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Ginzburg-Landau theory and Josephson effect in BCS-BEC crossover

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Introduction

Recent developments in the fields of confinement, cooling and control of intra-particles interaction brought a new focus on BCS-BEC crossover. Let's consider a system of Fermions at very low temperature. We know that under a certain temperature, both for charged that for neutral systems, appear mechanisms that permit the formation of pairs made by Fermions. Through an external magnetic field is possible to fix the intereaction between the Fermions and thus the size of the pair. When the Cooper pair size becomes very small compared to the avarege distance between particles, it is not the Fermionc behavior of the single particles but the Bosonic behavior of the entire pair to prevail in the system. Varying the pair interaction there is no simmetry breaking, so we have a continous transition passing from the BCS regime to the BEC one, just a crossover. In the beginning, the study of this model was aimed at understanding superconductivity in materials with extremely low electronic density. In a very diluite superconductor the Cooper pair is always smaller than the avarege distance between particles, so considerable a Boson. In the firtst chapter we will formally approach the problem through the Bogoliubov-de Gennes equations, from which we can easly derive the gap and the number equation under the conditions of zero temperature and no external potential. Then we moved to the critical temperature, near the phase transition point. A fondamental tool to study the phase transition is the Landau theory. We will explore the main feature of this theory and its application to superfluid neutral system: the Ginzuburg-Landau theory. Of particular importance is the work of C. A. R Sà de Melo, Mohit Randeria and Jan R. Engelbrecht [1]. They solved the microscopic number equation and the gap equation at the critical temperature. Using the mean-field approximation they obtain the dissocate temperature of the Cooper pair but not the bosonic condansation temperature. To obtain the right value in the BEC regime, as we will see in the second chapter, is necessary to go over the mean-field approximation and considering Gaussian fluctuations of the order parameter. At the end of the work they derived the Ginzburg-Landau functional and the relations for its parameters. What we wiil do is to study the behavior of the parameters in the mean-field and beyond-mean filed approximation. These parameters are linked to some charcteristic quantities by simple relation. Approaching the critical temperature from below one has that this quantities diverge or goes to 0 because of the pair breaking and the disappear of the superfluid phase. In this work we will study Ginzburg-Landau coherence lenght (diverges) and the critical frequency (goes to zero) for different values of the coupling. From their behavior we have extract some conditions to have a better superfluidity. This systems, for their nature, are best studied at very low temperature where the superfluidity is more consistent. We will investigate the temperature interval around T_c where the G-L theory is relaiable. In the last part of the chapter we will evaluate a relataion for the sound velocity directly from the Ginzuburg-Landau functional. In the second part instead we focus on another characterisctic superfluid phenomenon: the Josephson effect. This consists in a supercurrent that is established between two superfluids separeted by an insultator. The Josephson effect can be studied in two modes:

direct current (DC) or alternate current (AC). We will investigate the two possibilities for a Fermionic neutral system. For this part we will consider a bidimensional system, since the 3D case is has already been investigated far enough. We analyze the effects of fluctuations, evaluating in the mean-field and beyond mean-field case the composite Boson chemical potential (μ_B), the sound velocity (c_s), pressure (P) and the condensate fraction (n_0/n). Then we move to the study of Josephson current. In the first case we consider the critical current without the relative imbalance and fixed phase difference between the two reservoirs (DC Josephson current). Using the same procedure of [2] we evalutate how the ratio between the density critical current and the single particle tunneling probability changes along the crossover. Then comparing this relation with the one of the critical Josephson current we can understand how this latter changes along the crossover. We use the results in the AC Josephson effect replacing the coupling-dependent tunneling energy on the junction Josephson equations. We proceed with the study relative imbalance $z[t]$ and phase difference $\phi[t]$ time evolution behavior along the crossover.

Chapter 1

Introduction to the BCS-BEC crossover

There has been great excitement about the recent experimental and theoretical progress in the Bardeen-Cooper-Schrieffer (BCS) to Bose Einstein condensation (BEC) crossover in ultracold Fermi gases. The BCS–BEC crossover problem was of little direct experimental interest before the era of ultracold atoms, because of the difficulties encountered in the realization of ultra-cold systems with controllable interaction between particles . This situation changed dramatically with the realization of dilute gases of fermionic alkali atoms with (^{40}K, ^{6}Li) , such as can be cooled into the degenerate regime and their inter atomic interaction can be tuned via a Feshbach resonance. While the experimental realization of the phenomenon, we now call the "BEC–BCS crossover", was attained only in the last few years, theoretical considerations of this issue go back a lot further. In the first part of this chapter we review some of this history, while then we report the theoretical model that describe BCS-BEC crossover for a coupled Fermi system.

1.1 Historical background

In this part we shall give a brief review of the history of the BCS-BEC theory [3, 4]. While the many people worked on the theory of Bose-Einstein condensation in liquid helium in the years between London's proposal of this phenomenon in 1938 and the work of BCS [5] in 1957, nobody seems notice that the ^{4}He atom is actually a composite of six fermions. With hindsight this is hardly surprising, since the minimum energy scale relevant to dissociation of the atom into its fermionic components is several orders of magnitude greater than that involved in BEC of the liquid, and thus it is usually an excellent approximation, in the context of the latter, to treat the atom as a simple structureless boson. The first person to make the explicit suggestion that pairs of fermions (electrons) with an effectively attractive interaction might form a molecular-like object with bosonic statistics and thus undergo BEC appears to have been Ogg [6], in the context of a very specific superconducting system (an alkali metal-ammonia solution); however, Ogg speculated that this mechanism might more generally be the explanation of superconductivity. This idea was taken up a few years later by Schafroth [7] and amplified in the paper of Schafroth et al. [8]; however, it proved very difficult to use this approach to calculate specific experimental quantities. It treats the system as composed by non-overlapping composite bosons which undergo Bose-Einstein condensation at low temperature. Following the work of Bardeen Cooper and Schrieffer [5], furtherwork was done, mainly by Blatt and coworkers, along the lines developed in ref. [8]; see for example ref. [9]. This work emphasized the point of view that Cooper pairing in a weakly interacting Fermi gas could be viewed as a form of BEC (of pairs of electrons); the qualitative considerations developed in it foreshadow some of those that resurfaced subsequently in the context of the crossover problem. However, the interest in the Cooper pair and composite Bosons has been kept disjoint for some time, until theoretical interest arose for unifying them as two limiting (BCS and BEC) situations of a single theory where they share the same kind of broken symmetry. One important development that, at least with hindsight, pulls rather strongly in the opposite direction is the seminal paper of Yang [10] on off-diagonal long-range order (ODLRO). Yang showed that the generalized definition of BEC given by Penrose and Onsager [11] for a simple Bose system such as ^4He could be generalized to apply to a fermionic system provided one replaces the single-boson density matrix by the two-fermion one. However, it seems to have been some time before the full significance of this observation was appreciated by the community. Meanwhile, attempts were being made to apply BCS-like ideas to Fermi systems ^3He other than the electrons in metals. In the case of liquid and heavy nuclei, the situation seemed to be fairly close to the original BCS work, in the sense that the pairing interaction was likely to be so weak that the radius of any pairs formed would be much greater than the inter-fermion distance, just as it is in (pre-1970s) superconducting metals. The theory of the BCS-BEC crossover took shape initially through the work by Eagles [12] with possible applications to superconducting semiconductors, and later through the works by Leggett [13] and Nozieres and Schmitt-Rink [14] where the formal aspects of the theory were developed at zero temperature and above the critical temperature, respectively. The interest in the BCS-BEC crossover grew up with the advent of high-temperature (cuprate) superconductors in 1987, in which the size of the pairs appears to be comparable to the inter-particle spacing. Related interest in the BCS-BEC crossover soon spread to some problems in nuclear physics, but a real explosion of this activity appeared starting from 2003 with the advent of the fully controlled experimental realization essentially of all aspects of the BCS-BEC crossover in ultra-cold Fermi gases. This realization, in turn, has raised the interest in the crossover problem especially of the nuclear physics community, as representing an unprecedented tool to test fundamental and unanswered questions of nuclear many-body theory. The Fermi gas at the unitary limit (UL), where fermions of opposite spins interact via a contact interaction with infinite scattering length, was actually introduced as a simplified model of dilute neutron matter, and the possibility to realize this limit with ultra-cold atoms was hence regarded as extremely important for this field of nuclear physics. As these examples show, there are several aspects of the BCS-BEC crossover which are of broad joint interest to both ultra-cold atoms and nuclear communities.

1.2 BCS-BEC crossover

In this chapter we want to illustrate the fundamental characteristics of the BCS-BEC crossover phenomenon and the mechanism to tune the interaction between particles. We will consider a system of coupled Fermions at zero temperature in absence of an external potential. For simplicity we treat a single channel s-wave interaction. The Hamiltonian density of a dilute and interacting two-spin component Fermi gas in a box of volume V is given by:

$$\hat{H} = \sum_{\sigma} \int dr^d \hat{\Psi}^{\dagger}_{\sigma}(r) \Big[-\frac{\hbar^2}{2m} \nabla^2 - \mu_{\sigma} \Big] \hat{\Psi}_{\sigma}(r) + g \int dr^3 \hat{\Psi}^{\dagger}_{\sigma}(r) \hat{\Psi}^{\dagger}_{\downarrow}(r) \hat{\Psi}_{\downarrow}(r) \hat{\Psi}_{\uparrow}(r) \tag{1.1}$$

where the sum $\sigma =\uparrow, \downarrow$ is over the spin component, d is the dimension of the system, μ_{σ} is the single Fermion chemical potential and $g < 0$ the attractive interaction strength. The field operator $\hat{\Psi}_{\sigma}(r)$ destroys a Fermion of spin σ in the position r, while $\hat{\Psi}^{\dagger}_{\sigma}(r)$ creates a Fermion of spin σ in the position r. They are Grassman operators so they anticommute:

$$\Big[\hat{\Psi}_{\sigma}(\vec{r}, t), \hat{\Psi}^{\dagger}_{\sigma'}(\vec{r}^{\,\prime}, t^{\prime}) \Big] = \hat{\Psi}_{\sigma}(\vec{r}, t) \hat{\Psi}^{\dagger}_{\sigma'}(\vec{r}^{\,\prime}, t^{\prime}) + \hat{\Psi}^{\dagger}_{\sigma'}(\vec{r}^{\,\prime}, t^{\prime}) \hat{\Psi}_{\sigma}(\vec{r}, t) = \delta(\vec{r} - \vec{r}^{\,\prime}) \delta(\vec{t} - \vec{t}^{\,\prime}) \delta(\sigma - \sigma') \tag{1.2}$$

The total number of particles N=$N_{\uparrow}$+$N_{\downarrow}$ could be written in function of the field operator $\Psi_{\sigma}(r)$:

$$N = \int dr^3 < \hat{n}(r) > \tag{1.3}$$

where $\hat{n}(r)$ is the number density operator:

$$\hat{n}(r) = \frac{1}{V} \sum_{\sigma=\uparrow,\downarrow} \hat{\Psi}^{\dagger}(r) \hat{\Psi}_{\sigma}(r) \tag{1.4}$$

The Heisenberg equation of motions associated to (1.1) is:

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}_{\uparrow} = \Big[-\frac{\hbar^2}{2m} - \mu + g \hat{\Psi}^{\dagger}_{\downarrow} \hat{\Psi}_{\downarrow} \Big] \hat{\Psi}_{\uparrow} \tag{1.5}$$

The interacting term could be treated within the man-field approximation replacing $\Psi_{\sigma}(r) =< \Psi_{\sigma}(r) > + \delta\Psi_{\sigma}(r)$, where $\delta\Psi_{\sigma}(r)$ is considered a small fluctuations respect to the mean-field term. Neglecting all the term higher than the second order, it reads:

$$\hat{\Psi}^{\dagger}_{\downarrow} \hat{\Psi}_{\downarrow} \hat{\Psi}_{\uparrow} =< \hat{\Psi}^{\dagger}_{\downarrow} \hat{\Psi}_{\downarrow} > \hat{\Psi}_{\uparrow} + \hat{\Psi}^{\dagger}_{\downarrow} < \hat{\Psi}_{\downarrow} \hat{\Psi}_{\uparrow} > + < \hat{\Psi}^{\dagger}_{\downarrow} \hat{\Psi}_{\uparrow} > \hat{\Psi}_{\downarrow} \tag{1.6}$$

We introduce the gap function $\Delta(\vec{r}, t) = g < \Psi_{\downarrow}(\vec{r}, t) \Psi_{\uparrow}(\vec{r}, t) >$. We know that the gap function is the required energy to destroy the pair. It rappresent the order parameter of the superconductive phase transition and the term $\Theta(\vec{r}, t) =< \hat{\Psi}_{\downarrow}(\vec{r}, t) \hat{\Psi}_{\uparrow}(\vec{r}, t) >$ is the condensate wave function of the pair. Remembering also the definition of number density operator, the interacting term becomes:

$$g \hat{\Psi}^{\dagger}_{\downarrow} \hat{\Psi}_{\downarrow} \hat{\Psi}_{\uparrow} = g \Big[< n_{\downarrow}(\vec{r}, t) > \hat{\Psi}_{\uparrow} + \hat{\Psi}^{\dagger}_{\downarrow} \Delta(\vec{r}, t) + < \hat{\Psi}^{\dagger}_{\downarrow} \hat{\Psi}_{\uparrow} > \hat{\Psi}_{\downarrow} \Big] \tag{1.7}$$

Now we can apply another approximation in order to further simplify the situation. We neglect the third term, that rapresents the destruction process of a Fermion with spin $\sigma =\uparrow$ and creation of one with spin $\sigma =\downarrow$. We neglect also the Fock term $< \hat{\Psi}^{\dagger}_{\sigma} \hat{\Psi}_{\sigma} >$. In this way the Heisenberg equation of motion becomes:

$$i\hbar\frac{\partial}{\partial t}\hat{\Psi}_\uparrow = \Big[-\frac{\hbar^2}{2m} - \mu\Big]\hat{\Psi}_\uparrow + \Delta(\vec{r},t)\hat{\Psi}^\dagger_\downarrow \tag{1.8}$$

To solve it we rewrite the field operator using the standard stationary-state Bogoliubov transformation:

$$\Psi_\uparrow = \sum_k \Big[u_k(r)\hat{c}_{k\uparrow}e^{-iE_{k\uparrow}t/\hbar} + v_k^*(r)\hat{c}^\dagger_{k\downarrow}e^{iE_{k\downarrow}t/\hbar}\Big] \tag{1.9}$$

$$\Psi^\dagger_\downarrow = \sum_k \Big[u_k^*(r)\hat{c}^\dagger_{k\downarrow}e^{iE_{k\downarrow}t/\hbar} + v_k(r)\hat{c}_{k\uparrow}e^{-iE_{k\uparrow}t/\hbar}\Big] \tag{1.10}$$

where $\hat{c}_{k\uparrow}$,$\hat{c}_{k\downarrow}$,$\hat{c}^\dagger_{k\uparrow}$,$\hat{c}^\dagger_{k\downarrow}$ are the quasi-particle Fermi operators and $E_k = \hbar\omega_k$ is the energy of these quasi-particles. The coefficient $u_k(r)$ and $v_k(r)$ are renormalized as $u_k(r)^2 + v_k(r)^2 = 1$. Instead the quasi-particles operators follow this condition on the thermal average:

$$< \hat{c}^\dagger_{k\sigma}\hat{c}_{k`\sigma`} >= f(E_k)\delta_{k,k`}\delta_{\sigma,\sigma`} \tag{1.11}$$

where $f(E_k) = \frac{1}{e^{E_k/k_BT}+1}$ is the Fermi distribution function. Replacing the transformation in the gap function, mean-density and in the condensate number pairs one obtains:

$$\Delta(r) = -g\sum_k u_k(r)v_k^*\Big[1 - 2f(E_k)\Big] \tag{1.12}$$

$$n_\sigma = \sum_k \Big[|v_k(r)|^2 + f(E_k)(|u_k(r)|^2 - |v_k(r)|^2)\Big] \tag{1.13}$$

$$N_0 = \int d^dr|\Theta(r,t)|^2 = \int d^3r\sum_k |u_k(r)|^2|v_k(r)|^2\Big[1 - 2f(E_k)\Big]^2 \tag{1.14}$$

With this approach the condensate wave function becames indipendent of time, it's stationary. Instead replacing the Bogoliubov transformation in the Heisenberg equation of motion one obtains two equation, one for $u_k(r)$ and one for $v_k(r)$. They could be written in matrix form:

$$\begin{pmatrix} \hat{H}_0 & \Delta(r) \\ \Delta^*(x) & -\hat{H}_0 \end{pmatrix}\begin{pmatrix} u_k(r) \\ v_k(r) \end{pmatrix} = E_k\begin{pmatrix} u_k(r) \\ v_k(r) \end{pmatrix} \tag{1.15}$$

where $\hat{H}_0 = -\frac{\hbar^2\nabla^2}{2m} - \mu$. These are called Bogoliubov-de Gennes (BdG) equations which allow us to derive $u_k(r)$ and $v_k(r)$. To solve the BdG equation we proced considering that in our case of no external potential we could write $u_k(r)$ and $v_k(r)$ like plane wave:

$$u_k(r) = u_k e^{ik\cdot r} \tag{1.16}$$

$$v_k(r) = v_k e^{ik\cdot r} \tag{1.17}$$

Replacing these relations in the BdG equations one obtain the following solutions for u_k and v_k:

$$u_k = \frac{\Delta(r)v_k}{E_k - (\frac{\hbar^2k^2}{2m} - \mu)} \tag{1.18}$$

$$v_k = \frac{\Delta^*(r)u_k}{E_k + (\frac{\hbar^2k^2}{2m} - \mu)} \tag{1.19}$$

Substituing (1.19) in (1.18) one obtain the following relation for E_k:

$$E_k = \sqrt{(\frac{\hbar^2 k^2}{2m} - \mu)^2 + |\Delta(r)|^2} \tag{1.20}$$

This is the single particle exitacions energy. So the quasi-particle operator $\hat{b}_{k\sigma}$ could be associated to particles that mediate the excitations of the pair. To obtain indipendent relations for $u_k(r)$ and $v_k(r)$ it's necessary to open the matrix of BdG equations in the two equations:

$$E_k u_k = \Big(\frac{\hbar^2 k^2}{2m} - \mu\Big) u_k^2 + \Delta(r) v_k \tag{1.21}$$

$$E_k v_k = \Big(\frac{\hbar^2 k^2}{2m} - \mu\Big) v_k^2 + \Delta(r) u_k \tag{1.22}$$

We multiply (1.21) for u_k^* and (1.22) for v_k^* and then we make the difference between the two. Replacing (1.12) in $\Delta(r)$ and remembering that $u_k^2(r) + v_k^2(r) = 1$ one obtain the following relations:

$$u_k^2 = \frac{1}{2}\Big[\frac{(\frac{\hbar^2 k^2}{2m} - \mu)}{E_k} + 1\Big] \tag{1.23}$$

$$v_k^2 = \frac{1}{2}\Big[1 - \frac{(\frac{\hbar^2 k^2}{2m} - \mu)}{E_k}\Big] \tag{1.24}$$

Now we have all the information to evaluate the gap function, the mean density and the number of condensated paris. All these quantities depend on the temperature through the Fermi distribution function. Since we are looking to describe the main features of the BCS-BEC crossover we chose the temperature with the easiest case to treat. Fixing T=0 the Fermi distribution function $f(E_k)$ becomes zero. The gap function and the mean density relations reads:

$$\Delta(r) = -g \sum_k u_k(r) v_k(r) \tag{1.25}$$

$$n = \frac{1}{V} \sum_\sigma n_\sigma = \frac{2}{V} \sum_k |v_k|^2 \tag{1.26}$$

In the continuos limit the summation can be replace with the integral $\sum_k \rightarrow \frac{V}{(2\pi)^d} \int d^d r$. Then replacing (1.23) and (1.24) one obtain the number equation:

$$n = \frac{1}{(2\pi)^d} \int d^d k \Big[1 - \frac{(\frac{\hbar^2 k^2}{2m} - \mu)}{E_k}\Big] \tag{1.27}$$

and the familiar BCS gap equation:

$$\frac{1}{g} = -\frac{V}{(2\pi)^d} \int d^d k \frac{1}{2E_k} \tag{1.28}$$

Unfortunately due to the choice of the contact potential, the gap equation diverges in ultraviolet. This divergence is logarithmic in two dimensions and linear in three dimensions. We continue the discussion with the tridimensional case. The approach to the bidimensional one, as we will see in the third chapter, is pretty much the same but with some substantial physical difference. In 3D BCS one solves the problem inserting a cutoff to the integral. Since in this regime the interaction is weak, the single particles cannot have high energy otherwise they would not form pairs. Our gool is to describe the BCS-BEC crossover so in the BEC regime, where most of the partcles will condense, we have to consider strong coupling between all the particles. It's necessary to eliminate the ultraviolet cutoff in the integral. To solve the question one writes the Lippman-Schwinger equation approximated at the first order [13, 15]:

$$\frac{1}{g} = \frac{m}{4\pi\hbar^2 a_s} - \frac{1}{(2\pi)^3}\int d^3k \frac{1}{2\epsilon_k} \tag{1.29}$$

where the volume V was set equal at 1 for semplicity. The $\epsilon_k = \frac{\hbar^2 k^2}{2m}$ is the single particle energy. Instead a_s is the scattering lenght of the interacting process. For low energy interaction a_s is related to the elastic cross σ_e section through the relation $\sigma_e = 4\pi a_s^2$. As we can see the integral in (1.29) is divergent in the same way of the one in (1.28), but with opposite sign. So replacing this relation in (1.28) the divergence will disappear. Before to do that it's good to spend some words on (1.29). To describe the BCS-BEC crossover it's necessary to increase the attractive potential beween the Fermions. In the next section we will see in detail how to do it with an external magnetic field. Now let's see how the strenght interaction varies as the scattering lenght varies. In the BCS regime the streng coupling is $g \to 0^-$ and, as we said, we can consider the integral with the cutoff Γ. The (1.29) reads:

$$\frac{m}{4\pi\hbar^2 a_s} = \frac{1}{g} + \frac{m\Gamma}{2\pi^2\hbar^2} \tag{1.30}$$

The leading term on the right-hand side of the relation is 1/g that diverges negatively. This means that in the BCS regime $a_s \to 0^-$. In the deep BEC regime instead we would have $g \to -\infty$ and the divergent integral without the ultraviolet cutoff. In the BEC regime $a_s \to 0^+$. Generally the description is done in reference to the adimensional quantity $1/k_F a_s$ where k_F is the Fermi wave vector of the non-interacting system. In this way $1/k_F a_s$ passing from $-\infty$ to ∞ going from the BCS to the BEC regime. What happen in the middle where $1/k_F a_s = 0$? To find out we have to explicit the scattering lenght in the relation (1.30) in this way:

$$a_s = \frac{\pi g m}{4\pi^2\hbar^2 + 2gm\Gamma} \tag{1.31}$$

the limit $1/k_F a_s = 0$ corresponds to a positively or negatively divergent scattering length. The a_s diverges for the values of the cougling strenght:

$$g = -\frac{2\pi^2\hbar^2}{m\Gamma} = -g_0 \tag{1.32}$$

For semplicity of notation we can write g in unity of g_0. So for $g = -1^+$ we have $a_s = -\infty$ and for $g = -1^-$ the scattering lenght is $a_s = +\infty$. The fact that crossing g_0 the scattering lenght changes the sign means that around this value bounded states emerge. In the crossover region $|k_F a_s| > 1$, the pair size becomes of the order of interparticle spacing and thus the system con no longer be regardered as either a wealkly interactiong Bose or Fermi gas. In this region the pairs are so bounded to form molecules. In particular, the unitary limit, $1/k_F a_s = 0$, gives rise to a universal strongly interacting Fermi gas composed of molecules. In the next two chapters we will see that the unitary limit is a priveliged region because it is the meeting point between the BCS and the BEC properties. Clarified the link between the coupling strenght and the scattering lenght we can substitute the (1.29) in (1.28). One obtains the regularized gap equation:

$$-\frac{m}{4\pi\hbar^2 a_s} = \frac{1}{2}\int d^3k \Big(\frac{1}{E_k} - \frac{1}{\epsilon_k}\Big) \tag{1.33}$$

The gap equation and the number equation have to be solved simultaneosly in order to determine the gap energy and the chemical potential as functions of a_s. In three dimensions is convinent to introduce the following dimensionless quantities [16]:

$$x^2 = \frac{\hbar^2 k^2}{2m\Delta}$$

$$x_0 = \frac{\mu}{\Delta}$$

$$\xi_x = \frac{\frac{\hbar^2 k^2}{2m} - \mu}{\Delta} = x^2 - x_0$$

$$E_x = \frac{E_k}{\Delta} = \sqrt{\xi_x + 1}$$

In this way (1.33) and (1.27) reads:

$$-\frac{1}{a_s} = \frac{2}{\pi}(2m\Delta)^{1/2} I_1(x_0) \tag{1.34}$$

$$n = \frac{1}{2\pi^2}(2m\Delta)^{3/2} I_2(x_0) \tag{1.35}$$

where:

$$I_1(x_0) = \int_0^\infty dx \Big(\frac{1}{E_x} - \frac{1}{x^2}\Big)x^2 \tag{1.36}$$

$$I_2(x_0) = \int_0^\infty dx \Big(1 - \frac{\xi_x}{E_x}\Big)x^2 \tag{1.37}$$

These integrals had been evaluated in detail from Strinati et al. in [4, 16] with the use of complete elliptic integrals. Another very useful quantity to be plotted is the condensated fraction, namely the number of pairs that condense in the ground state over the total number of couples. Replacing (1.18) and (1.19) in (1.14) and using the adimensional formalism introduced above, one obtains:

$$\frac{N_0}{N} = \frac{\pi}{8\sqrt{2}} \frac{\sqrt{\frac{\mu}{\Delta} + \sqrt{1 + (\frac{\mu}{\Delta})^2}}}{I_2(\frac{\mu}{\Delta})} \tag{1.38}$$

In Fig.(1.1) we reported the Gap energy, the chemical potential and the condensated fraction as a function of $1/k_F a_s$, along the crossover from BCS to BEC regime. In the panel (a) of the figure we can see the gap energy growing monotonically. From the BCS to BEC regime the Fermions are more attracted to each other and so the pair requires more energy to be destroyed. In BCS theory the gap energy at zero temperature depends linearly on the critical temperature,$\Delta_0 = 1.764 k_B T_c$. This means that the critical temperature of a homogeneus Fermi system under mean-field theory increases as the gap energy. What we expected instead is that the critcal temperature in the BEC regime converges to the transition temperature of a BEC system of mass 2m: $T_c \simeq 0.218 E_F$. Another problem is related to the mean-field approximation There are cases when the BdG equations can be replaced by suitable non-linear differential equations for the gap parameter $\Delta(r)$, which are somewhat easier to solve numerically and conceptually more appealing than the BdG equations themselves. These non-linear dierential equations for $\Delta(r)$ are the Ginzburg-Landau (GL) equation for the Cooper-pair wave function and the Gross-Pitaevskii (GP) equation for the condensate wave function of composite bosons. As a matter of fact, it turns out that the GL and GP equations can be microscopically derived from the BdG equations in two characteristic limits; namely, the GL equation in the weak-coupling (BCS) limit close to Tc [17] and the GP equation in the strong-coupling (BEC) limit at T=0 [18]. Staying focus on the treatment at zero temperature under the mean-field approach we report the GP equations. From the BdG equations one obtains the GP equation for a gas of dilute composite bosons of mass 2m in terms of the order paramete $\Delta(r)$, in the form [4]:

$$\frac{-\hbar^2\nabla^2}{2m}\Delta(r) + \frac{4\pi a_B}{m_B}\Delta(r) = \mu_B\Delta(r) \tag{1.39}$$

where $m_B = 2m$ is the mass of the couple, μ_B is the chemical potential of the composite boson, defined by $\mu_B = 2\mu + \epsilon_0$ where $\epsilon_0 = \hbar^2/ma_s^2$ is the binding energy of the molecule. This is the energy that keep togheter the molecule. The term $\frac{4\pi a_B}{m_B}$ is the strenght of the repulsive interaction between the Bosons. The mean-field equation correctly recover the repulsion but with an incorrect scattering lenght $a_B = 2a_s$, which is overstimate compared with the exact result $a_B \simeq 0.6a_s$ obtained from four-body dimer-dimer calculations [19]. Fluctuations inside molecules become important at short distances. The energy of scale to consider and over which it no longer makes sense to speak of Fermion pairs, is given by the binding energy ϵ_0. In [20] K.Huang, Z.Qiang and L.Y showed in detail how to apply this limit and what they obtained is a scattering lenght $a_B = 0.56a_s$. In the next chapter we will see how including Gaussian thermal fluctuations to the order parameter can also solve the temperature question. Anyway the mean-field approach provides an intuitive and qualitatively reasonable description of the BCS-BEC crossover. Indeed, the increase of the coupling brings out the Bosons behavior of the couple in spite of Fermionic behavior of the single particles. In the BCS limit $\mu = E_F$, then it decreases and around the value of $1/k_F a_s = 0.5$ becomes negative. In the BEC regime the chemical potential becomes $\mu = \epsilon_0/2$ and thus μ_B=0. This is the value of the Bosons chemical potential at T=0 in BEC. But what characterized a Bose-Einstein condensate is the present of a macroscopic number of particles in the ground-state. In panel (c) we can see that from BCS to BEC this number increases and around the value of $1/k_F a_s = 0.5$ the 80% of the couples is condensated. This is in general the reference value for the BEC.

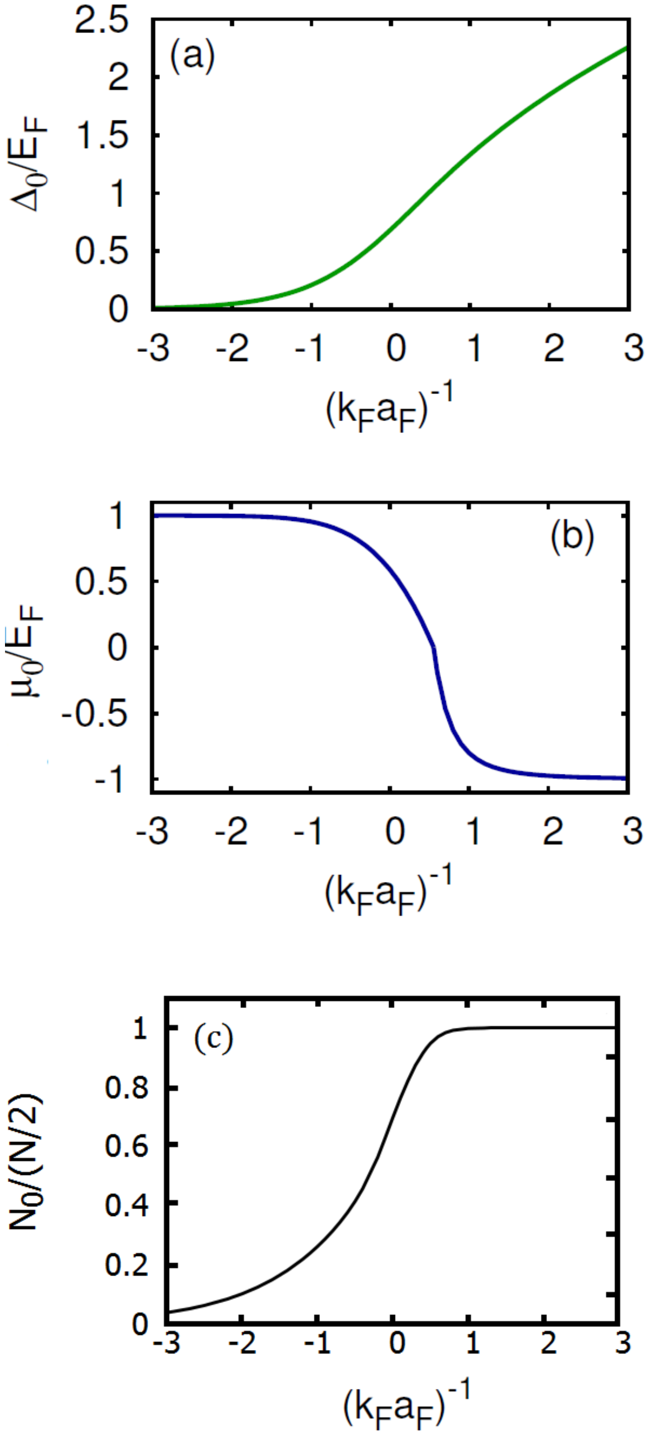

Figure 1.1: (a) Gap energy Δ_0, (b) chemical potential μ and (c) the condensate fraction $N_0/(N/2)$ vs $(k_F a_s)^{-1}$ at T=0 for homogenus system at mean-field level.

1.3 Tunable Interactions

In the following we describe the basic physics of magnetically tunable Feshbach resonances which allow to change the interaction between two Fermions simply by changing an external magnetic field [3,4,21]. As we shall see below, in general, one needs a two-channel model to describe a Feshbach resonance: two fermions in the "open channel" coupled to a bound state in the "closed channel". An energetically accessible reaction channel is referred to as an open channel, whereas a reaction channel forbidden by energy conservation is referred to as a closed channel. We have to consider the spin structure ot the atom, formed by the nuclear and the electronic one. What we are interested in is the possibility for this two to interact among each other. This allows not only the atom to change its spin, nuclear or electronic, which are therefore no longer constant in motion, but also during a collision the internal states of the atom can change opening up new channels for scattering. Quite generally, a Feshbach resonance in a two-particle collision appears whenever a bound state in a closed channel is coupled resonantly with the scattering continuum of an open channel. The ability to tune the scattering length by a change of an external magnetic field B relies on the difference in the magnetic moments of the closed and open channels. This difference allows the experimentalist to use an external magnetic field B as a knob to tune across a Feshbach resonance. The resulting interaction between atoms in the open channel can be described by a B-dependent scattering length that, in the vicinity of a resonance, has the form:

$$a(B) = a_{bg}\Big(1 - \frac{\Delta B}{B - B_0}\Big) \tag{1.40}$$

Here a_{bg} is the off-resonant background scattering length in the absence of the coupling to the closed channel while B and B_0 describe the width and position of the resonance expressed in magnetic field units. In the specific example of ^6Li the electron spin is essentially fully polarized and aligned in the same direction of the three lowest hyperfine states. Thus, two colliding ^6Li atoms are in a continuum spin-triplet state in the open channel. The closed channel has a singlet state that can resonantly mix with the open channel as a result of the hyperfine interaction that couples the electron spin to the nuclear one. A key property is that the singlet state supports a bound state, which we will seeto be the responsible for resonance. This molecular state has a different magnetic moment respect the two colliding atoms, the difference in energy can therefore be manipulated by a magnetic field:

$$\Delta E = \Delta\mu B \tag{1.41}$$

By modulating field B is possible to vary the difference between the energy of the particles in the scattering process and the energy of the bound state, the latter is closer to the scattering energy, the more likely it is for interacting particles to perform one transition to the bound state. This is the basic mechanism of Feshbach resonance. As we can see from Fig.(1.3,A) at the magnetic field value B_0 there is no more energy difference between the closed and the open channel. In this way two particles can form a bounded state. This situation results in a change of sign and divergence in the scatterig lenght, in line with that reported in the previous section. To get an intuitive feel for the scattering length, we do not need to understand the intricacies of the two-channel model of a Feshbach resonance. Instead, we can look at the much simpler single-channel problem of two particles with a short-range interaction. This simplified discussion is quite sufficient to understand much of the current experimental and theoretical literature on cold Fermi gases. The technical reason for the validity of this single-channel model is that most of the experiments are in the so-called broad resonance limit,

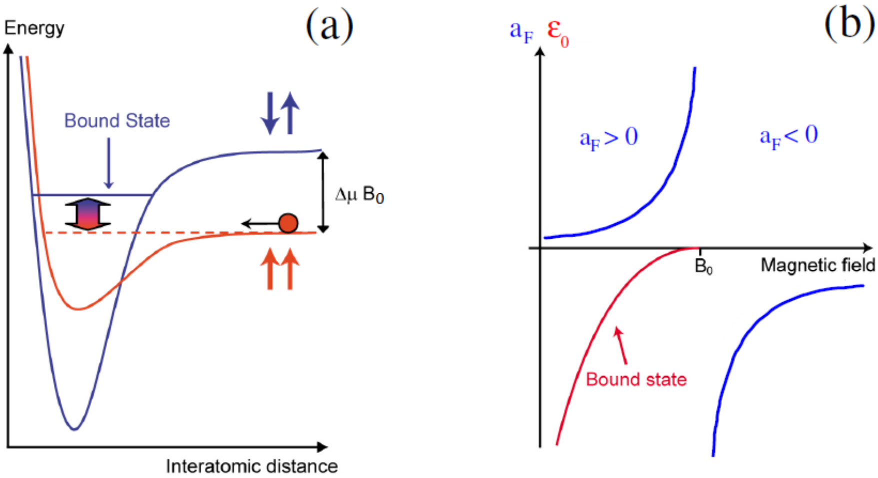

Figure 1.2: Coupling of closed and open dimer scattering channels, which can be displaced relative to each other by the coupling to a magnetic field B. (b) The corresponding scattering length vs the magnetic field.

where the effective range (which we do not discuss here) of the Feshbach resonance is much smaller than k_F^{-1}. This ensures that the fraction of closed-channel molecules is extremely small, a feature directly confirmed in experiments. Consider the problem of two fermions with spin $|\uparrow>$ and spin $|\downarrow>$ interacting with a two body potential with range r_0. The effective interaction is independent of the detailed shape of the potential thus we can examine it for the simplest model potential— a square well of depth V_0 and range r_0 — to get a better feel for the scattering length a_s as a function of V_0.

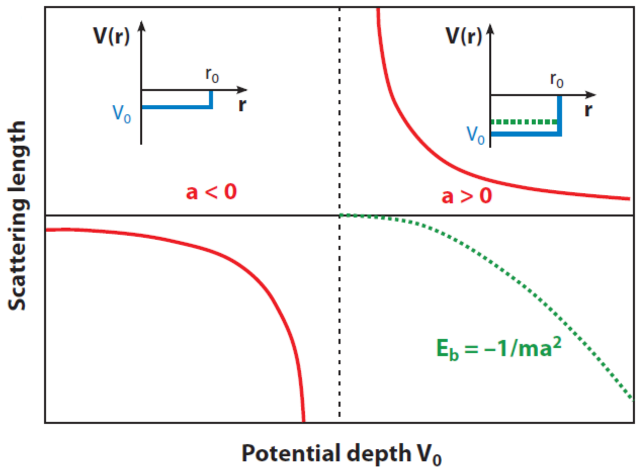

Figure 1.3: Coupling of closed and open dimer scattering channels, which can be displaced relative to each other by the coupling to a magnetic field B. (b) The corresponding scattering length vs the magnetic field.

As shown in Figure 2b, $a_s < 0$ for weak attraction, grows in magnitude with increasing V_0, and diverges to $-\infty$ at the threshold for the formation of a two-body bound state in vacuum. The threshold for a square well is $V_0 = \frac{2\pi^2}{mr_0^2}$, this is very similar to the relation (1.32). Once this bound state is formed, the scattering length changes sign and decreases with increasing V_0. Above the threshold, $a > 0$ has the simple physical interpretation as the size of the bound state, whose energy is given by $\epsilon_0 = -\hbar^2/ma_s^2$. In tridimensional systems it's possible yo have a bounded state only over a certain value of the coupling strenght and therefore a certain value of V_0.

Chapter 2

Ginzburg-Landau theory for 3D BCS-BEC crossover

The BCS and BEC theory describe system of particles in a superfluid phase. In both cases the phase transition happens through a spontaneaus simmetry breaking. Under a certain temperature, called critical temperature T_c, the system admits a non-zero value for the order parameter of the phase transition in the equilibrium state. This order parameter could be the Cooper pair wave function for the BCS theory, the condensated wave function for the BEC theory. In proximity of the critical temperature the order parameter is small and therefore also the energy associate to the superfluid part of the system. An important tool to study this situation is the Landau theory [22]. This is a phenominologial mean-field theory of phase transition. It is based on some assumptions:

- It must exist a uniform order parameter such that it is zero for $T > T_c$ and non-zero for $T < T_c$.
- The free energy is an analitic function of the order parameter.
- The free energy relation must satisfy the underlying symmetry of the system.
- Equilibrium state correspond to the absolute minima of the free energy.

Since the free energy is analytic it can be formally expanded in power of the order parameter. This approach is used to describe many case of phase transition, one of the most popular is the Ising model for the Ferromagnetic phase transition. One of the great accomplishments of this theory is its description of the non-analytic behavior at phase transitions in terms of the discontinuos jumps in the position of the absolute minimum of a function which is itself varying continuously with the quantity that control the phase transition; that could be the temperarute for superfluidity or the magnetic field for ferromagnetism. In this chapter we will apply this theory to the BCS-BEC crossover. Our starting point is of course the Ginzburg-Landau (G-L) theory [23]. This is a phenomenological theory which describes type-I superconductor. For a D-dimensional system of area L^D the super component F_s of the energy is given by:

$$F_s = \int_{L^D} d^D r \Big(a(T)|\Psi(r)|^2 + \frac{b}{2}|\Psi(r)|^4 + \gamma|\nabla\Psi(r)|^2 \Big) \tag{2.1}$$

where a(T), b, γ are the G-L parameters. The temperature dependence is only on the paramter a(T) because minimizing the energy functional the quadratic term becomes linear and its coefficient it must be zero in order to have a finite value of $\Psi(r)$ in the equilibrium state. In this way a(T) could be expandend around T_c in the form:

$$a(T) = \frac{da(T)}{dT}\Big|_{T=T_c}(T - T_c) \tag{2.2}$$

Later, a version of Ginzburg–Landau theory was derived from the Bardeen–Cooper–Schrieffer microscopic theory by Lev Gor'kov [17]. In the next section we will see in detail how it's possible to obtain a relation for the energy form-like the G-L functional starting from the density Hamiltonian of the system.

2.1 Ginzburg-Landau functional

Let's consider a 3D Fermionic neutral system in a unit volume with the Hamiltonian density:

$$H = \overline{\Psi}_\sigma(x)\Big[-\frac{\nabla^2}{2m} - \mu\Big]\Psi_\sigma(x) - g\overline{\Psi}_\uparrow(x)\overline{\Psi}_\downarrow(x)\Psi_\uparrow(x)\Psi_\downarrow(x) \tag{2.3}$$

where the first term is the free particle term and the second is the interaction term. This Hamiltonian described a system with single-channel s-wave interaction where $g > 0$ is the constant interaction. We use the functional integral formulation to study the finite temperature crossover. Like in [1] we proceed with the integral formulation of the effective action that appears in the definition of the partition function Z. This can be done introducing the Hubbard-Strotonovich bosonic field $\Delta(x)$. The Hamiltonian density becomes:

$$H = \overline{\Psi}_\sigma[-\frac{\nabla^2}{2m} - \mu]\Psi_\sigma + \frac{|\Delta|^2}{g} - \overline{\Delta}\Psi_\downarrow\Psi_\uparrow - \Delta\overline{\Psi}_\uparrow\overline{\Psi}_\downarrow \tag{2.4}$$

The action and partition function are related to the Hamiltonian density through the relation:

$$S = \int_0^\beta d\tau \int dx[\overline{\Psi}_\sigma(x)\partial_\tau\Psi_\sigma(x) + H] \tag{2.5}$$

$$Z = \int \mathcal{D}[\Delta,\overline{\Delta}]\mathcal{D}[\Psi_\sigma,\overline{\Psi}_\sigma]exp(-S_\sigma[\Delta,\overline{\Delta},\Psi_\sigma,\overline{\Psi}_\sigma]) \tag{2.6}$$

Integrating over the fermionic field Ψ_σ and obtain the effective action:

$$S_{eff}[\Delta(x)] = \int_0^\beta d\tau \int dx\frac{|\Delta(x)|^2}{g} - Tr\Big[lnG^{-1}[\Delta(x)]\Big] \tag{2.7}$$

It is written in terms of the Nambu propagator:

$$G^{-1}(x,x') = \begin{bmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu & \Delta(x) \\ \overline{\Delta}(x) & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{bmatrix} \times \delta(x - x') \tag{2.8}$$

and the trace in the S_{eff} is over space $\vec{x}$, imaginary time τ and Nambu indicies. The G-L theory works near the critical temperature where the order parameter is small. In the effective action it is the field $\Delta(x)$, that rappresents the gap energy, to be small. We can move to the momentum space and expand the effective action in function of $\Delta(q)$. Since we are looking to describe the system in its equilibrium state the first order term of the expansion is 0.

$$S_{eff}[\Delta,\overline{\Delta}] = \sum_{q} \frac{|\Delta(q)|^2}{\Pi(q)} + \frac{1}{2}\sum_{q_1,q_2,q_3} b_{1,2,3}\Delta_1\Delta_2^*\Delta_3\Delta_{1-2+3}^* + ... \tag{2.9}$$

where Π is the coefficient for the second order terms $|\Delta(q)|^2$ with all the grandient orders of $\Delta(q)$. To obtain a relation formally equal to the G-L funcional we have to consider only the quadratic term, the quadratic gradient term and the quartic term. So the coefficient Π reads:

$$\Pi^{-1}(q,0) = a + \frac{c|q|^2}{2m} + ... \tag{2.10}$$

The parameters have the following form:

$$a = -\frac{m}{4\pi a_s} + \sum_k [\frac{1}{2\epsilon_k} + \frac{tanh(\beta\xi_k/2)}{2\xi_k}] \tag{2.11}$$

$$b = \sum_k [\frac{tanh(\beta\xi_k/2)}{4\xi_k^3} - \frac{\beta sech^4(\beta\xi_k/2)}{8\xi_k^2}] \tag{2.12}$$

$$c = \sum_k [\frac{tanh(\beta\xi_k/2)}{4\xi_k^2} - \frac{\beta sech^2(\beta\xi_k/2)}{8\xi_k}] \tag{2.13}$$

where $\xi_k = \epsilon_k - \mu$ with $\epsilon_k = \hbar^2k^2/2m$. The exponent $S_{eff}[\Delta,\overline{\Delta}]$ appearing in the integral for the partition function (Eq.(2.6)) satisfies the assumptions of the Landau theory. The absolute minimum of $S_{eff}[\Delta,\overline{\Delta}]$ dominates Z in the stationary phase approximation [24], this mean that the minimum of S_{eff} corresponds to the equilibrium state of the system. So the bosonic field $\Delta(x)$ has to play the role of order parameter. We will see in the next chapter that the parameter a(T) is 0 at T=T_c, like in the Ginzburg-Landau functional. What it's generally done to study the system near the critical temperature is to expand it at the first order:

$$a(T) = \frac{da(T)}{dT}\Big|_{T=T_c}(T-T_c) = \alpha(T-T_c) \tag{2.14}$$

$$\alpha = \sum_k \frac{1}{2k_BT_c^2} sech^2[\frac{\epsilon_k}{2k_BT_c}] \tag{2.15}$$

We can easly see that α is positive so the parameter a(T) is negative below the critical temperature and positive above. In the next section we will see that the parameters b and c are always positive, so for $T > T_c$ we have the minimum for S_{eff} at $\Delta(x) = 0$, instead for $T < T_c$ we have the minimum for $\Delta(x) \neq 0$. So $\Delta(x)$ has the same behavior of the order parameter in the Landau theory. It's reasonable to use S_{eff} as Ginzuburg-Landau functional. Minimazing the S_{eff} one obtain the Ginzburg-Landau equation:

$$\frac{\delta S_eff[\Delta,\overline{\Delta}]}{\delta\overline{\Delta}} = 0 \tag{2.16}$$

$$\Big[\alpha(T-T_c) + b|\Delta(x,t)|^2 - \frac{\hbar^2c}{2m}\nabla^2\Big]\Delta(x,t) = 0 \tag{2.17}$$

In the standard Ginzburg-Landau equation appears the Cooper pair field $\Psi(x,t)$ instead of $\Delta(x,t)$. It's possible to pass to the standard one replaicing $\Delta(x,t) = \Psi(x,t)/\sqrt{2c}$. In this way one obtains:

$$[\overline{A}(T - T_c) + B|\Psi(x,t)|^2 - \frac{\Gamma}{2}\nabla^2]\Psi(x,t) = 0 \tag{2.18}$$

where:

$$\overline{A} = \frac{\alpha}{2c} \tag{2.19}$$

$$B = \frac{b}{4c^2} \tag{2.20}$$

$$\Gamma = \frac{\hbar^2 c}{4mc} = \frac{\hbar^2}{4m} \tag{2.21}$$

are the G-L parameters in the standard formulation.

2.2 Gap equation and Number equation

The G-L parameters depends by the critical temperature, the critical chemical potential and scattering lenght. To determine this triad of values we need to solve a system of equations that involves them. The first one is the gap equation calculated by minimizing the effective action $\partial S_{eff}(\Delta(x)/\partial\Delta(x)$=0 (saddle point condition):

$$\frac{1}{g} = \sum_k \frac{tanh(\xi_k/2T_0)}{2\xi_k} \tag{2.22}$$

We notice that the gap equation is indipendent by the field $\Delta(x)$. The T_0 appering in the gap equation is simply the temperature of the most stable state of the system. Replacing the summation with the integral we encounter a divergence and, like in the first chapter, this can be eliminated by replacing the Lippman-Schwinger equation. The regularized gap equation with the scattering lenght instead of the constant coupling g is:

$$-\frac{m}{4\pi\hbar^2 a_s} = \sum_k \Big[\frac{tanh(\xi_k/2T_0)}{2\xi_k} - \frac{1}{2\epsilon_k}\Big] \tag{2.23}$$

Fixing the scattering lenght $1/k_F a_s$ we have two unknown: μ and T_0. We need another equation, thus we introduce the number equation N=-$\partial\Omega/\delta\mu$, where Ω=$S_{eff}[\Delta = 0]/\beta$ is the grand potential. The number equation instead depends by the $\Delta(x)$. As first attempt we subtitute $\Delta(x) = 0$ everywhere, this condition corresponds to the mean-field approximation at critical temperature. The number equation obtained is:

$$n = n_0(\mu, T) = \sum_k \Big[1 - tanh(\frac{\xi_k}{2T})\Big] = 2f(E_k) \tag{2.24}$$

Where $f(E_k)$ is the Fermi function. Solving the Eq.(2.23) and Eq.(2.24) we can estimate the saddle point T_0 and μ_0, as a function of a_s. The n in the number equation is the number density of the system and in our case is fixed. What it's obtained is a T_0 that grows continuously without showing the phase transition from BCS to BEC. The problem is that the mean-field approximation is not enough to describe properly the system. We can check this fact evaluating the applicability of the mean-field approximation by the Ginzburg-Levanyuk criterion [25]. This is obtained studying the heat capacity around the critical temperature for mean-field and and with small fluctiations. The mean-field case shows a jump of the heat capacity at the critical temperature instead the beyond mean-field case has a divergence. Comparing the two cases it's possible to estimate the range of temperature where the effect of fluctuations are small respect to the mean-field jump. So the mean-field approximation is valid only if we are considering a temperature T that satisfies the relationship:

$$T - T_0 > Gi_{3D} T_0 \tag{2.25}$$

where Gi_{3D} is the Ginzuburg number:

$$Gi_{3D} = \Big(\frac{B T_0^{1/2}}{8\pi \Gamma^{3/2} \overline{A}^{3/2}}\Big)^2 \tag{2.26}$$

Using the relations obtained in the previous section for the G-L parameters we can study the behaviour of Gi along the crossover. In the BCS regime $T - T_0$ is about 10^{-14}, like ordinary superconductors.

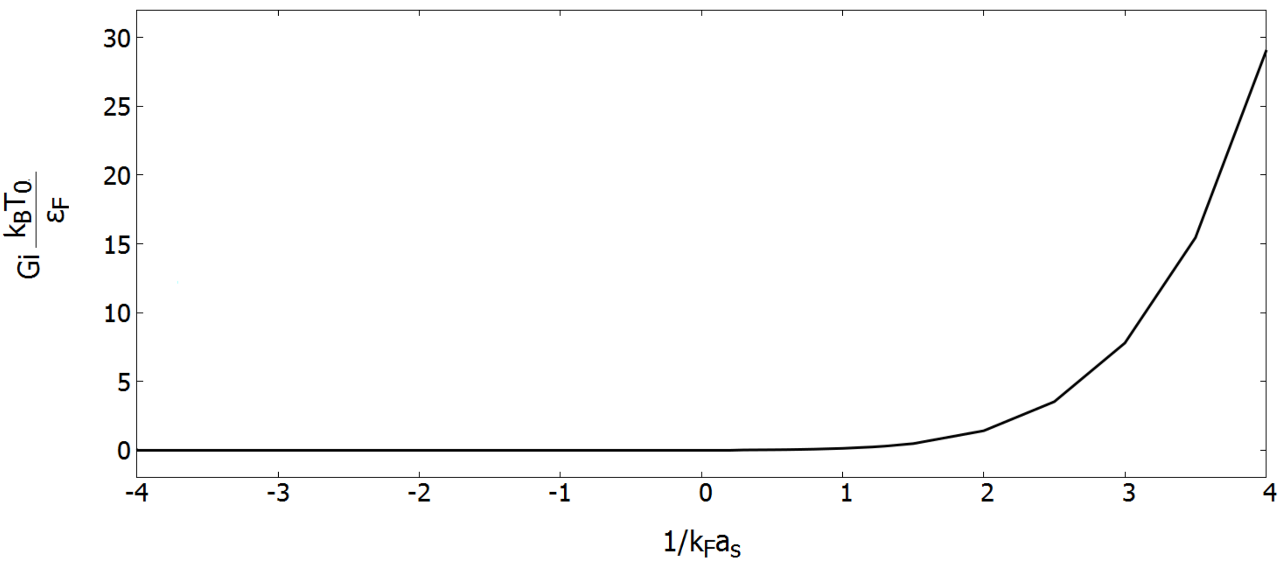

Figure 2.1: GiT_0 in function of the coupling

This value is much beyond the experimental sensitivity. In this case the mean-field approximation works very well. From BCS to BEC it grows constantly. This means that in the BEC regime the fluctuations became too important near the critical temperature and the mean-field approximantion is no more good to describe the system. In the strong coupling regime effects emerge due to the formation of very bounded pairs. To describe correctly the system in the BEC regime we look at Gaussian thermal fluctuations about the trivial saddle point. The action expanded to second order in $\Delta(x)$ is given by:

$$S_{Gauss} = S_{eff}[\Delta = 0] + \sum_{q,\omega_q} \Pi^{-1}(q, \omega_q) |\Delta(q, \omega_q)|^2 \tag{2.27}$$

where Π^{-1} is the same of Eq.(2.9) and $\omega_q = i2l\pi/\beta$. In this case we don't stop the expansion to the first order gradient term and quartic term but we will consider all the gradient orders of the quadratic term. Form S_{Gauss} we obtain a new thermodynamic potential $\Omega = \Omega_0 - \beta^{-1} \sum_{q,\omega_q} ln\Pi(q, \omega_q)$, which, differentiated by μ, gives the beyond mean-field number equation. Following Nozières Schimtt-Rink approach [26] one can rewrite Ω in term of phase shift defined by $\Pi(q, \omega_q \pm i0^+) = |\Pi(q, \omega)| exp[\pm i\delta(q, \omega)]$.

The number equation incorporating the effects of Gaussian fluctuations is given by:

$$n = n_0(\mu_c, T_c) + \sum_q \int_{-\infty}^{\infty} \frac{d\omega}{\pi} n_B(\omega) \frac{\partial \delta}{\partial \mu}(q, \omega) \tag{2.28}$$

where n_0 is Eq(2.24), $n_B = 1/[exp(\beta\omega) - 1]$ is the Bose function and $\delta(q, \omega) = -Arg(1 - \Pi^{-1}[q, \omega \pm i0^+)]$. Now, like in the mean-field case, we solve the system of (2.28) and (2.23). It's interesting to note including Gaussian fluctuations leads to a change of T_c, and therefore μ_c, in the mean-field part of n that are different respect to the pure mean-field case.

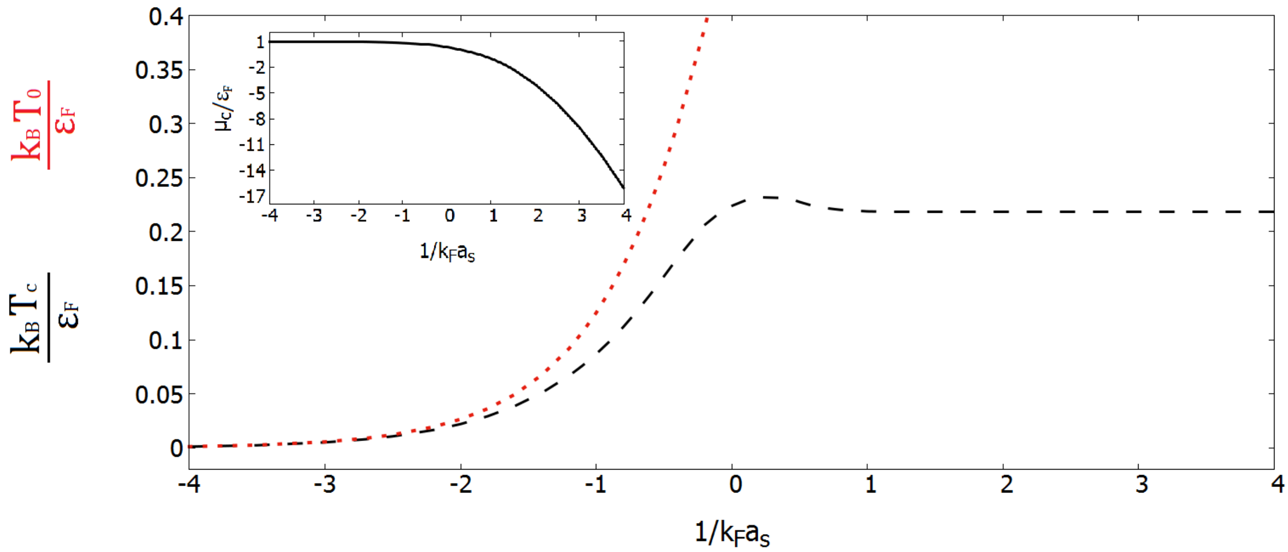

Figure 2.2: The critical temperature in function of the coupling. The red dotted line (T_0) is mean-field critical temperature while the black dashed line (T_c) is the beyond mean-field critical temperature. In the inset the critical chemical potential in function of the coupling.

In this way we can analyze the effects of fluctuations on critical temperature and critical chemical potential in function of the coupling. In the weak coupling regime the effect of the fluctuations is very small, just like we predicted from the Ginzburg-Levaniuk criterion. In the BEC regime the correction is fundamental to obtain the Bosonic condensation temperature $k_B T_B \simeq 0.218\epsilon_F$. In order to better understand the results we study the critical chemical potential. As we can see from the inset in Fig.2.2, over $1/k_F a_s = 0.35$ the chemical potential become negative. In the BEC regime we will have a large negative chemical potential $|\mu_c| >> T_c$. Applying this condition to the gap equation in the limit $1/k_F a_s \to \infty$, one obtains the relation $\mu_c = \hbar/(2ma_s^2)$. The Cooper pair size becomes much smaller than the avarege distance between the partcles because the attractive interaction becomes very strong. At this point another energy joins the game: the binding energy. This is the energy that holds the molecules togheter and is given by $\epsilon_0 = \hbar^2/(ma_s^2)$, so in the strong coupling limit $\mu_c = \epsilon_0/2$. As we said, one can obtain this condition starting from the gap equation, that it's invariant for any kind of approximation on the order parameter. Now we can also introduce the molecule chemical potential $\mu_B = 2\mu_c + \epsilon_0$ which is 0 in the strong coupling limit, like a Bosons system at critical temperature. This result is valid both for the mean-field and for the beyond mean-field approximation. So the critical temperature in the BEC limit founded with the man-field approximation is just related to pair-breaking temperature T_{dissoc} defined as the temperature at which the pairs dissocate. Instead the critical temperature founded with the fluctuations is the superfluid phase transition temperature of the system.

2.3 G-L parameters and characteristic quantities

Now we have all the tools to study the behavior of G-L parameters in function of the coupling. Using the relations defined in the previous sections one can substitute the values of critical temperature and critical chemical potential in the G-L parameter relations. We will focus about the differences between mean-field and beyond mean-field approximation. We founded that the better way to do this is to evaluate the relative difference of the parameters between the two approximations. Initially we defined the theory in function of the order parameter $\Delta(x)$, then we move to the standard formulation with $\Psi(x)$. In this last we founded that the quartic term Γ is a constant, so not dependent on the approximation used.

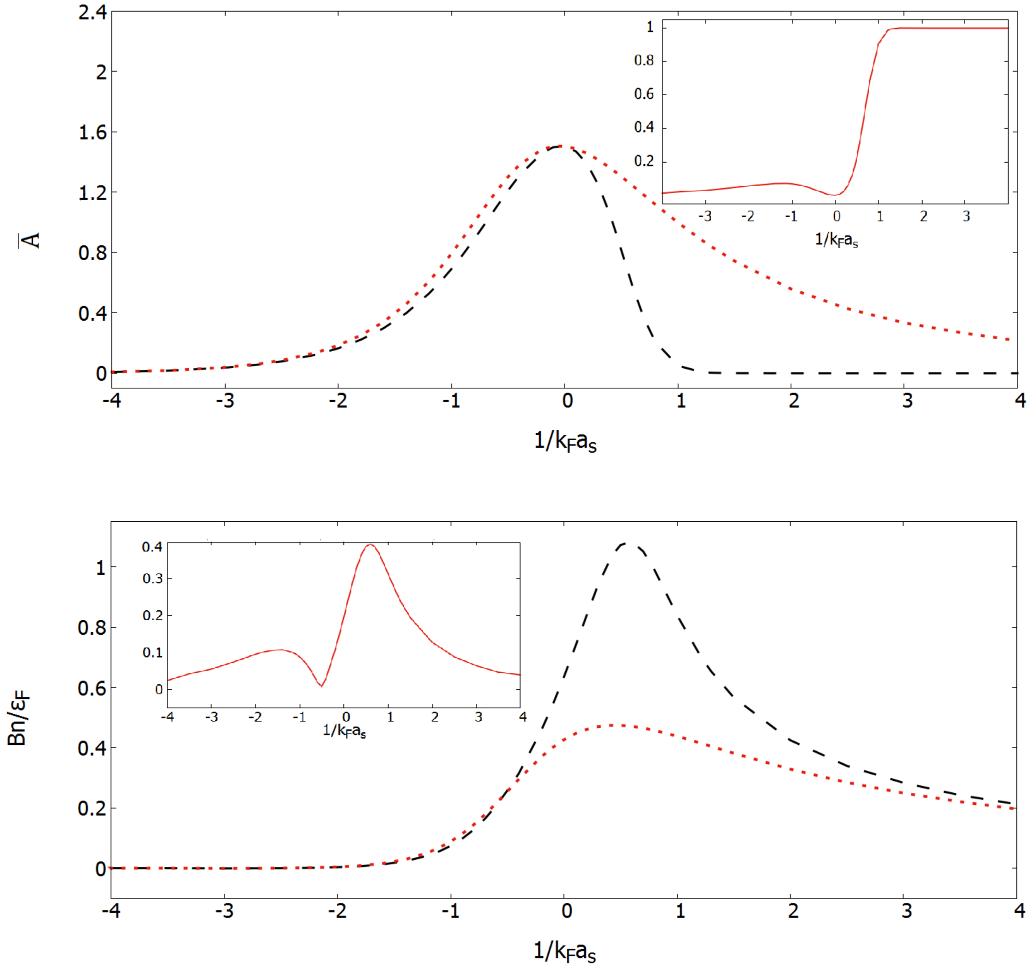

Figure 2.3: The beyond mean-field (black dashed line) and mean-field (red dotted line) parameters $\overline{A}$ and B as a function of the coupling. In the inset the relative error between the mean-field and beyond mean-field approximation in function of the coupling.

In the deep regions the parameter $\overline{A}$ follows what we previously said for the critical temperature. The fluctuations are neglibigle in the BSC limit and important in the BEC. But as we can seen from the inset Fig.(2.3)A), the relative error doesn't grows continuosly, there is a minimum around $1/k_F a_s = 0$. This means that around the unitary limit the coefficient of the quadratic term is very similar between the mean-field and the beyond mean-field approximation. Instead for the parameter B the fluctuations are neglibigle both for the BCS and BEC case, but are relevant for the unitary limit with a maximum around the value $1/k_F a_s \simeq 0.5$. From the BCS regime until the maximum value the relative error doesn't grow continuosly, there is a minimum around the value $1/k_F a_s = -0.5$. To better understand these results it's importat to see the behavior of some characteristic quantities: Ginzuburg-Landau critical coherence lenght [27] and the critical rotational frequency [28]. The Ginzburg-Landu coherence lenght is the distance form the system surface over which the Cooper pair wave function doesn't vary. In the BCS limit corresponds also to the Cooper pair size. Since we are considering neutral Fermions we looked for the neutral analogous of critical magnetic field. For superfluid netrual system it's possible to destroy superfluidity by rotating the system. It happens over a certain value of rotational frequency ω_c. We will evaluate these two quantities around the critical temperature.

$$\xi_{GL} = \sqrt{-\frac{\Gamma}{A(T)}} \tag{2.29}$$

$$\omega_c = \sqrt{\frac{12A(T)^2}{m\Gamma}} \tag{2.30}$$

We have taken a temperature from $3T_c/4$ to T_c in order to still have Δ small respect to the thermal energy. We will evaluate the critical coherence lenght and the critical rotational frequency at different coupling values. The first one diverge at T_c while the second one goes to 0. As we can see from Fig.(2.5) and Fig.(2.6), increasing the coupling, the distance between the point where mean-field and the beyond mean field curve diverges or go to 0, increases . For this reason in the case $1/k_F a_s = 4$ we report only the beyond mean-field curve. It's very interesting noting the scale of the curves. The G-L coherence lenght diverges much faster in the BCS and BEC regime than in the unitary limit. In the same way the critcal rotational frequency goes to 0 much faster in the BCS and BEC regime than in the unitary limit. This means that in the unitary limit, near the critical temperature, the superfluidity is much more "resistant" than in the BEC or BCS regime. Since these two quantities are functions of the parameters A and Γ, we can make some osservation. The coefficient Γ is a constant dependent on the mass of the particle. The lower the mass of particles, the lower the speed which the coherence lenght diverges or the critical rotational frequency goes to 0. On the other hand the parameter A(T) is more complicated to connect it directly to some physical quantities of the system. In order to identify some other physical quantites to associate with this better superfluidity one can investigate the density of the system, (2.28). In the BCS limit the mean-field term of n dominates while in the BEC regime is the beyond mean-field term dominating. We can identify these two terms with the density of the Fermionic couples and the Bosonic couples. In the unitary limit there is like a competition between the two behavior which benefits the superfluidity.

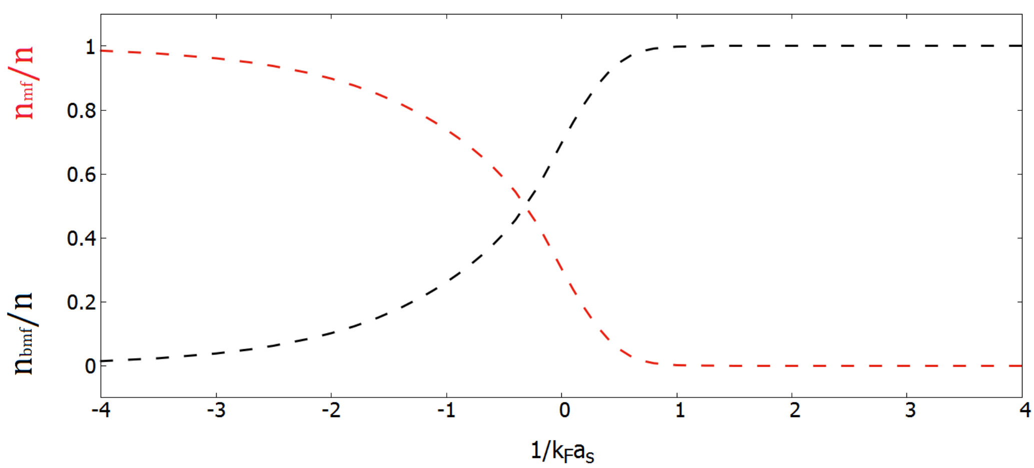

Figure 2.4: n_{mf} and n_{bmf} term of the beyond mean-field number equation as a function of the coupling.

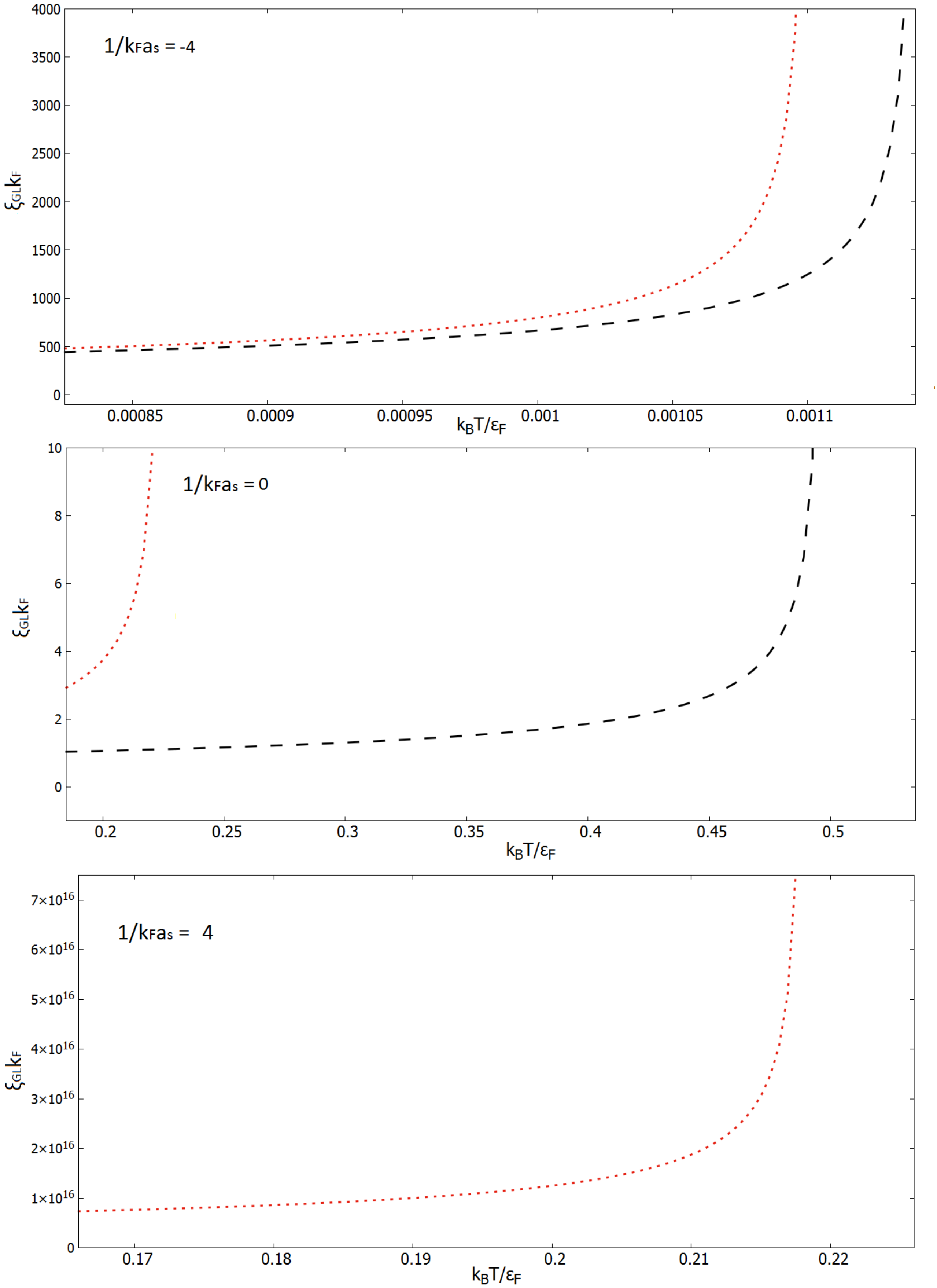

Figure 2.5: The G-L coherence lenght around the critical temperature at $1/k_F a_s = -4, 0, 4$. The black dashed line is with mean-field approximation while the red dotted line is with beyond mean-field approximation.

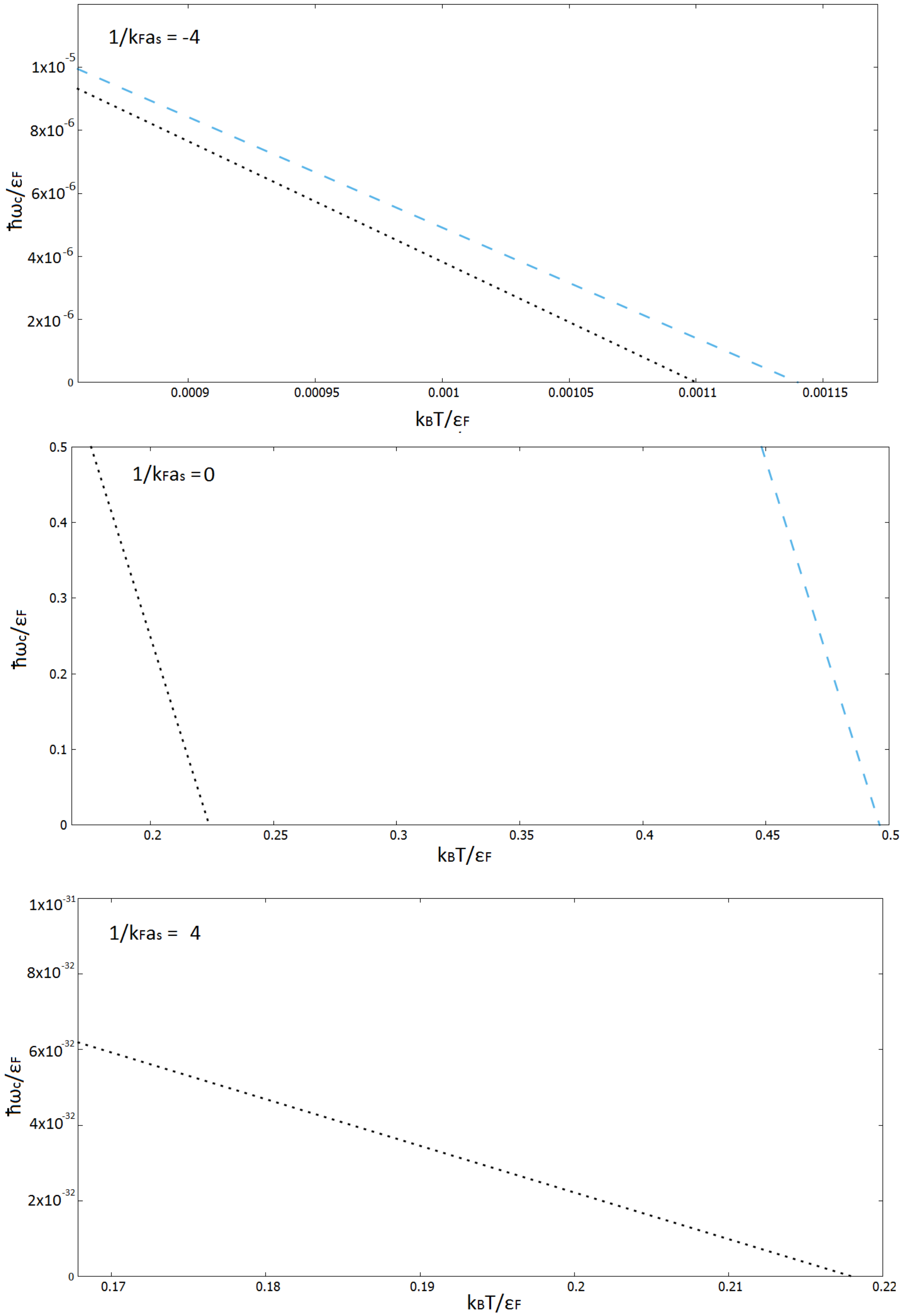

Figure 2.6: The critical rotational frequency around the critical temperature at $1/k_F a_s = -4, 0, 4$. The black dotted line is with beyond mean-field approximation while the blue dashed line is with mean-field approximation.

2.4 Ginzburg-Landau theory in the BEC regime at T=0

In the BEC regime the Fermi gas is dilute, $na_s^3 << 1$, and the order paramter Δ is much less than the binding energy of a diatomic molecule, $\epsilon_0 = \hbar/(ma_s^2)$. In this regime, since the chemical potential is negative, there is no Fermi surface [20]. The parameters in the Ginzburg-Landau equation given previosly are well defined even at zero temperature. Therefore the G-L theory can be applied from zero temperature to near T_c in the BEC regime. Now we will study the limits of this statement. The first thing to do is to check out the region where the order parameter Δ is much smaller than the binding energy. To do that we rewrite the gap equation and the number equation at T=0 in mean-field approximation since in this case the fluctuations are neglibile everywhere.

$$\frac{m}{4\pi a_s \hbar^2} = \sum_k \Big(\frac{1}{2\epsilon_k} - \frac{1}{2E_k} \Big) \tag{2.31}$$

$$n = \sum_k (1 - \frac{\xi_k}{E_k}) \tag{2.32}$$

where $\epsilon_k = \hbar^2 k^2/2m$, $\xi_k = \epsilon_k - \mu$ and $E_k = \sqrt{\xi_k^2 + \Delta^2}$. We have solved the system obtaining the values for the chemical potential and the gap energy. From Fig.(2.7) one can see that around $1/k_F a_s = 0.76$ the curves cross so we have to be over this value to consider the gap energy less than the binding energy. To locate the optiaml value over which we can extend the G-L theory we evaluate the gap energy using the G-L parameters. As we said at T=0 we can use the mean-field approximation and in this case it's easy to find the relation $|\Psi_0|^2 = -A(T)/B$. Replacing in $|\Delta|^2 = |\Psi|^2/2c$ we can compare this gap energy with the one obtained from (2.32). We founded that over $1/k_F a_s = 1.3$ there is a relative difference less than 5% between the two. It means that over this value it's reasonable to use the G-L theory at T=0. This is useful because generally superfluid systems are studied at law temperature through a microscopic approach.

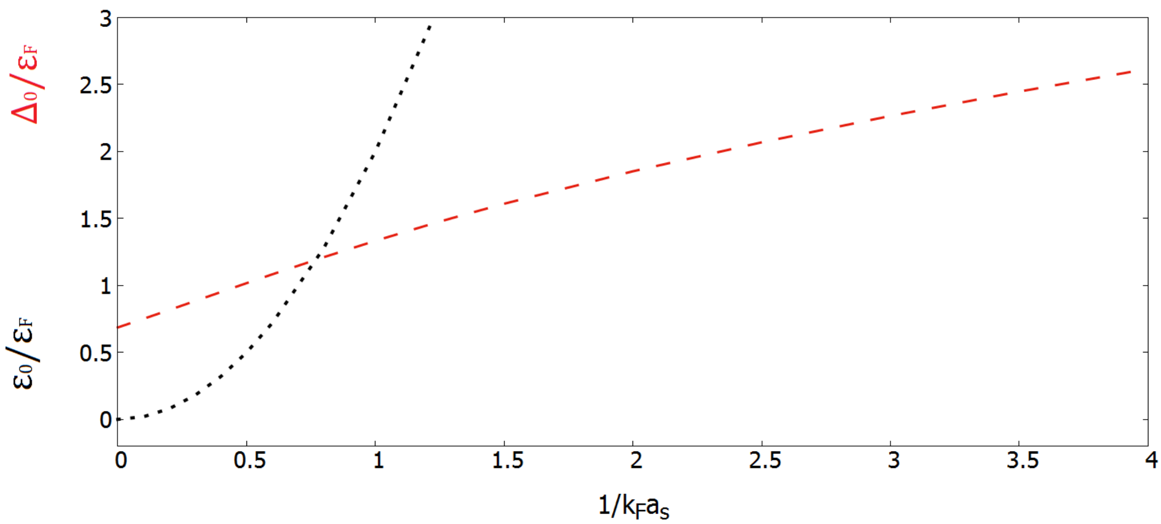

Figure 2.7: The red dashed line is the gap energy a T=0 while the black dotted line is the binding energy vs $1/k_F a_s$.

2.4.1 Sound velocity

The advantage to use G-L theory also at low temperature is that, once you know the behavior of the G-L parameter, the relation for this quantities are easier to evaluate respect to use the standard microscopic approach. In this section we derive the sound velocity starting from the G-L functional. The sound velocity is a quantity closely related to the dynamic of the system. To deal it in the G-L theory we have to consider the time evolution of the Ginzuburg-Landau equation (TDGL equation). This requires a careful examination of Gaussian fluctuations and a simple low frequency expansion is obtained only when $\Delta(T) << \omega$ [1]. In this case we needs to expand $Q(iq_l) = \Pi^{-1}(q = 0, iq_l) - \Pi^{-1}(0, 0)$ in powers of ω. A low frequency ($\omega << T_c$) expansion of Q is obtained both in the BCS and Bose limit where the condition $\omega << |\mu|$ is automatically satisfied. Instead in the unitary limit this is not possible because $|\mu| \simeq 0$. The condition $\Delta(T) << \omega$ with $\omega << min(T_c, |\mu(T_c)|)$ implies that our TDGL results are not valid in a region around the point $\mu(T_c) = 0$. So for $\Delta << \omega << min(k_B T_c, |\mu(T_c)|)$ the expansion $Q(\omega + i0^+) = -d\omega + ...$ yelds:

$$d = \sum_k \frac{tanh(\beta\xi_k/2)}{4\xi_k^2} + i\frac{\pi}{8}\kappa\beta\sqrt{\mu}\Theta(\mu) \tag{2.33}$$

where $\kappa = N(\epsilon_F)/\sqrt{\epsilon_F}$,$N(\epsilon_F)$ is the density of states at the Fermi energy and $\Theta(x)$ is the step function. We obtain the TDGL equation:

$$\Big(a + b|\Delta(x,t)|^2 - \frac{\hbar^2 c}{2m}\nabla^2 - i\hbar d\frac{\partial}{\partial t}\Big)\Delta(x,t) = 0 \tag{2.34}$$

To see all the steps in more detail, consult [1]. Anyway, since we are looking to expand the theory at T=0 in the BEC limit, where $\mu < 0$, the imaginary term of d is 0. Passing to the standard definition of the G-L theory with Ψ, instead of Δ, the coefficient of the time derivative term is $D = \hbar d/2c$. Below we will follow the same procedure to obtain the Bogoliubov spectrum from the Gross-Pitaevskii equation. Let's consider the standard TDGL equation:

$$iD\frac{\partial\Psi(\vec{x},t)}{\partial t} = A(T)\Psi(\vec{x},t) + B|\Psi(\vec{x},t)|^2\Psi(\vec{x},t) - \Gamma\nabla^2\Psi(\vec{x},t) \tag{2.35}$$

We write order parameter $\Psi(\vec{x},t)$ in the following way:

$$\Psi(\vec{x},t) = (\Psi_0 + \phi(\vec{x},t))e^{-i\overline{\mu}t} \tag{2.36}$$

where $\phi(\vec{x},t)$ rappresents a real small fluctuation with respect to the real and uniform configuration Ψ_0 and the imaginary expontial is the phase component of the order parameter. Inserting (2.36) in (2.35) we find:

$$iD(-i\overline{\mu}\Psi_0 - i\overline{\mu}\phi + \dot{\phi}) = A(T)\Psi_0 + A(T)\phi - \Gamma\nabla^2\phi + B\Psi_0^3 + 2b\Psi_0^2\phi + b\Psi_0^2\phi^* \tag{2.37}$$

where we neglect all terms of higher order than the second in ϕ. From now for the sake of notation we will omit the spatial and temporal dependence of $\Phi(\vec{x},t)$. To determine $\overline{\mu}$ we impose the uniform mean-field solution in the region $T < T_c$, $\Psi(\vec{x},t) = \Psi_0 e^{-i\overline{\mu}t}$. For for $T < T_c$ the condition $\overline{\mu} = 0$ holds. Applying that, and remembering that for the mean-field order parameter worth the relation

$A(T)\Psi_0 + B\Psi_0^3 = 0$, one obtain:

$$iD\dot{\phi} = -\Gamma^2\phi + B\Psi_0^2(\phi + \phi^*) \tag{2.38}$$

Now we split the field ϕ in a two-component wave:

$$\phi(\vec{x}, t) = Ae^{i(\vec{k}\vec{x}-\omega t)} + Be^{-i(\vec{k}\vec{x}-\omega t)} \tag{2.39}$$

Replacing this relation in (2.38) we can devide the resultung equation in its real and imaginary component. In this way we obtain a system of two indipendent equation, which determinant set to 0, gives us the dispersion relation between ω and $\vec{k}$. The dispersion relation is:

$$\omega = \sqrt{\frac{\Gamma k^2(\Gamma k^2 + B\Psi_0^2)}{D^2}} \tag{2.40}$$

In the limit of small $\vec{k}$ is linear like the phononic dispersion and in this case we know that the relation is $\omega = kc_s$, where c_s is the sound velocity. We have:

$$c_s = \sqrt{\frac{\Gamma B\Psi_0^2}{D^2}} \tag{2.41}$$

replacing $|\Psi_0|^2 = -A(T)/B$ one obtain:

$$c_s = \sqrt{\frac{-\Gamma A(T)}{D^2}} \tag{2.42}$$

Comparing the relation for d and c we can see that at T=0 there are equal. So definetly the sound velocity in the BEC regime is:

$$c_s = \frac{\epsilon_F}{n^{1/3}\hbar}\sqrt{\frac{1}{k_F\xi_{GL}(T)}} \tag{2.43}$$

This is a very simple relation that we can use to evaluate the sound velocity in a system made of coupled Fermions in the regime of strong couplig at very law temperature.

Chapter 3

Josephson effect in 2D BCS-BEC crossover

3.1 2D BCS-BEC crossover

Before to study the 2D Josephson equations reported in the first chapter we have to define in more detail the system we are going to study. We consider a two-dimensional (2D) attractive Fermi gas of ultracold and dilute two-spin component neutral atoms. In accordance with the Mermin-Wagner-Hohenberg theorem [29, 30] for $d \leq 2$ there cannot be spontanuous simmetry breaking and so finite condensate density at finite temperature. This is the first difference with the tridimensional case. The 2D BEC critical temperature is T=0. Nonetheless two-dimensional system can exhibit superfluidity at finite temperature. There could be a quasicondensate density under a certain temperature, this is called Berezinskii-Kosterlitz-Thouless (BKT) temperature [31, 32]. The BKT phase transition is a topologiacal phase transition that not require symmetry breaking. Under a certain temperature there is proliferation of disjointed vortices and, lowering further the temperature, under T_{BKT}, one has the formation of vortex-antivortic pairs that allow superfluidity. So there is a jump in the superfluid density, going discountinuosly from a finite value to zero at T_{BKT}. It's reasonable to think that in two-dimensional system, the role of quantum fluctuations should be crucial in describing several aspect of the system. What we are going to do in this first part is to evaluate the effects of thermal fluctuation in two-dimensional BCS-BEC crossover [33]. We will use the integral function procedure reported for the 3D case. The Hamiltonian density (2.4) and the partition function (2.6) are the starting point. Let's start with the mean-field case. We substitute $\Delta(x) = \Delta_0$ in (2.4) where Δ_0 real and then we integrate (2.6) over the Fermionic field Ψ_σ. In this way we obtian the mean-field partition function [34]:

$$Z_{mf} = exp\Big[-\frac{S_{mf}}{\hbar}\Big] = exp - \Big[-\beta\Omega_{mf}\Big] \tag{3.1}$$

where Ω_{mf} is the mean-field grand potential:

$$\beta\Omega_{mf} = \frac{S_{mf}}{\hbar} = -Tr\Big[ln(G_0^{-1})\Big] - \beta L^2 \frac{\Delta_0^2}{g} = \tag{3.2}$$

$$= -\sum_k [2ln(2cosh(\beta E_{sp}(k)/2) - \beta(\epsilon_k - \mu)] - \beta L^2 \frac{\Delta_0}{g} \tag{3.3}$$

with $\epsilon_k = \hbar^2 k^2/2m$, $E_{sp}(k) = \sqrt{(\epsilon_k - \mu)^2 + \Delta_0^2}$ and G_0^{-1} is (2.8) with $\Delta(x) = \Delta_0$. At zero temperature the mean-field grand potential Ω_{mf} becomes:

$$\Omega_{mf} = -\sum_k (E_{sp}(k) - \epsilon_k + \mu) - L^2 \frac{\Delta_0^2}{g} \tag{3.4}$$

We are looking for the most stable state of the system so we obtain the condition for Δ_0 that minimizes the grand potential.

$$\left(\frac{\partial \Omega_{mf}}{\partial \Delta_0} \right)_{\mu, L^2} = 0 \tag{3.5}$$

In this way one obtains the familiar BCS gap equation:

$$-\frac{1}{g} = \frac{1}{V} \sum_k \frac{1}{2E_{sp}} \tag{3.6}$$

In the continuum limit $\sum_k \to L^2 \int d^2k/(2\pi)^2$ the gap equation diverges logharitmically in the ultraviolet. As in the 3D case this problem is solved replacing the Lippman-Schwinger equation instead of $1/g$. In 2D the scattering lenght is nonnegative and $k_F a_{2D} >> 1$ corresponds to the BCS regime while $k_F a_{2D} << 1$ corresponds to the BEC regime. Since the scattering lenght it's always positive, in 2D BCS-BEC crossover always exists a bounded state between the Fermions for any value of the coupling g [16, 35]. It's possibile mapping the crossover with the binding energy ϵ_B instead of the scattering lenght a_{2D}. The binding energy can be written as:

$$\epsilon_B = \frac{\hbar^2}{m a_{2D}^2} \tag{3.7}$$

In this way the bound-state equation is:

$$-\frac{1}{g} = \frac{1}{V} \sum_k \frac{1}{\epsilon_k + \frac{\epsilon_B}{2}} \tag{3.8}$$

Subtracting this relation from (3.6), one obtains the the regularized gap equation:

$$\Delta_0 = \sqrt{2\epsilon_B \left(\mu + \frac{1}{2}\epsilon_B \right)} \tag{3.9}$$

Instead replacing (3.8) and (3.9) in (3.4) one obtains:

$$\Omega_{mf} = -\frac{m}{2\pi\hbar^2} \left(\mu^2 + \frac{1}{2}\epsilon_B \right)^2 \tag{3.10}$$

The number equation is derived from the zero-temperature thermodynamic relation:

$$n = -\frac{\partial \Omega_{mf}}{\partial \mu} \tag{3.11}$$

which gives a relation for the chemical potential μ as a function of the number density.

$$\mu = \frac{\pi \hbar^2 n}{m} - \frac{1}{2}\epsilon_B = \epsilon_F - \frac{\epsilon_B}{2} \tag{3.12}$$

Introducing $\mu_B = 2\mu + \epsilon_B/2$ as the chemical potential of composite Bosons (made of bound Fermionic pairs) one finds that $\mu_B = 2\epsilon_F$. So the Bosonic chemical potential μ_B is independent of the interaction between particles. Now we take into account the quantum fluctuations. To do that we have to take again (2.4) and (2.6) and subtitute the order parameter $\Delta(x)$ with:

$$\Delta(x) = \Delta_0 + \eta(x) \tag{3.13}$$

where $\eta(x)$ is the complex pairing field of bosonic fluctuations. The procedure is the same of the previous chapter. Expanding the effective action around the Δ_0 up to the quadratic (Gaussian) order in $\eta(x)$ one finds a new effective action, $S_{Gauss} = S_{eff}[\Delta_0] + S_g[\eta, \overline{\eta}]$. From the Gaussian effective action is possible to evaluate a new grand potential. The resulting gran potential reads:

$$\Omega = \Omega_{mf} + \Omega_g(\mu, \Delta_0) = \Omega_{mf} + \frac{1}{2\beta}\sum_q ln(det(\mathbb{M}(Q)) \tag{3.14}$$

where $\mathbb{M}(Q)$ is the inverse pair fluctuation propagator and its form is reported in detail in the supplement material of the work [33]. Using (3.11) is possbile to evaluate the equation of state that takes into account the effect of small Gaussian thermal fluctuation.

3.1.1 Properties of 2D BCS-BEC crossover system

To investigate the effects of thermal fluctuation in bidimensional BCS-BEC crossover we evaluate some characteristic quantities: the chemical potential of composite Bosons (μ_B), the sound velocity (c_s) and the pressure (P). The chemical potential, in the mean-field approximation, rescaled for the Fermi energy of the non interacting system is $\mu_B/\epsilon_F = 2$, while for the beyond mean-field case we numerically solve the gap and number equation reported in the previous section. For the sound velocity we use the thermodinamic relations:

$$c_s = \sqrt{\frac{n}{m}\frac{\partial\mu(n)}{\partial n}} \tag{3.15}$$

$$\tag{3.16}$$

where $\mu(n)$ is the single-particel chemical potential and n is the number density. To define the pressure we start from the thermodinamic relation:

$$U = TS - PV + \mu N \tag{3.17}$$

where U is the thermodinamic internal energy of the system, S is the entropy, V is the volume of the Fermi gas and N is the total number of Fermions. Imposing the condition T=0 and using the relation for the internal energy $U/V = \int_0^n dn\mu(n)$ one can find:

$$P = \mu(n)n - \int_0^n dn\mu(n) \tag{3.18}$$

We manage to write the sound velocity and pressure in function of $\mu(n)$. In the mean-field case it's easy to find:

$$\frac{c_{s,mf}}{v_F} = \frac{1}{\sqrt{2}} \tag{3.19}$$

$$\frac{P_{mf}}{P_F} = \frac{3}{2} \tag{3.20}$$

where v_F and P_F are the Fermi velocity and pressure of the non interacting system. As for the chemical potential they are indipendent of binding energy. Considering the Gaussian fluctuations instead we have that they decrease with increasing the energy coupling, as we can see from Fig(3.1). So also in 2D system the effects of fluctuations are more relevant in the limit of strong coupling. To identify a reference value for ϵ_B/ϵ_F that tell us where the system is considerable a Bose-Einstein condensed we study the single particle chemical potential. We will look for which value of $log(\epsilon_B/\epsilon_F)$ where the chemical potential becames negative. Physically we can say that under this value the system is in a BCS regime of weakly bound fermionic pairs, instead over this value the system is in a BEC regim of strongly bound fermionic pairs. Setting to 0 the equation (3.12) one finds that in the mean-field case the chemical potential is 0 when $\epsilon_B = 2\epsilon_F$ or $log(\epsilon_B/\epsilon_F) = 0.69$. Instead from the numerical solution of gap and number equation with Gaussian fluctuation we find that the single particle chemical potential becames negative over the value $\epsilon_B/\epsilon_F = 0.84$ or $log(\epsilon_B/\epsilon_F) = -0.17$. To further prove the goodness of this criterion, we can study the behavior of the condensate fraction density of couples.

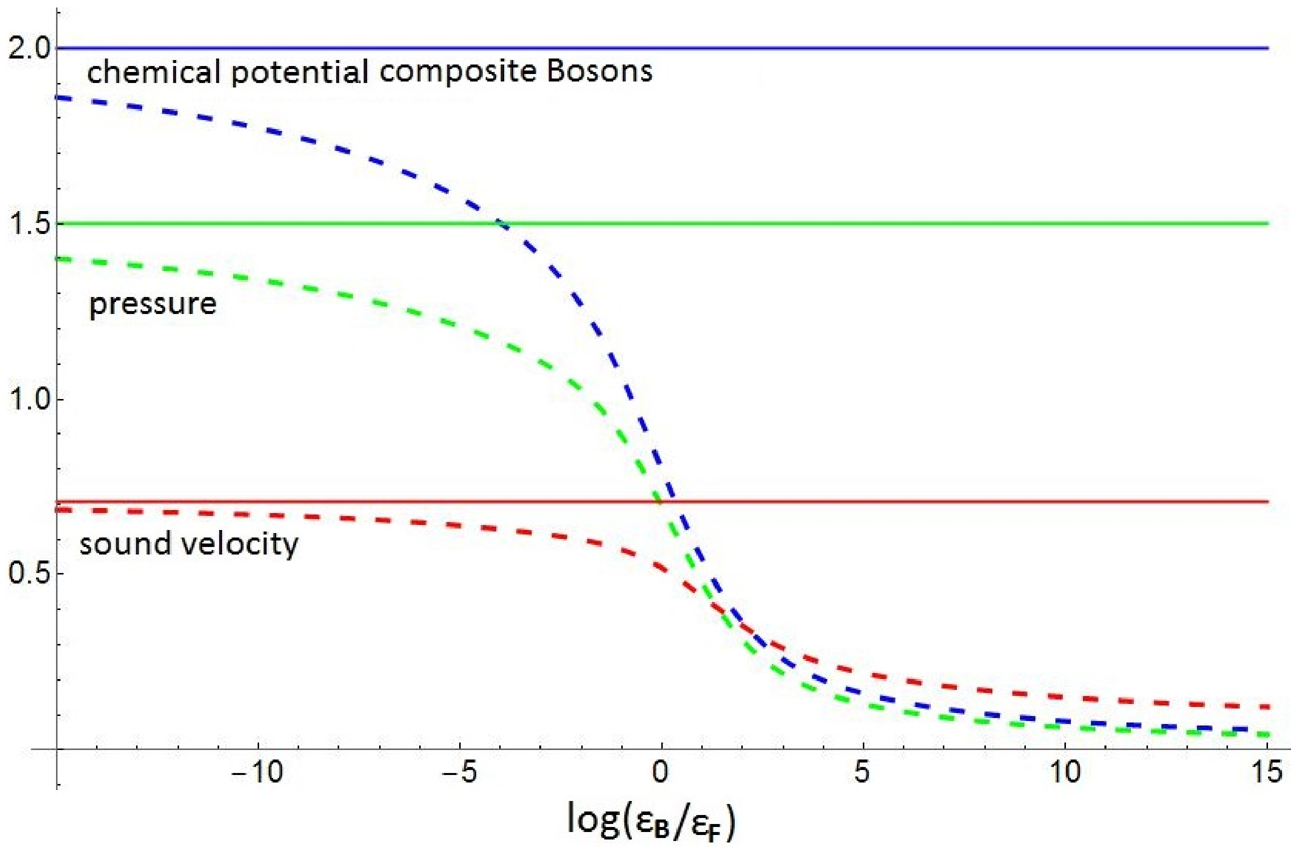

Figure 3.1: Chemical potential of composite Bosons μ_B/ϵ_F in blue, sound velocity c_s/v_F in red, pressure P/P_F in green, as function of the coupling. The continuous line is the constant mean-field value while the dashed line is considering Gaussian fluctuations.

To do that we will use Eq.(21) of the work [36] where the mean-field number of couples condensated is evaluated as the largest eigenvalue of the two-body density matrix written through the Bogoliubov rappresentation of the field operator ψ_σ. The relation is:

$$\frac{n_0}{n} = \frac{1}{4}\frac{\frac{\pi}{2} + arctan(\frac{\mu}{\Delta_0})}{\frac{\mu}{\Delta_0} + \sqrt{1 + \frac{\mu^2}{\Delta_0^2}}} \tag{3.21}$$

The effect of fluctuations for μ_B, c_s and P is relevant in the strong coupling limit and neglibible in the BCS regime. In the strong coupling limit we expect that the number of condensate couples is consistently larger than in the BCS regime, so we can assume that in the BEC limit the effect of fluctuations on the condensate fraction would be neglibible. In [37] N.Fukushima et al. calculate the effects of Gaussian fluctuations on the condensate fraction for a 3D BCS-BEC crossover. In particular they obtain a relation like:

$$n_0 = n_{c0} + n_{g0} \tag{3.22}$$

where n_{c0} is the mean-field term while n_{g0} rappresents the correction due to Gaussian fluctuations. They have concluded that n_{g0} can be neglect in all the crossover. We think that this results could be extended for the bidimensional case. So we will evaluate the beyond mean-field condensate fraction replacing in (3.21) the beyond mean-field chemical potential and gap energy. At the crossover values for the binding energy we obtain that, both for the mean-field and the beyond mean-field case, the number of particle condensated is 80%. This is a reasonable number of condensated particles to consider the system a Bose-Einstein condensate.

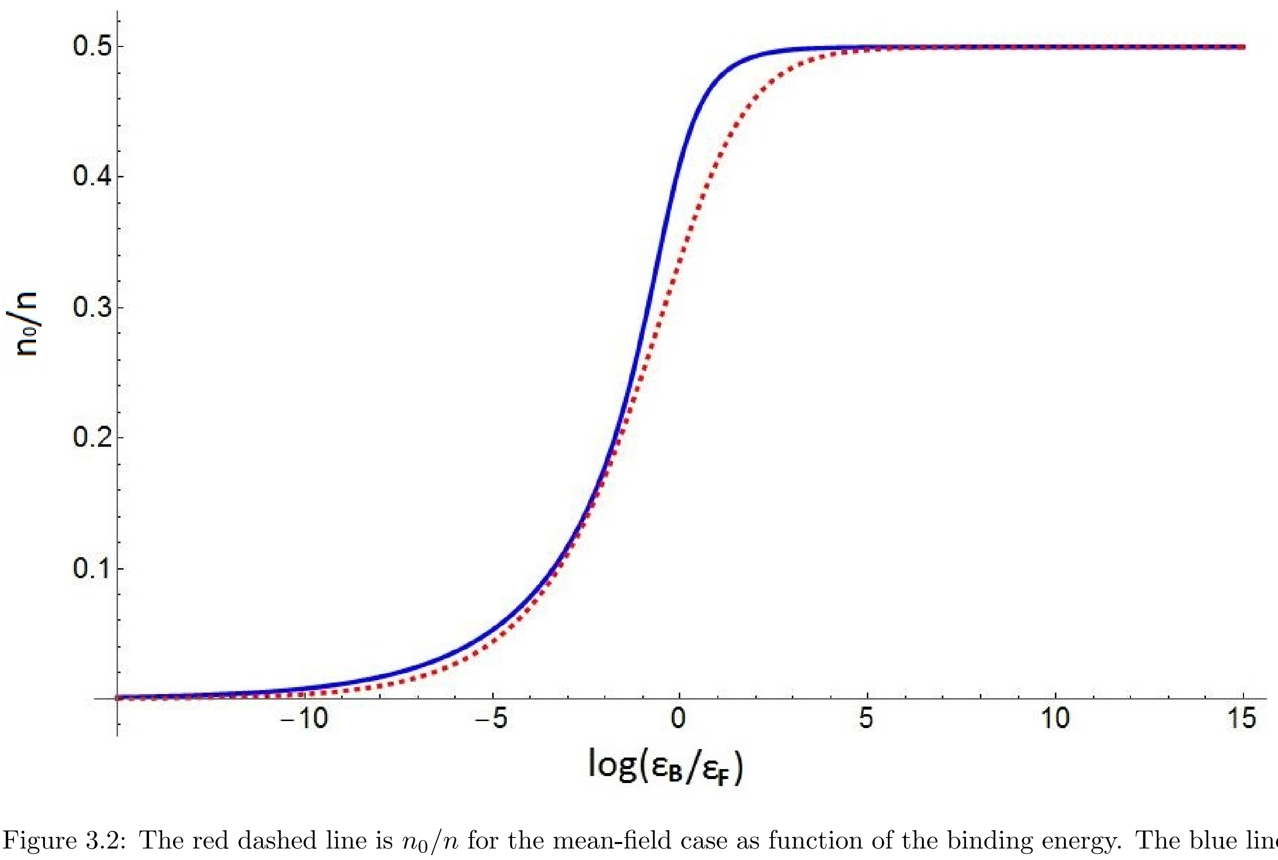

Figure 3.2: The red dashed line is n_0/n for the mean-field case as function of the binding energy. The blue line is n_0/n considering Gaussian fluctuations as function of the binding energy.

3.2 Josephson Effect

In 1962 physicist Brian Josephson predicted that the Cooper pairs could potentially tunnel cross an insulating layer [38]. This would create a coupling between the two superconducting states and create a current across the gap, which was experimentally confirmed later. This is the Josephson effect and it's a purely quantistic effect. To begin understanding the effects of this tunneling we need a setup to describe the quantum mechanical states related to the two sides of the junction.

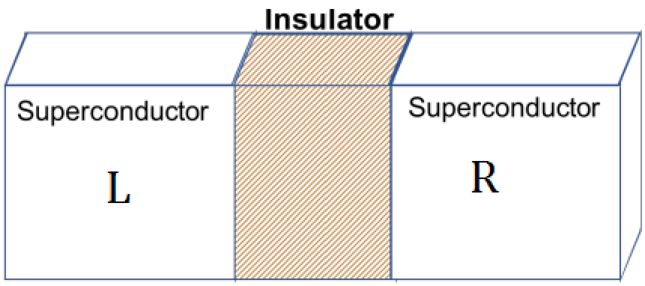

Figure 3.3: Josephson junction

Taking a look at Fig.(3.3), we'll describe the state of the superconductor on the left as Ψ_L and Ψ_R to describe the right. We will investigate the dynamical Josephson effect in BCS-BEC crossover based on a time dependent G-L equation. Let's consider a tridimensional Fermi gas of N atoms with two equally populated spin components and attractive inter-atomic strenght at zero temperature. At zero temperature where the superfluid density concides with the total density, the G-L order parameter describing the motion of Cooper pairs atom is defined as [39]:

$$\Psi(r,t) = \sqrt{\frac{n(r,t)}{2}} exp(i\theta(r,t)) \tag{3.23}$$

where $n(r,t)$ is the local atomic number density and $\theta(r,t)$ is the phase of the condensate wave function. Under an external potential U(r) acting on individual atoms, the non-linear time-dependent G-L equation (TDGLE) is:

$$i\hbar\frac{\partial}{\partial t}\Psi(r,t) = \Big[-\frac{\hbar^2\nabla^2}{4m} + 2U(r) + 2\mu/(n(r))\Big]\Psi(r,t) \tag{3.24}$$

Here m is the mass of one atom and $\mu(n(r))$ is the bulk chemical potential of homogeneus fluid with density n. The Josephson junction could be schematized like two reservoir of volume V_L and V_R such that $V_L + V_R = V$ in which N_L and N_R are enclosed at time 0, such that $N_L + N_R = N$. The two reservoirs are divided by a monodimensional double weel potential barrier whith size C. In the transverse directions we suppose the partcles are subjected to a strong harmonic potential that keep them confined. We look for a time-dependent solution of the TDGLE of the form:

$$\Psi(r,t) = \Psi_L(t)\Phi_A(r) + \Psi_B(t)\Phi_B(r) \tag{3.25}$$

where $\Phi_\alpha(r)$ is the quasi-stationary solution of the TDGLE localized in the region α. Inserting this relation in (3.24), after integrating over space and neglecting exponentially small termis, the system could be described by the following two-state model:

$$i\hbar\frac{\partial}{\partial t}\Psi_L = E_L\Psi_L - K\Psi_R \tag{3.26}$$

$$i\hbar\frac{\partial}{\partial t}\Psi_R = E_R\Psi_R - K\Psi_L \tag{3.27}$$

Here $E_\alpha = E^0_\alpha + E^I_\alpha$ is the energy in the region α and K is the tunneling energy:

$$E^0_\alpha = \int d^3r\Phi_\alpha\Big[\frac{\hbar^2}{4m}\nabla^2 + 2U(r)\Big] \tag{3.28}$$

$$E^I_\alpha = \int d^3r\Phi_\alpha 2\mu(2|\Psi_\alpha|^2\Phi_\alpha)\Phi_\alpha \tag{3.29}$$

$$K = \int d^3r\Phi_L\Big[-\frac{\hbar^2}{4m}\nabla^2 + U(r)\Big]\Phi_R \tag{3.30}$$

The tunneling term K describes phenomenologically the tunneling between the two region. Unfortunately a microscopic derivation of K in the full BCS-BEC crossover is not yet available. Under the assumption that U(r) can keep weakly connected the particles between the two reservoirs, we can write $\Psi_\alpha(t) = \sqrt{N_\alpha/2}exp(i\theta_\alpha(t))$ where $N_\alpha(t)$ and $\theta_\alpha(t)$ are the number of Fermions and the phase in the region α. Let's introduce the phase difference:

$$\phi(t) = \theta_R(t) - \theta_L(t) \tag{3.31}$$

and the relative number imbalance:

$$z(t) = \frac{N_L(t) - N_R(t)}{N} \tag{3.32}$$

Eq(3.26) and Eq(3.27) give:

$$\dot{z} = -\frac{2K}{\hbar}\sqrt{1-z^2}sin(\phi) \tag{3.33}$$

$$\dot{\phi} = \frac{2}{\hbar}\Big[\mu(\frac{N}{2V_L}(1+z)) - \mu(\frac{N}{2V_R}(1+z))\Big] + \frac{2K}{\hbar}\frac{z}{\sqrt{1-z^2}}cos(\phi) + \frac{E^0_L - E^0_R}{\hbar} \tag{3.34}$$

These are the atomic Josephson junction equations (AJJ) for the two dynamical variable $z(t)$ and $\phi(t)$ descibing the oscillations of N Fermi atoms tunneling in the superfluid state between the region L and the region R. The tunneling current is defined as:

$$I = -\dot{z}N/2 = (\frac{KN}{\hbar})\sqrt{1-z^2}sin(\phi) = I_0\sqrt{a-z^2}sin(\phi) \tag{3.35}$$

where I_0 is the critical current. In the limit of $z << 1$ the tunneling current reduces to the BCS case $I = I_0 sin(\phi)$ [40]. In deep BEC regime instead, where $\mu(n) \sim n$ the AJJ equations reduce to the bosonic Josephson junction equation (BJJ) introduced by Smerzi er al. [41]. This is the model we will adopt to study the Josephson effect in bidimensional BCS-BEC crossover.

3.3 DC Josephson effect and Tunneling Energy in 2D BCS-BEC crossover

We gonna start not including imbalance and fixing the phase difference. In this way we don't have to evaluate the oscillations of $z(t)$ and $\phi(t)$. This is the direct current mode (DC). Without oscillations of the relative imbalance and phase difference we focus on the tunneling energy K. Generally it's taken constant because a microscopic derivation of K in the full BCS-BEC crossover is not yet available. In a recent work M.Zaccanti and W.Zwerger [2] developed a model to describe Josephson tunneling between two superfluid reservois of ultracold atoms which account the dependence of the critical current on the coupling all along the crossover. It's possible to extend their result also to the bidimensional case. Let's consider a bidimensional setup of Josephson junction where a rectangular barrier connect two reservoirs like in the figure.

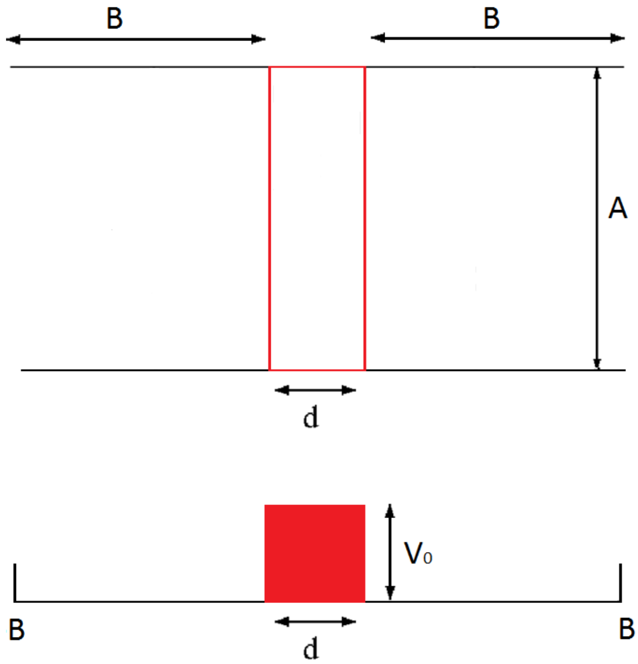

Figure 3.4: Two-dimensional Josephson junction setup where the rectangular barrier of height V_0 and size d divides the two reservoirs

Under the condition of barrier heights (V_0) considerably grater than chemical potential (μ), the superfluid current is:

$$I(\phi) = I_c sin(\phi) \tag{3.36}$$

where ϕ is the phase difference between the systems in the two regions and I_c is the maximum current possible. Assuming a homogeneus situation with transverse lenght A, the associated critical current density is:

$$\hbar j_c = \frac{\hbar I_c}{A} = B t_{cc}(\mu_B) n_0 \tag{3.37}$$

where $t_{cc}(\mu_B)$ is the transfer matrix element associated with coherent tunneling of bosons, n_0 is the density of condensated couples and 2B is the total longitudinal size of the system. For the junction geometry in Fig.(3.4) the matrix element con be written as [42]:

$$t_{cc}(\mu_B) = \frac{|t|(\mu_B)}{4k(\mu_B)B} \mu_B \tag{3.38}$$

Here $k = (\mu_B)\sqrt{2m\mu_B}/\hbar$ is the wave vector of a boson with mass 2m and chemical potential μ_B, while $|t|(\mu_B)$ is the associated single-boson transmission amplitude. The critical current can be written in the form:

$$\hbar j_c = \frac{\mu_B n_0}{2k(\mu_B)}|t|(\mu_B) \tag{3.39}$$

We can see that the dependence of the critical current on the strenght coupling is in μ_B and n_0 that are both bulk properties. The microscopic tunneling amplitude $|t|(\mu_B)$ depends on the interaction and determines the energy at which the tunneling process occurs. Under the condition of $V_0 >> \mu_B$ on the whole crossover, we are also supposing that the energy to tunnel doesn't changes so much respect to the barrier height. So we can consider this microscopic factor constant along the crossover. Until now the relations have been formulated considering the system in the BEC regime. Instead in the BCS regime the critical current of a Josephson junction is known to be equal to the current in the normal state at a finite voltage $eV = \pi\Delta_0/2$, where Δ_0 is the energy gap at zero temperature [38,43]. It's possible to connect the BCS current with the BEC one using the BCS relation of the condensate density n_0:

$$\frac{n_0}{n} = \frac{3\pi}{16}\frac{\Delta_0}{\epsilon_F} \tag{3.40}$$

In this way the BCS density current reads [2]:

$$\hbar j_c = \frac{\mu_F n_0}{2k_F}|t|^2(\mu_F) \tag{3.41}$$

where $|t|^2(\mu_F)$ is the trasmission probability of a single fermion at the Fermi energy $\mu_F \to \epsilon_F$. We note that Eq.(3.39) and Eq.(3.41) are very similar. In fact just replace $\mu_B \to 2\mu_F$ and $k_B \to 2k_F$ one can pass from one to the other. Considering the Fermi velocity of the noninteracting Fermi gas, $v_F = \sqrt{nv_F/m}$, one obtains:

$$\frac{j_c}{|t|(\mu_B)} = \frac{j_c}{|t|^2(\mu_F)} = \frac{nv_F}{8}\lambda_0\sqrt{\tilde{\mu}} \tag{3.42}$$

where $\lambda_0 = 2n_c/n$ is the condensate fraction and $\tilde{\mu} = \mu_F/\epsilon_F$ is the normalized chemical potential. The ratio between the tunneling amplitude of one pair and the trasmission probability of one fermion show a very weak dependence on the characteristic of the barrier for the assumpiont of high V_0. This means that one could use the density current BEC relation in the BCS regime, and viceversa, committing a small error. The dependence on the coupling is in the product $\lambda_0\sqrt{\tilde{\mu}}$. Its behavior in mean-field and beyond mean-field approximation is shown in Fig.(3.5). The beyond mean-field curve grows from the BCS regime to a maximum at the value $\log(\epsilon_B/\epsilon_F)$=0.26 and then decreases in the BEC regime. Instead the mean-field curve grows monotonically. To better understand the meaning of this product we can write (3.42) in the form of the critical current $\hbar I_c = KN$ where K is the tunneling energy and N the total number of Fermions.

$$\hbar I_c = \hbar A j_c = \hbar A|t|^2(\mu_F)\frac{nv_F}{8}\lambda_0\sqrt{\tilde{\mu}} \tag{3.43}$$

Replacing $n = N/(2AB)$ one obtains:

$$\hbar I_c = \hbar|t|^2(\mu_F)\frac{Nv_F}{16B}\lambda_0\sqrt{\tilde{\mu}} \tag{3.44}$$

Comparing with $\hbar I_c = KN$, K reads:

$$K = \hbar |t|^2(\mu_F)\frac{v_F}{16B}\lambda_0\sqrt{\tilde{\mu}} = K_0\lambda_0\sqrt{\tilde{\mu}} \tag{3.45}$$

where the factor $K_0 = \hbar|t|^2(\mu_F)v_F/16B$ encloses all the factors indipendent by the coupling. Fixed the setup of the system K_0 is a constant along the crossover. So the product $\lambda_0\sqrt{\tilde{\mu}}$ gives us information about the behavior of the tunneling energy and the critical current along the crossover. In the tridimensional case we have seen that the unitary limit is the better configuration to have a better superfluidity because of the competition between the Fermionic and the Bosonic density. In the 2D case at the value of $\log(\epsilon_B/\epsilon_F)$=0.26, where the current is mamximum, already the 85% of particles is condensated, so in the bidimensional case there should be onother criteria to determine the condition of better superfluidity. A possibility, always correlated to study of the condensate density, is that over $log(\epsilon_B/\epsilon_F) = 0.86$ most of the particles are already condensated and so becomes more difficult to take other couples in the ground state. In fact the slope of the curve n_0/n (Fig.3.2) decreases after that values. This could mean that in 2D the superfluidity is linked to the predisposition of the couples to condensate increasing the coupling. We will delve into this topic in the next section. As explained in [2] for the 3D case the origin of the nonmonotonic behavior of the critical current presented in the work is quite different from the interpretation given by Spuntarelli et al. [44]. There, the nonmonotonic behavior of the Josephson current was explanied through the Landau criterion with the existence of two types of critical velocities associated with two different excitatio branches: pair breaking on the BCS and phonons on the BEC. Instead in [2] the nonmonotonic behavior emerges from the competion between the bulk properties λ_0 and $\tilde{\mu}$. Zaccanti and Zwerger conclusions are also supported by recent experimantal studies which identify vortex rings and phonons, rather than fermionic pair-braking excitations, as dissipation mechanism in all the crossover. We think that it's possible to apply the same reasoning to the bidimensional case without many problems, since, both for 3D and 2D case, the potential barrier taken in exam is a one dimensional barrier, it extends only in one direction.

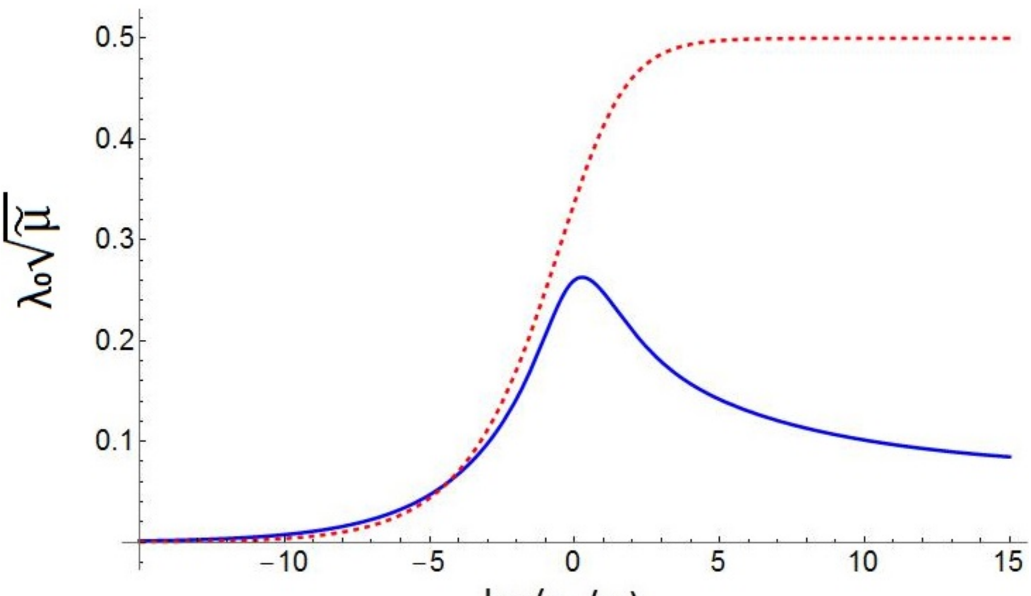

Figure 3.5: The dashed red line is the mean-fiedl product $\lambda_0\tilde{\mu}$. The blue line is the beyond mean-field product $\lambda_0\tilde{\mu}$.

3.4 AC Josephson effect in 2D BCS-BEC crossover

In the privious section we focus on the critical Josephson current and its behavior along the crossover restricting ourselves to the DC Josephson effect. In this last part we are going to see the effects on the time evolution of imbalance and phase difference, considering a tunneling energy dependent of the coupling. What we have to do is to substitute the tunneling energy K in the time evolution equations of imbalance and phase difference [39] with $K(\epsilon_B/\epsilon_F) = K_0\lambda_0(\epsilon_B/_F)\sqrt{\tilde{\mu}(\epsilon_B/\epsilon_F)}$. From now we will omit to write the dependence of ϵ_B/ϵ_F for simplicity of writing. We have to solve the following differential equations:

$$\dot{z} = -\frac{2K_0\lambda_0\sqrt{\tilde{\mu}}}{\hbar}\sqrt{1-z^2}sin(\phi) \tag{3.46}$$

$$\dot{\phi} = \frac{2}{\hbar}\Big[\mu\Big(\frac{N}{2V_A}(1+z)\Big) + \mu\Big(\frac{N}{2V_B}(1-z)\Big)\Big] + \frac{2K_0\lambda_0\sqrt{\tilde{\mu}}}{\hbar}\frac{z}{\sqrt{1-z^2}}cos(\phi) \tag{3.47}$$

We will solve numerically this system of differential equations inserting the beyond mean-field values of μ and n_0. Since in the mean-field case μ, in unit of the Fermi energy, is a constant and and n_0/n is very similar between the two cases, it's reasonable to think that the mean-field would not give us more interesting information then the beyond mean-field case. The first thing we do is to evaluate how $z[t]$ and $\phi[t]$ change along the crossover for a fixed setup system. On the base of [39] we chose to study a fermionic superfluid of ^{40}K atoms with the total density $n = 0.02 atoms/\mu m^2$ and a tunneling value of $K_0/k_B = 2*10^{-8}Kelvin$ and $z[0] = 0.5$, $\phi[0] = 0$. In Fig.(3.6) we show $z[t]$ and $\phi[t]$ for three different values of the coupling: $\log(\epsilon_B/\epsilon_F)$= -10, $\log(\epsilon_B/\epsilon_F)$= 0 and $\log(\epsilon_B/\epsilon_F)$= 10. For the unitary case the plot is until 20 ms because the oscillation frequency is too high. As we can see at unitary limit the oscillating frequency of $z[t]$ and $\phi[t]$ is higher, instead the maximum amplitude of $\phi[t]$ decreases regardeless of the tunneling energy. We also note that along all the crossover the oscillation frequency of $z[t]$ and $\phi[t]$ is the same.

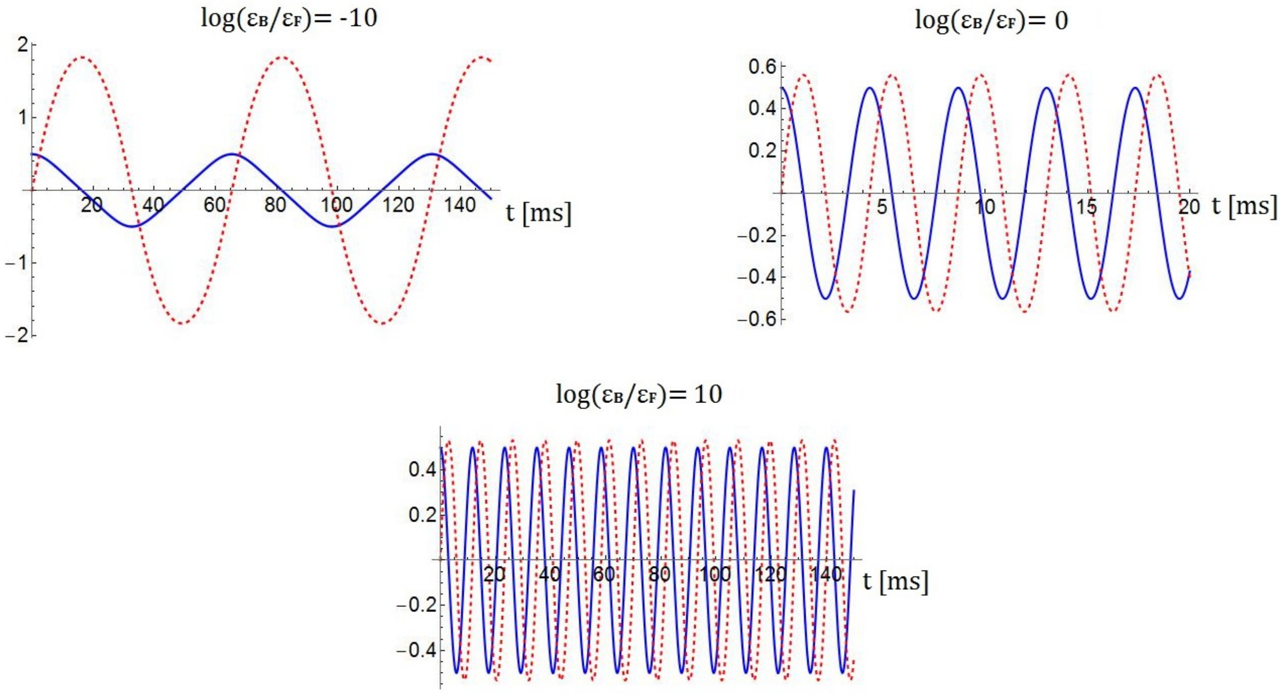

Figure 3.6: The red dashed line is $\phi[t]$ in $[rad]$. The blue line is z[t].

In the first panel of Fig.(3.7) we report the oscillation frequency in function of the binding energy. The behavior is very similar to the product $\lambda_0\sqrt{\tilde{\mu}}$ with a maximum at unitary. So the oscillation frequency seems to be very sensitive to variation of the tunneling energy. Instead the maximum phase difference has a monotonically beahvior, it seems to be more sensitive to the coupling strenght variation. In particular the phase difference in the deep BEC regime goes to the value $z[0] = 0.5$. In fact modifying the values of $z[0]$ and $\phi[0]$ we notice that in the BEC regime the maximum amplitude values of the imbalance and phase difference are the same and correspond to $Max(z[0], \phi[0])$. From BCS to BEC they adapt their maximum amplitude to reach this value in BEC regime. The oscillation frequency seems to be indipendent by the initial condition $z[0]$, $\phi[0]$. In the mean-field case instead the time evolution of $z[t]$ and $phi[t]$ have the same behavior of the beyond mean-field case until the value $\log(\epsilon_B/\epsilon_F)$= 0. Then the oscillation frequency continues to grow like the tunneling energy in Fig.(3.5).

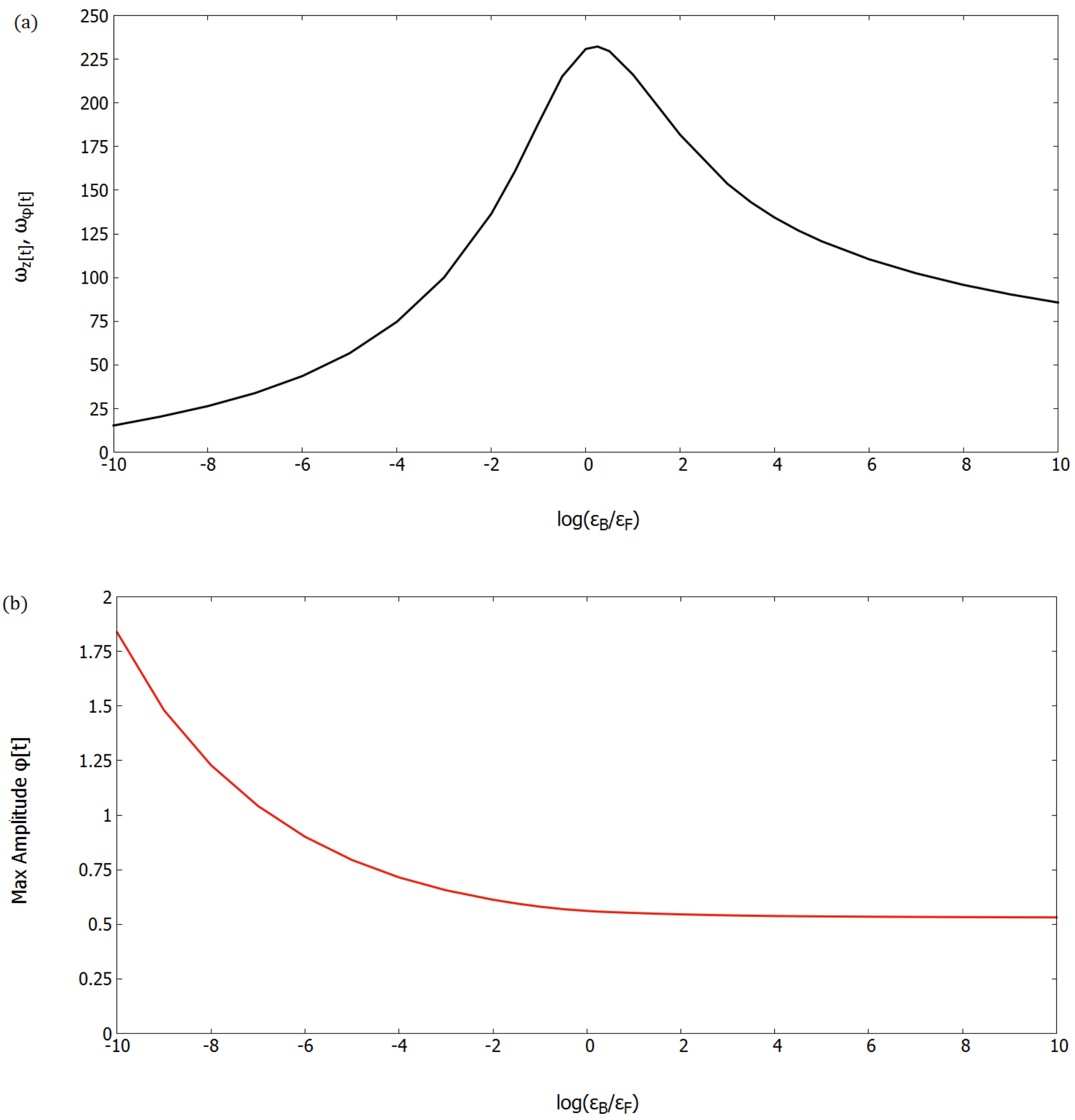

Figure 3.7: (a) Oscillation frequency of z[t] in Hz and ϕ[t] in rad*Hz. (b)Maximum amplitude ϕ[t] in $[rad]$.

To better understand the effect of variations of tunneling energy and coupling strenght in the beyond mean-field case we looked for two values of ϵ_B/ϵ_F which corresponds the same value of the tunneling energy. For example we have considered $\log(\epsilon_B/\epsilon_F)$= 5 and $\log(\epsilon_B/\epsilon_F)$= -3.57. In this way we can see the effects of the coupling strenght neglecting those of tunneling energy. The increase in binding energy leads to a dicrease of maximum amplitude of $\phi[t]$ and also a decrease of the oscillation frequency of $\phi[t]$ and $z[t]$. To understand the effects of tunneling energy we fixed the binding energy and vary the K_0. The increase in tunneling energy leads to a dicrease of maximum amplitude of $\phi[t]$ and an increase of the oscillation frequency in $\phi[t]$ and $z[t]$. This means that for the oscillation frequency there is a competition of opposite effects, from BCS to unitary, between the tunneling energy and the binding energy, and the effects linked to the tunneling energy seems wins over those of the binding energy. In the BEC regime K($\log(\epsilon_B/\epsilon_F)$) decreases so the two effects are no longer in opposition. For the phase difference instead it's the opposite. From BCS to unitary the two effect are in agreement and in the BEC are in opposition. In this case the effects linked to the binding energy get the better on those linked to the tunneling energy. We have to remember that the tunneling energy depends on the setup system throught the term K_0 and on the binding energy through the term $\lambda_0\sqrt{\tilde{\mu}}$. For K_0 fixed we can say is that an increase of binding energy take to different effects on the system depending on the regime of the crossover. These effects from BCS to unitary regime are charachterized by an increasing on the critical Josephson current and on the oscillation frequency of the relative number imbalance and of phase difference, while they lead to a decrease of the maximum amplitude of the phase difference. Instead from unitary to BEC an increase of the binding energy takes to a decrease of the critical current and oscillation frequency on $z[t]$ and $\phi[t]$ when the maximum amplitude of $\phi[t]$ continues to decrease until the value $Max(z[0], \phi[0])$. The turning point is when the number of condensated particles reaches the 80%−85% of the total particles, over which the system could be considered a Bose-Einstein condensate. We can reasonably conclude that there is a connection between the coupling strenght, the condensate fraction and the Josephson current. Increasing the coupling strenght from BCS the size of the couples decrease until it becomes less then the mean distance between the couples. In this situation the Bosonic nature of the couples begin to emerge. At this point the ground-state is still almost empty so the particles could easly condense. This is a condensate made of not strong-coupled Fermions, not still considerable a Bose-Einstein condensate. In this situation one has the most intense and oscillating Josephson current. This situation continue until the condensed fraction reaches the value of 0.80-0.85. Continuing increase the coupling strenght one has a system of strong-coupled Fermions in which most of the couples are already condensated, considerable as a Bose-Einstein condensate. And in this situation the Josephson current stabilezes to the BEC one.

Conclusions

The BCS-BEC crossover phenomenon is an excellent candidate to study purely quantum effects. In the second chapter we focus on the thermal fluctuations throught the use of the 3D Ginzburg-Landau theory. Their effect has been quantified through the numerical solution of the gap and number equation in mean-field and beyond mean-field approximation ((2.23),(2.24), (2.28)). We use the results in the Ginzubrg-Landau approach, which turned out to be a great tool to investigate characteristic quantities: Ginzburg-Landau coherence lenght or critical rotational frequency. Once determined the G-L paramters one can build very simple relations for ξ_{GL} and ω_c. From their study around the critical temperature (Fig.(2.5) and Fig.(3.10)) we got that the better configuration to have a more consistent superfluidity is the unitary limit. At $1/k_F a_s = 0$ there is a competition between the Fermionic and Bosonic density (Fig.(3.11)) which traslates in a more resistant superfluidity around the critical temperature. At this point we continued with the study of G-L theory because superfluid systems, by their nature, are best studied at low temperature where this phenomenon is more consistent. The use of G-L theory parameters could be extended to low temperature only in the BEC regime where the gap energy Δ_0 is less then the binding energy. In this regime also the sound velocity could be written though the G-L parameter, in particular it results to be proportional to the inverse of the coherence lenght. Generally to obtain this quantites one uses the microscopic approach that involve functional integral and Feynman diagram methods. The advantage of the G-L theory is that, once established the G-L parameters, the relations for the characteristic quantities are very easy to deal. Furthermore there are many other cases where we could apply the G-L theory. Would be very interesting adapt this model to charged particle systems. In this way one can study the critical magnetic field and its penetration lenght (λ). Evaluating the Ginzburg-Landau parameter $k = \lambda/\xi_{GL}$ it's possibile to determine where a system is type I or II superconductor.

In the last chapter we focus on onother purely quantum effect: 2D Josephson effect. We start with the effects of thermal fluctuations on the description of the system. Also for the 2D case the effects of fluctuations are predominant when the coupling strenght increases. Then we focus on the current of the AC and DC Jospehson effect. For the DC we impose no relative imbalance and fixed phase difference to the Josephson junction equations. In this way we study directly the maximum critical current. This last is proportional to the tunneling energy that we take dependent on the crossover [2]. When the binding energy is about equal to the Fermi energy the critical current shows a maximum. So it's reasonable to think that also in 2D there is some special mechanism that rule the superfluidity. In the last part we replace the K coupling dependent in the Josephon equation. We study in detail the effects along the crossover on the oscillating frequency on maximum amplitude of $z[t]$ and $\phi[t]$. Differently from the 3D, the competition between the Fermionic and Bosonic behavior seems involve directly the population of the ground-state energy. In fact the more intense and oscillating current occurs when the ground-state population is around the 80% of the total particles and in prossimity of the change of sign of the chemical potential. In this condition the condensated couples are characterized by a interaction strenght less then in the BEC regime. But in this region the condensate fraction increases rapidly with the strenght coupling. We thik the in a laboratory, analyzing different samples could be possible to prove these assumption.

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