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Singular moduli spaces on K3 surfaces and derived categories

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Introduction

Let X be a compact Riemann surface. One may ask if its complex structure can be recovered from the values of integrals of holomorphic forms; equivalently, does the period map determine a Riemann surface? This question, already proposed by Riemann, was given a positive answer by Torelli in 1914, after half of a century of work. The main tools appearing in a modern formulation of this problem are the cup product on the middle cohomology $H^1(X, \mathbb{Z})$ (an alternating form), marking, bilinearity and positivity relations.

These are indeed the starting points towards a generalization of Torelli theorem in higher dimension. Such a task, which looks difficult even for surfaces, turns out to be possible in the case of K3 surfaces, as we will see in Chapter 2. A *K3 surface* X is a (smooth) compact complex surface, whose canonical bundle is trivial and $H^1(X, \mathcal{O}_X) = 0$. The singular cohomology group of half dimension $H^2(X, \mathbb{Z})$ comes with a natural symmetric bilinear pairing, the cup product, and a weight two Hodge structure. The fundamental Global Torelli theorem states that two K3 surfaces X and Y are isomorphic if and only if there exists a Hodge isometry $H^2(X, \mathbb{Z}) \simeq H^2(Y, \mathbb{Z})$.

Now, given a smooth algebraic variety X over a field k, denote as $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves on X. Two algebraic varieties X and Y over k are said to be $\mathcal{D}$-equivalent if there exists a k-linear exact equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$. The derived category $\mathcal{D}^b(X)$ contains much geometric information on the variety X; for instance, if X has ample (anti-)canonical bundle, then it is uniquely determined by its derived category. However, in general, as in the case of K3 surfaces, the notion of being $\mathcal{D}$-equivalent is weaker than being isomorphic.

The derived version of Torelli theorem provides us with a cohomological criterion for $\mathcal{D}$-equivalence of K3 surfaces. One can endow the whole integral cohomology group $\tilde{H}(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ of a K3 surface X with a symmetric bilinear form extending the cup product and with a weight two Hodge structure. A beautiful result due to Mukai and Orlov states that two K3 surfaces X and Y are $\mathcal{D}$-equivalent if and only if there exists a Hodge isometry $\tilde{H}(X, \mathbb{Z}) \simeq \tilde{H}(Y, \mathbb{Z})$.

In proving such a theorem, several interesting tools come into play.

The first, to be introduced in Chapter 1, is Fourier-Mukai transform, which

establishes a bridge between the derived and cohomological worlds. Consider two smooth projective varieties X and Y. An object $\mathcal{P} \in \mathcal{D}^b(X \times Y)$ induces an exact functor

$$\Phi_{\mathcal{P}} \colon \mathcal{D}^b(X) \to \mathcal{D}^b(Y),$$

called *Fourier-Mukai transform with kernel* $\mathcal{P}$. From a geometric point of view, we can limit our attention to these functors: by virtue of a celebrated theorem of Orlov, any equivalence $F : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is of Fourier-Mukai type. A Fourier-Mukai transform $\Phi_{\mathcal{P}}$ induces a morphism $\Phi^H_{\mathcal{P}} \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ between the whole rational cohomology groups of X and Y; the key notion in this passage is the Mukai vector of a sheaf E on X, defined as $v(E) = \mathrm{ch}(E).\sqrt{\mathrm{td(X)}} \in H^*(X, \mathbb{Q})$.

The second is the concept of moduli space. Let us first consider the situation where the base scheme is a smooth connected projective algebraic curve $\mathcal{C}$ and the problem of classifying vector bundles on it. The set of isomorphism classes of coherent sheaves of fixed rank and degree (i.e. fixed Hilbert polynomial) on $\mathcal{C}$ cannot be parametrized by an algebraic variety. It is then natural to introduce the notions of (slope) stability and semistability. Using the Hilbert scheme and Mumford's Geometric Invariant Theory (GIT), eventually one constructs a (coarse) moduli space $\mathbf{M}(r, d)$; it is a projective variety whose closed points are in bijection with S-equivalence classes of semistable vector bundles on $\mathcal{C}$.

When passing to higher dimensional varieties, one has to adjust a bit the construction: vector bundles are replaced by torsion-free (or even pure) sheaves and the notion of slope (semi)stability by Gieseker (semi)stability. Then, for a projective variety X, the (coarse) moduli space of semistable sheaves with fixed Hilbert polynomial still exists.

The case of K3 surfaces is of particular interest. Let (X, H) be a polarized K3 surface; instead of fixing a Hilbert polynomial, the numerical invariants of the sheaves we want to parametrize will be encoded in a vector $v \in \tilde{H}(X, \mathbb{Z})$ of Hodge type $(1, 1)$. Denote as $\mathbf{M}_H(v)$ the moduli space of semistable sheaves with fixed Mukai vector v and as $\mathbf{M}^s_H(v)$ the open subscheme corresponding to stable sheaves. It was Mukai who first studied the geometry of these spaces in the 80's: $\mathbf{M}^s_H(v)$ is a smooth quasi-projective variety of dimension $\langle v, v\rangle + 2$ and it admits a symplectic form. If no strictly semistable sheaf exists, then $\mathbf{M}_H(v) = \mathbf{M}^s_H(v)$ is a projective holomorphically symplectic manifold; otherwise, $\mathbf{M}_H(v)$ is singular, and one may look for symplectic resolutions of the singularities. The different situations that can occur depend essentially on the divisibility of the Mukai vector and on the dimension of the moduli space.

More precisely, any Mukai vector v can be written as $v = mv_0$, for v_0 primitive and $m \in \mathbb{N}_0$ a multiplicity. Assume that the multiplicity $m = 1$, so that $v = v_0$ is primitive. The low-dimensional cases, that is $\langle v_0, v_0\rangle$ equals -2 or 0, were studied by Mukai in his seminal paper [21], whose results we present in Chapter 3. He obtains in particular that, if $v = v_0$ is isotropic and $\mathbf{M}^s_H(v)$ is non-empty and compact, then

$\mathbf{M}_H^s(v)$ is a K3 surface, whose period $H^2(\mathbf{M}_H^s(v), \mathbb{Z})$ can be expressed in terms of v. Without any assumption on $\langle v_0, v_0 \rangle$, but suitably choosing an ample divisor class H, a cornerstone result by Yoshioka, based on previous work by Beauville, Huybrechts and O'Grady among others, states that the moduli space $\mathbf{M}_H^s(v_0)$ is compact and deformation equivalent to a Hilbert scheme of points; in particular, it is non-empty.

Assume now that $m \geq 2$. Then the moduli space $\mathbf{M}_H(v)$ is singular and one has the following possibilities.

(1) If $\langle v_0, v_0 \rangle = -2$, then $\mathbf{M}_H(v)$ consists of a non-reduced point.

(2) If $\langle v_0, v_0 \rangle = 0$, it turns out that any semistable sheaf E with $v(E) = mv_0$ is S-equivalent to a direct sum $E_1 \oplus E_2 \oplus \cdots \oplus E_m$ of stable sheaves with $v(E_i) = v_0$. Therefore $\mathbf{M}_H(v) = S^m(\mathbf{M}_H(v_0))$ admits a symplectic resolution in terms of the Hilbert scheme $\mathbf{Hilb}^m(\mathbf{M}_H(v_0)) \to \mathbf{M}_H(v)$.

(3) If $m = 2$ and $\langle v_0, v_0 \rangle = 2$, by blowing-up the reduced singular locus, one obtains a projective symplectic resolution of $\mathbf{M}_H(v)$. The O'Grady's examples fall in this case.

(4) If $m \geq 3$ or $m = 2$ and $\langle v_0, v_0 \rangle \geq 4$, Kaledin, Sorger and Lehn in [16] proved that $\mathbf{M}_H(v)$ is locally factorial, and in particular does not admit a projective symplectic resolution.

The lack of a projective symplectic resolution in case (4), which will be discussed in Chapter 4 and will conclude this little work, can be the starting point for further possible developments in the theory. In particular, one may look for crepant categorical resolutions of the singularities, which should always exist and be a source of inspiration for a definition of hyperkähler category.

Chapter 1

Fourier-Mukai transforms

The first two sections of this chapter, dealing with derived categories and their application to Algebraic Geometry, are meant as a reminder. Even if only few basic facts are presented and proofs are omitted, we have decided to include these topics, as they provide the natural context for the central notion of Fourier-Mukai transforms. These are the object of the third section, which is divided into two parts. In the first, we state the main properties of these transforms and some criteria that decide if they are fully faithful functors or even equivalences. A celebrated theorem of Orlov, saying that all equivalences between derived categories of smooth projective varieties are of Fourier-Mukai type, is stated, but not proven. In the second, we study the cohomological transform induced by a Fourier-Mukai functor, its compatibilities with Hodge structures and with the Mukai pairing.

1.1 Derived categories

Let $\mathcal{A}$ be an abelian category and let $\mathrm{Kom}(\mathcal{A})$ be the category of complexes over $\mathcal{A}$. $\mathrm{Kom}(\mathcal{A})$ is still abelian and admits a shift functor. Moreover, for any integer i, one has a cohomological functor

$$H^i \colon \mathrm{Kom}(\mathcal{A}) \to \mathcal{A}.$$

Its importance is due to this fundamental result in homological algebra: any short exact sequence of complexes $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ yields a long exact cohomology sequence

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \dots.$$

One says that a complex morphism $f \colon A^\bullet \to B^\bullet$ is a *quasi-isomorphism* (or, in short, a *qis*) if $H^i(f) \colon H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism for any $i \in \mathbb{Z}$.

In homological algebra, it is common to consider a resolution of an object $A \in \mathcal{A}$, i.e. an exact complex $0 \to A \to I^\bullet$. This can be regarded as the datum of a quasi-

isomorphism

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \ldots \end{array}$$

The idea behind the notion of derived category is that an object should be identified with its resolutions; more generally, every quasi-isomorphism should become an isomorphism. This motivates the following

Definition 1.1.1. Let $\mathcal{A}$ be an abelian category. A *derived category* of $\mathcal{A}$ is a pair $(\mathcal{D}(\mathcal{A}), Q)$, where $\mathcal{D}(\mathcal{A})$ is a category and $Q\colon \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is a functor such that:

(i) if $f\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism, then $Q(f)$ is an isomorphism;

(ii) any functor $F\colon \mathrm{Kom}(\mathcal{A}) \to \mathcal{B}$ satisfying property (i) factors uniquely through Q, i.e. there exists a unique functor (up to isomorphism) $G\colon \mathcal{D}(\mathcal{A}) \to \mathcal{B}$ making the following diagram commute

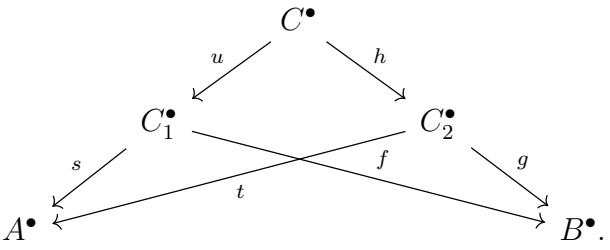
$$\begin{array}{ccc} \mathrm{Kom}(\mathcal{A}) & \xrightarrow{\quad Q \quad} & \mathcal{D}(\mathcal{A}) \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & \mathcal{B}. & \end{array}$$

The uniqueness of the derived category comes from the universal property. As far as the existence is concerned, one could obtain the derived category $\mathcal{D}(\mathcal{A})$ as the localisation of $\mathrm{Kom}(\mathcal{A})$ at the family of quasi-isomorphisms. However, the resulting description of morphisms is not so handy. To get a better understanding of $\mathcal{D}(\mathcal{A})$, we pass through the homotopy category. Recall that, for any abelian category $\mathcal{A}$, we can consider the homotopy category $\mathrm{K}(\mathcal{A})$, whose objects are just complexes over $\mathcal{A}$ and whose morphisms are morphisms of complexes up to homotopy. The advantage of the homotopy category is that the family S of quasi-isomorphisms is *localizing*. The derived category $\mathcal{D}(\mathcal{A})$ is then the localisation of the homotopy category at S. Let us briefly present what the derived category looks like.

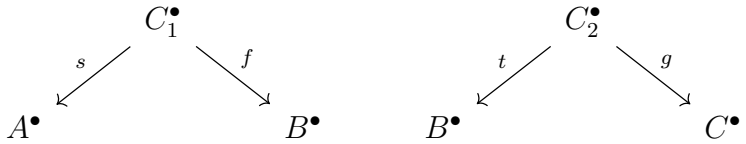
The class of objects of the derived category $\mathcal{D}(\mathcal{A})$ consists of complexes over $\mathcal{A}$. Given two complexes $A^\bullet, B^\bullet \in \mathcal{D}(\mathcal{A})$, morphisms from $A^\bullet$ to $B^\bullet$ in the derived category are equivalence classes of *roofs*

$$\begin{array}{ccc} & C^\bullet & \\ & {\scriptstyle s}\swarrow \quad \searrow{\scriptstyle f} & \\ A^\bullet & & B^\bullet \end{array}$$

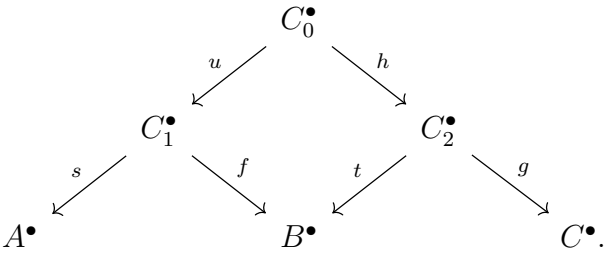
where $C^\bullet$ is a complex, s is a qis and f is a morphism from $C^\bullet$ to $B^\bullet$ in $\mathrm{Kom}(\mathcal{A})$. Two roofs are equivalent if they can be dominated in $\mathrm{K}(\mathcal{A})$ by a third roof, i.e. if there exists a commutative diagram of the form



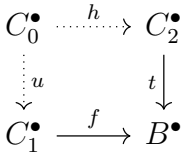
The composition of two roofs



is given by a commutative diagram of the form



It is in the definition of the composition of morphisms that the homotopy category is essential: indeed, by suitable choices of $C_0^\bullet$, of a quasi-isomorphism u and of a morphism h, the diagram



will be commutative in $\mathrm{K}(\mathcal{A})$, but not in $\mathrm{Kom}(\mathcal{A})$. The choice of $C_0^\bullet$ lies on the existence of the mapping cone $C(f)$ of a morphism $f\colon A^\bullet \to B^\bullet$. This is defined as the complex

$$C(f)^i = A^{i+1} \oplus B^i \quad \text{with} \quad d^i_{C(f)} = \begin{pmatrix} -d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}.$$

The mapping cone comes with natural maps

$$\tau\colon B^\bullet \to C(f) \quad \text{and} \quad \pi\colon C(f) \to A^\bullet[1]$$

given by the natural injection and projection respectively. These fit in a short exact sequence

$$0 \to B^\bullet \to C(f) \to A^\bullet[1] \to 0,$$

from which we deduce that $f\colon A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if $C(f)$ is acyclic. The mapping cone construction is of the greatest importance also for the definition of a triangulated structure on $\mathrm{K}(\mathcal{A})$ and on $\mathcal{D}(\mathcal{A})$. The shift functor $A^\bullet \mapsto A^\bullet[1]$ is induced by the shift functor of $\mathrm{Kom}(\mathcal{A})$. A triangle in $\mathrm{K}(\mathcal{A})$ (resp. in $\mathcal{D}(\mathcal{A})$)

$$A_1^\bullet \to A_2^\bullet \to A_3^\bullet \to A_1^\bullet[1]$$

is *distinguished* if it is isomorphic in $\mathrm{K}(\mathcal{A})$ (resp. in $\mathcal{D}(\mathcal{A})$) to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1].$$

Moreover, the cohomological functors H^i on $\mathrm{Kom}(\mathcal{A})$ descend to the homotopy category and to the derived category (in the latter case, this comes immediately from the universal property).

Remark 1.1.2. Performing these constructions, we have considered unbounded complexes. However, in practice, it is often more convenient to impose boundedness conditions. We denote $\mathrm{Kom}^+(\mathcal{A})$, $\mathrm{Kom}^-(\mathcal{A})$ and $\mathrm{Kom}^b(\mathcal{A})$ the categories of bounded below, bounded above and bounded complexes respectively. By dividing out first by homotopy equivalence and then by quasi-isomorphisms one obtains the categories $\mathrm{K}^*(\mathcal{A})$ and $\mathcal{D}^*(\mathcal{A})$ with $* = +, -, b$.

1.1.1 Derived functors

In this section we will briefly explain how to define the right derived functor of a left exact functor between abelian categories. Analogously, one can construct the left derived functor of a right exact functor.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let

$$F\colon \mathcal{A} \to \mathcal{B}$$

be a left exact functor. Assume that $\mathcal{A}$ has enough injectives; in other words, the full subcategory $\mathcal{I}_\mathcal{A}$ of $\mathcal{A}$ consisting of injective objects is cogenerating. In particular, this implies the existence of an equivalence $\iota\colon \mathrm{K}^+(\mathcal{I}_\mathcal{A}) \to \mathcal{D}^+(\mathcal{A})$. The functor F induces an exact functor $\mathrm{K}(F)\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{B})$ acting on complexes term by term. If the functor F were exact, $\mathrm{K}(F)$ would send quasi-isomorphisms of $\mathrm{K}^+(\mathcal{A})$ to quasi-isomorphisms of $\mathrm{K}^+(\mathcal{B})$ and, by the universal property of derived category, there would exist a functor $RF\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ making the following diagram commute

$$\begin{array}{ccc} \mathrm{K}^+(\mathcal{A}) & \xrightarrow{\mathrm{K}(F)} & \mathrm{K}^+(\mathcal{B}) \\ \big\downarrow{\scriptstyle Q_\mathcal{A}} & & \big\downarrow{\scriptstyle Q_\mathcal{B}} \\ \mathcal{D}^+(\mathcal{A}) & \xrightarrow{RF} & \mathcal{D}^+(\mathcal{B}). \end{array}$$

If F is just left exact, this is no longer true. The subcategory $\mathrm{K}^+(\mathcal{I}_\mathcal{A})$ allows us to work out the problem: informally speaking, it is "big enough" to represent, up to quasi-isomorphism, all the complexes in $\mathrm{K}^+(\mathcal{A})$ and the restriction of $\mathrm{K}(F)$ to it still behaves well. This happens because any acyclic complex of injective objects is sent to a complex that is still acyclic; thus quasi-isomorphisms in $\mathrm{K}^+(\mathcal{I}_\mathcal{A})$ are sent to quasi-isomorphisms in $\mathrm{K}^+(\mathcal{B})$. The situation is represented in the following diagram.

$$\begin{array}{ccccc} \mathrm{K}^+(\mathcal{I}_\mathcal{A}) & \hookrightarrow & \mathrm{K}^+(\mathcal{A}) & \xrightarrow{\mathrm{K}(F)} & \mathrm{K}^+(\mathcal{B}) \\ & \overset{\iota}{\searrow} & \downarrow{\scriptstyle Q_\mathcal{A}} & & \downarrow{\scriptstyle Q_\mathcal{B}} \\ & \overset{\iota^{-1}}{\nwarrow} & \mathcal{D}^+(\mathcal{A}) & & \mathcal{D}^+(\mathcal{B}) \end{array}$$

Definition 1.1.3. The *right derived functor* of F is the functor

$$RF \coloneqq Q_\mathcal{B} \circ \mathrm{K}(F) \circ \iota^{-1} \colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B}).$$

Remark 1.1.4. For any bounded below complex $A^\bullet \in \mathcal{D}^+(\mathcal{A})$, one sets $R^iF(A^\bullet) = H^i(RF(A^\bullet))$. This definition agrees with classical *higher derived functors* $R^iF \colon \mathcal{A} \to \mathcal{B}$ of homological algebra: this is clear just recalling the link between resolutions and quasi-isomorphisms.

Example 1.1.5. Assume that $\mathcal{A}$ is an abelian category having enough injectives. Let $A \in \mathcal{A}$ and consider the left exact covariant functor

$$\mathrm{Hom}(A, \ ) \colon \mathcal{A} \to \mathcal{A}b.$$

Its higher derived functors are denoted by $\mathrm{Ext}^i(A, \ )$ for $i \in \mathbb{Z}$. These have a nice (and extremely useful) interpretation in terms of the derived category. More precisely, for any $B \in \mathcal{A}$, one has

$$\mathrm{Ext}^i(A, B) \simeq \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(A, B[i]).$$

Indeed, suppose that $0 \to B \to I^0 \to I^1 \to \dots$ is an injective resolution of B. By definition, $\mathrm{Ext}^i(A, B)$ is the i-th cohomology group of the complex

$$\cdots \to \mathrm{Hom}(A, I^{i-1}) \to \mathrm{Hom}(A, I^i) \to \mathrm{Hom}(A, I^{i+1}) \to \dots$$

A morphism $f \in \mathrm{Hom}(A, I^i)$ is a cycle if and only if it defines a morphism of complexes $f^\bullet \colon A[-i] \to I^\bullet$

$$\begin{array}{ccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \dots \\ & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow & & \\ \dots & \longrightarrow & I^{i-1} & \longrightarrow & I^i & \longrightarrow & I^{i+1} & \longrightarrow & \dots \end{array}$$

On the other hand, f is a boundary if and only if $f^\bullet$ is homotopically trivial. Hence,

$$\mathrm{Ext}^i(A,B) \simeq \mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(A[-i], I^\bullet) \simeq \mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(A, I^\bullet[i]).$$

Recalling that $B \simeq I^\bullet$ in $\mathcal{D}^+(\mathcal{A})$, we are done if we prove that the natural map

$$\mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(A, J^\bullet) \to \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(A, J^\bullet)$$

is bijective whenever $J^\bullet$ is a complex of injective objects. Given a roof

$$\begin{array}{ccc} & C^\bullet & \\ {\scriptstyle \text{qis}}\swarrow & & \searrow \\ A & & J^\bullet, \end{array}$$

we have to show that there exists a unique morphism $A \to J^\bullet$ making the diagram commute up to homotopy, or, equivalently, that we have an isomorphism

$$\mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(A, J^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(C^\bullet, J^\bullet).$$

Complete the qis $C^\bullet \to A$ to a distinguished triangle $C^\bullet \to A \to D^\bullet \to C^\bullet[1]$ in $\mathrm{K}^+(\mathcal{A})$. Looking at the long exact sequence of groups obtained applying the functor $\mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(\ , J^\bullet)$, we can reduce to prove that, for $D^\bullet$ acyclic, $\mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(D^\bullet, J^\bullet) = 0$. This vanishing is a standard fact in homological algebra.

This interpretation allows us to define, for any triple of objects $A, B, C \in \mathcal{A}$, a natural composition for Ext-groups

$$\mathrm{Ext}^i(A,B) \times \mathrm{Ext}^j(B,C) \to \mathrm{Ext}^{i+j}(A,C) :$$

indeed, elements of

$$\mathrm{Ext}^i(A,B) \simeq \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(A, B[i])$$

and

$$\mathrm{Ext}^j(B,C) \simeq \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(B, C[j]) \simeq \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(B[i], C[i+j])$$

can be composed to an element of $\mathrm{Ext}^{i+j}(A,C) \simeq \mathrm{Hom}_{\mathcal{D}^+(\mathcal{A})}(A, C[i+j])$.

Remark 1.1.6. Looking carefully at the construction of the derived functor, one notes that some assumptions can be weakened. First of all, the functor that we want to derive may be directly given at the homotopy category level $F\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$, and not as a functor between the abelian categories $\mathcal{A}$ and $\mathcal{B}$. Second, since the category $\mathcal{A}$ does not always have enough injectives, we may need to enlarge the category $\mathcal{I}_\mathcal{A}$ of injectives to a category of objects adapted to the functor F. Let us be more precise.

Consider an exact functor

$$F\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}(\mathcal{B})$$

A triangulated subcategory $\mathcal{K}_F \subset \mathrm{K}^+(\mathcal{A})$ is *adapted* to the functor F if

(i) for any acyclic object $A^\bullet \in \mathcal{K}_F$, $F(A^\bullet)$ is acyclic;

(ii) any $A^\bullet \in \mathrm{K}^+(\mathcal{A})$ is quasi-isomorphic to a complex in $\mathcal{K}_F$.

Condition (ii) implies that, if S denotes the class of quasi-isomorphisms between objects of $\mathcal{K}_F$, we have an equivalence of categories $\iota\colon \mathcal{K}_F[S^{-1}] \to \mathcal{D}^+(\mathcal{A})$. Condition (i) says that the restriction of the functor F to $\mathcal{K}_F$ sends quasi-isomorphisms to quasi-isomorphisms in $\mathrm{K}(\mathcal{B})$, hence to isomorphisms in $\mathcal{D}(\mathcal{B})$. The universal property of localisation yields a functor $F'\colon \mathcal{K}_F[S^{-1}] \to \mathcal{D}(\mathcal{B})$. The situation is summarized in the following diagram

$$\begin{array}{ccccc}
\mathcal{K}_F & \hookrightarrow & \mathrm{K}^+(\mathcal{A}) & \xrightarrow{F} & \mathrm{K}(\mathcal{B}) \\
\downarrow & \overset{\iota^{-1}}{\leftarrow} & \downarrow{\scriptstyle Q_{\mathcal{A}}} & & \downarrow{\scriptstyle Q_{\mathcal{B}}} \\
\mathcal{K}_F[S^{-1}] & \xrightarrow{\iota} & \mathcal{D}^+(\mathcal{A}) & & \mathcal{D}(\mathcal{B}) \\
& \searrow^{F'} & & \nearrow &
\end{array}$$

The right derived functor of F is then defined as the composition

$$RF \coloneqq F' \circ \iota^{-1}\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}(\mathcal{B}).$$

This general approach applies in particular when we have a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$, but the class of injectives objects $\mathcal{I}_{\mathcal{A}}$ is not cogenerating. In this case, one considers a class $\mathcal{I}_F$ of F-adapted objects, i.e. an additive cogenerating subcategory of $\mathcal{A}$ such that, for any acyclic bounded below complex $A^\bullet \in \mathrm{K}^+(\mathcal{I}_F)$, the complex $F(A^\bullet)$ is acyclic. A useful F-adapted class is the class of F-acyclic objects, provided that it is cogenerating.

Such a construction is not just an attempt of reaching the largest possible generality, but it is really needed in concrete situations, as we will see in section 1.2.

We conclude this section with an important theorem on the composition of derived functors, whose proof can be found in [12, ch. 2].

Theorem 1.1.7. *Let $F_1\colon \mathcal{A} \to \mathcal{B}$ and $F_2\colon \mathcal{B} \to \mathcal{C}$ be left exact functors of abelian categories. Assume that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and that $F_1(\mathcal{I}_{\mathcal{A}})$ is contained in an F_2-adapted class $\mathcal{I}_{F_2}$ of $\mathcal{B}$.*

(i) The right derived functors $RF_1\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$, $RF_2\colon \mathcal{D}^+(\mathcal{B}) \to \mathcal{D}^+(\mathcal{C})$, $R(F_2 \circ F_1)\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{C})$ exist and there is a natural isomorphism

$$R(F_2 \circ F_1) \simeq RF_2 \circ RF_1.$$

(ii) For any complex $A^\bullet \in \mathcal{D}^+(\mathcal{A})$, there exists a spectral sequence

$$E_2^{p,q} = R^pF_2(R^qF_1(A^\bullet)) \Rightarrow E^{p+q} = R^{p+q}(F_2 \circ F_1)(A^\bullet).$$

1.2 Derived functors in Algebraic Geometry

Even if we could describe the theory in a more general setting (i.e. for noetherian schemes), we will focus our attention on projective varieties.

Definition 1.2.1. Let X be a projective variety over a field k and let $\mathrm{Coh}(X)$ be the abelian category of coherent sheaves on X. We define the *derived category* of X as

$$\mathcal{D}^b(X) \coloneqq \mathcal{D}^b(\mathrm{Coh}(X)).$$

Two projective varieties X and Y are said to be *derived equivalent* (or $\mathcal{D}$-*equivalent*) if there exists a k-linear exact equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$.

Usually, the category $\mathrm{Coh}(X)$ does not contain non-trivial injective sheaves, which makes difficult computing (left) derived functors. To work out this problem, one passes to the category $\mathrm{Qcoh}(X)$ of quasi-coherent sheaves on X. Then, to come back to $\mathcal{D}^b(X)$, one has to use two facts. The first is that, for noetherian schemes, $\mathcal{D}^b(X)$ is equivalent to the full subcategory $\mathcal{D}^b_{\mathrm{Coh}(X)}(\mathrm{Qcoh}(X)) \subset \mathcal{D}^b(\mathrm{Qcoh}(X))$ of complexes of quasi-coherent sheaves with coherent cohomology (see [12] Proposition 3.5: one uses that every quasi-coherent sheaf is limit of its coherent subsheaves). The second is the following general

Proposition 1.2.2. *Let $F\colon \mathrm{K}^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{B})$ be an exact functor admitting a right derived functor $RF\colon \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and assume that $\mathcal{A}$ has enough injectives.*

(i) Suppose that there exist $\mathcal{C} \subset \mathcal{B}$ a thick subcategory and an integer $n \in \mathbb{Z}$ such that, for all $A \in \mathcal{A}$, $R^iF(A) \in \mathcal{C}$ and $R^iF(A) = 0$ for $i < n$. Then RF takes values in the full triangulated subcategory $\mathcal{D}^+_{\mathcal{C}}(\mathcal{B}) \subset \mathcal{D}^+(\mathcal{B})$ of complexes with cohomology in $\mathcal{C}$.

(ii) If $RF(A) \in \mathcal{D}^b(\mathcal{B})$ for any object $A \in \mathcal{A}$, then $RF(A^\bullet) \in \mathcal{D}^b(\mathcal{B})$ for any complex $A^\bullet \in \mathcal{D}^b(\mathcal{A})$, i.e. RF descends to an exact functor

$$RF\colon \mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B}).$$

Proof. The assertion follows immediately from the spectral sequence

$$E_2^{p,q} = R^pF(H^q(A^\bullet)) \Rightarrow E^{p+q} = R^{p+q}(F)(A^\bullet). \qquad \square$$

We list the main derived functors that we will need, and sketch some of their properties.

Cohomology Let X be a projective variety over k. The functor of global sections $\Gamma(X, \ )\colon \mathrm{Qcoh}(X) \to \mathrm{Vec}(k)$ is left exact and yields the right derived functor

$$R\Gamma\colon \mathcal{D}^+(\mathrm{Qcoh}(X)) \to \mathcal{D}^+(\mathrm{Vec}(k)).$$

Recall that $H^i(X,F) = 0$ for any quasi-coherent sheaf F on X as soon as $i > \dim(X)$; if, in addition, F is coherent, then $H^i(X,F)$ are all finite dimensional (see [10, ch. III]). Hence, $R\Gamma$ descends to a functor

$$R\Gamma\colon \mathcal{D}^b(\mathrm{Coh}(X)) \to \mathcal{D}^b(\mathrm{Vec}_f(k)).$$

Direct image Let $f\colon X \to Y$ be a morphism of projective varieties. Consider the direct image functor $f_*\colon \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$. It is left exact and yields the right derived functor

$$Rf_*\colon \mathcal{D}^+(\mathrm{Qcoh}(X)) \to \mathcal{D}^+(\mathrm{Qcoh}(Y)).$$

Again, $R^i f_*(F) = 0$ for any quasi-coherent sheaf F on X as soon as $i > \dim(X)$; if, in addition, F is coherent, $R^i f_*(F)$ are all coherent sheaves on Y (see [10, ch. III]). Hence, Rf_* descends to an exact functor

$$Rf_*\colon \mathcal{D}^b(X) \to \mathcal{D}^b(Y).$$

When Rf_* is composed with other derived functors, it is worth remembering that the class of flabby sheaves is f_*-adapted.

Local Hom and dual For $F \in \mathrm{Qcoh}(X)$ a quasi-coherent sheaf, the covariant functor

$$\mathcal{H}om(F,\ )\colon \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X)$$

is left exact. If $F^\bullet \in \mathrm{Kom}^-(\mathrm{Qcoh}(X))$ is a bounded above complex of sheaves, one can extend such a functor by setting

$$\mathcal{H}om^\bullet(F^\bullet,\ )\colon \mathrm{K}^+(\mathrm{Qcoh}(X)) \to \mathrm{K}^+(\mathrm{Qcoh}(X))$$
$$\mathcal{H}om^i(F^\bullet, E^\bullet) = \prod_j \mathcal{H}om(F^j, E^{i+j}), \quad d = d_E - (-1)^i d_F.$$

This functor is exact, so that one can consider its right derived functor

$$R\mathcal{H}om^\bullet(F^\bullet,\ )\colon \mathcal{D}^+(\mathrm{Qcoh}(X)) \to \mathcal{D}^+(\mathrm{Qcoh}(X)).$$

This allows us to define the *dual* of a complex of sheaves $F^\bullet$ as

$$F^{\bullet\ \vee} = R\mathcal{H}om^\bullet(F^\bullet, \mathcal{O}_X).$$

Note that, even when $F^\bullet$ consists of a sheaf F concentrated in degree zero, its derived dual does not coincide with $\mathcal{H}om(F, \mathcal{O}_X)$, unless F is a locally free sheaf.

Tensor product For a coherent sheaf F, one can consider the right exact functor

$$F \otimes (\ )\colon \mathrm{Coh}(X) \to \mathrm{Coh}(X).$$

The category $\mathrm{Coh}(X)$ does not have enough projectives, but the class of locally free sheaves is adapted for the tensor product, so that the left derived functor exists. The

generalization to the case of a complex of sheaves $F^\bullet$ passes through the definition of an exact functor

$$F^\bullet \otimes (\ ) \colon \mathrm{K}^-(\mathrm{Coh}(X)) \to \mathrm{K}^-(\mathrm{Coh}(X))$$

as follows: for any bounded above complex $E^\bullet$, $F^\bullet \otimes E^\bullet$ is the total complex associated with the double complex $K^{i,j} = F^i \otimes E^j$. The triangulated subcategory of $\mathrm{K}^-(\mathrm{Coh}(X))$ consisting of complexes of locally free sheaves is adapted to this functor; we obtain

$$F^\bullet \otimes^L (\ ) \colon \mathcal{D}^-(X) \to \mathcal{D}^-(X).$$

If we assume that X is smooth, this derived functor descends to a functor of the bounded derived categories

$$F^\bullet \otimes^L (\ ) \colon \mathcal{D}^b(X) \to \mathcal{D}^b(X).$$

Inverse image Let $f \colon X \to Y$ be a morphism of projective varieties. The inverse image functor f^* is defined as the composition of the exact functor

$$f^{-1} \colon \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$$

and the right exact functor

$$\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} (\ ) \colon \mathrm{Coh}(X) \to \mathrm{Coh}(X).$$

It is possible to define its left derived functor Lf^*. In most of our applications, however, the morphism f will be flat, so that f^* is exact and there is no need to derive it.

We conclude this section listing some compatibility properties that are satisfied by the aforementioned functors and that we shall need afterwards.
If $F^\bullet$ and $E^\bullet$ are bounded complexes of sheaves on a projective variety X, then

$$F^{\bullet\,\vee} \otimes^L E^\bullet \simeq R\mathcal{H}om(F^\bullet, E^\bullet).$$

Given $f \colon X \to Y$ a projective morphism, for any $F^\bullet \in \mathcal{D}^b(X)$ and $E^\bullet \in \mathcal{D}^b(Y)$ one has the *projection formula*

$$Rf_*(F^\bullet) \otimes^L E^\bullet \xrightarrow{\sim} Rf_*(F^\bullet \otimes^L Lf^*(E^\bullet));$$

moreover, (Lf^*, Rf_*) is an adjoint pair, i.e. there exist functorial isomorphisms

$$\mathrm{Hom}(Lf^*F^\bullet, E^\bullet) \xrightarrow{\sim} \mathrm{Hom}(F^\bullet, Rf_*E^\bullet).$$

Duality Let $f \colon X \to Y$ be a morphism of smooth projective varieties. Define the *relative dimension* and the *relative dualizing bundle* as

$$\dim(f) = \dim(X) - \dim(Y), \quad \omega_f = \omega_X \otimes f^*\omega_Y.$$

Theorem 1.2.3 (Grothendieck-Verdier duality)**.** *For any $F^\bullet \in \mathcal{D}^b(X)$ and $E^\bullet \in \mathcal{D}^b(Y)$ there exist functorial isomorphisms*

$$Rf_* R\mathcal{H}om(F^\bullet, Lf^*(E^\bullet) \otimes \omega_f[\dim(f)]) \simeq R\mathcal{H}om(Rf_* F^\bullet, E^\bullet).$$

Applying the global sections functor on both sides one gets

$$R\mathrm{Hom}(F^\bullet, Lf^*(E^\bullet) \otimes \omega_f[\dim(f)]) \simeq R\mathrm{Hom}(Rf_* F^\bullet, E^\bullet),$$

and taking cohomology in degree zero gives

$$\mathrm{Hom}(F^\bullet, Lf^*(E^\bullet) \otimes \omega_f[\dim(f)]) \simeq \mathrm{Hom}(Rf_* F^\bullet, E^\bullet).$$

Consider the particular case of the structure morphism $f\colon X \to \mathrm{Spec}(k)$ of a smooth variety X. The relative dimension and the relative dualizing bundle are nothing but $\dim(X)$ and ω_X respectively. For any $E^\bullet, F^\bullet \in \mathcal{D}^b(X)$, one has

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}^b(X)}(F^\bullet, E^\bullet \otimes \omega_X[\dim(X)]) \\
&\qquad \simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(R\mathcal{H}om(E^\bullet, F^\bullet), \omega_X[\dim(X)]) \\
&\qquad \simeq \mathrm{Hom}_{\mathcal{D}^b(\mathrm{Spec}(k))}(R\Gamma(R\mathcal{H}om(E^\bullet, F^\bullet)), k) \\
&\qquad \simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(E^\bullet, F^\bullet)^\vee.
\end{aligned}$$

or, equivalently,

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(E^\bullet, F^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(F^\bullet, E^\bullet \otimes \omega_X[\dim(X)])^\vee.$$

These functorial isomorphisms express *Serre duality* for derived categories. If $F^\bullet$ is just a (shifted) sheaf $F[i]$, $E^\bullet = \mathcal{O}_X$ and n is the dimension of X, using Example 1.1.5 one recovers classical Serre duality:

$$\begin{aligned}
H^i(X, F) &= \mathrm{Ext}^i(\mathcal{O}_X, F) \simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{O}_X, F[i]) \\
&\simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(F[i], \omega_X[n])^\vee \simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(F, \omega_X[n-i])^\vee \simeq \mathrm{Ext}^{n-i}(F, \omega_X)^\vee.
\end{aligned}$$

1.3 Fourier-Mukai transforms

Let X and Y be two smooth projective varieties over a field k and denote by

$$q\colon X \times Y \to X \quad \text{and} \quad p\colon X \times Y \to Y$$

the two canonical projections.

Definition 1.3.1. Let $\mathcal{P} \in \mathcal{D}^b(X \times Y)$. The induced *Fourier-Mukai transform* is the functor

$$\begin{aligned}
\Phi_{\mathcal{P}}\colon \mathcal{D}^b(X) &\to \mathcal{D}^b(Y) \\
E^\bullet &\mapsto p_*(q^* E^\bullet \otimes \mathcal{P}).
\end{aligned}$$

We call the object $\mathcal{P}$ the *kernel* of the Fourier-Mukai transform $\Phi_{\mathcal{P}}$.

Remark 1.3.2. All the functors that appear in the definition of $\Phi_{\mathcal{P}}$ are intended as derived functors. In particular, $\Phi_{\mathcal{P}}$ is an exact functor.

Let us give a list of Fourier-Mukai functors.

Example 1.3.3. (i) The identity $\mathrm{id}\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ is naturally isomorphic to the Fourier-Mukai transform with kernel the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \subset X \times X$.

(ii) Let $f\colon X \to Y$ be a morphism. Then the push-forward $f_*\colon \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is naturally isomorphic to the Fourier-Mukai transform with kernel the structure sheaf $\mathcal{O}_{\Gamma_f}$ of the graph $\Gamma_f \subset X \times Y$. This kernel yields also a Fourier-Mukai transform $\Phi_{\Gamma_f}\colon \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ in the opposite direction, naturally isomorphic to the pull-back functor f^*.

(iii) Let $L \in \mathrm{Pic}(X)$. The autoequivalence of $\mathcal{D}^b(X)$ given by $E^\bullet \mapsto E^\bullet \otimes L$ is isomorphic to the Fourier-Mukai transform with kernel $\iota_* L$, where

$$\iota\colon X \to \Delta \subset X \times X$$

is the diagonal embedding.

(iv) The shift functor $T\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ can be regarded as the Fourier-Mukai transform with kernel $\mathcal{O}_\Delta[1]$.

(v) Let $\mathcal{P}$ be a coherent sheaf on $X \times Y$, flat over X. For any closed point $x \in X$, one has $\Phi_{\mathcal{P}}(\kappa(x)) = \mathcal{P}|_{\{x\}\times Y}$.

Fourier-Mukai transforms behave well in many respects: they always admit left and right adjoint - both of Fourier-Mukai type - and their composition is again a Fourier-Mukai transform.
Define, for any $\mathcal{P} \in \mathcal{D}^b(X \times Y)$, the objects

$$\mathcal{P}_L \coloneqq \mathcal{P}^\vee \otimes p^*\omega_Y[\dim(Y)] \quad \mathcal{P}_R \coloneqq \mathcal{P}^\vee \otimes q^*\omega_X[\dim(X)].$$

Proposition 1.3.4. *The Fourier-Mukai transform $\Phi_{\mathcal{P}_L}\colon \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ (resp. $\Phi_{\mathcal{P}_R}\colon \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$) is left (resp. right) adjoint to the Fourier-Mukai transform $\Phi_{\mathcal{P}}\colon \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$.*

Proof. For any $E^\bullet \in \mathcal{D}^b(X)$ and $F^\bullet \in \mathcal{D}^b(Y)$, one has

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{D}^b(X)}\left(\Phi_{\mathcal{P}_L}(F^\bullet), E^\bullet\right) \\
&\quad = \mathrm{Hom}_{\mathcal{D}^b(X)}\left(q_*(\mathcal{P}_L \otimes p^*F^\bullet), E^\bullet\right) \\
&\quad \simeq \mathrm{Hom}_{\mathcal{D}^b(X\times Y)}\left(\mathcal{P}_L \otimes p^*F^\bullet, q^*E^\bullet \otimes p^*\omega_Y[\dim(Y)]\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Grothendieck-Verdier duality)} \\
&\quad \simeq \mathrm{Hom}_{\mathcal{D}^b(X\times Y)}\left(\mathcal{P}^\vee \otimes p^*F^\bullet, q^*E^\bullet\right)
\end{aligned}$$

$$\simeq \mathrm{Hom}_{\mathcal{D}^b(X\times Y)}\left(p^*F^\bullet, \mathcal{P}\otimes q^*E^\bullet\right) \quad \text{(properties of the dual)}$$
$$\simeq \mathrm{Hom}_{\mathcal{D}^b(Y)}\left(F^\bullet, p_*(\mathcal{P}\otimes q^*E^\bullet)\right) \quad ((p^*,p_*) \text{ is an adjoint pair})$$
$$= \mathrm{Hom}_{\mathcal{D}^b(Y)}\left(F^\bullet, \Phi_{\mathcal{P}}(E^\bullet)\right).$$

An analogous proof shows that $\Phi_{\mathcal{P}_R}$ is right adjoint to $\Phi_{\mathcal{P}}$. □

Let X, Y and Z be three smooth projective varieties and denote π_{XY}, π_{YZ} and π_{XZ} the projections from $X\times Y\times Z$ to $X\times Y$, $Y\times Z$ and $X\times Z$ respectively. Given the objects $\mathcal{P}\in\mathcal{D}^b(X\times Y)$ and $\mathcal{Q}\in\mathcal{D}^b(Y\times Z)$, define

$$\mathcal{R} = \pi_{XZ*}\left(\pi_{XY}^*\mathcal{P}\otimes\pi_{YZ}^*\mathcal{Q}\right)\in\mathcal{D}^b(X\times Z)$$

Proposition 1.3.5. *The Fourier-Mukai transform*

$$\Phi_{\mathcal{R}}\colon \mathcal{D}^b(X)\to\mathcal{D}^b(Z)$$

is isomorphic to the composition of the Fourier-Mukai transforms with kernel $\mathcal{P}$ and $\mathcal{Q}$

$$\mathcal{D}^b(X)\xrightarrow{\Phi_{\mathcal{P}}}\mathcal{D}^b(Y)\xrightarrow{\Phi_{\mathcal{Q}}}\mathcal{D}^b(Z).$$

Proof. The proof is not hard and it is mainly an application of the projection formula. However, notations for all the possible involved projections cause some problems. Therefore, we would rather refer to [12] Proposition 5.10. □

The relation between arbitrary exact functors and Fourier-Mukai transforms is explained in the following highly non-trivial and celebrated

Theorem 1.3.6 (Orlov)**.** *Let X and Y be smooth projective varieties. Let*

$$F\colon\mathcal{D}^b(X)\to\mathcal{D}^b(Y)$$

be an exact functor which is fully faithful and admits left and right adjoints. Then there exists an object $\mathcal{P}$, unique up to isomorphism, such that $F\simeq\Phi_{\mathcal{P}}$. In particular, any equivalence $\mathcal{D}^b(X)\to\mathcal{D}^b(Y)$ is of Fourier-Mukai type.

We assume from now on that the base field k is algebraically closed. It is important to have at our disposal criteria to determine when a Fourier-Mukai functor $\Phi_{\mathcal{P}}\colon\mathcal{D}^b(X)\to\mathcal{D}^b(Y)$ is fully faithful or even an equivalence.

Proposition 1.3.7 (Bondal, Orlov)**.** *The functor $\Phi_{\mathcal{P}}$ is fully faithful if and only if, for any closed points $x,y\in X$, one has*

$$\mathrm{Hom}_{\mathcal{D}^b(Y)}\left(\Phi_{\mathcal{P}}(\kappa(x)),\Phi_{\mathcal{P}}(\kappa(y))[i]\right)=\begin{cases}k & \text{if } x=y \text{ and } i=0\\ 0 & \text{if } x\neq y \text{ or } i<0 \text{ or } i>\dim(X).\end{cases}$$

Proof. See [12] Proposition 7.1. □

Corollary 1.3.8. *Let $\mathcal{P}$ be a coherent sheaf on $X \times Y$ flat over X. Then $\Phi_{\mathcal{P}}$ is fully faithful if and only if the following conditions are satisfied:*

(i) For any point $x \in X$ one has $\mathrm{Hom}\left(\mathcal{P}|_{\{x\}\times Y}, \mathcal{P}|_{\{x\}\times Y}\right) \simeq k$;

(ii) If $x \neq y$, $\mathrm{Ext}^i\left(\mathcal{P}|_{\{x\}\times Y}, \mathcal{P}|_{\{x\}\times Y}\right) \simeq 0$ *for all i.*

Proof. Under the flatness assumption on the sheaf $\mathcal{P}$, by Example 1.3.3 (v) one has $\Phi_{\mathcal{P}}(\kappa(x)) \simeq \mathcal{P}|_{\{x\}\times Y}$ for any closed point $x \in X$. Then the result easily follows from the previous proposition. □

Proposition 1.3.9. *Suppose that the functor $\Phi_{\mathcal{P}}$ is fully faithful. It is an equivalence if and only if*

$$(i)\ \dim(X) = \dim(Y) \quad \textit{and} \quad (ii)\ \mathcal{P} \otimes q^*\omega_X \simeq \mathcal{P} \otimes p^*\omega_Y.$$

Proof. We shall prove just the "only if" direction, the other implication requiring some technical facts about triangulated categories (see [12, ch. 1]). By Proposition 1.3.4, $\Phi_{\mathcal{P}}$ admits left and right adjoints, with kernel $\mathcal{P}_L = \mathcal{P}^\vee \otimes p^*\omega_Y[\dim(Y)]$ and $\mathcal{P}_R = \mathcal{P}^\vee \otimes q^*\omega_X[\dim(X)]$ respectively. Since $\Phi_{\mathcal{P}}$ is an equivalence, a quasi-inverse is both left and right adjoint to $\Phi_{\mathcal{P}}$. By uniqueness of the Fourier-Mukai kernel in Orlov's theorem, $\mathcal{P}^\vee \otimes p^*\omega_Y[\dim(Y)] \simeq \mathcal{P}^\vee \otimes q^*\omega_X[\dim(X)]$, whence

$$\mathcal{P}^\vee \simeq \mathcal{P}^\vee \otimes q^*\omega_X \otimes p^*\omega_Y^\vee[\dim(X) - \dim(Y)]$$

Let us prove (i). Assume that equality does not hold, say $\dim X > \dim Y$. Let m be the maximal integer such that $H^m(\mathcal{P}^\vee)$ is non-zero. Then, as tensoring with invertible sheaves commutes with cohomology, we get

$$H^m(\mathcal{P}^\vee \otimes q^*\omega_X \otimes p^*\omega_Y^\vee[\dim(X) - \dim(Y)]) = H^{m+\dim(X)-\dim(Y)}(\mathcal{P}^\vee) \otimes q^*\omega_X \otimes p^*\omega_Y^\vee = 0$$

a contradiction. Hence $\dim(X) = \dim(Y)$. Therefore $\mathcal{P}^\vee \otimes p^*\omega_Y \simeq \mathcal{P}^\vee \otimes q^*\omega_X$ and, dualizing this relation, we obtain

$$\mathcal{P} \otimes q^*\omega_X \simeq \mathcal{P} \otimes p^*\omega_Y,$$

which is (ii). □

Corollary 1.3.10. *If X and Y are smooth projective varieties of the same dimension and with trivial canonical bundles ω_X and ω_Y, any fully faithful Fourier-Mukai transform $\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is an equivalence.*

1.3.1 Passage to cohomology

A smooth projective variety X over the field $\mathbb{C}$ of complex numbers can be considered as a complex manifold. Denote as $H^*(X, \mathbb{Q})$ the cohomology group of the constant sheaf $\mathbb{Q}$ on X. It comes with a natural ring structure, with product $\alpha.\beta$, for $\alpha, \beta \in H^*(X, \mathbb{Q})$. For any morphism $f\colon X \to Y$ of smooth projective varieties over $\mathbb{C}$, one has two homomorphisms in cohomology:

$$f^*\colon H^i(Y, \mathbb{Q}) \to H^i(X, \mathbb{Q})$$
$$f_*\colon H^i(X, \mathbb{Q}) \to H^{i+2\dim(Y)-2\dim(X)}(Y, \mathbb{Q})$$

the second being defined using Poincaré duality as follows: given $\alpha \in H^i(X, \mathbb{Q})$, $f_*\alpha$ is the unique element in $H^{i+2\dim(Y)-2\dim(X)}(Y, \mathbb{Q})$ such that, for every $\beta \in H^{2\dim(X)-i}(Y, \mathbb{Q})$ one has

$$\int_Y \beta.f_*\alpha = \int_X f^*\beta.\alpha.$$

For any cohomology class $\alpha \in H^*(X \times Y, \mathbb{Q})$ one introduces a *cohomological Fourier-Mukai transform*

$$\Phi^H_\alpha\colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$$
$$\beta \mapsto p_*(q^*\beta.\alpha).$$

Let us show how a Fourier-Mukai transform at the derived category level can induce a cohomological Fourier-Mukai transform. The key notion is introduced in the following

Definition 1.3.11. Let $F^\bullet \in \mathcal{D}^b(X)$ be a bounded complex of sheaves. We define the *Mukai vector* of $F^\bullet$ as

$$v(F^\bullet) \coloneqq \mathrm{ch}([F^\bullet]).\sqrt{\mathrm{td}(X)}.$$

where $[F^\bullet] = \sum(-1)^i[F^i]$ in the Grothendieck group $K(X)$ of X, ch denotes the Chern character and $\mathrm{td}(X)$ is the Todd class of the tangent sheaf of X.

Consider $\Phi_\mathcal{P}\colon \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ the Fourier-Mukai transform with kernel $\mathcal{P} \in \mathcal{D}^b(X \times Y)$. Its Mukai vector $v(\mathcal{P})$ defines a cohomological transform

$$\Phi^H_{v(\mathcal{P})}\colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q}).$$

It turns out that

$$v(\Phi_\mathcal{P}(F^\bullet)) = \Phi^H_{v(\mathcal{P})}(v(F^\bullet)) \quad \text{for all} \quad F^\bullet \in \mathcal{D}^b(X),$$

or, in other words, the following diagram commutes

$$\begin{array}{ccc} \mathcal{D}^b(X) & \xrightarrow{\Phi_\mathcal{P}} & \mathcal{D}^b(Y) \\ \downarrow v & & \downarrow v \\ H^*(X, \mathbb{Q}) & \xrightarrow{\Phi^H_{v(\mathcal{P})}} & H^*(Y, \mathbb{Q}). \end{array}$$

To simplify notation, in the following we will denote as $\Phi^H_{\mathcal{P}}$ the cohomological Fourier-Mukai transform induced by $\Phi_{\mathcal{P}}$.

Let us consider again some of the examples of the previous section. We would like to understand what the cohomological morphisms they induce look like.

Example 1.3.12. (i) $\Phi^H_{\mathcal{O}_\Delta} = \mathrm{id}$. To prove this, consider the diagonal embedding $\iota\colon X \to \Delta \subset X \times X$. As higher direct images of the structure sheaf $\mathcal{O}_X$ vanish, by the Grothendieck-Riemann-Roch formula one has

$$\mathrm{ch}(\mathcal{O}_\Delta).\mathrm{td}(X \times X) = \mathrm{ch}(\iota_!\mathcal{O}_X).\mathrm{td}(X \times X) = \iota_*(\mathrm{ch}(\mathcal{O}_X).\mathrm{td}(X)) = \iota_*(\mathrm{td}(X)).$$

Dividing by $\sqrt{\mathrm{td}(X \times X)}$ and recalling that

$$\iota^*\sqrt{\mathrm{td}(X \times X)} = \iota^*(p^*\sqrt{\mathrm{td}(X)}.q^*\sqrt{\mathrm{td}(X)}) = \mathrm{td}(X),$$

one gets

$$\begin{aligned} v(\mathcal{O}_\Delta) &= \mathrm{ch}(\mathcal{O}_\Delta).\sqrt{\mathrm{td}(X \times X)} = \iota_*(\mathrm{td}(X)).\sqrt{\mathrm{td}(X \times X)}^{-1} \\ &= \iota_*(\mathrm{td}(X).\iota^*\sqrt{\mathrm{td}(X \times X)}^{-1}) = \iota_*(\mathrm{td}(X).\mathrm{td}(X)^{-1}) = \iota_*(1). \end{aligned}$$

Finally, for any $\alpha \in H^*(X, \mathbb{Q})$, one has

$$\Phi^H_{\mathcal{O}_\Delta}(\alpha) = p_*(q^*\alpha.v(\mathcal{O}_\Delta)) = p_*(q^*\alpha.\iota_*(1)) = p_*\iota_*(\iota^*q^*\alpha) = \alpha$$

as $p \circ \iota = q \circ \iota = \mathrm{id}_X$.

(ii) If $L \in \mathrm{Pic}(X)$, the cohomological Fourier-Mukai transform induced by the autoequivalence $E^\bullet \mapsto E^\bullet \otimes L$ is the multiplication by $\mathrm{ch}(L) = \exp(c_1(L))$. In particular, it will not preserve the cohomological degree as soon as $c_1(L)$ is non-zero. To prove this, reasoning exactly as before, one finds

$$v(L) = \mathrm{ch}(\iota_*L).\sqrt{\mathrm{td}(X \times X)} = \iota_*(\mathrm{ch}(L)).$$

Hence, for any $\alpha \in H^*(X, \mathbb{Q})$,

$$\Phi^H_{\iota_*L}(\alpha) = p_*(q^*\alpha.v(\iota_*L)) = p_*(q^*\alpha.\iota_*(\mathrm{ch}(L))) = p_*\iota_*(\iota^*q^*(\alpha.\mathrm{ch}(L))) = \alpha.\mathrm{ch}(L).$$

(iii) The shift functor $T\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X)$ induces in cohomology the multiplication by -1. Indeed, $v(\mathcal{O}_\Delta[1]) = v(-\mathcal{O}_\Delta) = -v(\mathcal{O}_\Delta)$ implies $\Phi^H_{\mathcal{O}_\Delta[1]} = -\Phi^H_{\mathcal{O}_\Delta} = -\mathrm{id}$.

We would like to better understand the relations between $\Phi_{\mathcal{P}}$ and $\Phi^H_{\mathcal{P}}$. The main difficulty is that the map $v\colon \mathcal{D}^b(X) \to H^*(X, \mathbb{Q})$ is not surjective. However, surprisingly, we have the following

Proposition 1.3.13. *If $\mathcal{P} \in \mathcal{D}^b(X \times Y)$ defines an equivalence*

$$\Phi_{\mathcal{P}} \colon \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b(Y),$$

then the induced cohomological Fourier-Mukai transform

$$\Phi^H_{\mathcal{P}} \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$$

is an isomorphism of rational vector spaces.

Proof. It is not difficult to prove that, if $\Phi_{\mathcal{P}} \circ \Phi_{\mathcal{Q}} \simeq \Phi_{\mathcal{R}}$, then $\Phi^H_{\mathcal{P}} \circ \Phi^H_{\mathcal{Q}} \simeq \Phi^H_{\mathcal{R}}$.

Assume that $\Phi_{\mathcal{P}}$ is an equivalence. Its quasi-inverse, which is left and right adjoint to $\Phi_{\mathcal{P}}$, is still a Fourier-Mukai transform, whose kernel $\mathcal{P}_R$ is unique up to isomorphism by Orlov's theorem. One has

$$\Phi_{\mathcal{P}} \circ \Phi_{\mathcal{P}_R} \simeq \mathrm{id} \simeq \Phi_{\mathcal{O}_\Delta} \qquad \Phi_{\mathcal{P}_R} \circ \Phi_{\mathcal{P}} \simeq \mathrm{id} \simeq \Phi_{\mathcal{O}_\Delta}$$

Together with Example 1.3.12 (i), this implies that

$$\Phi^H_{\mathcal{P}} \circ \Phi^H_{\mathcal{P}_R} \simeq \Phi^H_{\mathcal{O}_\Delta} \simeq \mathrm{id} \qquad \Phi^H_{\mathcal{P}_R} \circ \Phi^H_{\mathcal{P}} \simeq \Phi^H_{\mathcal{O}_\Delta} \simeq \mathrm{id},$$

so that $\Phi^H_{\mathcal{P}}$ and $\Phi^H_{\mathcal{P}_R}$ are inverse to each other. □

Hodge struture For any complex projective manifold X, which is in particular a Kähler manifold, given an integer n we have a natural weight n Hodge structure on $H^n(X, \mathbb{Q})$, i.e. we can write

$$H^n(X, \mathbb{Q}) \otimes \mathbb{C} = H^n(X, \mathbb{C}) = \bigoplus_{r+s=n} H^{r,s}(X),$$

with $\overline{H^{r,s}(X)} = H^{s,r}(X)$. Moreover, one has $H^{r,s}(X) \simeq H^s(X, \Omega^r)$. As the Chern classes are of type (r, r), the Mukai vector of a sheaf is in $\bigoplus_r H^{r,r}(X) \cap H^{2r}(X, \mathbb{Q})$.

Proposition 1.3.14. *If $\Phi_{\mathcal{P}} \colon \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b(Y)$ is an equivalence, then the cohomological Fourier-Mukai transform $\Phi^H_{\mathcal{P}} \colon H^*(X, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$ yields isomorphisms*

$$\bigoplus_{r-s=i} H^{r,s}(X) \simeq \bigoplus_{r-s=i} H^{r,s}(Y)$$

for $i = -\dim X, \dots, 0, \dots, \dim X$.

Proof. By Proposition 1.3.13, it suffices to show that the $\mathbb{C}$-linear extension of $\Phi^H_{\mathcal{P}}$ satisfies

$$\Phi^H_{\mathcal{P}}\left(H^{r,s}(X)\right) \subset \bigoplus_{t-u=r-s} H^{t,u}(Y).$$

Using Künneth decomposition, we can write $v(\mathcal{P}) = \sum \alpha^{r',s'} \boxtimes \beta^{t,u}$, for $\alpha^{r',s'} \in H^{r',s'}(X)$ and $\beta^{t,u} \in H^{t,u}(Y)$. The class of $v(\mathcal{P})$ is algebraic, so, in the above

decomposition, only terms with $r' + t = s' + u$ will contribute. Let $\alpha \in H^{r,s}(X)$. One has

$$\Phi^H_{\mathcal{P}}(\alpha) = p_* \left(v(\mathcal{P}).q^*\alpha\right) = \sum p_* \left(q^*(\alpha^{r',s'}.\alpha)\right).\beta^{t,u}.$$

Let us focus on $p_*q^*\left(\alpha^{r',s'}.\alpha\right) \in H^{r+r'+s+s'-2\dim(X)}(Y,\mathbb{C})$. Note that $\alpha^{r',s'}.\alpha$ is non-zero only if $r+r' \leq \dim(X)$ and $s+s' \leq \dim(X)$. On the other hand, $p_*q^*\left(\alpha^{r',s'}.\alpha\right)$ is non-zero only if $r + r' + s + s' \geq 2\ \dim(X)$. We deduce that only terms with $(r + r', s + s') = (\dim(X), \dim(X))$ will contribute and that $p_*\left(q^*\left(\alpha^{r',s'}.\alpha\right)\right) \in H^0(Y,\mathbb{C}) \simeq \mathbb{C}$ is a complex scalar.

Hence,

$$\Phi^H_{\mathcal{P}}(\alpha) = \sum p_* \left(q^*(\alpha^{r',s'}.\alpha)\right) \beta^{t,u} \in \bigoplus H^{t,u}(Y)$$

with $t - u = s' - r' = \dim(X) - s - \dim(X) + r = r - s$. □

The Mukai pairing We conclude this section giving the definition of the *Mukai pairing* on $H^*(X,\mathbb{Q})$. We will show that it is compatible with the cohomological Fourier-Mukai transform induced by a derived equivalence.

Given $v = \sum_i v_i \in \oplus_i H^{2i}(X,\mathbb{Q}) = H^{\mathrm{ev}}(X,\mathbb{Q})$, we define its dual to be

$$v^\vee = \sum_i (-1)^i v_i \in \bigoplus_i H^{2i}(X,\mathbb{Q}).$$

We summarize the main properties of the dual in the following

Lemma 1.3.15. *(i) For any v and w in $H^{\mathrm{ev}}(X,\mathbb{Q})$, $(v.w)^\vee = v^\vee.w^\vee$.*

(ii) If $p\colon X \times Y \to Y$ is the second projection, then, for any $v \in H^{\mathrm{ev}}(X \times Y,\mathbb{Q})$,

$$p_*(v)^\vee = (-1)^{\dim(X)} p_*(v^\vee).$$

(iii) For any complex $E^\bullet \in \mathcal{D}^b(X)$,

$$v(E^{\bullet\ \vee}) = v(E^\bullet)^\vee . \exp\left(\frac{c_1(X)}{2}\right).$$

Proof. (i) Writing $v = \sum v_i$ and $w = \sum w_j$, it suffices to note that the k-th component of $v.w$ is $\sum v_i.w_{k-i}$, while the k-th component of $v^\vee.w^\vee$ is $\sum (-1)^i v_i.(-1)^{k-i} w_{k-i} = (-1)^k \sum v_i.w_{k-i}$.

(ii) If we write $v = \sum v_i$ with $v_i \in H^{2i}(X \times Y,\mathbb{Q})$, then $p_*v = \sum p_*v_i$, with $p_*v_i \in H^{2(i-\dim(X))}(Y,\mathbb{Q})$. Then

$$\begin{aligned} p_*(v^\vee) = p_*\left(\sum (-1)^i v_i\right) &= \sum (-1)^i p_*v_i \\ &= (-1)^{\dim(X)} \sum (-1)^{i-\dim(X)} p_*v_i = (-1)^{\dim(X)} (p_*v)^\vee. \end{aligned}$$

(iii) As $c_k(E^\vee) = (-1)^k c_k(E)$ for any locally free sheaf E, we have $\mathrm{ch}(E^{\bullet\vee}) = \mathrm{ch}(E^\bullet)^\vee$. Hence

$$v(E^{\bullet\vee}) = \mathrm{ch}(E^\bullet)^\vee.\sqrt{\mathrm{td}(X)} = v(E^\bullet)^\vee.\frac{\sqrt{\mathrm{td}(X)}}{\sqrt{\mathrm{td}(X)}^\vee}.$$

To conclude, it is enough to prove that $\sqrt{\mathrm{td}(X)} = \sqrt{\mathrm{td}(X)}^\vee.\exp(c_1(X)/2)$ or, equivalently, $\mathrm{td}(X) = \mathrm{td}(X)^\vee.\exp(c_1(X))$. By splitting principle, write

$$\mathrm{td}(X) = \prod_i \frac{a_i}{1-\exp(-a_i)}.$$

Then one has

$$\mathrm{td}(X)^\vee.\exp(c_1(X)) = \prod_i \frac{-a_i}{1-\exp(a_i)} \prod \exp(a_i) = \prod_i \frac{-a_i}{\exp(-a_i)-1} = \mathrm{td}(X).$$

□

Definition 1.3.16. The *Mukai pairing* is the bilinear form defined as

$$\langle v, v'\rangle := -\int_X \exp\left(\frac{c_1(X)}{2}\right).v^\vee.v' \quad \text{for any } v, v' \in H^{\mathrm{ev}}(X, \mathbb{Q}).$$

Recall that, for complexes $E^\bullet$ and $F^\bullet$ in $\mathcal{D}^b(X)$, we can define

$$\chi(E^\bullet, F^\bullet) = \sum_i (-1)^i \dim \mathrm{Ext}^i(E^\bullet, F^\bullet).$$

This generalizes the well-known notion of Euler characteristic of a coherent sheaf F on X: indeed, $\chi(\mathcal{O}_X, F) = \chi(F)$, as $\mathrm{Ext}^i(\mathcal{O}_X, F) = H^i(X, F)$.

Proposition 1.3.17. *Let $E^\bullet, F^\bullet \in \mathcal{D}^b(X)$. Then*

$$\chi(E^\bullet, F^\bullet) = -\langle v(E^\bullet), v(F^\bullet)\rangle.$$

Proof. It suffices a little calculation, which involves the Hirzebruch-Riemann-Roch theorem:

$$\begin{aligned} -\langle v(E^\bullet), v(F^\bullet)\rangle &= \int_X \exp\left(\frac{c_1(X)}{2}\right).v(E^\bullet)^\vee.v(F^\bullet) \\ &= \int_X v(E^{\bullet\ \vee}).v(F^\bullet) = \int_X \mathrm{ch}(E^{\bullet\vee}).\sqrt{\mathrm{td}(X)}.\mathrm{ch}(F^\bullet).\sqrt{\mathrm{td}(X)} \\ &= \int_X \mathrm{ch}(E^{\bullet\vee} \otimes F^\bullet).\mathrm{td}(X) = \chi(E^{\bullet\vee} \otimes F^\bullet) = \chi(E^\bullet, F^\bullet). \end{aligned}$$

□

Proposition 1.3.18. *Let $\Phi_{\mathcal{P}}\colon \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b(Y)$ be an equivalence. Then the induced cohomological Fourier-Mukai transform*

$$\Phi_{\mathcal{P}}^H\colon H^*(X,\mathbb{Q}) \to H^*(Y,\mathbb{Q})$$

is an isometry with respect to the Mukai pairing.

Proof. We need to check that, for any $v, v' \in H^*(X,\mathbb{Q})$,

$$\langle v, v'\rangle_X = \langle \Phi_{\mathcal{P}}^H(v), \Phi_{\mathcal{P}}^H(v')\rangle_Y.$$

As $\Phi_{\mathcal{P}}^H$ is an isomorphism by Proposition 1.3.13, it suffices to show that, for any $v \in H^*(X,\mathbb{Q})$ and $w \in H^*(Y,\mathbb{Q})$, one has

$$\langle \Phi_{\mathcal{P}}^H(v), w\rangle_Y = \langle v, \left(\Phi_{\mathcal{P}}^H\right)^{-1}(w)\rangle_X.$$

Recalling that $\left(\Phi_{\mathcal{P}}^H\right)^{-1} = \Phi_{\mathcal{P}_{\mathcal{R}}}^H$, let us perform the computations.

$$\begin{aligned}
\langle \Phi_{\mathcal{P}}^H(v), w\rangle_Y &= -\int_Y \exp\left(\frac{c_1(Y)}{2}\right).p_*(q^*v.v(\mathcal{P}))^\vee.w \\
&= (-1)^{\dim(X)+1}\int_{X\times Y} p^*\exp\left(\frac{c_1(Y)}{2}\right).(q^*v.v(\mathcal{P}))^\vee.p^*w \\
&= (-1)^{\dim(X)+1}\int_{X\times Y} p^*\exp\left(\frac{c_1(Y)}{2}\right).q^*v^\vee.v(\mathcal{P})^\vee.p^*w \\
&= (-1)^{\dim(X)+1}\int_{X\times Y} p^*\exp\left(\frac{c_1(Y)}{2}\right).q^*v^\vee.v(\mathcal{P}^\vee).\exp\left(\frac{c_1(X\times Y)}{2}\right)^{-1}.p^*w \\
&= (-1)^{\dim(X)+1}\int_{X\times Y} p^*\exp\left(\frac{c_1(Y)}{2}\right).q^*v^\vee.v(\mathcal{P}^\vee).q^*\exp\left(\frac{c_1(X)}{2}\right)^{-1}. \\
&\qquad\qquad .p^*\exp\left(\frac{c_1(Y)}{2}\right)^{-1}.p^*w \\
&= (-1)^{\dim(X)+1}\int_{X\times Y} q^*v^\vee.v(\mathcal{P}^\vee).q^*\exp\left(\frac{c_1(X)}{2}\right)^{-1}.p^*w \\
&= (-1)^{\dim(X)+1}\int_{X\times Y} q^*v^\vee.v(\mathcal{P}^\vee\otimes q^*\omega_X).q^*\exp\left(\frac{c_1(X)}{2}\right).p^*w \\
&= -\int_{X\times Y} q^*\exp\left(\frac{c_1(X)}{2}\right).q^*v^\vee.v(\mathcal{P}_R).p^*w \\
&= -\int_X \exp\left(\frac{c_1(X)}{2}\right).v^\vee.q_*(v(\mathcal{P}_R).p^*w) \\
&= \langle v, \left(\Phi_{\mathcal{P}}^H\right)^{-1}(w)\rangle_X.
\end{aligned}$$

□

Chapter 2

Torelli theorems

This chapter, devoted to Torelli theorems, is divided into two parts. The first is meant to be a sketchy introduction to K3 surfaces: we collect some basic facts on their singular cohomology groups, Hodge decomposition, ample and Kähler cone and period map, which allow us to state the Global Torelli theorem. This discussion motivates the second part, which deals with the Derived version of Torelli theorem. In the proof of this beautiful result, due to Mukai and Orlov, Fourier-Mukai transforms play a crucial role, as well as moduli spaces of stable sheaves, that we will introduce in the next chapter.

2.1 K3 surfaces

Definition 2.1.1. A *K3 surface* is a smooth compact connected complex surface X such that

$$\omega_X \simeq \mathcal{O}_X \quad \text{and} \quad H^1(X, \mathcal{O}_X) = 0.$$

Remark 2.1.2. The condition on the triviality of the canonical bundle means that there exists a (unique up to scalars) holomorphic differential 2-form that never vanishes on X. This form, which is closed and defines a non-degenerate alternating form on the tangent space T_xX to X at every point x, is then a *symplectic form*.

Remark 2.1.3. It is easy to determine the Euler characteristic $\chi(X, \mathcal{O}_X)$ of the structure sheaf. Indeed, by definition, one has $h^1(X, \mathcal{O}_X) = 0$. By Serre duality, $h^2(X, \mathcal{O}_X) = h^0(X, \omega_X) = h^0(X, \mathcal{O}_X) = 1$. Therefore,

$$\chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X) = 2.$$

Remark 2.1.4. Most of the K3 surfaces are not projective. However, they are always Kähler manifolds, but this is difficult to prove (see [3] Theorem 3).

Let us list some remarkable examples.

Example 2.1.5. Let us look for K3 surfaces among the smooth complete intersections X of degree $d_1, d_2, \ldots, d_n$ in the projective space $\mathbb{P}^{n+2}_{\mathbb{C}}$. We may assume that $2 \leq d_1 \leq d_2 \leq \cdots \leq d_n$ (if $d_1 = 1$, we are just considering complete intersections of degree $d_2, d_3, \ldots, d_n$ in $\mathbb{P}^{n+1}_{\mathbb{C}}$). By adjunction formula, the canonical bundle of X is

$$\omega_X = \mathcal{O}(-n-3+d_1+d_2+\cdots+d_n)|_X.$$

If we want it to be trivial, we need that $-n-3+d_1+d_2+\cdots+d_n = 0$; in particular, $n+3 \geq 2n$, i.e. $n \leq 3$. We have the following possibilities:

(i) if $n = 1$, then $d = 4$: we are looking for smooth quartics in $\mathbb{P}^3$;

(ii) if $n = 2$, then $d_1 = 2, d_2 = 3$: we are looking for smooth complete intersections of a quadric and a cubic in $\mathbb{P}^4$;

(iii) if $n = 3$, then $d_1 = d_2 = d_3 = 2$: we are looking for smooth complete intersections of three quadrics in $\mathbb{P}^5$.

It remains to show that $H^1(X, \mathcal{O}_X) = 0$. Let us do explicit calculations in case (ii). One has an exact sequence

$$0 \to \mathcal{O}(-2-3) \to \mathcal{O}(-2) \oplus \mathcal{O}(-3) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_X \to 0,$$

which can be split into the two short exact sequences

$$0 \to \mathcal{O}(-2-3) \to \mathcal{O}(-2) \oplus \mathcal{O}(-3) \to \mathcal{I} \to 0$$
$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_X \to 0.$$

Consider the two induced long exact cohomology sequences. In the latter, since $H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4})$ and $H^2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4})$ vanish, we deduce that $H^1(X, \mathcal{O}_X) \simeq H^2(\mathbb{P}^4, \mathcal{I})$. In the former, we have $H^2(\mathbb{P}^4, \mathcal{O}(-2) \oplus \mathcal{O}(-3)) = H^2(\mathbb{P}^4, \mathcal{O}(-2)) \oplus H^2(\mathbb{P}^4, \mathcal{O}(-3)) = 0 = H^3(\mathbb{P}^4, \mathcal{O}(-5))$, hence $H^2(\mathbb{P}^4, \mathcal{I}) = 0$.

Example 2.1.6. Let A be a two-dimensional complex torus. The involution $\iota\colon A \to A$ has 16 fixed points. Let $\varepsilon\colon \tilde{A} \to A$ be the blow-up at these points and denote as $E_1, E_2, \ldots E_{16}$ the exceptional divisors. The involution ι extends to an involution $\tilde{\iota}\colon \tilde{A} \to \tilde{A}$: it fixes pointwise the exceptional divisors and, on the other points, it acts as ι. Hence, the fixed points locus is a smooth divisor. Consider X the quotient surface of $\tilde{A}$ by $\tilde{\iota}$. $\pi\colon \tilde{A} \to X$ is a degree-two ramified cover, whose branch locus consists of 16 irreducible rational curves $C_1, C_2, \ldots, C_{16}$. This implies that the sum of the C_i's is divisible by 2 in $\mathrm{Pic}(X)$. The surface X we have obtained is called *Kummer surface* associated with A.

Conversely, assume that X is a K3 surface containing 16 disjoint smooth rational curves C_i whose sum is divisible by 2 in $\mathrm{Pic}(X)$ (actually, by a result of Nikulin, the

last condition is always satisfied). We claim that X is a Kummer surface. Indeed, one can construct the double cover $\pi\colon \tilde{A} \to X$ ramified along the curves C_i. Then, contracting the 16 exceptional curves $\pi^{-1}(C_i)$, we obtain a smooth surface A. As the canonical bundle is trivial and $\chi(\mathcal{O}_A) = 0$, by the classification of surfaces A is a complex torus. Hence, X is the Kummer surface associated with A.

2.1.1 Picard group and singular cohomology

Let X be a K3 surface. Several interesting facts can be deduced from the exponential sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\cdot 2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^{\times} \to 0$$

and from the following part of the induced long exact cohomology sequence

$$0 \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^{\times}) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \to \dots \tag{2.1}$$

First of all, 2.1 sheds light on the Picard group of X. Recalling that $H^1(X,\mathcal{O}_X) = 0$ by definition of K3 surface and that the Picard group $\mathrm{Pic}(X)$ can be identified with $H^1(X,\mathcal{O}_X^{\times})$, we have an injection

$$\mathrm{Pic}(X) \xhookrightarrow{c_1} H^2(X,\mathbb{Z}).$$

The image of c_1, isomorphic to $\mathrm{Pic}(X)$ in our case, is called the Néron-Severi group of X and it is denoted by $\mathrm{NS}(X)$. It is a finitely generated abelian group, whose rank, denoted by $\rho(X)$, is called the *Picard number* of X. For any $L, M \in \mathrm{Pic}(X)$, we can define their intersection product as

$$L.M = c_1(L) \cup c_1(M).$$

Remark 2.1.7. For the intersection product of L and M, we will use also the notation $(L.M)$ and, when $L = M$, (L^2).

Because of the triviality of the canonical bundle, the Riemann-Roch formula has a particularly simple expression:

$$\chi(X,L) = \frac{(L^2)}{2} + \chi(X,\mathcal{O}_X) = \frac{(L^2)}{2} + 2.$$

This immediately implies that the intersection product is even, but also that the Picard group is torsion-free: if L were a torsion element, we would have $\chi(X,L) = 2$; hence at least one of $h^0(X,L)$ or $h^2(X,L) = h^0(X,L^{\vee})$ would be non-zero, i.e. there would exist a non-zero section s of L or $L^{\vee}$. As $s^{\otimes \pm m}$ never vanishes ($L^{\otimes \pm m} \simeq \mathcal{O}_X$), so does s; therefore L would be trivial.
Something more can be said when the K3 surface X is projective.

Proposition 2.1.8. *Let X be a projective K3 surface. The intersection product on $\mathrm{Pic}(X) \otimes \mathbb{R}$ is non-degenerate and of signature $(1, \rho(X) - 1)$.*

Proof. As X is projective, there exists an ample line bundle H on X. Assume for a contradiction that there exists a non-trivial line bundle $L \in \mathrm{Pic}(X)$ such that $(L.M) = 0$ for any $M \in \mathrm{Pic}(X)$. In particular, $(L.H) = 0$. Then, neither L nor $L^\vee$ can be effective. By Serre duality and Riemann-Roch formula, $(L^2) = 2\chi(L) - 4 = -h^1(X, L) - 4 < 0$, a contradiction. Hence the intersection pairing is non-degenerate. The statement on the signature is Hodge index theorem (see [10, ch. V] Theorem 1.9). □

Secondly, 2.1 allows us to determine the singular cohomology groups of a K3 surface. Indeed, on the one hand, it immediately implies that $H^1(X, \mathbb{Z}) = 0$. On the other, as $\mathrm{Pic}(X)$ and $H^2(X, \mathcal{O}_X) \simeq \mathbb{C}$ are torsion-free, so is $H^2(X, \mathbb{Z})$. By universal coefficients theorems, which identify the torsion subgroups of $H^q(X, \mathbb{Z})$ and $H_{q-1}(X, \mathbb{Z})$, we get that $H_1(X, \mathbb{Z})$ is torsion-free, hence zero; we deduce, thanks to Poincaré duality, that $H^3(X, \mathbb{Z}) = 0$. Connectedness of X (with Poincaré duality once more) implies $H^0(X, \mathbb{Z}) \simeq H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$. To have a complete picture of the situation, it remains to determine the rank of the free abelian group $H^2(X, \mathbb{Z})$. To this aim, we use Noether formula, which establishes a relation between the holomorphic and topological Euler characteristic of X. In our case it reads as

$$12\chi(X, \mathcal{O}_X) = \omega_X^2 + \chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(X).$$

We deduce that $\chi_{\mathrm{top}}(X) = 24$. Now, the previous discussion, in terms of Betti numbers, yields $b_0(X) = b_4(X) = 1$, $b_1(X) = b_3(X) = 0$. Therefore,

$$b_2(X) = \chi_{\mathrm{top}}(X) - b_0(X) + b_1(X) + b_3(X) - b_4(X) = 24 - 2 = 22.$$

Let us focus now on the cup product

$$\cup \colon H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}.$$

Poincaré duality says that this bilinear form is unimodular. The Wu formula, which states that, for any $\alpha \in H^2(X, \mathbb{Z})$, $\alpha \cup \alpha \equiv \alpha \cup c_1(\omega_X) \pmod 2$, implies that the cup product is even. To determine the signature $(b_2(X)^+, b_2(X)^-)$ of its $\mathbb{R}$-bilinear extension to $H^2(X, \mathbb{R})$, recall that

$$b_2(X)^+ - b_2(X)^- = \frac{1}{3}(\omega_X^2 - 2\chi_{\mathrm{top}}(X)) = \frac{1}{3}(-2 \cdot 24) = -16.$$

Thus $(b_2(X)^+, b_2(X)^-) = (3, 19)$. By the classification of unimodular lattices, $(H^2(X, \mathbb{Z}), \cup)$ is isomorphic to the lattice

$$\Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where U is the hyperbolic plane and E_8 is the unique positive definite even unimodular lattice of rank 8. Λ is called *K3 lattice*.

2.1.2 Ample cone and Kähler cone

Let X be a K3 surface and assume that it is projective. Its Picard number $n = \rho(X)$ is then ≥ 1. Moreover, the intersection product on $\mathrm{Pic}(X)$ is symmetric and non-degenerate and its $\mathbb{R}$-bilinear extension to $V = \mathrm{Pic}(X) \otimes \mathbb{R}$ is of signature $(1, n-1)$ by Proposition 2.1.8. In a suitable basis of V, this bilinear form can be written as $x^2 = x_0^2 - x_1^2 - \cdots - x_n^2$. The cone

$$\left\{ x \in V \;\middle|\; x^2 > 0 \right\}$$

has two opposite connected components $\mathcal{P}^+$ and $\mathcal{P}^-$, corresponding to the two possible signs of x_0.

Remark 2.1.9. If $x \in \mathcal{P}^+$ and $y \in \overline{\mathcal{P}^+} \setminus \{0\}$, one has $x.y > 0$: imdeed, choosing an orthonormal basis whose first vector is $e_0 = x/\sqrt{x^2}$, we get $x.y = x^2 y_0 > 0$. Consequently, $x.y < 0$ if $y \in \overline{\mathcal{P}^-} \setminus \{0\}$. Therefore, if we pick x and y of strictly positive square, they are in the same connected component if and only if $x.y > 0$.

Define the *positive cone* of X to be

$$\mathrm{Pos}(X) \coloneqq \left\{ x \in \mathrm{Pic}(X) \otimes \mathbb{R} \;\middle|\; x^2 > 0, x.H > 0 \right\}$$

for some ample line bundle H and the *ample cone* of X to be

$$\mathrm{Amp}(X) \coloneqq \left\{ \sum_i \lambda_i L_i \;\middle|\; \lambda_i \in \mathbb{R}_{>0},\ L_i \text{ ample for all } i \right\}.$$

By Nakai-Moishezon criterion ([10, ch. V] Theorem 1.10), $\mathrm{Amp}(X) \subset \mathrm{Pos}(X)$; more precisely, a divisor class $L \in \mathrm{Pos}(X)$ is ample if and only if $(L.C) > 0$ for any irreducible curve C on X. We shall show that we can simplify this condition. Let $C \subset X$ be an integral curve and $g(C) = h^1(C, \mathcal{O}_C)$. Serre duality and the exact sequence

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

imply that

$$\chi(\mathcal{O}_X(C)) = \chi(\mathcal{O}_X(-C)) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_C) = 2 - (1 - g(C)) = g(C) + 1.$$

On the other hand, Riemann-Roch formula yields

$$\chi(\mathcal{O}_X(C)) = \frac{(C^2)}{2} + 2.$$

We deduce that

$$g(C) = \frac{1}{2}(C^2) + 1.$$

Hence, $(C^2) \geq -2$ and $(C^2) = -2$ if and only if C is a smooth rational curve. Define

$$\Delta \coloneqq \left\{ M \in \mathrm{Pic}(X) \;\middle|\; M^2 = -2 \right\}$$
$$\Delta^+ \coloneqq \left\{ M \in \Delta \;\middle|\; H^0(X, M) \neq 0 \right\}.$$

Note that any smooth rational curve lies in Δ^+, but there may be other classes: if we take for example C_1 and C_2 smooth rational curves meeting transversally at a single point, then $C_1 + C_2 \in \Delta^+$.

Proposition 2.1.10. *For a projective K3 surface X,*

$$\begin{aligned}\mathrm{Amp}(X) &= \left\{ x \in \mathrm{Pos}(X) \;\middle|\; x.M > 0 \text{ for any } M \in \Delta^+ \right\} \\ &= \{ x \in \mathrm{Pos}(X) \mid x.C > 0 \text{ for any } C \text{ smooth rational curve} \}.\end{aligned}$$

Proof. Clearly, we have inclusions

$$\begin{aligned}\mathrm{Amp}(X) &\subset \left\{ x \in \mathrm{Pos}(X) \;\middle|\; x.M > 0 \text{ for any } M \in \Delta^+ \right\} \\ &\subset \{ x \in \mathrm{Pos}(X) \mid x.C > 0 \text{ for any } C \text{ smooth rational curve} \}.\end{aligned}$$

As the points of $\mathrm{Pic}(X) \otimes \mathbb{Q}$ are dense in all these sets, it suffices to prove that, if $x \in \mathrm{Pos}(X) \cap (\mathrm{Pic}(X) \otimes \mathbb{Q})$ has strictly positive intersection with all smooth rational curves, it belongs to $\mathrm{Amp}(X)$. Up to multiplication with an integer, we may assume that x is a divisor class L. We will use the Nakai-Moishezon criterion to prove that L is ample. Let $C \subset X$ be an irreducible curve. If $(C^2) \geq 0$, then $C \in \overline{\mathrm{Pos}(X)} \setminus \{0\}$ and $L.C > 0$ by 2.1.9. If $(C^2) = -2$, C is a smooth rational curve, and by assumption we conclude that $L.C > 0$. □

As we have already said, most of K3 surfaces are not projective and, hence, do not have any ample divisor class. However, they are always Kähler manifolds. Recall that a *Kähler class* is the class in $H^2(X, \mathbb{R})$ of a non-degenerate positive real differential 2-forms of type $(1, 1)$. We can perform a construction similar to the one of the ample cone. On the real vector space $H^2(X, \mathbb{R}) \cap H^{1,1}(X)$, the cup product has signature $(1, 19)$. The set of classes of strictly positive square has two connected components; the component containing the Kähler classes is still called positive cone $\mathrm{Pos}(X)$ of X. Let $\Delta^+ \subset H^2(X, \mathbb{Z})$ denote the set of classes of effective divisors with square -2. As δ runs over the elements of Δ^+, the hyperplanes $\delta^\perp$ determine a paving of $\mathrm{Pos}(X)$. The connected components of $\mathrm{Pos}(X) \setminus \cup \delta^\perp$ are called the *chambers*. The *Kähler cone* is the chamber

$$\mathrm{Kah}(X) = \left\{ x \in \mathrm{Pos}(X) \;\middle|\; x.\delta > 0 \text{ for any } \delta \in \Delta^+ \right\}.$$

2.1.3 Hodge structure and period map

From the general theory of compact Kähler manifolds, we deduce that the singular cohomology group $H^2(X, \mathbb{Z})$ of a K3 surface X has a natural weight two Hodge

structure

$$H^2(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}\simeq H^2(X,\mathbb{C})\simeq H^{2,0}(X)\oplus H^{1,1}(X)\oplus H^{0,2}(X).$$

One may think at the Hodge decomposition in these terms. Via the de Rham isomorphism, the singular cohomology group $H^2(X,\mathbb{C})$ with complex coefficients can be identified with the de Rham cohomology group $H^2_{dR}(X,\mathbb{C})$ of classes of closed $\mathbb{C}$-valued differential 2-forms modulo exact forms. Let $H^{p,q}(X)\subset H^2_{dR}(X,\mathbb{C})$ be the subspace generated by classes of closed forms of type (p,q), $p+q=2$. Then the above decomposition says that any closed differential 2-form can be written as a sum of closed differential forms of type (p,q) up to an exact form.

We have already seen the importance of the cup product on $H^2(X,\mathbb{Z})$. Its $\mathbb{C}$-bilinear extension to $H^2(X,\mathbb{C})$ corresponds to the wedge product of differential forms followed by integration on X. According to the above description, $H^{2,0}(X)=\mathbb{C}[\omega]$, where ω is the symplectic form up to scalar. It clearly satisfies the relations

$$\int_X[\omega]\wedge[\omega]=0\quad\text{and}\quad\int_X[\omega]\wedge[\bar{\omega}]>0. \tag{2.2}$$

Remark 2.1.11. Actually, if we start from the lattice $(H^2(X,\mathbb{Z}),\cup)$, the datum of a differential form $\omega\in H^2_{dR}(X,\mathbb{C})$ satysfying 2.2 determines the Hodge decomposition:

$$H^{2,0}=\mathbb{C}[\omega]\quad H^{0,2}(X)=\mathbb{C}[\bar{\omega}]\quad H^{1,1}(X)=(H^{2,0}(X)\oplus H^{0,2}(X))^{\perp}.$$

Next, we define the period of a K3 surface. The Hodge decomposition plays a crucial role, but one has also to choose a suitable $\mathbb{Z}$-basis of $H^2(X,\mathbb{Z})$.

Definition 2.1.12. A *marked K3 surface* is a pair (X,σ), where X is a K3 surface and $\sigma\colon H^2(X,\mathbb{Z})\to\Lambda$ is an isometry.

The $\mathbb{C}$-linear extension of σ sends $H^{2,0}(X)$ to a one-dimensional subspace of $\Lambda\otimes\mathbb{C}$, which is identified to a point P of the projective space $\mathbb{P}(\Lambda\otimes\mathbb{C})$. P is called the *period* of (X,σ). As ω satisfies the relations 2.2 so does $\sigma_{\mathbb{C}}(\omega)$. Therefore the period P belongs to the subset

$$\Omega=\{\,x\in\mathbb{P}(\Lambda\otimes\mathbb{C})\mid x.x=0,x.\bar{x}>0\,\},$$

which is called the *period domain*.

Consider now $f\colon\mathcal{X}\to U$ a smooth family of K3 surfaces. A marking of f is an isometric isomorphism σ of the locally constant system $R^2f_*\underline{\mathbb{Z}}$ to the constant system Λ_U. This means that, for any $u\in U$, the induced map $\sigma_u\colon H^2(\mathcal{X}_u,\mathbb{Z})\to\Lambda$ is an isometry. We define the period map associated with (f,σ) as

$$\begin{aligned}\wp\colon U&\to\Omega\subset\mathbb{P}(\Lambda\otimes\mathbb{C})\\ u&\mapsto\wp(u)=\sigma_u(H^{2,0}(\mathcal{X}_u)).\end{aligned}$$

One has the following

Theorem 2.1.13 (Local Torelli)**.** *Let $0\in U$ and $\mathcal{X}\to U$ be a local universal deformation of $\mathcal{X}_0$. The period map $\wp\colon U\to\Omega$ is a local isomorphism at 0.*

Proof. See [5] Theorem 6.6. □

2.1.4 Global Torelli theorem

Let X and Y be K3 surfaces and $\varphi\colon H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ an isomorphism. We say that φ is a *Hodge isometry* if it preserves the cup product and the Hodge decomposition (i.e. the $\mathbb{C}$-linear extension of φ maps $H^{2,0}(X)$ to $H^{2,0}(Y)$). Moreover, we say that φ is *effective* if $\varphi(\mathrm{Kah}(X)) = \mathrm{Kah}(Y)$.

Theorem 2.1.14 (Global Torelli - Piatetski-Shapiro, Shafarevich)**.** *Two K3 surfaces X and Y are isomorphic if and only if there exists a Hodge isometry*

$$\varphi\colon H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z}).$$

If, moreover, φ is effective, there exists a unique isomorphism $u\colon Y \xrightarrow{\sim} X$ such that $\varphi = u^*$.

One direction is easy: if $u\colon Y \xrightarrow{\sim} X$ is an isomorphism, then the pull-back $u^*\colon H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is an effective Hodge isometry.

For the converse, we need the following three propositions, whose proof can be found in [3].

Proposition 2.1.15. *Torelli theorem holds when X is a Kummer surface.*

Proposition 2.1.16. *The set of periods of marked Kummer surfaces is dense in Ω.*

Proposition 2.1.17. *Let $f\colon \mathcal{X} \to U$ and $g\colon \mathcal{Y} \to U$ be two smooth families of K3 surfaces over the analytic variety U and $\Phi\colon R^2f_*\underline{\mathbb{Z}} \to R^2g_*\underline{\mathbb{Z}}$ an isomorphism of local systems such that, for any $u \in U$, $\Phi_u\colon H^2(\mathcal{X}_u,\mathbb{Z}) \to H^2(\mathcal{Y}_u,\mathbb{Z})$ is a Hodge isometry. Let $T \subset U$ and $0 \in U$ an accumulation point of T. Assume that, for any $t \in T$, there exists an isomorphism $u_t\colon \mathcal{Y}_t \to \mathcal{X}_t$ such that $u_t^* = \varphi_t$. Then*

(i) $\mathcal{X}_0$ and $\mathcal{Y}_0$ are isomorphic;

(ii) if, in addition, Φ_0 is effective, then there exists an isomorphism $u_0\colon \mathcal{Y}_0 \to \mathcal{X}_0$ such that $u_0^* = \varphi_0$.

Proof of Theorem 2.1.14. Let $f\colon \mathcal{X} \to U$ and $g\colon \mathcal{Y} \to V$ be two local universal families for X and Y, so that $X = \mathcal{X}_0$ for $0 \in U$ and $Y = \mathcal{Y}_{0'}$ for $0' \in V$. Up to restricting U (resp. V) to a simply connected neighbourhood of 0 (resp. $0'$), we may assume that the local system $R^2f_*\underline{\mathbb{Z}}$ on U (resp. $R^2g_*\underline{\mathbb{Z}}$ on V) is isomorphic to the constant sheaf associated with the K3 lattice Λ. Choose trivializations $\sigma\colon R^2f_*\underline{\mathbb{Z}} \to \Lambda_U$ and $\tau\colon R^2g_*\underline{\mathbb{Z}} \to \Lambda_V$ so that $\sigma_0 = \tau_0 \circ \varphi$. The induced period maps $\wp_U\colon U \to \Omega$ and $\wp_V\colon V \to \Omega$ satisfy $\wp_U(0) = \wp_V(0')$. The local Torelli theorem 2.1.13 allows us to identify $(U,0)$ and $(V,0')$ in such a way that the families (f,σ) and (g,τ) have the same period map. This means that the isomorphism $\Phi = \tau^{-1} \circ \sigma\colon R^2f_*\underline{\mathbb{Z}} \to R^2g_*\underline{\mathbb{Z}}$ induces a Hodge isometry at each point of U. By Proposition 2.1.16, there exists T a dense subset of U such that $\mathcal{X}_t$ is a Kummer surface. As Φ_0 is effective, so is Φ_t for t in a neighbourhood of 0. By Proposition 2.1.15, we deduce isomorphisms $u_t\colon \mathcal{Y}_t \to \mathcal{X}_t$ such that $u_t^* = \Phi_t$ for any $t \in T$. Proposition 2.1.17 implies the existence of $u\colon Y \to X$ such that $u^* = \Phi_0 = \varphi$. □

2.2 Derived equivalence of K3 surfaces

The aim of this section is to state and prove a derived version of Torelli theorem, i.e. a cohomological criterion that decides when two K3 surfaces have equivalent derived categories. In section 1.2 we have introduced the bounded derived category only for projective varieties. Therefore, all the involved K3 surfaces will be supposed projective.

Fix a K3 surface X and consider its whole cohomology group

$$H^*(X,\mathbb{Z}) = H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \subset H^{\mathrm{ev}}(X,\mathbb{Q}).$$

The Mukai pairing introduced in section 1.3.1 induces on $H^*(X,\mathbb{Z})$ a symmetric bilinear pairing

$$\langle \alpha, \beta \rangle = -\int_X \alpha^\vee.\beta \quad \text{for all} \quad \alpha, \beta \in H^*(X,\mathbb{Z}).$$

If $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $\beta = (\beta_0, \beta_1, \beta_2)$ for $\alpha_i, \beta_i \in H^{2i}(X,\mathbb{Z})$, the Mukai pairing is more explicitly given by

$$\langle \alpha, \beta \rangle = \alpha_2.\beta_2 - \alpha_0.\beta_4 - \alpha_4.\beta_0 \in H^4(X,\mathbb{Z}) \simeq \mathbb{Z}.$$

As usual, we have identified $H^4(X,\mathbb{Z})$ and $\mathbb{Z}$ via the fundamental cocycle of X. The group $H^*(X,\mathbb{Z})$ with the Mukai pairing is called the *Mukai lattice* and we will denote it by $\tilde{H}(X,\mathbb{Z})$. Moreover, $\tilde{H}(X,\mathbb{Z})$ has the following weight two Hodge decomposition:

$$\begin{aligned}
\tilde{H}^{2,0}(X,\mathbb{C}) &= H^{2,0}(X),\\
\tilde{H}^{1,1}(X,\mathbb{C}) &= H^0(X,\mathbb{C}) \oplus H^{1,1}(X) \oplus H^4(X,\mathbb{C}),\\
\tilde{H}^{0,2}(X,\mathbb{C}) &= H^{0,2}(X).
\end{aligned}$$

Since $c_1(X) = 0$ and $\mathrm{td}_2(X) = \chi(X,\mathcal{O}_X) = 2$, the Todd class $\mathrm{td}(X)$ of X is equal to $(1,0,2)$; thus, its positive square root $\sqrt{\mathrm{td}(X)}$ equals $(1,0,1)$. For a coherent sheaf E on X, $\mathrm{ch}(E) \in \tilde{H}(X,\mathbb{Z})$, because the intersection product on X is even. Recalling the Definition 1.3.11 of Mukai vector, we have

$$\begin{aligned}
v(E) = \mathrm{ch}(E).\sqrt{\mathrm{td}(X)} &= (r(E), c_1(E), \mathrm{ch}_2(E)).(1,0,1)\\
&= (r(E), c_1(E), \chi(E) - r(E)) \in \tilde{H}^{1,1}(X,\mathbb{Z}) \subset \tilde{H}(X,\mathbb{Z}).
\end{aligned}$$

Actually, also the Chern character of a sheaf on the product of two K3 surfaces X and Y lies in the integral cohomology group of $X \times Y$, as shown by the following

Lemma 2.2.1. *Let X and Y be two K3 surfaces. Then, the Mukai vector of any object $E^\bullet \in \mathcal{D}^b(X \times Y)$ is an integral cohomology class $v(E^\bullet) \in H^*(X \times Y, \mathbb{Z})$.*

Proof. We may assume that $E^\bullet$ is a sheaf E. It suffices to show $\mathrm{ch}(E)$ is integral: indeed

$$\sqrt{\mathrm{td}(X\times Y)} = q^*\sqrt{\mathrm{td}(X)}.p^*\sqrt{\mathrm{td}(Y)} = q^*(1,0,1).p^*(1,0,1) \in H^*(X\times Y,\mathbb{Z}).$$

Write

$$\mathrm{ch}(E) = (r(E), \mathrm{c}_1(E), \frac{1}{2}(c_1^2(E) - 2c_2(E)), \mathrm{ch}_3(E), \mathrm{ch}_4(E)).$$

It is clear that $r(E)$ and $c_1(E)$ are integral. Moreover, using the Künneth decomposition $H^2(X\times Y,\mathbb{Z}) = H^2(X,\mathbb{Z})\oplus H^2(Y,\mathbb{Z})$, we can write $c_1(E) = q^*\alpha + p^*\beta$ with $\alpha\in H^2(X,\mathbb{Z}), \beta\in H^2(Y,\mathbb{Z})$. Therefore, $c_1^2(E) = q^*\alpha^2 + 2q^*\alpha.p^*\beta + p^*\beta^2$, which is divisible by two, as the intersection pairing on X and Y is even. To conclude, it suffices to show that $\mathrm{ch}_3(E)$ and $\mathrm{ch}_4(E)$ are integral. We will use the Grothendieck-Riemann-Roch formula

$$\mathrm{ch}(p_!E) = p_*(\mathrm{ch}(E).q^*\mathrm{td}(X))$$

and the integrality of Chern characters on K3 surfaces. Using once more Künneth decomposition, we can write

$$\mathrm{ch}(E) = (e^{0,0}, e^{2,0}+e^{0,2}, e^{4,0}+e^{2,2}+e^{0,4}, e^{4,2}+e^{2,4}, e^{4,4})$$

for $e^{r,s}\in H^r(X,\mathbb{Q})\otimes H^s(Y,\mathbb{Q})$. We have already proven that $e^{0,0}, e^{2,0}, e^{0,2}, e^{4,0}, e^{2,2}, e^{0,4}$ are integral.

$$c_1(p_!E) = p_*(e^{4,2}+e^{2,4}+2e^{2,0}+2e^{0,2}) = \int_X e^{4,2}+2e^{0,2} \in H^2(Y,\mathbb{Z})$$

implies that $e^{4,2}$ is integral. Analogously, using Grothendieck-Riemann-Roch formula with respect to the projection q, one obtains that $e^{2,4}$ is integral. Hence, $\mathrm{ch}_3(E)\in H^6(X\times Y,\mathbb{Z})$. In the same way,

$$\mathrm{ch}_2(p_!E) = p_*(e^{4,4}+2e^{4,0}+2e^{2,2}+2e^{0,4}) = \int_X e^{4,4}+2e^{0,4} \in H^4(Y,\mathbb{Z})$$

implies that $\mathrm{ch}_4(E) = e^{4,4}$ is integral. □

Proposition 2.2.2. *If $\Phi_{E^\bullet}\colon \mathcal{D}^b(X)\to\mathcal{D}^b(Y)$ is an equivalence between the derived categories of two K3 surfaces, then the induced map in cohomology defines a Hodge isometry*

$$\Phi^H_{E^\bullet}\colon \tilde{H}(X,\mathbb{Z})\to\tilde{H}(Y,\mathbb{Z}).$$

Proof. At a cohomological level, $\Phi^H_{E^\bullet}(\alpha) = p_*(q^*\alpha.v(E^\bullet))$. If $\alpha\in\tilde{H}(X,\mathbb{Z})$, by Lemma 2.2.1 also $\Phi^H_{E^\bullet}(\alpha)$ is integral, i.e.

$$\Phi^H_{E^\bullet}(\tilde{H}(X,\mathbb{Z}))\subset\tilde{H}(Y,\mathbb{Z}).$$

Now, the inverse of $\Phi_{E^\bullet}$ is again a Fourier-Mukai equivalence and we can therefore apply the same argument. We eventually deduce $\Phi^H_{E^\bullet}\colon \tilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \tilde{H}(Y,\mathbb{Z})$.

The fact that $\Phi^H_{E^\bullet}$ respects the Mukai pairing and, when tensored with $\mathbb{C}$, maps $H^{2,0}(X)$ to $H^{2,0}(Y)$ comes from the more general Propositions 1.3.18 and 1.3.14, respectively. □

Let us discuss now some autoequivalences of the derived category $\mathcal{D}^b(X)$ of a K3 surface X.

Example 2.2.3. (i) Every line bundle L on X defines an autoequivalence

$$L \otimes (\ )\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X).$$

As shown in Example 1.3.3, it is a Fourier-Mukai transform inducing in cohomology the multiplication with the Chern character $\mathrm{ch}(L)$.

(ii) The structure sheaf $\mathcal{O}_X$ induces an autoequivalence of $\mathcal{D}^b(X)$ as follows. The mapping cone of the natural map $\mathcal{O}_{X\times X} \to \mathcal{O}_\Delta$ is an object $\mathcal{P}$ in $\mathcal{D}^b(X\times X)$. To keep track of the structure sheaf, we denote the Fourier-Mukai transform with kernel $\mathcal{P}$ as

$$T_{\mathcal{O}_X}\colon \mathcal{D}^b(X) \to \mathcal{D}^b(X).$$

It turns out that $T_{\mathcal{O}_X}$ is an equivalence (this is a general fact: for a K3 surface X, $\mathcal{O}_X$ is a spherical object and $T_{\mathcal{O}_X}$ is the associated spherical twist, see [12, ch. 8]). Note that $v(\mathcal{P}) = v(\mathcal{O}_{X\times X}[1] \oplus \mathcal{O}_\Delta) = v(\mathcal{O}_\Delta) - v(\mathcal{O}_{X\times X})$. Therefore, in cohomology, $T^H_{\mathcal{O}_X} = \Phi^H_{\mathcal{O}_\Delta} - \Phi^H_{\mathcal{O}_{X\times X}} = \mathrm{id} - \Phi^H_{\mathcal{O}_{X\times X}}$. Now, the Mukai vector $v(\mathcal{O}_{X\times X}) = \sqrt{\mathrm{td}(X\times X)} = p^*\sqrt{\mathrm{td}(X)}.q^*\sqrt{\mathrm{td}(X)}$, hence $\Phi^H_{\mathcal{O}_{X\times X}}(\alpha) = -\langle \alpha, \sqrt{\mathrm{td}(X)}\rangle \sqrt{\mathrm{td}(X)}$. Thus,

$$T^H_{\mathcal{O}_X}(\alpha) = \alpha + \langle \alpha, \sqrt{\mathrm{td}(X)}\rangle \sqrt{\mathrm{td}(X)}.$$

Notice that, in particular, $T^H_{\mathcal{O}_X}$ acts as the identity on $H^2(X,\mathbb{Z})$, and interchanges, up to a sign, $H^0(X,\mathbb{Z})$ and $H^4(X,\mathbb{Z})$.

Theorem 2.2.4 (Derived Torelli - Mukai, Orlov)**.** *Two K3 surfaces X and Y are derived equivalent if and only if there exists a Hodge isometry $\tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$.*

Proof. We have already shown the "only if " direction in Proposition 2.2.2. Let us prove the converse.

(i) Assume that there exists a Hodge isometry $\varphi\colon \tilde{H}(X,\mathbb{Z}) \xrightarrow{\sim} \tilde{H}(Y,\mathbb{Z})$ with $\varphi(0,0,1) = \pm(0,0,1)$. As the shift functor induces multiplication with -1 in cohomology, we may assume that $\varphi(0,0,1) = (0,0,1)$. Let $v = (r,\ell,s) =$

$\varphi(1,0,0)$. Notice that $v \in \tilde{H}^{1,1}(Y,\mathbb{Z})$ and in particular $\ell \in \mathrm{NS}(Y)$, i.e $\ell = c_1(L)$ for some line bundle L on Y.

$$-1 = \langle (1,0,0),(0,0,1)\rangle = \langle v,(0,0,1)\rangle = -r$$
$$0 = \langle (1,0,0),(1,0,0)\rangle = \langle v,v\rangle = \ell^2 - 2rs = \ell^2 - 2s \Rightarrow s = \frac{\ell^2}{2}.$$

Hence

$$\varphi(1,0,0) = \left(1,\ell,\frac{\ell^2}{2}\right).$$

Up to composition with the cohomological transform induced by tensoring with $L^\vee$, we may suppose that $\varphi(1,0,0) = (1,0,0)$ and $\varphi(0,0,1) = (0,0,1)$. Therefore, for any $\alpha \in H^2(X,\mathbb{Z})$, $\varphi(0,\alpha,0) = (r',\ell',s')$ will satisfy

$$0 = \langle (1,0,0),(0,\alpha,0)\rangle = \langle (1,0,0),(r',\ell',s')\rangle = -s'$$
$$0 = \langle (0,0,1),(0,\alpha,0)\rangle = \langle (0,0,1),(r',\ell',s')\rangle = -r'.$$

Hence, φ induces a Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$. By Torelli theorem 2.1.15, $X \simeq Y$, and in particular $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$.

(ii) Now, let us assume that $\varphi(0,0,1) = (r,\ell,s) = v$, with $r \neq 0$. Up to composition with the shift functor, we may assume that $r > 0$. One has $\langle v,v\rangle = 0$. Moreover, for $v' \coloneqq \varphi(-1,0,0)$, $\langle v,v'\rangle = 1$. By the general theory of moduli spaces of stable sheaves on K3 surfaces, to be introduced in the next chapter (but there are no circularities), there exist M a K3 surface and $\mathcal{E}$ a sheaf on $Y \times M$ such that $\Phi_{\mathcal{E}^\vee} \colon \mathcal{D}^b(Y) \to \mathcal{D}^b(M)$ is an equivalence. Consider the composition

$$\psi : \tilde{H}(X,\mathbb{Z}) \xrightarrow{\varphi} \tilde{H}(Y,\mathbb{Z}) \xrightarrow{\Phi^H_{\mathcal{E}^\vee}} \tilde{H}(M,\mathbb{Z}).$$

$\psi(0,0,1) = \Phi^H_{\mathcal{E}^\vee}(v) = (0,0,1)$ (this calculation will be done in Theorem 3.3.19). Then, by part (i), $\mathcal{D}^b(X) \simeq \mathcal{D}^b(M)$, hence, composing with the inverse of $\Phi_{\mathcal{E}^\vee}$, $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$.

(iii) Finally, assume that $\varphi(0,0,1) = (0,\ell,s) = v$, with $\ell \neq 0$. To reduce to case (ii), the aim is to use the spherical twist. However, as $T^H_{\mathcal{O}_Y}(v) = (0,\ell,s) + \langle (0,\ell,s),(1,0,1)\rangle (1,0,1) = (0,\ell,s) + (-s)(1,0,1) = (-s,\ell,0)$, we should ensure that $s \neq 0$. If $s \neq 0$, we are done. If this is not the case, consider a line bundle $L \in \mathrm{Pic}(Y)$ and the cohomological transform induced by tensoring with L. One has

$$\mathrm{ch}(L).(0,\ell,0) = \left(1,c_1(L),\frac{c_1(L)^2}{2}\right).(0,\ell,0) = (0,\ell,c_1(L).\ell).$$

As the intersection pairing is non-degenerate, $c_1(L).\ell \neq 0$ for some $L \in \mathrm{Pic}(Y)$, which concludes the proof. □

Chapter 3

Mukai theory of moduli spaces

Roughly speaking, moduli spaces are geometric spaces whose points parametrize algebraic objects up to a certain equivalence relation. Given a projective variety X, the set coherent sheaves on X with fixed Hilbert polynomial cannot be parametrized by an algebraic variety. To get around this problem, one introduces the notion of (semi)stability, which is presented in the first section. In the second, we outline the construction of the moduli space of semistable sheaves on X with fixed Hilbert polynomial. In particular, as it plays a prominent role, we introduce the Hilbert scheme, but, for space reasons, we have to take Mumford's Geometric Invariant Theory for granted. The third section is devoted to the theory of moduli spaces of semistable sheaves on a polarized K3 surface X. After introducing general machinery, we study in some detail the low dimensional cases (rigid and semirigid sheaves) and, just to be sure that we are not working on the empty set, we conclude with an existence result due to Mukai.

3.1 (Semi)stability

Let X be a projective scheme over $\mathbb{C}$ with a very ample invertible sheaf $\mathcal{O}(1)$; denote $H = c_1(\mathcal{O}(1))$ its first Chern class. For any coherent sheaf E on X, set $E(n) = E \otimes \mathcal{O}(n)$. Let $d = d(E)$ be the dimension of the support of E and $r = r(E)$ its rank (computed on its support). Then there exists a polynomial with rational coefficients

$$P_E(n) \coloneqq \chi(E(n)) = r\frac{n^d}{d!} + \ldots,$$

called the *Hilbert polynomial* of E. We define the *slope* and the *reduced Hilbert polynomial* of E as

$$\mu(E) = \frac{(c_1(E).H^{d-1})}{r} \quad \text{and} \quad p_E = \frac{P_E}{r} = \frac{\chi(E(n))}{r}$$

A coherent sheaf E is said to be *pure* of dimension d if, for all non-zero coherent subsheaves F of E, we have $d(F) = d$. For a sheaf E of maximal dimension, purity

is equivalent to torsion-freeness.

Definition 3.1.1. A coherent sheaf E is said to be *stable* (resp. *semistable*) if it is pure and, for any non-zero coherent subsheaf F of E, there exists N such that

$$p_F(n) < p_E(n) \qquad (\text{resp.} \quad p_F(n) \leq p_E(n))$$

for $n \geq N$; equivalently, if $p_F < p_E$ (resp. $p_F \leq p_E$) when polynomials are ordered lexicographically starting from the highest term. A semistable sheaf that is not stable is called *strictly semistable*. A *polystable* sheaf is a direct sum of stable sheaves with the same reduced Hilbert polynomial.

A coherent sheaf E is said to be μ-stable (resp. μ-semistable) if it is pure and, for any non-trivial coherent subsheaf F of E, one has

$$\mu(F) < \mu(E) \qquad (\text{resp.} \quad \mu(F) \leq \mu(E)) \,.$$

For E and F semistable sheaves, $\mathrm{Hom}(E, F) \neq 0$ implies $p_E \leq p_F$: if $\varphi \colon E \to F$ is a non-zero morphism of image I, then $p_E \leq p_I \leq p_F$.

The category of semistable sheaves with fixed reduced Hilbert polynomial p_0 is clearly abelian. It is also artinian: descending filtrations are stationary as the rank must decrease.

Every stable sheaf E is *simple*, meaning that the ring of endomorphisms $\mathrm{End}(E)$ equals $\mathbb{C}$: given a non-zero endomorphism $\varphi \colon E \to E$, $\mathrm{Ker}\, \varphi = 0$ (resp. $\mathrm{Coker}\, \varphi = 0$) as it is a proper subsheaf (resp. quotient sheaf) of E with the same reduced Hilbert polynomial; then $\mathrm{End}(E)$ is a finite division algebra over $\mathbb{C}$, hence it is $\mathbb{C}$.

For any semistable sheaf E with reduced Hilbert polynomial p, there exists a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

whose successive quotients E_i/E_{i-1} are stable with reduced Hilbert polynomial p for $i = 1, 2, \ldots, n$. Indeed, if E is stable, we are done. Otherwise, there exists a proper non-trivial subsheaf with the same reduced Hilbert polynomial p. If even this is not stable, we continue in the same way; as the category is artinian, we finally find a stable subsheaf E_1 with $p_{E_1} = p$. Then apply the same procedure to E/E_1. This filtration, called *Jordan-Hölder filtration*, is not unique, but the polystable sheaf $\mathrm{gr}(E) = \bigoplus (E_i/E_{i-1})$ is unique. To semistable sheaves E and E' are *S-equivalent* if $\mathrm{gr}(E) \simeq \mathrm{gr}(E')$.

Let us conclude this section spelling out what stability means for sheaves on a smooth projective complex surface X. Fix $\mathcal{O}(1)$ an ample sheaf on X with first Chern class $c_1(\mathcal{O}(1)) = H$ and consider a coherent sheaf E on X.

(0) If E is zero-dimensional, namely it is supported on a finite number of closed points, it is obviously pure. Its Hilbert polynomial $P_E = \dim\, H^0(X, E) = r$ is constant and then $p_E = 1$. Hence, E is always semistable, and it is stable if and only if it is the structure sheaf of a reduced point.

(1) If E is supported on a smooth connected curve $\mathcal{C}$, then the notion of (semi)stability of E on X coincide with the notion of μ-(semi)stability on $\mathcal{C}$.

(2) Assume that E is two-dimensional of rank $r = r(E)$. E is pure if and only if it is torsion free. By Hirzebruch-Riemann-Roch formula, the Hilbert polynomial of E can be computed as follows:

$$\begin{aligned} P_E(n) = \chi(E(n)) &= \int_X \mathrm{ch}(E(n)).\mathrm{td}(X) = \int_X \mathrm{ch}(E).\mathrm{ch}(\mathcal{O}(n)).\mathrm{td}(X) \\ &= \int_X (r, c_1(E), \mathrm{ch}_2(E)) \,.\, \left(1, nH, \frac{n^2}{2}(H^2)\right).\mathrm{td}(X) \\ &= \frac{r(H^2)}{2}n^2 + \left((c_1(E).H) + \frac{r}{2}(c_1(X).H)\right)n + \chi(E). \end{aligned}$$

Therefore, the reduced Hilbert polynomial of E is

$$p_E(n) = \frac{(H^2)}{2}n^2 + \left(\frac{(c_1(E).H)}{r} + \frac{(c_1(X).H)}{2}\right)n + \frac{\chi(E)}{r},$$

while its slope is

$$\mu_E = \frac{(c_1(E).H)}{r}.$$

Hence, E is stable (resp. semistable) if and only if, for any non-zero coherent subsheaf F, either $\mu_F < \mu_E$ (resp. $\leq$) or, in case of equality,

$$\frac{\chi(F)}{r(F)} < \frac{\chi(E)}{r(E)} \quad (\text{resp. } \leq).$$

It is clear in this concrete example the chain of implications (valid in general)

$$\mu\text{-stable} \Rightarrow \text{stable} \Rightarrow \text{semistable} \Rightarrow \mu\text{-semistable}.$$

3.2 The moduli space $\mathbf{M}_H(P)$

Definition 3.2.1. Let $\mathbf{M}$ be a functor from the category of schemes of finite type over $\mathbb{C}$ to the category of sets. A *coarse moduli space* for $\mathbf{M}$ consists of a scheme M of finite type over $\mathbb{C}$ and a morphism of functors $f\colon \mathbf{M} \to \mathrm{Hom}(\ , M)$ satisfying the following universal property: for any morphism of functors $g\colon \mathbf{M} \to \mathrm{Hom}(\ , N)$, there exists a unique morphism $\varphi\colon M \to N$ making this diagram commute:

$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{f} & \mathrm{Hom}(\ , M) \\ & \searrow^{g} & \downarrow \\ & & \mathrm{Hom}(\ , N). \end{array}$$

M is a *fine moduli space* if it represents the functor $\mathbf{M}$, i.e. $\mathbf{M} \simeq \mathrm{Hom}(\ , M)$.

Let X be a projective scheme over $\mathbb{C}$ and let H be the first Chern class of a fixed very ample invertible sheaf $\mathcal{O}(1)$. Let P be a polynomial with rational coefficients. A *family of semistable sheaves* on X parametrized by an algebraic variety S is a coherent sheaf F on $X \times S$, flat over S, such that, for any closed point $s \in S$, the sheaf $F(s)$ on X is semistable. By flatness assumption, the Hilbert polynomial $P_{F(s)}$ is locally constant.

Consider the functor $\underline{\mathbf{M}}_H(P)$ associating with any algebraic variety S the set $\underline{\mathbf{M}}_H(P)(S)$ of isomorphism classes of families of semistable sheaves parametrized by S and with Hilbert polynomial P. We have the following

Theorem 3.2.2. *The functor $\underline{\mathbf{M}}_H(P)$ admits a coarse moduli space $\mathbf{M}_H(P)$ satisfying the following properties:*

(i) $\mathbf{M}_H(P)$ is a projective variety;

(ii) closed points of $\mathbf{M}_H(P)$ are in 1 - 1 correspondence with S-equivalence classes of semistable sheaves with Hilbert polynomial P;

(iii) under this correspondence, the set of stable sheaves is identified to an open subset $\mathbf{M}_H^s(P)$ of $\mathbf{M}_H(P)$.

As we will see, in the construction of such a moduli space, the Hilbert scheme plays an essential role. In the following section, we shall briefly recall its definition and the main properties it satisfies.

3.2.1 Hilbert scheme

Fix $\mathcal{H}$ a locally free sheaf on X. For any algebraic variety S, define on $X \times S$ the sheaf $\mathcal{H}_S = \mathcal{H} \boxtimes \mathcal{O}_S$. Consider the functor $\underline{\mathbf{Hilb}}^P(\mathcal{H})$ whose sections over S are coherent quotient sheaves

$$\mathcal{H}_S \to E \to 0$$

flat over S, such that the Hilbert polynomial of $E(s)$ is P for any closed point $s \in S$. With any morphism $f\colon S' \to S$ of algebraic varieties, we associate the map

$$\begin{aligned} \underline{\mathbf{Hilb}}^P(\mathcal{H})(S) &\to \underline{\mathbf{Hilb}}^P(\mathcal{H})(S') \\ E &\mapsto (\mathrm{id} \times f)^*(E). \end{aligned}$$

An important theorem of Grothendieck states that the functor $\underline{\mathbf{Hilb}}^P(\mathcal{H})$ is representable by a projective algebraic variety, which we denote as $\mathbf{Hilb}^P(\mathcal{H})$. In particular, $\mathbb{C}$-rational points of $\mathbf{Hilb}^P(\mathcal{H})$ correspond bijectively to coherent quotient sheaf $\mathcal{H} \to E \to 0$ on X, with Hilbert polynomial P. We shall often write a closed point as $[q\colon \mathcal{H} \to E]$. It is useful to fix notations also for the corresponding short exact sequence

$$0 \to G \xrightarrow{j} \mathcal{H} \xrightarrow{q} E \to 0.$$

Lemma 3.2.3. *Let $[q\colon \mathcal{H} \to E]$ be a $\mathbb{C}$-rational point of the Hilbert scheme $\mathbf{Hilb}^P(\mathcal{H})$. The Zariski tangent space $T_{[q]}\mathbf{Hilb}^P(\mathcal{H})$ to the Hilbert scheme at the point $[q]$ is isomorphic to $\mathrm{Hom}(G, E)$.*

Proof. Let $D = \mathrm{Spec}(\mathbb{C}[\varepsilon]/(\varepsilon^2))$ be the affine scheme associated with the algebra of dual numbers. The closed embedding $\iota\colon \mathrm{Spec}(\mathbb{C}) \to D$ corresponds to the surjective homomorphism of algebras $\mathbb{C}[\varepsilon]/(\varepsilon^2) \to \mathbb{C}$. By [10, ch. II] Exercise 2.8, a tangent vector at a point $[q]$ is a morphism in $\mathrm{Hom}(D, \mathbf{Hilb}^P(\mathcal{H}))$, whose composition with ι is $[q]$. Equivalently, by the commutativity of the diagram

$$\begin{array}{ccc} \underline{\mathbf{Hilb}}^P(\mathcal{H})(D) & \xrightarrow{\sim} & \mathrm{Hom}(D, \mathbf{Hilb}^P(\mathcal{H})) \\ \downarrow{\scriptstyle (\mathrm{id}\times\iota)^*} & & \downarrow{\scriptstyle (\ )\circ\iota} \\ \underline{\mathbf{Hilb}}^P(\mathcal{H})(\mathrm{Spec}(\mathbb{C})) & \xrightarrow{\sim} & \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}), \mathbf{Hilb}^P(\mathcal{H})), \end{array}$$

it is a coherent quotient sheaf $\mathcal{E}$ of $\mathcal{H}_D$, flat over D and such that $(\mathrm{id} \times \iota)^*(\mathcal{E}) = E$. Now, let $\mathcal{E} = \mathcal{H}_D/\mathcal{G}$ be such a sheaf. We show that this gives a homomorphism $G \to E$. Applying the bifunctor $\otimes_{\mathbb{C}[\varepsilon]/(\varepsilon^2)}$ to the short exact sequences

$$0 \to \mathbb{C} \xrightarrow{\cdot\varepsilon} \mathbb{C}[\varepsilon]/(\varepsilon^2) \to \mathbb{C} \to 0 \quad \text{and}$$
$$0 \to \mathcal{G} \to \mathcal{H}_D \to \mathcal{E} \to 0,$$

thanks to flatness of $\mathcal{E}$, one obtains a commuting diagram of short exact sequences (see [11] for more details)

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G & \to & \mathcal{G} & \to & G & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{H} & \to & \mathcal{H}_D & \to & \mathcal{H} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E & \to & \mathcal{E} & \to & E & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}.$$

The homomorphism we are looking for goes from the right top corner G to the left bottom corner E. We do diagram chasing. Let $x \in G$ and lift it to an element of $\mathcal{G}$. Since $\mathcal{H}_D \simeq \mathcal{H} \oplus \varepsilon\mathcal{H}$, we can write the lifting as $x + \varepsilon t \in \mathcal{G}$, for $t \in \mathcal{H}$. Two liftings differ by something of the form εz, $z \in G$. Thus, t is not uniquely determined, but its image $\bar{t} \in E$ is. Sending x to $\bar{t}$ defines a morphism $\varphi \in \mathrm{Hom}(G, E)$.

We remark that the zero morphism is obtained when $\mathcal{E} = E \oplus \varepsilon E$ and the quotient morphism $\mathcal{H}_D \to \mathcal{E}$ is given by the matrix

$$\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}. \qquad \square$$

Now, consider the functor $\underline{\mathrm{Aut}}(\mathcal{H})$ whose sections over an algebrac variety S are the automorphisms of $\mathcal{H}_S$. It is represented by the algebraic variety $\mathrm{Aut}(\mathcal{H})$. We have a natural morphism of functors

$$\sigma \colon \underline{\mathbf{Hilb}}^P(\mathcal{H}) \times \underline{\mathrm{Aut}}(\mathcal{H}) \to \underline{\mathbf{Hilb}}^P(\mathcal{H}) :$$

for any algebraic variety S, σ_S is given by the action from the right

$$\begin{aligned} \sigma_S \colon \underline{\mathbf{Hilb}}^P(\mathcal{H})(S) \times \underline{\mathrm{Aut}}(\mathcal{H})(S) &\to \underline{\mathbf{Hilb}}^P(\mathcal{H})(S) \\ ([q \colon \mathcal{H}_S \to E], \quad g \quad) &\mapsto [q \circ g \colon \mathcal{H}_S \to E]. \end{aligned}$$

Lemma 3.2.4. *Let us fix $[q \colon \mathcal{H} \to E]$ a closed point of the Hilbert scheme and consider the orbit map*

$$\begin{aligned} \mathrm{Aut}(\mathcal{H}) &\to \mathbf{Hilb}^P(\mathcal{H}) \\ g &\mapsto [q \circ g]. \end{aligned}$$

The tangent map at the identity is given by

$$\begin{aligned} \mathrm{End}(\mathcal{H}) &\to \mathrm{Hom}(G, E) \\ \varphi &\mapsto -q \circ \varphi \circ j. \end{aligned} \tag{3.1}$$

Proof. Keep the notations introduced in the proof of Lemma 3.2.3. The tangent space to $\mathrm{Aut}(\mathcal{H})$ at the identity consists of automorphisms of $\mathcal{H}_D$ that are sent to the identity by the map $\mathrm{Hom}(D, \mathrm{Aut}(\mathcal{H})) \to \mathrm{Hom}(\mathrm{Spec}(\mathbb{C}), \mathrm{Aut}(\mathcal{H}))$. It can be identified to $\mathrm{End}(\mathcal{H})$ via the map

$$\begin{aligned} \mathrm{End}(\mathcal{H}) &\xrightarrow{\sim} \mathrm{Aut}(\mathcal{H}_D) \\ \varphi &\mapsto \mathrm{id} + \varepsilon\varphi. \end{aligned}$$

We want to determine the image of any $\varphi \in \mathrm{End}(\mathcal{H})$ under the composition

$$\mathrm{Aut}(\mathcal{H}_D) \xrightarrow{0\times\mathrm{id}} \underline{\mathbf{Hilb}}^P(\mathcal{H})(D) \times \mathrm{Aut}(\mathcal{H}_D) \xrightarrow{\sigma_D} \underline{\mathbf{Hilb}}^P(\mathcal{H})(D).$$

The automorphism $\mathrm{id} + \varepsilon\varphi$ sends the zero tangent vector to the vector

$$\mathcal{H} \oplus \varepsilon\mathcal{H} \xrightarrow{\begin{pmatrix} q & 0 \\ q\varphi & q \end{pmatrix}} E \oplus \varepsilon E.$$

We compute the homomorphism $G \to E$ corresponding to this D-point as we have done in the proof of Lemma 3.2.3. Let $x \in G$ and consider a lifting $x + \varepsilon t \in \mathcal{G}$. One has that $q\varphi(x) + q(t) = 0$, that is $\varphi(x) + t \in G$. Hence the class of t in E equals the class of $-\varphi(x)$. The required homomorphism is then $q \circ (-\varphi) \circ j$. $\square$

3.2.2 Construction of $\mathbf{M}_H(P)$

Let us briefly see how to use the Hilbert scheme to construct the moduli space $\mathbf{M}_H(P)$. For a thorough presentation of this construction, consult [14, ch. 4]. Consider the family of semistable sheaves on X with fixed Hilbert polynomial P. It can be shown that, for m big enough, any semistable sheaf E with Hilbert polynomial P satisfies the following conditions:

(i) $E(m)$ is generated by its global sections;

(ii) the cohomology groups $H^i(X, E(m))$ vanish for every $i > 0$.

Set $N = P(m)$. Consider V an N-dimensional complex vector space and define on X the locally free sheaf $\mathcal{H} \coloneqq V \otimes \mathcal{O}_X(-m)$. Condition (ii) implies that $\dim H^0(X, E(m)) = \chi(E(m)) = N$. Condition (i) yields a surjective morphism

$$\mathcal{H} \xrightarrow{q} E \to 0$$

obtained composing the canonical evaluation map $H^0(X, E(m)) \otimes \mathcal{O}_X(-m) \to E$ with an isomorphism $V \to H^0(X, E(m))$. This defines a closed point

$$[q\colon \mathcal{H} \to E] \in \mathbf{Hilb}^P(\mathcal{H}).$$

Such a point, in fact, belongs to the subset $\mathcal{R} \subset \mathbf{Hilb}^P(\mathcal{H})$ of all those quotients $[q\colon \mathcal{H} \to E]$, where E is semistable and the induced map

$$V = H^0(\mathcal{H}(m)) \to H^0(E(m))$$

is an isomorphism. Hence, $\mathcal{R}$ parametrizes all semistable sheaves with Hilbert polynomial P, but with a certain ambiguity due to the choice of a basis of $H^0(X, E(m))$. In order to identify them, we will consider the action of a linearly reductive algebraic group on $\mathbf{Hilb}^P(\mathcal{H})$. The tools of Geometric Invariant Theory (GIT) will allow us to construct the moduli space we are looking for.

As we have already seen, the group of automorphisms of $\mathcal{H}$, which is isomorphic to $\mathrm{GL}(V)$, acts on $\mathbf{Hilb}^P(\mathcal{H})$ from the right:

$$\begin{aligned} \mathbf{Hilb}^P(\mathcal{H}) \times \mathrm{GL}(V) &\to \mathbf{Hilb}^P(\mathcal{H}) \\ ([q], g) &\mapsto [q].g = [q \circ g]. \end{aligned}$$

Lemma 3.2.5. *The stabilizer subgroup* $\mathrm{GL}(V)_{[q]}$ *of a point* $[q]$ *is isomorphic to the group* $\mathrm{Aut}(E)$ *of automorphisms of* E.

Proof. Pick $\varphi \in \mathrm{Aut}(E)$ and consider the diagram

$$\begin{array}{ccc} V \otimes \mathcal{O}_X & \xrightarrow{q(m)} & E(m) \\ \vdots\, {\scriptstyle g\otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \varphi(m)} \\ V \otimes \mathcal{O}_X & \xrightarrow{q(m)} & E(m). \end{array}$$

Taking global sections, we obtain $g = H^0(q(m))^{-1} \circ H^0(\varphi(m)) \circ H^0(q(m)) \in \mathrm{GL}(V)$. Thus, we have built a group homomorphism

$$\begin{aligned} \mathrm{Aut}(E) &\to \mathrm{GL}(V) \\ \varphi &\mapsto H^0(q(m))^{-1} \circ H^0(\varphi(m)) \circ H^0(q(m)). \end{aligned}$$

This morphism is injective as $E(m)$ is globally generated. Its image is the stabilizer subgroup $\mathrm{GL}(V)_{[q]}$: indeed $g \in \mathrm{GL}(V)_{[q]}$ if and only if there exists an automorphism φ of E such that $q \circ g = \varphi \circ q$. □

As the centre of $\mathrm{GL}(V)$ is contained in the stabilizer of any point $[q] \in \mathcal{R}$, the action of $\mathrm{GL}(V)$ descends to an action of $\mathrm{PGL}(V)$. Let $\mathcal{R}^{ss}$ (resp. $\mathcal{R}^s$) be the set of semistable (resp. stable) points for the action of $\mathrm{PGL}(V)$.

Proposition 3.2.6. *Let $[q\colon \mathcal{H} \to E]$ be a closed point of $\mathcal{R}$. The following are equivalent:*

(i) the sheaf E is semistable (resp. stable) and the morphism

$$H^0(q(m))\colon V \to H^0(X, E(m))$$

is an isomorphism;

(ii) the point $[q]$ is semistable (resp. stable) for the action of $\mathrm{PGL}(V)$.

We have all the needed machinery to prove Theorem 3.2.2.

Proof. Define $\mathbf{M}_H(P)$ as the GIT-quotient $\mathcal{R}^{ss} /\!\!/ \mathrm{PGL}(V)$. By construction, it is a projective variety, which proves (i). The quotient map $\pi\colon \mathcal{R}^{ss} \to \mathbf{M}_H(P)$ is a good quotient. Again by geometric invariant theory (see [17, ch. 6, §1]), the subset $\mathcal{R}^s$ is open in $\mathcal{R}^{ss}$, and it is the inverse image of an open subset $\mathbf{M}_H^s(P)$ of $\mathbf{M}_H(P)$. The restriction $\pi\colon \mathcal{R}^s \to \mathbf{M}_H^s(P)$ is a geometric quotient. In particular, we have proven (iii). It remains to prove that we have a bijection between S-equivalence classes of semistable sheaves and $\mathbb{C}$-rational points of $\mathbf{M}_H(P)$. First, we show that a semistable sheaf E and $\mathrm{gr}(E)$ define the same point in $\mathbf{M}_H(P)$. It suffices to show that, for E_1 a subsheaf of E with the same reduced Hilbert polynomial, E and $E_1 \oplus E/E_1$ define the same point in $\mathbf{M}_H(P)$. In the vector space $\mathrm{Ext}^1(E/E_1, E_1)$ consider the line $(E_t)_{t\in\mathbb{C}}$ spanned by (the class of) the short exact sequence $0 \to E_1 \to E \to E/E_1 \to 0$. We get a morphism

$$\begin{aligned} \mathbb{A}^1 &\to \mathbf{M}_H(P) \\ t &\mapsto \pi(E_t). \end{aligned}$$

As $E_t \simeq E$ for $t \neq 0$, this map is constant on $\mathbb{A}^1 \setminus \{0\}$. By separatedness of $\mathbf{M}_H(P)$ the map must be constant. Hence, E and $E_0 \simeq E_1 \oplus E/E_1$ define the same point in $\mathbf{M}_H(P)$.

It remains to prove that two distinct polystable sheaves define distinct points of $\mathbf{M}_H(P)$. To this aim, it suffices to show that the orbit of a point $[q\colon \mathcal{H} \to E]$ is closed for E polystable. Write

$$E = \bigoplus_i E_i^{\oplus n_i}$$

with E_i stable. Let $[q'\colon \mathcal{H} \to E']$ be a point in the closure of the orbit of $[q]$. If we prove that $E' \simeq E$, we are done. By GIT, we can find a 1-parameter subgroup $\lambda\colon \mathbb{C}^\times \to \mathrm{PGL}(V)$ such that $\lim_{t\to 0} \lambda(t).q = q'$. This corresponds to a family $\mathcal{E}$ of semistable sheaves flat over $\mathbb{A}^1$ such that

$$\mathcal{E}_t \simeq E \quad \text{for } t \neq 0 \quad \text{and} \quad \mathcal{E}_0 \simeq E'.$$

For any i, consider the map $t \mapsto \dim \mathrm{Hom}(E_i, \mathcal{E}_t)$. It is upper semicontinuous by flatness of $\mathcal{E}$. For any $t \neq 0$, $\dim \mathrm{Hom}(E_i, \mathcal{E}_t) = \dim \mathrm{Hom}(E_i, E) = n_i$. Hence, $n_i' = \dim \mathrm{Hom}(E_i, E') \geq n_i$. As E_i is stable, the evaluation map $E_i \otimes \mathrm{Hom}(E_i, E') \to E'$ is injective. We deduce that

$$\bigoplus_i E_i^{\oplus n_i'} \subset E'.$$

As E and E' have the same rank, $n_i = n_i'$ and $E \simeq E'$. □

Proposition 3.2.7. *Let $[q\colon \mathcal{H} \to E] \in \mathcal{R}^s$ be a stable point for the action of $\mathrm{PGL}(V)$. We have a canonical isomorphism*

$$T_{[E]}\mathbf{M}_H^s(P) \simeq \mathrm{Ext}^1(E, E).$$

Proof. The stabilizer of any point in $\mathcal{R}^s$ under the $\mathrm{GL}(V)$ action is $\mathbb{C}^\times$; therefore, the action of $\mathrm{PGL}(V)$ on $\mathcal{R}^s$ is free. Hence, by GIT (see [17, ch. 8, §3]), the tangent space

$$T_{[E]}\mathbf{M}_H^s(P)$$

is isomorphic to the normal space of the orbit of $[q]$ at $[q]$. This is identified with the cokernel of the linear map 3.1, which we compute as follows.

Applying the functor $\mathrm{Hom}(\ , E)$ to the short exact sequence

$$0 \to G \xrightarrow{j} \mathcal{H} \xrightarrow{q} E \to 0.$$

we get in cohomology

$$\mathrm{Hom}(\mathcal{H}, E) \xrightarrow{j^*} \mathrm{Hom}(G, E) \to \mathrm{Ext}^1(E, E) \to \mathrm{Ext}^1(\mathcal{H}, E).$$

Note that the natural map

$$\mathrm{Hom}(\mathcal{H}, \mathcal{H}) \xrightarrow{q_*} \mathrm{Hom}(\mathcal{H}, E)$$

is an isomorphism, since $H^0(X, \mathcal{H}) \to H^0(X, E)$ is an isomorphism. Moreover, by our choice of m,

$$\mathrm{Ext}^1(\mathcal{H}, E) = \mathrm{Ext}^1(\mathcal{O}_X(-m)^{\oplus N}, E) \simeq H^1(X, E(m))^{\oplus N} = 0.$$

We conclude that $\mathrm{Ext}^1(E, E)$ is the cokernel of the composition

$$\mathrm{Hom}(\mathcal{H}, \mathcal{H}) \xrightarrow{q_*} \mathrm{Hom}(\mathcal{H}, E) \xrightarrow{j^*} \mathrm{Hom}(G, E),$$

which is exactly, up to a sign, the linear map 3.1. □

3.3 Mukai theory of moduli spaces

Throughout this section, X will be an algebraic complex K3 surface. Following the seminal article [21] by Mukai, we will introduce the tools that are needed for the study of moduli spaces of sheaves on X with fixed numerical invariants. In particular, we will understand the moduli spaces of stable sheaves with primitive Mukai vector v of square -2 or 0.

3.3.1 Generalities

Consider an ample divisor class on X; it can be identified with its first Chern class H. The construction of the moduli space would require the choice of a Hilbert polynomial. In the case of K3 surfaces, in order to fix the numerical invariants of the sheaves we want to parametrize, it turns out that the choice of a Mukai vector in $\tilde{H}(X, \mathbb{Z})$ is more convenient. In section 2.2 we have already computed the Mukai vector of a coherent sheaf E on X

$$v(E) = \mathrm{ch}(E).\sqrt{\mathrm{td}(X)} = (r(E), c_1(E), \chi(E) - r(E)) \in \tilde{H}^{1,1}(X, \mathbb{Z}) \subset \tilde{H}(X, \mathbb{Z}),$$

and in Proposition 1.3.17 we have proven that, for any E and F coherent sheaves on X, one has

$$\chi(E, F) = \dim \mathrm{Hom}(E, F) - \dim \mathrm{Ext}^1(E, F) + \dim \mathrm{Ext}^2(E, F) = -\langle v(E), v(F) \rangle.$$

Therefore, the Hilbert polynomial of a sheaf E with Mukai vector v will be

$$P_E(n) = \chi(E(n)) = \chi(\mathcal{O}(-n), E) = -\langle v(\mathcal{O}(-n)), v(E) \rangle = -\langle v(\mathcal{O}(-n)), v \rangle.$$

We shall use the notation $\mathbf{M}_H(v)$ for the moduli space of sheaves on X whose Mukai vector is v and that are semistable with respect to H. We will denote as $\mathbf{M}_H^s(v)$ the open subset of $\mathbf{M}_H(v)$ parametrizing stable sheaves only.

Due to the triviality of the canonical bundle of a K3 surface, Serre duality is not only particularly simple in form, but provides us with an effective tool of study. It says that, for any E and F coherent sheaves on X,

$$\mathrm{Hom}(E,F) \simeq \mathrm{Ext}^2(F,E)^\vee \quad \text{and} \quad \mathrm{Ext}^1(E,F) \simeq \mathrm{Ext}^1(F,E)^\vee.$$

If $E = F$, the latter relation states the existence of a non-degenerate bilinear form

$$\omega_{[E]} \colon \mathrm{Ext}^1(E,E) \times \mathrm{Ext}^1(E,E) \to \mathbb{C},$$

given as follows: to $\alpha \in \mathrm{Ext}^1(E,E) \simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(E,E[1])$ and $\beta \in \mathrm{Ext}^1(E,E) \simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(E[1],E[2])$, we associate $\mathrm{tr}(\beta \circ \alpha) \in H^2(X,\mathcal{O}_X) \simeq \mathbb{C}$. Whenever E is a stable sheaf, Mukai showed that the form $\omega_{[E]}$ is alternating and glues to a symplectic form ω on $\mathbf{M}_H^s(v)$.

From the above discussion we deduce the following

Corollary 3.3.1. *(i) For any E, F coherent sheaves on X,*

$$\langle v(E), v(F)\rangle = \dim\ \mathrm{Ext}^1(E,F) - \dim\ \mathrm{Hom}(E,F) - \dim\ \mathrm{Hom}(F,E).$$

(ii) For any coherent sheaf E on X,

$$\dim\ \mathrm{Ext}^1(E,E) = \langle v(E)^2\rangle + 2\ \dim\ \mathrm{End}(E).$$

Thus, $\dim\ \mathrm{Ext}^1(E,E)$ *is an even integer. Furthermore, if E is simple,* $\dim\ \mathrm{Ext}^1(E,E) = \langle v(E)^2\rangle + 2$ *and then* $\langle v(E)^2\rangle \geq -2$.

Recall that, by Lemma 3.2.4, the tangent space to $\mathbf{M}_H^s(v)$ at a stable point $[E]$ is isomorphic to $\mathrm{Ext}^1(E,E)$. Hence, as $\mathbf{M}_H^s(v)$ is smooth at $[E]$ and E is simple, Corollary 3.3.1 says that the dimension of the moduli space at $[E]$ is $\langle v(E)^2\rangle + 2$. The following lemma is handy for having estimates on the dimension of this tangent space.

Lemma 3.3.2. *Consider $0 \to G \to E \to F \to 0$ a short exact sequence of sheaves on X.*

(i) If $\mathrm{Hom}(G,F) = 0$, *then* $\dim\ \mathrm{Ext}^1(F,F) + \dim\ \mathrm{Ext}^1(G,G) \leq \dim\ \mathrm{Ext}^1(E,E)$.

(ii) If $\mathrm{Ext}^1(F,E) = 0$, *then* $\dim\ \mathrm{Ext}^1(F,F) + \dim\ \mathrm{Ext}^1(E,E) \leq \dim\ \mathrm{Ext}^1(G,G)$.

Proof. The proof is rather technical, see [21] Propositions 2.7 and 2.10. □

Let E be a coherent sheaf on X and denote $\tilde{E} = \mathcal{H}om(\mathcal{H}om(E,\mathcal{O}_X),\mathcal{O}_X)$ its double dual, which is always a locally free sheaf. Assume that E is torsion-free. Then the natural map $E \to \tilde{E}$ is injective, and has cokernel M of finite length. As $\tilde{E}$ is locally free, $\mathrm{Ext}^1(M,\tilde{E}) \simeq \mathrm{Ext}^1(\tilde{E},M)^\vee \simeq H^1(X,M)^\vee = 0$. Moreover,

dim $\mathrm{Ext}^1(M,M) = 2$ dim $\mathrm{End}(M)$, because the vector $v(M) = (0,0,a)$ is isotropic. From Lemma 3.3.2 (ii) and the fact that length(End(M)) $\geq$ length(M) ([21]Lemma 2.13) we deduce the inequalities

$$\begin{aligned} &\dim \mathrm{Ext}^1(\tilde{E},\tilde{E}) + 2 \ \mathrm{length}(M) \\ &\leq \dim \mathrm{Ext}^1(\tilde{E},\tilde{E}) + 2 \dim \mathrm{End}(M) \leq \dim \mathrm{Ext}^1(E,E). \end{aligned} \tag{3.2}$$

Proposition 3.3.3. *Let $0 \to G \to E \to F \to 0$ be an exact sequence of torsion-free sheaves on X. Then*

$$\frac{\langle v(G)^2\rangle}{r(G)} + \frac{\langle v(F)^2\rangle}{r(F)} - \frac{\langle v(E)^2\rangle}{r(E)} = \frac{r(F)r(G)}{r(E)}\left(\frac{c_1(F)}{r(F)} - \frac{c_1(G)}{r(G)}\right)^2. \tag{3.3}$$

Proof. The additivity of Mukai vector and of rank yields

$$\begin{aligned} &\frac{\langle v(G)^2\rangle}{r(G)} + \frac{\langle v(F)^2\rangle}{r(F)} - \frac{\langle v(E)^2\rangle}{r(E)} \\ &\quad = \frac{\langle v(G)^2\rangle}{r(G)} + \frac{\langle v(F)^2\rangle}{r(F)} - \left(\frac{\langle v(G)^2\rangle + \langle v(F)^2\rangle + 2\langle v(G), v(F)\rangle}{r(E)}\right) \\ &\quad = \frac{r(F)}{r(E)r(G)}\langle v(G)^2\rangle + \frac{r(G)}{r(E)r(F)}\langle v(F)^2\rangle - \frac{2\langle v(G), v(F)\rangle}{r(E)} \\ &\quad = \frac{r(F)r(G)}{r(E)}\frac{\langle v(G)^2\rangle}{r(G)^2} + \frac{r(F)r(G)}{r(E)}\frac{\langle v(F)^2\rangle}{r(F)^2} - \frac{r(F)r(G)}{r(E)}\frac{2\langle v(G), v(F)\rangle}{r(F)r(G)} \\ &\quad = \frac{r(F)r(G)}{r(E)}\langle\left(\frac{v(F)}{r(F)} - \frac{v(G)}{r(G)}\right)^2\rangle. \end{aligned}$$

We conclude by observing that

$$\frac{v(F)}{r(F)} - \frac{v(G)}{r(G)} = \left(0, \frac{c_1(F)}{r(F)} - \frac{c_1(G)}{r(G)}, \frac{s(F)}{r(F)} - \frac{s(G)}{r(G)}\right). \qquad \square$$

Corollary 3.3.4. *Keep the notations of the previous proposition. If X is algebraic with Picard number $\rho(X) = 1$, we have*

$$\frac{\langle v(G)^2\rangle}{r(G)} + \frac{\langle v(F)^2\rangle}{r(F)} \geq \frac{\langle v(E)^2\rangle}{r(E)}$$

and equality holds if and only if $c_1(F)/r(F) = c_1(G)/r(G)$.

Proof. By assumption, Pic(X) is generated by a line bundle on X of positive square. The right handside of the equation 3.3 is then non-negative, and it vanishes if and only if $c_1(F)/r(F) = c_1(G)/r(G)$. $\square$

Corollary 3.3.5. *Keep the notations of Proposition* 3.3.3. *Assume that G and F have the same slope with respect to an ample line bundle H. Then*

$$\frac{\langle v(G)^2\rangle}{r(G)} + \frac{\langle v(F)^2\rangle}{r(F)} \leq \frac{\langle v(E)^2\rangle}{r(E)}$$

and equality holds if and only if $c_1(F)/r(F) = c_1(G)/r(G)$.

Proof. Look at 3.3. By the Hodge index theorem

$$\left(H.\frac{c_1(F)}{r(F)} - \frac{c_1(G)}{r(G)}\right) = 0 \Rightarrow \left(\frac{c_1(F)}{r(F)} - \frac{c_1(G)}{r(G)}\right)^2 \leq 0$$

and equality holds if and only if $c_1(F)/r(F) = c_1(G)/r(G)$. □

Proposition 3.3.6. *Let E be a semistable sheaf and*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

a Jordan-Hölder filtration of E. Denote by $F_i = E_i/E_{i-1}$ the successive stable quotients. Then

$$\sum_{i=1}^{n} \frac{\langle v(F_i)^2\rangle}{r(F_i)} \leq \frac{\langle v(E)^2\rangle}{r(E)} \tag{3.4}$$

and equality holds if and only if $c_1(F_i)/r(F_i) = c_1(E)/r(E)$ for every $i = 1, 2, \ldots, n$.

Proof. Having the same reduced Hilbert polynomial of E, the stable sheaves F_i have also the same slope $\mu(E)$. By induction, also the E_i's have the same slope. Applying repeatedly Corollary 3.3.5 to the short exact sequences $0 \to E_{i-1} \to E_i \to F_i \to 0$, we get the result. □

3.3.2 Rigid and semirigid sheaves

The aim of this section is the study of low dimensional moduli spaces $\mathbf{M}_H^s(v)$. The case of a primitive Mukai vector v will be particularly important. A vector $v \in \tilde{H}(X, \mathbb{Z})$ is called *primitive* if $\tilde{H}(X, \mathbb{Z})/\mathbb{Z}v$ is a free abelian group.

Definition 3.3.7. A sheaf E on X is called *rigid* if $\mathrm{Ext}^1(E, E) = 0$.

Assume that E is a simple sheaf. By Corollary 3.3.1, we have obvious equivalences

$$E \text{ is rigid} \Leftrightarrow \langle v(E), v(E)\rangle = -2 \Leftrightarrow \langle v(E), v(E)\rangle < 0 \tag{3.5}$$

Proposition 3.3.8. *If a sheaf E is rigid and torsion free, then E is locally free.*

Proof. The result follows immediately by using the inequality 3.2. □

Corollary 3.3.9. *Let E be a rigid stable bundle. If F is a semistable sheaf and $v(F) = v(E)$, then F is isomorphic to E.*

Proof. As E is rigid, $\chi(E,E) = 2\dim \mathrm{End}(E) > 0$; since $\chi(F,E) = -\langle v(F), v(E)\rangle = -\langle v(E), v(E)\rangle = \chi(E,E)$, $\chi(F,E) > 0$, so that $\mathrm{Hom}(E,F)$ and $\mathrm{Hom}(F,E)$ cannot both vanish. In particular, there exists a non-zero morphism between E and F.

Suppose that $f\colon E \to F$ is such a morphism. The condition $v(F) = v(E)$ implies that E and F have the same reduced Hilbert polynomial. By stability of E, f is injective. Moreover, E and F also have the same Hilbert polynomial, and then $\mathrm{Coker} f = 0$. This proves that f is an isomorphism.

The case $f\colon F \to E$ goes analogously, first showing the surjectivity of f. □

Corollary 3.3.10. *Let v be a vector of $\tilde{H}^{1,1}(X,\mathbb{Z})$ with $\langle v, v\rangle = -2$. Then the moduli space $\mathbf{M}_H^s(v)$ is empty or a reduced point.*

Proof. Assume that $\mathbf{M}_H^s(v)$ is not empty. Then, by Corollary 3.3.9, $\mathbf{M}_H(v)$ consists of a single point $[E]$. The tangent space to $\mathbf{M}_H(v)$ at $[E]$ is canonically isomorphic to $\mathrm{Ext}^1(E,E) = 0$. Hence $\mathbf{M}_H(v)$ is reduced. □

For a simple sheaf E, $\dim \mathrm{Ext}^1(E,E)$ is an even integer. Thus, after considering the case of rigid sheaves, we try to describe sheaves such that $\dim \mathrm{Ext}^1(E,E) = 2$, which deserve a particular name.

Definition 3.3.11. A sheaf E on X is called *semirigid* if it is simple and satisfies one of the following equivalent conditions:

(i) $\dim \mathrm{Ext}^1(E,E) = 2$;

(ii) $v(E)$ is isotropic for the Mukai pairing.

Example 3.3.12. Let F be a simple rigid vector bundle of rank r. We will construct semirigid sheaves from it. Pick a closed point $x \in X$. Let $V = F(x)$ be the fibre of F at x and define

$$\tilde{F} = F \otimes V^\vee.$$

It is a rigid vector bundle of rank r^2. As soon as $r > 1$, it is no longer simple, as $\mathrm{End}(\tilde{F}) \simeq \mathrm{End}(F) \otimes \mathrm{End}(V^\vee) \simeq \mathrm{End}(V^\vee)$. Furthermore, one has $\tilde{F}(x) = V \otimes V^\vee = \mathrm{End}(V)$. Consider the map

$$f\colon \tilde{F} \to \tilde{F}(x) \simeq \mathrm{End}(V) \xrightarrow{\mathrm{tr}} \kappa(x),$$

where tr is the trace map of $\mathrm{End}(V)$. The kernel E of f fits in the short exact sequence

$$0 \to E \xrightarrow{j} \tilde{F} \xrightarrow{f} \kappa(x) \to 0. \tag{3.6}$$

We claim that E is semirigid.

To see that E is simple, we proceed as follows. Take $\varphi \in \mathrm{End}(E)$ an endomorphism of E. Applying the functor $\mathrm{Hom}(\ ,\tilde{F})$ to 3.6, we obtain the exact sequence

$$\mathrm{Hom}(\tilde{F},\tilde{F}) \xrightarrow{j^*} \mathrm{Hom}(E,\tilde{F}) \to \mathrm{Ext}^1(\kappa(x),\tilde{F}) \simeq H^1(X,\kappa(x))^\vee = 0.$$

Therefore, there exists $\tilde{\varphi} \in \mathrm{End}(\tilde{F})$ such that $j \circ \varphi = \tilde{\varphi} \circ j$. If we prove that $\tilde{\varphi}$ is the multiplication with a constant, we are done by injectivity of j. The endomorphism $\tilde{\varphi} \in \mathrm{End}(V^\vee)$ induces the endomorphisms $\mathrm{id} \otimes \tilde{\varphi}$ of the fibre $\tilde{F}(x) \simeq V \otimes V^\vee$ and $\lambda \cdot \mathrm{id}$ of $\kappa(x)$ making the diagram

$$\begin{array}{ccccc}
\tilde{F} & \longrightarrow & \tilde{F}(x) \simeq V \otimes V^\vee & \xrightarrow{\mathrm{tr}} & \kappa(x) \\
\downarrow{\scriptstyle \tilde{\varphi}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\tilde{\varphi}} & & \downarrow{\scriptstyle \lambda\cdot\mathrm{id}} \\
\tilde{F} & \longrightarrow & \tilde{F}(x) \simeq V \otimes V^\vee & \xrightarrow{\mathrm{tr}} & \kappa(x)
\end{array}$$

commute. Let us spell out what this compatibility means. Let $\mathcal{V} = \{ v_1, v_2, \dots v_r \}$ be a basis of V and $\mathcal{V}^* = \{ v_1^*, v_2^*, \dots v_r^* \}$ its dual basis. We associate with $\tilde{\varphi}$ the matrix $(a_{ij})_{1\leq i,j\leq r}$ with respect to $\mathcal{V}^*$. For any i and j, we have on the one hand

$$\lambda \cdot \mathrm{tr}(v_i \otimes v_j^*) = \lambda \cdot \delta_{ij},$$

and on the other

$$\mathrm{tr}(v_i \otimes \tilde{\varphi}(v_j^*)) = \sum_{l=1}^{r} a_{lj}\mathrm{tr}(v_i \otimes v_l^*) = a_{ij}.$$

The compatibility condition implies $\tilde{\varphi} = \lambda \cdot \mathrm{id}$, as desired.

It remains to compute $\langle v(E)^2 \rangle$. As $v(E) = v(\tilde{F}) - v(\kappa(x))$,

$$\langle v(E)^2 \rangle = \langle v(\tilde{F})^2 \rangle + \langle v(\kappa(x))^2 \rangle - 2\langle v(\tilde{F}), v(\kappa(x)) \rangle = -2r^2 + 0 + 2r^2 = 0.$$

This shows that E is a semirigid sheaf on X, called the *semirigid sheaf associated to* F. Note that E is locally free except at the point x. In particular, Proposition 3.3.8 is no longer true for semirigid sheaves.

Proposition 3.3.13. *Let E be a stable semirigid sheaf and F a semistable sheaf with $v(F) = v(E)$. If E is not isomorphic to F, then $\mathrm{Ext}^i(E,F) = \mathrm{Ext}^i(F,E) = 0$ for $i = 0, 1, 2$.*

Proof. Observe that $\chi(E,F) = \chi(F,E) = -\langle v(E), v(F) \rangle = 0$; thus, if $\mathrm{Hom}(E,F) = \mathrm{Hom}(F,E) = 0$, then $\mathrm{Ext}^1(E,F) = \mathrm{Ext}^1(F,E) = 0$. The same argument used in the proof of Corollary 3.3.9 shows that every morphism between E and F is either zero or an isomorphism. Hence, if E is not isomorphic to F, $\mathrm{Hom}(E,F) = \mathrm{Hom}(F,E) = 0$, whence the conclusion. □

Proposition 3.3.14. *Let X be a projective K3 surface with Picard number $\rho(X) = 1$ and consider a simple torsion-free sheaf E on X. Assume that E is either rigid or semirigid and that $v = v(E)$ is primitive in the Mukai lattice. Then E is stable.*

Proof. Under the hypotheses of the proposition, there are no strictly semistable sheaves with Mukai vector v. Indeed, if this were not the case, we could pick E' a semistable sheaf with $v(E') = v$ and choose F a non-zero proper subsheaf with the same reduced Hilbert polynomial. As $\rho(X) = 1$, this would imply $r(E')v(F) = r(F)v(E')$, a contradiction with primitivity of v.

Therefore, it suffices to show that E is semistable. Suppose that E is not so. Then, among the subsheaves of E with maximal reduced Hilbert polynomial, choose F_1 of maximal rank. The quotient $F_2 = E/F_1$ is torsion-free and $\mathrm{Hom}(F_1, F_2) = 0$ by our choice of F_1. Hence, by Lemma 3.3.2, one has

$$\dim \mathrm{Ext}^1(F_1, F_1) + \dim \mathrm{Ext}^1(F_2, F_2) \leq \dim \mathrm{Ext}^1(E, E). \tag{3.7}$$

The inequality $\dim \mathrm{Ext}^1(E, E) = \langle v(E)^2 \rangle + 2 \leq 2$ forces $\dim \mathrm{Ext}^1(F_i, F_i) \leq 2$ for $i = 1, 2$, whence $\langle v(F_i)^2 \rangle = \dim \mathrm{Ext}^1(F_i, F_i) - 2 \dim \mathrm{End}(F_i) \leq 0$ for $i = 1, 2$. Since $r(F_i) < r(E)$, by Corollary 3.3.4 we have

$$\begin{aligned}\langle v(E)^2 \rangle &\leq r(E) \left(\frac{\langle v(F_1)^2 \rangle}{r(F_1)} + \frac{\langle v(F_2)^2 \rangle}{r(F_2)} \right) \\ &= \langle v(F_1)^2 \rangle + \langle v(F_2)^2 \rangle + \frac{r(F_2)}{r(F_1)} \langle v(F_1)^2 \rangle + \frac{r(F_1)}{r(F_2)} \langle v(F_2)^2 \rangle \leq \langle v(F_1)^2 \rangle + \langle v(F_2)^2 \rangle.\end{aligned}$$

We obtain

$$\begin{aligned}&\dim \mathrm{Ext}^1(F_1, F_1) + \dim \mathrm{Ext}^1(F_2, F_2) \\ &= \langle v(F_1)^2 \rangle + 2 \dim \mathrm{End}(F_1) + \langle v(F_2)^2 \rangle + 2 \dim \mathrm{End}(F_2) \\ &\geq \langle v(E)^2 \rangle + 2 \dim \mathrm{End}(F_1) + 2 \dim \mathrm{End}(F_2) \\ &= \dim \mathrm{Ext}^1(E, E) + 2 \dim \mathrm{End}(F_1) + 2 \dim \mathrm{End}(F_2) - 2 \\ &> \dim \mathrm{Ext}^1(E, E),\end{aligned}$$

which contradicts 3.7. □

Proposition 3.3.15. *Let E be a semirigid sheaf with $v(E) = (r, \ell, s)$. Assume that ℓ is ample and E is stable with respect to ℓ. If r divides s and $v(E)$ is primitive, then E is μ-stable with respect to ℓ.*

Proof. Assume for a contradiction that E is not μ-stable. Among the proper quotients of E with slope $\mu(E)$, choose E' of minimal rank. E' is then μ-stable, and in particular simple. Denote $v(E') = (r', \ell', s')$ its Mukai vector. Since $\mu(E) = \mu(E')$,

$$\left(\ell.\ell' - \frac{r'}{r} \ell \right) = 0.$$

Moreover, $\ell^2 = 2rs$ as E is semirigid. Therefore, we have

$$\begin{aligned}\langle v(E')^2\rangle = \ell'^2 - 2r's' &= \left(\left(\ell' - \frac{r'}{r}\ell\right) + \frac{r'}{r}\ell\right)^2 - 2r's' \\ &= \left(\ell' - \frac{r'}{r}\ell\right)^2 + \left(\frac{r'}{r}\ell\right)^2 - 2r's' \\ &= \left(\ell' - \frac{r'}{r}\ell\right)^2 + 2r'\left(\frac{r'}{r}s - s'\right).\end{aligned}$$

Since $v(E)$ is primitive and r divides s, r and ℓ must be coprime. Hence $\ell' - r'\ell/r$ is not zero. As its intersection with the ample sheaf ℓ is zero, by Hodge index theorem we deduce

$$\left(\ell' - \frac{r'}{r}\ell\right)^2 < 0.$$

On the other hand, the stability of E implies that the integer $r's/r - s' < 0$. We deduce that $\langle v(E')^2\rangle < -2r' \leq -2$, against the simpleness of E'. □

Proposition 3.3.16. *Let $v = (r, \ell, s) \in \tilde{H}^{1,1}(X, \mathbb{Z})$ be a primitive isotropic vector and E a sheaf with Mukai vector v. Suppose that ℓ is ample and E is strictly semistable with respect to ℓ. Consider*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E \quad \text{for } n \geq 2$$

a Jordan-Hölder filtration of E. Then the successive quotients $F_i = E_i/E_{i-1}$ are rigid for $i = 1, 2, \ldots, n$.

Proof. By Proposition 3.3.6, we have that $\langle v(F_i)^2\rangle \leq 0$ for all i. Since v is primitive, equality is not attained for any i. Hence F_i is rigid. □

Corollary 3.3.17. *Let $v = (r, \ell, s) \in \tilde{H}^{1,1}(X, \mathbb{Z})$ be a primitive isotropic vector. Then the complement of $\mathbf{M}^s_\ell(v)$ in the moduli space $\mathbf{M}_\ell(v)$ of semistable sheaves with Mukai vector v is a zero-dimensional set.*

The remaining part of this section is devoted to the proof of two beautiful theorems by Mukai. By the way, at the end, we will have all the needed tools to conclude the proof of the Derived Torelli theorem 2.2.4.

Theorem 3.3.18. *Let $v \in \tilde{H}^{1,1}(X, \mathbb{Z})$ be a primitive isotropic vector. Suppose that $\mathbf{M}^s_H(v)$ contains a compact connected component M all of whose elements are locally free sheaves. Then $\mathbf{M}^s_H(v)$ is compact and irreducible.*

Proof. Since $\mathbf{M}^s_H(v)$ is smooth, M is irreducible. We will show that every semistable sheaf F with Mukai vector $v(F) = v$ belongs to M: the chain of inclusions $M \subset \mathbf{M}^s_H(v) \subset \mathbf{M}_H(v) \subset M$ implies the result. We will do the proof assuming that there

exists a universal family on $X \times \mathbf{M}_H^s(v)$. Consider $\mathcal{E}$ the restriction of such a family to $X \times M$; it induces a Fourier-Mukai transform

$$\begin{aligned} \Phi_{\mathcal{E}^\vee} \colon \mathcal{D}^b(X) &\to \mathcal{D}^b(M) \\ F &\mapsto p_*(\mathcal{E}^\vee \otimes q^* F), \end{aligned}$$

where, as usual, q (resp. p) denotes the projection from $X \times M$ onto the first (resp. the second) factor. Denote $\Phi^i_{\mathcal{E}^\vee}(F) = H^i(\Phi_{\mathcal{E}^\vee}(F))$ for $i = 0, 1, 2$. The fibre of $\Phi^i_{\mathcal{E}^\vee}(F)$ at a point $[E] \in M$ is

$$\Phi^i_{\mathcal{E}^\vee}(F) \otimes \kappa([E]) = H^i(X, \mathcal{E}^\vee|_{X \times \{[E]\}} \otimes F) = H^i(X, E^\vee \otimes F) \simeq \mathrm{Ext}^i(E, F).$$

Hence, if $[E]$ is in the support of $\Phi^i_{\mathcal{E}^\vee}(F)$ for some i, then $\mathrm{Ext}^i(E, F) \neq 0$ and, as $\langle v(F), v(E) \rangle = 0$, we have a non-zero morphism between E and F, which is in fact an isomorphism. We deduce that $\Phi^i_{\mathcal{E}^\vee}(F)$ is supported at most at one point. One can deduce that $\Phi^0_{\mathcal{E}^\vee}(F) = \Phi^1_{\mathcal{E}^\vee}(F) = 0$ (see [21] Proposition 2.26). Now, by Grothendieck-Riemann-Roch theorem, the cohomology class

$$\begin{aligned} \mathrm{ch}(\Phi_{\mathcal{E}^\vee}(F)) &= \mathrm{ch}(\Phi^0_{\mathcal{E}^\vee}(F)) - \mathrm{ch}(\Phi^1_{\mathcal{E}^\vee}(F)) + \mathrm{ch}(\Phi^2_{\mathcal{E}^\vee}(F)) \\ &= p_*(\mathrm{ch}(\mathcal{E}^\vee \otimes q^* F).\mathrm{td}(X \times M)).\mathrm{td}(M)^{-1} \\ &= p_*(\mathrm{ch}(\mathcal{E}^\vee).q^*\mathrm{ch}(F).q^*\mathrm{td}(X).p^*\mathrm{td}(M)).\mathrm{td}(M)^{-1} \\ &= p_*(\mathrm{ch}(\mathcal{E}^\vee).q^*\sqrt{\mathrm{td}(X)}.q^* v(F)) \end{aligned}$$

depends just on the Mukai vector of F, and not on F itself. For a sheaf $E \in M$, $\mathrm{ch}(\Phi_{\mathcal{E}^\vee}(E)) \neq 0$. Hence, for all sheaves F in $\mathbf{M}_H(v)$, $\mathrm{ch}(\Phi_{\mathcal{E}^\vee}(F)) \neq 0$, which implies $F \simeq E$ for some $E \in M$. □

Theorem 3.3.19. *Let v be a primitive isotropic vector. Assume that $\mathbf{M}_H^s(v)$ is compact and let $M \subset \mathbf{M}_H^s(v)$ be a connected component. Then M is a K3 surface. Moreover, there exists a Hodge isometry*

$$H^2(M, \mathbb{Z}) \simeq v^\perp / \mathbb{Z} v.$$

Proof. We will do the proof under the assumption that $\mathbf{M}_H^s(v)$ is fine. Let $\mathcal{E}$ be the restriction to $M \times X$ of the universal family and consider the Fourier-Mukai transform

$$\Phi_{\mathcal{E}} \colon \mathcal{D}^b(M) \to \mathcal{D}^b(X).$$

$\mathcal{E}$ is flat over M and we have

(i) for any $[E] \in M$, $\mathrm{Hom}(\Phi_{\mathcal{E}}(\kappa([E])), \Phi_{\mathcal{E}}(\kappa([E]))) = \mathrm{Hom}(E, E) \simeq \mathbb{C}$;

(ii) if $[E] \neq [F]$, then $\mathrm{Ext}^i(\Phi_{\mathcal{E}}(\kappa([E])), \Phi_{\mathcal{E}}(\kappa([F]))) = \mathrm{Ext}^i(E, F) = 0$ for $i = 0, 1, 2$.

By Corollary 1.3.8, $\Phi_{\mathcal{E}}$ is fully faithful. As the canonical bundles of X and M are trivial, by Corollary 1.3.10, $\Phi_{\mathcal{E}}$ is an equivalence. By Proposition 1.3.14, the induced cohomological transform $\Phi^H_{\mathcal{E}}$ yields an isomorphism

$$H^{0,1}(M) \oplus H^{1,2}(M) \simeq H^{0,1}(X) \oplus H^{1,2}(X).$$

Recalling the Hodge numbers of a K3 surface, the right handside is zero. This implies in particular the vanishing of $H^{0,1}(M) \simeq H^1(M, \mathcal{O}_M)$. Hence M is a K3 surface.

By Proposition 2.2.2, $\Phi_{\mathcal{E}}$ descends to an isometry $\Phi^H_{\mathcal{E}} \colon \tilde{H}(M,\mathbb{Z}) \to \tilde{H}(X,\mathbb{Z})$. Let us compute $\Phi^H_{\mathcal{E}}((0,0,1))$. Note that $(0,0,1)$ is the Mukai vector of a skyscraper sheaf $\kappa([E])$ for some $[E] \in M$. Hence,

$$\Phi^H_{\mathcal{E}}((0,0,1)) = \Phi^H_{\mathcal{E}}(v(\kappa([E]))) = v(\Phi_{\mathcal{E}}(\kappa([E]))) = v(\mathcal{E}|_{\{[E]\}\times X}) = v(E) = v.$$

This means that $\Phi^H_{\mathcal{E}}(H^4(M,\mathbb{Z})) = \mathbb{Z}v$. Moreover, $(0,0,1)^\perp = H^2(M,\mathbb{Z}) \oplus H^4(M,\mathbb{Z})$. Thus, we have

$$\begin{array}{ccc} (0,0,1)^\perp & \xrightarrow{\sim} & v^\perp \\ \downarrow & & \downarrow \\ H^2(M,\mathbb{Z}) & \xrightarrow{\sim} & v^\perp/\mathbb{Z}v, \end{array}$$

as desired. □

In the proof of the previous theorem, we have seen that, if $v = (r,\ell,s)$ is a primitive isotropic vector and a universal family exists on $X \times \mathbf{M}_H(v)$, then we have a derived equivalence between X and a connected component M of $\mathbf{M}_H(v)$. M is again a K3 surface and the induced cohomological transform sends v to the fundamental class of M. The following results show that a universal family exists if we can find $v' \in \tilde{H}^{1,1}(X,\mathbb{Z})$ with $\langle v, v'\rangle = 1$. This fills the gap in the proof of the Derived Torelli theorem 2.2.4.

Proposition 3.3.20. *If the* g.c.d.$(r, \ell.H, s) = 1$*, then* $\mathbf{M}^s_H(v)$ *is compact. Moreover,* $\mathbf{M}_H(v)$ *is fine.*

Proof. Assume for a contradiction that there exists a strictly semistable sheaf E with Mukai vector v. Pick a non-zero proper subsheaf F of E, with Mukai vector $v(F) = (r', \ell', s')$ and the same reduced Hilbert polynomial. This implies that

$$\frac{\ell.H}{r} = \frac{\ell'.H}{r'} \quad \text{and} \quad \frac{s}{r} = \frac{s'}{r'}$$

Writing $1 = ar + b(\ell.H) + cs$ and multiplying by r', we get $r' = arr' + b(\ell.H)r' + csr' = arr' + br(\ell'.H) + crs' = r(ar' + b(\ell'.H) + cs')$, a contradiction as $0 < r' < r$. Therefore, the closed points of the moduli space $\mathbf{M}_H(v)$ are in bijection with isomorphism classes of stable sheaves on X. For the proof of the existence of a universal family, we refer to [14], Remark 4.6.8. □

Lemma 3.3.21. *Let $v = (r, \ell, s) \in \tilde{H}^{1,1}(X, \mathbb{Z})$ and assume that there exists a vector $v' \in \tilde{H}^{1,1}(X, \mathbb{Z})$ with $\langle v, v' \rangle = 1$. Then there exists an ample class H with* g.c.d.$(r, \ell.H, s) = 1$.

Proof. Write $v = (r', \ell', s')$. Suppose that the integer a divides $r, (\ell.\ell')$ and s. Then it also divides $(\ell.\ell') - rs' - r's = \langle v, v' \rangle = 1$. Hence $a = \pm 1$. A priori, ℓ' is not ample, but adding $(nr) \cdot H'$ for n big enough and H' an ample class, we obtain an ample divisor H with g.c.d.$(r, \ell.H, s) = 1$. □

3.3.3 An existence result

Let us fix X a K3 surface. In this section, whose reference is [21, §5], we shall prove the following existence result.

Theorem 3.3.22. *Let $v = (r, \ell, s) \in \tilde{H}^{1,1}(X, \mathbb{Z})$ be a primitive isotropic vector with $r \geq 1$. For any ample divisor H on X, there exists a simple sheaf E with Mukai vector $v(E) = v$ that is μ-semistable with respect to H.*

This theorem is equivalent to the following stronger version:

Theorem 3.3.23. *Fix m a divisor class of X. Then the sheaf E can be chosen in such a way that, for every torsion-free quotient sheaf F of E with $\mu(F) = \mu(E)$, one has*

$$\frac{c_1(F).m}{r(F)} \geq \frac{c_1(E).m}{r(E)}.$$

Indeed, for n big enough, $nH + m$ is ample. By Theorem 3.3.22, there exists a sequence of simple sheaves E_n with Mukai vector $v(E_n) = v$ and μ-semistable with respect to $H + m/n$. We deduce the existence of a simple sheaf E which is μ-semistable for infinitely many $H + m/n$ and for H (see [21] Appendix 1). Hence, if F is a torsion-free quotient sheaf of E with the same slope with respect to H, we have

$$\frac{1}{n}\frac{c_1(F).m}{r(F)} + \mu(F) \geq \frac{1}{n}\frac{c_1(E).m}{r(E)} + \mu(E) \implies \frac{c_1(F).m}{r(F)} \geq \frac{c_1(E).m}{r(E)}.$$

The proof of Theorems 3.3.22 and 3.3.23 will be done by induction on the rank r.

Base of the induction If $r = 1$, the isotropic vector v is of the form

$$v = \left(1, \ell, \frac{\ell^2}{2}\right).$$

As $v \in \tilde{H}^{1,1}(X, \mathbb{Z})$, $v = \mathrm{ch}(\mathcal{O}(\ell))$. Now, the ideal sheaf $\mathcal{I}$ of a point has Mukai vector $v(\mathcal{I}) = (1, 0, 0)$. If we take $E = \mathcal{O}(\ell) \otimes \mathcal{I}$, then $v(E) = \mathrm{ch}(\mathcal{O}(\ell)).v(\mathcal{I}) = v$ and E is clearly μ-stable with respect to any ample line bundle H.

Now, let us fix $v = (r, \ell, s)$ and assume that Theorems 3.3.22 and 3.3.23 are true for any Mukai vector of rank $< r$.

Step 1 Assume that $-r < s < 0$ and $\ell.H = 0$. Then there exists a simple μ-semistable sheaf with Mukai vector $v(E) = v$.

We prove this claim using the following construction. Let F be a coherent sheaf on X for which

(i) the canonical evaluation map $f\colon H^0(X, F) \otimes \mathcal{O}_X \to F$ is injective and

(ii) $H^2(X, F) = 0$.

We will construct a sheaf E from F such that

$$r(E) = -s(F), \quad c_1(E) = c_1(F), \quad s(E) = -r(F).$$

Such a sheaf E is called the *reflection* of F.

Consider the short exact sequence

$$0 \to H^0(X, F) \otimes \mathcal{O}_X \xrightarrow{f} F \to \bar{F} \to 0.$$

Because of the vanishing of $H^1(X, \mathcal{O}_X)$ and $H^2(X, F)$, in the induced long exact sequence we have

$$0 \to H^1(X, F) \xrightarrow{g} H^1(X, \bar{F}) \to H^0(X, F) \otimes H^2(X, \mathcal{O}_X) \simeq H^0(X, F)^\vee \to 0.$$

Moreover, $H^0(X, \bar{F}) = H^2(X, \bar{F}) = 0$. Construct an exact sequence

$$0 \to \bar{F} \to E \to H^1(X, F) \otimes \mathcal{O}_X \to 0$$

such that the boundary map $H^1(X, F) \otimes H^0(X, \mathcal{O}_X) \to H^1(X, \bar{F})$ is g. The Mukai vector of E is

$$\begin{aligned} v(E) &= v(\bar{F}) + h^1(F)v(\mathcal{O}_X) \\ &= v(F) - h^0(F)v(\mathcal{O}_X) + h^1(F)v(\mathcal{O}_X) \\ &= v(F) - \chi(F)v(\mathcal{O}_X) \\ &= (r(F), c_1(F), \chi(F) - r(F)) - (\chi(F), 0, \chi(F)) = (-s(F), c_1(F), -r(F)), \end{aligned}$$

as desired.

Lemma 3.3.24. *For F and E as above,* $\mathrm{End}(F) = \mathrm{End}(E)$. *In particular, if F is simple, so is E.*

Proof. See [21] Proposition 2.25. □

Proof of Step 1. By induction hypothesis, there exists a simple μ-semistable sheaf F with isotropic Mukai vector $v(F) = (-s, \ell, -r)$. As $\mu(F) = 0$, the canonical morphism $H^0(X, F) \otimes \mathcal{O}_X \to F$ is injective: indeed, any morphism $\mathcal{O}_X \to F$ is injective by stability of $\mathcal{O}_X$. If we put $m = -\ell$, by Theorem 3.3.23, we can take F so that

$$-\frac{(c_1(G).\ell)}{r(G)} \geq -\frac{(\ell^2)}{r(F)}$$

for any torsion-free quotient sheaf G of F with $\mu(G) = \mu(F)$. As $\ell^2 = 2rs < 0$, $c_1(G).\ell < 0$. It follows that $\mathrm{Hom}(F, \mathcal{O}_X) = 0$: if $f\colon F \to \mathcal{O}_X$ were a non-zero morphism, by μ-stability of $\mathcal{O}_X$ it would be surjective, but $c_1(\mathcal{O}_X) = 0$. By Serre duality, $H^2(X, F) = 0$.

Consider E the reflection of F. Then $v(E) = (r, \ell, s)$ and we have the exact sequences

$$0 \to H^0(X, F) \otimes \mathcal{O}_X \to F \to \bar{F} \to 0,$$
$$0 \to \bar{F} \to E \to H^1(X, F) \otimes \mathcal{O}_X \to 0.$$

From the former we deduce that of $\bar{F}$ is torsion-free and μ-semistable. The latter yields then torsion-freeness and μ-semistability of E. Finally, as F is simple, so is E by Lemma 3.3.24. □

For the next steps, we will need this definition. A quasi-polarized K3 surface (X, H) is *monogonal* if there exists a smooth elliptic curve C on X such that $(H.C) = 1$. If we set $g = \frac{1}{2}(H^2) + 1$, we get

$$(H - gC)^2 = (H^2) + g^2(C^2) - 2g(H.C) = (2g - 2) + 0 - 2g = -2$$
$$(C.H - gC) = (C.H) - g(C^2) = 1 - 0 = 1.$$

Hence, there exists an effective divisor D linearly equivalent to $H - gC$. As $D^2 = -2$, D is a smooth rational curve in X. If $\rho(X) = 2$, then $\mathrm{Pic}(X)$ is generated by C and D. A divisor $aC + b(C + D)$ on X is ample if and only if $a > b > 0$.

Step 2 Suppose that X is monogonal with Picard number $\rho(X) = 2$. Then there exists a simple torsion-free sheaf E on X with Mukai vector $v(E) = v$.

Proof. Write $\ell = aC + b(C + D)$ for some $a, b \in \mathbb{Z}$. Let b' be an integer congruent to b modulo r and $|b'| \leq r/2$. Then, take an integer a' congruent to a modulo r, with $r/2 < |a'| \leq 3r/2$ and $a'b' < 0$ if $b' \neq 0$, $-r < a' \leq 0$ if $b' = 0$. Consider $\ell' = a'C + b'(C + D)$. By construction,

$$\ell' = \ell \pmod{r} \quad \text{and} \quad \ell'^2 = 2a'b' = 2ab = \ell^2 \pmod{2r}.$$

As ℓ^2 is divisible by $2r$, so is ℓ'^2; therefore $s' = \ell'^2/2r$ is an integer. If we show the existence of a simple torsion-free sheaf E' with $v(E') = (r, \ell', s')$, then we are done: the sheaf

$$E \coloneqq E' \otimes \mathcal{O}\left(\frac{\ell - \ell'}{r}\right)$$

is simple, torsion-free and with Mukai vector

$$v(E) = v(E').\mathrm{ch}\left(\mathcal{O}\left(\frac{\ell-\ell'}{r}\right)\right)$$
$$= \left(r, \ell', \frac{\ell'^2}{2r}\right).\left(1, \frac{\ell-\ell'}{r}, \frac{\ell^2+\ell'^2-2\ell\ell'}{2r^2}\right) = (r, \ell, s).$$

We distiguish two cases:

(1) $b' \neq 0$. By our choice of a' and b', $-3r^2/4 \leq a'b' < 0$, whence $-3r/4 \leq s' < 0$. Let us consider the divisor $H = a'C - b'(C+D)$; either H or $-H$ is ample. Since $H.\ell' = 0$ and $-r < s' < 0$, there exists a simple torsion-free sheaf E' with $v(E) = (r, \ell', s')$ by Step 1.

(2) $b' = 0$. In this case, $s' = 0$. Primitivity of v implies that r and a' are coprime. It can be shown that there exists a vector bundle G on the elliptic curve C of rank $-a'$ and degree r, which is generated by its global sections and has $H^1(X, G) = 0$ ([1] and [21] Lemma 5.3); this vector bundle can be considered as a sheaf on X supported on C. Let E' be the kernel of the natural surjective homomorphism $H^0(X, G)\otimes\mathcal{O}_X \to G$. E' is clearly torsion-free. As $\dim H^0(X, G) = \dim H^0(C, G) = \chi(G) = \deg(G) = r$, the rank of E' equals r. The surjectivity of the natural morphism $\mathbb{C} \simeq \mathrm{End}(G) \to \mathrm{End}(E')$ comes from $H^1(X, G) = 0$ and shows that E' is simple. □

Next, let us consider the situation where H is primitive and $\ell = kH$, for k an integer. The idea is to regard the K3 surface (X, H) as part of a family of polarized K3 surfaces of degree $d = (H^2)$. Recall that, for any even positive integer d, there exists a coarse moduli space $\mathcal{M}_d$ for polarized K3 surfaces of degree d. It is an irreducible quasi-projective scheme whose closed points are in bijection with isomorphism classes of polarized K3 surfaces of degree d. For any family $(\mathcal{X}, \mathcal{L}) \to U$ of polarized K3 surfaces of degree $d = \mathcal{L}_t^2$, there exists a morphism $U \to \mathcal{M}_d$ sending u to the isomorphism class of the fibre $(\mathcal{X}_u, \mathcal{L}_u)$.

For any polarized K3 surface (X, H) in $\mathcal{M}_d$, we can consider the moduli spaces $\mathbf{M}^s_{X,H}(v)$, $\mathbf{M}_{X,H}(v)$ and $\mathbf{Spl}_X(v)$ of stable, semistable and simple sheaves respectively, with Mukai vector v. The families $\{\mathbf{M}^s_{X,H}(v)\}_{(X,H)\in\mathcal{M}_d}$ and $\{\mathbf{Spl}_X(v)\}_{(X,H)\in\mathcal{M}_d}$ are smooth over an étale covering of $\mathcal{M}_d$. The family $\{\mathbf{M}_{X,H}(v)\}_{(X,H)\in\mathcal{M}_d}$ is proper over $\mathcal{M}_d$.

Step 3 There exists an open subset U of $\mathcal{M}_d$ such that, for any polarized K3 surface $(X, H) \in U$, the corresponding moduli space $\mathbf{M}^s_{X,H}(v) \neq \emptyset$.

Proof. By Step 2, if (X, H) is monogonal with Picard number $\rho(X) = 2$, $\mathbf{Spl}_X(v) \neq \emptyset$. By smoothness of $\{\mathbf{Spl}_X(v)\}_{(X,H)\in\mathcal{M}_d}$ over an étale covering of $\mathcal{M}_d$, there exists an open subset V of $\mathcal{M}_d$ such that, for any polarized K3 surface $(X', H') \in V$, there exists a simple torsion-free sheaf E' with Mukai vector $v(E') = (r, kH', s)$. The set

of polarized K3 surfaces with Picard number 1 is dense in $\mathcal{M}_d$. Hence, we can find $(X', H') \in V$ with $\rho(X) = 1$ and a simple semirigid torsion-free sheaf E' whose Mukai vector is primitive. By Proposition 3.3.14, E' is stable, i.e. $\mathbf{M}^s_{X',H'}(v) \neq \emptyset$. By smoothness of $\{\mathbf{M}^s_{X,H}(v)\}_{(X,H)\in\mathcal{M}_d}$, there exists an open neighbourhood U of (X', H') with the required property. □

Step 4 The open subset U of $\mathcal{M}_d$ found in the previous step coincides with $\mathcal{M}_d$. This means that, for any polarized K3 surface (X, H), there exists a sheaf E with Mukai vector $v(E) = v$, which is stable with respect to H.

Proof. In Step 3 we have found a dense open subset U of $\mathcal{M}_d$ such that, for any $(X, H) \in U$, $\mathbf{M}_{X,H}(v)$ is non-empty. By properness of $\{\mathbf{M}_{X,H}(v)\}_{(X,H)\in\mathcal{M}_d}$, $\mathbf{M}_{X,H}(v) \neq \emptyset$ for any $(X, H) \in \mathcal{M}_d$. Fix $(X_0, H_0) \in \mathcal{M}_d$; we want to show that $\mathbf{M}^s_{X_0,H_0}(v) \neq \emptyset$. As the moduli space $\mathcal{M}_d$ is connected, we can find a curve T joining (X_0, H_0) to an element (X_1, H_1) in U. This corresponds to a family $(\mathcal{X}, \mathcal{H}) \to T$ whose fibres over t_0 and t_1 are (X_0, H_0) and (X_1, H_1) respectively. For such a family, it is possible to construct an algebraic space $\mathcal{M}_{\mathcal{H}}(v)$ smooth and proper over T such that $\mathcal{M}_{\mathcal{H}}(v)_t = \mathbf{M}_{\mathcal{X}_t,\mathcal{H}_t}(v)$ for any $t \in T$. The function $t \mapsto \dim \mathcal{M}_{\mathcal{H}}(v)_t$ is upper semicontinuous. Since

$$\dim \mathcal{M}_{\mathcal{H}}(v)_t \geq \dim \mathbf{M}^s_{\mathcal{X}_t,\mathcal{H}_t}(v) = 2$$

for any $(\mathcal{X}_t, \mathcal{H}_t) \in U \cap T$, we have $\dim \mathcal{M}_{\mathcal{H}}(v)_{t_0} \geq 2$. By Corollary 3.3.17, the complement of $\mathbf{M}^s_{X_0,H_0}(v)$ in $\mathbf{M}_{X_0,H_0}(v)$ is discrete. Hence, $\mathbf{M}^s_{X_0,H_0}(v) \neq \emptyset$. □

Step 5 There exists a simple torsion-free sheaf E with Mukai vector $v(E) = (r, \ell, s)$ which is μ-semistable with respect to H.

Proof. Consider the divisor $nrH + \ell$. It is ample for n big enough. We claim that, for any such n, there exists a simple sheaf E_n with Mukai vector v and μ-semistable with respect to $nrH + \ell$. As tensoring with an invertible sheaf preserves simpleness and μ-semistability, it is sufficient to find a simple sheaf E'_n, semistable with respect to $nrH + \ell$ and with isotropic Mukai vector

$$v(E'_n) = v(E_n \otimes \mathcal{O}(nH)) = (r, \ell, s).\left(1, nH, \frac{n^2(H^2)}{2}\right) = (r, rnH + \ell, s').$$

But the existence of E'_n is exactly the content of Step 4. As the sequence of ample $\mathbb{Q}$-divisors $\{H + \ell/rn\}_n$ converges to H, there exists a simple sheaf E μ-semistable with respect to H and with Mukai vector $v = (r, \ell, s)$. □

Chapter 4

Higher dimensional moduli spaces

This chapter is devoted to the study of higher dimensional moduli spaces. It will be convenient to write the Mukai vector $v = mv_0$, for $m \in \mathbb{N}_0$ a multiplicity and v_0 a primitive vector. In the first section, we introduce the technical notion of v-general ample divisor; it ensures that, if the multiplicity $m = 1$, no strictly semistable sheaf exists and hence that the moduli space $\mathbf{M}_H(v)$ is smooth. In the second, we state a fundamental existence theorem due to Yoshioka; we do not prove this difficult result, but we show how to reduce the problem to the important class of elliptic K3 surfaces. In the last section, based on a beautiful article by Kaledin, Sorger and Lehn, we study the singular moduli spaces $\mathbf{M}_H(v)$, for $m \geq 3$ and $\langle v_0, v_0 \rangle \geq 2$ or $m = 2$ and $\langle v_0, v_0 \rangle \geq 4$. These are proven to be locally factorial symplectic varieties; in particular, they do not admit projective symplectic resolutions.

4.1 v-general ample divisors

Fix $v = (r, \ell, s) \in \tilde{H}^{1,1}(X, \mathbb{Z})$ a Mukai vector. The aim of this section is to determine conditions on an ample class H such that the following condition holds:

$$\begin{gathered} \text{for any sheaf } E \text{ with } v(E) = v \text{ that is semistable with respect to } H \\ \text{and for any non-zero proper coherent subsheaf } F \text{ of } E, \\ \text{if } p_F = p_E, \text{ then } v(F) \in \mathbb{Q}v(E). \end{gathered} \tag{4.1}$$

An ample divisor satisfying 4.1 is said to be *v-general*. As an application of this notion, we have the following

Proposition 4.1.1. *If $v \in \tilde{H}^{1,1}(X, \mathbb{Z})$ is primitive and H is v-general, $\mathbf{M}_H^s(v)$ is compact.*

Proof. Assume that there exists a strictly semistable sheaf E with Mukai vector $v(E) = v$, and choose F a non-zero proper subsheaf with the same reduced Hilbert polynomial. Condition 4.1 implies that $v(F) \in \mathbb{Q}v(E)$, a contradiction as v is primitive. □

Let us start from the case of one-dimensional pure sheaves. The fixed Mukai vector will be of the form $v = (0, \ell, s)$; we shall assume that $s \neq 0$ and ℓ effective. Let E be a pure one-dimensional sheaf with Mukai vector $v(E) = (0, c_1(E), \chi(E)) = v$. Its Hilbert polynomial with respect to an ample class H is

$$P_E(n) = \chi(E(n)) = (H.\ell)n + \chi(E).$$

Hence, E is semistable with respect to H if and only if

$$\frac{\chi(F)}{(H.c_1(F))} \leq \frac{\chi(E)}{(H.c_1(E))}$$

for any non-zero proper subsheaf F of E. Choose an ample divisor H that does not satisfy condition 4.1. This implies that there exist E semistable with respect to H and F non-zero proper subsheaf of E such that

$$\frac{\chi(F)}{(H.c_1(F))} = \frac{\chi(E)}{(H.c_1(E))} \quad \text{and} \quad \frac{c_1(F)}{\chi(F)} \neq \frac{c_1(E)}{\chi(E)}.$$

Set $\xi_{E,F} = \chi(F)c_1(E) - \chi(E)c_1(F) \neq 0$. Associating with any $\xi \in H^2(X, \mathbb{Z})$ a *wall*

$$W_\xi := \mathrm{Amp}(X) \cap \xi^\perp,$$

the condition $(\xi_{E,F}.H) = 0$ means that $H \in W_{\xi_{E,F}}$. We would like to count the number of walls $W_{\xi_{E,F}}$ that can occur. By the Hodge index theorem

$$(\xi_{E,F})^2 = \chi(F)^2(c_1(E))^2 - 2\chi(E)\chi(F)(c_1(F).c_1(E)) + \chi(E)^2(c_1(E))^2 < 0. \quad (4.2)$$

It can be shown (see [25]) that the number of choices for $c_1(F)$ is finite and only depends on v. Hence, also the number of $\chi(F)$ satisfying the quadratic inequality 4.2 is finite. Eventually, the number of choices for $\xi_{E,F}$ is finite and only depends on v.

In conclusion, if we choose $H \in \mathrm{Amp}(X)$ out of a finite number of hyperplanes, then it satisfies condition 4.1.

Next, let us treat the case of torsion-free sheaves. The fixed Mukai vector will be of the form $v = (r, \ell, s)$, $r > 0$. A sheaf E with Mukai vector $v(E) = (r(E), c_1(E), \chi(E) - r(E)) = v$ is semistable if and only if, for every non-zero proper subsheaf F with Mukai vector $v(F) = (r(F), c_1(F), \chi(F) - r(F))$,

$$\frac{(c_1(F).H)}{r(F)} \leq \frac{(c_1(E).H)}{r(E)} \quad \text{or}$$
$$\frac{(c_1(F).H)}{r(F)} = \frac{(c_1(E).H)}{r(E)} \quad \text{and} \quad \frac{\chi(F)}{r(F)} \leq \frac{\chi(E)}{r(E)}.$$

Choose H so that condition 4.1 does not hold. Hence, there exists a semistable sheaf E with Mukai vector $v(E) = v$ admitting a non-zero proper subsheaf F with the same reduced Hilbert polynomial. In particular

$$\frac{(c_1(F).H)}{r(F)} = \frac{(c_1(E).H)}{r(E)} \quad \text{and} \quad \frac{c_1(F)}{r(F)} \neq \frac{c_1(E)}{r(E)}.$$

Set $\xi_{E,F} \coloneqq r(E)c_1(F) - r(F)c_1(E) \neq 0$. Since $(\xi_{E,F}.H) = 0$, by the Hodge index theorem

$$(\xi_{E,F}^2) = r(E)^2(c_1(F)^2) + r(F)^2(c_1(E)^2) - 2r(F)r(E)(c_1(F), c_1(E)) < 0.$$

Moreover, thanks to Bogomolov inequality, it can be shown that $\xi_{E,F}$ satisfies also the inequality

$$(\xi_{E,F}^2) \geq -\frac{r(E)^2}{4}(v(E)^2 - 2r(E)^2)$$

(see [14] Theorem 4.C.3). As before, define a wall as

$$W_\xi = \{\, L \in \mathrm{Amp}(X) \mid (\xi.L) = 0 \,\}.$$

It can be shown ([14] Lemma 4.C.2) that the union of walls W_ξ, with $-(r^2/4)(\langle v^2\rangle - 2r^2) \leq \xi^2 < 0$, is locally finite. Hence H is v-general as soon as we choose it out of this locally finite union of hyperplanes.

4.2 Yoshioka's existence result

As we have already seen in section 3.3.3, the question of non-emptyness of the moduli space is far from being trivial. A cornerstone result in this sense is the following

Theorem 4.2.1 (Yoshioka)**.** *Let (X, H) be a polarized K3 surface and let $v = (r, \ell, s) \in \tilde{H}^{1,1}(X, \mathbb{Z})$ be a primitive Mukai vector. Assume that H is v-general and suppose that $r > 0$ or ℓ is ample. Then $\mathbf{M}_H^s(v)$ is deformation equivalent to $\mathbf{Hilb}^{\frac{1}{2}\langle v,v\rangle+1}(X)$.*

The proof of this theorem in its full generality is rather involved and technical. For the details, one may consult [25]. The exposition in this thesis owes almost everything to the notes by I. Vogt for the MIT-NEU graduate seminar on Moduli of Sheaves on K3 surfaces ([24]). In the argument, a fundamental role is played by the class of elliptic K3 surfaces.

Definition 4.2.2. A K3 surface X is said to be *elliptic* if it admits a proper morphism $\pi\colon X \to \mathbb{P}^1$ such that X_y is a smooth curve of genus one for all but finitely many $y \in \mathbb{P}^1$. The morphism π is called an *elliptic fibration*.

The following lemma allows us to recognize such surfaces by looking at their Picard lattice.

Lemma 4.2.3. *Let X be a K3 surface whose Picard lattice* $\mathrm{Pic}(X)$ *contains a copy U of the hyperbolic lattice. Then X is an elliptic K3 surface.*

Proof. By assumption, there exist two divisor classes $D, F \in \mathrm{Pic}(X)$ such that $(D^2) = (F^2) = 0$ and $D.F = 1$. As $(2F + D)^2 = 4$, we can assume that $2F + D$ is ample. The complete linear system $|F|$ yields a rational morphism

$$\varphi_F \colon X \dashrightarrow \mathbb{P}(H^0(X, F)^\vee).$$

As $(-F.2F + D) = -1$, by Nakai-Moishezon criterion $-F$ cannot be effective and so $h^0(X, \mathcal{O}_X(-F)) = 0$. From Serre duality and the Riemann-Roch theorem

$$\chi(X, F) = h^0(X, \mathcal{O}_X(F)) - h^1(X, \mathcal{O}_X(F)) + h^0(X, \mathcal{O}_X(-F)) = \frac{(F^2)}{2} + 2 = 2,$$

we deduce that $h^0(X, \mathcal{O}_X(F)) \geq 2$; hence φ_F takes values in a projective space of dimension at least one.

Let C be a curve in $|F|$. Note that, by adjunction formula, $2g(C)-2 = (F^2) = 0$. Then C is an elliptic curve. Twisting by $\mathcal{O}_X(F)$ the short exact sequence that defines C, we get

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(F) \to \mathcal{O}_C(F) \simeq \mathcal{N}_{C/X} \to 0$$

and the exact sequence in cohomology

$$0 \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X(F)) \to H^0(X, \mathcal{N}_{C/X}) \to H^1(X, \mathcal{O}_X) = 0. \qquad (4.3)$$

We claim that $|F|$ is base point free, so that φ_F is a morphism. Assume that $|F|$ has a base point. Then the complete linear system corresponding to $\mathcal{N}_{C/X}$ has a base point, too. The triviality of the canonical bundle ω_X implies, by adjunction formula, that $\omega_C \simeq \mathcal{N}_{C/X}$. Since the canonical bundle of an elliptic curve is trivial, ω_C is base point free, a contradiction.
Moreover, looking at 4.3, from $h^0(C, \omega_C) = 1$ we deduce that $h^0(X, \mathcal{O}_X(F)) = 2$. We have proven that the morphism

$$\varphi_F \colon X \to \mathbb{P}^1$$

is an elliptic fibration. Let us conclude by showing that this morphism has a section. Consider the divisor class $D - F$, whose square $(D - F)^2$ equals -2. Its opposite $F - D$ is not effective, as $(2F + D.F - D) = -2 + 1 < 0$. Hence, by Riemann-Roch theorem, $h^0(X, \mathcal{O}_X(D - F)) \geq (D - F)^2/2 + 2 = 1$, so that $D - F$ is effective. Therefore, in the linear system $|D - F|$ there is a rational curve, which allows us to define a section of φ_F. □

Remark 4.2.4. Let X be an elliptic K3 surface with an elliptic fibration $\pi\colon X \to \mathbb{P}^1$. Let σ be the class of a section and f the class of a fibre. Clearly, $(\sigma^2) = -2$, $(f^2) = 0$ and $(\sigma.f) = 1$. Then $\sigma + nf$ has square $-2 + 2n$. This implies that X has polarizations of any degree.

We limit ourselves to show that it is enough to prove the Theorem 4.2.1 for an elliptic K3 surface. More precisely, we will show that the moduli space $\mathbf{M}^s_{X,H}(v)$ is deformation equivalent to the moduli space $\mathbf{M}^s_{Y,H'}(w)$, where Y is an elliptic K3 surface. We need the following lemma, which sheds light on the construction of moduli spaces in families.

Lemma 4.2.5. *Let $\pi\colon (\mathcal{X}, \mathcal{L}) \to T$ be a smooth family of polarized K3 surfaces over a smooth connected curve T. Let $v = (r, k\mathcal{L}, s) \in R^*\pi_*\underline{\mathbb{Z}}$ be a primitive Mukai vector. Assume that, for some $t_0 \in T$, the Picard number $\rho(\mathcal{X}_{t_0}) = 1$. Then there exists an algebraic space $\mathcal{M}^s_{\mathcal{L}}(v)$ smooth and proper over T such that, for any $t \in T$, $\mathcal{M}^s_{\mathcal{L}}(v)_t = \mathbf{M}^s_{\mathcal{X}_t,H_t}(v)$, for some H_t a general ample divisor on $\mathcal{X}_t$. Moreover, if we choose a finite subset T_0 of T and, for each $t \in T_0$, we fix any ample divisor H'_t such that $\mathbf{M}_{\mathcal{X}_t,H'_t}(v) = \mathbf{M}^s_{\mathcal{X}_t,H'_t}(v)$, then we can construct $\mathcal{M}^s_{\mathcal{L}}(v)$ so that $\mathcal{M}^s_{\mathcal{L}}(v)_t = \mathbf{M}^s_{\mathcal{X}_t,H'_t}(v)$ for any $t \in T_0$.*

Moduli spaces of sheaves that are fibres of the morphism $\mathcal{M}^s_{\mathcal{L}}(v) \to T$ of the lemma are deformation equivalent.

Proposition 4.2.6. *Let X_1 and X_2 be K3 surfaces. Let $v_1 = (r, \ell_1, s_1) \in \tilde{H}^{1,1}(X_1, \mathbb{Z})$ and $v_2 = (r, \ell_2, s_2) \in \tilde{H}^{1,1}(X_2, \mathbb{Z})$ be primitive Mukai vectors such that*

(1) $r > 0$; *(2)* $\text{g.c.d}(r, \ell_1) = \text{g.c.d}(r, \ell_2) = \xi$*;*
(3) $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle$; *(4)* $s_1 \equiv s_2 \pmod{\xi}$.

For $i = 1, 2$, let H_i be a v_i-general polarization. Then $\mathbf{M}^s_{H_1}(v_1)$ and $\mathbf{M}^s_{H_2}(v_2)$ are deformation equivalent.

Proof. Up to tensoring with a sufficiently high power of H_i, which gives an isomorphism $\mathbf{M}^s_{H_i}(v_i) \simeq \mathbf{M}^s_{H_i}(v_i.\text{ch}(H_i^{\otimes n_i}))$, we may assume that ℓ_1 and ℓ_2 are ample. As the moduli space of polarized K3 surfaces of degree $d = (\ell_1^2)$ is connected, there exists a curve joining (X_1, ℓ_1) to a polarized elliptic K3 surface (X'_1, ℓ'_1). Regard this as a smooth family $(\mathcal{X}_1, \mathcal{L}_1) \to T$ with fibres (X_1, ℓ_1) and (X'_1, ℓ'_1) over t_1 and t'_1. We have already fixed H_1 a v_1-general ample divisor on X_1 and we choose H'_1 a v'_1-general ample divisor on X'_1. By Lemma 4.2.5, the moduli spaces $\mathbf{M}^s_{X_1,H_1}(v_1)$ and $\mathbf{M}^s_{X'_1,H'_1}(v'_1)$ are deformation equivalent. We can apply the same argument to (X_2, ℓ_2). Therefore, we may suppose that (X_1, ℓ_1) and (X_2, ℓ_2) are elliptic K3 surfaces. In fact, as elliptic K3 surfaces admit polarizations of any degree, we may assume that $X_1 = X_2 = X$ and $\sigma + n_i f = \ell_i/\xi$, for σ divisor class of a section and

f divisor class of a fibre. Now, the condition $\langle v_1, v_1\rangle = \langle v_2, v_2\rangle$ implies

$$\begin{aligned}
\ell_1^2 - 2rs_1 &= \ell_2^2 - 2rs_2 \\
\xi^2(\sigma + n_1 f)^2 - 2rs_1 &= \xi^2(\sigma + n_2 f)^2 - 2rs_2 \\
\xi^2(2n_1 - 2) - 2rs_1 &= \xi^2(2n_2 - 2) - 2rs_2 \\
\xi^2 n_1 - rs_1 &= \xi^2 n_2 - rs_2 \\
r(s_2 - s_1) &= \xi^2(n_2 - n_1).
\end{aligned}$$

Consider now the difference

$$\begin{aligned}
v_2 - v_1 &= (0, \ell_2 - \ell_1, s_2 - s_1) \\
&= (0, \xi(\sigma + n_2 f) - \xi(\sigma + n_1 f), s_2 - s_1) \\
&= (0, (n_2 - n_1)\xi f, s_2 - s_1) \\
&= \left(0, \frac{(s_2 - s_1)rf}{\xi}, s_2 - s_1\right) \\
&= (r, \xi(\sigma + n_1 f), a_1).\left(0, \frac{(s_2 - s_1)f}{\xi}, 0\right).
\end{aligned}$$

As $f^2 = 0$, we obtain that

$$v_2 = v_1.\mathrm{ch}\left(\frac{(s_2 - s_1)f}{\xi}\right).$$

Then, if $H_1 = H_2$, we are done. If $H_1 \neq H_2$, consider a smooth family $(\mathcal{X}, \mathcal{L}) \to T$ of K3 surfaces over a smooth connected curve T such that the fibre $(\mathcal{X}_{t_0}, \mathcal{L}_{t_0})$ over t_0 has Picard number 1 and (X, ℓ_1) is the fibre over t_1. Again by Lemma 4.2.5, the moduli spaces $\mathbf{M}^s_{\mathcal{X}_{t_0},\mathcal{L}_{t_0}}(v_1)$ and $\mathbf{M}^s_{X,H_1}(v_1)$ are deformation equivalent. This concludes the proof. □

Now, we show how to reduce the proof of Theorem 4.2.1 to the case of elliptic K3 surfaces. Let (X, H) be a polarized K3 surface and let $v = (r, \ell, s) \in \tilde{H}^{1,1}(X, \mathbb{Z})$ be a primitive Mukai vector for which H is v-general. The idea is to produce an elliptic K3 surface Y and a Mukai vector $w \in \tilde{H}^{1,1}(Y, \mathbb{Z})$ satisfying the hypotheses of Proposition 4.2.6. Let $\xi = \text{g.c.d.}(r, \ell)$ and write $v = (\xi r', \xi \ell', s)$. Define

$$\begin{aligned}
n &\coloneqq r'k - \frac{\ell'^2}{2} \\
b &\coloneqq s - \xi r' - \xi k
\end{aligned}$$

for $k \in \mathbb{Z}$. By choosing k large enough, we may assume that $n > 1$ and that b and r are relatively prime (recall that ξ and s are coprime). Let Λ' be an even rank 3 lattice whose intersection matrix is given by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2n \end{pmatrix}.$$

This lattice admits an embedding into the K3 lattice Λ. By surjectivity of the period map, there exists a K3 surface Y whose Picard lattice is isomorphic to Λ'. Since Λ' contains a copy of the hyperbolic lattice U, by Lemma 4.2.3 Y is an elliptic K3 surface. Call σ the class of a section, f the class of a fibre and ζ_n the last basis class, with $(\zeta_n, \sigma) = (\zeta_n, f) = 0$ and $\zeta_n^2 = -2n$. As $n > 1$, all the fibres are irreducible. Set

$$w = (\xi r', \xi(f - \zeta_n), b + \xi r') = (r, \xi(f - \zeta_n), s - \xi k).$$

Then

$$\begin{aligned}\langle w, w\rangle &= (\xi(f - \zeta_n))^2 - 2r(s - \xi k) = -2n\xi^2 - 2rs + 2r\xi k \\ &= (-2r'k + \ell'^2)\xi^2 - 2rs + 2r\xi k = -2r\xi k + \ell^2 - 2rs + 2r\xi k = \langle v, v\rangle.\end{aligned}$$

It is easy to show that other conditions of Proposition 4.2.6 hold. Hence, choosing H' w-general, one concludes by Proposition 4.2.6 that $\mathbf{M}^s_{X,H}(v)$ and $\mathbf{M}^s_{Y,H'}(w)$ are deformation equivalent.

4.3 Singular moduli spaces

This section is devoted to present a result of the beautiful paper [16] by Kaledin, Sorger, Lehn. Let $v \in \tilde{H}(X, \mathbb{Z})$ and decompose it as $v = mv_0$, where v_0 is primitive and $m \in \mathbb{N}_0$ is a multiplicity. We shall assume that $v_0 = (r_0, \ell_0, s_0)$ has the following properties

$$(*) \begin{cases} \text{Either } r_0 > 0 \text{ and } \ell_0 \in \mathrm{NS}(X) \\ \quad \text{or } r_0 = 0, \ell_0 \in \mathrm{NS}(X) \text{ is effective and } s_0 \neq 0; \\ \langle v_0, v_0\rangle \geq 2. \end{cases}$$

Double is the motivation for the assumption $(*)$. On the one hand, by the discussion in section 4.1, it guarantees that the set of v-walls is either locally finite or finite, and hence that v-general ample divisors exist. On the other, by Theorem 4.2.1, $(*)$ implies that the moduli space $\mathbf{M}_H(v_0)$ is non-empty, except when $r_0 = 0$ and ℓ_0 is not ample. Actually, as reported in [16], Yoshioka, in an unpublished note, has proven non-emptyness of the moduli space even if $r_0 = 0$ and ℓ_0 is not ample.

Given a Mukai vector v satisfying assumptions $(*)$ and a v-general ample divisor H, the following situations can occur for the moduli space $\mathbf{M}_H(v)$.

(A) $m = 1$: Yoshioka Theorem 4.2.1 states that $\mathbf{M}_H(v_0)$ is a smooth symplectic variety that is deformation equivalent to $\mathbf{Hilb}^{\frac{1}{2}\langle v_0, v_0\rangle + 1}(X)$.

(B) $m = 2$ and $\langle v_0, v_0\rangle = 2$: the moduli space $\mathbf{M}_H(v)$ has dimension 10 and its singular locus has codimension 2. By blowing up the singular locus, we obtain a projective symplectic resolution of the singularities. The O'Grady examples fall in this case.

(C) $m \geq 3$ or $m = 2$ and $\langle v_0, v_0 \rangle \geq 4$: $\mathbf{M}_H(v)$ has singular locus of codimension ≥ 4 and does not admit a projective symplectic resolution of the singularities.

Our aim is to give reason for the lack of a projective symplectic resolution in case (C). Therefore, even if some statements hold in a more general context, from now on we shall assume that

$$m \text{ and } v_0 \text{ satisfy conditions of case (C).}$$

With respect to the original article [16], under this assumption, results on the local structure of the moduli space are easier to state (one does not have to take into account the exceptions, which correspond to case (B)). For this observation, I am deeply indebted with Y. Lin; to his notes [19], again in the context of the MIT-NEU graduate student seminar, also our approach to regularity results owes a lot.

4.3.1 Local description of the moduli space

In this section we present some regularity results. We will need conditions S_k of Serre and regularity R_k in codimension k, which we recall here for the convenience of the reader.

(S_k): A ring A satisfies condition S_k if, for every prime ideal $\mathfrak{p} \subset A$,

$$\text{depth } A_{\mathfrak{p}} \geq \min\{k, \text{ht}(\mathfrak{p})\}.$$

(R_k): A ring A satisfies condition R_k if, for every prime ideal $\mathfrak{p} \subset A$ of height $\text{ht}(\mathfrak{p}) \leq k$, $A_{\mathfrak{p}}$ is regular.

We shall keep the notations introduced in section 3.2. For any closed point of $\mathbf{M}_H(v)$, choose as a representative of the corresponding S-equivalence class a polystable sheaf $E \in \mathcal{R}^{ss}$. The orbit of $[q\colon \mathcal{H} \to E]$ under the action of $\text{PGL}(V)$ is closed, and the stabilizer subgroup is canonically isomorphic to $\text{PAut}(E) = \text{Aut}(E)/\mathbb{C}^\times$. By Luna's slice theorem (see [23] Proposition 1.23), there exists a $\text{PAut}(E)$-invariant subscheme $S \subset \mathcal{R}^{ss}$ containing $[q]$ such that

$$(\text{PGL}(V) \times S) /\!\!/ \text{PAut}(E) \to \mathcal{R}^{ss} \quad \text{and} \quad S /\!\!/ \text{PAut}(E) \to \mathbf{M}_H(v)$$

are étale. The Zariski tangent space $T_{[q]}S$ is isomorphic to $\text{Ext}^1(E, E)$.

We would like to describe the local structure of S at $[q]$ and of $\mathbf{M}_H(v)$ at $[E]$. To this aim, let $\mathbb{C}[\text{Ext}^1(E, E)]$ be the ring of polynomial functions on the affine space $\text{Ext}^1(E, E)$ and consider its completion

$$A := \mathbb{C}[\text{Ext}^1(E, E)]^\wedge$$

at the maximal ideal $\mathfrak{m}$ of functions vanishing at 0. Let

$$\mathrm{Ext}^2(E,E)_0 = \mathrm{Ker}\left(\mathrm{Ext}^2(E,E) \xrightarrow{\mathrm{tr}} H^2(X,\mathcal{O}_X)\right).$$

The automorphism group $\mathrm{Aut}(E)$ acts on $\mathrm{Ext}^1(E,E)$ and $\mathrm{Ext}^2(E,E)_0$ by conjugation; as the action of scalar multiples of the identity is trivial, we have in fact an action of the projective automorphism group $\mathrm{PAut}(E)$. There exists a $\mathrm{PAut}(E)$-equivariant map, called *Kuranishi map*,

$$\kappa\colon \mathrm{Ext}^2(E,E)_0^\vee \to \mathbb{C}[\mathrm{Ext}^1(E,E)]^\wedge$$

such that, for every linear form $\varphi \in \mathrm{Ext}^2(E,E)_0^\vee$ and $e \in \mathrm{Ext}^1(E,E)$,

$$\kappa(\varphi)(e) = \frac{1}{2}\varphi(e\cup e) + \text{higher order terms in } e.$$

Moreover, letting I be the ideal generated by the image of κ, one has the following ring isomorphisms

$$\hat{\mathcal{O}}_{S,[q]} \simeq A/I \quad \text{and} \quad \hat{\mathcal{O}}_{\mathbf{M}_H(v),[E]} \simeq (A/I)^{\mathrm{PAut}(E)}.$$

Unfortunately, it is really hard to compute explicitly the Kuranishi map and hence to study directly $\hat{\mathcal{O}}_{S,[q]}$ or $\hat{\mathcal{O}}_{\mathbf{M}_H(v),[E]}$. The idea is to pass to the tangent cone to S at $[q]$, where things are easier to describe. This passage corresponds to considering just the quadratic part of the Kuranishi map

$$\begin{aligned}\kappa_2\colon \mathrm{Ext}^2(E,E)_0^\vee &\to S^2(\mathrm{Ext}^1(E,E)^\vee)\\ \varphi &\mapsto \left(e \mapsto \frac{1}{2}\varphi(e\cup e)\right).\end{aligned}$$

This $\mathbb{C}$-linear map determines a morphism of affine spaces

$$\begin{aligned}\mu\colon \mathrm{Ext}^1(E,E) &\to \mathrm{Ext}^2(E,E)_0\\ e &\mapsto \mu(e) = \frac{1}{2}(e\cup e).\end{aligned}$$

The ideal J generated by the image of κ_2 is the ideal of null-fibre $F = \mu^{-1}(0)$. One should think of $F = \mu^{-1}(0)$ as the tangent cone to S at $[q]$ and the vector space $\mathrm{Ext}^2(E,E)_0$ should be considered as the obstruction space.

Remark 4.3.1. Note that, if E is stable, then $\mathrm{Ext}^2(E,E)$ is one-dimensional and the obstruction space automatically vanishes; the tangent cone is then the whole tangent space, and we obtain smoothness of the moduli space at $[E]$.

Remark 4.3.2. The way one may think of the obstruction space is explained in this example. Consider the plane nodal cubic curve $V(y^2-x^2-x^3) \subset \mathrm{Spec}(\mathbb{C}[x,y])$. At the origin $(0,0)$, the tangent cone has equation $y^2-x^2=0$ in the tangent space $\mathbb{C}^2$. Hence, a tangent vector belongs to the tangent cone if and only if it lies in the preimage of 0 under the map

$$\begin{aligned} \mu\colon \mathbb{C}^2 &\to \mathbb{C} \\ (x,y) &\mapsto y^2-x^2. \end{aligned}$$

In this situation, the obstruction space is the target $\mathbb{C}$.

The relation between the ideals $I \subset A$ and $J \subset \mathbb{C}[\mathrm{Ext}^1(E,E)]$ introduced before is as follows. The graded ring $\mathrm{gr}\ A = \bigoplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is canonically isomorphic to $\mathbb{C}[\mathrm{Ext}^1(E,E)]$. With any ideal $\mathfrak{a} \in A$ we can associate the ideal $\mathrm{in}(\mathfrak{a}) \subset \mathrm{gr}\ A$ generated by the initial part $\mathrm{in}(f)$ of all the elements $f \in \mathfrak{a}$. By the expression of κ,

$$J \subset \mathrm{in}(I).$$

Thus, we have the following inequalities:

$$\begin{aligned} \dim(F) &= \dim\ (\mathrm{gr}\ A)/J \\ &\geq \dim\ (\mathrm{gr}\ A)/\mathrm{in}(I) = \dim\ \mathrm{gr}(A/I) = \dim\ (A/I) \\ &\geq \dim\ \mathrm{Ext}^1(E,E) - \dim\ \mathrm{Ext}^2(E,E)_0. \end{aligned} \tag{4.4}$$

The last inequality is due to Krull's theorem as A is of dimension $\dim\ \mathrm{Ext}^1(E,E)$ and I can be generated by $\dim\ \mathrm{Ext}^2(E,E)_0$ elements.

Proposition 4.3.3. *The null-fibre F is an irreducible normal complete intersection of dimension* $\dim\ \mathrm{Ext}^1(E,E) - \dim\ \mathrm{Ext}^2(E,E)_0$. *Moreover, it satisfies R_3.*

Proof. This proposition, which is the technical core of [16], holds more generally for a class of symplectic momentum maps. We cannot enter into details here. We limit ourselves to explain why our situation falls in this class. The polystable sheaf E can be written as

$$E = \bigoplus_{i=1}^{s} W_i \otimes E_i,$$

with stable sheaves E_i and vector spaces W_i of dimension n_i. If we set

$$W_{ij} = \mathrm{Hom}(W_i, W_j) \quad \text{and} \quad V_{ij} = \mathrm{Ext}^1(E_i, E_j),$$

we have decompositions

$$\mathrm{End}(E) = \bigoplus_i W_{ii}, \quad \mathrm{Ext}^1(E,E) = \bigoplus_{i,j} W_{ij} \otimes V_{ij}, \quad \mathrm{Ext}^2(E,E) = \bigoplus_i W_{ii}.$$

The automorphism group

$$\mathrm{Aut}(E) = \prod_i \mathrm{Aut}(W_i) \simeq \prod_i \mathrm{GL}(n_i).$$

acts on $\mathrm{Ext}^1(E,E)$ by conjugation on the first factor in each direct summand. As scalar multiples of the identity act trivially, we have an action of $\mathrm{PAut}(E)$. On $\mathrm{Ext}^1(E,E) = \bigoplus_{i,j} W_{ij} \otimes V_{ij}$ Serre duality yields a symplectic form ω. The summands $W_{ij} \otimes V_{ij}$ and $W_{ab} \otimes V_{ab}$ are orthogonal with respect to ω, unless $i = b$ and $j = a$, in which case

$$\begin{aligned} \omega \colon (W_{ij} \otimes V_{ij}) \otimes (W_{ji} \otimes V_{ji}) &\to \mathbb{C} \\ (A \otimes e) \otimes (A' \otimes e') &\mapsto \mathrm{tr}(A'A)\mathrm{tr}(e' \cup e). \end{aligned}$$

Moreover, in this decomposition, the quadratic map $\mu \colon \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2(E,E)_0$ is given by

$$\mu\left(\sum_{i,j}\sum_k A_{ij}^k \otimes e_{ij}^k\right) = \sum_{i,j}\sum_{k,l} A_{ij}^k A_{ji}^l \mathrm{tr}(e_{ij}^k \cup e_{ji}^l). \tag{4.5}$$

We can recover the target $\mathrm{Ext}^2(E,E)_0$ of the map μ as follows: from the exact sequence

$$0 \to \mathbb{C}^\times \to \prod_i \mathrm{GL}(n_i) \to \mathrm{PAut}(E) \to 0,$$

one deduces a short exact sequence of associated Lie algebras

$$0 \to \mathbb{C} \to \bigoplus_i \mathfrak{gl}(n_i) \to \mathrm{Lie}\ \mathrm{PAut}(E) \to 0.$$

Dualizing and recalling that $\mathfrak{gl}(n_i)^\vee \simeq \mathrm{End}(E_i)^\vee \simeq \mathrm{Ext}^2(E_i,E_i)$, one obtains

$$\begin{aligned} (\mathrm{Lie}\ \mathrm{PAut}(E))^\vee &= \mathrm{Ker}\left(\bigoplus_i \mathfrak{gl}(n_i)^\vee \to \mathbb{C}\right) \\ &= \mathrm{Ker}\left(\mathrm{Ext}^2(E,E) \xrightarrow{\mathrm{tr}} \mathbb{C}\right) \simeq \mathrm{Ext}^2(E,E)_0. \end{aligned}$$

In conclusion, on the affine space $\mathrm{Ext}^1(E,E)$ we have a symplectic form ω which is invariant for the action of $\mathrm{PAut}(E)$. The expression 4.5 says that

$$\mu \colon \mathrm{Ext}^1(E,E) \to (\mathrm{Lie}\ \mathrm{PAut}(E))^\vee$$

is a momentum map. □

From the properties of the tangent cone we can deduce various regularity results on $\mathcal{R}^{ss}$ and $\mathbf{M}_H(v)$.

Proposition 4.3.4. *Let H be a v-general ample divisor and $E = \bigoplus_{i=1}^{s} E_i^{\oplus n_i}$ a polystable sheaf. Consider a point $[q\colon \mathcal{H} \to E] \in \mathcal{R}^{ss}$ and a slice $S \subset \mathcal{R}^{ss}$ to the orbit of $[q]$. Then $\mathcal{O}_{S,[q]}$ is a normal complete intersection domain of dimension*

$$\dim \operatorname{Ext}^1(E,E) - \dim \operatorname{Ext}^2(E,E)_0$$

that has property R_3.

Proof. By Proposition 4.3.3, the null-fibre $F = \operatorname{Spec}(\operatorname{gr} A/J)$ is an irreducible normal complete intersection of dimension $\dim \operatorname{Ext}^1(E,E) - \dim \operatorname{Ext}^2(E,E)_0$. Thus, we have equality at all places of 4.4. Furthermore, since $F = \operatorname{Spec}(\operatorname{gr} A/J)$ is reduced and irreducible, the equality of dimensions implies $J = \operatorname{in}(I)$. It follows that

$$\operatorname{gr}(\hat{\mathcal{O}}_{S,[q]}) = \operatorname{gr}(A/I) = \operatorname{gr}A/\operatorname{in}(I) = \Gamma(F, \mathcal{O}_F)$$

is a normal complete intersection. In particular, $\operatorname{gr}(\hat{\mathcal{O}}_{S,[q]})$ is Cohen-Macaulay, hence satisfies condition S_k for all $k \in \mathbb{N}$. Moreover it is regular in codimension 3. Observing that $\operatorname{gr}(\hat{\mathcal{O}}_{S,[q]}) = \operatorname{gr}(\mathcal{O}_{S,[q]})$, we deduce that also $\operatorname{gr}(\mathcal{O}_{S,[q]})$ has all these properties. By Lemma 4.3.5, $\mathcal{O}_{S,[q]}$ is a normal complete intersection which satisfies R_3. □

Lemma 4.3.5. *Let $(B, \mathfrak{m})$ be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $B/\mathfrak{m} \simeq \mathbb{C}$. Let* gr B *denote the graded ring associated to the $\mathfrak{m}$-adic filtration of B. Then* $\dim(B) = \dim(\operatorname{gr} B)$ *and, if* gr B *is an integral domain or normal or a complete intersection, then the same is true for B. Moreover, if* gr B *satisfies R_k and S_{k+1} for some $k \in \mathbb{N}$, then B satisfies R_k.*

Proposition 4.3.6. *Let H be a v-general ample divisor. Then $\mathcal{R}^{ss}$ is normal and locally a complete intersection of dimension $\langle v, v\rangle + 1 + N^2$. It has property R_3 and hence is locally factorial.*

Proof. Let $[q] \in \mathcal{R}^{ss}$ be a point with closed orbit, and let S be a $\operatorname{PAut}(E)$-invariant slice through $[q]$. By Proposition 4.3.4, the local ring $\mathcal{O}_{S,[q]}$ is a normal complete intersection with property R_3. Being normal or locally a complete intersection or having property R_k are open properties (EGA IV 19.3.3, 6.12.9). Hence, there exists an open neighbourhood U of $[q]$ in S that is normal, locally a complete intersection and has property R_3. The natural morphism $\operatorname{PGL}(V) \times S \to \mathcal{R}^{ss}$ is smooth. Therefore, every closed orbit in $\mathcal{R}^{ss}$ has an open neighbourhood that is normal, locally a complete intersection and with property R_3. Finally, every $\operatorname{PGL}(V)$-orbit of $\mathcal{R}^{ss}$ meets such an open neighbourhood. It follows that $\mathcal{R}^{ss}$ is normal, locally a complete intersection and regular in codimension 3. The following theorem of Grothendieck implies that $\mathcal{R}^{ss}$ is locally factorial. □

Theorem 4.3.7 (Grothendieck, [9], Exp. XI Cor. 3.14)**.** *Let B a noetherian local ring. If B is a complete intersection and regular in codimension ≤ 3, then B is factorial.*

Then, a result of Drezet ([6], Theorem A) implies that

Theorem 4.3.8. *Let H be a v-general ample divisor. Then $\mathbf{M}_H(v)$ is locally factorial.*

4.3.2 Irreducibility

Theorem 4.3.9. *Let X be a projective K3 surface with an ample divisor H. Let $v \in \tilde{H}(X, \mathbb{Z})$. Suppose that $M \subset \mathbf{M}_H(v)$ is a connected component parametrizing stable sheaves only. Then $\mathbf{M}_H(v) = M$.*

Proof. For the complete proof, we have to refer to [16], Theorem 4.1. If a universal family $\mathcal{E}$ exists on $X \times M$, the same proof of Theorem 3.3.18 yields the result. One has to slightly modify the functor Φ and consider

$$\begin{aligned} \Phi \colon \mathcal{D}^b(X) &\to \mathcal{D}^b(M) \\ F &\mapsto p_* \mathcal{H}om(\mathcal{E}, q^* F). \end{aligned}$$

In this way, it is no longer necessary to assume that elements of M are locally free. □

Theorem 4.3.10. *Let H be a v-general ample divisor. Then $\mathbf{M}_H(v)$ is a normal irreducible variety of dimension $2 + \langle v, v \rangle$.*

Proof. By Proposition 4.3.6, R^{ss} is normal, and so is $\mathbf{M}_H(v)$ as a GIT-quotient.

Let us prove irreducibility by induction on the multiplicity m. If $m = 1$, the moduli space $\mathbf{M}_H(v) = \mathbf{M}_H^s(v_0)$, hence it is smooth. By Theorem 4.3.9, $\mathbf{M}_H^s(v_0)$ is irreducible. Condition $(*)$ implies that $\mathbf{M}_H^s(v_0)$ is non-empty. Suppose that $m \geq 2$ and that the statement of the theorem has been proven for all moduli spaces $\mathbf{M}_H(m'v_0)$, $1 \leq m' < m$. For any decomposition $m = m' + m''$ with $1 \leq m' \leq m''$, consider the morphism

$$\begin{aligned} \varphi(m', m'') \colon \mathbf{M}_H(m'v_0) \times \mathbf{M}_H(m''v_0) &\to \mathbf{M}_H(mv_0) \\ ([E'], [E'']) &\mapsto [E' \oplus E''] \end{aligned}$$

and denote by $Y(m', m'')$ its image; it is an irreducible (hence connected) subscheme of $\mathbf{M}_H(mv_0)$ by induction hypothesis. Let us consider the strictly semistable locus of $\mathbf{M}_H(mv_0)$, which is covered by the $Y(m', m'')$, $1 \leq m' \leq m''$. As all of them contain a point of the form $[E_0^{\oplus m}]$, for $[E_0] \in \mathbf{M}_H(v_0)$, the strictly semistable locus is connected. Let $\mathcal{C}$ be the connected component of $\mathbf{M}_H(mv_0)$ that contains the strictly semistable locus; it is irreducible because of normality of $\mathbf{M}_H(mv_0)$. Thus, if $\mathcal{C}$ is the unique connected component, we are done. Assume for a contradiction that there exists another connected component; then it would parametrize stable sheaves only. By Theorem 4.3.9, this last component would equal $\mathbf{M}_H(mv_0)$, a contradiction. □

Remark 4.3.11. The proof works without any change also in case (B).

4.3.3 Singularities

Proposition 4.3.12. *The moduli space $\mathbf{M}_H(v)$ has a non-empty singular locus $\mathbf{M}_H(v)_{\mathrm{sing}}$, which equals the strictly semistable locus. The irreducible components of $\mathbf{M}_H(v)_{\mathrm{sing}}$ correspond to integers m', $1 \leq m' \leq m/2$, and have codimension $2m'(m-m')\langle v_0, v_0\rangle - 2$ respectively. In particular,* $\operatorname{codim} \mathbf{M}_H(v)_{\mathrm{sing}} \geq 4$.

Proof. Keep the same notations introduced in the proof of Theorem 4.3.10. The strictly semistable locus is the union of the varieties $Y(m', m'')$, $1 \leq m' \leq m''$, $m' + m'' = m$. As the maps

$$\varphi(m', m'') \colon \mathbf{M}_H(m'v_0) \times \mathbf{M}_H(m''v_0) \to Y(m', m'') \subset \mathbf{M}_H(mv_0)$$

are finite and surjective,

$$\begin{aligned}
\operatorname{codim}(Y(m', m'')) &= \dim(\mathbf{M}_H(mv_0)) - \dim(Y(m', m'')) \\
&= \dim(\mathbf{M}_H(mv_0)) - \dim(\mathbf{M}_H(m'v_0)) - \dim(\mathbf{M}_H(m''v_0)) \\
&= 2 + m^2\langle v_0, v_0\rangle - (2 + m'^2\langle v_0, v_0\rangle) - (2 + m''^2\langle v_0, v_0\rangle) \\
&= 2m'm''\langle v_0, v_0\rangle - 2.
\end{aligned}$$

Hence, $\operatorname{codim}(Y(m', m'')) \geq 4$.

We know that $\mathbf{M}_H(v)$ is smooth at the points corresponding to stable sheaves. Hence, it remains to prove that the points corresponding to strictly semistable sheaves are singular. Note that, as $\varphi(m', m'')$ is dominant, the image $V(m', m'')$ of the open subset $U(m', m'') = \mathbf{M}_H^s(m'v_0) \times \mathbf{M}_H^s(m''v_0)$ is dense in $Y(m', m'')$. If we prove that the points belonging to $V(m', m'')$ are singular, we are done: indeed, recalling that the singular locus $\mathbf{M}_H(v)_{\mathrm{sing}}$ is closed in $\mathbf{M}_H(v)$, we obtain

$$\begin{aligned}
\bigcup_{m', m''} Y(m', m'') \supset \mathbf{M}_H(v)_{\mathrm{sing}} &\supset \overline{\bigcup_{m', m''} V(m', m'')} \\
&= \bigcup_{m', m''} \overline{V(m', m'')} = \bigcup_{m', m''} Y(m', m'').
\end{aligned}$$

Let $[E = E' \oplus E'']$ be a point in $Y(m', m'')$, E' and E'' being stable sheaves with Mukai vectors $v(E') = m'v_0$ and $v(E'') = m''v_0$ respectively. It is clear that

$$\begin{aligned}
\operatorname{PAut}(E) &= \left(\operatorname{Aut}(E') \times \operatorname{Aut}(E'')\right)/\mathbb{C}^\times \simeq \mathbb{C}^\times, \\
\operatorname{Ext}^2(E, E)_0 &= \operatorname{Ker}(\operatorname{Ext}^2(E, E) \xrightarrow{\mathrm{tr}} \mathbb{C}) \simeq \mathbb{C}.
\end{aligned}$$

Thus, the Kuranishi map is completely described by a $\operatorname{PAut}(E)$-invariant function $f \in \mathbb{C}[\operatorname{Ext}^1(E, E)]^\wedge$. Therefore,

$$\hat{\mathcal{O}}_{\mathbf{M}_H(v),[E]} \simeq \left(\mathbb{C}[\operatorname{Ext}^1(E, E)]^\wedge/(f)\right)^{\operatorname{PAut}(E)} \simeq \left(\mathbb{C}[\operatorname{Ext}^1(E, E)]^\wedge\right)^{\mathbb{C}^\times}/(f).$$

Now, $\mathbb{C}^\times$ acts on the four summands of

$$\mathrm{Ext}^1(E,E) = \mathrm{Ext}^1(E',E') \oplus \mathrm{Ext}^1(E',E'') \oplus \mathrm{Ext}^1(E'',E') \oplus \mathrm{Ext}^1(E'',E'')$$

with weights $0, 1, -1, 0$. Therefore,

$$\mathrm{Ext}^1(E,E) /\!\!/ \mathbb{C}^\times = \mathrm{Ext}^1(E',E') \times \mathcal{C} \times \mathrm{Ext}^1(E'',E''),$$

where $\mathcal{C}$ denotes the cone of matrices of rank ≤ 1 in $M_{d\times d}(\mathbb{C})$ and

$$\begin{aligned} d &= \dim\ \mathrm{Ext}^1(E',E'') \\ &= \frac{1}{2}\left(\dim\ \mathrm{Ext}^1(E,E) - \dim\ \mathrm{Ext}^1(E',E') - \dim\ \mathrm{Ext}^1(E'',E'')\right) \\ &= \frac{1}{2}\left(2\ \dim\ \mathrm{End}(E) + m^2\langle v_0,v_0\rangle - (2+m'^2\langle v_0,v_0\rangle) - (2+m''^2\langle v_0,v_0\rangle)\right) \\ &= m'm''\langle v_0,v_0\rangle \geq 2. \end{aligned}$$

As a consequence, $\hat{\mathcal{O}}_{\mathbf{M}_H(v),[E]}$ is singular, as the quotient of a singular local ring by a non-zero divisor cannot become regular. □

Remark 4.3.13. The same proof holds for case (B). One deduces in particular that $\mathrm{codim}\ \mathbf{M}_H(v)_{\mathrm{sing}} = 2m'm''\langle v_0,v_0\rangle - 2 = 2\cdot 2 - 2 = 2$.

4.3.4 Symplectic resolutions

We need to introduce a bit of terminology.

Definition 4.3.14. A complex manifold X is called *symplectic* if it admits a symplectic form, i.e. a closed holomorphic 2-form ω that is non-degenerate at every point.

The moduli spaces $\mathbf{M}_H(v)$ for $v = v_0$ primitive Mukai vector and H v-general ample divisor provide us with a supply of symplectic manifolds. We would like to extend this definition to the singular case, so to take into account also the other moduli spaces we have met. We proceed as follows. Recall that, for an algebraic variety X, a *resolution of the singularities* of X is a proper birational morphism $\sigma\colon X' \to X$, where X' is smooth; if σ is a projective morphism, we say that the resolution is *projective*.

Definition 4.3.15. A variety X has a *symplectic singularity* at a point x if x admits an open neighbourhood U such that:

(i) U is normal;

(ii) the smooth part U_{reg} of U admits a symplectic 2-form ω;

(iii) for any resolution $\sigma\colon U' \to U$, the pull-back of ω to $\sigma^{-1}(U_{\mathrm{reg}})$ extends to a holomorphic 2-form on U'.

A *symplectic variety* X is a normal variety whose smooth part admits a symplectic form and whose singularities are symplectic.

Note that, if $x \in X$ is a symplectic singularity, the extension of $\sigma^*(\omega)$ given in (iii) may be degenerate at some point out of $\sigma^{-1}(U_{\mathrm{reg}})$. In the case it is non-degenerate at any point of U', it is a symplectic form on U' and we say that the resolution $\sigma\colon U' \to U$ of the singularities is *symplectic*.

Let us go back to case (C).

Theorem 4.3.16. *The moduli space $\mathbf{M}_H(v)$ is a symplectic variety of dimension $2 + \langle v, v\rangle$. The singular locus is non-empty and has codimension at least 4. All singularities are symplectic, but no open neighbourhood of a singular point admits a projective symplectic resolution.*

Proof. We have already seen that the smooth part $\mathbf{M}_H^s(v)$ of the moduli space admits a symplectic form. As the singular locus has codimension ≥ 4, by Flenner's theorem on extendability of differential forms ([7]), all the singularities are symplectic. This proves that $\mathbf{M}_H(v)$ is a symplectic variety.

Now, let $[E] \in \mathbf{M}_H(v)$ be a singular point and $U \subset \mathbf{M}_H(v)$ an open neighbourhood of $[E]$. Assume for a contradiction that there exists a projective symplectic resolution $\sigma\colon U' \to U$ of the singularities of U. Call $Z' \subset U'$ the exceptional locus of σ so that $Z = \sigma(Z') \subset U$ is the singular locus of U. Due to a result of Kaledin ([15] Lemma 2.11), $2 \cdot \mathrm{codim}(Z') \geq \mathrm{codim}(Z)$; the exceptional locus Z' has then codimension ≥ 2. On the other hand, since $\mathbf{M}_H(v)$ is locally factorial, the exceptional locus should be a divisor, a contradiction. □

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