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Finite Abelian Descent Obstruction

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Abstract

Let X be a smooth projective geometrically connected variety over a number field k. If X has adelic points but no k-rational points, then it is said to have failed the Hasse principle. Most of the time this failure can be accounted for by the Brauer-Manin obstruction, that comes in the form of the Brauer set, which sits in the middle of the inclusions

$$X(k) \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \subset X(\mathbb{A}_k)_\bullet.$$

In this paper, we apply the theory of descent to investigate another form of obstruction to the Hasse principle. This involves studying the category $\mathcal{A}b(X)$ of X-torsors under finite abelian étale group schemes defined over k. Let $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ denote the set cut out by restrictions coming from the finite abelian étale coverings of X, we informally view them as the set of points of $X(\mathbb{A}_k)_\bullet$ that 'survives' all X-torsors in $\mathcal{A}b(X)$. Our main result will be to show that when $X = C$ is a curve, this set coincides with the Brauer set. In other words, $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet^{\mathrm{Br}}$.

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Introduction

In 1900, Hilbert's tenth problem asked, "given a polynomial Diophantine equation $f(x_1, ..., x_n) = 0$ with integer coefficients, does there exist an algorithm which can determine if f has a solution in $\mathbb{Z}$?" This problem can of course be extended to the case where we consider a system $f_1, ..., f_m$ of Diophantine equations, and ask if we are able to determine if there is a common integral solution to these equations. Such a system defines a variety over $\mathbb{Q}$, and therefore we can reformulate the problem as

Question 0.1 (Hilbert's 10th Problem over $\mathbb{Q}$).[1] Does there exist an algorithm for deciding whether a variety over $\mathbb{Q}$ has a rational point?

This problem generated further interest in the existence of rational points on a variety. A few years later, after the creation of the p-adic numbers by Hensel, there was suspicion by Hasse that considering the p-adic numbers for all prime numbers p could play an important role in number theory. To put it roughly, Hasse promotes the view that we can study a problem over $\mathbb{Q}$ by studying it in $\mathbb{R}$ and in all the local fields $\mathbb{Q}_p$. This is known as the *local-global principle*, or, as we now call it, the *Hasse principle*.

What led Hasse towards such a perspective? Well, despite the unclear foundations of the p-adic numbers at that time, it was Minkowski's work on quadratic forms that provided the motivation. Simply putting, to determine if a quadratic form has a solution in $\mathbb{Q}$, we only have to determine if it has a solution in $\mathbb{R}$ (this is usually obvious), and a solution in $\mathbb{Q}_p$ for every p, which can be done by analytic tools such as Hensel's lemma. This gives us the famous result

Theorem 0.2 (Hasse-Minkowski). Let $Q(x_1, ..., x_n)$ be a quadratic form with rational coefficients. Then

(1) for $c \in \mathbb{Q}^\times$, the equation $Q(x_1, ..., x_n) = c$ has a solution in $\mathbb{Q}$ if and only if it has a solution in $\mathbb{R}$ and every $\mathbb{Q}_p$.

(2) the equation $Q(x_1, ..., x_n) = 0$ has a solution in $\mathbb{Q}$ besides $(0, ..., 0)$ if and only if it has a solution in $\mathbb{R}$ and every $\mathbb{Q}_p$ besides $(0, ..., 0)$.

With such a powerful tool in hand, it would be natural to try to extend it to higher order forms. Unfortunately, this principle cannot hold in general, even in the immediate case of cubic forms. In the famous example of Selmer [Sel51], the cubic equation

$$3x^3 + 4y^3 + 5z^3 = 0$$

has a solution other than $(0, 0, 0)$ in $\mathbb{R}$ and every $\mathbb{Q}_p$, but its only solution in $\mathbb{Q}$ is $(0, 0, 0)$. We will see a quartic counterexample later on, at the end of Chapter 1.

[1] Refer to [Poo01] for a survey about this problem. For the original Hilbert's 10th problem, such an algorithm cannot exist [Mat70].

Nonetheless this new obstacle did not hold mathematicians back, because the focus has now shifted to a new question:

Question 0.3. What accounts for the failure of the Hasse principle?

This innocuous-looking problem gave rise to a novel approach to the study of rational points: *obstructions.* To explain the meaning of this term, we first note that when a smooth projective variety X over $\mathbb{Q}$ to have a point in $\mathbb{R}$ and every $\mathbb{Q}_p$, we obtain a point in

$$\prod_p X(\mathbb{Q}_p) \times X(\mathbb{R}).$$

The properness of X ensures we have an equality between this product and the adelic points of X, i.e.,

$$X(\mathbb{A}_\mathbb{Q}) = \prod_p X(\mathbb{Q}_p) \times X(\mathbb{R}).$$

Trivially, we have an embedding of $X(\mathbb{Q})$ into $X(\mathbb{A}_\mathbb{Q})$ via $a \longmapsto (a, a, ...)$. An obstruction of rational points is a set S such that

$$X(\mathbb{Q}) \subset S \subset X(\mathbb{A}_\mathbb{Q}).$$

And if X fails the Hasse principle and $S = \emptyset$, we say that S *accounts* for this failure.

The earliest known obstruction was discovered in the 1970s by Manin [Man70], and it is called the Brauer-Manin obstruction. The set S in this case is called the Brauer set, containing adelic points that vanish under the Brauer-Manin pairing with all points of the Brauer group $\mathrm{Br}(X)$ of X, see Chapter 1.4.

For a long time, the Brauer-Manin obstruction accounted for all failures of the Hasse principle. It wasn't until the late 1990s when Skorobogatov [Sko99] constructed an example of a surface with no rational points, even though there was no Brauer-Manin obstruction. This provides us with a glimpse of the difficulty of working with varieties of dimensions greater than 1, since many aspects of their behaviours are not so well-understood. That being said, the main goal of this paper is to show that the Brauer-Manin obstruction accounts for all failures of the Hasse principle when the variety under consideration is a smooth projective geometrically connected curve C over a number field.

Our approach is as follows: we apply techniques developed by Colliot-Thélène and Sansuc on descent theory to define new obstruction sets (Definition 3.5). The idea revolves around the twists of torsors under certain finite group schemes. The resulting obstructions are called *finite descent obstructions.* We will prove that in the case of C, these obstruction sets coincide with the Brauer set (Theorem 3.20), and therefore the Brauer-Manin obstruction accounts for the obstruction against rational points on C.

Contents

1 The Brauer-Manin Obstruction

The Hasse principle (in the case of points in $\mathbb{Q}$) is the statement that for a smooth and projective variety X, the set $X(\mathbb{Q})$ is nonempty if and only if for every prime $p \leq \infty, X(\mathbb{Q}_p)$ is also nonempty. Certain classes of varieties are known to satisfy the Hasse principle, for example, the degree 2 hypersurfaces in $\mathbb{P}^n$ [Poo01]. The goal of this chapter is to outline one of the possible reasons the Hasse principle might be false in the case of smooth projective geometrically connected curves over a number field. Such a failure is due to the Brauer-Manin obstruction, discovered by Manin [Man70], and it is named so because it arises from the Brauer group of the curve.

1.1 Finer Topologies

The issue of a topological space having too few open sets to accurately compute the cohomology groups of certain coherent sheaves was the reason Grothendieck and Artin developed the theory of étale cohomology. It required the relaxation of the requirement that a covering set of a space consists of only its subsets.

Let E be one of the following classes of morphisms: (Zar), (ét) and (fl); which denote, respectively, the class of all open immersions, étale morphisms of finite type, and flat morphisms locally of finite type. Elements of E will be called *E-morphisms*. Fix a base scheme X and a class E. Let $\mathbf{C}/X$ be a full subcategory of the category of schemes over X, $\mathbf{Sch}/X$, such that

(i) $\mathbf{C}/X$ is closed under fiber products;

(ii) for any $Y \longrightarrow X$ in $\mathbf{C}/X$ and any E-morphism $U \longrightarrow Y$, the composite $U \longrightarrow X$ is in $\mathbf{C}/X$.

Definition 1.1. An *E-covering* of an object Y of $\mathbf{C}/X$ is a family of E-morphisms $(g_i : U_i \longrightarrow Y)_{i \in I}$ such that $Y = \bigcup g_i(U_i)$. The class of all such coverings of all such objects is the *E-topology* on $\mathbf{C}/X$. The category $\mathbf{C}/X$ together with the E-topology is the *E-site* over X, denoted by $(\mathbf{C}/X)_E$, or simply, X_E. The *small E-site* on X is $(E/X)_E$, where E/X is the full subcategory of $\mathbf{Sch}/X$ whose objects are schemes Y over X such that the structure morphism $Y \longrightarrow X$ is an E-morphism. And in the case where all E-morphisms are locally of finite type, the *big E-site* on X is $(\mathbf{LFT}/X)_E$, where $\mathbf{LFT}/X$ is the full subcategory of $\mathbf{Sch}/X$ of X-schemes whose structure morphism is an E-morphism that is locally of finite type.

One easily checks that $\mathbf{C}/X$, together with the family of E-coverings, is a *Grothendieck topology*, where the usual notion of an open set is replaced with an E-morphism. The theory of presheaves and sheaves extend naturally into this new situation. We note the following two presheaves:

Example 1.2. For any $U \longrightarrow X$ in $\mathbf{C}/X$, the presheaf $\mathbb{G}_a$ associates to U the additive abelian group $\Gamma(U, \mathcal{O}_U)$ while the presheaf $\mathbb{G}_m$ associates to U the multiplicative abelian group $\Gamma(U, \mathcal{O}_U)^\times$, both with obvious restriction maps.

Remarks 1.3. (a) We can view the (Zar)-topology on a scheme as the Zariski topology in the usual sense, up to identification of any open immersion with its image. Hence we refer to the (Zar)-topology simply as the Zariski topology. Respectively, the étale and flat topology refer to the (ét)- and (fl)-topology.
(b) Since every open immersion is étale and flat, we see that the étale and flat topologies are finer than the Zariski topology.
(c) By the same reasoning, the flat topology is finer than the étale topology.

By the *Zariski site* X_{Zar} on X we always mean the small (Zar)-site $((\text{Zar})/X)_{\text{Zar}}$; by the *étale site* $X_{\text{ét}}$ we always mean the small (ét)-site $((\text{ét})/X)_{\text{ét}}$, and by the *flat site* X_{fl}, we always mean the big (fl)-site $(\mathbf{LFT}/X)_{\text{fl}}$.

Another distinction between the Zariski sites and the étale and flat sites is given by the following result (see [Mil80], II.1.5), which makes it easier to check if a presheaf is a sheaf.

Proposition 1.4. Let P be a presheaf of abelian groups on the étale or flat site on X. Then P is a sheaf if and only if it satisfies the following two conditions:

(a) for any U in $\mathbf{C}/X$, the restriction of P to the usual Zariski topology on U is a sheaf;

(b) for any covering $(U' \longrightarrow U)$ with U, U' both affine, $P(U) \longrightarrow P(U') \rightrightarrows P(U' \times_U U')$ is exact.

We denote by $\mathbf{S}(X_E)$ the category of sheaves on the site $(\mathbf{C}/X)_E$ and remark that this is an abelian category with enough injectives (see [Mil80], III.1). Therefore we are able to define the right derived functors of any left exact functor from $\mathbf{S}(X_E)$ into an abelian category.

Definition 1.5. The functor $\Gamma(X, -) : \mathbf{S}(X_E) \longrightarrow \mathbf{Ab}$, with $\Gamma(X, F) = F(X)$, is left exact and its right derived functors are written

$$R^i\Gamma(X, -) = H^i(X, -) = H^i(X_E, -).$$

The group $H^i(X_E, F)$ is called the i^{th}-*cohomology group* of X_E with values (or coefficients) in F. For instance, *étale cohomology* involves taking X_E as the étale site and *flat cohomology* when X_E is the flat site.

The two presheaves mentioned in Example 1.2 are actually sheaves on the Zariski, étale and flat sites. The reason is that both these presheaves are defined by *commutative group schemes*, i.e., as functors from $\mathbf{Sch}/X$ to $\mathbf{Sets}$, they factor through the category $\mathbf{Ab}$ (see [Mil80], II.1.7). The sheaf $\mathbb{G}_m$ is of particular importance to us, which we will see in the next section. For now, we finish off with the following important result, which tells us that the étale topology is sufficiently (see Remark

1.3(c)) fine to compute interesting cohomology groups of a smooth group scheme, i.e., for the rest of this paper, we can work equally well with either étale or flat cohomology.

Theorem 1.6. If G is a smooth, quasi-projective, commutative group scheme over a scheme X, then the canonical maps

$$H^i(X_{\text{ét}}, G) \longrightarrow H^i(X_{\text{fl}}, G)$$

are isomorphisms.

Proof. See [Mil80], III.3.9. □

1.2 The Brauer Group

We review the classical construction of the Brauer group, following the content in [Mil13], IV.2. Let k be a field, a k-algebra A is said to be *central* if its center $Z(A)$ is k. If A is both central and simple then we call it a *central simple algebra*. We note that every simple k-algebra is central simple over a finite extension of k. Let A and B be central simple algebras over k. We say that A and B are *similar*, and denote it by $A \sim B$, if

$$A \otimes_k M_n(k) \cong B \otimes_k M_m(k)$$

for some positive integers m, n. By [Mil13], IV.2.2, this is an equivalence relation. Let the set of similarity classes of central simple algebras over k be denoted by $\mathrm{Br}(k)$, and write $[A]$ for the similarity class of A. We define an operation between two classes $[A]$ and $[B]$ by

$$[A][B] = [A \otimes_k B].$$

It is easy to verify that this operation is well-defined, and the associativity and commutativity of this operation follows from that of the tensor product. For each n, $[M_n(k)]$ would be an identity element since we have that $[A][M_n(k)] = [A \otimes_k M_n(k)] = [A]$ and by [Mil13], IV.2.9 we also have that $[A][A^{\text{opp}}] = [A \otimes_k A^{\text{opp}}] = [M_n(k)]$, where A^{opp} is the *opposite algebra*, i.e., multiplication of the underlying ring is performed in the reverse order. Trivially, all the $M_n(k)$ belong to the same class. Hence $\mathrm{Br}(k)$ is an abelian group, called the *Brauer group* of k.

We describe an alternative way to define $\mathrm{Br}(k)$. Let $[A], [A']$ be similarity classes in Br(X) and let L be an extension of k, not necessarily finite. By the following observations,

$$M_n(k) \otimes L \cong M_n(L),$$

$$(A \otimes_k L) \otimes_L (A' \otimes_k L) = (A \otimes_k A') \otimes_k L,$$

we see that the map $A \longmapsto A \otimes_k L$ defines a homomorphism $\mathrm{Br}(k) \longrightarrow \mathrm{Br}(L)$. Let $\mathrm{Br}(L/k)$ denote the kernel of this map, which consists of the similarity classes represented by central simple algebras A such that the L-algebra $A \otimes_k L$ is a matrix algebra. For such an A, we say that it is a *split* central simple algebra. By [Mil13], IV.2.17, we see that $\mathrm{Br}(k) = \bigcup \mathrm{Br}(K/k)$ where K runs over all the finite extensions of k contained in some fixed $\bar{k}$.

Given any finite Galois extension L of k, [Mil13], IV.3.14 tells us that we have an isomorphism of abelian groups $\mathrm{Br}(L/k) \cong H^2(\mathrm{Gal}(L/k), L^\times)$. More importantly, we have, for a separable algebraic closure $\bar{k}$ of k, the canonical isomorphism

$$\mathrm{Br}(k) \cong H^2(\mathrm{Gal}(\bar{k}/k), \bar{k}^\times).$$

We use the above definitions of the Brauer group to compute $\mathrm{Br}(k)$ for some fields k.

Proposition 1.7. If k is algebraically closed, then $\mathrm{Br}(k) = 0$.

Proof. This follows from the fact that there are no nonsplit central simple algebra over k, hence the Brauer group is trivial. □

Proposition 1.8. The Brauer group of $\mathbb{R}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Proof. By Proposition 1.7 we know that $\mathrm{Br}(\mathbb{C}/\mathbb{R}) \cong \mathrm{Br}(\mathbb{R})$. Hence

$$\mathrm{Br}(\mathbb{R}) \cong (\mathbb{C}^\times)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}/\mathrm{Nm}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) \cong \mathbb{R}^\times/\mathbb{R}_{>0} \cong \mathbb{Z}/2\mathbb{Z}$$

by [Wei97], VI.6.2. □

In the case of local fields, by this we mean the finite extensions of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, we have a nicer result which takes some machinery to construct. We provide a sketch. For a local field K and a finite unramified extension L/K, let G denote the Galois group, which is isomorphic to the Galois group of the corresponding residue field extension, hence G is cyclic. Let U_L denote the group of units of the ring of integers of L and we begin with the exact sequence

$$0 \longrightarrow U_L \longrightarrow L^\times \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the latter map is the normalized valuation of L. By [Mil13], III.1.1, we have that $H^2(G, U_L) = 0 = H^3(G, U_L)$. This gives us an isomorphism $H^2(G, L^\times) \cong H^2(G, \mathbb{Z})$. The cohomology of $\mathbb{Z}$ follows from the exact sequence of G-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$

The groups $H^r(G, \mathbb{Q})$ are torsion for $r > 0$ by [Mil13], II.4.3 and uniquely divisible, hence zero. Thus we have an isomorphism $H^1(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(G, \mathbb{Z})$. The group $H^1(G, \mathbb{Q}/\mathbb{Z})$ can also be interpreted as $\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ and since G is cyclic, let σ be its generator we have the following homomorphism

$$\mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}, \quad f \longmapsto f(\sigma).$$

Definition 1.9. The composite

$$H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\sim} H^2(G, \mathbb{Z}) \xleftarrow{\sim} H^1(G, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

is called the *local invariant map*

$$\mathrm{inv}_{L/K} : H^2(\mathrm{Gal}(L/K), L^\times) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

We omit the proof of the following important result concerning the Brauer group of a local field:

Theorem 1.10. Let k be a local field, then $\mathrm{inv}_{\bar{k}/k}$ is an isomorphism, i.e.,

$$H^2(\mathrm{Gal}(\bar{k}/k), \bar{k}^\times) \cong \mathbb{Q}/\mathbb{Z}.$$

Of the three descriptions of the Brauer group $\mathrm{Br}(k)$ we have so far, the third one is probably the most appealing. This is also known as the *cohomological Brauer group* of the field k, and it is a natural question to ask if they can be generalized to schemes. In fact, in place of using central simple algebras as in the first two definitions, the case of schemes uses the so-called *Azumaya algebra*, which, loosely speaking, is a generalization of central simple algebras to be defined over R, a commutative ring which need not be a field. The bad news is that these definitions in general may not be equivalent. For a scheme X, we have the canonical injective homomorphism (see [Mil80], IV.2.5)

$$\mathrm{Br}(X) \longrightarrow H^2(X_{\text{ét}}, \mathbb{G}_m),$$

where the Brauer group of X on the left is defined using similarity classes of Azumaya algebras in the same way as before. It would be interesting[2] to know when such a map is also surjective, but fortunately for the purposes of this paper, the schemes we consider would always have the above map as an isomorphism, so both definitions coincide.

Definition 1.11. Let X be a smooth projective geometrically connected k-variety, then

$$\mathrm{Br}(X) \coloneqq H^2(X_{\text{ét}}, \mathbb{G}_m)$$

is called the *(cohomological) Brauer group* of X. As previously mentioned, $\mathbb{G}_m$ is the abelian sheaf on $X_{\text{ét}}$, sending U to the multiplicative abelian group $\Gamma(U, \mathcal{O}_U)^\times$.

1.3 The Cohomological Group of Torsors

Aside from the Brauer group there are several other groups that play a central role in class field theory. In this section we develop some of them that will be of greater use in the latter chapters. In this section, unless otherwise stated, we shall assume

[2] For the interested reader, this happens when X is a regular and quasi-projective variety over a field, see [Poo17], 6.6.19.

that K is a number field, and X is a smooth projective geometrically connected curve defined over K.

Recall that the *Picard group* Pic_X of X is the quotient of the group of divisors of X by the subgroup of principal divisors. Since the degree of a divisor depends only on its linear equivalence class ([Har77], II.6.10), we have a surjective homomorphism $\mathrm{Pic}_X \longrightarrow \mathbb{Z}$. Let Pic_X^0 denote the kernel of this map.

Definition 1.12. The group Pic_X^0 is isomorphic to the group of closed points of an abelian variety associated to X called the *Jacobian variety*, or simply the *Jacobian*, J_X of X.

If one has the feeling that this seems more like a result than a definition, this is because the actual construction of the Jacobian variety of a curve carries heavier content than what's being shown, see [C-S86], VII.1-4. In fact, Pic_X^0 can be interpreted as the group $H^1(X, \mathcal{O}_X^\times)$ of isomorphism classes of invertible sheaves on X, and the Jacobian variety is being represented as a particular functor from schemes over K to the abelian groups. Nonetheless, we note the following useful result:

Theorem 1.13. The tangent space to J_X at 0 is canonically isomorphic to $H^1(X, \mathcal{O}_X)$; therefore, the dimension of J_X is equal to the genus of X.

For a group G and a G-module M, we let $Z^1(G, M)$ denote the abelian group of *1-cocycles* $G \longrightarrow M$. Note that if the action of G on M is trivial, then this group, and indeed $H^1(G, M)$, is simply $\mathrm{Hom}(G, M)$.

Definition 1.14. Let $H \leq G$ be a subgroup. The map

$$Z^1(G, M) \longrightarrow Z^1(H, M), \; f \longmapsto f|_H$$

descends to a homomorphism

$$\mathrm{res} : H^1(G, M) \longrightarrow H^1(H, M)$$

called the *restriction* homomorphism.

We shift our attention slightly to another group-theoretic concept. Given a group G, we recall that a *homogeneous space* Y for G is a nonempty manifold or topological space on which G acts transitively.

Definition 1.15. The homogeneous space Y is a *G-principal homogeneous space*, or a *G-torsor*, if Y is nonempty and it is equipped with a map $Y \times G \longrightarrow Y$ such that for all $y \in Y$ and $g, h \in G$, we have

(i) $y \cdot 1 = y$;

(ii) $y \cdot (gh) = (y \cdot g) \cdot h$;

(iii) the map $Y \times G \longrightarrow Y \times Y$ given by $(y, g) \longmapsto (y, y \cdot g)$ is an isomorphism in the appropriate category.

To suit this idea according to our needs, we fix a base scheme S and consider the flat site on S. Y is now defined as an S-scheme, and let G be a group scheme (not necessarily commutative) that is also defined over S. We say that Y is an S-torsor under G (or a G-torsor over S), equipped with a morphism $Y \times_S G \longrightarrow Y$ satisfying the above three conditions. We note that G itself is an S-torsor under G, called the *trivial torsor.*

Proposition 1.16. Y is trivial if and only if $Y(S)$ is nonempty.

Proof. If $y_0 \in Y$ is an S-rational point, then

$$G \longrightarrow Y,\ g \longmapsto y_0 \cdot g$$

is an isomorphism. Conversely, if Y is trivial and $f : G \longrightarrow Y$ is the corresponding isomorphism, we first note that since G is a group scheme, $G(S)$ contains at least the identity section g_0. Then $f(g_0)$ is an S-rational point of Y. $\square$

In this section we have so far introduced three seemingly unrelated concepts: the Jacobian variety, restriction homomorphisms, and torsors. Now we put them together.

Again, K is a number field and let J_X be defined over K. The base scheme we are considering is Spec K, so Galois cohomology coincides with étale cohomology ([Mil80, II.1.7]), and therefore with flat cohomology by Theorem 1.6. For convenience, the flat cohomology groups $H^i(\mathrm{Gal}(\overline{K}/K)_{\mathrm{fl}}, J_X(\overline{K}))$ will be denoted by $H^i(K, J_X)$. Given a finite place v of K, we have the embedding $\overline{K} \hookrightarrow \overline{K_v}$. Since the action of $\mathrm{Gal}(\overline{K_v}/K_v)$ on $\overline{K_v}$ restricts to an action on $\overline{K}$, we have the embedding $\mathrm{Gal}(\overline{K_v}/K_v) \hookrightarrow \mathrm{Gal}(\overline{K}/K)$, which induces the restricted homomorphism

$$\mathrm{res}_v : H^1(K, J_X) \longrightarrow H^1(K_v, J_X).$$

A K-torsor under J_X would now be a smooth variety on which J_X acts freely and transitively by morphisms over K. Two K-torsors Y and Y' under J_X are *equivalent* if there exists a K-isomorphism $\Phi : Y \longrightarrow Y'$ such that the diagram

$$\begin{array}{ccc} Y \times_K J_X & \longrightarrow & Y \\ \downarrow & & \downarrow \Phi \\ Y' \times_K J_X & \longrightarrow & Y' \end{array}$$

commutes. This leads us to the following theorem, which puts in place all we have defined so far together (refer to Chapter 2, Theorem 2.13 for a proof):

Theorem 1.17. The group $H^1(K, J_X)$ can be identified with the equivalence classes of K-torsors under J_X.

It follows that the map res_v is a homomorphism of torsors, sending a K-torsor Y under J_X to the base extension $Y \times_K K_v$. The group $H^1(K, J_X)$ is called the *Weil-Châtelet group*, denoted by $\mathrm{WC}(J_X(K))$.

We say that a K-torsor Y under J_X *fails the Hasse principle* if it has a rational point over each completion K_v but no K-rational point. By Proposition 1.16, the presence of a K-rational point on Y is equivalent to being the trivial element in $H^1(K, J_X)$. Therefore, if Y is not isomorphic to J_X, then it fails the Hasse principle if and only if it belongs to $\ker(\mathrm{res}_v)$ for every v, i.e., Y has points everywhere locally. We capture this information nicely in the following definition:

Definition 1.18. The *Tate-Shafarevich group* of the abelian variety J_X is defined to be

$$Ш(J_X(K)) := \bigcap_v \ker(H^1(K, J_X) \longrightarrow H^1(K_v, J_X)).$$

Hence, the non-trivial elements of $Ш(J_X(K))$ correspond to K-torsors under J_X that fail the Hasse principle.

Conjecture 1.19 (Tate-Shafarevich). For every abelian variety A over a global field k, the group $Ш(A(k))$ is finite.

Even for the case of an elliptic curve E over $\mathbb{Q}$, this conjecture remains open, and it remains so even if we assume E to have complex multiplication. In fact, even in this special case, it is not known if the ℓ-primary torsion subgroup of $Ш(E(\mathbb{Q}))$ is finite for almost all primes ℓ. For a general genus 1 curve X over $\mathbb{Q}$, it is a torsor of its Jacobian E, which in this case is an elliptic curve. It is known that if $Ш(E(\mathbb{Q}))$ is finite, then there is an algorithm for determing whether $X(\mathbb{Q})$ is empty. A famous result by Cassels also showed that, for a global field k, if $Ш(E(k))$ is finite, then its order is a square.

We now introduce another important class of groups in the theory of abelian varieties, which also arises from the idea of the Weil-Châtelet groups.

For an abelian group G and an integer $n \geq 2$, let $G[n]$ denote the kernel of the multiplication-by-n map from G to itself. Now we consider the situation where G is taken to be the Jacobian variety J_X defined over a number field K and the multiplication-by-n map is the isogeny $[n]$.

Definition 1.20. The *n-Selmer group* $\mathrm{Sel}^n(J_X(K))$ of J_X over K is defined to be

$$\bigcap_v \ker(H^1(K, J_X[n]) \longrightarrow H^1(K_v, J_X)[n]).$$

We will revisit the Tate-Shafarevich group and the Selmer group at the end of the paper.

1.4 Sketch of the Obstruction

To finish off this chapter we will explain what the Brauer-Manin obstruction is in the case of curves, by first constructing its Brauer set. We mainly follow the notes on rational points by Poonen [Poo17].

Let K be a number field and $\mathbb{A}_K$ be its *adèle ring*, i.e., the restricted product $\prod'_v(K_v, \mathcal{O}_v)$ over all places v, where $K_v = \mathcal{O}_v$ if v is infinite. Let

$$F : (\mathbf{Sch}/K)^{\mathrm{opp}} \longrightarrow \mathbf{Sets}$$

be a functor. For each K-algebra L, write $F(L)$ in place of $F(\mathrm{Spec}\ L)$ and fix a K-variety Y.

Suppose $P \in F(Y)$. For each K-algebra L, define the map $\mathrm{ev}_P : Y(L) \longrightarrow F(L)$ as follows: given $y \in Y(L)$, the corresponding morphism Spec $L \longrightarrow Y$ induces a map $F(Y) \longrightarrow F(L)$, sending P to $\mathrm{ev}_P(y)$. Therefore the diagram

$$\begin{array}{ccc} Y(K) & \hookrightarrow & Y(\mathbb{A}_K) \\ {\scriptstyle \mathrm{ev}_P}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{ev}_P} \\ F(K) & \xrightarrow[\varphi]{} & F(\mathbb{A}_K) \end{array}$$

commutes. Define the set

$$Y(\mathbb{A}_K)^P := \{y \in Y(\mathbb{A}_K) : \mathrm{ev}_P(y) \in \mathrm{Im}(\varphi)\} \subset Y(\mathbb{A}_K).$$

Then the commutativity of the above diagram tells us that $Y(K) \subset Y(\mathbb{A}_K)^P$. In other words, the element P puts constraints on the locus in $Y(\mathbb{A}_K)$ where the K-points can lie. By imposing all such constraints we have the subset

$$Y(\mathbb{A}_K)^F := \bigcap_{P \in F(Y)} Y(\mathbb{A}_K)^P$$

which still contains $Y(K)$.

Definition 1.21. The *Brauer set* of the nice curve X is defined to be $X(\mathbb{A}_K)^{\mathrm{Br}}$.

Now for $P \in \mathrm{Br}(X)$, given a K-algebra L and a point $x \in X(L)$, the corresponding map Spec $L \longrightarrow X$ induces a morphism $\mathrm{Br}(X) \longrightarrow \mathrm{Br}(L)$, mapping P to $\mathrm{ev}_P(x)$, which we shall now write as $P(x)$ for simplicity.

Proposition 1.22. If $(x_v) \in X(\mathbb{A}_K)$ and $P \in \mathrm{Br}(X)$, then $P(x_v) = 0$ for almost all v.

Sketch of proof. By [Poo17], 6.6.11 we know that there exists a finite set S of places of K such that we have an $\mathcal{O}_S$-model $\mathcal{X}$ for X and a $\mathcal{P}$ in $\mathrm{Br}(\mathcal{X})$ that maps to P under the map $\mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(X)$ induced by the inclusion of $X \cong \mathcal{X}_K$ into $\mathcal{X}$. By choosing S such that $x_v \in \mathcal{X}(\mathcal{O}_v)$ for all $v \notin S$, we see that $P(x_v)$ comes from an element $\mathcal{P}(x_v) \in \mathrm{Br}(\mathcal{O}_v)$. But [Poo17], 6.9.3 tells us that $\mathrm{Br}(\mathcal{O}_v) = 0$. ☐

Corollary 1.23 (Brauer-Manin pairing). We have a well-defined pairing

$$\mathrm{Br}(X) \times X(\mathbb{A}_K) \longrightarrow \mathbb{Q}/\mathbb{Z},\ (P, (x_v)) \longmapsto \sum_v \mathrm{inv}_v(P(x_v)),$$

where $\mathrm{inv}_v : \mathrm{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z}$ is the local invariant map (cf. Definition 1.9).

Proof. From the construction given at the start of the section, set $Y = X$, $L = K_v$, and $F = \mathrm{Br}$. This implies that $P(x_v)$ lies in $\mathrm{Br}(K_v)$ and so the image of $\mathrm{Br}(X) \times X(\mathbb{A}_K)$ lies in $\mathbb{Q}/\mathbb{Z}$. The finiteness of the sum is a consequence of Proposition 1.22. ☐

Therefore we see that for a fixed $P \in \mathrm{Br}(X)$, we have a map

$$X(\mathbb{A}_K) \longrightarrow \mathbb{Q}/\mathbb{Z},\ (x_v) \longmapsto (P, x_v) := \sum_v \mathrm{inv}_v(P(x_v)).$$

Proposition 1.24. For $x \in X(K) \subset X(\mathbb{A}_K)$, via the diagonal embedding, we have $(P, x) = 0$.

Proof. It follows from the commutativity of

$$\begin{array}{ccccccccc} & & X(K) & \hookrightarrow & X(\mathbb{A}_K) & & & & \\ & & \downarrow {\scriptstyle \mathrm{ev}_P} & & \downarrow & & & & \\ 0 & \longrightarrow & \mathrm{Br}(K) & \longrightarrow & \bigoplus_v \mathrm{Br}(K_v) & \xrightarrow{\sum_v \mathrm{inv}_v} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

where the bottom row is the fundamental exact sequence ([Tat67], VII.9.6). ☐

We now define the set

$$X(\mathbb{A}_K)^P := \{(x_v) \in X(\mathbb{A}_K) : (P, (x_v)) = 0\},$$

which is naturally followed by

$$X(\mathbb{A}_K)^{\mathrm{Br}} = \bigcap_{P \in \mathrm{Br}(X)} X(\mathbb{A}_K)^P.$$

This gives us an explicit description of the Brauer set (cf. Definition 1.21 and compare it with the construction at the start of the section), and the above proposition tells us that $X(K) \subset X(\mathbb{A}_K)^{\mathrm{Br}}$. Also, for any subset $S \subset \mathrm{Br}(X)$, we set $X(\mathbb{A}_K)^S := \bigcap_{P \in S} X(\mathbb{A}_K)^P$, which gives us the following inclusions:

$$X(K) \subset X(\mathbb{A}_K)^{\mathrm{Br}} \subset X(\mathbb{A}_K)^S \subset X(\mathbb{A}_K).$$

Definition 1.25. We say that there is a *Brauer-Manin obstruction to the Hasse principle for* X if $X(\mathbb{A}_K) \neq \emptyset$ but $X(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset$. Such an obstruction is said to be the *only one* if the implication

$$X(K) = \emptyset \implies X(\mathbb{A}_K)^{\mathrm{Br}} = \emptyset$$

holds.

For elliptic curves, by definition, they always have a rational point over $\mathbb{Q}$. However, a general genus 1 curve need not. One such example is the well-known projective curve

$$2Y^2Z^2 = Z^4 - 17X^4$$

discovered by Lind [Lin40] and Reichardt [Rei42] independently, which fails the Hasse principle. One of the main problems of the theory of rational points is whether the Brauer-Manin obstruction is the only one for a particular class of curves of interest. An example of a variety whose failure of the Hasse principle cannot be accounted for by the Brauer-Manin obstruction can be found in [Sko99]. In the work of Scharaschkin [Sch99], he proved that given a nice curve over a number field, if both its Jacobian and the associated Tate-Shafarevich group are finite, then the Brauer-Manin obstruction fully explains the failure of the Hasse principle. In fact, some major results related to obstructions require the Tate-Shafarevich group to be finite, which, as we have already seen (cf. Conjecture 1.19), is itself a wildly open problem.

Conjecture 1.26 (Colliot-Thélène). Let X be a smooth projective geometrically connected variety over a number field K. Suppose that X is rationally connected[3]. Then the Brauer-Manin obstruction to the Hasse principle is the only obstruction.

See [P-V04] for the history of this conjecture as well as sources for evidence motivating the statement. We emphasise that the notion of a variety being smooth, projective, and geometrically connected has many interesting and convenient properties which we will continue to see throughout this paper, particularly in the case of curves. To this end, it would make sense to introduce an adjective for such varieties.

Definition 1.27. We say that a variety X over a number field K is *nice* if it is smooth, projective, and geometrically connected.

[3]Every two points of X are connected by a rational curve contained in X.

2 Descent Theory

The idea of descent can be summarised as follows: given a base field k, one can attempt to construct a variety X over k by first constructing its analogue X' over some field extension k' of k. Then the problem now is to decide if X' is the base extension of some variety over k. In this chapter, we introduce standard descent theory with focus on the general fpqc descent. We then look at a special case by considering descent over schemes that are the spectrum of a field. This concept has an important relationship with the application of torsors, which was introduced in the last chapter and will be treated with greater detail here.

2.1 Galois Descent

Let S be a topological space and $\{S_i\}$ be an open cover of S. The usual gluing problem on sheaves can be restated by introducing the disjoint union $S' := \coprod S_i$. Let $\pi : S' \longrightarrow S$ be an 'open covering morphism' (cf. Definition 1.1) such that on each S_i, the map $S_i \longrightarrow S$ is the inclusion map. To give a sheaf $\mathcal{F}_i$ on each S_i is the same as giving a single sheaf $\mathcal{F}'$ on S'. The problem of gluing now asks if there exists a sheaf $\mathcal{F}$ on S such that the sheaf $\pi^{-1}\mathcal{F}$ on S' is isomorphic to the given $\mathcal{F}'$. Alternatively, we can view this problem as an attempt to 'descend' the sheaf $\mathcal{F}'$ on S' to the appropriate sheaf $\mathcal{F}$ on S.

Let $S^{''}$ denote the fiber product $S' \times_S S'$, which equals the disjoint union $\coprod S_{ij}$ over all i, j, where $S_{ij} := S_i \cap S_j = S_i \times_S S_j$. Let p_1, p_2 denote projections from $S^{''}$ to the first and second coordinates respectively. The sheaf $p_1^{-1}\mathcal{F}'$ on $S^{''}$ restricted to the piece S_{ij} corresponds to the sheaf $\mathcal{F}_i|_{S_{ij}}$. Therefore, to ask for an isomorphism

$$\phi_{ij} : \mathcal{F}_i|_{S_{ij}} \longrightarrow \mathcal{F}_j|_{S_{ij}}$$

is equivalent to asking if we have an isomorphism

$$\phi : p_1^{-1}\mathcal{F}' \longrightarrow p_2^{-1}\mathcal{F}'$$

of sheaves on S'. Let $S^{'''} := S' \times_S S' \times_S S'$, and let $p_{ij} : S^{'''} \longrightarrow S^{''}$ be the projection to the ith and jth coordinates, where $1 \leq i < j \leq 3$. Then $p_{13}^{-1}\phi$ is an isomorphism of sheaves on S'', satisfying the cocycle condition

$$p_{13}^{-1}\phi = p_{23}^{-1}\phi \circ p_{12}^{-1}\phi.$$

The idea of descent usually begins with the case of quasi-coherent sheaves (see [Poo17], 4.2), but in our situation, we are concerned with the problem of descending schemes. This is Grothendieck's theory of *faithfully flat descent*. Here we let $p : S' \longrightarrow S$ be the more general fpqc morphism of schemes in place of the Zariski open covering morphism and let X' be an S'-scheme. The problem is as follows: under what conditions is X' isomorphic to an S'-scheme of the form p^*X for some S-scheme X (here $p^*X := p^{-1}X = X \times_S S'$)?

Definition 2.1. A *descent datum* on an S'-scheme X' is an S''-isomorphism

$$\phi : p_1^* X' \longrightarrow p_2^* X'$$

satisfying the cocycle condition. The pair (X', ϕ) is an object of the category of *schemes with descent data*. If X is an S-scheme, then p^*X has a canonical descent datum ϕ_X. We say that ϕ is *effective* if $(X', \phi) \cong (X, \phi_X)$.

Definition 2.2. Let X' be an S'-scheme, and let $\phi : p_1^*X' \longrightarrow p_2^*X'$ be a descent datum. An open subscheme $U' \subset X'$ is called *stable under* ϕ if ϕ induces a descent datum on U', i.e., if ϕ restricts to an isomorphism $p_1^*U' \longrightarrow p_2^*U'$ of S''-schemes.

We recall that a scheme is *quasi-affine* if it is an open subscheme of an affine scheme and it is quasi-compact. A morphism $f : X \longrightarrow S$ is quasi-affine if $f^{-1}(S_0)$ is quasi-affine for each affine open subscheme S_0 of S. The following result is the descent theorem for schemes, refer to [Gro95], B1, Theorem 2, for a proof in the case of quasi-coherent sheaves:

Theorem 2.3. Let $p : S' \longrightarrow S$ be an fpqc morphism of schemes.

(i) The functor $X \longmapsto p^*X$ from S-schemes to S'-schemes with descent data is fully faithful.

(ii) The functor $X \longmapsto p^*X$ from quasi-affine S-schemes to quasi-affine S'-schemes with descent data is an equivalence of categories.

(iii) Suppose that S and S' are affine. Then a descent datum ϕ on an S'-scheme X' is effective if and only if X' can be covered by quasi-affine open subschemes which are stable under ϕ.

Before moving on, we say a few words about base extensions. If X is an S-scheme and $S' \longrightarrow S$ is a morphism, then the *base extension* $X_{S'}$ is the S'-scheme $X \times_S S'$. For a field k, let X be a k-scheme, and let $\sigma \in \mathrm{Aut}(k)$. The base extension of X by the morphism $\sigma^* : \mathrm{Spec}\ k \longrightarrow \mathrm{Spec}\ k$ induced by σ is a new k-scheme ${}^{\sigma}X$. Since σ^* is an isomorphism of schemes, X and ${}^{\sigma}X$ are isomorphic as abstract schemes, but in general not isomorphic as k-schemes.

Now we look at the case where $S = \mathrm{Spec}\ k$. Let k'/k be a finite Galois extension and denote let $S' = \mathrm{Spec}\ k'$. Since $S' \longrightarrow S$ is fpqc, we can use part (iii) of the above theorem to say something about descending k'-schemes to k-schemes. This is known as *Galois descent*, which was developed by Weil.

Let $G = \mathrm{Gal}(k'/k)$. The left action of G on k' induces a right action of G on S', so each $\sigma \in G$ induces an automorphism σ^* of S'. We give the following results (see [Poo17], 4.4. and [BLR90], 6.2B for proof and details):

Proposition 2.4. Let X' be a k'-scheme. Giving a descent datum on X' is equivalent to the following two data:

(i) Giving a collection of k'-isomorphisms $f_\sigma : {}^\sigma X' \longrightarrow X'$ for $\sigma \in G$ satisfying the 'cocycle condition' $f_{\sigma\tau} = f_\sigma \cdot {}^\sigma(f_\tau)$ for all $\sigma, \tau \in G$.

(ii) Giving a right action of G on X' compatible with the right action of G on S', i.e., to give a collection of isomorphisms $\tilde{\sigma} : X' \longrightarrow X'$ for $\sigma \in G$ such that

$$\begin{array}{ccc} X' & \xrightarrow{\tilde{\sigma}} & X' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\sigma^*} & S' \end{array}$$

commutes for each $\sigma \in G$ and $\widetilde{\sigma\tau} = \tilde{\tau}\tilde{\sigma}$ for all $\sigma, \tau \in G$.

Corollary 2.5. Let k'/k be a finite Galois extension of fields. Let X' be a quasi-projective k'-scheme. Suppose that we are given k'-isomorphisms $f_\sigma : {}^\sigma X' \longrightarrow X'$ for $\sigma \in G$ satisfying $f_{\sigma\tau} = f_\sigma \cdot {}^\sigma(f_\tau)$ for all $\sigma, \tau \in G$. Then $X' = X_{k'}$ for a k-scheme X.

The above corollary gives us a sufficient condition for descending a quasi-projective variety over a Galois extension k'/k. Note that in the case where the scheme X' to be descended to k is defined over the separable closure k_s instead of a finite Galois extension of k, assuming X' is finitely presented, we use the fact that k_s is the direct limit of its finite Galois subextension of k. This means that we reduce X' to the case where it is the base extension of a scheme over a finite Galois extension of k before applying Galois descent.

Using the characterization of k'-schemes with descent data mentioned in Proposition 2.4(i), we have:

Proposition 2.6. Let X' with $(f_\sigma)_{\sigma \in G}$ and Y' with $(g_\sigma)_{\sigma \in G}$ be k'-schemes with descent data. An isomorphism between X' and Y' is a map

$$h : X' \longrightarrow Y'$$

such that $f_\sigma = h^{-1} g_\sigma {}^\sigma h$ for all $\sigma \in G$.

Remark 2.7. The fact that k'/k is Galois gives us an isomorphism

$$k' \otimes_k k' \xrightarrow{\sim} \prod_{\sigma \in G} k', \ a \otimes b \longmapsto (a \cdot {}^\sigma b)_{\sigma \in G}.$$

This induces an isomorphism

$$S'' \cong \coprod_{\sigma \in G} k' =: S' \times G.$$

A finite and faithfully flat morphism of schemes $p : S' \longrightarrow S$ equipped with a finite group G of automorphisms of S' as an S-scheme, with action of G on the right, is called a *Galois covering with Galois group G* if the morphism

$$S' \times G \longrightarrow S''$$

is an isomorphism of schemes described as follows: rewrite the left side as

$$S' \times G = \coprod_{\sigma \in G} S \times \{\sigma\}$$

and on every piece $S \times \{\sigma\}$, we have the map $(s, \sigma) \longmapsto (s, s \cdot \sigma)$ for all $s \in S$.

2.2 Torsors and Twists

Here we introduce the notion of twists with two main goals: to classify all the G-torsors up to k-isomorphism, and to apply it to torsors, obtaining the so-called twisted torsor. Most of the content in this section are from the book by Skorobogatov [Sko01].

Let X be a quasi-projective k-variety, and let k'/k be a Galois extension of fields. Again, we denote by G the Galois group $\mathrm{Gal}(k'/k)$.

Definition 2.8. A k'/k-*twist* of X is a k-variety Y such that there exists an isomorphism $\phi : X_{k'} \longrightarrow Y_{k'}$. A *twist* of X is a k_s/k-twist of X, where k_s is the separable closure of k. Two k'/k-twists Y, Y' of X are *isomorphic* if $Y \cong Y'$ as k-varieties.

The set of k-isomorphism classes of k'/k-twists of X then forms a pointed set, with the class of X being the obvious neutral element. The action of G on k' induces a map from G to the automorphism group $\mathrm{Aut}(X_{k'})$.

Theorem 2.9. There is a natural bijection of pointed sets

$$\frac{\{k'/k\text{-twists of } X\}}{k\text{-isomorphism}} \longrightarrow H^1(G, \mathrm{Aut}(X_{k'})).$$

Proof. We may reduce to the case where k'/k is a finite Galois extension, and take direct limits on both sides if necessary. This allows us to apply Galois descent. For each $\sigma \in G$, we identify ${}^{\sigma}X_{k'}$ with $X_{k'}$. To give a k'/k-twist of X is to descend $X_{k'}$ to a k-variety. Using the fact that X is quasi-projective and applying Theorem 2.3(ii), this is the same as giving a descent datum on $X_{k'}$, which, by Theorem 2.4(i), is the same as giving a 1-cocycle $\varphi : G \longrightarrow \mathrm{Aut}(X_{k'})$ given by $\varphi(\sigma) = f_\sigma$ and so

$$f_{\sigma\tau} = \varphi(\sigma\tau) = \varphi(\sigma) \cdot \sigma(\varphi(\tau)) = f_\sigma \cdot {}^{\sigma}(f_\tau)$$

as required. By Theorem 2.3(i), two such twists are k-isomorphic if and only if the descent data are isomorphic, which by Proposition 2.6 holds if and only if the 1-cocycles are cohomologous. ☐

Remark 2.10. Explicitly, the 1-cocycle associated to the k'/k-twist Y is constructed as follows: choose a k'-isomorphism $\phi : X_{k'} \longrightarrow Y_{k'}$, and define

$$f_\sigma \coloneqq \phi^{-1}({}^{\sigma}\phi) \in \mathrm{Aut}(X_{k'}).$$

However, given a 1-cocycle and the associated isomorphism class of k'/k-twist, there is no natural way to select a twist from that class. Therefore it makes no sense to speak of 'the twist associated to a cohomology class'.

In Chapter 1.3 we spoke briefly about torsors, mainly in the case of the K-torsors under the abelian variety J_X, where K is a number field. Our aim now is to provide a proof of Theorem 1.17 by involving the concept of twists which we have just developed. We do so in a more general setting, by considering k-torsors under an algebraic group, for an arbitrary field k.

Recall that an *algebraic group* G over a field k is a group scheme of finite type over k, and it is *affine* if it embeds in GL_n for some $n \geq 0$. We shall always assume that G is smooth. Denote by $\mathbf{G}$ the *trivial G-torsor over k*, which in this case refers to the underlying variety of G equipped with the right action of G by translation. We have the following definition of a G-torsor (this is slightly different from the one given previously, cf. Definition 1.15):

Definition 2.11. A *G-torsor over k* (or a *k-torsor under G*) is a k-variety X equipped with a right action of G such that X_{k_s} equipped with its right G_{k_s}-action is isomorphic to $\mathbf{G}_{k_s}$. If X, X' are G-torsors, the map $\varphi : X \longrightarrow X'$ is a *morphism of G-torsors* if it is a G-equivariant morphism of k-schemes, i.e., $\varphi(x) \cdot g = \varphi(x \cdot g)$ for all $x \in X, g \in G$.

From the definition we see that up to isomorphism, a G-torsor X can be identified with a twist of $\mathbf{G}$, and vice versa. Hence the isomorphism classes of G-torsors over k are in one-to-one correspondence with the isomorphism classes of twists of $\mathbf{G}$. Furthermore, if we have a G-equivariant automorphism $\varphi : \mathbf{G} \longrightarrow \mathbf{G}$, then for $g \in \mathbf{G}$,

$$\varphi(g) = \varphi(e \cdot g) = \varphi(e) \cdot g.$$

This implies that $\varphi(e) \in \mathbf{G} = G(k)$. Therefore, we conclude that we have 1-1 correspondence between the groups $G(k)$ and $\mathrm{Aut}(\mathbf{G})$ (in fact, this is an isomorphism), by identifying an element in $G(k)$ with the image of e under φ.

We can also consider fpqc descent involving algebraic groups via the following result due to Chow [Cho57]:

Theorem 2.12. Every algebraic group over a field k is quasi-projective.

We now have everything we need to prove the next result, which is the generalized form of Theorem 1.17.

Theorem 2.13. We have a bijection

$$\frac{\{G\text{-torsors over } k\}}{k\text{-isomorphism}} \longrightarrow H^1(k, G) := H^1(k, G(k_s)).$$

Proof. We know that the set on the left can be identified with isomorphism classes of the twists of $\mathbf{G}$. Theorem 2.12 allows us to perform Galois descent (see the paragraph after Corollary 2.5) and hence we can invoke Theorem 2.9 to further identify the set with the cohomology group $H^1(k, \mathrm{Aut}(\mathbf{G}_{k_s}))$. Since we have another 1-1 correspondence between $\mathrm{Aut}(\mathbf{G}_{k_s})$ with $G(k_s)$, we are done. □

The above bijection is a special case of a technique known as 'twisting by Galois descent', see [Sko01], I.2.1. Roughly speaking, for any quasi-projective k-variety F endowed with an action of G, we define a continuous 1-cocycle

$$\sigma : \mathrm{Gal}(k_s/k) \longrightarrow G(k_s).$$

The 'twist' of F by σ, denoted by F_σ, is defined as the quotient of F_{k_s} by the *twisted (left) action* ρ of $\mathrm{Gal}(k_s/k)$ on F_{k_s} given by $(g, s) \longmapsto \sigma_g \cdot {}^g s$. This action is well-defined, indeed, we have

$$\rho(g_1 g_2, s) = \sigma_{g_1 g_2} \cdot {}^{g_1 g_2} s = (\sigma_{g_1} \cdot {}^{g_1}\sigma_{g_2}) \cdot {}^{g_1 g_2} s = \sigma_{g_1} \cdot {}^{g_1}\rho(g_2, s) = \rho(g_1, \rho(g_2, s)),$$

for $g_1, g_2 \in \mathrm{Gal}(k_s/k)$ and $s \in F_{k_s}$. In the case where $F = G$ where G acts on itself by conjugations we have the notion of the *inner twist* G_σ of G. This is an algebraic group over k which is a twist of G as an algebraic group.

The following construction is important for the application of torsors, this is otherwise known as *twisting by fppf descent* (see [Sko01], I.2.3 for more details). We consider everything over a base scheme X, which is a generalization of the case in [Poo17], 5.12 where the base scheme is taken to be Spec k.

Lemma 2.14. Let P be a right X-torsor under an X-group scheme G, and F be an affine X-scheme equipped with a left action of G that is compatible with the projection to X. Then the quotient of $P \times_X F$ by the action of G given by $(p, f) \longmapsto (pg^{-1}, gf)$ exists as an affine X-scheme, i.e., there exists a morphism of X-schemes $P \times_X F \longrightarrow Y$ whose fibres are orbits of G.

The quotient described above is called the *contracted product* of P and F with respect to G, and it is denoted by $P \times^G F$. This is a fiber product taken over X. It is also called *the twist of F by P*, denoted by ${}_PF$. Note that P has the structure of a left X-torsor under ${}_PG$, so that ${}_PG$ acts on ${}_PF$ on the left.

Example 2.15 (Inverse torsor). In the case where F is a left X-torsor under G, we first consider the *inverse torsor* F' of F: which is isomorphic to F as an X-scheme, and it is a right torsor under G with respect to the action $f' \cdot g := g^{-1} f'$, for $f' \in F', g \in G$. Likewise, F' is also a left X-torsor under ${}_{F'}G$. Therefore, F is equipped with the structure of a right X-torsor under ${}_{F'}G$ with respect to the action $f \cdot g' := g'^{-1} f$ which gives us

$$G \curvearrowright F \curvearrowleft {}_{F'}G, \;\; {}_{F'}G \curvearrowright F' \curvearrowleft G.$$

Then the contracted product $P \times^G F$ is a right X-torsor under ${}_{F'}G$, and a left X-torsor under ${}_PG$. If $P = F'$, then $F' \times^G F$ is given by the quotient of $F' \times_X F$ by the group action

$$(f_1, f_2) \longmapsto (f_1 \cdot g^{-1}, g \cdot f_2) = (gf_1, gf_2)$$

for $g \in G, f_1, f_2 \in F$. By the transitive action of G on F, it follows that by fixing some $f \in F$, the set $\{(f, f') : f' \in F\}$ are the equivalence classes of $F' \times^G F$. Hence $F' \times^G F \cong G$.

Example 2.16 (Twisted torsor). Let k be a field, S a k-scheme, and G an affine algebraic group over k. Then G_S is an fppf group scheme over S that is affine over S. Suppose that $f : Z \longrightarrow S$ is a right G_S-torsor and $T \longrightarrow \text{Spec } k$ is a right G-torsor. Define

$$Z_\tau := {}_TZ = Z \times_S^{G_S} T_S = Z \times_k^G T$$

where $\tau \in H^1(k, G)$ is the cohomology class of T, and let $f_\tau : Z_\tau \longrightarrow S$ be its structure morphism. Then Z_τ is a right ${}_TG$-torsor (or a G_τ-torsor) over S, called a *twisted torsor*.

As seen in the example above, it is sometimes more natural to consider G-torsors as a morphism into the base scheme rather than as an object on which G acts. Indeed, this is useful in situations where we are considering different bases. If G is defined over a field k but we are considering a torsor as an S-scheme under G, then we are really considering it as a G_S-torsor. Throughout the rest of the paper, we will stick to speaking about a G-torsor, despite the abuse of language.

2.3 Fundamental Descent Theorems

We conclude this chapter with two theorems that will play a role in the next chapter, which will be the main part of this paper.

We begin with the usual setup: let k be a field, X a k-variety, and G a smooth algebraic group over k. Let $f : Z \longrightarrow X$ be a G-torsor over X and $\zeta \in H^1(X, G)$ be its cohomology class (cf. Theorem 2.13). For $x \in X(k)$, the fiber $Z_x \longrightarrow \{x\}$ will be a G-torsor over k, and we let $\zeta(x)$ denote its class in $H^1(k, G)$. This is equivalent to saying that x determines a morphism

$$H^1(X, G) \longrightarrow H^1(k, G), \ \zeta \longmapsto \zeta(x)$$

induced by the map Spec $k \longrightarrow X$. Therefore the torsor $Z \longrightarrow X$ gives rise to an evaluation map

$$X(k) \longrightarrow H^1(k, G), \ x \longmapsto \zeta(x).$$

In other words, $Z \longrightarrow X$ can be thought of as a family of torsors parameterized by X, and $\zeta(x)$ gives the class of the fiber above x.

Remark 2.17. The evaluation map $X(k) \longrightarrow H^1(k, G)$ can also be described as the pull-back of the torsor $Z \longrightarrow X$ through the map Spec $k \longrightarrow X$ corresponding to a point in $X(k)$.

The above construction enables us to partition $X(k)$ according to the class of the fiber above each point x:

$$X(k) = \coprod_{\tau \in H^1(k,G)} \{x \in X(k) : \zeta(x) = \tau\}.$$

And the right-hand side is reinterpreted by the first main result of this section:

Theorem 2.18. Let G be a smooth affine algebraic group over k. Let $f : Z \longrightarrow X$ be a G-torsor with cohomology class $\zeta \in H^1(X, G)$. For each $\tau \in H^1(k, G)$, let $f_\tau : Z_\tau \longrightarrow X$ be the twisted torsor of f constructed in Example 2.16. Then

$$X(k) = \coprod_{\tau \in H^1(k,G)} f_\tau(Z_\tau(k)).$$

Proof. From the previous construction, this is the same as showing

$$\{x \in X(k) : \zeta(x) = \tau\} = f_\tau(Z_\tau(k)).$$

For each $x \in f_\tau(Z_\tau(k))$, the fiber $Z_{\tau x}$ is a G'_τ-torsor (where $G'_\tau := {}_{T'}G$, see Example 2.15) over k with a k-rational point. By Proposition 1.16, we know that $(Z_\tau)_x = Z_x \times^G T'$ belongs to the same class as the trivial G'_τ-torsor over k. By taking the contracted product with T on the right side we get $Z_x \cong T$, this follows from the fact that $T' \times^G T$ is the trivial G'_τ-torsor (again, see Example 2.15). Hence $\zeta(x) = \tau$ as required. ☐

We mention a result of the evaluation map over a local field.

Proposition 2.19. Let k be a local field and X a proper k-variety. Let G be a finite étale algebraic group over k, and $f : Z \longrightarrow X$ a G-torsor over X. Then the image of $X(k) \longrightarrow H^1(k, G)$ is finite.

The proof can be found in [Poo17], 8.4.3. Furthermore, if char $k = 0$, a theorem of Borel and Serre [B-S64], Theorem 6.1, tells us that $H^1(k, G)$ is finite.

Now we assume that k is a global field, and for each place v of k, Chapter 1.3 gives us the restriction map between flat cohomology groups

$$H^1(k, G) \longrightarrow H^1(k_v, G).$$

For $\tau \in H^1(k, G)$, let $\tau_v \in H^1(k_v, G)$ be its image.

Definition 2.20. The (Z, G)-*Selmer set*[4] is the following subset of $H^1(k, G)$:

$$\mathrm{Sel}_{Z,X}(k, G) := \{\tau \in H^1(k, G) : \tau_v \in \mathrm{Im}(X(k_v) \longrightarrow H^1(k_v, G)) \text{ for all places } v\},$$

[4]In the case where $f : Z \longrightarrow X$ is an isogeny between abelian varieties viewed as a torsor under $\ker(f)$, this terminology is compatible with the notion of the Selmer group given in Definition 1.20, refer to [Sil92], V.4 for more details.

which we will denote by $\mathrm{Sel}_Z(k, G)$ if it is clear that we are considering X-torsors.

By applying Theorem 2.18 over each k_v, we know that τ_v is in the image of the evaluation map if and only if the cohomology class ζ of Z is mapped to $\zeta(x) = \tau$ if and only if $Z \cong T$ as G-torsors over k_v if and only if Z_τ is the trivial G_τ-torsor over k_v if and only if $Z_\tau(k_v) \neq \emptyset$. Hence

$$\mathrm{Sel}_Z(k, G) = \{\tau \in H^1(k, G) : Z_\tau(k_v) \neq \emptyset \text{ for all places } v\}$$
$$\supset \{\tau \in H^1(k, G) : Z_\tau(k) \neq \emptyset\},$$

where the extra inclusion is obvious. In particular, we have

$$X(k) = \coprod_{\tau \in \mathrm{Sel}_Z(k,G)} f_\tau(Z_\tau(k)).$$

We now arrive at the second fundamental result of standard descent theory:

Theorem 2.21. If X is a proper variety over a global field k, then $\mathrm{Sel}_Z(k, G)$ is finite.

Sketch of proof. Let F be the component group of G, then G being affine implies that F is a finite group scheme. For a suitable finite nonempty subset $S \subset \Omega_k$ containing the infinite places, [Poo17], 3.2 allows to spread out G to a smooth finite type separated group scheme $\mathcal{G}$ over $\mathcal{O}_{k,S}$, spread out X to a proper scheme $\mathcal{X}$ over $\mathcal{O}_{k,S}$ and spread out Z to a $\mathcal{G}$-torsor over $\mathcal{X}$. Let $\tau \in H^1(k, G)$. For $v \notin S$, we have the commutative diagram

$$\begin{array}{ccccc} H^1(k,G) & \xrightarrow{\tau \mapsto \tau_v} & H^1(k_v,G) & \longleftarrow & H^1(\mathcal{O}_v,\mathcal{G}) \\ & & \varphi \uparrow & & \uparrow \\ & & X(k_v) & = & \mathcal{X}(\mathcal{O}_v) \end{array}$$

where the equality is by the valuative criterion for properness, and it shows that if τ_v comes from $X(k_v)$, then τ_v also comes from $H^1(\mathcal{O}_v, \mathcal{G})$, so $\mathrm{Sel}_Z(k, G)$ is contained in the set of *unramified* torsors outside S (see [Poo17], 6.5). By Proposition 2.19, the image of φ is finite, so the image of $\mathrm{Sel}_Z(k, G)$ in $\prod_{v \in S} H^1(k_v, F)$ is finite. From [Poo17], 6.5.13, we conclude that $\mathrm{Sel}_Z(k, G)$ is finite. □

3 Descent Obstructions

For a smooth projective geometrically connected (or nice, cf. Definition 1.27) variety X over a number field k, we introduce new constraints on the location of k-rational points among the adelic points. This will result in the following chain of inclusions of obstructions on X

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}.$$

We see that the Brauer-Manin obstruction is generally the strongest of the lot, but when $X = C$ is a curve, we will prove that the sets above are all equal to one another. In particular, we have

$$C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}},$$

that is, all results that we will deduce about finite abelian descent obstructions on curves also apply to the Brauer-Manin obstruction. Most of the content in this chapter, as well as the next, will follow closely the work on finite descent obstructions by Stoll [Sto07].

3.1 Torsors under Finite Étale Group Schemes

Utilising the content on torsors and twists which we have developed from the previous sections, we now study X-torsors under finite étale group schemes. The purpose is to understand the images of the rational adelic points in the product $\prod_v H^1(k_v, G)$, where k is a number field; this will be the case from now on.

Let X be a nice variety over k. We define the category $\mathcal{C}ov(X)$. Its objects are quadruples (Y, G, μ, π), with the following descriptions:

(i) Y is an X-torsor under a finite étale (smooth) group scheme G;

(ii) $\mu : Y \times G \longrightarrow Y$ is a k-morphism describing a right action of G on Y;

(iii) $\pi : Y \longrightarrow X$ is a finite étale k-morphism such that the diagram

$$\begin{array}{ccc} Y \times G & \xrightarrow{\mu} & Y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow[\pi]{} & X \end{array}$$

is cartesian (i.e., identifies $Y \times G$ with the fiber product $Y \times_X Y$).

For brevity, we will write (Y, G) in place of an object (Y, G, μ, π), with the maps μ and π being understood. For $(Y, G), (Y', G') \in \mathcal{C}ov(X)$, a morphism

$$\alpha \in \mathrm{Hom}_{\mathcal{C}ov(X)}((Y', G'), (Y, G))$$

is given by a pair of k-morphisms of schemes

$$\phi : Y' \longrightarrow Y,\ \gamma : G' \longrightarrow G$$

such that the following diagram commutes

$$\begin{array}{ccccc} Y' \times G' & \xrightarrow{\mu'} & Y' & \xrightarrow{\pi'} & X \\ {\scriptstyle \alpha := \phi \times \gamma}\downarrow & & {\scriptstyle \phi}\downarrow & & \| \\ Y \times G & \xrightarrow[\mu]{} & Y & \xrightarrow[\pi]{} & X \end{array}$$

Remarks 3.1. (a) The "$\mathcal{C}ov$" in $\mathcal{C}ov(X)$ is an abbreviation for "cover", since we can view the étale morphism π as a cover of X.
(b) One easily observes that the map γ is uniquely determined by ϕ, since if $y' \in Y'$ and $g' \in G'$, there is a unique $g \in G$ such that $\phi(y') \cdot g = \phi(y' \cdot g')$. Hence we must have that $\gamma(g') = g$.

We will denote by $\mathcal{A}b(X)$ the full subcategory of $\mathcal{C}ov(X)$, whose objects are the torsors (Y, G) such that G is abelian. Note that if $X' \longrightarrow X$ is a k-morphism of smooth projective varieties, then we can pull back X-torsors under G to obtain X'-torsors under G. This define covariant functors $\mathcal{C}ov(X) \longrightarrow \mathcal{C}ov(X')$ and $\mathcal{A}b(X) \longrightarrow \mathcal{A}b(X')$.

The following three constructions related to twists are described for $\mathcal{C}ov(X)$, but they are also valid for $\mathcal{A}b(X)$:

- If $(Y, G) \in \mathcal{C}ov(X)$ is an X-torsor and ξ is a cohomology class in $H^1(k, G)$, then we can construct the twist (Y_ξ, G_ξ) of (Y, G) by ξ, where G_ξ is the inner twist of G. The structure maps are denoted by μ_ξ and π_ξ.

- Twists are transitive in the following sense: if $(Y, G) \in \mathcal{C}ov(X)$ is an X-torsor and $\xi \in H^1(k, G), \eta \in H^1(k, G_\xi)$, then there exists a $\zeta \in H^1(k, G)$ such that

  $$((Y_\xi)_\eta, (G_\xi)_\eta) \cong (Y_\zeta, G_\zeta).$$

  Conversely, if ξ and ζ are given, then there exists an $\eta \in H^1(k, G_\xi)$ such that the relation above holds.

- If $(\phi, \gamma) : (Y', G') \longrightarrow (Y, G)$ is a morphism and $\xi \in H^1(k, G')$, then we get an induced morphism $(Y'_\xi, G'_\xi) \longrightarrow (Y_{\gamma*\xi}, G_{\gamma*\xi})$, where $\gamma*$ is the induced map $H^1(k, G') \longrightarrow H^1(k, G)$.

Before proceeding further, we note a particular subtlety concerning the v-adic components of $X(\mathbb{A}_k)$ at the infinite places. Since X is projective, we know that

$$X(\mathbb{A}_k) = \prod_v X(k_v).$$

However, if we attempt to study the constraints on where the k-rational points can lie inside of $X(\mathbb{A}_k)$, the information we get only tells us on which connected component a point can lie at the infinite places [Sto07]. For this reason, we make a slight modification by replacing the v-adic component of $X(\mathbb{A}_k)$ with its set of connected components, for infinite v,

$$X(\mathbb{A}_k)_\bullet := \prod_{v \text{ finite}} X(k_v) \times \prod_{v \text{ infinite}} \pi_0(X(k_v)).$$

This gives us a canonical surjection from $X(\mathbb{A}_k)$ onto $X(\mathbb{A}_k)_\bullet$, which is an isomorphism (in fact, an equality) if X is a reduced finite scheme. For a finite extension K/k, we shall assume that the canonical map $X(\mathbb{A}_k)_\bullet \longrightarrow X(\mathbb{A}_K)_\bullet$ is an inclusion, even though in general this is not true at the infinite places. For example, let $X = E$ be an elliptic curve, and $k = \mathbb{Q}, K/\mathbb{Q}$ be a finite extension. Then the number of connected components of $E(\mathbb{Q}_\infty) = E(\mathbb{R})$ could possibly be 2, but an elliptic curve over $\mathbb{C}$ is a torus and so

$$|\pi_0(E(\mathbb{R}))| \geq |\pi_0(E(\mathbb{C}))| = 1,$$

proving that we may not have an inclusion of connected components at the infinite places.

We are now set to explore what we are mainly interested in: the information we can obtain from the various torsors regarding the image of $X(k)$ in $X(\mathbb{A}_k)_\bullet$.

Definition 3.2. Let $(Y, G) \in \mathcal{C}ov(X)$ be an X-torsor. We say that a point $P \in X(\mathbb{A}_k)_\bullet$ *survives* (Y, G), if it lifts to a point in $Y_\xi(\mathbb{A}_k)_\bullet$ for some twist (Y_ξ, G_ξ) of (Y, G).

To understand the intuition behind the definition, we can adopt a cohomological approach. An X-torsor Y is an element of $H^1(X_{\text{ét}}, G)$. Recall (cf. Remark 2.17) that the pull-back of $Y \longrightarrow X$ through the map Spec $k \longrightarrow X$ corresponding to a point in $X(k)$ induces the evaluation map

$$X(k) \longrightarrow H^1(k, G).$$

Therefore we get a similar map on adelic points:

$$X(\mathbb{A}_k)_\bullet \longrightarrow \prod_v H^1(k_v, G).$$

Together with the canonical restriction map

$$H^1(k, G) \longrightarrow \prod_v H^1(k_v, G),$$

we get the commutative diagram

$$\begin{array}{ccc} X(k) & \longrightarrow & H^1(k,G) \\ \downarrow & & \downarrow \\ X(\mathbb{A}_k)_\bullet & \longrightarrow & \prod_v H^1(k_v,G) \end{array}$$

A point $P \in X(\mathbb{A}_k)_\bullet$ survives (Y,G) if and only if its image in $\prod_v H^1(k_v,G)$ is in the image of the restriction map. To see this, suppose $P = (x_v)$ lifts to a point $P_\xi \in Y_\xi(\mathbb{A}_k)_\bullet$. We know that the image of $\xi \in H^1(k,G)$ in $\prod_v H^1(k_v,G)$ is $(\xi(x_v))$, and that the image of P in $\prod_v H^1(k_v,G)$ is precisely the image of P_ξ in the composite

$$Y_\xi(\mathbb{A}_k)_\bullet \longrightarrow X(\mathbb{A}_k)_\bullet \longrightarrow \prod_v H^1(k_v,G).$$

Consider the v-adic component P_{ξ_v} of P_ξ. We have

$$P_{\xi_v} \longmapsto x_v \longmapsto \xi(x_v)$$

which proves that the image of P_ξ in $\prod_v H^1(k_v,G)$ is $(\xi(x_v))$ as required. The converse is clear from this argument.

Remark 3.3. One easily observes that the preimage in $H^1(k,G)$ of the image of $X(\mathbb{A}_k)_\bullet$ is precisely the (Y,G)-Selmer set.

We end this section with the following results are some basic properties of the survivability of a point:

Lemma 3.4.

(a) If $(\phi,\gamma) : (Y',G') \longrightarrow (Y,G)$ is a morphism in $\mathcal{C}ov(X)$, and if $P \in X(\mathbb{A}_k)_\bullet$ survives (Y',G'), then P also survives (Y,G).

(b) If $(Y',G) \in \mathcal{C}ov(X)$ is the pull-back of $(Y,G) \in \mathcal{C}ov(X)$ under a morphism $\psi : X' \longrightarrow X$, then $P \in X'(\mathbb{A}_k)_\bullet$ survives (Y',G) if and only if $\psi(P)$ survives (Y,G).

(c) If $(Y,G) \in \mathcal{C}ov(X)$ and $\xi \in H^1(k,G)$, then $P \in X(\mathbb{A}_k)_\bullet$ survives (Y,G) if and only if P survives (Y_ξ, G_ξ).

Proof.

(a) Since P survives (Y',G'), there exist a $\xi \in H^1(k,G')$, a $Q \in Y'_\xi(\mathbb{A}_k)_\bullet$ and a morphism

$$\pi'_\xi : Y'_\xi(\mathbb{A}_k)_\bullet \longrightarrow X(\mathbb{A}_k)_\bullet$$

such that $\pi'_\xi(Q) = P$. We also have an induced morphism

$$\phi_\xi : Y'_\xi \longrightarrow Y_{\gamma * \xi}$$

defined over X. Hence $\pi_{\gamma*\xi}(\phi_\xi(Q)) = \pi'_\xi(Q) = P$, where $\pi_{\gamma*\xi} : Y_{\gamma*\xi} \longrightarrow X$.

(b) Assume that P survives (Y', G), then there exist $\xi \in H^1(k, G), Q \in Y'_\xi(\mathbb{A}_k)_\bullet$ such that $\pi'_\xi(Q) = P$. There is a morphism $\Psi_\xi : Y'_\xi \longrightarrow Y_\xi$ over ψ, hence we have that $\pi_\xi(\Psi_\xi(Q)) = \psi(P)$, so $\psi(P)$ survives (Y, G). Conversely, assume that $\psi(P)$ survives (Y, G), then there exist $\xi \in H^1(k, G)$ and $Q \in Y_\xi(\mathbb{A}_k)_\bullet$ such that $\pi_\xi(Q) = \psi(P)$. The twist (Y'_ξ, G_ξ) is the pull-back of (Y_ξ, G_ξ) under ψ, as seen in the commutative diagram

$$\begin{CD} Y'_\xi := Y_\xi \times_X X' @>{\pi'_\xi}>> X' \\ @V{\Psi_\xi}VV @VV{\psi}V \\ Y_\xi @>>{\pi_\xi}> X \end{CD}$$

and so there exists $Q' \in Y'_\xi(\mathbb{A}_k)_\bullet$ mapping to Q in Y_ξ and mapping to P in X'. Hence P survives (Y', G).

(c) This is clear since every twist of (Y, G) is also a twist of (Y_ξ, G_ξ) and vice versa.

□

3.2 Finite Descent Obstructions

Our goal here is to introduce a new form of obstruction arising from adelic points surviving a particular class of torsors. Such an obstruction is known as finite descent obstruction, named so because such points are descended from the adelic points of twisted torsors under a finite étale group scheme. We will also say a few words regarding the geometrical connectedness of the variety X.

What can be said about the survivability of the k-rational points of X (via the embedding into $X(\mathbb{A}_k)_\bullet$)? By Chapter 2.3, the first fundamental descent theorem (Theorem 2.18) tells us that every point $P \in X(k)$ can be lifted to a point in $Y_\xi(k)$ for any X-torsor Y and some $\xi \in H^1(k, G)$ and by the commutative diagram

$$\begin{CD} Y_\xi(k) @>>> Y_\xi(\mathbb{A}_k)_\bullet \\ @VVV @VVV \\ X(k) @>>> X(\mathbb{A}_k)_\bullet \end{CD}$$

it is clear that the image of P in $X(\mathbb{A}_k)_\bullet$ survives (Y, G). So P survives every X-torsor. However, the other fundamental theorem (Theorem 2.21) forces $\mathrm{Sel}_Y(k, G)$ to be finite, hence there are only finitely many twists (Y_ξ, G_ξ) such that Y_ξ has points everywhere locally. Our task of locating k-rational points within the adelic points has therefore been shifted to that of studying the set of adelic points that survive every torsor in a suitable subclass of torsors.

The above discussion motivates the following definitions:

Definition 3.5. Let X be a nice k-variety. We set

(1) $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} := \{P \in X(\mathbb{A}_k)_\bullet : P \text{ survives all } (Y, G) \in \mathcal{C}ov(X)\}$;

(2) $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} := \{P \in X(\mathbb{A}_k)_\bullet : P \text{ survives all } (Y, G) \in \mathcal{A}b(X)\}$,

where the "f" in the superscript stands for "finite", since we are dealing with torsors under finite group schemes only.

Note that for a given X-torsor Y, the set of points in $X(\mathbb{A}_k)_\bullet$ surviving Y is given as the union of the images of the maps

$$\pi_\xi : Y_\xi(\mathbb{A}_k)_\bullet \longrightarrow X(\mathbb{A}_k)_\bullet,$$

where ξ runs over all cohomology classes in $H^1(k, G)$. By Theorem 2.21, we know that this union is finite. Since $Y_\xi \longrightarrow X$ is proper, one can easily show (see [Poo17], 2.6 and Exercise 8.7) that $Y_\xi(k_v) \longrightarrow X(k_v)$ is a proper map of topological spaces (i.e., the inverse image of any compact subset of $X(k_v)$ is compact) for each v and thus $Y_\xi(k_v)$ is compact. It follows that $\pi_\xi(Y_\xi(\mathbb{A}_k)_\bullet)$ is closed in $X(\mathbb{A}_k)_\bullet$. Hence

$$\bigcup_{\xi \in H^1(k,G)} \pi_\xi(Y_\xi(\mathbb{A}_k)_\bullet)$$

is closed in $X(\mathbb{A}_k)_\bullet$.

The sets given in Definition 3.5 are therefore closed in $X(\mathbb{A}_k)_\bullet$ and this gives us the following chain of inclusions

$$X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet,$$

where $\overline{X(k)}$ is the topological closure of $X(k)$ in $X(\mathbb{A}_k)_\bullet$.

Much like how the Brauer-Manin obstruction discussed in Chapter 1.4 comes in the form of the Brauer set $X(\mathbb{A}_k)^{\text{Br}}$ (which from now on should be written as $X(\mathbb{A}_k)_\bullet^{\text{Br}}$, and we also have $\overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{Br}}$), one would expect that the finite descent obstructions come in the form of the sets $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ and $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. This motivates us to provide an interpretation of them similar to that of $X(\mathbb{A}_k)_\bullet^{\text{Br}}$.

Fix a finite étale group scheme G over k and choose a point $P = (x_v) \in X(\mathbb{A}_k)_\bullet$, so we have $x_v \in X(k_v)$. If ξ is the cohomology class of Y in $H^1(X_{\text{ét}}, G)$, we denote by $\xi(x_v)$ the class of the k_v-torsor $Y_{x_v} \longrightarrow \{x_v\}$ in $H^1(k_v, G)$. Thus P determines the morphism which we shall loosely call the "evaluation map"

$$\text{ev}_{P,G} : H^1(X_{\text{ét}}, G) \longrightarrow \prod_v H^1(k_v, G).$$

Together with the usual restriction map

$$\operatorname{res}_G : H^1(k, G) \longrightarrow \prod_v H^1(k_v, G),$$

we have

$$X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \bigcap_G \{P \in X(\mathbb{A}_k)_\bullet : \operatorname{Im}(\operatorname{ev}_{P,G}) \subset \operatorname{Im}(\operatorname{res}_G)\},$$

where G runs through all finite étale group schemes over k. We obtain $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ by restricting G to abelian group schemes. Therefore these sets are cut out by restrictions coming from the various finite coverings of X.

Definition 3.6. We say that there is a *finite descent obstruction to the Hasse principle for* X if $X(\mathbb{A}_k)_\bullet \neq \emptyset$ but $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \emptyset$. In the case of $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, we call it *finite abelian descent obstruction.*

In the reformulation of $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$ above, we can restrict to (Y, G) with Y connected over k if X is connected. The reason is that if Y is not connected, we can replace Y with one of its connected component Y_0, and let $G_0 \subset G$ be the stabilizer of this component. Then (Y_0, G_0) is again a torsor of the same kind as (Y, G), which gives us a morphism $(Y_0, G_0) \longrightarrow (Y, G)$. By Lemma 3.4(a), if $P \in X(\mathbb{A}_k)_\bullet$ survives (Y_0, G_0), then it also survives (Y, G).

Remarks 3.7. (a) However, we cannot restrict to geometrically connected torsors when X is geometrically connected. This is because there can be obstructions coming from the fact that a suitable geometrically connected torsor does not exist.
(b) Suppose that X is geometrically connected. If there is a torsor $(Y, G) \in \mathcal{C}ov(X)$ such that Y and all twists Y_ξ are k-connected, but not geometrically connected, then $X(\mathbb{A}_k)_\bullet = \emptyset$. The analogous statement holds for the abelian version.
(c) Let $\overline{X} = X \times_k \bar{k}$. Then X being geometrically connected yields the following results concerning its geometric (étale) fundamental group $\pi_1(\overline{X})$ over $\bar{k}$:

(i) If $\pi_1(\overline{X})$ is trivial, then $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$.

(ii) If the abelianization $\pi_1(\overline{X})^{\text{ab}}$ is trivial, then $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet$.

(iii) If $\pi_1(\overline{X})$ is abelian, then $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

The latter two remarks are respectively Lemma 5.5 and Lemma 5.8 of [Sto07].

We have already seen the natural inclusion $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ by Definition 3.5, and so this tells us that finite descent obstruction is *stronger* than finite abelian descent obstruction. Since the main focus of this paper is on the latter obstruction, at the end of this chapter, we will explore how the strength of this obstruction compares to that of the obstruction arising from the Brauer set. Therefore, it would help by noting some elementary properties of the set $X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, which hold similarly for $X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. With the exception of the immediate proposition, the other statements are given without proof, see [Sto07], Chapter 5.

Proposition 3.8. If $\psi : X' \longrightarrow X$ is a morphism, then $\psi(X'(\mathbb{A}_k)_\bullet^{\text{f-ab}}) \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

Proof. Let $P \in X'(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, and let $(Y, G) \in \mathcal{A}b(X)$ be an X-torsor. This implies that P survives the pull-back (Y', G) of (Y, G) under ψ and so by Lemma 3.4(b), $\psi(P)$ survives (Y, G). Since (Y, G) was arbitrary, $\psi(P) \in X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. □

Proposition 3.9. If $X = X_1 \coprod X_2 \cdots X_n$ is a disjoint union, then

$$X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \coprod_{j=1}^{n} X_j(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

Proposition 3.10. We have

$$(X \times Y)(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \times Y(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

Proposition 3.11. If K/k is a finite extension, then

$$X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet \cap X(\mathbb{A}_K)_\bullet^{\text{f-ab}}.$$

The intersection is viewed as the pull-back of $X(\mathbb{A}_K)_\bullet^{\text{f-ab}}$ under the canonical map $X(\mathbb{A}_k)_\bullet \longrightarrow X(\mathbb{A}_K)_\bullet$, which may not be injective at the infinite places, as we have already seen.

3.3 The Albanese Variety

Let V be a k-variety. We can associate to V an abelian variety that gives rise to another form of obstruction. Our goal here is to compare this obstruction to that coming from finite abelian descent.

Let $f : V \longrightarrow A$ be a rational map[5] from V into an abelian variety A. We say that A is *generated* by (V, f) if A is the group generated by $f(V)$. After a suitable translation, we may assume that the image of V in A goes through the origin, and in this case one sees that (V, f) generates A if and only if the smallest abelian subvariety of A that contains $f(V)$ is equal to A.

Definition 3.12. An *Albanese variety* of V is a pair (Alb_V^0, f) consisting of an abelian variety Alb_V^0 and a rational map $f : V \longrightarrow \mathrm{Alb}_V^0$, called the *Albanese morphism*, such that:

(i) (V, f) generates Alb_V^0;

(ii) for every rational map $g : V \longrightarrow B$ of V into an abelian variety B, there exists a homomorphism $g_* : \mathrm{Alb}_V^0 \longrightarrow B$ and a constant $c \in B$ such that $g = g_* f + c$.

[5]This arrow is not the usual way to denote a rational map but in the case where V is smooth, f will extend to a morphism defined on all of V, which will be what we want it to be.

For convenience we usually refer directly to Alb_V^0 as the Albanese variety of V.

Observe that the homomorphism g_* is uniquely determined by g. Suppose we have $g = g'f + c'$, for another homomorphism g'. Let u be a generic point of Alb_V^0 and $p_1, ..., p_n$ be independent generic points of V such that $u = \sum_{i=1}^n f(p_i)$. We have $g(p_i) = g_* f(p_i) + c = g' f(p_i) + c'$. By the linearity of g_* (and of g'), taking the sum gives us

$$g_*(u) = g'(u) + c_0$$

for some $c_0 \in B$. Then both g_* and g' sends 0_B to 0_A and so $c_0 = 0$. Hence $g_* = g'$, and we say that g_* is the homomorphism *induced* by g.

In the following result we note the existence and uniqueness of the Albanese variety, see [Lan59].

Theorem 3.13. Let V be a variety. Then there exists an Albanese variety Alb_V^0 of V, uniquely determined up to a birational isomorphism, and the canonical map f is determined up to a translation.

Remarks 3.14. (a) The notation Alb_V^0 of the Albanese variety of V comes from the fact that this is the dual of the *Picard variety* Pic_V^0, which was introduced in Chapter 1.3 as being isomorphic to J_V.
(b) The Albanese variety can be viewed as the generalization of the Jacobian variety of a curve to higher dimensions. In other words, if V is a curve, then $\mathrm{Alb}_V^0 \cong J_V$. This is a consequence of the Abel-Jacobi theorem.

To avoid the technicalities of the dual abelian variety, we emphasise the fact that the following theorem is only concerned with the case of curves, but we state the rest of it in full generality. It is a deep result, and we provide at best an outline of the proof.

Theorem 3.15. Let C be a nice (cf. Definition 1.27) curve over k. Let $A = \mathrm{Alb}_C^0$ be its Albanese variety, and let $V = \mathrm{Alb}_C^1$ be the torsor under A that parametrizes the divisor classes of degree 1 on C. Then there is a canonical map $\phi : C \longrightarrow V$, and we have

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}).$$

Sketch of proof. Since C is a curve, by Remark 3.14(b), we have $\mathrm{Alb}_C^0 \cong J_C \cong \mathrm{Pic}_C^0$. The existence of ϕ is a result of Serre [Ser60] stating that there is a universal object among morphisms from a nice variety X to torsors under (semi-) abelian varieties: the *Albanese torsor* Alb_X^1. In fact, ϕ sends a point $P \in X$ to its divisor class. Now, observe that the inclusion

$$\phi(C(\mathbb{A}_k)_\bullet^{\text{f-ab}}) \subset V(\mathbb{A}_k)_\bullet^{\text{f-ab}}$$

is given by Proposition 3.8. So it suffices to prove the other inclusion. By [Ser88], VI.2, all connected finite abelian unramified coverings of $\overline{C} = C \times_k \bar{k}$ are obtained through pull-backs from isogenies into $\overline{V} \cong \bar{A}$. Then after some work (using spectral sequences) in [Sto07], Theorem 6.4, we have that the induced homomorphism

$$\phi^* : H^1(V_{\text{ét}}, G) \longrightarrow H^1(C_{\text{ét}}, G)$$

is an isomorphism. Let $P \in C(\mathbb{A}_k)_\bullet$ such that $\phi(P) \in V(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, and let $(Y, G) \in \mathcal{A}b(C)$. By the isomorphism ϕ^*, there exists a $(W, G) \in \mathcal{A}b(V)$ such that Y is the pull-back of W under ϕ. By assumption, $\phi(P)$ survives (W, G). Since G is abelian, it is equal to all its inner twists, and so we can assume further that the lift of $\phi(P)$ is in (W, G). Hence the lift of P is in (Y, G) and since (Y, G) was arbitrary, $P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. □

Remarks 3.16. (a) In fact, we do not need X to be nice in order for the canonical map into the Albanese torsor to exist, it just has to be smooth and geometrically connected. This is a consequence of Galois descent. However, if we further include the condition that X is proper, then the Albanese variety is really an abelian variety, not just a semi-abelian one, see [Wit80].
(b) The result in the preceding theorem will hold for a general nice variety X instead of curves if all finite étale abelian covering of $\overline{X}$ can be obtained as pull-backs of isogenies into the Albanese variety of X. For this, it is necessary and sufficient that the *Néron-Severi group* $\mathrm{NS}_X := \mathrm{Pic}_X/\mathrm{Pic}_X^0$ of X is torsion-free, see [Ser88], VI.20.

For arbitrary varieties X, let A be its Albanese variety and let V be a k-torsor under A such that we get the map $\phi : X \longrightarrow V$ as in Theorem 3.15. We can define a set $X(\mathbb{A}_k)_\bullet^{\text{Alb}}$, which we call the *Albanese set*, consisting of the adelic points on X surviving all torsors that are pull-backs of V-torsors. For $(W, A) \in \mathcal{A}b(V)$, let (Y, A) be its pull-back under ϕ. Applying Lemma 3.4(b), we have

$$P \in V(\mathbb{A}_k)_\bullet^{\text{f-ab}} \Longleftrightarrow \phi^{-1}(P) \in X(\mathbb{A}_k)_\bullet^{\text{Alb}}$$

and so

$$X(\mathbb{A}_k)_\bullet^{\text{Alb}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}).$$

By Proposition 3.8 we have the obvious inclusion $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\text{Alb}}$. Therefore we can say that finite abelian descent obstruction is at least as strong as the obstruction arising from adelic points surviving the pull-backs of torsors under the Albanese variety. We shall refer to the latter obstruction as the *Albanese obstruction*. Theorem 3.15 tells us that both obstructions are equal in the case of curves.

3.4 Comparing Obstructions

In this section we will prove the main result of this paper: for a nice curve over k, the obstructions coming from the Brauer set and finite abelian descent are equivalent.

Let X be a nice k-variety, let A be its Albanese variety, and denote by V the Albanese torsor such that there is a canonical map $\phi : X \longrightarrow V$. If $V(k) \neq \emptyset$, then V is the trivial torsor, and so there is an *n-covering* $W_n \longrightarrow V$ of V, i.e., a

V-torsor under the torsion group $A[n]$ (see [Sko01], III, for a proper treatment on n-coverings). So the non-existence of any such W_n is an obstruction against rational points on V and therefore on X.

If an n-covering of V exists, we can pull it back to a torsor $(Y_n, A[n]) \in \mathcal{A}b(X)$, and we say that a point $P \in X(\mathbb{A}_k)_\bullet$ *survives the n-covering of X* if it survives $(Y_n, A[n])$. If for some n no n-covering exist, then by definition no point in $X(\mathbb{A}_k)_\bullet$ survives the n-covering of X. If we denote the set of adelic points surviving the n-covering of X by $X(\mathbb{A}_k)_\bullet^{\text{n-ab}}$, then we have

$$X(\mathbb{A}_k)_\bullet^{\text{Alb}} = \bigcap_{n\geq 1} X(\mathbb{A}_k)_\bullet^{\text{n-ab}},$$

where the intersection is taken over all n such that an n-covering exists. In particular, for a curve C, Theorem 3.15 tells us that

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \bigcap_{n\geq 1} C(\mathbb{A}_k)_\bullet^{\text{n-ab}}.$$

The new interpretation of $X(\mathbb{A}_k)_\bullet^{\text{Alb}}$ would enable us to establish deeper connections between finite abelian descent obstructions and other obstructions related to the Brauer group of X.

Recall (cf. Chapter 1.4) that the Brauer set $X(\mathbb{A}_k)_\bullet^{\text{Br}}$ of X is defined to be the intersection of the sets

$$X(\mathbb{A}_k)_\bullet^{P} = \{(x_v) \in X(\mathbb{A}_k)_\bullet : (P, (x_v)) = 0\}$$

across all points $P \in \text{Br}(X)$. We define the *algebraic part* of $\text{Br}(X)$:

$$\text{Br}_1(X) := \ker(\text{Br}(X) \longrightarrow \text{Br}(X \times_k \bar{k})) \subset \text{Br}(X)$$

and set $X(\mathbb{A}_k)_\bullet^{\text{Br}_1} = X(\mathbb{A}_k)_\bullet^{\text{Br}_1(X)}$.

The work on descent theory by Colliot-Thélène and Sansuc [CoS87] and extended by Skorobogatov states that $X(\mathbb{A}_k)_\bullet^{\text{Br}_1}$ is equal to the set obtained from finite descent obstructions with respect to torsors under k-groups G of multiplicative type, which includes all finite abelian k-groups. Therefore, we have the following result:

Theorem 3.17. Let X be a smooth projective geometrically connected variety, then

$$X(\mathbb{A}_k)_\bullet^{\text{Br}} \subset X(\mathbb{A}_k)_\bullet^{\text{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

Proof. See, for example, [Sko01], Theorem 6.1.1. □

Also, we have:

Proposition 3.18. Let X be a proper and geometrically integral variety over k. Then there is an exact sequence

$$\mathrm{Br}(k) \xrightarrow{\alpha} \mathrm{Br}_1(X) \xrightarrow{\phi} H^1(k, \mathrm{Pic}_X) \longrightarrow H^3(k, \bar{k}^\times).$$

Proof. See [Poo17], Corollary 6.7.8. □

Note that we can invoke this result as X is both smooth and geometrically connected (since it is nice), hence geometrically integral ([Poo17], Proposition 3.5.67). Furthermore, [Poo17], Remark 6.7.10 tells us that for a local or global field k, the last term in the exact sequence is 0. Thus ϕ is surjective. If we set $\mathrm{Br}_0(X) = \mathrm{Im}(\alpha)$ (just to be consistent with the notation in [Sto07]), we have

$$\mathrm{Br}_1(X)/\mathrm{Br}_0(X) \cong H^1(k, \mathrm{Pic}_X).$$

Now let $\varphi : H^1(k, \mathrm{Pic}^0_X) \longrightarrow H^1(k, \mathrm{Pic}_X)$ be the canonical map and for each $n \geq 1$, we define the restriction

$$\varphi_n : H^1(k, \mathrm{Pic}^0_X)[n] \longrightarrow H^1(k, \mathrm{Pic}_X).$$

Using the fact that both ϕ and φ are group homomorphisms, we denote by $\mathrm{Br}_{1/2}(X)$ the subgroup of $\mathrm{Br}_1(X)$ that maps into the image of φ and by $\mathrm{Br}_{1/2,n}(X)$ the subgroup of $\mathrm{Br}_1(X)$ that maps into the image of φ_n. It is then clear that

$$\mathrm{Br}_{1/2}(X) = \bigcup_{n\geq 1} \mathrm{Br}_{1/2,n}(X).$$

And so

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}} = \bigcap_{n\geq 1} X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}.$$

For a fixed $n \geq 1$, the next theorem ([Sto07], Theorem 7.2) establishes a relation between the obstruction coming from an n-covering and that coming from the subgroup of the algebraic part of the Brauer group mapping into $\mathrm{Im}(\varphi_n)$. This in turn induces a relation between the Albanese set and the set defined above.

Theorem 3.19. Let X be a smooth projective geometrically connected variety, and let $n \geq 1$. Then

$$X(\mathbb{A}_k)_\bullet^{\text{n-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}.$$

In particular,

$$X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}.$$

Proof. Let $P \in X(\mathbb{A}_k)_\bullet^{\text{n-ab}}$ and $b \in \mathrm{Br}_{1/2,n}(X)$. We want to show that under the Brauer-Manin pairing between $X(\mathbb{A}_k)_\bullet$ and $\mathrm{Br}(X)$, we have $(P, b) = 0$. From the sequence of maps

$$\mathrm{Br}_{1/2,n}(X) \subset \mathrm{Br}_1(X) \longrightarrow \frac{\mathrm{Br}_1(X)}{\mathrm{Br}_0(X)} \cong H^1(k, \mathrm{Pic}_X) \longleftarrow H^1(k, \mathrm{Pic}^0_X)[n],$$

let b' denote the image of b in $H^1(k, \mathrm{Pic}_X)$, and let $b'' \in H^1(k, \mathrm{Pic}^0_X)[n]$ be an element mapping to b', which exists by the definition of $\mathrm{Br}_{1/2,n}(X)$.

Now, as before, let A be the Albanese variety of X, and let V be the Albanese torsor with a canonical map $\phi : X \longrightarrow V$. Since V is a torsor under A, we have that $\mathrm{Alb}^0_V \cong \mathrm{Alb}^0_A = A = \mathrm{Alb}^0_X$. Therefore, taking their duals we get

$$\mathrm{Pic}^0_X \cong \mathrm{Pic}^0_A \cong \mathrm{Pic}^0_V.$$

Since $P \in X(\mathbb{A}_k)^{\text{n-ab}}_{\bullet} \xrightarrow{\phi} V(\mathbb{A}_k)^{\text{n-ab}}_{\bullet}$, we know that the latter set contains $\phi(P)$ and hence it is nonempty. By the definition of $V(\mathbb{A}_k)^{\text{n-ab}}_{\bullet}$, it follows that V admits a torsor of the form $(W_n, A[n])$. This implies that there is some twist of $(W_n, A[n])$ such that $\phi(P)$ lifts to it. We can assume that $(W_n, A[n])$ is already this twist, so there is some $Q \in W_n(\mathbb{A}_k)_{\bullet}$ such that $\gamma(Q) = \phi(P)$, where $\gamma : W_n \longrightarrow V$ is the covering map associated to $(W_n, A[n])$.

Let $(Y_n, A[n]) \in \mathcal{A}b(X)$ be the pull-back of $(W_n, A[n])$ to X. By Lemma 3.4(b), the diagram

$$\begin{array}{ccc} Y_n & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow \phi \\ W_n & \xrightarrow[\gamma]{} & V \end{array}$$

tells us that there is some $R \in Y_n(\mathbb{A}_k)_{\bullet}$ such that $\pi(R) = P$. Using the fact that the Picard functor is contravariant, we have the induced commutative diagram

$$\begin{array}{ccccccc} \mathrm{Pic}_{Y_n} & \longleftarrow & \mathrm{Pic}^0_{Y_n} & \longleftarrow & \mathrm{Pic}^0_{W_n} & \xleftarrow{\cong} & \mathrm{Pic}^0_A \\ \pi^* \uparrow & & \pi^* \uparrow & & \uparrow \gamma^* & & \uparrow \cdot n \\ \mathrm{Pic}_X & \longleftarrow & \mathrm{Pic}^0_X & \xleftarrow{\cong} & \mathrm{Pic}^0_V & \xleftarrow{\cong} & \mathrm{Pic}^0_A \end{array}$$

where the map on the rightmost column is multiplication-by-n. Applying $H^1(k, -)$ to the diagram, we start from $b'' \in H^1(k, \mathrm{Pic}^0_X)$, whose image corresponds to a 1-cocycle

$$f : \mathrm{Gal}(\bar{k}/k) \longrightarrow n \cdot \mathrm{Pic}^0_X$$

in $H^1(k, \mathrm{Pic}^0_{W_n})$. Since W_n is an $A[n]$-torsor, we see that f must be the trivial homomorphism. Hence $\pi^*(b') = 0 \in H^1(k, \mathrm{Pic}_{Y_n}) \cong \mathrm{Br}_1(Y_n)/\mathrm{Br}_0(Y_n)$. Treating b' as an element in $\mathrm{Br}(X)$, we can identify it with b. Therefore,

$$(P, b) = (\pi(R), b') = (\pi^*(b'), R) = 0$$

as desired. The second statement easily follows from the first. □

Combining the preceding theorem with Theorem 3.17, we obtain the chain of inclusions

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}} \subset X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}.$$

However, again by the descent theory of Colliot-Thélène and Sansuc, the last inclusion is in fact an equality, since we actually have $X(\mathbb{A}_k)_\bullet^{\text{n-ab}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2,n}}$. Indeed, let $M = \mathrm{Pic}^0_X[n]$, and let $\lambda : M \longrightarrow \mathrm{Pic}_X$ be the inclusion, and note that its dual is $\mathrm{Alb}^0_X[n]$. Then the n-coverings of X are exactly the *torsors of type* λ, see [Sko01]. We have $\mathrm{Br}_\lambda = \mathrm{Br}_{1/2,n}$, and the result follows from Theorem 6.1.2(a) of [Sko01].

It is therefore natural to ask if there are any other equalities amongst the remaining inclusions. For the case of curves, we arrive at the main result of this paper ([Sto07], Corollary 7.3):

Theorem 3.20. Let C be a smooth projective geometrically connected curve over k. Then

$$C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C(\mathbb{A}_k)_\bullet^{\mathrm{Br}_{1/2}}.$$

In particular,

$$C(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

Proof. Note that the curve $\overline{C} = C \times_k \bar{k}$ has trivial Brauer group. This is a consequence of Tsen's theorem (see, for example, [Bri18], Chapter III). Hence $\mathrm{Br}_1(C) = \mathrm{Br}(C)$. Also, we have the short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0_C \longrightarrow \mathrm{Pic}_C \longrightarrow \mathrm{NS}_C \longrightarrow 0.$$

Since C is a curve, we have $\mathrm{NS}_C = \mathbb{Z}$. Taking $H^1(k, -)$ throughout, we get an exact sequence

$$H^1(k, \mathrm{Pic}^0_C) \longrightarrow H^1(k, \mathrm{Pic}_C) \longrightarrow H^1(k, \mathbb{Z}).$$

Now, $\mathrm{Gal}(\bar{k}/k)$ acts trivially on $\mathbb{Z}$, so $H^1(k, \mathbb{Z}) = \mathrm{Hom}(\mathrm{Gal}(\bar{k}/k), \mathbb{Z})$. The absolute Galois group is the inverse limit of finite Galois extensions of K of k, and since $\mathbb{Z}$ is torsion-free,

$$\mathrm{Hom}(\mathrm{Gal}(K/k), \mathbb{Z}) = 0.$$

Thus $H^1(k, \mathbb{Z}) = 0$ and we have that $H^1(k, \mathrm{Pic}^0_C) \longrightarrow H^1(k, \mathrm{Pic}_C)$ is a surjection. It follows that $\mathrm{Br}_{1/2}(C) = \mathrm{Br}_1(C)$. □

Remark 3.21. For an arbitrary nice k-variety X, if we impose the condition that $H^1(k, \mathrm{Pic}^0_X) \longrightarrow H^1(k, \mathrm{Pic}_X)$ is a surjection, then

$$X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(\mathbb{A}_k)_\bullet^{\mathrm{Alb}}.$$

In general it is not true that $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. For example, a smooth cubic surface X in $\mathbb{P}^3$ has trivial geometric fundamental group and thus $X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(\mathbb{A}_k)_\bullet$ by Remark 3.7(c)(i). It may have points everywhere locally but no rationl points since $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}_1} = \emptyset$ (see [CKS87]).

4 Obstructions on Curves

Having established Theorem 3.20 (which was the main result that we wanted to prove) in the previous chapter, we get a glance of the interesting aspects of curves that we can look into. Our goal of this chapter is to use this theorem to show that, under suitable conditions, the absence of rational points on a curve is explainable by the Brauer-Manin obstruction.

4.1 Properties of Curves by Genus

We introduce some notions about nice (cf. Definition 1.27) varieties related to the finite descent obstructions described in the previous chapter. With particular interest in the case of curves, we establish basic properties connecting these notions with their genera.

Let $X = A$ be an abelian variety over k, then

$$\prod_{v \text{ finite}} \{0\} \times \prod_{v \text{ infinite}} A(k_v)^0 = A(\mathbb{A}_k)_{\mathrm{div}}$$

is the divisible subgroup of $A(\mathbb{A}_k)$. Hence we have

$$A(\mathbb{A}_k)_\bullet / nA(\mathbb{A}_k)_\bullet = A(\mathbb{A}_k)/nA(\mathbb{A}_k).$$

Taking inverse limits, we obtain $A(\mathbb{A}_k)_\bullet = \widehat{A(\mathbb{A}_k)}$, where the latter is the profinite completion of $A(\mathbb{A}_k)$.

Let $\widehat{A(k)}$ be the profinite completion of the Mordell-Weil group $A(k)$. By [Ser71], Theorem 3, the natural map $\widehat{A(k)} \longrightarrow \widehat{A(\mathbb{A}_k)} = A(\mathbb{A}_k)_\bullet$ is an injection and therefore induces an isomorphism with the topological closure $\overline{A(k)}$ of $A(k)$ in $A(\mathbb{A}_k)_\bullet$. We have an exact sequence

$$0 \longrightarrow A(k)/nA(k) \longrightarrow \mathrm{Sel}^n(A(k)) \longrightarrow Ш(A(k))[n] \longrightarrow 0.$$

involving the n-Selmer group and the Tate-Shafarevich group (cf. Chapter 1.3). If $n|m$, we have canonical maps

$$A(k)/mA(k) \longrightarrow A(k)/nA(k),$$

$$\mathrm{Sel}^m(A(k)) \longrightarrow \mathrm{Sel}^n(A(k)),$$

$$Ш(A(k))[m] \longrightarrow Ш(A(k))[n].$$

Forming the projective limit $\widehat{\mathrm{Sel}}(A(k)) := \varprojlim \mathrm{Sel}^n(A(k))$ and the Tate module $TШ(A(k))$, we again have an exact sequence

$$0 \longrightarrow \widehat{A(k)} \longrightarrow \widehat{\mathrm{Sel}}(A(k)) \longrightarrow TШ(A(k)) \longrightarrow 0.$$

Remark 4.1. If $Ш(A(k))$ is finite (cf. Conjecture 1.19), or, more generally, if the divisible subgroup $Ш(A(k))_{\mathrm{div}}$ is trivial, then the Tate module vanishes. This implies that $\widehat{A(k)} \cong \widehat{\mathrm{Sel}}(A(k))$.

Now, for a nice variety X, recall that we have the following chain of inclusions

$$X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset X(\mathbb{A}_k)_\bullet^{\text{f-ab}} \subset X(\mathbb{A}_k)_\bullet.$$

We are interested in when certain equalities hold when X is a curve, which motivates the following definitions:

Definition 4.2. Let X be a nice k-variety. We say that X is

(1) *good* if $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$,

(2) *very good* if $\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$,

(3) *excellent with respect to all coverings* if $X(k) = X(\mathbb{A}_k)_\bullet^{\text{f-cov}}$,

(4) *excellent with respect to abelian coverings* if $X(k) = X(\mathbb{A}_k)_\bullet^{\text{f-ab}}$.

Remark 4.3. These definitions certainly tell us something more. We have the implication (3) $\Longrightarrow$ (1) which further implies

$$X(k) = \emptyset \iff X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \emptyset.$$

This is a very weak statement, whereby any variety fulfilling any of the statements in the above definition would certainly satisfy this, and so would any variety for which the Hasse principle holds. Nonetheless, if the statement holds for X and if $X(k)$ is nonempty, then Theorem 5.17 of [Sto07] ensures that there is a feasible procedure in determining the k-rational points on X.

Let $X = C$ be a curve. When C has genus 0, it follows from the Riemann-Roch theorem that there is an embedding of X as a degree 2 curve in $\mathbb{P}^2$, which is known to satisfy the Hasse principle. Therefore we have

$$C(k) = \emptyset \implies C(\mathbb{A}_k)_\bullet = \emptyset$$

and so all intermediate sets are also empty. On the other hand, if $C(k) \neq \emptyset$, then we can identify C with $\mathbb{P}^1$ as follows: let $P \in C(k)$, and visualize $\mathbb{P}^1$ as the set of lines in $\mathbb{P}^2$ passing through P. Map the line L in $\mathbb{P}^2$ to the point $Q = L \cap C$ not equal to P. The existence of Q is due to Bézout's theorem, which ensures that L intersects C at exactly 2 points. If L is the tangent line to C at P, choose $Q = P$. We now obtain a parametrization of $C(k)$. By the Strong Approximation theorem ([Poo17], Theorem 5.10.6), $C(k)$ is dense in $C(\mathbb{A}_k)_\bullet$, so

$$\overline{C(k)} = C(\mathbb{A}_k)_\bullet$$

and thus curves of genus 0 are very good.

If C is of genus strictly greater than 1, by Falting's theorem we know that $C(k)$ is finite and so $C(k) = \overline{C(k)}$. Therefore C is good (resp. very good) if and only if it is excellent with respect to all coverings (resp. excellent with respect to abelian coverings).

Finally, when C has genus 1, refer to the discussion at the end of Chapter 1 for the case where C has no rational points. We are interested in the situation where $C = A$ is an elliptic curve. Then $\pi_1(\bar{A})$ is abelian and so by Lemma 3.7(c)(iii), we have

$$A(\mathbb{A}_k)_\bullet^{\text{f-cov}} = A(\mathbb{A}_k)_\bullet^{\text{f-ab}}.$$

Moreover, we can restrict the abelian coverings to the multiplication-by-n endomorphisms of A, which gives us

$$A(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \widehat{\text{Sel}}(A(k)).$$

Since the cokernel of the canonical map

$$\overline{A(k)} \cong \widehat{A(k)} \longrightarrow \widehat{\text{Sel}}(A(k))$$

is $T\text{Ш}(A(k))$, we have the following results (cf. Remark 4.1):

Theorem 4.4. Let A be an elliptic curve, then

$$A \text{ is very good} \iff \text{Ш}(A(k))_{\text{div}} = 0,$$

$$A \text{ is excellent w.r.t. abelian coverings} \iff A(k) \text{ is finite and } \text{Ш}(A(k))_{\text{div}} = 0.$$

4.2 Further Results

In this section, we explore sufficient conditions for nice curves over k with positive genus to be excellent with respect to coverings. The results we pick up along the way will motivate a conjecture which we will talk about towards the end.

One of the important results we have gathered on curves so far is Theorem 3.15, so we begin with its immediate consequences.

Corollary 4.5. Let C be a nice curve over k. Let A be its Albanese variety and V be its Albanese torsor (which are respectively J_C and Pic^1_C).

(1) If $C(\mathbb{A}_k)_\bullet = \emptyset$, then $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k) = \emptyset$.

(2) If $C(\mathbb{A}_k)_\bullet \neq \emptyset$ and $V(k) \neq \emptyset$ (i.e., C has a k-rational divisor class of degree 1), then there is a k-defined embedding $\iota : C \hookrightarrow A$, and we have

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \iota^{-1}(\widehat{\text{Sel}}(A(k))).$$

If $\text{Ш}(A(k))_{\text{div}} = 0$, we have

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \iota^{-1}(\overline{A(k)}).$$

(3) If $C(\mathbb{A}_k)_\bullet \neq \emptyset$ and $V(k) = \emptyset$, then via the canonical map $\phi : C \longrightarrow V$, we have

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}).$$

Let $\xi \in Ш(A(k))$ be the element corresponding to V. By assumption, $\xi \neq 0$. Then if $\xi \notin Ш(A(k))_{\text{div}}$, we have $C(k) = C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \emptyset$.

Similar statements also hold true for a general nice variety X, by replacing $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ with $X(\mathbb{A}_k)_\bullet^{\text{Alb}}$. Indeed, we have already seen (cf. the paragraph after Remarks 3.16) that

$$X(\mathbb{A}_k)_\bullet^{\text{Alb}} = \phi^{-1}(V(\mathbb{A}_k)_\bullet^{\text{f-ab}}).$$

In the following, for a nice k-curve C, we let ι denote an embedding of C into its Jacobian (if it exists). We say that *the absence of rational points is explained by the Brauer-Manin obstruction* if $C(\mathbb{A}_k)_\bullet^{\text{Br}} = \emptyset$. By Theorem 3.20, we conclude that this absence is explained by the Brauer-Manin obstruction when C is excellent with respect to abelian coverings and $C(k) = \emptyset$.

Corollary 4.6. Let C be a curve of genus at least 1. Assume that $Ш(J_C(k))_{\text{div}} = 0$ and that $J_C(k)$ is finite. Then C is excellent with respect to abelian coverings. Furthermore if $C(k) = \emptyset$, the absence of rational points in explained by the Brauer-Manin obstruction.

Proof. By Corollary 4.5(2), we have

$$C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = \iota^{-1}(\overline{J_C(k)}) = \iota^{-1}(J_C(k)) = C(k).$$

as desired. □

In fact, the statement that $C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(k)$ holds even for finite subschemes Z of C.

Theorem 4.7. Let C be a curve of genus at least 1, and let $Z \subset C$ be a finite subscheme. Then the image of $Z(\mathbb{A}_k)_\bullet$ in $C(\mathbb{A}_k)_\bullet$ meets $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ in $Z(k)$. More generally, if $P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ is such that $P_v \in Z(k_v)$ for a set of places v of k of density 1, then $P \in Z(k)$.

We refer the reader to [Sto07], Theorem 8.2 for a proof. The next two results are direct applications of this theorem which reflect the well-behaved nature of a curve being excellent with respect to coverings.

Proposition 4.8. Let K/k be a finite extension, and let C be a k-curve of genus at least 1. If C_K is excellent with respect to all coverings or abelian coverings, then so is C.

Proof. By Proposition 3.11 and the canonical identification $C_K(K) = C(K)$, we have

$$C(k) \subset C(\mathbb{A}_k)_\bullet^{\text{f-cov}} \subset C(\mathbb{A}_k)_\bullet \cap C(\mathbb{A}_K)_\bullet^{\text{f-cov}} = C(\mathbb{A}_k)_\bullet \cap C(K).$$

Since $C(K)$ has to be finite in order to equal $C(\mathbb{A}_K)_\bullet^{\text{f-cov}}$, the above inclusion implies that $C(k)$ is also finite. Let $Z = C(k)$ be the finite subscheme as in Theorem 4.7. Then we have $P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$ such that $P_v \in C(k_v)$ for all finite places of k and by Theorem 4.7, we conclude that $P \in C(k)$. □

Proposition 4.9. Let $\phi : C \longrightarrow X$ be a non-constant morphism over k from the curve C of genus at least 1 into a variety X. If X is excellent with respect to all coverings or abelian coverings, then so is C. In particular, if $X(\mathbb{A}_k)_\bullet^{\text{f-ab}} = X(k)$ and $C(k) = \emptyset$, then the absence of rational points on C is explained by the Brauer-Manin obstruction.

Proof. Let $P \in C(\mathbb{A}_k)_\bullet^{\text{f-cov}}$. By Proposition 3.8, $\phi(P) \in X(\mathbb{A}_k)_\bullet^{\text{f-cov}} = X(k)$. Let $Z \subset C$ be the preimage of $\phi(P)$ in C, this is a subscheme of C and it is finite since ϕ is non-constant. We easily see that P is in the image of $Z(\mathbb{A}_k)_\bullet$ in $C(\mathbb{A}_k)_\bullet$, and by Theorem 4.7,

$$P \in C(\mathbb{A}_k)_\bullet^{\text{f-ab}} \cap Z(\mathbb{A}_k)_\bullet = Z(k) \subset C(k).$$

The same proof works for $C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$. □

In light of the preceding statement (and Theorem 4.4), the following essential result is naturally implied:

Theorem 4.10. Let $C \longrightarrow A$ be a non-constant morphism over k of a curve C into an abelian variety A. Assume that $\text{Ш}(A(k))_{\text{div}} = 0$ and that $A(k)$ is finite. Then C is excellent with respect to abelian coverings. In particular, if $C(k) = \emptyset$, then the absence of rational points on C is explained by the Brauer-Manin obstruction.

This enables us to produce many examples of curves C defined over $\mathbb{Q}$ that are excellent with respect to abelian coverings, see [Sto07], Example 8.7. At the same time, Corollary 8.8 of the same paper tells us that the modular curves $X_0(N), X_1(N), X(N)$ of positive genus are also excellent with respect to abelian coverings.

Most of the results we have seen so far in this section provides us with curves C of positive genera such that

$$C(k) = C(\mathbb{A}_k)_\bullet^{\text{Br}_{1/2}} \subset C(\mathbb{A}_k)_\bullet.$$

The amount of sufficient conditions for such curves to be excellent with respect to abelian coverings seems to indicate to us that perhaps a condition slightly weaker would hold in general for all nice curves, even those of genus 0. With what we have discussed in the previous section, it would not be absurd to make the following claim, after which we will discuss the consequences of it validity:

Conjecture 4.11. If C is a nice curve over a number field k, then C is very good.

By Theorem 4.4, the conjecture implies that for an elliptic curve E, we have $\text{Ш}(E(k))_{\text{div}} = 0$. We have already seen from Conjecture 1.19 that $\text{Ш}(A(k))$ for an abelian variety A is most probably finite, and therefore it would have trivial divisible subgroup. Furthermore, for curves of higher genera, it is equivalent to saying that they are excellent with respect to abelian coverings.

Another important implication of the conjecture is related to the discussion in Remark 4.3, where a curve C being very good would imply that

$$C(k) = \emptyset \iff C(\mathbb{A}_k)_\bullet^{\text{f-cov}} = \emptyset.$$

As already mentioned, this weaker variant of the Hasse principle would allow us to algorithmically determine whether a given nice curve over a number field k has rational points or not.

But perhaps the most striking consequence of Conjecture 4.11 is one that is related to the Brauer-Manin obstruction. By Theorem 3.20, we know that

$$C(\mathbb{A}_k)_\bullet^{\text{Br}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet^{\text{Br}_{1/2}}.$$

Since C is very good, we have the stronger equality $\overline{C(k)} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}}$, and so we have the inclusions

$$C(k) \subset \overline{C(k)} = C(\mathbb{A}_k)_\bullet^{\text{Br}} = C(\mathbb{A}_k)_\bullet^{\text{f-ab}} = C(\mathbb{A}_k)_\bullet^{\text{Br}_{1/2}} \subset C(\mathbb{A}_k)_\bullet,$$

which therefore implies that the Brauer-Manin obstruction is the only obstruction against rational points on nice curves over number fields, i.e.,

$$C(k) = \emptyset \iff C(\mathbb{A}_k)_\bullet^{\text{Br}} = \emptyset.$$

References

[B-S64] A. Borel and J.-P. Serre, *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv. 39, 1964, pp. 111–164 (French).

[BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge [Results in Mathematics and Related Areas. 3rd Series.] A Series of Modern Surveys in Mathematics, vol. 21, Springer-Verlag, Berlin, 1990.

[Bri18] M. Bright, *Rational points in families of varieties*, Online lecture notes, http://www.boojum.org.uk/maths/articles/mainz.pdf, 2018.

[Cho57] W. Chow, *On the projective embedding of homogeneous varieties*, Algebraic Geometry and Topology. A Symposium in Honor of S. Lefschetz, Princeton University Press, Princeton, NJ, 1957, pp. 122–128.

[CKS87] J.-L Colliot-Thélène, D. Kanevsky, J.-J. Sansuc, *Arithmétique des surfaces cubiques diagonales*, Diophantine Approximation and Transcendence Theory (Bonn, 1985), Lecture Notes in Math., vol. 1290, Springer, Berlin, 1987, pp. 1–108 (French).

[CoS87] J.-L Colliot-Thélène, J.-J. Sansuc, *La descente sur les variétés rationnelles*, Journées de Géométrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979, Sijthoff Noordhoff, Alphen aan den Rijn, 1980, pp. 223–237 (French).

[C-S86] G. Cornell, J. Silverman, *Arithmetic Geometry*, Springer-Verlag, 1986.

[Gro95] A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique. I. Généralités. Descente par morphismes fidèlement plats*, Séminaire Bourbaki, Vol. 5, Exp. No. 190, Soc. Math. France, Paris, 1995, pp. 299– 327 (French).

[Har77] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, Springer-Verlag, New York, 1977.

[Lan59] S. Lang, *Abelian varieties*, Interscience Tracts in Pure and Applied Mathematics. No. 7. Interscience Publishers, Inc., New York, 1959.

[Lin40] C.-E. Lind, *Untersuchungen über die rationalen Punkte der ebenen kubischen Kurven vom Geschlecht Eins*, Thesis, University of Uppsala, 1940.

[Man70] Y. I. Manin, *Le groupe de Brauer-Grothendieck en géométrie diophantienne*, Actes du Congrès International des Mathématiciens (Nice, 1970) Gauthier-Villars, Paris, 1971, pp. 401–411.

[Mat70] Y. Matiyasevich, *The Diophantineness of enumerable sets.* (Russian) Dokl. Akad. Nauk SSSR 191 (1970), pp. 279–282.

[Mil80] J. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33, Princeton University Press, Princeton, N.J., 1980.

[Mil13] J. Milne, *Class Field Theory*, www.jmilne.org/math/CourseNotes/CFT310.pdf, 2013.

[P-V04] B. Poonen, J.F. Voloch, *Random Diophantine equations*, Arithmetic of Higher-Dimensional Algebraic Varieties (Palo Alto, CA, 2002), 2004, pp. 175–184.

[Poo01] B. Poonen, *Computing rational points on curves*, Number Theory for the Millennium, III (Urbana, IL, 2000), 2002, pp. 149–172.

[Poo17] B. Poonen, *Rational Points on Varieties*, Graduate Studies in Mathematics 186, American Mathematical Society, 2017.

[Rei42] H. Reichardt, *Einige im Kleinen überall lösbare, im Grossen unlösbare diophantische Gleichungen*, J. Reine Angew. Math. 184, 1942, pp. 12–18.

[Sch99] V. Scharaschkin, *Local-Global Problems and the Brauer-Manin Obstruction*, Thesis, 1999.

[Sel51] E. Selmer, *The Diophantine equation* $ax^3 + by^3 + cz^3 = 0$, Acta Arithmetica 85 (1951), pp. 203–362.

[Ser60] J.-P. Serre, *Morphismes universels et variétés d'Albanese*, Séminaire Claude Chevalley, tome 4, 1958/1959, 1960, pp. Exposé 10 (French).

[Ser71] J.-P. Serre, *Sur le groupes de congruence des variétés abéliennes*, II, Izv. Akad. Nauk SSSR Ser. Mat 35, 731-737, 1971.

[Ser88] J.-P. Serre, *Algebraic Groups and Class Fields*, Graduate Text in Mathematics, Springer-Verlag, 1988.

[Sko99] A. N. Skorobogatov, *Beyond the Manin obstruction*, Invent. Math. 135, 1999, no. 2, pp. 399–424.

[Sko01] A. N. Skorobogatov, *Torsors and Rational Points*, Cambridge Tracts in Mathematics, vol. 144, Cambridge University Press, Cambridge, 2001.

[Sto07] M. Stoll, *Finite Descent Obstructions and Rational Points on Curves*, Algebra Number Theory 1, 2007, no. 4, pp. 349–391.

[Wit80] O. Wittenberg, *On Albanese Torsors and the Elementary Obstruction*, Math. Ann. 340, no. 4, 805–838, 2008.