UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Classical-Quantum dualities

Thesis supervisor

Prof. Marco Matone

Candidate

Ideal Majtara

Academic Year 2020/2021

Contents

Introduction

Quantum mechanics appears radically different from classical mechanics and its principles, such as the superposition principle or the uncertainty principle, seem incompatible with classical physics. Usually, these principles are directly encoded in the mathematical structure of quantum mechanics which is based on the algebra of operators in a Hilbert spaces. In the standard approach the classical theory is considered an approximation of quantum mechanics, valid when quantum effects are negligible, and it is formally recovered in the classical limit $\hbar \to 0$.

Actually, when classical mechanics is reformulated in the language of statistical mechanics it shows many similarities with quantum theory. Indeed, in this form a classical theory can be completely reformulated in the mathematical framework of quantum mechanics leading to the problem of analyzing what characteristics of quantum mechanics belong to its formalism, and what are genuine physical effects.

In this thesis we will see that classical mechanics has a deeper structure, which is hidden in the usual phase space formulation. Indeed, when classical theories are formulated in the Hilbert space formalism the formal structure naturally leads to consider a dual approach, where the classical theory reduces to the quantum theory when a suitable limit is taken.

The first indication that this may be possible, comes from the path integral formulation of QFT. In the path integral formulation the probability amplitude for a quantum process is given by assigning to each configuration ϕ a weight $\exp(iS[\phi]/\hbar)$, where $S[\phi]$ is the classical action of the configuration ϕ, and by summing over all possible configurations of the system. In QFT all observable quantities are obtained from vacuum expectation values of the quantum fields that give the correlators of the theory. These correlators can be completely derived using the generating functional of the theory. Given a field theory described by a classical action $S[\phi]$, where $\phi = (\phi_1, \ldots, \phi_n)$ are the fields of the theory, the generating functional is

$$Z[J] = \int \mathcal{D}\phi \ \exp\left(iS[\phi]/\hbar\right) \exp\left(i \int J\phi/\hbar\right) \ , \tag{0.0.1}$$

where J is an external source. The definition of $Z[J]$ corresponds to the definition of the functional Fourier transform of $\exp(iS[\phi]/\hbar)$. Then, it is natural to consider the inverse Fourier transform

$$\exp(iS[\phi]/\hbar) = \int \mathcal{D}J \ Z[J] \exp\left(-i \int J\phi/\hbar\right) \ . \tag{0.0.2}$$

In this way we can consider $\exp(iS[\phi]/\hbar)$ as the generating functional of a dual theory $Z[J]$. The domain of the dual path integral is the space of the sources J. Because fields ϕ are in general distributions this forces J to be a test function (which belongs to Schwartz space for tempered distributions or to the space of smooth functions with compact support). Equation (0.0.2) is essentially the Fourier duality for the path integral. In the path integral formulation of a quantum theory, the classical limit

is obtained using a stationary phase argument: when $\hbar \to 0$, the phase $S[\phi]$ in the path integral oscillates rapidly and only the classical configurations ϕ such that $\delta S[\phi]/\delta\phi = 0$ give a non-vanishing contribution. The fact that $\exp(iS[\phi]/\hbar)$ is given by the classical theory $S[\phi]$ suggests that it may be possible to find a "quantum limit" where the quantum theory is given when we take the limit $\epsilon \to 0$ for some dual constant ϵ. This also gives an alternative way of quantization: starting from the classical theory and then taking the zero order in $\epsilon \to 0$ we "quantize" the theory.

The aim of this thesis is to verify if the heuristic idea, suggested by the path integral formalism, of a "quantum limit", where the classical theory reduces to the quantum one when a dual constant ϵ goes to zero, reversing the standard interpretation in which classical mechanics is seen as an approximation of quantum mechanics, can be realized for general theories and to give a physical interpretation of this limit. This approach also shows how the quantum concepts emerge from a classical context and that, in some cases, the differences between quantum mechanics and classical mechanics are very subtle.

The natural approach to analyze this problem is to consider some formulations of classical mechanics in the mathematical framework of Hilbert spaces where we can directly compare the classical and the quantum structure and from which a path integral can be derived. In the thesis we do an analysis to verify if the formal dual constant ϵ can be considered as a new fundamental constant and if it can introduced with a physical argument that leads naturally to consider the quantum theory as a limit of the classical theory. The physical meaning of this dual limit and its relation with the classical limit and with the quantization procedure will be carefully analyzed.

We observe that obtaining the quantum theory as a limit of the classical one gives an alternative to the usual quantization and may be used to understand in a deeper way the obstruction to quantization. For example, in the context of field theory, if we start with a well defined classical theory and reformulate it in the Hilbert space, using the limit we may obtain a quantum theory free from singularities or, if singularities are present, we may verify if they are present already at the classical level, and in this case they may be due to a "bad choice" of the space in which the theory is formulated, or, if they arise only in the limit in which the theory becomes quantum, we can see directly what is the origin of these singularities. In the literature are present various examples of "quantum limit", however this is usually treated only as a useful approximation and no systematic analysis of this limit is present, in particular for theories which are crucial at a fundamental level, such as gauge theories or gravity. In the thesis we try to give a coherent presentation of the results which concern the idea of a quantum limit and try to extend to general theories these ideas.

The thesis is organized as follows. In the first chapter, we consider the well-know Koopman-von Neumann (KvN) formulation of classical mechanics which expresses the classical theory using an operatorial formalism and substituting the quantum commutation relation $[\hat{x}, \hat{p}] = i\hbar$ with the classical one $[\hat{x}, \hat{p}] = 0$. In this way we can derive a path integral representation for classical mechanics. We show that, after a change of coordinate, the KvN path integral acquires a form very similar to the quantum path integral which is directly related to the so called Wigner-Weyl (WW) formalism.

The WW formalism is then introduced in the second chapter. In this formalism, an isomorphism, called Weyl transform, between functions on the classical phase space and self-adjoint operators in Hilbert space is introduced. This isomorphism is usually used to reformulate quantum mechanics in the classical phase, transporting all the structures of the Hilbert space in equivalent phase space structures. Actually, although less known, also the opposite procedure is possible, thus we can transport the classical commutative structure and the Poisson bracket dynamics in equivalent structures on the Hilbert space. This formulation naturally allows to consider a quantum limit, considering the classical

structure as a deformation of the quantum one, where the role of $\hbar$ is reversed because it measures classical corrections to the quantum theory.

In the third chapter, we analyse the dynamics of several classical systems in the WW formalism to explore their relation with the quantum theory. In particular, we show that the classical harmonic oscillator dynamics coincides with the classical one and we analyse its classical properties. After this, we move to systems with general potentials. We see that for non-quadratic potential another limit, which is not directly related to the $\hbar$ limit, naturally appears, which involve a length scale instead of an action and gives a sort of quantization procedure. In the general case, in particular if a non-trivial metric is present, also a momentum scale appears which forces to consider a joint limit both in position and in momentum, which seems a factorization of the $\hbar$ limit and seems of purely classical nature.

Finally, in the fourth chapter we extend the WW formalism to field theories to see in particular its relation with the Fourier duality (0.0.2) observed in the QFT context.

Chapter 1

Koopman von Neumann formulation of classical mechanics

We start the analysis reformulating the classical theory using a quantum mechanics language. The simplest way is to use the Koopman-von Neumann (KvN) operatorial formulation, that expresses classical mechanics in terms of dynamics of wavefunctions in a Hilbert space, which, however, differs from the quantum Hilbert space. From this formalism it is possible to derive a path integral formulation which has an interesting relation with the ordinary quantum path integral and that naturally leads to consider the WW formulation of classical mechanics.

1.1 From phase space to Hilbert space

Classical mechanics is usually formulated in terms of a Poisson algebra in a phase space manifold M. In this formalism the pure states of the system are the points of M and the observables are smooth functions on M, $F \in C^\infty(M)$. The expectation value of F on a state $x \in M$ is given simply by $F(x)$. The phase space M is endowed with a Poisson bracket which is a bilinear operator $\{\ ,\ \}$ on $C^\infty(M)$ that satisfies the following properties

1. $\{f, g\} = -\{g, f\}$ (skew-symmetry) ,
2. $\{f, gh\} = \{f, g\}h + g\{f, h\}$ (Leibniz rule) ,
3. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ (Jacobi identity) .

Using the Poisson bracket we can define the evolution of an observable f

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} \ , \tag{1.1.1}$$

where H is the hamiltonian function, which defines the dynamics of the system and is a conserved quantity. This formulation is the classical analogue of the Heisenberg picture in quantum mechanics and in fact the properties of the Poisson bracket are the same of the commutator. The key difference is that the algebra of classical observables is commutative while in the quantum case the observables are the self-adjoint operators of an Hilbert Space, that in general do not commute. A concrete representation of the phase space formalism can be obtained introducing coordinates on M. For simplicity, we restrict to the symplectic case, where M is an even-dimensional manifold. This means that it is locally diffeomorphic to $\mathbb{R}^{2n}$ and we can introduce a pair of coordinates (q, p) which are the generalized positions and momenta that label the point of M. It can be shown (Darboux's theorem)

that it is always possible to choose a set of coordinates such that the Poisson bracket reduces to the canonical one

$$\{f,g\} = \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial g}{\partial q^j}\frac{\partial f}{\partial p_j} \ , \tag{1.1.2}$$

where we have used Einstein convention on repeated indices. This result means that, neglecting topological problems, we can always work with $M = \mathbb{R}^{2n}$ and in the following we consider only this case. In canonical coordinates the evolution of the pure states (q,p) are given by Hamilton's equations

$$\frac{\mathrm{d}q^j}{\mathrm{d}t} = \frac{\partial H}{\partial p_j} \ , \ \frac{\mathrm{d}p_j}{\mathrm{d}t} = -\frac{\partial H}{\partial q^j} \ . \tag{1.1.3}$$

This formalism can be easily extended to classical statistical systems characterized by a probability density in phase space $\rho(q,p)$ that gives the probability to find a particle in the state (q,p) if a measurement is performed and that satisfies the following properties

1. $\int dqdp \ \rho(q,p) = 1$,
2. $\rho(q,p) \geq 0$.

The expectation value of an observable f on the mixed state ρ is given by

$$\langle f \rangle_\rho = \int dqdp \ f(q,p)\rho(q,p) \ . \tag{1.1.4}$$

We can embed the set of pure states M in the set of the mixed states given by the probability distributions on M associating to the phase space point (q_0,p_0) the Dirac delta $\delta(q-q_0)\delta(p-p_0)$. Substituting in (1.1.4) we obtain

$$\langle f \rangle_{(q_0,p_0)} = \int dqdp \ f(q,p)\delta(q-q_0)\delta(p-p_0) = f(q_0,p_0) \ . \tag{1.1.5}$$

The evolution of the phase space probability density $\rho(q,p,t)$ is given by the Liouville's theorem which states that ρ satisfies, along any solution $(q(t),p(t))$ of (1.1.3), the Liouville equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\partial \rho}{\partial q^j}\dot{q}^j + \frac{\partial \rho}{\partial p_j}\dot{p}_j + \frac{\partial \rho}{\partial t} = \{\rho, H\} + \frac{\partial \rho}{\partial t} = 0 \ . \tag{1.1.6}$$

The meaning of this equation is that ρ is conserved along the trajectories of the system in phase space. Introducing the liouvillian operator

$$\mathbb{L} = i\{H, \ \} = i\frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j} - i\frac{\partial H}{\partial p_j}\frac{\partial}{\partial q^j} \ , \tag{1.1.7}$$

and denoting $\dot{\rho} = \partial_t \rho$ we can rewrite equation (1.1.5) in the form

$$i\dot{\rho} = \mathbb{L}\rho \ , \tag{1.1.8}$$

which has a structure that resembles the Schrödinger equation.

1.2 Koopman-von Neumann postulates for classical mechanics

The above analysis shows that the statistical formulation of classical mechanics in phase space is very similar to quantum mechanics in Schrödinger picture. Then, it is natural to ask if classical

mechanics, in its statistical version, can be formulated in the formalism of Hilbert spaces. This was done by Koopman and von Neumann in [7] and [8].

In analogy with quantum mechanics we introduce a separable complex Hilbert space $\mathcal{H}$ and denote by $\langle\phi|\psi\rangle$ its inner product. States $|\psi\rangle$ correspond to rays in $\mathcal{H}$, i.e. normalized vectors $\langle\psi|\psi\rangle = 1$ defined up to a phase. Observables $\hat{A}$ are given by the self-adjoint operators on $\mathcal{H}$ and the expectation value on a state $|\psi\rangle$ is defined by

$$\langle\hat{A}\rangle_\psi = \langle\psi|\hat{A}|\psi\rangle \ . \tag{1.2.1}$$

In particular, an observable admits a orthonormal basis of eigenstates $|\lambda_n\rangle$ (in the general case a Dirac completeness) and also in the classical case we interpret the expectation value $\langle\Pi(\lambda_n)\rangle_\psi = |\langle\lambda_n|\psi\rangle|^2$ on the projector $\Pi(\lambda_n) = |\lambda_n\rangle\langle\lambda_n|$ as the probability to get the value λ_n in the state $|\psi\rangle$, that is the Born rule. The time evolution is given by a unitary representation of the time translations $\hat{U}(t)$. By the Stone's theorem we can always find a self-adjoint operator $\hat{L}$ that generates the time translations

$$\hat{U}(t) = \exp(-i\hat{L}t) \ . \tag{1.2.2}$$

The evolution of the states, in Schrödinger picture, is given by $|\psi(t)\rangle = \hat{U}(t)\,|\psi\rangle$. Taking the derivative with respect to t, we obtain the Schrödinger equation

$$i\frac{d}{dt}\,|\psi(t)\rangle = \hat{L}\,|\psi(t)\rangle \ . \tag{1.2.3}$$

The above postulates are the same of quantum mechanics. The difference between the two cases lies in the commutation relations between positions and momenta operators. Considering for simplicity a one-dimensional system, let $\hat{x}$ and $\hat{p}$ be the position and momentum operators respectively. In the quantum case we assume the canonical quantization prescription, that substitutes the classical Poisson bracket $\{\ ,\ \}$ with the commutator $[\ ,\ ]/i\hbar$

$$[\hat{x},\hat{p}] = i\hbar \ . \tag{1.2.4}$$

From the axioms of quantum mechanics (that are also the classical ones) follows that non-trivial commutation between two observables $\hat{A}$ and $\hat{B}$ implies an uncertainty relation between the two observables which limits the accuracy of simultaneous measurements of $\hat{A}$ and $\hat{B}$. In the classical case we can determine with arbitrary precision the initial conditions in phase space (x,p) and this means that, in the Hilbert space formalism, $\hat{x}$ and $\hat{p}$ commute

$$[\hat{x},\hat{p}] = 0 \ . \tag{1.2.5}$$

Another consequence of (1.2.5) is that $\hat{x}$ is not a complete set of commuting observables meaning that given a state $|\psi\rangle$ the spatial wavefunction $\psi(x) = \langle x|\psi\rangle$ does not uniquely determine the state. We obtain a complete set if we include the momentum, obtaining the phase space wavefunction $\psi(x,p) = \langle x,p|\psi\rangle$. We notice that, according to the Born rule, $|\psi(x,p)|^2$ is the probability density to find a particle in (x,p) when a measurement is done on the system, that is the definition of the classical probability density $\rho(q,p)$ in phase space. Thus, we have $\rho(q,p) = |\psi(x,p)|^2$. We can also introduce the Dirac completeness

$$\int dxdp\,|x,p\rangle\langle x,p| = 1 \ . \tag{1.2.6}$$

The fact that $(\hat{x},\hat{p})$ is a complete set of observables leads to a deep difference between the classical case

and the quantum case. Assuming that physical observables are functions of $\hat{x}$ and $\hat{p}$ only, it means that the only pure states are those localized in a phase space point, $\psi(x,p) = \delta(x-x_0)\delta(p-p_0)$. To understand this fact we use the algebraic approach, defining a state ψ as a linear functional on the space of observables whose value $\langle \hat{A} \rangle_\psi$ corresponds to the expectation value of an observable $\hat{A}$ in the state ψ. A state is called mixed if it can be written as a convex combination of other states otherwise is called pure. Physically, a mixed state is a state obtained mixing different ensembles. The uncertainty associated to this state is not intrinsic but epistemic, because depends on our ignorance on the real state of the system. Considering an observable of the form $f(\hat{x},\hat{p})$, its expectation value on a generic state $|\psi\rangle$ is

$$\langle f(\hat{x},\hat{p})\rangle_\psi = \langle \psi | f(\hat{x},\hat{p}) | \psi \rangle = \int dxdpdx'dp' \ \psi^*(x',p')\psi(x,p) \langle x',p'|f(\hat{x},\hat{p})|x,p\rangle = \tag{1.2.7}$$

$$= \int dxdp \ |\psi(x,p)|^2 f(x,p) = \int dxdp \ \rho(x,p) f(x,p) = \tag{1.2.8}$$

$$= \int dxdp \ \rho(x,p) \langle f(\hat{x},\hat{p})\rangle_{(x,p)} \ , \tag{1.2.9}$$

where we used $\langle f(\hat{x},\hat{p})\rangle_{(x,p)} = f(x,p)$. This is exactly the rule (1.1.4) and, because ρ is a probability distribution, we have that, in the algebraic sense, ψ is a mixed state because it is a convex combination of the states (x,p), in agreement with the determinism of classical mechanics which implies that, in principle, we can have complete knowledge on the initial conditions of the system.

We now consider equation (1.2.3) in the representation (x,p). In the Heisenberg picture the observables evolve according to Heisenberg equation

$$\hat{A}(t) = \hat{U}^\dagger(t)\hat{A}\hat{U}(t) \rightarrow \ i\frac{\mathrm{d}\hat{A}}{\mathrm{d}t} = [\hat{A}(t), \hat{L}] \ . \tag{1.2.10}$$

Consistency with the classical case requires that $\hat{x}$ and $\hat{p}$ satisfy Hamilton's equations (we omit the time dependence)

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = -i[\hat{x},\hat{L}] = \frac{\partial H}{\partial p}(\hat{x},\hat{p}) \ , \tag{1.2.11}$$

$$\frac{\mathrm{d}\hat{p}}{\mathrm{d}t} = -i[\hat{p},\hat{L}] = -\frac{\partial H}{\partial x}(\hat{x},\hat{p}) \ . \tag{1.2.12}$$

Equations (1.2.11) and (1.2.12) imply that the commutator of $\hat{L}$ with $\hat{x}$ and $\hat{p}$ is nonvanishing. Then $\hat{L}$ cannot be a classical observable that is only function of $\hat{x}$ and $\hat{p}$ because they commute. We notice that in the classical case the hamiltonian is not the generator of the time translations, because it is a function of $\hat{x}$ and $\hat{p}$ only. Thus we are forced to introduce two auxiliary parameters $\hat{\lambda}_x$ and $\hat{\lambda}_p$ that are canonically conjugated to $\hat{x}$ and $\hat{p}$ respectively

$$[\hat{x},\hat{\lambda}_x] = i \ , \tag{1.2.13}$$

$$[\hat{p},\hat{\lambda}_p] = i \ , \tag{1.2.14}$$

and the other commutators are vanishing. Using (1.2.13) and (1.2.14) we can get equations (1.2.11) and (1.2.12) by setting

$$\hat{L} = \frac{\partial H}{\partial p}(\hat{x},\hat{p})\,\hat{\lambda}_x - \frac{\partial H}{\partial x}(\hat{x},\hat{p})\,\hat{\lambda}_p \ . \tag{1.2.15}$$

Since in the (x,p) representation $\langle x,p|\hat{x}|\psi\rangle = x\psi(x,p)$ and $\langle x,p|\hat{p}|\psi\rangle = p\psi(x,p)$, in order to satisfy

(1.2.13) and (1.2.14) we have

$$\langle x,p|\hat{\lambda}_x|\psi\rangle = -i\frac{\partial\psi}{\partial x}(x,p) \ , \ \ \langle x,p|\hat{\lambda}_p|\psi\rangle = -i\frac{\partial\psi}{\partial p}(x,p) \ . \tag{1.2.16}$$

The action of $\hat{L}$ on a state $|\psi\rangle$ in the (x,p) representation reads

$$\langle x,p|\hat{L}|\psi\rangle = i\left(\frac{\partial H}{\partial x}\frac{\partial\psi}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial\psi}{\partial x}\right)(x,p) = i\{H,\psi\}(x,p) = \mathbb{L}\psi(x,p) \ , \tag{1.2.17}$$

therefore $\hat{L}$ corresponds to the liouvillian operator of equation (1.2.3) and $\psi(x,p,t)$ satisfies the Liouville equation

$$i\dot{\psi}(x,p,t) = \mathbb{L}\psi(x,p,t) \ . \tag{1.2.18}$$

Because, $\rho(x,p) = |\psi(x,p)|^2$, equation (1.2.18) implies (we omit the dependence on x,p,t)

$$i\dot{\rho} = \psi^*\mathbb{L}\psi + \psi\mathbb{L}\psi^* = \psi^*\{H,\psi\} + \psi\{H,\psi^*\} = \{H,\rho\} \ , \tag{1.2.19}$$

where we have used the Leibniz rule for the Poisson bracket. This also proves that $\rho(x,p) = |\psi(x,p)|^2$ satisfies the classical Liouville equation, implying that the KvN formulation is completely equivalent to the standard phase space approach.

1.3 Classical path integral from the KvN formalism

In order to investigate the relations with quantum mechanics and with the QFT Fourier duality (0.0.2), in this section we want to apply the KvN formalism to derive a path integral representation for classical transition amplitudes. Because the duality (0.0.2) appears in the QFT context, we consider directly field theories. For simplicity, we set $\hbar = 1$.

Consider a classical scalar field ϕ. We associate to this field a field operator $\hat{\phi}$ and its (classical) conjugate momentum $\hat{\nu}$. The classical nature of these fields implies that they commute

$$[\hat{\phi}(\boldsymbol{x},t),\hat{\nu}(\boldsymbol{y},t)] = 0 \ , \forall t \ . \tag{1.3.1}$$

To simplify the notation we denote the integral $\int f(\bar{x})g(\bar{x})\ d^3x$ by $f\cdot g$. To obtain the path-integral we consider a hamiltonian H, of the form

$$H[\hat{\phi},\hat{\nu}] = \frac{1}{2}\hat{\nu}\cdot\hat{\nu} + V[\hat{\phi}] \ . \tag{1.3.2}$$

The equations of motion read

$$\frac{d}{dt}\hat{\phi} = \frac{\delta H}{\delta\nu}[\hat{\phi},\hat{\nu}] = \hat{\nu} \ , \tag{1.3.3}$$

$$\frac{d}{dt}\hat{\nu} = -\frac{\delta H}{\delta\phi}[\hat{\phi},\hat{\nu}] = -\frac{\delta V}{\delta\phi}[\hat{\phi}] \ . \tag{1.3.4}$$

As said in the previous section, we cannot obtain these equations from the commutator of $\hat{\phi}$ and $\hat{\nu}$, hence we introduce some auxiliary fields $\hat{\pi}$ and $\hat{\lambda}$ that are conjugated respectively to $\hat{\phi}$ and $\hat{\nu}$

$$[\hat{\phi}(\boldsymbol{x},t),\hat{\pi}(\boldsymbol{y},t)] = i\delta(\bar{x}-\bar{y}) \ , \tag{1.3.5}$$

$$[\hat{\nu}(\boldsymbol{x},t),\hat{\lambda}(\boldsymbol{y},t)] = i\delta(\bar{x}-\bar{y}) \ , \tag{1.3.6}$$

where (in natural units) π has the dimension of $\dot{\phi}$ and λ has the dimension of ϕ. The generator of the classical time evolution $\hat{L}$ is the liouvillian operator

$$\hat{L} = \hat{\pi} \cdot \hat{\nu} - \frac{\delta V}{\delta \phi}[\hat{\phi}] \cdot \hat{\lambda} \ . \tag{1.3.7}$$

Using $\hat{L}$ we can define the unitary time evolution operator

$$U(t) = \exp(-i\hat{L}t) \ . \tag{1.3.8}$$

Introducing the field in Heisenberg picture as

$$\hat{\phi}(\boldsymbol{x}, t) = U^{\dagger}(t)\hat{\phi}(\boldsymbol{x})U(t) \ , \tag{1.3.9}$$

we get

$$\frac{d}{dt}\hat{\phi} = \frac{[\hat{\phi}, \hat{L}]}{i} = \hat{\nu} \ , \tag{1.3.10}$$

$$\frac{d}{dt}\hat{\nu} = \frac{[\hat{\nu}, \hat{L}]}{i} = -\frac{\delta V}{\delta \phi}[\hat{\phi}] \ , \tag{1.3.11}$$

which are the equations of motion (1.3.3), (1.3.4).

Following the same steps of the quantum case, we can derive the path integral associated to this operatorial formulation. To do this we have to consider paths in the space (ϕ, λ). We notice that this is not the usual phase space but a new configuration space where we have a "gauge" field λ. As in the quantum case, we define a classical vacuum state $|0_{cl}\rangle$ which is invariant under time translation. The quantum fields $\hat{\phi}(\boldsymbol{x}, t), \hat{\lambda}(\boldsymbol{x}, t)$ can be seen, at every time t, as a collection of commuting quantum operators labelled by $\bar{x}$, to which corresponds a Dirac completeness[1]

$$\int d\phi d\lambda \, |\phi, \lambda\rangle \, \langle \phi, \lambda| = 1 \ , \tag{1.3.12}$$

where we have defined the functional measure

$$d\phi d\lambda = \prod_{\boldsymbol{x}} d\phi(\boldsymbol{x}) d\lambda(\boldsymbol{x}) \ . \tag{1.3.13}$$

We start from the transition amplitude

$$\langle \phi', \lambda', t | \phi, \lambda \rangle = \langle \phi', \lambda' | \exp(-i\hat{L}t) \, |\phi, \lambda\rangle \ . \tag{1.3.14}$$

As in the quantum case we do a time slicing, splitting the interval $[0, t]$ in N parts of length $\varepsilon = t/N$

[1] With the ket $|\phi, \lambda, t\rangle$ we mean a common generalized eigenstate of the collection of quantum operators that forms the quantum fields at time t, $\hat{\phi}(\boldsymbol{x}, t), \hat{\lambda}(\boldsymbol{x}, t)$. The eigenvalues relative to this common state form the classical fields $\phi(\bar{x}, t), \lambda(\bar{x}, t)$.

and inserting N Dirac completeness

$$\langle\phi',\lambda'|\exp(-i\hat{L}t)\,|\phi,\lambda\rangle = \lim_{N\to+\infty}\langle\phi',\lambda'|\left[\exp(-i\hat{L}\varepsilon)\right]^N|\phi,\lambda\rangle = \tag{1.3.15}$$

$$= \lim_{N\to+\infty}\langle\phi',\lambda'|\left[\exp\left(-i\varepsilon\hat{\pi}\cdot\hat{\nu}\right)\exp\left(i\varepsilon\frac{\delta V}{\delta\phi}[\hat{\phi}]\cdot\hat{\lambda}\right)\right]^N|\phi,\lambda\rangle = \tag{1.3.16}$$

$$= \lim_{N\to+\infty}\left(\prod_{n=1}^{N-1}\int d\phi d\lambda\right)\prod_{n=0}^{N-1}\langle\phi_{n+1},\lambda_{n+1}|\exp\left(-i\varepsilon\hat{\pi}\cdot\hat{\nu}\right)\exp\left(i\varepsilon\frac{\delta V}{\delta\phi}[\hat{\phi}]\cdot\hat{\lambda}\right)|\phi_n,\lambda_n\rangle\ . \tag{1.3.17}$$

In the previous equations we have denoted $\phi_N=\phi'$, $\phi_0=\phi$, $t_n=n\varepsilon$ and in (1.3.15) we have used the Trotter product formula

$$\exp\left(\hat{A}+\hat{B}\right) = \lim_{N\to+\infty}\left[\exp\left(\hat{A}/N\right)\exp\left(\hat{B}/N\right)\right]^N\ . \tag{1.3.18}$$

To get the path integral we consider the matrix element

$$\langle\phi_{n+1},\lambda_{n+1}|\exp\left(-i\varepsilon\hat{\pi}\cdot\hat{\nu}\right)\exp\left(i\varepsilon\frac{\delta V}{\delta\phi}[\hat{\phi}]\cdot\hat{\lambda}\right)|\phi_n,\lambda_n\rangle = \tag{1.3.19}$$

$$= \int d\nu d\pi\,\langle\phi_{n+1},\lambda_{n+1}|\exp\left(-i\varepsilon\hat{\pi}\cdot\hat{\nu}\right)|\pi,\nu\rangle\,\langle\pi,\nu|\exp\left(i\varepsilon\frac{\delta V}{\delta\phi}[\hat{\phi}]\cdot\hat{\lambda}\right)|\phi_n,\lambda_n\rangle = \tag{1.3.20}$$

$$= \int d\nu d\pi\ \exp\left[i\varepsilon\left(\frac{\delta V}{\delta\phi}[\phi]\cdot\lambda-\pi\cdot\nu\right)\right]\langle\phi_{n+1},\lambda_{n+1}|\pi,\nu\rangle\,\langle\pi,\nu|\phi_n,\lambda_n\rangle = \tag{1.3.21}$$

$$= \int d\nu d\pi\ \exp\left[i\varepsilon\left(\frac{\delta V}{\delta\phi}[\phi]\cdot\lambda-\pi\cdot\nu+\frac{(\phi_{n+1}-\phi_n)}{\varepsilon}\cdot\pi-\frac{(\lambda_{n+1}-\lambda_n)}{\varepsilon}\cdot\nu\right)\right] = \tag{1.3.22}$$

$$= \int d\nu d\pi\ \exp\left[i\varepsilon\left(\frac{\delta V}{\delta\phi}[\phi]\cdot\lambda-\pi\cdot\nu+\dot{\phi}_n\cdot\pi-\dot{\lambda}_n\cdot\nu\right)\right]\ , \tag{1.3.23}$$

where we have defined $\dot{a}_n=\frac{(a_{n+1}-a_n)}{\varepsilon}$.

To integrate out the momenta (ν,π) is convenient to change coordinates defining

$$\chi_+ = \frac{\pi+\nu}{\sqrt{2}}\ ,\ \chi_- = \frac{\pi-\nu}{\sqrt{2}}\ , \tag{1.3.24}$$

$$\pi = \frac{\chi_++\chi_-}{\sqrt{2}}\ ,\ \nu = \frac{\chi_+-\chi_-}{\sqrt{2}}\ . \tag{1.3.25}$$

Substituting in (1.3.23) we get

$$\int d\nu d\pi\ \exp\left[i\varepsilon\left(\frac{\delta V}{\delta\phi}[\phi]\cdot\lambda-\pi\cdot\nu+\dot{\phi}_n\cdot\pi-\dot{\lambda}_n\cdot\nu\right)\right] = \tag{1.3.26}$$

$$= \int d\chi_+ d\chi_-\ \exp\left\{i\varepsilon\left[\frac{\delta V}{\delta\phi}[\phi]\cdot\lambda-\frac{1}{2}\left(\chi_++\frac{\sqrt{2}}{2}(\dot{\lambda}_n-\dot{\phi}_n)\right)^2+\frac{1}{2}\left(\chi_-+\frac{\sqrt{2}}{2}(\dot{\lambda}_n+\dot{\phi}_n)\right)^2-\dot{\lambda}_n\cdot\dot{\phi}_n\right]\right\}\ , \tag{1.3.27}$$

which is a gaussian integral. Integrating and substituting in (1.3.16) we finally obtain the path integral

$$\int\limits_{\substack{\phi\to\phi'\\ \lambda\to\lambda'}}\mathcal{D}\phi\mathcal{D}\lambda\ \exp\left(i\int_0^t\frac{\delta V}{\delta\phi}[\phi_t]\cdot\lambda_t-\dot{\phi}_t\cdot\dot{\lambda}_t\ dt\right) = \int\limits_{\substack{\phi\to\phi'\\ \lambda\to\lambda'}}\mathcal{D}\phi\mathcal{D}\lambda\ \exp\left[i\int_0^t\left(\ddot{\phi}_t+\frac{\delta V}{\delta\phi}[\phi_t]\right)\cdot\lambda_t\right]\ . \tag{1.3.28}$$

We observe that in the path integral weight appear the classical equations of motion $\ddot{\phi}+\frac{\delta V}{\delta\phi}[\phi]$. We

denote, as in the spatial case, $A \cdot B = \int d^4x A(x)B(x)$. The classical path integral takes the form

$$\int\limits_{\substack{\phi\to\phi'\\ \lambda\to\lambda'}} \mathcal{D}\phi\mathcal{D}\lambda \ \exp\left(i\frac{\delta S}{\delta\phi}[\phi]\cdot\lambda\right) \ . \tag{1.3.29}$$

The path integral derived here is related to the paths from (ϕ, λ) to (ϕ', λ'). Using the Gell-Mann and Low theorem we can relate this to the vacuum-vacuum amplitude[2] $\langle 0_{cl}|0_{cl}\rangle$ that coincides with (1.3.29) in the limit[3] $t \to +\infty$ and assuming an adiabatic interaction. In this limit the path integral becomes independent of the boundary conditions and we obtain the classical partition function

$$Z_{cl} = \langle 0_{cl}|0_{cl}\rangle = \int \mathcal{D}\phi\mathcal{D}\lambda \ \exp\left(-i\frac{\delta S}{\delta\phi}[\phi]\cdot\lambda\right) = \int \mathcal{D}\phi \ \delta\left(\frac{\delta S}{\delta\phi}[\phi]\right) = \tag{1.3.30}$$

$$= \int \mathcal{D}\phi \ \delta(\phi - \phi_{cl}) \det\left[\frac{\delta^2 S}{\delta\phi\delta\phi}[\phi_{cl}]\right]^{-1} \ , \tag{1.3.31}$$

where ϕ_{cl} is the classical path, that solves the classical equations of motion. The weight of the classical path integral is, up to a constant, just a Dirac delta localized on the classical path, as the classical path integral formulated in [5]. This means that only the classical path contributes, as expected. The term $\det\left[\frac{\delta^2 S}{\delta\phi\delta\phi}[\phi_{cl}]\right]^{-1}$ is just a constant[4] and is related to the so called Van Vleck-Pauli-Morette factor

$$\sqrt{-\frac{1}{2\pi i\hbar}\frac{\partial^2 S}{\partial x_i \partial x_f}[x_{cl}]} \ , \tag{1.3.32}$$

which appears in the semiclassical approximation of the path integral in quantum mechanics. In the free case, this corresponds to the well-known factor

$$\left(\frac{m}{2\pi i\hbar t}\right)^{n/2} \ , \tag{1.3.33}$$

where n is the dimension of the configuration space. The above analysis shows that the square of this factor is present already at the classical level and suggests that quantum mechanics requires a "factorization" of the classical dynamics as we show in the next section.

1.4 Relation with the quantum path integral

We want to see the relation between the classical partition function Z_{cl} and the quantum one. We start considering the expression of the classical partition function where λ is not integrated out

$$Z_{cl} = \int \mathcal{D}\phi\mathcal{D}\lambda \ \exp\left(i\frac{\delta S}{\delta\phi}[\phi]\cdot\lambda\right) \ . \tag{1.4.1}$$

We observe that the phase of the path integral has the form of the first order Taylor expansion around the configuration ϕ with increment given by the field λ. We show that this indeed is related to the quantum partition function if λ is treated as a "small" parameter, in a suitable sense. Let Z be the

[2]The nature of this classical vacuum is not clear at this stage. In the following chapters we will show that for free theories it essentially coincide with the quantum Fock vacuum. In the interacting case we will see that it is related to the classical Boltzmann distribution.

[3]It is convenient to replace the interval $[0, t]$ with $[-t, t]$ to have a symmetric limit.

[4]This is true if, given the boundary conditions, the classical solution is unique, otherwise we have to sum over all solutions. We ignore here problems which arise if multiple solution are present.

quantum partition function

$$Z = \int \mathcal{D}\phi \ \exp\left(iS[\phi]\right) \ , \tag{1.4.2}$$

and consider the quantum probability associated to this amplitude, i.e. its square modulus

$$|Z|^2 = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \ \exp\left(iS[\phi_1] - iS[\phi_2]\right) \ . \tag{1.4.3}$$

Note that the change of variables

$$\phi = \frac{\phi_1 + \phi_2}{2} \ , \quad \lambda = \phi_1 - \phi_2 \ , \tag{1.4.4}$$

$$\phi_1 = \phi + \frac{\lambda}{2} \ , \quad \phi_2 = \phi - \frac{\lambda}{2} \ , \tag{1.4.5}$$

leads to the following equation for the quantum probability

$$|Z|^2 = \int \mathcal{D}\phi \mathcal{D}\lambda \ \exp\left(iS[\phi + \lambda/2] - iS[\phi - \lambda/2]\right) \ . \tag{1.4.6}$$

This equation shows that the quantum probability (square of the amplitude) is given by considering a configuration ϕ and looking at all the possible fluctuations λ around ϕ. In particular, the weight of the path integral is given by the difference of the action between the extrema of a "segment" with midpoint ϕ and length λ. The form of (1.4.6) is similar to the form of the classical partition function Z_{cl} given by (1.4.1) and indeed there is a simple relation between them. We consider the phase of the quantum probability

$$S[\phi + \lambda/2] - S[\phi - \lambda/2] \ , \tag{1.4.7}$$

and we treat λ as a "small" parameter, $\lambda << \phi$. Therefore, we can expand equation (1.4.7) around ϕ with respect to λ

$$S[\phi + \lambda/2] - S[\phi - \lambda/2] = S[\phi] - S[\phi] + \frac{\delta S}{\delta \phi} \cdot \lambda + O(\lambda)^3 \ = \frac{\delta S}{\delta \phi} \cdot \lambda + O(\lambda)^3, \tag{1.4.8}$$

which shows that at first order the classical and quantum phases which enter in the path integral are equivalent. However, at this level it is difficult to interpret it as a classical, or quantum, limit because in the path integral we have to integrate over all possible "fluctuations" λ and this means that the assumption $\lambda << \phi$ in general is not satisfied. The approximation shows that the classical and quantum theories are equivalent in the regime in which λ is small which can be possible only if in the path integral we restrict to observable quantities which are, in some sense, localized in λ. We can also do the opposite procedure expressing the classical partition function in terms of the "quantum" variables ϕ_1, ϕ_2

$$Z_{cl} = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \ \exp\left(i \frac{\delta S}{\delta \phi} \left[\frac{\phi_1 + \phi_2}{2}\right] \cdot \lambda\right) \ . \tag{1.4.9}$$

It is interesting to consider (1.4.9) in the case of particle mechanics. Consider a classical particle with action

$$S[x] = \int_0^t \frac{1}{2} m \dot{x}^2 - V(x) \ dt \ . \tag{1.4.10}$$

Setting, (λ_x, x) at $t = 0$ and (λ_y, y) at t as boundary conditions, we have that the path integral representation of the classical propagator $K(\lambda_x, x, \lambda_y, y, t)$ is

$$K(\lambda_x, x, \lambda_y, y, t) = \int\limits_{\substack{x\to y \\ \lambda_x\to\lambda_y}} \mathcal{D}z\mathcal{D}\lambda_z \ \exp\left(i\int_0^t \frac{\delta S}{\delta z}\,[z]\,\lambda\ dt\right)\ . \tag{1.4.11}$$

Introducing coordinates z_1 and z_2, analogous to the fields ϕ_1 and ϕ_2

$$z = \frac{z_1 + z_2}{2}\ , \quad \lambda = z_1 - z_2\ , \tag{1.4.12}$$

$$z_1 = z + \frac{\lambda}{2}\ , \quad z_2 = z - \frac{\lambda}{2}\ , \tag{1.4.13}$$

and making the change of coordinates in the path integral, we obtain

$$K_{cl}(x_1, y_1, x_2, y_2, t) = \int\limits_{\substack{x_1\to y_1 \\ x_2\to y_2}} \mathcal{D}z_1\mathcal{D}z_2 \ \exp\left(i\int \frac{\delta S}{\delta z}\left[\frac{z_1 + z_2}{2}\right](z_1 - z_2)\ dt\right)\ . \tag{1.4.14}$$

Using (1.4.10) we can calculate explicitly the variation of S around a path z. For an increment ϵ, which is a path $\epsilon(t)$ such that $\epsilon(0) = \epsilon(t) = 0$, the variation of the action around the path z is

$$\delta S[z]\epsilon = \frac{\mathrm{d}}{\mathrm{d}\lambda}S[z + \lambda\epsilon]\Big|_{\lambda=0} = \int_0^t m\dot{x}\dot{\epsilon} + V'(x)\epsilon\ dt \tag{1.4.15}$$

and by definition of functional derivative we have

$$\delta S[z]\epsilon = \int_0^t \frac{\delta S}{\delta z}[z]\epsilon\ dt\ , \tag{1.4.16}$$

which implies that (1.4.15) is exactly the argument of the exponential in (1.4.14) for an increment $\epsilon = z_1 - z_2$ in a point $z = (z_1 + z_2)/2$. We observe that, by definition of increment, we are forced to impose the boundary conditions $x_1 = x_2 = x$ and $y_1 = y_2 = y$ in order to have $z_1 = z_2$ at the boundary, meaning that the fluctuation $\epsilon = z_1 - z_2$ vanishes at the boundary. Substituting in (1.4.14) we obtain

$$K_{cl}(x_1, y_1, x_2, y_2, t) = \int\limits_{x\to y} \mathcal{D}z_1\mathcal{D}z_2 \ \exp\left(i\int \frac{1}{2}m(\dot{z}_1^2 - \dot{z}_2^2) + V'\left(\frac{z_1 + z_2}{2}\right)(z_1 - z_2)\ dt\right)\ . \tag{1.4.17}$$

In this case it is natural to interpret $(x + y)/2$ as the center of a region of radius $|x - y|$ in which the particle is localized . We observe that the kinetic part is the same of the quantum case (considering the modulus square of the amplitude) and this is due to the fact that the kinetic term is quadratic in $\dot{z}$. Indeed, for a quadratic potential we have $V'[z_1 + z_2)/2](z_1 - z_2) = V(z_1) - V(z_2)$ and the classical dynamics coincides with the quantum one.

In the following we will see that for quadratic hamiltonians the dynamics of the classical theory is completely equivalent to the quantum one, in particular this means that also free classical field theories are dynamically equivalent to the quantum one, the difference arises when a non trivial interaction is introduced. We notice that for a generic potential in the approximation of "small distances" $(x-y) << L$, for some length scale L, this term reduces to the quantum one as previously observed. This approximation seems different from the ordinary classical limit which is usually considered to relate the quantum theory with the classical one. In this formalism, the two theories seems equivalent, we

can start with one theory and in the small λ regime we approach the other. This fact conceptually resembles a sort of "universality" of the theories, classical or quantum, which becomes the same in the "infrared" limit $\lambda \to 0$. In the case of particle mechanics $\lambda = (x - y)$ meaning that this limit involves a length scale. It is natural to ask if there is a relation between this approximation and the standard classical limit $\hbar \to 0$.

We will show that a dual limit, in which the classical theory reduces to the quantum one when $\hbar \to 0$, exists, but the approximation (1.4.8) is not equivalent to this dual limit and has a different nature.

The key property of the quantum path integral which gives the transition probability is that it can be factorized in two path integrals which give the probability amplitude. This is the property which characterizes quantum dynamics and allows to restrict the dynamics in such a way that a wavefunction description in the quantum Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ is possible. In the classical case, this factorization cannot be obtained in general, in particular, the dynamics is defined only on the "square" of $\mathcal{H} \simeq L^2(R^{2n})$, as we will see in the next chapter using Wigner-Weyl formalism from which the path integral (1.4.14) can be directly derived and which makes more clear the meaning of the formal approximation (1.4.8).

Chapter 2

Wigner-Weyl formalism for classical mechanics

The KvN formalism shows that classical statistical mechanics can be reformulated in a way that strongly resembles quantum mechanics. In particular the only differences between the quantum case and the classical one lies in the commutation relation between $\hat{x}$ and $\hat{p}$ and in the form of the generator of time translations $\hat{L}$ that differs from the ordinary Hamiltonian and requires the introduction of auxiliary parameters. From this picture it emerged that it is possible to generally recover quantum mechanics using the approximation (1.4.8) but the nature of this approximation is unclear and it is not clear if it has a relation with the limit $\hbar \to 0$.

Starting again from the Liouville equation, a more suggestive formulation, from the dynamical point of view, can be given using the so called Wigner-Weyl (WW) formalism which was originally introduced to formulate quantum mechanics in phase space. In the following sections we review the WW formalism for quantum mechanics and then we apply it to classical mechanics. This gives interesting relations between classical and quantum dynamics, it explicitly shows the obstructions that separates classical dynamics from the quantum one and that the limit $\hbar \to 0$ admits a dual interpretation in which it measures "classical corrections" to the quantum dynamics. Finally, it clarifies the nature of the approximation (1.4.8) that we will see gives an alternative "quantization" procedure.

2.1 Wigner-Weyl formalism

In this section we introduce the WW formalism which allows to reformulate quantum mechanics in phase space and can also be used to reformulate classical mechanics in a way which resembles quantum mechanics. For simplicity, in the following we consider a particle with phase space $(q, p) \in \mathbb{R}^2$, the formalism can be easily generalized to arbitrary dimension simply replacing the one-dimensional variables with their higher dimensional counterpart.

The problem of quantization of a classical systems leads Hermann Weyl to introduce a map between functions on the classical phase space M and symmetrized self-adjoint operators on the quantum Hilbert space $\mathcal{H} = L^2(\mathbb{R})$. For this purpose in [9] Weyl introduced the so called Weyl transform

$$\Phi[f] \equiv \hat{f} = \int dadb\ \tilde{f}(a,b) \exp[i(a\hat{x} + b\hat{p})/\hbar] \ , \tag{2.1.1}$$

$$\tilde{f}(a,b) = \frac{1}{(2\pi\hbar)^2} \int dqdp\ f(q,p) \exp[-i(aq + bp)/\hbar] \ , \tag{2.1.2}$$

where $\hat{x}, \hat{p}$ are the quantum position and momentum operators respectively. The Weyl transform asso-

ciates to a function $f(q,p)$ of the phase space a self-adjoint operator $\hat{f}$ which for a polynomial $P(q,p)$ corresponds to the completely symmetrized polynomial operator $Symm[P(\hat{x},\hat{p})]$, which defines the Weyl quantization rule. For example $\Phi(qp) = (\hat{x}\hat{p}+\hat{p}\hat{x})/2, \Phi(qp^2) = (\hat{x}\hat{p}^2+\hat{x}\hat{p}\hat{x}+\hat{p}^2\hat{x})/3$. It is interesting to notice that this transform is invertible giving a completely identification of the mathematical structures of the classical phase space with the structures of the quantum Hilbert space. The inverse transformation is called Wigner transform and is defined by

$$W[\hat{f}](q,p) \equiv f(q,p) \equiv \Phi^{-1}[\hat{f}](q,p) = \int dy \ \exp(-ipy/\hbar) \langle q+y/2|\hat{f}|q-y/2\rangle \ . \tag{2.1.3}$$

The map (2.1.3) was first introduced in [10] by Wigner for density matrices $\hat{\rho}$ to study the quantum corrections to classical statistical mechanics. If it is applied to a pure state $|\psi\rangle\langle\psi|$ the Wigner transform gives the so called Wigner function

$$\psi(q,p) = \int dy \ \exp(-ipy/\hbar)\psi^*(q-y/2)\psi(q+y/2) \ . \tag{2.1.4}$$

that satisfies the property $\int dqdp \ \psi(q,p) = 1$ and is a real function, but in general it is not positive definite so it cannot be interpreted as a probability distribution in phase space and is called a "quasiprobability distribution". Nevertheless, the marginal probability distributions of $\psi(q,p)$ obtained integrating out p or q correspond exactly to the probability of finding a particle with position q or momentum p respectively.

In the computations it is useful to use a more explicit form of the Weyl transform. To do this we consider the integral kernel of an operator $\hat{f}$ of the Hilbert space $\mathcal{H}$ which is a distribution $\hat{f}(x,y)$ such that for every $\psi \in \mathcal{H}$

$$\hat{f}\psi(x) = \int dy \hat{f}(x,y)\psi(y) \ . \tag{2.1.5}$$

In Dirac notation $\langle x|\psi\rangle = \psi(x)$ and

$$\hat{f}\psi(x) = \ \langle x|\hat{f}|\psi\rangle = \int dy \, \langle x|\hat{f}|y\rangle \, \langle y|\psi\rangle \ , \tag{2.1.6}$$

which implies the integral kernel $\hat{f}(x,y)$ corresponds to the (generalized) matrix element $\langle x|\hat{f}|y\rangle$. Using integral kernels the Weyl transform takes the form

$$\Phi[f](x,y) = \frac{1}{2\pi\hbar}\int dp \exp[ip(x-y)/\hbar] f\left(\frac{x+y}{2}, p\right) \ . \tag{2.1.7}$$

The Weyl transform cannot be seen as a quantization map, i.e. the Dirac canonical quantization prescription $[\Phi(f),\Phi(g)] = i\hbar\Phi(\{f,g\})$ is not verified for all functions f,g of the phase space[1]. However, it can be used to give a phase space formulation of quantum mechanics that is completely equivalent to the standard Hilbert space formulation. The observables $\hat{f}$ of the Hilbert space are mapped in phase space function $f(q,p) = W[\hat{f}](q,p)$ called the "Weyl symbol" of $\hat{f}$. The product of two operators $\hat{f}\hat{g}$ is not given by the ordinary pointwise product of their symbols $fg(q,p) = f(q,p)g(q,p)$ because the operator product is not commutative. To reproduce the non-commutative structure on symbols the pointwise product is replaced by a non-commutative and non-local product called "star product"

[1]This is due to the so called "Groenewold–Van Hove theorem" which shows that it is not possible to satisfy the canonical quantization prescription for all polynomials in q and p.

obtained mapping the operator product in phase space.

$$f \star g = W(\Phi[f]\Phi[g]) \ , \tag{2.1.8}$$

and which posses all the properties of the operator composition, in particular it is bilinear and associative. Furthermore, the star product is globally equivalent to the pointwise product, in the following sense

$$\int dxdp \ f \star g(q,p) = \int dqdp \ f(q,p)g(q,p) \ , \tag{2.1.9}$$

By the star product it is also possible to transport the dynamics of the Hilbert space mapping the commutator of the Heisenberg equation in a bracket on phase space called the "Moyal" bracket

$$\{f,g\}_\star = \frac{f \star g - g \star f}{i\hbar} \ , \tag{2.1.10}$$

which makes $(C^\infty(M), \star, \{\ ,\ \}_\star)$ a Poisson algebra and which gives the quantum dynamics on the phase space

$$\dot{f} = \{f, H\}_\star \ , \tag{2.1.11}$$

where H is the hamiltonian of the system. The hermitian conjugate of an operator $\hat{A}^\dagger$ is mapped in the complex conjugate of its symbol

$$W(\hat{A}^\dagger) = W(\hat{A})^* = A^* \ , \tag{2.1.12}$$

in particular we obtain that complex conjugation reverses the order in the star product

$$(f \star g)^* = g^* \star f^* \ . \tag{2.1.13}$$

The expectation value of an observable $\hat{f}$ in a state $\hat{\rho}$ is given by

$$\langle A \rangle_\rho = \mathrm{Tr}\left[\hat{\rho}\hat{A}\right] = \int dx \ \Phi[\rho \star A](x,x) = \frac{1}{2\pi\hbar}\int dqdp \ \rho \star A(q,p) = \frac{1}{2\pi\hbar}\int dqdp \ \rho(q,p)A(q,p) \tag{2.1.14}$$

where in the last passage we have used (2.1.9).

Finally, the star product can be written as the bidifferential operator

$$f \star g = f \exp\left(i\frac{\hbar}{2}\left[\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right]\right) g \ , \tag{2.1.15}$$

which can be expanded in powers of $\hbar$

$$f \star g = fg + i\frac{\hbar}{2}\{f,g\} + O(\hbar^2) \ , \tag{2.1.16}$$

Equation (2.1.16) shows that the star product $\star$ can be seen as a deformation of the pointwise product to which are added symplectic corrections in powers of $\hbar$ that vanish in the limit $\hbar \to 0$ which can be seen as the classical limit. This is also true for dynamics because the Moyal bracket is

$$\{f,g\}_\star = \frac{f \star g - g \star f}{i\hbar} = \{f,g\} + O(\hbar) \ , \tag{2.1.17}$$

implying

$$\lim_{\hbar\to 0}\{f,g\}_\star = \{f,g\}\ , \tag{2.1.18}$$

so that the Moyal bracket reduces to the classical Poisson bracket in the limit $\hbar \to 0$ and the classical dynamics is recovered.

The above limits are a precise mathematical realization of the idea that classical mechanics can be seen as a limit case of quantum mechanics, when the quantum effects are negligible. This is measured by the parameter $\hbar$ which quantifies the quantum corrections to the classical result. Actually, later we show that is possible to do the opposite and see quantum mechanics as a limit case of classical mechanics.

2.2 Classical mechanics in the Wigner-Weyl formalism

Because the Wigner-Weyl map is invertible it allows to transport the phase space structures in completely equivalent Hilbert space structures and vice versa. It is natural to consider the inverse approach of that followed in the formulation of quantum mechanics in phase space, transporting the classical states, and the algebraic structure of the phase space in the Hilbert space. The idea to apply the WW formalism to classical mechanics was first considered, indirectly, in [6] where using a Fourier transform the Schrödinger equation is derived from the Boltzmann equation. Recently a review of this formalism was considered in [3] where it was used to calculate the quantum eigenstates using classical Gibbs ensembles. In that place, to compute these states, the formal approximation (1.4.8) is used in the case of Hamiltonians of the form $H(x,p) = p^2/2m + V(x)$. In [2] it is explicitly showed that classical mechanics can be seen as a deformation of quantum mechanics when $\hbar \to 0$. In this section we show that the WW formalism applies to classical mechanics and in the following section we explore its consequences.

To better understand the relation between quantum mechanics and the WW classical mechanics is useful to consider first the formalism of density matrices in quantum mechanics. A generic state in quantum mechanics is defined as a density matrix i.e. a self-adjoint operator $\hat{\rho}$ in the Hilbert space $\mathcal{H}$ that satisfies the following properties

$$1.\ \mathrm{Tr}\,\hat{\rho} = 1\ , \tag{2.2.1}$$

$$2.\ \langle\psi|\hat{\rho}|\psi\rangle \geq 0, \quad \forall\,|\psi\rangle \in \mathcal{H}\ , \tag{2.2.2}$$

a state $\hat{\rho}$ is pure if $\hat{\rho}^2 = \hat{\rho}$ otherwise it is mixed. Pure states are one-dimensional projectors of the form

$$\hat{P}_\psi = |\psi\rangle\langle\psi|\ , \tag{2.2.3}$$

for some $\psi \in \mathcal{H}$ and they are in a one-to-one correspondence with the rays[2] of the Hilbert space $\mathcal{H}$. The expectation value of an observable $\hat{A}$ on a state $\hat{\rho}$ is given by

$$\left\langle \hat{A} \right\rangle_\rho = \mathrm{Tr}\Big[\hat{\rho}\hat{A}\Big]\ . \tag{2.2.4}$$

The evolution for density matrices can be obtained from the Heisenberg picture where states $\hat{\rho}$ do not evolve and the time evolution given by a hamiltonian $\hat{H}$ for an observables $\hat{A}$ is

$$\hat{A}(t) = \hat{U}^\dagger(t)\hat{A}\hat{U}(t), \quad \hat{U}(t) = \exp\Big(-i\hat{H}t/\hbar\Big). \tag{2.2.5}$$

[2]The image of $\hat{P}_\psi$ is the one-dimensional subspace generated by ψ i.e. is a ray of $\mathcal{H}$.

The expectation value of $\hat{A}(t)$ on the state $\hat{\rho}$ is given by

$$\left\langle \hat{A} \right\rangle_\rho (t) = \left\langle \hat{A}(t) \right\rangle_\rho = \mathrm{Tr}\Big[\hat{A}(t)\hat{\rho}\Big] = \mathrm{Tr}\Big[\hat{\rho}\hat{U}^\dagger(t)\hat{A}\hat{U}(t)\Big] = \mathrm{Tr}\Big[\hat{U}(t)\hat{\rho}\hat{U}^\dagger(t)\hat{A}\Big] \ , \tag{2.2.6}$$

where in the last step we have used the cyclicity of the trace. Therefore, the evolution of a density matrix $\hat{\rho}$ is

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}\hat{U}^\dagger(t) \ . \tag{2.2.7}$$

and taking the time derivative we obtain the von Neumann equation

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)] \ . \tag{2.2.8}$$

For later analysis we derive (2.2.8) in matrix elements. Consider a hamiltonian operator of the form $\hat{H} = \hat{p}^2/2m + \hat{V}(x)$. Equation (2.2.8) becomes

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(t) = \frac{1}{2m}[\hat{p}^2, \hat{\rho}(t)] + [\hat{V}(x), \hat{\rho}(t)] \ . \tag{2.2.9}$$

We have

$$\langle x|\hat{A}\hat{B}|y\rangle = \int dz \, \langle x|\hat{A}|z\rangle \ \langle z|\hat{B}|y\rangle = \int dz \hat{A}(x,z)\hat{B}(z,y) \ . \tag{2.2.10}$$

and

$$\hat{x}(x,y) = \ \langle x|\hat{x}|y\rangle = y\delta(x-y) \ , \tag{2.2.11}$$

$$\hat{p}(x,y) = \ \langle x|\hat{x}|y\rangle = i\hbar\frac{\partial}{\partial y}\delta(x-y) \ . \tag{2.2.12}$$

Substituting (2.2.11), (2.2.12) in (2.2.9) we get

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(x,y,t) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\hat{\rho}\,(x,y,t) + (V(x) - V(y))\,\hat{\rho}\,(x,y,t) \ . \tag{2.2.13}$$

We now consider classical mechanics. The expectation value of a classical observable f on a state ρ is given by

$$\langle f\rangle_\rho = \int dqdp \ \rho(q,p)f(q,p) = \int dqdp \ \rho \star f(q,p) = \int dqdp \ W[\hat{\rho}\hat{f}](q,p) = \tag{2.2.14}$$

$$= \int dqdpdy \ \exp(-ipy/\hbar)\hat{\rho}\hat{f}(q+y/2, q-y/2) = 2\pi\hbar\int dq\hat{\rho}\hat{f}(q,q) = 2\pi\hbar\,\mathrm{Tr}\Big[\hat{\rho}\hat{f}\Big] \ , \tag{2.2.15}$$

that coincides with the quantum mechanics rule up to a normalization factor $2\pi\hbar$. We notice that ρ has the dimension of $\hbar^{-1}$ while $\hat{\rho}$ is dimensionless. It is convenient to make ρ dimensionless inserting a $2\pi\hbar$ normalization factor in the classical expectation value

$$\langle f\rangle_\rho = \frac{1}{2\pi\hbar}\int dqdp \ \rho(q,p)f(q,p) = \mathrm{Tr}\Big[\hat{\rho}\hat{f}\Big] \ . \tag{2.2.16}$$

In this way the classical probability distribution is given by $\rho/2\pi\hbar$. In particular we have

$$\mathrm{Tr}[\hat{\rho}] = \frac{1}{2\pi\hbar}\int dqdp \ \rho(q,p) = 1 \tag{2.2.17}$$

and from the properties of the Weyl transform $\hat{\rho}$ is self-adjoint. However, we have for a $|\psi\rangle \in \mathcal{H}$

$$\langle\psi|\hat{\rho}|\psi\rangle = \mathrm{Tr}[\hat{\rho}\,|\psi\rangle\langle\psi|] = \frac{1}{2\pi\hbar}\int dqdp\ \rho(q,p)\psi(q,p)\ , \tag{2.2.18}$$

that in general can be negative because the Wigner function $\psi(q,p)$ is not positive definite. For this reason $\hat{\rho}$ is called a "quasi-density matrix". This means that $\hat{\rho}$ in general cannot be interpreted as a quantum density matrix. The situation is completely symmetric with respect to the quantum case where the Wigner function does not in general correspond to a classical phase space distribution.

The algebraic structure of classical mechanics is given by the pointwise product

$$fg(q,p) = f(q,p)g(q,p) \tag{2.2.19}$$

and by the Poisson braket

$$\{f,g\} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\ . \tag{2.2.20}$$

Using the Wigner-Weyl transform, the pointwise product is mapped in a commutative product on $\mathcal{H}$

$$\hat{f}\circ\hat{g} = \Phi[W(\hat{f})W(\hat{g})] = \Phi[W(\hat{g})W(\hat{f})]\ , \tag{2.2.21}$$

and the global equivalence between the star product and the pointwise product in the phase space implies that the commutative product $\circ$ is globally equivalent to the operator product in the Hilbert space $\mathcal{H}$

$$\mathrm{Tr}\Big[\hat{f}\circ\hat{g}\Big] = \mathrm{Tr}\Big[\hat{f}\hat{g}\Big]\ . \tag{2.2.22}$$

Finally, the classical Poisson bracket is mapped in a Hilbert space Poisson bracket

$$[\![\hat{f},\hat{g}]\!] = i\hbar\Phi\left[\Big\{W(\hat{f}),W(\hat{g})\Big\}\right]\ , \tag{2.2.23}$$

which has the same properties of the classical Poisson bracket, in particular it is bilinear, skew-symmetric and satisfies

$$1.\ [\![\hat{f},\hat{g}\circ\hat{h}]\!] = [\![\hat{f},\hat{g}]\!]\circ\hat{h} + \hat{g}\circ[\![\hat{f},\hat{h}]\!] \quad \text{(Leibniz rule)}\ , \tag{2.2.24}$$

$$2.\ [\![\hat{f},[\![\hat{g},\hat{h}]\!]]\!] + [\![\hat{g},[\![\hat{h},\hat{f}]\!]]\!] + [\![\hat{h},[\![\hat{f},\hat{g}]\!]]\!] = 0 \quad \text{(Jacobi identity)}\ , \tag{2.2.25}$$

and which in general differs from the commutator $[\hat{f},\hat{g}]$.

To obtain the dynamics it is useful to consider the following properties of the Weyl transform that follow by the properties of the Fourier transform and by the chain rule for derivatives

$$\Phi\left[\frac{\partial f}{\partial p}\right](x,y) = -\frac{i}{\hbar}(x-y)\Phi[f](x,y)\ , \tag{2.2.26}$$

$$\Phi\left[\frac{\partial f}{\partial q}\right](x,y) = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\Phi[f](x,y)\ , \tag{2.2.27}$$

$$\Phi[pf](x,y) = -\frac{i\hbar}{2}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\Phi[f](x,y)\ , \tag{2.2.28}$$

$$\Phi[qf](x,y) = \frac{1}{2}(x+y)\Phi[f](x,y)\ . \tag{2.2.29}$$

Considering the classical Liouville equation

$$\dot{\rho}(q,p,t) = V'(q)\frac{\partial \rho}{\partial p} - \frac{p}{m}\frac{\partial \rho}{\partial q} \ , \tag{2.2.30}$$

for a hamiltonian $H(q,p) = p^2/2m + V(q)$, and applying the Weyl transform (in matrix elements) to both members we obtain, using the above properties

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(x,y,t) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\hat{\rho}\,(x,y,t) + (x-y)V'\left(\frac{x+y}{2}\right)\hat{\rho}\,(x,y,t) \ . \tag{2.2.31}$$

Equation (2.2.31) is the classical equation of motion in the WW formalism. The structure of this equation is very similar to the von Neumann equation but presents a crucial difference in the potential term because the dependence on x and y cannot be separated. In the following section we do a more sistematic analysis of this equation and study its relation with the quantum dynamics.

2.3 Wigner-Weyl classical dynamics, Hilbert-Schimdt space and path integral representation

As we noticed, the potential term in (2.2.31) seems in some sense non separable, that is, it is not clear if it can be written as a difference of a function of x only and a function of y only. This non separability has an important consequence in deriving a path integral for the WW classical dynamics. In the standard derivation of the path integral in quantum mechanics we consider the matrix element

$$\langle x_f, t_f | x_i, t_i \rangle = \langle x_f | \hat{U}(t_f - t_i) | x_i \rangle \ , \tag{2.3.1}$$

where $\hat{U}(t) = \exp\left(-i\hat{H}t/\hbar\right)$ is the time evolution operator that gives the quantum dynamics $|\psi(t)\rangle = \hat{U}(t)\,|\psi\rangle$ in $\mathcal{H}$. From the time evolution operator we obtain the evolution of density matrices $\hat{\rho}$

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}\hat{U}^\dagger(t) \tag{2.3.2}$$

In particular, for a pure state $|\psi\rangle\langle\psi|$ we have

$$|\psi\rangle\langle\psi|\,(t) = \hat{U}(t)\,|\psi\rangle\langle\psi|\,\hat{U}^\dagger(t) = |\psi(t)\rangle\langle\psi(t)| \ , \tag{2.3.3}$$

meaning that the quantum dynamics sends pure states in pure states. Viceversa, if we start from the von Neumann equation

$$i\hbar\frac{\partial}{\partial t}\hat{\rho}(t) = [\hat{H}, \hat{\rho}(t)] \ , \tag{2.3.4}$$

we can define the quantum liouvillian as $\mathcal{L}_Q\hat{\rho} = [\hat{H}, \hat{\rho}]$ which is a map on the space $\mathcal{B}(\mathcal{H})$ of the bounded operators of $\mathcal{H}$. Therefore, the evolution for a density matrix $\hat{\rho}$ is given by the map

$$\mathcal{U}_Q(t)\hat{\rho} = \exp(-i\mathcal{L}_Q t/\hbar)\rho = \hat{U}(t)\hat{\rho}\hat{U}^\dagger(t) \ , \tag{2.3.5}$$

where we have used the Hadamard formula

$$\exp([A,\ ])X = \exp(A)X\exp(-A) \ . \tag{2.3.6}$$

The dynamics on $\mathcal{H}$ can be obtained using (2.3.3). We can define $\hat{U}(t)$ as the map that sends the ray $|\psi\rangle$ of $\mathcal{H}$ associated to the one-dimensional projector $|\psi\rangle\langle\psi|$ in the ray $|\psi(t)\rangle$ associated to $\mathcal{U}_Q(t)\,|\psi\rangle\langle\psi|$. $\hat{U}(t)$ is well defined precisely because it sends a pure state in a pure state. It is important to notice that we need $\hat{U}(t)$ to derive the standard path integral representation on $\mathcal{H} = L^2(\mathbb{R})$. It is natural to ask if from the classical WW dynamics we can define a time evolution on $\mathcal{H}$. First, consider the classical liouvillian operator in phase space

$$\mathbb{L}\rho = i\hbar\{H, \rho\}\ . \tag{2.3.7}$$

We can map it in a operator on $\mathcal{B}(\mathcal{H})$ using the Wigner-Weyl transforms

$$\mathcal{L}_{Cl}\hat{\rho} = [\![\hat{H}, \hat{\rho}]\!]. \tag{2.3.8}$$

This operator is the classical liouvillian mapped in the quantum Hilbert space $\mathcal{H}$ and its integral kernel corresponds exactly to the right hand side of the classical WW equation (2.2.31). If a dynamics can be defined in $\mathcal{H}$ the map $\mathcal{U}_{Cl} = \exp(-i\mathcal{L}_{Cl}t)$ have to preserve the one-dimensional projector. To verify if this can be true we notice that a one-dimensional projector is an operator of rank-one, i.e. its image is a one-dimensional vector subspace of $\mathcal{H}$ (which is precisely the ray associated to the projector). The maps that preserve rank-one self-adjoint operators are called "rank-one preservers" and are completely classified by the following result (see [11])

Theorem. *Let $\mathcal{H}$ be a complex Hilbert space. If a linear map $\mathcal{U}$ on $\mathcal{B}(\mathcal{H})$ is a rank-one preserver map then it has one of the following forms*

1. $\mathcal{U}\hat{\rho} = \hat{U}\hat{\rho}\hat{U}^\dagger$, *for some $\hat{U}$ linear (or antilinear) bounded operator of $\mathcal{H}$* , (2.3.9)
2. $\mathcal{U}\hat{\rho} = F(\hat{\rho})\,|\psi\rangle\langle\psi|$, *for some linear functional $F : \mathcal{B}(\mathcal{H}) \to \mathbb{R}$ and some $|\psi\rangle \in \mathcal{H}$* . (2.3.10)

The above result implies that the only non trivial way to define a dynamics on $\mathcal{H}$ starting from $\mathcal{B}(\mathcal{H})$ is to use an evolution $\mathcal{U}(t) = \exp(i\mathcal{L}t/\hbar)$ on $\mathcal{B}(\mathcal{H})$ given by a commutator $\mathcal{L} = [\hat{H},\ ]$ for some operator $\hat{H}$ on $\mathcal{H}$. Actually, there is a way to derive a path integral without restricting the dynamics to $\mathcal{H}$ which shows what is the obstruction to get a dynamics on $\mathcal{H}$. To do this, we consider the space of Hilbert-Schmidt vectors $B(\mathcal{H})_{HS} \subset B(\mathcal{H})$. An operator $\hat{A} \in B(\mathcal{H})$ is an Hilbert-Schmidt (HS) vector if

$$\mathrm{Tr}\Big[\hat{A}^\dagger\hat{A}\Big] < +\infty\ , \tag{2.3.11}$$

and $B(\mathcal{H})_{HS}$ admits an inner product

$$\langle A|B\rangle = \mathrm{Tr}\Big[\hat{A}^\dagger\hat{B}\Big]\ , \tag{2.3.12}$$

which makes it an Hilbert space where a ket is an HS vector. A classical state (quasi-density matrix) $\hat{\rho}$ can be mapped in this space taking the square root $|\psi\rangle \equiv \hat{\psi}$ such that $\hat{\psi}^\dagger\hat{\psi} = \hat{\rho}$. This is a normalized state in $B(\mathcal{H})_{HS}$ because

$$\langle\psi|\psi\rangle_{HS} = \mathrm{Tr}\Big[\hat{\psi}^\dagger\hat{\psi}\Big] = \mathrm{Tr}[\hat{\rho}] = 1 \tag{2.3.13}$$

We notice that $\hat{\psi}$ is not self-adjoint because $\hat{\rho}$ is not positive definite in the classical case. The operators

$\hat{A}$ define two representations on $B(\mathcal{H})_{HS}$

$$\hat{A}_L \left|\psi\right\rangle = \hat{A}\hat{\psi} \ , \tag{2.3.14}$$

$$\hat{A}_R \left|\psi\right\rangle = \hat{\psi}\hat{A} \ , \tag{2.3.15}$$

The HS hermitian conjugate of the left and right representations of an operator $\hat{A}$ is

$$\left\langle \hat{A}_L\phi\middle|\psi\right\rangle_{HS} = \mathrm{Tr}\Big[(\hat{A}\hat{\phi})^\dagger\hat{\psi}\Big] = \mathrm{Tr}\Big[\hat{\phi}^\dagger\hat{A}^\dagger\psi\Big] = \ \langle\phi|(\hat{A}^\dagger)_L|\psi\rangle_{HS} \ , \tag{2.3.16}$$

$$\left\langle \hat{A}_R\phi\middle|\psi\right\rangle_{HS} = \mathrm{Tr}\Big[(\hat{\phi}\hat{A})^\dagger\hat{\psi}\Big] = \mathrm{Tr}\Big[\hat{\phi}^\dagger\psi\hat{A}^\dagger\Big] = \ \langle\phi|(\hat{A}^\dagger)_R|\psi\rangle_{HS} \ , \tag{2.3.17}$$

thus it is simply the left and right representation of $\hat{A}^\dagger$ respectively. Using these representations the quantum liouvillian becomes

$$\mathcal{L}_Q \left|\psi\right\rangle = (\hat{H}_L - \hat{H}_R)\left|\psi\right\rangle \ . \tag{2.3.18}$$

We observe that the quantum and the classical liouvillians are HS self-adjoint operators. Indeed, given two HS states $|\psi\rangle$ and $|\phi\rangle$, using the Leibniz rule, in the quantum case we have

$$\langle\mathcal{L}_Q\phi|\psi\rangle_{HS} = \mathrm{Tr}\Big[[\hat{H},\hat{\phi}]^\dagger\psi\Big] = -\,\mathrm{Tr}\Big[[\hat{H},\hat{\phi}^\dagger]\psi\Big] = \mathrm{Tr}\Big[\hat{\phi}^\dagger[\hat{H},\hat{\psi}]\Big] - \mathrm{Tr}\Big[[\hat{H},\hat{\phi}^\dagger\psi]\Big] = \tag{2.3.19}$$

$$= \mathrm{Tr}\Big[\hat{\phi}^\dagger[\hat{H},\hat{\psi}]\Big] = \langle\phi|\mathcal{L}_Q\psi\rangle_{HS} = \ \langle\phi|\mathcal{L}_Q|\psi\rangle_{HS} \ , \tag{2.3.20}$$

where $\hat{\phi}$ and $\hat{\rho}$ are in the domain of $\mathcal{L}_Q$. Analogously, in the classical case

$$\langle\mathcal{L}_{Cl}\phi|\psi\rangle_{HS} = \mathrm{Tr}\left[[\![\hat{H},\hat{\phi}]\!]^\dagger\psi\right] = -\,\mathrm{Tr}\left[[\![\hat{H},\hat{\phi}^\dagger]\!]\circ\psi\right] = \mathrm{Tr}\left[\hat{\phi}^\dagger\circ[\![\hat{H},\hat{\psi}]\!]\right] - \mathrm{Tr}\left[[\![\hat{H},\hat{\phi}^\dagger\circ\psi]\!]\right] \tag{2.3.21}$$

$$= \mathrm{Tr}\left[\hat{\phi}^\dagger[\![\hat{H},\hat{\psi}]\!]\right] = \langle\phi|\mathcal{L}_{Cl}\psi\rangle_{HS} = \ \langle\phi|\mathcal{L}_{Cl}|\psi\rangle_{HS} \ , \tag{2.3.22}$$

where we have used the global equivalence between the commutative product $\circ$ and the operator product. The term $\mathrm{Tr}\left[[\![\hat{H},\hat{\phi}^\dagger\circ\psi]\!]\right]$ vanishes by the assumption that the phase space (quasi)distributions ϕ and ψ vanishes at infinity faster than the derivatives of H

$$\mathrm{Tr}\left[[\![\hat{H},\hat{\phi}^\dagger\circ\psi]\!]\right] = \frac{i}{2\pi}\int dqdp\{H,\phi^*\psi\} = \frac{i}{2\pi}\int dqdp\left[\frac{\partial H}{\partial q}\frac{\partial\phi^*\psi}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial\phi^*\psi}{\partial q}\right] = \tag{2.3.23}$$

$$= \frac{i}{2\pi}\int dqdp\left[\frac{\partial}{\partial p}\left(\frac{\partial H}{\partial q}\phi^*\psi\right) - \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p}\phi^*\psi\right)\right] = \frac{i}{2\pi}\left[\int dq\frac{\partial H}{\partial q}\phi^*\psi\Big|_{p=-\infty}^{p=+\infty} - \int dp\frac{\partial H}{\partial p}\phi^*\psi\Big|_{q=-\infty}^{q=+\infty}\right] = 0 \ . \tag{2.3.24}$$

We denote with $\mathcal{L}_\alpha$ the generic liouvillian, classical $\alpha = Cl$ or quantum $\alpha = Q$. The fact that the liouvillian $\mathcal{L}_\alpha$ is self-adjoint in $\mathcal{H}_{HS}$ implies that the HS evolution operator

$$\mathcal{U}_\alpha(t) = \exp[-i\mathcal{L}_\alpha t/\hbar] \ , \tag{2.3.25}$$

is a HS unitary operator. In the HS space the quantum and classical theories becomes very similar and we can easily derive a path integral representation for classical dynamics[3]. Consider the (generalized)

[3]This path integral representation was first derived in [4], where a different nomenclature is used: the HS formalism is called "Liouville space", the HS vectors are called "superkets" and the HS operators are called "superoperators".

HS vectors $|x,y\rangle = |y\rangle\langle x|$ which form a completeness in $B(\mathcal{H})_{HS}$. Defining

$$\langle x,y|\psi\rangle_{HS} = \hat{\psi}(x,y) = \hat{\rho}^{\frac{1}{2}}(x,y) \ , \tag{2.3.26}$$

$$\langle x,y|\mathcal{V}_{Cl}|\psi\rangle_{HS} = V'\left(\frac{x+y}{2}\right)(x-y)\hat{\psi}(x,y) \ , \tag{2.3.27}$$

$$\langle x,y|\mathcal{V}_{Q}|\psi\rangle_{HS} = [V(x)-V(y)]\hat{\psi}(x,y) \ , \tag{2.3.28}$$

$$\mathcal{K}_{Cl} = \mathcal{K}_{Q} = \frac{1}{2m}\left(\hat{p}_L^2 - \hat{p}_R^2\right) \ , \tag{2.3.29}$$

$$\Delta\mathcal{V} = \mathcal{V}_{Cl} - \mathcal{V}_{Q} \ , \tag{2.3.30}$$

the liouvillian becomes (both in the quantum case and in the classical case)

$$\mathcal{L}_\alpha = \mathcal{K}_\alpha + \mathcal{V}_\alpha \ . \tag{2.3.31}$$

Formally we can consider the von Neumann equation (2.2.13) and the classical WW equation (2.2.31) as a Schrödinger equation in $B(\mathcal{H})_{HS}$ with "hamiltonian" $\mathcal{L}_Q$ and $\mathcal{L}_{Cl}$ respectively. Therefore the matrix element

$$K_{Cl}[x_i,y_i,t_i,x_f,y_f,t_f] = \langle x_f,y_f|\mathcal{U}_{Cl}(t)|x_i,y_i\rangle_{HS} = \langle x_f,y_f|\exp[-i\mathcal{L}_{Cl}(t_f-t_i)/\hbar]|x_i,y_i\rangle_{HS} \ , \tag{2.3.32}$$

which is the WW classical propagator, admits the following path integral representation

$$K_{Cl}[x_i,y_i,t_i,x_f,y_f,t_f] = \int\limits_{\substack{x_i\to x_f\\ y_i\to y_f}} \mathcal{D}x\mathcal{D}y \ \exp\left[i\left(S[x]-S[y]+\Delta S[x,y]\right)/\hbar\right] \tag{2.3.33}$$

where

$$S[x] = \int_{t_i}^{t_f} dt \ \frac{1}{2}m\dot{x}^2 + V(x) \ , \tag{2.3.34}$$

is the classical action and

$$\Delta S[x,y] = \int_{t_i}^{t_f} dt \ \Delta V(x,y) = \int_{t_i}^{t_f} dt \ V'\left(\frac{x+y}{2}\right)(x-y) - V(x) + V(y) \ , \tag{2.3.35}$$

is an "interaction" which measures the obstruction to have a separable dynamics, i.e. to have a liouvillian that is the commutator $[\hat{H},\ ]$ of some hamiltonian operator. Indeed, in the quantum case the term ΔV vanishes and the path integral (2.3.33) can be factorized. We observe that this path integral is exactly the one given by equation (1.4.17) which was derived in the first chapter from the KvN formalism. Equation (2.3.33) suggests that we can obtain a quantum limit eliminating the "non-separability" given by the obstruction ΔS. Thus, we have to analyse under which conditions we can neglect this term. We observe that ΔV vanishes for quadratic potentials which means that at the quadratic level classical and quantum dynamics are equivalent and the difference between them appears when we consider non-linear dynamics.

The non-separability is related to the fact that the classical liouvillian $\mathcal{L}_{Cl}$ has not a commutator structure. Thus, we ask if the classical liouvillian $\mathcal{L}_{Cl}$ can be seen as a deformation of the quantum liouvillian $\mathcal{L}_Q = [\hat{H},\ ]$ for some parameter ϵ in such a way that the limit $\epsilon \to 0$ gives the quantum dynamics. We show that this is possible, at least formally, simply setting $\epsilon = \hbar$. To verify this, we can proceed as done in the case of the Moyal bracket, following essentially the derivation given in [2].

The Moyal bracket is given by the bidifferential operator

$$\{f,g\}_\star = \frac{2}{\hbar} f \sin\left(\frac{\hbar}{2}\left[\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right]\right) g \equiv f P_\star g \ , \tag{2.3.36}$$

where we observe that

$$\{f,g\} = f\left[\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right] g \ . \tag{2.3.37}$$

Formally, we can invert this expression, at operatorial level, obtaining

$$\{f,g\} = \frac{2}{\hbar} f \arcsin\left(\frac{\hbar}{2} P_\star\right) g \ , \tag{2.3.38}$$

We are interested in the first order term. Formally, inverting the sin series we get

$$\arcsin x = x + \frac{x^3}{6} + o(x^5) \ , \tag{2.3.39}$$

and using $x = \hbar P_\star/2$ we obtain

$$\{f,g\} = f P_\star g + \frac{\hbar^2}{24} f P_\star^3 g + O(\hbar^4) \ . \tag{2.3.40}$$

By definition of the WW Poisson bracket (with $\epsilon = \hbar$) and of the Moyal bracket we have

$$\Phi[\{f,g\}] = i\hbar[\![f,g]\!], \quad \Phi[\{f,g\}_\star] = i\hbar[f,g] \ . \tag{2.3.41}$$

Therefore, applying the Weyl transform to both members of (2.3.40) and multiplying by $i\hbar$ we obtain

$$[\![\hat{f},\hat{g}]\!] = [\hat{f},\hat{g}] + O(\hbar^2) \ , \tag{2.3.42}$$

In this way when we take the limit $\hbar \to 0$ we obtain exactly the quantum commutator[4] and we recover the quantum theory. In a similar way we also recover the operator product $\hat{f}\hat{g}$ from the commutative product $\hat{f} \circ \hat{g}$. This is a realization of the quantum limit, where classical mechanics is seen as "exact" and the quantization is given taking the limit $\hbar \to 0$. This suggests that, at the dynamical level, the two theories, classical and quantum, are on equal footing and the limit $\hbar \to 0$ can be approached from both sides, and is a sort of interface between classical and quantum dynamics. The interpretation is similar to the IR limit of the renormalization group, in which different theories in the UV reduce to the same theory in the IR.

Finally, we observe that in the quantum limit $\epsilon \to 0$ the interaction ϵ vanishes and the classical propagator factorizes as

$$K_{Cl}[x_i,y_i,t_i,x_f,y_f,t_f] = K[x_i,x_f,t_i,t_f]K^*[y_i,y_f,t_i,t_f] \ , \tag{2.3.43}$$

where

$$K[x_i,x_f,t_i,t_f] = \int\limits_{x_i \to x_f} \mathcal{D}x \ \exp\left[iS[x]/\hbar\right] \ , \tag{2.3.44}$$

[4]In (2.3.42) we may expect an extra factor $\hbar$ in the higher order terms. Actually applying the Weyl map to $fP_\star^3 g$ one of this factor is absorbed and we get an order $\hbar^2$. For the full computation see [2].

is the quantum propagator associated to the evolution $\hat{U}(t)$ in the quantum Hilbert space $\mathcal{H}$. Thus, in the quantum limit we have that the dynamics of x and y (which essentially corresponds to the dynamics of bras and kets) is decoupled and the dynamics can be restricted, for one-dimensional projectors $|\psi\rangle\langle\psi|$, to the Hilbert space $\mathcal{H}$.

Chapter 3

Classical systems in the Wigner-Weyl formalism

In this section we analyse the classical dynamics of various systems and we investigate the relations between the classical evolution and the quantum one. We start analysing the harmonic oscillator whose dynamics, as seen in the previous chapter, is the same in the classical and in the quantum case. Then, we consider general potentials and we see that there is a natural approximation, related to the formal approximation (1.4.8) encountered in the KvN formalism, which gives a sort of "small-distance" quantum limit alternative to the limit $\hbar \to 0$.

3.1 Classical harmonic oscillator in the WW formalism

We start the analysis considering first the one-dimensional harmonic oscillator. There are several reasons for considering this system. First, we have seen that the quantum and the classical dynamics coincide in the quadratic case. The only difference between the quantum case and the classical case is that the classical states are in general not positive definite as operators in $\mathcal{H}$. Therefore, it is natural to explore the relations between classical and quantum states in this case. Furthermore, the harmonic oscillator is the starting point of quantum field theories thus this is also useful to study the relation between classical and quantum field theories.

The standard analysis of the quantum harmonic oscillator is carried out using the Hilbert space $\mathcal{H}$ of pure states. To see the relations between the classical and the quantum case we have to reformulate the problem using the formalism of (quasi-)density matrix. The hamiltonian operator of an harmonic oscillator is

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \ , \tag{3.1.1}$$

where m is the mass and ω is the frequency of the oscillator. Using $\hat{x}$ and $\hat{p}$ we can define the creation and annihilation operators $\hat{a}^\dagger, \hat{a}$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + i \frac{\hat{p}}{m\omega} \right) \ , \tag{3.1.2}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - i \frac{\hat{p}}{m\omega} \right) \ . \tag{3.1.3}$$

The ground state of $\hat{H}_0$ is the vacuum state $|0\rangle$. The operators $\hat{a}$, $\hat{a}^\dagger$ and the vacuum state $|0\rangle$ satisfy

the following properties

$$1.\ [\hat{H}_0, \hat{a}] = -\hbar\omega\hat{a}, \quad [\hat{H}_0, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger, \quad [\hat{a}, \hat{a}^\dagger] = 1\ , \tag{3.1.4}$$

$$2.\ \hat{a}\,|0\rangle = 0\ . \tag{3.1.5}$$

Using properties (3.1.4) and (3.1.5) follows that the states $|n\rangle = 1/\sqrt{n!}(\hat{a}^\dagger)^n\,|0\rangle$ are eigenstates of $\hat{H}_0$ with eigenvalues $\mathcal{E}_n = \hbar\omega\left(n + \frac{1}{2}\right)$. Therefore the states $|n\rangle$ are discrete excitations of the field vacuum state $|0\rangle$. In the case of free QFT they precisely correspond to the asymptotic particles states. The "Fock vacuum" condition (3.1.5) defines $|0\rangle$ as a pure state in the quantum Hilbert space. This can be reformulated in the language of density matrices using the vacuum density matrix $\hat{\rho}_0 = |0\rangle\langle 0|$. the Fock condition becomes

$$\hat{a}\hat{\rho}_0 = 0 \iff \hat{\rho}_0\hat{a}^\dagger = 0\ . \tag{3.1.6}$$

We observe that $\hat{\rho}_0$ can be interpreted as a classical state. Indeed a quasi-density matrix defines a classical state if its Wigner transform is a semi-positive definite function in phase space. In general the Wigner transform does not preserve positivity of density matrices, however, for pure gaussian state, such as $\hat{\rho}_0$, the Wigner transform is positive definite, thus defines a classical state. We can compute explicitly the classical state corresponding to $\hat{\rho}_0$ using the Wigner transform. The vacuum state in the x-representation is

$$\psi_0(x) = \langle x|0\rangle = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{m\omega x^2}{2\hbar}\right]\ , \tag{3.1.7}$$

and its Wigner transform reads

$$\rho_0(q,p) = W[\hat{\rho}_0](x,p) = \frac{1}{2\pi\hbar}\int dy \exp(-ipy/\hbar)\psi_0(q+y/2)\psi_0^*(q-y/2) = \tag{3.1.8}$$

$$= \frac{1}{\pi\hbar}\exp\left[-\frac{2}{\hbar\omega}\left(\frac{1}{2}m\omega^2 q^2 + \frac{p^2}{2m}\right)\right]\ , \tag{3.1.9}$$

which is the Boltzmann distribution of a classical harmonic oscillator at the temperature $T_0 = \mathcal{E}_0/k_B$, where $\mathcal{E}_0 = \hbar\omega/2$ is the zero-point energy of the quantum vacuum. This implies that we can treat $\hat{\rho}_0$ both as a quantum pure state or as a thermal classical state at the "vacuum temperature" T_0. We observe that this result can be extended to arbitrary dimension. For a system of N non interacting one-dimensional harmonic oscillators with frequencies $\omega_j,\ j = 1,\ldots,N$ and mass $m_j = m,\ \forall j$ the vacuum state wavefunction is the tensor product

$$\psi_0^{(N)}(x) = \langle x|0\rangle = \prod_j \left(\frac{m\omega_j}{2\hbar}\right)^{\frac{1}{4}} \exp\left[-\frac{m\sum_j \omega_j x_j^2}{2\hbar}\right]\ , \tag{3.1.10}$$

where $x = (x_1,\ldots,x_N)$, and the Wigner transform becomes

$$\rho_0^{(N)}(q,p) = W[\hat{\rho}_0^{(N)}](q,p) = \frac{1}{(\pi\hbar)^N}\exp\left[-\frac{2}{\hbar\omega_j}\left(\sum_j \frac{1}{2}m\omega_j^2 q_j^2 + \frac{p_j^2}{2m}\right)\right]\ . \tag{3.1.11}$$

This is still a classical state, however, it cannot be directly interpreted as a system of N particles in thermal equilibrium for some temperature T because is a product of Boltzmann distributions at the temperatures $\beta_j = 2/\hbar\omega_j$. We observe that performing a change of coordinates we can obtain an interpretation similar to the one-dimensional case. Following the analogy with the one-dimensional

case we can consider the vacuum energy $\mathcal{E}_0$

$$\mathcal{E}_0 = \frac{1}{2}\sum_j \hbar\omega_j \ , \tag{3.1.12}$$

and define the "vacuum temperature"

$$\beta_0 = \frac{2N}{\sum_j \hbar\omega_j} \ . \tag{3.1.13}$$

Making the coordinate change

$$q'^i = \sqrt{\frac{\hbar\omega_i N}{\mathcal{E}_0}}q^i, \quad p'_j = \sqrt{\frac{\mathcal{E}_0}{\hbar\omega_j N}}p_j \ , \tag{3.1.14}$$

which is a canonical transformation, we obtain

$$\rho_0^{(N)}(x',p') = W[\hat{\rho}_0^{(N)}](q,p) = \frac{1}{(\pi\hbar)^N}\exp\left[-\beta_0\left(\sum_j \frac{1}{2}m\omega_0^2 q'^2_j + \frac{p'^2_j}{2m}\right)\right] \ . \tag{3.1.15}$$

meaning that the state $\rho_0^{(N)}$ can be interpreted as the Boltzmann distribution of a system of N oscillators with mass m and frequency $\omega_0 = \mathcal{E}_0/N\hbar$ which is the mean of the frequencies ω_j at temperature β_0.

Finally, the density matrix associated to the n-th quantum excited state $|n\rangle$

$$\hat{\rho}_n = \frac{1}{n!}(\hat{a}^\dagger)^n\hat{\rho}_0\hat{a}^n = \frac{(\hat{a}^\dagger)^n\hat{\rho}_0\hat{a}^n}{\mathrm{Tr}[(\hat{a}^\dagger)^n\hat{\rho}_0\hat{a}^n]} \ . \tag{3.1.16}$$

In the case of an excited state the associated Wigner function is not in general positive definite so it cannot be interpreted as a classical state. To find an analogous of (3.1.16) at the classical level we can use an algebraic approach, simply substituting the quantum algebraic structure with the corresponding classical structure, i.e. making the substitutions

$$\hat{f}\hat{g} \to fg \ , \quad [\hat{f},\hat{g}] \to i\hbar\{f,g\} \ . \tag{3.1.17}$$

To derive the explicit form of the classical excited states $\hat{\rho}_n$ it is easier to start from the phase space and then map the result using the Weyl map. Applying the formal substitution (3.1.17) to (3.1.4), the classical creation and annihilation operators a, a^* in phase space can be defined by the conditions

$$\{H_0, a\} = -i\omega a, \quad \{H_0, a^*\} = i\omega\hat{a}^*, \quad \{a, a^*\} = -i \ , \tag{3.1.18}$$

The solutions of (3.1.18) are simply

$$a = \sqrt{\frac{m\omega}{2}}\left(q + i\frac{p}{m\omega}\right) \ , \tag{3.1.19}$$

$$a^* = \sqrt{\frac{m\omega}{2}}\left(q - i\frac{p}{m\omega}\right) \ . \tag{3.1.20}$$

We observe that $|a|^2 = H_0/\omega$. The classical analogue of the quantum excited states can be obtained

applying the formal rule (3.1.17) to (3.1.16) using as vacuum state the Boltzmann distribution (3.1.8)

$$\rho_n = \frac{|a|^{2n}\rho_0}{\int dxdp\ |a|^{2n}\rho_0} = \frac{H_0^n \exp(-\beta H_0)}{\int dxdp\ H_0^n \exp(-\beta H_0)} = \frac{H_0^n \exp(-\beta H_0)}{Z_{Cl}\langle H_0^n\rangle} \tag{3.1.21}$$

Actually, we can slightly generalize the vacuum state using a Boltzmann distribution at an arbitrary temperature $T = 1/(k_B\beta)$

$$\rho_{0,\beta} = \frac{1}{Q_{Cl}} \exp\left[-\beta\left(\frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m}\right)\right] , \tag{3.1.22}$$

where Q_{Cl} is the classical partition function

$$Q_{Cl} = \int dqdp\ \exp(-\beta H) = \frac{2\pi}{\beta\omega} . \tag{3.1.23}$$

We observe that $1/\beta$ introduce a natural energy scale and can be seen as the classical analogue of the quantum of energy $\hbar\omega$.

To compute $\langle H_0^n\rangle$ we take the derivatives with respect to β of Q_{Cl}

$$\langle H_0^n\rangle = \frac{1}{Z_{Cl}}\left(-\frac{\partial^n}{\partial\beta^n}\right) Q_{Cl} = \frac{n!}{\beta^n} , \tag{3.1.24}$$

and the states $\rho_{n,\beta}$ associated to $\rho_{0,\beta}$ read

$$\rho_{n,\beta} = \frac{\beta\omega}{2\pi}\frac{(\beta H_0)^n}{n!}\exp(-\beta H_0) , \tag{3.1.25}$$

It is interesting to compute the expectation value of the energy in the state $\rho_{n,\beta}$. We obtain

$$\langle H_0\rangle_n = \frac{\beta^{n+1}\omega}{2\pi n!}\int dxdp\ H_0^{n+1}\exp(-\beta H_0) = \frac{\beta^{n+1}\omega}{2\pi n!} Z_{Cl}\langle H_0^{n+1}\rangle = \frac{1}{\beta}(n+1) , \tag{3.1.26}$$

meaning that the states $\rho_{n,\beta}$ can be interpreted as discrete excitations of energy $1/\beta$ of the classical vacuum $\rho_{0,\beta}$ which has "zero-point energy" $1/\beta$. Using the vacuum temperature $\beta = \beta_0/2 = 1/\hbar\omega$ we obtain the quantum spectrum, with the only difference that the classical zero-point energy exceed the quantum one by $\hbar\omega/2$. We observe that in classical mechanics the value of β is not fixed and that the energy does not depend on the oscillator frequency. It is interesting to notice that the classical states (3.1.25), derived using only the classical algebra of phase space, have an interesting relation with the Wigner function of the quantum eigenstates. Indeed, using $\beta = 1/\hbar\omega$, the state $\rho_{n,\beta}$ corresponds to a quantum quasiprobability distribution obtained from the gaussian "smearing" of the Wigner function[1], which is called "Husimi function".

Consider a phase space region given by Δx and Δp, and the gaussian smearing (also called "Weierstrass transform")

$$Q(x,p) = \frac{1}{2\pi\Delta x\Delta p}\int dx'dp'\ \exp\left[-\frac{(x-x')^2}{2\Delta x^2} - \frac{(p-p')^2}{2\Delta p^2}\right] W(x',p') . \tag{3.1.27}$$

The Husimi function is defined as the smearing (3.1.27) with $\Delta x\Delta p = \hbar/2$, thus it depends only on the parameter Δx. The Husimi function is normalized and satisfies the bound $0 \le Q \le 1/\pi\hbar$, which

[1] see for example [1].

in particular implies that it is positive definite. However, the expectation value given by Q treated as a phase space distribution does not correspond in general to the quantum one and the rule to obtain it is more complicated. For this reason Q is considered a quantum quasi-probability distribution. Using the above derivation it is natural to consider Q a classical state obtained from the smearing of W over a particular phase space region W. Indeed, the Husimi function for the eigenstates of the harmonic oscillator are exactly given by (3.1.25) with $\beta = \hbar\omega$. We notice that taking the double limit $\Delta x \to 0$, $\Delta p \to 0$ the transform (3.1.27) reduces to the identity thus we obtain the Wigner functions of the quantum oscillator. This can be naturally seen as a quantum limit in which the classical states reduce to the quantum one when a length scale Δx and a momentum scale Δp becomes sufficiently small. We will see that we obtain a similar limit at the level of the WW equations of motion when we perform an approximation similar to (1.4.8).

3.2 Duality between classical and quantum Boltzmann distributions for the Harmonic oscillator

The states $\rho_{\beta,n}$ were derived using an arbitrary (inverse) temperature β. In the case of $\beta = \beta_0 = 2/\hbar\omega$ we have shown that the classical state $\rho_0 = \rho_{0,\beta_0}$ corresponds to the quantum vacuum state $|0\rangle\langle 0|$. Actually, we now show that this correspondence is also valid for the quantum Boltzmann distribution of the quantum harmonic oscillator at a generic temperature β and the previous result is recovered in the zero-temperature limit $\beta \to +\infty$. Consider the harmonic oscillator quantum Boltzmann distribution

$$\hat{\rho}_B = \frac{\exp(-it\hat{H}_0/\hbar)}{\mathrm{Tr}\Big[\exp(-\tau\hat{H}_0/\hbar)\Big]} \ . \tag{3.2.1}$$

Defining an euclidean time $\tau = \hbar\beta$, the kernel of $\hat{\rho}_B$ can be obtained applying a Wick rotation $t \to -i\tau$ to the propagator of $\hat{H}_0$. The propagator for the harmonic oscillator is given by the path integral

$$\langle x|\exp(-it\hat{H}_0/\hbar)|y\rangle = \int\limits_{x\to y} \mathcal{D}z \ \exp(iS[z]/\hbar) = \int\limits_{x\to y} \mathcal{D}z \ \exp\left[\frac{i}{\hbar}\int_0^t dt \ \frac{1}{2}m\dot{z}^2 - \frac{1}{2}m\omega^2 z^2\right] \ , \tag{3.2.2}$$

which can be explicitly computed. The result is

$$\langle x|\exp(-it\hat{H}_0/\hbar)|y\rangle = \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega t)}}\exp\left\{i\frac{m\omega}{2\hbar}\left[(x^2+y^2)\mathrm{cotan}(\omega t) - \frac{2xy}{\sin(\omega t)}\right]\right\} \ , \tag{3.2.3}$$

and applying the Wick rotation we get the euclidean propagator

$$\langle x|\exp(-\tau\hat{H}_0/\hbar)|y\rangle = \sqrt{\frac{m\omega}{2\pi\hbar\sinh(\omega\tau)}}\exp\left\{-\frac{m\omega}{2\hbar}\left[(x^2+y^2)\mathrm{cotanh}(\omega\tau) - \frac{2xy}{\sinh(\omega\tau)}\right]\right\} \ . \tag{3.2.4}$$

The quantum partition function $Z_Q(\beta) = \mathrm{Tr}\Big[\exp(-\tau\hat{H}_0/\hbar)\Big]$ is

$$\mathrm{Tr}\Big[\exp(-\tau\hat{H}_0/\hbar)\Big] = \sum_n \exp(-\tau\mathcal{E}_n/\hbar) = \exp(-\tau\omega/2)\sum_n \exp(-\tau n\omega) = \tag{3.2.5}$$

$$= \frac{1}{2\sinh(\omega\tau/2)} = \sqrt{\frac{1}{2[\cosh(\omega\tau)-1]}} \ . \tag{3.2.6}$$

Is useful to introduce the identities

$$\tanh(x/2) = \text{cotanh}(x) - \frac{1}{\sinh(x)} \ , \tag{3.2.7}$$

$$\text{cotanh}(x/2) = \text{cotanh}(x) + \frac{1}{\sinh(x)} \ . \tag{3.2.8}$$

then the kernel of $\hat{\rho}_B$ reads

$$\hat{\rho}_B(x,y) = \sqrt{\frac{m\omega}{\pi\hbar}\tanh(\omega\tau)}\exp\left\{-\frac{m\omega}{2\hbar}\left[(x^2+y^2)\text{cotanh}(\omega\tau) - \frac{2xy}{\sinh(\omega\tau)}\right]\right\} \ . \tag{3.2.9}$$

The Wigner transform of $\hat{\rho}_B$ is

$$W[\hat{\rho}_B](q,p) = \frac{1}{2\pi\hbar}\int dy \exp(-ipy/\hbar)\hat{\rho}_B(q+y/2, q-y/2) \ , \tag{3.2.10}$$

and substituting (3.2.9) we obtain, using relations (3.2.7) and (3.2.8)

$$W[\hat{\rho}_B](q,p) = \frac{1}{\pi\hbar}\tanh(\omega\tau/2)\exp\left[-\frac{2}{\hbar\omega}\tanh(\omega\tau/2)\left(\frac{1}{2}m\omega^2q^2 + \frac{p^2}{2m}\right)\right] \ . \tag{3.2.11}$$

This can be interpreted as the Boltzmann distribution of a classical harmonic oscillator. Indeed, defining the dual temperature T_D

$$\beta_D = \frac{1}{k_B T_D} = \frac{2}{\hbar\omega}\tanh\left(\frac{\omega\tau}{2}\right) = \frac{1}{\mathcal{E}_0}\tanh(\beta\mathcal{E}_0) \ , \tag{3.2.12}$$

where $\mathcal{E}_0$ is the quantum vacuum energy. Equation (3.2.11) becomes

$$W[\hat{\rho}_B](q,p) = \frac{\beta_D\omega}{2\pi}\exp\left[-\beta_D\left(\frac{1}{2}m\omega^2q^2 + \frac{p^2}{2m}\right)\right] = \rho_{0,\beta_D}(q,p) \ , \tag{3.2.13}$$

which is exactly the Boltzmann distribution of a classical harmonic oscillator at temperature $T_D = 1/(k_B\beta_D)$ and $Q_{Cl} = 2\pi/\beta_D\omega$ is the classical partition function. Relation (3.2.13) means that a quantum harmonic oscillator at temperature $T = 1/(k_B\beta)$ can be equivalently interpreted as a classical harmonic oscillator at a different dual temperature $T_D = 1/(k_B\beta_D)$. In particular, at high temperature $T \to +\infty$ we have $T_D \sim T$ so the quantum oscillator corresponds to a classical one at the same (high) temperature. At low temperature $T \to 0$, as we expect from the analysis done for the vacuum state, the dual temperature T_D tends to the finite value $T_0 = \mathcal{E}_0/k_B$. Indeed, when $T \to 0$ the quantum Boltzmann distribution tends to the vacuum state

$$\hat{\rho}_B(x,y) \underset{T\to 0}{\to} \sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}(x^2+y^2)\right] = \langle x|0\rangle\,\langle 0|y\rangle \tag{3.2.14}$$

and we recover the result found in the previous section.

We conclude this section observing that the quantum Boltzmann distribution $\hat{\rho}_B(\beta)$ admits the path integral representation

$$\langle x|\hat{\rho}_B(\beta)|y\rangle = \frac{\langle x|\exp(-\beta\hat{H}_0)|y\rangle}{Z_Q(\beta)} = \frac{1}{Z_Q(\beta)}\int_{x\to y}\mathcal{D}z\ \exp(-S_E[z,\beta]/\hbar) \ , \tag{3.2.15}$$

where $S_E[z,\beta] = \int_0^\tau d\tau\ \dot{z}^2/2 + m\omega^2z^2/2 = \int_0^\tau d\tau H_0(z,\dot{z})$ is the euclidean action and $Z_Q(\beta)$ is the

quantum partition function at temperature β

$$Z_Q(\beta) = \int\limits_{z(0)=z(\tau)} \mathcal{D}z \ \exp(-S_E[z,\beta]/\hbar) \ . \tag{3.2.16}$$

Inserting (3.2.15) in the definition of the Wigner transform we obtain

$$\frac{\exp(-\beta_D H_0)}{Z_{Cl}(\beta_D)} = \frac{1}{Z_Q(\beta)} \int dy \int\limits_{x-y/2 \to x+y/2} \mathcal{D}z \ \exp[-(S_E[z,\beta] + ipy)/\hbar] \ . \tag{3.2.17}$$

We observe that given a path $z : x - y/2 \to x + y/2$ we have $\int_0^\tau \dot{z} = z(\tau) - z(0) = y$. Using this fact, the expression (3.2.20) reads

$$\frac{\exp(-\beta_D H_0)}{Z_{Cl}(\beta_D)} = \frac{1}{Z_Q(\beta)} \int dy \int\limits_{x-y/2 \to x+y/2} \mathcal{D}z \ \exp\left[-(S_E[z] + i\int p\dot{z})/\hbar\right] \ . \tag{3.2.18}$$

Fixed a position x, the domain of the path integral in (3.2.20) consists of all paths of extrema z_1, z_2 whose midpoint is $\bar{z} = (z_1 + z_2)/2$ is x. Denoting with $\bar{z} = x$ the domain, the path integral in (3.2.18) can be rewritten in the form

$$\frac{\exp(-\beta_D H_0)}{Z_{Cl}(\beta_D)} = \frac{1}{Z_Q(\beta)} \int\limits_{\bar{z}=x} \mathcal{D}z \ \exp\left[-(S_E[z,\beta] + i\int p\dot{z})/\hbar\right] \ . \tag{3.2.19}$$

Thanks to invariance under euclidean time translation, we can replace the interval $[0,\tau]$ with the symmetric interval $[-\tau/2, \tau/2]$[2]. Taking the limit $\tau \to +\infty$ the quantum Boltzmann distribution reduces to the vacuum state

$$\rho_0(x,p) \equiv \frac{\exp(-\beta_0 H_0)}{Z_{Cl}(\beta_0)} = \frac{1}{Z_Q} \int\limits_{\bar{z}=x} \mathcal{D}z \ \exp\left[-(S_E[z] + i\int p\dot{z})/\hbar\right] \ , \tag{3.2.20}$$

where $\beta_0 = 1/(k_B T_0) = 1/\mathcal{E}_0$ is the classical vacuum temperature and $S_E[z] = \int_{-\infty}^{+\infty} H_0(z,\dot{z})$ and the paths are defined $\forall \tau \in \mathbb{R}$. The equation (3.2.20) shows that for an harmonic oscillator the classical vacuum state ρ_0 can be expressed as a sort of Functional Fourier transform of $\exp[-(S_E[z]/\hbar)]$, where the momentum p plays the role of a "source" term.

3.3 Wigner-Weyl classical dynamics and local quantum approximation

In the previous chapter we have shown that the classical dynamics in the quantum Hilbert space $\mathcal{H}$, obtained applying the Weyl transform to the Liouville equation in the phase space, is governed by the equation

$$i\hbar \frac{\partial}{\partial t}\hat{\rho}(x,y,t) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\hat{\rho}\,(x,y,t) + (x-y)V'\left(\frac{x+y}{2}\right)\hat{\rho}\,(x,y,t) \ . \tag{3.3.1}$$

As noticed this equation is similar to the von Neumann equation, in the integral kernel representation

$$i\hbar \frac{\partial}{\partial t}\hat{\rho}(x,y,t) = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)\hat{\rho}\,(x,y,t) + (V(x) - V(y))\,\hat{\rho}\,(x,y,t) \ , \tag{3.3.2}$$

[2]The values $\tau < 0$ are not physical but what has physical meaning, thanks to the translational invariance, is only the interval length which, divided by $\hbar$, corresponds to the quantum temperature of the system

and, as seen using the path integral representation, there is an obstruction term $\Delta V(x,y) = V'((x+y)/2)(x-y) - V(x) + V(y)$ which does not allow to factorize the classical dynamics. Therefore, in general the Hilbert-Schmidt evolution is not given by an underlying evolution in the quantum Hilbert space $\mathcal{H}$. Despite the classical dynamics cannot be restricted to pure states there is a natural way to relate the above equations treating the quantity $|x-y|$ as a perturbative parameter, as we already saw in the context of the KvN theory. To see this we first introduce an arbitrary length scale L and we consider the limit in which $|x-y| << L$. In this limit we can consider the expansion

$$V(x)-V(y) = V\left(\frac{x+y}{2}+\frac{x-y}{2}\right)-V\left(\frac{x+y}{2}-\frac{x-y}{2}\right) = V'\left(\frac{x+y}{2}\right)(x-y)+O((x-y)^3)\ , \quad (3.3.3)$$

which shows that at first order in $\epsilon = |x-y|$ the classical potential term coincides with the quantum one and the dynamics is given by a commutator allowing for a wavefunction description. Therefore, it is natural to interpret this limit as a "quantum limit" where the classical dynamics reduces to the quantum one when $\epsilon/L \to 0$. In this way quantization is achieved using a limit process and quantum mechanics is seen as an "approximation" of classical mechanics.

We observe that we have introduced the limit in a pure formal way without giving a physical argument for the introduction of a length scale L. A natural choice of this scale can be to use the Compton wavelength $\lambda_c = \hbar/mc$ of the particle, and in this way $L \to \infty \Rightarrow \epsilon/L \to 0$ corresponds to the limit $\hbar \to \infty$ which seems dual[3] with respect to the usual $\hbar \to 0$. The approximation performed in (3.3.3) admits another interpretation which does not require the introduction of the length scale L. For this purpose we consider again the expansion (3.3.3) taking into account the higher order terms

$$V(x) - V(y) = V'(x_S)(x-y) + \frac{1}{6}V'''(x_S)(x-y)^3 + O((x-y)^5)\ , \quad (3.3.4)$$

where we have included the first term after the leading order and we have defined $x_S = (x+y)/2$. We can consider the "small-distance" limit as an approximation where we keep only the leading order and we neglect the higher order terms. This requires that the leading order in $|x-y|$ gives the main contribution, thus a necessary condition is

$$|V'(x_S)(x-y)| >> |\frac{1}{6}V'''(x_S)(x-y)^3|\ . \quad (3.3.5)$$

We can fix x_s and solve this inequality with respect to $\epsilon = |x-y|$. We obtain

$$|x-y| << \sqrt{\left|\frac{6V'(x_S)}{V'''(x_S)}\right|} \equiv L(x_S)\ , \quad (3.3.6)$$

thus, the length scale $L = L(x_S)$ in this way is given by the dynamics of the system and in general becomes a local scale which depends on the midpoint x_S. We observe that a similar condition is used in the WKB approximation. Furthermore, this resembles the equivalence principle, where the local inertial frame approximation is possible in a sufficiently small neighborhood of a spacetime point x, whose radius depends on the spacetime curvature in that point. To see this we observe that the scale $L(x_S)$ can be interpreted as a scale which gives a "local quantum approximation". First, we have that a generic quasi-density matrix $\hat{\rho}$, thanks to self-adjointness, can always be written as a combination

[3]This is not the only possibility. We can obtain the same result using $m \to 0$ or $c \to 0$ or a combination of these limits.

of pure states

$$\hat{\rho} = \sum_n p_n \, |\psi_n\rangle\langle\psi_n| \ , \tag{3.3.7}$$

for a suitable set of wavefunctions ψ_n and weight p_n. In the quantum case the state $\hat{\rho}$ is interpreted as a statistical mixture where p_n are the probability that the system is in the state $|\psi_n\rangle$, which is due to the ignorance of the actual state of the system. Then the matrix element $\hat{\rho}(x,x)$ can be interpreted as the avarage of the quantum probabilities, relative to the pure states ψ_n to find the system in the position x when a measurement is performed, and is called the "population" of the state x

$$\hat{\rho}(x,x) = \sum_n p_n |\psi_n(x)|^2 \ . \tag{3.3.8}$$

The off-diagonal elements $\hat{\rho}(x,y)$ are called "coherences" and measure the interference effects between two points x, y due to the state $\hat{\rho}$. Indeed, for a pure state $\psi(x)$ the off-diagonal element is simply the product $\psi(x)\psi^*(y)$ and measures how the wavefunction is delocalized giving correlations between different points of space. In the case of a state with support localized in a region of radius R around a point x, the matrix element $\hat{\rho}(x,y)$ vanishes when $|x-y| > R$ and in this case the physical effects are only due to the behaviour of the wavefunction in the region $|x-y| < R$. In the classical case the state is in general a quasi-density matrix elements, thus, we cannot in general interpret it as a quantum statistical mixture. However, also in this case the fact that the support of $\hat{\rho}(x,y)$ is localized in a region $|x-y| < R$ implies that all observables quantities are related only to that region[4]. Then the small distance limit can naturally arise in an experimental situation where at $t = 0$ the system is localized at a point x_S with an uncertainty $\Delta x << L(x_S)$ through a measurement of position. For times $t > 0$ sufficiently small such that the system remains localized in a neighborhood of x_S of radius $L(x_S)$, we can equivalently apply the classical or quantum dynamics to the system and the result are indistinguishable[5]. In this way $L(x_S)$ acquires the meaning of the scale under which the quantum and classical dynamical laws are equivalent.

3.4 WW dynamics of a charged particle and local quantum approximation

In this section we extend the above analysis to the case of a system where a static electromagnetic field is present. We consider a particle in the three-dimensional space with charge e and mass m in a static electromagnetic field given by the potential $(\phi(q), A_i(q))$, with $i = 1,2,3$ and $q \in \mathbb{R}^3$. The hamiltonian of this system is

$$H(q,p) = \frac{1}{2m}(p - eA)^2 + e\phi(q) \ . \tag{3.4.1}$$

The classical Liouville equation is

$$\dot{\rho} = \{H, \rho\} = \left[-e(p-eA)^j \frac{\partial A_j}{\partial q^i} + e\frac{\partial \phi}{\partial q^i}\right] \frac{\partial \rho}{\partial p_i} - (p-eA)^i \, \frac{\partial \rho}{\partial q_i} \ . \tag{3.4.2}$$

[4]This is true for small time intervals around the time t when the system is prepared. In general, evolution will tend to delocalize the system leading to an analysis of the region outside the one where the system was initially prepared. Thus, in general we require a state localized in a spacetime neighborhood.

[5]The argument does not concern the initial state of the system given by the preparation which is subject to the uncertainty principle. This analysis clearly shows that the dynamical laws which give the evolution of the state and the kinematical statements which define the state allowed by the theory have to be treated separately.

The correspondent equation in Hilbert space is obtained applying, to both members of (3.4.2), the three-dimensional version of the Weyl transform, which, using integral kernels, is

$$\langle x|\hat{\rho}|y\rangle = \hat{\rho}(x,y) = \Phi[\rho](x,y) = \frac{1}{(2\pi\hbar)^3}\int d^3p \exp[ip(x-y)/\hbar]\rho\left(\frac{x+y}{2},p\right) \ . \tag{3.4.3}$$

For this purpose we consider the obvious generalization of the rules (2.2.26) - (2.2.29), derived in the one-dimensional case

$$\Phi\left[\frac{\partial\rho}{\partial p^i}\right](x,y) = -\frac{i}{\hbar}(x-y)^i\Phi[\rho](x,y) \ , \tag{3.4.4}$$

$$\Phi\left[\frac{\partial\rho}{\partial q^i}\right](x,y) = \left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)\Phi[\rho](x,y) \ , \tag{3.4.5}$$

$$\Phi[p^i\rho](x,y) = -\frac{i\hbar}{2}\left(\frac{\partial}{\partial x^i}-\frac{\partial}{\partial y^i}\right)\Phi[\rho](x,y) \ , \tag{3.4.6}$$

$$\Phi[q^i\rho](x,y) = \frac{1}{2}(x+y)^i\Phi[\rho](x,y) \ . \tag{3.4.7}$$

It is useful to consider x^i, y^i and the correspondent derivatives $\partial/\partial x^i, \partial/\partial y^i$ as operators $\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y$ on the Hilbert-Schimdt space $\mathcal{H}_{HS} \simeq L^2(R^{2n}, d^nxd^ny)$. As said in the previous section, in this space a (quasi-)density matrix $\hat{\rho}$ becomes a ket[6] $|\hat{\rho}\rangle$ of $\mathcal{H}_{HS}$ through the identification $\langle x,y|\hat{\rho}\rangle = \hat{\rho}(x,y)$. To simplify the calculation we introduce the following notation: for an operator $\hat{f} = \hat{f}(\hat{q},\hat{p})$ in $\mathcal{H} = L^2(\mathbb{R}^{2n})$ we define the following operators on $\mathcal{H}_{HS}$

$$\hat{f}_S = \frac{\hat{f}(\hat{x},\hat{p}_x)+\hat{f}(\hat{y},\hat{p}_y)}{2} \ , \tag{3.4.8}$$

$$\hat{f}_A = \frac{\hat{f}(\hat{x},\hat{p}_x)-\hat{f}(\hat{y},\hat{p}_y)}{2} \ , \tag{3.4.9}$$

the rules (3.4.4) - (3.4.7) can be reformulated in the following way

$$p^i \to \quad \langle x,y|\hat{p}^i_A|\rho\rangle = -\frac{i\hbar}{2}\left(\frac{\partial}{\partial x^i}-\frac{\partial}{\partial y^i}\right)\hat{\rho}(x,y) \ , \tag{3.4.10}$$

$$q^i \to \quad \langle x,y|\hat{x}^i_S|\rho\rangle = \frac{1}{2}(x+y)^i\hat{\rho}(x,y) \ , \tag{3.4.11}$$

$$i\hbar\frac{\partial}{\partial p_i} \to \quad \langle x,y|2\hat{x}^i_A|\rho\rangle = (x-y)^i\hat{\rho}(x,y) \ , \tag{3.4.12}$$

$$-i\hbar\frac{\partial}{\partial q_i} \to \quad \langle x,y|2\hat{p}^i_S|\rho\rangle = -i\hbar\left(\frac{\partial}{\partial x^i}+\frac{\partial}{\partial y^i}\right)\hat{\rho}(x,y) \ . \tag{3.4.13}$$

We observe that the symmetric variables $\hat{x}^i_S, \hat{p}^i_S$ and antisymmetric ones $\hat{x}^i_A, \hat{p}^i_A$ form a representation of the CCR[7]

$$[\hat{x}^i_A, \hat{p}^j_A] = \frac{i\hbar}{2}\delta^{ij} \quad , \quad [\hat{x}^i_S, \hat{p}^j_S] = \frac{i\hbar}{2}\delta^{ij} \quad , \tag{3.4.14}$$

and the other commutator vanish. Finally, applying the above rules to (3.4.2) we obtain

$$\langle x,y|\mathcal{L}_{Cl}|\hat{\rho}\rangle = 2\,\langle x,y|\left\{-\frac{e}{m}\left[\hat{p}_A - eA(\hat{x}_S)\right]^j\frac{\partial A_j}{\partial q^i}(\hat{x}_S) + e\frac{\partial\phi}{\partial q^i}(\hat{x}_S)\right\}\hat{x}^i_A + \frac{e}{m}\left[\hat{p}_A - eA(\hat{x}_S)\right]_i\hat{p}^i_S|\hat{\rho}\rangle \ . \tag{3.4.15}$$

[6] More precisely, we have to take the (commutative) square root $\hat{\psi}$ such that $\hat{\psi}^\dagger \circ \hat{\psi} = \hat{\rho}$ in order to have a true HS state. This does not modify the result of the computation, thus we can neglect this problem and work direclty with $\hat{\rho}$.

[7] The factor 1/2 is due to the choice of the normalization in (3.4.8) and (3.4.9).

We observe that classical equation of motion must be gauge invariant and this also implies that the WW equation (3.4.15) must be gauge invariant. In this respect note that gauge invariance is not manifest from (3.4.15). To check gauge invariance, it is easier to first see the form of the gauge transformation in phase space and then, using the Weyl transform, derive the correspondent transformation in the Hilbert space. The Liouville equation in phase space (3.4.2) is invariant under the gauge transformation

$$A'_j(q) = A_j(q) + \partial_j \Lambda(q) \ , \qquad p'_j = p_j - e\partial_j \Lambda(q) \ , \tag{3.4.16}$$

implying that ρ transforms with the "pull-back"

$$\rho'(q,p) \equiv \rho(q, p - e\nabla\Lambda(q)) \ . \tag{3.4.17}$$

Taking the Weyl transform of ρ' we obtain

$$\hat{\rho}'(x,y) = \frac{1}{2\pi\hbar}\int dp \exp[ip(x-y)/\hbar]\rho\left(x_s, p - e\nabla\Lambda(x_s)\right) = \tag{3.4.18}$$

$$= \frac{1}{2\pi\hbar}\int dp \exp\left[i(p - e\nabla\Lambda(x_s)/\hbar\right]\rho(x_s,p) = \hat{\rho}(x,y)\exp\left[-ie\nabla\Lambda(x_s)(x-y)\right] \ . \tag{3.4.19}$$

which gives the gauge transformation in the WW formalism. We observe that when $x - y \sim 0$ this transformation reduces to

$$\hat{\rho}'(x,y) \simeq \hat{\rho}(x,y)\exp\left[-ie(\Lambda(x) - \Lambda(y))\right] \tag{3.4.20}$$

This is precisely the gauge transformation of the quantum density matrix. Indeed, the quantum wavefunction transform as $\psi'(x) = \psi(x)\exp(ie\Lambda(x))$ and a generic density matrix can be always written as a convex combination of pure states

$$\hat{\rho}(x,y) = \sum_n \psi(x)\psi^*(y) \ , \tag{3.4.21}$$

which implies that we have the gauge transformation given by (3.4.20)

$$\hat{\rho}'(x,y) = \sum_n \psi'(x)\psi'^*(y) = \left(\sum_n \psi(x)\psi^*(y)\right)\exp\left[-ie(\Lambda(x) - \Lambda(y))\right] = \hat{\rho}(x,y)\exp\left[-ie(\Lambda(x) - \Lambda(y))\right] \ . \tag{3.4.22}$$

According to the interpretation discussed in the above section this means that the classical gauge transformations can be approximated by the quantum one when is satisfied the condition

$$\left|\frac{\partial\Lambda}{\partial q^j}(x_S)x^j_A\right| >> \left|\frac{1}{6}\frac{\partial^3\Lambda}{\partial q^j\partial q^k\partial q^l}(x_S)x^j_A x^k_A x^l_A\right| \ . \tag{3.4.23}$$

We now want to verify if we obtain a quantum commutator when we consider the local quantum approximation. As done for the case of the scalar potential we consider the approximation $|x_A| = |x - y| << L(x_S)$, where $L(x_S)$ depends on the form of the potentials ϕ and A_i of the system. Inside the matrix element we have the first order approximation

$$\langle x,y|\frac{\partial f}{\partial q^i}(\hat{x}_S)\hat{x}^i_A|\hat{\rho}\rangle \simeq \langle x,y|\hat{f}_A|\hat{\rho}\rangle \ , \tag{3.4.24}$$

$$\langle x,y|f(\hat{x}_S)|\hat{\rho}\rangle \simeq \langle x,y|\hat{f}_S|\hat{\rho}\rangle \ . \tag{3.4.25}$$

which is the local quantum approximation defined in the previous section. Using this approximation, equation (3.4.15) becomes

$$\langle x,y|\mathcal{L}_{Cl}|\hat{\rho}\rangle = 2\,\langle x,y|-\frac{e}{m}\left(\hat{p}_A - eA_S\right)_j A^j_A + \frac{e}{m}\left(\hat{p}_A - eA_S\right)_i \hat{p}^i_S + e\phi_A|\hat{\rho}\rangle \ . \tag{3.4.26}$$

This expression can be simplified using the following relations

$$2\hat{p}^i_A\hat{p}_{i,S} = \frac{1}{2}(\hat{p}_x - \hat{p}_y)^i(\hat{p}_x + \hat{p}_y)^j = \frac{1}{2}(\hat{p}^i_x\hat{p}^i_x - \hat{p}^i_y\hat{p}^i_y) = \left[\hat{p}^i\hat{p}^i\right]_A \ , \tag{3.4.27}$$

$$2\hat{p}^j_A\hat{A}_{j,A} + 2\hat{A}_{j,S}\hat{p}^j_S = \frac{1}{2}(\hat{p}_x - \hat{p}_y)^j\left[A(\hat{x}) - A(\hat{y})\right]_j + \frac{1}{2}\left[A(\hat{x}) + A(\hat{y})\right]_j(\hat{p}_x + \hat{p}_y)^j = \left[\hat{p}^j A(\hat{q})_j + A(\hat{q})_j\hat{p}^j\right]_S \ . \tag{3.4.28}$$

Using these relations we obtain

$$\langle x,y|\mathcal{L}_{Cl}|\hat{\rho}\rangle = 2\,\langle x,y|\left[\frac{1}{2m}p^jp_j + e\phi(x) + \frac{e^2}{2m}A^j(x)A_j(x)\right]_A - \frac{e}{m}\left[\hat{A}^j(x)\hat{p}_j + \hat{p}_j\hat{A}_j(x)\right]_S|\hat{\rho}\rangle \ . \tag{3.4.29}$$

The operator (3.4.29) corresponds[8] to the quantum liouvillian operator $\mathcal{L}_Q\,|\hat{\rho}\rangle = [\hat{H},\hat{\rho}]$ for the hamiltonian operator

$$\hat{H} = \frac{1}{2m}\left[\hat{p} - eA(\hat{x})\right]^2 + e\phi(\hat{x}) \ . \tag{3.4.30}$$

Therefore, we have that, as for the case of the scalar potential, the classical dynamics in this approximation reduces to quantum dynamics. We observe that the hamiltonian operator $\hat{H}$ obtained with the local quantum approximation does not correspond in general to the Weyl mapping $\Phi(H)$ of H because the expression is not completely symmetric in $\hat{x}$ and $\hat{p}$. This is interesting because implies that the quantum local approximation selects a non-trivial ordering which does not correspond to the Weyl ordering which is obtained for example taking the formal limit $\hbar \to 0$.

3.5 Hamiltonians with non trivial metric and superalgebra structure

In the previous section we have seen that for hamiltonians with generic static electromagnetic fields and scalar potentials we have that the classical dynamics reduces to the quantum one when we take the local quantum approximation. In this way the approximation can be used to quantize the theory using a limit procedure. A natural question is to understand which types of hamiltonians can be "quantized" in this way. We observe that in the physical applications , especially at the fundamental level, the hamiltonians are usually quadratic in the momenta. The more general hamiltonian of this form in the phase space $\mathbb{R}^{2n}$ is

$$H(q,p) = \frac{1}{2}K_{ij}(q)p^ip^j + A_j(q)p^j + V(q) \ , \tag{3.5.1}$$

where $K_{ij}(q)$ is a symmetric non-degenerate matrix[9] and all coefficients are function of the position q. We observe that thanks to non-degeneracy the matrix $K_{ij}(q)$ has an inverse $K^{ij}(q)$. Thus, we can

[8] The presence of the symmetric term linear in the momentum apparently seems not to be a commutator. However, we should recall that we are working with integral kernels so we are taking matrix elements. Indeed, the action of the momentum on the bras is the conjugate one $\langle\psi|\hat{p}|y\rangle = i\hbar\partial/\partial y\psi^*(y) = -(-i\hbar\partial/\partial y\psi^*(y))$, thus we have an extra minus sign when the momentum acts from the right which cancels the sign of the commutator.

[9] Actually, in the general case this matrix can be degenerate. This is related to the presence of hamiltonian constraints which arises for example in gauge theories. For simplicity, we consider only the non-degenerate case.

always define $A^i(q) = K^{ij}(q)A_j$ and complete the square to rewrite the hamiltonian as

$$H(q,p) = \frac{1}{2}K_{ij}(q)p'^i(q)p'^j(q) + V(q) - \frac{1}{2}K_{ij}A^i(q)A^j(q) \ , \quad p' = p - A \ , \tag{3.5.2}$$

which implies that we obtain essentially a particle in a electromagnetic field (with charge $e = 1$) for the potentials $\phi(q) = V(q) - 1/2K_{ij}A^i(q)A^j(q)$ and A^j. Therefore, the only difference with respect to the hamiltonian (3.4.1) comes from the "metric" term $K_{ij}(q)$. We want to verify if even in this case the local quantum approximation leads to a quantum commutator. Before doing the computation we observe that in the case of the electromagnetic field the key fact that leads to equation (3.4.29) is that after the approximation we can apply the relations (3.4.27) and (3.4.28) where we have cancellations of the mixed terms of the form $\hat{f}(\hat{x},\hat{p}_x)\hat{g}(\hat{y},\hat{p}_y)$. This is precisely what characterizes the quantum commutator which has no mixed term in x, y.

In the case of non trivial metric $K_{ij}(q)$ the classical liouvillian is

$$\mathcal{L}_{Cl}\rho = i\hbar\{H,\rho\} = i\hbar\left[\frac{1}{2}\frac{\partial K_{jk}}{\partial q^i}p^j p^k + \frac{\partial A_j}{\partial q^i}p^j + \frac{\partial V}{\partial q^i}\right]\frac{\partial \rho}{\partial p_i} - i\hbar\left[K_{ij}(q)p^j + A_i\right]\frac{\partial \rho}{\partial q_i} \ , \tag{3.5.3}$$

and applying the Weyl map we obtain

$$\begin{aligned}&\langle x,y|\Phi[\mathcal{L}_{Cl}\rho]\rangle = \\ &= \langle x,y|2\left[\frac{1}{2}\frac{\partial K_{jk}}{\partial q^i}(\hat{x}_S)\hat{p}^j_A\hat{p}^k_A + \frac{\partial A_j}{\partial q^i}(\hat{x}_S)\hat{p}^j_A + \frac{\partial V}{\partial q^i}(\hat{x}_S)\right]\hat{x}^i_A + 2\left[K_{ij}(\hat{x}_S)\ \hat{p}^j_A + A_i(\hat{x}_S)\right]\hat{p}^i_S|\hat{\rho}\rangle \ ,\end{aligned} \tag{3.5.4}$$

that, using the approximation $|(x-y)^i| << l$, reads

$$\langle x,y|\Phi[\mathcal{L}_{Cl}\rho]\rangle \simeq 2\,\langle x,y|\frac{1}{2}\hat{p}^i_A\hat{p}^j_A\hat{K}_{ij,A} + \hat{K}_{ij,S}\ \hat{p}^i_A\hat{p}^j_S + \hat{p}^j_A\hat{A}_{j,A} + \hat{A}_{j,S}\ \hat{p}^j_S + \hat{V}_A|\hat{\rho}\rangle \ . \tag{3.5.5}$$

In this case it is not possible to obtain a quantum commutator because of the presence of the term $\hat{p}^i_A\hat{p}^j_A\hat{K}_{ij,A}$ which gives non vanishing mixed terms in $\hat{p}_x, \hat{p}_y$. Then the local equivalence between the classical and the quantum evolution seems to break in presence of non trivial "metric" $K_{ij}(q)$. Actually, also in this case we can obtain a commutator but the local quantum approximation has to be improved in such a way to allow for approximations also in the momenta. To do this we have to understand the nature of the cancellations which are obtained in the case of trivial metric. Therefore we analyze again the rules (3.4.10) - (3.4.13) . They suggest that there is a sort of "grading" given by the "symmetric" and "antisymmetric" variables and in fact cancellations correspond to operations that preserve this grading. To make precise this statement we recall that from the algebra of operators in the quantum Hilbert space $\mathcal{H}$ we can define for every operator $\hat{a}$ the "left" and "right" representations

$$a_L\,|\rho\rangle = \hat{a}\hat{\rho} \ , \quad \forall\hat{\rho} \in \mathcal{H}_{HS} \ , \tag{3.5.6}$$

$$a_R\,|\rho\rangle = \hat{\rho}\hat{a} \ , \quad \forall\hat{\rho} \in \mathcal{H}_{HS} \ , \tag{3.5.7}$$

which are operators in $\mathcal{H}_{HS}$. We observe that the two representations commute, $a_L b_R = b_R a_L$. Considering the kernels of the left and right representations of the variables $\hat{x}, \hat{p}$ we obtain

$$(\hat{x}_L\hat{\rho})(x,y) = x\rho(x,y) \ , \quad (\hat{x}_R\hat{\rho})(x,y) = y\rho(x,y) \ , \tag{3.5.8}$$

$$(\hat{p}_L\hat{\rho})(x,y) = -i\hbar\frac{\partial}{\partial x}\rho(x,y) \ , \quad (\hat{p}_R\hat{\rho})(x,y) = i\hbar\frac{\partial}{\partial y}\rho(x,y) \ . \tag{3.5.9}$$

Using the left and right representations we can define the commutator and anticommutator associated to an operator $\hat{a}$

$$a^- |\rho\rangle = (a_L - a_R) |\rho\rangle = [\hat{a}, \hat{\rho}] \ , \quad a^+ |\rho\rangle = (a_L + a_R) |\rho\rangle = \{\hat{a}, \hat{\rho}\} \equiv \hat{a}\hat{\rho} + \hat{\rho}\hat{a} \ , \tag{3.5.10}$$

Using these definitions the rules (3.4.4) - (3.4.7) become

$$p_i \to \frac{1}{2} p_i^+ \ , \tag{3.5.11}$$

$$q_i \to \frac{1}{2} q_i^+, \tag{3.5.12}$$

$$i\hbar \frac{\partial}{\partial p_i} \to q_i^- \ , \tag{3.5.13}$$

$$-i\hbar \frac{\partial}{\partial q_i} \to p_i^- \ . \tag{3.5.14}$$

We define the vector spaces $L^+(\mathcal{H})$ and $L^-(\mathcal{H})$ as the set of all self-adjoint operators a^+ and a^- respectively. These spaces are isomorphic because a^+ and a^- "projects" to the same operator $\hat{a}$ in the Hilbert space $\mathcal{H}$ and the space $L^-(\mathcal{H})$ simply corresponds to the space of inner derivations of $\mathcal{H}$. We observe that the quantum dynamics is given by an element of L^-. The classical dynamics is given by an "entangled" element which does not in general belong to $L^\pm$. We observe that these spaces are disjoint, therefore they are in direct sum. Indeed, suppose $a^- = b^+$ for some self-adjoint operators $\hat{a}$ and $\hat{b}$. We have

$$(a^- |\rho\rangle)^\dagger = -[\hat{a}, \hat{\rho}] = -\{\hat{b}, \hat{\rho}\} = -(b^+ |\rho\rangle)^\dagger = -(a^- |\rho\rangle)^\dagger \ \Rightarrow a^- = b^+ = 0 \ . \tag{3.5.15}$$

Then we can define the space

$$L(\mathcal{H}) = L^-(\mathcal{H}) \oplus L^+(\mathcal{H}) \ . \tag{3.5.16}$$

This space has a superalgebra structure where the parity is given by the sign $+, -$. To see this, we rewrite a generic element a^ϵ of parity $\epsilon = +1, -1$ as $a^\epsilon = a_L + \epsilon a_R$ and we define the graded commutator and anticommutator[10]

$$[a^{\epsilon_1}, b^{\epsilon_2}] = a^{\epsilon_1} b^{\epsilon_2} - b^{\epsilon_2} a^{\epsilon_1} \ , \tag{3.5.17}$$

$$\{a^{\epsilon_1}, b^{\epsilon_2}\} = a^{\epsilon_1} b^{\epsilon_2} + b^{-\epsilon_2} a^{-\epsilon_1} \ . \tag{3.5.18}$$

We have to verify if these brackets are well defined as operators in L and respect the grading. This can be verified directly using the definition of a^ϵ and the result is simply

$$[a^{\epsilon_1}, b^{\epsilon_2}] = [a, b]^{-\epsilon_1 \epsilon_2} \ , \tag{3.5.19}$$

$$\{a^{\epsilon_1}, b^{\epsilon_2}\} = \{a, b\}^{\epsilon_1 \epsilon_2} \ , \tag{3.5.20}$$

because the mixed terms of the form $a_L b_R$ cancel out. In the case in which only commutators appear the relation (3.5.19) is simply a consequence of the Jacobi identity. In the other case it can be seen as a graded version of the Jacobi identity. We observe that these cancellations are precisely the ones arising in the case of a hamiltonian of the form (3.4.1) and which allows, in the small distance limit,

[10]The graded commutator is simply the ordinary commutator for HS operators, the non trivial fact is that it is well-defined in L. The graded anticommutator instead is different from the ordinary anticommutator because it requires to reverse the grading in the second term.

to approximate the liouvillian in $\mathcal{H}$ with a commutator, i.e. an element of $L^-(\mathcal{H})$. This shows that the graded brackets are well defined operations in L and each of these brackets corresponds to a superalgebra[11], because the product of parities $\epsilon_1\epsilon_2$ appears giving a $\mathbb{Z}_2$ action. Applying the graded commutator to the coordinates x, p we obtain the graded CCR

$$[x^{\epsilon_1}, p^{\epsilon_2}] = (i\hbar)^{-\epsilon_1\epsilon_2} \ . \tag{3.5.21}$$

Note that $1^- = 0$ and $1^+ = 2$ meaning that coordinates with opposite parity are conjugated and coordinates with the same parity commute. As an example we consider again the case of a charged particle (with trivial metric)

$$i\hbar\frac{\partial}{\partial t}|\rho\rangle = \left\{-\frac{e}{m}\left[p^+/2 - eA(x^+/2)\right]^j \frac{\partial A_j}{\partial q^i}(x^+/2)x_i^- + \frac{1}{m}\left[p^+/2 - eA(x^+/2)\right]^i p_i^-\right\}|\rho\rangle \ . \tag{3.5.22}$$

The small scale limit corresponds to the following approximations

$$\partial_i f(x^+/2)x_i^- \simeq f(x)^- \ , \quad g(x^+/2) \simeq \frac{1}{2}g(x)^+ \ , \tag{3.5.23}$$

and in this limit (3.5.22) becomes

$$i\hbar\frac{\partial}{\partial t}|\rho\rangle = \left\{-\frac{e}{2m}\left[p - eA(x)\right]^{+,i} A(x)_i^- + \frac{1}{2m}\left[p - eA(x)\right]^{+,i} p_i^-\right\}|\rho\rangle \ , \tag{3.5.24}$$

that can be reduced to a commutator thanks to the identity

$$a^+a^- = \frac{1}{2}\{a^+, a^-\} = \frac{1}{2}\{a, a\}^- = (a^2)^- \ . \tag{3.5.25}$$

Using $a = p - eA$ the identity (3.5.25) allows to rewrite (3.5.24) as a commutator

$$i\hbar\frac{\partial}{\partial t}|\rho\rangle = \left\{\frac{1}{2m}\left[p - eA(x)\right]^2\right\}^- |\rho\rangle \ . \tag{3.5.26}$$

We observe that in this case the computation is performed at a pure operator level and this means that the approximations (3.5.23) have to be interpreted considering the action of the operators on the states. As discussed in the case of the scalar potential the local quantum approximation can be interpreted as a constraint on the state $\hat{\rho}$ which has to be sufficiently localized inside a region of radius $L(x)$, at point $x \simeq \langle\hat{x}\rangle_\rho$, meaning that $\Delta x = |\langle\Delta\hat{x}\rangle_\rho| << L(x)$. The scale $L(x)$ is defined in such a way that when the operators $f(x)^-$ and $f(x)^+/2$ are expanded with respect to x^-, the first order terms $f'(x^+)x^-$ and $f(x^+/2)$ are dominant with respect to the higher order terms if $\Delta x << L(x)$, thus, it is valid the approximation (3.5.23).

We can now consider the case of a non-trivial metric. For simplicity, we consider the case in which no potentials are present, i.e. a hamiltonian of the form

$$H(q,p) = \frac{1}{2}K_{ij}(q)p^ip^j \ . \tag{3.5.27}$$

Applying the Weyl transform we obtain the correspondent equation in the Hilbert space $\mathcal{H}$

$$i\hbar\frac{\partial}{\partial t}|\rho\rangle = \left[\frac{1}{8}\frac{\partial K_{ij}}{\partial q^k}(x^+/2)p^{+,i}p^{+,j}x^{-,k} + \frac{1}{2}K_{ij}(x^+/2)p^{+,i}p^{-,j}\right]|\rho\rangle \ . \tag{3.5.28}$$

[11]This is not a supercommutative algebra, because the sign we obtain exchanging a with b does not depend on parity.

As usual we consider the approximation

$$\frac{\partial K_{ij}}{\partial q^k}(x^+/2)x^{-,k} \simeq K^-_{ij}(x) \ , \quad K_{ij}(x^+/2) \simeq \frac{1}{2}K^+_{ij}(x) \ , \tag{3.5.29}$$

and from the symmetry of $K_{ij}(q)$ we have for the term in $p^{+,i}p^{-,j}$

$$K_{ij}(x^+/2)p^{+,i}p^{-,j} = \frac{1}{2}K_{ij}(x^+/2)\{p^{+,i},p^{-,j}\} = \frac{1}{2}K_{ij}(x^+/2)\{p^i,p^j\}^- = K_{ij}(x^+/2)(p^ip^j)^- \ . \tag{3.5.30}$$

After these substitutions equation (3.5.28) reads

$$i\hbar\frac{\partial}{\partial t}\left|\rho\right\rangle = \left[\frac{1}{8}K^-_{ij}(x)p^{+,i}p^{+,j} + \frac{1}{4}K^+_{ij}(x)(p^ip^j)^-\right]\left|\rho\right\rangle \ . \tag{3.5.31}$$

This equation is not yet in a form compatible with the superalgebra structure. This is due to the quadratic term in the momentum $K^-_{ij}(x)p^{+,i}p^{+,j}$. We observe that in the other cases the quadratic terms in the momentum appear only in the form given in (3.5.30). In the case of the position x^- this correspond to the case of quadratic potentials where the cancellation is exact. For position we used the approximation

$$f'(x^+/2)x^- \simeq f^-(x) \ , \quad f(x^+/2) \simeq \frac{1}{2}f^+(x) \ . \tag{3.5.32}$$

Formally, these expressions are obtained by expanding around x^+ treating x^- as a "small parameter". Then we can do the same, at least formally, for the momentum introducing the approximations

$$f'(p^+/2)p^- \simeq f^-(x) \ , \quad f(p^+/2) \simeq \frac{1}{2}f^+(p) \ . \tag{3.5.33}$$

Let us define the function of p

$$f^{ij}(p) = p^ip^j \ . \tag{3.5.34}$$

Using the approximations defined in (3.5.33) we have

$$\frac{1}{4}p^{+,i}p^{+,j} = f^{ij}(p^+/2) \simeq \frac{1}{2}f^{+,ij}(p) = \frac{1}{2}(p^ip^j)^+ \ . \tag{3.5.35}$$

Substituting in equation (3.5.31)

$$i\hbar\frac{\partial}{\partial t}\left|\rho\right\rangle = \left[\frac{1}{4}K^-_{ij}(x)(p^ip^j)^+ + \frac{1}{4}K^+_{ij}(x)(p^ip^j)^-\right]\left|\rho\right\rangle \ . \tag{3.5.36}$$

Equation (3.5.36) has exactly the form of the graded anticommutator defined in (3.5.18)

$$K^-_{ij}(x)(p^ip^j)^+ + K^+_{ij}(x)(p^ip^j)^- = \{K^-_{ij}(x),(p^ip^j)^+\} = \{K_{ij}(x),p^ip^j\}^- = 2[K_{ij}(x)p^ip^j + p^ip^jK_{ij}(x)]^- \ . \tag{3.5.37}$$

Therefore, equation (3.5.36) becomes

$$i\hbar\frac{\partial}{\partial t}\left|\rho\right\rangle = \left[\frac{1}{2}K_{ij}(x)p^ip^j + \frac{1}{2}p^ip^jK_{ij}(x)\right]^-\left|\rho\right\rangle \ , \tag{3.5.38}$$

which is the von Neumann equation for the hamiltonian

$$\hat{H} = \frac{1}{2}K_{ij}(\hat{x})\hat{p}^i\hat{p}^j + \frac{1}{2}\hat{p}^i\hat{p}^jK_{ij}(\hat{x}) \ . \tag{3.5.39}$$

The relation (3.5.35) can be interpreted exactly in the same way as done for position, i.e. as a condition on the state. In this case we require the state to be localized in the momentum space around a value $p \simeq \langle \hat{p} \rangle_\rho$ inside a region of radius $P(p)$ defined in such a way that the difference in the action of the operators $f(p^+/2)$ and $f^+(p)/2$ are higher order corrections which can be neglected when $|\langle \Delta\hat{p} \rangle_\rho| << P(p)$. We observe that in the case in which the localization is required both in x and p we encounter a problem because if $L(x)P(p) < \hbar/2$ we violate the uncertainty principle. In the classical framework this is allowed because the uncertainty principle holds only for positive definite states, but in general the classical state can be negative definite. This shows that the local quantum approximation gives the quantum dynamics but not the quantum Hilbert space which requires states which satisfy the uncertainty principle. Therefore, in this formalism quantization is achieved extending the dynamics which we derive for localized state to the quantum Hilbert space.

Chapter 4

Wigner-Weyl classical field theory

The Wigner-Weyl formalism gives a representation of classical dynamics very similar to the quantum one. In particular, using quadratic hamiltonians the classical and the quantum evolution are completely equivalent[1]. The dynamical obstruction between the classical and the quantum case is given by the evolution due to non-quadratic interaction terms. In particular in the WW representation the classical dynamics seems more general because it is not given by a commutator implying that pure states are not preserved. In this section we apply the WW formalism to classical field theories. We show that we can formulate a scattering theory analogous to the one derived in QFT and again we recover QFT when we can neglect the obstruction which does not allow the factorization of the path integral. Finally, at the path integral level, we want to verify if it is possible to find a relation similar to the classical-quantum Fourier duality observed in the context of QFT.

4.1 Interaction picture for Wigner-Weyl classical dynamics

In this section we show that we can derive an interaction picture for the WW dynamics which is crucial for a perturbative treatment of QFT. In order to study dynamics of general hamiltonians it is convenient to work again in the Hilbert-Schmidt space $\mathcal{H}_{HS}$ where states $|\Psi\rangle$ are given taking the square root $|\Psi\rangle = \hat{\Psi}$ of the (quasi-)density matrix $\hat{\rho} = \hat{\Psi}^\dagger\hat{\Psi}$. We observe that the states $|\psi\rangle\langle\psi|$ are idempotent $|\psi\rangle\langle\psi|^2 = |\psi\rangle\langle\psi|$, thus, they are Hilbert-Schimdt states. Let $\hat{\rho}_n = |n\rangle\langle n|$ be the density matrix associated to the n-th eigenstate of the harmonic oscillator $|n\rangle$. Because this is pure, it is an element of $\mathcal{H}_{HS}$

$$\langle\hat{\rho}_n|\hat{\rho}_n\rangle_{HS} = \mathrm{Tr}\Big[\hat{\rho}_n^\dagger\hat{\rho}_n\Big] = 1 < +\infty \ , \tag{4.1.1}$$

and we denote it by $|n,n\rangle = \hat{\rho}_n$. More generally the operators $|n\rangle\langle m|$ are HS states and form an orthonormal basis of $\mathcal{H}_{HS}$ which we denote by $|n,m\rangle$

$$\langle n,m|n',m'\rangle_{HS} = \mathrm{Tr}\left[|m\rangle\langle n|\,|n'\rangle\langle m'|\right] = \delta_{n,n'}\delta_{m,m'} \ . \tag{4.1.2}$$

Let us rewrite these states using the left and right representations of the annihilation and creation operators $\hat{a}, \hat{a}^\dagger$

$$|n,m\rangle = \frac{1}{\sqrt{n!m!}}(\hat{a}^\dagger)^n\,|0\rangle\langle 0|\,\hat{a}^m = \frac{1}{\sqrt{n!m!}}(\hat{a}_L^\dagger)^n(\hat{a}_R)^m\,|0,0\rangle \ . \tag{4.1.3}$$

[1]However, there are still differences at the level of the states because a classical state is in general a quasi-density matrix, thus it cannot always be interpreted as a quantum state.

Using the HS formalism we can give a unified treatment of the classical and the quantum case. The dynamics in $\mathcal{H}_{HS}$ is given by the Schrodinger-like equation

$$i\hbar\frac{\partial}{\partial t}\left|\Psi\right\rangle = \mathcal{L}_Q\left|\Psi\right\rangle \ , \tag{4.1.4}$$

where $\mathcal{L}$ is the liouvillian operator (classical or quantum). For a harmonic oscillator the classical liouvillian operator coincides with the quantum one

$$\mathcal{L}_0 = \mathcal{L}_{0,Cl} = \mathcal{L}_{0,Q} = (\hat{H}_0)_L - (\hat{H}_0)_R \ . \tag{4.1.5}$$

The HS time evolution operator is given by the exponential

$$\mathcal{U}(t) = \exp(-i\mathcal{L}_\alpha t/\hbar) \ . \tag{4.1.6}$$

The observable quantities are given by the expectation values

$$\langle\hat{A}\rangle_\rho = \mathrm{Tr}\Big[\hat{\rho}\hat{A}\Big] = \mathrm{Tr}\Big[\hat{\Psi}^\dagger\hat{\Psi}\hat{A}\Big] = \ \langle\Psi|\hat{A}_R|\Psi\rangle_{HS} = \ \langle\Psi|\hat{A}_L^\dagger|\Psi\rangle_{HS} \ . \tag{4.1.7}$$

We can introduce an interaction picture considering a hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{int}$ where $\hat{H}_{int}$ is an interaction term. Because the liouvillian is linear in $\hat{H}$ we have that the liouvillian $\mathcal{L}$ of $\hat{H}$ is

$$\mathcal{L}_\alpha = \mathcal{L}_{0,\alpha} + \mathcal{L}_{int,\alpha} \ . \tag{4.1.8}$$

We observe that the interacting classical liouvillian $\mathcal{L}_{int,Cl}$ is different in general from the quantum one $\mathcal{L}_{int,Q}$. In the interaction picture the time evolution of the HS states is

$$\left|\Psi(t)\right\rangle_I = \mathcal{U}_0^\dagger(t)\mathcal{U}(t)\left|\Psi\right\rangle \ , \tag{4.1.9}$$

and operators $\mathcal{O}$ of $\mathcal{H}_{HS}$ evolve with the free hamiltonian

$$\mathcal{O}_I(t) = \mathcal{U}_0^\dagger(t)\mathcal{O}\mathcal{U}_0(t) \ . \tag{4.1.10}$$

In the following, to simplify the notation we drop the index I in the evolution of the HS operators $\mathcal{O}$ and we denote the generic liouvillian with $\mathcal{L}$, dropping the index α. As in the usual case, we can define the interaction picture operator

$$\left|\Psi(t')\right\rangle_I \equiv \mathcal{U}_I(t',t)\left|\Psi(t)\right\rangle_I = \mathcal{U}_0^\dagger(t)\mathcal{U}(t'-t)\mathcal{U}_0(t)\left|\Psi(t)\right\rangle_I \ , \tag{4.1.11}$$

which satisfies the equation

$$i\hbar\frac{\partial}{\partial t}\mathcal{U}_I(t,t_0) = \mathcal{L}_{int}(t)\mathcal{U}_I(t,t_0) \ . \tag{4.1.12}$$

Imposing the initial condition $U_I(t_0,t_0) = 1$, the solution of equation (4.1.13) is the time ordered exponential

$$\mathcal{U}_I(t,t_0) = \mathrm{T}\text{-}\exp[-i\mathcal{L}_{int}(t)/\hbar] = \sum_n \frac{(-i)^n}{\hbar^n n!}\int_{t_0}^t dt_1\cdots\int_{t_0}^t dt_n \ \mathrm{T}[\mathcal{L}_{int}(t_1)\cdots\mathcal{L}_{int}(t_n)] \ , \tag{4.1.13}$$

which allows a perturbative treatment of the theory. We observe that the free evolution is given by the quantum rule for mixed states

$$\mathcal{U}_0(t)\,|\Psi\rangle = \hat{U}_0(t)\hat{\Psi}\hat{U}_0^\dagger(t)\ . \tag{4.1.14}$$

This implies that the quantum interacting liouvillian $\mathcal{L}_{int,Q}$ in interaction picture is

$$\mathcal{L}_{int,Q}(t)\,|\Psi\rangle = \mathcal{U}_0^\dagger(t)\mathcal{L}_{int,Q}\mathcal{U}_0(t)\,|\Psi\rangle = \hat{U}_0^\dagger(t)[\hat{H}_{int},\hat{U}_0(t)\hat{\Psi}\hat{U}_0^\dagger(t)]\hat{U}_0(t) = [\hat{H}_{int}(t),\hat{\Psi}]\ . \tag{4.1.15}$$

Therefore, it is the quantum liouvillian of the interaction hamiltonian in the interaction picture $\hat{H}_{int}(t) = \hat{U}_0^\dagger(t)\hat{H}_{int}\hat{U}_0^\dagger(t)$. In the classical case we do not have a commutator structure thus the argument above does not work. Actually, the result depends on the Poisson algebra structure only. Indeed the interaction liouvillian (both classical and quantum) can be rewritten using the Hadamard formula

$$\mathcal{L}_{int}(t) = \mathcal{U}_0^\dagger(t)\mathcal{L}_{int}\mathcal{U}_0(t) = \exp\left(i[\mathcal{L}_0,\ ]_{HS}t/\hbar\right)\mathcal{L}_{int}\ , \tag{4.1.16}$$

where $[\ ,\ ]_{HS}$ denotes the commutator between HS operators. This follows taking the time derivative of $\mathcal{L}_{int}(t)$ which gives the free HS Heisenberg equation

$$\frac{\partial}{\partial t}\mathcal{L}_{int}(t) = \frac{[\mathcal{L}_{int}(t),\mathcal{L}_0]_{HS}}{i\hbar}\ . \tag{4.1.17}$$

We denote with $\mathcal{L}(\hat{A})$ the liouvillian[2] associated to an operator $\hat{A}$ and with $[\hat{A},\hat{B}]_\alpha$ the general commutator, classical ($\alpha = Cl$) or quantum ($\alpha = Q$). Thanks to Jacobi identity we have the homomorphism relation

$$[\mathcal{L}(\hat{A}),\mathcal{L}(\hat{B})]_{HS}\,|\Psi\rangle = [\hat{A},[\hat{B},\hat{\Psi}]_\alpha]_\alpha - [\hat{B},[\hat{A},\hat{\Psi}]_\alpha]_\alpha = [[\hat{A},\hat{B}]_\alpha,\Psi]_\alpha = \mathcal{L}([\hat{A},\hat{B}]_\alpha)]\,|\Psi\rangle\ . \tag{4.1.18}$$

Therefore, defining

$$\mathcal{L}_{int}^{(n)} = \mathcal{L}([\hat{H}_0,\ ]_\alpha^n\hat{H}_{int})\ , \tag{4.1.19}$$

we obtain

$$[\mathcal{L}_0,\ ]_{HS}^n\mathcal{L}_{int} = [\mathcal{L}_0,\ ]_{HS}^{n-1}\mathcal{L}_{int}^{(1)} = \cdots = \mathcal{L}_{int}^{(n)}\ , \tag{4.1.20}$$

$$\exp\left(i[\mathcal{L}_0,\ ]_{HS}t/\hbar\right)\mathcal{L}_{int} = \mathcal{L}(\exp\left[i\mathcal{L}(\hat{H}_0)t/\hbar\right]\hat{H}_{int}) = \mathcal{L}(\hat{H}_{int}(t)) = [\hat{H}_{int}(t),\ ]_\alpha\ , \tag{4.1.21}$$

where in (4.1.21) we have used the fact that the evolution of an observable $\hat{A}$ in interaction picture is the same in the classical and in the quantum case and is given by

$$\hat{A}(t) = \hat{U}_0^\dagger(t)\hat{A}\hat{U}_0(t) = \exp\left[i\mathcal{L}(\hat{H}_0)t/\hbar\right]\hat{A}\ . \tag{4.1.22}$$

4.2 Classical field theory in the WW formalism

In this section we reformulate classical field theory in the Wigner-Weyl formalism. In the previous chapter we saw that, in the case of a classical system with n degrees of freedom and phase space $M = \mathbb{R}^{2n}$, an isomorphism between the functions on M and the self-adjoint operators of the quantum

[2] In classical mechanics this corresponds to the hamiltonian vector field associated to a hamiltonian function A.

Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ is given by the Weyl transform

$$\Phi[f] \equiv \hat{f} = \int d^n a d^n b \ \tilde{f}(a,b) \exp[i(a \cdot \hat{x} + b \cdot \hat{p})/\hbar] \ , \tag{4.2.1}$$

$$\tilde{f}(a,b) = \frac{1}{(2\pi\hbar)^{2n}} \int dq dp \ f(x,p) \exp[-i(a \cdot x + b \cdot p)/\hbar] \ , \tag{4.2.2}$$

which maps a classical function f of M in a self-adjoint operator on $\mathcal{H}$. Because Φ is an isomorphism, this allows to transport all the mathematical structures of the phase space in equivalent structures on the Hilbert space and vice versa. We observe that the Weyl map is obtained taking the ordinary Fourier transform $\tilde{f}(a,b)$ of $f(x,p)$ and then taking the operator-valued inverse Fourier transform where the classical variables x, p are substituted by the quantum non-commuting operators $\hat{x}, \hat{p}$. We can formally extend this construction to field theories. Consider the space $\mathcal{C}$ of the spatial configurations (i.e. at a fixed time t) of some classical field ϕ. Defining the conjugate momenta $\pi = \partial\mathcal{L}/\partial\dot{\phi}$, where $\mathcal{L}$ is the lagrangian density of the system, we can consider the phase space $\mathcal{M}$ of fields which is the set of all spatial configurations (ϕ, π) of the classical fields. A function on $\mathcal{M}$ is a functional $F[\phi, \pi]$ of the fields. Therefore, a natural generalization of the Weyl transform is

$$\Phi[F] \equiv \hat{F} = \int_{\mathcal{S}} dJ dI \ \tilde{F}[J,I] \exp\left[i \left(\int J\hat{\phi} + \int I\hat{\pi} \right) /\hbar \right] \ , \tag{4.2.3}$$

$$\tilde{F}[J,I] = \frac{1}{N} \int_{\mathcal{M}} d\phi d\pi \ F[\phi,\pi] \exp\left[-i \left(\int J\phi + \int I\pi \right) /\hbar \right] \ , \tag{4.2.4}$$

where N is a normalization constant. This definition formally extends the Weyl map to field theories. We observe that the Fourier transform becomes a functional Fourier transform and to evaluate it we have to perform a functional integral over the field phase space $\mathcal{M}$. The measure $d\phi d\pi$ can be defined using a discretization process. First, we discretize the space $\mathbb{R}^3$, working on a lattice with spacing a and we introduce a IR cutoff, considering a finite volume V of this space and we define the discretized measure

$$\int (d\phi d\pi)_{a,V} = \prod_{\boldsymbol{x_i} \in V} \int d\phi(\boldsymbol{x}_i) d\pi(\boldsymbol{x}_i) \ . \tag{4.2.5}$$

We can make this measure dimensionless, using a constant α of the dimension of $\phi\pi$ and making the substitution $d\phi(\boldsymbol{x}_i)d\pi(\boldsymbol{x}_i) \to d\phi(\boldsymbol{x}_i)d\pi(\boldsymbol{x}_i)/\alpha$. Introducing a length scale L in the case of a scalar field we have $[\phi] = \hbar^{\frac{1}{2}} L^{-1}$ and $[\pi] = \hbar^{\frac{1}{2}} L^2 c^{-1}$, thus we can use a constant $\alpha = \hbar L/c$. All the manipulations we do in the following make sense if a UV and IR cutoffs are understood. The measure $d\phi d\pi$ is obtained taking the limit $a \to 0, V \to +\infty$ and this leads to the same problems encountered in the definition of a measure for the definition of an Hilbert space and a path integral in QFT. Because the configurations on $\mathcal{M}$ are distributions, we expect that the configurations (J, I) on the dual space $\mathcal{S}$ must be test functions belonging to Schwartz space. Indeed, following the axiomatic approach, the field operators $\hat{\phi}(x), \hat{\pi}(x)$, are operator-valued distribution thus they make sense only if they are smeared with some test function, $\hat{\phi}(J) = \int dx \ \phi(x)J(x)$. We can bypass all these problems defining the Wigner-Weyl map in an algebraic way using the formal rules to pass from phase space to Hilbert space.

The map (4.2.3) formally allows to associate to a classical functional $F[\phi, \rho]$ an operator $\hat{F}$. It is natural to ask if using this map we can reformulate classical field theories in a way similar to QFT. For simplicity, we study the case of a scalar field ϕ.

Considering an ensemble of spatial configurations of the field ϕ and its momentum π at time

$t = 0$, we can introduce a phase space functional probability distribution $\rho = \rho[\phi, \pi, t]$ which gives the probability to find the system in a spatial configuration (ϕ, π) at time t. The evolution of ρ is given by the functional Liouville equation

$$\dot{\rho} = \{H, \rho\} = \int d^3x \frac{\delta H}{\delta \phi(\boldsymbol{x})} \frac{\delta \rho}{\delta \pi(\boldsymbol{x})} - \frac{\delta H}{\delta \pi(\boldsymbol{x})} \frac{\delta \rho}{\delta \phi(\boldsymbol{x})} \ . \tag{4.2.6}$$

The expectation value of a functional $F = F[\phi, \pi]$ in the state $\rho(t)$ at time t is given formally by the integral

$$\int d\phi d\pi \ F[\phi, \pi] \rho[\phi, \pi, t] \ . \tag{4.2.7}$$

Using the Weyl transform we can map this expectation value in a trace in the quantum Hilbert space

$$\mathrm{Tr}\left[\hat{\rho}(t)\hat{F}\right] \ . \tag{4.2.8}$$

Following the same steps of the case of finite degrees of freedom, we have that the liouvillian $\mathcal{L}_{Cl} = [\![\hat{H}, \ ]\!]$ is a self-adjoint operator in the HS space $\mathcal{H}_{HS}$ and taking the square root $\hat{\psi}$ we can rewrite (4.2.9) as an expectation value in the HS space

$$\mathrm{Tr}\left[\hat{\rho}(t)\hat{F}\right] = \langle \psi(t)|\hat{F}_R|\psi(t)\rangle_{HS} \ , \tag{4.2.9}$$

where $\hat{F}_R$ denotes the right-action representation of the observables of the quantum Hilbert space $\mathcal{H}$ and $|\psi(t)\rangle$ is given by the HS time evolution operator

$$|\psi(t)\rangle = \mathcal{U}_{Cl}(t)\,|\psi\rangle = \exp(-i\mathcal{L}_{Cl}t/\hbar) \ . \tag{4.2.10}$$

Using the fact that $\mathcal{L}_{Cl}$ is self-adjoint in HS, and therefore $\mathcal{U}_{Cl}(t)$ is unitary, we can move the evolution to the observable $\hat{F}_R$

$$\langle \psi(t)|\hat{F}_R|\psi(t)\rangle_{HS} = \langle \psi|\mathcal{U}^\dagger_{Cl}(t)\hat{F}_R\mathcal{U}_{Cl}(t)|\psi\rangle_{HS} = \langle \psi|\hat{F}_{R,H}(t)|\psi\rangle_{HS} \ , \tag{4.2.11}$$

which defines the Heisenberg picture $\hat{F}_{R,H}(t)$. In particular considering the pointwise functionals $\boldsymbol{x}[\phi] = \phi(\boldsymbol{x})$ and $\boldsymbol{x}[\pi] = \pi(\boldsymbol{x})$ which gives the time-zero fields ϕ, π at the position $\boldsymbol{x}$ we can map them in operators $\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{x})$ which satisfies the CCR

$$[\hat{\phi}(\boldsymbol{x}), \hat{\pi}(\boldsymbol{y})] = i\hbar\delta^{(3)}(\boldsymbol{x} - \boldsymbol{y}) \ . \tag{4.2.12}$$

We can associate to such operators their "left" and "right" representations $\hat{\phi}_{L,R}(\boldsymbol{x})$ and the time evolution for these representations is given by the Heisenberg picture

$$\hat{\phi}_{L,R}(x) \equiv \mathcal{U}^\dagger_{Cl}(t)\hat{\phi}_{L,R}(\boldsymbol{x})\mathcal{U}_{Cl}(t) \ , \tag{4.2.13}$$

$$\hat{\pi}_{L,R}(x) \equiv \mathcal{U}^\dagger_{Cl}(t)\hat{\pi}_{L,R}(\boldsymbol{x})\mathcal{U}_{Cl}(t) \ . \tag{4.2.14}$$

We start the analysis from free field theories. Consider a relativistic[3] classical free scalar field $\phi^{(0)}$

[3] We use the signature $(+, -, -, -)$ for the metric.

with mass m. Its hamiltonian is

$$H = H[\phi, \pi] = \int d^3x \frac{1}{2c}\pi^2 + \frac{1}{2}\left(\partial_i\phi\partial^i\phi + \frac{m^2c^2}{\hbar^2}\phi^2\right) , \tag{4.2.15}$$

where $\partial_i = \partial/\partial x^i$. The field ϕ_0 admits a plane wave decomposition

$$\phi^{(0)}(x) = \int d^3k \frac{\sqrt{\hbar c}}{(2\pi)^3 2\omega_k}\left(a_k e^{-ikx} + a_k^* e^{ikx}\right) , \tag{4.2.16}$$

where the four-vector k^μ is on-shell has the dimension of an inverse length and $k^0 c = \omega_k = \sqrt{\boldsymbol{k}^2c^2 + m^2c^4/\hbar^2}$. Using the Weyl transform this is mapped to a quantum free scalar field

$$\hat{\phi}^{(0)}(x) = \int d^3k \frac{\sqrt{\hbar c}}{(2\pi)^3 2\omega_k}\left(\hat{a}_k e^{-ikx} + \hat{a}_k^\dagger e^{ikx}\right) , \tag{4.2.17}$$

where $\hat{a}_k$ corresponds to the Weyl transform of a_k and using the CCR for $\hat{\phi}$ we obtain the usual algebra for bosonic creation and annihilation operators with covariant normalization

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = (2\pi)^3 2\omega_k \delta^{(3)}(\boldsymbol{k} - \boldsymbol{k}') , \tag{4.2.18}$$

and the operators $\hat{a}_k$ annihilates the vacuum state $|0\rangle$ which corresponds to the ground state of $\hat{H}$ and, as shown in the previous section, can be interpreted as a classical Boltzmann distribution at temperature $\beta_0 = \int d^3k/\rho_{vac}$ where ρ_{vac} is the energy density of the ground state

$$\rho_{vac} = \int d^3k \ \frac{\hbar\omega_k}{2} , \tag{4.2.19}$$

which is UV divergent. In the free case the left and right representations $\hat{\phi}_{R,L}(\boldsymbol{x})$ evolve simply with the quantum evolution rule

$$\hat{\phi}^{(0)}_{L,R}(x) = \mathcal{U}_0^\dagger(t)\hat{\phi}_{L,R}(\boldsymbol{x})\mathcal{U}_0(t) = [(\hat{U}_0)^\dagger(t)\phi(\boldsymbol{x})\hat{U}_0(t)]_{L,R} = [\hat{\phi}^{(0)}(x)]_{L,R} , \tag{4.2.20}$$

where $\mathcal{U}_0(t)$ is the HS free time evolution operator which is the same in the classical and in the quantum case and $\hat{U}_0(t)$ is the free time evolution operator in $\mathcal{H}$.

4.3 Scattering matrix and LSZ formula for WW classical field theory

In the previous section we saw that classical free field theories admit a formulation similar to QFT. In this section we want to consider the interacting case. In this case the algebraic structure of classical fields differs from the quantum one because the interaction has non-quadratic terms. We are interested in considering scattering experiments, where some free particles are prepared at a time in the far past, ideally at $t = -\infty$, and evolving in time they interact in a non-trivial way and finally become again free particles in the far future, ideally at $t = +\infty$ where they are detected. Therefore, we assume that the Hamiltonian $\hat{H}$ has an interaction term such that in the far past and in the far future the field representations $\hat{\phi}_{L,R}(x)$ tends to a free field

$$\hat{\phi}_{L,R}(x) \underset{t\to-\infty}{\longrightarrow} \hat{\phi}^{(in)}_{L,R}(x) , \tag{4.3.1}$$

$$\hat{\phi}_{L,R}(x) \underset{t\to+\infty}{\longrightarrow} \hat{\phi}^{(out)}_{L,R}(x) , \tag{4.3.2}$$

which defines the "in" and "out" free fields $\hat{\phi}^{(in)}(x)$ and $\hat{\phi}^{(out)}(x)$. As for QFT, assuming that the interacting field satisfies the CCR implies that these limits can be only understood as weak limits i.e. at the level of matrix elements. As shown in the previous section, thanks to the equivalence of quantum dynamics and classical dynamics at the quadratic level we have that the behaviour of a free scalar classical field is the same of a scalar quantum free field. In the free case we have the relation

$$\hat{a}(k)_{L,R} = i \int d^3x \; \hat{\phi}^{(0)}_{L,R}(x) \overset{\leftrightarrow}{\partial}_0 e^{ikx} \; , \tag{4.3.3}$$

where $\overset{\leftrightarrow}{\partial}_0 = \overset{\rightarrow}{\partial}_0 - \overset{\leftarrow}{\partial}_0$. In the interacting case we use these relations as the definition of creation and annihilation operators at time t

$$\hat{a}_{L,R}(k,t) \equiv i \int d^3x \; \hat{\phi}_{L,R}(x) \overset{\leftrightarrow}{\partial}_0 e^{ikx} \; , \tag{4.3.4}$$

which in general do not satisfy the CCR. Asimptotically, we have that the HS fields $\hat{\phi}_{L,R}(x)$ tend to the free fields $\hat{\phi}_{in}$ and $\hat{\phi}_{out}$ which define two asymptotic Fock spaces through asymptotic creation and annihilation operators $\hat{a}_{k,in} = \lim_{t\to-\infty} \hat{a}_k(t)$, $\hat{a}_{k,out} = \lim_{t\to+\infty} \hat{a}_k(t)$ which act on the full vacuum state $|\Omega\rangle$. In the ordinary quantum scattering experiments we are interested in the scattering amplitudes between scattering states in the far past and in the far future. Consider a state of n free particles in the far past

$$|p_1, \dots, p_n\rangle_{in} = \hat{a}^\dagger_{p_1,in} \cdots \hat{a}^\dagger_{p_1,in} |\Omega\rangle = \left(\prod_{j=0}^{n} \hat{a}_{p_j,in} \right) |\Omega\rangle \; , \tag{4.3.5}$$

and a set of m particles in the far future

$$|q_1, \dots, q_m\rangle_{out} = \hat{a}^\dagger_{q_1,out} \cdots \hat{a}^\dagger_{q_m,in} |\Omega\rangle = \left(\prod_{j=0}^{n} \hat{a}_{q_j,out} \right) |\Omega\rangle \; . \tag{4.3.6}$$

We are interested in the scattering amplitude

$$\langle (q_1, \dots, q_m)_{out} | (p_1, \dots, p_n)_{in} \rangle \equiv \langle q_1, \dots, q_m | \hat{S} | p_1, \dots, p_n \rangle \; , \tag{4.3.7}$$

which defines the matrix elements of the scattering matrix $\hat{S}$. In the HS space the scattering states can be rewritten

$$|p_1, \dots, p_n\rangle_{in,HS} = \left(\prod_{j=0}^{n} \hat{a}_{p_j,in} \right)_L \left(\prod_{k=0}^{n} \hat{a}_{p_k,in} \right)_R |\Omega\rangle_{HS} \; , \tag{4.3.8}$$

$$|q_1, \dots, q_n\rangle_{out,HS} = \left(\prod_{j=0}^{m} \hat{a}_{q_j,in} \right)_L \left(\prod_{k=0}^{n} \hat{a}_{p_k,in} \right)_R |\Omega\rangle_{HS} \; , \tag{4.3.9}$$

where $|\Omega\rangle_{HS} = |\Omega\rangle\langle\Omega|$.The HS matrix element $\langle q_1, \dots, q_m | \hat{S} | p_1, \dots, p_n \rangle_{HS}$ gives the probability (i.e. square modulus of the amplitude) to have a scattering between the scattering states $|p_1, \dots, p_n\rangle$ and

$|q_1,\dots,q_m\rangle$. Repeating the same steps used in ordinary QFT we can derive the HS "LSZ formula"

$$\langle q_1,\dots,q_m|\hat{S}|p_1,\dots,p_n\rangle_{HS} = \tag{4.3.10}$$

$$(-1)^{n+m}\int\prod_{j=0}^{n}\prod_{k=0}^{n} d^4x_j d^4y_j d^4x'_k d^4y'_k \exp\left[-ip_j(x-y)_j + iq_k(x'-y')_k\right] \tag{4.3.11}$$

$$(\Box_{x_j}+m^2)(\Box_{y_j}+m^2)(\Box_{x'_k}+m^2)(\Box_{y'_k}+m^2) \tag{4.3.12}$$

$$\langle\Omega|\mathrm{T}[\hat{\phi}_L(x_1)\hat{\phi}_L(x'_1)\cdots\hat{\phi}_L(x_n)\hat{\phi}_L(x'_n)\hat{\phi}_R(y_1)\hat{\phi}_R(y'_1)\cdots\hat{\phi}_R(y_m)\hat{\phi}_R(y'_m)]|\Omega\rangle_{HS}\ , \tag{4.3.13}$$

where the Klein-Gordon operators $(\Box+m^2)$ cancel the poles of the propagators in the external legs. This formula has the same structure of the ordinary QFT formula, but the correlators here involves "left" and "right" fields $\hat{\phi}_L$ and $\hat{\phi}_R$, and is valid both for quantum and classical dynamics. This allows to consider classical scattering processes using the same formalism of QFT, in particular we can compute the classical correlators using a path integral representation. Following a derivation analogous to the finite case, given a scalar field theory with action $S[\phi]$, the classical generating functional is given by the path integral

$$Z_{Cl}[J_L,J_R] = \int \mathcal{D}\phi_L\mathcal{D}\phi_R \exp\left[i(S[\phi_L]-S[\phi_R]+\Delta S[\phi_L,\phi_R]+J_L\phi_L-\phi_R J_R)/\hbar\right]\ , \tag{4.3.14}$$

which is antisymmetric in J_L, J_R. As in the case of particle mechanics we have the obstruction term

$$\Delta S[\phi_L,\phi_R] = V'\left(\frac{\phi_L+\phi_R}{2}\right)(\phi_L-\phi_R) - V(\phi_L)+V(\phi_R)\ , \tag{4.3.15}$$

which does not allow to factorize the path integral. Taking the functional derivatives with respect to the sources J_L, J_R we obtain the classical correlation functions in the HS space

$$\langle\Omega|\mathrm{T}[\hat{\phi}_L(x_1)\cdots\hat{\phi}_L(x_n)\hat{\phi}_R(y_1)\cdots\hat{\phi}_R(y_m)]|\Omega\rangle_{HS} = (-i\hbar)^n(i\hbar)^m\frac{\delta^{n+m}Z[J_L,J_R]}{\delta J_L(x_1)\cdots\delta J_L(x_n)\delta J_R(y_1)\cdots\delta J_R(y_m)}\ . \tag{4.3.16}$$

We observe that in the scattering amplitude (4.3.10) the right and left states coincide. This is due to the fact that only the states $\sim|n,n\rangle$ are involved in the process, which are eigenstates of the free hamiltonian. A general scattering experiments involves states which in general can be wavepackets of the scattering states (4.3.5) and (4.3.6) or, if the assumption of purity is dropped, mixed states, which are quantum ensembles of wavepackets. In the quantum case we can make the assumption that our measurements select eigenstates of the Hamiltonian and we can, in principle, neglect these problems. However, in the classical case, the scattering states (4.3.5) and (4.3.6) cannot be interpreted as classical states because their Wigner function is not positive definite and the previous assumption is not allowed and we are forced to consider the scattering of generic density matrices. Given a generic density matrix $\hat{\rho}$ and taking the square root $\hat{\psi}$ and expanding in the basis $|n,m\rangle$ we obtain (for simplicity we work in QM, the generalization to QFT is straightforward)

$$|\psi\rangle = \hat{\psi} = \sum_{n,m}\langle n|\psi|m\rangle\,|n\rangle\langle m| = \sum_{n,m}\langle n,m|\psi\rangle_{HS}\,|n,m\rangle\ . \tag{4.3.17}$$

We recall that for pure states $|\psi\rangle\langle\psi|$ the HS square root is again $|\psi\rangle\langle\psi|$ thanks to idempotence $|\psi\rangle\langle\psi|^2 = |\psi\rangle\langle\psi|$. Expanding this state in the basis of scattering states we obtain

$$|\psi\rangle\langle\psi| = \sum_{n,m} \langle n|\psi\rangle \langle\psi|m\rangle |n\rangle\langle m| = \sum_{n,m} \langle n|\psi\rangle |n,m\rangle \ , \tag{4.3.18}$$

This implies that, if the prepared states are wavepackets of states $|n\rangle$, in the scattering we have contributions from the "off-diagonal states" where the left and right particles are not the same. Therefore, when wavepackets, or more in general mixed states, are used in the computation, we have states $|n,m\rangle$ with n "left particles" and m "right particles". In the quantum case this set of "particles" does not interact and this leads, in the case of pure states, to a factorization of the scattering probability, and the problem reduces only to determine the scattering amplitude. In the classical case, because the dynamics does not preserve pure states, we have interactions between the two sets of particles, left and right, which is precisely given by the obstruction (4.3.15) which appears in the path integral (4.3.14). Therefore, in the classical case, even when the asymptotic classical state corresponds to a pure quantum state[4], although we can define a "scattering amplitude" taking the square root of the probability, this does not have a simple structure, in particular is not given by a LSZ formula, because of additional correlations due to the "interaction" between the left and right representations, which are absent in the quantum case.

4.4 Analysis of the WW path integral for field theory

We conclude the analysis of this chapter briefly considering the relation between the quantum path integral and the classical one. In particular, we show that we can rewrite the classical partition function Z_{Cl} in a way which resembles the classical-quantum duality observed in the context of QFT, although we do not have precisely the Fourier duality. Indeed, we show that we can obtain the classical partition function Z_{Cl} summing over a sort of "quantum fluctuations".

To see this, we start from the definition of the classical partition function

$$Z_{Cl} = \int \mathcal{D}\phi_L \mathcal{D}\phi_R \exp\left[i(S[\phi_L] - S[\phi_R] + \Delta S[\phi_L, \phi_R])/\hbar\right] \ . \tag{4.4.1}$$

We rename $\phi_L = \phi$ and $\phi_R = \varphi$ and we consider the following functional

$$Z_{eff}[\varphi] = \int \mathcal{D}\phi \exp\left[i(S[\phi] + \Delta S[\phi, \varphi])/\hbar\right] \ , \tag{4.4.2}$$

where φ is a fixed classical background field. The functional (4.4.2) is the quantum partition function for a theory $S[\varphi]$ which is perturbed with an interaction $\Delta S[\phi, \varphi]$ which couples the quantum field ϕ with the classical field φ. We can define an effective action $S_{eff}[\varphi]$ as

$$S_{eff}[\varphi] = -i\hbar \log Z_{eff}[\varphi] \ . \tag{4.4.3}$$

Substituting (4.4.3) in (4.4.1) we obtain

$$Z_{Cl} = \int \mathcal{D}\phi \exp\left[-i(S[\varphi] - S_{eff}[\varphi])/\hbar\right] . \tag{4.4.4}$$

The term[5] $\Delta S_{eff}[\varphi] = S[\varphi] - S_{eff}[\varphi]$ can naturally be interpreted as a quantum fluctuation where

[4] Meaning that its Wigner function is positive definite, which is the case of gaussian pure states.

[5] This does not correspond to the obstruction term ΔS.

only the off-shell contribution is considered. Indeed, in the quadratic case we have $\Delta S = 0$ which implies that $S_{eff}[\varphi] \sim S[\varphi_{Cl}]$, where φ_{Cl} is the solution of the classical equation of motion. Therefore, $\Delta S_{eff}[\varphi] = S[\varphi] - S[\varphi_{Cl}]$, meaning that the on-shell contribution is subtracted. Summing over all possible fluctuations gives the classical partition function.

Conclusions

In this thesis we have tried to verify if it is possible to derive the quantum theory as a limit of the classical theory. To do this, we have reformulated classical mechanics using an operatorial formalism in such a way to directly compare the classical and the quantum theories. Starting from the KvN formalism we have shown that it is possible to find a path integral representation of classical mechanics and that this representation leads to consider the Wigner-Weyl formalism. The Wigner-Weyl formalism allows a systematic analysis of the obstruction to obtain a quantum theory starting from the classical one. In particular, we have seen that using the classical path integral derived from this formalism, we can obtain a quantum theory if we are able to factorize the path integral in two amplitudes. We have shown that formally the limit $\hbar \to 0$, which usually gives the classical limit, admits a dual interpretation as a quantum limit. Indeed, starting from the classical algebraic structure in the quantum Hilbert space obtained applying the WW formalism, we have that in the limit $\hbar \to 0$ the classical structure reduces to the quantum one, given by the non-commutative operator product.

The factorization of the path integral is always possible for quadratic hamiltonians and, indeed, in this case classical and quantum dynamics are the same. Because of this fact and because the properties of the harmonic oscillator are crucial for QFT, in this case we have also explored the relations between the classical states and the quantum one. We have shown that the quantum eigenstates obtained from the creation and annihilation operators $\hat{a}, \hat{a}^\dagger$ have a classical counterpart which is essentially given by the classical states $\rho_n = H_0^n e^{-\beta H_0}/Q_n$ where H_0 is the classical hamiltonian of the harmonic oscillator and $Q_n = \int H_0^n e^{-\beta H_0}$ is the normalization of the state. We have that the states ρ_n correspond to a gaussian smearing over a phase space region $\Delta x \Delta p = \hbar$ of the quantum Wigner functions W_n associated to the quantum excited states. In the limit in which $\Delta x \Delta p \to 0$ we have that the gaussian smearing disappears and we obtain the quantum eigenstates, i.e. the classical states reduce to the quantum ones in this limit.

We have then analyzed the dynamics for arbitrary non-quadratic potentials. We have seen that in this case the factorization of the path integral is not possible because of obstructions due to non quadratic potentials. At the operator level this obstruction is related to the impossibility of writing the classical liouvillian operator $\mathcal{L}_{Cl} |\hat{\rho}\rangle = [\![\hat{H}, \hat{\rho}]\!]$, which is the generator of the classical evolution, as a quantum commutator $\mathcal{L}_Q |\hat{\rho}\rangle = [\hat{H}, \hat{\rho}]$. We have shown that in the "local quantum approximation" that is in the limit $\Delta x << L(x)$ with $L(x)$ a local length scale which depends on the potentials, the classical dynamics can be approximated with the quantum one i.e. we can obtain a quantum commutator. In this way we obtain the quantum dynamical law from an approximation of the classical one. This limit can be interpreted as a limit on the states of the systems which has to be sufficiently localized in order to neglect the higher order terms and reduce the classical dynamics to a quantum one. Precisely, if the state $\hat{\rho}$ is localized in $x \simeq \langle \hat{x} \rangle_\rho$ we want its uncertainty to be sufficiently small in such a way that $| \langle \Delta \hat{x} \rangle_\rho | << L(x)$. Next we have analysed the case in which a static electromagnetic field is introduced

and we have shown that also in this case the approximation applies.

When a non trivial metric is introduced in the hamiltonian, the local quantum approximation in the position is not sufficient to obtain a commutator and we have also to introduce a momentum scale $P(p)$ for a state with momentum localized in p and to require a localization in the momentum space, $|\langle \Delta\hat{p} \rangle_\rho| << P(p)$. As we said, this in general may lead to a violation of the uncertainty principle and so the procedure requires to use classical states highly localized in phase space, which correspond to non-positive definite quasi-density matrices. Therefore, this limit concerns the dynamics and the quantization procedure consists in extending the commutator founded for localized classical states to the whole space of quantum states.

At an algebraic level, the cancellations which arise when the local quantum approximation is performed and which allow to obtain a quantum commutator are related to a superalgebra structure of the space of HS operators

$$[a^{\epsilon_1}, b^{\epsilon_2}] = a^{\epsilon_1} b^{\epsilon_2} - b^{\epsilon_2} a^{\epsilon_1} = [a, b]^{-\epsilon_1 \epsilon_2} \ , \tag{4.4.5}$$

$$\{a^{\epsilon_1}, b^{\epsilon_2}\} = a^{\epsilon_1} b^{\epsilon_2} + b^{-\epsilon_2} a^{-\epsilon_1} = \{a, b\}^{\epsilon_1 \epsilon_2} \ . \tag{4.4.6}$$

The key fact is that, after the approximation, in the classical liouvillian there appear the graded brackets (4.4.5), (4.4.6) which gives the quantum commutator.

Finally, we have extended the Wigner-Weyl formalism to field theories. We have verified that for a classical field theory we can define field operators which evolve classically and we can use them to derive a scattering theory similar to the QFT one. We have also shown that in the classical case we can define an interaction picture, which allows for a perturbative treatment at the operator level, and the LSZ formula, which express the classical scattering probability in terms of classical correlators. Finally, we have shown that we can derive a path integral for classical field theory and that to get a QFT we need to factorize this path integral, as in the case of particle mechanics. Using the expression of the classical partition function we could recover a relation which resembles, conceptually, the classical-quantum Fourier duality

$$\exp(iS[\phi]/\hbar) = \int \mathcal{D}J \ Z[J] \exp\left(i \int J\phi/\hbar\right) \ , \tag{4.4.7}$$

in which a classical object is seen as a sum over the quantum contributions. Precisely, this relation is given by

$$Z_{Cl} = \int \mathcal{D}\varphi \exp\left(i\Delta S_{eff}[\varphi]/\hbar\right) \ . \tag{4.4.8}$$

where the term $\Delta S_{eff}[\varphi]$ is obtained from the quantum path integral and can be interpreted as a contribution due to quantum fluctuations.

Bibliography

[1] N. Atakishiyev et al., "The Husimi distribution function and superposition of q-harmonic oscillator stationary states", 2014.

[2] A J Bracken, *Quantum mechanics as an approximation to classical mechanics in Hilbert space*, *Journal of Physics A: Mathematical and General* 36.23 (May 2003), L329–L335.

[3] Pieter W. Claeys and Anatoli Polkovnikov, *Quantum eigenstates from classical Gibbs distributions*, *SciPost Phys.* 10 (2021), 14.

[4] H. Elze, G. Gambarotta, and F. Vallone, *A path integral for classical dynamics, entanglement, and Jaynes-Cummings model at the quantum-classical divide*, *International Journal of Quantum Information* 09 (2010), 203–224.

[5] E. Gozzi, M. Reuter, and W. D. Thacker, *Hidden BRS invariance in classical mechanics. II*, *Phys. Rev. D* 40 (10 Nov. 1989), 3363–3377.

[6] Satio Hayakawa, *Atomism and Cosmology*, *Progress of Theoretical Physics Supplement* E65 (Jan. 1965), 532–541.

[7] B. O. Koopman, *Hamiltonian Systems and Transformation in Hilbert Space. Proceedings of the National Academy of Sciences of the United States of America* 17 5 (1931), 315–8.

[8] J. v. Neumann, *Zur Operatorenmethode In Der Klassischen Mechanik*, *Annals of Mathematics* 33.3 (1932), 587–642.

[9] H. Weyl, *Quantenmechanik und Gruppentheorie*, *Zeitschrift fur Physik* 46.1-2 (Nov. 1927), 1–46.

[10] E. Wigner, *On the Quantum Correction For Thermodynamic Equilibrium*, *Phys. Rev.* 40 (5 June 1932), 749–759.

[11] Hong You, Shaowu Liu, and Guodong Zhang, *Rank one preserving R-linear maps on spaces of self-adjoint operators on complex Hilbert spaces*, *Linear Algebra and its Applications* 416.2 (2006), 568–579.