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ALGANT Master Thesis in Mathematics

Zeroes of p-adic L-functions: the Ellenberg-Jain-Venkatesh conjecture

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Dedicated to Fabrizio Mori,
my much needed guiding light.

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Introduction

In 1914 Hilbert and Polya conjectured the existence of a spectral interpretation of the zeroes of the Riemann Zeta function. Today, there is a lot of evidence that makes this conjecture convincing. Firstly, in some numerical experiments, Montgomery and Odlyzko found empirically that the distribution of the spacings between zeroes of the Riemann zeta function is the same as the Gaussian Unitary Ensemble (GUE), a probability measure arising in random matrix theory; this is expected to happen for all Dirichlet L-functions. Secondly, in various families of L-functions, the statistics that low-lying zeroes follow are, empirically, the same statistics of the eigenvalue distributions of a compact classical group.

Some interesting developments of these studies involve many different classes of zeta and L-functions, one of them being p-adic L-functions.

Thanks to the p-adic Weierstrass preparation theorem, it is known that p-adic L-functions have a finite number of zeroes, called the "lambda invariant" of the function. The study of this quantity is, in some way, analogous to certain questions about low-lying zeroes studied in the literature on Dirichlet L-functions. This allows an adaption and a restatement of many conjectures and studies from the complex setting to the p-adic one, where they can be studied with different tools. This approach has already been successful in other cases, with the main example being the Function Field analogue, where Artin introduced another zeta function and a spectral interpretation of its zeroes has been proved.

In this thesis we will introduce Dirichlet and p-adic L-functions and their main properties; we will discuss what is known and what is expected about the zeroes of both classes of L-functions. After this, we will focus on the study of the Ellenberg-Jain-Venkatesh conjecture (2011) about the zeroes of family of p-adic L-functions corresponding to imaginary quadratic fields; we will support this study also by doing some numerical experiments on these zeroes.

Let's now focus on the contents of the chapters.

Chapter 1 : Zeroes of Dirichlet L-functions

In the first chapter we will introduce Dirichlet L-functions by using the Mellin transform

$$f \mapsto \frac{1}{\Gamma(s)} \int_0^\infty f(t) t^s \frac{dt}{t};$$

we will prove their analytic continuation to $\mathbb{C}$ and the link between their special values and generalized Bernoulli numbers

$$L(\chi, 1-k) = -\frac{B_{k,\chi}}{k}.$$

Then, we will define the distributions linked with the zeroes of these functions; we will have a look at the main properties of these zeroes and at the main conjectures about them, in particular RH and GRH. We will also study many matrix groups models (for example GUE) and we will show how groups like $U(N)$,$USp(2N)$ and $SO(N)$ can generate statistics related to low lying zeroes. One important instance of this is the study of the family $\mathcal{F}$ of Dirichlet L-function with a quadratic character that is (empirically) related to the statistics of $USp(2N)$; a comparison between the distribution of the lowest zeroes for L-functions in this family and the one resulting from $USp(2N)$ is showed in Figure 1. As we will show in Section 1.4 all quadratic characters χ can be associated to a fundamental discriminant d; in Figure 1 we are studying a subfamily of $\mathcal{F}$ obtained considering only certain values of d.

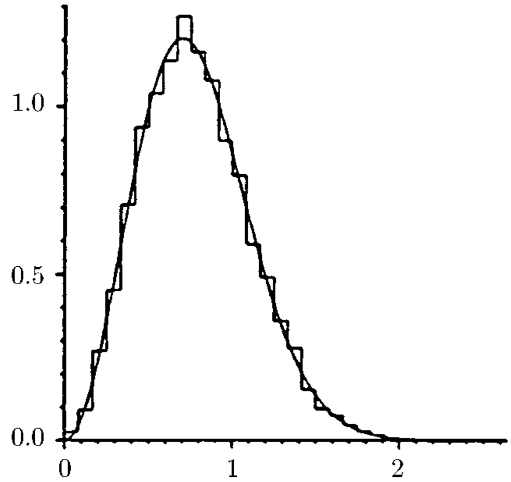

Figure 1: First zero above 0 for $L(s;\chi_d)$ with $10^{12} < |d| < 10^{12} + 200000$.
N. Katz, P. Sarnak, "Zeroes of zeta functions and symmetry"

The continue function in Figure 1 is the density of $\nu_1(Sp)$ that we will construct in Section 1.3 and that studies the distribution of the angle associated to the first eigenvalue of matrices in the group $USp(2N)$ for arbitrary large N. Above the continue function are represented the frequencies of imaginary parts of zeroes obtained computing the first zero in a subfamily of $\mathcal{F}$.

Chapter 2: Zeroes of p-adic L-functions

The second chapter will show Mazur's construction of p-adic L-functions that creates an appropriate measure on $\mathbb{Z}_p^\times$ and consider it as a function; this allow us to carry over to the p-adic setting the construction of the first chapter and to show the parallel between these two classes of L-functions in a more compelling way.

We will start by introducing some basics of p-adic analysis to study continuous functions, the convergence of series on $\mathbb{Z}_p$ and Mahler's theorem that will allow us to express continuous functions as series of binomial coefficients.

We will also define and study the main properties of p-adic measures proving that a measure on $\mathbb{Z}_p$ can be expressed as a formal series and by introducing some useful operations to work on them.

After that we will be able to create the measures μ_χ on $\mathbb{Z}_p$ that have k-th moments equal to

$$\int_{\mathbb{Z}_p} x^k \cdot \mu_\chi = L(\chi, -k),$$

this can be done by defining its Mahler transform $\mathcal{A}_{\mu_\chi}(T)$ using Bernoulli numbers; after that we will restrict μ_χ to $\mathbb{Z}_p^\times$ and obtain the measure that Mazur called p-adic L-function. We can consider this measure as a function in $p-1$ different ways using the identification of $s \in \mathbb{C}_p$ as the continuous homomorphism between $\mathbb{Z}_p^\times$ and $\mathbb{C}_p^\times$ that maps $x \mapsto \omega(x)^i \langle x \rangle^s$. In this way, each function $L_{p,i}(\chi, s)$ we will obtain will be interpolating the associated Dirichlet L-function in a different range following the formula

$$L_{p,i}(\chi, -k) = (1 - \chi\omega^{-k}(p)p^k) \cdot L(\chi, -k), \quad -k \equiv i \pmod{p-1}.$$

Finally, we will discuss the finiteness of the zeroes of these functions using Iwasawa theory and the p-adic Weierstrass preparation theorem; this will allow us to define the lambda invariant of a p-adic L-function. We will prepare to study this quantity by introducing also Newton polygon.

Chapter 3: The Ellenberg-Jain-Venkatesh conjecture

In the third chapter we will study Ellenberg-Jain-Venkatesh conjecture about the family of p-adic L-functions of imaginary quadratic fields

Conjecture 0.0.1. *Amongst imaginary quadratic fields K in which p does not split, the proportion with an associated p-adic L-function $L_{p,0}(\chi, s)$ that has lambda invariant equal to r is*

$$p^{-r} \prod_{t>r} (1 - p^{-t}). \tag{1}$$

This conjecture links the distribution of the lambda invariant in this family and the statistics of random p-adic matrices.
In fact, the quantity contained in the conjecture appears from the study of the distribution of the associated polynomial $P_A(T)$ of random matrices A from $GL_n(\mathbb{Z}_p)$. The polynomial $P_A(T)$ is obtained by taking the characteristic polynomial of A, discarding all the roots that are not p-adically close to 1 (i.e. those roots α for which $\alpha - 1$ is a unit) and making a change of variable from T to $T - 1$. We will prove that the fraction of elements in $GL_n(\mathbb{Z}_p)$ that has an associated polynomial $P_A(T)$ of a fixed degree r approaches the value in the conjecture $\left(p^{-r} \prod_{t>r}(1 - p^{-t})\right)$ as $n \to \infty$.

Finally, we will create a Sage script to study numerically the Ellenberg-Jain-Venkatesh conjecture. We will compute the first terms of the power series that characterise the p-adic L-function by computing special values of this function with generalized Bernoulli numbers and interpolating them with a polynomial; this polynomial will agree with the power series modulo a power of p. This is enough to use Netwon Polygons and to find the number of zeroes and their absolute values.

We will use this script to compute the 3-adic lambda invariant of the admissible imaginary quadratic fields with discriminant smaller than 10^5 and compare the statistics we found with the conjectured ones. The following table contains the results we have obtained.

λ	0	1	2	3	4	5	6
predicted	0.5601	0.2800	0.1050	0.0363	0.0122	0.0041	0.0013
observed - (a)	0.62380	0.2538	0.0819	0.0272	0.0094	0.0025	0.0009
observed - (b)	0,61214	0.2605	0.0833	0.0296	0.0092	0.0035	0.0013

We have considered as two different cases (a) the fields where the prime 3 splits and (b) the ones where the prime 3 does not split.

We will also compute others statistics regarding the zeroes of the p-adic L-functions in this family, using Newton polygon; we will mainly focus on studying the distribution of the lower zero of each function, in particular by noticing that (empirically) they repel the point 0. This behaviour has been observed numerically also for the complex case (*Hazelgrave phenomenon*) and shows another interesting connection between the study of the p-adic and the complex version of the same family.

Contents

Chapter 1

Zeroes of Dirichlet L-functions

For almost two centuries many different mathematicians studied the Riemann Zeta function and its properties; they revealed many new theorems and evolved the study of Number Theory and of Mathematics. We will study some of the results that were obtained and we will have a look at some questions that still remain open. We will explain why studying the zeroes of this function is important and why introducing other similar functions can be useful for this study.

In this chapter we will define Riemann Zeta and Dirichlet L-functions, we will introduce the Mellin transform of a function and we will use this tool to study the analyticity and some special values of these functions.

Then, we will study the zeroes of these functions; we will define zero-free regions and explicit the most important ones, we will state the Generalized Riemann Hypothesis in order to study the distributions of the zeroes in Riemann Zeta and in Dirichlet L-functions on the critical line.

After this we will introduce some random matrix group models studying the eigenvalues in unitary matrices and studying their spacings and the smallest ones. This models are linked trough numerical experiments with the distributions of zeroes that we are studying.

We will also focus on the conjecture regarding the family of Dirichlet L-functions with a quadratic character to show a compelling example of the use of these models to describe in an effective way the zeroes of a family of Dirichlet L-functions.

1.1 Riemann Zeta and Dirichlet L-functions

In 1859, Riemann published a very influential paper (B. Riemann, Uber die Anzahl der Primzahlen unter einer gegebenen Große, Monatshefte der Berliner Akademie der Wissenschaften 1859, 671–680) where he related the distribution of prime numbers with the study of the zeroes of the complex function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

This function, that is called the Riemann Zeta function, converges absolutely on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$ and diverges on $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \leq 1\}$. Riemann also proved that this function has analytical continuation to $\mathbb{C} \setminus \{1\}$ and has a simple pole of residue 1 in $s = 1$.

In this section we will start our journey: we will prove some of the most known properties of this function; we will explain why these properties are important and we will talk about the reasons that make interesting to study the zeroes of Riemann Zeta.

In 1831 Dirichlet introduced for the first time Dirichlet characters and Dirichlet L-series as a useful tool for the study of arithmetic progressions; these functions are considered a very important generalization of Riemann Zeta and the study of their properties is as important as the one on Riemann Zeta. We will now introduce them and, during the rest of the section, we will study them alongside Riemann Zeta to show the similarities between these functions.

Definition 1.1.1. Let $N > 1$ be a positive integer. A *Dirichlet character* modulo N is a group homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times,$$

extended to a function $\chi : \mathbb{Z} \to \mathbb{C}$ by sending elements that are not coprime with N to zero.

Definition 1.1.2. The Dirichlet character modulo N obtain by extended the identity morphism is called *principal*.

Definition 1.1.3. We define the *Dirichlet L-function* of a Dirichlet character χ as

$$L(\chi, s) := \sum_{n \geq 1} \chi(n) n^{-s}.$$

Since $|\chi(n)| \in \{0, 1\}$ for all n, this series converges absolutely for $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1\}$. As for $\zeta(s)$, also $L(\chi, s)$ has an analytical continuation: if χ is not principal, then it can be extended to $\mathbb{C}$, otherwise it can only be extended to $\mathbb{C} \smallsetminus \{1\}$.

We can see a strong similarity between $\zeta(s)$ and $L(\chi, s)$; if we consider the defining formula of Dirichlet L-functions and we assume that $\chi(n) = 1$ for every n then we obtain $\zeta(s)$. This means that, sometimes, can be useful to think to $\zeta(s)$ as a Dirichlet L-functions $L(\chi, s)$ where χ is a special Dirichlet character "modulo 1" obtained by lifting the identity group homomorphism. Keeping this idea in mind will guide us in studying and proving formulas and properties that are valid for both of these functions.

We will now introduce Euler's Gamma and the Mellin transform of a function, in order to prove the existence of analytic extensions of $\zeta(s)$ and $L(\chi, s)$. Riemann proved these results in a different way but, by following this alternative path, we can create a compelling parallel with the study of p-adic L-functions in Chapter 2.

Definition 1.1.4. Euler's Gamma function is given by

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} \mathrm{d}t, \ \mathrm{Re}(s) > 0.$$

where $t^s := e^{s \log(t)}$ with $\log(t)$ the ordinary real natural logarithm of t.

Definition 1.1.5. The *Mellin transform* of a C^∞ function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $f(x) = O(\frac{1}{e^x})$ for $x \to \infty$ is defined as

$$L(f, s) := \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(t) \mathrm{d}t, \ \ \mathrm{Re}(s) > 0;$$

Lemma 1.1.6. *The Mellin transform $L(f, s)$ of a function f as in* 1.1.5 *has an analytical continuation to $\mathbb{C}$ and for every $n \in \mathbb{Z}_{\geq 0}$ we have*

$$L(f, -n) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(0).$$

Proof. Using integrations by parts we can rewrite the function

$$\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} f(t) \mathrm{d}t = \frac{1}{\Gamma(s)} \left(0 - \int_0^\infty \frac{t^s f'(t)}{s} \mathrm{d}t \right)$$

Using the identity $\Gamma(s)s = \Gamma(s+1)$ we can rewrite this as

$$-\frac{1}{\Gamma(s+1)} \int_0^\infty t^s f'(t) \mathrm{d}t,$$

obtaining the identity

$$L(f, s) = -L(f', s+1);$$

we notice that the right hand side now converges for $\mathrm{Re}(s) > -1$. Iterating this identity we obtain that $L(f, s)$ has an analytic continuation to all $\mathbb{C}$. We also find that

$$L(f, -n) = (-1)^{n+1} L\left(\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}} f, 1 \right) = (-1)^{n+1} \int_0^\infty \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}} f(t) \mathrm{d}t = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(0).$$

□

This lemma shows that, if a function is obtained as the Mellin transform of another one, then we can say that it can be extended to $\mathbb{C}$ and it is easy to compute some of its values.

Computing *special values* in Riemann Zeta is a very important problem. One of the first instances of this is the *Basel problem* posed in 1650 by Mengoli. This problem consists in finding the value of

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots.$$

Euler solved this problem in 1734 in his paper *De summis serierum reciprocarum* showing that

$$\zeta(2) = \frac{\pi^2}{6},$$

and giving a closed formula for the computation of all the special values of Riemann Zeta in even integers $\zeta(2n)$.

This special values are not the only important ones. In this thesis we will mainly focus with the study of the Riemann Zeta and Dirichlet L-functions on the negative integers; this will help us in future chapters for the construction of p-adic L-functions and for the study of the so-called *trivial zeroes* of $\zeta(s)$ and $L(\chi, s)$.

We cannot use the Mellin transform to obtain directly $\zeta(s)$ as the Mellin transform of another function because $\zeta(s)$ has a simple pole in $s = 1$ and does not have an analytic continuation on $\mathbb{C}$; therefore, we want to obtain $(s-1)\zeta(s)$ as $L(f, s)$ for a suitable f and then use Lemma 1.1.6. The suitable function we want to consider is $f(t) := \frac{t}{e^t - 1}$. Since Lemma 1.1.6 will link some special value of the Mellin transform of f to the n-th derivatives of this function computed in zero, is a good idea to study them first. These values are strongly linked with the coefficients in the Taylor expansion of f that are often called *Bernoulli numbers*.

Definition 1.1.7. The *Bernoulli numbers* B_n are defined as the coefficients in the Taylor expansion of $\frac{t}{e^t-1}$, that is

$$\frac{t}{e^t - 1} = \sum_{n \geq 0} B_n \frac{t^n}{n!}$$

These numbers can be easily obtained by computing the derivatives of f and evaluating them at 0. To have a better idea of their behaviour we can compute the first Bernoulli numbers

$B_0 = 1$	$B_1 = -\frac{1}{2}$	$B_2 = \frac{1}{6}$	$B_3 = 0$	$B_4 = -\frac{1}{30}$
$B_5 = 0$	$B_6 = \frac{1}{42}$	$B_7 = 0$	$B_8 = -\frac{1}{30}$	$B_9 = 0$

Looking at this first values we can notice that the odd Bernoulli numbers, except for B_1, always seem to be zero. We can prove that this is true by studying the parity of the function $f(t) + \frac{1}{2}t$.

Lemma 1.1.8. *The function $f(t) + \frac{1}{2}t = \frac{t}{e^t-1} + \frac{1}{2}t$ is even, therefore $B_1 = -\frac{1}{2}$ and $B_n = 0$ for n odd, greater than* 1.

Proof.

$$\frac{t}{e^t - 1} + \frac{1}{2}t = t\left(\frac{1}{e^t - 1} + \frac{1}{2}\right) = t\left(\frac{2 + e^t - 1}{2(e^t - 1)}\right) = \frac{t}{2}\left(\frac{e^t + 1}{e^t - 1}\right) =$$

$$= \frac{t}{2}\left(\frac{e^t + 1}{e^t - 1}\right)\frac{e^{-t/2}}{e^{-t/2}} = \frac{t}{2}\left(\frac{e^{t/2} + e^{-t/2}}{e^{t/2} - e^{-t/2}}\right) = \frac{t}{2}\left(\frac{\cosh(t/2)}{\sinh(t/2)}\right),$$

and the last function is clearly even.

□

Now that we know more about Bernoulli numbers and about the function f, we can study the Mellin transform of f.

$$
\begin{aligned}
L(f, s-1) &= \frac{1}{\Gamma(s-1)} \int_0^\infty \frac{t}{e^t - 1} t^{s-1} \frac{\mathrm{d}t}{t} \\
&=^* \frac{s-1}{\Gamma(s)} \int_0^\infty \sum_{n \geq 1} e^{-nt} t^s \frac{\mathrm{d}t}{t} \\
&=^{**} \frac{(s-1)}{\Gamma(s)} \left(\sum_{n \geq 1} n^{-s} \right) \int_0^\infty e^{-v} v^s \frac{\mathrm{d}v}{v} \\
&= (s-1)\zeta(s);
\end{aligned}
$$

where in the starred $(*)$ equality we used the following algebraic identity

$$\frac{1}{e^t - 1} = \frac{e^{-t}}{1 - e^{-t}} = \frac{1}{1 - e^{-t}} - 1 = \sum_{n=1}^{\infty} e^{-nt},$$

and in the double starred $(**)$ equality we applied the substitution $v = nt$.

Using 1.1.6, this implies that $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$ and we can compute the special values of $\zeta(1-n)$ for $n \in \mathbb{N}_{>1}$

$$\zeta(1-n) = (-1)^{1-n} \frac{B_n}{n} = -\frac{B_n}{n},$$

where the last equality is justified because $B_n = 0$ for n odd and greater than 1, as shown in 1.1.8.

We will now try to obtain similar results for Dirichlet L-functions. This time, since for all the non principal characters we want to prove that $L(\chi, s)$ has an analytical extension to all $\mathbb{C}$, we will obtain $L(\chi, s)$ directly as the Mellin transform of a function f_χ. The definition of this function will use the generalized Bernoulli numbers $B_{n,\chi}$.

Definition 1.1.9. The *generalized Bernoulli numbers* associated to χ, a Dirichlet character modulo N, are defined by

$$F_\chi(t) := \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) \frac{te^{at}}{e^{Nt} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}.$$

As with classical Bernoulli numbers we want to investigate if some $B_{k,\chi}$ are trivially zero. To do this we want to change the sign of t in the defining series of these numbers and to study the results.

$$
\begin{aligned}
F_\chi(-t) = \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) \frac{(-t)e^{a(-t)}}{e^{N(-t)} - 1} = - \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) e^{Nt} \frac{te^{-at}}{1 - e^{Nt}} = \\
= - \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) \frac{te^{(N-a)t}}{1 - e^{Nt}} = \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(-1)\chi(-a) \frac{te^{(N-a)t}}{e^{Nt} - 1} = \chi(-1) F_\chi(t);
\end{aligned}
$$

from these computations we obtain that $F_\chi(t)$ is odd or even depending on the values of the Dirichlet character $\chi(-1)$. This immediately implies the following lemma;

Lemma 1.1.10. *If $\chi(-1) = 1$ then we know that*

$$B_{k,\chi} = 0 \text{ for } k \equiv 1 \pmod 2$$

otherwise, if $\chi(-1) = -1$ then we know that

$$B_{k,\chi} = 0 \text{ for } k \equiv 0 \pmod 2;$$

Using these numbers we can now define f_χ as

$$f_\chi(t) := \sum_{a \in (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times} \chi(a) \frac{e^{at}}{e^{Nt} - 1} = \sum_{k \geq 1} \frac{B_{k,\chi}}{k} \frac{t^{k-1}}{(k-1)!}.$$

Defining f_χ using the generalized Bernoulli numbers allow us to easily computes the special values of its Mellin transform because it is easy to compute the n-th derivatives evaluated in 0 of f_χ and this makes easier to use Lemma 1.1.6.

This definition of f_χ still does not allow us to easily see that the Mellin transform of f_χ is linked with the Dirichlet L-function of χ. We want to find a different expression for f_χ that makes easier to compute $L(f,s)$.

Lemma 1.1.11. *Let ζ be a primitive N-th root of unity; we can rewrite the function f_χ as*

$$f_\chi(t) = \frac{1}{G(\chi^{-1})} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \frac{\chi^{-1}(a)}{e^t \zeta^a - 1},$$

where

$$G(\chi) := \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a)\zeta^a.$$

Proof.

$$\begin{aligned} f_\chi(t) &= \frac{1}{G(\chi^{-1})} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \frac{\chi^{-1}(a)}{e^t \zeta^a - 1} \\ &= \frac{-1}{G(\chi^{-1})} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \frac{\chi^{-1}(a)}{1 - e^t \zeta^a} \\ &= \frac{-1}{G(\chi^{-1})} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi^{-1}(a) \sum_{k \geq 0} e^{tk} \zeta^{ak} \\ &= \frac{-1}{G(\chi^{-1})} \sum_{k \geq 0} e^{tk} \left(\sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi^{-1}(a) \zeta^{ak} \right) \end{aligned}$$

We can recognize that the inner sum is equal the Gauss sum $\tau(k, \chi^{-1})$; we can therefore use Theorem 3.4.1 at page 98. of [9] that says

$$\tau(k,\chi) = \begin{cases} \chi^{-1}(k)G(\chi) & \text{if } \gcd(k,N) = 1 \\ 0 & \text{if } \gcd(k,N) > 1 \end{cases}.$$

Using this we can rewrite the main quantity as

$$\begin{aligned} f_\chi(t) &= \frac{-1}{G(\chi^{-1})} \sum_{k \geq 0} e^{tk} \tau(k, \chi^{-1}) \\ &= \frac{-1}{G(\chi^{-1})} \sum_{\gcd(k,N)=1} e^{tk} \chi(k) G(\chi^{-1}) \\ &= - \sum_{\gcd(k,N)=1} e^{tk} \chi(k) \end{aligned}$$

Now we can split this sum grouping up the indexes considering their residue modulo N;

$$\begin{aligned} f_\chi(t) &= - \sum_{a \in (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times} \sum_{k \geq 0} e^{t(kN+a)} \chi(kN + a) \\ &= - \sum_{a \in (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times} \chi(a) e^{ta} \sum_{k \geq 0} e^{t(kN)} \\ &= - \sum_{a \in (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times} \chi(a) e^{ta} \frac{1}{1 - e^{tN}} \\ &= \sum_{a \in (\mathbb{Z}/\mathbb{N}\mathbb{Z})^\times} \frac{\chi(a) e^{ta}}{e^{tN} - 1}. \end{aligned}$$

□

With this new expression of f_χ we can now compute $L(f,s)$ using directly the definition of Mellin transform.

Proposition 1.1.12. *The Dirichlet L-function of a non principal Dirichlet character χ is the Mellin transform of $f_\chi(t)$, therefore*

$$L(\chi,s) = \chi(-1)L(f_\chi,s)$$

Proof.

$$\begin{aligned}
L(f_\chi,s) &= \frac{1}{\Gamma(s)G(\chi^{-1})}\int_0^\infty \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \frac{\chi^{-1}(a)}{e^t\zeta^a-1} t^s\frac{\mathrm{d}t}{t}\\
&=^* \frac{1}{\Gamma(s)G(\chi^{-1})}\int_0^\infty \sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi^{-1}(a)\sum_{k\geq 1} e^{-kt}\zeta^{-ka}t^s\frac{\mathrm{d}t}{t};
\end{aligned}$$

in the starred $(*)$ equality we used the following algebraic identity

$$\frac{1}{e^t\zeta^a-1} = \frac{e^{-t}\zeta^{-a}}{1-e^{-t}\zeta^{-a}} = \frac{1}{1-e^{-t}\zeta^{-a}} - 1 = \sum_{k=1}^{\infty} e^{-kt}\zeta^{-ka};$$

now, back at the main equality chain

$$\begin{aligned}
&= \frac{1}{\Gamma(s)G(\chi^{-1})}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times}\chi^{-1}(a)\sum_{k\geq 1}\zeta^{-ka}\int_0^\infty e^{-kt}t^s\frac{\mathrm{d}t}{t}\\
&= \frac{1}{\Gamma(s)G(\chi^{-1})}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times}\chi^{-1}(a)\sum_{k\geq 1}\zeta^{-ka}k^{-s}\int_0^\infty e^{-v}v^s\frac{\mathrm{d}v}{v},
\end{aligned}$$

using the definition of Euler's Gamma we can simplify it with the integral on the right and obtain

$$\begin{aligned}
&= \frac{1}{G(\chi^{-1})}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times}\chi^{-1}(a)\sum_{k\geq 1}\zeta^{-ka}k^{-s}\\
&= \frac{1}{G(\chi^{-1})}\sum_{k\geq 1}k^{-s}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times}\chi^{-1}(a)\zeta^{-ka}.
\end{aligned}$$

To conclude the proof we have to show that, for $k\in\mathbb{N}_{\geq 1}$ we have that

$$\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times}\chi^{-1}(a)\zeta^{-ka} = \tau(-k,\chi^{-1}) = \overline{\chi^{-1}(-k)}G(\chi^{-1}) = \chi(-1)\chi(k)G(\chi^{-1});$$

in the last two equalities we used known properties of Gauss Sums, see [9] page 99, Th 3.4.1.

□

Now that we have proved that Dirichlet L-functions are obtainable as the Mellin transform of f_χ we can conclude the section by applying Lemma 1.1.6 and obtaining the two results that we wanted: the analytic continuation and the special values formula for negative integers.

Corollary 1.1.13. *$L(\chi,s)$ has an analytic continuation to $\mathbb{C}$, and for every $k\in\mathbb{N}_{\geq 1}$*

$$L(\chi,1-k) = -\frac{B_{k,\chi}}{k}$$

We notice that we were able to simplify the formula eliminating $\chi(-1)$, by applying Lemma 1.1.10.

1.2 Distribution of zeroes

The zeroes in Riemann Zeta are one of the most studied aspects of this function; to understand why, we should have a look at the story of the proof and of the improvements of the Prime Number Theorem.

Theorem 1.2.1 (Prime Number Theorem). *Let $\pi(x)$ be the number of primes smaller than x. Then, for $x \to \infty$*

$$\pi(x) \sim \frac{x}{\log(x)}.$$

This theorem was conjectured by Legendre in 1798 and was proved in 1896 by Hadamard and de la Vallée Poussin. The fact that $\{s \in \mathbb{C} \setminus \{1\} \mid \operatorname{Re}(s) \geq 1\}$ is a zero-free region was a crucial ingredient in this proof; After that year the study of Riemann Zeta zeroes improved and it was proved that a wider zero-free region existed; this brought new improved versions of the Prime Number Theorem where $\pi(x)$ was studied with a more precise error term. In fact, de la Valléè Poussin showed that, for $x \to \infty$

$$\pi(x) - \frac{x}{\log x} \sim \frac{x}{(\log x)^2};$$

in his proof he used that, for some constant $c > 0$, $\{s \in \mathbb{C} \setminus \{1\} \mid \operatorname{Re}(s) > 1 - \frac{c}{\log(|Im(s)|+2)}\}$ is a zero-free region. As we can see, the zeroes of Riemann Zeta $\zeta(s)$ yield impressive informations concerning prime numbers distributions.

In this chapter we will introduce some of the main results about which are the zeroes of Riemann Zeta and of Dirichlet L-functions, which are the so-called *trivial zeroes* and what are the known and the conjectured zero-free regions for $\zeta(s)$ and $L(\chi, s)$.

Lemma 1.2.2. *[9] Let $N \in \mathbb{Z}_{\geq 2}$ and let χ be a Dirichlet character modulo N. Then*

$$L(\chi, s) = \prod_p (1 - \chi(p)p^{-s})^{-1} \neq 0,$$

for $\{s \in \mathbb{C} \mid \operatorname{Re} s > 1\}$.

This is called *Euler product* and it is an alternative formula for $L(\chi, s)$ for $\operatorname{Re}(s) > 1$. Lemma 1.2.2 give us the first zero-free region of $L(\chi, s)$ that is $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1\}$.

To extend this region we want to introduce an important formula regarding these functions. There is a functional equation that links $\zeta(s)$ to $\zeta(1-s)$ and $L(\chi, s)$ to $L(\overline{\chi}, 1-s)$. This is helpful because it links the zeroes in the region $\{s \in \mathbb{C} \mid \operatorname{Re}(s) < 0\}$ with the ones in the region $\{s \in \mathbb{C} \mid \operatorname{Re}(s) > 1\}$, that is easier to study thanks to the Euler product and thanks to the absolute convergence of $\zeta(s)$.

Before introducing the functional equation we have to recall a couple of useful properties of Euler's Gamma

Lemma 1.2.3. *[9] $\Gamma(s) \neq 0$ for $s \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$. Moreover, $\Gamma(s)$ has a simple pole for $s \in \{0, -1, -2, \dots\}$.*

We also need to give some more definitions about Dirichlet characters and to define what a conductor of a Dirichlet L-function is.

Definition 1.2.4. Given χ a Dirichlet character modulo N and d a positive divisor of N, we say that χ is *induced* by a Dirichlet character χ' modulo d if $\chi(a) = \chi'(a)$ for every $a \in \mathbb{Z}$ with $\gcd(a, N) = 1$. In this definition we consider χ_0 as a Dirichlet character modulo 1, where $\chi_0(a) = 1$ for $a \in \mathbb{Z}$.

Definition 1.2.5. The *conductor* of a Dirichlet character χ modulo N is the smallest positive divisor d of N such that χ is induced by a character modulo d.

Definition 1.2.6. A Dirichlet character modulo N is called *primitive* if its conductor is N.

Now we can state the functional equation using the function $\xi(\chi, s)$

Definition 1.2.7. Let χ be a primitive Dirichlet character. We define the function $\xi(\chi, s)$ as

$$\xi(\chi, s) := \left(\frac{N}{\pi}\right)^{(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(\chi, s)$$

where N is the modulus of χ, a is 0 if $\chi(-1) = 1$ and a is 1 if $\chi(-1) = -1$.

Theorem 1.2.8. *[14]*

$$\xi(\chi, s) = \frac{G(\chi)}{i^a\sqrt{N}}\xi(\overline{\chi}, 1-s)$$

where a and N as in 1.2.7 and G is the Gauss sum of χ, a primitive Dirichlet character.

We can use this functional equation and the properties of Γ to study the zeroes of $L(\chi, s)$ also in the region $\{s \in \mathbb{C} \mid \operatorname{Re}(s) < 0\}$. To do this we have to distinguish the two cases we have found in the definition of $\xi(\chi, s)$.

Definition 1.2.9. A Dirichlet character modulo N is called *odd* if $\chi(-1) = -1$ while it is called *even* if $\chi(-1) = 1$.

It is easy to see that all the Dirichlet characters are even or odd, since χ is obtained by extending a group homomorphism, so $\chi(-1) \in \{-1, 1\}$ because $\chi(-1)^2 = \chi(1) = 1$. This also shows that a, in the definition of $\xi(\chi, s)$, is well defined.

Looking at the functional equation we can see that the only zeroes of $L(\chi, s)$ in the region $\{s \in \mathbb{C} \mid \operatorname{Re}(s) < 0\}$ are the same of the poles in $\Gamma(\frac{s+a}{2})$, since we know that $L(\overline{\chi}, 1-s)$ and $\Gamma(\frac{1-s+a}{2})$ has no zeroes in this region. This zeroes are called *trivial zeroes* and they are the set $\{-2, -4, -6, \dots\}$ for even Dirichlet characters and the set $\{-1, -3, -5, \dots\}$ for the odd Dirichlet characters. Since $\zeta(s)$ can be obtained as the L-function of an even character we can see that it has trivial zeroes in $\{-2, -4, -6, \dots\}$. We already knew about these zeroes of $\zeta(s)$ thanks to the results of the previous section, when we proved that odd Bernoulli numbers are zero (except for B_1) and we showed the link between these numbers and the special values of $\zeta(s)$ at negative integers.

When we refer to a subset of $\mathbb{C}$ as a zero-free region we often omit considering trivial zeroes. This means that, with this functional equation, we can say that in $\zeta(s)$ and $L(\chi, s)$ there is a zero-free region that is $\{s \in \mathbb{C} \mid \operatorname{Re}(s) < 0 \vee \operatorname{Re}(s) > 1\}$. This region can be improved to $\{s \in \mathbb{C} \mid \operatorname{Re}(s) \leq 0 \vee \operatorname{Re}(s) \geq 1\}$.

Now we know that $\zeta(s)$ and $L(\chi, s)$ do not have non-trivial zeroes except for the *critical strip* $\{s \in \mathbb{C} \mid 0 < \operatorname{Re}(s) < 1\}$. This bound can be improved to even smaller regions of the complex plane and there are two very famous conjectures about where non-trivial zeroes of $\zeta(s)$ and of $L(\chi, s)$ can be.

Riemann Hypothesis (RH). *All the non-trivial zeroes of $\zeta(s)$ are in*

$$\mathcal{R} := \{s \in \mathbb{C} \mid \operatorname{Re}(s) = \frac{1}{2}\}.$$

Generalized Riemann Hypothesis (GRH). *For every Dirichlet character χ, all the non-trivial zeroes of $L(\chi, s)$ are in*

$$\mathcal{R} := \{s \in \mathbb{C} \mid \operatorname{Re}(s) = \frac{1}{2}\}.$$

From now on, we assume that these two conjectures are true and that all the non trivial zeroes of $\zeta(s)$ and $L(\chi, s)$ lie on the line $\mathcal{R}$. Therefore we will express the non-trivial zeroes of these functions as

$$\frac{1}{2} + i\gamma,$$

with $\gamma \in \mathbb{R}$;.

Even if we assume these conjectures to be true there are still other questions about the zeroes we can ask. We would like to study if their number is finite or not, where they are on this line, their distribution and if they tend to repel or attract each other.

This problems have been studied for a long period of time and even Riemann, when he was still alive, apparently already knew the answers of some of this questions. At the time it was already known that the zeroes on this line are infinite and also an important result about their density

Lemma 1.2.10. *Let $\{\gamma_i\}$ be the sequence of the imaginary parts of the ordered non-trivial zeroes of $\zeta(s)$, then*

$$\#\{i \mid 0 \leq \gamma_i \leq T\} \sim \frac{T \log T}{2\pi}.$$

Since they are infinite and they are countable we can order the different γ of all the zeroes of $\zeta(s)$:

$$\cdots \leq \gamma_{-1} \leq 0 \leq \gamma_1 \leq \gamma_2 \leq \ldots.$$

Another known result that is an immediate consequence of the functional equation of Theorem 1.2.8, is that if $\zeta(s) = 0$ then also $\zeta(\bar{s}) = 0$ and this imply that $\gamma_i = -\gamma_{-i}$; therefore we can focus only on the positive index zeroes $\{\gamma_i\}_{i\geq 1}$.

Since we know the density of these zeroes we want to re-normalize these zeroes and to study the statistical laws associated to

$$\tilde{\gamma}_i := \frac{\gamma_i \log \gamma_i}{2\pi},$$

in this way we obtain that

$$\#\{j \mid 0 \leq \tilde{\gamma}_i \leq T\} \sim T.$$

From this sequence we want to create and to study another sequence: the sequence associated to the consecutive spacings δ_i

$$\delta_i := \tilde{\gamma}_{i+1} - \tilde{\gamma}_i;$$

this sequence allow us to have an insight also on how the zeroes are spaced in the line and if they tend to repulse or attract each other. We can also generalize this idea and create the sequence of the k-th spacing as

$$\delta_i^{(k)} = \tilde{\gamma}_{i+k} - \tilde{\gamma}_i$$

The study of the sequence $\{\delta_i\}$ is a different approach to the study of zeroes of $\zeta(s)$ and $L(\chi, s)$; it is clear that we are not focusing anymore on which are the zero-free regions on $\mathbb{C}$ but we are focusing on the study of the density and of the spacings between zeroes. This different approach to this study uses also different tools; one of them is the numerical approach.

Starting in 1980 Odlyzko [22] has made impressive numerical studies of these zeroes and of their local spacings; he computed the sequences of $\{\tilde{\gamma}_i\}$ and of $\{\delta_i\}$ for very large value of T and for many different Dirichlet L-functions. Studies like this one where we analyse the distributions of "big enough" zeroes are called studies about the high-lying zeroes.

He discovered that these sequences often follow a common behaviour; in Figure 1.1., from Katz and Sarnak's *"Zeroes of Zeta functions and symmetry"* [16], we can find the histogram of the spacings δ_i for $i \sim 10^{20}$ of the zeroes in Riemann Zeta.

Odlyzko noticed that the interpolating function in the figure is common to all the numerical studies of spacings of high-lying zeroes. So it empirically appears that the spacings δ_i of high-lying zeroes of each Dirichlet L-function follow a common distribution function. This empirical fact is called the *Montgomery-Odlyzko law.*

1.3 Random Matrix Models

Many different theories try to explain the reasons behind Montgomery-Odlyzko law. The main one, that we are going to study more in detail in this section, is the existence of a spectral interpretation on the zeroes of $\zeta(s)$ and of $L(\chi, s)$.

The idea of linking the zeroes of these functions to a group of matrices and to their eigenvalues goes back to 1914 when Hilbert and Polya first conjectured such a link. Today, one of the many evidences that makes this conjecture convincing are the already mentioned Odlyzo's numerical experiments. These experiments show how all the zeroes in all the different L-functions might

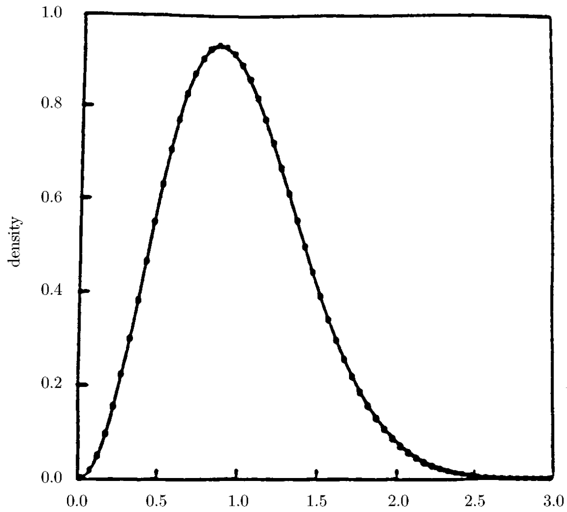

Figure 1.1: The histogram of spacings δ_i for $10^{20} \leq i \leq 10^{20} + 7 \cdot 10^6$

have a common origin or nature (maybe a spectral one); they also gave us a good approximation of the mysterious common interpolating function.

To better understand the nature of this function and why a spectral interpretation of the zeroes is expected, we have to talk about the distributions of eigenvalues of matrix in unitary matrix groups. We start by defining the *Gaussian Unitary Ensemble* (GUE).

Definition 1.3.1. A matrix A of size N with complex entries $a_{i,j}$ is a *Unitary matrix* if

$$A^{-1} = \overline{A^T},$$

where $B := \overline{A^T}$ is the matrix obtained as the complex conjugate of the transpose matrix of A, i.e. the entry $b_{i,j}$ is equal to $\overline{a_{j,i}}$. The group of all the unitary matrices of size N is called $U(N)$.

The eigenvalues of a unitary matrix A of size N lie all on the unit circle and are exactly N.

Remark 1.3.2. *Let take a matrix $A \in U(N)$, then every eigenvalue λ of A has $|\lambda_i| = 1$.*

Proof. We know that

$$A\overline{A^T} = I.$$

Let's take an eigenvector v with associated eigenvalue λ; we can compute

$$\overline{v^T A^T} Av = \overline{v^T} v = ||v||^2;$$

but we also know that

$$\overline{v^T A^T} Av = \left(\overline{\lambda v^T}\right)(\lambda v) = \overline{\lambda}\lambda ||v||^2 = |\lambda|^2 ||v||^2,$$

but this means that $||v||^2 = |\lambda|^2||v||^2$, therefore $|\lambda| = 1$. □

As we did for zeroes in L-functions we want to study the distribution of this eigenvalues; in particular we want to obtain a sequence of their spacings and then try to compare it with the spacings of high-lying zeroes in Dirichlet L-functions ($\{\delta_i^{(k)}\}$); moreover we want to create a distribution associated to this spacings and compare it with the distribution function observed in the Montgomery-Odlyzko law in Figure 1.1.

Fixing an element A in $U(N)$, we can express the N eigenvalues as $e^{i\theta_j(A)}$ and order the angles as

$$0 \leq \theta_1(A) \leq \theta_2(A) \leq \cdots \leq \theta_N(A) < 2\pi$$

We can define some new measures related to this values; the k-th consecutive spacings $\mu_k(A)$ are measures on $[0,\infty)$

$$\mu_k(A)[a,b] = \frac{\#\{1 \le j \le N \mid \frac{N}{2\pi}(\theta_{j+k} - \theta_j) \in [a,b]\}}{N}.$$

The definition of these measures recall the definition of the sequence $\{\delta_i^{(k)}\}$ because it effectively is a way to study the differences $\theta_{j+k} - \theta_j$ of the angles of the eigenvalues of A.

We can mimic this construction with other matrix groups that are not $U(N)$. Even if this construction works for many matrix groups we will focus on a couple of them. The first one will be crucial in the conjectures that we are going to study.

Definition 1.3.3. The group $USp(2N)$ is defined as the group of all the unitary matrices of size $2N$ satisfying

$$A^t J A = J,$$

where $J = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}$.

Definition 1.3.4. The group $SO(N)$ is defined as the group of all the unitary (!) matrices of size N satisfying

$$A^t A = I.$$

Let $G(N)$ be any one of the three groups $U(N)$,$USp(2N)$, $SO(N)$. We can construct, for any matrix $A \in G(N)$ the measures $\mu_k(A)$ as shown previously. The measures we created are not very satisfying yet because they just describe the behaviour of a finite number of eigenvalues that are related to a single matrix; this still seems quite different from the distribution of infinite zeroes of $\zeta(s)$ and $L(\chi,s)$ that we want to describe. The idea we want to develop is trying to take matrices of arbitrary big size and to study their associated measures; in this way we will obtain an arbitrary large number of eigenvalues and every $\mu_k(A)$ will recall more the distributions of the previous section.

Katz and Sarnak [15] followed this idea showing that there exists a measure that can be obtained as the limit of measures $\mu_k(A)$ with larger and larger size of A.

Theorem 1.3.5. *Fix $k \ge 1$. There are measures $\mu_k(\mathrm{GUE})$ such that for any G*

$$\lim_{N\to\infty} \int_{G(N)} \mu_k(A)\mathrm{d}A = \mu_k(\mathrm{GUE}),$$

where $\mathrm{d}A$ denotes the total mass one Haar measure on $G(N)$.

Moreover, they proved [15] that it is possible to get measures $\mu_k(A)$ that are arbitrary close to $\mu_k(\mathrm{GUE})$ just by taking a typical A of big enough size.

Theorem 1.3.6. *For a typical (in measure) matrix $A \in G(N)$ it is true that*

$$\lim_{N\to\infty} \int_{G(N)} D(\mu_k(A), \mu_k(\mathrm{GUE}))\mathrm{d}A = 0$$

where

$$D(\nu_1,\nu_2) := \sup\{|\nu_1(I) - \nu_2(I)| \mid I \subset \mathbb{R} \textit{ an interval}\}.$$

The measures $\mu_k(\mathrm{GUE})$, that we have just defined, are very important for the study of distribution of zeroes in $\zeta(s)$ and in $L(\chi,s)$. In fact, it has been numerically observed that the interpolation function of Figure 1.1 that Odlyzko has found by studying the neighbour spacings in Riemann Zeta and that it also seems to be common to all Dirichlet L-functions, strongly agrees with $\mu_1(\mathrm{GUE})$. This support the idea that zeroes of Dirichlet L-functions have a spectral interpretation; the fact that the spacings between these zeroes and the spacings between eigenvalues in G are (numerically) linked suggest that also other strongest connections may exist between these two objects. Since we have now identified the function in Figure 1.1 we can restate the

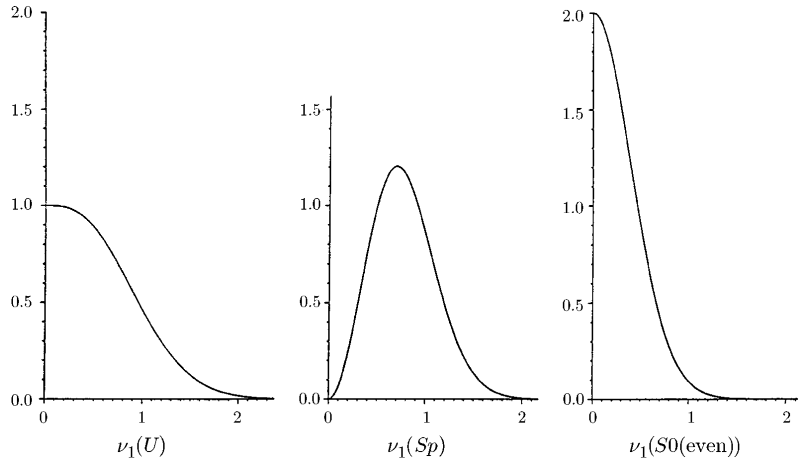

Figure 1.2

Montgomery-Odlyzko Law by saying that (empirically) the spacings δ_i of high-lying zeroes of a Dirichlet L-function follow the common distribution $\mu_1(\text{GUE})$.

There are also other probability distributions we can define with this matrices groups to study different others behaviours of the zeroes of Dirichlet L-functions and of the eigenvalues of matrices. The measures that we have studied up until now, so all the $\mu_k(A)$, do not depend on G when $N \to \infty$. There are other distributions that are sensitive to the particular group we choose; one important example of this is the distribution of the eigenvalue nearest to 1.

As we can see, this measure is different from the ones we have studied up until now, because it does not consider only one fixed matrix A and then studies the behaviour of all the different eigenvalues of the matrix but it study many different matrices by considering only the first eigenvalue of each one of them. This kind of study does not seem to match with the study of spacings between zeroes of $L(\chi, s)$ with arbitrary large imaginary part (i.e. high-laying zeroes) that we did in the last section; to match this new study we should consider only one zero from many different L-functions. We will see, in the next section, that the new measures we are going to introduce are linked to a different kind of study and we will be using them to study the so-called low-lying zeroes.

For $k \geq 1$ we can define the distribution of the k-th eigenvalues of the matrices in $G(N)$.

$$\nu_k(G(N))[a,b] = \text{Haar}\{A \in G(N) \mid \frac{N}{2\pi}\theta_k(A) \in [a,b]\},$$

again we are considering the total mass one Haar measure on $G(N)$. We can generalize this measure by not studying a fixed size N but by considering this measure on arbitrarily big values of N. Katz and Sarnak [15] proved that

Theorem 1.3.7. *There are measures $\nu_k(G)$ on $[0, \infty)$, which depend on G, such that*

$$\lim_{N\to\infty} \nu_k(G(N)) = \nu_k(G)$$

These measures depend on the group we choose and they are generally different from each other. In Figure 1.2 we can see some examples of the different $\nu_1(G)$ in these groups; $SO(\text{even})$ is obtained as the limit on $SO(2N)$ and $\nu_1(Sp)$ is obtained as the limit on $GSp(2N)$.

Looking at Figure 1.2, from Katz and Sarnak's *"Zeroes of Zeta functions and symmetry"* [16], we can notice that $\nu_1(Sp)$ has a peculiar property that differentiates it from the other two distributions: the fact that the density of $\nu_1(Sp)$ vanishes at $s = 0$. This means that the eigenvalues of a typical large A in $USp(2N)$ are repelled by 1.

1.4 The family of Dirichlet L-functions with a quadratic character

In this section we want to introduce a family of Dirichlet L-functions and we want to talk about some numerical studies on it. In particular we will analyse many different statistics about the zeroes of the $L(\chi, s)$ in this family and we will compare the numerical results with some of the measures we defined in the last section. In this way we can have a better overview on how random matrix models describe the behaviour of zeros and why a spectral interpretation of zeroes is expected.

The zeroes that we will mainly focus on during the study in this chapter are the ones closest to the points $s = \frac{1}{2}$. These zeroes are called the *low-lying zeroes*. If we have a family of Dirichlet L-functions $\mathcal{F}$ we can write all the non trivial zeroes of a function $f \in \mathcal{F}$ as

$$\frac{1}{2} + i\gamma_f,$$

with $\gamma_f \in \mathbb{R}$ and, for every $f \in \mathcal{F}$, order them as

$$\cdots \leq \gamma_f^{(-2)} \leq \gamma_f^{(-1)} \leq 0 \leq \gamma_f^{(1)} \leq \gamma_f^{(2)} \leq \cdots.$$

We are interested on the study of the k-th closest zeroes to $s = \frac{1}{2}$; therefore, after fixing a $k \geq 1$ we will study the distributions of the numbers

$$\frac{\gamma_f^{(k)} \log(c_f)}{2\pi},$$

where c_f is the conductor of the associated character to $L(\chi, f)$ and f varies over $\mathcal{F}$.

To study these distributions we can define the subfamily $\mathcal{F}_N$ as

$$\mathcal{F}_N := \{L(\chi, s) \in \mathcal{F} \mid c_\chi \leq N\}$$

If all the subfamilies $\mathcal{F}_N$ for $N \in \mathbb{N}$ are finite we can study the distribution of the k-th zero nearest to $\frac{1}{2}$ in each $\mathcal{F}_N$ by defining the distribution

$$\nu_k(\mathcal{F}_N)[a, b] := \frac{\#\{L(\chi, s) \in \mathcal{F}_N \mid \frac{\gamma_f^{(k)} \log c_\chi}{2\pi} \in [a, b]\}}{\#\mathcal{F}_N}.$$

The study of low-lying zeroes seems to agree, in its definition, with the measures $\nu_k(G(N))$: both try to analyse a whole family/group by studying a little information from every element of the family/group.

This study has big differences with the study of high-lying zeroes; in fact, the study of high-lying zeroes recalls more the measures $\mu_k(G)$ that focus on the distribution of many different zeroes in the same L-function.

To continue the link between the study of low-lying zeroes and the measures $\nu_k(G(N))$ we would like the limit of $\nu_k(\mathcal{F}_N)$ for $N \to \infty$ to exist. This is non trivial and the existence and the nature of this limit strongly depend on the family $\mathcal{F}$ we are considering.

For chosen families there are many studies, both theoretical ones and numerical ones, that show that $\nu_k(\mathcal{F})$, the limit of $\nu_k(\mathcal{F}_N)$, exists and it is equal to $\nu_k(G)$ for one of the G we have mentioned in the last chapter. Some examples of this can be the study of all the families of L-functions with a character χ such that $\chi^k = 1$ ([16]).The interesting aspect of this equality is the fact that, since $\nu_k(G)$ depends from G, we can associate to the family $\mathcal{F}$ a group G that represent, in some way, the behaviour of the zeroes of the L-functions in the family.

One compelling example of this are the numerical studies on the family $\mathcal{F}$ of Dirichlet L-functions associated to a primitive quadratic Dirichlet character. To study this family and the subfamilies $\mathcal{F}_N$ we have to define and introduce an alternative description of the quadratic Dirichlet characters.

Definition 1.4.1. A Dirichlet character is called *quadratic* if the associated group homomorphism χ is quadratic, i.e. $\chi^2 = 1$ and χ is not principal. In other words, a quadratic Dirichlet character is a rational and non-principal Dirichlet character.

Primitive quadratic Dirichlet characters are often associated to quadratic number fields; in fact, is possible to construct an interesting bijection between the set of all these fields and the set of all the primitive quadratic characters. To show this we have to introduce the concept of Kronecker symbol $\left(\frac{D}{n}\right)_K$ with D a fundamental discriminant.

Definition 1.4.2. An integer $D \in \mathbb{Z}$ is a *fundamental discriminant*, if either

(a) D is square-free and $D \equiv 1 \pmod 4$, or

(b) $D = 4d$ with d square-free such that $d \equiv 2$ or $d \equiv 3 \pmod 4$.

This subset of $\mathbb{Z}$ is the set of all the integer numbers that arises as the discriminant of a quadratic number field. For each fundamental discriminant we can define a related Kronecker symbol $\left(\frac{D}{n}\right)_K$.

We recall the definition of Legendre symbol, with p prime, is

$$\left(\frac{a}{p}\right)_L := \begin{cases} 1 \ \text{ if } a \in ((\mathbb{Z}/p\mathbb{Z})^\times)^2 \\ -1 \ \text{ if } a \notin ((\mathbb{Z}/p\mathbb{Z})^\times)^2 \\ 0 \ \text{ if } p|a \end{cases}$$

Definition 1.4.3. Let $D \in \mathbb{Z}$ be a fundamental discriminant, then its associated *Kronecker symbol* $\chi_D(n) = \left(\frac{D}{n}\right)_K$ is the Dirichlet character modulo $|D|$ with the following properties

1. $\chi_D(p) = 0$ when $p|D$;
2. $\chi_D(2) = \begin{cases} \text{if } D \equiv 1 \pmod 8 \\ -1 \ \text{ if } D \equiv 5 \pmod 8 \end{cases}$
3. for every odd prime p we have that $\chi_D(p) = \left(\frac{D}{p}\right)_L$;
4. $\chi_D(-1) = \begin{cases} 1 \ \text{ if } D > 0 \\ -1 \ \text{ if } D < 0 \end{cases}$
5. χ_D is completely multiplicative;

Lemma 1.4.4. *[20] For every fundamental discriminant D, the Kronecker symbol χ_D is a primitive quadratic character $\pmod{|D|}$. Moreover, every primitive quadratic character can be obtained as χ_D with D a fundamental discriminant in a unique way.*

Proof. We want to study the behaviour of $\left(\frac{p}{n}\right)_K$ for all p primes and to do this we will consider different cases, depending on the congruence of p modulo 4.

When $p \equiv 1 \pmod 4$ we want to show that

$$\left(\frac{p}{n}\right)_K = \left(\frac{n}{p}\right)_L \quad \text{for all } n \in \mathbb{N}.$$

This is clear if we consider q an odd prime, since it is just implied by the definition of $\left(\frac{p}{q}\right)_K$ and by quadratic reciprocity. It is also easy to show for $n = 2$ and $n = -1$.

When $p \equiv 3 \pmod 4$ we want to show that

$$\left(\frac{-p}{n}\right)_K = \left(\frac{n}{p}\right)_L \quad \text{for all } n \in \mathbb{N}.$$

This can be done in a similar way to the case $p \equiv 1 \pmod 4$.

Let's now consider d_1 and d_2, two coprime fundamental discriminants and let $d := d_1 d_2$. We assume $\left(\frac{d_1}{n}\right)_K$ and $\left(\frac{d_2}{n}\right)_K$ to be two primitive quadratic characters modulo $|d_1|$ and $|d_2|$. We notice that, for q odd prime,

$$\left(\frac{d}{q}\right)_K = \left(\frac{d}{q}\right)_L = \left(\frac{d_1}{q}\right)_L \left(\frac{d_2}{q}\right)_L = \left(\frac{d_1}{q}\right)_K \left(\frac{d_2}{q}\right)_K,$$

and it easy to show that this equality also hold for 2 and -1. Since these two functions are totally multiplicative and agree on all primes and on -1 we can conclude that they agree for every $n \in \mathbb{N}$, i.e.

$$\left(\frac{d}{n}\right)_K = \left(\frac{d_1}{n}\right)_K \left(\frac{d_2}{n}\right)_K \quad \text{for all } n \in \mathbb{N}.$$

Since we assumed that both $\left(\frac{d_1}{n}\right)_K$ and $\left(\frac{d_2}{n}\right)_K$ to be primitive we know that their product is a primitive character modulo $|d|$.

To conclude this proof we have to consider two facts:

1. For every odd prime q there is a unique quadratic character modulo q ([20], pg 296).
2. Let χ a character modulo $d_1 d_2$, with $\gcd(d_1, d_2) = 1$ and both greater than 1, obtained as the product of the two characters χ_1 modulo d_1 and χ_2 modulo d_2. Then χ is primitive if and only if both χ_1 and χ_2 are primitive ([20], Lemma 9.3).

Using this facts it is clear that we have considered all the primitive quadratic characters and all of them can be written uniquely as Kronecker symbols. $\square$

This means that there is a bijection

$$\{\text{Kronecker symbols}\} \longleftrightarrow \{\text{Primitive quadratic Dirichlet characters}\}$$

Since we can also create a bijection between the sets of Kronecker symbols, fundamental discriminants and quadratic number fields. This allow us to create a more interesting bijection

$$\{\text{Quadratic number fields}\} \longleftrightarrow \{\text{Primitive quadratic Dirichlet characters}\}$$

where we can map K, a quadratic number field to χ_D, where $D = \text{Disc}(K)$ and we can map χ_D to the quadratic number field that has discriminant equal to D.

Now that we have a better insight about the quadratic Dirichlet characters, we want to study the family $\mathcal{F}$ of Dirichlet L-functions $L(\chi_D, s)$ such that χ_D is a primitive quadratic Dirichlet character; this family is also called the family of Dirichlet L-functions associated to a quadratic number field, because of what we have just proved.

To study the zeroes of this family we can apply the procedure we have defined for a generic family. First, we write all the zeroes of $L(\chi, s)$ as $\frac{1}{2} + i\gamma_f^{(j)}$ and order them

$$\cdots \leq \gamma_f^{(-2)} \leq \gamma_f^{(-1)} \leq 0 \leq \gamma_f^{(1)} \leq \gamma_f^{(2)} \leq \cdots.$$

For $j \geq 1$ we define

$$\tilde{\gamma}_f^{(j)} := \frac{\gamma_f^{(j)} \log D_f}{2\pi},$$

where D_f is the conductor of χ_D, therefore D.

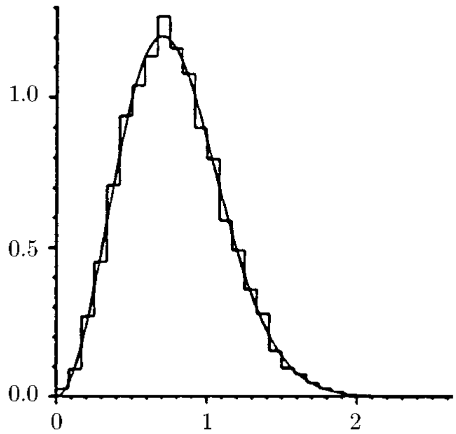

Figure 1.3: Frequencies of the imaginary parts of the first zero above $\frac{1}{2}$ in $L(\chi_D, s)$ with $10^{12} < |D| < 10^{12} + 2 \cdot 10^5$. The continue function above is the density of $\nu_1(Sp)$.

We let $\mathcal{F}_N$ be the subfamily of all the Dirichlet L-function $L(\chi_D, s)$ with $|D| \leq N$; now we can define

$$\nu_k(\mathcal{F}_N)[a, b] = \frac{\#\{L(\chi, s) \in \mathcal{F}_N \mid \tilde{\gamma}_\chi^{(k)} \in [a, b]\}}{\#\mathcal{F}_N}.$$

Katz, Sarnak [15], Odlyzko [22] , Rubinstein [24] and Hazelgrave made many numerical experiments and theoretical discoveries on this family of Dirichlet L-functions and they studied, in particular, what is the behaviour of $\nu_k(\mathcal{F}_N)$ as $N \to \infty$ trying to compare it with $\nu_k(G)$ for many different G.

The first important discovery is that (empirically) the G that best fits the limits of these measure is USp; therefore we can conjecture that

Conjecture 1.4.5.

$$\lim_{N \to \infty} \nu_j(\mathcal{F}_N) = \nu_j(Sp).$$

In particular Rubinstein's numerical experiments establish that for N of the size of 10^{12} the measures $\nu_j(\mathcal{F}_N)$ and $\nu_j(Sp)$ fit together very well. For example, in Figure 1.3 we can see the comparison between the distribution of $\nu_1(\mathcal{F}')$ and $\nu_1(Sp)$, where $\mathcal{F}'$ is the subfamily of functions with a conductor D such that $10^{12} < |D| < 10^{12} + 2 \cdot 10^5$.

Another important discovery is that the low-lying zeroes of this family repel the point $s = \frac{1}{2}$; this is referred to as the *Hazelgrave phenomenon*. As we remarked in Section 1.3, $G = USp$ is the only group, of the ones we studied, such that $\nu_1(G)$ vanishes in 0; this suggests that the Hazelgrave phenomenon is a manifestation of the group USp that we associated to this family.

Chapter 2

Zeroes of p-adic L-functions

In the mid 1950's Leopoldt established various p-adic analogues of the classical complex analytic class number formulas. He and Kubota introduced p-adic analogues of certain Dirichlet L-functions attached to cyclotomic extensions of $\mathbb{Q}$. Later, in the late 1960's Iwasawa discovered a close link between these p-adic functions of Kubota-Leopoldt and his work on towers of cyclotomic fields.

In this chapter we will introduce p-adic integration theory and Mazur's formulation of p-adic L-functions as Mellin transform. We want to follow this path because it turns out to be a more convenient basic definition than the previous formulation by Iwasawa. This is, also, the main reason why we proved the main properties of Dirichlet L-functions using Mellin transform in Chapter 1; we hope that the common elements in the complex constructions and in the p-adic ones will explain better the reasons behind certain proofs and patterns.

This construction is based on many results by Leopoldt so we will begin this chapter by introducing some of these about p-adic Measure Theory, studying how distributions, measures and integration work. In fact, we will see how the most natural way to construct p-adic L-functions is to create suitable p-adic measures on $\mathbb{Z}_p^\times$.

2.1 p-adic Analysis

In this section we introduce the formalism of the theory of p-adic analysis that we will use later when creating p-adic L-functions.

We consider L a complete field that is a finite extension of $\mathbb{Q}_p$ and we call $|.|_p$ the valuation on L that extends the p-adic valuation $|.|$ on $\mathbb{Q}_p$ where $|p| = p^{-1}$.

The natural starting point for speaking about p-adic Analysis is to introduce and to study the continuous functions f on $\mathbb{Z}_p$.

Definition 2.1.1. A function $f : \mathbb{Z}_p \to L$ is a continuous function if

$$\forall x \in \mathbb{Z}_p \; \forall \epsilon > 0 \; \exists \delta > 0 \mid |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

We want to study the special properties of continuity in these p-adic spaces and find a better way to express continuous functions. The first important tool that p-adic analysis offer us is the compactness of $\mathbb{Z}_p$. This tool suggest us to think about properties of continuous functions on compact subsets of $\mathbb{R}$; in particular we want to adapt the Heine-Cantor theorem for continuous functions defined on closed intervals $[a, b] \in \mathbb{R}$ by creating a p-adic analogue of it.

Definition 2.1.2. A function $f : \mathbb{Z}_p \to L$ is uniformly continuous if

$$\forall \epsilon > 0 \; \exists \delta > 0 \mid \forall x, y \in \mathbb{Z}_p \, , \; |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

The important aspect of this definition is that the δ we obtain fixing a value of ϵ is the same for every $x \in \mathbb{Z}_p$ and does not change from point to point.

Theorem 2.1.3 (Heine-Cantor). *If $f : \mathbb{Z}_p \to L$ is continuous, then it is also uniformly continuous.*

Proof. Fix $\epsilon > 0$. For every $x \in \mathbb{Z}_p$ let δ_x be the δ we obtain using the continuity definition. We can express $\mathbb{Z}_p$ as union of open balls as

$$\mathbb{Z}_p = \bigcup_{x \in \mathbb{Z}_p} B(x, \delta_x),$$

and we can obtain a finite subcovering using $\mathbb{Z}_p$ compactness. Let δ be the minimum of the δ_x in this finite subcovering. We notice that this δ satisfies the definition of uniform continuity, since taking y such that $|x - y| < \delta$ means that, thanks to the peculiar topology generated by the non-archimedean valuation, there exists a ball in the subcovering we created that contains both x and y; therefore $|f(x) - f(y)| < \epsilon$. □

This theorem allow us to consider smaller (dense) subsets in $\mathbb{Z}_p$ to define continuous function on $\mathbb{Z}_p$. This process is called p-adic interpolation and will be very useful to obtain approximations of p-adic L-functions starting from some special values of Dirichlet L-functions.

Lemma 2.1.4. *Let $f : A \to L$ be a uniformly continuous function where A is a dense subset of $\mathbb{Z}_p$ endowed with the subset topology. Then f uniquely extends to a continuous function $f : \mathbb{Z}_p \to L$.*

Proof. Let $s \in \mathbb{Z}_p$, then $s = \lim_n a_n$ with $\{a_n\}$ a Cauchy sequence with $a_n \in A$ that we can obtain using the density of A in $\mathbb{Z}_p$. We consider the function $f : \mathbb{Z}_p \to L$ where $f(s) := \lim_n f(a_n)$. This is a good definition: if there are two different sequences $\{a_n\}$ and $\{b_n\}$ that tends to s we can define $l_1 := \lim_n f(a_n)$ and $l_2 := \lim_n f(b_n)$; they exist thanks to uniform continuity of f (image of Cauchy sequences under uniform continuous functions is Cauchy and L is complete) and they are the same because

$$|l_1 - l_2| = |l_1 - f(a_n) + f(a_n) - f(b_n) + f(b_n) - l_2|$$

for any value of n, now thanks to valuation properties this is smaller than

$$|l_1 - f(a_n)| + |f(a_n) - f(b_n)| + |f(b_n) - l_2|$$

but the first and the last term tends to zero for $n \to \infty$ and the second one too, thanks to the continuity of f because

$$|a_n - b_n| = |a_n - s + s - b_n| \leq |a_n - s| + |s - b_n|$$

so $|a_n - b_n|$ tends to zero for $n \to \infty$; this allows us to say that $l_1 = l_2$.

We have obtained the function $f : \mathbb{Z}_p \to L$ which coincides with the given function f on A and it is continuous; this last claim can be easily shown by studying $|f(s_1) - f(s_2)|$ in the same way we did for $|l_1 - l_2|$. □

The second important tool we want to use is the non-archimedean behaviour of the valuation $|.|_L$. We will use this tool to prove a basic result about L-valued infinite series.

Remark 2.1.5. *The L-valued infinite series $\sum_n a_n$ converges in L if and only if $a_n \to 0$*

Proof. The convergence of the series always implies that the general term tends to zero, conversely the general term tending to 0 makes the sequence of partial sums $\{S_n\}$ a Cauchy sequence in L, because

$$|S_m - S_n| = |a_m + a_{m-1} + \cdots + a_{n+1}| \leq \max_{n+1 \leq i \leq m} |a_i| \to 0,$$

so the series converges in L by completeness. □

Using all the results we have just proved we can give a very interesting description for all continuous functions $f : \mathbb{Z}_p \to L$. Mahler proved an important structure theorem about $\mathcal{C}(\mathbb{Z}_p, L)$ that allows us to write continuous functions as infinite linear combinations of special polynomials; in other words, he found an infinite base of $\mathcal{C}(\mathbb{Z}_p, L)$ that consists of all the polynomials $\binom{x}{n}$.

Definition 2.1.6. For $x \in \mathbb{Z}_p$ and $n \geq 1$ let

$$\binom{x}{n} := \frac{x(x-1)\dots(x-n+1)}{n!}$$

and let $\binom{x}{0} := 1$.

Theorem 2.1.7. *For $n \geq 1$*

$$\binom{x}{n} \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p).$$

Proof. Since $\mathbb{Q}_p$ is a topological field, we know that polynomials are continuous functions since addition and multiplication are continuous. This implies that $f(x) := \binom{x}{n}$ is in $\mathcal{C}(\mathbb{Q}_p, \mathbb{Q}_p)$. We know that $\mathbb{N}$ is a dense subset of $\mathbb{Z}_p$ where for every $n \in \mathbb{N}$ we have that $|f(n)|_p \leq 1$. Since the absolute value is continuous we know that

$$|f(x)| = \lim_{i\to\infty} |f(n_i)| \leq 1.$$

This implies that for all $x \in \mathbb{Z}_p = \overline{\mathbb{N}}$ also $f(x) \in \mathbb{Z}_p$. This implies that $f \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$. $\square$

Theorem 2.1.8. *[2] Let $f : \mathbb{Z}_p \to L$ be a continuous function. There exists a unique sequence $\{a_n\}$ with $a_n \in L$ such that*

$$f(x) = \sum_{n\geq 0} a_n \binom{x}{n}, \quad \textit{with } \lim_{n\to\infty} a_n = 0;$$

moreover, we also know that for $n \in \mathbb{N}$ the value of a_n is

$$a_n := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k).$$

Proof. To prove this theorem we have to show that (1) the right hand side is a continuous function and that (2) it is exactly f.

We start the proof by defining the n-th finite difference operator

$$f^{[n]}(x) := \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k),$$

in particular, we notice that $a_n = f^{[n]}(0)$.

We will now prove many intermediate identities that will be useful to prove (1) and (2). The first one is

$$f^{[n]}(x) = \sum_{k=0}^{m} \binom{m}{k} f^{[n+k]}(x-m).$$

We know that, with $m \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{m} \binom{m}{k} f^{[k]}(x-m) &= \sum_{k=0}^{m} \binom{m}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x-m+j) \\ &= \sum_{j=0}^{m} (-1)^j f(x-m+j) \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{k}{j}\end{aligned}$$

Since

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{k}{j} = \begin{cases} 0 \text{ if } 0 \leq j \leq m-1, \\ (-1)^j \text{ if } j = m, \end{cases}$$

it follows that $f(x) = \sum_{k=0}^{m} \binom{m}{j} f^{[k]}(x-m)$ and then, applying the n-th difference operator, we obtain the desired identity.

This identity implies that

$$f^{[n]}(m) = \sum_{k=0}^{m} \binom{m}{k} f^{[n+k]}(0) = \sum_{k=0}^{m} \binom{m}{k} a_{n+k}.$$

We can obtain the following crucial identity by considering the definition of $f^{[n]}(m)$ and the identity we have just found

$$\sum_{k=0}^{m} \binom{m}{k} a_{n+k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k+m),$$

valid for every $m, n \in \mathbb{N}$.

Now that we have proved this main identity we can go back to proving (1) and (2), starting from (1).

We know that f is uniformly continuous. Let's fix $\epsilon = p^{-s}$ and take the corresponding $\delta_\epsilon = p^{-t}$, so we have that

$$|x-y| \le p^{-t} \Rightarrow |f(x) - f(y)| \le p^{-s}$$

Using the main identity with $m = p^t$ we find

$$a_{n+p^t} = -\sum_{k=1}^{p^t-1} \binom{p^t}{k} a_{n+k} + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left(f(k+p^t) - f(k) \right).$$

All the binomials in the first sum are divisible by p, in other words $p | \binom{p^t}{k}$ with $1 \le k \le p^t - 1$. Using the non-archimedean properties of the valuation we can say that

$$|a_{n+p^t}| \le \max_{1 \le k \le p^t-1} \{ \frac{1}{p^s}, \frac{1}{p} |a_{n+k}| \}.$$

Using $\mathbb{Z}_p$ compactness and the fact that f is uniformly continuous, we can say that f is bounded and without loss of generality (i.e. multiplying f for a suitable number) say that $|f(x)| \le 1$ and, therefore, $|a_n| \le 1$.

With this, we have $|a_n| \le \frac{1}{p}$ for $n \ge p^t$. We can iterate this reasoning by using the main identity on $m = 2p^t$ and using the improved bound on $|a_n|$ to obtain that $|a_n| \le \frac{1}{p^2}$ for $n \ge 2p^t$. Repeating this argument $s-1$ times we find that

$$|a_n| \le \frac{1}{p^s} \text{ for } n \ge sp^t,$$

this implies that $\lim_n a_n = 0$ and that the right hand side is continuous.

To prove the last point we notice that the two functions agrees on $\mathbb{N}$, a dense subset of $\mathbb{Z}_p$. This is immediate by using the main identity with $n = 0$ and $m \in \mathbb{N}$

$$\sum_{k=0}^{m} \binom{m}{k} a_{n+k} = f(m).$$

□

This alternative description of each continuous function is called the *Mahler expansion* of f and the a_n are called the *Mahler coefficients* of f. This theorem allow us to have a good description of all the functions in $\mathcal{C}(\mathbb{Z}_p, L)$.

Let's finish this section about p-adic analysis by giving a look at the set $\mathcal{C}(\mathbb{Z}_p, L)$ as a whole. The space of continuous functions $\mathcal{C}(\mathbb{Z}_p, L)$ is a L-vector space and is equipped with the supremum norm

$$||f|| := \sup_{x \in \mathbb{Z}_p} |f(x)|.$$

Since $\mathbb{Z}_p$ is compact we know that continuous functions are bounded; this implies that the supremum norm is finite for every continuous f.

2.2 p-adic Measure Theory

In the past section we have introduced many results about p-adic Analysis; now, it is time to focus on p-adic Measure Theory. As we have already said, these two topics are more related than one might think: we will use the theorems we proved to define distributions, measures and integrals; we will also build many tools to work with measures and modify them; this will be very important for our work because being able to construct suitable measures will be the key to define p-adic L-functions in the next section.

In this section we will prove many results that works for both $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$, therefore we fix G to be either $\mathbb{Z}_p$ or $\mathbb{Z}_p^\times$, only for this section, to avoid repetition.

We start this section by defining a p-adic analogue of the real step functions, used in the theory of Riemann integration for real analytic functions.

Definition 2.2.1. A function $f : G \to L$ is *locally constant* if for every $x \in G$ there exists U_x, an open neighbourhood of x, such that f is constant on U_x.

We denote with $LC(G, L)$ the L-vector space of all the locally constant functions.

Definition 2.2.2. A distribution μ is an element of the dual space $LC(G, L)^*$, i.e. a linear functional $\mu : LC(G, L) \to L$.

The space of all the distributions μ on G is denoted by $\mathrm{Dist}(G, L)$. The image of a locally constant function f trough μ is often denoted with the symbol

$$\int_G f \cdot \mu := \mu(f).$$

Now we are able to compute integrals of locally constant functions on G. We have described the locally constant functions as a p-adic analogue of real step functions; in the same way we obtain real Riemann integral on continuous functions as the limit of the integrals on real step functions, we would like to understand if knowing the values of integrals of locally constant functions can help us to define an integral for continuous functions. Answering to this question leads us to defining *measures*.

Definition 2.2.3. A measure μ is an element of the continuous dual of $\mathcal{C}(G, L)$, i.e. a linear functional $\mu : \mathcal{C}(G, L) \to L$ that is continuous with respect to the topology on $\mathcal{C}(G, L)$ induced by the supremum norm.

The image of continuous functions trough μ is often denoted with the symbol

$$\int_G f \cdot \mu := \mu(f);$$

also, we will often write $\mu(U)$, where U is an open subgroup of G, when referring to $\mu(1_U)$, where 1_U is the characteristic function of U in G, i.e. $f(u) = 1$ if $u \in U$ and $f(u) = 0$ otherwise.

We denote with $\mathrm{Meas}(G, L)$ the space of measures μ on G. Since any locally constant function is continuous, any measure is also a distribution and we have that $\mathrm{Meas}(G, L) \subset \mathrm{Dist}(G, L)$.

From now on we want to focus on studying $\mathrm{Meas}(G, \mathcal{O}_L)$, i.e. the space of the measures that are valued in $\mathcal{O}_L$. We start by giving an alternative description of it.

If we take f locally constant, there is some open subgroup H of G such that f can be viewed as a function on G/H; in this case we know that

$$\int_G f \cdot \mu = \sum_{[a] \in G/H} f(a)\mu(a + H).$$

Also, the integral of a continuous function f can be defined by considering it as the limit of locally constant functions. For this reason we can construct the following isomorphism;

Definition 2.2.4. Let R be a ring and A be a R-module. We define the *group algebra* $R[A]$ as the free R-module with generatores $[a]$ indexed by $a \in A$. It is easy to show that $R[A]$ has an R-algebra structure by considering the multiplication $[a_1] \cdot [a_2] := [a_1 a_2]$ for any $a_1, a_2 \in A$.

Lemma 2.2.5. *There is an isomorphism of $\mathcal{O}_L$-modules*

$$\mathrm{Meas}(G, \mathcal{O}_L) \cong \varprojlim_H \mathcal{O}_L[G/H],$$

where the limit is over all open subgroups of G.

Proof. Let μ be a measure and let H be an open subgroup of G. Then we can send μ to the element of $\varprojlim_H \mathcal{O}_L[G/H]$ given by

$$(\lambda_H) := \left(\sum_{[a] \in G/H} \mu(a+H)[a] \right)_H ;$$

this element belong to the projective limit thanks to the additivity property of μ.

Conversely, given $(\lambda_H) \in \varprojlim_H \mathcal{O}_L[G/H]$, where $\lambda_H = \sum_{[a]\in G/H} c_a[a]$ we can define the measure μ giving images for the open compact sets

$$\mu(a+H) = c_a.$$

Since λ_H are all compatible under natural projection maps we can say that μ is an additive functions, therefore μ is a distribution. Using the fact that $c_a \in \mathcal{O}_L$ it is also easy to show that μ is a measure.

Since the two maps we have constructed between $\mathrm{Meas}(G, \mathcal{O}_L)$ and $\varprojlim_H \mathcal{O}_L[G/H]$ are both $\mathcal{O}_L$-linear maps and their compositions are the identities of the two spaces we can say that they are isomorphisms. □

Another important way to determine a measure μ is to find the integral of all binomial polynomials $\int_G \binom{x}{n} \cdot \mu$; in this way we can use Mahler's theorem and determine μ uniquely. We can encode all these informations into a single power series in $\mathcal{O}_L[[T]]$ where the n-th coefficient is the integral of $\binom{x}{n}$.

Definition 2.2.6. Let $\mu \in \mathrm{Meas}(\mathbb{Z}_p, \mathcal{O}_L)$. We define its *Mahler transform* to be

$$\mathcal{A}_\mu(T) := \sum_{n \geq 0} \left(\int_{\mathbb{Z}_p} \binom{x}{n} \cdot \mu \right) T^n \in \mathcal{O}_L[[T]].$$

This can be seen as the generating series of the values of the integrals of binomial functions $\binom{x}{n}$. An equivalent way to define it is

$$\mathcal{A}_\mu := \int_{\mathbb{Z}_p} (1+T)^x \cdot \mu(x).$$

Theorem 2.2.7. *The Mahler transform gives an $\mathcal{O}_L$-modules isomorphism*

$$\mathrm{Meas}(\mathbb{Z}_p, \mathcal{O}_L) \cong \mathcal{O}_L[[T]].$$

Proof. We can define an inverse to the transform. Let take $g(T) = \sum_{n\geq 0} b_n T^n \in \mathcal{O}_L[[T]]$. Let $H \subset \mathbb{Z}_p$ be an open subgroup, for each $[a] \in \mathbb{Z}_p/H$ we can consider the characteristic function 1_{a+H}. We can use Mahler's theorem on this function to obtain that

$$1_{a+H}(x) = \sum_{n\geq 0} c_n^{[a]} \binom{x}{n},$$

with $c_n^{[a]} \in \mathcal{O}_L$. This means that we can also define

$$\mu_{[a]} := \sum_{n \geq 0} c_n^{[a]} b_n,$$

and

$$\mu_H := \sum_{[a] \in \mathbb{Z}_p/H} \mu_{[a]}[a] \in \mathcal{O}_L[G/H].$$

So we obtain the element (μ_H), we want to show that this is an element of $\varprojlim_H \mathcal{O}_L[G/H]$. We consider $J \subset H$ another open subgroup and study μ_H and μ_J; we notice that 1_{a+U} is the sum of all the cosets of V contained in $a + U$ and that the definition of $\mu_{[a]}$ is linear in $c_n^{[a]}$.

We conclude this proof noticing that the Mahler transform of (μ_H) is exactly $g(T)$.

□

In the next sections, as we have already mentioned, we will work with measures to construct specific ones that will allow us to obtain p-adic L-functions; the best way to do that is introducing first some important tools that will help us to manipulate measures. Using the Mahler transform we will also study how this operations behave on the associated power series.

The first operation we want to introduce is the *Multiplication by x*; let $\mu \in \mathrm{Meas}(\mathbb{Z}_p, \mathcal{O}_L)$, then

$$\int_{\mathbb{Z}_p} f(x) \cdot x\mu(x) := \int_{\mathbb{Z}_p} (xf(x)) \cdot \mu(x).$$

Let's now look at the behaviour of this operation through Mahler transform

Lemma 2.2.8. *Let's denote with ∂ the differential operator $(1+T)\frac{\mathrm{d}}{\mathrm{d}T}$. Then we have*

$$\mathcal{A}_{x\mu} = \partial \mathcal{A}_\mu$$

Proof.

$$x\binom{x}{n} = (x-n)\binom{x}{n} + n\binom{x}{n} = (n+1)\binom{x}{n+1} + n\binom{x}{n}.$$

This imply that

$$\int_{\mathbb{Z}_p} \binom{x}{n} \cdot x\mu = \int_{\mathbb{Z}_p} \left((n+1)\binom{x}{n+1} + n\binom{x}{n}\right) \cdot \mu.$$

□

This operation allow us to easily study the moments of a measure. A natural question about measures is the k-th moment of $\mu \in \mathrm{Meas}(\mathbb{Z}_p, \mathcal{O}_L)$, i.e. the integral $\int_{\mathbb{Z}_p} x^k \cdot \mu$. To answer to this question we have to notice that

$$\int_{\mathbb{Z}_p} 1 \cdot \mu = \mathcal{A}_\mu(0),$$

which is clearly implied from the construction of Mahler transform. Now, we can use this identity and iterate Lemma 2.2.8 to obtain

$$\int_{\mathbb{Z}_p} x^k \cdot \mu = \partial^k \mathcal{A}_\mu(0).$$

Another important operation is the restriction of a measure μ on $\mathbb{Z}_p$ to a measure $\mathrm{Res}_X(\mu)$ on an open compact subset $X \subset \mathbb{Z}_p$. This is defined as

$$\int_X f \cdot \mathrm{Res}_X(\mu) := \int_{\mathbb{Z}_p} (f 1_X) \cdot \mu,$$

where 1_X is the characteristic function of X on $\mathbb{Z}_p$, i.e. $f(x) = 1$ if $x \in X$ and $f(x) = 0$ otherwise. Let's study two specific cases of X; the first case is the one where $X = a + p^n\mathbb{Z}_p$, this characteristic function can be written in an explicit way as

$$1_{a+p^n\mathbb{Z}_p}(x) = \frac{1}{p^n} \sum_{\zeta^{p^n}=1} \zeta^{x-a},$$

in this case we can also compute the Mahler transform of this measure we have just defined

$$\mathcal{A}_{\mathrm{Res}_{1_{a+p^n\mathbb{Z}_p}}(\mu)}(T) = \frac{1}{p^n}\sum_{\zeta^{p^n}=1} \zeta^{-a}\mathcal{A}_\mu\left((1+T)\zeta - 1\right).$$

The second case is $X = \mathbb{Z}_p^\times$. This is can easily obtained from the last formula subtracting to μ the measure $\mathrm{Res}_{p\mathbb{Z}_p}(\mu)$, obtaining the following Mahler transform

$$\mathcal{A}_{\mathrm{Res}_{\mathbb{Z}_p^\times}}(\mu)(T) = \mathcal{A}_\mu(T) - \frac{1}{p}\sum_{\zeta^p=1} \mathcal{A}_\mu\left((1+T)\zeta - 1\right).$$

We will now define the operator ϕ that acts on measures and on formal series as

$$\int_{\mathbb{Z}_p} f(x)\cdot\phi(x) := \int_{\mathbb{Z}_p} f(px)\cdot\mu(x),$$

and

$$\phi(\mathcal{A}_\mu)(T) = \mathcal{A}_{\phi(\mu)}(T) := \mathcal{A}_\mu\left((1+T)^p - 1\right).$$

Finally, we can also define an "inverse" operator for ϕ. This operator is called ψ and is defined as

$$\int_{\mathbb{Z}_p} f(x)\cdot\psi(\mu) := \int_{p\mathbb{Z}_p} f(p^{-1}x)\cdot\mu(x);$$

this time is not immediate to see how ψ behave with the Mahler transform but we can notice something interesting about the relation between ϕ and ψ,

$$\psi\circ\phi(\mu) = \mu;\quad \phi\circ\psi(\mu) = \mathrm{Res}_{p\mathbb{Z}_p}(\mu),$$

this allow us to express the restriction to $\mathbb{Z}_p^\times$ in an alternative way as

$$\mathrm{Res}_{\mathbb{Z}_p^\times}(\mu) = (1-\phi\circ\psi)(\mu).$$

Now we have a clear view of what measures are and how they behave, in particular how they can be expressed as formal series in $\mathcal{O}_L[[T]]$, and which are the main operations we can use on them. We are now ready to proceed with the next section where we will define p-adic L-functions starting from suitable measures that we will construct using the tools we just introduced.

2.3 p-adic L-functions

Now it is the moment to use all the tools we have created in the last sections to introduce p-adic L-functions.

In the last chapters we have computed the special values of Dirichlet L-functions on negative integers showing the link between them and generalized Bernoulli numbers; from the definition of these numbers is easy to see that all these special values are rational. It can be proved that these special values are p-adically continuous. This give us the main idea behind the new class of functions we are introducing: trying to obtain a p-adic continuous function that interpolates the values $L(\chi, k)$ for $k \leq -1$. This would allow us to have a strong link between the complex and the p-adic L-function, since they share many values, and would allow us to study a new object with the new tools of p-adic analysis.

To associate a p-adic L-function to each Dirichlet character χ we have to introduce a useful fact about these characters.

Lemma 2.3.1. *[9] Given a primitive Dirichlet character χ modulo N with $N = Dp^k$ and where D and p are coprime. We can express χ as*

$$\chi = \nu\cdot\theta,$$

where ν and θ are two Dirichlet characters modulo D and p^k.

This simple fact will be really important in our study because it will be easier to start by constructing an associated measure only for Dirichlet characters modulo D (where $p \not| D$) and then generalize this for every Dirichlet character using the last lemma.

We start with this study by stating the following theorem.

Theorem 2.3.2. *Let $D > 1$ be any integer coprime to p, and let χ denote a primitive Dirichlet character of conductor D. There exists a unique measure $\mu_\chi \in \mathrm{Meas}(\mathbb{Z}_p^\times, \mathcal{O}_L)$ such that for all $k > 0$, we have*

$$\int_{\mathbb{Z}_p^\times} x^k \cdot \mu_\chi = (1 - \chi(p)p^k)L(\chi, -k);$$

where $L(\chi, s)$ is the Dirichlet L-function of character χ.

The next pages will be the proof of this result.

Recall the result from the last chapter: we can write the Dirichlet L-function $L(\chi, s)$ as the Mellin transform $L(f_\chi, s)$ with

$$f_\chi(t) = \frac{-1}{G(\chi^{-1})} \sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \frac{\chi^{-1}(a)}{e^t \epsilon^a - 1},$$

where ϵ is a primitive D-th root of unity, contained in a suitable extension of L. (We introduced a scaling factor of -1 in f_χ to simplify this proof).

We want to rewrite $f_\chi(t)$ as $F_\chi(T)$ with the substitution $e^t - 1 = T$. This allow us to rewrite the function as

$$F_\chi(T) = \frac{-1}{G(\chi^{-1})} \sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \frac{\chi^{-1}(a)}{(T+1)\epsilon^a - 1} =^* \frac{-1}{G(\chi^{-1})} \sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \chi^{-1}(a) \sum_{k \geq 0} \frac{\epsilon^{ka}}{(\epsilon^a - 1)^{k+1}} T^k,$$

where the starred equality (*) is given by the following algebraic identity

$$\frac{1}{(T+1)\epsilon^a - 1} = \frac{1}{\epsilon^a - 1} \cdot \frac{1}{1 + \frac{T\epsilon^a}{\epsilon^a - 1}} = \frac{1}{\epsilon^a - 1} \sum_{k \geq 0} \left(\frac{\epsilon^a T}{\epsilon^a - 1} \right)^k = \sum_{k \geq 0} \frac{\epsilon^{ka}}{(\epsilon^a - 1)^{k+1}} T^k.$$

With this change of variable we notice that the derivative $\mathrm{d}/\mathrm{d}t$ becomes the operator $\partial = (1+T)\mathrm{d}/\mathrm{d}T$; in particular we have $f_\chi^{(k)}(0) = (\partial^k F_\chi)(0)$.

Moreover we notice that $F_\chi(T)$ is an element of $\mathcal{O}_L[[T]]$, with L sufficiently large finite complete extension of $\mathbb{Q}_p$ that contains ϵ; this happens because

i. We know that $G(\chi^{-1})$ is a p-adic unit; we can tell this thanks to a known property of Gauss sums is that $G(\chi)G(\chi^{-1}) = \chi(-1)D \in \mathbb{Z}_p^\times$. [9]

ii. We also know that $\epsilon^a - 1 \in \mathcal{O}_L^\times$. This happens because, since $\chi^{-1}(a) \neq 0$, then b is coprime with p; this implies that ϵ^b is a root of unity of order coprime with p and distinct from 1.

Since $F_\chi(T) \in \mathcal{O}_L[[T]]$, there is a measure $\mu_\chi \in \mathrm{Meas}(\mathbb{Z}_p, \mathcal{O}_L)$ such that has Mahler transform equal to $F_\chi(T)$.

Using all the results we have just proved we can see that this new measure μ_χ contains many informations about the Dirichlet L-function $L(\chi, s)$; in fact, we can compute the k-th moment of this measure and notice that

$$\int_{\mathbb{Z}_p} x^k \cdot \mu_\chi = (\partial^k F_\chi)(0) = f_\chi^{(k)}(0) = (-1)^{k+1} \frac{B_{k,\chi}}{k} = L(\chi, -k).$$

The measure we want to obtain is the restriction of μ_χ to $\mathbb{Z}_p^\times$, we can use the properties of ϕ and ψ to easily restrict it.

Proposition 2.3.3. *We have that $\psi(F_\chi) = \chi(p)F_\chi$. Moreover, for every $k \geq 0$, it is true that*

$$\int_{\mathbb{Z}_p^\times} x^k \cdot \mu_\chi = (1 - \chi(p)p^k)L(\chi, -k).$$

Proof. We start the proof by studying

$$\mathcal{A}_{\mathrm{Res}_{\mathbb{Z}_p^\times}(\mu_\chi)}(T) = \frac{1}{p}\sum_{\zeta^p=1}\frac{1}{(1+T)\zeta\epsilon^a-1} = \frac{-1}{p}\sum_{\zeta^p=1}\sum_{n\geq 0}(1+T)^n\epsilon^{na}\zeta^n =$$

where in the second equality we expanded summands as geometric series

$$= \frac{-1}{p}\sum_{n\geq 0}(1+T)^n\epsilon^{na}\sum_{\zeta^p=1}\zeta^n =^{*} -\sum_{n\geq 0}(1+T)^{pn}\epsilon^{pan} = \frac{1}{(1+T)^p\epsilon^{pa}-1};$$

the starred inequality is justified by the behaviour of the sums of p-th roots of unity

$$\sum_{\zeta^p=1}\zeta^n = \begin{cases} p & \text{if } p|n \\ 0 & \text{if } p\not| n \end{cases}.$$

It follows that

$$(\phi\circ\psi)(F_\chi) = \frac{-1}{pG(\chi^{-1})}\sum_{\zeta^p=1}\sum_{a\in(\mathbb{Z}/D\mathbb{Z})^\times}\frac{\chi(a)^{-1}}{(1+T)\zeta\epsilon^a-1} =$$

$$= \frac{-1}{G(\chi^{-1})}\sum_{a\in(\mathbb{Z}/D\mathbb{Z})^\times}\frac{\chi(a)^{-1}}{(1+T)^p\epsilon^{pa}-1} = \chi(p)\phi(F_\chi).$$

We obtain now the first claim thanks to injectivity of ϕ. For the second claim we note that

$$\mathrm{Res}_{\mathbb{Z}_p^\times}(\mu_\chi) = (1-\phi\circ\psi)(\mu_\chi) = \mu_\chi - \chi(p)\phi(\mu_\chi);$$

this implies

$$\int_{\mathbb{Z}_p} x^k\cdot\phi(\mu_\chi) = p^k\int_{\mathbb{Z}_p} x^k\cdot\mu_\chi = p^kL(\chi,-k);$$

We can conclude saying that

$$\int_{\mathbb{Z}_p^\times} x^k\cdot\mu_\chi = \int_{\mathbb{Z}_p} x^k\cdot\mu - \int_{\mathbb{Z}_p} x^k\cdot(\phi\circ\psi)\mu_\chi = L(\chi,-k)-\chi(p)p^kL(\chi,-k).$$

□

This conclude the proof of Theorem 2.3.2.

Now, as we said at the beginning of the section we will try to generalize this construction to every Dirichlet character. To do this we will prove this extended version of Theorem 2.3.2.

Theorem 2.3.4. *Let ν be a Dirichlet character modulo D, with D coprime with p and let $\mu_\nu \in \mathrm{Meas}(\mathbb{Z}_p^\times, \mathcal{O}_L)$ be its associated measure. Then, for every primitive Dirichlet character θ modulo p^n, $n \geq 0$ and for all $k > 0$, we have*

$$\int_{\mathbb{Z}_p^\times}\theta(x)x^k\cdot\mu_\nu = (1-\chi(p)p^k)L(\chi,-k),$$

where $\chi := \theta\nu$. Moreover, it exists a measure $\mu_\chi \in \mathrm{Meas}(\mathbb{Z}_p^\times, \mathcal{O}_L)$ such that

$$\int_{\mathbb{Z}_p^\times} x^k\cdot\mu_\chi = (1-\chi(p)p^k)L(\chi,-k),$$

It is implicit that in this theorem we are considering a complete field L finite extension of $\mathbb{Q}_p$ that contains the values of ν.

Proof. To show this theorem we start with the following quantity.

$$\int_{\mathbb{Z}_p}\theta(x)(1+T)^x\cdot\mu_\nu$$

In this quantity, μ_ν is the measure associated to the ν defined on all $\mathbb{Z}_p$.

$$\begin{aligned}
\int_{\mathbb{Z}_p} \theta(x)(1+T)^x \cdot \mu_\nu &= \sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \theta(a) \int_{a+p^n\mathbb{Z}_p} (1+T)^x \cdot \mu_\nu \\
&= \sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \theta(a) \left(\frac{1}{p^n} \sum_{\zeta^{p^n}=1} \zeta^{-1} \mathcal{A}_\nu((1+T)\zeta-1) \right) \\
&= \sum_{\zeta^{p^n}=1} \mathcal{A}_\nu((1+T)\zeta-1) \left(\frac{1}{p^n} \sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \theta(a)\zeta^{-a} \right).
\end{aligned}$$

If we write $\zeta = e^{2i\pi\frac{c}{p^n}}$ we can see that the factor in the parenthesis on the right is a generalized Gauss sum divided by p^n. This allow us to rewrite the parenthesis as

$$\left(\frac{1}{p^n} \sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \theta(a)\zeta^{-a} \right) = \frac{1}{p^n} G(c,\theta) = \frac{1}{p^n}\theta^{-1}(c)G(\theta) = \frac{\theta^{-1}(c)}{G(\theta^{-1})};$$

where the first equality is a known property of Gauss sums [9] and the second equality is given by the formula $G(\theta)G(\theta^{-1}) = \theta(-1)p^n$.

$$\int_{\mathbb{Z}_p} \theta(x)(1+T)^x \cdot \mu_\nu = \frac{1}{G(\theta^{-1})} \sum_{a\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \theta^{-1}(a)\mathcal{A}_\mu((1+T)\zeta^a-1)$$

We have already proved that

$$\mathcal{A}_\nu = \frac{-1}{G(\nu^{-1})} \sum_{b\in(\mathbb{Z}/D\mathbb{Z})^\times} \frac{\nu^{-1}(b)}{(1+T)\epsilon^b - 1}$$

we can rewrite the main quantity as

$$\int_{\mathbb{Z}_p} \theta(x)(1+T)^x \cdot \mu_\nu = \frac{-1}{G(\nu^{-1})G(\theta^{-1})} \sum_{b\in(\mathbb{Z}/D\mathbb{Z})^\times} \sum_{c\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \frac{\nu^{-1}(b)\theta^{-1}(c)}{(1+T)\epsilon^b\zeta^c - 1}.$$

To better rewrite the right hand side we want to change the double sum in a common sum modulo Dp^n, in order to do this we will rewrite the product $\epsilon^b\zeta^c$ as η^a; we will rewrite $\nu^{-1}(b)\theta^{-1}(c)$ as $\nu^{-1}(p^n)\theta^{-1}(D)\nu^{-1}(b)\nu^{-1}(c)$ and we will rewrite

$$\begin{aligned}
G((\nu\theta)^{-1}) &= \sum_{a\in(\mathbb{Z}/Dp^n\mathbb{Z})^\times} \nu\theta^{-1}(a)\eta^a \\
&= \nu^{-1}(p^n)\theta^{-1}(D) \left(\sum_{b\in(\mathbb{Z}/D\mathbb{Z})^\times} \nu^{-1}(b)\zeta^b \right) \left(\sum_{c\in(\mathbb{Z}/p^n\mathbb{Z})^\times} \theta^{-1}(c)\epsilon^c \right) \\
&= \nu^{-1}(p^k)\theta^{-1}(D)G(\nu^{-1})G(\theta^{-1}).
\end{aligned}$$

With this change in the main sum we can rewrite the main quantity as

$$\begin{aligned}
\int_{\mathbb{Z}_p} \theta(x)(1+T)^x \cdot \mu_\nu &= \frac{-1}{G(\nu\theta^{-1})} \sum_{a\in(\mathbb{Z}/Dp^n\mathbb{Z})^\times} \frac{\nu\theta^{-1}(a)}{(1+T)\eta^a - 1} \\
&= \frac{-1}{G(\chi^{-1})} \sum_{a\in(\mathbb{Z}/Dp^n\mathbb{Z})^\times} \frac{\chi^{-1}(a)}{(1+T)\eta^a - 1}.
\end{aligned}$$

Now that we have obtained this quantity we can define the measure μ_χ as the measure on $\mathbb{Z}_p$ that has $\frac{-1}{G(\chi^{-1})}\sum_{a\in(\mathbb{Z}/Dp^n\mathbb{Z})^\times}\frac{\chi^{-1}(a)}{(1+T)\eta^a-1}$ as its Mahler transform.

To finish this proof we have to study the moments of the restriction to $\mathbb{Z}_p^\times$ of this measure. We have already done this computation in Proposition 2.3.3.

□

We can finally introduce the definition of p-adic L-function for a Dirichlet character χ.

Definition 2.3.5. Given a primitive Dirichlet character χ, we will call the associated measure μ_χ the *p-adic L-function of χ*.

This definition can result a little obscure for the reader; in particular it is probably not clear why are we referring to a measure with the name "function", how can we interpret this measure in such a way? The main idea we had at the start of the section was to obtain a function on $\mathbb{Z}_p$ that interpolates all the values $L(\chi,-k)$ for $k\geq 0$ and a fixed χ; the measure μ_χ we constructed contains all the information we need to work. We won't be able to construct such a function but we will obtain something really close using the *Mellin transform*.

To define it we need to recall a couple of facts about $\mathbb{Z}_p$.

Definition 2.3.6. Since we are assuming p to be odd we have a composition $\mathbb{Z}_p^\times \cong \mu_{p-1}\times(1+p\mathbb{Z}_p)$. Let

$$\omega:\mathbb{Z}_p^\times\to\mu_{p-1},$$

$$\langle\cdot\rangle:\mathbb{Z}_p^\times\to 1+p\mathbb{Z}_p,$$

where ω is the Teichmuller lift of the reduction modulo p of x denoting the projection to the first factor, while $\langle x\rangle=\omega^{-1}(x)x$ is the projection to the second factor. In fact we know that if $x\in\mathbb{Z}_p^\times$ we have that $x=\omega(x)\langle x\rangle$.

Definition 2.3.7. The *p-adic exponential* is defined by

$$\exp_p(x):=\sum_{i=0}^{\infty}\frac{x^i}{i!}.$$

The *p-adic logarithm* is defined by

$$\log_p(1+x):=\sum_{i=1}^{\infty}(-1)^{i+1}\frac{x^i}{i}.$$

The *p- adic power function* is defined as

$$\langle a\rangle^s:=\exp_p(s\cdot\log_p(\langle a\rangle)).$$

With these functions we can define the *Mellin transform* of a measure.

Definition 2.3.8. For any measure $\mu\in\mathrm{Meas}(\mathbb{Z}_p^\times,\mathcal{O}_L)$ and $i\in\mathbb{Z}$ we define the *Mellin transform* of μ at i as the function with $s\in\mathbb{Z}_p$ such that

$$\mathrm{Mel}_{\mu,i}(s)=\int_{\mathbb{Z}_p^\times}\omega^i(x)\langle x\rangle^s\cdot\mu(x).$$

From this definition we can notice that the Mellin transform of a measure μ consists of $p-1$ different functions on $\mathbb{Z}_p$.

This transform allow us to define p-adic L-function in the following way.

Definition 2.3.9 (p-adic L-function)**.** Given χ a Dirichlet character, we define the p-adic L-function associated to χ at i as

$$L_{p,i}(\chi,s):=\mathrm{Mel}_{\mu_\chi,i}(s).$$

We generally say that these functions are the $p-1$ *branches* of the p-adic L-function associated to χ. Each one of these functions does not interpolate all the values of $L(\chi, -k)$ for $k \geq 0$ but the i-th function interpolate all the values of $L(\chi, -k)$ for $k \equiv i \pmod{p-1}$ and $k \geq 0$. This means that, even if we were not able to find a unique function, we managed successfully to transfer all the information in the special values at negative integers of $L(\chi, s)$ to the p-adic setting. This is an immediate corollary of the study we have completed about the moments of the measure μ_χ when restricted to $\mathbb{Z}_p^\times$ in Theorem 2.3.2.

Corollary 2.3.10. *For all $k \in \mathbb{N}$ such that $-k \equiv i \pmod{p-1}$, we have that*

$$L_{p,i}(\chi, -k) = (1 - \chi(p)p^k)L(\chi, -k).$$

Proof. For $-k \equiv i \pmod{p-1}$ we know that $\omega^i(x)\langle x\rangle^{-k} = (\omega(x)\langle x\rangle)^{-k} = x^{-k}$. This implies that

$$\mathrm{Mel}_{\mu_\chi, i}(-k) = \int_{\mathbb{Z}_p^\times} \omega^i(x)\langle x\rangle^{-k} \cdot \mu_\chi(x) = \int_{\mathbb{Z}_p^\times} x^{-k} \cdot \mu_\chi(x).$$

□

The main reason we created $p-1$ different functions instead of a single one should not be seen as a weakness of our path but as a strength. In fact, not having a single function is more natural in the p-adic setting and we were able to adapt to this peculiar behaviour by changing our approach to the topic.

Let's explain more the last passage. In the complex setting we know that the domain of a Dirichlet L-function associated to a primitive Dirichlet character is $\mathbb{C}$. We can identify $\mathbb{C}$ with the set of continuous homomorphisms

$$\mathrm{Hom}_{\mathrm{cts}}(\mathbb{R}_{>0}^\times, \mathbb{C}^\times),$$

considering s as the power function $x \mapsto x^s$. This may seems an unusual identification for $\mathbb{C}$ but after expressing $L(\chi, s)$ as a Mellin transform in 1.1.12 this description is much more natural.

If we repeat the same reasoning in the p-adic setting we notice that we expressed $L_{p,i}(\chi, s)$ as a Mellin transform so it make sense to consider the domain $\mathbb{Z}_p$ as

$$\mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{C}^\times);$$

since $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ we can see it as the disjoint union of $p-1$ sets; this makes defining $p-1$ different functions much more natural that trying to define a single one!

We will now focus on the zeroes of $L_{p,i}(\chi, s)$. In the last chapter we introduced many results and conjectures about the zeroes of Dirichlet L-functions $L(\chi, s)$, explaining also why studying these zeroes is so important. One of the reasons that make p-adic L-functions interesting is to face similar problems to the ones associated to Dirichlet L-functions by using p-adic tools.

We also have slightly different questions about the zeroes that in the Dirichlet L-functions we have already proved or do not have an interesting analogous but that are interesting in the p-adic environment. For example we have clear results about the infinity, even about the density, of zeroes in Dirichlet L-functions but we still don't know if the number of zeroes of $L_{p,i}(\chi, s)$ are finite or not. We will try to study the valuation of the zeroes of $L_{p,i}(\chi, s)$ and we will try to repeat the studies about low-lying zeroes adapting the distributions ν_k and the families of L-functions.

We will now introduce some of the new tools that will us to better describe the zeroes of $L_{p,i}(\chi, s)$ and in general its behaviour.

Theorem 2.3.11. *There is a unique $g_\chi \in \overline{\mathbb{Z}_p}[[T]]$ such that*

$$g_\chi((1+p)^{-s} - 1) = L_{p,i}(\chi, s).$$

You can find the proof of this theorem in [20], Theorem 7.10.

This theorem has many different applications; since we want to use it to study the zeroes of $L_{p,i}(\chi, s)$ we can introduce two other important results that will synergize with this description of $L_{p,i}(\chi, s)$ as a formal series and will allow us to obtain a practical way to study the number and the valuation of the zeroes of each p-adic L-function.

Let L be a complete field that is a finite extension of $\mathbb{Q}_p$. We call $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_L$ and let π be a generator of $\mathfrak{m}$.

Definition 2.3.12. Let $P(T)$ be a polynomial in $\mathcal{O}_L[T]$. We say that $P(T)$ is *distinguished* if

$$P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0,$$

with $a_i \in \mathfrak{m}$ for $i \in \{0, 1, \ldots, n-1\}$.

Theorem 2.3.13 (p-adic Weierstrass Preparation Theorem). *Let*

$$f(T) = \sum_{i=0}^{\infty} a_i T^i \in \mathcal{O}_L[[T]],$$

then f can be uniquely written as

$$f(T) = \pi^{\mu} P(T) Q(T),$$

where $\mu \in \mathbb{N}$, $Q(T) \in \mathcal{O}_L[[T]]$ is a unit and $P(T)$ is a distinguished polynomial.

Moreover, if $\mu = 0$ then there exists $n \geq 0$ such that $a_i \in \mathfrak{m}$ for $i \in \{0, \ldots, n-1\}$ but $a_n \notin \mathfrak{m}$. This implies that we can write f as

$$f(T) = P(T)Q(T),$$

with where the degree of $P(T)$ is exactly n.

The p-adic Weierstrass Preparation Theorem implies that the number of zeroes of any non-zero formal $\mathcal{O}_L$-series is finite. This happens because $Q(T)$ is a unit in $\mathcal{O}_L[[T]]$, therefore it does not have any zero. This means that the zeroes of $f(T)$ are exactly the zeroes of $P(T)$.

Using this theorem for the study of $L_{p,i}(\chi, s)$ will be really effective also thanks to the following lemma of Ferrero and Washington [10].

Lemma 2.3.14. *Let χ be a Dirichlet character and $L_{p,i}(\chi, s)$ the associated p-adic L-function at i. Then, in the expression obtained from Theorem 2.3.13,*

$$\mu = 0.$$

In the cases where $\mu = 0$, we can simply observe the smaller coefficients of the series and find the first one that is not in $\mathfrak{m}$ to obtain n, the number of zeroes of the series. We define

$$\lambda := deg(P(T)) = \min\{i \mid a_i \text{ is a unit in } \mathcal{O}_L\}.$$

We have answered to the first important question about the zeroes of p-adic L-functions proving their finiteness. This prevent us from having a clear analogue of questions about high lying zeroes, since any brunch $L_{p,i}(\chi, s)$ only have finite. On the other hand, this allow us to create a new question that does not have a complex counterpart: the number of zeroes in each $L_{p,i}(\chi, s)$; Studying the value λ_χ and their distribution will be a central point in the next chapter.

The main other property of zeroes that we have not studied yet is the order. This bring us to the introduction of the last important tool of this section: the *Newton Polygon*.

Newton polygons are really useful for studying zeroes of power series over complete non-archimedean fields. In particular, they are able to give information about the number and about the p-adic valuations of zeroes of a formal series, knowing only their coefficients.

Let f be

$$f = a_0 + a_1 x + a_2 x^2 + \cdots \in L[[x]],$$

the Newton polygon associated to f is $\mathrm{NP}(f)$ and is defined as the lower convex hull of

$$\mathcal{S} = \{(i, \mathrm{ord}(a_i)) \mid i \in \mathbb{N}\};$$

the convex hull of a set of points is the union of all the linear segments connecting two points in the set that are strictly lower than any other point in the set. Every segment in this hull is called a *slope* and we define its *multiplicity* as the length of the projection of this segment on the horizontal axis.

We will now introduce a theorem that shows the very useful relation between the p-adic valuations of the roots of $f(x)$ and the slopes of $\mathrm{NP}(f)$.

Theorem 2.3.15. *[5] Let $f(x) \in L[[x]]$ and suppose λ is a slope of $\mathrm{NP}(f)$ of finite multiplicity m_λ. Then f has precisely m_λ roots α_i of order $\mathrm{ord}(\alpha_i) = -\lambda$ and we have a factorisation*

$$f(x) = P(x)Q(x), \quad P(x) = \prod_{i=1}^{m_\lambda}(x - \alpha_i),$$

where $Q(x) \in L[[x]]$ is such that $\mathrm{NP}(Q)$ does not have a side of slope λ.

When using this theorem to study $L_{p,i}(\chi, s)$ we can use the results we have already mentioned to obtain even more information. For example, we know that the set of the finite slopes will be finite; this happens because the sum of the multiplicity of all the finite slopes otherwise would be infinite while the number of zeroes of $L_{p,i}(\chi, s)$ is finite. This means that, using the slopes of the Newton Polygon, we can obtain informations about the order of the zeroes just by studying the orders of the smaller coefficients of f until we find one that has order zero.

Chapter 3

The Ellenberg-Jain-Venkatesh conjecture

In this chapter we will introduce the Ellenberg-Jain-Venkatesh conjecture about the family of p-adic L-functions associated to imaginary quadratic fields. This conjecture avoids the case $p = 2$ because it behaves differently and it creates statistics that do not align with all the ones related to odd primes. For the rest of the chapter we will only consider odd primes.

We have already showed how to link quadratic characters to quadratic number fields in chapter 1. We have also studied an extended version of this family for Dirichlet L-functions; we have explained what is the conjecture in the complex setting and what are the empirical facts that Odlyzko, Hazelgrave and others observed with their numerical experiments.

The main focus of the conjecture is about the distribution of the λ-invariants of $L_{p,0}(\chi, s)$.

Conjecture 3.0.1. *[6] Amongst the quadratic characters χ linked to imaginary quadratic fields K in which p does not split, the proportion with $\lambda(\chi) = r$ is*

$$p^{-r} \prod_{t>r} (1 - p^{-t}),$$

where $\lambda(\chi)$ denotes the λ-invariant of $L_{p,0}(\chi, s)$.

In particular, this also implies that the λ-invariant would be unbounded in this family.

The aim of this conjecture is to study the zeroes of the functions in this family modelling them with matrices. The λ-invariant, as we have already said, is a completely new element in the study of L-functions that we have not faced in the complex setting. If we want to study this quantity we have to change also our algebraic approach and to use different matrix groups and study different aspects of them.

To understand the reasons behind the conjecture and the peculiar quantity that it mentions, we have to study p-adic matrix groups and to introduce a new approach to low lying zeroes; in fact, one of the key ideas in this conjecture is considering all the zeroes in $L_{p,0}(\chi, s)$ as low lying zeroes.

3.1 p-adic Random Matrix Models

We will now introduce new matrix groups, new tools and we will be obtaining new distributions that will (empirically) match the zeroes distribution of the family that we are studying.

We will analyse two matrix groups: $\mathrm{GL}_n(\mathbb{Z}_p)$ and $\mathrm{GSp}_{2n}(\mathbb{Z}_p)$.

Definition 3.1.1. Let ω be the standard nondegenerate symplectic form on $\mathbb{Z}_p^{2n}$:

$$\omega(x, y) = \sum_{1 \le i \le n} (x_i y_{i+n} - x_{i+n} y_i),$$

where $x = (x_1, \dots, x_{2n})$ and $y = (y_1, \dots, y_{2n})$ are two elements of $\mathbb{Z}_p^{2n}$. We can now define, for each $\alpha \in \mathbb{Z}_p^\times$, the group

$$\mathrm{GSp}_{\alpha,2n}(\mathbb{Z}_p) := \{g \in \mathrm{GL}_{2n}(\mathbb{Z}_p) \mid \omega(gx, gy) = \alpha\omega(x, y)\}.$$

The generalized symplectic group $\mathrm{GSp}_{2n}(\mathbb{Z}_p)$ is defined as

$$\bigcup_{\alpha \in \mathbb{Z}_p^\times} \mathrm{GSp}_{\alpha,2n}(\mathbb{Z}_p),$$

and each $\mathrm{GSp}_{\alpha,2n}(\mathbb{Z}_p)$ is a coset.

For the rest of the chapter we will assume that $\alpha \in \mathbb{Z}_p^\times$ does not reduce to $1 \pmod p$ when talking about cosets of the symplectic group; we will do this because considering also these α changes the distributions we are going to study, as can be seen in [12].

When studying these groups and their associated distributions is important to mention the measures that we are considering. In $\mathrm{GL}_n(\mathbb{Z}_p)$ we are considering the Haar probability measure that is the limit of the normalized counting measures on each $GL_n(\mathbb{Z}/p^r\mathbb{Z})$ for $r \to \infty$. In $\mathrm{GSp}_{\alpha,2n}(\mathbb{Z}_p)$ we are considering the unique probability measure invariant under $\mathrm{Sp}_{2n}(\mathbb{Z}_p)$. We will refer to both measures as μ.

We want to introduce P_A, the *associated polynomial* to a matrix A; our goal is to investigate the statistics of P_A where A is a random matrix chosen in $\mathrm{GL}_n(\mathbb{Z}_p)$ or in a coset of the symplectic group. To investigate the statistics of P_A we will study the direct sum decompositions of $\mathbb{Z}_p^n = X \oplus Y$, where $A - 1$ acts nilpotently on X and invertibly on Y. Let's explain this more in detail.

Definition 3.1.2. Let $V = \mathbb{Z}_p^n$, a free finite rank $\mathbb{Z}_p$-module and $A \in \mathrm{GL}(V)$. We can set

$$Z := \{v \in V \mid (A-1)^N v \xrightarrow{N\to\infty} 0\},$$

$$P_A(T) := det(((1+T)\cdot \mathrm{id} - A)_{|Z}) \in \mathbb{Z}_p[T],$$

we will refer to $P_A(T)$ as the *associated polynomial* to A.

We can give a better description of the construction of this polynomial explaining the defining formula. We can obtain $P_A(T)$ by considering the characteristic polynomial of A, discarding all the roots α of $f_A(T)$ such that $\alpha - 1$ is a unit and then applying a change of variable. As we can see, in this definition we discard all the roots that are not p-adically close to 1; this matches the idea that we want to study low-lying zeroes distributions. Moreover, we can already notice a first property of P_A.

Lemma 3.1.3. *Let $A \in \mathrm{GL}_n(\mathbb{Z}_p)$. Then $P_A(T)$ is distinguished.*

This lemma is an immediate consequence of the fact that a matrix $N \in \mathrm{GL}_n(\mathbb{Z}_p)$ such that its reduction $\tilde{N} \in \mathrm{GL}_n(\mathbb{F}_p)$ is nilpotent have a distinguished characteristic polynomial. This happens because the characteristic polynomial of $\tilde{N}$ is x^n, that means that all the coefficients of the characteristic polynomial of N are divisible by p except for the leading one.

The main idea in this conjecture is to link the distribution of the number of roots of the associated polynomial P_A in a matrix group to the distribution of the λ-invariant in the family of p-adic L-functions that we are studying. To analyse the distributions of these polynomials we introduce the constant ρ and a crucial theorem.

Definition 3.1.4. Let ρ denote the infinite product

$$\rho := \prod_{i\geq 1}(1 - p^{-i}).$$

Theorem 3.1.5. *Let S be an open subset of the distinguished polynomials of degree r. Define*

$$e_S := \mu(\{A \in \mathrm{GL}_r(\mathbb{Z}_p) \mid P_A \in S\}),$$

where we recall that μ is the Haar probability measure on $\mathrm{GL}_r(\mathbb{Z}_p)$ that is the limit of the normalized counting measures on each $GL_r(\mathbb{Z}/p^m\mathbb{Z})$ for $m \to \infty$. For each n, let $P(n,S)$ be the subset of $\mathrm{GL}_n(\mathbb{Z}_p)$ of the elements such that $P_A \in S$; similarly, let $Q(n,S)$ be the subset of $\mathrm{GSp}_{\alpha,2n}(\mathbb{Z}_p)$ of the elements such that $P_A \in S$, where $\alpha \in \mathbb{Z}_p^\times$ and it does not reduce to $1 \pmod p$. Then

$$\lim_{n\to\infty} \mu(P(n,S)) = \lim_{n\to\infty} \mu(Q(n,S)) = \rho e_S.$$

We can notice that in e_S we are taking matrices of size r which have a associated polynomial of degree r, i.e. all the roots of the characteristic polynomial are p-adically close to 1. Instead, in $P(n,S)$ and $Q(n,S)$ we are taking matrices of arbitrary size that always have exactly r roots that are p-adically close to 1.

We will now prove Theorem 3.1.5. We will do this in two steps: we will prove it for $S = S_0 = \{1\}$ and then we will extend the proof to any S.

Proposition 3.1.6.
$$\lim_{n\to\infty} \mu(P(n,S_0)) = \lim_{n\to\infty} \mu(Q(n,S_0)) = \rho.$$

We will prove the propositions for the two different groups by giving a proof that satisfy both cases.

Proof. In this proof we replace α with its reduction modulo p, therefore α will be an element of $\mathbb{F}_p^\times$ different from 1.

We want to define three different random variables:

X_1 : The number of fixed vectors of a random matrix $A \in \mathrm{GL}_n(\mathbb{F}_p)$.

X_2 : The number of fixed vectors of a random matrix $A \in \mathrm{GSp}_{\alpha,2n}(\mathbb{F}_p)$.

X_3 : The number of vector annihilated by a random matrix of size n and entries in $\mathbb{F}_p$.

In each case "random" refers to the counting measure.

We notice that we are studying the cardinality of the eigenspaces related to 1 and of the kernel of these matrices; since these are subspaces they will be able to assume only specific values of cardinality (i.e. p powers).

The distribution of each X_i defines a probability measure $\mu_{i,n}$, where n is the size of the matrices considered in X_i, on the set $\{1, p, p^2, \dots\}$. We can consider μ_i as any weak limit of $\mu_{i,n}$ for $n \to \infty$. We notice that μ_3 is easy to compute, in particular

$$\mu_3(\{1\}) = \lim_{n\to\infty} \frac{\#\mathrm{GL}_n(\mathbb{F}_p)}{\#\mathrm{Mat}_{n\times n}(\mathbb{F}_p)} = \lim_{n\to\infty} \frac{p^{n^2}\prod_{i\le n}(1-p^{-i})}{p^{n^2}} = \lim_{n\to\infty} \prod_{i\le n}(1-p^{-i}) = \rho.$$

We now want to prove that $\mu_1 = \mu_2 = \mu_3$; in this way we would know that $\mu_1(\{1\}) = \mu_2(\{1\}) = \rho$.

In order to do this we have to study the behaviour of these random variables X_i and we analyse them by computing some interesting expected values. For all $t < n/2$ we have that

$$E[(X_1 - 1)(X_1 - p)\dots(X_1 - p^{t-1})] = 1$$

and we also have that
$$E[(X_2 - 1)(X_2 - p)\dots(X_2 - p^{t-1})] = 1.$$

While for X_3, with the same conditions, we have that

$$E[(X_3 - 1)(X_3 - p)\dots(X_3 - p^{t-1})] = (1 - p^{-n})(1 - p^{1-n})\dots(1 - p^{t-n-1}).$$

As $n \to \infty$ these three expected values agrees and they all approaches 1. We will now show these three claims:

1. Using Burnside's Lemma we notice that this expected value is exactly the number of orbits of $\mathrm{GL}(\mathbb{F}_p^n)$ on injective maps $\mathbb{F}_p^t \to \mathbb{F}_p^n$, which is 1.

2. We need a symplectic analogue of Burnside's Lemma for this case. See [7] for a proof of this case.
3. The left hand side represent the number of injections into the kernel of a random matrix of size n; this is equal to the number of injections $\mathbb{F}_p^t \to \mathbb{F}_p^n$ multiplied by p^{-tn}.

We claim that the μ_i have the same moments, i.e. for each $k \in \mathbb{N}$ it is true that

$$E[X_1^k] = E[X_2^k] = E[X_3^k].$$

This follows from the equalities we have just obtained; in fact,

$$P(X_i^k \geq p^n) \leq Cp^{-n^2},$$

for $i \in \{1, 2, 3\}$ when $k > 2n$, that grantees the equality of the moments. This is obtained using elementary convergence estimates to carry our switches of limits and summation.

Even if it is not generally true that the equality between two probability distributions is not implied by the equality of the moments, it has been checked by Fouvry and Kluners [11] that this happens in this case. Therefore $\mu_1 = \mu_2 = \mu_3$.

This is enough to conclude the proof, since $\mu_1(\{1\}) = \rho$ and

$$\mu_1(\{1\}) = \lim_{n\to\infty} \mu(\{A \in \mathrm{GL}_n(\mathbb{F}_p) \mid \text{the only fixed vetctor by } A \text{ is } 0\}) =$$

$$= \lim_{n\to\infty} \mu(\{A \in \mathrm{GL}_n(\mathbb{F}_p) \mid P_A = 1\}) = \lim_{n\to\infty} \mu(P(n, \{1\}));$$

with the same identities on $\mathrm{GSp}_{\alpha,2n}(\mathbb{F}_p)$ we can show that this works also with μ_2 and $Q(n, \{1\})$. $\square$

We will now prove the second half of the main theorem splitting the two cases. We will start by proving that the statement is true for $\mathrm{GL}_n(\mathbb{Z}_p)$ and, after that, we will try to adjust the first proof for $\mathrm{GSp}_{\alpha,2n}(\mathbb{Z}_p)$.

Proof of Theorem 3.1.5 for GL_n. We can suppose that S, the open subset of distinguished polynomials of degree r, is the preimage of a finite set $\overline{S}$ of polynomials of degree r in $\mathbb{Z}/p^m\mathbb{Z}$ for some $m \geq 1$. Therefore, we have the equality

$$e_S = e_{\overline{S}} = \mu(\{C \in \mathrm{GL}_r(\mathbb{Z}/p^m\mathbb{Z}) \mid P_C \in \overline{S}\}).$$

We consider V to be a finite rank free module over $\mathbb{Z}_p$ and $\overline{V} := V/p^mV$. Let

$$\mathcal{Q} := \{X \subset \overline{V}, Y \subset \overline{V}, A_X \in \mathrm{GL}(X), A_Y \in \mathrm{GL}(Y)\},$$

where X, Y are subgroups such that $X \oplus Y = \overline{V}$, $A_Y - 1$ is invertible and $A_X - 1$ is nilpotent. We claim that there is a bijection between $\mathcal{Q}$ and $\mathrm{GL}(\overline{V})$ sending

$$(X, Y, A_X, A_Y) \to A_X \oplus A_Y.$$

We can see that this is a bijection by writing an inverse. Let take $A \in \mathrm{GL}(\overline{V})$; we can consider the following two sequences

$$\ker(A-1) \subset \ker(A-1)^2 \subset \dots$$
$$\mathrm{Im}(A-1) \supset \mathrm{Im}(A-1)^2 \supset \dots$$

both these sequences must stabilize. If we consider N such that $X := \ker(A-1)^N = \ker(A-1)^{N+1}$ and $Y := \mathrm{Im}(A-1)^N = \mathrm{Im}(A-1)^{N+1}$ we obtain that $A-1$ is nilpotent on X and invertible on Y. Moreover we know that $X \cap Y = \{0\}$ and $\#X \cdot \#Y = \#\overline{V}$, therefore $X \oplus Y = \overline{V}$.

Using this bijection, we want to study $\mu(P(n, S))$. Since X and Y are free $\mathbb{Z}/p^m\mathbb{Z}$-modules we can consider $P_{A|X}$ and obtain that $\mu(\{A \in \mathrm{GL}(\overline{V}) \mid P_A \in \overline{S}\})$ is equal to

$$\frac{\#\left\{\begin{smallmatrix}\text{Splitting of } \overline{S} \\ \text{as } X \oplus Y\end{smallmatrix}\right\} \cdot \#\{A_Y \in \mathrm{GL}(Y) \mid A_Y - 1 \text{ invertible}\} \cdot \#\{A_X \in \mathrm{GL}(X) \mid P_A \in \overline{S}\}}{\#\,\mathrm{GL}(\overline{V})},$$

using the fact that the number of different splittings are

$$\frac{\#\mathrm{GL}_n(\mathbb{Z}/p^m\mathbb{Z})}{\#\mathrm{GL}_{n-r}(\mathbb{Z}/p^m\mathbb{Z})\#\mathrm{GL}_r(\mathbb{Z}/p^m\mathbb{Z})},$$

we can rewrite this quantity as

$$\frac{\#\{A_Y \in \mathrm{GL}_{n-r}(\mathbb{Z}/p^m\mathbb{Z}) \mid A_Y - 1 \text{ invertible}\}}{\#\mathrm{GL}_{n-r}(\mathbb{Z}/p^m\mathbb{Z})} \cdot \frac{\#\{A_X \in \mathrm{GL}_r(\mathbb{Z}/p^m\mathbb{Z}) \mid P_A \in \overline{S}\}}{\#\mathrm{GL}_r(\mathbb{Z}/p^m\mathbb{Z})}.$$

We can immediately rewrite this, factor by factor, as

$$P(n-r,\{1\}) \cdot e_S;$$

it is enough to consider the limit for $n \to \infty$ and Proposition 3.1.6 to obtain that

$$\lim_{n\to\infty} \mu(P(n,S)) = \rho \cdot e_S.$$

□

Now, we conclude this section by proving Theorem 3.1.5 for $\mathrm{GSp}_{\alpha,2n}(\mathbb{Z}_p)$ by proceeding along similar lines.

Proof of Theorem 3.1.5 for $\mathrm{GSp}_{\alpha,2n}$. We take $\overline{V} := (\mathbb{Z}/p^m\mathbb{Z})^{2n}$ and we take it endowed with the symplectic form we mentioned in 3.1.1. Let

$$\mathcal{Q} := \{X \subset \overline{V}, Y \subset \overline{V}, A_X \in \mathrm{GSp}_\alpha(X), A_Y \in \mathrm{GSp}_\alpha(Y)\},$$

so that $(A-1)(A-\alpha)$ is nilpotent on X and it is invertible on Y and so that $X \oplus Y = \overline{V}$ in a orthogonal direct sum with respect to the symplectic form.

Again, we claim that there is a bijection between $\mathcal{Q}$ and $\mathrm{GSp}_\alpha(\overline{V})$ sending

$$(X, Y, A_X, A_Y) \to A_X \oplus A_Y.$$

We can see that this is a bijection by writing an inverse exactly as for the other case but taking $B := (A-1)(A-\alpha)$ instead of just $(A-1)$ in the construction. Let's take $A \in \mathrm{GL}(\overline{V})$; we can consider the following two sequences

$$\ker(B) \subset \ker(B)^2 \subset \dots$$

$$\mathrm{Im}(B) \supset \mathrm{Im}(B)^2 \supset \dots$$

both these sequences must stabilize. If we consider N such that $X := \ker(B)^N = \ker(B)^{N+1}$ and $Y := \mathrm{Im}(B)^N = \mathrm{Im}(B)^{N+1}$ we obtain that B is nilpotent on X and invertible on Y. Moreover we know that $X \cap Y = \{0\}$ and $\#X \cdot \#Y = \#\overline{V}$, therefore $X \oplus Y = \overline{V}$.

This time we also have to show that X and Y are orthogonal respect to the symplectic form i.e. $\omega(x,y) = 0$ for all $x \in X, y \in Y$. Since $y \in \mathrm{Im}(B)$ we can write it as $y = Bz$ and consider

$$\omega(x,y) = \alpha\omega(A^{-2}Bx, z).$$

Iterating this for each N we obtain that $y = B^N z_N$ and

$$\omega(x,y) = \alpha^N \omega(A^{-2N}B^N x, z_N),$$

for some $z_N \in Y$; taking N large enough makes the right-hand side zero.

Since $P_A \in \overline{S}$ if and only if $P_{A|X} \in \overline{S}$ we can see that $\deg(P_A) = r$ if and only if the rank of X over $\mathbb{Z}/p^m\mathbb{Z}$ is $2r$. We also know that the two spaces

$$X_1 = \bigcup_N \ker(A-1)^N, \quad X_2 = \bigcup_N \ker(A-\alpha)^N$$

satisfy $X_1 \cap X_2 = 0$ because $A - \alpha$ is invertible on X_1 and $A - 1$ is invertible on X_2 (we are using that α is not congruent to 1 modulo p). We can break X as a (non-orthogonal) direct sum of X_1 and X_2 that are two free $\mathbb{Z}/p^m\mathbb{Z}$-modules; moreover we can prove that they have the same rank. This can be seen using the fact that

$$\omega(x_1, (A-1)x_2) = \omega((\alpha A^{-1} - 1)x_1, x_2);$$

therefore, the maps $X_1 \to \mathrm{Hom}(X_2, \mathbb{Z}/p^m\mathbb{Z})$ and $X_2 \to \mathrm{Hom}(X_1, \mathbb{Z}/p^m\mathbb{Z})$ both given by ω are isomorphisms.

We will now use the following lemma that we will prove later

Lemma 3.1.7.

$$\mu(\{A \in \mathrm{GSp}_{\alpha,2r}(\mathbb{Z}_p) \mid (A-1)(A-\alpha) \textit{ is nilpotent and } P_A \in \overline{S}\}) = e_S.$$

Assuming this we can end the proof as in the last case. We have that $\mu(\{A \in \mathrm{GSp}_{2n}(\mathbb{Z}/p^m\mathbb{Z}) \mid P_A \in \overline{S}\})$ is equal to

$$\frac{\#\{A_Y \in \mathrm{GSp}_{\alpha,2n-2r}(\mathbb{Z}/p^m\mathbb{Z}) \mid A_Y - 1 \text{ invertible}\}}{\#\mathrm{Sp}_{2n-2r}(\mathbb{Z}/p^m\mathbb{Z})} \cdot \frac{\#\{A_X \in \mathrm{GSp}_{\alpha,2r}(\mathbb{Z}/p^m\mathbb{Z}) \mid P_A \in \overline{S}\}}{\#\mathrm{Sp}_{2r}(\mathbb{Z}/p^m\mathbb{Z})}.$$

that we can immediately rewrite this quantity, factor by factor, as

$$Q(2n-2r, \{1\}) \cdot e_S;$$

it is enough to consider the limit for $n \to \infty$ and Proposition 3.1.6 to obtain that

$$\lim_{n\to\infty} \mu(Q(n,S)) = \rho \cdot e_S.$$

$\square$

Proof of Lemma 3.1.7. Let $X = (\mathbb{Z}/p^m\mathbb{Z})^{2r}$ and $\mathrm{GSp}_\alpha(X) := \mathrm{GSp}_{\alpha,2r}(\mathbb{Z}/p^m\mathbb{Z})$. The set we are computing the measure of, can be written as the disjoint union of subgroups $X_1 \subset X$ that have the same rank as $\mathbb{Z}/p^m\mathbb{Z}$-modules and that are isomorphic as groups to $(\mathbb{Z}/p^m\mathbb{Z})^r$:

$$C := \bigcup_{X_1} \{A \in \mathrm{GSp}_\alpha(X) \text{ s. t. } A \text{ preserves } X_1 \text{ and } P_{A|X_1} \in S.\}$$

We know that if $(A-1)(A-\alpha)$ is nilpotent, then A determines a subgroup $X_1 \cong (\mathbb{Z}/p^m\mathbb{Z})^r \subset X = (\mathbb{Z}/p^m\mathbb{Z})^{2r}$ on which A acts unipotently, and P_A is the characteristic polynomial of $A|X_1$. Vice versa, every matrix A in C has $(A-1)(A-\alpha)$ nilpotent: since $P_{A|X_1} \in S$ it is true that $A-1$ acts nilpotently on X_1 and that $A - \alpha$ acts nilpotently on X/X_1 by duality.

In [19] we can see that the action of $\mathrm{GSp}(X)$ on isotropic subgroups that are isomorphic to $(\mathbb{Z}/p^m\mathbb{Z})^r$ is transitive. This means that the size of C is $\mathrm{GSp}(X)/\mathrm{Stab}(X_1)$. On the other hand, the action on X_1 and on the symplectic form gives us the following homomorphism

$$\mathrm{stab}(X_1) \to \mathrm{GL}(X_1) \times \mathbb{F}_p^\times,$$

which is a surjection. This implies that each set in the disjoint union that defines C has size $\frac{e_S}{p-1}\#\mathrm{Stab}(X_1)$, because it is the preimage under a group homomorphism of a set of relative measure $\frac{e_S}{p-1}$. We can conclude the proof computing the cardinality of C that is

$$\frac{\#\,\mathrm{GSp}(X)}{\#\mathrm{Stab}(X_1)} \cdot \left(\frac{e_S}{p-1}\#\mathrm{Stab}(X_1)\right) = e_S \frac{\#\,\mathrm{GSp}(X)}{p-1} = e_S \#\,\mathrm{GSp}_\alpha(X).$$

$\square$

Now that we have finished the proof of this theorem we can use it to explain the origin of the expression that appears in the conjecture.

We will do this by applying this theorem on S_r, the set of all the distinguished polynomials of degree r. This because we want to link the proportion of the matrices with r eigenvalues p-adically close to 1 with the proportion of p-adic L-functions in the family such that $\lambda(\chi) = r$. We link these two quantity because we expect a possible spectral interpretation of these zeroes that can possibly assign a matrix to each p-adic L-function where it exists a bijection between the zeroes of the function and the "low" eigenvalues of the matrix.

Applying Theorem 3.1.5 on S_r, the set of all distinguished polynomials of degree r, we obtain that

Corollary 3.1.8.

$$\lim_{n\to\infty} \mu(P(n, S_r)) = \lim_{n\to\infty} \mu(Q(n, S_r)) = p^{-r} \prod_{i>r} (1 - p^{-i}).$$

Proof. After applying Theorem 3.1.5 on S_r, we obtain that the probability of $\deg P_A = r$ is the measure of the subset of $\mathrm{GL}_r(\mathbb{Z}_p)$ of the matrices A such that the characteristic polynomial of $A - 1$ is distinguished. This happens if and only if the reduction $\tilde{A} \in \mathrm{GL}_r(\mathbb{F}_p)$ is unipotent (see [25]). Therefore

$$e_{S_r} = \frac{\{A \in \mathrm{GL}_r(\mathbb{F}_p) \mid A \text{ is unipotent}\}}{\#\mathrm{GL}_r(\mathbb{F}_p)} =^* \frac{p^{r^2-r}}{p^{r^2} \prod_{i\le r}(1 - p^{-i})} = p^{-r} \prod_{i\le r} (1 - p^{-i})^{-1},$$

where the starred $(*)$ equality is proven in [13]; this implies that

$$\rho e_{S_r} = p^{-r} \prod_{i>r} (1 - p^{-i}).$$

□

3.2 Numerical studies: λ-invariant

Now that we have studied the theoretical origins of the conjecture, we want to compute the λ-invariant of many different functions in this family: in this way we can obtain empirical evidences that support the conjecture.

In the past chapters we have already introduced all the tools we need to face this study but we will recall all of them now.

The main idea is to study $L_{p,0}(\chi, s)$ by approximating the associated formal series $g_\chi((1 + p)^{-s} - 1)$ with a polynomial. To do this we evaluate the function in many points that are easier to compute (the already mentioned special values); we can do this with the following formula that we obtain by combining 2.3.10 and 1.1.13,

$$g_{\chi_D}((1+p)^{k-1} - 1) = L_{p,0}(\chi_D, 1-k) = (1 - \chi_D(p)p^{k-1})L(\chi_D, 1-k) = (1 - \chi_D(p)p^{k-1})\frac{-B_{k,\chi}}{k}.$$

After computing many points with this formula we interpolate them with Lagrange polynomial and obtain the approximating polynomial for the formal series; at this point we can study this approximating polynomial using Newton polygon and computing the valuations of the coefficients of the polynomial to find the associated λ-invariant. A more precise analysis of the p-adic precision can be made to show that the first N coefficients of the polynomial agree with the first N coefficients of g_{χ_D} modulo p^M for some large M; a useful reference to understand how many points we need to compute to obtain a good interpolation polynomial can be found in [8].

By looking at this formula we can now, also, understand why the conjecture does not consider the cases where p splits in $K = \mathbb{Q}[\sqrt{m}]$; if p splits then we know that m is a quadratic residue modulo p (see [4], Theorem 25). This means that, using the definition we gave in Chapter 1, $\chi(p) = 1$; this implies that there is a trivial zero for all the p-adic L-functions defined in this way because we know that, for $k = 1$, we have

$$g_{\chi_D}((1+p)^{1-1} - 1) = (1 - \chi_D(p)p^{1-1})(-B_{1,\chi}) = -(1 - \chi_D(p))B_{1,\chi} = 0;$$

this trivial zero in $s = 0$ would change the statistics of λ-invariants by adding 1 to it in all the cases where m is a quadratic residue modulo p. This zero is often called *"exceptional zero"*.

Instead of ignoring all these cases and not considering them in our statistics we will study them, separately from the other cases, subtracting 1 from the λ-invariant we compute, therefore studying the behaviour of all their non-trivial zeroes.

We will now look at this procedure more in detail by commenting the SAGE script we are going to use for the computation and explaining step by step what is being computed.

The function *chi* compute the value of $\chi_D(a)$ using the definition we gave in Definition 1.4.3.

```
In [1]: def chi(D,a):   #Quadratic Dirichlet Character
v2=0
while a%2==0:
  v2+=1
  a=a/2
odd_part=jacobi_symbol(D,a)
chi_2=1
if D%2==0: chi_2=0
elif D%8==5: chi_2=-1
return odd_part*(chi_2^v2)
```

Now that we can easily access to the values $\chi_D(a)$ we can create a function to obtain the approximating polynomial.

```
In [2]: def PolynomialApprox(d,p):
triv=kronecker(d,p)
points=[((1+p)^(i-1)-1,(-1)*QuadraticBernoulliNumber(i, d)/i*...
...*(1-prechi(d,p)*p^(i-1))) for i in range(1,10*p,p-1)];
f = PolynomialRing(QQ, 'x').lagrange_polynomial(points);
if triv==1:
  coeff=list(f)[1:]
else: coeff=list(f)
return(coeff,triv)
```

In this function we compute $\left(\frac{D}{p}\right)_K$ to understand if we are studying a case with or without trivial zero. Then, after computing enough points with the special values formula we found, we obtain the Lagrange approximation polynomial.

If $\left(\frac{D}{p}\right)_K = 1$ we know that $s = 0$ will be a zero of infinite order for the function, therefore, thanks to the Newton polygon, we know that the formal series does not have a constant term. In this case we want to study the polynomial divided by x; in this way we are subtracting 1 to the λ-invariant we are computing and we are still recovering all the informations about other zeroes from Newton polygon because the coefficients remains the same ones just shifted by one position.

We can now easily compute the λ-invariant just by finding the first coefficient that is not divisible by p.

```
In [3]: def LambdaOnly(coeff,p):
i=0
v=QQ.valuation(p)
while i<15 :
  if v(coeff[i])<=0: return i
  else: i+=1
return(15)
```

At this point we can just use this program to compute the λ-invariant of $L_{p,0}(\chi_D, s)$ for a big number of negative fundamental discriminants D, observe the proportion of cases where $\lambda(K) = r$ for $r \in \mathbb{N}$ and compare this with the conjectured quantity $p^{-r} \prod_{i>r}(1 - p^{-i})$.

In the following table we record the results of computing the 3-adic λ-invariants for all the negative fundamental discriminants D with absolute value smaller than 10^5. As already mentioned, we split the results in two cases. In the case (a) we observe the value of λ for $L_{3,0}(\chi_D, s)$ for negative values of D with absolute value smaller than 10^5 for which $\left(\frac{D}{3}\right)_K \neq 1$; in the case (b) we observe the value of $\lambda - 1$ for $L_{3,0}(\chi_D, s)$ for the negative values of D with absolute value smaller than 10^5 for which $\left(\frac{D}{3}\right)_K = 1$.

λ	0	1	2	3	4	5	6
predicted	0.5601	0.2800	0.1050	0.0363	0.0122	0.0041	0.0013
observed - (a)	0.62380	0.2538	0.0819	0.0272	0.0094	0.0025	0.0009
observed - (b)	0.61214	0.2605	0.0833	0.0296	0.0092	0.0035	0.0013

We have also repeated this study for the case $p = 5$, by computing all the 5-adic λ-invariant for all the negative fundamental discriminants with absolute value smaller than $5 \cdot 10^4$. In the following table we will be splitting, again, the results in two cases. In the case (a) we observe the value of λ for $L_{5,0}(\chi_D, s)$ for the negative values of D with absolute value smaller than $5 \cdot 10^4$ for which $\left(\frac{D}{5}\right)_K \neq 1$; in the case (b) we observe the value of $\lambda - 1$ for $L_{5,0}(\chi_D, s)$ for the negative values of D smaller than $5 \cdot 10^4$ for which $\left(\frac{D}{5}\right)_K = 1$.

λ	0	1	2	3	4	5
predicted	0.7603	0.1901	0.0396	0.0080	0.0016	0.00032
observed - (a)	0.7952	0.1651	0.0313	0.0069	0.0012	0.00033
observed - (b)	0.7826	0.1727	0.0376	0.0063	0.0006	0.00016

From this two tables it may seem that the conjecture consistently *overestimates* the fraction of L-functions with large λ-invariants. Further data on shorter ranges at higher values of D show that the observed probabilities shift towards the predicted ones. Computational studies that compute the first 10^6 negative fundamental discriminants made by Ellenberg, Jain, Venkatesh shows that the numerical results tend to get closer to the prediction when considering wider ranges [6].

This also supports the idea that the distribution of $\lambda - 1$ among L-functions with a trivial zero agrees with the distribution of λ among L-functions without a trivial zero.

3.3 Numerical studies: The p-adic Hazelgrave phenomenon and $\nu_k(\mathcal{F})$

We want to extend our studies to other properties of these zeroes. At the beginning of the chapter we have mentioned that one of the ideas hidden in this conjecture is to consider this zeroes as low lying zeroes; this means that studying distributions of this family that are linked to low lying zeroes might give us interesting results.

In order to do this we want to recall the numerical experiments on the complex analogue of this family that we studied in Section 1.4. We have mentioned two important results of these experiments:

1. The *Hazelgrave phenomenon*. In the complex family of Dirichlet L-functions with a quadratic character, low lying zeroes repel the point $s = \frac{1}{2}$.

2. Conjecture 1.4.5:
$$\lim_{N\to\infty} \nu_j(\mathcal{F}_N) = \nu_j(\mathrm{Sp}).$$

Therefore, we want to define and study some analogous distributions in the family $\mathcal{F}$ of p-adic L-functions that we are studying.

We can consider every function $f = L_{p,0}(\chi_D, s)$ in $\mathcal{F}$ that have at least one non-trivial zero and study its non-trivial zeroes $\{\alpha_i\}_{i=1}^{\lambda}$; we write the absolute values of the zeroes $\{|\alpha_i|_p\}_{i=1}^{\lambda}$ and, since the zeroes of f are finite, we can take the minimum of these absolute values and call it $\gamma_D^{(1)}$. More generally, we can consider every function f that have at least k non-trivial zeroes and repeat the construction to consider the k-th lower absolute value of a zero and call it $\gamma_D^{(k)}$.

We let $\mathcal{F}_N$ be the subfamily of all the p-adic L-functions $L_{p,0}(\chi_D, s)$ with $-N \le D \le 0$ and that have at least k non-trivial zero; now we can define the following distribution

$$\nu_k(\mathcal{F}_N)[a,b] = \frac{\#\{L_{p,0}(\chi, s) \in \mathcal{F}_N \mid \gamma_D^{(k)} \in [a,b]\}}{\#\mathcal{F}_N}.$$

In the code we used for computing λ-invariants we are already computing a polynomial that approximates the formal series associated to $L_{p,0}(\chi, s)$; we can use the Newton polygon of this polynomial to discover other important informations about the zeroes, in particular their absolute values. We know that the slopes of the Newton polygon are equal to the orders of the zeroes of the function multiplied by -1. Since $|x|_p = p^{-ord_p(x)}$ we know that the first slope of the Newton polygon correspond to the zero with the smallest absolute value; therefore, to compute statistics about $\gamma_D^{(1)}$ and $\nu_1(\mathcal{F}_N)$ we just need to study the first slope of the Newton polygon for each function in $\mathcal{F}_N$.

Using the SAGE built-in function *.newton_polygon(p)*, we obtain immediately the slopes of the polygon and the orders of the zeroes.

In the following table we record the results of computing the value of the first slope (i.e. the value of $-\operatorname{ord}_p(\alpha)$) on the family of the 3-adic L-functions $L_{3,0}(\chi_D, s)$ for $-5 \cdot 10^4 \le D \le 0$ for which $\left(\frac{D}{3}\right)_K \neq 1$ and that have at least one non-trivial zero.

first slope	1/7	1/6	1/5	1/4	1/3	2/5	1/2	2/3	1	4/3
occurrences	2	11	25	62	215	1	635	28	2284	1

first slope	3/2	2	5/2	3	4	5	6	7	8	10	13
occurrences	31	763	1	246	70	29	8	6	2	1	1

We can extend this study also to the cases where $\left(\frac{D}{3}\right)_K = 1$ and we notice that the behaviour of low lying zeroes near the point $s = 0$ is substantially the same, giving another evidence towards the conjecture that the non-trivial zeroes in a function of the family with a trivial zero behave in the same way of the zeroes of the functions without trivial zeroes.

We can observe this by looking at the following table; we record the results of computing the value of the first non-trivial slope (i.e. that is not infinite) on the family of the 3-adic L-functions $L_{3,0}(\chi_D, s)$ for $-5 \cdot 10^4 \le D \le 0$ for which $\left(\frac{D}{3}\right)_K = 1$ and that have at least one non-trivial zero.

first slope	1/8	1/7	1/6	1/5	1/4	1/3	1/2	2/3
occurrences	2	4	13	32	115	327	1096	42

first slope	3/4	1	3/2	2	3	4	5
occurrences	1	3915	27	1210	320	36	1

To show the distributions of the first zeroes $\gamma_D^{(1)}$ in $\mathcal{F}_N$ we can obtain a plot of the relative occurrences of each $\gamma_D^{(1)}$ after clustering the possible values, grouping together nearby ones, we show these plots in Figure 3.1 and Figure 3.2.

We have also studied the case $p = 5$, computing the value of the first slope on the family of the 5-adic L-functions $L_{5,0}(\chi_D, s)$ for $-5 \cdot 10^4 \le D \le 0$. In the first table are contained the frequencies of the first slopes for the functions $L_{5,0}(\chi_D, s)$ for which $\left(\frac{D}{5}\right)_K \neq 1$ and that have at least one non-trivial zero while in the second table are contained the frequencies of the first slopes for the functions $L_{5,0}(\chi_D, s)$ for which $\left(\frac{D}{5}\right)_K = 1$ and that have at least one non-trivial zero.

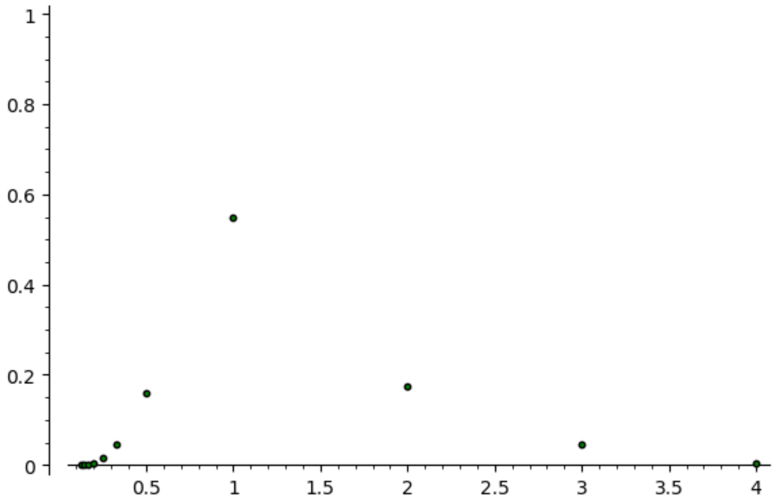

Figure 3.1: Distributions of $\gamma_D^{(1)}$ in $\mathcal{F}_{10^5}$ for functions that have at least a zero and for which $\left(\frac{D}{3}\right)_K \neq 1$

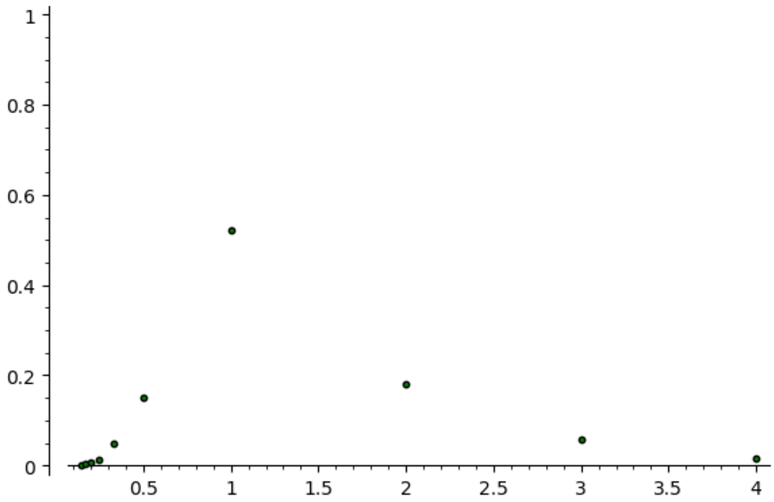

Figure 3.2: Distributions of $\gamma_D^{(1)}$ in $\mathcal{F}_{10^5}$ for functions that have at least a non-trivial zero and for which $\left(\frac{D}{3}\right)_K = 1$

first slope	1/5	1/4	1/3	1/2	2/3	1	2	3
occurrences	3	9	53	231	3	1330	181	5

first slope	1/5	1/4	1/3	1/2	1	3/2	2	3	4
occurrences	1	3	30	191	919	1	185	35	11

Again, to show the distributions of the first zeroes $\gamma_D^{(1)}$ in $\mathcal{F}_N$ we can obtain a plot of the relative occurrences of each $\gamma_D^{(1)}$ after clustering the values, grouping together nearby ones, we show these plots in Figure 3.3 and Figure 3.4.

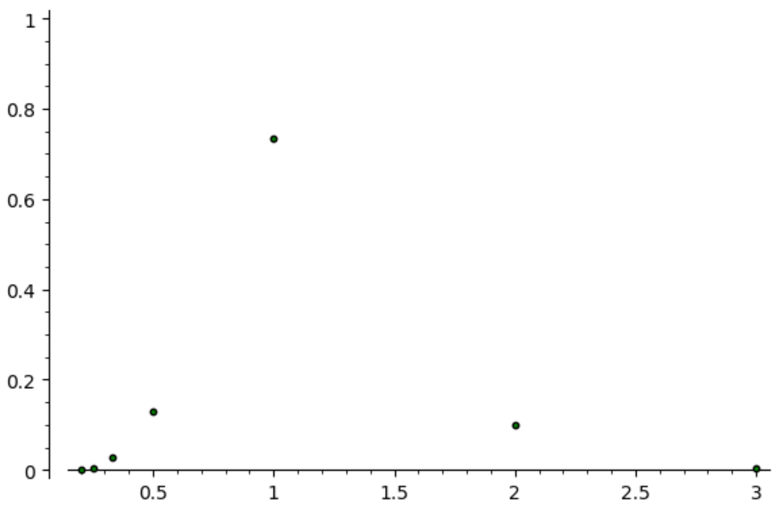

Figure 3.3: Distributions of $\gamma_D^{(1)}$ in $\mathcal{F}_{5\cdot 10^4}$ for functions that have at least a zero and for which $\left(\frac{D}{5}\right)_K \neq 1$

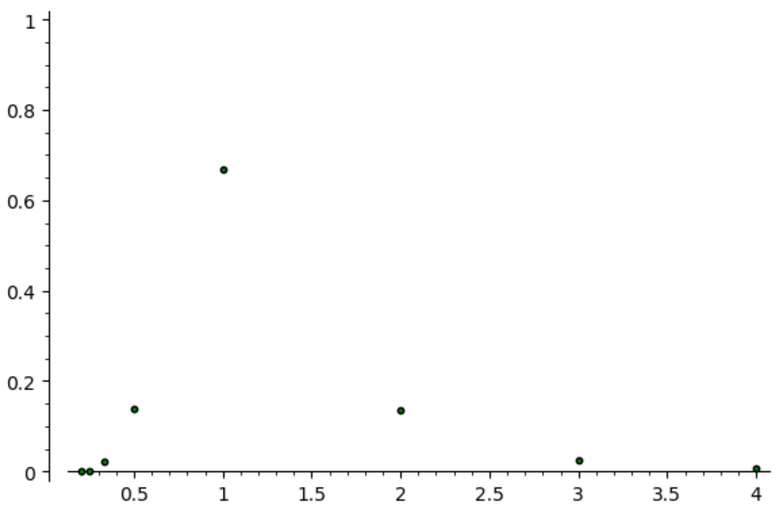

Figure 3.4: Distributions of $\gamma_D^{(1)}$ in $\mathcal{F}_{5\cdot 10^4}$ for functions that have at least a non-trivial zero and for which $\left(\frac{D}{5}\right)_K = 1$

These frequencies show that low lying zeroes in $\mathcal{F}$ (empirically) repel the point $s = 0$: this is a compelling similarity with the complex case that resemble the Hazelgrave phenomenon! With these computations we have noticed that this peculiar behaviour of low lying zeroes seems to appear also in this p-adic analogue of the original family; therefore, we can say that we have observed the presence of a *p-adic Hazelgrave phenomenon*.

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