sum_{\mu=1}^{n} \left(\partial_c \phi^\mu \partial_d \phi^\mu - \frac{1}{2} \sum_{a,b} g_{dc} g^{ab} \partial_a \phi^\mu \partial_b \phi^\mu \right) \tag{3.5}$$

Proof. Consider Polyakov action:

$$W = \frac{1}{2} \int \sqrt{\det g} g^{ab} \partial_a \phi^\mu \partial_b \phi^\mu d^2\xi = \frac{1}{2} \int \sqrt{\det g} \sum_{a,b,\mu} g^{ab} \partial_a \phi^\mu \partial_b \phi^\mu d^2\xi$$

We calculate the variation of $W(\phi, g)$ w.r.t. g, that is to say:

$$\frac{d}{dt} W(\phi, g + tr)|_{t=0} = 0 \qquad \text{with } r|_{\partial D} \equiv 0,$$

we obtain

$$
\begin{aligned}
0 &= \frac{1}{2}\frac{d}{dt}\int_D \sqrt{\det(g+tr)}\sum_{a,b,\mu}(g+tr)^{-1}_{ab}\partial_a\phi^\mu\partial_b\phi^\mu d^2\xi\Bigg|_{t=0} \\
&= \frac{1}{2}\int_D \frac{1}{2}\frac{det'(g+tr)r}{\sqrt{det(g+tr)}}\left(\sum_{a,b,\mu}(g+tr)^{-1}_{ab}\partial_a\phi^\mu\partial_b\phi^\mu\right)d^2\xi \quad + \\
&\quad + \frac{1}{2}\int_D \sqrt{\det(g+tr)}\left(\sum_{a,b,\mu} -(g+tr)^{-1}\frac{d}{dt}(g+tr)(g+tr)^{-1}\right)_{ab}\partial_a\phi^\mu\partial_b\phi^\mu \\
&= \frac{1}{2}\int_D \frac{1}{2}\sqrt{\det g}tr(g^{-1}r)\sum_{a,b,\mu} g^{ab}\partial_a\phi^\mu\partial_b\phi^\mu \quad + \\
&\quad + \frac{1}{2}\int_D \sqrt{\det g}\sum_{a,b,\mu}(-g^{-1}rg^{-1})_{ab}\partial_a\phi^\mu\partial_b\phi^\mu d^2\xi \\
&= \frac{1}{2}\int_D \sqrt{\det g}\sum_{a,b,\mu}\left[\frac{1}{2}tr(g^{-1}r)g^{ab} - (g^{-1}rg^{-1})_{ab}\right]\partial_a\phi^\mu\partial_b\phi^\mu d^2\xi.
\end{aligned}
$$

We now estimate separately some terms:

$$
(g^{-1}r)_{cd} = \sum_{j=1}^{d}(g^{-1})_{cj}r_{jd} \quad \text{then} \quad tr(g^{-1}r) = \sum_{i=1}^{d}\sum_{j=1}^{d} g^{ij}r_{ji};
$$

another term is $(g^{-1}rg^{-1})_{cd}$:

$$
(g^{-1}rg^{-1})_{cd} = \sum_{i=1}^{d}(g^{-1}r)_{ci}(g^{-1})_{id} = \sum_{i,j} g^{ci}r_{ji}g^{id}.
$$

We replace these new expression to the previous ones and, doing the products and collecting

$$
0 = \frac{1}{2}\int_D \left[\frac{1}{2}\sum_{i,j}\sum_{a,b,\mu}\sqrt{detg}\left(\frac{1}{2}g^{ij}r_{ji}g^{ab} - g^{aj}r_{ij}g^{ib}\right)\right]\partial_a\phi^\mu\partial_b\phi^\mu d^2\xi.
$$

In $\sum_{i,j}\sum_{a,b,\mu} g^{aj}r^{ij}g^{ib}$ I can exchange i and j because the matrix $g^{-1}rg^{-1}$ is symmetric, and then picking up r_{ij}, we have:

$$
\begin{aligned}
0 &= \frac{1}{2}\int_D \left[\frac{1}{2}\sum_{i,j}\sum_{a,b,\mu}\sqrt{\det g}\left(\frac{1}{2}g^{ij}r_{ji}g^{ab} - g^{aj}r^{ij}g^{ib}\right)\right]\partial_a\phi^\mu\partial_b\phi^\mu d^2\xi \\
&= \frac{1}{2}\int_D \sum_{a,b,\mu}\sum_{i,j}\sqrt{\det g}\left(\frac{1}{2}g^{ij}r_{ji}g^{ab} - g^{ai}r_{ji}g^{jb}\right)\partial_a\phi^\mu\partial_b\phi^\mu d^\xi \\
&= \frac{1}{2}\int_D \sum_{a,b,\mu}\sum_{i,j}\sqrt{\det g}\left(\frac{1}{2}g^{ij}g^{ab} - g^{ai}g^{jb}\right)r_{ji}\partial_a\phi^\mu\partial_b\phi^\mu d^2\xi.
\end{aligned}
$$

This equality is valid for every r_{ij}, then by the Fundamental lemma in the calculus of variation

$$\sum_{a,b,\mu} \left(\frac{1}{2} g^{ij} g^{ab} - g^{ai} g^{jb} \right) \partial_a \phi^\mu \partial_b \phi^\mu = 0$$

for all i, j. With a little trick we obtain the formula (3.5):
multiplying by g_{ic}, summing over i, and recalling that $\sum_b g^{ab} g_{bc} = \delta_{ac}$ we have:

$$\begin{aligned} 0 &= \sum_i \sum_{a,b,\mu} \left(g^{ij} g_{ic} g^{ab} - g_{ic} g^{ai} g^{jb} \right) \partial_a \phi^\mu \partial_b \phi^\mu \\ &= \sum_{a,b,\mu} \left(\frac{1}{2} \delta_{cj} g^{ab} \right) \partial_a \phi^\mu \partial_b \phi^\mu - \sum_{a,b,\mu} \left(\delta_{ac} g^{jb} \right) \partial_a \phi^\mu \partial_b \phi^\mu; \end{aligned}$$

but δ_{ac} is equal to zero except when $a = c$, therefore

$$= \sum_{a,b,\mu} \left(\frac{1}{2} \delta_{cj} g^{ab} \right) \partial_a \phi^\mu \partial_b \phi^\mu - \sum_{b,\mu} g^{jb} \partial_c \phi^\mu \partial_b \phi^\mu.$$

The same method leads us to the conclusion: multiplying by g_{dj} and summing over j, we have

$$0 = \sum_j \sum_{a,b,\mu} \left(\frac{1}{2} \delta_{cj} g_{dj} g^{ab} \right) \partial_a \phi^\mu \partial_b \phi^\mu - \sum_{b,\mu} g_{dj} g^{jb} \partial_c \phi^\mu \partial_b \phi^\mu.$$

We now study separately the two addends. The second one is $\sum_{b,\mu} g_{dj} g^{jb} \partial_c \phi^\mu \partial_b \phi^\mu = \sum_{b,\mu} \delta_{db} \partial_c \phi^\mu \partial_b \phi^\mu = \sum_\mu \partial_c \phi^\mu \partial_d \phi^\mu$ being $\delta_{db} \neq 0$ only if $b = d$; the first term contains δ_{cj}, then:

$$\sum_{j,a,b,\mu} \frac{1}{2} \delta_{cj} g_{dj} g^{ab} \partial_a \phi^\mu \partial_b \phi^\mu = \sum_{a,b,\mu} \frac{1}{2} g_{dc} g^{ab} \partial_a \phi^\mu \partial_b \phi^\mu.$$

Reccollecting all the quantities we can conclude:

$$0 = \sum_{a,b,\mu} \frac{1}{2} g_{dc} g^{ab} \partial_a \phi^\mu \partial_b \phi^\mu - \sum_\mu \partial_c \phi^\mu \partial_d \phi^\mu.$$

□

3.2.1 Polyakov Action advantages

For first thing we recall that (h_{ab}) is a metric on $\phi(D)$ induced by the Euclidean metric of $\mathbb{R}^d$, that clearly depends on ϕ; moreover the metric is not intrinsically determined on D: we have $A(\phi) = A(\phi \circ \psi)$ for any $\psi : D \to D$ diffeomorphism. Then we notice that if there wasn't the square root in our action functional, this could be more similar to the action studied in the previous chapter: $\int_D |\nabla \phi(x)|^2 \, dx$.

Polyakov action $W(\phi,(g_{ab}))$ depends on two independent variables, a Riemannian metric on D (g) and the parameterization ϕ. With respect to Nabu-Goto action, ϕ is no more under the square root, and Euler-Lagrange equations for W are linear equations for ϕ. The crucial fact is that Polyakov action (in Section 3.3) will allow us to use the results about the Feynman-Kač measure of the previous chapter in order to define the Polyakov measure. The construction strongly depends on the fact that $W(\phi,g)$ can be expressed in function of the Laplace-Beltrami operator Δ_g.

Another advantage of using W lies in the fact that, if the metric is conformally euclidean, we can write W such that it becomes invariant only under conformal diffeomorphism, with a coordinates change (isothermal coordinates).

The metric is conformally euclidean

Definition 3.2.4 (Conformal metrics[3])**.** *Let D be a region (open connected subset) of $\mathbb{R}^d$ and g, f two metrics defined on D, and we suppose the metrics are expressed in the same coordinates; we shall say that the two metric are conformal, if exists a positive function $\lambda(\xi)$ such that $g(\xi)=\lambda(\xi)f(\xi), \quad \forall \xi \in D$.*

Definition 3.2.5 (Conformally euclidean metric)**.** *The metric g is said to be **conformally euclidean** if it is conformal to the euclidean metric δ_{ab}: i.e. if there exist a positive function $\lambda(\xi)$ and local coordinates around any point such that $\lambda(\xi)g_{ab}(\xi)=\delta_{ab}(\xi), \quad \xi \in D$.*

Proposition 3.2.6. *If the energy-momentum tensor is equal to zero, then $g(\xi)$ and $h(\xi)$ are conformal metrics, where $h(\xi)$ is the metric given by the jacobian of the parameterization $\phi(\xi)$. In other words the metric (g_{ab}) on D and the induced metric on $\phi(D)$ are proportional.*

Proof. $h_{c,d}=\sum_\mu \partial_c\phi^\mu\partial_d\phi^\mu$, then if we consider the formula (3.5), follows:

$$g_{cd}\sum_{a,b,\mu}\frac{1}{2}g^{ab}\partial_a\phi^\mu\partial_b\phi^\mu = h_{cd}$$

If $\lambda \equiv \sum_{a,b,\mu}\frac{1}{2}g^{ab}\partial_a\phi^\mu\partial_b\phi^\mu$, the previous equation is equal to $g_{cd}=\lambda h_{cd}$; if the function λ is positive, g and h are conformal.
In fact λ is positive because all addends are positive. □

Fact 3.2.7. *Polyakov Action is invariant under conformal transformations:*

$$W(\phi,g)=W(\phi,\lambda g) \qquad \forall\lambda>0 \tag{3.6}$$

Proof. Obvious: it is sufficient to note that the λs cancel out:

$$W(\phi,\lambda g)=\frac{1}{2}\int\sqrt{\lambda^d g}(\lambda g)^{ab}\partial_a\phi^\mu\partial_b\phi^\mu d^2\xi,$$

[3]Sometimes the change of coordinates is included in the definition, and there isn't the assumption that the metrics are expressed in terms of the same coordinates.

but $d = 2$ and λ is positive then $\sqrt{\lambda^2 g} = \lambda\sqrt{g}$ and $(\lambda g)^{ab} = \left((\lambda g)^{-1}\right)_{ab} = \lambda^{-1}\left(g^{-1}\right)_{ab} = \lambda g^{ab}$. Note that this is possible because $d = 2$ ($\phi : D \subset \mathbb{R}^2 \to \mathbb{R}^n$). □

In String Theory the conformal invariance is called **Weyl invariance**.

Remark 3.2.8. *At first Polyakov suppose to work in a euclidean space (immersed manifolds). In [1] Polyakov conclude immediately $g_{ab} = h_{ab}$ and not $g_{ab} = \lambda h_{ab}$.*
Maybe he left out λ because of the Weyl Invariance.

It's useful to explain better what is the meaning of the sentence "note that it is possible because *dim* $= 2$ ($\phi : D \to \mathbb{R}^n$)".

Fact 3.2.9. *Let g and h metrics, if g and h are conformal ("$g \sim h$") then $\lambda = \left(\frac{det(g)}{det(h)}\right)^{\frac{1}{d}}$ where $\lambda : D \to \mathbb{R}^{>}$ is the positive function such that $g = \lambda h$.*

Proof. $g \sim h \Leftrightarrow g_{ab} = \lambda h_{ab} \quad \forall a, b = 1, \dots, d$, then $\det g = \lambda^d \det h \Rightarrow \lambda = \left(\frac{\det(g)}{\det(h)}\right)^{\frac{1}{d}}$.
Moreover $g \sim h \Leftrightarrow \frac{g_{ab}}{(\det g)^{\frac{1}{d}}} = \frac{h_{ab}}{(\det h)^{\frac{1}{d}}}$. Then if $h_{ab} = \delta_{ab}$, the metric is conformally euclidean and it happens iff $\frac{g_{ab}}{(\det g)^{\frac{1}{d}}} = \delta_{ab}$ □

Corollary 3.2.10. $d = 2$
If $g \sim h$ then the Nabu-Goto action A is equal to the Polyakov Action W.

Proof. Recall that $A = \int \sqrt{h}\, d^2\xi$ and $W = \frac{1}{2}\int \sqrt{g} g^{ab} \partial_a \phi^\mu \partial_b \phi^\mu d^2\xi$, and $h_{ab} = \partial_a \phi^\mu \partial_b \phi^\mu$.
$g \sim h \Leftrightarrow g_{ab} = \lambda h_{ab} \quad \forall a, b$.
Therefore

$$\begin{aligned} W &= \frac{1}{2}\int \lambda\sqrt{g}\left((\lambda h)^{-1}\right)_{ab} \partial_a \phi^\mu \partial_b \phi^\mu d^2\xi \\ &= \frac{1}{2}\int \sqrt{g}\lambda^{-1}\left(h^{-1}\right)_{ab} h_{ab} d^2\xi \quad \text{but } \lambda^{-1} \text{ is equal to } \frac{\sqrt{h}}{\sqrt{g}} \\ &= \frac{1}{2}\int \sqrt{h} h^{ab} h_{ab} d^2\xi = \frac{1}{2}\int \sqrt{h}\,\mathrm{tr}(\mathbb{I}) d^2\xi \\ &= \frac{1}{2}\int 2\sqrt{h} d^2\xi = \int \sqrt{h} d^2\xi = A \end{aligned}$$

□

Theorem 3.2.11. *It is possible to change coordinates in such a way that the metric becomes conformally euclidean. A the change of coordinates $f = (f_1, f_2)$ can be given, following Polyakov, by the solution of $(\partial_a f_1, \partial_a f_2) = (-\epsilon_{ac}\sqrt{g} g^{cb} \partial_b f_2, \epsilon_{ac}\sqrt{g} g^{cb} \partial_b f_1)$, where $\epsilon_{ac} = \begin{cases} 1 \text{ if } (a,c) = (1,2) \\ 0 \text{ if } a = c \\ -1 \text{ if } (a,c) = (2,1) \end{cases}$.*

Remark 3.2.12. *The theorem can be stated in this way:* Every metric g on a 2-dimensional surface, is conformally euclidean: exist local coordinates for the surface such that the metric is conformally euclidean.*(See Theorem 13.1.1. page 110 [5])*
This theorem is what we need in fact our hypothesis are: (D, g) a 2-dim Riemannian manifold. We will follow only the initial part of the proof of the statement (for a complete proof look in [5]) to show that f can be made to satisfy exactly the conditions imposed by Polyakov.

Proof. First notice that

$$\begin{aligned} &g_{11}d(x^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2 = \\ &= \left(\sqrt{g_{11}}dx^1 + \frac{g_{12} + i\sqrt{g}}{\sqrt{g_{11}}}dx^2\right)\left(\sqrt{g_{11}}dx^1 + \frac{g_{12} - i\sqrt{g}}{\sqrt{g_{11}}}dx^2\right). \end{aligned}$$

We start by looking for a complex-valued function λ

$$g_{11}d(x^1)^2 + 2g_{12}dx^1dx^2 + g_{22}(dx^2)^2 = |\lambda|^{-2}(d(f_1)^2 + d(f_2)^2).$$

Even more we look for a λ such that

$$\lambda\left(\sqrt{g_{11}}dx^1 + \frac{g_{12} + i\sqrt{g}}{\sqrt{g_{11}}}dx^2\right) = d(f_1) + id(f_2)$$

where i is the imaginary unit. In fact taking the complex conjugate of that expression and multiplying for the former equation[4] we get

$$|\lambda|^2(g_{11}d(x^1)^2 + g_{12}dx^1dx^2 + g_{22}d(x^2)^2) = d(f_1)^2 + d(f_2)^2$$

So, from our condition we get

$$\begin{cases} \lambda\sqrt{g_{11}} & = \frac{\partial f_1}{\partial x^1} + i\frac{\partial f_2}{\partial x^1} \\ \lambda\frac{g_{12}+i\sqrt{g}}{\sqrt{g_{11}}} & = \frac{\partial f_1}{\partial x^2} + i\frac{\partial f_2}{\partial x^2} \end{cases}$$

That is, eliminating λ from the computation

$$(g_{12} + i\sqrt{g})\left(\frac{\partial f_1}{\partial x^1} + i\frac{\partial f_2}{\partial x^1}\right) = g_{11}\left(\frac{\partial f_1}{\partial x^2} + i\frac{\partial f_2}{\partial x^2}\right)$$

Separating the real and imaginary parts, the equation splits in the system

$$\frac{\partial f_2}{\partial x^1} = \frac{g_{12}\frac{\partial f_1}{\partial x^1} - g_{11}\frac{\partial f_1}{\partial x^2}}{\sqrt{g}} \qquad \frac{\partial f_2}{\partial x^2} = \frac{g_{22}\frac{\partial f_1}{\partial x^1} - g_{12}\frac{\partial f_1}{\partial x^2}}{\sqrt{g}}$$

$$\frac{\partial f_1}{\partial x^1} = \frac{g_{11}\frac{\partial f_2}{\partial x^2} - g_{12}\frac{\partial f_2}{\partial x^1}}{\sqrt{g}} \qquad \frac{\partial f_1}{\partial x^2} = \frac{g_{12}\frac{\partial f_2}{\partial x^2} - g_{22}\frac{\partial f_2}{\partial x^1}}{\sqrt{g}}$$

which is exactly the condition imposed by Polyakov (it suffices to replace g_{ij} with its expression in terms of g^{cd}, and to consider $-f$ instead of f). □

[4] In the sense of symmetric tensor product

Fact 3.2.13. *In the isothermal coordinates*[5] *z Polyakov Action becomes:*

$$W(z) = \frac{1}{2} \int \partial_a \phi^\mu(z) \partial_b \phi^\mu(z) \, d^2 z$$

Proof. Immediate by the facts that, if $\xi = f(z) = (f_1(z), f_2(z))$ is the change which makes the metric conformally euclidean, the volume form $\sqrt{g} d^2\xi$ becomes $\lambda^{-1}(z) d^2 z$ (λ is the conformal factor) and $\sum_{a,b} g^{ab} \partial_a \phi^\mu \partial_b^\mu = |\nabla \phi^\mu|_g^2$ it's invariant under change of coordinates. □

3.3 Polyakov Measure

Recall our purpose: Polyakov in [1] says the "the most immediate problem is to define the proper measure for the summation over continuous surfaces".
We now try to define the measure

$$e^{-W(\phi, g)} d\phi.$$

where we have fixed the metric g.[6]

In this section we assume that $M = D$ [7] is a compact d-dimensional Riemannian manifold, $\{g_{ab}\}_{ab}$ a riemannian metric (then it is positive-definite).

Lemma 3.3.1. *Suppose that (M, g) is a compact Riemannian manifold embedded (but non necessarily isometrically) in some $\mathbb{R}^n$. Then the metric is uniformely definite-positive, i.e. there exist an $\alpha > 0$ (depending only on M, g and the embedding) such that*

$$g^{ab}(x) \xi^a \xi^b \geq \alpha ||\xi||^2$$

for each $x \in M$ and $\xi \in T_x M$.

Proof. Assume that for all $\alpha > 0$ exists $x \in M$ and $0 \neq \xi \in T_x M$ such that $0 < g^{ab}(x) \xi_a \xi_b \leq \alpha |\xi|^2$. If we consider $\eta = \frac{\xi}{|\xi|}$, the condition is $0 < g^{ab}(x) \eta_a \eta_b \leq \alpha$. Then we can take η in the "unitary tangent space" which is compact; we denote it by TUM.
Define the continuous function $f(x, \eta) = g^{ab}(x) \eta_a \eta_b$. Take $\alpha = 1/n$ with $n \in \mathbb{N}^>$, then $\exists x_n \in M$ and $\eta \in T_{x_n} M$ such that $f(x_n, \eta) \leq \frac{1}{n}$. Therefore $f(TUM) \cap [0, 1/n] \neq \emptyset$ for all $n > 0$. Recall that a continuous image of a compact is compact, then $f(TUM)$ has to be a compact in $\mathbb{R}$ which means that is closed and limited. $\{0\} = \cap_{n \geq 1} [0, 1/n] \Rightarrow 0 \in f(TUM)$, this is impossible because $g^{ab}(x) > 0 \, \forall x \in M \Rightarrow f(x, \eta) > 0 \, \forall x \in M$ and $\eta \in TU_x M$. We can conclude that g^{ab} is uniformely positive-definite. □

[5] Isothermal coordinates on a Riemannian manifold are local coordinates where the metric is conformal to the Euclidean metric.

[6] In section 3.4 there is the statement of the complete problem.

[7] Recall that D and $\phi(D)$ are diffeomorphic manifolds, then have the same properties as manifolds.

Lemma 3.3.1 assures that the Polyakov action $W\left(\left(\phi^{\mu}\right)_{\mu},\left(g_{ab}\right)_{ab}\right) =$
$= \frac{1}{2}\int \sqrt{g}g^{ab}\partial_a\phi^{\mu}\partial_b\phi^{\mu}d^2\xi$ is positive.

We recall some definition which will be useful in the understanding of the sequel.

If (M,g) is a Riemannian manifold and f is a smooth function on M we can define the **gradient** of f as the only vector field ∇f such that

$$\langle \nabla f, X\rangle = df X$$

for each vector field X over M.

Moreover if X is a smooth vector field over M, we can define its **divergence** as the only smooth function $\nabla \cdot X$ on M such that in local coordinates it holds

$$\nabla \cdot X = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}X^i)$$

This definition is chosen in such a way that the classical divergence theorem holds.

We can now define the Laplace-Beltrami operator on a Riemannian manifold. If f is a smooth function on M we set

$$\Delta_g f = \nabla \cdot \nabla f\,.$$

In coordinates we can state the definition as follows:

Definition 3.3.2 (Laplace-Beltrami operator)**.** *Let $\phi,\psi : D \to \mathbb{R}$, we set:*

$$\langle -\Delta_g\phi,\psi\rangle = \int_D \sqrt{g}g^{ab}\partial_a\phi\partial_b\psi\; d\xi.$$

Now consider the following space:

$$L^2(M,\mathbb{R}^n) = \left\{\phi : M \to \mathbb{R}^d \mid \int_M |\phi|^2 dV < +\infty\right\}.$$

This is a scalar product space with the definition

$$\langle \phi,\psi\rangle = \int_M \langle\phi(x),\psi(x)\rangle dV = \int_M \phi(x)\cdot\psi(x)dV,$$

where dV is the volume form $\sqrt{g}d^dx$ if M is $d-$dimensional.
In this space we let Δ_g act componentwise. We claim that

$$\langle -\Delta_g\phi,\psi\rangle = \sum_{\mu}\langle -\Delta_g\phi^{\mu},\psi^{\mu}\rangle = \sum_{\mu}\int_M \langle -\Delta_g\phi^{\mu}(x),\psi^{\mu}(x)\rangle dV =$$
$$= \sum_{\mu}\int_M \langle \nabla\phi^{\mu}(x),\nabla\psi^{\mu}(x)\rangle dV$$

In terms of the Laplace-Beltrami operator Polyakov action can be written:

$$W\left(\left(\phi^{\mu}\right)_{\mu},\left(g_{ab}\right)_{ab}\right)=\frac{1}{2}\langle-\Delta_{g}\phi,\phi\rangle=\frac{1}{2}\tilde{C}(\phi,\phi),$$

where we denote by $\tilde{C}(\phi,\psi)$ the quantity $\int_D \sqrt{g}g^{ab}\partial_a\phi^{\mu}\partial_b\psi^{\mu}\ d\xi$.[8]

We are interested in defining the measure

$$e^{-W(\phi,g)}d[\phi]d[g],$$

where $d[\phi]$ and $d[g]$ are formal measure over all parameterizations and all metrics. If g is fixed, then this measure becomes a sort of gaussian:

$$e^{-\frac{1}{2}\langle-\Delta_{g}\phi,\phi\rangle}\,d\phi. \tag{3.7}$$

Remark 3.3.3. *This is a generalization of the Feynman-Kač measure, and the metric is no more the euclidean one. In that case we had a problem (see Remark 2.7.5). If the metric $g_{ab}=\delta_{ab}$ then $g^{ab}=\delta^{ab}$ and we have $\tilde{C}(\phi,\phi)=\langle\Delta_{g}\phi,\phi\rangle=0\Leftrightarrow$ [9] $\int_D|\nabla\phi|_{g}^{2}d\xi=0\Rightarrow|\nabla\phi|_{g}=0\Rightarrow d\phi=0\Rightarrow\phi\equiv cost$. These possibility must be prevented because, $e^{-\frac{1}{2}\tilde{C}(\phi,\phi)}d^{\infty}\phi$ won't be a probability measure, as has already been noticed.*

To obviate this eventuality we assume that $\phi\in S\equiv\{\phi:D\to\mathbb{R}^{n}|\phi\neq$ costant not null $\}$, in particular, ϕ is supposed to be in $L^{2}\cap S$. It is convenient to call $L^{2}\cap S=L_{0}^{2}$.

Notation 3.3.4. *We will denote by $C=C_{g}=(-\Delta_{g})^{-1}$ the covariance matrix, then $\tilde{C}(\phi,\psi)=\langle C^{-1}\phi,\psi\rangle$.*

We try to use the same tecnique used in the definition of the Feynman-Kač measure: the basic idea of the proof is to diagonalize the Laplace-Beltrami operator.

3.3.1 Diagonalization of the Laplace-Beltrami operator

The following theorems are stated without proof.

Theorem 3.3.5 (Sturm-Liouville decomposition)**.** *Let M be a Riemannian manifold which is also compact; there exists a complete orthonormal basis $e_0,e_1,e_2,\ldots$ of $L^{2}(M)$ of eigenfunction e_k of Δ_g with λ_k as eigenvalue such that $0=\lambda_0\leq\lambda_1\leq\lambda_2\leq\ldots$*

Theorem 3.3.6 (Weyl's asymptotic formula)**.** *Let M be a d-dimensional Riemannian manifold, and $N(\lambda)$ be the number of eigenvalues, counted with multiplicity, such that $0=\lambda_0\leq\lambda_1,\ldots\leq\lambda$; then, as λ tends to $+\infty$*

$$N(\lambda)\sim\omega_{d}\lambda^{\frac{d}{2}}\frac{vol(M)}{(2\pi)^{d}} \tag{3.8}$$

[8] If now on we take $g_{ab}=\delta_{ab}$ and D a compact region of $\mathbb{R}^{d}$, we obtained the generalization of the measure for (2.8) in the case of a vector field with domain different from the torus.

[9] Here we use the closedness (compactness and absence of boundary).

and also, as $m \to +\infty$ the m-th eigenvalue

$$(\lambda_m)^{\frac{d}{2}} \sim \frac{(2\pi)^d}{\omega_d} \frac{m}{vol(M)}.$$

Having an orthonormal basis we can identify ϕ with its Fourier coefficients: $\phi_k = \int_D \phi e_k \, dV$ where dV is the volume form, and $k \in \mathbb{N}$ (following the enumeration given by the Sturm-Liouville theorem).

Remark 3.3.7. *The Sturm-Liouville theorem tells us that* 0 *is an eigenvalue and there is an orthonormal basis for the kernel. A function in the kernel satisfies the condition seen in remark 3.3.3 (i.e. $\tilde{C}(\phi, \phi) = 0$) then is constant. Therefore the Kernel has dimension one and is the set of all constant functions. So we are confirmed in our choise of L^2_0 as the domain, but we would like to write it differently. Observe that $1 = e_0$ then $\phi_0 = \int_D \phi \, dV$ which is a constant and we call it "mean", then we can choose $L^2_0(D) = \left\{\phi \in L^2(D) | \phi \text{ has null mean}\right\}$.*

3.3.2 Definition of the measure

We have just seen that the Laplace-Beltrami operator C can be diagonalized, then the construction of the measure can be developed as in the euclidean case. If λ_k is the eigenvalue with eigenfunction e_k for all $k \in \mathbb{N}$ and $\{e_k\}_k$ is an orthonormal basis, we can identify ϕ with its Fourier coordinates $(\phi_k)_k$, where $\phi_k = \int_D \phi e_k dV$. If D is a d-dimensional riemannian manifold, we can identify the space $L^2_0(D)$ with the space $\ell^2_0(\mathbb{N}) = \{(\phi_k)_{k\in\mathbb{N}} \in \ell^2(D) | \phi_0 = 0\} \equiv \ell^2(\mathbb{N}\backslash\{0\})$.

In analogy with (2.12) we have:

$$\begin{aligned} e^{-\frac{1}{2}\tilde{C}(\phi,\phi)} d^\infty\phi = e^{-\frac{1}{2}\langle -\Delta_g\phi,\phi\rangle} d^\infty\phi = e^{-\frac{1}{2}\sum_{k\in\mathbb{N}^>} \lambda_k\phi_k^2} d\phi_1 d\phi_2 \dots d\phi_n \dots \\ = \bigotimes_{k\in\mathbb{N}^>} \frac{e^{-\frac{1}{2}\lambda_k\phi_k^2}}{\sqrt{2\pi(\lambda_k)^{-1}}} d\phi_k. \end{aligned}$$

The last product it's very similar to a product of gaussian measure. If those formal equalityies had sense, the definition which gives sense to (3.7) could be provided analogously to the euclidean case:

$$e^{-\frac{1}{2}\tilde{C}(\phi,\phi)} \, d\phi = e^{-\frac{1}{2}\langle -\Delta_g\phi,\phi\rangle_{L^2}} \, d\phi = \mathcal{N}(0, (-\Delta_g)^{-1}) \, d\phi = \mathcal{N}(0, C_g) \, d\phi$$

Finally, using theorem (2.3.15), the measure is:

$$\mu(d\phi) = e^{-\frac{1}{2}\tilde{C}(\phi,\phi)} \, d\phi \tag{3.9}$$

3.3.3 Concentration of the measure

Notice that we don't have an explicit expression for e_k, but Weyl's asymptotic formula is useful to check if it is still true that the space of concentration of the measure is not L_0. In such a eventuality, the relative solution (find the space of concentation of the measure among H^s-spaces) can be used again.

The measure of L_0^2 is still zero because the proof given to (2.14) at page 25 is based only in the fact that $\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\sigma_k$ is infinite, than can easily be generalized. We recall that $\sigma_k = \frac{1}{\lambda_k}$ $(k \in \mathbb{N}^{>})$ by definition.

Lemma 3.3.8. *Let $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots$ all the ordered eigenvalues of the Laplace-Beltrami operator, the $\sum_{k=0}^{+\infty}\sigma_k = +\infty$.*

Proof. By Weyl's asymptotic formula we have $\lambda_n \sim \frac{(2\pi)^2 n^{2/d}}{(\omega_d vol(D))^{2/d}}$, therefore for n big enought, we can assume it is sufficient $n \geq m$ for a fixed m, the required sum is:

$$\sum_{n\geq m}\frac{1}{\lambda_n} \sim \sum_{n=m}^{\infty}\frac{(\omega_d vol(D))^{2/d}}{(2\pi)^2 n^{2/d}} = \frac{(\omega_d vol(D))^{2/d}}{(2\pi)^2}\sum_{n=m}^{\infty}\frac{1}{n^{2/d}} \sim K\int_m^{+\infty}\frac{dx}{x^{\frac{2}{d}}},$$

where $K = \frac{(\omega_d vol(D))^{2/d}}{(2\pi)^2}$ is a positive constant depending only on the metric g and the dimension d of the manifold. The integral converges if and only if $\frac{2}{d} > 1$ but $d \geq 2$. □

3.4 Complete Polyakov Measure

The characterization, given by Polyakov in [1], of the measure is that it must count all surfaces of a given area with the same weight, and he says that this condition leads to expression (14) (second page in [1]) for the measure.

Remark 3.4.1 (Expression (14))**.** *Set $S = \phi(D)$ Let ψ a functional on S, the measure is given by*

$$\int \psi(S)\, d\mu(S) = \int e^{-\lambda\int\sqrt{g}\, d^2\xi}\,[dg_{ab}(\xi)]\times\int e^{-\frac{1}{2}\int_D\sqrt{g}g^{ab}\partial_a\phi^\mu\partial_b\phi^\mu\, d^2(\xi)}\, d\phi(\xi)\times\psi[\phi(\xi)],$$

where $[Dg_{ab}]$ is an integration measure over all possible metrics, λ is an arbitrary parameter and the other terms are known. The expression above can be written in this way:

$$\int \psi(\phi(\xi))e^{-\frac{1}{2}\int_D\sqrt{g}g^{ab}\partial_a\phi^\mu\partial_b\phi^\mu\, d^2(\xi)}e^{-\lambda\int\sqrt{g}d^2\xi}\, dg\, d\phi.$$

In this chapter we have restricted our attention to the rigorous definition of the "gaussian part" of this measure. In other words we succeed in defining the integration over the space of all change of coordinates, but we are not sure if there exists a rigorous definition for the part of the measure which allows to integrate over the space of the metrics.
In every book or article we have read all passages are heuristic, but all report the same procedure. The idea is to use gauge's invariance[10] and the fact that in two dimensions all metrics are conformally euclidean, to change the integration

[10] Weyl's invariance and invariance under diffeomorphism (change of coordinates)

over the space of all metric in a integration over the space of all gauge's invariance. In this way we also eliminate the possibility that the integration of all possible metrics diverges because we integrate only over configuration (ϕ, g) non equivalent (we fix the metric and consider only conformal diffeomorphism). In [9] and in [11] is used the Faddeev-Popov procedure to define formally Polyakov Measure, but we aren't able to give a rigorous definition of the terms employed, which are also of difficult comprehension; we are even not sure that this is possible to do.

Appendix A

Weyl's asymptotic formula for $\mathbb{T}^2$

We present a proof for the Weyl's asymptotic formula for the 2-dimensional torus (see formula (2.11) at page 23 and consider $d = 2,\ m \equiv N(n^2) \sim \pi n^2$). We stated it without proof.

Lemma A.0.2. *Let m be the number of $k \in \mathbb{Z}^2$ such that $|k| \leq n$; then $m \sim \pi n^2$.*

Proof. The basic idea of the proof is that the estimate for m is exactly the estimate of the number of integers m contained in the unitary circle of radius n in $\mathbb{R}^2$.
We set:

$$X(r) = \{k \in \mathbb{R}^2 \mid |k| \leq r\},$$

and in particular $|X(n)| = m$. If $k = (k_1, k_2)$ we set

$$X_1(r) = \left\{ k \in \mathbb{R}^2 \mid \left[k_1 - \frac{1}{2}, k_1 + \frac{1}{2}\right) \times \left[k_2 - \frac{1}{2}, k_2 + \frac{1}{2}\right) \cap B_r \neq \varnothing \right\}.$$

$X_1(r)$ is the set of all $k \in \mathbb{Z}^2$ such that the square, of side 1 centered in k, intersect B_r, which is the ball of radius r. We also consider the set of all $k \in \mathbb{Z}^2$ such that the square of side 1 centered in k is entirely contained in B_r:

$$X_2(r) = \left\{ k \in \mathbb{R}^2 \mid \left[k_1 - \frac{1}{2}, k_1 + \frac{1}{2}\right) \times \left[k_2 - \frac{1}{2}, k_2 + \frac{1}{2}\right) \subseteq B_r \right\},$$

Trivially $X_2(r) \subseteq X(r) \subseteq X_1(r)$. Moreover we notice that the union of all the sqares of side 1 centered in k, $k \in X_2(r)$, is contained B_r, on the contrary the union of all the sqares of side 1 centered in k, $k \in X_1(r)$ contains B_r; this establishes the formula (see figure A.1):

$$|X_2(r)| \leq \pi r^2 \leq |X_1(r)|.$$

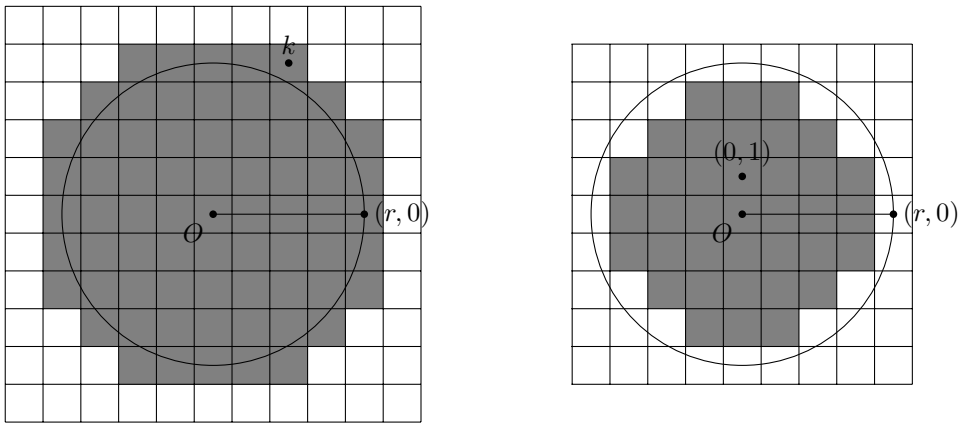


Figure A.1: Respectively $X_1(r)$ and $X_2(r)$ are the sets which contain exactly the centers ($= k = (k_1, k_2) \in \mathbb{Z}^2$) of the squares colored in the two figures.

Furthermore, if $k \in X(r)$ is an integer point ($k \in \mathbb{Z}^2$) contained also in B_r, the unitary square centered in k is entirely contained in $B_{r+\frac{\sqrt{2}}{2}}$. In fact a point in such a square is distant from k at the most $\frac{\sqrt{2}}{2}$, then at the most $|k| + \frac{\sqrt{2}}{2} \leq r + \frac{\sqrt{2}}{2}$ from the origin. Therefore $X(r) \subseteq X_2(r + \frac{\sqrt{2}}{2})$, that is

$$|X(r)| \leq \left| X_2\left(r + \frac{\sqrt{2}}{2}\right)\right| \leq \pi\left(r + \frac{\sqrt{2}}{2}\right)^2.$$

Analogically with X_1 we have the estimate

$$|X(r)| \geq \pi\left(r - \frac{\sqrt{2}}{2}\right)^2.$$

Then

$$\pi(r - \frac{\sqrt{2}}{2})^2 \leq |X(r)| \leq \pi(r + \frac{\sqrt{2}}{2})^2 \Rightarrow$$
$$\Rightarrow \pi(-\frac{1}{2} - 2\frac{\sqrt{2}}{2}r) \leq |X(r)| - \pi r^2 \leq \pi(\frac{1}{2} + \frac{\sqrt{2}}{2}r)$$

and consequently

$$\left||X(r)| - \pi r^2\right| \leq \frac{1}{2} + \frac{\sqrt{2}}{2}r \Rightarrow |X(r)| = \pi r^2 + O(r).$$

In particular, we recall that $m = |X(n)|$, and follows the lemma:

$$m = \pi n^2 + O(n)\,.$$

□

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