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Order statistics of random walks

A test of universality

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Dedicated to Paola, Gianfranco,
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Introduction

When dealing with complex systems we are required to use a statistical approach. In this context we are usually interested in the expectation values and the variances of some relevant observables since they provide information about the average behavior of the system. However, in some circumstances the knowledge of rare events statistics is much more important. For example tsunamis, tornadoes, financial crisis happen rarely but they have remarkable consequences in our lives.
For this reason the **Extreme Value Statistics** (EVS), that is the branch of statistics studying the extreme distributions of a set of random variables, plays a crucial role in many fields. Just for citing some:

Environmental science with applications to Ecology [1], the study of Hydrology and water resources [2], large wildfires [3] and so on.

Finance where one may be interested on foreseeing extreme events [4] or find the optimal time for selling a stock price [5].

Random matrix theory where the EVS is exploited for shedding light on the behavior of the largest eigenvalue [6, 7].

Statistical Physics where significant results have been found for the ground state distribution of disordered systems [8, 9].

Miscellaneous Others applications can be found in the *Engineering* [10] and in the evolution of *athletics records* [11].

These are just some of the reasons that drove many researchers to investigate the statistics of extreme events. So far, thanks to the contributions made by *Gumbel* [10], *Tippett* [12], *Fréchet* [13], *Gnedenko* [14] and so on, we have an exact theory of EVS for *i.i.d.* random variables. In particular, as we will show in the next chapter, in the thermodynamic limit three universality classes can be recognized for the **maximum distribution**. This theory is very accurate and it can be extended also to the case of *weakly correlated* random variables i.e., when the correlation length is much smaller than the system size.

Nevertheless several interesting systems involve a set of *strongly correlated* random variables. Although few problems can be solved, a background theory still doesn't exist (even for the mean value). In this context good "laboratories" for studying the EVS are the random walks, indeed all the positions together represent a set of strongly correlated random variables $\{X_0, \dots, X_n\}$.

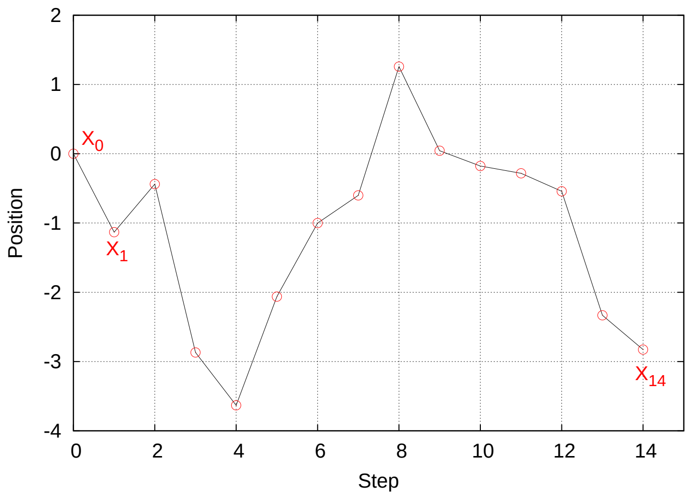
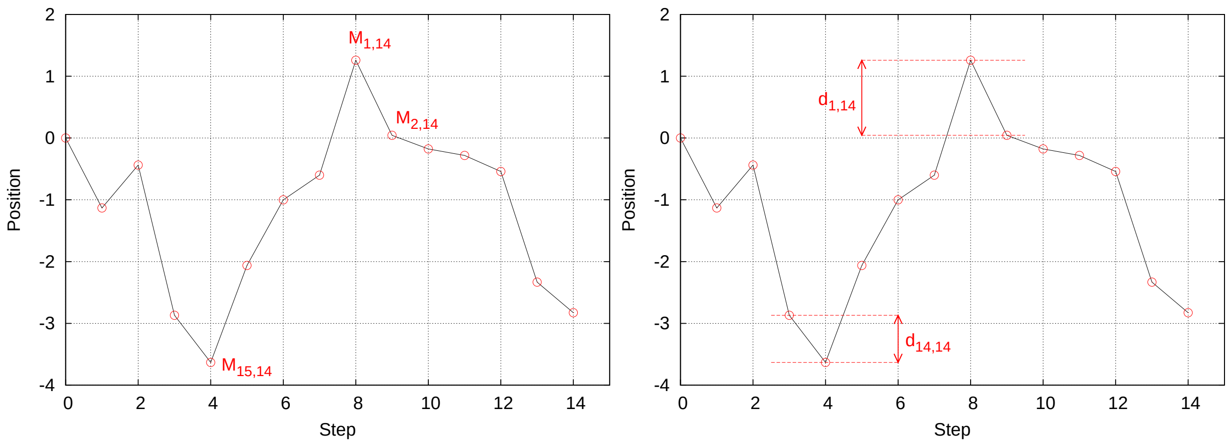


Figure 1: Random walk realization.

The analysis of a random walk requires the definition of two sets of random variables:

Maxima $\{M_{1,n}, \dots, M_{n+1,n}\}$ obtained by rearranging the positions in a decreasing order of magnitude, so that $M_{1,n} \equiv X_{\max}$ is the first maximum and $M_{n+1,n} \equiv X_{\min}$ is the first minimum.

Gaps $\{d_{1,n}, \dots, d_{n,n}\}$ which are the gaps between successive maxima: $d_{k,n} := M_{k,n} - M_{k+1,n}$.



(a) Maxima (b) Gaps

Figure 2: Random variables sets for order statistics.

The branch of EVS whose aim is to compute the maxima and gaps distributions is the **order statistics**. Some progress in this field has been recently made by *Schehr* and *Majumdar* in [15]. Using an exponential function as jump distribution of the random walk, they showed that in the thermodynamic limit[1] the **gap statistics**

[1] number of random variables n goes to infinity.

exhibits a rich universal behavior[2]. Basically they analytically demonstrated that the asymptotic behavior of the k-th gap mean value is *stationary* (independent on n) and **universal**:

$$\frac{\langle d_{k,\infty}\rangle}{\sigma} \overset{k\gg 0}{\approx} \frac{1}{\sqrt{2\pi k}} \tag{1}$$

Furthermore they proved that for *typical fluctuations*, i.e. fluctuations around the expectation value $\delta \sim k^{-\frac{1}{2}}$, the full k-th gap distribution scales as:

$$\Pr\left(d_{k,\infty} = \delta\right) \overset{\text{t.f.}}{\approx} \frac{\sqrt{k}}{\sigma} P\left(\frac{\delta\sqrt{k}}{\sigma}\right) \tag{2}$$

where the scaling function appearing in (2) is:

$$P(t) = 4\left[\frac{2}{\sqrt{2\pi}}\left(1+2t^2\right) - e^{2t^2} t\left(3+4t^2\right)\operatorname{erfc}\left(\sqrt{2}t\right)\right]$$

and it shows a surprising power law tail going as t^{-4}. Moreover they show numerical evidences for the universality of (2) starting from different jump distributions.

Thesis outlines

In this work we will extend the study of the gap statistics of a random walk made in [15], to the entire class of **Gamma distributions**. In particular we will try to retrieve the universality of the k-th gap mean value (1) and then prove or disprove the claim of universality made for the full k-th gap distribution in the scaling limit of typical fluctuations (2). Furthermore we will shed light on the relation between typical/large fluctuations and moments. Indeed if the typical fluctuations are found to be universal, as consequence some moments might be universal.

[2]it means that there is no dependence on the jump distribution.

Chapter 1

Extreme value statistics of *i.i.d.* random variables

Let's consider a physical system described by a set of *i.i.d.* random variables $\{X_1 \dots X_n\}$. The variables share the same parent distribution $p(x)$ with both the mean μ and the variance σ^2 finite. We are interested on studying the *mean* and the *extreme* statistics of the system.

For studying the average behavior we consider the observable **mean value**:

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{1.1}$$

We want to investigate the asymptotic behavior of this quantity in the large n limit. For proceeding it is necessary to know the *joint probability distribution* of the set. By definition of *i.i.d.* variables, each variable is independent from the others, hence:

$$P_n(x_1, \dots, x_n) = \prod_{i=1}^{n} p(x_i) \tag{1.2}$$

Straightforwardly the probability distribution of the mean value is given by:

$$P_{\overline{X}}(\overline{x}) = \int dx_1 \cdots \int dx_n \delta \left(\frac{1}{n} \sum_{i=1}^{n} x_i - \overline{x} \right) P_n(x_1, \dots, x_n)$$

Let's evaluate now the characteristic function of (1.1):

$$\begin{aligned} f_{\overline{X}}(k) &= \left\langle e^{ik\overline{X}} \right\rangle = \int d\overline{x} e^{ik\overline{x}} P_{\overline{X}}(\overline{x}) \\ &= \int d\overline{x} e^{ik\overline{x}} \int dx_1 \cdots \int dx_n \delta \left(\frac{1}{n} \sum_{i=1}^{n} x_i - \overline{x} \right) P_n(x_1, \dots, x_n) \\ &= \int dx_1 \cdots \int dx_n e^{i\frac{k}{n} \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} p(x_i) = \left[\int dx e^{i\frac{k}{n}x} p(x) \right]^n = \left[\left\langle e^{i\frac{k}{n}x} \right\rangle \right]^n \end{aligned}$$

In the large n limit:

$$f_{\overline{X}}(k) \overset{n\gg 0}{\approx} \left[1 + i\frac{k}{n}\langle x\rangle - \frac{k^2}{2n^2}\left\langle x^2\right\rangle\right]^n \xrightarrow{n\to\infty} e^{ik\mu}e^{-\frac{k^2\sigma^2}{2n}}$$

but this is the characteristic function of a Gaussian distribution with mean μ and variance σ^2/n, therefore:

$$P_{\overline{X}}(\overline{x}) \xrightarrow{n\to\infty} N\left[\mu, \sigma^2/n\right] = \frac{1}{\sqrt{2\pi\sigma^2/n}}e^{-\frac{1}{2}\left(\frac{\overline{x}-\mu}{\sigma/\sqrt{n}}\right)^2}$$

So it is evident that in the large n limit, whatever the parent distribution $p(x)$, the probability distribution of the mean value approaches to a Gaussian. This result is known under the name of **central limit theorem**.

Let's concern now on the *extreme* behaviour of the system. In order to do so, we define the observable **maximum**:

$$\hat{X} = \max\left(\{X_1, \ldots, X_n\}\right)$$

Let $X^\star$ be the superior limit of the domain of the probability distribution $p(x)$. One intuitively expects that in the large n limit, the maximum approaches to this value: $\hat{X} \to X^\star$. Therefore a rough way for estimating the mean value μ_n of the maximum distribution is setting the constraint that only one random variable (i.e. the maximum) stays in the interval $[\mu_n, X^\star]$ [16]:

$$P_>\left(\mu_n\right) = \int_{\mu_n}^{X^\star} p(x)dx = \frac{1}{n} \tag{1.3}$$

We want now to evaluate the asymptotic behavior of the maximum distribution. In order to do so it is convenient to work with the cumulative distribution $F_n\left(m\right)$ which is the probability that the maximum stays below m. Clearly this is also the probability that all the random variables do not exceed m, hence:

$$F_n\left(m\right) = \Pr\left[\hat{X} \le m\right] = \left[P_<(m)\right]^n = \left[\int_{-\infty}^{m} p(x)dx\right]^n$$

In order to avoid trivial results, it is necessary to define a rescaled variable which remains constant in the thermodynamic limit:

$$Y = \frac{m - b_n}{a_n} \tag{1.4}$$

In this fashion one finds the limiting distribution to be:

$$\lim_{n,m\to\infty} F_n\left(a_nY + b_n\right) = G_i(Y) \qquad\qquad i = 1,2,3 \tag{1.5}$$

As suggested in (1.5), there are three different limiting distributions which define three universality classes. They are selected on the basis of the asymptotic behavior of $p(x)$ close to the superior limit $X^\star$.

1.1 Gumbel distribution

This class is selected when the parent distribution decreases faster than any power law and no bounds are set to the domain, that is $X^\star$ can be infinity.
The limiting distribution is the so called **Gumbel's law** [10]:

$$G_1(Y) = e^{-e^{-Y}} \tag{1.6}$$

The coefficient $b_n = \mu_n$ can be evaluated using (1.3) while a_n is given by

$$a_n = \frac{\int_{b_n}^{X^\star} (x - b_n)\, p(x) dx}{\int_{b_n}^{\infty} p(x) dx}$$

and it can be interpreted as the average distance between $\hat{X}$ and b_n.

Example 1.1.1 (Exponential distribution)**.**
Let the parent distribution be $p(x) = e^{-x}$ whose domain is the positive real axis. The cumulative distribution reads:

$$F_n(x) = \left[\int_0^x e^{-x'} dx'\right]^n = \left(1 - e^{-x}\right)^n = \left(1 - \frac{1}{n} e^{-(x - \log n)}\right)^n$$

hence in the large n, x limit:

$$G_1(a_n x + b_n) = e^{-e^{-(x - \log n)}}$$

and comparing with (1.4) and (1.6), it turns out that:

$$b_n = \log n \qquad\qquad a_n = 1$$

Example 1.1.2 (Normal distribution)**.**
Another relevant example is the one of a Gaussian distribution. For instance let's consider a bunch of n non-interacting Brownian particles starting from $x = 0$ at $t = 0$. Their positions are drew from the parent distribution:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \qquad\qquad \sigma = \sqrt{2Dt}$$

Using Gumbel theory, the coefficients are found to be [16]:

$$a_n = \frac{\sigma}{\sqrt{2 \log n}} \qquad\qquad b_n = \sigma\sqrt{2\log n} - \frac{\sigma \log(\log n)}{2\sqrt{2\log n}} + O\left(\frac{1}{\sqrt{\log n}}\right)$$

so the average position of the maximum, that is the farthest Brownian particle from the origin, is:

$$x_{\max}(t) \sim \sqrt{4Dt \log n}$$

1.2 Fréchet distribution

This class includes all the parent distributions decreasing with a power law $p(x) \overset{x\to\infty}{\sim} x^{-\alpha-1}$, $\alpha > 0$ and unbounded domain ($X^\star = \infty$).
The limiting distribution is the *Fréchet* one [13]:

$$G_2(Y) = \begin{cases} e^{-Y^{-\alpha}} & Y > 0 \\ 0 & Y \leq 0 \end{cases}$$

In this case the coefficient b_n is always zero while $a_n = \mu_n$ can be computed from (1.3) and it grows as $n^{1/\alpha}$.

Example 1.2.1 (Cauchy's law)**.**
The Cauchy distribution

$$p(x) = \frac{1}{\pi\,(1 + x^2)}$$

is part of the *Fréchet* universality class, with coefficients:

$$\alpha = 1 \qquad\qquad a_n \sim \frac{n}{\pi}$$

1.3 Weibull distribution

In this class are included all the distributions with bounded domain ($X^\star < \infty$) and cumulative distribution scaling as x^α when $x \to X^\star$. The limiting distribution is the *Weibull* one [13, 17]

$$G_3(Y) = \begin{cases} 1 & Y > 0 \\ e^{-|Y|^\alpha} & Y \leq 0 \end{cases}$$

The probability distribution is clearly centered in the (finite) superior limit, hence $b_n = X^\star$ whereas a_n is provided by the relation:

$$\int_{X^\star - a_n}^{X^\star} p(t)dt = \frac{1}{n} \tag{1.7}$$

Example 1.3.1 (Uniform distribution)**.**
The easiest example is the uniform distribution:

$$p(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{elsewhere} \end{cases}$$

The limiting distribution is a *Weibull* function with $\alpha = 1$, $b_n = 1$ and using (1.7) $a_n = 1/n$.

1.4 Fisher-Tippet-Gnedenko theorem

This theorem allows to unify the three universality classes by adding a parameter $\gamma \in \mathbb{R}$ [14] :

$$G_\gamma(Y) = e^{-(1+\gamma Y)^{-\frac{1}{\gamma}}} \qquad (1+\gamma Y > 0) \tag{1.8}$$

According to the value of γ, the three limiting distributions are included:

$\boldsymbol{\gamma \to 0}$ *Gumbel* distribution.

$\boldsymbol{\gamma > 0}$ *Fréchet* distribution with $\alpha = \gamma^{-1}$.

$\boldsymbol{\gamma < 0}$ *Weibull* distribution with $\alpha = -\gamma^{-1}$.

Remark 1.
It can be shown that if a random variable X is distributed according to a *Fréchet* distribution, then two new random variables distributed according to a Gumbel and a Weibull distributions, are given by:

$$\alpha \log X \quad \longrightarrow \quad \textit{Gumbel} \qquad\qquad -\frac{1}{X} \quad \longrightarrow \quad \textit{Weibull}$$

1.5 Order statistics

So far we concerned on the first maximum of the set. However one might be interested also on the statistics of a general order maximum and on the gaps between successive maxima i.e., the order statistics. This requires the definition of the sets of *maxima* and *gaps*.

1.5.1 Maxima

Rearranging the set $\{X_1, \ldots, X_n\}$ in a decreasing magnitude order:

$$\{M_{1,n}, \ldots, M_{n,n}\} \qquad X_{\max} \equiv M_{1,n} > M_{2,n} > \cdots > M_{n,n} \equiv X_{\min}$$

The full joint probability reads [18]:

$$P_M(m_1, \ldots, m_n) = n! \prod_{i=1}^{n} p(m_i) \prod_{i=1}^{n-1} \Theta(m_i - m_{i+1})$$

where the differences from (1.2) are the $n!$ factor counting all the possible permutations and the Θ functions ensuring the order of the variables. The cumulative distribution is given by:

$$\begin{aligned}
F_{k,n}(m) &= \prod_{i=1}^{k-1} \int_{-\infty}^{+\infty} dm_i \int_{-\infty}^{m} dm_k \prod_{j=k+1}^{n} \int_{-\infty}^{+\infty} dm_j P_M(m_1, \ldots, m_n) \\
&= n! \prod_{i=1}^{k-1} \int_{m_{i+1}}^{\infty} p(m_i) dm_i \int_{-\infty}^{m} p(m_k) dm_k \prod_{j=k+1}^{n} \int_{-\infty}^{m_{j-1}} p(m_j) dm_j
\end{aligned}$$

by deriving with respect to m one obtains the k-th maximum distribution:

$$f_{k,n}(m) = n!p(m) \int_m^{\infty} p(m_{k-1})dm_{k-1} \prod_{i=1}^{k-2} \int_{m_{j+1}}^{\infty} p(m_j)dm_j \\ \times \int_{-\infty}^{m} p(m_{k+1})dm_{k+1} \prod_{j=k+2}^{n} \int_{-\infty}^{m_{j-1}} p(m_j)dm_j \tag{1.9}$$

On defining:

$$P(m_j) = \int_{-\infty}^{m_j} p(m_{j+1})dm_{j+1} \qquad \Rightarrow \qquad 1 - P(m_j) = \int_{m_j}^{\infty} p(m_{j+1})dm_{j+1}$$

relation (1.9) becomes:

$$f_{k,n}(m) = \frac{n!}{(k-1)!(n-k)!} p(m) \left[P(m)\right]^{n-k} \left[1 - P(m)\right]^{k-1}$$

and this is the probability distribution of a general k-th order maximum.
We already know that setting $k = 1$ and taking the large n limit, the asymptotic behavior is well described by the *Fisher-Tippett-Gnedenko theorem* (1.8). For general k it can be shown that the asymptotic behavior of the cumulative distribution is given by:

$$F_{k,n}(m) \quad \xrightarrow[(m-a_n)/b_n \text{ fixed}]{n,m\to\infty} \quad G_i^{(k)}\left(\frac{m-a_n}{b_n}\right) = G_i^{(k)}(Y)$$

where the scaling function is now defined as:

$$G_i^{(k)}(Y) = G_i(Y) \sum_{j=0}^{k-1} \frac{[-\log G_i(Y)]^j}{j!} = \frac{1}{\Gamma(k)} \int_{-\log G_i(Y)}^{\infty} e^{-t} t^{k-1} dt$$

and the G_i are the well-known scaling functions of the leading maximum.

1.5.2 Gaps

The **gaps** are random variables helpful for studying near-extreme crowding phenomena:

$$\{d_{1,n}, \ldots, d_{n-1,n}\} \qquad\qquad d_{k,n} := M_{k,n} - M_{k+1,n}$$

In the large n limit, the asymptotic joint distribution of the k-th gap for typical fluctuations is found to be [19, 20]:

$$p_{k,n}(\delta) \overset{n \gg 0}{\approx} \frac{1}{a_n} q_i\left(\frac{\delta}{a_n}\right) \qquad\qquad i = \{1, 2, 3\}$$

where the q_i depend on the universality classes.

Gumbel

In the Gumbel class the scaling function q_1 is:

$$q_1(\delta) = ke^{-k\delta}$$

Recalling example (1.1.1), we have $a_n = 1$, so the k-th gap distribution is:

$$p_{k,n}(\delta) \overset{n \gg 0}{\approx} q_1(\delta)$$

the first moment is given by:

$$\langle d_{k,n} \rangle \overset{n \gg 0}{\approx} k \int_0^\infty d\delta \delta e^{-k\delta} = \frac{1}{k}$$

Fréchet

In the Fréchet class the scaling function q_2 is:

$$q_2(\delta) = \frac{\alpha^2}{\Gamma(k)} \int_0^\infty e^{-x^{-\alpha}} x^{-\alpha-1} (x+\delta)^{-\alpha k-1} dx \overset{\delta \gg 0}{\approx} \frac{\alpha}{\Gamma(k)} \delta^{-\alpha k-1}$$

Recalling example (1.2.1), we have $\alpha = 1$ and $a_n \sim n/\pi$ so the k-th gap distribution reads:

$$\begin{aligned} p_{k,n}(\delta) &\overset{n \gg 0}{\approx} \frac{\pi}{n\Gamma(k)} \int_0^\infty e^{-x^{-1}} x^{-2} \left(x + \frac{\pi}{n}\delta\right)^{-k-1} dx \\ &= \left(\frac{n}{\pi}\right)^k k(k+1) \delta^{-k-1} \mathrm{U}\left(k+1, 0, \frac{n}{\pi\delta}\right) \end{aligned}$$

the first moment is given by:

$$\langle d_{k,n} \rangle \overset{n \gg 0}{\approx} \left(\frac{n}{\pi}\right)^k k(k+1) \int_0^\infty d\delta \delta^{-k} \mathrm{U}\left(k+1, 0, \frac{n}{\pi\delta}\right) = \frac{n}{\pi k(k-1)}$$

Weibull

In the Weibull class the scaling function q_3 is:

$$q_3(\delta) = \frac{\alpha^2}{\Gamma(k)} \int_0^\infty e^{-(x+\delta)^\alpha} x^{\alpha k-1} (x+\delta)^{\alpha-1} dx \overset{\delta \gg 0}{\approx} \frac{\Gamma(k\alpha)}{\Gamma(k)} \left(\frac{\alpha}{\delta}\right)^{2-k\alpha} \delta^{\alpha(2-\alpha k)} e^{-\delta^\alpha}$$

Recalling example (1.3.1), we have $\alpha = 1$ and $a_n = 1/n$ so the k-th gap distribution reads:

$$p_{k,n}(\delta) \overset{n \gg 0}{\approx} \frac{n}{\Gamma(k)} \int_0^\infty e^{-x-n\delta} x^{k-1} dx = ne^{-n\delta}$$

the first moment is given by:

$$\langle d_{k,n} \rangle \overset{n \gg 0}{\approx} n \int_0^\infty d\delta \delta e^{-n\delta} = \frac{1}{n}$$

Chapter 2

Extreme value statistics of correlated random variables

We study now the EVS of a set of correlated random variables. If the correlation is **weak** then we can consider the coarse-grained system and so retrieve a new set of *i.i.d.* variables. Indeed let $\{X_1, \dots, X_n\}$ be a set of weakly correlated random variables, by definition the correlation function decays fast with space:

$$C_{ij} = \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle \sim e^{-\frac{|i-j|}{\zeta}}$$

where $\zeta \ll n$. So dividing the system in blocks of size ζ, the random variables inside are still correlated but the blocks are not, hence the new set of $n' = n/\zeta$ random variables is now *i.i.d.* and so its statistics is well-known.

On the other hand if the correlation within variables is **strong**, a coarse-graining process is no more feasible since $\zeta \gtrsim n$. This framework lacks of a general theory, but few particular problems can be solved.

2.1 Wiener processes

Let's consider a free Brownian particle starting from the origin. Its dynamics is described by a Wiener process, so the Langevin equation reads:

$$\frac{dx}{d\tau} = \eta(\tau) \qquad \Longrightarrow \qquad x(\tau) = \int_0^\tau \eta(s) ds \tag{2.1}$$

$\eta(\tau)$ is the white noise:

$$\langle \eta(\tau) \rangle = 0 \qquad\qquad \left\langle \eta(\tau)\eta(\tau') \right\rangle = 2D\delta(\tau - \tau')$$

We are interested on the first maximum distribution over a time interval $\tau \in [0, t]$:

$$M(t) := \max_{0 \le \tau \le t} [x(\tau)]$$

Mean value and correlation can be easily computed from (2.1):

$$\langle x(t) \rangle = 0 \qquad\qquad \left\langle x(t)x(t') \right\rangle = 2D \min\left(t, t'\right)$$

Thus the positions are strongly correlated in time.
As usual it is convenient to compute the *cumulative distribution*, i.e. the probability that the Brownian particle stays below z:

$$F(z,t) = \text{Prob}\left[M(t) \leq z\right] = \text{Prob}\left[x(\tau) \leq z, 0 \leq \tau \leq t\right]$$

For proceeding we need the probability distribution $P(x,t|z)$ of the random variable x. This is provided by the **Fokker-Planck equation** which is in this case a diffusion equation. Adding the initial condition that the particle starts at the origin and the absorbing condition preventing the particle to reach z, the *Cauchy problem*

$$\begin{cases} \frac{\partial}{\partial t} P(x,t|z) = D\frac{\partial^2}{\partial x^2} P(x,t|z) \\ P(x,0|z) = \delta(x) \\ P(-\infty,t|z) = P(z,t|z) = 0 \end{cases}$$

can be solved using the method of images:

$$P(x,t|z) = \frac{1}{\sqrt{4\pi Dt}}\left[e^{-\frac{x^2}{4Dt}} - e^{-\frac{(x-2z)^2}{4Dt}}\right] \tag{2.2}$$

Integrating (2.2), the cumulative distribution is:

$$\begin{aligned} F(z,t) &= \int_{-\infty}^{z} P(x,t|z)dx = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{z}\left[e^{-\frac{x^2}{4Dt}} - e^{-\frac{(x-2z)^2}{4Dt}}\right]dx \\ &= \frac{2}{\sqrt{\pi}}\int_0^{\frac{z}{\sqrt{4Dt}}} e^{-u^2}du = \text{erf}\left(\frac{z}{\sqrt{4Dt}}\right) \end{aligned}$$

deriving with respect to z we get the maximum distribution:

$$P_M(z,t) := \frac{d}{dz}F(z,t) = \frac{1}{\sqrt{\pi Dt}}e^{-\frac{z^2}{4Dt}}\Theta(z)$$

The mean value of the maximum

$$\langle M(t)\rangle = \int_{-\infty}^{+\infty} zP_M(z)dz = 2\sqrt{\frac{Dt}{\pi}}$$

grows as the square root of the time. Similar calculations produce:

$$\text{Var}\left[M(t)\right] = \frac{4\left(2\pi - 1\right)}{\pi}Dt$$

2.2 Ornstein-Uhlenbeck processes

Let's add now an elastic force to the previous Brownian particle (2.1). The Langevin equation reads:

$$\frac{dx}{d\tau} = -\mu x + \eta(\tau) \tag{2.3}$$

Equation (2.3) can be integrated on multiplying by $e^{\mu\tau}$:

$$\frac{dx}{d\tau}e^{\mu\tau} + \mu x e^{\mu\tau} = \frac{d}{d\tau}\left(xe^{\mu\tau}\right) = \eta(\tau)e^{\mu\tau} \qquad \Longrightarrow \qquad x(t) = e^{-\mu t}\int_0^t \eta(\tau)e^{\mu\tau}d\tau$$

Quick calculations provide the mean value and the correlation of the position:

$$\langle x(t)\rangle = 0 \qquad\qquad \langle x(t_1)x(t_2)\rangle = \frac{D}{\mu}\left[e^{-\mu|t_1-t_2|} - e^{-\mu(t_1+t_2)}\right]$$

As a test of consistency, taking the limit for $\mu \to 0$ we should retrieve the Wiener process correlation, indeed:

$$\begin{aligned}\langle x(t_1)x(t_2)\rangle &\overset{\mu\ll 1}{\sim} 2D\frac{1-\mu|t_1-t_2|-1+\mu(t_1-t_2)}{\mu} \\ &= 2D\left[|t_1-t_2|+t_1-t_2\right] = 2D\min(t_1,t_2)\end{aligned}$$

The characteristic time of the elastic force is $\tau_p = \mu^{-1}$. Assuming it to be finite ($\mu > 0$), one easily notices that for times way larger than τ_p, the correlation decays exponentially:

$$\langle x(t_1)x(t_2)\rangle \sim e^{-\mu|t_1-t_2|}$$

This means that after a $t \gg \tau_p$, the system ends up to be weakly correlated and so it can be analyzed using EVS theory for *i.i.d.* variables. Since the parent distribution of the random variables is Gaussian, we expect a *Gumbel* limiting distribution. For retrieving this result, let's start from the Cauchy problem:

$$\begin{cases}\dfrac{\partial}{\partial t}P\left(x,t|z\right) = \mu\dfrac{\partial}{\partial x}\left[xP\left(x,t|z\right)\right] + D\dfrac{\partial^2}{\partial x^2}P\left(x,t|z\right) \\ P\left(x,0|z\right) = \delta(x) \\ P\left(-\infty,t|z\right) = P(z,t|z) = 0\end{cases}$$

The harmonic potential prevent us to solve it using the method of images, however we can proceed by expanding the solution in its eigenstates [21]:

$$P(x,t|z) = \sum_\lambda a_\lambda e^{-\lambda t}D_{\lambda/\mu}\left(-\sqrt{2\mu}x\right)e^{-\mu\frac{x^2}{2}}$$

where $D_p(x)$ is the parabolic cylinder function satisfying:

$$\frac{d^2}{dx^2}D_p(x) + \left(p+\frac{1}{2}-\frac{x^2}{4}\right)D_p(x) = 0$$

The absorbing condition implies:

$$D_{\lambda/\mu}\left(-\sqrt{2\mu}z\right) = 0 \qquad\qquad \forall\lambda$$

and this fixes the eigenvalues. In the large t limit the leading term of the expansion is the one related to the smallest eigenvalue $\lambda_0(z)$:

$$P(x,t|z) \overset{t\gg 0}{\sim} e^{-\lambda_0(z)t}$$

it can be shown that for large z the eigenvalue behaves as [22]:

$$\lambda_0(z) \xrightarrow{z\to\infty} \frac{2}{\sqrt{\pi}}\mu^{\frac{3}{2}} z e^{-\mu z^2}$$

therefore the cumulative probability reads:

$$F(z,t) \sim e^{-\lambda_0(z)t} = e^{-e^{-\mu z^2 + \log\left(\frac{2t\mu^{\frac{3}{2}} z}{\sqrt{\pi}}\right)}} = G_1\left[\sqrt{4\mu\log t}\left(z - \sqrt{\frac{\log t}{\mu}}\right)\right]$$

So we recover a *Gumbel* distribution with coefficients:

$$a_t = \frac{1}{\sqrt{4\mu\log t}} \qquad\qquad b_t = \sqrt{\frac{\log t}{\mu}} = \langle M(t)\rangle$$

As result the mean value of the maximum distribution increases very slowly with time. A deeper analysis shows that the Brownian particle doesn't feel the potential for times way smaller than τ_p, hence it freely diffuses:

$$\langle M(t)\rangle \sim \begin{cases} \sqrt{t} & t \ll \tau_p \\ \sqrt{\log t} & t \gg \tau_p \end{cases}$$

Final remarks

In this chapter we presented two systems of strongly correlated variables where one can get some information about the first maximum distribution. Other solvable cases can be found in the *1D fluctuating surface* [23] and in the evaluation of the *largest eigenvalue distribution* in Random Matrix Theory [6, 7].

Chapter 3

Order statistics of random walks

In this chapter I will present the techniques used for studying the *order statistics* of a set of strongly correlated random variables. The set is provided by the positions of a one-dimensional random walker.

Definition 3.0.1 (Random Walk)**.**
Let $\{\eta_0, \ldots, \eta_n\}$ be a set of *i.i.d.* random variables. The partial sum X_j up to the step j is defined by:

$$X_j = \sum_{i=0}^{j} \eta_i \qquad\qquad j \leq n$$

The set of partial sums $\{X_0, \ldots, X_n\}$ is called **random walk** and it represents a set of strongly correlated random variables.

We can visualize it as a particle moving through the positions X_j at the times j. The *i.i.d.* variables η_j represent then the jumps from a position to another:

$$X_j - X_{j-1} = \eta_j$$

By definition the jumps η_j share the same parent distribution $f(\eta)$ that is intuitively called **jump distribution**. In the further analysis we will assume the jump distribution to be continuous and symmetric with zero mean and finite variance σ^2.

3.1 Relevant quantities

The study of the order statistics of the random walk requires the sets of **maxima** $\{M_{1,n} \ldots, M_{n+1,n}\}$ and **gaps** $\{d_{1,n}, \ldots, d_{n,n}\}$. Given that in [15] it has been found that the gap statistics shows a rich universal behavior, our analysis will revolve around the two following quantities:

1. the mean value of the k-th gap;

2. the probability distribution of the k-th gap.

The first quantity can be computed from the second one, but it can also be extracted from results of the k-th maximum statistics. The strategy we followed is rather simple: find a way for computing the cumulative distribution $F_{k,n}(x)$ of the k-th maximum/gap and then derive for obtaining the probability distribution. Thus the challenging step is to compute the cumulative distribution. The procedures followed for the two quantities are almost the same and they require the introduction of some well-defined auxiliary variables.

3.1.1 Mean value of the k-th gap

Let $q_{k,n}(x)$ and $r_{k,n}(x)$ be two auxiliary quantities related by:

$$r_{k,n}(x) = q_{n-k,n}(x) \tag{3.1}$$

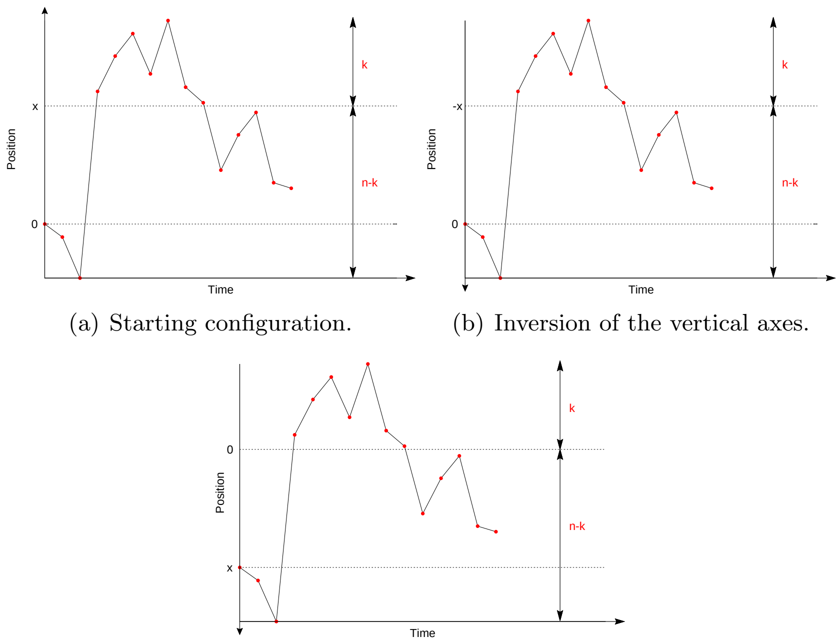
with:

$$q_{0,0}(x) = r_{0,0}(x) = 1 \qquad\qquad q_{k,n}(x) = r_{k,n}(x) = 0 \quad (\text{if } k > n)$$

The quantity $q_{k,n}(x)$ can be defined in two different ways:

1. the probability that starting in $x_0 = 0$ there are k points above x and so $n - k$ points below it;
2. the probability that starting in $x_0 = x$ there are k points below 0 and so $n - k$ points above it.

The connection between the two definition is well explained in figure (3.1).



(a) Starting configuration.

(b) Inversion of the vertical axes.

(c) Vertical shift of magnitude $+x$.

Figure 3.1: Visual representation of the equivalence of the two $q_{k,n}(x)$ definitions.

The cumulative distribution of the k-th maximum can be expressed as a combination of these auxiliary quantities. Indeed it sets the bound for the k-th maximum (and so for the further orders too) to stay below the value of x. As consequence it can be expressed as the sum of all the possible configurations of the lower order maxima $(1, \dots, k-1)$, which are free to be smaller or greater than x. Let's treat separately the positive/negative cases.

$x > 0$ The cumulative distribution is nothing but the sum of the probabilities $q_{m,n}(x)$ where m varies from 0 (all the maxima are smaller than x) to $k-1$ (the first $k-1$ maxima are above x):

$$F_{k,n}(x) = \sum_{m=0}^{k-1} q_{m,n}(x)$$

$x < 0$ This case is analog to the previous one though there is at least the starting point X_0 above x. The cumulative distribution reads:

$$F_{k,n} = \sum_{m=0}^{k-2} r_{m,n}(-x)$$

all together:

$$F_{k,n}(x) = \Pr\left[M_{k,n} \leq x\right] = \begin{cases} \sum_{m=0}^{k-1} q_{m,n}(x) & x > 0 \\ \sum_{m=0}^{k-2} r_{m,n}(-x) & x < 0 \end{cases}$$

Taking the derivative with respect to x, the k-th maximum distribution is achieved:

$$P_M\left(M_{k,n} = x\right) = \frac{\partial}{\partial x} F_{k,n}(x) = \sum_{m=0}^{k-1} \frac{\partial}{\partial x} q_{m,n}(x) + \sum_{m=0}^{k-2} \frac{\partial}{\partial x} r_{m,n}(-x)$$

Hence the mean value of the k-th maximum is:

$$\langle M_{k,n} \rangle = \sum_{m=0}^{k-1} \int_0^\infty x \frac{\partial}{\partial x} q_{m,n}(x) dx + \sum_{m=0}^{k-2} \int_0^\infty x \frac{\partial}{\partial x} r_{m,n}(-x) dx \tag{3.2}$$

Exploiting the linearity of the first moment, the mean value of the k-th gap is simply:

$$\langle d_{k,n} \rangle = \langle M_{k,n} \rangle - \langle M_{k+1,n} \rangle = -\int_0^\infty x \frac{\partial}{\partial x} \left[q_{k,n}(x) + r_{k-1,n}(-x)\right] dx \tag{3.3}$$

It is now clear the crucial role played by the auxiliary variables $q_{k,n}$ and $r_{k,n}$. An equation for them is provided by the *Markov chain backward equation*, related to the first step of the random walk:

$$\begin{aligned} q_{k,n}(x) &= \int_0^\infty q_{k,n-1}(x') f(x'-x) dx' + \int_{-\infty}^0 r_{k-1,n-1}(-x') f(x'-x) dx' \\ &= \int_0^\infty q_{k,n-1}(x') f(x'-x) dx' + \int_0^\infty r_{k-1,n-1}(x') f(x'+x) dx' \end{aligned} \tag{3.4}$$

where the first (second) term represents the probability to jump in $x' > 0$ ($x' < 0$) from $x > 0$. For $r_{k,n}$, using (3.1):

$$r_{k,n}(x) = \int_0^\infty r_{k-1,n-1}(x') f(x'-x) dx' + \int_0^\infty q_{k,n-1}(x') f(x'+x) dx' \tag{3.5}$$

3.1.2 Probability distribution of the k-th gap

Almost the same procedure is pursued for evaluating the probability distribution of the k-th gap. Let's start by considering the joint probability that the k-th maximum has the value y and the $(k+1)$-th one assumes the value $x < y$:

$$P(M_{k,n} = y; M_{k+1,n} = x) = P_{k,n}(x, y)$$

The cumulative distribution $S_{k,n}(x, y)$ can be defined as:

$$S_{k,n}(x, y) = \Pr\left[M_{k,n} > y, M_{k+1,n} < x\right] = \int_y^{\infty} dt \int_{-\infty}^{x} du P_{k,n}(t, u)$$

hence:

$$P_{k,n}(x, y) = -\frac{\partial^2}{\partial x \partial y} S_{k,n}(x, y) \tag{3.6}$$

The probability that the gap between the two maxima is $\delta = y - x$ will be:

$$P\left(d_{k,n} = \delta\right) = P_{k,n}(\delta) = \int_{\mathbb{R}^2} dx dy P_{k,n}(x, y) \Theta(y - x) \delta\left(y - x - \delta\right)$$

using (3.6):

$$P_{k,n}(d_{k,n} = \delta) = -\int_{\mathbb{R}^2} \frac{\partial^2 S_{k,n}(x, y)}{\partial x \partial y} \Theta(y - x) \delta(y - x - \delta) dx dy \tag{3.7}$$

As we have done before, for computing the cumulative distribution $S_{k,n}(x, y)$ we introduce two auxiliary variables $Q_{k,n}(x, \delta)$ and $R_{k,n}(x, \delta)$, related by:

$$R_{k,n}(x, \delta) = Q_{n-k,n}(x, \delta) \tag{3.8}$$

with:

$$Q_{0,0}(x, \delta) = R_{0,0}(x, \delta) = 1 \qquad\qquad Q_{k,n}(x, \delta) = R_{k,n}(x, \delta) = 0 \quad (\text{if } k > n)$$

Analogously the quantity $Q_{k,n}(x, \delta)$ is defined as the probability that the random walk, starting at $x_0 = x$, has k points in $]-\infty, -\delta]$ and $n-k$ in $[0, \infty[$. The cumulative distribution is related to these quantities by:

$$S_{k,n}(x, y) = \begin{cases} Q_{k,n}(x, y - x) & x > 0 \\ 0 & x < 0 \wedge y > 0 \\ R_{k-1,n}(-y, y - x) & x < 0 \wedge y < 0 \end{cases}$$

Then (3.7) becomes:

$$\begin{aligned} P_{k,n}(\delta) = &-\int_0^{\infty} dx \int_x^{\infty} dy \frac{\partial^2}{\partial x \partial y} Q_{k,n}(x, y - x) \delta(y - x - \delta) \\ &-\int_{-\infty}^{0} dx \int_x^{0} dy \frac{\partial^2}{\partial x \partial y} R_{k-1,n}(-y, y - x) \delta(y - x - \delta) \end{aligned} \tag{3.9}$$

In the same way, a backward equation can be defined for each auxiliary variable:

$$\begin{aligned} Q_{k,n}(x,\delta) &= \int_0^\infty Q_{k,n-1}(x',\delta)f(x-x')dx' \\ &\quad + \int_0^\infty R_{k-1,n-1}(x',\delta)f(x'+x+\delta)dx' \\ R_{k,n}(x,\delta) &= \int_0^\infty R_{k-1,n-1}(x',\delta)f(x-x')dx' \\ &\quad + \int_0^\infty Q_{k,n-1}(x',\delta)f(x'+x+\delta)dx' \end{aligned} \tag{3.10}$$

Remark 2.
In the study of the k-th gap distribution we will often use the rescaled gap γ, defined by:

$$\gamma = \frac{\delta}{b}$$

Therefore, it is equivalent to use the variable δ or γ since they only differ of a positive constant.

3.2 Mathematical tools

During the analysis we encountered several mathematical issues like recursiveness of the equations and heavy computations. The two following tools allowed us to lighten these problems.

3.2.1 Generating function method

In order to remove the recursiveness in the equations, we made large use of the probability generating function method.

Definition 3.2.1 (Probability generating function)**.**
Given a sequence of many indexes $f_{n_1,\ldots,n_r}$ the associated probability generating function is:

$$G(z_1,\ldots,z_r) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} z_1^{n_1} \cdots z_r^{n_r} f_{n_1,\ldots,n_r}$$

where the $\{z_1,\ldots,z_r\}$ are complex variables whose modulus must be confined in the interval $[0,1]$.

Although this method let us proceed with the calculations, eventually one is required to invert the generating function for retrieving the needed result. For this purpose a general recipe doesn't exist, anyway some techniques as the Maclaurin expansion, guessing the scaling function and the *Bromwich formula* (C) are helpful for the scope.

The double generating functions of the k-th maximum auxiliary variable are:

$$\tilde{q}(s,z;x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} s^n z^k q_{k,n}(x) \qquad \tilde{r}(s,z;x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} s^n z^k r_{k,n}(x)$$

In the following these quantities will be defined by linear differential equations. Boundaries conditions for the coefficients are provided by the generating function representations of (3.4) and (3.5):

$$\begin{aligned}\tilde{q}(s,z;x) &= 1 + s\int_0^\infty \tilde{q}(s,z;x')f(x'-x)dx' + sz\int_0^\infty \tilde{r}(s,z;x')f(x'+x)dx' \\ \tilde{r}(s,z;x) &= 1 + sz\int_0^\infty \tilde{r}(s,z;x')f(x'-x)dx' + s\int_0^\infty \tilde{q}(s,z;x')f(x'+x)dx'\end{aligned} \tag{3.11}$$

The same arguments and equations are valid for the gaps' auxiliary variables as long as one makes the following substitutions:

$$\tilde{q}\,(s,z;x) \mapsto \tilde{Q}\,(s,z;x,\delta) \qquad \tilde{r}\,(s,z;x) \mapsto \tilde{R}\,(s,z;x,\delta)$$

$$f\,(x'-x) \mapsto f\,(x'-x) \qquad f\,(x'+x) \mapsto f\,(x'+x+\delta)$$

so:

$$\begin{aligned}\tilde{Q}(s,z;x,\delta) &= 1 + s\int_0^\infty \tilde{Q}(s,z;x',\delta)f(x-x')dx' \\ &\quad + sz\int_0^\infty \tilde{R}(s,z;x',\delta)f(x'+x+\delta)dx' \\ \tilde{R}(s,z;x,\delta) &= 1 + sz\int_0^\infty \tilde{R}(s,z;x',\delta)f(x-x')dx' \\ &\quad + s\int_0^\infty \tilde{Q}(s,z;x',\delta)f(x'+x+\delta)dx'\end{aligned} \tag{3.12}$$

Remark 3.
In the generating function framework the limit for $s \to 1$ corresponds to $n \to \infty$ while the limit for $z \to 1$ corresponds to $k \to \infty$.

3.2.2 Symmetry

The relations between auxiliary variables (3.1) and (3.8) produce in the generating function representation a global symmetry coded in the **involution** ϕ:

$$\phi : (s,z) \mapsto \left(sz, \frac{1}{z}\right) \qquad \phi^2 = \mathbb{I}$$

Let $\tilde{q}\,(s,z;x)$ and $\tilde{r}\,(s,z;x)$ be the generating functions of the auxiliary variables, then:

$$\tilde{r}\,(s,z;x) = (\tilde{q}\circ\phi)\,(s,z,x) = \tilde{q}\left(sz, \frac{1}{z}, x\right)$$

and exploiting the involution property $\phi\circ\phi = \mathbb{I}$:

$$\tilde{q}\,(s,z;x) = (\tilde{q}\circ\mathbb{I})\,(s,z,x) = (\tilde{q}\circ\phi\circ\phi)\,(s,z,x) = (\tilde{r}\circ\phi)\,(s,z,x) = \tilde{r}\left(sz, \frac{1}{z}, x\right)$$

As consequence, we can work only with one of the auxiliary variables and then automatically extract the other by symmetry.

Remark 4.
The same symmetry holds for the gaps' auxiliary variables $\tilde{Q}, \tilde{R}$.

3.3 Final remarks

It is evident how the analysis deeply hinge on the auxiliary variables, which are defined by the *Wiener-Hopf* integrals (3.4), (3.5) and (3.10). These integral equations are pretty hard to solve for arbitrary jump distributions $f(x)$, however for the whole class of **Gamma distributions**, one can reduce them in recurrent differential equations.
In the next chapter we will report the analysis made with an exponential jump distribution in [15]. In the fifth and in the sixth chapters instead we will show the original results we obtained for a first order and a general order Gamma distributions.

Chapter 4

Exponential distribution

We consider now an exponential (or zeroth order Gamma distribution) jump distribution:

$$f_0(x) = \frac{1}{2b} e^{-\frac{|x|}{b}} \tag{4.1}$$

with:

$$\langle x \rangle = 0 \qquad\qquad \sigma^2 = \left\langle x^2 \right\rangle = 2b^2$$

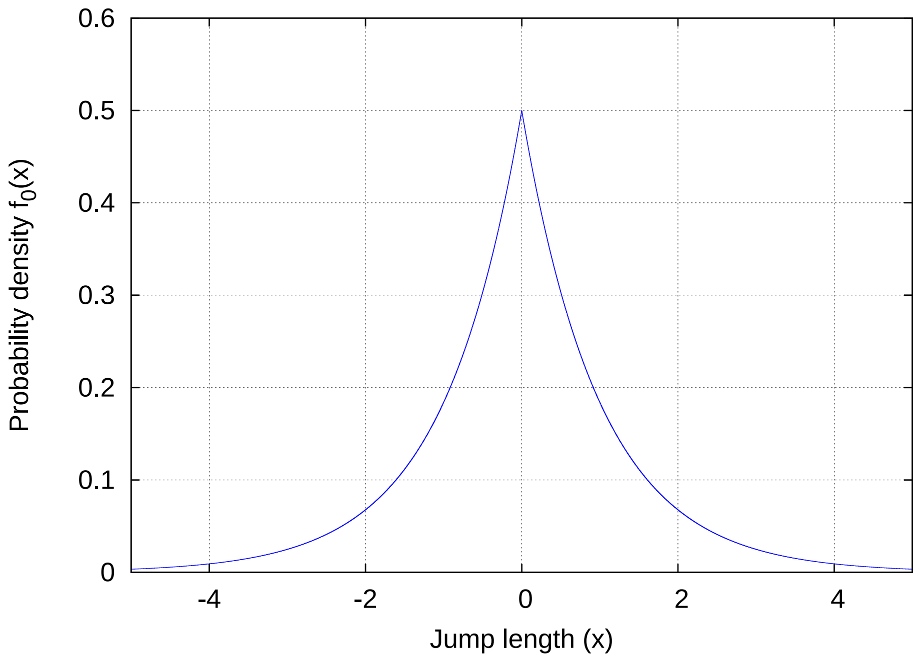


Figure 4.1: Exponential distribution (4.1) with $b = 1$.

Computing the second derivative, an useful relation can be isolated for this distribution [15]

$$b^2 f_0''(x) - f_0(x) = -\delta(x) \tag{4.2}$$

4.1 Mean value of the k-th gap

Relation (4.2) is very helpful for reducing the Wiener-Hopf integrals (3.4), (3.5) in two differential equations. Indeed deriving two times (3.4) with respect to x and using relation (4.2), one finds:

$$\begin{aligned} q''_{k,n}(x) &= \int_0^\infty q_{k,n-1}(x') f''_0(x'-x)dx' + \int_0^\infty r_{k-1,n-1}(x') f''_0(x'+x)dx' \\ &= \frac{1}{b^2}\int_0^\infty q_{k,n-1}(x')\left[f_0(x'-x) - \delta(x'-x)\right]dx' \\ &\quad + \frac{1}{b^2}\int_0^\infty r_{k-1,n-1}(x')\left[f_0(x'+x) - \delta(x'+x)\right]dx' \\ &= \frac{1}{b^2}\left[q_{k,n}(x) - q_{k,n-1}(x)\right] \end{aligned}$$

so we remain with the recurrent differential equation:

$$b^2 \frac{d^2}{dx^2} q_{k,n}(x) = q_{k,n}(x) - q_{k,n-1}(x) \tag{4.3}$$

The recursiveness can be removed using the double generating function:

$$\tilde{q}(s,z;x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} s^n z^k q_{k,n}(x)$$

hence:

$$b^2 \frac{\partial^2}{\partial x^2}\tilde{q}(s,z;x) = (1-s)\tilde{q}(s,z;x) - 1$$

which is a linear differential equation with solution:

$$\tilde{q}(s,z;x) = a(s,z)e^{-\sqrt{1-s}\frac{x}{b}} + \frac{1}{1-s} \tag{4.4}$$

By symmetry:

$$\tilde{r}(s,z;x) = a'(s,z)e^{-\sqrt{1-sz}\frac{x}{b}} + \frac{1}{1-sz} \qquad a'(s,z) = a\left(sz, \frac{1}{z}\right) \tag{4.5}$$

In order to evaluate the amplitude $a(s,z)$ let's combine the first of (3.11) with (4.1) and (4.4) so:

$$\begin{aligned} ae^{-\frac{\sqrt{1-s}}{b}x} + \frac{1}{1-s} &= 1 + \frac{s}{1-s}\int_0^\infty f_0(x'-x)dx' + as\int_0^\infty e^{-\frac{\sqrt{1-s}}{b}x'} f_0(x'-x)dx' \\ &\quad + \frac{sz}{1-sz}\int_0^\infty f_0(x'+x)dx' + a'sz\int_0^\infty e^{-\Lambda(sz)x'} f_0(x'+x)dx' \\ &= \frac{1}{1-s} + \frac{e^{-\frac{x}{b}}}{2}\frac{s(z-1)}{(1-s)(1-sz)} + a'sz\frac{e^{-\frac{x}{b}}}{2(\sqrt{1-sz}+1)} \\ &\quad - \frac{a}{2}\left[e^{-\frac{x}{b}}\left(1+\sqrt{1-s}\right) - 2e^{-\frac{\sqrt{1-s}}{b}x}\right] \end{aligned}$$

where we made use of results from (A). Finally we recover the equation:

$$\frac{a}{\sqrt{1-s}-1}+\frac{a'z}{\sqrt{1-sz}+1}+\frac{z}{1-sz}-\frac{1}{1-s}=0 \tag{4.6}$$

Applying the symmetry to the (4.6) we obtain a second equation and so a closed system[1]

$$\begin{cases} \dfrac{a}{\sqrt{1-s}-1}+\dfrac{a'z}{\sqrt{1-sz}+1}+\dfrac{z}{1-sz}-\dfrac{1}{1-s}=0 \\ \dfrac{a}{\sqrt{1-s}+1}+\dfrac{a'z}{\sqrt{1-sz}-1}-\dfrac{z}{1-sz}+\dfrac{1}{1-s}=0 \end{cases}$$

which has solution:

$$\begin{cases} a(s,z)=\dfrac{1}{\sqrt{(1-s)(1-sz)}}-\dfrac{1}{1-s} \\ a'(s,z)=\dfrac{1}{\sqrt{(1-s)(1-sz)}}-\dfrac{1}{1-sz}\equiv a\left(sz,\dfrac{1}{z}\right) \end{cases}$$

so finally:

$$\begin{cases} \tilde{q}(s,z;x)=\dfrac{1}{1-s}+\left(\dfrac{1}{\sqrt{(1-s)(1-sz)}}-\dfrac{1}{1-s}\right)e^{-\sqrt{1-s}\frac{x}{b}} \\ \tilde{r}(s,z;x)=\tilde{q}\left(sz,\dfrac{1}{z};x\right) \end{cases}$$

Now that the generating functions of the auxiliary quantities are well-known, we can easily compute the first moment of the k-th maximum distribution (3.2). In order to compute these integrals, it is better to switch to the generating function representation:

$$\begin{aligned} \int_0^\infty x\frac{\partial}{\partial x}\tilde{q}(s,z;x)dx &= \frac{\sigma}{\sqrt{2}}\left(\frac{1}{(1-s)^{3/2}}-\frac{1}{1-s}\frac{1}{\sqrt{1-sz}}\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} s^n z^m \frac{\sigma}{\sqrt{2\pi}}\left(2\delta_{m,0}\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+1\right)}-\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)}\right) \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} s^n z^m \int_0^\infty x\frac{\partial}{\partial x}q_{m,n}(x)dx \end{aligned}$$

where the Taylor function expansion around $s,z=0$ has been used for returning to the original form. Comparing:

$$\int_0^\infty x\frac{\partial}{\partial x}q_{m,n}(x)dx=\begin{cases} \frac{\sigma}{\sqrt{2}}\left(\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+1)}-1\right) & (m=0) \\ -\frac{\sigma}{\sqrt{2\pi}}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} & (m>0) \end{cases}$$

Similarly, for $r_{m,n}$, using (3.1):

$$\int_0^\infty x\frac{\partial}{\partial x}r_{m,n}(x)dx=\begin{cases} \frac{\sigma}{\sqrt{2}}\left(\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+1)}-1\right) & (m=n) \\ -\frac{\sigma}{\sqrt{2\pi}}\frac{\Gamma\left(n-m+\frac{1}{2}\right)}{\Gamma(n-m+1)} & (m<n) \end{cases}$$

[1]otherwise we could have obtained it combining the second of (3.11) with (4.1) and (4.5).

On substituting into (3.2) we obtain:

$$\langle M_{k,n}\rangle = \frac{\sigma}{\sqrt{2}}\left[\frac{2}{\sqrt{\pi}}\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+1\right)} - 1 - \frac{1}{\sqrt{\pi}}\sum_{m=1}^{k-1}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} - \frac{1}{\sqrt{\pi}}\sum_{m=0}^{k-2}\frac{\Gamma\left(n-m+\frac{1}{2}\right)}{\Gamma(n-m+1)}\right] \tag{4.7}$$

By exploiting the noteworthy relation

$$\sum_{m=0}^{n}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m+1\right)} = 2\frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(n+1\right)}$$

we can rewrite (4.7) as:

$$\begin{aligned}\langle M_{k,n}\rangle &= \frac{\sigma}{\sqrt{2\pi}}\left[\sum_{m=0}^{n}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} - \sum_{m=0}^{k-1}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} - \sum_{m=0}^{k-2}\frac{\Gamma\left(n-m+\frac{1}{2}\right)}{\Gamma(n-m+1)}\right]\\ &= \frac{\sigma}{\sqrt{2\pi}}\left[\sum_{m=k}^{n}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} - \sum_{m=0}^{k-2}\frac{\Gamma\left(n-m+\frac{1}{2}\right)}{\Gamma(n-m+1)}\right]\\ &= \frac{\sigma}{\sqrt{2\pi}}\left[\sum_{m=k}^{n}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} - \sum_{m=n-k+2}^{n}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)}\right]\end{aligned} \tag{4.8}$$

Assuming now $k < n-k+2$, i.e. $k < 1+n/2$ that is we concern on the first half of the maxima (by symmetry the second would produce the same results), the two sums in (4.8) can be subtracted, so:

$$\langle M_{k,n}\rangle = \frac{\sigma}{\sqrt{2\pi}}\sum_{m=k}^{n-k+1}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} \tag{4.9}$$

Equation (4.9) is the exact value of the first moment of k-th maximum distribution. In the large n limit, one finds the following behaviour:

$$\langle M_{k,n}\rangle \overset{n\gg 0}{\approx} \frac{\sigma}{\sqrt{2\pi}}2\sqrt{n} = \sigma\sqrt{\frac{2n}{\pi}} \qquad \Rightarrow \qquad \frac{\langle M_{k,n}\rangle}{\sigma} \overset{n\gg 0}{\approx} \sqrt{\frac{2n}{\pi}}$$

we notice that it loses the dependence on k.
From (4.9) it is straightforward to extract the k-th gap mean value using linearity:

$$\begin{aligned}\langle d_{k,n}\rangle = \langle M_{k,n}\rangle - \langle M_{k+1,n}\rangle &= \frac{\sigma}{\sqrt{2\pi}}\left[\sum_{m=k}^{n-k+1}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)} - \sum_{m=k+1}^{n-k}\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m+1)}\right]\\ &= \frac{\sigma}{\sqrt{2\pi}}\left[\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+1\right)} + \frac{\Gamma\left(n-k+\frac{3}{2}\right)}{\Gamma(n-k+2)}\right]\end{aligned}$$

in the large n limit, given that

$$\frac{\Gamma(n+a)}{\Gamma(n+b)} \xrightarrow{n\to\infty} n^{a-b} \tag{4.10}$$

we have:

$$\lim_{n\to\infty}\frac{\langle d_{k,n}\rangle}{\sigma}=\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\left[\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)}+\frac{1}{\sqrt{n}}\right]=\frac{1}{\sqrt{2\pi}}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+1\right)} \tag{4.11}$$

in the large k limit, using (4.10):

$$\frac{\langle d_{k,\infty}\rangle}{\sigma}\overset{k\gg 0}{\approx}\frac{1}{\sqrt{2\pi k}}+O\left(k^{-1}\right) \tag{4.12}$$

and we notice that the mean value becomes **stationary**, that is it doesn't depend on n. Using *Pollaczek-Wendel identity* this result can be proved to be **universal**, i.e. to hold for arbitrary symmetric and continuous jump distributions $f(x)$ [15]:

$$\lim_{n\to\infty}\langle d_{k,n}\rangle=\frac{\sigma}{\sqrt{2\pi}}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)}-\frac{1}{\pi k}\int_0^\infty\frac{dq}{q^2}\left[\hat{f}_0(q)^k-\frac{1}{\left(1+\frac{\sigma^2}{2}q^2\right)^k}\right] \tag{4.13}$$

where $\hat{f}_0(q)$ is the Fourier transform of the jump distribution. For large values of k one finds again (4.12).

4.2 Probability distribution of the k-th gap

We are now interested on evaluating the asymptotic behavior of the full k-th gap distribution. As we already stated, the gaps auxiliary variables $(Q_{k,n},R_{k,n})$ share the same properties of the maxima ones $(q_{k,n},r_{k,n})$, so as beforehand we find the following recurrent differential equation:

$$b^2\frac{\partial^2}{\partial x^2}Q_{k,n}(x,\delta)=Q_{k,n}(x,\delta)-Q_{k,n-1}(x,\delta)$$

switching to the generating function representation:

$$\tilde{Q}(s,z;x,\delta)=A(s,z;\delta)e^{-\sqrt{1-s}\frac{x}{b}}+\frac{1}{1-s}$$

Injecting this expression into (3.12) and using the symmetry, we get a linear system for the coefficients A and $B\equiv A\left(sz,\frac{1}{z};\delta\right)$:

$$\begin{cases}\dfrac{1}{1-s}+\dfrac{A}{1-\sqrt{1-s}}-\dfrac{ze^{-\frac{\delta}{b}}}{1-sz}-\dfrac{Bze^{-\frac{\delta}{b}}}{1+\sqrt{1-sz}}=0\\[2ex]\dfrac{z}{1-sz}+\dfrac{zB}{1-\sqrt{1-sz}}-\dfrac{e^{-\frac{\delta}{b}}}{1-s}-\dfrac{Ae^{-\frac{\delta}{b}}}{1+\sqrt{1-s}}=0\end{cases} \tag{4.14}$$

which has solutions $(\gamma\equiv\delta/b)$:

$$\begin{cases}A(s,z;\gamma)=\dfrac{\frac{sz}{\sqrt{1-sz}}-\frac{s\sqrt{1-sz}}{1-s}\cosh\gamma-\frac{s}{1-s}\sinh\gamma}{\left(\sqrt{1-s}+\sqrt{1-sz}\right)\cosh\gamma+\left(1+\sqrt{(1-sz)(1-s)}\right)\sinh\gamma}\\[2ex]B(s,z;\gamma)=A(sz,1/z,\gamma)\end{cases} \tag{4.15}$$

Now that the generating functions of the auxiliary variables are known, we can compute the generating functions of the gap distribution using (3.9):

$$\begin{aligned}\tilde{P}(s,z;\gamma) = &-\frac{1}{b}\int_0^{+\infty} dx \int_x^{+\infty} dy \delta\left(\gamma - \frac{\delta}{b}\right) \frac{\partial^2}{\partial x \partial y}\tilde{Q}(s,z;x,\gamma) \\ &-\frac{z}{b}\int_{-\infty}^{0} dx \int_x^{0} dy \delta\left(\gamma - \frac{\delta}{b}\right) \frac{\partial^2}{\partial x \partial y}\tilde{R}(s,z;-y,\gamma)\end{aligned} \tag{4.16}$$

In order to do so let's start by evaluating the double derivatives of the auxiliary variables ($\gamma = (y-x)/b$):

$$\begin{aligned}\frac{\partial^2}{\partial x \partial y}\tilde{Q}(s,z;x,\gamma) &= -\frac{e^{-\lambda x}}{b^2}\left[\frac{\partial^2}{\partial \gamma^2} + \lambda b \frac{\partial}{\partial \gamma}\right] A(s,z;\gamma) \\ \frac{\partial^2}{\partial x \partial y}\tilde{R}(s,z;-y,\gamma) &= -\frac{e^{\eta y}}{b^2}\left[\frac{\partial^2}{\partial \gamma^2} + \eta b \frac{\partial}{\partial \gamma}\right] B(s,z;\gamma)\end{aligned} \tag{4.17}$$

where for simplicity:

$$\lambda = \frac{\sqrt{1-s}}{b} \qquad \eta = \frac{\sqrt{1-sz}}{b}$$

Injecting (4.17) into (4.16) and computing the integrals, one finally obtains:

$$\begin{aligned}\tilde{P}(s,z;\gamma) = &\frac{1}{\lambda b^2}\frac{\partial^2 A}{\partial \gamma^2}(s,z;\gamma) + \frac{1}{b}\frac{\partial A}{\partial \gamma}(s,z;\gamma) + \frac{z}{\eta b^2}\frac{\partial^2 B}{\partial \gamma^2}(s,z;\gamma) \\ &+ \frac{z}{b}\frac{\partial B}{\partial \gamma}(s,z;\gamma)\end{aligned} \tag{4.18}$$

On substituting the values obtained in (4.15) for the coefficients and taking the limit of $s \to 1$ we have:

$$\tilde{p}(z;\gamma) = \sum_{k=1}^{\infty} z^k p_k(\gamma) = \frac{8z}{b} e^{-2\gamma} \frac{u(z) - v(z)e^{-2\gamma}}{[u(z) + v(z)e^{-2\gamma}]^3} \tag{4.19}$$

where we introduced the functions $u(z) = \sqrt{1-z} + 1$, $v(z) = \sqrt{1-z} - 1$. The next step is to extract the $p_k(\gamma)$ from the generating function. This is not trivial, but using scaling arguments we can extract the asymptotic behavior in the large k limit ($z \to 1$) for *typical* and *large* fluctuations.

1. **Typical fluctuations** $\delta, \gamma \sim 1/\sqrt{k}$
   Here we consider fluctuations around the mean value. In order to do so we guess a scaling form for $p_k(\delta)$ where $\sqrt{k}\delta$ is fixed:

$$p_k(\delta) = \frac{1}{\sigma}\sqrt{k} P\left(\sqrt{k}\frac{\delta}{\sigma}\right) \tag{4.20}$$

   Looking at its generating function:

$$\tilde{p}(z;\delta) = \frac{1}{\sigma}\sum_{k=1}^{\infty} z^k \sqrt{k} P\left(\sqrt{k}\frac{\delta}{\sigma}\right)$$

by defining $z = e^{-t}$ the limit for $z \to 1$ corresponds to $t \to 0$ hence:

$$\tilde{p}(z;\delta) = \frac{1}{\sigma}\sum_{k=1}^{\infty} e^{-kt}\sqrt{k}P\left(\sqrt{k}\frac{\delta}{\sigma}\right) \approx \frac{1}{\sigma}\int_1^{\infty} e^{-kt}\sqrt{k}P\left(\sqrt{k}\frac{\delta}{\sigma}\right) dk$$

making the change of variable $x = k\left(\frac{\delta}{\sigma}\right)^2$:

$$\tilde{p}(z;\delta) \overset{\text{t.f.}}{\approx} \frac{1}{\sigma}\left(\frac{\sigma}{\delta}\right)^3 \int_{\left(\frac{\delta}{\sigma}\right)^2}^{\infty} e^{-\left(\frac{\sigma}{\delta}\right)^2 xt}\sqrt{x}P\left(\sqrt{x}\right) dx$$

So, given that $\delta \to 0$:

$$\tilde{p}(z;\delta) \overset{\text{t.f.}}{\approx} \frac{\sigma^2}{\delta^3}\int_0^{\infty} e^{-\frac{\sigma^2 t}{\delta^2}x}\sqrt{x}P\left(\sqrt{x}\right) dx = \frac{\sigma^2}{\delta^3}F\left(\frac{\sigma^2 t}{\delta^2}\right) \tag{4.21}$$

We expect then a scaling of this form for the generating function of (4.20). In order to prove or disprove it, let's look at the asymptotic behaviour for $\delta, t \to 0$ ($z \to 1$) with fixed ratio t/δ^2 of (4.19):

$$\frac{8z}{b}e^{-2\gamma}\frac{u(z) - v(z)e^{-2\gamma}}{[u(z) + v(z)e^{-2\gamma}]^3} \overset{\text{t.f.}}{\approx} \frac{16}{b}\frac{1}{\left(2\sqrt{1-z} + 2\frac{\delta}{b}\right)^3}$$

Now since for $t \to 0$ $e^{-t} \approx 1 - t$, then $t \approx 1 - z$:

$$\frac{16}{b}\frac{1}{\left(2\sqrt{t} + 2\frac{\delta}{b}\right)^3} = \frac{2b^2}{\delta^3}\frac{1}{\left(1 + \frac{b\sqrt{t}}{\delta}\right)^3} = \frac{\sigma^2}{\delta^3}\frac{1}{\left(1 + \sqrt{\frac{\sigma^2 t/\delta^2}{2}}\right)^3}$$

where it has been made the substitution $b = \sigma/\sqrt{2}$. Comparing with (4.21) the scaling function turns out to be:

$$F(\lambda) = \left(1 + \sqrt{\frac{\lambda}{2}}\right)^{-3} \qquad \lambda = \frac{\sigma^2 t}{\delta^2}$$

and from (4.21), it can be interpreted as the Laplace transform of $\sqrt{x}P\left(\sqrt{x}\right)$:

$$\mathscr{L}_{x\to\lambda}\left[\sqrt{x}P(\sqrt{x})\right] := \int_0^{\infty} e^{-x\lambda}\sqrt{x}P(\sqrt{x})dx = \left(1 + \sqrt{\frac{\lambda}{2}}\right)^{-3}$$

The inversion of this Laplace transform is not trivial, however expressing it as:

$$\left(1 + \sqrt{\frac{\lambda}{2}}\right)^{-3} = \frac{1}{2}\int_0^{\infty} y^2 e^{-y} e^{-y\sqrt{\frac{\lambda}{2}}} dy$$

one has:

$$\begin{aligned} \sqrt{x}P\left(\sqrt{x}\right) &= \mathscr{L}^{-1}_{\lambda\to x}\left[\frac{1}{2}\int_0^{\infty} y^2 e^{-y} e^{-y\sqrt{\frac{\lambda}{2}}} dy\right] \\ &= \frac{1}{2}\int_0^{\infty} y^2 e^{-y}\mathscr{L}^{-1}_{\lambda\to x}\left[e^{-y\sqrt{\frac{\lambda}{2}}}\right] dy \end{aligned} \tag{4.22}$$

So the problem is reduced on evaluating the inverse Laplace transform of the function $e^{-y\sqrt{\frac{\lambda}{2}}}$. Using **Bromwich formula** (C):

$$\mathscr{L}^{-1}_{\lambda\to x}\left[e^{-y\sqrt{\frac{\lambda}{2}}}\right] = \frac{y}{2\sqrt{2\pi}x^{3/2}}e^{-\frac{y^2}{8x}} \tag{4.23}$$

Hence matching (4.22) with (4.23):

$$\begin{aligned} P(\sqrt{x}) &= \frac{1}{4\sqrt{2\pi}x^2}\int_0^\infty y^3 e^{-\left(y+\frac{y^2}{8x}\right)}dy = \frac{4e^{2x}}{\sqrt{2\pi}x^2}\int_{2x}^\infty (\tilde{y}-2x)^3 e^{-\frac{\tilde{y}^2}{2x}}d\tilde{y} \\ &= 4\left[\frac{2}{\sqrt{2\pi}}(1+2x) - e^{2x}\sqrt{x}\,(3+4x)\,\mathrm{erfc}\left(\sqrt{2x}\right)\right] \end{aligned}$$

setting $t := \sqrt{x}$:

$$P(t) = 4\left[\frac{2}{\sqrt{2\pi}}\left(1+2t^2\right) - e^{2t^2}t\left(3+4t^2\right)\mathrm{erfc}\left(\sqrt{2}t\right)\right] \tag{4.24}$$

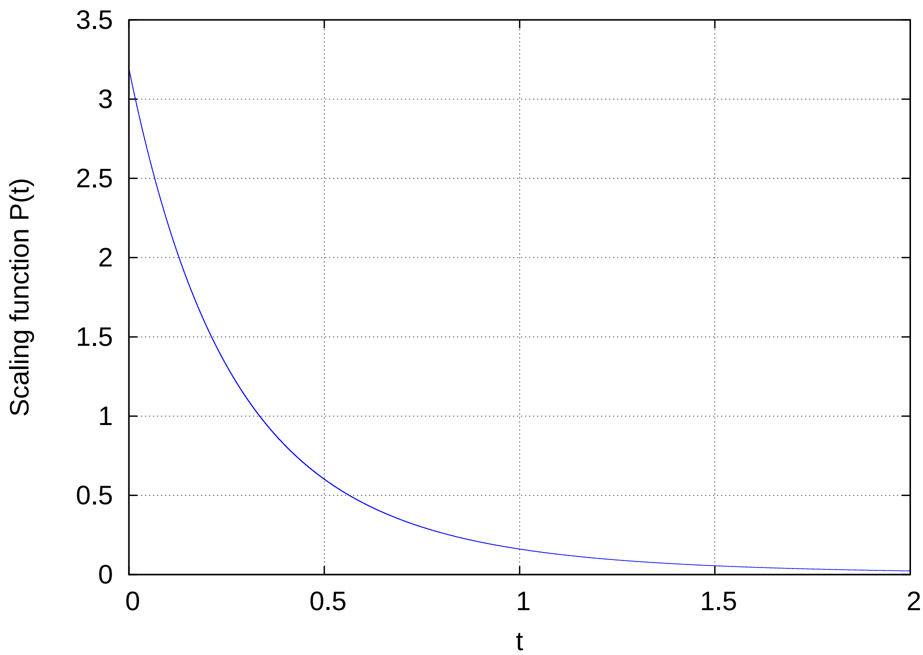


Figure 4.2: Scaling function for typical fluctuations (4.24).

which has the asymptotic behaviour:

$$P(t) \approx \begin{cases} \sqrt{\frac{32}{\pi}} & t\to 0 \\ \frac{3}{\sqrt{8\pi}}t^{-4} & t\to\infty \end{cases} \tag{4.25}$$

showing a surprising power law tail.

2. **Large fluctuations** $\delta, \gamma \gg 0$
   Numerical results suggest that, in the large k, δ limits, $p_k(\delta)$ scales as:

$$p_k(\delta) \approx \varphi_0(\delta) k^{-3/2} \tag{4.26}$$

Switching to the generating function representation, the large k limit corresponds to the $z \to 1$ limit. Hence developing $\tilde{p}(z;\delta)$ around $z = 1$:

$$\tilde{p}(z;\delta) = \sum_{k=1}^{\infty} z^k p_k(\delta) \overset{z\to 1}{\approx} \tilde{p}_1(\delta) + \tilde{p}_2(\delta)\sqrt{1-z} \tag{4.27}$$

In this limit, $p_k(\delta)$ is related to the coefficient of $\sqrt{1-z}$ in the expansion, indeed:

$$\begin{aligned} \tilde{p}_2(\delta)\sqrt{1-z} &= -\frac{\tilde{p}_2(\delta)}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma\left(k-\frac{1}{2}\right)}{\Gamma(k+1)} z^k \\ &= \tilde{p}_2(\delta) - \frac{\tilde{p}_2(\delta)}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\Gamma\left(k-\frac{1}{2}\right)}{\Gamma(k+1)} z^k \end{aligned}$$

so:

$$p_k(\delta) \approx -\frac{\tilde{p}_2(\delta)}{2\sqrt{\pi}} \frac{\Gamma\left(k-\frac{1}{2}\right)}{\Gamma(k+1)} \approx -\frac{\tilde{p}_2(\delta)}{2b\sqrt{\pi}} k^{-\frac{3}{2}}$$

comparing with (4.26), $\varphi_0(\delta)$ turns out to be:

$$\varphi_0(\delta) = -\frac{\tilde{p}_2(\delta)}{2\sqrt{\pi}} \tag{4.28}$$

finally, expanding (4.19) we find:

$$\tilde{p}_2(\delta) = -\frac{16}{b} e^{2\frac{\delta}{b}} \frac{e^{4\frac{\delta}{b}} + 4e^{2\frac{\delta}{b}} + 1}{\left(1 - e^{2\frac{\delta}{b}}\right)^4}$$

hence by virtue of (4.28):

$$\varphi_0(\gamma) = \frac{8}{b\sqrt{\pi}} e^{2\gamma} \frac{e^{4\gamma} + 4e^{2\gamma} + 1}{(1 - e^{2\gamma})^4} \qquad \gamma = \frac{\delta}{b}$$

with asymptotic behaviour:

$$\varphi_0(\gamma) \approx \begin{cases} \frac{3}{b\sqrt{\pi}} \gamma^{-4} & \gamma \to 0 \\ \frac{8}{b\sqrt{\pi}} e^{-2\gamma} & \gamma \to \infty \end{cases} \tag{4.29}$$

Collecting (4.20) and (4.26):

$$p_k(\gamma) \overset{k\gg 0}{\approx} \begin{cases} \frac{1}{b}\sqrt{\frac{k}{2}} P\left(\sqrt{\frac{k}{2}}\gamma\right) & \gamma \sim 1/\sqrt{k} \\ \varphi_0(\gamma) k^{-3/2} & \gamma \gg 0 \end{cases}$$

The results are consistent since the asymptotic behaviours of the two regimes match when they move "toward each other":

$$\begin{cases} \frac{1}{b}\sqrt{\frac{k}{2}}P\left(\sqrt{\frac{k}{2}}\gamma\right) \xrightarrow{\gamma\gg 0} \frac{1}{b}\sqrt{\frac{k}{2}}\frac{3}{\sqrt{8\pi}}\frac{4}{k^2}\gamma^{-4} = \frac{3}{b\sqrt{\pi}}\gamma^{-4}k^{-3/2} \\ \varphi_0(\gamma)k^{-3/2} \xrightarrow{\gamma\sim 0} \frac{3}{b\sqrt{\pi}}\gamma^{-4}k^{-3/2} \end{cases}$$

This proves that the chosen scaling forms are correct.

4.3 Alternative approaches

The analysis we made for studying the k-th gap statistics for an exponential jump distribution is no more feasible for further order Gamma distributions since it turns out to be computationally prohibitive. In these cases we are required to follow alternative paths for extracting the asymptotic behavior of the needed quantities. These alternative paths need to be tested for verifying their consistency. In order to do so in this section we will check them for the exponential jump distribution. Indeed if the final results match with the ones we already got, then these approaches are trustful and we could use them in the next chapters.

4.3.1 Generating function of the k-th gap mean value

Instead of passing through the computation of the k-th maximum mean value (3.2), we start from the evaluation of the k-th gap (3.3) generating function:

$$\left\langle \tilde{d}(s,z)\right\rangle = -\int_0^\infty x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}\sum_{k=1}^{n} s^n z^k \left[q_{k,n}(x) + r_{k-1,n}(x)\right] dx \tag{4.30}$$

For achieving an expression with generating functions, we need to work on the two R.H.S. addends.

1. Starting with the first[2]

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=1}^{n} s^n z^k q_{k,n}(x) &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} s^n z^k q_{k,n}(x) - q_{0,n}s^n\right] \\ &= \tilde{q}(s,z;x) - \sum_{n=0}^{\infty} q_{0,n}s^n\end{aligned}$$

the last term can be rewritten as:

$$\sum_{n=0}^{\infty} q_{0,n}s^n = \sum_{n=0}^{\infty} r_{n,n}s^n = \sum_{n=0}^{\infty} r_n s^n = \hat{r}(s;x) \tag{4.31}$$

A differential equation for $\hat{r}$ is provided by (4.3) setting $k = n$ and summing over n:

$$b^2\frac{d^2}{dx^2}\sum_{n=1}^{\infty} s^n r_n(x) = \sum_{n=1}^{\infty} s^n r_n(x) - \sum_{n=1}^{\infty} s^n r_{n-1}(x)$$

[2] the sum over k starts from 1 since the zeroth gap doesn't exist.

this produces the well-known differential equation:

$$b^2 \frac{\partial^2}{\partial x^2}\hat{r}(s;x) = (1-s)\hat{r}(s;x) - 1$$

with solution:

$$\hat{r}(s;x) = c(s)e^{-\frac{\sqrt{1-s}}{b}x} + \frac{1}{1-s} \tag{4.32}$$

in order to evaluate the coefficient let's use the (3.5) with $k = n$:

$$r_n(x) = \int_0^\infty r_{n-1}(x')f(x'-x)dx'$$

summing it over n:

$$\hat{r}(s;x) = 1 + s\int_0^\infty \hat{r}(s;x')f(x'-x)dx' \tag{4.33}$$

hence injecting (4.32) and (4.1) into (4.33):

$$c(s) = \frac{\sqrt{1-s}-1}{1-s} = \frac{1}{\sqrt{1-s}} - \frac{1}{1-s}$$

The first R.H.S. term of (4.30) is now evaluated.

2. The second addend reads:

$$\sum_{n=0}^{\infty}\sum_{k=1}^{n} s^n z^k r_{k-1,n} = z\sum_{n=0}^{\infty}\sum_{k=0}^{n-1} s^n z^k r_{k,n} = z\sum_{n=0}^{\infty} s^n \left[\sum_{k=0}^{n} z^k r_{k,n} - z^n r_n\right]$$
$$= z\tilde{r}(s,z;x) - z\sum_{n=0}^{\infty}(sz)^n r_n$$

where the last sum:

$$\sum_{n=0}^{\infty}(sz)^n r_n = \hat{r}(sz;x)$$

coincides with (4.31) where $s \mapsto sz$, so:

$$\hat{r}(sz;x) = c(sz)e^{-\frac{\sqrt{1-sz}}{b}x} + \frac{1}{1-sz} \qquad c(sz) = \frac{1}{\sqrt{1-sz}} - \frac{1}{1-sz}$$

Finally putting all together into (4.30):

$$\left\langle \tilde{d}(s,z)\right\rangle = \int_0^\infty x\frac{\partial}{\partial x}\left[\hat{r}(s;x) - \tilde{q}(s,z;x) + z\hat{r}(sz;x) - z\tilde{r}(s,z;x)\right]dx \tag{4.34}$$

The integral reads:

$$\left\langle \tilde{d}(s,z)\right\rangle = \frac{\sigma}{\sqrt{2}}\left[\frac{a-c(s)}{\sqrt{1-s}} + z\frac{a'-c(sz)}{\sqrt{1-sz}}\right]$$
$$= \frac{\sigma}{\sqrt{2}}\left[\frac{1}{(1-s)\sqrt{1-sz}} + \frac{z}{\sqrt{1-s}(1-sz)} - \frac{1}{1-s} - \frac{z}{1-sz}\right]$$

In the $s \to 1$ limit:

$$\left\langle \tilde{d}(s,z) \right\rangle \overset{s\to 1}{\approx} \frac{\sigma}{\sqrt{2}} \left(\frac{1}{\sqrt{1-z}} - 1 \right) \frac{1}{1-s} = \frac{1}{1-s} \sum_{k=1}^{\infty} z^k \left\langle d_{k,\infty} \right\rangle \tag{4.35}$$

where $\langle d_{k,\infty} \rangle = \lim_{n\to\infty} \langle d_{k,n} \rangle$. Using Taylor expansion we can easily extract $\langle d_{k,\infty} \rangle$ from (4.35):

$$\frac{\sigma}{\sqrt{2}} \left(\frac{1}{\sqrt{1-z}} - 1 \right) = \frac{\sigma}{\sqrt{2}} \left(\frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)} z^k - 1 \right) = \frac{\sigma}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)} z^k$$

hence comparing:

$$\langle d_{k,\infty} \rangle = \frac{\sigma}{\sqrt{2\pi}} \frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma(k+1)}$$

This is the same result we got in (4.11) so taking the large k limit we will for sure end up in (4.12). As consequence this approach is consistent and it is simpler since it doesn't require to compute the k-th maximum mean value.

4.3.2 Asymptotic linear system

For computing the full k-th gap distribution we will here consider the $s \to 1$ limit before of solving the linear systems. The easiest way to proceed is to develop the coefficients A, B up to the second order:

$$A \overset{s\to 1}{\approx} \frac{A^{(1)}}{1-s} + \frac{A^{(2)}}{\sqrt{1-s}} \qquad\qquad B \overset{s\to 1}{\approx} \frac{B^{(1)}}{1-s} + \frac{B^{(2)}}{\sqrt{1-s}}$$

substituting into (4.14):

$$\begin{cases} \frac{1}{1-s} + \frac{1}{1-\sqrt{1-s}} \left(\frac{A^{(1)}}{1-s} + \frac{A^{(2)}}{\sqrt{1-s}} \right) \\ \qquad - \frac{ze^{-\gamma}}{1-sz} - \frac{ze^{-\gamma}}{1+\sqrt{1-sz}} \left(\frac{B^{(1)}}{1-s} + \frac{B^{(2)}}{\sqrt{1-s}} \right) = 0 \\ \frac{z}{1-sz} + \frac{z}{1-\sqrt{1-sz}} \left(\frac{B^{(1)}}{1-s} + \frac{B^{(2)}}{\sqrt{1-s}} \right) \\ \qquad - \frac{e^{-\gamma}}{1-s} - \frac{e^{-\gamma}}{1+\sqrt{1-s}} \left(\frac{A^{(1)}}{1-s} + \frac{A^{(2)}}{\sqrt{1-s}} \right) = 0 \end{cases} \tag{4.36}$$

where as usual $\gamma = \delta/b$. Expanding now (4.36) around $s = 1$ up to the second order:

$$\begin{cases} \frac{A^{(1)} + 1 - B^{(1)} e^{-\gamma} \left(1 - \sqrt{1-z}\right)}{1-s} - \frac{A^{(1)} + A^{(2)} - B^{(2)} e^{-\gamma} \left(1 - \sqrt{1-z}\right)}{\sqrt{1-s}} = 0 \\ \frac{B^{(1)} \left(1 + \sqrt{1-z}\right) - \left(A^{(1)} + 1\right) e^{-\gamma}}{1-s} - \frac{e^{-\gamma} \left(A^{(1)} - A^{(2)}\right) + B^{(2)} \left(1 + \sqrt{1-z}\right)}{\sqrt{1-s}} = 0 \end{cases}$$

All the terms are independent so we finally get a four equations linear system in the four variables $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}$:

$$\begin{cases} A^{(1)} + 1 - B^{(1)} e^{-\gamma} \left(1 - \sqrt{1-z}\right) = 0 \\ A^{(1)} + A^{(2)} - B^{(2)} e^{-\gamma} \left(1 - \sqrt{1-z}\right) = 0 \\ B^{(1)} \left(1 + \sqrt{1-z}\right) - \left(A^{(1)} + 1\right) e^{-\gamma} = 0 \\ e^{-\gamma} \left(A^{(1)} - A^{(2)}\right) + B^{(2)} \left(1 + \sqrt{1-z}\right) = 0 \end{cases}$$

which has solution:

$$\begin{array}{ll} A^{(1)} = -1 & A^{(2)} = \frac{\cosh\gamma + \sqrt{1-z}\sinh\gamma}{\sqrt{1-z}\cosh\gamma + \sinh\gamma} \\ B^{(1)} = 0 & B^{(2)} = \frac{1}{\sqrt{1-z}\cosh\gamma + \sinh\gamma} \end{array} \tag{4.37}$$

On substituting (4.37) into (4.18) we get:

$$\begin{aligned} &\frac{z\left[e^{\gamma}\sqrt{1-z}\left(1+\sqrt{1-z}\right) + e^{-\gamma}\sqrt{1-z}\left(1-\sqrt{1-z}\right)\right]}{\sqrt{1-z}\left[\sqrt{1-z}\cosh\gamma + \sinh\gamma\right]^3}\frac{1}{1-s} \\ &= \frac{z\left[e^{\gamma}\left(1+\sqrt{1-z}\right) + e^{-\gamma}\left(1-\sqrt{1-z}\right)\right]}{\left[\sqrt{1-z}\cosh\gamma + \sinh\gamma\right]^3}\frac{1}{1-s} \\ &= \frac{8z}{b}e^{-2\gamma}\frac{\left(\sqrt{1-z}+1\right) - \left(\sqrt{1-z}-1\right)e^{-2\gamma}}{\left[\left(\sqrt{1-z}+1\right) + \left(\sqrt{1-z}-1\right)e^{-2\gamma}\right]^3}\frac{1}{1-s} \\ &= \frac{8z}{b}e^{-2\gamma}\frac{u(z) - v(z)e^{-2\gamma}}{\left[u(z) + v(z)e^{-2\gamma}\right]^3}\frac{1}{1-s} \end{aligned}$$

which perfectly matches with (4.19), therefore the approach is consistent.

Chapter 5

First order Gamma distribution

We consider now a first order Gamma distribution:

$$f_1(x) = \frac{|x|}{2b^2} e^{-\frac{|x|}{b}} \tag{5.1}$$

with:

$$\langle x \rangle = 0 \qquad \sigma^2 = \langle x^2 \rangle = 6b^2$$

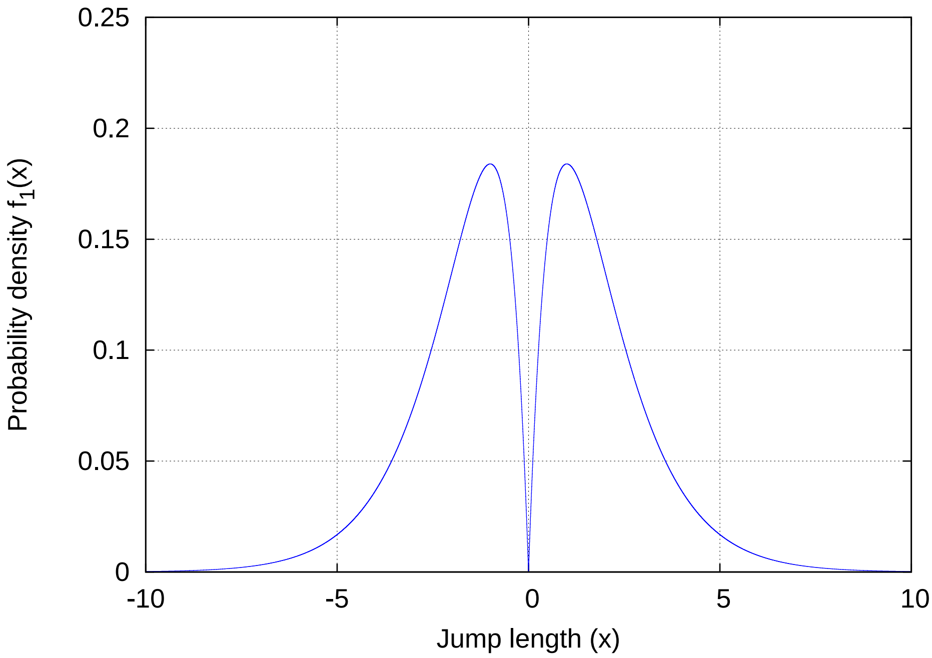


Figure 5.1: First order Gamma distribution (5.1) with $b = 1$.

It can be shown that a relation analog to (4.2) holds:

$$b^4 \frac{d^4}{dx^4} f_1(x) - 2b^2 \frac{d^2}{dx^2} f_1(x) + f_1(x) = \left[1 + b^2 \frac{d^2}{dx^2}\right] \delta(x) \tag{5.2}$$

5.1 Mean value of the k-th gap

Exploiting relation (5.2) the Wiener-Hopf integral (3.4) can be reduced to:

$$b^4 \frac{\partial^4}{\partial x^4} q_{k,n}(x) - 2b^2 \frac{\partial^2}{\partial x^2} q_{k,n}(x) + q_{k,n}(x) = b^2 \frac{\partial^2}{\partial x^2} q_{k,n-1}(x) + q_{k,n-1}(x)$$

Switching to the generating function representation:

$$b^4 \frac{\partial^4}{\partial x^4} \tilde{q}(s,z;x) - b^2 (s+2) \frac{\partial^2}{\partial x^2} \tilde{q}(s,z;x) + (1-s)\tilde{q}(s,z;x) = 1$$

This linear differential equation is easily solvable. Performing the change of variable

$$\tilde{q} = g + \frac{1}{1-s}$$

we recover an homogeneous equation for g:

$$b^4 \frac{\partial^4 g}{\partial x^4} - b^2 (s+2) \frac{\partial^2 g}{\partial x^2} + (1-s)g = 0$$

the solution is a superposition of exponentials, so setting $g(s,z;x) = A(s,z)e^{-\lambda x}$ we obtain the **characteristic equation**:

$$b^4 \lambda^4 - b^2 (s+2) \lambda^2 + 1 - s = 0$$

which has the four real solutions $(0 < s < 1)$:

$$\lambda_1 = \sqrt{\tfrac{s+2-\sqrt{s(s+8)}}{2b^2}} \qquad \lambda_2 = \sqrt{\tfrac{s+2+\sqrt{s(s+8)}}{2b^2}}$$

$$\lambda_3 = -\sqrt{\tfrac{s+2-\sqrt{s(s+8)}}{2b^2}} \qquad \lambda_4 = -\sqrt{\tfrac{s+2+\sqrt{s(s+8)}}{2b^2}}$$

Remark 5.
In the range $0 < s < 1$ both the radicands are always positive, so all the solutions are real.

The general solution is a linear combination of them:

$$g(s,z;x) = A_1(s,z)e^{-\lambda_1 x} + A_2(s,z)e^{-\lambda_2 x} + A_3(s,z)e^{-\lambda_3 x} + A_4(s,z)e^{-\lambda_4 x}$$

For avoiding divergences we need to set $A_3 = A_4 = 0$. Returning to $\tilde{q}$:

$$\tilde{q}(s,z;x) = A_1(s,z)e^{-\lambda_1 x} + A_2(s,z)e^{-\lambda_2 x} + \frac{1}{1-s}$$

By symmetry:

$$\tilde{r}(s,z;x) = B_1(s,z)\, e^{-\eta_1 x} + B_2(s,z)\, e^{-\eta_2 x} + \frac{1}{1-sz}$$

where:

$$\eta_1 = \sqrt{\frac{sz+2-\sqrt{sz(sz+8)}}{2b^2}} \qquad \eta_2 = \sqrt{\frac{sz+2+\sqrt{sz(sz+8)}}{2b^2}}$$

and $B_i \equiv A_i\left(sz, \frac{1}{z}\right)$. For evaluating the amplitudes (A_1, A_2, B_1, B_2), let's use relations (3.11) and the results from (A):

$$
\begin{aligned}
\tilde{q}(s,z;x) &= 1 + s\int_0^\infty \left[A_1(s,z)e^{-\lambda_1 x'} + A_2(s,z)e^{-\lambda_2 x'} + \frac{1}{1-s}\right] f(x'-x)dx' \\
&\quad + sz\int_0^\infty \left[B_1(s,z)e^{-\eta_1 x'} + B_2(s,z)e^{-\eta_2 x'} + \frac{1}{1-sz}\right] f(x'+x)dx' \\
&= 1 + A_1 s\frac{2be^{-\lambda_1 x}\left(\lambda_1^2 b^2+1\right) - \left(\lambda_1 b+1\right)^2\left[b + x\left(1-\lambda_1 b\right)\right]e^{-\frac{x}{b}}}{2b\left(\lambda_1^2 b^2 - 1\right)^2} \\
&\quad + A_2 s\frac{2be^{-\lambda_2 x}\left(\lambda_2^2 b^2+1\right) - \left(\lambda_2 b+1\right)^2\left[b + x\left(1-\lambda_2 b\right)\right]e^{-\frac{x}{b}}}{2b\left(\lambda_2^2 b^2 - 1\right)^2} \\
&\quad + \frac{s}{1-s}\left(1 - \frac{e^{-\frac{x}{b}}}{2b}(x+b)\right) + B_1 sz\frac{e^{-\frac{x}{b}}\left[b + x\left(1+\eta_1 b\right)\right]}{2b\left(\eta_1 b+1\right)^2} \\
&\quad + B_2 sz\frac{e^{-\frac{x}{b}}\left[b + x\left(1+\eta_2 b\right)\right]}{2b\left(\eta_2 b+1\right)^2} + \frac{sz}{1-sz}\frac{e^{-\frac{x}{b}}}{2b}(x+b)
\end{aligned}
$$

exploiting the non-trivial relation:

$$
K\frac{\Lambda^2 b^2 + 1}{\left(\Lambda^2 b^2 - 1\right)^2} = 1 \qquad\qquad K = \{s, sz\} \quad \Lambda = \{\lambda, \eta\}
$$

One gets the equation:

$$
\begin{aligned}
&-A_1 s\frac{b + x\left(1-\lambda_1 b\right)}{\left(\lambda_1 b - 1\right)^2} - A_2 s\frac{b + x\left(1-\lambda_2 b\right)}{\left(\lambda_2 b - 1\right)^2} + B_1 sz\frac{b + x\left(1+\eta_1 b\right)}{\left(\eta_1 b + 1\right)^2} \\
&+ B_2 sz\frac{b + x\left(1+\eta_2 b\right)}{\left(\eta_2 b + 1\right)^2} + (x+b)\left[\frac{sz}{1-sz} - \frac{s}{1-s}\right] = 0
\end{aligned}
$$

which can conveniently be rewritten as:

$$
\begin{aligned}
&\left[-\frac{A_1 sb}{\left(1-\lambda_1 b\right)^2} - \frac{A_2 sb}{\left(1-\lambda_2 b\right)^2} + \frac{B_1 szb}{\left(\eta_1 b+1\right)^2} + \frac{B_2 szb}{\left(\eta_2 b+1\right)^2} + \frac{bs\left(z-1\right)}{\left(1-s\right)\left(1-sz\right)}\right] \\
&+ x\left[-\frac{A_1 s}{1-\lambda_1 b} - \frac{A_2 s}{1-\lambda_2 b} - \frac{B_1 sz}{\eta_1 b+1} - \frac{B_2 sz}{\eta_2 b+1} + \frac{s\left(z-1\right)}{\left(1-s\right)\left(1-sz\right)}\right] = 0
\end{aligned}
$$

Since the condition must be true whatever the value of x, we need to set both the contents of the square brackets to be zero:

$$
\begin{cases}
\dfrac{A_1}{\left(1-\lambda_1 b\right)^2} + \dfrac{A_2}{\left(1-\lambda_2 b\right)^2} - \dfrac{B_1 z}{\left(\eta_1 b+1\right)^2} - \dfrac{B_2 z}{\left(\eta_2 b+1\right)^2} + \dfrac{1-z}{\left(1-s\right)\left(1-sz\right)} = 0 \\
\dfrac{A_1}{1-\lambda_1 b} + \dfrac{A_2}{1-\lambda_2 b} - \dfrac{B_1 z}{\eta_1 b+1} - \dfrac{B_2 z}{\eta_2 b+1} + \dfrac{1-z}{\left(1-s\right)\left(1-sz\right)} = 0
\end{cases}
$$

The other two equations are provided by symmetry, hence we obtain the linear system:

$$\begin{cases} \dfrac{A_1}{1-\lambda_1 b} + \dfrac{A_2}{1-\lambda_2 b} - \dfrac{B_1 z}{\eta_1 b+1} - \dfrac{B_2 z}{\eta_2 b+1} + \dfrac{1-z}{(1-s)\,(1-sz)} = 0 \\ \dfrac{A_1}{(1-\lambda_1 b)^2} + \dfrac{A_2}{(1-\lambda_2 b)^2} - \dfrac{B_1 z}{(\eta_1 b+1)^2} - \dfrac{B_2 z}{(\eta_2 b+1)^2} + \dfrac{1-z}{(1-s)\,(1-sz)} = 0 \\ \dfrac{B_1 z}{1-\eta_1 b} + \dfrac{B_2 z}{1-\eta_2 b} - \dfrac{A_1}{\lambda_1 b+1} - \dfrac{A_2}{\lambda_2 b+1} - \dfrac{1-z}{(1-s)\,(1-sz)} = 0 \\ \dfrac{B_1 z}{(\eta_1 b-1)^2} + \dfrac{B_2 z}{(\eta_2 b-1)^2} - \dfrac{A_1}{(\lambda_1 b+1)^2} - \dfrac{A_2}{(\lambda_2 b+1)^2} - \dfrac{1-z}{(1-s)\,(1-sz)} = 0 \end{cases}$$

which has solution:

$$\begin{cases} A_1 = \dfrac{\lambda_2\eta_1\eta_2\left(\lambda_1^2 b^2-1\right)^2(1-z)}{(\lambda_1-\lambda_2)\,(\lambda_1+\eta_1)\,(\lambda_1+\eta_2)\,(1-s)\,(1-sz)} \\ A_2 = -\dfrac{\lambda_1\eta_1\eta_2\left(\lambda_2^2 b^2-1\right)^2(1-z)}{(\lambda_1-\lambda_2)\,(\lambda_2+\eta_1)\,(\lambda_2+\eta_2)\,(1-s)\,(1-sz)} \\ B_1 = -\dfrac{\lambda_1\lambda_2\eta_2\left(\eta_1^2 b^2-1\right)^2(1-z)}{z\,(\eta_1-\eta_2)\,(\eta_1+\lambda_1)\,(\eta_1+\lambda_2)\,(1-s)\,(1-sz)} \\ B_2 = \dfrac{\lambda_1\lambda_2\eta_1\left(\eta_2^2 b^2-1\right)^2(1-z)}{z\,(\eta_1-\eta_2)\,(\eta_2+\lambda_1)\,(\eta_2+\lambda_2)\,(1-s)\,(1-sz)} \end{cases}$$

Now that the auxiliary variables are known, let's compute the first moment of the k-th gap starting from its generating function (4.34). For proceeding we need first to evaluate the quantity $\hat{r}(s;x)$:

$$\hat{r}(s;x) = C_1(s)e^{-\lambda_1 x} + C_2(s)e^{-\lambda_2 x} + \frac{1}{1-s}$$

Using (4.33) we get an equation for the coefficients:

$$C_1\frac{b+x\,(1-\lambda_1 b)}{(1-\lambda_1 b)^2} + C_2\frac{b+x\,(\lambda_2 b-1)}{(\lambda_2 b-1)^2} + \frac{x+b}{1-s} = 0 \tag{5.3}$$

Equation (5.3) has to be true whatever the value of x, hence:

$$\begin{cases} \dfrac{C_1}{(\Lambda_1 b-1)^2} + \dfrac{C_2}{(\Lambda_2 b-1)^2} + \dfrac{1}{1-l} = 0 \\ \dfrac{C_1}{1-\Lambda_1 b} + \dfrac{C_2}{1-\Lambda_2 b} + \dfrac{1}{1-l} = 0 \end{cases} \tag{5.4}$$

The system (5.4) has solution:

$$\begin{cases} C_1(s) = -\dfrac{(\lambda_1 b-1)^2\,\lambda_2}{(1-s)(\lambda_2-\lambda_1)} \\ C_2(s) = -\dfrac{(\lambda_2 b-1)^2\,\lambda_1}{(1-s)(\lambda_1-\lambda_2)} \end{cases}$$

Therefore, on substituting in (4.34):

$$\left\langle \tilde{d}(s,z) \right\rangle = \frac{A_1 - C_1(s)}{\lambda_1} + \frac{A_2 - C_2(s)}{\lambda_2} + z\frac{B_1 - C_1(sz)}{\eta_1} + z\frac{B_2 - C_2(sz)}{\eta_2}$$

Taking the $s \to 1$ limit:

$$\left\langle \tilde{d}(s,z) \right\rangle \overset{s\to 1}{\approx} \sigma \left[\frac{\sqrt{z+2+\sqrt{z(z+8)}} + \sqrt{z+2-\sqrt{z(z+8)}}}{2\sqrt{3}\sqrt{1-z}} - \sqrt{\frac{2}{3}} \right] \frac{1}{1-s}$$

hence:

$$\begin{aligned} \sum_{k=1}^{\infty} z^k \left\langle d_{k,\infty} \right\rangle &= \sigma \left[\frac{\sqrt{z+2+\sqrt{z(z+8)}} + \sqrt{z+2-\sqrt{z(z+8)}}}{2\sqrt{3}\sqrt{1-z}} - \sqrt{\frac{2}{3}} \right] \\ &\overset{z\to 1}{\approx} \frac{\sigma\sqrt{6}}{2\sqrt{3}\sqrt{1-z}} = \frac{\sigma}{\sqrt{2}\sqrt{1-z}} \end{aligned} \tag{5.5}$$

We expects the scaling $\left\langle d_{k,\infty} \right\rangle \overset{k\gg 0}{\approx} \frac{A}{\sqrt{k}}$, so:

$$A \sum_{k=1}^{\infty} \frac{z^k}{\sqrt{k}} = \frac{A\sqrt{\pi}}{\sqrt{1-z}} \tag{5.6}$$

comparing (5.6) with (5.5):

$$A = \frac{\sigma}{\sqrt{2\pi}}$$

so:

$$\frac{\left\langle d_{k,\infty} \right\rangle}{\sigma} \overset{k\gg 0}{\approx} \frac{1}{\sqrt{2\pi k}}$$

The asymptotic behavior of the k-th gap mean value coincides with the one computed from an exponential jump distribution. This is a check of the reliability of the *Pollaczek-Wendel* identity (4.13).

5.2 Probability distribution of the k-th gap

We want now to extract the full k-th gap distribution and check if the claims of universality for typical fluctuations holds for a first order Gamma distribution. From prior results, the generating function of the auxiliary variable $Q_{k,n}(x,\delta)$ reads:

$$\tilde{Q}(s,z;x,\delta) = A_1(s,z;\delta)e^{-\lambda_1 x} + A_2(s,z;\delta)e^{-\lambda_2 x} + \frac{1}{1-s}$$

Injecting this expression in the integral equation (3.12), we obtain an equation for the amplitudes:

$$\begin{aligned} &-\frac{x+b}{1-s} - \frac{A_1\left[b + x(1-\lambda_1 b)\right]}{(\lambda_1 b - 1)^2} - \frac{A_2\left[b + x(1-\lambda_2 b)\right]}{(\lambda_2 b - 1)^2} + \frac{ze^{-\frac{\delta}{b}}}{1-sz}(b+x+\delta) \\ &B_1 z e^{-\frac{\delta}{b}} \frac{b + (x+\delta)(\eta_1 b + 1)}{(\eta_1 b + 1)^2} + B_2 z e^{-\frac{\delta}{b}} \frac{b + (x+\delta)(\eta_2 b + 1)}{(\eta_2 b + 1)^2} = 0 \end{aligned} \tag{5.7}$$

Since (5.7) must be zero whatever the value of x, it splits in two independent equations. Moreover using symmetry other two equations are provided, hence we get a four equations linear systems in the four variables A_1, A_2, B_1, B_2 (in the following $\gamma := \frac{\delta}{b}$):

$$\begin{cases} \frac{A_1}{1-\lambda_1 b} + \frac{A_2}{1-\lambda_2 b} + \frac{B_1 z e^{-\gamma}}{\eta_1 b+1} + \frac{B_2 z e^{-\gamma}}{\eta_2 b+1} + \frac{1}{1-s} - \frac{z e^{-\gamma}}{1-sz} = 0 \\ \frac{B_1 z}{1-\eta_1 b} + \frac{B_2 z}{1-\eta_2 b} - \frac{A_1 e^{-\gamma}}{\lambda_1 b+1} - \frac{A_2 e^{-\gamma}}{\lambda_2 b+1} + \frac{z}{1-sz} - \frac{e^{-\gamma}}{1-s} = 0 \\ \frac{A_1}{(\lambda_1 b-1)^2} + \frac{A_2}{(\lambda_2 b-1)^2} - \frac{B_1 z e^{-\gamma}}{\eta_1 b+1}\left[\frac{1}{\eta_1 b+1} + \gamma\right] \\ \quad - \frac{B_2 z e^{-\gamma}}{\eta_2 b+1}\left[\frac{1}{\eta_2 b+1} + \gamma\right] + \frac{1}{1-s} - \frac{z e^{-\gamma}}{1-sz}(1+\gamma) = 0 \\ \frac{B_1 z}{(\eta_1 b-1)^2} + \frac{B_2 z}{(\eta_2 b-1)^2} - \frac{A_1 e^{-\gamma}}{\lambda_1 b+1}\left[\frac{1}{\lambda_1 b+1} + \gamma\right] \\ \quad - \frac{A_2 e^{-\gamma}}{\lambda_2 b+1}\left[\frac{1}{\lambda_2 b+1} + \gamma\right] + \frac{z}{1-sz} - \frac{e^{-\gamma}}{1-s}(1+\gamma) = 0 \end{cases} \tag{5.8}$$

The solutions of (5.8) are rather messy, too much to be handled. So we need first to consider the asymptotic behavior of the system, as explained in (4.3.2):

$$\begin{array}{ll} A_1(s,z;\gamma) \approx \frac{A_1^{(1)}(z;\gamma)}{1-s} + \frac{A_1^{(2)}(z;\gamma)}{\sqrt{1-s}} & A_2(s,z;\gamma) \approx \frac{A_2^{(1)}(z;\gamma)}{1-s} + \frac{A_2^{(2)}(z;\gamma)}{\sqrt{1-s}} \\ B_1(s,z;\gamma) \approx \frac{B_1^{(1)}(z;\gamma)}{1-s} + \frac{B_1^{(2)}(z;\gamma)}{\sqrt{1-s}} & B_2(s,z;\gamma) \approx \frac{B_2^{(1)}(z;\gamma)}{1-s} + \frac{B_2^{(2)}(z;\gamma)}{\sqrt{1-s}} \end{array} \tag{5.9}$$

on substituting relations (5.9) into (5.8) and expanding around $s = 1$, we obtain four equations of the general form:

$$\frac{P\left(A_i^{(j)}, B_k^{(l)}\right)}{1-s} + \frac{Q\left(A_i^{(j)}, B_k^{(l)}\right)}{\sqrt{1-s}} + \mathscr{O}(1) = 0 \qquad (i,j,k,l = 1,2)$$

where P, Q are polynomials linear in the amplitudes $\left(A_i^{(j)}, B_k^{(l)}\right)$. Given that each term of the expansion is independent the linear system of four equations splits in a linear system of eight equations in the eight amplitudes defined in (5.9). This system has solution:

$$\begin{array}{ll} A_1^{(1)} = -1 & A_1^{(2)} = A_1^{(2)}(z;\gamma) \\ A_2^{(1)} = 0 & A_2^{(2)} = A_2^{(2)}(z;\gamma) \\ B_1^{(1)} = 0 & B_1^{(2)} = B_1^{(2)}(z;\gamma) \\ B_2^{(1)} = 0 & B_2^{(2)} = B_2^{(2)}(z;\gamma) \end{array} \tag{5.10}$$

where the second row is still too heavy to be presented.
Now that the amplitudes are known, let's evaluate the generating function of the gap distribution in the $s \to 1$ limit $\tilde{p}(z;\gamma)$. In order to do so, we start from relation

(4.16). The double derivatives of the auxiliary variables are:

$$\begin{aligned}\frac{\partial^2}{\partial x\partial y}\tilde{Q}(s,z;x,\gamma) &= -\frac{e^{-\lambda_1 x}}{b^2}\left[b\lambda_1\frac{\partial}{\partial\gamma}+\frac{\partial^2}{\partial\gamma^2}\right]A_1\left(s,z;\gamma\right)\\ &\quad -\frac{e^{-\lambda_2 x}}{b^2}\left[b\lambda_2\frac{\partial}{\partial\gamma}+\frac{\partial^2}{\partial\gamma^2}\right]A_2\left(s,z;\gamma\right)\\ \frac{\partial^2}{\partial x\partial y}\tilde{R}(s,z;-y,\gamma) &= -\frac{e^{\eta_1 y}}{b^2}\left[b\eta_1\frac{\partial}{\partial\gamma}+\frac{\partial^2}{\partial\gamma^2}\right]B_1\left(s,z;\gamma\right)\\ &\quad -\frac{e^{\eta_2 y}}{b^2}\left[b\eta_2\frac{\partial}{\partial\gamma}+\frac{\partial^2}{\partial\gamma^2}\right]B_2\left(s,z;\gamma\right)\end{aligned} \tag{5.11}$$

Injecting (5.11) into (4.16) we obtain:

$$\begin{aligned}\tilde{P}\left(s,z;\gamma\right) = \frac{1}{b}\Bigg[&\left(\frac{\partial}{\partial\gamma}+\frac{1}{\lambda_1 b}\frac{\partial^2}{\partial\gamma^2}\right)A_1\left(s,z;\gamma\right)+\left(\frac{\partial}{\partial\gamma}+\frac{1}{\lambda_2 b}\frac{\partial^2}{\partial\gamma^2}\right)A_2\left(s,z;\gamma\right)\\ &+ze^{-\eta_1 b\gamma}\left(\frac{\partial}{\partial\gamma}+\frac{1}{\eta_1 b}\frac{\partial^2}{\partial\gamma^2}\right)B_1\left(s,z;\gamma\right)\\ &+ze^{-\eta_2 b\gamma}\left(\frac{\partial}{\partial\gamma}+\frac{1}{\eta_2 b}\frac{\partial^2}{\partial\gamma^2}\right)B_2\left(s,z;\gamma\right)\Bigg]\end{aligned} \tag{5.12}$$

Since we know the amplitudes in the $s \to 1$ limit (5.10), we need to study the asymptotic behavior of (5.12). In order to so we have to inject the (5.10) into (5.12) and substitute for the roots their limiting behavior, that are found to be:

$$\lambda_1 b \overset{s\to 1}{\approx} \frac{\sqrt{1-s}}{\sqrt{3}} \qquad \lambda_2 b \overset{s\to 1}{\approx} \sqrt{3}$$

$$\eta_1 b \overset{s\to 1}{\approx} \frac{\sqrt{2+z-\sqrt{z(z+8)}}}{\sqrt{2}} \qquad \eta_2 b \overset{s\to 1}{\approx} \frac{\sqrt{2+z+\sqrt{z(z+8)}}}{\sqrt{2}}$$

All the derivatives with respect to γ of (5.12) acting on (5.10) share, for $s \to 1$, the same asymptotic behaviour $\frac{1}{\sqrt{1-s}}$. As consequence all the terms of (5.12) scale as $\frac{1}{\sqrt{1-s}}$ with except for the one containing the inverse of the root λ_1 which scales as $\frac{1}{1-s}$. This is the leader term and it is the only one relevant in the limit $s \to 1$:

$$\tilde{P}\left(s,z;\gamma\right) \overset{s\to 1}{\approx} \frac{1}{\lambda_1 b^2}\frac{\partial^2}{\partial\gamma^2}A_1\left(s,z;\gamma\right)\Bigg|_{s\to 1} = \tilde{p}\left(z;\gamma\right)\frac{1}{1-s}$$

where $\tilde{p}\left(z;\gamma\right)$ is the generating function of the gap distribution in the large $s \to 1$ limit:

$$\tilde{p}\left(z;\gamma\right) = \sum_{k=1}^{\infty} z^k p_{k,\infty}\left(\gamma\right) \tag{5.13}$$

Remark 6.
The quantity $\tilde{p}\left(z;\gamma\right)$ is still to heavy for being transcribed in the document.

We investigate now the two scaling regimes of typical and large fluctuations.

1. **Typical fluctuations** $\delta, \gamma \sim 1/\sqrt{k}$
   For testing the claim of universality, let's consider the limit of $z \to 1$ keeping the ratio $(z-1)/\gamma^2$ constant for selecting typical fluctuations. Equation (5.13) approaches to:

$$\tilde{p}\,(z;\gamma) \overset{\text{t.f.}}{\approx} \frac{1}{b} \frac{18\sqrt{3}}{\left(\sqrt{3}+3\frac{\sqrt{t}}{\gamma}\right)^3} \frac{1}{\gamma^3}$$

   where $t = 1 - z$. Substituting now:

$$b = \frac{\sigma}{\sqrt{6}} \qquad\qquad \gamma = \frac{\delta}{b} = \sqrt{6}\frac{\delta}{\sigma}$$

   We obtain exactly the same asymptotic behaviour as for the exponential distribution

$$\tilde{p}\,(z;\gamma) \overset{\text{t.f.}}{\approx} \frac{\sigma^2}{\delta^3} \left(1 + \sqrt{\frac{\lambda}{2}}\right)^{-3}$$

   and so the same scaling function (2).

2. **Large fluctuations** $\delta, \gamma \gg 0$
   Expanding $\tilde{p}(z;\gamma)$ around $z = 1$ as in (4.27), we can extract the scaling function $\varphi_1\,(\gamma)$. It turns out to be rather complicate, anyway its asymptotic behavior is:

$$\varphi_1(\gamma) \approx \begin{cases} \frac{9}{b}\sqrt{\frac{3}{\pi}}\gamma^{-4} & \gamma \to 0 \\ \frac{C}{b}\gamma^2 e^{-2\gamma} & \gamma \to \infty \end{cases} \tag{5.14}$$

where C is a large numerical coefficient. As before, combining the two fluctuations regimes:

$$p_k(\gamma) \overset{k\gg 0}{\approx} \begin{cases} \frac{1}{b}\sqrt{\frac{k}{6}} P\left(\sqrt{\frac{k}{6}}\gamma\right) & \gamma \sim 1/\sqrt{k} \\ \varphi_1(\gamma) k^{-\frac{3}{2}} & \gamma \gg 0 \end{cases}$$

it turns out that their asymptotic behavior match when approaching to each other

$$\begin{cases} \frac{1}{b}\sqrt{\frac{k}{6}} P\left(\sqrt{\frac{k}{6}}\gamma\right) \xrightarrow{\gamma\gg 0} \frac{1}{b}\sqrt{\frac{k}{6}}\frac{3}{\sqrt{8\pi}}\frac{36}{k^2}\gamma^{-4} = \frac{9}{b}\sqrt{\frac{3}{\pi}}\gamma^{-4}k^{-\frac{3}{2}} \\ \varphi_1(\gamma)k^{-\frac{3}{2}} \xrightarrow{\gamma\to 0} \frac{9}{b}\sqrt{\frac{3}{\pi}}\gamma^{-4}k^{-\frac{3}{2}} \end{cases}$$

as a proof that the chosen scaling functions are correct.

Chapter 6

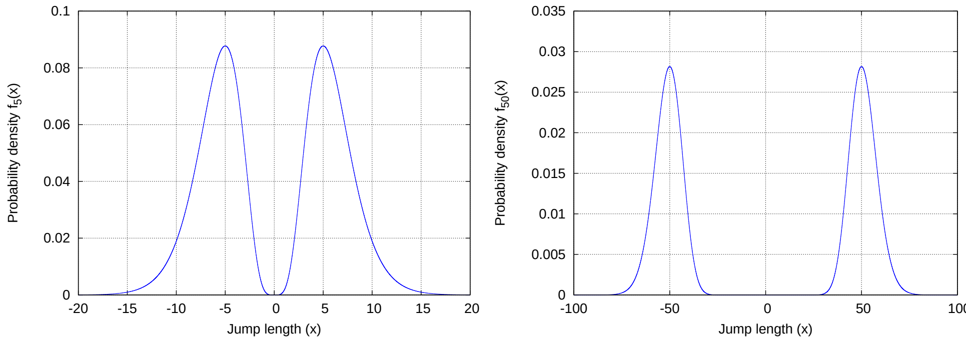
General p-th order Gamma distribution

We consider now a general p-th order Gamma distribution:

$$f_p(x) = \frac{|x|^p}{2p!b^{p+1}} e^{-\frac{|x|}{b}} \qquad\qquad p \in \mathbb{N} \tag{6.1}$$

By symmetry, the mean value is zero, while the variance is:

$$\sigma^2 = \frac{1}{2p!b^{p+1}} \int_{-\infty}^{+\infty} x^2 |x|^p e^{-\frac{|x|}{b}} dx = \frac{1}{p!b^{p+1}} \int_0^{\infty} x^{p+2} e^{-\frac{x}{b}} dx$$
$$= \frac{b^2}{p!} \int_0^{\infty} y^{p+2} e^{-y} dy = \frac{b^2}{p!} (p+2)! = b^2 (p+1)(p+2)$$



(a) $p = 5$ (b) $p = 50$

Figure 6.1: Graphs of some general order Gamma distributions with $b = 1$.

In the prior cases for reducing the Wiener-Hopf integrals (3.4) and (3.10) in recurrent differential equation, we exploited some identities valid for the jump distributions, see (4.2) and (5.2). The aim is now to find an identity for the general p-th order Gamma distribution. We start on evaluating the derivative of (6.1):

$$\frac{d}{dx} f_p = \frac{\sigma(x)}{b} [f_{p-1} - f_p] \tag{6.2}$$

where $\sigma(x)$ is the sign function:

$$\sigma(x) = \Theta(x) - \Theta(-x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

whose derivative is:

$$\frac{d}{dx}\sigma(x) = 2\delta(x)$$

The second derivative of (6.1) reads:

$$\begin{aligned} \frac{d^2}{dx^2} f_p &= \frac{2\delta(x)}{b}\left[f_{p-1} - f_p\right] + \frac{1}{b^2}\left[f_p - 2f_{p-1} + f_{p-2}\right] \\ &= \frac{\delta(x)}{b^2}\left[\delta_{p,1} - \delta_{p,0}\right] + \frac{1}{b^2}\left[f_p - 2f_{p-1} + f_{p-2}\right] \end{aligned} \tag{6.3}$$

Or case by case:

$$f_p'' = \begin{cases} f_0 - \delta(x) & p = 0 \\ f_1 - 2f_0 + \delta(x) & p = 1 \\ f_p - 2f_{p-1} + f_{p-2} & p \geq 2 \end{cases}$$

Deriving an even number of times (6.1), we obtain a combination of lower order distributions and derivatives of delta functions. By defining the rescaled linear differential operator

$$D^2 = b^2 \frac{d^2}{dx^2} \tag{6.4}$$

we can rewrite (6.3) as:

$$D^2 f_p = (-1)^{p+1} \binom{1}{p} \delta(x) + \sum_{l=0}^{2} \binom{2}{l} (-1)^l f_{p-l}$$

Applying (6.4) a second time it can be shown that (B):

$$D^4 f_p = (-1)^{p+1} \left[\binom{1}{p} D + \binom{3}{p}\right] \delta(x) + \sum_{l=0}^{4} \binom{4}{l} (-1)^l f_{p-l}$$

similarly:

$$D^6 f_p = (-1)^{p+1} \left[\binom{1}{p} D^2 + \binom{3}{p} D + \binom{5}{p}\right] \delta(x) + \sum_{l=0}^{6} \binom{6}{l} (-1)^l f_{p-l}$$

Thus we can generalize for k times:

$$D^{2k} f_p = (-1)^{p+1} \sum_{m=0}^{k-1} \binom{2k-2m-1}{p} D^{2m}\delta(x) + \sum_{l=0}^{2k} \binom{2k}{l} (-1)^l f_{p-l} \tag{6.5}$$

We look now for a linear combination of even derivatives that removes the lower order distributions terms, i.e. the second R.H.S. term in (6.5). From equations (4.2) and (5.2) we get the hint of taking a combination of the sort:

$$\sum_{k=0}^{p+1} \binom{p+1}{k} (-1)^k D^{2k} f_p = \left(1 - D^2\right)^{p+1} f_p \tag{6.6}$$

Summing the (6.5) in this way indeed, one finds that the lower order distributions term cancels while the term of delta functions assumes a simple form. These calculations involving sums and binomials are rather heavy, it is way faster to work in the Fourier space. Let's evaluate the Fourier transform of (6.1):

$$\begin{aligned}
\hat{f}_p(s) = \mathscr{F}\left[f_p(x)\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-isx} f_p(x) dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2p! b^{p+1}} \int_{-\infty}^{+\infty} e^{-isx} |x|^p e^{-\frac{|x|}{b}} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2p!} \int_{-\infty}^{+\infty} e^{-ibsx} |x|^p e^{-|x|} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2p!} \left[\int_0^{\infty} x^p e^{-(1+ibs)x} dx + (-1)^p \int_{-\infty}^{0} x^p e^{(1-ibs)x} dx\right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{2p!} \left[\int_0^{\infty} x^p e^{-(1+ibs)x} dx + \int_0^{\infty} x^p e^{-(1-ibs)x} dx\right]
\end{aligned}$$

The first R.H.S. integral reads:

$$\begin{aligned}
\int_0^{\infty} x^p e^{-(1+ibs)x} dx &= -\frac{1}{1+ibs} x^p e^{-(1+ibs)x}\Big|_0^{\infty} + \frac{p}{1+ibs} \int_0^{\infty} x^{p-1} e^{-(1+ibs)x} dx \\
&= \frac{p}{1+ibs} \int_0^{\infty} x^{p-1} e^{-(1+ibs)x} dx \\
&\ \vdots \\
&= \frac{p!}{(1+ibs)^p} \int_0^{\infty} e^{-(1+ibs)x} dx = \frac{p!}{(1+ibs)^{p+1}}
\end{aligned}$$

so the Fourier transform of a p-th order Gamma distribution is:

$$\hat{f}_p(s) = \frac{1}{2\sqrt{2\pi}} \left[\frac{1}{(1+ibs)^{p+1}} + \frac{1}{(1-ibs)^{p+1}}\right]$$

The Fourier transform of the R.H.S. of (6.6) is simply:

$$\begin{aligned}
\mathscr{F}\left[\left(1 - D^2\right)^{p+1} f_p(x)\right] &= \left(1 + b^2 s^2\right)^{p+1} \hat{f}_p(s) \\
&= \frac{\left(1 + b^2 s^2\right)^{p+1}}{2\sqrt{2\pi}} \left[\frac{1}{(1+ibs)^{p+1}} + \frac{1}{(1-ibs)^{p+1}}\right] \\
&= \frac{1}{2\sqrt{2\pi}} \left[(1+ibs)^{p+1} + (1-ibs)^{p+1}\right] \\
&= \mathscr{F}\left[\frac{1}{2}\left[(1-D)^{p+1} + (1+D)^{p+1}\right] \delta(x)\right]
\end{aligned} \tag{6.7}$$

Therefore it is evident from (6.7 that (6.1) satisfies:

$$\left(1-D^2\right)^{p+1} f_p(x) = \frac{1}{2}\left[(1-D)^{p+1} + (1+D)^{p+1}\right]\delta(x) \tag{6.8}$$

As a check, setting $p = 0, 1$, one easily recovers (4.2) and (5.2).

Remark 7.
Since the computational effort is rather heavy, for the p-th order Gamma distribution we will skip the k-th gap mean value and concern only on the full k-th gap distribution. Anyway the mean value can be computed *a posteriori* from it.

6.1 Probability distribution of the k-th gap

Using (6.8) we get a recurrent differential equation for the auxiliary variable $Q_{k,n}$:

$$\left(1-D^2\right)^{p+1} Q_{k,n}(x,\delta) = \frac{1}{2}\left[(1-D)^{p+1} + (1+D)^{p+1}\right] Q_{k,n-1}(x,\delta)$$

switching to the generating function representation:

$$\left(1-D^2\right)^{p+1}\left[\tilde{Q}\left(s,z;x,\delta\right) - 1\right] = \frac{s}{2}\left[(1-D)^{p+1} + (1+D)^{p+1}\right]\tilde{Q}\left(s,z;x,\delta\right)$$

or rather:

$$\left\{\left(1-D^2\right)^{p+1} - \frac{s}{2}\left[(1+D)^{p+1} + (1-D)^{p+1}\right]\right\}\tilde{Q}\left(s,z;x,\delta\right) = 1$$

This is a non-homogeneous linear differential equation of order $2\,(p+1)$. The change of variable

$$\tilde{Q}\left(s,z;x,\delta\right) = \tilde{f}\left(s,z;x,\delta\right) + \frac{1}{1-s}$$

let us put it in a homogeneous form:

$$\left\{\left(1-D^2\right)^{p+1} - \frac{s}{2}\left[(1+D)^{p+1} + (1-D)^{p+1}\right]\right\}\tilde{f}\left(s,z;x,\delta\right) = 0 \tag{6.9}$$

The solution is a superposition of exponential functions of the fashion:

$$\tilde{f}(s,z;x) = A(s,z,\delta)e^{-\lambda(s)x} \tag{6.10}$$

where the coefficient $\lambda(s)$ must have positive real part in order to avoid divergences. On substituting (6.10) into (6.9), we get the **characteristic equation** for the roots:

$$\left(1-t^2\right)^{p+1} - \frac{s}{2}\left[(1+t)^{p+1} + (1-t)^{p+1}\right] = 0 \tag{6.11}$$

where we implicitly defined the rescaled root $t = b\lambda$.
This is a polynomial equation of order $2\,(p+1)$, so from the *fundamental theorem of algebra*, it has $2\,(p+1)$ complex roots. Since the polynomial is even than if t is a solution also $-t$ it is. Moreover, given that all the coefficients are real, if t is a solution also $t^\star$ it is. Therefore, in the Gauss plane, the roots are symmetric with

respect to both the Imaginary and the Real axis that is, taking $a, b > 0$ such that $t = a + ib$ is a solution, all the four possible combinations $t = \pm a \pm ib$ are solutions. Let $t_m(s)$ with $m = 1, \dots, 2(p+1)$ be the roots of (6.11). We order them such that the first half has positive real part while the second half has negative real part. The general solution for $\tilde{q}$ is then:

$$\tilde{Q}\left(s, z; x, \delta\right) = \frac{1}{1-s} + \sum_{m=1}^{2(p+1)} A_m(s, z, \delta) e^{-t_m(s)\frac{x}{b}}$$

For avoiding divergences, we set all the coefficients A_m related to a negative real part roots to be zero:

$$A_m(s, z, \delta) = 0 \qquad\qquad \forall m > p+1$$

so:

$$\tilde{Q}\left(s, z; x, \delta\right) = \frac{1}{1-s} + \sum_{m=1}^{p+1} A_m(s, z, \delta) e^{-t_m(s)\frac{x}{b}} \tag{6.12}$$

and using symmetry:

$$\tilde{R}\left(s, z; x, \delta\right) = \tilde{Q}\left(sz, \frac{1}{z}; x, \delta\right) = \frac{1}{1-sz} + \sum_{m=1}^{p+1} B_m\left(s, z; \delta\right) e^{-t_m(sz)\frac{x}{b}} \tag{6.13}$$

where $B_m \equiv A\left(sz, \frac{1}{z}; \delta\right)$. The coefficients A_m, B_m can be computed using relations (3.12). Two general forms of integrals appear from these relations:

$$\begin{aligned} I_p^+(\Lambda, \delta) &= \int_0^\infty e^{-\Lambda\frac{x'}{b}} f_p\left(x' + x + \delta\right) dx' = \frac{e^{-y-\gamma}}{2\left(\Lambda+1\right)^{p+1}} \sum_{k=0}^{p} \frac{\left[\left(y+\gamma\right)\left(\Lambda+1\right)\right]^k}{k!} \\ I_p^-(\Lambda) &= \int_0^\infty e^{-\Lambda\frac{x'}{b}} f_p\left(x - x'\right) dx' = -\frac{e^{-y}}{2\left(1-\Lambda\right)^{p+1}} \sum_{r=0}^{p} \frac{\left[y\left(1-\Lambda\right)\right]^r}{r!} \\ &\qquad + \frac{e^{-\Lambda y}}{2\left(1-\Lambda\right)^{p+1}} + \frac{e^{-\Lambda y}}{2\left(1+\Lambda\right)^{p+1}} \end{aligned} \tag{6.14}$$

where $\Lambda \in \mathbb{C}$ has non negative real part. The computation has been carried out in (A).

Let's define:

$$y = x/b \qquad\qquad t_m \equiv t_m(s) \qquad\qquad u_m \equiv t_m(sz)$$

We are ready now to substitute the formal solutions of the auxiliary variables (6.12) and (6.13) into (3.12):

$$\begin{aligned} \frac{1}{1-s} + \sum_{m=1}^{p+1} A_m e^{-t_m y} = 1 + \frac{s}{1-s} I_p^-\left(0\right) + s \sum_{m=1}^{p+1} A_m I_p^-\left(t_m\right) + \frac{sz}{1-sz} I_p^+\left(0\right) \\ + sz \sum_{m=1}^{p+1} B_m I_p^+\left(u_m\right) \end{aligned}$$

using relations (6.14), we get:

$$
\begin{aligned}
&\sum_{m=1}^{p+1}\left\{s\left[\frac{e^{-t_m y}}{2\left(1-t_m\right)^{p+1}}+\frac{e^{-t_m y}}{2\left(1+t_m\right)^{p+1}}\right.\right.\\
&\qquad\left.\left.-\frac{e^{-y}}{2\left(1-t_m\right)^{p+1}}\sum_{\nu=0}^{p}\frac{y^\nu}{\nu!}\left(1-t_m\right)^\nu\right]-e^{-t_m y}\right\}A_m\\
&\qquad+sz\frac{e^{-y-\gamma}}{2}\sum_{m=1}^{p+1}\frac{B_m}{\left(1+u_m\right)^{p+1}}\sum_{\nu=0}^{p}\frac{\left(y+\gamma\right)^\nu}{\nu!}\left(1+u_m\right)^\nu\\
&\qquad-\frac{s}{1-s}\frac{e^{-y}}{2}\sum_{\nu=0}^{p}\frac{y^\nu}{\nu!}+\frac{sz}{1-sz}\frac{e^{-y-\gamma}}{2}\sum_{\nu=0}^{p}\frac{\left(y+\gamma\right)^\nu}{\nu!}=0
\end{aligned}
\tag{6.15}
$$

This equation contains polynomials and exponentials in y. Since it has to be zero whatever the value of y, we need to separate the independent terms. However the terms containing $\sum_\nu c_\nu\left(y+\gamma\right)^\nu$ make it difficult to select the different powers of y:

$$
\sum_{\nu=0}^{p}c_\nu\left(y+\gamma\right)^\nu=\sum_{\nu=0}^{p}\sum_{l=0}^{\nu}c_\nu\binom{\nu}{l}y^l\gamma^{\nu-l}=\sum_{\nu=0}^{p}\sum_{l=0}^{\nu}a_{\nu,l}
\tag{6.16}
$$

this is a sum over the triangle $0\leq l\leq\nu\leq p$, so we can use symmetry and sum over the complementary triangle, hence:

$$
\sum_{\nu=0}^{p}\sum_{l=0}^{\nu}a_{\nu,l}=\sum_{\nu=0}^{p}\sum_{l=\nu}^{p}a_{l,\nu}=\sum_{\nu=0}^{p}\sum_{l=\nu}^{p}c_l\binom{l}{\nu}y^\nu\gamma^{l-\nu}
$$

Using this trick for every term similar to (6.16), the equation (6.15) becomes:

$$
\begin{aligned}
&\sum_{m=1}^{p+1}\left\{s\left[\frac{e^{(1-t_m)y}}{\left(1-t_m\right)^{p+1}}+\frac{e^{(1-t_m)y}}{\left(1+t_m\right)^{p+1}}\right.\right.\\
&\qquad\left.\left.-\frac{1}{\left(1-t_m\right)^{p+1}}\sum_{\nu=0}^{p}\frac{y^\nu}{\nu!}\left(1-t_m\right)^\nu\right]-2e^{(1-t_m)y}\right\}A_m\\
&\qquad+sze^{-\gamma}\sum_{m=1}^{p+1}\frac{B_m}{\left(1+u_m\right)^{p+1}}\sum_{\nu=0}^{p}\sum_{l=\nu}^{p}\frac{\left(1+u_m\right)^l}{l!}\binom{l}{\nu}y^\nu\gamma^{l-\nu}\\
&\qquad-\frac{s}{1-s}\sum_{\nu=0}^{p}\frac{y^\nu}{\nu!}+\frac{sz}{1-sz}e^{-\gamma}\sum_{\nu=0}^{p}\sum_{l=\nu}^{p}\frac{1}{l!}\binom{l}{\nu}y^\nu\gamma^{l-\nu}=0
\end{aligned}
$$

rewriting it in such a way the independent terms are distinguished:

$$\begin{aligned}&\sum_{m=1}^{p+1} e^{(1-t_m)y}\left\{s\left[\frac{1}{(1-t_m)^{p+1}}+\frac{1}{(1+t_m)^{p+1}}\right]-2\right\}A_m\\&\quad+\sum_{\nu=0}^{p} y^\nu\left[-\sum_{m=1}^{p+1}\frac{sA_m}{(1-t_m)^{p+1}}\frac{1}{\nu!}(1-t_m)^\nu\right.\\&\quad+sze^{-\gamma}\sum_{m=1}^{p+1}\frac{B_m}{(1+u_m)^{p+1}}\sum_{l=\nu}^{p}\frac{(1+u_m)^l}{l!}\binom{l}{\nu}\gamma^{l-\nu}\\&\quad\left.-\frac{s}{1-s}\frac{1}{\nu!}+\frac{sz}{1-sz}e^{-\gamma}\sum_{l=\nu}^{p}\frac{1}{l!}\binom{l}{\nu}\gamma^{l-\nu}\right]=0\end{aligned} \tag{6.17}$$

Equation (6.17) splits in $p+2$ equations, $p+1$ for the different $p+1$ orders of y and an additional one for the exponential terms in y:

$$\sum_{m=1}^{p+1} e^{(1-t_m)y}\left\{s\left[\frac{1}{(1-t_m)^{p+1}}+\frac{1}{(1+t_m)^{p+1}}\right]-2\right\}A_m=0$$

Since all the $p+1$ exponential terms $e^{(1-t_m)y}$ are independent, the sum is zero if every single term is zero, so we obtain the condition:

$$\frac{1}{(1-t_m)^{p+1}}+\frac{1}{(1+t_m)^{p+1}}=\frac{2}{s}$$

which can be easily put in shape of the roots' equation (6.11). This means that the procedure is self-consistent. On the other hand, we get an equation for general ν that is:

$$\begin{aligned}&\sum_{m=1}^{p+1}\frac{A_m}{(1-t_m)^{p-\nu+1}}-\sum_{m=1}^{p+1}\left\{\frac{ze^{-\gamma}}{(1+u_m)^{p-\nu+1}}\sum_{l=0}^{p-\nu}\frac{[(1+u_m)\gamma]^l}{l!}\right\}B_m\\&\quad+\frac{1}{1-s}-\frac{ze^{-\gamma}}{1-sz}\sum_{l=0}^{p-\nu}\frac{\gamma^l}{l!}=0\end{aligned} \tag{6.18}$$

By symmetry we get from (6.18):

$$\begin{aligned}&\sum_{m=1}^{p+1}\frac{zB_m}{(1-u_m)^{p-\nu+1}}-\sum_{m=1}^{p+1}\left\{\frac{e^{-\gamma}}{(1+t_m)^{p-\nu+1}}\sum_{l=0}^{p-\nu}\frac{[(1+t_m)\gamma]^l}{l!}\right\}A_m\\&\quad+\frac{z}{1-sz}-\frac{e^{-\gamma}}{1-s}\sum_{l=0}^{p-\nu}\frac{\gamma^l}{l!}=0\end{aligned} \tag{6.19}$$

As proof of the truthfulness, setting $p=0,1$, equations (6.18) and (6.19) can be compared with the ones for the zeroth and the first order Gamma distributions.

Now that we have the equations for the coefficients of the auxiliary variables, let's reflect a little about the generating function of the k-th gap distribution. The double derivation of (6.12) and (6.13) lead to ($\gamma = (y-x)/b$):

$$\frac{\partial^2}{\partial x \partial y}\tilde{Q}(s,z;x,\gamma) = -\frac{1}{b^2}\sum_{m=1}^{p+1} e^{-t_m(s)\frac{x}{b}}\left[t_m(s)\frac{\partial}{\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right] A_m(s,z;\gamma)$$
$$\frac{\partial^2}{\partial x \partial y}\tilde{R}(s,z;-y,\gamma) = -\frac{1}{b^2}\sum_{m=1}^{p+1} e^{t_m(sz)\frac{y}{b}}\left[t_m(sz)\frac{\partial}{\partial\gamma} + \frac{\partial^2}{\partial\gamma^2}\right] B_m(s,z;\gamma)$$

Injecting them into (4.16):

$$\tilde{P}(s,z;\gamma) = \frac{1}{b}\sum_{m=1}^{p+1}\left[\left(\frac{\partial}{\partial\gamma} + \frac{1}{t_m(s)}\frac{\partial^2}{\partial\gamma^2}\right)A_m(s,z;\gamma) + ze^{t_m(sz)\gamma}\left(\frac{\partial}{\partial\gamma} + \frac{b}{t_m(sz)}\frac{\partial^2}{\partial\gamma^2}\right)B_m(s,z;\gamma)\right] \tag{6.20}$$

So in the moment we have the coefficients we have also the k-th gap distribution. However the system made of (6.18) and (6.19) can't be analytically solved, hence we consider the large n limit following the procedure explained in (4.3.2).

6.1.1 Large n limit

We assume the coefficients to scale as:

$$A_m \approx \frac{A_m^{(1)}}{1-s} + \frac{A_m^{(2)}}{\sqrt{1-s}} \qquad B_m \approx \frac{B_l^{(1)}}{1-s} + \frac{B_l^{(2)}}{\sqrt{1-s}} \tag{6.21}$$

Thus we need to investigate on the asymptotic behavior of the roots. They are known to satisfy relation (6.11) that, as we already pointed out, has $p+1$ solutions with positive real part. In the $s \to 1$ limit it turns out that all of them are (at first order) constant, with the exception of one that approaches to zero as $c_1\sqrt{1-s}$ (let us label it as the first):

$$t_1(s) \overset{s\to 1}{\approx} c_1\sqrt{1-s} \qquad t_m(s) \overset{s\to 1}{\approx} c_m \qquad u_l(sz) \overset{s\to 1}{\approx} d_l(z) \tag{6.22}$$

The coefficients c_m and $d_l(z)$ can be computed by setting $s = 1$ in their respective roots' equations:

$$c_m = m\text{-root}\left\{\left(1-t^2\right)^{p+1} - \frac{1}{2}\left[(1-t)^{p+1} + (1+t)^{p+1}\right] = 0 \mid \mathcal{R}(t) > 0\right\}$$
$$d_l(z) = l\text{-root}\left\{\left(1-t^2\right)^{p+1} - \frac{z}{2}\left[(1-t)^{p+1} + (1+t)^{p+1}\right] = 0 \mid \mathcal{R}(t) > 0\right\} \tag{6.23}$$

where the m, l indexes run respectively over $m = 2, \ldots, p+1$ and $l = 1, \ldots, p+1$. The scaling found in (6.22) can be analytically proved by setting $t = c_1\sqrt{\epsilon}$ into (6.11), with $\epsilon = 1-s$ arbitrarily small:

$$\left(1-c_1^2\epsilon\right)^{p+1} - \frac{1-\epsilon}{2}\left[\left(1-c_1\sqrt{\epsilon}\right)^{p+1} + \left(1+c_1\sqrt{\epsilon}\right)^{p+1}\right] = 0$$

Using Taylor expansion, by virtue that $\epsilon \ll 1$:

$$2 - c_1^2\left[p^2 + 3p + 2\right] = 0$$

solving:

$$c_1 = \pm\sqrt{\frac{2}{(p+1)\,(p+2)}} = \pm\frac{b\sqrt{2}}{\sigma}$$

so in agree with the numerical results we expect only one solution (with positive real part) scaling as $c_1\sqrt{1-s}$ in the $s \to 1$ limit. On substituting (6.21) and (6.22) into (6.18) and developing around $s = 1$, we get:

$$\frac{A_1^{(1)} + \sum_{m=2}^{p+1} J_m(\nu)A_m^{(1)} - \sum_{l=1}^{p+1} L_l(\nu)B_l^{(1)} + 1}{1-s} + \frac{J_1^{(2)}(\nu)A_1^{(1)} + A_1^{(2)} + \sum_{m=2}^{p+1} J_m(\nu)A_m^{(2)} - \sum_{l=1}^{p+1} L_l(\nu)B_l^{(2)}}{\sqrt{1-s}} = 0$$

where:

$$\begin{cases} J_1^{(2)}(\nu) = c_1\,(p-\nu+1) \\ J_m(\nu) = \dfrac{1}{(1-c_m)^{p-\nu+1}} \\ L_l(\nu) = \dfrac{ze^{-\gamma}}{(1+d_l)^{p-\nu+1}} \displaystyle\sum_{i=0}^{p-\nu} \frac{\left[(1+d_l)\,\gamma\right]^i}{i!} \end{cases} \tag{6.24}$$

Doing the same for (6.19):

$$\frac{-F_1^{(1)}(\nu)A_1^{(1)} - \sum_{m=2}^{p+1} F_m(\nu)A_m^{(1)} + \sum_{l=1}^{p+1} G_l(\nu)B_l^{(1)} - e^{-\gamma}\sum_{i=0}^{p-\nu}\frac{\gamma^i}{i!}}{1-s} + \frac{-F_1^{(2)}(\nu)A_1^{(1)} - F_1^{(1)}(\nu)A_1^{(2)} - \sum_{m=2}^{p+1} F_m(\nu)A_m^{(2)} + \sum_{l=1}^{p+1} G_l(\nu)B_l^{(2)}}{\sqrt{1-s}} = 0$$

where:

$$\begin{cases} F_1^{(1)}(\nu) = e^{-\gamma} \displaystyle\sum_{i=0}^{p-\nu} \frac{\gamma^i}{i!} \\ F_1^{(2)}(\nu) = -c_1 e^{-\gamma} \left[\dfrac{\gamma^{p-\nu+1}}{(p-\nu)!} + (p-\nu+1-\gamma)\displaystyle\sum_{i=0}^{p-\nu}\frac{\gamma^i}{i!}\right] \\ F_m(\nu) = \dfrac{e^{-\gamma}}{(1+c_m)^{p-\nu+1}} \displaystyle\sum_{i=0}^{p-\nu} \frac{\left[(1+c_m)\,\gamma\right]^i}{i!} \\ G_l(\nu) = \dfrac{z}{(1-d_l)^{p-\nu+1}} \end{cases} \tag{6.25}$$

Collecting everything, we have finally a system of $4(p+1)$ equations:

$$\begin{cases} \vdots \\ A_1^{(1)} + \sum_{m=2}^{p+1} J_m(\nu) A_m^{(1)} - \sum_{l=1}^{p+1} L_l(\nu) B_l^{(1)} = -1 \\ F_1^{(1)}(\nu) A_1^{(1)} + \sum_{m=2}^{p+1} F_m(\nu) A_m^{(1)} - \sum_{l=1}^{p+1} G_l(\nu) B_l^{(1)} = -e^{-\gamma} \sum_{i=0}^{p-\nu} \frac{\gamma^i}{i!} \\ J_1^{(2)}(\nu) A_1^{(1)} + A_1^{(2)} + \sum_{m=2}^{p+1} J_m(\nu) A_m^{(2)} - \sum_{l=1}^{p+1} L_l(\nu) B_l^{(2)} = 0 \\ F_1^{(2)}(\nu) A_1^{(1)} + F_1^{(1)}(\nu) A_1^{(2)} + \sum_{m=2}^{p+1} F_m(\nu) A_m^{(2)} - \sum_{l=1}^{p+1} G_l(\nu) B_l^{(2)} = 0 \\ \vdots \end{cases} \tag{6.26}$$

in the $4(p+1)$ variables $A_1^{(1)}, A_1^{(2)}, A_m^{(1)}, A_m^{(2)}, B_l^{(1)}, B_l^{(2)}$. It is easy to see that the first two equations represent a closed subsystem of $2(p+1)$ equations in the $2(p+1)$ variables with the (1) superscript. It is also easy to prove that a solution for it is given by:

$$\begin{cases} A_1^{(1)} = -1 \\ A_m^{(1)} = B_l^{(1)} = 0 \qquad \forall m, l \end{cases}$$

so the system (6.26) simplifies in:

$$\begin{cases} \vdots \\ A_1^{(2)} + \sum_{m=2}^{p+1} J_m(\nu) A_m^{(2)} - \sum_{l=1}^{p+1} L_l(\nu) B_l^{(2)} = J_1^{(2)}(\nu) \\ F_1^{(1)}(\nu) A_1^{(2)} + \sum_{m=2}^{p+1} F_m(\nu) A_m^{(2)} - \sum_{l=1}^{p+1} G_l(\nu) B_l^{(2)} = F_1^{(2)}(\nu) \\ \vdots \end{cases} \tag{6.27}$$

We want now to show that in the $s \to 1$ limit, the k-th gap distribution only depends, at first order, on the coefficient $A_1^{(2)}$. Injecting (6.21) and (6.22) into (6.20) and developing around $s = 1$, with $\gamma = \delta/b$:

$$\begin{aligned} \tilde{P}(s, z; \gamma) \overset{s \to 1}{\approx} \frac{1}{b} \Bigg[&\left(\frac{1}{c_1 \sqrt{1-s}} \frac{\partial^2}{\partial \gamma^2} + \frac{\partial}{\partial \gamma} \right) \left[-\frac{1}{1-s} + \frac{A_1^{(2)}(z,\gamma)}{\sqrt{1-s}} \right] \\ &+ \sum_{m=2}^{p+1} \left(\frac{1}{c_m} \frac{\partial^2}{\partial \gamma^2} + \frac{\partial}{\partial \gamma} \right) \frac{A_m^{(2)}(z,\gamma)}{\sqrt{1-s}} + z \sum_{l=1}^{p+1} e^{d_l \gamma} \left(\frac{1}{d_l} \frac{\partial^2}{\partial \gamma^2} + \frac{\partial}{\partial \gamma} \right) \frac{B_l^{(2)}(z,\gamma)}{\sqrt{1-s}} \Bigg] \end{aligned}$$

It is then clear that there is a leader term depending on $A_1^{(2)}$:

$$\tilde{P}(s, z; \gamma) \overset{s \to 1}{\approx} \frac{1}{bc_1} \frac{1}{1-s} \frac{\partial^2}{\partial \gamma^2} A_1^{(2)}(z, \gamma) = \tilde{p}(z, \gamma) \frac{1}{1-s}$$

So we are interested on the quantity:

$$\tilde{p}\,(z,\gamma) = \frac{1}{bc_1}\frac{\partial^2}{\partial\gamma^2}A_1^{(2)}(z,\gamma) \tag{6.28}$$

Even though we only need $A_1^{(2)}(z,\gamma)$, system (6.27) still can't be solved. We look then explicitly for *typical fluctuations*.

6.1.2 Typical fluctuations

We consider now the $z \to 1$ and $\gamma \to 0$ limits, keeping fixed the ratio $\mu = \frac{\gamma}{\sqrt{1-z}}$. In this context the roots d_l behave as:

$$d_1(z) \overset{z\to 1}{\approx} c_1\sqrt{1-z} \qquad\qquad d_m(z) \overset{z\to 1}{\approx} c_m \tag{6.29}$$

Driven from the $p = 0, 1$ cases, we guess for the coefficients the scaling forms:

$$\begin{aligned} A_1^{(2)} &\approx \frac{A_{1,0}^{(2)}}{\sqrt{t}} & \qquad B_1^{(2)} &\approx \frac{B_{1,0}^{(2)}}{\sqrt{t}} \\ A_m^{(2)} &\approx A_{m,0}^{(2)} & B_m^{(2)} &\approx B_{m,0}^{(2)} \end{aligned} \tag{6.30}$$

In this limit, the coefficients (6.24) and (6.25) behave as:

$$\begin{cases} L_1(\nu) = 1 - \left[c_1\,(p-\nu+1) + \mu\delta_{\nu,p}\right]\sqrt{t} + O(t) \\ L_m(\nu) = \dfrac{1}{(1+c_m)^{p-\nu+1}} + \mu\left[\dfrac{c_m}{(1+c_m)^{p-\nu+1}} - \delta_{\nu,p}\right]\sqrt{t} + O(t) \\ F_1^{(1)}(\nu) = 1 - \mu\sqrt{t}\delta_{\nu,p} + O(t) \\ F_1^{(2)}(\nu) = -c_1\,(p-\nu+1) + c_1\mu\sqrt{t} + O(t) \\ F_m(\nu) = \dfrac{1}{(1+c_m)^{p-\nu+1}} + \mu\left[\dfrac{c_m}{(1+c_m)^{p-\nu+1}} - \delta_{\nu,p}\right]\sqrt{t} + O(t) \\ G_1(\nu) = 1 + c_1\,(p-\nu+1)\sqrt{t} + O(t) \\ G_m(\nu) = \dfrac{1}{(1-c_m)^{p-\nu+1}} + O(t) \end{cases} \tag{6.31}$$

Therefore, injecting (6.29), (6.30) and (6.31) into the system (6.27) and expanding around $t = 0$, we get:

$$\begin{cases} \vdots \\ A_{l,0}^{(2)} = B_{l,0}^{(2)} \\ \displaystyle\sum_{m=2}^{p+1}\left[J_m(\nu) - S_m(\nu)\right]A_{m,0}^{(2)} + \left[J_1^{(2)}(\nu) + \mu\delta_{\nu,p}\right]A_{1,0}^{(2)} = J_1^{(2)}(\nu) \\ \vdots \end{cases} \tag{6.32}$$

where we defined:

$$S_m(\nu) = \frac{1}{(1+c_m)^{p-\nu+1}}$$

Summing now over ν the second half of (6.32), we get the relation:

$$\sum_{m=2}^{p+1} A_{m,0}^{(2)} \left[\sum_{\nu=0}^{p} J_m(r) - \sum_{\nu=0}^{p} S_m(\nu) \right] + \left[\sum_{\nu=0}^{p} J_1^{(2)}(\nu) + \mu \right] A_{1,0}^{(2)} = \sum_{\nu=0}^{p} J_1^{(2)}(\nu) \tag{6.33}$$

Three sums over ν appear, let's evaluate them:

1.

$$\sum_{\nu=0}^{p} J_m(\nu) = \sum_{\nu=0}^{p} \frac{1}{(1-c_m)^{p-\nu+1}} = \sum_{\nu=0}^{p} \left(\frac{1}{1-c_m} \right)^{p-\nu+1}$$

for the sake of simplicity, we define:

$$t = \frac{1}{1-c_m}$$

then let $l = p - \nu$:

$$\sum_{\nu=0}^{p} J_m(\nu) = \sum_{\nu=0}^{p} t^{p-\nu+1} = t \sum_{l=0}^{p} t^l = \frac{t\left(1 - t^{p+1}\right)}{1-t}$$

coming back to the definition of t, we finally obtain:

$$\sum_{\nu=0}^{p} J_m(\nu) = \frac{1}{c_m} \left[\frac{1}{(1-c_m)^{p+1}} - 1 \right]$$

2.

Very similar result for the second sum, where we get:

$$\sum_{\nu=0}^{p} S_m(\nu) = \frac{1}{c_m} \left[1 - \frac{1}{(1+c_m)^{p+1}} \right]$$

3.

$$\sum_{\nu=0}^{p} J_1^{(2)}(\nu) = c_1 \sum_{\nu=0}^{p} (p - \nu + 1) = c_1 \left[(p+1)^2 - \frac{p\,(p+1)}{2} \right] = \frac{1}{c_1}$$

Let's then substitute what we got into (6.33):

$$\sum_{m=2}^{p+1} \frac{A_{m,0}^{(2)}}{c_m} \left[\frac{1}{(1+c_m)^{p+1}} + \frac{1}{(1-c_m)^{p+1}} - 2 \right] + \frac{1 + c_1\mu}{c_1} A_{1,0}^{(2)} = \frac{1}{c_1} \tag{6.34}$$

We now recognize the expression in the square brackets as the roots' equation (6.23) for the c_m, then the first term is zero whatever the values of m and p, hence (6.34) reduces to:

$$A_{1,0}^{(2)} = \frac{1}{1 + c_1\mu} \tag{6.35}$$

Let's work a little bit with μ from its definition:

$$\mu = \frac{\gamma}{\sqrt{t}} = \frac{\delta}{b\sqrt{t}} = \frac{\delta}{\sqrt{t}\sigma}\frac{\sqrt{2}}{c_1} = \sqrt{\frac{2}{\lambda}}\frac{1}{c_1} \qquad \lambda = \frac{\sigma^2 t}{\delta^2} \tag{6.36}$$

Finally on substituting (6.35) and (6.36) into (6.30), the coefficient $A_1^{(2)}$ becomes:

$$A_1^{(2)} \overset{\text{t.f.}}{\approx} \frac{1}{1+\sqrt{\frac{2}{\lambda}}}\frac{1}{\sqrt{t}}$$

Now we can straightforwardly compute the asymptotic behavior of the k-th gap distribution:

$$\begin{aligned}\tilde{p}(z,\gamma) &\overset{\text{t.f.}}{\approx} \frac{1}{\sqrt{t}}\frac{1}{bc_1}\frac{\partial^2}{\partial\gamma^2}\frac{1}{1+c_1\frac{\gamma}{\sqrt{t}}} = \frac{2c_1}{b}\frac{1}{\left(1+c_1\frac{\gamma}{\sqrt{t}}\right)^3}\frac{1}{t^{\frac{3}{2}}} \\ &= \frac{2\sqrt{2}}{\sigma}\frac{1}{\left(1+\sqrt{\frac{2}{\lambda}}\right)^3}\frac{1}{t^{\frac{3}{2}}} = \frac{\lambda^{\frac{3}{2}}}{\sigma}\frac{1}{\left(1+\sqrt{\frac{\lambda}{2}}\right)^3}\frac{1}{t^{\frac{3}{2}}} = \frac{\sigma^2}{\delta^3}\frac{1}{\left(1+\sqrt{\frac{\lambda}{2}}\right)^3}\end{aligned}$$

which is exactly what we already obtained for the $p = 0, 1$ cases but now it holds for general p. So, for typical fluctuations, we still find (2) as scaling function of the k-th gap distribution.

6.1.3 Large fluctuations

Already from the $p = 1$ case the large fluctuations scaling function $\varphi_1(\gamma)$ has a large complicate form, too much to be presented in the document. So we don't expect here to find the shape of the general p-th order scaling function $\varphi_p(\gamma)$. We can anyway compute its asymptotic behavior for small and large γ. We have just proved that for typical fluctuations the k-th gap distribution is universal for the entire class of Gamma distributions. So, given that the two scaling behaviors for typical and large fluctuations have to match when moving toward each other, the small γ behavior of the scaling function $\varphi_p(\gamma)$ is fixed to be:

$$\lim_{\gamma\to 0}\varphi_p(\gamma)\, k^{-3/2} = \lim_{\gamma\to\infty} p_k(\gamma) \tag{6.37}$$

We need then to compute the limit:

$$\lim_{\gamma\to\infty} p_k(\gamma) = \lim_{\gamma\to\infty}\frac{1}{\sigma}\sqrt{k}P\left(b\sqrt{k}\frac{\gamma}{\sigma}\right) \qquad \sigma = b\sqrt{(p+1)(p+2)}$$

Using (4.25) this limit turns out to be:

$$\lim_{\gamma\to\infty} p_k(\gamma) = \frac{3}{b}\sqrt{\frac{[(p+1)(p+2)]^3}{8\pi}}\gamma^{-4}k^{-3/2} \tag{6.38}$$

So from (6.37) and (6.38), for small γ the scaling function behaves as:

$$\varphi_p(\gamma) \overset{\gamma\to 0}{\approx} \frac{3}{b}\sqrt{\frac{[(p+1)(p+2)]^3}{8\pi}}\gamma^{-4} \tag{6.39}$$

Now it only remains to compute the large γ behavior for $\varphi_p(\gamma)$. We know that formally the scaling function $\varphi_p(\gamma)$ can be calculated from the coefficient $\tilde{p}_2(\gamma)$ related to the term going as $\sqrt{1-z}$ in the series expansion around $s, z = 1$ of the full k-th gap distribution:

$$\tilde{p}(z,\gamma) \overset{z\to 1}{\approx} \tilde{p}_1(\gamma) + \tilde{p}_2(\gamma)\sqrt{1-z} \tag{6.40}$$

since $\tilde{p}(z,\gamma)$ is given by (6.28), we have also that:

$$\tilde{p}(z,\gamma) \overset{z\to 1}{\approx} \frac{1}{bc_1}\frac{\partial^2}{\partial\gamma^2}\left[\alpha_1(\gamma) + \beta_1(\gamma)\sqrt{1-z}\right] \tag{6.41}$$

where:

$$A_1^{(2)}(z,\gamma) \overset{z\to 1}{\approx} \alpha_1(\gamma) + \beta_1(\gamma)\sqrt{1-z}$$

hence matching (6.40) with (6.41) we get:

$$\tilde{p}_2(\gamma) = \frac{1}{bc_1}\frac{\partial^2}{\partial\gamma^2}\beta_1(\gamma) \tag{6.42}$$

Using the same reasonings as for (4.28), the scaling function $\varphi_p(\gamma)$ is related to $\tilde{p}_2(\gamma)$ by:

$$\varphi_p(\gamma) = -\frac{\tilde{p}_2(\gamma)}{2\sqrt{\pi}} \tag{6.43}$$

Therefore the large γ behavior of $\varphi_p(\gamma)$ is related to the large γ behavior of the coefficient $\beta_1(\gamma)$. In order to evaluate it, let's start by expanding the system (6.27) around $z = 1$. Driven from numerical results, we consider for the amplitudes the following scaling forms:

$$A_l^{(2)} \overset{z\to 1}{\approx} \alpha_l(\gamma) + \beta_l(\gamma)\sqrt{1-z} \qquad\qquad B_l^{(2)} \overset{z\to 1}{\approx} \tau_l(\gamma) + \omega_l(\gamma)\sqrt{1-z} \tag{6.44}$$

the coefficients of the system (depending on z) behave in this limit:

$$\begin{cases} L_1(\nu) \overset{z\to 1}{\approx} e^{-\gamma}\displaystyle\sum_{i=0}^{p-\nu}\frac{\gamma^i}{i!} - c_1 e^{-\gamma}\left[\frac{\gamma^{p-\nu+1}}{(p-\nu)!} + (p-\nu+1-\gamma)\sum_{i=0}^{p-\nu}\frac{\gamma^i}{i!}\right]\sqrt{1-z} \\ \qquad = F_1^{(1)}(\nu) + F_1^{(2)}(\nu)\sqrt{1-z} \\ L_m(\nu) \overset{z\to 1}{\approx} \dfrac{e^{-\gamma}}{(1+c_m)^{p-\nu+1}}\displaystyle\sum_{i=0}^{p-\nu}\frac{[(1+c_m)\gamma]^i}{i!} = F_m(\nu) \\ G_1(\nu) \overset{z\to 1}{\approx} 1 + c_1(p-\nu+1)\sqrt{1-z} = 1 + J_1^{(2)}(\nu)\sqrt{1-z} \\ G_m(\nu) \overset{z\to 1}{\approx} \dfrac{1}{(1-c_m)^{p-\nu+1}} = J_m(\nu) \end{cases} \tag{6.45}$$

Injecting (6.44) and (6.45) into (6.27) we obtain:

$$
\begin{cases}
\vdots \\
\alpha_1 + \sum_{m=2}^{p+1} J_m(\nu)\alpha_m - F_1^{(1)}(\nu)\,\tau_1 - \sum_{m=2}^{p+1} F_m(\nu)\,\tau_m = J_1^{(2)}(\nu) \\
\beta_1 + \sum_{m=2}^{p+1} J_m(\nu)\beta_m - F_1^{(1)}(\nu)\,\omega_1 - F_1^{(2)}(\nu)\,\tau_1 - \sum_{m=2}^{p+1} F_m(\nu)\,\omega_m = 0 \\
F_1^{(1)}(\nu)\alpha_1 + \sum_{m=2}^{p+1} F_m(\nu)\alpha_m - \tau_1 - \sum_{m=2}^{p+1} J_m(\nu)\tau_m = F_1^{(2)}(\nu) \\
F_1^{(1)}(\nu)\beta_1 + \sum_{m=2}^{p+1} F_m(\nu)\beta_m - \omega_1 - J_1^{(2)}(\nu)\,\tau_1 - \sum_{m=2}^{p+1} J_m(\nu)\omega_m = 0 \\
\vdots
\end{cases}
\tag{6.46}
$$

Let's consider now the large γ limit. The coefficients in (6.46) behave for large γ as:

$$
\begin{aligned}
&F_1^{(1)}(\nu) \overset{\gamma\gg 0}{\approx} \frac{e^{-\gamma}\gamma^{p-\nu}}{(p-\nu)!} \qquad && F_1^{(2)}(\nu) \overset{\gamma\gg 0}{\approx} -c_1\frac{e^{-\gamma}\gamma^{p-\nu}}{(p-\nu)!} \\
&F_m(\nu) \overset{\gamma\gg 0}{\approx} \frac{e^{-\gamma}\gamma^{p-\nu}}{(p-\nu)!}\frac{1}{1+c_m}
\end{aligned}
\tag{6.47}
$$

We also need to set a reasonable scaling for the variables α, β, τ and ω. From the $p = 0, 1$ cases we got the hint to set:

$$
\alpha_l \overset{\gamma\gg 0}{\approx} A_l \qquad \beta_l \overset{\gamma\gg 0}{\approx} B_l\gamma^{2p}e^{-2\gamma} \qquad \tau_l \overset{\gamma\gg 0}{\approx} T_l\gamma^{p}e^{-\gamma} \qquad \omega_l \overset{\gamma\gg 0}{\approx} W_l\gamma^{p}e^{-\gamma}
\tag{6.48}
$$

where the coefficients A_l, B_l, T_l and W_l are constant. On substituting (6.47) and (6.48) into (6.46) and considering the large γ limit we get:

$$
\begin{cases}
\vdots \\
A_1 + \sum_{m=2}^{p+1} J_m(\nu)A_m = J_1^{(2)}(\nu) \\
B_1 + \sum_{m=2}^{p+1} J_m(\nu)B_m - \frac{\delta_{\nu,0}}{p!}W_1 + \frac{c_1\delta_{\nu,0}}{p!}T_1 - \frac{\delta_{\nu,0}}{p!}\sum_{m=2}^{p+1}\frac{W_m}{1+c_m} = 0 \\
\frac{\delta_{\nu,0}}{p!}A_1 + \frac{\delta_{\nu,0}}{p!}\sum_{m=2}^{p+1}\frac{A_m}{1+c_m} - T_1 - \sum_{m=2}^{p+1} J_m(\nu)T_m = -\frac{c_1\delta_{\nu,0}}{p!} \\
W_1 + J_1^{(2)}(\nu)\,T_1 + \sum_{m=2}^{p+1} J_m(\nu)W_m = 0 \\
\vdots
\end{cases}
\tag{6.49}
$$

The fact that in (6.49) the dependence on γ has been lost means that in (6.48) we chose the correct scaling. Indeed for $p = 0$ (6.49) reduces to:

$$\begin{cases} A = 1 \\ B - W + T = 0 \\ A - T = -1 \\ W + T = 0 \end{cases} \qquad \Longrightarrow \qquad \begin{cases} A = 1 \\ B = -4 \\ T = 2 \\ W = -2 \end{cases}$$

Therefore from (6.48):

$$\beta(\gamma) \overset{\gamma \gg 0}{\approx} -4e^{-2\gamma}$$

Substituting into (6.42) and using (6.43) we get:

$$\varphi_0(\gamma) \overset{\gamma \gg 0}{\approx} \frac{8}{b\sqrt{\pi}} e^{-2\gamma}$$

that perfectly matches with the large γ behavior in (4.29). Also for $p = 1$ one gets for $\varphi_1(\gamma)$ results in agree with (5.14). Therefore the system (6.49) is consistent. So we can affirm that in the large γ limit, combining (6.48) with (6.42) and (6.43), the scaling function $\varphi_p(\gamma)$ behaves as:

$$\varphi_p(\gamma) \overset{\gamma \gg 0}{\approx} -\frac{2B_1}{bc_1\sqrt{\pi}} \gamma^{2p} e^{-2\gamma} \tag{6.50}$$

where B_1 is a coefficient that (in theory) can be extracted from (6.49). Collecting (6.39) and (6.50), the asymptotic behavior of $\varphi_p(\gamma)$ reads:

$$\varphi_p(\gamma) \approx \begin{cases} \frac{3}{b}\sqrt{\frac{[(p+1)(p+2)]^3}{8\pi}}\gamma^{-4} & \gamma \to 0 \\ \\ -\frac{2B_1}{bc_1\sqrt{\pi}}\gamma^{2p}e^{-2\gamma} & \gamma \to \infty \end{cases}$$

6.2 Large p limiting distribution

Let's consider the class of Gamma functions f_p as a sequence in p and let's look for the $p \to \infty$ limit. For studying the convergence of f_p it is helpful to determine the maximum. From (6.2) the condition for a stationary point is:

$$f_{p-1}(x) = f_p(x) \qquad \Leftrightarrow \qquad x = \pm bp$$

both are maxima since:

$$\frac{d^2}{dx^2} f_p(\pm bp) = -\frac{e^{-p}p^{p-2}}{2b^3(p-1)!} < 0$$

The Gamma distribution assumes in the maxima the value of:

$$f_p(\pm bp) = \frac{b^p p^p}{2p! b^{p+1}} e^{-p} = \frac{1}{2b}\frac{p^p}{p!}e^{-p} \overset{p \gg 0}{\sim} \frac{1}{\sqrt{p}}$$

Taking the limit of large p, the two maxima of the Gamma distributions become sharper and sharper and f_p converges to the limiting distribution [24]:

$$f_\infty(x) = \frac{1}{2}\left[\delta\left(x - bp\right) + \delta\left(x + bp\right)\right] \tag{6.51}$$

where the values $\pm bp$ are seen as constant. In this framework it is convenient to define the rescaled variable $\tilde{x} = \frac{x}{bp}$, hence (6.51) can be rewritten as:

$$f_\infty(\tilde{x}) = \frac{1}{2bp}\left[\delta\left(\tilde{x} - 1\right) + \delta\left(\tilde{x} + 1\right)\right] \tag{6.52}$$

In the large p limit then one retrieves the discrete symmetric random walk where the walker can only move with steps of length ± 1. In this case the order statistics is not well defined because of a **degeneracy problem**. The walker indeed can pass through the same position for several times and this is problematic for ordering the positions. However using the (6.52) in the Wiener-Hopf integral (3.4) one manages

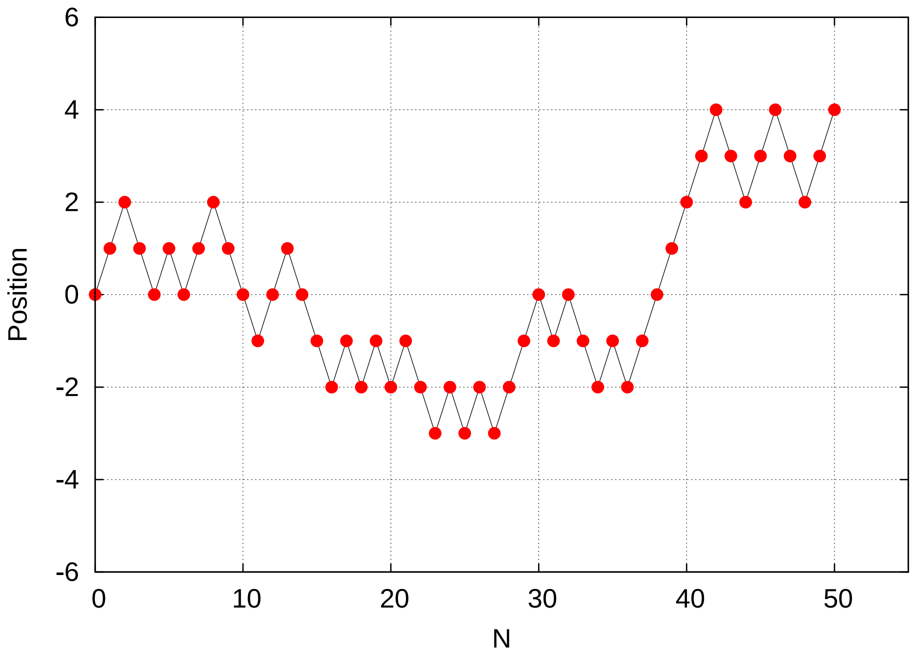


Figure 6.2: Degeneracy problem in the large p limit.

to obtain a recurrent equation for $q_{k,n}(x)$. Therefore, using the same approach as for finite p, one may be able to extract some results for this pathologic case.

Chapter 7

Moments of the k-th gap distribution

We saw that for any order p of the Gamma jump distribution, the k-th gap distribution has always the same asymptotic behavior for typical fluctuations. Since the moments are strictly related to the distribution, let's see what are the consequences of this result on them.

Let the general p-th order Gamma distribution (6.1) be the jump distribution of the random walk. The θ-moment of the k-th gap distribution, in the large n limit, is defined as:

$$\left\langle d_{k,\infty}^{\theta} \right\rangle = \int_0^{\infty} \delta^{\theta} p_k(\delta)\, d\delta \tag{7.1}$$

given that $\delta = \gamma b$, we can rewrite (7.1) as:

$$\frac{\left\langle d_{k,\infty}^{\theta} \right\rangle}{\sigma^{\theta}} = bc(p)^{-\theta/2} \int_0^{\infty} \gamma^{\theta} p_k(\gamma) d\gamma \tag{7.2}$$

where we defined the quantity $c(p) = (p+1)(p+2)$. Both the typical and the large fluctuations contribute to this integral. By virtue that

$$p_k(\gamma) \overset{k \gg 0}{\approx} \begin{cases} \frac{1}{b}\sqrt{\frac{k}{c(p)}} P\left(\sqrt{\frac{k}{c(p)}}\gamma\right) & \gamma \sim 1/\sqrt{k} \\ \\ \varphi_p(\gamma) k^{-3/2} & \gamma \gg 0 \end{cases} \tag{7.3}$$

we can separate the two fluctuations regimes using an $\epsilon > 0$ arbitrarily small:

$$\frac{\left\langle d_{k,\infty}^{\theta} \right\rangle}{\sigma^{\theta}} \overset{k \gg 0}{\approx} bc(p)^{-\theta/2} \left[\frac{1}{b}\sqrt{\frac{k}{c(p)}} \int_0^{k^{\epsilon-1/2}} \gamma^{\theta} P\left(\sqrt{\frac{k}{c(p)}}\gamma\right) d\gamma + k^{-3/2} \int_{k^{\epsilon-1/2}}^{\infty} \gamma^{\theta} \varphi_p(\gamma) d\gamma \right]$$

on making the change of variables $\sqrt{\frac{k}{c(p)}}\gamma \to \gamma$ we get:

$$\frac{\left\langle d_{k,\infty}^{\theta} \right\rangle}{\sigma^{\theta}} \overset{k \gg 0}{\approx} k^{-\theta/2} \int_0^{\frac{k^{\epsilon}}{\sqrt{c(p)}}} \gamma^{\theta} P(\gamma)\, d\gamma + bc(p)^{-\theta/2} k^{-3/2} \int_{k^{\epsilon-1/2}}^{\infty} \gamma^{\theta} \varphi_p(\gamma) d\gamma \tag{7.4}$$

Relation (7.4) is the θ-moment of a general p-th order Gamma distribution in which the contributions of typical and large fluctuations are split. We are going to show now that in the large k limit one of the two contributions always dominates on the other, with except for a borderline case. In order to do so let's evaluate the asymptotic behavior for large k of the two addends of (7.4).
Starting with the typical fluctuations:

$$k^{-\theta/2}\int_0^{\frac{k^\epsilon}{\sqrt{c(p)}}}\gamma^\theta P(\gamma)d\gamma \overset{k\gg 0}{\approx} k^{-\theta/2}\begin{cases}1 & \theta<3\\ \epsilon\log k & \theta=3\\ k^{\epsilon(\theta-3)} & \theta>3\end{cases} \tag{7.5}$$

while for the large ones:

$$bc(p)^{-\theta/2}k^{-3/2}\int_{k^{\epsilon-1/2}}^{\infty}\gamma^\theta\varphi_p(\gamma)d\gamma \overset{k\gg 0}{\approx} k^{-3/2}\begin{cases}k^{\frac{3-\theta}{2}}k^{\epsilon(\theta-3)} & \theta<3\\ \left(\frac{1}{2}-\epsilon\right)\log k & \theta=3\\ 1 & \theta>3\end{cases} \tag{7.6}$$

Relations (7.5) and (7.6) suggest that three situations need to be distinguished:

$\theta < 3$

$$\frac{\left\langle d_{k,\infty}^\theta\right\rangle}{\sigma^\theta} \overset{k\gg 0}{\approx} k^{-\theta/2} + k^{-\theta/2+\epsilon(\theta-3)} \approx k^{-\theta/2} \tag{7.7}$$

This is the case of the first and second moments. Here the typical fluctuations dominate.

$\theta = 3$

$$\frac{\left\langle d_{k,\infty}^3\right\rangle}{\sigma^3} \overset{k\gg 0}{\approx} \epsilon k^{-3/2}\log k + k^{-3/2}\left(\frac{1}{2}-\epsilon\right)\log k = \frac{1}{2}k^{-3/2}\log k \tag{7.8}$$

In the borderline case of the third moment the two fluctuations give a contribution of the same order.

$\theta > 3$

$$\frac{\left\langle d_{k,\infty}^\theta\right\rangle}{\sigma^\theta} \overset{k\gg 0}{\approx} k^{-\theta/2+\epsilon(\theta-3)} + k^{-3/2} \approx k^{-3/2} \tag{7.9}$$

From the fourth moment on instead the large fluctuations dominate.

In the last chapter we proved that for the class of Gamma distributions, the asymptotic k-th gap distribution is universal in the regime of typical fluctuations. Therefore from (7.7) the first two moments are universal too. The third moment is universal as well since the two fluctuations give a contribution of the same order and one of them (the typical) are universal (7.8). From the fourth moment on instead the moments are non-universal since dominated by large fluctuations (7.9). Let's demonstrate these statements by computing the moments using (7.4):

First moment

$$\frac{\langle d_{k,\infty}\rangle}{\sigma} \overset{k\gg 0}{\approx} \frac{1}{\sqrt{k}}\int_0^{\frac{k^\epsilon}{\sqrt{c(p)}}} \gamma P(\gamma)d\gamma \approx \frac{1}{\sqrt{k}}\int_0^{\infty} \gamma P(\gamma)d\gamma = \frac{1}{\sqrt{2\pi k}}$$

Second moment

$$\frac{\left\langle d^2_{k,\infty}\right\rangle}{\sigma^2} \overset{k\gg 0}{\approx} \frac{1}{k}\int_0^{\frac{k^\epsilon}{\sqrt{c(p)}}} \gamma^2 P(\gamma)d\gamma \approx \frac{1}{k}\int_0^{\infty} \gamma^2 P(\gamma)d\gamma = \frac{1}{2k}$$

Third moment

$$\frac{\left\langle d^3_{k,\infty}\right\rangle}{\sigma^3} \overset{k\gg 0}{\approx} k^{-3/2}\int_0^{\frac{k^\epsilon}{\sqrt{c(p)}}} \gamma^3 P(\gamma)d\gamma + bc(p)^{-3/2}k^{-3/2}\int_{k^{\epsilon-1/2}}^{\infty} \gamma^3 \varphi_p(\gamma)d\gamma$$

Here the integrals are logarithmically divergent, so we develop the integrands keeping only the divergent terms (the others are negligible in the large k limit):

$$\begin{aligned}
\frac{\left\langle d^3_{k,\infty}\right\rangle}{\sigma^3} &\overset{k\gg 0}{\approx} k^{-3/2}\int^{\frac{k^\epsilon}{\sqrt{c(p)}}} \gamma^3 \frac{3}{\sqrt{8\pi}}\gamma^{-4}d\gamma + bc(p)^{-3/2}k^{-3/2}\int_{k^{\epsilon-1/2}} \gamma^3 \frac{3c(p)^{3/2}}{b\sqrt{8\pi}}\gamma^{-4}d\gamma \\
&= \frac{3}{\sqrt{8\pi}}k^{-3/2}\int^{\frac{k^\epsilon}{\sqrt{c(p)}}} \frac{d\gamma}{\gamma} + \frac{3}{\sqrt{8\pi}}k^{-3/2}\int_{k^{\epsilon-1/2}} \frac{d\gamma}{\gamma} \\
&= \frac{3}{\sqrt{8\pi}}\left[\epsilon\log k - \frac{1}{2}\log c(p) - \left(\epsilon - \frac{1}{2}\right)\log k\right] k^{-3/2} \\
&\approx \frac{3}{4\sqrt{2\pi}}k^{-3/2}\log k
\end{aligned}$$

Further order moments ($\boldsymbol{\theta > 3}$)

$$\begin{aligned}
\frac{\left\langle d^\theta_{k,\infty}\right\rangle}{\sigma^\theta} &\overset{k\gg 0}{\approx} bc(p)^{-\theta/2}k^{-3/2}\int_{k^{\epsilon-1/2}}^{\infty} \gamma^\theta \varphi_p(\gamma)d\gamma \approx bc(p)^{-\theta/2}k^{-3/2}\int_0^{\infty} \gamma^\theta \varphi_p(\gamma)d\gamma \\
&= D_\theta\left(p\right) k^{-\frac{3}{2}}
\end{aligned}$$

where we defined the non-universal coefficient $D_\theta\left(p\right)$:

$$D_\theta\left(p\right) = bc(p)^{-\theta/2}\int_0^{\infty} \gamma^\theta \varphi_p(\gamma)d\gamma \tag{7.10}$$

Summarizing the universality of typical fluctuations (in the context of Gamma distributions) has the remarkable consequence that the first three moments are universal as well. Therefore, in this scaling limit, the mean μ, the variance Ψ^2 and the skewness Υ of the k-th gap distribution remain invariant for any p:

$$\begin{aligned}
\mu &= \left.\frac{\langle d_{k,\infty}\rangle}{\sigma}\right|_{k\gg 0} = \frac{1}{\sqrt{2\pi k}} \\
\Psi^2 &= \left[\frac{\left\langle d^2_{k,\infty}\right\rangle}{\sigma^2} - \left(\frac{\langle d_{k,\infty}\rangle}{\sigma}\right)^2\right]_{k\gg 0} = \frac{\pi - 1}{2\pi k} \\
\Upsilon &= \left.\frac{\left\langle d^3_{k,\infty}\right\rangle}{\left\langle d^2_{k,\infty}\right\rangle^{3/2}}\right|_{k\gg 0} = \frac{3}{2\sqrt{\pi}}\log k
\end{aligned}$$

The further moments all show a characteristic scaling as $k^{-3/2}$ but they present a coefficient depending on the order p of the distribution.
Recollecting the results we got:

$$\frac{\left\langle d_{k,\infty}^{\theta}\right\rangle}{\sigma^{\theta}} \overset{k\gg 0}{\approx} \begin{cases} \frac{1}{\sqrt{2\pi k}} & \theta = 1 \\ \frac{1}{2k} & \theta = 2 \\ \frac{3}{4\sqrt{2\pi}} k^{-3/2}\log k & \theta = 3 \\ D_{\theta}\left(p\right) k^{-3/2} & \theta \geq 4 \end{cases}$$

7.1 Non-universal coefficients

The non-universal coefficients $D_{\theta}\left(p\right)$ can be computed using (7.10). This relation requires the large fluctuations scaling functions $\varphi_p\left(\gamma\right)$ to be known.

7.1.1 Exponential distribution

In this case we could explicitly compute $\varphi_0\left(\gamma\right)$:

$$\varphi_0\left(\gamma\right) = \frac{8}{b\sqrt{\pi}} e^{2\gamma} \frac{e^{4\gamma} + 4e^{2\gamma} + 1}{\left(1 - e^{2\gamma}\right)^4}$$

By virtue that $c\left(0\right) = 2$, relation (7.10) becomes for $p = 0$:

$$D_{\theta}\left(0\right) = \frac{2^{3-\theta/2}}{\sqrt{\pi}} \int_0^{\infty} \gamma^{\theta} e^{2\gamma} \frac{e^{4\gamma} + 4e^{2\gamma} + 1}{\left(1 - e^{2\gamma}\right)^4} d\gamma$$

This integral can be computed even analytically for some θ. Few non-universal coefficient of the exponential distribution are listed in table (7.1).

$\boldsymbol{D_4\left(0\right)} = \frac{\pi^{\frac{3}{2}}}{4} \approx 1.39$	$\boldsymbol{D_5\left(0\right)} = \frac{15\zeta(3)}{4\sqrt{2\pi}} \approx 1.80$
$\boldsymbol{D_6\left(0\right)} = \frac{\pi^{\frac{7}{2}}}{16} \approx 3.44$	$\boldsymbol{D_7\left(0\right)} = \frac{315\zeta(5)}{16\sqrt{2\pi}} \approx 8.14$
$\boldsymbol{D_8\left(0\right)} = \frac{\pi^{\frac{11}{2}}}{24} \approx 22.6$	$\boldsymbol{D_9\left(0\right)} = \frac{2835\zeta(7)}{16\sqrt{2\pi}} \approx 71.3$
$\boldsymbol{D_{10}\left(0\right)} = \frac{3\pi^{\frac{15}{2}}}{64} \approx 251$	$\boldsymbol{D_{11}\left(0\right)} = \frac{155925\zeta(9)}{64\sqrt{2\pi}} \approx 974$

Table 7.1: Non-universal coefficients for the exponential distribution.

7.1.2 First order Gamma distribution

Also in this case $\varphi_1(\gamma)$ could be exactly computed though it has a large confusing formalism that prevent us to transcribe it. Anyway we could numerically evaluate some non-universal coefficients, listed in table (7.2):

$\boldsymbol{D_4(1)} \approx 2.31$	$\boldsymbol{D_5(1)} \approx 1.83$	$\boldsymbol{D_6(1)} \approx 2.30$	$\boldsymbol{D_7(1)} \approx 3.66$
$\boldsymbol{D_8(1)} \approx 6.80$	$\boldsymbol{D_9(1)} \approx 14.3$	$\boldsymbol{D_{10}(1)} \approx 33.3$	$\boldsymbol{D_{11}(1)} \approx 85.0$

Table 7.2: Non-universal coefficients for the first order Gamma distribution.

From table (7.1) and (7.2) the non-universality of the coefficients is evident.

Conclusions

In this work we studied the gap statistics of random walks. The goal was to check and extend the results obtained for an exponential jump distribution in [15]. In particular, starting from a p-th order Gamma jump distribution, we aimed to:

1. recover the universality of the k-th gap mean value, demonstrated with the *Pollaczek-Wendel* identity in (4.13);

2. prove/disprove the claim that the k-th gap distribution is universal for typical fluctuations.

We were able to fulfill these tasks. Indeed we could demonstrate that, for the whole class of Gamma distributions, the asymptotic k-th gap distribution has the same scaling form for typical fluctuations (2). Moreover we shed light on the intimate link between typical/large fluctuations and moments. These results let us prove that the asymptotic behavior of the first, the second and the third moments are related to typical fluctuations and so they retain the same form for any order p of the Gamma jump distribution (7.3).

$\frac{\langle d_{k,\infty}\rangle}{\sigma} \overset{k\gg 0}{\approx} \frac{1}{\sqrt{2\pi k}}$	$\frac{\langle d^2_{k,\infty}\rangle}{\sigma^2} \overset{k\gg 0}{\approx} \frac{1}{2k}$	$\frac{\langle d^3_{k,\infty}\rangle}{\sigma^3} \overset{k\gg 0}{\approx} \frac{3}{4\sqrt{2\pi}} k^{-3/2} \log k$

Table 7.3: First, second and third moment.

This has the remarkable consequence that the asymptotic distribution of the k-th gap has always the same mean μ, variance Ψ^2 and skewness Υ (7.4).

$\mu = \frac{1}{\sqrt{2\pi k}}$	$\Psi^2 = \frac{\pi-1}{2\pi k}$	$\Upsilon = \frac{3}{2\sqrt{\pi}} \log k$

Table 7.4: Mean, variance and skewness of the asymptotic k-th gap distribution.

Summarizing, the validity of the results obtained in [15] for a $p = 0$ Gamma jump distribution (exponential), have been here extended to a general order p, i.e. to the whole class of Gamma distributions.

Furthermore we focused on the asymptotic behavior of the k-th gap distribution in the framework of large fluctuations. We could figure out that large fluctuations are linked to non-universal moments and so we could get some information about them. We found that they all share the same scaling with k independently on the order p of the distribution and on the order θ of the moment.

We also extracted an exact formula (7.10) for computing the non-universal coefficients $D_\theta(p)$, that requires the knowledge of the large fluctuations scaling function $\varphi_p(\gamma)$. This function is rather demanding to calculate, however we could compute it exactly for $p = 0, 1$ and discern its asymptotic behavior for a general p-th order Gamma distribution.

We conclude this manuscript by mentioning some open problems related to these topics that may be subject of future research.

- At first it would be interesting to check whether the universality class we isolated for the Gamma distributions is unique. If this would be the case then, starting from any symmetric and continuous jump distribution, one would eventually recover the same asymptotic behavior for the k-th gap distribution (2) and the same moments in (7.3).
- Understand the origin of the scaling $k^{-3/2}$ in the non-universal moments.
- Because of degeneracy the case of the symmetric discrete random walk is pathologic for the order statistics. However we showed in section (6.2) that it emerges as limiting case taking the $p \to \infty$ limit of the Gamma jump distribution. For this reason we wonder if the order statistics of a symmetric discrete random walk may be studied using the same analysis we did for finite p.

Appendix A

Noteworthy integrals

In the chapters (4), (5) and (6) we had to deal with integrals of Gamma distributions of the form:

$$\begin{aligned} &\bullet \quad I_p^+ (\Lambda, \delta) = \int_0^\infty e^{-\Lambda \frac{x'}{b}} f_p \left(x' + x + \delta \right) dx' \\ &\bullet \quad I_p^- (\Lambda) = \int_0^\infty e^{-\Lambda \frac{x'}{b}} f_p \left(x' - x \right) dx' \end{aligned} \tag{A.1}$$

where $f_p(x)$ is a p-th order Gamma distribution. In the following we will show the proceeding for solving them.

Computation of $I_p^+ (\Lambda, \delta)$

We start by carrying on the integration for obtaining a recurrent relation:

$$\begin{aligned} I_p^+ (\Lambda, \delta) &= \int_0^\infty e^{-\Lambda \frac{x'}{b}} f_p \left(x' + x + \delta \right) dx' \\ &= \frac{1}{2p! b^{p+1}} \int_0^\infty e^{-\Lambda \frac{x'}{b}} \left| x' + x + \delta \right|^p e^{-\frac{\left| x' + x + \delta \right|}{b}} dx' \\ &= \frac{e^{-y-\gamma}}{2p! \Lambda} (y + \gamma)^p + \frac{1}{\Lambda} I_{p-1}^+ (\Lambda, \delta) - \frac{1}{\Lambda} I_p^+ (\Lambda, \delta) \end{aligned}$$

where we defined $y = x/b$ and $\gamma = \delta/b$. We obtain then the recursive relation:

$$I_p^+ (\Lambda, \delta) = \frac{e^{-y-\gamma}}{2p! (\Lambda + 1)} (y + \gamma)^p + \frac{1}{\Lambda + 1} I_{p-1}^+ (\Lambda, \delta) \tag{A.2}$$

In order to solve it let's use the generating function method. For the sake of simplicity we define $A_p \equiv I_p^+ (\Lambda, \delta)$, so:

$$\tilde{A}(z) = \sum_{p=0}^{\infty} z^p A_p$$

In the generating function representation, (A.2) reads:

$$\sum_{p=1}^{\infty} z^p A_p = \frac{e^{-y-\gamma}}{2 (\Lambda + 1)} \sum_{p=1}^{\infty} z^p \frac{(y + \gamma)^p}{p!} + \frac{1}{\Lambda + 1} \sum_{p=1}^{\infty} z^p A_{p-1}$$

which can be rewritten as:

$$\tilde{A}(z) - A_0 = \frac{e^{-y-\gamma}}{2\left(\Lambda+1\right)}\left[e^{z(y+\gamma)}-1\right] + \frac{z}{\Lambda+1}\tilde{A}(z)$$

A_0 is defined as:

$$A_0 = \frac{1}{2}e^{-y-\gamma}\int_0^\infty e^{-(1+\Lambda)y'}dy' = \frac{e^{-y-\gamma}}{2\left(\Lambda+1\right)}$$

hence:

$$\tilde{A}(z) = \frac{e^{-y-\gamma}}{2\left(\Lambda+1-z\right)}e^{z(y+\gamma)}$$

using Taylor expansion around $z=0$ we can easily get back to A_p:

$$\tilde{A}(z) = \sum_{p=0}^{\infty} z^p A_p = \frac{e^{-y-\gamma}}{2}\sum_{p=0}^{\infty}\frac{z^p}{\left(\Lambda+1\right)^{p+1}}\sum_{k=0}^{p}\frac{\left(y+\gamma\right)^k}{k!\left(\Lambda+1\right)^{-k}}$$

so comparing:

$$I_p^+(\Lambda,\gamma) = \frac{e^{-y-\gamma}}{2\left(\Lambda+1\right)^{p+1}}\sum_{k=0}^{p}\frac{1}{k!}\left(y+\gamma\right)^k\left(\Lambda+1\right)^k$$

Computation of $I_p^-\left(\Lambda\right)$

As before let's explicitly integrate the second of (A.1):

$$\begin{aligned}
I_p^-\left(\Lambda\right) &= \int_0^\infty e^{-\Lambda\frac{x'}{b}}f_p\left(x-x'\right)dx' \\
&= \frac{1}{2p!b^{p+1}}\int_0^\infty e^{-\Lambda\frac{x'}{b}}\left|x-x'\right|^p e^{-\frac{\left|x-x'\right|}{b}}dx' \\
&= \frac{1}{2p!}\int_0^\infty e^{-\Lambda y'}\left|y-y'\right|^p e^{-\left|y-y'\right|}dy' \\
&= \frac{1}{2p!}\int_0^y e^{-\Lambda y'}\left(y-y'\right)^p e^{-\left(y-y'\right)}dy' \\
&\quad + \frac{1}{2p!}\int_y^\infty e^{-\Lambda y'}\left(y'-y\right)^p e^{-\left(y'-y\right)}dy' = I_{1,p}^-\left(\Lambda\right) + I_{2,p}^-\left(\Lambda\right)
\end{aligned}$$

We need to study the two integrals $I_{1,p}^-\left(\Lambda\right)$ and $I_{2,p}^-\left(\Lambda\right)$ separately:

$I_{1,p}^-\left(\Lambda\right)$

Let's extract a recurrent relation:

$$\begin{aligned}
I_{1,p}^-\left(\Lambda\right) &= \frac{1}{2p!}\int_0^y e^{-\Lambda y'}\left(y-y'\right)^p e^{-\left(y-y'\right)}dy' \\
&= -\frac{1}{2p!\Lambda}e^{-\Lambda y'}\left(y-y'\right)^p e^{-\left(y-y'\right)}\Big|_0^y \\
&\quad + \frac{1}{2p!\Lambda}\int_0^y e^{-\Lambda y'}\left[-p\left(y-y'\right)^{p-1}e^{-\left(y-y'\right)} + \left(y-y'\right)^p e^{-\left(y-y'\right)}\right] \\
&= \frac{e^{-y}}{2p!\Lambda}y^p - \frac{1}{\Lambda}I_{1,p-1}^-\left(\Lambda\right) + \frac{1}{\Lambda}I_{1,p}^-\left(\Lambda\right)
\end{aligned}$$

Defining now $A_p \equiv I^-_{1,p}(\Lambda)$:

$$A_p = \frac{e^{-y}}{2(\Lambda-1)}\frac{y^p}{p!} - \frac{1}{\Lambda-1}A_{p-1}$$

using generating function method:

$$\sum_{p=1}^{\infty} z^p A_p = \frac{e^{-y}}{2(\Lambda-1)}\sum_{p=1}^{\infty} z^p \frac{y^p}{p!} - \frac{1}{\Lambda-1}\sum_{p=1}^{\infty} z^p A_{p-1}$$

rearranging:

$$\tilde{A}(z) - A_0 = \frac{e^{-y}}{2(\Lambda-1)}\left[e^{zy}-1\right] - \frac{z}{\Lambda-1}\tilde{A}(z) \tag{A.3}$$

A_0 is provided by:

$$A_0 = \frac{e^{-y}}{2}\int_0^y e^{(1-\Lambda)y'}dy' = \frac{e^{-y}}{2(\Lambda-1)}e^{(1-\Lambda)y'}\Big|_y^0 = \frac{e^{-y}}{2(\Lambda-1)}\left[1-e^{(1-\Lambda)y}\right]$$

so, substituting on (A.3), we finally obtain:

$$\tilde{A}(z) = \frac{e^{-y}}{2}\frac{e^{zy}}{\Lambda-1+z} - \frac{e^{-\Lambda y}}{2}\frac{1}{\Lambda-1+z}$$

using Taylor expansion around $z=0$ we can invert the generating function method:

$$\tilde{A}(z) = -\frac{e^{-y}}{2}\sum_{p=0}^{\infty}\frac{z^p}{(1-\Lambda)^{p+1}}\sum_{r=0}^{p}\frac{1}{r!}y^r(1-\Lambda)^r + \frac{e^{-\Lambda y}}{2}\sum_{p=0}^{\infty}\frac{z^p}{(1-\Lambda)^{p+1}}$$

hence finally:

$$I^-_{1,p}(\Lambda) = \frac{e^{-\Lambda y}}{2(1-\Lambda)^{p+1}} - \frac{e^{-y}}{2(1-\Lambda)^{p+1}}\sum_{r=0}^{p}\frac{1}{r!}y^r(1-\Lambda)^r \tag{A.4}$$

$I^-_{2,p}(\Lambda)$

Let's extract a recurrent relation:

$$\begin{aligned}
I^-_{2,p}(\Lambda) &= \frac{1}{2p!}\int_y^{\infty} e^{-\Lambda y'}\left(y'-y\right)^p e^{-(y'-y)}dy' \\
&= -\frac{1}{2p!\Lambda}e^{-\Lambda y'}\left(y'-y\right)^p e^{-(y'-y)}\Big|_y^{\infty} \\
&\quad + \frac{1}{2p!\Lambda}\int_y^{\infty} e^{-\Lambda y'}\left[p\left(y'-y\right)^{p-1}e^{-(y'-y)} - \left(y'-y\right)^p e^{-(y'-y)}\right]dy' \\
&= \frac{1}{\Lambda}I^-_{2,p-1}(\Lambda) - \frac{1}{\Lambda}I^-_{2,p}(\Lambda)
\end{aligned}$$

By defining $A_p \equiv I^-_{2,p}(\Lambda)$:

$$A_p = \frac{1}{\Lambda+1}A_{p-1}$$

using generating function method:

$$\sum_{p=1}^{\infty} z^p A_p = \frac{1}{\Lambda+1} \sum_{p=1}^{\infty} z^p A_{p-1}$$

rearranging:

$$\tilde{A}(z) - A_0 = \frac{z}{\Lambda+1} \tilde{A}(z) \tag{A.5}$$

A_0 is provided by:

$$A_0 = \frac{e^y}{2} \int_y^{\infty} e^{-(1+\Lambda)y'} dy' = -\frac{e^y}{2\,(\Lambda+1)} e^{-(1+\Lambda)y'} \Big|_y^{\infty} = \frac{e^{-\Lambda y}}{2\,(\Lambda+1)}$$

so, substituting on (A.5), we finally obtain:

$$\tilde{A}(z) = \frac{e^{-\Lambda y}}{2} \frac{1}{\Lambda + 1 - z}$$

using Taylor expansion around $z = 0$ we can invert the generating function method:

$$\tilde{A}(z) = \frac{e^{-\Lambda y}}{2} \sum_{p=0}^{\infty} \frac{z^p}{(\Lambda+1)^{p+1}}$$

hence:

$$I_{2,p}^{-}(\Lambda) = \frac{e^{-\Lambda y}}{2\,(\Lambda+1)^{p+1}} \tag{A.6}$$

Finally, merging the two solutions (A.4) and (A.6) we have:

$$I_p^{-}(\Lambda) = \frac{e^{-\Lambda y}}{2\,(1-\Lambda)^{p+1}} - \frac{e^{-y}}{2\,(1-\Lambda)^{p+1}} \sum_{r=0}^{p} \frac{1}{r!} y^r \,(1-\Lambda)^r + \frac{e^{-\Lambda y}}{2\,(1+\Lambda)^{p+1}}$$

Summarizing we obtained for the two integrals:

- $I_p^{+}(\Lambda,\gamma) = \frac{e^{-y-\gamma}}{2\,(\Lambda+1)^{p+1}} \sum_{k=0}^{p} \frac{1}{k!} (y+\gamma)^k \,(\Lambda+1)^k$
- $I_p^{-}(\Lambda) = \frac{e^{-\Lambda y}}{2\,(1-\Lambda)^{p+1}} - \frac{e^{-y}}{2\,(1-\Lambda)^{p+1}} \sum_{r=0}^{p} \frac{1}{r!} y^r \,(1-\Lambda)^r + \frac{e^{-\Lambda y}}{2\,(1+\Lambda)^{p+1}}$

where $y = x/b$.

Appendix B

Sum identities

We exhibit here the main sum identities we made use during the analysis.

Generating functions

We recall here the initial and boundaries conditions for the auxiliary variables $q_{k,n}$ and $r_{k,n}$:

$$q_{0,0} = r_{0,0} = 1 \qquad\qquad q_{n+1,n} = r_{-1,n} = 0$$

Often during the analysis we were required to switch to the generating function representations of $q_{k,n}$ and $r_{k,n}$ for proceeding with the calculations:

$$\tilde{q}(s,z;x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} s^n z^k q_{k,n}(x) \qquad\qquad \tilde{r}(s,z;x) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} s^n z^k r_{k,n}(x)$$

In order to do so, the main identities we used are:

- $\displaystyle\sum_{n=1}^{\infty}\sum_{k=0}^{n} s^n z^k q_{k,n}(x) = \tilde{q}(s,z;x) - q_{0,0} = \tilde{q}(s,z;x) - 1$
- $\displaystyle\sum_{n=1}^{\infty}\sum_{k=0}^{n} s^n z^k r_{k,n}(x) = \tilde{r}(s,z;x) - r_{0,0} = \tilde{r}(s,z;x) - 1$
- $$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=0}^{n} s^n z^k q_{k,n-1}(x) &= s\sum_{n=0}^{\infty}\sum_{k=0}^{n+1} s^n z^k q_{k,n}(x) \\ &= s\tilde{q}(s,z;x) + sz\sum_{n=0}^{\infty} s^n z^n q_{n+1,n}(x) = s\tilde{q}(s,z;x)\end{aligned}$$
- $$\begin{aligned}\sum_{n=1}^{\infty}\sum_{k=0}^{n} s^n z^k r_{k-1,n-1}(x) &= sz\sum_{n=0}^{\infty}\sum_{k=-1}^{n} s^n z^k r_{k,n}(x) \\ &= sz\tilde{r}(s,z;x) + s\sum_{n=0}^{\infty} s^n r_{-1,n}(x) = sz\tilde{r}(s,z;x)\end{aligned}$$

Action of the differential operator D^2

We consider:

$$D^2 f_p = (-1)^{p+1} \binom{1}{p} \delta(x) + \sum_{l=0}^{2} \binom{2}{l} (-1)^l f_{p-l}$$

where $D^2 = b^2 \partial_x^2$ is the rescaled differential operator and f_p a p-th order Gamma function. On applying D^2 a second time we get:

$$\begin{aligned} D^4 f_p &= (-1)^{p+1} \binom{1}{p} b^2 \delta''(x) + b^2 \sum_{l=0}^{2} \binom{2}{l} (-1)^l f''_{p-l} \\ &= (-1)^{p+1} \binom{1}{p} b^2 \delta''(x) + \sum_{l=0}^{2} \binom{2}{l} (-1)^l \left[\delta(x) \left[\delta_{p-l,1} - \delta_{p-l,0} \right] \right. \\ &\quad \left. + \sum_{m=0}^{2} \binom{2}{m} (-1)^m f_{p-l-m} \right] \end{aligned}$$

The first term in the R.H.S. sum gives:

$$\begin{aligned} \delta(x) \sum_{l=0}^{2} \binom{2}{l} (-1)^l \left[\delta_{p-l,1} - \delta_{p-l,0} \right] &= \delta(x) \sum_{l=0}^{2} \binom{2}{l} (-1)^l \left[\delta_{l,p-1} - \delta_{l,p} \right] \\ &= \delta(x) \left[\binom{2}{p-1} (-1)^{p-1} - \binom{2}{p} (-1)^p \right] \\ &= (-1)^{p+1} \delta(x) \left[\binom{2}{p-1} + \binom{2}{p} \right] = (-1)^{p+1} \binom{3}{p} \delta(x) \end{aligned}$$

while the second one:

$$\sum_{l=0}^{2} \sum_{m=0}^{2} \binom{2}{l} \binom{2}{m} (-1)^{l+m} f_{p-l-m} = \sum_{l=0}^{4} \binom{4}{l} (-1)^l f_{p-l}$$

hence, recollecting everything:

$$D^4 f_p = (-1)^{p+1} \left[\binom{1}{p} D + \binom{3}{p} \right] \delta(x) + \sum_{l=0}^{4} \binom{4}{l} (-1)^l f_{p-l}$$

Appendix C

Bromwich formula

The Bromwich formula is an useful relation for inverting the **Laplace transform** [25]. Let $f(x)$ be a function, we define $g(x)$ as:

$$g(x) = f(x)e^{-\gamma x}\Theta(x) \tag{C.1}$$

Exploiting the Fourier transform properties, we can write:

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{g}(k)e^{-ikx}dk = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} \int_{-\infty}^{+\infty} g(y)e^{iky}dydk$$

using (C.1):

$$f(x)e^{-\gamma x} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ikx} \int_{0}^{\infty} f(y)e^{-\gamma y}e^{iky}dydk$$
$$\Rightarrow \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(\gamma-ik)x} \int_{0}^{\infty} f(y)e^{-(\gamma-ik)y}dydk$$

on making the change of variables $t = \gamma - ik$ then $dt = -idk$, so:

$$f(x) = \frac{i}{2\pi} \int_{\gamma+i\infty}^{\gamma-i\infty} e^{tx} \int_{0}^{\infty} f(y)e^{-ty}dydt = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{tx} \int_{0}^{\infty} f(y)e^{-ty}dydt$$

On noticing that

$$\int_{0}^{\infty} f(y)e^{-ty}dy = \mathscr{L}_{y\to t}\left[f(y)\right] = \tilde{f}(t)$$

is the Laplace transform, we have:

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{tx}\tilde{f}(t)dt$$

and this is the **Bromwich formula**. The integration path is a vertical line centered in an arbitrary real value γ that must be at the right of all the singularities of $\tilde{f}(t)$.

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