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Gravitational waves
sourced by primordial scalar perturbations

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Abstract

Primordial enhanced scalar perturbations can source a significant amount of gravitational waves (GWs). Existing studies compute the SGWB produced in the RD-era by the primordial enhanced scalar perturbations with the Green formalism. In this work we review this result by using both Green function method and the in-in formalism, and eventually we go over by evaluating the subdominant contribution from primordial enhanced-scalar & tensor perturbations.

To my parents and friends.

Contents

Introduction

It is astonishing how in the course of the previous and present century the potentials of cosmology were revealed. From a modern viewpoint, the detection of the cosmic microwave background (CMB) radiation is a milestone in cosmology analogous to the Hubble's discovery of the expansion of the universe for our ancestors of early 20th century. The implications of this detection not only concern to astrophysics, but they extend to fundamental physics and cosmology. It is the personal opinion of the - yet inexperienced - author that the amount of physical informations about the universe we can extract from the CMB is extraordinary. They cover the greatest temporal distance ever experienced, letting us measure indirect effects of the fundamental physics well beyond the reach of our particle accelerators.

It would not be fair, if not for the purpose of this introduction, not mentioning others important milestones of cosmology. For the author purpose, it will be sufficient to highlight the recent detection of gravitational waves (GWs) on Sept. 2015 [1]. After few decades of experimental attempts, the first detection of a gravitational wave gave us a new powerful probe for the exploration and understanding of our universe. Analogously to the CMB, the implications of this detection concern not only astrophysics but fundamental physics and cosmology. The latter refers to the fact that due to the weakness of gravity, the GWs decouple from the rest of matter/radiation components of the universe soon after production. Moreover, if we indicate with Γ the rate of interactions for the GWs and with H the Hubble rate [2]

$$\left(\frac{\Gamma}{H}\right) \sim \left(\frac{T}{M_P}\right)^3 \tag{1}$$

with M_P the Plank mass and T the universe temperature. The weaker the interaction rate of GWs is, the higher is the energy scale they drop out of equilibrium and decouple from the other components of the universe. The estimate above shows that for $T < M_P$ the GWs interaction rate is smaller than the Hubble rate, therefore GWs propagate freely in the early universe as soon as they are generated. As well as the CMB is a thermal relic of the $T \sim MeV$, the relic gravitational waves are a potential source of informations about the universe at epochs and energy scales unreachable by any other means.

The author mentioned the CMB radiation to emphasize that the potential detection of the stochastic gravitational wave background (SGWB) is in principle comparable to the discovery of CMB, which we may consider marking the beginning of modern cosmology. Current GW interferometers, like LIGO and VIRGO, put upper limits on the SGWB amplitude [3] although to the present day there are no direct measurements. The detection of this signal is in the potential reach of near-future instruments both earth based (e.g. KAGRA) and space based (e.g. LISA).

One of the puzzles affecting the modern cosmology is the nature of the dark matter. In the past, all the observations of dark matter regarded its gravitational interactions. The newborn window on the universe, provided by the GWs detectors, constitutes a new tool to investigate the dark matter.

In the last few years, some interest reappeared in the possibility that part of the dark matter in the universe is made of primordial black holes (PBHs), for instance see [4] for a review. In particular, some scenarios where PBHs are based on the fundamental physics of inflation have important consequences for cosmology. The formation of these PBHs is accompanied with GWs in the potential reach of present and near-future GW detectors [5], whose signal may contain a signature of these processes [6]. Specifically, the inflationary processes we have in mind produce some enhancement of the primordial nearly scale-invariant scalar perturbations (see sec. 5.8) which unavoidably features the gravitational waves produced in the late RD era from them. This way, the GWs represent an investigation technique to test these mechanism of PBHs production and, broadly speaking, to study the dark matter.

This work studies the production of GWs during radiation dominated (rd) era, sourced by first order primordial scalar & tensor perturbations. Existing studies compute the SGWB produced by the primordial scalar perturbations with the Green function method. We review this result with the in-in formalism and eventually we evaluate a subdominant contribution, from a source made of primordial scalar & tensor perturbations. Eventually, we numerically evaluate these contributions under the assumption of a monochromatic enhancement in the spectrum of primordial scalar perturbations.

Outline

This work is organized as follows

Chap. 1
: We introduce the FLRW metric, the observed universe and the conformal frame.

Chap. 2
: We study the causal structure of the FLRW spacetime, first in general and later for physical cosmological solutions. In order to introduce the inflation theory we mention the horizon problem, also rephrasing it in terms of CMB properties, and its solution. We find the necessary condition for accelerated expansion, eventually implementing the inflation-stage with a single slow-rolling scalar field theory.

Chap. 3
: We illustrate the relation between quantum fluctuations and primordial perturbations. We use the case-study of the quantum fluctuations of a massless field during a de Sitter expansion, to show the canonical quantization procedure and to discuss the boundary conditions on cosmological observables.

Chap. 4
: This chapter is devoted to introduce the in-in formalism. The machinery is introduced in close analogy with the in-out operatorial formulation of QFT, but emphasizing the conceptual and practical differences. The presentation ends with the eq. (4.26) which we use in the following.

Chap. 5
: The proper formalization of the GW concept is discussed. We are concerned with early-universe GWs. We introduce the stochastic GW background and we focus on cosmological GW sources. A brief digression about the stress-energy tensor for the GWs precedes the calculation for the primordial relic GW power spectrum. It follows an important discussion about cosmological perturbations & classicality. The part 2 of the chapter is devoted to second-order RD era sourced GWs from primordial first-order scalar perturbations. Eventually we specialize this to the case of enhanced scalar perturbations.

Chap. 6

This chapter complements the previous. A subdominant contribution to the 2-point GW function, from a source made of first-order enhanced-scalar & tensor perturbations, is calculated. The power spectrum of the second-order sourced GWs from primordial enhanced-scalar & tensor perturbations is given. Further we calculate the GW density parameter and we discuss the results.

Chap. 7

The conclusions of this thesis are presented here. Moreover, we sketch some prospects for interesting related future works.

Conventions

The conventions we use are reported trough the text, we mention here the one for Fourier transform, see eq. (3.1). Unless explicitly stated, we use natural units in which $c = \hbar = k_B \equiv 1$

Chapter 1

The FLRW geometry

The goal of this chapter is to introduce notation and main concepts of Cosmology (e.g. the Hubble rate, the different epochs of the universe, the conformal coordinates, etc.) that will be useful in the following.

The standard model of cosmology is the so-called hot Big Bang (hBB) model which, on the basis of the cosmological principle, represents the observable patch of the universe as spatially homogeneous and isotropic. The hypothesis of homogeneity and isotropy is an abstraction but nonetheless it serves as "ground state" upon which studying fluctuations in the early universe, which to our present understanding gave origin to the structures we observe. Its role in cosmology is analogous to the principle of inertia in newtonian mechanics.

This patch is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which is indeed maximally symmetric in its spatial part

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right] \tag{1.1}$$

here c is the speed of light; the function $a(t)$ has the dimension of a length and it is called *scale factor*; the κ is a real parameter; the r is a generalized radial coordinate and $d\Omega$ is the solid angle element. The real parameter κ is a fixed quantity (not depending on time) related to the curvature of the geometry. We can use the freedom of rescaling coordinates to conventionally constrain the curvature parameter to three equivalence classes $\kappa \in \{-1, 0, 1\}$. They are associated with the global topology of the FLRW geometry. We speak of an *open* ($\kappa \leq 0$) or *closed* universe ($\kappa > 0$), where this terminology comes from the fact that the former spacetime does not have boundaries while the latter does.
Now on, unless explicitly stated, we will use natural units where $c \equiv 1$.

The equations of motion, relating the energy-matter content of the universe to its geometry, are the Einstein equations. The simplest source, whose symmetries are consistent with the cosmological principle, is the perfect fluid. Despite modelling the bulk energy-matter content of the universe in such a way might seem reductive, it is in fact a good approximation as eventually confirmed by the cosmological observations. The energy momentum tensor of the perfect fluid in its rest frame, also called *comoving frame*, is given by

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p, p, p) \tag{1.2}$$

here we omitted the dependence of the energy density and pressure (scalar) fields from the cosmic time t and the position $\vec{x}$. We shall follow this notational practise for convenience in all this work, unless a possible ambiguity occurs.
The equation of state for the perfect fluid is given by

$$p = w\rho \tag{1.3}$$

and one typically assumes $w =$ const., in which case the pressure is directly proportional to the energy density and we have a *barotropic fluid.*
The non-trivial independent Einstein equations for the FLRW metric with the perfect fluid are two. For historical reasons, they are also known as Friedmann equations. They are

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{\kappa}{a^2} \tag{1.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3\,p) \tag{1.5}$$

These are supplemented by the conservation law $\nabla^\mu\, T_{\mu\nu} = 0$, coming from diffeomorphism invariance, below

$$\dot{\rho} = -3\frac{\dot{a}}{a}\,(1+w)\,\rho \tag{1.6}$$

where we used the equation of state (1.3); we note that it can be obtained equivalently by combining eq. (1.4) and eq. (1.5).

Here G is the Newton gravitational constant and we have defined the *Hubble rate* $H(t) \equiv \frac{\dot{a}}{a}$, the latter has the dimension of a frequency and it characterizes the expansion rate of the universe. We carefully explain it in sec. 2.1.1, when we talk about causality and the horizon problem.

Before solving the eq.s (1.4)-(1.6), we introduce the *density parameter* $\Omega(t)$ as the ratio between the energy density over the *critical density* $\rho_c \equiv \frac{3H^2}{8\pi G}$. The first Friedman equation rewrites to

$$\Omega - 1 = \frac{\rho}{\rho_c} - 1 = \frac{\kappa}{a^2\,H^2} \tag{1.7}$$

Defining the density parameter is twofold: first, this quantity can be easily measured from direct observations; second, the eq. (1.4) rewritten above let us appreciate that the magnitude of Ω determines the sign of the curvature parameter κ. Recalling the previous discussion, we see that ρ_c is the energy density required for the universe to have zero curvature $\kappa = 0$. In this case, we talk of flat-universe since the equal time hyper-surfaces have euclidean geometry. We will always neglect the curvature κ/a^2 with respect to the energy density term, effectively setting $\kappa = 0$.

The eq. (1.3) closes the Friedmann equations. We can solve first the continuity equation, finding $\rho(t) \propto a^{-3(1+w)}$, and eventually the scale factor over time, which for $w \neq -1$ reads

$$\rho \propto a^{-3(1+w)}\,, \qquad\qquad a(t) \propto t^{\frac{2}{3(1+w)}} \tag{1.8}$$

conversely in the case $w = -1$ we have $H^2 \propto \rho = const.$, which imply the solution $a \propto e^{\pm H\,t}$.
The energy density ρ is positive-definite (being the Hamiltonian density). A fluid with equation of state $w < 0$ is called *exotic* since both relativistic and non-relativistic ordinary matter have non-negative pressure. E.g. radiation ($w = 1/3$) and dust ($w = 0$) are ordinary relativistic and non-relativistic

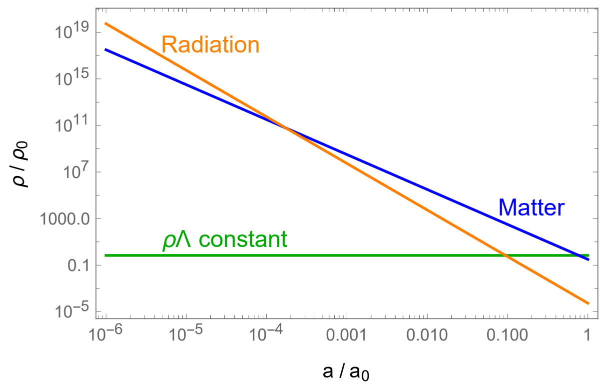


Figure 1.1: $\rho(a)$ for dust, radiation, vacuum.

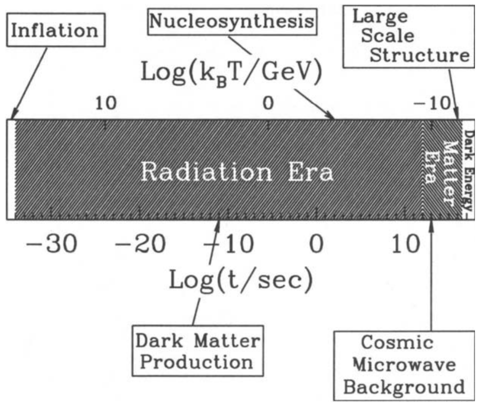


Figure 1.2: History timeline of the universe.(from [7])

fluids. A fluid for which $w = -1$ is called in many ways: for example vacuum energy or "cosmological constant". The latter is of historical importance, since the cosmological constant was introduced by A. Einstein when he first presented the static universe model. From the fig. 1.1, we can appreciate how the various components of the energy density scale with $a(t)$; in the picture we considered the behaviour for dust ($w = 0$), radiation ($w = 1/3$) and vacuum energy ($w = -1$).

1.1 The observed universe

We briefly discussed the FLRW spacetime and the tools we use to describe its evolution. Now it is worth spending some time talking about the observed universe. We indicate the present time t_0 and physical quantities evaluated today with a '0' subscript. We shall use the updated cosmological parameters and experimental data from the PLANK 2018 collaboration [8].

The present-day scale factor $a_0 = 1$ is set to one. Direct observations quantify the expansion rate of the universe providing for the Hubble constant the value $H_0 = 100\,h\,\frac{\mathrm{km/s}}{\mathrm{Mpc}}$, where the adimensional number h accounts for experimental and model-dependent uncertainties related to the actual measure of H_0. It is of the order of unity ($h = (6.74 \pm 0.05) \times 10^{-1}$ for ΛCDM-model [8]).
The present critical density, we defined in the preceding paragraph, is (from [8])

$$\rho_{0c} = 1.69 \times 10^{-8}\, h^2 \;\; erg\; cm^{-3} \tag{1.9}$$

Let us make a brief digression. There are many techniques we can use to measure the actual energy-matter content of the universe; by means of astronomical surveys, one usually estimates the contribution to the density given by electromagnetically interacting matter (visible). From 1960s scientists discovered, studying velocity curves of galaxies (typically of spiral ones), that there is almost everywhere a significant amount of matter which interacts gravitationally (and moreover tends to cluster) but not light emitting: this kind of matter was called *dark matter*. From observations, the amount of dark matter in galaxies is estimated to be about 5 times bigger than ordinary matter, called *baryonic*. Usually the DM observed in universe is dubbed "cold dark matter" (CDM), referring to the fact that we are dealing with non-relativistic particles. Although in the past scientists thought that the entire observed amount of CDM was just ordinary matter in the form black holes and brown dwarfs (very faint stars), it was soon realized that primordial nucleosynthesis models, whose predictions about abundances of light elements are extremely accurate, provide strong superior constrains on the total contribution of these two to ρ_{0dm}, that are much smaller than the measured one. The most studied particle physics candidates for dark matter are the lightest neutral supersymmetric particle (stable under R-parity) and the axion. Although at the moment of this writing, no dark matter particle has been detected yet.

If we consider only the baryonic matter, the density parameter is $\Omega_{0b} \equiv \frac{\rho_{0b}}{\rho_{0c}} = 0.049 \pm 0.007$ [8]. Also accounting the dark matter, we sum to $\Omega_m = \frac{\rho_{0b}+\rho_{0dm}}{\rho_{0c}} = 0.315 \pm 0.007$ [8]. We considered non-relativistic contributions to the density parameter, the relativistic species in the universe today are two: photons (radiation) and neutrinos; the former is straightforward to calculate since the CMB is the closest-perfect black body spectrum ever observed, we can easily evaluate the total energy density associated with this plankian distribution having the maximum temperature at $\mathrm{T_{CMB}} = 2.726 \pm 0.005$ K as

$$\rho_{0r} = 2 \int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \, \frac{pc}{e^{pc/k_B \, T_{\mathrm{CMB}}} - 1} = \frac{k_B^4}{(c\hbar)^3} \frac{\pi^2}{15} \, T_{CMB}^4 \simeq (4.18 \pm 0.03) \times 10^{-13} \;\; erg \; cm^{-3} \tag{1.10}$$

where we have momentary reintroduced cgs units. The 2 in front counts the polarisations of the photon. As we can see, the photon contribution today is very tiny and can be neglected, being $\approx 10^{-4}$ smaller than the matter one. The neutrinos are of three types (electronic, muonic and tauonic) and their masses are bounded to $\sum m_{\nu_i} < 0.12\,\mathrm{eV}$ [8]. If we assume the neutrinos today are in the ultra-relativistic regime, the contribution to the total density parameter is of the same order of that of radiation; however there is evidence suggesting that for one of three species, this approximation does not hold, e.g. see [7]. Anyway the relative contribution from a non-relativistic species ν is (from [7])

$$\Omega_\nu \simeq \frac{m_\nu}{94\,h^2\,\mathrm{eV}} \tag{1.11}$$

using the mass-constrain, we can bound the contribution from a neutrino with $\Omega_\nu < 0.002$.

By observing the ‘recession’ of galaxies from us, it was further noticed that the velocity of expansion increases: that is, the expansion is accelerated. By inspecting the second Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\,(\rho + 3\,p) \; > \; 0 \qquad \rightarrow \qquad p \; < \; -\frac{1}{3}\rho \tag{1.12}$$

we realize that the necessary condition for accelerated expansion is $w < -1/3$. The dominant contribution to the cosmic energy-matter content of the observed universe comes from a fluid with equation of state $w = -1.03 \pm 0.03$ (eq. (50) of [8]). This substance is not observed in the electromagnetic spectrum. Also it does not gravitationally cluster, as ordinary matter and dark matter. It is called *dark energy.*
The observed density parameter is very close to the critical one, in fact it is measured to [8]

$$1 - \Omega_0 = 0.001 \pm 0.002 \tag{1.13}$$

The fig. 1.1 shows that among the various contributions, the component which will dominate at late times (and in fact today) is the cosmological constant. During its evolution the universe underwent many phases. The recent period of matter domination was preceded by a phase in which the main component of ρ was coming from radiation (e.g. more in general from relativistic particles) as can be appreciated by looking at Figure 1.2.

1.2 Conformal coordinates

The eq.s(1.4)-(1.6) were given in terms of the cosmic time t. It is worth to mention that there exist a coordinate set in which the line element of eq. (1.1) ($\kappa = 0$) is

$$ds^2 = a^2(\tau)\,\left[-d\tau^2 + \delta_{ij}\,dx^i\,dx^j\right] \tag{1.14}$$

These are called *conformal coordinates.* The FLRW metric in such coordinates is related by a conformal transformation, that is a local change of scale, to the metric of flat Minkowski in cartesian coordinates.

The change between the physical $(t, \vec{x})$ and conformal $(\tau, \vec{x})$ coordinates does not affect the spatial axes. These two sets are related each other by

$$a^2\,d\tau^2 = dt^2 \tag{1.15}$$

The equation above is integrated by taking the positive solution (e.g. as t increases, τ does as well) and setting the integration constant such that at the Big Bang singularity ($t = 0$) there is $\tau = 0$. We shall call the resulting time-like coordinate τ as the *conformal time*

$$\tau \equiv \int \frac{dt}{a(t)} \tag{1.16}$$

The quantity τ has the physical meaning of maximal distance of causal connection that two events can ever experience, measured on the comoving grid.

For completeness we report the Friedmann eq.s in both physical (left) and conformal (right) coordinates

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\,\rho & \qquad \mathcal{H}^2 \equiv \left(\frac{a'}{a}\right)^2 &= \frac{8\pi G}{3}\,\rho\,a^2 \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\,(\rho + 3\,p) & \qquad \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 &= -\frac{4\pi G}{3}\,(\rho + 3\,p)\;a^2 \\ \dot{\rho} &= -3\frac{\dot{a}}{a}\,(1+w)\,\rho & \qquad \rho' &= -3\frac{a'}{a}\,(1+w)\,\rho \end{aligned} \tag{1.17}$$

and the behaviour of the scale factor eq. (1.8) in conformal time

$$a(\tau) \propto \tau^{\frac{2}{1+3w}} \tag{1.18}$$

Table 1.1: Some quantities in conformal coordinates

	$w = -1$ Inflation era	$w = 0$ Matter era	$w = 1/3$ Radiation era
$\rho(a) \propto$	$const.$	a^{-3}	a^{-4}
$a(\tau) \propto$	τ^{-1}	τ^2	τ
$\mathcal{H} =$	$-\tau^{-1}$	$2\tau^{-1}$	τ^{-1}
$\mathcal{H}' =$	τ^{-2}	$-2\tau^{-2}$	$-\tau^{-2}$

Chapter 2

The inflationary universe

In this chapter we present in detail one of the conundrums of the standard hBB model: the horizon problem. We first introduce the causal properties of FLRW spacetimes in general, eventually we rephrase the horizon problem in light of the properties of the cosmic microwave background radiation. In sec. 2.2 we introduce the inflationary solution and the simplest dynamical implementation thereof, by means of a single real scalar field (called inflaton) slow-rolling its potential $V(\varphi)$.

The standard model of the hot Big Bang is based upon the assumptions that the laws of Physics today were also valid in the early universe, that the Cosmological principle holds, and that at early times there were some "initial conditions" which gave rise to the structures we observe nowadays. However, these initial conditions must be unnaturally fine-tuned to account for the homogeneity, isotropy, and flatness of the observed universe. As a representative example, we anticipate that the amount of tuning required to explain the observed flatness is of about one part over $\approx 10^{60}$.

The theory of inflation is today considered a paradigm of the modern Cosmology. It get rid of the fine-tuning providing a dynamical solution to evolve a wide variety of initial conditions into an homogeneous and isotropic, nearly-flat universe. In fact it is the clearest explanatory and "predictive" framework we have now to understand the structures in our universe, as they are originated from the evolution of primordial cosmological perturbations (we study in the next chapter.)

2.1 The shortcomings of the hot Big Bang model

2.1.1 The horizon problem

The horizon problem is one of the conundrums of which the hBB suffers. It comes from a very intrinsic property of the causal structure of FLRW spacetimes: the existence of the time of creation t_{BB}. The causal structure of a geometry is found by inspecting the trajectories of massless particles. Let us consider a photon travelling along a radial path ($d\Omega^2 = 0$) in a spatially-flat FLRW spacetime. It is straightforward to find the eq. of motion, by solving the line element eq. (1.14) with null interval

$$ds^2 = a(\tau)^2 \left[-d\tau^2 + \, dr^2\right] = 0 \tag{2.1}$$

The light cones of FLRW spacetime in conformal coordinates are those of ordinary Minkowski space. However, the time variable here is bounded and starts from $\tau_{BB} = 0$. In Minkowski space, the light

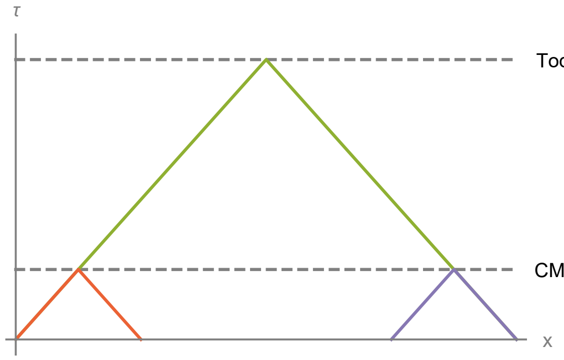


Figure 2.1: Light cones of FLRW in conformal coordinates for $t_{\rm ls}$ photons.

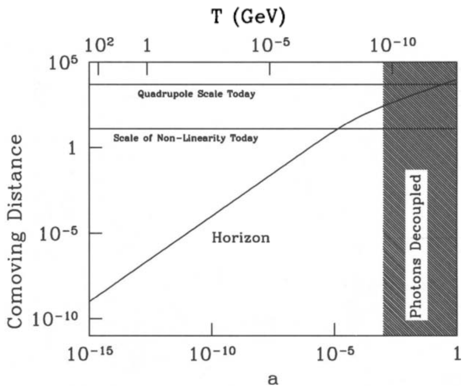


Figure 2.2: The comoving Hubble radius as function of a (from [7]).

cones of two points x_1, x_2 separated by arbitrary distance $|x_2 - x_1|$ always intersect in the past. This condition is not met in the case of FLRW spacetime, as illustrated in the fig. 2.1.

Let us return to physical coordinates to introduce few quantities, they will be useful later. The comoving distance a massless particle travels from the beginning of time $t = 0$ up to the time t is called *comoving horizon*

$$d_H(t) \equiv \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da'}{a'} \frac{1}{aH} \tag{2.2}$$

on the right we wrote the comoving horizon as the logarithmic integral of the *comoving Hubble radius* $(aH)^{-1}$. Massive particles move slower than light, therefore the comoving horizon is also called *particle* horizon; it represents the maximal comoving distance that the particles can ever travelled from the Big Bang up to the time t. At that time, an observer would measure the physical distance $a(t)\, d_H(t)$.

Both the comoving Hubble radius and the comoving horizon are functions of time. Given two points separated by l on the comoving grid, the comoving particle horizon establishes if they have ever been in causal contact up to the time $\bar{t}$ ($l \leq d_H(\bar{t})$), while the comoving Hubble radius tells us whether they are at $\bar{t}$ in causal contact ($l \leq (aH)^{-1}$). In an expanding universe, it is conceivable that points no more in contact today have been in causal communication in the past (e.g. $(aH)^{-1} \leq l \leq d_H$). In the conventional hot Big Bang model ($w \geqslant 0$), we see from eq. (1.5) that

$$\frac{\mathrm{d}}{\mathrm{dt}}\,(aH)^{-1} = -\frac{\ddot{a}}{(aH)^2} \geqslant 0 \tag{2.3}$$

the comoving Hubble radius increases with time, as well as the comoving particle horizon.

The cosmic microwave background radiation is a thermal relic of the universe when it was about $t_{ls} \approx 300000$ yrs.-old. The subscript here stands for 'last-scattering': the instant in which photons were last-scattered by matter. Since then, the photons practically free-streamed though the universe and what we observe today by looking the CMB is the red-shifted snapshot of the universe at time $t_{\rm ls}$. This radiation is observed along any directions of the sky, approximately with the same temperature.[1] It

[1]this statement is not strictly correct. We are implicitly disregarding our proper motion with respect to a comoving frame, which otherwise results in a dipole anisotropy

has a near-perfect blackbody spectrum with peak intensity at about $T_{CMB} \sim 2.7\ K$ in the microwave band.

The horizon problem, in terms of the CMBR, is the observation that we measure a very high degree of isotropy on all the scales: even for comoving modes of wavelength l that at the time of emission $t_{\rm ls}$ were not in causal contact each others

$$(aH)^{-1}(t_{\rm ls}) \ < \ l \ < \ (aH)^{-1}(t_0) \tag{2.4}$$

namely a mode l which we observe in the Hubble radius today ($l \ < \ (aH)^{-1}(t_0)$) was not causally connected at the time of emission $t_{\rm ls}$, that is $(aH)^{-1}(t_{\rm ls}) \ < \ l$.
The situation is qualitatively illustrated in fig. 2.1: when looking the CMB temperature today from opposite directions of the sky, we observe the same temperature despite the past-light cones of the photons do not overlap at the time of emission. Since these photons never had occasion to thermalize, it is not conceivable they have the same temperature. The more accurate RD-matter dominated (MD) era transition, as well as photon decoupling and scale references, are accounted in fig. 2.2.

Before we illustrate the solution to the horizon problem (sec. 2.2), it is worth mentioning another flag related with the conventional hBB model: the flatness problem.

2.1.2 The flatness problem

In the previous chapter, we observed that the density parameter is very close to one eq. (1.13). This does not mean that the curvature parameter κ vanishes but rather that the curvature term (κa^{-2}) of eq. (1.4) is negligible *today* with respect to the energy density ($\frac{8\pi G}{3}\rho_0$)

$$\frac{\kappa}{a^2} \ll \frac{8\pi G}{3}\rho_0 \tag{2.5}$$

From eq. (1.8) we see that the latter decreases with the scale factor more rapidly than the curvature term, which scales as a^{-2} in eq. (1.4).

We would expect that, at a certain time, the effects of a non-zero curvature parameter κ would violate the condition eq. (2.5). We may calculate the initial tuning of the density parameter we would need to obtain the present flatness eq. (1.13)

$$\frac{1-\Omega}{\Omega} \approx \left(\frac{1-\Omega_0}{\Omega_0}\right)\left(\frac{T}{T_P}\right)^{-2} \times 10^{-60} \qquad\qquad (T_P \approx 10^{29}\,\mathrm{GeV}) \tag{2.6}$$

where T is the temperature of the universe at time t_i.
We see that assuming the initial condition is given at the Plank scale, the tuning of the curvature parameter would be very high; even if we would set the initial conditions at the energy scale of Big Bang nucleosynthesis, which is not in accordance with the observed universe at that time, the initial tuning should be of the order of 10^{-5}. The required fine-tuning of initial conditions represents a so-called *naturalness* problem. This is not a flag of wrongness, rather it tells us that the theory is valid only for privileged conditions and therefore lacks of generality. As we will see, the amount of tuning is ameliorated by the enormous expansion of the universe during inflation.

2.2 The inflationary solution

The problems briefly cited in the previous section are inconceivable with the conventional Big Bang picture. The crucial point about the horizon problem relies on the fact that, for a universe RD or

MD, the comoving Hubble radius is always non-decreasing function of time as a consequence of the decelerated expansion. On the contrary, if we suppose a period of accelerated expansion $\ddot{a} > 0$, also called *inflation*, from eq. (2.3) we have that the comoving Hubble radius decreases

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[(aH)^{-1}\right] = \frac{\mathrm{d}}{\mathrm{dt}}\left[(\dot{a})^{-1}\right] = -\frac{a\left(H^2+\dot{H}\right)}{(aH)^2} \; < \; 0 \qquad \Leftrightarrow \qquad H^2+\dot{H} > 0 \tag{2.7}$$

For what concerns the horizon problem, it is solved if the value of the comoving Hubble radius at the onset of inflation - henceforth it decreases - is greater than the biggest observable comoving size today, as represented in fig. 2.3.
The equation above inserted in the Friedmann eq.s (1.17) implies that during the inflationary stage, the dominant contribution must satisfy

$$\rho + 3\,p \; < \; 0 \tag{2.8}$$

that is, the equation of state fulfil $w < -\frac{1}{3}$. The eq. (2.8) is a necessary condition for accelerated expansion. We recall that this substance is exotic, since it deviates from the equation of state of ordinary fluids.

The great success of inflation theories is to provide a dynamical explanation for the physical origin of the primordial density perturbations. The quantum fluctuations of light fields during inflation are stretched due to the (quasi-) exponential expansion resulting in macroscopic density perturbations. Upon horizon re-entry in the post-inflationary universe, they are the initial conditions which eventually grew up to form structures via gravitational collapse.

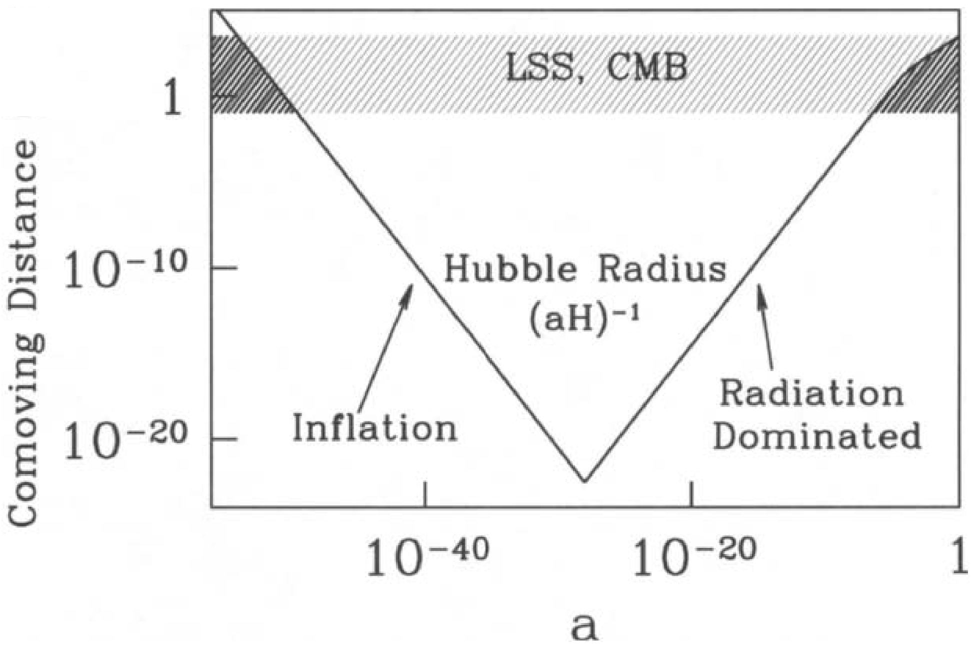


Figure 2.3: The behaviour of the comoving Hubble radius to solve the horizon problem. LSS stands for large-scale structures. (from [7])

2.3 Scalar field inflation and slow roll

To obtain an accelerated expansion in a dynamical way, the standard procedure is to assume that the universe has undergone a period in which the dominant contribution to energy-matter density was provided by some field, called the inflaton (φ), with negative pressure. We do not know the microscopic details of the field driving inflation. However, we implement the inflaton with an effective field description in terms of a single scalar field slow-rolling (SFSR) its potential.

Let us indicate with $\varphi(t, \vec{x})$ the real scalar field - the inflaton. The classical lagrangian is the sum of the kinetic and potential $V(\varphi)$ term[2]

$$\mathcal{L}_\varphi = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \mathrm{V}(\varphi) \tag{2.9}$$

The theory is defined by the Einstein-Hilbert action for gravity plus the inflaton lagrangian above minimally coupled

$$S = \int d^4x\,\sqrt{-g}\,\left[\frac{M_p^2}{2}R\;+\;\mathcal{L}_\varphi\right] \tag{2.10}$$

where $\sqrt{-g}$ is the determinant of the metric, R is the Ricci scalar. Here we neglect other light fields active during inflation but not relevant to the present purpose.

Let us recall the symmetries of the unsourced FLRW geometry, e.g. spatial isotropy and homogeneity. They allow us to extract the homogeneous (i.e. $\vec{x}$-independent) part from the scalar field sourcing inflation

$$\varphi(t, \vec{x}) = \varphi_0(t) + \delta\varphi(t, \vec{x}) \tag{2.11}$$

In principle this decomposition is not unique, however we anticipate that we can uniquely fix it when dealing with the quantum theory by assuming that the classical dynamic is solved exactly by the homogeneous part. For this reason, we shall call it also 'background'. If we focus on background dynamics, the homogeneous components of the stress-energy tensor

$$T_{\mu\nu} = \partial_\mu\varphi\,\partial_\nu\varphi\;-\;g_{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\,\partial_\beta\varphi - V(\varphi)\right) \tag{2.12}$$

source the Einstein equation, analogously to the Friedmann eq. (1.17) where now the energy density ($\rho = T^0{}_0$) and pressure ($p = \frac{1}{3}T^i{}_i$) are given by

$$\begin{aligned}\rho &= \frac{1}{2}\dot{\varphi}_0^2\;+\;V(\varphi_0)\\ p &= \frac{1}{2}\dot{\varphi}_0^2\;-\;V(\varphi_0)\end{aligned} \tag{2.13}$$

[2] the sign in front of the canonically-normalized kinetic term is due to our signature convention (-,+,+,+)

We understand that a *necessary* condition to obtain an inflationary stage by means of a scalar field is requiring the potential $V(\varphi_0)$ to be greater than the kinetic term $\dot{\phi}_0^2$ and doing so for a sustained amount of time. The time dependence of the field ϕ_0 is realized by stating the field slowly rolls along its potential, whose qualitative shape may be fig. 2.4. This condition, when applied to the Friedmann equations lead to

$$H^2 = \frac{8\pi G}{3}\, V(\varphi_0)\left(1 + \frac{1}{3}\epsilon_\varphi\right) \simeq \frac{8\pi G}{3}\, V(\varphi_0) \tag{2.14}$$

where $\epsilon_\varphi \equiv \frac{3}{2}\frac{\dot{\phi}_0^2}{V(\varphi_0)}$. Recalling the necessary condition for accelerated expansion eq. (2.8), we realize that the quantity ϵ must be small. Notice that we neglected the curvature term $-\frac{\kappa}{a^2} \propto a^{-2}$ as it scales with a and after some time it becomes subleading with respect to the field energy density.

To sustain the inflating regime sufficiently long, it is also necessary for the acceleration of the field to be smaller than the field temporal derivative in an Hubble time $\ddot{\varphi}_0 \ll H\dot{\varphi}_0$. This condition is quantified by looking the Klein-Gordon equation for the background inflaton

$$-V' = 3H\,\dot{\varphi}_0\left(1 + \frac{1}{3}\eta_\varphi\right) \simeq 3H\,\dot{\varphi}_0 \tag{2.15}$$

where we defined $\eta_\varphi = \frac{\ddot{\varphi}_0}{H\,\dot{\varphi}_0}$. The approximations in the r.h.s. of eq.s (2.14)-(2.15) are valid as long as both parameters are small $\epsilon_\varphi \ll 1\,,\ \eta_\varphi \ll 1$.
In literature, the slow-roll parameters are defined in a slight different way:

$$\begin{aligned} \epsilon &\equiv \frac{M_p^2}{2}\left(\frac{V'}{V}\right)^2 = -\frac{\dot{H}}{H^2} \\ \eta &\equiv M_p^2\left(\frac{V''}{V}\right) = \frac{1}{3}\frac{V''}{H^2} \end{aligned} \tag{2.16}$$

they are related to the ones defined previously by $\epsilon_\varphi \simeq \epsilon\,,\ \eta_\varphi \simeq \eta - \epsilon$.
The conditions for obtaining a slow-roll inflationary stage are therefore

$$\epsilon \ll 1, \quad |\eta| \ll 1 \tag{2.17}$$

We remark that demanding the conditions above is a sufficient (though not a necessary) condition for the universe to inflate enough.

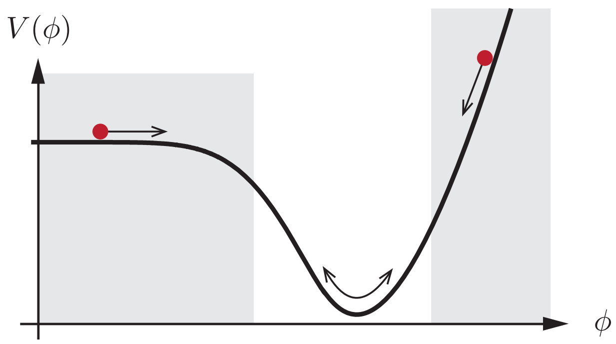


Figure 2.4: Qualitative shape for a slow roll potential. Inflation starts in the left grey region. There, the slow-roll conditions eq. (2.17) are qualitatively satisfied by the slope and concavity of the curve.

The ϵ slow-roll parameter is proportional to the variation rate of the Hubble rate. Also, it measures the deviation of the eq. of state from a pure de Sitter expansion, that is the exponential expansion (see below). In the limit $\epsilon \to 0$ the Hubble rate is constant and we can integrate the Friedmann equation for the scale factor

$$a(t) \;\propto\; e^{H\,t} \tag{2.18}$$

In the slow-roll regime, the ϵ parameter may be slightly different from zero and therefore the universe does not experience an exact de Sitter stage. In fact, from eq. (2.16) we see that the Hubble rate decreases with time however it is important to notice that during the *quasi de Sitter expansion* this decrease is slow-roll suppressed.

Chapter 3

Inflation and primordial perturbations

This chapter starts by showing qualitatively that initial conditions are tied to inflaton quantum fluctuations. In sec. 3.1 we study the fluctuations of a free massless field in a de Sitter stage. We use this example to present the canonical quantization and to discuss the boundary conditions for the cosmological fields. In sec. 3.2 we introduce the metric perturbations and we calculate the power spectrum of the primordial scalar perturbation. The study of the primordial tensor perturbation is postponed to chapter 5.

The current understanding of the structures of our universe is the following: they have originated from assigned conditions set at some initial time and then evolving up to the present era. The inflationary theory, besides solving the problems of the conventional hBB model (some of which were discussed before), also provides a framework in which one explains naturally the origin of these initial perturbations as zero-point fluctuations of quantum fields during the accelerated expansion phase.

QM & initial conditions We first address conceptually the relation between quantum fluctuations of the inflaton and the cosmological initial conditions. Let us recall eq. (2.11)

$$\varphi(t,\vec{x}) = \varphi_0(t) + \delta\varphi(t,\vec{x})$$

Quantum mechanics gives to $\delta\varphi(t,\vec{x})$ new meaning. In the previous chapter we discovered that the homogeneous part of the inflaton $\varphi_0(t)$ governs the accelerated expansion by means of Friedmann eq. (2.14). Therefore the slow-rolling inflaton, measuring the amount of inflation occurred, can be regarded as a clock. From elementary physics, a quantum-mechanical clock has some intrinsic variance due to the Heisenberg uncertainty principle. In light of this analogy $\varphi \sim$ clock, one realizes that the $\delta\varphi(t,\vec{x})$ represents such intrinsic *fluctuations*. These local deviations from the average expansion time - i.e. $\leftrightarrow \varphi_0$ - after the end of inflation, *eventually* appear as perturbations of the mean local density and geometry. The first are captured by $\zeta(t,\vec{x})$, which we study in this chapter.

To concretely understand this mechanism, let us consider the dynamics of a fourier mode as inflation progresses. We decompose the fluctuation $\delta\varphi(t,\vec{x})$ in momentum space

$$\delta\varphi(t,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \delta\varphi(t,\vec{p})\, e^{i\vec{p}\cdot\vec{x}} \tag{3.1}$$

Recall that the typical comoving length of causal correlation is the Hubble radius $(aH)^{-1}$. If $\lambda \propto \frac{1}{p}$ is the wavelength of the mode, at very early times it is well within the horizon. The Hubble radius decreases as inflation proceeds, therefore at a certain time the mode leaves the horizon henceforth no causal physics can affect its dynamics. We term the two regimes respectively as sub-Hubble and super-Hubble. The visible mode re-enters the horizon after the inflationary stage during either the radiation or the matter era. This behaviour is illustrated in fig. 2.3.

As a preliminary example, we study the dynamics of a light field in a de Sitter expanding background. It is the occasion to acquire confidence with the causal behaviour, by inspecting the sub/super-Hubble equation of motion. Moreover we introduce the canonical quantization and we discuss boundary conditions for the cosmological fields.

3.1 Massless scalar field in de Sitter

Let us start by finding the classical Euler-Lagrange equation for a free massless scalar field minimally coupled to gravity during a de Sitter stage. We neglect the contribution of the scalar field to the total energy-momentum tensor which sources the geometry: such field is called 'spectator'.
Recall the metric in conformal coordinates eq. (1.14). The action for the matter field is

$$\begin{aligned}\mathcal{S}_\varphi &= \int d^4x\, \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi\right)\\ &= \int d\tau d^3\vec{x}\;\; a^2\left(\frac{1}{2}(\varphi')^2 - \frac{1}{2}(\nabla\varphi)^2\right)\end{aligned}\tag{3.2}$$

where on the second line we used the metric determinant $\sqrt{-g} = a^4$ in conformal coordinates. The kinetic term is not in canonical form, we define the field $v \equiv a\,\varphi$; in term of which we write the canonically normalized action (hint: integrate by parts)

$$\mathcal{S}_v = \int d\tau d^3\vec{x}\;\left(\frac{1}{2}v'^2 - \frac{1}{2}(\nabla v)^2 + \frac{1}{2}\left(\mathcal{H}^2 + \mathcal{H}'\right)v^2\right)\tag{3.3}$$

here the Hubble rate and its derivative were given in eq. (1.17).
We decompose the v field in fourier modes

$$v(\tau,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\; v(\tau,\vec{p})\; e^{i\vec{p}\cdot\vec{x}}\tag{3.4}$$

Given the SO(3)-invariance of the action eq. (3.3), the modes $v(\tau,\vec{p})$ are independent of the momentum direction; therefore we use interchangeably $v(\tau,\vec{p}) \leftrightarrow v_p$.
The Euler-Lagrange equation, in momentum space, is given by

$${v_p}'' + \Big(p^2 - \left(\mathcal{H}^2 + \mathcal{H}'\right)\Big)v_p = 0\tag{3.5}$$

which is the de Sitter case of the *Mukhanov-Sasaki equation*. For ultra-short wavelengths, it reduces to the ordinary Klein-Gordon eq. in Minkowski space as we expect - i.e. the equivalence principle - since the characteristic comoving length over which one experiences geometry deviations from flatness is $\propto (aH)^{-1}$.

As a preliminary, let us recall the scale factor of de Sitter expansion as function of conformal time

$$a(\tau) = -\frac{1}{\tau\, H}\tag{3.6}$$

where we fixed the integration constant of eq. (1.16) such that $\tau \to 0^-$ as the expansion progresses.

3.1.1 Canonical quantization

Let us promote the field $v(\tau, \vec{x})$ to a quantum operator $\hat{v}(\tau, \vec{x})$ acting in some Hilbert space of states $|\Psi\rangle$. We work in Heisenberg picture (the operators depend on time).

The conjugate momentum $\pi(\tau, \vec{x})$ now becomes an operator

$$\pi(\tau, \vec{x}) \equiv \frac{\delta \mathcal{S}_v}{\delta v'(\tau, \vec{x})} = v'(\tau, \vec{x}) \qquad \Rightarrow \qquad \hat{\pi}(\tau, \vec{x}) = \hat{v}'(\tau, \vec{x}) \tag{3.7}$$

The canonical quantization amounts to set the commutator of the field and its conjugate momentum to satisfy the equal-time commutation rules (ccr)

$$\Big[\hat{v}(\tau, \vec{p}_1),\, \hat{\pi}(\tau, \vec{p}_2)\Big] = i(2\pi)^3\, \delta^{(3)}\left(p_1 - p_2\right) \tag{3.8}$$
$$\Big[\hat{v}(\tau, \vec{p}_1),\, \hat{v}(\tau, \vec{p}_2)\Big] = \Big[\hat{\pi}(\tau, \vec{p}_1),\, \hat{\pi}(\tau, \vec{p}_2)\Big] = 0$$

where $\delta^{(3)}(p)$ is the Dirac delta in three-dimensional momentum space.

As for the harmonic oscillator, let us introduce the ladder operators $(\hat{a},\ \hat{a}^\dagger)$ in terms of which we write

$$\hat{v}(\tau, \vec{p}) \equiv v_p(\tau)\, \hat{a}(\vec{p}) \ + \ v_p^*(\tau)\, \hat{a}^\dagger(-\vec{p}) \tag{3.9}$$

where the $v_p(\tau)$ is called *mode function* (positive frequency mode). We used the same notation of eq. (3.5) because in fact these (complex) functions satisfy the classical eq. of motion.
The (3.8) in terms of ladder operators are the familiar commutation rules of the harmonic oscillator

$$\Big[\hat{a}(\vec{p}_1),\, \hat{a}^\dagger(\vec{p}_2)\Big] = (2\pi)^3 \delta^{(3)}\left(p_1 - p_2\right) \tag{3.10}$$
$$\Big[\hat{a}(\vec{p}_1),\, \hat{a}(\vec{p}_2)\Big] = 0$$
$$v_p\, v_p'^* - v_p'\, v_p^* = i \tag{3.11}$$

the latest is also called *wroskian*, as it is a normalization condition for the mode functions.

3.1.2 Mode functions

We have quantized the modes. Let us turn our attention to the initial conditions of eq. (3.5). They are assigned at asymptotically-early time $\tau \to -\infty$. A general solution is the linear combination of the two independents

$$v_p(\tau) = c_1 \left(1 - \frac{i}{p\,\tau}\right) e^{-ip\tau} + c_2 \left(1 + \frac{i}{p\,\tau}\right) e^{ip\tau} \tag{3.12}$$

To find out the coefficients $c_1,\ c_2$ we shall choose the vacuum state $|0\rangle$ of the free theory. It is defined by the two conditions: it must satisfy $\hat{a}(\vec{p})|0\rangle = 0$ for all the sub-Hubble modes and it must be the field configuration such that at the initial time the expectation value of the Hamiltonian operator $\hat{H}$ is minimized. These requirements, along with the normalization condition eq. (3.11), single out the Bunch-Davies solution (modulo a global phase)

$$v_p(\tau) = \frac{1}{\sqrt{2p}} \left(1 - \frac{i}{p\tau}\right) e^{-ip\tau} \tag{3.13}$$

In the far sub-Hubble regime ($|p\tau| \gg 1$) the mode reduces to a plane wave (as the harmonic oscillator), on the contrary when a mode crosses the horizon its amplitude starts decreasing with over the conformal time as $\propto 1/|p\tau|$. The solution is consistent with the discussion we made below the Mukhanov-Sasaki eq. (3.5), when we consider the physical field φ. We shall return to the Bunch-Davies vacuum and the relation between classical & quantum theory in chapter 5.

Before we study inflaton fluctuations, let us digress about an important tool which we use to characterize the statistical properties of the fluctuations.

3.1.3 The power spectrum

Let us consider the position operator $\hat{x}$ of a quantum-mechanical harmonic oscillator. The expectation value of $\hat{x}$ over the vacuum vanishes. Analogously, the expectation value $\langle 0|\hat{v}|0\rangle$ vanishes, as one can immediately verify by considering eq. (3.9). To obtain a non-trivial expectation value over the vacuum, one should average a product of operators at least quadratic in the fields.
Let us consider an observable $\hat{\mathcal{O}}$, we define the *power spectrum* $\mathcal{P}_{\hat{\mathcal{O}}}$ (of $\hat{\mathcal{O}}$) as the function

$$\left\langle 0 \,\middle|\, \hat{\mathcal{O}}(\tau,\vec{p}_1)\, \hat{\mathcal{O}}(\tau,\vec{p}_2) \,\middle|\, 0 \right\rangle \equiv (2\pi)^3 \delta^{(3)}\left(p_1 + p_2\right)\, \frac{2\pi^2}{p_1^3}\, \mathcal{P}_{\hat{\mathcal{O}}}(\tau,\vec{p}_1) \tag{3.14}$$

The power spectrum is proportional to the variance of the equal time 2-point function. We use the convention where the $\frac{1}{p^3}$ is factored out from $\mathcal{P}_{\hat{\mathcal{O}}}$, it is worth to mention that in literature sometimes one refers to the power spectrum as the quantity $(\frac{2\pi^2}{p^3}\,\mathcal{P})$ as a whole.

Example: power spectrum of the free massless scalar field in de Sitter

Let us calculate the power spectrum of the massless scalar field φ we studied above.
The field φ is given in terms of $v \equiv a\varphi$. We use eq. (3.9) and eq. (3.10)

$$\begin{aligned}\langle 0|\; \hat{v}(\tau,\vec{p}_1)\; \hat{v}(\tau,\vec{p}_2)\; |0\rangle &= \langle 0| \left(v_{p_1}(\tau)\, \hat{a}(\vec{p}_1) \;+\; v^*_{p_1}(\tau)\, \hat{a}^\dagger(-\vec{p}_1)\right) \left(v_{p_2}(\tau)\, \hat{a}(\vec{p}_2) \;+\; v^*_{p_2}(\tau)\, \hat{a}^\dagger(-\vec{p}_2)\right) |0\rangle \\ &= v_{p_1}(\tau)\, v^*_{p_2}(\tau)\; \langle 0|\hat{a}(\vec{p}_1)\, \hat{a}^\dagger(-\vec{p}_2)|0\rangle \\ &= v_{p_1}(\tau)\, v^*_{p_2}(\tau)\; (2\pi)^3 \delta^{(3)}\left(p_1 + p_2\right)\end{aligned} \tag{3.15}$$

The power spectrum depends on the mode functions of the canonically-normalized field v. Using eq. (3.13) and the power-spectrum definition, we obtain

$$\begin{aligned}\mathcal{P}_{\hat{\varphi}}(\tau,\vec{p}_1) &= \frac{1}{a^2}\mathcal{P}_{\hat{v}}(\tau,\vec{p}_1) = \frac{1}{a^2}\frac{p_1^3}{2\pi^2}\; |v_{p_1}(\tau)|^2 \\ &= \left(\frac{H_{\text{inf}}}{2\pi}\right)^2 \left(1 + (p_1\tau)^2\right)\end{aligned} \tag{3.16}$$

where we used eq. (3.6) on the second line. The power spectrum above depends from both momentum and time, but in the combination $p\tau$. Such dependence, as well as the power spectrum, is dubbed "scale-invariant". Moreover, in the super-Hubble regime ($|p\tau| \to 0$) the spectrum is constant. A power spectrum independent from both time and momentum is called 'flat' or white-noise spectrum.

3.2 Primordial inflationary perturbations

In the previous section the field energy-momentum $T^{(\varphi)}_{\mu\nu}$ was assumed not changing the dynamics of the background de Sitter geometry. In this section we study quantum fluctuations of the inflaton.
By assumption, this condition is not verified for the field driving inflation since this time the most significant contribution sourcing the expansion is in fact coming from the inflaton. Inflaton and metric are tied each others by the Einstein equation. Therefore inflaton fluctuations $\delta\varphi$ will induce perturbations on the background metric $\delta g_{\mu\nu}$: we shall treat the metric perturbations on the same footing of the inflaton fluctuations. Therefore, the first thing to do is to properly define the metric perturbations.

3.2.1 The perturbed metric I act

It is natural to study metric perturbations in the context of linearised gravity. During the inflation stage, the background metric is the quasi de Sitter of sec. 2.3, which is governed by the Friedmann eq. (2.14). Analogously to eq. (2.11), we write the metric tensor as

$$g_{\mu\nu}(\tau,\vec{x}) = g^{(\mathrm{B})}_{\mu\nu} + \delta g_{\mu\nu} \qquad\qquad \delta g_{\mu\nu} \ll g^{(\mathrm{B})}_{\mu\nu} \tag{3.17}$$

where $g^{(\mathrm{B})}$ is the *background metric* and $\delta g_{\mu\nu}$ are little deviations from the background metric.
The condition $\delta g_{\mu\nu} \ll g^{(\mathrm{B})}_{\mu\nu}$ means that one is allowing for (a) weak deviations from the background metric and (b) a restricted frame of references set where eq. (3.17) holds.

The components of metric tensor are coordinate-dependent and the decomposition eq. (3.17) does not uniquely define the metric perturbations. There may exist other coordinate systems in which the metric could still be written as $g^{(\mathrm{B})}_{\alpha\beta} + \delta g_{\alpha\beta}$ but nonetheless the components $\delta g_{\alpha\beta}$ would be different. Moreover, the $\delta g_{\alpha\beta}$ may not satisfy the weakness condition $\delta g_{\mu\nu} \ll g^{(\mathrm{B})}_{\mu\nu}$. To properly explain the points above it is mandatory to talk about gauge invariance.

Linearised theory and gauge invariance The linearised theory around a background can be formalized by means of a diffeomorphism $\phi : \mathcal{M}_b \to \mathcal{M}_p$ between the background spacetime $\mathcal{M}_b$ and the physical spacetime $\mathcal{M}_p$. The spacetimes $\mathcal{M}_b$ and $\mathcal{M}_p$ represents the same abstract geometry, but we imagine the tensor fields defined within to be different. Let us indicate with x^μ, y^α the coordinates of the background and physical spacetimes, respectively. On $\mathcal{M}_p$ we use the metric $g_{\alpha\beta}$ to write down the Einstein equations, while on $\mathcal{M}_b$ we have the metric tensor $g^{(\mathrm{B})}_{\mu\nu}$ of eq. (1.14). The correspondence allows us to define the perturbation as the difference between the pulled back physical metric $(\phi^* g)_{\mu\nu}$ and the background metric $g^{(\mathrm{B})}_{\mu\nu}$

$$\delta g_{\mu\nu} = (\phi^* g)_{\mu\nu} - g^{(\mathrm{B})}_{\mu\nu} \tag{3.18}$$

A priori the perturbation δg does not met the conditions (a)-(b), however if in the physical manifold the deviations from the abstract background geometry are small then for some diffeomorphisms we will have $\delta g_{\mu\nu} \ll g^{(\mathrm{B})}_{\mu\nu}$. The linearised Einstein equations are the pulled-back Einstein equations from the physical $\mathcal{M}_p$ to the background $\mathcal{M}_b$.

The many permissible diffeomorphisms lead to physically-equivalent perturbations, related each others by gauge transformations. From an highbrow point of view, we think of the gauge transformations as a one-parameter family of diffeomorphisms $\Psi_\xi : \mathcal{M}_b \to \mathcal{M}_b$ generated by the vector field

$\xi^\mu(x)$ (i.e. the gauge parameters). For small transformations ξ, the gauge transformed $\delta g^{(\xi)}_{\mu\nu}$ is given by

$$\delta g^{(\xi)}_{\mu\nu} = \delta g_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)} \tag{3.19}$$

where the round bracket is a shorthand to indicate the index symmetrization and the covariant derivative is calculated using the background metric. In order to preserve (a)-(b), one requires $\left|\nabla_{(\mu}\xi_{\nu)}\right| \leq |\delta g_{\mu\nu}|$.

Let us focus to the case of interest, where the background is the cosmological FLRW spacetime. By anticipating the discussion of chapter 5, the way in which one defines the perturbations is by exploiting a separation between the typical scales/frequencies of background and perturbations. Moreover, in the case of the curved FLRW background the symmetries of the background geometry further helps us in reducing the ambiguity of what pertains to $g^{(\mathrm{B})}$ and what to δg.

The background geometry is spatially homogeneous and isotropic. It is common procedure to decompose the metric perturbation on hyper-surfaces of constant time in irreducible parts with respect to spatial rotations [9, A.2.1]. Under SO(3), the 00-component is a scalar, the 0i-components form a three-vector and the ij-components form a two-index (symmetric) spatial tensor.

$$\delta g_{00} = a^2\,(-2\Phi) \tag{3.20}$$
$$\delta g_{0i} = a^2\,(S_i + \partial_i\, B) \tag{3.21}$$
$$\delta g_{ij} = a^2\,(-2\Psi\,\delta_{ij} + D_{ij}\,E + \partial_{(i}\,F_{j)} + h_{ij}) \tag{3.22}$$

where $D_{ij} \equiv \left(\partial_i\,\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)$. Here we have further decomposed the three-vector g_{0i} in real curl-free and divergenceless vectors, according to the Helmholtz theorem. Also, the spatial (symmetric) tensor perturbation δg_{ij} was decomposed in trace and traceless irreducible parts.

The functions are classified depending upon their behaviour under local spatial rotations: scalar or spin-0 field (Φ, B, Ψ, E), vector or spin-1 field (S_i, F_i) and tensor or spin-2 field (h_{ij}). Let us recall the constrains for the functions above

$$\begin{aligned} \partial_i\,S_i &= 0 \qquad & \partial_i\,F_i &= 0 \\ \partial_i\,h_{ij} &= 0 \qquad & \delta^{ij}\,h_{ij} &= 0 \end{aligned} \tag{3.23}$$

The degrees of freedom contained in the metric perturbations are easy to calculate. First, we have to count the number of arbitrary functions one has introduced in eq.s (3.20)-(3.23). It is straightforward to check that this is $10 = 4\ (scalars) + 3\times 2\ (vectors) + 6\ (tensors) - 6\ (constrains)$. Lastly, these arbitrary functions are not independent each others, for instance they satisfy the conservation equation $\nabla^\mu\,G_{\mu\nu} = 0$ for the Einstein tensor. Therefore the metric perturbation contains 6 degrees of freedom. Alternatively, one may find more immediate to calculate the six degrees of freedom of the metric from $6 = 10\ (rank\text{-}2\,symmetric\,tensor) - 4\ (coord.\,transf.)$. Further, we anticipate that out of these six only two are propagating degrees of freedom represented by the gravitational waves and contained in the perturbation h^{TT}_{ij} (where 'TT' stands for transverse-traceless). The vector perturbations S_i and F_i are not created by inflation, and we shall ignore them.

In conclusion, the metric perturbation contains (a) non-physical gauge degrees of freedom, (b) physical non-propagating degrees of freedom (in the scalar & vector fields), and (c) physical propagating degrees of freedom (in the tensor field). We will see that only the tensor perturbation of the metric h_{ij} obeys a wave-equation, while the other metric perturbations obey Poisson-like equations. Therefore *only* the tensor part represents degrees of freedom that can propagate in vacuum.

Primordial scalar spectrum

We are ready to calculate the power spectrum of the primordial scalar metric perturbation. The scalar degrees of freedom of the metric are contained in Φ, B, Ψ, E

$$g_{\mu\nu} = a^2 \begin{pmatrix} -(1+2\Phi) & \partial_i B \\ \partial_i B & (1-2\Psi)\,\delta_{ij} + D_{ij}E \end{pmatrix} \tag{3.24}$$

The inverse metric is the solution of

$$g^{\mu\alpha}\, g_{\alpha\nu} = \delta^{\mu}_{\nu} \tag{3.25}$$

It is straightforward to express the inverse metric as a power series in the perturbation: it is sufficient to factor out from Φ, B, Ψ, E a factor ϵ and Taylor expand the equation above, and eventually set $\epsilon \to 1$. The inverse metric to first order in perturbations is

$$g^{\mu\nu} = a^{-2} \begin{pmatrix} -(1-2\Phi) & \partial^i B \\ \partial^i B & (1+2\Psi)\,\delta^{ij} - D^{ij}E \end{pmatrix} \tag{3.26}$$

where the spatial indices i, j are raised by means of δ^{ij}.

Longitudinal gauge The cosmological perturbation theory can be formulated in terms of gauge-invariant quantities (e.g. the Bardeen potentials) [10] or one may perform the calculations in a specific gauge choice. This last approach often allows to simply the calculations by exploiting particular gauges at the price of loosing manifest gauge-invariance.
An infinitesimal gauge transformation $x^\mu \to x^\mu + \delta x^\mu$ with parameters

$$\begin{aligned} \delta x^0 &= \xi^0 \\ \delta x^i &= \partial^i \beta + v^i \qquad\qquad (\partial_i v^i = 0)\,, \end{aligned} \tag{3.27}$$

transform the metric perturbation with eq. (3.19), and the inflaton field eq. (2.11) as

$$\begin{aligned} \Phi &\to \Phi - \xi^{0\prime} - \mathcal{H}\xi^0 \\ B &\to B + \xi^0 + \beta' \\ \Psi &\to \Psi - \frac{1}{3}\nabla^2\beta + \mathcal{H}\xi^0 \\ E &\to E + 2\beta \end{aligned} \tag{3.28}$$

$$\delta\varphi \to \delta\varphi - \delta x^0\, \partial_0(\varphi_0) \tag{3.29}$$

We define the *longitudinal gauge* (also called conformal newtonian) as that gauge where the coordinates are chosen such that it holds

$$E = B = 0 \qquad\qquad \text{(longitudinal gauge)} \tag{3.30}$$

Therefore three scalar degrees of freedom are contained in Φ, Ψ, $\delta\varphi$. Two of them are non-propagating. In the following section we find that combination of the quantities Φ, Ψ, $\delta\varphi$ which is associated with the propagating degree of freedom.

3.2.2 The comoving curvature and the primordial scalar perturbations

The metric perturbations are sourced by inflaton fluctuations though Einstein equations. To describe the evolution of the primordial perturbations we need to find some combination of the quantities $\delta\varphi$ and Φ, Ψ. At the very beginning of the $\delta\varphi$ evolution, we expect such a quantity to be proportional to $\delta\varphi$ and after some time, so that Φ, Ψ gets sourced, it will be a combination of the two fluctuations. A candidate is the quantity ($\vec{k}$-momentum space)

$$\zeta = -\frac{ik^i\, \delta T^0{}_i\, H}{k^2(\rho + p)} - \Psi \tag{3.31}$$

where H is the Hubble rate and $\delta T^0{}_i$ is the perturbed inflaton stress-energy.
This gauge-invariant quantity is called *comoving curvature perturbation*[1]. The key feature of ζ is that it is constant for super-Hubble modes if the cosmological perturbations are adiabatic[2] [11], [12]. The perturbations are called *adiabatic* (or also curvature pert.s) if given any scalar quantity Q, its perturbations are described as a unique perturbation in expansion with the background

$$H\frac{\delta Q}{\dot{Q}} = H\delta t \qquad \text{(adiabatic perturbations)} \tag{3.32}$$

That is there are no relative variations between different perturbations: any δQ originates from the same curvature perturbation ζ. In the SFSL inflation theory we focus the perturbations are adiabatic.

The $\delta T^0{}_i$ is obtained by perturbing eq. (2.12). Recalling the results of B, we specialize eq. (3.31) to

$$\zeta \simeq -H\frac{\delta\varphi}{\dot{\varphi}_0} - \Psi \tag{3.33}$$

By the heuristic argument above, we can relate the scalar perturbation of the metric Ψ over super-Hubble scales with the amplitude of the quantum fluctuations $\delta\varphi$ at horizon crossing

$$\zeta = const. \qquad \rightarrow \qquad \Psi\Big|_{\substack{after\\ inflation}} \propto H\frac{\delta\varphi}{\dot{\varphi}_0}\Big|_{\substack{horizon\\ crossing}} \tag{3.34}$$

In weak field approximation, the scalar perturbation Ψ is related to the newtonian gravitational potential. After the inflationary stage, this potential enters the poisson equation governing the gravitational collapse which eventually forms structures. Here we see concretely that the initial conditions are set once the mode exits the comoving horizon, as we discussed previously in abstract. Therefore we must solve for the inflaton fluctuation during inflation. Recall the Einstein equations at first order in perturbations (B.9)-(B.12) and the linear Klein-Gordon eq. (B.6)

$$G_{00} = \frac{1}{M_p^2}T_{00} \qquad \nabla^2\Phi - 3\mathcal{H}\Phi' = \frac{1}{M_p^2}\left[\frac{1}{2}\varphi_0'\delta\varphi' + a^2\left(\frac{1}{2}\frac{\partial V}{\partial\varphi_0}\delta\varphi + \Phi V\right)\right] \tag{B.9}$$

$$G_{0i} = \frac{1}{M_p^2}T_{0i} \qquad \Phi' + \mathcal{H}\,\Phi = \frac{1}{M_p^2}\frac{1}{2}\varphi_0'\delta\varphi \tag{B.10}$$

$$G_{ij} = \frac{1}{M_p^2}T_{ij} \quad (i \neq j) \qquad \partial_i\,\Psi = \partial_i\,\Phi \tag{B.11}$$

$$G_{ii} = \frac{1}{M_p^2}T_{ii} \qquad \Phi'' + 3\mathcal{H}\Phi' = \frac{1}{M_p^2}\left[\frac{1}{2}\varphi_0'\delta\varphi' - a^2\left(\frac{1}{2}\frac{\partial V}{\partial\varphi_0}\delta\varphi + \Phi V\right)\right] \tag{B.12}$$

[1]to linear order, in the comoving gauge, indeed it measures the spatial curvature of comoving temporal hypersurfaces
[2]and if on super-Hubble scales it holds $\partial_i\partial_j\left(\delta T_{ij} - \frac{1}{3}\delta_{ij}T\right) = 0$ with $T \equiv T_{kk}$

The eq. (B.11) tells us that to describe the scalar degrees of freedom we can use only φ and Ψ (or Φ). We replace[3] $\Phi \to \Psi$ and we simplify the equations above to

$$\Phi'' + \nabla^2\Phi = \frac{1}{M_p^2}\,\varphi_0'\delta\varphi' \tag{3.35}$$

$$\Phi' + \mathcal{H}\,\Phi = \frac{1}{M_p^2}\frac{1}{2}\varphi_0'\delta\varphi \tag{3.36}$$

$$\Phi'' + 6\mathcal{H}\Phi' - \nabla^2\Phi = -\frac{2a^2}{M_p^2}\left(\frac{1}{2}\frac{\partial V}{\partial\varphi_0}\delta\varphi + \Phi V\right) \tag{3.37}$$

We use the lowerscript $\mathbf{k}$ on a quantity as shorthand for indicating the fourier mode. For super-Hubble wavelengths in l.h.s. of eq. (B.10) we have $\Psi' \ll \mathcal{H}\Psi$, moreover using eq. (2.14) ($\epsilon = \frac{3}{2}\frac{\dot\phi_0^2}{V(\varphi_0)}$)

$$\Psi_{\mathbf{k}} = \epsilon\,\mathcal{H}\frac{\delta\varphi_{\mathbf{k}}}{\varphi_0'} \qquad \Big(|k\tau| \ll 1\Big) \tag{3.38}$$

The super-Hubble comoving curvature perturbation is proportional to the inflaton fluctuation

$$\zeta_{\mathbf{k}} = -(1+\epsilon)\mathcal{H}\frac{\delta\varphi_{\mathbf{k}}}{\varphi_0'} \qquad \Big(|k\tau| \ll 1\Big) \tag{3.39}$$

We can simplify the linearised Klein-Gordon eq. (B.6) in super-Hubble regime

$$\begin{aligned}
\delta\varphi_{\mathbf{k}}'' + 2\mathcal{H}\delta\varphi_{\mathbf{k}}' + k^2\delta\varphi_{\mathbf{k}} - 4\varphi_0'\,\Phi_{\mathbf{k}}' &= -a^2\,V''\,\delta\varphi_{\mathbf{k}} - 2a^2\,\Phi_{\mathbf{k}}\,V' \\
\Rightarrow \qquad \delta\varphi_{\mathbf{k}}'' + 2\mathcal{H}\delta\varphi_{\mathbf{k}}' + a^2\left(V''\,\delta\varphi_{\mathbf{k}} + 2\Phi_{\mathbf{k}}\,V'\right) &= 0 \\
\Rightarrow \qquad \delta\varphi_{\mathbf{k}}'' + 2\mathcal{H}\delta\varphi_{\mathbf{k}}' + \left(a^2\,V'' - 6\epsilon\,\mathcal{H}^2\right)\delta\varphi_{\mathbf{k}} &= 0 \\
\Rightarrow \qquad \delta\varphi_{\mathbf{k}}'' + 2\mathcal{H}\delta\varphi_{\mathbf{k}}' + 3\mathcal{H}\Big(\eta - 2\epsilon\,\mathcal{H}\Big)\delta\varphi_{\mathbf{k}} &= 0
\end{aligned} \tag{3.40}$$

where we used the eq.s (2.15)-(2.16) and eq. (3.38). The differential equation is easier to solve if expressed in terms of $v \equiv a^{-1}\delta\varphi$

$$v'' - \frac{1}{\tau^2}\Big(\nu^2 - \frac{1}{4}\Big) = 0 \qquad \left(\nu^2 \equiv \frac{9}{4} + 7\epsilon - 3\eta\right) \tag{3.41}$$

The explicit solution of the eq. above is given in terms of Hankel functions. The physical solution is properly normalized by choosing a Bunch-Davies vacuum in the asymptotic past (we will return to this later). In terms of v, the physical fluctuation (modulo a phase) in super-Hubble regime is

$$|v_{\mathbf{k}}| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} + o\left(\epsilon, \eta\right) \tag{3.42}$$

The spectrum of primordial comoving curvature perturbations

$$\mathcal{P}_\zeta(p) = \frac{1}{M_p^2}\frac{1}{2\,\epsilon}\left(\frac{H}{2\pi}\right)^2\left(\frac{p}{aH}\right)^{\mathcal{N}_\zeta - 1} \equiv \mathcal{A}_\zeta^2\left(\frac{p}{aH}\right)^{\mathcal{N}_\zeta - 1} \tag{3.43}$$

[3] we took the physical solution of eq. (B.11) out of the most general one ($\Phi = \Psi + f(\tau)$)

for example, at the horizon crossing (herefrom ζ is constant) we have

$$\mathcal{P}_\zeta(p) = \frac{1}{M_p^2}\frac{1}{2\,\epsilon_\star}\left(\frac{H_\star}{2\pi}\right)^2 \tag{3.43b}$$

The $\mathcal{N}_\zeta$ is called spectral index, the $\mathcal{A}_\zeta^2$ represents the typical magnitude of the modes at horizon-crossing. The power spectrum of the comoving curvature perturbation has small deviations ($\mathcal{N}_\zeta - 1 = 2\eta_\star - 6\epsilon_\star \sim 1$) from a scale-independent one, whose deviation depends of slow-roll parameters. The scale dependence of the spectrum follows from the time-dependence of the Hubble parameter.

It is worth to anticipate that there is yet another signature of the inflationary stage: an intrinsic primordial gravitational waves background. We will study the tensor part of the metric perturbation in chapter 5. Before, however, it is mandatory to discuss the tools to set up our calculations in a methodical and clever way: the in-in formalism.

Chapter 4

The in-in formalism for cosmology

In this chapter we introduce the CTP (closed-time-path) formalism for cosmology, also called in-in formalism. The in-in formulation in terms of quantum operators and states closely parallels the traditional operatorial formulation of flat space quantum field theory, in terms of in-out states. Therefore, after a brief digression about the most important differences when dealing with cosmological observables (sec. 4.1), we review the operatorial formulation of fields in interaction (sec. 4.2) and of perturbation theory (sec. 4.2.1). In retracing these steps, it is mandatory to emphasize how the operatorial formulation of ordinary in-out QFT is related to in-in (sec. 4.3). This chapter ends with the eq. (4.26), which is of fundamental importance later.
This brief introduction is oriented to practical applications. The interested reader can refer to literature [13] [14] [15], and the reviews [16] [17] [18].

Let us remark again a crucial result of the previous chapters. In the inflationary picture, the cosmological perturbations arise naturally from the uncertainty principle.

$$\left\langle (\Delta x)^2 \right\rangle \left\langle (\Delta p_x)^2 \right\rangle \geq \hbar^2/4 \tag{4.1}$$

The observed perturbations arise as zero-point quantum fluctuations - they saturate the uncertainty relation - parametrically amplified by the rapid expansion. This process is strictly quantum mechanical. Today, however, the cosmological perturbations appear as classical large macroscopic fluctuations - i.e. they have large occupation numbers and the classicality corresponds to an uncertainty much bigger than the minimal one allowed by the rules of quantum mechanics [19].
In light of this, we can frame the application of CTP formalism to cosmology as, amongst the manifold reasons, a tool in the study of cosmological perturbations which benefits of the machinery of QFT. For example, it is applied to the study of statistical properties cosmological correlators [13] and the evaluation of quantum corrections [15].

An original result of this work is the application of the in-in formalism to the calculation of sourced gravitational waves from second-order primordial scalar perturbation, e.g. to semi-classical stochastic GWs production.

4.1 S-matrix and CTP at comparison, for cosmology

In particle physics one is generally interested in scattering amplitudes, that is $\mathcal{S}$-matrix elements $\langle out|\mathcal{S}|in\rangle$. Recall that a scattering amplitude is the probability amplitude that an initial state $|in\rangle$, consisting of certain particles in momentum eigenstates, is observed in a given final configuration $|out\rangle$ after the system has evolved over a finite period in which there are interactions. For example, the initial state may be $|in\rangle \equiv |\mathrm{e}^-_{\mathbf{p_1}}, \gamma_{\mathbf{p_2}}, ...\rangle$ - composed by a free electron with momentum $\vec{p}_1$ etc. - and the final state $|out\rangle \equiv |\mu^-_{\mathbf{q_1}}, \gamma_{\mathbf{q_2}}, ...\rangle$.

The assumption that the interactions take place for a finite amount of time $-T < t < T$ is of fundamental importance. This condition is surely verified in typical collider experiments, where free particles in initial approximate momentum eigenstates are collided against each others. Moreover it guarantees a well-defined formulation of the theory: in fact if there were interactions all the time, we would not be able to define stable initial states unless we would solve exactly the full theory; which is not feasible discarding some toy-models exceptions. The assumption above let us set the initial and final states at respectively asymptotic past/future-infinity, when no interactions occurred.
Therefore the *asymptotic states* are eigenstates of the free theory, which are known exactly.

In cosmology - we have in mind the primordial perturbations - there are important differences with respect to both the conventional use of in-out states and the flat space in-in formalism (e.g. used in condensed matter QFT)

- We do not prepare the initial/final states, such as in particle accelerators.
  At most one can assume the existence of *one* in-state in the asymptotic-past. Consider the mode $\phi_{\mathbf{k}}$. At early times, the physical wavelength $\mathbf{k}$ is very small and we feel in flat space (see eq. (3.5)), so we use as vacuum for these high energy modes the one we would use in the interacting theory in this approximately Minkowski space. There are few subtleties [20, see §3.5] but, for our purpose, the solution is using as $|in\rangle$ a "Bunch-Davies" vacuum.

- The basic observables are correlation functions, rather than S-matrix elements.
  The in-in formalism takes the initial state of some fields and it calculates the vacuum expectation value of a certain observable $\hat{\mathcal{O}}(\tau)$ at the late-time τ, expressed in terms of these fields, i.e. $\langle\Omega|\,\hat{\mathcal{O}}(\tau)\,|\Omega\rangle$. This is an initial-valued problem, rather than a boundary-valued one. See sec. 4.3.

- The background spacetime is curved.
  This implies that in principle the results depends on the causal structure of spacetime.

The implementation of the in-in formalism for studying cosmological perturbations is similar to the way one study fields in interaction in quantum field theory. One has to give up the purpose of finding an exact solution of the full interacting theory, so that we build a perturbative description of the interaction. We extract from the full hamiltonian the quadratic (free) part, whose solutions are known. We define the interaction picture which allows us to solve the interacting theory in terms of the free field solutions, in a manifestly-covariant way. We end up with the Dyson series, master eq. (4.24), which is the starting point for our calculations in perturbation theory.

Notations In favour of notational neatness, we discard the hat symbol over quantum operators and we omit the spacetime dependence of fields where no ambiguity can arise, contrariwise we will indicate the three-dimensional spatial coordinate with boldface $\vec{x} \leftrightarrow \mathbf{x}$.

For example, in eq. (2.11) we decomposed the field ϕ into the homogeneous part, also termed background, plus the perturbation. With our new notation, henceforth we strip to the essential by indicating

$$\phi \;=\; \phi_0 + \delta\phi \tag{4.2}$$

where $\phi_0 \;\leftrightarrow\; \phi_0(\tau)\,\hat{\mathbb{I}}$.

4.2 Dynamics around the background

We indicate with τ the time variable, it is a good idea to think of it as the conformal time.
Under the decomposition eq. (4.2), the hamiltonian operator $H\,[\phi,\pi]$ writes [1]

$$H\,[\phi,\pi] = H\,[\phi_0,\pi_0] + (linear) + \tilde{H}[\delta\phi,\delta\pi,\tau] \tag{4.3}$$

where the linear term is proportional to $\delta\phi$ (and $\delta\pi$) and vanishes due to the eq.s of motion.
In Heisenberg picture, the full Hamiltonian generates the time evolution of the fields

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\phi = [\phi, H] \qquad\qquad i\frac{\mathrm{d}}{\mathrm{d}\tau}\pi = [\pi, H] \tag{4.4}$$

The $\tilde{H}[\delta\phi,\delta\pi]$ is the *perturbed hamiltonian* for the perturbations, we used a different symbol since in general the functional dependence from ϕ,π differs from that of H. Note here the explicit time dependence, it comes from the background solutions $\phi_0,\ \pi_0$.

Perturbation dynamics. It is clear that the hamiltonian $\tilde{H}[\delta\phi,\delta\pi]$ generates the time evolution of perturbations. In fact, by using eq. (4.2)-(4.3) in eq. (4.4) we have

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\delta\phi = \left[\delta\phi, \tilde{H}\right] \qquad\qquad i\frac{\mathrm{d}}{\mathrm{d}\tau}\delta\pi = \left[\delta\pi, \tilde{H}\right] \tag{4.5}$$

The formal solution of the eq. above is given in terms of the unitary operator $U(\tau,\tau_0)$ as

$$\delta\phi(\tau) = U^{-1}(\tau,\tau_0)\,\delta\phi(\tau_0)\,U(\tau,\tau_0) \qquad\qquad \delta\pi(\tau) = U^{-1}(\tau,\tau_0)\,\delta\pi(\tau_0)\,U(\tau,\tau_0) \tag{4.6}$$

the time τ_0 is when we assign the initial condition (right) to the differential equation (left) which defines the time-evolution operator

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau,\tau_0) = U(\tau,\tau_0)\;\tilde{H}\,[\delta\phi(\tau),\,\delta\pi(\tau),\tau] \qquad\qquad U(\tau_0,\tau_0) = \mathbb{I} \tag{4.7}$$

In words, the dynamics of the background fields (ϕ_0, π_0) is governed by the Hamiltonian $H\,[\phi_0,\pi_0]$ while the time evolution of the perturbations is generated by the perturbed Hamiltonian.

[1] square brackets indicate functional dependence

The perturbed Hamiltonian starts quadratic in the fluctuations $\delta\phi$, $\delta\pi$. The quadratic terms lead to linear equations of motion, whose solutions are those of the free theory we know exactly. In general however $\tilde{H}$ contains also higher-order terms, and the resulting equations of motion are non-linear and an exact solution is not feasible. Since we can solve exactly the free dynamics (e.g. that from quadratic terms), we want to factorize the free theory. To this end let us extract *the quadratic part* H_0 (*free*) from the perturbed Hamiltonian

$$\tilde{H}\left[\delta\phi, \delta\pi, \tau\right] = H_0\left[\delta\phi, \delta\pi, \tau\right] + H_{\text{int}}\left[\delta\phi, \delta\pi, \tau\right] \tag{4.8}$$

the H_{int} is the *interaction Hamiltonian* which contains terms of order higher than the quadratic one. If the H_I were turned off, the theory for the perturbations $\delta\phi$, $\delta\pi$ would be free and we may express the solution of eq. (4.7) in the closed form[2] $e^{-i\int_{\tau_0}^{\tau} d\tau' H_0(\tau')}$. In the general case, we would like to use perturbation theory to write the general solution in terms of the free one. We introduce the interaction picture, in which this can be done covariantly for fields.

Interaction picture. We indicate interaction-picture fields as $\delta\phi^I$. They are defined such that their evolution is governed by the free theory

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\delta\phi^I \equiv \left[\delta\phi^I, H_0\right] \qquad\qquad i\frac{\mathrm{d}}{\mathrm{d}\tau}\delta\pi^I \equiv \left[\delta\pi^I, H_0\right] \tag{4.9}$$

The solution to the equation above is given in terms of

$$\delta\phi^I(\tau) = U_0^{-1}(\tau,\tau_0)\,\delta\phi^I(\tau_0)\,U_0(\tau,\tau_0) \qquad\qquad \delta\pi^I(\tau) = U_0^{-1}(\tau,\tau_0)\,\delta\pi^I(\tau_0)\,U_0(\tau,\tau_0) \tag{4.10}$$

where the unitary operator U_0 fulfils the differential equation

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}U_0(\tau,\tau_0) = U_0(\tau,\tau_0)\,H_0\left[\delta\phi^I(\tau),\,\delta\pi^I(\tau),\tau\right] \qquad\qquad U_0(\tau_0,\tau_0) = \mathbb{I} \tag{4.11}$$

The interaction picture is completely determined by comparing the fields at the reference time τ_0 with the correspondents in Heisenberg picture, that is

$$\delta\phi^I(\tau_0) = \delta\phi(\tau_0) \qquad\qquad \delta\pi^I(\tau_0) = \delta\pi(\tau_0) \tag{4.12}$$

From eq. (4.9) we have that the value of the Hamiltonian H_0 is the same over the phase-space trajectory of the solution

$$H_0\left[\phi^I(\tau),\,\pi^I(\tau),\,\tau\right] = H_0\left[\phi^I(\tau_0),\,\pi^I(\tau_0),\,\tau\right] \tag{4.13}$$

this way we see that the time-dependence of H_0 is only explicit. The eq. (4.11) for U_0 becomes[3]

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}U_0(\tau,\tau_0) = U_0(\tau,\tau_0)\,H_0\left[\delta\phi_0^I,\,\delta\pi_0^I,\tau\right] \tag{4.14}$$

By using eq.s (4.6) and (4.10) it is straightforward to see that any operator is related to the corresponding one in interaction-picture by

$$Q(\tau) = \left(U_0^{-1}(\tau,\tau_0)\,U(\tau,\tau_0)\right)^{-1} Q^I(\tau)\left(U_0^{-1}(\tau,\tau_0)\,U(\tau,\tau_0)\right) \tag{4.15}$$

[2]this holds in ordinary flat-space QFT, for arbitrary times τ. However, in a non-static background QFT this does not hold unless $\tau \sim \tau_0$. The interested reader is again referred to §3.5 ”Adiabatic vacuum” of [20]

[3]we abbreviate $\phi_0^I \equiv \phi^I(\tau_0)$.

We remark that (a) the interaction-picture $Q^I(\tau)$ is a functional of the interaction-picture fields, which evolve like with the free Hamiltonian; and (b) the operator $Q(\tau)$ in the two pictures (lhs & rhs) is evaluated at the same time.
We define the unitary operator $U_I(\tau,\tau_0) \equiv \big(U_0^{-1}(\tau,\tau_0)\, U(\tau,\tau_0)\big)$, it is straightforward to see that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\, U_I(\tau,\tau_0) &= U_0^{-1}\,\frac{\mathrm{d}U}{\mathrm{d}\tau} - U_0^{-1}\,\frac{\mathrm{d}U_0}{\mathrm{d}\tau}\, U_0^{-1}\, U \\
&= -iU_0^{-1}\Bigg\{U\Big[\Big(i\frac{\mathrm{d}U}{\mathrm{d}\tau}\Big)\, U^{-1}\Big]\, U^{-1} - \Big(i\frac{\mathrm{d}U_0}{\mathrm{d}\tau}\Big)\, U_0^{-1}\Bigg\}U \\
&= -iU_0^{-1}\Bigg\{U\,\tilde{H}[\delta\phi(\tau),\delta\pi(\tau),\tau]\, U^{-1} - H_0[\delta\phi_0,\delta\pi_0,\tau]\Bigg\}U \\
&= -iU_0^{-1}\Bigg\{\tilde{H}[\delta\phi(\tau_0),\delta\pi(\tau_0),\tau] - H_0[\delta\phi_0^I,\delta\pi_0^I,\tau]\Bigg\}U \\
&= -iU_0^{-1}\Bigg\{\tilde{H}[\delta\phi_0^I,\delta\pi_0^I,\tau] - H_0[\delta\phi_0^I,\delta\pi_0^I,\tau]\Bigg\}U \\
&= -iU_0^{-1}\Bigg(H_{\mathrm{int}}[\delta\phi_0^I,\delta\pi_0^I,\tau]\Bigg)\, U \\
&= -i\, H_I(\tau)\; U_I(\tau,\tau_0)
\end{aligned} \tag{4.16}$$

where we used eq.s (4.7)-(4.14) the third line; the eq.s from third one to (4.6)-(4.12) the fourth one and again in the fifth one; the eq. (4.8) in sixth line. In the last line, we have defined

$$H_I(\tau) \equiv U_0^{-1}\; H_{\mathrm{int}}\left[\delta\phi_0^I,\; \delta\pi_0^I,\tau\right]\; U_0 \tag{4.17}$$

$$= H_{\mathrm{int}}^I\left[\delta\phi^I(\tau),\; \delta\pi^I(\tau),\tau\right] \tag{4.18}$$

which is the interaction Hamiltonian in interaction picture. The differential equation, along with the boundary condition, for the $U_I(\tau,\tau_0)$ operator is

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\, U_I(\tau,\tau_0) = H_I(\tau)\, U_I(\tau,\tau_0) \qquad\qquad U_I(\tau_0,\tau_0) = \mathbb{I} \tag{4.19}$$

4.2.1 Perturbative solution for $U_I(\tau,\tau_0)$

Let us find a solution to the eq. above in perturbation theory. Assume the interaction Hamiltonian is proportional to the coupling constant $\lambda < 1$. We seek for a representation of the U_I operator in the form of a series

$$U_I(\tau,\tau_0) = \sum_{k=0}^{\infty} U_I^{(k)}(\tau,\tau_0) \tag{4.20}$$

where the term $U_I^{(k)}(\tau,\tau_0)$ is assumed proportional to the coupling λ^k.
Inserting this expression in eq. (4.19) gives

$$\begin{aligned}
i\frac{\mathrm{d}}{\mathrm{d}\tau}\, U_I^{(k+1)}(\tau,\tau_0) &= H_I(\tau)\, U^{(k)}(\tau,\tau_0) \\
\Rightarrow U_I^{(k+1)}(\tau,\tau_0) &= (-i)\int_{\tau_0}^{\tau} d\tau'\; H_I(\tau')\, U^{(k)}(\tau',\tau_0)
\end{aligned} \tag{4.21}$$

The initial condition $U_I(\tau_0, \tau_0) = \mathbb{I}$ is obviously satisfied by $U_I^{(0)}$. We have for example

$$
\begin{aligned}
U_I^{(2)}(\tau, \tau_0) &= (-i) \int_{\tau_0}^{\tau} d\tau_1 \; H_I(\tau_1)\, U^{(1)}(\tau_1, \tau_0) \\
&= (-i)^2 \int_{\tau_0}^{\tau} d\tau_1 \; H_I(\tau_1) \Big(\int_{\tau_0}^{\tau_1} d\tau_2 \; H_I(\tau_2) \Big) \\
&= (-i)^2 \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \; H_I(\tau_1)\, H_I(\tau_2)
\end{aligned}
$$

That is, the solution is given in terms of the so-called *Dyson series*

$$
\begin{aligned}
U_I(\tau, \tau_0) &= 1 + (-i) \int_{\tau_0}^{\tau} d\tau_1 \; H_I(\tau_1) + (-i)^2 \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \; H_I(\tau_1)\, H_I(\tau_2) \\
&\quad + (-i)^3 \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \int_{\tau_0}^{\tau_2} d\tau_3 \; H_I(\tau_1)\, H_I(\tau_2)\, H_I(\tau_3) + ... \\
&= \mathbb{T} \left\{ \exp\left(-i \int_{\tau_0}^{\tau} d\tau' H_I(\tau') \right) \right\}
\end{aligned}
\tag{4.22}
$$

where in the last line, the exponential function must be interpreted as a formal power series of its argument, and we introduced the time-ordering operator. It re-organizes the products of operators in the integrand so that early time operators are on the right, that is the operators are written from left to right in the decreasing order of time arguments. The factorial of the exponential expansion is cancelled by the integration over the squared domain.

So far, modulo few remarks in the footnotes, we have introduced the interaction picture in the same way one does in flat QFT. We conclude this section emphasizing once again that very fundamental property of interaction picture is that the operators in such representation are functional of the free fields, whose time-evolution is known exactly.

4.3 Boundary-value vs. initial-value problem

Before discussing the prescription we use to pick up the $|\text{in}\rangle$ state, it is worth to remark the key point about it. In the conventional formulation of scattering amplitudes, both the in and out states are specified in the asymptotic regions (past/future-infinity). The solution of the interacting theory is therefore a boundary-value problem rather than an initial-value problem, as it is the case for the in-in formalism.

Vacuum state In traditional S-matrix formalism, the boundary conditions are set when the interactions are negligible and the theory behaves freely - i.e. $|in\rangle$ at early time, when we prepare the particles to collide, and $|out\rangle$ at late time when the interactions no more occur. So, we solve the full interacting theory in terms of the free theory, whose evolution is trivial. In the CTP formalism for cosmology, we would like to define the $|in\rangle$ analogously - i.e. at early time, when the theory is free and a stable state $|in\rangle$ can be found - however there are few differences: (a) the gravitational interaction is universal, e.g. gravitational interactions are always on; (b) moreover in general, the asymptotic in region is not static, e.g. we may not be able to define a vacuum state in the sense of ordinary QFT. The CTP formalism applied to cosmology is complicated by the background geometry. A discussion about quantum field theory in curved space is far beyond the purpose of this work, the interested

reader can refer to the literature, for example [20].
For the case of cosmological perturbations, specifying the in-state amount to choosing a Bunch-Davies vacuum $|in\rangle = |0\rangle$ (sec. 3.13). The vacuum state of the full theory $|\Omega\rangle$ differs from the vacuum state of the free theory $|0\rangle$. In Schrödinger picture, how is $|in, \tau_0\rangle = |0\rangle$ related to $|\Omega, \tau_0\rangle$?
Analogously to ordinary QFT, accounting for initial conditions amount to adding a (small) ϵ imaginary part to the time variable $\tau \to \tau(1 - i\epsilon)$ [16]. Physical results are independent of the ϵ-dependence, which we drop at the end of the calculation by taking the limit $\epsilon \to 0$. This prescription amounts to integrate over times from the real axis to a complex contour. Moreover let us relate the vacuum of the interacting theory with that of the free theory. We would find that the vacua of free and interacting theory are proportional. For example, the expectation value of the operator $Q(\tau)$

$$\langle\Omega|\, Q(\tau)\, |\Omega\rangle = \frac{\langle 0|\, U_I^\dagger(\tau,\tau_0)\, Q^{(I)}(\tau)\, U_I(\tau,\tau_0)\, |0\rangle}{|\langle\Omega|0\rangle|^2} \tag{4.23}$$

In ordinary QFT, the constant of proportionality between the free/interacting vacuum is a phase. Moreover, the amplitude $\langle 0|\Omega\rangle$ has physical meaning, as for example represents quantum vacuum fluctuations diagrams. However, in the in-in formalism all vacuum fluctuation diagrams automatically cancel [15]. That is, we have $|\langle\Omega|0\rangle|^2 = 1$

4.4 The in-in master equation

In the case of cosmological perturbations, we start from the master equation

$$\Big\langle \Omega \Big| \, Q(\tau) \, \Big| \Omega \Big\rangle = \Big\langle 0 \Big| \, \bar{\mathbb{T}}\Big[\exp\Big(i\int_{\tau_0}^{\tau} d\tau' H_I(\tau')\Big)\Big]\; Q^{(I)}(\tau)\; \mathbb{T}\Big[\exp\Big(-i\int_{\tau_0}^{\tau} d\tau' H_I(\tau')\Big)\Big] \,\Big|\, 0 \Big\rangle \tag{4.24}$$

Commutator form The are few variants of the in-in formula above. In the following we will use the so-called *commutator form*, which is obtained by applying to eq. (4.24) the Baker-Hausdorff lemma (above)

$$\exp\{iA\lambda\}\, B \exp\{-iA\lambda\} = B + i\lambda\,[A,B] + \left(\frac{(i\lambda)^2}{2!}\right)[B,[B,A]] + \left(\frac{(i\lambda)^3}{3!}\right)[B,[B,[B,A]]] + ... \tag{4.25}$$

here A, B are hermitian operators and λ is a real parameter. By considering $A \leftrightarrow H_I$, $B \leftrightarrow Q$ we arrive to the commutator form of the eq. (4.24), that is

$$\langle Q(\tau)\rangle = \sum_{N=0}^{\infty} i^N \int_{\tau_0}^{\tau} d\tau_N \int_{\tau_0}^{\tau_N} d\tau_{N-1} ... \int_{\tau_0}^{\tau_2} d\tau_1 \Big\langle \Big[H_I(\tau_1), \Big[H_I(\tau_2), ... \Big[H_I(\tau_N), Q(\tau)\Big]...\Big]\Big]\Big\rangle \tag{4.26}$$

where on the left we omitted the vacuum state $|\Omega\rangle$ and analogously for $|0\rangle$ on the r.h.s.

Chapter 5

Cosmological gravitational waves

We start by introducing the concept of GW in the cosmological curved space. In sec. 5.1 we identify the GWs as the sub-Hubble modes of the tensor perturbation. In sec. 5.2 we derive a rough relation between the observed GW frequency and its production time. Early universe GWs are expected to have a stochastic character, as explained in sec. 5.3. In sec. 5.4 there is a brief digression about the stress-energy tensor of the GWs. The study of early universe GWs starts in sec. 5.6, where we report the primordial GWs spectrum. It follows the sec. 5.7 where we discuss the quantum-to-classical transition of the cosmological perturbations. The Part II is dedicated to second-order GWs sourced in the RD era from primordial first-order enhanced scalar perturbations. In sec. 5.8, the enhancement ansatz is justified in the context of inflationary models. The calculation for the GW power spectrum is presented starting from sec. 5.9.

Gravitational waves in curved space

In Section 3.2.1, the metric perturbations were decomposed in scalar, vector and tensor fields. The tensor perturbation contains the propagating degrees of freedom of the metric, called gravitational waves (GWs). For a general spacetime, the concept of gravitational wave as tensor perturbation has meaning if and only if one assumes the existence of a separation between the typical variations length & time scales of the perturbation and those of the background. If we denote with $\lambda^{(\mathrm{B})}$ ($T^{(\mathrm{B})}$) these quantity for the background spacetime $g^{(\mathrm{B})}_{\mu\nu}$, and with $\lambda^{(h)}$ ($T^{(h)}$) those of the tensor perturbation, the concept of "gravitational wave" is well defined when the condition below is met.

$$\lambda^{(h)} \ll \lambda^{(\mathrm{B})} \qquad\qquad T^{(h)} \ll T^{(\mathrm{B})} \tag{5.1}$$

This way, the GWs can be consistently regarded as ripples propagating in the slowly-varying background spacetime, i.e. see [21] [2] .

To stress the importance of both the inequalities in (5.1), let us consider an example. The GW event detected by the LIGO/VIRGO collaboration contained a wave packet of modes with wavelengths in the interval $\lambda \in [1.2, 8.6] \times 10^3$ km [1]. The detector is embedded in the background geometry given by the Minkowski spacetime plus some newtonian corrections, from the Earth gravitational field. The condition $\lambda^{(h)} \ll \lambda^{(\mathrm{B})}$ is not fulfilled, in that the newtonian potential varies appreciably over kms. However, the background metric in the frequency bandwidth $f \in$ 100-1000 Hz is static, and actual GW measurements are possible.

Said that, in the cosmological context we can safely refer to the (sub-Hubble) tensor modes h_{ij} as GWs. In fact, the typical length scale of the FLRW background metric is the Hubble radius $\propto (aH)^{-1}$ and in sec. 5.2 we will see that, for the wavelengths typical of the GWs of interest, it holds $\lambda \ll (a_* H_*)^{-1}$ at the emission time t_*.

Stochastic gravitational wave background

The GWs of the example have been produced by the coalescence of an astrophysical binary system. This work focusses on a different type of sources, which produce a GW signal not circumscribable either in time or from a direction of the sky. These GWs are produced by a large number of unresolved uncorrelated sources, so that the GW signal today can be characterized only statistically - we return to this point in sec. 5.3. It is called *stochastic gravitational wave background* (SGWB). One should not confuse this with the background geometry of sec. 3.2.1. Many phenomena contribute to the SGWB, acting either in early universe or in recent epochs. For example, some astrophysical processes from our galaxy may contribute to this background.

However, we focus on cosmological sources of SGWs. The amount of information it is possible to read via this signal it is very appealing and the detection of this background would be a milestone in the modern cosmology [22]. In fact, the cosmological GWs potentially convey informations about the primordial universe. For instance, in sec. 5.6 we will see that a prediction of the inflation theory is the existence of a relic primordial spectrum of gravitational waves.

5.1 The perturbed metric |Act 2

Recall the perturbed metric eq. (3.20)-(3.22), and in particular the tensor perturbation h_{ij} and its properties eq. (3.23)

$$\delta g_{ij} = a^2 \left(-2\Psi\, \delta_{ij} + D_{ij}\, E + \partial_{(i}\, F_{j)} + h_{ij}\right)$$
$$\partial_i\, h_{ij} = 0 \qquad\qquad \delta^{ij}\, h_{ij} = 0$$

The physical propagating degrees of freedom of the metric are two. They are embedded in the spin-2 transverse traceless tensor perturbation h_{ij}, whose excitations are GWs. It is customary to consider the GWs as the 'true' degrees of freedom of the metric, because they propagate even in vacuum (5.4). An infinitesimal gauge transformation, like eq. (3.19), leaves the tensor perturbation unaffected to linear order

$$h_{ij} \to h_{ij} \tag{5.2}$$

so the field h_{ij} is gauge-invariant. That is, the transverse-traceless gauge-invariant field h_{ij} contains physical propagating degrees of freedom only.

In absence of sources, the action of the free GWs is obtained by expanding the Einstein-Hilbert action. To the lowest order of the longitudinal-gauge linearised theory, we have

$$S^{(\mathrm{GW})}_{free} = \frac{M_p^2}{2} \int d\tau d^3\vec{x}\ \frac{a^2}{4}\Big((h'_{ij})^2 - (\nabla h_{ij})^2\Big) \tag{5.3}$$

It is straightforward to calculate the Euler-Lagrange/Einstein equation of motion for the GWs

$$h''_{ij} + 2\mathcal{H}\, h'_{ij} - \nabla^2\, h_{ij} = 16\pi G\, T_{ij} \tag{5.4}$$

where we used this opportunity to anticipate the possibility for a non-vanishing source T_{ij}.

The action eq. (5.3) describes GWs freely propagating over the background FLRW spacetime. In light of the concept of GW we have discussed above, the solutions in the comoving frame are plain waves. Also, we mention that the general solution of the free eq. (5.4) ($T_{ij} = 0$) for the tensor perturbation is given exactly in terms of spherical Bessel functions.

In general, most of the cosmological sources in the RD & late eras act for a limited time. This implies that the spectral bandwidth of the resulting GWs is bounded. Moreover, after the source turns off $T_{ij} \to 0$ the GWs propagate freely - we always neglect non-linearity effects in the propagation - and the amplitude of h_{ij} decays with its mass-dimension during the cosmological expansion. In light of this, we can roughly relate the time at which a mode was produced with the GW frequency observed today.

5.2 Frequency/Time-of-Emission relation for SGWB modes

Due to the non-linear dynamics, the sourced GWs can in principle be contributed by scalar perturbations from very different scales. However, in practise the second-order GWs are primarily sourced when the first-order scalar perturbations of same scale (i.e. same λ of the GW) re-enter the Hubble radius (during RD) [23] [24]. For a GW mode produced at the time t_e with frequency $f(t_e) \propto (a(t_e)\,\lambda)^{-1}$, it holds the relation

$$(aH)^{-1}\,(t_e) \approx \lambda \tag{5.5}$$

which is understood in some parametrical sense.

For definiteness, we focus on GWs produced during the radiation era and we neglect the present dark age. To find the sought-after frequency/time-of-emission relation, all we need to do is to rescale the observed frequency of the GW to the frequency at the emission time t_e, where (5.5) holds. We need to account for the different scaling of a physical lengths during the RD/MD era - see Tab. 1.1. That is, we have to account of the radiation-matter transition, see fig. 1.2. Let us assume the transition from the radiation to the matter dominated era to occur at t_{eq} abruptly.
The Hubble rate $H(t_e)$ is related to the present Hubble constant H_0 by

$$H(t_e) = H_0 \left(\frac{H(t_e)}{H_{eq}}\right)\left(\frac{H_{eq}}{H_0}\right) = H_0 \frac{t_0}{t_e} \tag{5.6}$$

analogously, the frequency of the wave at the emission is related to the present by

$$f(t_e) = \frac{a(t_{eq})}{a(t_e)}\frac{a(t_0)}{a(t_{eq})} f = \left(\frac{t_e}{t_{eq}}\right)^{-1/2}\left(\frac{t_0}{t_{eq}}\right)^{2/3} f \tag{5.7}$$

Putting all the pieces together, e.g. inserting (5.6)-(5.7) in (5.5), one obtains the final result. In order to get a numerical value, we consider the equality in (5.5) and we express it in terms of the matter-radiation redshift (defined in the footnote) ($f = f(t_0)$)

$$
\begin{aligned}
t_e &= \left(\frac{H_0}{f}\right)^2 t_0 \,(1+z_{eq})^{-1/2} \\
&\simeq \left(\frac{f}{\mathrm{mHz}}\right)^{-2} \; 3.6\times 10^{-14} \;\; sec.
\end{aligned}
\tag{5.8}
$$

In the last, recent values of the equivalence redshift[1] $z_{eq} = 3379 \pm 22$ [8] and the age of the universe $t_0 = (4.351 \pm 0.006) \times 10^{17}$ [8] have been used.

The GWs with today frequency $\sim$ mHz are contributed from processes which originated GWs in the RD era. The reader is invited to note that the RD-era sourced GWs - which we will study soon Part II - are in the potential spectral capability of present/future GWs detectors, see [2].

Ansatz consistency We were interested in GWs produced during RD era, in fact we accounted for the radiation-matter transition (z_{eq}). For consistency, the time of emission should lie in the radiation era, that is $t_e \leq t_{eq}$. Using eq. (5.8) we obtain the bound

$$
f \geqslant f_c \equiv H_0 \left(\frac{t_0}{t_{eq}}\right)^{1/3} = (1+z_{eq})^{1/2} \approx 10^{-16} \;\; \mathrm{Hz}
\tag{5.9}
$$

We conclude that modes with frequency $f < f_c$ are not produced during the radiation era with this mechanism.

5.3 Stochastic character of the cosmological gravitational wave background

We claimed that the GWs produced from sources acting in the early universe are expected to appear as part of a stochastic background. Let us address this point and justify it quantitatively. As we said, a stochastic variable can only be characterized statistically. The reader familiar with statistical mechanics knows that by 'statistically' one means an ensemble average. However, we inhabit *one* realisation of the ensemble - our universe - so it is worth to make a brief digression about the ergodic hypothesis.

In statistical mechanics, it customary to invoke the ergodic hypothesis to swap ensemble average with time average over the phase-space trajectory of a single realisation of the ensemble [25]. In cosmology, this implies that by observing a region of the universe for long enough time one has access to the many realisations of the ensemble. We will not enter into the details of the technical conditions for the primordial GWs to assume a stochastic character and for the ergodic theorem to hold, the interested reader can see [22] [26].

Said that, the cosmological GWB is expected to have stochastic character because the causal production of GWs happens at a time when the comoving Hubble radius is smaller than the present value. The GWs background signal today is composed by the superposition of many uncorrelated GWs. It is straightforward to calculate the minimum number of independent GW signals an observer would measure today, since we can use the results of the previous section - i.e. we combine eq. (5.6)

[1]The redshift z is defined as $1 + z_e = \frac{a(t_0)}{a(t_e)}$.

and eq. (5.8). The number of independent GWs of frequency f which arrive to the observer today is of the order of

$$\frac{H_0^{-1}}{H_e^{-1}} \geqslant \left(\frac{f}{\mathrm{mHz}}\right)^2 \times 10^{31} \tag{5.10}$$

meaning that a typical signal of interest - i.e. produced in RD era with frequencies $\lesssim$ Hz - is the superposition of uncorrelated GWs produced in at least $\sim 10^{26}$ regions. As noted by B. Allen in [22], a such GW signal from the early universe cannot be resolved beyond its stochastic nature because for instance the necessary angular resolution for a detector is too small ($\leqslant 10^{-12}\, deg$).

Before addressing concretely our case-study, let us terminate the 'conceptual' part of this chapter by digressing about the contribution of stochastic GWs to the universe energetics.

5.4 The effective stress-energy tensor of the gravitational waves

In general relativity (GR) a local measure of gravitational energy does not exist. It is customary to ascribe the impossibility of attaching an energy-stress tensor to gravitational fields to the Equivalence Principle. What this means, in practise, is that one can eliminate the first derivatives of the metric tensor by a coordinate change along any world line, but for a typical 'stress-energy object' it holds (b). However, the GWs carry energy & momentum with their propagation somewhat analogously to the electromagnetic waves. We will see that it is possible attach an 'effective' stress-energy tensor $T_{\mu\nu}^{(\mathrm{GW})}$ to GWs. The key point here is in the relation between the background metric and its perturbations.

A comprehensive discussion about the energy-momentum tensor associated to the GWs requires a more sophisticated viewpoint than the linearised theory[2] - as discussed in [21, see §35.7,§35.13,§35.15]. The interested reader is referred to the literature, while here we will report the essential results.

However before that, let us try to understand heuristically what structure the sought-after GW stress-energy must have. We start by recalling the general properties of a stress-energy tensor

(a) The stress-energy tensor is the proper generalization of the densities of energy, momentum - and the fluxes-of - in the continuous media (fields), where these quantities are functions of points and they combine in the tensor $T_{\mu\nu}(x^\alpha)$.

(b) It is symmetric ($T_{\mu\nu} = T_{\nu\mu}$) and it is related to locally conserved quantities ($\nabla^\mu T_{\mu\nu} = 0$). In a field theory (e.g. ϕ) with Euler-Lagrange second-order differential equations, the stress-energy tensor depends on the squares of first-order derivatives ($\sim \partial\phi\, \partial\phi$).

(c) From a lagrangian, one derives the stress-energy tensor as a Noether current. The resulting $T_{\mu\nu}$ is the unique symmetric tensor[3] (well-transforming under arbitrary coordinate changes), conserved due to the equations of motion and free of second (and higher) derivatives of the field variable.

(d) In GR, the stress-energy tensor for the matter field ϕ follows from diffeomorphism invariance of the matter action. It enters in the Einstein equation

$$G_{\mu\nu} \;=\; 8\pi G\, T_{\mu\nu}$$

on the r.h.s as source for the geometry.

[2] where h_{ij} is infinitesimal, and effects higher than first order are neglected

[3] the canonical stress-energy tensor is not a priori symmetrical. However it can be made symmetric without altering the equations of motion. The symmetry is required for a proper formulation of the angular momentum conservation

Second, we refresh the concepts of pag.33 about GWs & tensor perturbation. The GWs are small ripples propagating in the background metric: the variations of the background and of the tensor perturbation occur on very different length & time scales in virtue of (5.1).

It is conceivable that there is an energy/momentum transfer between GWs and background. We seek for an equation relating GWs with background geometry. In light of the point (d), a good starting point is the Einstein equation since it contains both the background geometry and the tensor perturbation. Because of the separation of scales (5.1), one expects that for this purpose the linear terms in h_{ij} are irrelevant - to convince himself/herself one can go in momentum space. Therefore we understand that we need to build a physical quantity which is a functional of the tensor perturbation and moreover it is at least bilinear in h_{ij}. Also, if we average a term like $h(\mathbf{x})\, h(\mathbf{y})$ over space, the convolved fourier amplitude may contribute to long wavelength $\lambda^{(\mathrm{B})} \gg \lambda$ mode dynamics.

This heuristic argument was guided by the properties (a)-(d) and by an analogy between the dynamics of the gravitational waves & background and that of a ferromagnetic system, with magnetic moments & mean-field magnetization $m(\mathbf{x})$. From the field-theoretical formulation viewpoint, these systems share non-linear dynamics described by the fields $h_{ij}(\mathbf{x})$, $m(\mathbf{x})$ respectively.
Anyway, the energy-momentum tensor of GWs is defined by analysing the Einstein equation and by identifying an object which has some desirable properties (a)-(d) shared by any stress-energy tensor. This "effective" $T^{(\mathrm{GW})}_{\mu\nu}$ sources the background geometry as consequence of the GR non-linearity. It is defined in eq. (35.61) of [21] by averaging a quantity bilinear in h_{ij} - as we guessed

$$T^{(\mathrm{GW})}_{\mu\nu} \equiv -\frac{1}{8\pi}\left\{\left\langle R^{(2)}_{\mu\nu}(h)\right\rangle - \frac{1}{2} g^{(\mathrm{B})}_{\mu\nu}\left\langle R^{(2)}(h)\right\rangle\right\} \tag{5.11}$$

$$= \frac{M_p^2}{4}\left\langle -\partial_\mu\, h_{ij}\, \partial_\nu\, h_{ij} + \frac{1}{2}\, g^{(\mathrm{B})}_{\mu\nu}\; {g^{(\mathrm{B})}}^{\alpha\beta} \partial_\alpha\, h_{ij}\, \partial_\beta\, h_{ij}\right\rangle \tag{5.12}$$

The terms in curly brackets are those parts of Ricci tensor and Ricci scalar bilinear in the tensor perturbation. The second line is specialized to our study of gravitational waves in the longitudinal gauge [21]. For example, $R^{(2)}_{\mu\nu}(h)$ is sum of terms containing two h_{ij} fields and analogously for the Ricci scalar. The angle brackets $\langle A\rangle$ denote an average procedure which extracts out of A the part which varies on scales - much larger of λ - to which the background geometry is sensible.

The averaging procedure is also called coarse-graining. A more familiar example of this procedure occurs for instance in ferromagnets, where to account of non-linear interactions of the magnetisations with the macroscopic magnetic field (which themselves affect) one usually uses a mean-field approach - where the 'gross' effects of microscopics is accounted with coarse-graining.
The averaging procedure for the GW stress-energy is twofold: first, it allows to capture enough information about 'the geometry' so that the quantity eq. (5.11) is gauge-invariant and becomes a 'well-defined' tensor; second, it extracts from bilinear terms like $h\,h$ the long-scale behaviour. There are many explicit averaging schemes, however all of them should be performed on sizes l

$$\lambda \ll l \ll \lambda^{(\mathrm{B})} \tag{5.13}$$

much bigger than the typical wavelength λ of a GW mode (of interest) and much smaller than the typical scale of variation of the background geometry $\lambda^{(\mathrm{B})}$.

It is customary to say that the stress-energy eq. (5.11) is defined in a "smeared-out" sense. The $T^{(\mathrm{GW})}_{\mu\nu}$ enters into the Einstein equation for the background, in fact it acts as a source curving the background geometry. This is a non-linear effect [21]. Again, the reader is invited to appreciate the analogy with the magnetic system.

Note about eq. (5.12) For clarity, let us delineate the steps which lead to eq. (5.12). The protocol is simple, but the calculations are rather long. For example, let us concentrate on $R^{(2)}_{\mu\nu}(h)$: (a) one starts from the perturbed metric eq. (3.17) for the tensor perturbation ($\Phi = B = \Psi = E = B_i = F_i = 0$); (b) one extracts a parameter, say ϵ, out of h_{ij}, and calculates all the relevant quantities (inverse metric, Levi-Civita symbols, Riemann, Einstein tensor, etc.) in series - for example $R_{\mu\nu} = \sum_{n=0}^{\infty} \epsilon^n R^{(n)}_{\mu\nu}$; (c) in the final result, we collect the terms depending on the powers of ϵ, $\epsilon^2, (...)$. The $R^{(n)}_{\mu\nu}$ is by construction a sum of terms each containing the field h_{ij} n-times. Finally, one sets $\epsilon \to 1$.
By following this procedure, in the longitudinal gauge, we obtained eq. (5.12).

5.4.1 Stress-energy of stochastic GWs

Let us turn our attention to stochastic GWs, and we reflect about the averaging procedure. For a stochastic GWs background, the average over several wavelengths is the same as a time average at a given point. To understand this, the reader may imagine how the average is performed in practise. One takes a photography of a spatial region with size $l \times l \times l$ of eq. (5.13). In this picture, the SGWs are frozen distributed isotropically and the average procedure amounts to sum the amplitude interferences over the many spatial points. On the other hand, one may sits at a fixed spatial point and wait some time such that all the waves contained in the previous photography ($N \sim (\frac{l}{\lambda})^3$) arrives to his/her detector.
Therefore by trading the spatial average with a time average, we recovered the ensemble average of sec. 5.3 and the meaning of the angle brackets is the same. For more details see [2, §2.2]. This explains a posteriori why we used the same symbol.

Energy density Let us calculate the energy density $\rho^{(\mathrm{GW})} \equiv -T^{(\mathrm{GW})}_{00}$ of GWs using eq. (5.12). Recall that the effective stress-energy eq. (5.12) sources the background geometry, therefore the scalar quantity $\rho^{(\mathrm{GW})}$ is on equal footing of the background energy density of eq. (1.2).

$$\rho^{(\mathrm{GW})} = \frac{M_p^2}{4} \left\langle \frac{{h'_{ij}}^2}{2} + \frac{{\nabla h_{ij}}^2}{2} \right\rangle \tag{5.14}$$

We take advantage of this opportunity, to write the momentum-space energy density of SGWs [2]. We will need this formula later. We use eq. (A.4) for the fourier transform of the tensor perturbation $h_{ij}(\tau, \mathbf{x})$ to write eq. (5.14) in momentum space (see Appendix B.8)

$$\rho^{(\mathrm{GW})}(\tau, \mathbf{k}) \simeq \frac{M_p^2}{4} k^2 \sum_{\lambda} \frac{1}{2} \overline{(h_\lambda(\tau, \mathbf{k}))^2} \simeq \frac{M_p^2}{4} \sum_{\lambda} \frac{1}{2} \overline{(h'_\lambda(\tau, \mathbf{k}))^2} \tag{5.15}$$

Here we swapped the ensemble average with the time average. The over-line denotes the time-average over many GW periods. Notice that in our case, the two polarisations contribute equally to the sum. Moreover, because of (5.1) $k \gg \frac{a'}{a}$ the GWs - which are sub-Hubble modes - are approximately free and we used the equipartition theorem.

5.5 The two contributions to the cosmological SGWB: vacuum GWs and sourced GWs

Strictly speaking, the calculation we made in sec. 5.3 holds only for sourced GWs in the radiation era. It is worth to mention that also GWs sourced during MD era at least $z_p \gtrsim 17$ - for instance see [26]

- are expected to have a stochastic character. These arguments both use the increasing monotony of the Hubble radius during RD/MD eras, see fig. 2.3.

A remarkable fact about the inflationary theories is the mechanism which amplifies zero-point quantum fluctuations to macroscopic perturbations. The tensor perturbation h_{ij} is another light-field experiencing the accelerated expansion phase, and a general prediction of the inflationary theories is a spectrum of gravitational waves. We will cover this in details later, but for the moment we anticipate that also these GWs are expected to have a stochastic character.
It is useful to distinguish the contributions to the SGWB from the early universe in

- Vacuum gravitational waves - they arise from zero-point quantum fluctuations of h_{ij} occurring during the inflation era. In the following FLRW phases they appear as an irreducible component (i.e. always present) of the SGWB.
- Sourced gravitational waves - they are classically produced though non-linear gravitational interactions. From the early universe, the sourced GWs may be produced either in the inflationary stage or in the following ones. We shall focus on GWs production in the RD era.

For completeness, we will calculate the contribution from vacuum GWs to the SGWB in sec. 5.6. Starting from pag.46 we dedicate our efforts in studying the contribution to the SGWB from sourced GWs generated in the radiation dominated era by the primordial scalar perturbation ζ.

In Section 3.2 we restricted our attention to the primordial scalar perturbation, and the resulting spectrum from the inflationary stage $\mathcal{P}_\zeta$. In the following, we calculate the linear contribution to the SGWB from the SFSR inflation theory of sec. 2.3.

5.6 Vacuum GWs from inflation

The mechanism upon which zero-point quantum fluctuations of the tensor perturbation are parametrically amplified to large scales by the inflationary stage is analogous to that presented in sec. 3.2. The calculation of the primordial spectrum of linear GWs is easier than the corresponding case for the scalar perturbation of sec. 3.2.2. Because, by evaluating the Einstein equation we would discover indeed that the stress-energy tensor of the inflaton sources the GWs to higher orders. One obtains the quadratic action for the tensor fluctuations by expanding the Einstein-Hilbert action

$$S^{(\mathrm{GW})} = \frac{M_p^2}{2} \int d\tau d^3\vec{x} \;\; \frac{a^2}{4} \Big[(h'_{ij})^2 - (\nabla h_{ij})^2 \Big] \tag{5.16}$$

The starting point is the action in momentum space. The (5.16) in momentum space is

$$S^{(\mathrm{GW})} = \sum_\lambda \int \frac{d\tau \, d^3\vec{k}}{(2\pi)^3} \, \frac{a^2}{2} \, M_p^2 \Big(h'_\lambda(\tau,\, \mathbf{k}) \, h'_\lambda(\tau,\, -\mathbf{k}) + k^2 \, h_\lambda(\tau,\, \mathbf{k}) \, h_\lambda(\tau,\, -\mathbf{k}) \Big) \tag{5.17}$$

Here, the polarisation index λ appears. The reader is invited to look the Appendix A for conventions and details.

The protocol we use to calculate the power spectrum of the vacuum GWs from inflation is similar to the one we followed in sec. 3.1. In the following, we avoid repetitions but rather the differences are pointed out. The calculation strategy is summarized below

(a) We canonically normalize the kinetic term, by defining the field $v^\lambda_{\mathbf{k}} \equiv a\,\frac{M_p}{2}\,h^\lambda_{\mathbf{k}}$ (see eq. (3.3)).
(b) We promote the classical field to a quantum operator $v^\lambda_{\mathbf{k}} \to \hat{v}^\lambda_{\mathbf{k}}$ and we quantize it (see sec. 3.1.1). Imposing the canonical commutation relation will lead to a boundary condition on the mode functions (see eq. (3.11)).
(c) The mode functions (eq. (3.12)) are completely fixed by choosing the vacuum state. In particular, we will use the Bunch-Davies vacuum (see eq. (3.13)).
(d) Finally, we compute the power spectrum of primordial gravitational waves from inflation.

Step (a) With a little amount of algebra, it is straightforward to bring the kinetic term of eq. (5.17) in canonical form (e.g. its prefactor is 1/2)

$$S^{(\mathrm{GW})} = \sum_\lambda \int \frac{d\tau\, d^3\vec{k}}{(2\pi)^3} \left[\frac{1}{2}\, v'^\lambda_{\mathbf{k}}\, v'^\lambda_{\mathbf{-k}} - \frac{1}{2}\left(k^2 - \mathcal{H}' - \mathcal{H}^2\right)\, v^\lambda_{\mathbf{k}}\, v^\lambda_{\mathbf{-k}} \right] \tag{5.18}$$

This is very close to eq. (3.3), with the only difference that here the field $v^\lambda_{\mathbf{k}}$ depends on the polarisation index λ. Therefore we can use results of sec. 3.1 by using the correspondence $v_{\mathbf{k}} \Rightarrow v^\lambda_{\mathbf{k}}$ and by accounting of the polarisation sum, when necessary. It is straightforward to generalize the steps (b)-(c) to the present case, for future convenience let us write down the corresponding eq. (3.9) & eq. (3.13)

$$\hat{v}^\lambda(\tau,\vec{p}) \equiv v_{p,\lambda}(\tau)\,\hat{a}^\lambda(\vec{p}) \;+\; v^*_{p,\lambda}(\tau)\,\hat{a}^{\dagger\lambda}(-\vec{p}) \tag{5.19}$$

$$v_{p,\lambda}(\tau) = \frac{1}{\sqrt{2p}}\left(1 - \frac{i}{p\tau}\right) e^{-ip\tau} \tag{5.20}$$

Step (d) - Power spectrum of primordial GWs The power spectrum of the tensor perturbation h_{ij} is related to the mode functions eq. (5.20), analogously to eq. (3.15). Here we have two independent circular polarisations $\lambda = \pm 1$. For an unpolarised background, we sum over the polarisations (obtaining a factor 2). The power spectrum of the tensor perturbation $\mathcal{P}_T(p)$ is

$$\mathcal{P}_T(p) \equiv \sum_\lambda \mathcal{P}_{h_\lambda}\,, \qquad\qquad \mathcal{P}_T(p) = 2\,\frac{4}{M_p^2}\left(\frac{H}{2\pi}\right)^2 \left(\frac{p}{aH}\right)^{\mathcal{N}_T} \tag{5.21}$$

where $\mathcal{N}_T = -2\epsilon$ is the tensor spectral index, in terms of the slow roll parameter ϵ.
At the horizon exit, the r.h.s becomes

$$\mathcal{P}_T(p) = \frac{2}{\pi^2}\left(\frac{H_\star}{M_p}\right)^2 \tag{5.21b}$$

where we have indicated with the star-subscript the Hubble parameter at that time. Again, we remark that the scale dependence of the spectrum follows from the time-dependence of the Hubble parameter. This 'irreducible' component of the SGWB signal is typically too weak for being directly observed by present GWs detectors [22] - we will return to this point in sec. 5.8.

5.7 Cosmological perturbations and classicality

In the previous section we talked about vacuum GWs generated during the inflationary phase. It is worth to remark again that this process (as that of sec. 3.2.2) is purely quantum-mechanical. Indeed

M_p contains $\hbar$ and, as one may check, in the classical limit $\hbar \to 0$ the power spectra vanish. Undoubtedly the field amplitudes we measure today are dominated by classical dynamics[4], therefore it is mandatory to discuss the process of quantum-to-classical transition. This point is well established and the interested reader can find a deeper discussion in literature [19] [28] [29] [30] therefore we shall constrain the discussion to the main aspects only.

The indispensable characteristics for a system to behave classically are two: quantum decoherence and classical correlation [30]. The first refers to the vanishing of quantum interference: generic quantum states are not mutually exclusive, contrariwise to the classical configurations. The last refers to a definite correlation of the points in phase-space: the classical dynamics is the Euler-Lagrange trajectory. The classical behaviour of the cosmological perturbations emerges as result of the time-evolution of the quantum fluctuations during the inflationary expansion. It is instructive to address this question using the Schrödinger viewpoint, formulating the 'classicality' as a property of the time-evolved quantum state.

The Hamiltonian operator has a key role in this approach. Let us start by eq. (5.17): we remember that the tensor perturbation in expressed in terms of the canonically-normalized field $v^\lambda_{\mathbf{k}}$. Also, this is related to the ladder operators as

$$v^\lambda_{\mathbf{k}} = \frac{1}{\sqrt{2k}}\left(a^\lambda_{\mathbf{k}} + a^{\dagger\lambda}_{-\mathbf{k}}\right) \qquad\qquad \pi^\lambda_{\mathbf{k}} = -i\sqrt{\frac{k}{2}}\left(a^\lambda_{\mathbf{k}} - a^{\dagger\lambda}_{-\mathbf{k}}\right) \tag{5.22}$$

It is straightforward to verify that the hamiltonian[5], in terms of the annihilation/creation operators, is given by

$$\begin{aligned} \mathrm{H} = \frac{1}{2}\sum_\lambda \int \frac{d^3k}{(2\pi)^3}\, k \Bigg[&\left(a^\lambda_{\mathbf{k}}\, a^{\dagger\lambda}_{\mathbf{k}} + a^{\dagger\lambda}_{\mathbf{k}}\, a^\lambda_{\mathbf{k}}\right) \\ &- i\frac{\mathcal{H}}{k}\left(a^{\dagger\lambda}_{\mathbf{k}}\, a^{\dagger\lambda}_{-\mathbf{k}} - a^\lambda_{\mathbf{k}}\, a^\lambda_{-\mathbf{k}}\right)\Bigg] \end{aligned} \tag{5.23}$$

This is the hamiltonian operator of a collection of free harmonic oscillators with a coupling term coming from the background expansion - with the prefactor $\frac{\mathcal{H}}{k}$. The last line is responsible for the time evolution into a "squeezed state", as we shall see very soon.
The time-evolution of the system is governed by the Schrödinger equation and it can be regarded as a Bogoliubov transformation which maps the operators $A \equiv \left\{a^\lambda_{\mathbf{k}}(\tau_0),\, a^{\dagger\lambda}_{-\mathbf{k}}(\tau_0)\right\}$ into $B \equiv \left\{a^\lambda_{\mathbf{k}}(\tau),\, a^{\dagger\lambda}_{-\mathbf{k}}(\tau)\right\}$ also preserving the canonical commutation relations, that is

$$\begin{pmatrix} a^\lambda_{\mathbf{k}} \\ a^{\dagger\lambda}_{-\mathbf{k}} \end{pmatrix}' = \begin{pmatrix} -ik & \mathcal{H} \\ \mathcal{H} & ik \end{pmatrix}\begin{pmatrix} a^\lambda_{\mathbf{k}} \\ a^{\dagger\lambda}_{-\mathbf{k}} \end{pmatrix} \qquad \Rightarrow \qquad \begin{cases} a^\lambda_{\mathbf{k}}(\tau) = p_{k,\lambda}(\tau)\, a^\lambda_{\mathbf{k}}(\tau_0) + q_{k,\lambda}(\tau)\, a^{\dagger\lambda}_{-\mathbf{k}}(\tau_0) \\ a^{\dagger\lambda}_{-\mathbf{k}}(\tau) = p^*_{k,\lambda}(\tau)\, a^{\dagger\lambda}_{-\mathbf{k}}(\tau_0) + q^*_{k,\lambda}(\tau)\, a^\lambda_{\mathbf{k}}(\tau_0) \end{cases} \tag{5.24}$$

where the functions $p_k(\tau)$, $q_k(\tau)$ are called 'Bogoliubov coefficients'.
If we plug eq. (5.22) in the canonical quantization rule eq. (3.8), we obtain the wronskian condition for the Bogoliubov coefficients

$$|p_{k,\lambda}(\tau)|^2 - |q_{k,\lambda}(\tau)|^2 = 1 \tag{5.25}$$

[4] e.g. the reader interested to a concrete example of quantum corrections to GR can look [27, §22.4.1]

[5] we use eq. (5.17) expressed in terms of $v^\lambda_{\mathbf{k}}$. The eq. (5.18) is classically equivalent, since these two differs by a total-derivative

The solution is given in terms of the real functions r_k, φ_k, called respectively the squeezing parameter and the squeezing angle, and defined as

$$\begin{aligned} p_{k,\lambda}(\tau) &= e^{-i\theta_k(\tau)} \cosh\left[r_k(\tau)\right] \\ q_{k,\lambda}(\tau) &= e^{i\theta_k(\tau)+2\varphi_k(\tau)} \sinh\left[r_k(\tau)\right] \end{aligned} \tag{5.26}$$

The squeezing variables (θ_k is a global phase) are known from quantum optics and one can understand their meaning in connection with the uncertainty principle. In fact, they are related to the field/conjugate-momentum quadrature operator $(\Delta v)^2$ (where $\Delta v \equiv \hat{v} - \langle \hat{v} \rangle$), and to the Heisenberg uncertainty relation by eq.(34) of [19]

$$\left\langle (\Delta v)^2 \right\rangle \left\langle (\Delta \pi)^2 \right\rangle = \frac{\hbar^2}{4} \Big(1 + \sin^2(2\varphi_k) \sinh^2(2r_k) \Big) \geqslant \frac{\hbar^2}{4} \tag{5.27}$$

The presence of the interaction term $\sim \mathcal{H}$ implies that a state with definite occupation number is no more an eigenstate of hamiltonian and in fact the annihilation/creation operators mix each others during the time evolution. In light of eq. (5.23), one can revisit the prescription for the initial state $|in\rangle$ of sec. 4.1: in the asymptotic-past - when the interaction (line 2) is negligible - one picks as initial state the ground state of the free hamiltonian (line 1), which we called the Bunch-Davies vacuum. More importantly, due to the presence of the last term of eq. (5.23) the time-evolution leads the wave-function of the initial ground state to smear over time - the squeezing parameter r_k increases over time from the initial value $r_k(\tau_0) = 0$ to large value, i.e. $|r_k(\tau_\star)| \gg 1$ at horizon-crossing. The resulting $|r_k| \gg 1$ state is called *squeezed state* because, in fact, in a v–π diagram it is squeezed along the axis with slope $\tan(\varphi_k)$, see similarly fig. 5.1. Also, the fluctuations normal to this axis are exponentially small.

Classicality for cosmologists The feature of squeezed states is that they exhibit dramatically the uncertainty relation, by allowing one variable to have arbitrary small uncertainty while the conjugate variable has a compensating large uncertainty - so that the Heisenberg relation is satisfied. In this sense, a squeezed state is a special quantum mechanical state in that it may be viewed as a coherent superposition of many localized wave packets. However, it is very different from the "traditional" classical-like coherent state, where the uncertainty is the minimal allowed by QM.
The classicality of cosmological perturbations is in a very different sense: the squeezed states are classical in the sense of the WKB approximation. This means that a squeezed state can be approximated in its evolution as a classical phase-space distribution, as long as one measures classical quantities. The crucial point here is that the ordinary interactions of the cosmological perturbations with the universe are of this type - they 'measure' field amplitudes - and these processes destroy the quantum coherence, which anyway decays with time as consequence of the monotonic squeezing increase.
The analogy of the part II in [29] is illuminating. It is between the classical dynamics of the inverted harmonic oscillator[6] and that of the squeezed quantum state eq. (5.26). In the phase-space Fig.(5.1), the classical dynamics evolves any state (i.e. the circle) into a squeezed trajectory (i.e. the squeezed shape above the circle). The quantum-mechanical system eq. (5.23) mirrors this behaviour, in fact the fluctuations normal to the axis defined by φ_k in the v–π plane are exponentially small.

[6] it is an harmonic oscillator with complex-frequency $\tilde{\omega} = i\omega$. It is inverted in that the potential is upside-down

Squeezing parameter for Bunch-Davies Let us take advantage of the previous sections to evaluate the squeezing parameter. We recall the v_λ decomposition in terms of ladder operators[7]

$$\hat{v}^\lambda(\tau,\vec{k}) \equiv v_{k,\lambda}(\tau)\, a^\lambda_{\mathbf{k}}(\tau_0) \;+\; v^*_{k,\lambda}(\tau)\, a^{\dagger\lambda}_{-\mathbf{k}}(\tau_0) \tag{5.28}$$

$$\hat{\pi}^\lambda(\tau,\vec{k}) \equiv -i\Big[w_{k,\lambda}(\tau)\, a^\lambda_{\mathbf{k}}(\tau_0) \;-\; w^*_{k,\lambda}(\tau)\, a^{\dagger\lambda}_{-\mathbf{k}}(\tau_0)\Big] \tag{5.29}$$

The mode functions $v_{k,\lambda}$, $w_{k,\lambda}$ are related to the Bogoliubov coefficients $p_{k,\lambda}$, $q_{k,\lambda}$ by the relation

$$v_{k,\lambda} = \frac{p_{k,\lambda} + q^*_{k,\lambda}}{\sqrt{2k}} \qquad\qquad w_{k,\lambda} = \sqrt{\frac{k}{2}}\big(p_{k,\lambda} - q^*_{k,\lambda}\big) = i\big(v'_{p,\lambda} - \mathcal{H}\, v_{p,\lambda}\big) \tag{5.30}$$

It is straightforward to invert the above equations and to calculate the Bogoliubov coefficients, using the Bunch-Davies mode function eq. (5.20)

$$p_{k,\lambda} = e^{-i(k\tau+\delta_k)}\,\cosh(r_k) \tag{5.31}$$

$$q_{k,\lambda} = e^{i(k\tau+\frac{\pi}{2})}\,\sinh(r_k) \tag{5.32}$$

We can read off the squeezing parameter/angle by comparison with eq. (5.26) obtaining

$$\sinh(r_k) = \frac{1}{2k\tau} \tag{5.33}$$

$$\varphi_k = \frac{\pi}{4} - \frac{1}{2}\text{arctanh}\Big(\frac{1}{2k\tau}\Big) \tag{5.34}$$

From eq. (5.33), at the beginning - when $|k\tau_0| \gg 1 \;\; \forall\, k$ - the squeezing parameter vanishes and the state is of minimum uncertainty. As time elapses, because of the interaction $\sim \frac{\mathcal{H}}{k}$ in eq. (5.23) the state evolves and eventually it becomes very squeezed along the coordinate-direction.

[7] this is equivalent to eq. (5.19), but here we have explicitly written the time dependence of the operators in light of the Bogoliubov transformation. For notational convenience, we defined $v_{k,\lambda}(\tau_0) \equiv \frac{w_{k,\lambda}(\tau_0)}{k} = \frac{1}{\sqrt{2k}}$

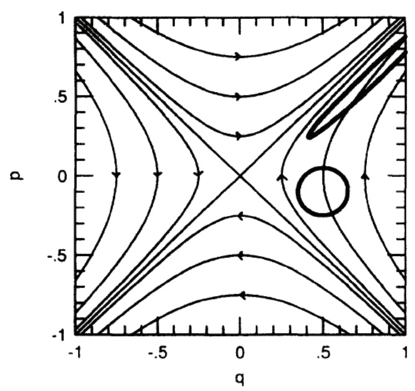


Figure 5.1: The phase space trajectories for a classical inverted harmonic oscillator. In the horizontal/vertical axis is reported the coordinate/conjugate momentum in cartesian coordinates. The field lines represent the normal coordinates $b_\pm \propto (p \pm q)$. Time-evolution maps any initial trajectory (e.g. the circle) into the squeezed shape trajectory (above it). (from [29])

In conclusion, in the previous example we saw that the squeezing parameter increases monotonically with time. At the horizon crossing, it has a large value and continues to increase considerably as long as a perturbation stays in the super-Hubble regime. Aside of the example, this reflects the fact that the growing mode of eq. (3.5) becomes more and more dominant with respect to the decaying mode. After the second horizon crossing, during either RD or MD era, the squeezed state is practically indistinguishable from a classical stochastic field with mode function eq. (5.20).

In-in for the sourced SGWB In the RD era, there is no net squeezing and the dominant features of the solution are the oscillations of the squeezed state. As pointed out by Polarski & Starobinski [19], new quantum fluctuations correlating both the two modes (growing and decaying) can be generated by local processes during RD/MD. Their amplitude is much bigger than those of the residual decaying mode from the inflationary stage but much smaller than the growing mode. In light of this, there is not any conceptual problem in the use of the CTP formalism to the study of cosmological perturbations in the RD/MD eras - one may be tempted to say that after inflation $[h, \pi] = 0$ and it would be pointless to use non-commuting quantum fields for the perturbations - since the resulting observable is always 'automatically' dominated by the classical dynamics. Indeed, later when we calculate quantum expectation values we recover the very same results of classical ensemble averages, which we have discussed so far.

Let us remark the key points of this section

(a) Semi-classicality of the cosmological perturbations is the sense of the WKB approximation. This means that the state can be approximated in its evolution by a classical phase-space distribution. The necessary condition for this to hold is $|r_k(\tau)| \gg 1$ [19].

(b) The aftermath of accelerated expansion is a wave-function very smeared in the coordinate space. It is this fact, that one has in mind when saying "the inflation stretches quantum fluctuations to macroscopic scales". The expansion produces a squeezed state.

(c) The time-evolution as well as the interactions of the perturbations with the universe destroys quantum coherence, effectively increasing the squeezing parameter. The squeezed state satisfies even more the WKB criterion for semi-classicality as it evolves. This classical behaviour is very different from that of coherent state.

(d) The physical origin of the fluctuations is quantum mechanical but their known physical effects are indistinguishable from the fluctuations of a classical stochastic field. QM is reflected only in the initial conditions. What is left for observation is the prediction about the statistics of the cosmological perturbations.

Part II
Sourced gravitational waves from enhanced scalar perturbations

So far, we have introduced the concepts we needed to concretely address our case study.
We are interested in the gravitational waves produced in the radiation dominated era, more specifically in those GWs sourced by the primordial scalar perturbations.

It is known that the equation of motion for GWs at second-order in perturbation theory is sourced by terms quadratic in the first-order primordial scalar perturbation ζ, see [31] [32] [23] [33]. However, this contribution to the cosmological SGWB is typically weaker than the primordial (vacuum) one from inflation. In this thesis we study an exception of this, that arises when the spectrum of primordial scalar perturbation is enhanced with respect to the standard scale-invariant case, see sec. 5.8. In the present chapter, we discuss the second-order evolution equation for the GWs during the RD era.
The conventional approach is to solve the Einstein equation by means of the Green function [23] [33]. It is instructive to review this method as a benchmark. Then we will use the in-in formalism to calculate the second-order sourced SGWB from first-order enhanced scalar perturbations in the RD era. As far as we know, this application of the operatorial CTP formalism - that is to GW production in the RD era - is unprecedented in literature. The outline of the calculation is presented in page 47.

5.8 Enhanced scalar perturbations

The simplest models of inflation produce a nearly scale invariant spectrum of primordial perturbations and of gravitational waves. Under this assumption, the amount of measured scalar perturbations at the large CMB scales, and the upper limits on the tensor perturbations at those scales, imply that the inflationary SGWB cannot be observed in the current and near future GW interferometers. This conclusion applies to the tensor modes directly produced during inflation, and to the GW sourced non-linearly (at horizon re-entry, during the RD era) by the primordial perturbations produced during inflation. This leads us to consider specific inflationary mechanisms that produce blue signals [34], [35] (so that, at small scales / large frequencies the signal can be much greater than at the CMB scales) or a bump at some specific scales (for instance, if a feature is present in the inflaton potential, therefore the spectrum of primordial perturbations is modified only at the scales that left the horizon when the inflaton moved past this feature). In this work we consider this second possibility. For computational simplicity we assume that the feature is extremely sharp, and it can be well described by a Dirac δ-function, that we add to the (nearly) scale invariant signal present at the other scales

$$\mathcal{P}_\zeta(p) = \mathcal{P}_\zeta^{(0)}(p) + A_s\, p_*\, \delta(p - p_*) \tag{5.35}$$

where $\mathcal{P}_\zeta^{(0)}(p)$ is the standard nearly invariant contribution, while the second term accounts for the feature. The adimensional parameter A_s accounts for the strength of the enhancement and p_* is the norm of the momentum in which we assume the power spectrum is peaked.

Summary of the calculation strategy

As endpoint, one is interested in the power spectrum & energy density of the sourced GWs from the enhanced primordial scalar perturbations. The discussion is divided in three sections. Their common ground is clearly the protocol to find the GW source, which is presented first. Secondly, we briefly review the traditional Green function method, and finally in sec. 5.11 we use the in-in formalism.

In what follows, only the key points and main results are presented. The detailed calculations can be found in Appendix B. The procedure is outlined below

- sec. 5.9 - The source. Expanding the action

(a) We calculate the source of the second-order gravitational waves - e.g. the r.h.s. of the Einstein equation. The approach to find the evolution equation for the GW is the one from the action viewpoint. One finds the interaction lagrangian by expanding eq. (2.10) in the perturbations. The Einstein equation for the GW is obtained by extremizing the action.

(b) For convenience, we trade the tensor perturbation for the field $v^\lambda_{\mathbf{k}} \equiv a\,\frac{M_p}{2}\,h^\lambda_{\mathbf{k}}$. The action in terms of v is canonically normalized. Also, we use the transfer function (see eq. (5.41)) to express the interaction lagrangian as a functional of the fields v, ζ in radiation domination.

- sec. 5.10 - Reference solution. Review of the Green method

(c) The conventional approach to this problem is to solve the sourced Einstein equation for the second-order GWs by means of the Green function method, e.g. see [23], [33]. This standard approach is reviewed.

- sec. 5.11 - CTP solution. The in-in for the sourced GWs

(d) The classical fields are promoted to canonically quantized operators. As preliminary for the perturbative solution, one studies the unsourced problem (free GWs) to find the ground state. In practise, it means we find the mode functions for the unsourced second-order GWs during the RD era.

(e) The sourced GW power spectrum is calculated in perturbation theory. It is found from the v.e.v. of the 2-point GW function using eq. (4.26). In practise, the perturbative solution amounts to calculating the $N = 2$ term of the r.h.s. (4.26) w.r.t. the free field solutions of the point (d). The Green method solution is recovered and we discuss the gravitational wave power spectrum under the case of enhanced primordial scalar perturbations.

5.9 The source. Expanding the action

The gravitational waves are sourced at second and higher order in the cosmological perturbations. The precise form of the source can be extracted from the lagrangian eq. (2.10) (also below)

$$S \equiv \int d\tau\; L \equiv \int d^4x\; \mathcal{L} = \int d^4x\; \sqrt{-g}\left[\frac{M_p^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V\right] \tag{5.36}$$

from the terms, proportional to the h_{ij} field, that are n-linear with $n \geqslant 3$. For instance, the cubic terms start at third-order in the cosmological perturbations. Among those, one is interested in the ones containing the h_{ij} field because they contribute to the second-order Euler-Lagrange equation for the gravitational waves (i.e. think of performing the variations).

Let us find the interaction lagrangian by expanding the action in the perturbations. The first terms of the expansion, beyond the quadratic lagrangian for the free GWs eq. (5.16), are cubic in the perturbations. They are schematically grouped in the two categories

$$h\,\Phi\,\Phi\,, \qquad h\,\Phi\,\delta\varphi\,, \qquad h\,\delta\varphi\,\delta\varphi \qquad\qquad \text{(dominant)} \tag{5.37}$$
$$h\,h\,\Phi\,, \qquad h\,h\,\delta\varphi \qquad\qquad \text{(sub-dominant)} \tag{5.38}$$

notice that in virtue of the background/linear Einstein equations (i.e. see eq. (B.10)), eventually all the terms of these two categories are either of the form $h\,\Phi\,\Phi$ or $h\,h\,\Phi$. Here we omitted indices and derivatives, which are present in such a way to contract among each others to obtain scalar quantities - for example $h_{ij}\partial_i\Phi\partial_j\Phi$, etc.
We dub (5.37) as 'dominant interactions' because clearly they source gravitational waves whose power spectrum is mostly featured by the enhancement of eq. (5.35) - it can be easily inferred by thinking the scalar perturbation as containing a $\sim\sqrt{A_s}$ enhanced component. The dominant contribution to the sourced GWs power spectrum is analysed in the present chapter; while the subdominant contribution is studied in the next one.

The interaction lagrangian for the dominant source is (for details see Appendix B, sec. B.2)

$$\mathcal{L}^{\text{int}} = -\frac{1}{2}\,M_p^2\,a^2 h_{ij}\left[-2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right] \tag{5.39}$$

For completeness, we report below the action $S^{(\text{GW})}$ for the gravitational waves in presence of the (dominant) source during the radiation dominated era

$$S = \int d^4x\sqrt{-g}\Big[\frac{M_p^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V\Big] \supset S^{(\text{GW})} = S^{(\text{GW})}_{free} + S^{(\text{GW})}_{int}$$

where

$$S^{(\text{GW})} = \frac{M_p^2}{2}\int d^4x\left\{\frac{a^2}{4}\,\left[h'_{ij}h'_{ij} - \partial_k h_{ij}\,\partial_k h_{ij}\right] - a^2 h_{ij}\left[-2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right]\right\} \tag{5.40}$$

5.9.1 The second-order GW evolution equation

We end this section by giving the evolution equation for the second-order gravitational waves. It the Euler-Lagrange equation, obtained by extremizing the action above. As crosscheck, in Appendix B we derive the GW evolution equation in the 'geometrical way' by calculating the Einstein equation. Clearly, the two methods are equivalent. The result is recalled from the Appendix:

Einstein equation

ORDER 2 - The Einstein equation for the induced GWs is (dominant contribution)

$$h''_{ij} + 2\mathcal{H}\,h'_{ij} - \nabla^2 h_{ij} = -2\left[-2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\right]^{TT} \tag{B.23}$$

Clearly, the tensor perturbation appearing in the l.h.s. is second-order in the cosmological perturbation theory, while the scalar field appearing in the r.h.s. is first-order. The 'TT' remembers us that only the transverse-traceless part of the source matters for the evolution of the gravitational waves - recall that the h_{ij}-field is transverse & traceless, see eq. (3.23).

5.9.2 The transfer function. The canonically normalized action

The scalar field Φ (or equivalently Ψ because of eq. (B.11)) is related to the curvature perturbation ζ by eq. (3.31). In particular the mode $\Phi_{\mathbf{k}}$ in the RD era is proportional to the comoving curvature $\zeta_{\mathbf{k}}(\tau_\star)$ at the time $k\tau_\star = 1$ when the mode exited the horizon - see eq. (3.34).

The detailed relation between the late-time potential Φ - e.g. the metric scalar perturbation observed during the RD/MD era - and the primordial value must account for the adequate form of $\delta T^0{}_i$, $(\rho + p)$, H at late-time in the eq. (3.31). The wavelength-dependent evolution of the perturbations though the epoch of horizon re-enter and over is described by the *transfer function* T.
For the modes crossing the horizon during the RD era, the relation between the scalar perturbation Φ and the comoving curvature perturbation ζ is (from [7])

$$\Phi(\tau,\,\mathbf{k}) = \frac{2}{3}\,T\,(\tau,\,k)\;\zeta(\mathbf{k}) \tag{5.41}$$

$$T\,(\tau,\,k) \equiv T(k\tau) \;\equiv\; \frac{9}{(k\tau)^2}\left[\frac{\sin\left(k\tau/\sqrt{3}\right)}{k\tau/\sqrt{3}} - \cos\left(\frac{k\tau}{\sqrt{3}}\right)\right] \tag{5.42}$$

The transfer function depends on the combination $k\tau$, as we emphasized by defining the quantity $T(k\tau)$. The conformal time - that during the inflation is negative - in eq. (5.42) is normalized such that the radiation era begins at $\tau = 0$. We assume a sharp instantaneous transition between inflation and RD era at the time $\tau = \tau_* < 0$. The proper time matching-condition would require using $\tau - \tau_*$ in eq. (5.42) instead of τ, however with the hindsight that the physical results (at time $\ln|\tau/\tau_*| \gg 1$) are independent of τ_* we may discard this difference. In fact, we will see in sec. 5.11 that indeed it is the case.

The canonically normalized action

For later convenience, it useful to canonically normalize the action for the sourced GWs. A theory is 'canonically normalized' when the coefficient of the kinetic term $(\partial h_{ij})^2$ is $1/2$. This is twofold: first, this way it is easier to solve the GW e.o.m. with the traditional Green function method; second, it guarantees the right normalization in the canonical quantization procedure (next section). The action (5.40) expressed in terms of the field $v^\lambda_{\mathbf{k}} \equiv a\,\frac{M_p}{2}\,h^\lambda_{\mathbf{k}}$ is canonically normalized.

It is convenient to express the late-time RD era scalar perturbation Φ in terms of the comoving perturbation ζ, which is conserved. The following expressions are worth to be well imprinted in mind.

The action $S^{(\mathrm{GW})} = S^{(\mathrm{GW})}_{free} + S^{(\mathrm{GW})}_{int}$ for the sourced second-order GWs is

$$S^{(\mathrm{GW})}_{free} = \sum_\lambda \int \frac{d\tau\, d^3\vec{k}}{(2\pi)^3} \left[\frac{1}{2} v'^{\lambda}_{\mathbf{k}}\, v'^{\lambda}_{\textbf{-k}} - \frac{1}{2}\left(k^2 - \mathcal{H}' - \mathcal{H}^2\right) v^{\lambda}_{\mathbf{k}}\, v^{\lambda}_{\textbf{-k}} \right] \tag{5.43}$$

$$S^{(\mathrm{GW})}_{int} = \frac{4}{9} M_p \sum_\lambda \int \frac{d\tau\, d^3\vec{k}}{(2\pi)^3}\, a(\tau) \int \frac{d^3\vec{q}}{(2\pi)^3}\, v^{\lambda *}(\tau,\, \mathbf{k})\, \zeta(\mathbf{q})\, \zeta(\mathbf{k}-\mathbf{q})\, \mathcal{Q}_\lambda(\tau,\, \mathbf{k},\, \mathbf{q}) \tag{5.44}$$

with

$$\begin{aligned}\mathcal{Q}_\lambda(\tau,\, \mathbf{k},\, \mathbf{q}) \equiv \mathrm{e}^*_\lambda(\hat{k},\, \mathbf{q}) \Big\{ & 3\, T(\tau,\, q)\, T(\tau,\, |\mathbf{k}-\mathbf{q}|) + \frac{1}{\mathcal{H}^2} T'(\tau,\, q)\, T'(\tau,\, |\mathbf{k}-\mathbf{q}|) \\ & + \frac{1}{\mathcal{H}} \partial_\tau \left[T(\tau,\, q)\, T(\tau,\, |\mathbf{k}-\mathbf{q}|) \right] \Big\}\end{aligned} \tag{5.45}$$

The quantity $\mathrm{e}^*_\lambda(\hat{k},\, \mathbf{q})$ is introduced for notational convenience, and it is defined below.

$$\mathrm{e}^*_\lambda(\hat{k},\, \mathbf{q}) \equiv \sum_{ij} \mathrm{e}^*_{ij,\lambda}(\hat{k})\, q_i\, q_j \tag{5.46}$$

5.10 Reference solution. Review of the Green method

The conventional approach to this problem is to solve the Einstein equation for the second-order GWs by means of the Green function, i.e. see [23] [33]. The reader can find the details of this section in B.4 of Appendix B. In radiation domination, the evolution equation for the second-order GWs is

$$\left(\partial_\tau^2 + k^2\right) v_\lambda(\tau, \mathbf{k}) \;=\; \mathcal{S}_\lambda(\tau, \mathbf{k}) \tag{5.47}$$

$$\mathcal{S}_\lambda(\tau, \mathbf{k}) = \frac{4}{9} M_p\, a(\tau) \int \frac{d^3\vec{q}}{(2\pi)^3}\, \zeta(\mathbf{q})\, \zeta(\mathbf{k}-\mathbf{q})\, \mathcal{Q}_\lambda(\tau,\, \mathbf{k},\, \mathbf{q}) \tag{5.48}$$

The solution of the equation above is given in terms of the convolution between the Green function and the source. That is, it is of the form

$$v^\lambda(\tau, \mathbf{k}) = \int_0^\infty d\tau'\, \mathcal{G}_k(\tau, \tau')\, \mathcal{S}_\lambda(\tau', \mathbf{p}) \tag{5.49}$$

here $\mathcal{G}$ is the Green function. It is defined as the kernel of the differential operator appearing in the l.h.s. of (5.47), see eq. (B.37). The advantage of using the field v is now self-explanatory: the differential operator of the GW equation is those of the harmonic oscillator. Its Green function is

$$\mathcal{G}_k(\tau, \tau') = \theta(\tau - \tau')\, \frac{\sin\left[k(\tau - \tau')\right]}{k} \tag{5.50}$$

where the θ-function is the Heaviside step-function - it accounts of causality, namely that the source at time τ' can only influence the GW at times $\tau > \tau'$.

The 2-point GW function can be evaluated by means of the formal solution in (5.49). The involved calculation can be found in details in sec. B.4.2 of Appendix B. The result is expressed in the following simple form

$$\Big\langle v_\lambda(\tau,\mathbf{k}_1)\, v_\sigma(\tau,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\left(\frac{M_p}{2}\right)^2\frac{1}{81}$$
$$\iint_{\mathcal{D}} dx\,dy\,\mathcal{P}_\zeta(x\,k_1)\,\mathcal{P}_\zeta(y\,k_1)\left(\frac{x}{y}\right)^2\Big[1-\Big(\frac{1+x^2-y^2}{2x}\Big)^2\Big]^2$$
$$\times\Big[\sin(k_1\tau)\,\mathcal{I}_s\,(x,y)+\cos(k_1\tau)\,\mathcal{I}_c\,(x,y)\Big]^2 \quad (5.51)$$

The integration over the adimensional variables is performed in the domain $\mathcal{D}=\Big\{(x,y)\in\mathbb{R}^+\times\mathbb{R}^+\mid |1-x|<y<1+x\Big\}$ illustrated in Figure B.1. The integrand is symmetric around the axis $x=y$ and depends quadratically from the primordial scalar power spectrum. The latter was the naive expectation we guessed for the dominant interactions (5.37).

Also, in the integrand we introduced the adimensional functions $\mathcal{I}_{c,s}$. The τ-dependence in (5.51) is contained in the trigonometric functions as well as in these functions. However, as we discuss in the Appendix, given the late-time τ when we measure the GW power spectrum there is no practical difference between integrating in $[0,\tau]$ vs. $[0,+\infty]$. In the latter case, the functions $\mathcal{I}_{c,s}$ can be expressed in closed form as (see eq.(D.8) of [36])

$$\mathcal{I}_c(x,y)=\left(\frac{1}{H_{\rm inf}\,k\tau_*^2}\right)\left[-36\pi\frac{\left(d^2+s^2-2\right)^2}{\left(s^2-d^2\right)^3}\theta(s-1)\right]_{s\equiv\frac{x+y}{\sqrt{3}}\;,\;d\equiv\frac{|x-y|}{\sqrt{3}}} \quad (5.52)$$

$$\mathcal{I}_s(x,y)=\left(\frac{1}{H_{\rm inf}\,k\tau_*^2}\right)\left[-36\frac{\left(d^2+s^2-2\right)}{\left(s^2-d^2\right)^2}\left(\frac{\left(d^2+s^2-2\right)}{\left(s^2-d^2\right)}\log\frac{\left(1-d^2\right)}{|s^2-1|}+2\right)\right]_{s\equiv\frac{x+y}{\sqrt{3}}\;,\;d\equiv\frac{|x-y|}{\sqrt{3}}} \quad (5.53)$$

The (5.51) is the power spectrum of second-order gravitational waves sourced by the interaction type (5.37), which is quadratic in the first-order primordial scalar perturbation. This result depends on the primordial scalar power spectrum as initial-condition. So far, we have not specialized to the case eq. (5.35) of enhanced primordial perturbation. It will be our reference standard for the CTP-calculation in the next section. We postpone the detailed discussion about these results to page 66.

5.11 The sourced GWB from primordial scalar perturbations with the operatorial CTP formalism

In chapter 4 we described in detail the operatorial CTP formalism. The common point between the traditional Green function method and the in-in formalism is the perturbative approach to solve the physical problem. In other words, the solution of the theory in presence of the interactions is given as a perturbative expansion in terms of the free theory solution. In the case of the Green function method, the free theory solution was represented by the Green function. In fact, the starting point has been to find the Green function in the RD era. In the case of the in-in formalism, the approach is the one presented in chapter 4 and solving the free theory consists, in practise, in finding the mode functions for the free fields in the RD era.

It is of fundamental importance to remember what we discussed in section 5.7, where we justified the application of the in-in formalism to this case study. The production of gravitational waves in the radiation dominated era is classical process. The classical correlations which are created in the GW-field by the sources (5.37) are encoded in the promoted quantum GW-field. Potentially the in-in formalism carries a bigger amount of informations with respect to the Green function method which treats the fields as classical commuting objects, see [15]. However the quantum dynamics, which ultimately is linked to the non-vanishing of the commutators, is exponentially suppressed with respect to the classical one [29]. Therefore in the application of the in-in formalism, it would be absolutely pointless to carry on inessential terms in the calculation which eventually we know are not of any importance. For example, $[\zeta, \zeta]$ is one of these terms as we will see more deeply later.

The route we follow is: first, the classical theory is promoted to a canonically-quantized quantum field theory. The solution for the ground state of the free theory is the basis for the solution in presence of interactions. In practise, it means to find the GW mode functions in the RD era. As a preliminary, we evaluate some tools (the GW commutator & the 2-point function) on the free-theory. Finally, we set up the in-in calculation for the 2-point gravitational wave function.

5.11.1 The quantum theory. The mode functions in RD era

It is straightforward to promote the classical action to the quantum theory: one promotes the classical fields v, ζ to quantum operators. Two comments are mandatory

- the complex conjugate field has to be considered as the hermitian conjugate. The implications of self-adjointness for the tensor perturbation (and analogously for ζ) are discussed in Appendix A.1.1;
- due to the non-commutative character of the quantum fields, ambiguities may arise when promoting to the quantum theory. In this case, one has to account for the proper generalization of the classical term to eliminate any ambiguity - e.g. in eq. (5.44), ambiguity occurs for $\zeta(\mathbf{q})\,\zeta(\mathbf{k}-\mathbf{q})$, so one first symmetrizes and then promotes the classical field to a quantum operator.

The quantum theory is obtained by the classical one with the prescriptions above accounted for

$$L^{(\mathrm{GW})}_{free} = \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3} \left[\frac{1}{2} v^{\lambda\dagger\prime}_{\mathbf{k}} v^{\lambda\prime}_{\mathbf{k}} - \frac{1}{2}\left(k^2 - \mathcal{H}' - \mathcal{H}^2\right) v^{\lambda\dagger}_{\mathbf{k}} v^{\lambda}_{\mathbf{k}} \right] \tag{5.54}$$

$$L^{(\mathrm{GW})}_{int} = \frac{4}{9} M_p\, a(\tau) \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{d^3\vec{q}}{(2\pi)^3} \left(\hat{v}_\lambda(\tau,\mathbf{k})\right)^\dagger \hat{\zeta}(\mathbf{q})\, \hat{\zeta}(\mathbf{k}-\mathbf{q})\, \mathcal{Q}_\lambda(\tau,\mathbf{k},\mathbf{q}) \tag{5.55}$$

here we used the identities (A.11) and (B.91). Also, we have indicated with hats the operators in the interaction lagrangian.

The $\hat{\zeta}(\mathbf{q})$ field The dynamics of the first-order scalar perturbation Φ is already accounted for, as it is evident by the (5.55) expressed in terms of the time-independent curvature perturbation $\zeta(\mathbf{q})$. As well, in the quantum theory, the operator $\hat{\zeta}$ is conserved: it commutes with the hamiltonian (Schrödinger equation). Concerning only the first-order perturbation then, strictly speaking, $\hat{\zeta}$ is not a dynamical field at all and we should write $\hat{\zeta}(\mathbf{q}) = \zeta(\mathbf{q})\,\hat{I}$, where $\zeta(\mathbf{q})$ is the stochastic classical variable and $\hat{I}$ is the identity operator. However, we will consider $\hat{\zeta}(\mathbf{q})$ as a quantized field. This is not wrong, since treating an external current injection as an heavy quantum field is allowed by the quantum field theory (e.g. see Mott scattering, etc.), however the reader should forgive the abuse of notation in writing things like $[\zeta,\, \zeta]$: this usage shall be seen in the perspective that this thesis serves as the ground basis for future investigations, see chapter 7.

The canonical quantization procedure is similar to those presented previously, see page 41. It is only necessary to generalize including the presence of the polarisation index. For example, the commutation relation (3.8) and the wronskian condition (3.11) read

$$\Big[v_\lambda(\tau,\vec{p}_1),\, \pi_\sigma(\tau,\vec{p}_2)\Big] = i\,(2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}\,(p_1 - p_2) \tag{5.56}$$

$$v_{p,\lambda}\, v'^{*}_{p,\lambda} - v'_{p,\lambda}\, v^{*}_{p,\lambda} = i \tag{5.57}$$

The conjugate-momentum is defined as $\pi^\lambda_{\mathbf{k}}(\tau) = {v^\lambda_{\textbf{-k}}}'(\tau)$. In eq. (5.57) there is no sum over the repeated index: it holds for each λ. Remember that the function $v_{p,\lambda}$ depends from the magnitude of the momentum but not from its direction.

The ground state of the free theory. Mode functions

As we discussed earlier, the perturbative approach consists in writing the solution of the full interacting theory in terms of the known free-theory solution. In the in-in formalism, solving the free quantum theory amounts to finding the ground state. In practise, this amounts to finding the field mode function.

The solution of the free theory in our case is obtained by solving the unsorced evolution equation (B.34) for the field mode function

$$\left(\partial_\tau^2 + k^2\right) v_{k,\lambda} \;=\; 0 \tag{5.58}$$

The initial-conditions for this equation are fixed by matching the inflationary solution - the Bunch-Davies mode function, see eq. (5.20) - at the transition between the inflationary stage and the RD era. Let us assume that at a certain time $\tau = \tau_* < 0$, the accelerated expansion ends and the universe instantaneously enters into the RD era. We fix the initial-conditions for eq. (5.58) by matching the inflationary Bunch-Davies solution eq. (5.20) with the most general one of eq. (5.58) at the interface $\tau = \tau_*$. This is identical to a tunneling problem of elementary quantum mechanics, when one calculates transmission/reflection coefficients. Indeed, one is searching for a smooth continuation of the Bunch-Davies mode function during the radiation dominated era.
Welding the field/conjugate-momentum amplitudes at the time $\tau = \tau_*$ - we note that this is equivalent to the matching of the field amplitude and with its first time derivative - we get

$$v_{k,\lambda}(\tau) = \begin{cases} \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right) e^{-ik\tau} & \tau \leqslant \tau_* \\[2ex] \frac{e^{-ik\tau_*}}{\sqrt{2k}}\left[-\frac{1+2ik\tau_* - 2(k\tau_*)^2}{2(k\tau_*)^2}\, e^{-ik(\tau-\tau_*)} + \frac{1}{2(k\tau_*)^2}\, e^{ik(\tau-\tau_*)}\right] & \tau > \tau_* \end{cases} \tag{5.59}$$

Free-theory. The GW commutator and the 2-point function

Armed with the free theory solution eq. (5.59), we are almost ready to set up the calculation with the in-in formalism. As last preliminary, let us calculate some tools which are useful in the following: the commutator and the 2-point function of the free theory

It is straightforward to express the commutator of the GW free field in terms of the mode function, for example using the field expansion in terms of ladder operators. During the radiation dominated era, we have

$$\begin{aligned}\Big[v_\lambda(\tau,\,\mathbf{k})\,,\;v_\sigma(\tau',\,\mathbf{k}')\Big] &= (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k+k')\;\left[v_{k,\lambda}(\tau)\,v^*_{k,\lambda}(\tau') - v^*_{k,\lambda}(\tau)\,v_{k,\lambda}(\tau')\right]\\ &= (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k+k')\;\frac{\sin\left[k(\tau-\tau')\right]}{ik}\end{aligned} \tag{5.60}$$

We note the strong similarity with the propagator eq. (5.50). Also, we note that this quantity depends on the difference $k(\tau'-\tau)$ but it does not depend the initial-conditions assigned at τ_*.

Analogously, we calculate the free-theory 2-point GW correlator. It is the vacuum expectation value

$$\begin{aligned}\Big\langle v_\lambda(\tau,\,\mathbf{k})\,v_\sigma(\tau',\,\mathbf{k}')\Big\rangle &= (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k+k')\;v_{k,\lambda}(\tau)\,v^*_{k,\lambda}(\tau')\\ &= (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k+k')\;\left(\frac{H_{\rm inf}}{\sqrt{2k}}\right)^2\frac{a(\tau)}{k\tau}\frac{a(\tau')}{k\tau'}\;\frac{\sin(k\tau)\,\sin(k\tau')}{k^2}\Big(1+\mathrm{O}(k\tau_*)\Big)\end{aligned} \tag{5.61}$$

The unwritten contributions are sub-dominant for the modes that are outside the horizon during the transition from the inflationary era to the radiation dominated era, e.g. $k\tau_*\ll 1$. Also, this expression holds for the times of interest $\tau\sim\tau'\gg\tau_*$.

5.11.2 The sourced GW power spectrum with the in-in formalism

We are ready to set up the calculation for the 2-point GW correlator using the in-in formalism. Let us recall the master equation eq. (4.26) from chapter 4

$$\langle Q(\tau)\rangle = \sum_{N=0}^{\infty} i^N \int_{\tau_0}^{\tau} d\tau_N \int_{\tau_0}^{\tau_N} d\tau_{N-1} ... \int_{\tau_0}^{\tau_2} d\tau_1 \,\Big\langle \Big[H_I(\tau_1),\,\Big[H_I(\tau_2),...\Big[H_I(\tau_N),\,Q(\tau)\Big]...\Big]\Big]\Big\rangle \tag{5.62}$$

In our case, we are interested in the equal-time 2-point function $Q(\tau)\equiv v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)$. The interaction hamiltonian has to be calculated from the lagrangian (5.55). Since the interaction lagrangian does not involve derivative couplings, the interaction hamiltonian is simply

$$\begin{aligned}\hat{H}_I(\tau) &= -\hat{L}^{\rm int}(\tau)\\ &= -\frac{4}{9}M_p\,a(\tau)\sum_\lambda\int\frac{d^3\vec{k}}{(2\pi)^3}\int\frac{d^3\vec{q}}{(2\pi)^3}\;(\hat{v}_\lambda(\tau,\,\mathbf{k}))^\dagger\;\hat{\zeta}(\mathbf{q})\;\hat{\zeta}(\mathbf{k}-\mathbf{q})\;\mathcal{Q}_\lambda(\tau,\,\mathbf{k},\,\mathbf{q})\end{aligned} \tag{5.63}$$

It is mandatory to comment on the role of τ_0 in eq. (5.62). We first discussed the role of the initial time τ_0 in chapter 4, when we talked about the in-in formalism in full generality. Recall that the time

τ_0 was the initial time when one sets the state $|in\rangle$. In particular, we defined τ_0 as the time when all the interactions were assumed to be turned off and the theory was regarded as free in Minkowski (e.g. we can neglect the line-2 of eq. (5.23)). In chapter 4 we argued that the proper choice of τ_0 required to take it in the asymptotic-past. In the present application of the in-in formalism, one should not be tempted to consider $\tau_0 = -\infty$ since it would be wrong. We are considering the dominant contribution to the sourced SGWB during the RD era, and the initial conditions on the free field are assigned at the time $\tau_0 = \tau_*$ when all the second-order GW modes of interest are sub-Hubble (yet unsourced) - it is the *source*, namely the first-order scalar/tensor perturbation, that is super-Hubble at the time τ_* when we quantize the GWs and we find the ground state of the free theory.

Notation All the time integrations start from $\tau_0 = \tau_*$. This is understood and henceforth we will omit it. Also, it is understood that all the operators in the r.h.s. of eq. (5.62) are in interaction picture, therefore we will omit the relative superscript.

The $N = 0$ contribution is the free theory. The $N = 1$ contains an odd number of v fields in the r.h.s of eq. (5.62), it gives zero after we take the vacuum expectation value. The dominant contribution is encoded in the $N = 2$ term. For the sake of brevity, the main results are reported. The interested reader is again referred to the Appendix B. Focussing on the $N = 2$ term, we have

$$\begin{aligned}\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle \equiv -\int^{\tau} d\tau_2 \int^{\tau_2} d\tau_1\,\left(\frac{4}{9}M_p\right)^2 a(\tau_1)\,a(\tau_2)\sum_{\alpha\beta}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\,\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{d^3\vec{p}_2}{(2\pi)^3}\,\frac{d^3\vec{q}_2}{(2\pi)^3}\\ \times\,\mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\beta(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\\ \times\Big\langle\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\,\Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,,\,v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big]\Big]\Big\rangle \quad (5.64)\end{aligned}$$

Let us briefly outline how to handle this expression. One starts from the last line, the v.e.v. containing the commutators. The ζ and v_λ fields mutually commute. The ζs in the inner commutator simply exit from it. Also, an expression like $[A, BC]$ can be reduced further, see app.B.6, and expressed in terms of eq. (5.60). What is left is a sum of terms involving the GW field and the scalar perturbation ζ, the latter with the two possible structures:

$$\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle \quad (5.64a)$$

$$\begin{aligned}&\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle\Big[\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\,\zeta(\mathbf{q}_2)\Big]\;,\;\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\Big\rangle\Big[\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big]\;,\\ &\Big\langle\zeta(\mathbf{p}_2-\mathbf{q}_2)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\Big\rangle\Big[\zeta(\mathbf{q}_1)\,,\,\zeta(\mathbf{q}_2)\Big]\;,\;\Big\langle\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\Big\rangle\Big[\zeta(\mathbf{q}_1)\,,\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big]\end{aligned} \quad (5.64b)$$

The contributions proportional to (5.64a) depend quadratically of the enhanced primordial spectrum, while those proportional to (5.64b) linearly. Moreover, the terms in (5.64b) contain the scalar field commutator.

In sec. 5.7, we argued that the quantum state of a primordial perturbation is squeezed in coordinate space. The amount of squeezing increases over time and after the first horizon-crossing, during the inflation, is very large. This in turns implies the semi-classical behaviour. More importantly for the present purpose, in practise it means that the commutator $[\zeta, \zeta]$ is much smaller than the $\langle\zeta\,\zeta\rangle$: first, as it is written (the ζ time-independent) it holds $[\zeta, \zeta] = 0$ exactly. As discussed in the disclaimer of page 52, we include terms like this in the perspective of a future study. Second, even if we allow for a time-dependence of the primordial perturbation, the field commutator is indeed proportional to the squeezing parameter $e^{-2\,r_k(\tau)} \ll 1$ [28]. Last, as sufficient but unnecessary condition, the presence of the enhancement (5.35) in the 2-point scalar function renders parametrically bigger the terms (5.64a) in comparison with (5.64b).

In summary, if we take into account the detailed structure of the contributions above we understand that the (5.64b) terms are subdominant with respect to those proportional to (5.64a). In light of the initial discussion, by now we neglect any term proportional to (5.64b).
The relevant contribution in eq. (5.64) is (see Appendix B.6)

$$
\begin{aligned}
\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = &-\int^{\tau} d\tau_2 \int^{\tau_2} d\tau_1\,\left(\frac{4}{9}M_p\right)^2 a(\tau_1)\,a(\tau_2)\sum_{\alpha\beta}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3}\\
&\times\,\mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\beta(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\,\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle\\
\times\Bigg\{&\Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\;,\,v_\lambda(\tau,\,\mathbf{k}_1)\Big]\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,,\,v_\sigma(\tau,\,\mathbf{k}_2)\Big]+\Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\;,\,v_\sigma(\tau,\,\mathbf{k}_2)\Big]\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,,\,v_\lambda(\tau,\,\mathbf{k}_1)\Big]\Bigg\}
\end{aligned}
\tag{5.65}
$$

The terms in curly brackets are related one to each other by $(\lambda, \mathbf{k}_1) \leftrightarrow (\sigma, \mathbf{k}_2)$. The following simplifications consist in replacing the explicit expression for the GW commutators eq. (5.60), Wick-expand the scalar 4-point function and simplify the rest. The reader can find the details in the Appendix B, page 96. A compact, simplified version of this expression is

$$
\begin{aligned}
\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= (2\pi)^3\delta^{(3)}(k_1+k_2)\,\frac{2\pi^2}{k_1^3}\,\left(\frac{4}{9}M_p\right)^2 k_1\int_0^{\tau} d\tau_2\int_0^{\tau} d\tau_1\;a(\tau_1)\,a(\tau_2)\\
&\times\sum_{\alpha\beta}\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\frac{2\pi^2}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\,\sin\left[k_1(\tau_2-\tau)\right]\,\sin\left[k_1(\tau_1-\tau)\right]\\
&\times\Big\{\mathcal{Q}_\sigma(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\lambda(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1)+\mathcal{Q}_\lambda(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\sigma(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big\}
\end{aligned}
\tag{5.66}
$$

It is worth to note few things. The presence of the overall momentum conserving δ-function, let us read off easily the GW power spectrum. The GW-scalar interaction (5.37) corrects the free GW power spectrum in eq. (5.61). Further simplifications let us rewrite this in the form

$$
\begin{aligned}
\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\,\frac{2\pi^2}{k_1^3}\,\left(\frac{4}{9}M_p\right)^2 k_1\\
&\times\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\frac{2\pi^2}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\,\frac{1}{4}\,q_1^4\,\sin^4\theta\\
&\times\left\{\frac{1}{(2k_1)^2}\,\Big[\cos(k_1\tau)\,\mathcal{I}_c\,(x,y)+\sin(k_1\tau)\,\mathcal{I}_s\,(x,y)\Big]^2\right\}
\end{aligned}
\tag{5.67}
$$

which agrees with the classical result obtained with the Green function method in eq. (B.59).

The final result for the dominant contribution to the 2-point GW function from enhanced scalar perturbations is

$$\Big\langle v_\lambda(\tau,\mathbf{k}_1)\,v_\sigma(\tau,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}$$
$$\times\left\{\theta\left(x_\star-\frac{1}{2}\right)\left(\frac{M_p}{2}\right)^2\left(\frac{A_s\,x_\star}{9}\right)^2\left[1-\frac{1}{(2x_\star)^2}\right]^2\Big[\sin(k_1\tau)\,\mathcal{I}_s\left(x_\star,x_\star\right)+\cos(k_1\tau)\,\mathcal{I}_c\left(x_\star,x_\star\right)\Big]^2\right\} \tag{5.68}$$

5.11.3 Comments and remarks

The result eq. (5.68) will be discussed in sec. 6.3. We conclude this chapter by retracing the steps of the calculation, in order to elucidate some key point more in details. The in-in formalism was applied to study the second-order sourced GWs from primordial first-order enhanced scalar perturbations. We are not aware of the precise physical mechanism which has been responsible, during inflation, of producing the enhancement in the scalar spectrum. The sourced GWs depend on $\mathcal{P}_\zeta$, whatever accurate form it has.

The GWs production happens in the RD era, where the cosmological perturbation dynamics is classical. The second-quantization 'imprints' (i.e. loops, etc.) are present but are way far from driving the dynamics of the fields, and quantum coherence is rapidly destroyed by the time-evolution, see sec. 5.7. The in-in formalism treats the second-order GW field as a non-commuting quantum field, and the first-order primordial scalar & tensor perturbations as classical current insertions. In particular it is the case of the scalar commutator, which we argued - mostly guided by the discussion about quantum coherence of the cosmological perturbations - to be subdominant with respect to the 2-point scalar correlator. This consideration holds having in mind to treat the field $\hat{\zeta}$ in light of the discussion in page 52 - that is, allowing for a time-dependence for the comoving curvature perturbation. Otherwise, a boldly observation that the quantum field $\hat{\zeta}(\mathbf{q})$ is time-independent would immediately imply that its commutator vanishes identically. This is nothing but the trivial awareness that the production of second-order GWs is made from a source in terms of first-order fields, whose dynamics is completely solved by using the first-order Einstein/Klein-Gordon equations of motion.

Recap and remarks

Finally, to remove any further doubts we remark the most important points so far

- The tensor perturbation contains the GWs. If one expands the tensor perturbation h_{ij} in the cosmological perturbative series - i.e. see eq. (C.18) - the Einstein equation for the GWs can be studied order-by-order and, in the RD era, one would discover that second-order GWs are sourced by first-order perturbations.

- The first-order primordial perturbations are the initial conditions for the universe, their dynamics freeze after they leave the horizon during the inflation. The transfer function let one relates the first-order perturbation in the RD era with the primordial value, at horizon crossing. It accounts for the evolution of the first-order perturbations with time, due to their mass-dimension and the expansion of the universe.

  - The dynamics of the first-order fields is assumed to be given. For instance it is evident by expressing the perturbation $\Phi(\tau,\mathbf{q})$ in terms of the time-independent comoving perturbation $\zeta(\mathbf{q})$. Eventual refinements of the primordial first-order fields dynamics can be accounted by modifying the transfer function, for example see [37].

- In the present chapter, we neglected a subdominant source term in the cubic interaction lagrangian. It is the subject of the next chapter. This last approximation was justified in a fundamental way by the presence of the enhancement in the primordial scalar spectrum eq. (5.35).

CTP SOLUTION. THE IN-IN FOR THE SOURCED GWS

- We are concerned with the dynamics of the second-order GWs. It is the second-order tensor perturbation which is treated as a canonically-quantized field. To study the production of these GWs we used the in-in formalism. The perturbative approach is based on the 'free' theory. Here 'free' is in quote marks because one means 'in absence of the first-order sources', meanwhile the universe expansion is always present. So 'free' is in the sense of a quadratic-action QFT in the expanding background spacetime, contrariwise to the Minkowski QFT.

- In this respect, the relevant quantum field in eq. (5.63) is the second-order GW field (in terms of the field v_λ). And for the first-order scalar perturbation it holds $\hat{\zeta}(\mathbf{q}) = \zeta(\mathbf{q})\,\mathbb{I}$, where $\zeta(\mathbf{q})$ is the classical stochastic real-valued primordial comoving curvature perturbation. In other words, strictly speaking the interaction hamiltonian operator is of the form $\sim \hat{v}_\lambda\, J^\lambda$ where J^λ is a classical current insertion.
  However, as discussed in page 52, when we wrote $[\zeta, \zeta]$ we mean that we are considering the ζ-field as if it would be a dynamical field in order to reuse this calculation in future investigations - where its dynamics matter. Some readers may find more satisfying to conceive this treatment of the external classical source as a quantum object in light of the more familiar analogy from quantum electrodynamics (QED). For instance, there is a one-to-one correspondence of this problem with the semi-quantic treatment of the Mott scattering in QED. The scattering of an electron with a nucleus is allowed to be described with one or the other of the vertices in fig. 5.2: (left) as the interaction of a quantized light field (electron) with a much heavier quantized one (nucleus) - which it does not move, and therefore has not dynamics; or (right) as the interaction between the quantized light field with the classical electromagnetic current generated by the charge.

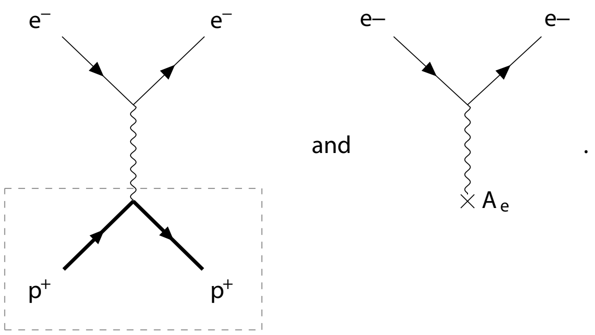
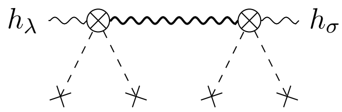


Figure 5.2: Diagrammatic representation of the interactions for the Mott scattering. (from [27])

The result of the in-in calculation obviously match the classical one obtained with the Green function method. This is clear, because the quantum treatment incorporates the classical dynamics. This process was studied in semi-classical approximation and second-quantization effects (like loops, radiative corrections, etc.) are not present: all that it can happens is the proliferation of classical interactions in tree-diagrams, see [27, §7.1.1]. In fact, the Feynman pictorial representation of the sourced contribution to the 2-point GW function is given by the following tree-diagram (see the caption in fig. 6.1)



In the context of GWs production in radiation dominated era, the advantages of the in-in approach, with respect to the traditional Green functional method, are in the more schematic way to efficiently organize the perturbative calculation. Clearly, it is another story if one is interested in the study of perturbations in a regime when quantum corrections matter - e.g. like during inflation.

We postpone the discussion about the result we have obtained (i.e. eq. (5.68)) to the end of chapter 6, where we evaluate the sub-dominant contribution to the 2-point GW function from the interaction (5.38)

Chapter 6

The sourced GWB from primordial scalar perturbations with the operatorial CTP formalism: the subdominant contribution

In the previous chapter, we calculated the power spectrum of those second-order GWs produced from the interaction (5.37). When the first-order primordial scalar perturbation is enhanced, these terms mostly feature the GW power spectrum. In the following, we calculate the contribution from the interaction (5.38), containing the first-order primordial scalar & tensor perturbations, using the in-in formalism. The final omni-comprehensive GW power spectrum is given and the results are discussed.

In page 48, we pointed out that the terms of the interaction lagrangian sourcing the second-order GWs can be grouped in two types. This separation was motivated by the fact that some terms produce second-order GWs whose power spectrum is mostly featured. Accordingly they were called 'dominant', see (5.37). Equivalently, this is put in another way: the power spectrum of the sourced GWs from the terms (5.37) depends quadratically on the enhancement ($\sim A_s^2$) and it is independent from the primordial tensor spectrum. On the other hand, the terms (5.38) contributes to the power spectrum of the sourced GWs with a linear dependence on the enhancement ($\sim A_s$) and a linear dependence on the primordial tensor spectrum (i.e., the one in (5.21b)).

The main goal of this chapter is to quantify the sub-dominant contribution from (5.38) to the second-order GW power spectrum. The calculation is performed by means of the in-in formalism, and it parallels those presented previously. What follows is outlined below

sec. 6.1 - The source. Expanding the action |Act 2

(a) We find the precise interaction lagrangian for the subdominant terms (5.38). We write the sourced Einstein equation for the second-order GWs, comprehensive of both the dominant and the sub-dominant interactions.

sec. 6.2 - CTP solution. The in-in for the sourced GWs |Act 2

(b) The promotion of the sub-dominant interaction to the quantum theory does not present difficulties. The dominant/sub-dominant interaction vertices do not mix each other. The sub-dominant contribution is studied singularly and, using eq. (4.26), one finds its effect on the GW power spectrum.

(c) Each contribution (dominant & sub-dominant) of the sourced GW power spectrum from enhanced scalar perturbations is numerically evaluated. The discussion is complemented by calculating the associated GW energy density.

6.1 The source. Expanding the action |Act 2

To find the sub-dominant sources, one expands the action in the perturbations.
The reader will find the details in sec. C.1 of Appendix C. Let us denote the interaction lagrangian for the sub-dominant source terms (5.38) with $\mathcal{L}^{\rm int}|_{h^2\Phi}$, it is

$$\mathcal{L}^{\rm int}|_{h^2\Phi} = -\frac{M_p^2}{2}a^2\left[\frac{1}{2}h_{ij}h_{ij}\left(\Phi''+2\mathcal{H}\Phi'-\nabla^2\Phi\right)+\left(\partial_k h_{ij}\right)^2\Phi\right] \tag{C.13}$$

For completeness, the reader will find below the action $S^{(\rm GW)}$ describing the evolution of second-order gravitational waves, in the radiation dominated era, produced by the primordial first-order perturbations

$$\begin{aligned}S^{(\rm GW)} = \frac{M_p^2}{2}\int d^4x\Bigg\{&\frac{a^2}{4}\left(h'_{ij}h'_{ij}-\partial_k h_{ij}\,\partial_k h_{ij}\right)-a^2h_{ij}\left[-2\partial_i\Phi\partial_j\Phi-\partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right]\\ &-a^2\left[\frac{1}{2}h_{ij}h_{ij}\left(\Phi''+2\mathcal{H}\Phi'-\nabla^2\Phi\right)+\left(\partial_k h_{ij}\right)^2\Phi\right]\Bigg\}\end{aligned} \tag{6.1}$$

Remember not to confuse the (first-order) primordial tensor perturbation h_{ij} appearing in the interactions with the dynamical (second-order) GW field. The primordial (first-order) scalar&tensor perturbations are assigned non-dynamical fields. In fact, the action above is to be considered as a functional of the second-order GWs $h_{ij}^{(2)}$ (and not of the primordial $\Phi^{(1)}, h_{ij}^{(1)}$) - here we have explicitly written in superscript the perturbative order (see also page 111).

6.1.1 The second-order GW evolution equation |Act 2

The evolution equation for the second-order gravitational waves is obtained performing the variation of the action eq. (6.1). This way, one obtains the Euler-Lagrange equation for the second-order GW field h_{ij}. Alternatively, it can be found in 'the geometrical way' by calculating the Einstein equation (see Appendix C.2)

Einstein equation

ORDER 2 - The Einstein equation for the induced GWs is

$$\begin{aligned}h''_{ij}+2\mathcal{H}\,h'_{ij}-\nabla^2\,h_{ij} = -2&\left[-2\partial_i\Phi\partial_j\Phi-\partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right]^{TT}\\ &+4\left[\partial_k\left(\Phi\,\partial_k h_{ij}\right)\right]^{TT}-\,2\,h_{ij}\left[\Phi''+2\mathcal{H}\Phi'-\nabla^2\Phi\right]\end{aligned} \tag{C.17}$$

The reader is invited to note a couple of things: (a) the sub-dominant terms (5.38) corrects the Einstein eq. (B.23) with the source written in the second line of (6.1); (b) the fields appearing on the r.h.s. are the first-order primordial scalar & tensor perturbations.

6.1.2 The canonically normalized theory

The late-time RD era scalar perturbation Φ is expressed in terms of the comoving perturbation ζ. As well, one may express the first-order tensor perturbation h_λ accordingly - for instance, by introducing a 'transfer function', i.e. $h_\lambda(\tau,\,\mathbf{k}) \propto T_h(\tau\,k)\,h_\lambda(\mathbf{k})$, which relates the late-time (first order) tensor perturbation in radiation domination to its primordial value (at the first horizon crossing). We will not proceed this way because this information is already encoded in the free field solutions eq. (5.59), where one makes the slight abuse of notation of considering the quantity indicated with the same symbol h_λ as the first-order perturbation when necessary - e.g. remember that (5.59) was derived for the second-order modes. In fact, the transfer function in RD era for the first-order GW field would be obtained by solving precisely eq. (5.58), e.g. see §5.3.3 & Ex. 13 of [7]. Also, we are going to express the lagrangian in terms of the field v.
The interaction lagrangian is

$$L_{int}^{(\mathrm{GW})} = L_{int}^{(\mathrm{GW})}|_{h\,\Phi\Phi} + L_{int}^{(\mathrm{GW})}|_{h\,h\Phi} \tag{6.2}$$

$$L_{int}^{(\mathrm{GW})}|_{h\,\Phi\Phi} = \frac{4}{9}M_p \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3}\,a(\tau) \int \frac{d^3\vec{q}}{(2\pi)^3}\, v^{\lambda *}(\tau,\,\mathbf{k})\,\zeta(\mathbf{q})\,\zeta(\mathbf{k}-\mathbf{q})\,\mathcal{Q}_\lambda(\tau,\,\mathbf{k},\,\mathbf{q}) \tag{6.3}$$

$$L_{int}^{(\mathrm{GW})}|_{h\,h\Phi} = -\sum_{\lambda,\,\sigma} \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\, v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\,\zeta^*(\mathbf{k}_1+\mathbf{k}_2)\,\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) \tag{6.4}$$

where

$$\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) \equiv \frac{2}{3}\,|k_1+k_2|^2\;\mathrm{e}^{\lambda}_{ij}(\hat{k}_1)\,\mathrm{e}^{\sigma}_{ij}(\hat{k}_2)$$
$$\left[T''\left(|k_1+k_2|\tau\right) + \frac{2\mathcal{H}}{|k_1+k_2|}T'\left(|k_1+k_2|\tau\right) + \left(1 - \frac{2\,k_1\cdot k_2}{(k_1+k_2)^2}\right)T\left(|k_1+k_2|\tau\right)\right] \tag{6.5}$$

The function $\mathcal{R}_{\lambda\sigma}$ is defined in close analogy with the $\mathcal{Q}_\lambda$ in (5.45). It satisfies the properties

$$\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) = \mathcal{R}_{\sigma\lambda}(\tau,\,\mathbf{k}_2,\,\mathbf{k}_1) \tag{6.6}$$
$$\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) = \mathcal{R}_{\sigma\lambda}(\tau,\,\text{-}\mathbf{k}_1,\,\text{-}\mathbf{k}_2) \tag{6.7}$$

Moreover, the τ-dependence of the transfer function is intended in the sense we have previously discussed (e.g., we put effectively $\tau_* \to 0$). To refresh this point, the reader is invited to see page 49.

6.2 The sourced GWB from primordial scalar perturbations with the operatorial CTP formalism |Act 2

It is easy to promote the classical theory to the quantum case. It is sufficient to remember the prescriptions of page 52. In the previous chapter we saw that the quantum fluctuations of the first-order scalar/tensor perturbation are unimportant, and it is adequate to treat these fields as classical (stochastic) variables. The interaction quantum lagrangian for the sub-dominant terms (5.38) is

$$\hat{L}_{int}^{(\mathrm{GW})}|_{h\,h\Phi} \;=\; -2\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\;\hat{v}_\lambda(\tau,\,\mathbf{k}_1)\;v_\sigma(\tau,\,\mathbf{k}_2)\;\zeta^*(\mathbf{k}_1+\mathbf{k}_2)\;\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) \tag{6.8}$$

notice that this operator is hermitian. Further doubts will be clarified with the following remark

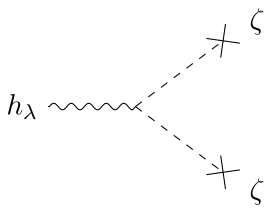
- Two v_λ fields appear in the classical lagrangian (6.4). Using the expansion, e.g. see (C.18), the interaction lagrangian is of the form

$$L_{int}^{(\mathrm{GW})}|_{h\,h\Phi} \;=\; -2\int v_\lambda^{(2)}\;v_\sigma^{(1)}\;\zeta^{*(1)}\;\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) \quad\Rightarrow\quad \hat{L}_{int}^{(\mathrm{GW})}|_{h\,h\Phi}$$

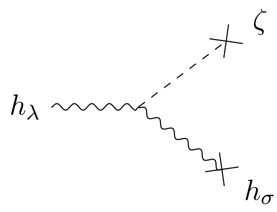
  where we expressed explicitly the perturbative $n-$order of each field. The reader is invited to note the factor 2 appearing here. The quantum lagrangian eq. (6.8) is obtained promoting the second-order GW field to a quantum operator. While the first-order fields ($v^{(1)}$, $\zeta^{(1)}$) are classical (non-quantized) stochastic variables. For instance $v_\sigma^{(1)}(\tau,\mathbf{k}_2) = v_{k_2,\sigma}(\tau)\,\mathbb{I}$, where $v_{k_2,\sigma}(\tau)$ is related to its primordial value - as we mentioned before - trough eq. (B.70) and eq. (5.61). Again, the reader should forgive the abuse of notation, which is made for the sake of brevity.

Feynman digrams for the interactions

It is useful to represent the interactions (5.55)-(6.8) in terms of the corresponding Feynman diagrams. They are respectively associated with the vertices (a)-(b) represented below



(a)

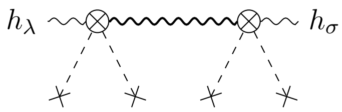


(b)

The cross-mark denotes that the corresponding field is a classical source. The $\zeta\zeta$ ($h_\sigma\zeta$) are current insertions and these vertices are both of the form $\hat{h}_\lambda\,J^\lambda$. Notice that, in literature, it is also common to indicate the current insertion with the symbol $\sim\!\!\sim\!\!\sim\!\otimes$

One can benefit of the diagrammatic representation to read off the structures contributing to the 2-point GW function. They are all the diagrams one can write down with the vertices above. There are only two tree-level contributions - remember that the loops are always exponentially suppressed with respect to the classical tree-level diagrams in the post-inflationary era - which are represented in fig. 6.1 (below).

The diagram Fig. (6.1a) is the contribution from the dominant interaction we calculated previously, i.e. eq. (B.101). The diagram Fig. (6.1b) is the contribution from the sub-dominant interaction. It is the present case study. Moreover, notice there are not diagrams made of both the vertex (a) and (b): the dominant/sub-dominant interactions do not mix at tree-level. This follows from the fact that the first-order fields are stochastic (there would be a term like $\langle h_\alpha\rangle \sim 0$).

(6.1a) 

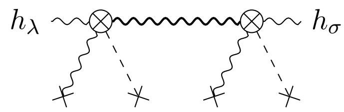
, (6.1b) 

Figure 6.1: Diagrammatic representation of the contributions to the $\langle h_\lambda(\mathbf{k}_1)\, h_\sigma(\mathbf{k}_2)\rangle$ of the sourced second-order gravitational waves produced by primordial scalar & tensor perturbations. The cross denotes the insertion of the classical ζ, h_λ currents. The internal wavy line denotes the graviton propagator. The 4-momentum integration over the internal line is reduced to the integration over $\vec{q}_1$ because of the vertex insertion.

The subdominant interactions do not involve derivative coupling. Therefore, the interaction hamiltonian is simply

$$\begin{aligned} \hat{H}_I(\tau) &= -\hat{L}^{\text{int}}(\tau) \\ &= \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\; \hat{v}_\lambda(\tau,\,\mathbf{k}_1)\; v_\sigma(\tau,\,\mathbf{k}_2)\; \zeta^*(\mathbf{k}_1+\mathbf{k}_2)\; \mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) \end{aligned} \tag{6.9}$$

The diagram (6.1b) is contained in the N=2 term of the expansion of eq. (4.26). Accordingly with our previous notation, we denote this contribution to the 2-point GW function with $\delta_2^{(\text{b})}$. It is

$$\begin{aligned} \delta_2^{(\text{b})}\Big\langle \hat{v}_\lambda(\tau,\,\mathbf{k}_1)\,\hat{v}_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= -4\int^\tau d\tau_2 \int^{\tau_2} d\tau_1 \sum_{\alpha\beta\gamma\delta}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3} \\ &\times \mathcal{R}_{\alpha\beta}(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{R}_{\gamma\delta}(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\,\Big\langle \zeta^*(\mathbf{p}_1+\mathbf{q}_1)\,\zeta^*(\mathbf{p}_2+\mathbf{q}_2)\Big\rangle\,\Big\langle v_\beta(\tau_1,\,\mathbf{q}_1)\,v_\delta(\tau_2,\,\mathbf{q}_2)\Big\rangle \\ &\times \Big\langle \Big[\hat{v}_\alpha(\tau_1,\,\mathbf{p}_1)\,,\;\Big[\hat{v}_\gamma(\tau_2,\,\mathbf{p}_2)\,,\;\hat{v}_\lambda(\tau,\,\mathbf{k}_1)\,\hat{v}_\sigma(\tau,\,\mathbf{k}_2)\Big]\Big]\Big\rangle \end{aligned} \tag{6.10}$$

The involved calculation can be found in all details in Appendix C.3. It follows closely the one we performed in the previous chapter/appendix. The final result can be expressed in a relatively neat form. To do so, one introduces the functions

$$\begin{aligned} \mathcal{I}_c(x,\,y) &\equiv \frac{k_1}{|k_1-q_1|^2}\int_0^\tau d\tau_1\,\Big(\frac{a(\tau_1)}{k_1\tau_1}\,\sin(q_1\tau_1)\,\sin(k_1\tau_1)\,R(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\Big)\Big|_{(x,y)} \\ &= \frac{k_1}{(y\,k_1)^2}\int_0^\tau d\tau_1\,\frac{a(\tau_1)}{k_1\tau_1}\,\sin(x\,k_1\tau_1)\,\sin(k_1\tau_1)\,R(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\Big|_{(x,y)} \end{aligned} \tag{C.39}$$

$$\begin{aligned} \mathcal{I}_s(x,\,y) &\equiv -\frac{k_1}{|k_1-q_1|^2}\int_0^\tau d\tau_1\,\Big(\frac{a(\tau_1)}{k_1\tau_1}\,\sin(q_1\tau_1)\,\cos(k_1\tau_1)\,R(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\Big) \\ &= -\frac{k_1}{(y\,k_1)^2}\int_0^\tau d\tau_1\,\frac{a(\tau_1)}{k_1\tau_1}\,\sin(x\,k_1\tau_1)\,\cos(k_1\tau_1)\,R(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\Big|_{(x,y)} \end{aligned} \tag{C.40}$$

where we recalled from the Appendix C the scalar function R. It is is related to $\mathcal{R}$ by the relation $\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) \equiv \mathrm{e}^{\lambda}_{ij}(\hat{k}_1)\,\mathrm{e}^{\sigma}_{ij}(\hat{k}_2)\;R(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2)$. It has mass dimension 2, therefore the $\mathcal{I}_{c,s}$ are adimensional quantities.

The (6.10) is expressed in terms of them in the compact form below.

$$\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle \equiv (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\left(\frac{H_{\rm inf}}{\sqrt{2}}\right)^2\,\frac{1}{2(2\pi)^2}\iint_{\mathcal{D}} dxdy\,\frac{\mathcal{P}_\zeta(x\,k_1)}{(xy)^2}\,y^4$$
$$\times\left[1+6\Big(\frac{1+x^2-y^2}{2x}\Big)^2+\Big(\frac{1+x^2-y^2}{2x}\Big)^4\right]\left[\cos(k_1\tau)\,\mathit{I\!I}_c(x,\,y)+\sin(k_1\tau)\,\mathit{I\!I}_s(x,\,y)\right]^2 \tag{6.11}$$

The domain of integration is the one we have already encountered in eq. (B.57), namely $\mathcal{D}=\Big\{(x,y)\in\mathbb{R}^+\times\mathbb{R}^+\;|\;|1-x|<y<1+x\Big\}$. It is illustrated in fig. B.1.
The reader is invited to notice that this result is indeed proportional to the primordial values of the scalar ($\sim\mathcal{P}_\zeta$) and tensor ($\sim\mathcal{P}_h\propto H_{\rm inf}^2$) perturbation.

The 2-point function of GWs sourced from the interaction (5.38)

Finally, let us specialize this result to case-study of enhanced primordial scalar perturbations. Using eq. (5.35), it is easy to prove that the sub-dominant contribution to the second-order GWs sourced, from the primordial first-order enhanced scalar perturbation and first-order tensor perturbation, is given by ($x_\star=p_*/k_1$)

$$\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\left(\frac{H_{\rm inf}}{\sqrt{2}}\right)^2\,\frac{1}{2(2\pi)^2}\,\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star} dy\,y^2$$
$$\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right]\left[\cos(k_1\tau)\,\mathit{I\!I}_c(x,\,y)+\sin(k_1\tau)\,\mathit{I\!I}_s(x,\,y)\right]^2 \tag{6.12}$$

6.3 Results. The power spectrum and energy density of the GWs sourced from enhanced scalar perturbations

The case-study of the last couple of chapters is the power spectrum of second-order gravitational waves sourced by the first-order primordial scalar & tensor perturbations, where the primordial scalar power spectrum is featured with a monochromatic peak, superimposed to the standard (nearly) scale-invariant power spectrum. By studying the GW evolution equation, one finds that there are two second-order sources: one containing quadratically the first-order enhanced scalars (5.37), and another containing both the first-order enhanced scalar and the first-order primordial tensor perturbation (5.38). The power spectrum of second-order GWs is mostly ‘featured’ from the waves sourced from (5.37) - in the sense we have previously explained.
The tensor power spectrum is related to the 2-point GW function by

$$\mathcal{P}_T(p)=2\,\mathcal{P}_{h_\lambda}(p)\quad,\qquad\Big\langle h_\lambda(\tau,\,\mathbf{k}_1)\,h_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle\equiv(2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\,\frac{2\pi^2}{k_1^3}\,\mathcal{P}_{h_\lambda}(\tau,k_1) \tag{6.13}$$

The dominant and sub-dominant contributions to the power spectrum of the second-order gravitational waves can be read off from eq. (B.106) and eq. (C.47). They are respectively ($x_\star \equiv p_*/k_1$)

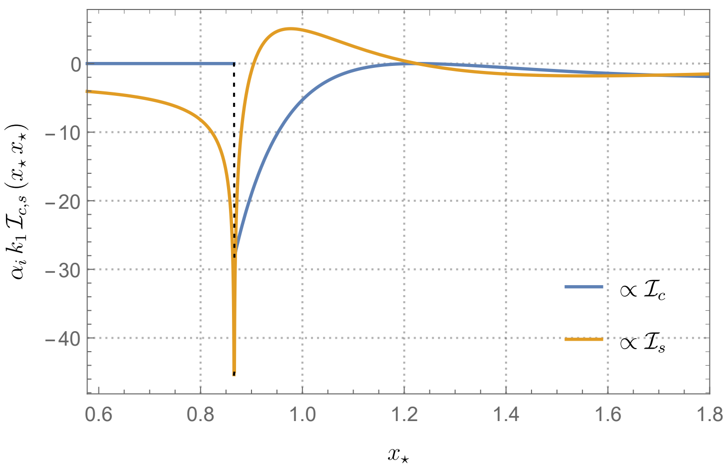
$$\begin{aligned}\delta_2\mathcal{P}_h(\tau,\, k_1) \;\simeq\; & \theta\left(x_\star - \frac{1}{2}\right)\,\frac{1}{a^2(\tau)}\left(\frac{A_s\, x_\star}{9}\right)^2\left[1-\frac{1}{(2x_\star)^2}\right]^2 \\ & \times\Big[\sin(k_1\tau)\,\mathcal{I}_s\,(x_\star, x_\star) - \cos(k_1\tau)\,\mathcal{I}_c\,(x_\star, x_\star)\Big]^2\end{aligned} \tag{6.14}$$

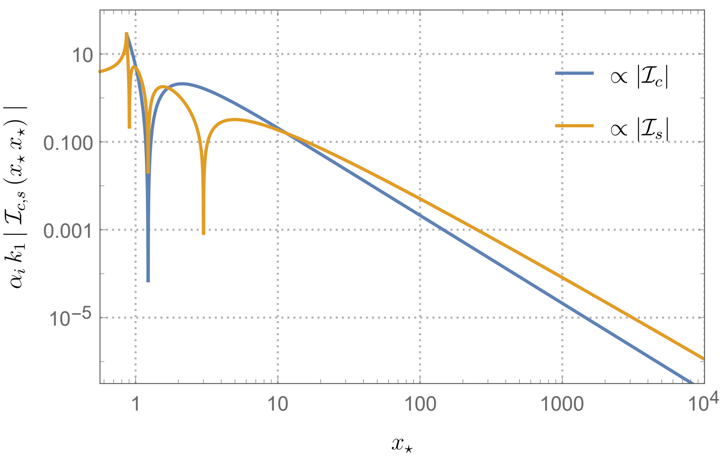
$$\begin{aligned}\delta_2^{(b)}\mathcal{P}_h(\tau, k_1) \simeq \frac{2}{\pi^2}\left(\frac{H_{\rm inf}}{M_p}\right)^2\,\frac{1}{a^2(\tau)}\,\frac{1}{8}\,\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star} dy\; y^2\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right] \\ \times\Big[\cos(k_1\tau)\,\mathcal{II}_c(x_\star,\, y)+\sin(k_1\tau)\,\mathcal{II}_s(x_\star,\, y)\Big]^2\end{aligned} \tag{6.15}$$

Let us analyse the results we have obtained.
We are interested in the behaviour of the GW power spectrum in the radiation domination at the late-time $\tau \gg \tau_*$, when we approximate the scale factor with $a(\tau) \simeq \frac{\tau}{H_{\rm inf}\,\tau_*^2}$. The enhanced scalar spectrum introduces a length scale p_*. From eq.s(6.14)-(6.15) one sees that the power spectra depends the conformal time, the magnitude of the momentum k_1 and, parametrically, from the enhancement scale p_* and its strength A_s. The momentum dependence in eq. (6.14) is both explicit (e.g. in the trigonometric functions) and implicit, though the adimensional variable $x_\star = p_*/k_1$. As preliminary, one has to investigate the behaviour of the $\mathcal{I}_{c,s}\,\&\,\mathcal{II}_{c,s}$-functions

6.3.1 The behaviour of the $\mathcal{I}_{c,s}\,\&\,\mathcal{II}_{c,s}$-functions with respect to $x_\star = p_*/k$

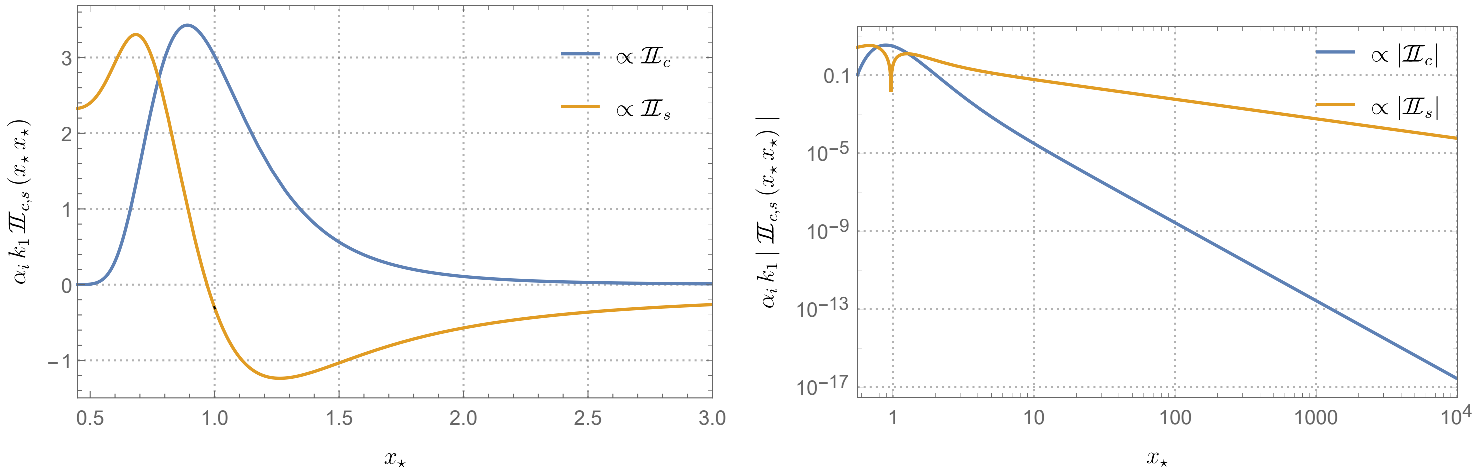
The behaviour of the functions $\mathcal{I}_{c,s}(x_\star,\, x_\star)$ eq.s(B.42)-(B.43) is shown below: (left) in a neighbourhood of $x_\star \simeq 1 \leftrightarrow k_1 \simeq p_*$ and (right) for large values of the $x_\star$-variable (remember that $\alpha_i \equiv H_{\rm inf}\tau_*^2$)





The narrow peaks in the $\mathcal{I}_{c,s}$-functions are caused by the oscillatory behaviour of the transfer function. Notice that we represented $\alpha_i\, k_1\, \mathcal{I}_{c,s}$ in order to get rid the initial scale factor normalization. Sometimes this point is not mentioned in literature since the final results are independent of this.

The functions $\mathcal{II}_{c,s}(x_\star, y)$ appearing in eq. (6.15) depend on two independent variables. However, by inspecting the integrand (6.15) one would discover that a significant contribution to the sum comes from a narrow neighbourhood of $y = x_\star$. Therefore it is useful to plot the quantities $\mathcal{I\!I}_{c,s}(x_\star, x_\star)$ as well. We make use of the approximations of Appendix C.3.2, the qualitative behaviour of the functions is shown below



6.3.2 Results. Close-up on the dominant contribution $\delta_2\mathcal{P}_h$

The first thing we notice in (6.14) is the presence of the θ-function. It implies that no contribution comes from the waves with momentum $k_1 > 2\,p_*$. This is a very intuitive result, which can be evinced by the diagram (6.1a) in fig. 6.1. In other words, it is not possible to produce correlated gravitational waves with momentum greater than those injected by the scalar currents.

The late-time $\tau \gg \tau_*$ power spectrum (6.14) is an oscillatory function on the time scales $T \sim 1/k_1$. One may get rid of the oscillations (e.g. from $\sin(k_1\tau)$, $\cos(k_1\tau)$) by considering the r.m.s. value of waves. From the experimental point of view, the resulting averaged power spectrum is what an observer would obtain as the result of a measuring process of time duration $\Delta t \geqslant 1/k_1$.
We indicate with $\overline{\delta_2\mathcal{P}_h}$ the time-averaged power spectrum, which we calculate by time-averaging over the scale of oscillation of the trigonometric functions $\sin(k_1\tau)$, $\cos(k_1\tau)$. We expand the square in the second line of (6.14). Then we use that

$$\overline{\cos^2(k_1\tau)} \;=\; \overline{\sin^2(k_1\tau)} \;=\; \frac{1}{2} \qquad\qquad \overline{\cos(k_1\tau)} \;=\; \overline{\sin(k_1\tau)} \;=\; 0 \tag{6.16}$$

so that

$$\begin{aligned}\overline{\delta_2\mathcal{P}_h(\tau,\, k_1)} \;=\; \theta\left(x_\star - \frac{1}{2}\right)\, \frac{1}{a^2(\tau)}\, \left(\frac{A_s\, x_\star}{9}\right)^2 \left[1 - \frac{1}{(2x_\star)^2}\right]^2 \\ \times \frac{1}{2}\left[\mathcal{I}_s^2\,(x_\star, x_\star) + \mathcal{I}_c^2\,(x_\star, x_\star)\right]\end{aligned} \tag{6.17}$$

Armed with this result, we can calculate the contribution to density parameter Ω from the second-order GWs produced by the enhanced scalar perturbations.

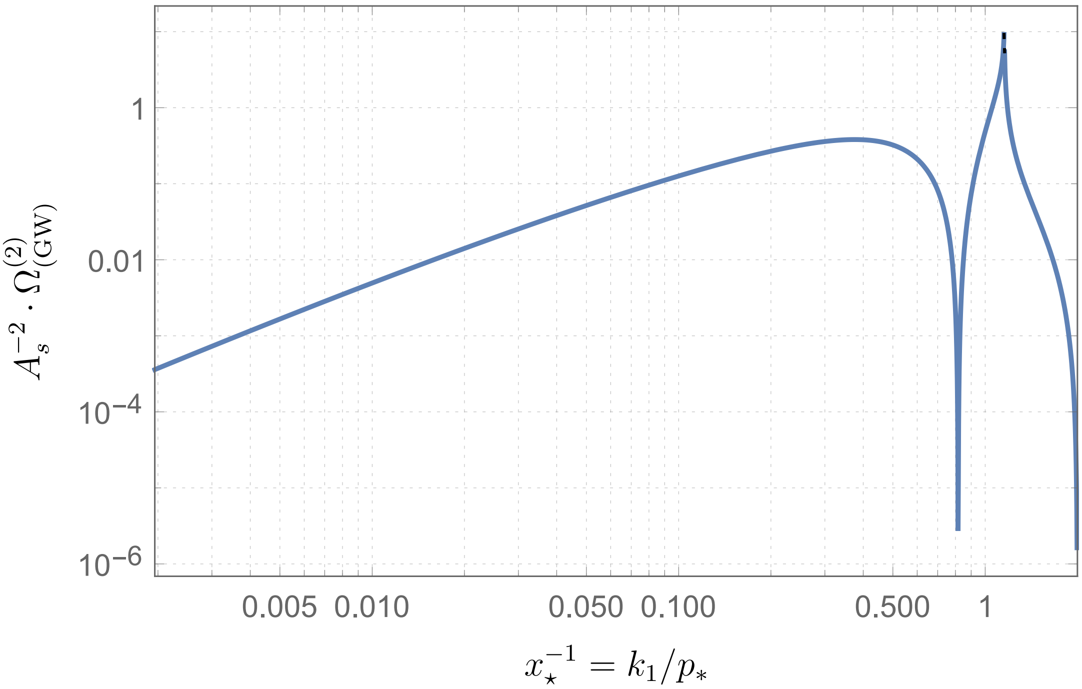


Figure 6.2: Log-Log plot of the dominant contribution (from 5.37) to the second-order GWs density parameter from primordial enhanced scalar perturbations. Remember that A_s measures the strength of the scalar enhancement, see page 46.

The density parameter from the sourced GWs: dominant contribution

Using the definition eq. (B.110), the late-time density parameter of the sourced second-order GWs from the dominant source (5.37) is

$$\begin{aligned}\Omega^{(2)}_{(\mathrm{GW})}(k_1) &= \frac{1}{12}\theta\left(x_\star - \frac{1}{2}\right) k_1^2 \frac{a^{-4}}{H^2}\left(\frac{A_s\, x_\star}{9}\right)^2 \left[1 - \frac{1}{(2x_\star)^2}\right]^2 \left[\, \mathcal{I}_s^2 + \mathcal{I}_c^2 \,\right]_{x_\star = p_*/k} \\ &= \theta\left(x_\star - \frac{1}{2}\right) \frac{(A_s\, x_\star)^2}{3 \cdot 18^2} \left[1 - \frac{1}{(2x_\star)^2}\right]^2 \left[\, (\alpha_i\, k_1\, \mathcal{I}_s)^2 + (\alpha_i\, k_1\, \mathcal{I}_c)^2 \,\right]_{x_\star = p_*/k_1}\end{aligned} \tag{6.18}$$

The reader may notice that this quantity is τ-independent, in fact at late-time the quantity $\frac{a^{-4}}{H^2} = \frac{\tau^2}{\alpha_i^{-2}\tau^2} = \alpha_i^2$ is constant. Once a GW mode is produced, it propagates freely and its amplitude scales with time as the radiation. So in a radiation dominated universe, where $\rho_c \propto H^2 \propto a^{-4}$, the GW parameter density is constant in time. As we mentioned in page 117, the normalization α_i does not appear in the final results.

The behaviour of the $A_s^{-2}\, \Omega^{(2)}_{(\mathrm{GW})}(k_1)$ vs. $x_\star^{-1} = k_1/p_*$ is shown in fig. 6.2; this result is consistent with the literature (i.e. see [38]). We notice a suppression in the density parameter for $x_\star^{-1} = \frac{\sqrt{3}}{2}$, coming from the logarithm in eq. (B.61), and a resonant-like enhancement for $x_\star^{-1} = 2 - \frac{\sqrt{3}}{2}$.

6.3.3 Results. Close-up on the sub-dominant contribution $\delta_2^{(\mathrm{b})}\mathcal{P}_h$

Again, we are interested in the density parameter of the GWs in the presence of the enhancement in the primordial scalar perturbation. An observer inhabiting the late time RD universe, who probes the GW density parameter, would typically measure the dominant contribution from the enhancement eq. (6.18). Depending on the scale he/she is probing however one may be able to observe the sub-dominant contribution, e.g. those from (5.38). Following closely the steps of Appendix B.8.2, let us calculate the density parameter for the sub-dominant contribution. We indicate the late-time density parameter of the sourced second-order GWs from the sub-dominant (5.38) as $\Omega^{(2b)}_{(\mathrm{GW})}(k_1)$. Using the definition eq. (B.110), at late-times one has

$$\Omega^{(2b)}_{(\mathrm{GW})}(k_1) \simeq \frac{\Omega^{(0)}_{(\mathrm{GW})}}{4}\,\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star} dy\, y^2\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right] \times\left[(\alpha_i\, k_1\, \mathcal{I}_c)^2+(\alpha_i\, k_1\, \mathcal{I}_s^2)\right] \tag{6.19}$$

The quantity above is τ-independent, as it should be accordingly with the previous discussion. In the second line we placed the α_i-normalization near the $\mathcal{I}$-functions to emphasize that it simplifies. Also, we have factorized the first-order density parameter contribution in front (see eq. (B.114)). Again, we notice that the final result is independent from the normalization α_i. The (C.53) is similar to (6.18). This time, there is no θ-function because the interaction vertex contains both the primordial first-order scalar and tensor perturbation. The first-order tensor modes are populated over a broad range of momenta, as it is evident from the scale-invariant flat spectrum eq. (5.21b). Rephrasing this in terms of the diagram, there is not any condition because the tensor perturbation in the interaction vertex can always inject a correlated-graviton with arbitrary momentum.

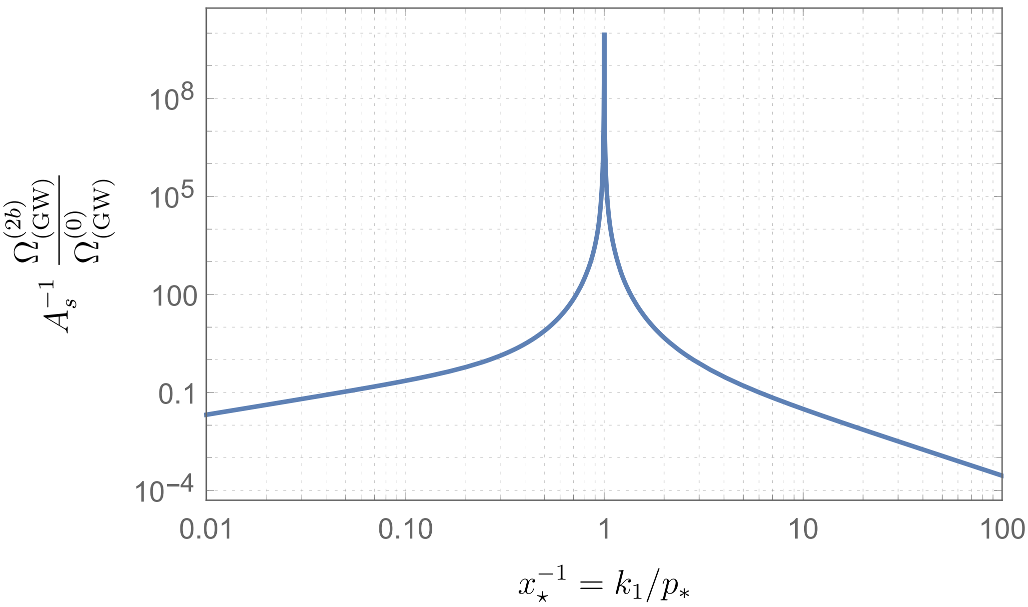


Figure 6.3: Log-Log plot of the sub-dominant contribution (from 5.38) to the second-order GWs density parameter from the primordial enhanced scalar perturbation and the tensor perturbation.

The sub-dominant contribution to the second-order GW density parameter is monotonically increasing for $k_1 < p_*$, it has a spike around $k_1 \sim p_*$ after which it decreases steadily. A spike appears for $x_\star = 1 \Leftrightarrow k_1 = p_*$. It is believed that this behaviour is an artefact of approximating the enhancement in the primordial scalar spectrum with a δ-function. In fact, if one starts from (6.11) and calculates the GW density parameter in the case where the primordial scalar spectrum is scale-invariant (for instance for $\mathcal{P}_\zeta = A_s$), then one would obtain a constant: that is, $\Omega^{(2b)}_{\rm (GW)} = \mathrm{O}(10^2)\, A_s\, \Omega^{(0)}_{\rm (GW)}$. The result above for $k_1 \simeq p_*$ should be considered with some caution. A further analysis, for example using as enhancement a smoother gaussian/power-law function, may be performed to quantify the size of the neighbourhood of $x_\star = 1$ where the δ-function is no longer a reliable *physical* approximation [38].

Let us remark that we dubbed the contribution $\delta_2^{(\mathrm{b})}\mathcal{P}_h$ from the interaction (5.38) as 'sub-dominant' in that it is suppressed by a factor A_s^{-1} with respect to the one of eq. (6.14). We may compare the two results (6.18) & (6.19) for the density parameter of the GWs with some extent. Let us assume that for $x_\star^{-1} \approx 0.01$ one is sufficiently away from the unphysical behaviour of the spike of fig. 6.3. Comparing the two contributions to the GW density parameter, at $x_\star^{-1} \approxeq 0.01$ would give

$$\Omega^{(2)}_{\rm (GW)} \approxeq 10^{-3}\, A_s^2 \qquad\qquad \Omega^{(2b)}_{\rm (GW)} \approxeq 10^{-2}\, A_s\, \Omega^{(0)}_{\rm (GW)} \tag{6.20}$$

Therefore, as a merely consistency condition one would obtain the bound

$$10^{-2}\, A_s\, \Omega^{(0)}_{\rm (GW)} \lesssim 10^{-3}\, A_s^2 \qquad \Rightarrow \qquad \Omega^{(0)}_{\rm (GW)} \lesssim 0.1 \cdot A_s \tag{6.21}$$

which may be considered either as a constrain over A_s or an indicator that the monochromatic enhancement $\delta(p - p_*)$ is not a good approximation if the condition above is not fulfilled.

Finally, a more quantitative analysis of this problem requires both a smoother approximation for the enhancement and the careful calculation of all the other subdominant contributions to the GW density parameter (e.g. like $\mathcal{P}_\zeta^{(0)}\, \mathcal{P}_\zeta^{(\mathrm{enhan.})}$) and it is postponed to future investigations.

Chapter 7

Conclusions

In this chapter we draw the conclusions of this thesis, and we present some prospects for interesting related future work.

(a) The main purpose of this thesis is the study of the contribution to the SGWB from the GWs produced in the RD era by primordial enhanced scalar perturbations. The gravitational waves are sourced by the primordial scalar & tensor perturbations starting from second-order in the perturbative series.

(b) The (first-order) primordial scalar & tensor spectra are input functions. An increased power at a given scale (a "bump") is assumed to be present in the former. While several mechanisms have been proposed in the literature to generate this bump during inflation (see for instance [6], [39], [40]), the GW that we study are produced when these modes re-enter the horizon well after inflation, and so it is insensitive to the specific mechanism responsible for the creation of the primordial modes, but only to its outcome. For this reason, the (first-order) primordial scalar & tensor spectra are taken as input functions in our computations.

(c) The traditional approach to this problem is by means of the Green function method. We implemented the in-in formalism to study the classical production of second-order GWs in the RD era by primordial first-order perturbations. The application of CTP formalism to such problems benefits of the tools and techniques of QFT.

(d) The sourced GWs are contributed both by the scalar and the tensor perturbation. In presence of the scalar enhancement, the latter contribution is typically expected to be subdominant with respect to the one from first-order scalars only. Employing the in-in formalism we computed for the first time the contribution from the leading process involving one tensor mode in the source.

7.1 Issues and further aspects for future investigation

(a) Non-gaussianity of the primordial perturbations
Typical models of slow roll inflation generate primordial perturbations that are Gaussian with high accuracy (see for instance [41] for a review). It means that the wavefunction of the initial Bunch-Davies vacuum is gaussian, and the time evolution preserves its gaussianity [19]. In more practical terms, it means that the theory is completely determined with the two-point function. For instance, any n-point function with $n > 2$ can be reduced by means of the Wick theorem to sums of products of the 2-point correlator. Also, the $(2n+1)$-point functions vanish. On the other hand, typical models of slow roll inflation do not generate a bump in the power spectrum. For these specific models, the scalar perturbations are expected to be non-Gaussian at the scales of the bump [6]. This can modify the power of the GW sourced by these perturbations. A few studies exist in the literature for specific types of non-Gaussianity [42], [6], [43], [44], [45] but a general study is stil missing.
For instance, there are two interesting aspects for further studies

- non-gaussianity in the second-order sourced $\langle h_\lambda\, h_\sigma\, h_\rho \rangle$
- non-gaussianity from the three-point scalar function $\langle \zeta\, \zeta\, \zeta \rangle$

A large enhancement may give importance to the term $\langle \zeta\, \zeta\, \zeta \rangle$.

(b) The GWs sourced during the inflation & first horizon-crossing
The gravitational waves produced in the RD era do not depend on the precise mechanism which produced the enhanced in the primordial scalar spectrum. However, second-order gravitational waves are produced also during the inflationary stage. Therefore it is conceivable that the mechanism which enhanced the scalar spectrum leaves a sign over the second-order sourced gravitational waves produced during inflation. Moreover, it is likely that the second-order inflationary GWs can be distinguished in those classically produced around the time of the first horizon-crossing of the modes and those produced earlier. In the latter, depending on the squeezing parameter of the state, the quantum behaviour may be relevant and a comprehensive treatment of both the second-order GWs & the sources may be necessary.

(c) Perturbative series consistency
The solution for the gravitational waves produced by the sources $(\zeta\, \zeta)$ & $(\zeta\, h)$ have been calculated by means of the perturbation theory, starting from the free (unsourced) solution for the second-order GWs. It is conceivable that the perturbative solution would break down for large values of the enhancement parameters. When this happens, the sources cannot longer be regarded as a small perturbation of the free theory. In practise, such a study for the perturbative-solution consistency may be performed by evaluating the higher-order contributions in the perturbative series (4.24). For instance, in the case of the dominant contribution from the $\zeta\, \zeta$-source, one may evaluate the 2-point GW function from the diagram associated with $N = 4$ of (5.62) (e.g. $\propto A_s^4$) and compare this contribution with the $N = 2$ one we have calculated in eq. (5.68). The consistency of the perturbative solution requires the former to be smaller than the latter.

In the long term, this work may be revalued as a playground to investigate these aspects. These studies are rather involving both from the point of view of the theory and from the computational viewpoint. In this context, the CTP method may emerge as a clean and efficient way to account of these effects, while providing a simple & flexible calculation protocol.

Appendices

Appendix A

Polarisation vectors

In this Appendix are collected the properties and identities related to the polarisation vectors for the tensor perturbation h_{ij}.

A.1 Properties of the polarisation vectors

In sec. 3.2.1 we decomposed the metric perturbations in scalar, vector and tensor fields. Gravitational waves are embedded in the tensor perturbation h_{ij}. We recall below the properties eq. (3.23) of the GWs field

$$h_{ij}(\tau,\vec{x}) = h_{ji}(\tau,\vec{x}) \qquad (symmetric) \tag{A.1}$$

$$\delta^{ij}\, h_{ij}(\tau,\vec{x}) = 0 \qquad (traceless) \tag{A.2}$$

$$\partial_i\, h_{ij}(\tau,\vec{x}) = 0 \qquad (transverse) \tag{A.3}$$

We write the tensor perturbation in momentum space in terms of the *polarisation vectors* $\mathrm{e}^{\lambda}_{ij}$

$$h_{ij}(\tau,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}} \sum_{\lambda} h_{\lambda}(\tau,\, \mathbf{k})\, \mathrm{e}^{\lambda}_{ij}(\hat{k}) \tag{A.4}$$

The polarisation basis is a finite-dimensional representation of the $\vec{k}-$little group. Here λ is the internal index labelling the independent vectors associated with the two GWs degrees of freedom. The properties eq.s(A.1)-(A.3) are rephrased in terms of the polarisation vectors as both algebraic and differential constraints

$$\mathrm{e}^{\lambda}_{ij}(\hat{k}) = \mathrm{e}^{\lambda}_{ji}(\hat{k}) \qquad (symmetric) \tag{A.5}$$

$$\delta^{ij}\, \mathrm{e}^{\lambda}_{ij}(\hat{k}) = 0 \qquad (traceless) \tag{A.6}$$

$$p_i\, \mathrm{e}^{\lambda}_{ij}(\hat{k}) = 0 \qquad (transverse) \tag{A.7}$$

$$\sum_{i,j} \mathrm{e}^{\lambda}_{ij}(\hat{k})\, \mathrm{e}^{\sigma}_{ij}(-\hat{k}) = \delta^{\lambda\sigma} \qquad (orthonormal) \tag{A.8}$$

The last equation is the orthonormal condition for the basis vector.

A.1.1 Reality/hermitian condition

The perturbation h_{ij} is part of the metric tensor. In the classical theory, therefore the tensor perturbation is a real-valued field.

$$h_{ij}(\tau, \vec{x}) = h^*_{ij}(\tau, \vec{x}) \qquad \rightarrow \qquad h_\lambda(\tau,\, \mathbf{k})\, \mathrm{e}^\lambda_{ij}(\hat{k}) = h^*_\lambda(\tau,\, -\mathbf{k})\, \mathrm{e}^{*\lambda}_{ij}(-\hat{k}) \tag{A.9}$$

This implies that we can assign the amplitudes $h_\lambda(\tau,\, \mathbf{k})$ on the half $\mathbb{R}^3-$momentum space.

In the quantum theory, the classical field $h_{ij}(\tau, \mathbf{x})$ is promoted to quantum operator $\hat{h}_{ij}(\tau, \mathbf{x})$. The polarisation vectors are classical functions, therefore the self-adjoint condition on the quantum field implies

$$\hat{h}_{ij}(\tau, \vec{x}) = \hat{h}^\dagger_{ij}(\tau, \vec{x}) \qquad \rightarrow \qquad \hat{h}_\lambda(\tau,\, \mathbf{k})\, \mathrm{e}^\lambda_{ij}(\hat{k}) = \hat{h}^\dagger_\lambda(\tau,\, -\mathbf{k})\, \mathrm{e}^{*\lambda}_{ji}(-\hat{k}) \tag{A.10}$$

explicitly

$$\hat{h}_\lambda(\tau,\, \mathbf{k}) = \hat{h}^\dagger_\lambda(\tau,\, -\mathbf{k}) \tag{A.11}$$

$$\mathrm{e}^\lambda_{ij}(\hat{k}) = \mathrm{e}^{*\lambda}_{ji}(-\hat{k}) \tag{A.12}$$

Explicit relations for the linear/helicity bases. It is customary to take eq. (A.12) holding also for eq. (A.9). Therefore we shall include as defining property among the eq.s(A.5)-(A.8) also the reality condition

$$\mathrm{e}^\lambda_{ij}(\hat{k}) = \mathrm{e}^{*\lambda}_{ji}(-\hat{k}) \qquad\qquad (\textit{reality}) \tag{A.12}$$

Let us introduce two explicit representations for the polarisation vectors: the *linear basis* and the *helicity basis*.

A.2 The Linear basis of polarisations

The polarisation vectors in the linear basis are real-valued. The defining properties render this basis analogous to the one used to describe linearly polarised light. In fact, the explicit polarisation tensor $e^\lambda_{ij}(\hat{k})$ is built similarly. Consider two directions $\hat{\mathbf{u}}(\hat{k})$, $\hat{\mathbf{v}}(\hat{k})$, mutually orthogonal and perpendicular to the direction of the wave vector $\vec{k}$. The two tensors

$$\begin{aligned} \mathrm{e}^{(+)}_{ij}(\hat{k}) &= \frac{1}{\sqrt{2}}\Big[u_i(\hat{k})\, u_j(\hat{k}) - v_i(\hat{k})\, v_j(\hat{k})\Big] \\ \mathrm{e}^{(\times)}_{ij}(\hat{k}) &= \frac{1}{\sqrt{2}}\Big[u_i(\hat{k})\, v_j(\hat{k}) + v_i(\hat{k})\, u_j(\hat{k})\Big] \end{aligned} \tag{A.13}$$

are obviously real. Also, they satisfy eq.s(A.5)-(A.8). The polarisation $\mathrm{e}^{(+)}_{ij}(\hat{k})$, $\mathrm{e}^{(\times)}_{ij}(\hat{k})$ are called respectively 'plus' and 'cross' as they are related to the way in which the positions of test masses vary when the GW passes.

For example, let us verify the orthonormal condition eq. (A.8). Recall the explicit conditions $\mathbf{\hat{u}}(\hat{k})$, $\mathbf{\hat{v}}(\hat{k})$ fulfil

$$\mathbf{\hat{u}}(\hat{k}) \cdot \mathbf{\hat{v}}(\hat{k}) \equiv u_i\, v_i = 0 \tag{A.14}$$
$$u_i\, k_i = v_i\, k_i = 0 \tag{A.15}$$

It is straightforward to check that

$$\mathrm{e}^{(+)}_{ij}(\hat{k})\, \mathrm{e}^{(+)}_{ij}(\hat{k}) = \mathrm{e}^{(\times)}_{ij}(\hat{k})\, \mathrm{e}^{(\times)}_{ij}(\hat{k}) = \frac{1}{2}\Big[u_i u_j\, u_i u_j - u_i u_j\, v_i v_j - v_i v_j\, u_i u_j + v_i v_j\, v_i v_j\Big] = 1 \tag{A.16}$$
$$\mathrm{e}^{(+)}_{ij}(\hat{k})\, \mathrm{e}^{(\times)}_{ij}(\hat{k}) = \mathrm{e}^{(\times)}_{ij}(\hat{k})\, \mathrm{e}^{(+)}_{ij}(\hat{k}) = \frac{1}{2}\Big[u_i u_j\, u_i v_j + u_i u_j\, v_i u_j - v_i v_j\, u_i v_j - v_i v_j\, v_i u_j\Big] = 0 \tag{A.17}$$

A.2.1 Explicit representation

It is worth to write down the polarisation vectors in coordinates using a cartesian frame of reference. We pick an arbitrary direction, for example the z-axis $\hat{e}_z$, and we write in components the vectors $\mathbf{\hat{u}}(\hat{k})$, $\mathbf{\hat{v}}(\hat{k})$ using

$$\mathbf{\hat{u}}(\hat{k}) = \frac{\hat{k} \times \hat{e}_z}{\left|\hat{k} \times \hat{e}_z\right|} \tag{A.18}$$
$$\mathbf{\hat{v}}(\hat{k}) = \frac{\hat{k} \times \mathbf{\hat{u}}(\hat{k})}{\left|\hat{k} \times \mathbf{\hat{u}}(\hat{k})\right|} = \left(\hat{k} \cdot \hat{e}_z\right) \hat{k} - \hat{e}_z \tag{A.19}$$

In the last, the identity $\vec{v}_1 \times (\vec{v}_2 \times \vec{v}_3) = (\vec{v}_1 \cdot \vec{v}_3)\, \vec{v}_2 - (\vec{v}_1 \cdot \vec{v}_2)\, \vec{v}_3$ was used. Also, we used $\left|\hat{k} \times \mathbf{\hat{u}}(\hat{k})\right| = 1$.

The explicit components are easily found to be (we set $k_\perp \equiv |\hat{k} \times \hat{e}_z| = \sqrt{\hat{k}_1^2 + \hat{k}_2^2}$)

$$u_i = \frac{1}{k_\perp} \epsilon_{i\alpha\beta} \hat{k}^\alpha \delta^{\beta 3} = \frac{1}{k_\perp} \begin{pmatrix} k_2 \\ -k_1 \\ 0 \end{pmatrix} \tag{A.20}$$
$$v_i = \frac{1}{k_\perp} \epsilon_{i\rho\sigma} \epsilon_{\sigma\alpha\beta} \hat{k}^\rho \hat{k}^\alpha\, \delta^{\beta 3} = \frac{1}{k_\perp} \begin{pmatrix} k_2\, k_3 \\ k_2\, k_3 \\ -k_\perp^2 \end{pmatrix} \tag{A.21}$$

They will turn out useful in the following sec. A.4.

A.3 The Helicity basis of polarisations

The polarisation vectors in the helicity basis are complex-valued. It is worth again to mention the analogy with the circular polarisation basis used for light. Let us recall that the metric perturbations are classified according to their *helicity* h: a perturbation of defined helicity h has its amplitude multiplied by a phase $e^{ih\theta}$ under a local rotation of θ around the direction $\vec{k}$ of propagation.

For the tensor perturbation, the helicity assumes the values $\propto \pm 1$. It is straightforward to give an explicit form for the helicity basis vectors by using the linear polarisations eq. (A.13)

$$\begin{aligned} \mathrm{e}_{ij}^{(\mathrm{L})} &= \frac{1}{\sqrt{2}}\left[\mathrm{e}_{ij}^{(+)} - i\,\mathrm{e}_{ij}^{(\times)}\right] \\ \mathrm{e}_{ij}^{(\mathrm{R})} &= \frac{1}{\sqrt{2}}\left[\mathrm{e}_{ij}^{(+)} + i\,\mathrm{e}_{ij}^{(\times)}\right] \end{aligned} \tag{A.22}$$

Here we have used the label L (R) instead of the corresponding helicity -2 $(+2)$. In the end, it is a matter of convention. However, the rationale behind this usage is in the fact that GWs are in a spin-2 massless field. And for massless particles, the concepts of helicity and chirality are tangled each other. The reader can check by himself the validity of eq.s(A.5)-(A.8) either as a consequence of the corresponding properties for the linear basis or explicitly by choosing a reference frame.

A.4 "Polarisation-trace" identity

The purpose of this section is to prove the identity

$$\mathrm{e}_{ab}^{\lambda}(\hat{k})\,\mathrm{e}_{cd}^{\lambda}(-\hat{k}) = \frac{1}{4}\left[\delta_{ac} + \hat{k}_a\,\hat{k}_c - i\lambda\,\epsilon_{ace}\,\hat{k}^e\right]\left[\delta_{bd} + \hat{k}_b\,\hat{k}_d - i\lambda\,\epsilon_{bdf}\,\hat{k}^f\right] \tag{A.23}$$

where no sum over the index $\lambda = \pm 1$ is to be performed, and $\hat{k}^e = \delta^{ei}\,\hat{k}_i$. The polarisation vector $\mathrm{e}^{(\mathrm{L/R})}$ is in the helicity base, but it is normalized so that it carries eigenvalue normalized in ± 1 (instead of being ± 2).

The proof is easier if we write the helicity polarisations in terms of the linear basis, and eventually as products of the vectors $u_i,\ v_i$. First, we introduce the complex vector

$$\vec{\epsilon}\,(\hat{k}) \equiv \frac{\hat{\mathbf{u}} - i\,\hat{\mathbf{v}}}{\sqrt{2}} \tag{A.24}$$

in terms of the vectors of sec. A.2. After a little amount of algebra, it is easy to write the polarisation vectors in terms of eq. (A.24). For instance, in the case of $\mathrm{e}^{(\mathrm{L})}$ one has

$$\begin{aligned} \mathrm{e}_{ij}^{(L)}(\hat{k}) &= \frac{1}{\sqrt{2}}\left(\mathrm{e}_{ij}^{(+)} - i\,\mathrm{e}_{ij}^{(\times)}\right) = \frac{1}{2}\Big[\Big(u_i(\hat{k})\,u_j(\hat{k}) - v_i(\hat{k})\,v_j(\hat{k})\Big) - i\,\Big(u_i(\hat{k})\,v_j(\hat{k}) + v_i(\hat{k})\,u_j(\hat{k})\Big)\Big] \\ &= \frac{u_i - iv_i}{\sqrt{2}}\frac{u_j - iv_j}{\sqrt{2}} \equiv \epsilon_i(\hat{k})\,\epsilon_j(\hat{k}) \end{aligned}$$

and analogously for $\mathrm{e}_{ij}^{(R)}(\hat{k}) = \epsilon_i^*(\hat{k})\,\epsilon_j^*(\hat{k})$

For example we consider the $\lambda = -1 \Leftrightarrow L$ case, the l.h.s of the equation eq. (A.23) is therefore

$$\mathrm{e}_{ij}^{(L)}(\hat{k})\,\mathrm{e}_{ij}^{(L)}(-\hat{k}) = \epsilon_a(\hat{k})\,\epsilon_b(\hat{k})\,\epsilon_c(-\hat{k})\,\epsilon_d(-\hat{k}) \tag{A.25}$$

Second, let us work the product $\epsilon_a(\hat{k})\,\epsilon_c(-\hat{k})$. For notational neatness, we suppress the $\hat{k}-$dependence

$$\begin{aligned} \epsilon_a(\hat{k})\,\epsilon_c(-\hat{k}) &= \frac{1}{2}\,(u_a + i\lambda v_a)\,(-u_c + i\lambda v_c) \\ &= -\frac{1}{2}\left[u_a\,u_c + v_a\,v_c - i\lambda\,(u_a\,v_c - v_a\,u_c)\right] \end{aligned} \tag{A.26}$$

We need the terms: (a) $u_a\, u_c$, (b) $v_a\, v_c$ and (c) $\Big(u_a\, v_c - v_a\, u_c\Big)$

$$
\begin{aligned}
u_a\, u_c &= \frac{1}{k_\perp^2}\epsilon_{a\alpha\beta}\epsilon_{c\gamma\rho}\delta^{3\beta}\delta^{3\rho}\hat{k}^\alpha \hat{k}^\gamma \\
&= \frac{1}{k_\perp^2}\epsilon_{a\alpha 3}\epsilon_{c\gamma 3}\hat{k}^\alpha \hat{k}^\gamma = \frac{1}{k_\perp^2}\begin{pmatrix} k_2^2 & -k_2\, k_1 & 0 \\ -k_1\, k_2 & k_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned} \tag{a}
$$

$$
\begin{aligned}
v_b\, v_d &= \frac{1}{k_\perp^2}\epsilon_{b\rho\sigma}\epsilon_{\sigma\alpha\beta}\epsilon_{d\rho'\sigma'}\epsilon_{\sigma'\alpha'\beta'}\delta^{\beta 3}\delta^{3\beta'}\,\hat{k}^\rho \hat{k}^\alpha \hat{k}^{\rho'}\hat{k}^{\alpha'} \\
&= \frac{1}{k_\perp^2}\Big(\hat{k}^3\hat{k}^b - \delta_{b3}\Big)\Big(\hat{k}^3\hat{k}^d - \delta_{d3}\Big) \\
&= \frac{k_3^2}{k_\perp^2}\begin{pmatrix} k_1^2 & k_1 k_2 & 0 \\ k_1 k_2 & k_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & k_1 k_3 \\ 0 & 0 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & -k_\perp^2 \end{pmatrix}
\end{aligned} \tag{b}
$$

$$
u_a\, v_c - v_a\, u_c = \epsilon_{ija}\hat{k}^a = \begin{pmatrix} 0 & k_3 & -k_2 \\ -k_3 & 0 & k_1 \\ k_2 & -k_1 & 0 \end{pmatrix} \tag{c}
$$

In eq. (b) we used the identity $\epsilon_{b\rho\sigma}\epsilon_{\sigma\alpha 3} = \delta_{\rho 3}\delta_{b\alpha} - \delta_{b3}\delta_{\rho\alpha}$. Recall that $k_\perp^2 = 1 - \hat{k}_3^2 = \hat{k}_1^2 + \hat{k}_2^2$. If we sum the first two contributions we get

$$
\begin{aligned}
&\frac{1}{k_\perp^2}\begin{pmatrix} k_2^2 & -k_2\, k_1 & 0 \\ -k_1\, k_2 & k_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \left(\frac{1}{k_\perp^2} - 1\right)\begin{pmatrix} k_1^2 & k_1 k_2 & 0 \\ k_1 k_2 & k_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & k_1 k_3 \\ 0 & 0 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & -k_\perp^2 \end{pmatrix} \\
&= \frac{1}{k_\perp^2}\begin{pmatrix} k_2^2 & -k_2\, k_1 & 0 \\ -k_1\, k_2 & k_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{k_\perp^2}\begin{pmatrix} k_1^2 & k_1 k_2 & 0 \\ k_1 k_2 & k_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} k_1^2 & k_1 k_2 & k_1 k_3 \\ k_1 k_2 & k_2^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & -k_\perp^2 \end{pmatrix} \\
&= \frac{1}{k_\perp^2}\begin{pmatrix} k_1^2 + k_2^2 & 0 & 0 \\ 0 & k_1^2 + k_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} k_1^2 & k_1 k_2 & k_1 k_3 \\ k_1 k_2 & k_2^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & -k_\perp^2 \end{pmatrix}
\end{aligned}
$$

or, equivalently

$$
\begin{aligned}
u_a\, u_c + v_a\, v_c &= \frac{1}{u^2}\begin{pmatrix} k_1^2 + k_2^2 & 0 & 0 \\ 0 & k_1^2 + k_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} k_1^2 & k_1 k_2 & k_1 k_3 \\ k_1 k_2 & k_2^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & -u^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} k_1^2 & k_1 k_2 & k_1 k_3 \\ k_1 k_2 & k_2^2 & k_2 k_3 \\ k_3 k_1 & k_3 k_2 & k_3^2 \end{pmatrix} = \delta_{ij} + \hat{k}_i\,\hat{k}_j
\end{aligned} \tag{a+b}
$$

In conclusion we have

$$
\begin{aligned}
\epsilon_a(\hat{k})\,\epsilon_c(-\hat{k}) &= -\frac{1}{2}\Big[u_a\, u_c + v_a\, v_c - i\lambda\,(u_a\, v_c - v_a\, u_c)\Big] \\
&= -\frac{1}{2}\Big[\delta_{ac} + \hat{k}_a\,\hat{k}_c - i\lambda\,\epsilon_{ace}\,\hat{k}^e\Big]
\end{aligned} \tag{A.27}
$$

Replacing the identity above in eq. (A.25) we have the result eq. (A.23). The proof is completed.

$$\begin{aligned} \mathrm{e}_{ab,\lambda}(\hat{k})\, \mathrm{e}_{cd,\lambda}(-\hat{k}) &= \epsilon_a(\hat{k})\, \epsilon_c(-\hat{k})\, \epsilon_b(\hat{k})\, \epsilon_d(-\hat{k}) \\ &= \frac{1}{4}\Big[\delta_{ac} + \hat{k}_a\, \hat{k}_c - i\lambda\, \epsilon_{ace}\, \hat{k}^e\Big]\Big[\delta_{bd} + \hat{k}_b\, \hat{k}_d - i\lambda\, \epsilon_{bdf}\, \hat{k}^f\Big] \end{aligned} \tag{A.28}$$

This result is consistent with [46, App. A].

A.5 "Polarisation-trace" identity II: a longer proof for eq. (B.51)

In page 92, we used the identity above to obtain eq. (B.50). There is yet another way to obtain eq. (B.51) without using eq. (A.23). We can make use of the fact that, if we choose a spherical coordinate set in which to perform the integral, and moreover we orient the z-axis parallel to the wave-vector $\vec{k}$, then the only dependence of the integrand from the azimuthal angle ϕ is contained in the polarisation vectors.
Doing so, we can rewrite the linear polarisations as

$$\mathrm{e}^{\lambda}_{ij}(\hat{k})\, \mathrm{e}^{\sigma}_{kl}(-\hat{k})\, p_i p_j p_k p_l \tag{A.29}$$

where one uses the reality condition $\mathrm{e}^{(R)}_{ij}(-\hat{k}) = \mathrm{e}^{(L)}_{ij}(\hat{k})$ and expresses the helicity basis in term of the linear polarization one (A.22). But this is

$$\begin{aligned} \mathrm{e}^{(+)}_{ij}(\hat{k})\; p_i\, p_j &= \frac{1}{\sqrt{2}}\Big[u_i(\hat{k})\, u_j(\hat{k}) - v_i(\hat{k})\, v_j(\hat{k})\Big]\; p_i\, p_j \\ &= \frac{p^2}{\sqrt{2}} \sin^2\theta \left[\cos^2\phi - \sin^2\phi\right] \\ &= \frac{p^2}{\sqrt{2}} \sin^2(\theta) \cos(2\phi) \end{aligned} \tag{A.30}$$

$$\begin{aligned} \mathrm{e}^{(\times)}_{ij}(\hat{k})\; p_i\, p_j &= \frac{1}{\sqrt{2}}\Big[u_i(\hat{k})\, v_j(\hat{k}) + v_i(\hat{k})\, u_j(\hat{k})\Big]\; p_i\, p_j \\ &= \frac{p^2}{\sqrt{2}} \sin^2\theta\; 2\cos\phi\, \sin\phi \\ &= \frac{p^2}{\sqrt{2}} \sin^2(\theta)\; \sin(2\phi) \end{aligned} \tag{A.31}$$

where we used

$$\begin{aligned} \vec{p}\cdot\hat{k} &= p\cos\theta \\ \hat{u}(\hat{k})\cdot\vec{p} &= p\sin\theta\cos\phi \\ \hat{v}(\hat{k})\cdot\vec{p} &= p\sin\theta\sin\phi \end{aligned} \tag{A.32}$$

The eq. (A.29) then becomes

$$\begin{aligned} \frac{1}{2}\Big[\mathrm{e}^{(+)}_{ij}(\hat{k}) + i\lambda \mathrm{e}^{(\times)}_{ij}(\hat{k})\Big]\; \Big[\mathrm{e}^{(+)}_{kl}(\hat{k}) - i\sigma \mathrm{e}^{(\times)}_{kl}(\hat{k})\Big] p_i p_j p_k p_l &= \frac{1}{4}\Big[\mathrm{e}^{(+)}\mathrm{e}^{(+)} + \lambda\sigma \mathrm{e}^{(\times)}\mathrm{e}^{(\times)} + i(\lambda-\sigma)\mathrm{e}^{(+)}\mathrm{e}^{(\times)}\Big] \\ &= \frac{p^4}{4}\sin^4(\theta)\, \Big[\delta_{\lambda\sigma} + (1-\delta_{\lambda\sigma})\, e^{\pm 4\, i\, \phi}\Big] \\ &= \frac{p^4}{4}\sin^4(\theta) \begin{cases} \cos^2(2\phi) + \sin^2(2\phi) & \lambda = \sigma \\ \cos^2(2\phi) - \sin^2(2\phi) \pm 2i\, \cos(2\phi)\sin(2\phi) & \lambda \neq \sigma \end{cases} \end{aligned} \tag{A.33}$$

In conclusion, the case $\lambda \neq \sigma$ implies that the vanishing of the integral $\int_0^{2\pi} e^{\pm 4\,i\,\phi} = 0$ and we recover the same result we would obtain using eq. (A.23).

Appendix B

Cosmological gravitational waves

The purpose of this appendix is to support the contents of the chapter 5. It provides additional details for the calculations which have been first introduced there but, for the sake of readability, were omitted.

B.1 The perturbed metric

Let us recall here some fundamental quantities we use in the following. We start from the longitudinal-gauge (see eq. (3.30)) perturbed line-element in conformal coordinates

$$ds^2 = a^2 \left\{-\left(1+2\Phi\right) d\tau^2 + \left[\left(1-2\Psi\right)\delta_{ij} + h_{ij}\right] dx^i dx^j\right\} \tag{B.1}$$

In the study of the first-order GWs as well as the second-order sourced GWs we are concerned with the first-order scalar perturbations. We always neglect second-order scalar perturbations and, for convenience, we consider from the beginning to hold $\Psi = \Phi$. Analogously, we always neglect the second-order vector perturbation - the first are zero in virtue of the gauge choice.
In compact form, we write the perturbed metric tensor as

$$g_{\mu\nu} = a^2 \begin{pmatrix} -(1+2\Phi) & 0 \\ 0 & (1-2\Phi)\,\delta_{ij} + h_{ij} \end{pmatrix} \qquad g^{\mu\nu} \simeq a^{-2} \begin{pmatrix} -(1-2\Phi) & 0 \\ 0 & (1+2\Phi)\,\delta^{ij} + h^{ij} \end{pmatrix} \tag{B.2}$$

on the right, we have written the inverse-metric to the lowest order in the perturbations. Note that $h^{ij} \equiv -h_{ij}$.

Metric determinant Under the assumption $\Psi = \Phi$, we expand the square root of minus the metric determinant in terms of the scalar & tensor perturbations. This expansion is based on the algebraic identities

$$\det \mathbb{A} = e^{\,\mathrm{Tr}[\ln \mathbb{A}]}\,, \quad e^x = 1 + x + x^2 + x^3 + \mathrm{O}\left(x^4\right)\,, \quad \ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \mathrm{O}\left(x^4\right) \tag{B.3}$$

The longitudinal-gauge (eq. (3.30)) metric determinant is

$$\sqrt{-g} = a^4 \left[1 - 2\Phi - 2\Phi^2 + 4\Phi^3 - 2\Phi^4 - \frac{1}{4}h_{ij}h_{ij} - \frac{1}{2}h_{ij}h_{ij}\Phi + 4\Phi^5 + \mathrm{O}\begin{pmatrix}\text{higher} \\ \text{orders}\end{pmatrix}\right] \tag{B.4}$$

the unwritten contributions are n-linear terms with $n > 5$.

Klein-Gordon equation

Background - The background Klein-Gordon equation is

$$\varphi_0'' + 2\mathcal{H}\,\varphi_0' + a^2\,V' = 0 \tag{B.5}$$

Linear - The Klein-Gordon equation at first order in the perturbations is

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \partial_i\partial^i\delta\varphi - \varphi_0'\Phi' - 3\varphi_0'\,\Psi' = -a^2\,V''\,\delta\varphi - 2a^2\,\Psi\,V' \tag{B.6}$$

Einstein equation

Background - The non-trivial components of the background Einstein equation are

$$G_{00} = \frac{1}{M_p^2}T_{00} \qquad 3M_p^2\mathcal{H}^2 = \frac{1}{2}{\varphi_0'}^2 + a^2V \tag{B.7}$$

$$G_{ii} = \frac{1}{M_p^2}T_{ii} \qquad M_p^2\left[\mathcal{H}^2 - 2\frac{a''}{a}\right] = \frac{1}{2}{\varphi_0'}^2 - a^2V \tag{B.8}$$

Linear - The Einstein equation at first order in the perturbations is

$$G_{00} = \frac{1}{M_p^2}T_{00} \qquad \nabla^2\Phi - 3\mathcal{H}\Phi' = \frac{1}{M_p^2}\left[\frac{1}{2}\varphi_0'\delta\varphi' + a^2\left(\frac{1}{2}\frac{\partial V}{\partial\varphi_0}\delta\varphi + \Phi V\right)\right] \tag{B.9}$$

$$G_{0i} = \frac{1}{M_p^2}T_{0i} \qquad \delta\varphi = \frac{2\,M_p^2}{\varphi_0'}\left(\Phi' + \mathcal{H}\,\Phi\right) \tag{B.10}$$

$$G_{ij} = \frac{1}{M_p^2}T_{ij} \quad (i \neq j) \qquad \Psi = \Phi \tag{B.11}$$

$$G_{ii} = \frac{1}{M_p^2}T_{ii} \qquad \Phi'' + 3\mathcal{H}\Phi' = \frac{1}{M_p^2}\left[\frac{1}{2}\varphi_0'\delta\varphi' - a^2\left(\frac{1}{2}\frac{\partial V}{\partial\varphi_0}\delta\varphi + \Phi V\right)\right] \tag{B.12}$$

B.2 The source. Expanding the action

In this section we report in full detail the steps that we omitted in sec. 5.9. We divided the calculation in two steps: the contribution from the geometry $\sim \sqrt{-g}\,R$ and the other from the inflaton. Our theory is defined by the action eq. (2.10), which we recall below for completeness

$$S \equiv \int d^4x\mathcal{L} = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - V\right] \tag{B.13}$$

1/2 | Let us start from the geometry contribution: we have only terms like $h\Phi\Phi$. If we expand the metric determinant with eq. (B.4), this contribution is of the form

$$\frac{M_p^2}{2}\sqrt{-g}\,R \supset \frac{M_p^2}{2}a^4\,\left[R|_{h\,\Phi^2} - 2\Phi R|_{h\,\Phi} - 2\Phi^2 R|_h\right] \tag{B.14}$$

where for instance $R|_{h\Phi^2}$ is the Ricci scalar at third-order in perturbations which contains the fields h, Φ, Φ - and derivatives thereof. In order to have a scalar, there are few possible structures

$$\begin{aligned} R|_{h\,\Phi^2} &= C_1\,(a)\,\partial_i\Phi\partial_j\Phi h_{ij} + C_2\,(a)\,\Phi\partial_i\partial_j\Phi h_{ij} \\ R|_{h\,\Phi} &= 0 \\ R|_h &= 0 \end{aligned} \tag{B.15}$$

In principle, one should allow also for time derivatives. However, if the reader would retrace the steps to obtain the Ricci scalar - e.g. as the contraction of the Ricci tensor $R = g^{\mu\nu}R_{\mu\nu}$, which itself is the contraction of the Riemann tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$ - one would realize that (a) the term $R|_{h\,\Phi^2}$ contains two derivatives and (b) if either one or both are temporal then, given the properties of the tensor perturbation h_{ij}, it would vanish. In fact the direct calculation confirms the one we wrote is the correct form, with $C_1 = -\frac{6}{a^2}$ and $C_2 = -\frac{8}{a^2}$.
Finally, the contribution from the geometry is

$$\begin{aligned} \frac{M_p^2}{2}\sqrt{-g}\,R &\supset \frac{M_p^2}{2}a^4\left(-\frac{6}{a^2}\partial_i\Phi\partial_j\Phi h_{ij} - \frac{8}{a^2}\Phi\partial_i\partial_j\Phi h_{ij}\right) \\ &= -M_p^2\,a^2\,h_{ij}\,(3\partial_i\Phi\partial_j\Phi + 4\Phi\partial_i\partial_j\Phi) \end{aligned} \tag{B.16}$$

2/2 | The contribution from the matter action is easy. The background fields are homogeneous and the tensor perturbation is in the spatial components h_{ij}. The potential $\sqrt{-g}V$ has not any of the terms we are interested, because it does not contain the tensor perturbation *linearly*. Therefore we conclude that the desired contribution, to the interaction we are looking for, comes from the kinetic term. Also, the tensor perturbation must come from the inverse metric $g^{\mu\nu}$, that is

$$\begin{aligned} &-\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \supset \sqrt{-g}\frac{1}{2}\frac{h_{ij}}{a^2}\partial_i\varphi\partial_j\varphi \supset a^4\frac{1}{2}\frac{h_{ij}}{a^2}\partial_i\delta\phi\partial_j\delta\phi \\ &\qquad = \frac{a^2}{2}h_{ij}\left(\frac{2\,M_p^2}{\varphi_0'}\right)^2\partial_i\Big(\Phi' + \mathcal{H}\Phi\Big)\,\partial_j\Big(\Phi' + \mathcal{H}\Phi\Big) \end{aligned} \tag{B.17}$$

here notice that the determinant was taken at the background level $\sqrt{-g} \to a^4$ and we used the background equation eq. (B.10).

Summing the two contributions, we finally obtain the desired interaction lagrangian third-order in the cosmological perturbations

$$\begin{aligned} \mathcal{L}^{\text{int}} &= -M_p^2\,a^2\,h_{ij}\left[3\partial_i\Phi\partial_j\Phi + 4\Psi\partial_i\partial_j\Phi - \frac{2\,M_p^2}{\varphi_0'^2}\partial_i\Big(\Phi' + \mathcal{H}\Phi\Big)\,\partial_j\Big(\Phi' + \mathcal{H}\Phi\Big)\right] \\ &= -M_p^2\,a^2\,h_{ij}\left[3\partial_i\Phi\partial_j\Phi + 4\Phi\partial_i\partial_j\Phi - \frac{\mathcal{H}^2}{\mathcal{H}^2 - \mathcal{H}'}\partial_i\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\,\partial_j\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\right] \end{aligned} \tag{B.18}$$

The last line was obtained with $2\, M_p^2/\varphi_0'^2 = \frac{1}{\mathcal{H}^2-\mathcal{H}'}$, which in turn can be found by combining eq.s(B.7)-(B.8). Specializing to the radiation dominated era, we obtain

$$\text{RD era}: \qquad \mathcal{H} = \frac{1}{\tau}\,, \quad \frac{\mathcal{H}^2}{\mathcal{H}^2-\mathcal{H}'} = \frac{1}{2} \tag{B.19}$$

$$\Rightarrow \qquad \mathcal{L}^{\text{int}} = -\frac{1}{2}\, M_p^2\, a^2 h_{ij}\left[-2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right] \tag{B.20}$$

where we have integrated by parts.

B.3 Equation of motion for the sourced GW

The Euler-Lagrange equation for the GW is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu h_{ij})} - \frac{\partial \mathcal{L}}{\partial h_{ij}} = 0 \tag{B.21}$$

Starting from the action of eq. (5.40), we have

$$-\left(\frac{a^2}{2}\, h'_{ij}\right)' + \frac{a^2}{2}\,\nabla^2 h_{ij} + a^2\left[2\partial_i\Phi\partial_j\Phi + \partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right]^{\text{TT}} = 0 \tag{B.22}$$

or

$$h''_{ij} + 2\mathcal{H}\, h'_{ij} - \nabla^2 h_{ij} = -2\left[-2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right]^{\text{TT}} \tag{B.23}$$

where the 'TT' on the source term emphasizes that only the transverse and trace-less part of the source is projected over the equation for h_{ij}.

To check consistency with eq.s(13)-(14) of [23], which is rather obscure in the form above, we add and subtract the term $(4\Phi\partial_i\partial_j\Phi + 2\partial_i\Phi\,\partial_j\Phi)$ and we simplify eq. (B.23) to

$$h''_{ij} + 2\mathcal{H}\, h'_{ij} - \nabla^2 h_{ij} = -2\left[-4\partial_i\left(\Phi\partial_j\Phi\right) + 4\Phi\partial_i\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right]^{TT} \tag{B.24}$$

The first term in square bracket is longitudinal, and so it simplifies away in the TT projection. Said it differently, if we go into momentum space the total derivative factorizes the same momentum as the l.h.s. namely the momentum of the GW. When we contract this with the GW polarization operator (to get the TT part) we obtain zero. Therefore eq. (B.24) is equivalent to

$$h''_{ij} + 2\mathcal{H}\, h'_{ij} - \nabla^2 h_{ij} = -2\left[4\Phi\partial_i\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right]^{TT} \tag{B.25}$$

B.3.1 Crosscheck from the Einstein equations

As crosscheck, let us reproduce the eq. (B.25) in the "geometrical way" - that is, from the linearised Einstein equation. We recall the stress-energy tensor eq. (2.12). It is easier to use the Einstein equation with the up-down indices. For our purpose, we need the $\mathcal{O}(h_{ij})$ and $\mathcal{O}(\Phi^2)$ terms in this computation. We have

$$G^i{}_j\Big|_{TT} = \frac{1}{2a^2}h''_{ij} + \frac{a'}{a^3}h'_{ij} - \frac{1}{2a^2}\partial_k\partial_k\, h_{ij} \tag{B.26}$$

$$-\frac{T^i{}_j}{M_p^2}\Big|_{TT} = 0 \tag{B.27}$$

$$G^i{}_j\Big|_{TT} = \frac{2}{a^2}\partial_i\Phi\partial_j\Phi + \frac{4}{a^2}\Phi\partial_i\partial_j\Phi = \frac{2}{a^2}\Big(\partial_i\partial_j\left(\Phi^2\right) - \partial_i\Phi\partial_j\Phi\Big) \tag{B.28}$$

$$-\frac{T^i{}_j}{M_p^2}\Big|_{TT} = -\frac{\delta\partial_i\varphi\,\delta\partial_j\varphi}{a^2M_p^2} = -\frac{4\,M_p^2}{a^2\,\varphi_0'^2}\,\partial_i\Big(\Phi' + \mathcal{H}\Phi\Big)\partial_j\Big(\Phi' + \mathcal{H}\Phi\Big) \tag{B.29}$$

Putting all together and using again the background equations for $2\,M_p^2/\varphi_0'^2 = \frac{1}{\mathcal{H}^2-\mathcal{H}'}$ we get

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \partial_k\partial_k\, h_{ij} + 4\Big(\partial_i\partial_j\left(\Phi^2\right) - \partial_i\Phi\,\partial_j\Phi\Big) - \frac{4\mathcal{H}^2}{\mathcal{H}^2-\mathcal{H}'}\partial_i\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right) = 0 \tag{B.30}$$

and once we detail to the radiation phase using (B.19) we obtain

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \partial_k\partial_k\, h_{ij} = -2\left[2\,\partial_i\partial_j\left(\Phi^2\right) - 2\,\partial_i\Phi\,\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\right] \tag{B.31}$$

or

$$h''_{ij} + 2\mathcal{H}\,h'_{ij} - \partial_k\partial_k\, h_{ij} = -2\left[4\Phi\partial_i\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\right]^{TT} \tag{B.32}$$

where we also recalled that we only want the TT part of the source. This is precisely eq. (B.25). This agrees with eqs. (13) and (14) of [23], once we consider that the authors use a different normalization for h_{ij} (their field is 2 times the h_{ij} we use here).

B.4 Perturbative solution with the Green function method

Let us consider the momentum space Einstein equation for the GW, using eq. (A.8) we have

$$\begin{aligned} h''_\lambda(\tau,\,\mathbf{k}) + \frac{2}{\tau}\,h'_\lambda(\tau,\,\mathbf{k}) + k^2\,h_\lambda(\tau,\,\mathbf{k}) = \\ -\,2\sum_{i,j}\mathrm{e}^*_{ij,\lambda}(\hat{k})\int d^3x\; e^{-i\vec{k}\cdot\vec{x}}\left[4\Phi\partial_i\partial_j\Phi + 2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}} + \Phi\right)\right] \end{aligned} \tag{B.33}$$

The field h_{ij} is transverse-traceless: only the TT-part of the source in r.h.s. of eq. (B.23) matter. Here we dropped the 'TT' label, because in momentum space the factor $\mathrm{e}^*_{ij,\lambda}(\hat{k})$ projects the longitudinal component only out of the source.

It is more convenient to solve the equation above in terms of the field $v^\lambda(\tau, \mathbf{k}) = \frac{M_p}{2}\, a(\tau)\, h^\lambda(\tau, \mathbf{k})$. The equation in terms of v_λ is

$$\left(\partial_\tau^2 + k^2\right) v_\lambda(\tau, \mathbf{k}) \;=\; \mathcal{S}_\lambda(\tau, \mathbf{k}) \tag{B.34}$$

$$\mathcal{S}_\lambda(\tau, \mathbf{k}) = \frac{4}{9} M_p\, a(\tau) \int \frac{d^3\vec{q}}{(2\pi)^3}\, \zeta(\mathbf{q})\, \zeta(\mathbf{k}-\mathbf{q})\, \mathcal{Q}_\lambda(\tau,\, \mathbf{k},\, \mathbf{q}) \tag{B.35}$$

where also we have expressed the scalar perturbation Φ in terms of the curvature perturbation ζ, by means of eq. (5.41). The function $\mathcal{Q}_\lambda$ was given in eq. (5.45).

The classical equation of motion can be solved in perturbation theory by using the Green function method. The solution is written as a weighted time-integral of the source, where the weight-function is the Green function. The Green function is by definition the kernel of the differential operator of the homogeneous (e.g. $\mathrm{S}_\lambda = 0$) equation (B.34).

B.4.1 Green function

Let us evaluate the Green function. The Green function for the field v^λ is easy to find, because the homogeneous Einstein equation eq. (B.34) in the radiation dominated era is

$$\left(\partial_\tau^2 + k^2\right) v^\lambda(\tau, \mathbf{k}) \;=\; 0 \tag{B.36}$$

This is the equation of motion of a classical harmonic oscillator. Note that it holds irrespective of the tensor mode being sub/super-Hubble. The Green function is the solution of

$$\left(\partial_\tau^2 + k^2\right) \mathcal{G}_k^\lambda(\tau, \tau') = \delta\left(\tau - \tau'\right) \tag{B.37}$$

The differential operator is invariant under rotations. It means that the solution depends on the modulo $k \equiv |\mathbf{k}|$ as we have indicated. This equation can be solved either directly in the sense of the distributions or by going in the frequency-domain using

$$\begin{aligned} \mathcal{G}_k^\lambda\left(\tau,\, \tau'\right) &= \int \frac{dE}{2\pi}\, e^{iE(\tau-\tau')}\, \mathcal{G}_k^\lambda(E) \\ \delta\left(\tau - \tau'\right) &= \int \frac{dE}{2\pi}\, e^{iE(\tau-\tau')} \end{aligned} \tag{B.38}$$

Here we used the translational invariance of the r.h.s. in eq. (B.37), which imply that the solution depends on the difference $(\tau' - \tau)$, rather than τ', τ independently. The frequency-domain solution is simply the algebraic reciprocal (left)

$$\mathcal{G}_k^\lambda(E) = -\frac{1}{E^2 - k^2} \qquad\qquad \mathcal{G}_k^\lambda(\tau, \tau') = \theta\left[\tau - \tau'\right]\, \frac{\sin\left[k(\tau - \tau')\right]}{k} \tag{B.39}$$

The inverse Fourier transform - with the adequate boundary prescriptions - is the sought-after τ-space Green function. The step-function θ is usually ascribed to causality. From the mathematical point of view, it derives from the contour choice.

B.4.2 The 2-point GW function with the Green function method

In this section we calculate the 2-point GW correlator with the Green function method.
The solution of eq. (B.34) is the convolution between the Green function and the source, that is

$$\begin{aligned} v^{\lambda}(\tau,\mathbf{k}) &= \int_{-\infty}^{\infty} d\tau'\, \mathcal{G}_k^{\lambda}(\tau,\tau')\, \mathrm{S}_{\lambda}(\tau',\mathbf{k}) \\ &= \int_{-\infty}^{\tau} d\tau' \,\frac{\sin\left[k(\tau-\tau')\right]}{k}\, \mathrm{S}_{\lambda}(\tau',\mathbf{k}) \\ &= \frac{\sin(k\tau)}{k}\int^{\tau} d\tau' \cos(k\tau')\, \mathrm{S}_{\lambda}(\tau',\mathbf{k}) \;-\; \frac{\cos(k\tau)}{k}\int^{\tau} d\tau' \sin(k\tau')\, \mathrm{S}_{\lambda}(\tau',\mathbf{k}) \end{aligned} \tag{B.40}$$

where in the last line we used the trigonometric identity $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$ for future convenience. The conformal time is here conventionally normalized such that the RD era begins at $\tau = 0$. We assume to turn on the source during the radiation era. Therefore the lower integration extreme of eq. (B.40) is implicitly considered as $\tau_0 = 0$. The solution eq. (B.40) is rewritten as

$$v^{\lambda}(\tau,\mathbf{k}) = \frac{M_p}{9}\frac{1}{k^2}\int \frac{d^3\vec{q}}{(2\pi)^3}\,\zeta(\mathbf{q})\,\zeta(\mathbf{k}-\mathbf{q})\,\mathrm{e}^*_{\lambda}(\hat{k},\,\mathbf{q})\,\Big[\sin(k\tau)\,\mathcal{I}_s\left(x,y\right) + \cos(k\tau)\,\mathcal{I}_c\left(x,y\right)\Big]_{x=\frac{q}{k},\,y=\frac{|\mathbf{k}-\mathbf{q}|}{k}} \tag{B.41}$$

where $\mathrm{e}^*_{\lambda}(\hat{k},\,\mathbf{q})$ is given by (5.46) and further we defined the adimensional functions

$$\begin{aligned}\mathcal{I}_c\left(x,y\right) &\equiv -4k \\ &\times \int_0^{\tau} d\tau' a(\tau')\,\sin(k\tau')\left[2T\left(k\tau' x\right)T\left(k\tau' y\right) + \left(\frac{k\,x\,T'\left(k\tau' x\right)}{\mathcal{H}} + T\left(k\tau' x\right)\right)\left(\frac{k\,y\,T'\left(k\tau' y\right)}{\mathcal{H}} + T\left(k\,\tau'\,y\right)\right)\right]\end{aligned} \tag{B.42}$$

$$\begin{aligned}\mathcal{I}_s\left(x,y\right) &\equiv 4k \\ &\times \int_0^{\tau} d\tau' a(\tau')\,\cos(k\tau')\left[2T\left(k\tau' x\right)T\left(k\tau' y\right) + \left(\frac{k\,x\,T'\left(k\tau' x\right)}{\mathcal{H}} + T\left(k\tau' x\right)\right)\left(\frac{k\,y\,T'\left(k\tau' y\right)}{\mathcal{H}} + T\left(k\,\tau'\,y\right)\right)\right]\end{aligned} \tag{B.43}$$

For convenience, we encoded the momentum dependence of $\mathcal{I}_c$, $\mathcal{I}_s$ in x, y defined contextually in eq. (B.41). Also, notice the appearance of the transfer function $T(k\tau)$ (see eq. (5.42)), the prime here denotes the derivative with respect to the argument - i.e. $T'(z) \equiv \partial_z\, T(z)$.
It is worth to note that the functions $\mathcal{I}_c$, $\mathcal{I}_s$ do indeed depend also on the conformal time τ, because of the τ'-integral domain. However, we are interested to the solution at late-time $\tau \gg 1$ when this dependence can be neglected and, for any practical purpose, the τ'-integrals can be effectively evaluated between $[0,\infty]$, for instance see [36].

The 2-point function reads

$$\begin{aligned}&\Big\langle v_{\lambda}(\tau,\mathbf{k}_1)\, v_{\sigma}(\tau,\mathbf{k}_2)\Big\rangle = \\ &\left(\frac{M_p}{9}\frac{1}{k_1^2}\right)^2 \int \frac{d^3\vec{q}_1}{(2\pi)^3}\int \frac{d^3\vec{q}_2}{(2\pi)^3}\,\Big\langle \zeta(\mathbf{q}_1)\,\zeta(\mathbf{k}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{k}_2-\mathbf{q}_2)\Big\rangle\; \mathrm{e}^*_{\lambda}(\hat{k}_1,\,\mathbf{q}_1)\,\mathrm{e}^*_{\sigma}(\hat{k}_2,\,\mathbf{q}_2) \\ &\quad\times\Big[\sin(k_1\tau)\,\mathcal{I}_s\left(x_1,y_1\right) + \cos(k_1\tau)\,\mathcal{I}_c\left(x_1,y_1\right)\Big]\Big[\sin(k_2\tau)\,\mathcal{I}_s\left(x_2,y_2\right) + \cos(k_2\tau)\,\mathcal{I}_c\left(x_2,y_2\right)\Big]\end{aligned} \tag{B.44}$$

where $(x_i,\, y_i) = \left(\frac{q_i}{k_i},\, \frac{|\mathbf{q}_i - \mathbf{k}_i|}{k_i}\right)$ with $i = 1,\, 2$. The simplification of eq. (B.44) will proceed in two steps.

Wick expansion First, we consider the 4-point scalar correlator $\langle\zeta\,\zeta\,\zeta\,\zeta\rangle$. Let us recall the definition eq. (3.14) for the power spectrum - in this context, the angle brackets must be considered in the sense of ensemble average

$$\Big\langle \zeta(\mathbf{p}_1)\,\zeta(\mathbf{p}_2)\Big\rangle = (2\pi)^3\delta^{(3)}(p_1+p_2)\,\frac{2\pi^2}{p_1^3}\,\mathcal{P}_\zeta(\tau,\mathbf{p}_1) \tag{B.45}$$

We are concerned with gaussian scalar perturbations (ansatz). We use the Wick theorem to write the 4-point scalar correlator in terms of products of 2-point correlators. Also, we write the resulting expression in terms of the scalar power spectrum above. We have

$$\begin{aligned}\Big\langle \zeta(\mathbf{q}_1)\,\zeta(\mathbf{k}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{k}_2-\mathbf{q}_2)\Big\rangle &= \Big\langle \zeta(\mathbf{q}_1)\,\zeta(\mathbf{k}_1-\mathbf{q}_1)\Big\rangle\Big\langle \zeta(\mathbf{q}_2)\,\zeta(\mathbf{k}_2-\mathbf{q}_2)\Big\rangle\\ &+\Big\langle \zeta(\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\Big\rangle\Big\langle \zeta(\mathbf{k}_1-\mathbf{q}_1)\,\zeta(\mathbf{k}_2-\mathbf{q}_2)\Big\rangle\\ &+\Big\langle \zeta(\mathbf{q}_1)\,\zeta(\mathbf{k}_2-\mathbf{q}_2)\Big\rangle\Big\langle \zeta(\mathbf{k}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\Big\rangle\end{aligned} \tag{B.46}$$

Let us consider the three terms in the r.h.s. separately

$$\begin{aligned}\Big\langle \zeta(\mathbf{q}_1)\,\zeta(\mathbf{k}_1-\mathbf{q}_1)\Big\rangle\Big\langle \zeta(\mathbf{q}_2)\,\zeta(\mathbf{k}_2-\mathbf{q}_2)\Big\rangle &= (2\pi)^3(2\pi)^3\,\delta^{(3)}(k_1)\delta^{(3)}(k_2)\\ &\times\frac{2\pi^2}{q_1^3}\frac{2\pi^2}{q_2^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(q_2)\end{aligned} \tag{B.47}$$

$$\begin{aligned}\Big\langle \zeta(\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\Big\rangle\Big\langle \zeta(\mathbf{k}_1-\mathbf{q}_1)\,\zeta(\mathbf{k}_2-\mathbf{q}_2)\Big\rangle &= (2\pi)^3(2\pi)^3\,\delta^{(3)}(q_1+q_2)\delta^{(3)}(k_1+k_2-q_1-q_2)\\ &\times\frac{2\pi^2}{q_1^3}\frac{2\pi^2}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\end{aligned} \tag{B.48}$$

$$\begin{aligned}\Big\langle \zeta(\mathbf{q}_1)\,\zeta(\mathbf{k}_2-\mathbf{q}_2)\Big\rangle\Big\langle \zeta(\mathbf{k}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\Big\rangle &= (2\pi)^3(2\pi)^3\,\delta^{(3)}(q_1+k_2-q_2)\delta^{(3)}(k_1-q_1+q_2)\\ &\times\frac{2\pi^2}{q_1^3}\frac{2\pi^2}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\end{aligned} \tag{B.49}$$

The first term gives zero for non-vanishing momentum, while the second and the third terms give equal contributions - as one can verify by exploiting the dummy integration variables in eq. (B.44). The reader may notice that of the two δ-functions, one can always be factorized because it depends from the external momenta, i.e. $\delta^{(3)}(k_1+k_2)$. Indeed, the remaining δ-function reduces the two integrations $d^3\vec{q}_1\,d^3\vec{q}_2\,\delta^{(3)}(q_1+q_2)\to d^3\vec{q}_1$ to a non-trivial one to be performed.

Vector identity The second simplification we make regards the polarisation vectors. Once we take into account of the deltas in eq. (B.44), we have the product

$$\mathrm{e}^*_\lambda(\hat{k}_1,\,\mathbf{q}_1)\,\mathrm{e}^*_\sigma(-\hat{k}_1,\,-\mathbf{q}_1) = \delta_{\lambda\sigma}\sum_{ij}\sum_{kl}\,\mathrm{e}^*_{ij,\lambda}(\hat{k}_1)\,\mathrm{e}^*_{kl,\lambda}(-\hat{k}_1)\;q_{1i}\,q_{1j}\,q_{1k}\,q_{1l}$$

on the r.h.s. we used the definition eq. (5.46). We use the identity eq. (A.23), to obtain

$$\begin{aligned}\sum_{ij}\sum_{kl}\,\mathrm{e}^*_{ij,\lambda}(\hat{k}_1)\,\mathrm{e}^*_{kl,\lambda}(-\hat{k}_1)\;q_{1i}\,q_{1j}\,q_{1k}\,q_{1l} &= \frac{1}{4}\left[q_1^2-\Big(\hat{k}_1\cdot\vec{q}_1\Big)^2\right]^2\\ &=\frac{1}{4}\,q_1^4\,\Big(1-\cos^2\theta\Big)^2\\ &=\frac{1}{4}\,q_1^4\,\sin^4\theta\end{aligned} \tag{B.50}$$

where θ is the angle between the wavevector and the dummy integration variable $\vec{q}$.
In conclusion, we put all the pieces together and obtain

$$\Big\langle v_\lambda(\tau,\mathbf{k}_1)\, v_\sigma(\tau,\mathbf{k}_2)\Big\rangle = (2\pi)^3\, \delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)$$
$$2\left(\frac{M_p}{9}\frac{1}{k_1^2}\right)^2 \int \frac{d^3\vec{q}_1}{(2\pi)^3}\, \frac{2\pi^2}{q_1^3}\, \frac{2\pi^2}{|\mathbf{k}_1-\mathbf{q}_1|^3}\, \mathcal{P}_\zeta(q_1)\, \mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\, \frac{1}{4}\, q_1^4\, \sin^4\theta$$
$$\times \Big[\sin(k_1\tau)\,\mathcal{I}_s\,(x_1,y_1)+\cos(k_1\tau)\,\mathcal{I}_c\,(x_1,y_1)\Big]\Big[\sin(k_1\tau)\,\mathcal{I}_s\,(x_1,y_1)+\cos(k_1\tau)\,\mathcal{I}_c\,(x_1,y_1)\Big] \quad \text{(B.51)}$$

We go in polar coordinates and (trivially) integrate over the azimuthal angle, obtaining

$$\Big\langle v_\lambda(\tau,\mathbf{k}_1)\, v_\sigma(\tau,\mathbf{k}_2)\Big\rangle = (2\pi)^3\, \delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\, \left(\frac{M_p}{9}\frac{1}{k_1^2}\right)^2$$
$$\int_0^\infty dq_1 \int_0^\pi d\theta\; q_1^2 \sin\theta\, \frac{2\pi^2}{q_1^3}\, \frac{1}{|\mathbf{k}_1-\mathbf{q}_1|^3}\, \mathcal{P}_\zeta(q_1)\, \mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\, \frac{1}{4}\, q_1^4\, \sin^4\theta$$
$$\times \Big[\sin(k_1\tau)\,\mathcal{I}_s\,(x_1,y_1)+\cos(k_1\tau)\,\mathcal{I}_c\,(x_1,y_1)\Big]^2 \quad \text{(B.52)}$$

or

$$\Big\langle v_\lambda(\tau,\mathbf{k}_1)\, v_\sigma(\tau,\mathbf{k}_2)\Big\rangle = (2\pi)^3\, \delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\, \left(\frac{M_p}{9}\right)^2\, \frac{1}{4\,k_1}$$
$$\iint dq_1 d\theta\; q_1^2 \sin\theta\, \frac{1}{|\mathbf{k}_1-\mathbf{q}_1|^3}\, \mathcal{P}_\zeta(q_1)\, \mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\; q_1\, \sin^4\theta$$
$$\times \Big[\sin(k_1\tau)\,\mathcal{I}_s\,(x_1,y_1)+\cos(k_1\tau)\,\mathcal{I}_c\,(x_1,y_1)\Big]^2 \quad \text{(B.53)}$$

This result can be written in compact form by performing the variable change

$$\begin{cases} x \equiv \dfrac{q_1}{k_1} \\ y \equiv \dfrac{|\mathbf{q}_1-\mathbf{k}_1|}{k_1} \end{cases} \qquad \Rightarrow \qquad q_1^2\;\, dq_1\, d(\cos\theta) = k_1^3\, x\, y\;\; dx\, dy \quad \text{(B.54)}$$

note that these variables were first introduced for the functions $\mathcal{I}_{c,s}$. One can check that

$$\begin{aligned} y^2 &= 1 + \left(\frac{q_1}{k_1}\right)^2 - 2\frac{q_1}{k_1}\cos\theta \\ &= 1 + x^2 - 2x\cos\theta \end{aligned} \qquad \Rightarrow \qquad \cos\theta = \frac{1+x^2-y^2}{2x} \quad \text{(B.55)}$$

so that in the new variables the expression eq. (B.53) is

$$\Big\langle v_\lambda(\tau,\mathbf{k}_1)\, v_\sigma(\tau,\mathbf{k}_2)\Big\rangle = (2\pi)^3\, \delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k^3}\, \left(\frac{M_p}{9}\right)^2\, \frac{1}{4\,k_1}$$
$$\iint_{\mathcal{D}} dx\, dy\; k_1^3\, x\, y\; \frac{1}{(y\,k_1)^3}\; \mathcal{P}_\zeta(x\,k_1)\, \mathcal{P}_\zeta(y\,k_1)\; k_1 x\, \left[1 - \left(\frac{1+x^2-y^2}{2x}\right)^2\right]^2$$
$$\times \Big[\sin(k_1\tau)\,\mathcal{I}_s\,(x_1,y_1)+\cos(k_1\tau)\,\mathcal{I}_c\,(x_1,y_1)\Big]^2 \quad \text{(B.56)}$$

or

$$\Big\langle v_\lambda(\tau,\mathbf{k}_1)\, v_\sigma(\tau,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\left(\frac{M_p}{2}\right)^2\frac{1}{81}$$
$$\iint_{\mathcal{D}} dx\,dy\,\mathcal{P}_\zeta(x\,k_1)\,\mathcal{P}_\zeta(y\,k_1)\,\left(\frac{x}{y}\right)^2\left[1-\left(\frac{1+x^2-y^2}{2x}\right)^2\right]^2$$
$$\times\left[\sin(k_1\tau)\,\mathcal{I}_s\,(x,y)+\cos(k_1\tau)\,\mathcal{I}_c\,(x,y)\right]^2 \quad \text{(B.57)}$$

The domain of integration $\mathcal{D} = \Big\{(x,y)\in\mathbb{R}^+\times\mathbb{R}^+ \mid |1-x|<y<1+x\Big\}$ is shown in fig. B.1. Notice that the integral is symmetric around the axis $y = x$.

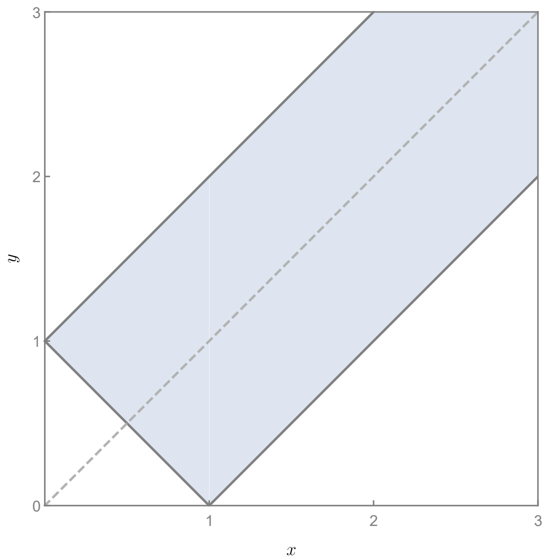


Figure B.1: The shadowed region is the domain of integration for the expression eq. (B.57).

The tensor 2-point correlator We have calculated the 2-point gravitational wave correlator in terms of the field v_λ. This choice was motivated by convenience. The 2-point GW correlator in terms of the tensor perturbation h_{ij} is given by

$$\Big\langle v^\lambda(\tau,\mathbf{k}_1)\, v^\sigma(\tau,\mathbf{k}_2)\Big\rangle = \left(\frac{M_p}{2}\,a(\tau)\right)^2\,\Big\langle h^\lambda(\tau,\mathbf{k}_1)\, h^\sigma(\tau,\mathbf{k}_2)\Big\rangle \quad \text{(B.58)}$$

so that we have

$$\Big\langle h_\lambda(\tau,\mathbf{k}_1)\, h_\sigma(\tau,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\frac{1}{a(\tau)^2}\,\frac{1}{81}$$
$$\iint_{\mathcal{D}} dx\,dy\,\mathcal{P}_\zeta(x\,k_1)\,\mathcal{P}_\zeta(y\,k_1)\,\left(\frac{x}{y}\right)^2\left[1-\left(\frac{1+x^2-y^2}{2x}\right)^2\right]^2$$
$$\times\left[\sin(k_1\tau)\,\mathcal{I}_s\,(x,y)+\cos(k_1\tau)\,\mathcal{I}_c\,(x,y)\right]^2 \quad \text{(B.59)}$$

This result is consistent with literature, i.e. see eq. (43) of [36]. The closed form for the functions $\mathcal{I}_c$, $\mathcal{I}_s$ is obtained by time-integrating between $[0,\infty]$. They are (see eq.(D.8) of [36])

$$\mathcal{I}_c(x,y) = \left(\frac{1}{H_{\rm inf}\, k\tau_*^2}\right)\left[-36\pi\frac{\left(d^2+s^2-2\right)^2}{\left(s^2-d^2\right)^3}\theta(s-1)\right]_{s\equiv\frac{x+y}{\sqrt{3}}\ ,\ d\equiv\frac{|x-y|}{\sqrt{3}}} \tag{B.60}$$

$$\mathcal{I}_s(x,y) = \left(\frac{1}{H_{\rm inf}\, k\tau_*^2}\right)\left[-36\frac{\left(d^2+s^2-2\right)}{\left(s^2-d^2\right)^2}\left(\frac{\left(d^2+s^2-2\right)}{\left(s^2-d^2\right)}\log\frac{\left(1-d^2\right)}{|s^2-1|}+2\right)\right]_{s\equiv\frac{x+y}{\sqrt{3}}\ ,\ d\equiv\frac{|x-y|}{\sqrt{3}}} \tag{B.61}$$

B.5 The RD era mode functions

Let us concretely address the problem of page 53 for the mode functions during the radiation era.

First of all, we need a smooth connection for the scale factor between the inflation stage (we assume de Sitter) and the next radiation dominated era. We assume that an abrupt transition between inflation and radiation era takes place at τ_* (which is a negative number). During inflation we have (neglecting slow-roll corrections) (see Tab.1.1)

$$a = -\frac{1}{H_{\rm inf}\,\tau}\ , \qquad\qquad \mathcal{H} = -\frac{1}{\tau} \tag{B.62}$$

if we indicate with $H_{\rm inf}$ the (constant) value of the Hubble rate during inflation, we can specialize the RD solution - remember $\rho \propto a^{-4}$ and $H \propto \rho^{1/2} \propto a^{-2}$ - for the sought-after scale factor

$$\frac{a'}{a} = a\,H = H_{\rm inf}\,\frac{a_*^2}{a} \qquad \Rightarrow \qquad a\left(\tau\right) = C_1 + a_*^2\, H_{\rm inf}\,\tau = a_*\Big[1 + a_*\, H_{\rm inf}\,\left(\tau-\tau_*\right)\Big] \tag{B.63}$$

Armed with the solution above, we can find the continued mode functions in the radiation domination by matching with those coming from the inflation.

Recall the Bunch-Davies mode function

$$v_{k,\lambda}(\tau) = \frac{1}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right)e^{-ik\tau} \tag{B.64}$$

This is the mode function during the inflationary era - for instance, it is the solution of eq. (3.5) in de Sitter, neglecting slow-roll corrections. The mode function during the radiation domination fulfils eq. (5.58), that is

$$\left(\partial_\tau^2 + k^2\right) v^\lambda(\tau,\mathbf{k}) \ = \ 0 \qquad\qquad \Rightarrow \qquad v_{k,\lambda}(\tau) = C_1\, e^{-ik\tau} + C_2\, e^{ik\tau} \tag{B.65}$$

The initial-condition for eq. (B.65) is given by matching the field & conjugated-momentum amplitudes at the interface $\tau = \tau_* < 0$. This is tantamount to match the amplitude of the field and its first derivative. Therefore

$$\begin{cases} v_{k,\lambda}(\tau) = C_1\, e^{-ik\tau} + C_2\, e^{ik\tau} \\ v'_{k,\lambda}(\tau) = -ik\left(C_1\, e^{-ik\tau} - C_2\, e^{ik\tau}\right) \end{cases} \qquad \Rightarrow \qquad \begin{cases} C_1 = \dfrac{1}{2}\left(v_{k,\lambda}(\tau_*) + \dfrac{v'_{k,\lambda}(\tau_*)}{-ik}\right)e^{ik\tau_*} \\ C_2 = \dfrac{1}{2}\left(v_{k,\lambda}(\tau_*) - \dfrac{v'_{k,\lambda}(\tau_*)}{-ik}\right)e^{-ik\tau_*} \end{cases} \tag{B.66}$$

using the eq. (B.64) as input for $v_{k,\lambda}(\tau_*)$, $v'_{k,\lambda}(\tau_*)$ we eventually obtain

$$v_{k,\lambda}(\tau) \;=\; \frac{e^{-ik\tau_*}}{\sqrt{2k}}\left[-\frac{1+2ik\tau_*-2(k\tau_*)^2}{2(k\tau_*)^2}\,e^{-ik(\tau-\tau_*)}+\frac{1}{2(k\tau_*)^2}\,e^{ik(\tau-\tau_*)}\right] \qquad \tau \geqslant \tau_* \tag{B.67}$$

This is the desired mode function. It represents the basis for our perturbative approach. It worth to note that it still satisfies eq. (5.57) exactly. This is not a case: this condition is preserved by the general solution of eq. (B.65).

Due to the non-linear dynamics, the sourced GWs can in principle be contributed by scalar perturbations from very different scales. However, in practise the second-order GWs are primarily sourced when the first-order scalar perturbations of same scale (i.e. same λ of the GW) re-enter the Hubble radius (during RD) [23] [24]. For the modes λ of interest, the second horizon crossing happens for $\tau \gg \tau_*$.
All the GW modes of interest are contained in the horizon at the beginning of the radiation era. At late times $\tau \gg \tau_*$, the mode function for these modes can be simplified. In order to do this, we first approximate the scale factor at late times $\tau \gg \tau_*$ obtaining

$$a(\tau) \;\simeq\; a_*^2\,H_{\rm inf}\,\tau \;=\; \frac{1}{H_{\rm inf}^2\,\tau_*^2}\,H_{\rm inf}\,\tau \;=\; \frac{\tau}{H_{\rm inf}\tau_*^2}\,, \qquad \tau \gg \tau_* \tag{B.68}$$

Then, we can simplify the eq. (B.67) for these modes ($|k\tau_*| \ll 1$) finally obtaining

$$v_{k,\lambda}(\tau) = \frac{H_{\rm inf}}{\sqrt{2k}}\frac{a(\tau)}{\tau}\left[-\frac{1}{2k^2}\,e^{-ik\tau}+\frac{1}{2k^2}\,e^{ik\tau}\right] \tag{B.69}$$

where we have neglected the inessential overall phase $\mathrm{e}^{-ik\tau_*}$. It is worth to notice that this late-time mode function is purely imaginary, and we can make it real by a *time-independent* phase rotation.
In summary, the mode function for the tensor perturbation is

$$v_{k,\lambda}(\tau) = \begin{cases} \frac{1}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right)e^{-ik\tau} & \tau \leqslant \tau_* \\[2ex] \frac{e^{-ik\tau_*}}{\sqrt{2k}}\left[-\frac{1+2ik\tau_*-2(k\tau_*)^2}{2(k\tau_*)^2}\,e^{-ik(\tau-\tau_*)}+\frac{1}{2(k\tau_*)^2}\,e^{ik(\tau-\tau_*)}\right] & \tau > \tau_* \\[2ex] \frac{H_{\rm inf}}{\sqrt{2k}}\frac{a(\tau)}{\tau}\left[-\frac{1}{2k^2}\,e^{-ik\tau}+\frac{1}{2k^2}\,e^{ik\tau}\right] & \tau \gg \tau_*\,,\ |k\tau_*| \ll 1 \end{cases} \tag{B.70}$$

B.6 The sourced GWB from primordial scalar perturbations with the operatorial CTP formalism

The purpose of this section is to support sec. 5.11.2, where we present synthetically the application of the in-in formalism to the study of second-order sourced GWs from first-order primordial scalar perturbations, during the radiation era. The observable of interest is the 2-point GW function. In order to efficiently organize the many steps of the long calculation, we divide them in the subsections

1. The v.e.v. contributions. All the possible structures

2. Wick expansion

3. The dominant contribution

It is easier to work in terms of the normalized field v^λ. At the end of the calculation, we will express the final result in terms of the tensor perturbation h_λ. We need to implement eq. (4.26) to our actual problem. It means writing eq. (5.64), which we report below for convenience

$$\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = -\int^{\tau} d\tau_2 \int^{\tau_2} d\tau_1\;\left(\frac{4}{9}M_p\right)^2 a(\tau_1)\,a(\tau_2)\sum_{\alpha\beta}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3}$$
$$\times\,\mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\beta(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)$$
$$\times\,\Big\langle\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big]\Big]\Big\rangle \quad \text{(B.71)}$$

The v.e.v. contributions. All the possible structures

Let us demonstrate that the possible structures in the vacuum expectation value are those presented in page 55. Let us start from the innermost commutator. The ζ and v_λ fields mutually commute. The two ζs trivially exit the commutator. Moreover, using the identity

$$\Big[A,\,BC\Big] = \Big[A,\,B\Big]C + B\Big[A,\,C\Big] \quad \text{(B.72)}$$

the innermost commutator becomes

$$\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,\Big(\Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\;,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big]\,v_\sigma(\tau,\,\mathbf{k}_2) + v_\lambda(\tau,\,\mathbf{k}_1)\Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\;,\;v_\sigma(\tau,\,\mathbf{k}_2)\Big]\Big)$$
$$= \zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,\Big(C_1\,v_\sigma(\tau,\,\mathbf{k}_2) + C_2\,v_\lambda(\tau,\,\mathbf{k}_1)\Big) \quad \text{(B.73)}$$

where we read off the quantities C_1, C_2 defined as

$$C_1 \equiv \Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\;,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big]$$
$$C_2 \equiv \Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\;,\;v_\sigma(\tau,\,\mathbf{k}_2)\Big] \quad \text{(B.74)}$$

The vev of eq. (B.71) becomes

$$\Big\langle\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,\Big(C_1\,v_\sigma(\tau,\,\mathbf{k}_2) + v_\lambda(\tau,\,\mathbf{k}_1)C_2\Big)\Big]\Big\rangle =$$
$$C_1\,\Big\langle\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big]\Big\rangle$$
$$+C_2\,\Big\langle\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,v_\lambda(\tau,\,\mathbf{k}_1)\Big]\Big\rangle \quad \text{(B.75)}$$

Let us simplify separately the two vacuum expectation values in the r.h.s. of eq. (B.75). We start from the one proportional to C_1. It is easy to verify - e.g. iterating the identity eq. (B.72) - that the v.e.v. multiplying C_1 is

$$\Big\langle\Big[\Big(v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\Big)\,,\;v_\sigma(\tau,\,\mathbf{k}_2)\,\Big(\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big)\Big]\Big\rangle =$$
$$\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,,\;v_\sigma(\tau,\,\mathbf{k}_2)\Big]\,\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle$$
$$+\,\Big\langle v_\sigma(\tau,\,\mathbf{k}_2)\,v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\Big[\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big]\Big\rangle \quad \text{(B.76)}$$

and analogously the one which multiples C_2 is

$$\Big\langle\Big[\Big(v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\Big)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\,\Big(\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big)\Big]\Big\rangle=$$
$$\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big]\,\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle$$
$$+\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\Big[\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big]\Big\rangle \tag{B.77}$$

Note the strong similarity between the two. Indeed, they are related each other by $(\lambda,\mathbf{k}_1)\leftrightarrow(\sigma,\mathbf{k}_2)$.

Finally, we are left to prove that the commutator $[\zeta\,\zeta\,,\;\zeta\,\zeta]$ is sum of terms like (b) of pag.55. Using the identity below

$$\Big[AB,CD\Big]=A\Big[B,C\Big]D+AC\Big[B,D\Big]+\Big[A,C\Big]DB+C\Big[A,D\Big]B \tag{B.78}$$

we have

$$\Big[\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big]=$$
$$\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,\Big[\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\zeta(\mathbf{q}_2)\Big]+\zeta(\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\Big[\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big]$$
$$+\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\Big[\zeta(\mathbf{q}_1)\,,\;\zeta(\mathbf{q}_2)\Big]+\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\Big[\zeta(\mathbf{q}_1)\,,\;\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big] \tag{B.79}$$

as we wrote in page 55. We note that there are not terms like $[\zeta,\,\zeta]^2$.

In conclusion, eq. (B.75) is the sum of two terms. Schematically, we have

$$(a)\;\;\Big[v^\dagger,\,v\Big]\,\Big\langle\zeta\,\zeta\,\zeta\,\zeta\Big\rangle \qquad\qquad (b)\;\;\Big\langle v^\dagger\,v\Big\rangle\,\Big[\zeta\,\zeta\,,\;\zeta\,\zeta\Big]$$

The (a) terms are proportional to $(\mathcal{P}_\zeta)^2$, as we demonstrate in the next paragraph. The (b) terms are subdominant with respect to (a), as we discussed in page 55. Inserting all the expressions into the eq. (B.73), after further simplifications one gets

$$\Big\langle\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,,\;\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\,\Big(C_1\,v_\sigma(\tau,\,\mathbf{k}_2)+v_\lambda(\tau,\,\mathbf{k}_1)C_2\Big)\Big]\Big\rangle=$$
$$\Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big]\,\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,,\;v_\sigma(\tau,\,\mathbf{k}_2)\Big]\,\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle$$
$$+\,\Big[v^\dagger_\beta(\tau_2,\,\mathbf{p}_2)\;\;,\;v_\sigma(\tau,\,\mathbf{k}_2)\Big]\,\Big[v^\dagger_\alpha(\tau_1,\,\mathbf{p}_1)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big]\,\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle \tag{B.80}$$

The second and third line are related each others by $(\lambda,\mathbf{k}_1)\leftrightarrow(\sigma,\mathbf{k}_2)$.

Wick expansion

We can reduce the 4-point scalar correlator $\langle\zeta\,\zeta\,\zeta\,\zeta\rangle$ into 2-point functions. Let us recall the definition eq. (3.14) for the power spectrum. The procedure is analogous to that of page 92. The only exception is the meaning of the angle brackets, which this time must be considered as indicating the vacuum expectation value $\langle\cdots\rangle\equiv\langle 0|\cdots|0\rangle$.

We can recycle eq. (B.46) with the replacement $\mathbf{k}\to\mathbf{p}$, that is

$$\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle=\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\Big\rangle\,\Big\langle\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle$$
$$+\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\Big\rangle\,\Big\langle\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle$$
$$+\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle\,\Big\langle\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\Big\rangle \tag{B.81}$$

using eq.s(B.47)-(B.49), we have

$$\begin{aligned}\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\Big\rangle\Big\langle\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle &= (2\pi)^3(2\pi)^3\;\delta^{(3)}(p_1)\delta^{(3)}(p_2)\\ &\quad\times\frac{2\pi^2}{q_1^3}\frac{2\pi^2}{q_2^3}\;\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(q_2)\end{aligned} \tag{B.82}$$

$$\begin{aligned}\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\Big\rangle\Big\langle\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle &= (2\pi)^3(2\pi)^3\;\delta^{(3)}(q_1+q_2)\delta^{(3)}(p_1+p_2-q_1-q_2)\\ &\quad\times\frac{2\pi^2}{q_1^3}\frac{2\pi^2}{|\mathbf{p}_1-\mathbf{q}_1|^3}\;\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{p}_1-\mathbf{q}_1|)\end{aligned} \tag{B.83}$$

$$\begin{aligned}\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle\Big\langle\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\Big\rangle &= (2\pi)^3(2\pi)^3\;\delta^{(3)}(q_1+p_2-q_2)\delta^{(3)}(p_1-q_1+q_2)\\ &\quad\times\frac{2\pi^2}{q_1^3}\frac{2\pi^2}{|\mathbf{p}_1-\mathbf{q}_1|^3}\;\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{p}_1-\mathbf{q}_1|)\end{aligned} \tag{B.84}$$

The first term in the r.h.s. of eq. (B.81) is non-vanishing only for $p_1 = p_2 = k_1 = k_2 = 0$. We neglect it since we are considering finite external momenta $k_1,\, k_2 \neq 0$. We rearrange the δ-functions of eq. (B.84) as

$$\delta^{(3)}(q_1+p_2-q_2)\delta^{(3)}(p_1-q_1+q_2) = \delta^{(3)}(p_1+p_2)\delta^{(3)}(p_1-q_1+q_2) \tag{B.85}$$

notice that aside of a δ-function, the two non-vanishing contributions in the r.h.s of eq. (B.81) are equal.

Dominant contribution

This way, the integral eq. (B.71) becomes

$$\begin{aligned}\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= -\int^{\tau} d\tau_2\int^{\tau_2} d\tau_1\;\left(\frac{4}{9}M_p\right)^2\;a(\tau_1)\,a(\tau_2)\sum_{\alpha\beta}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3}\\ &\quad\times\mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\beta(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\\ &\quad\times\Bigg\{\Big[v_\beta^\dagger(\tau_2,\,\mathbf{p}_2)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big]\Big[v_\alpha^\dagger(\tau_1,\,\mathbf{p}_1)\,,\;v_\sigma(\tau,\,\mathbf{k}_2)\Big]\;\Big\langle\zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle\\ &\quad+\Big((\lambda,\mathbf{k}_1)\leftrightarrow(\sigma,\mathbf{k}_2)\Big)\Bigg\}\end{aligned} \tag{B.86}$$

We use eq. (A.11) to express the commutator with the hermitian conjugate field in a form like eq. (5.60), in particular

$$\Big[v_\beta^\dagger(\tau_2,\,\mathbf{p}_2)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big] = \Big[v_\beta(\tau_2,\,-\mathbf{p}_2)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big] \tag{B.87}$$

We replace the explicit expression for the commutator and we factor the ζ-vev, obtaining

$$\begin{aligned}\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= -\int^{\tau} d\tau_2 \int^{\tau_2} d\tau_1\, \left(\frac{4}{9}M_p\right)^2 a(\tau_1)\,a(\tau_2) \sum_{\alpha\beta}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3} \\ &\times \mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\beta(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\,\Big\langle \zeta(\mathbf{q}_1)\,\zeta(\mathbf{p}_1-\mathbf{q}_1)\,\zeta(\mathbf{q}_2)\,\zeta(\mathbf{p}_2-\mathbf{q}_2)\Big\rangle \\ &\times (2\pi)^6\left\{\delta_{\beta\lambda}\delta_{\alpha\sigma}\,\delta^{(3)}(k_1-p_2)\delta^{(3)}(k_2-p_1)\frac{\sin\left[k_1(\tau_2-\tau)\right]}{ik_1}\,\frac{\sin\left[k_2(\tau_1-\tau)\right]}{ik_2} + \Big((\lambda,\mathbf{k}_1)\leftrightarrow(\sigma,\mathbf{k}_2)\Big)\right\}\end{aligned} \tag{B.88}$$

or, after we insert the explicit expression for the vev

$$\begin{aligned}\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= -\int^{\tau} d\tau_2 \int^{\tau_2} d\tau_1\, \left(\frac{4}{9}M_p\right)^2 a(\tau_1)\,a(\tau_2) \sum_{\alpha\beta}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3} \\ &\times \mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\beta(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2) \\ &\times (2\pi)^6\,\delta^{(3)}(p_1+p_2)\,\frac{2\pi^2}{q_1^3}\frac{2\pi^2}{|\mathbf{p}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{p}_1-\mathbf{q}_1|)\left\{\delta^{(3)}(q_1+q_2)+\delta^{(3)}(p_1-q_1+q_2)\right\} \\ &\times (2\pi)^6\,\frac{\sin\left[k_1(\tau_2-\tau)\right]}{ik_1}\,\frac{\sin\left[k_2(\tau_1-\tau)\right]}{ik_2}\left\{\delta_{\beta\lambda}\delta_{\alpha\sigma}\,\delta^{(3)}(k_1-p_2)\delta^{(3)}(k_2-p_1) + \Big((\lambda,\mathbf{k}_1)\leftrightarrow(\sigma,\mathbf{k}_2)\Big)\right\}\end{aligned} \tag{B.89}$$

The δ-functions in the last line imply that $\mathbf{p}_1+\mathbf{p}_2=\mathbf{k}_1+\mathbf{k}_2$. We can factorize a delta in front of the integral, accounting for the total momentum conservation. They remain 3 δ-functions and the final number of independent integrations is reduced to one, which for convenience is written in the $\vec{q}_1$ variable.
In light of this, we conveniently rewrite the expression in a form where it is easy to read off the power spectrum

$$\begin{aligned}\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= (2\pi)^3\delta^{(3)}(k_1+k_2)\,\frac{2\pi^2}{k_1^3} \\ &\left(\frac{4}{9}M_p\right)^2 k_1\int^{\tau} d\tau_2\int^{\tau_2} d\tau_1\, a(\tau_1)\,a(\tau_2)\sum_{\alpha\beta}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\,d^3\vec{q}_1\,d^3\vec{p}_2\,d^3\vec{q}_2 \\ &\times \mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\beta(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\,\frac{1}{q_1^3}\frac{2\pi^2}{|\mathbf{p}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{p}_1-\mathbf{q}_1|)\sin\left[k_1(\tau_2-\tau)\right]\,\sin\left[k_1(\tau_1-\tau)\right] \\ &\times\left\{\delta^{(3)}(q_1+q_2)+\delta^{(3)}(p_1-q_1+q_2)\right\}\left\{\delta_{\beta\lambda}\delta_{\alpha\sigma}\,\delta^{(3)}(k_1-p_2)\delta^{(3)}(k_2-p_1) + \Big((\lambda,\mathbf{k}_1)\leftrightarrow(\sigma,\mathbf{k}_2)\Big)\right\}\end{aligned} \tag{B.90}$$

This expression may be further simplified using the properties

$$\mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1) = \mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{p}_1-\mathbf{q}_1) \tag{B.91}$$

$$\mathcal{Q}_\alpha(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1) = \mathcal{Q}_\alpha(\tau_1,\,-\mathbf{p}_1,\,-\mathbf{q}_1) \tag{B.92}$$

In fact, these properties are shared also by the other factors of integrand in eq. (B.90). Let us integrate over the variables with label '2'. The key point is that we are going to obtain terms of the four types

$$\begin{array}{cc} \mathcal{Q}_\sigma(\tau_1,\,\mathbf{k}_2,\,\mathbf{q}_1)\,\mathcal{Q}_\lambda(\tau_2,\,\mathbf{k}_1,\,-\mathbf{q}_1) & \mathcal{Q}_\sigma(\tau_1,\,\mathbf{k}_2,\,\mathbf{q}_1)\,\mathcal{Q}_\lambda(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1-\mathbf{k}_2) \\ \mathcal{Q}_\lambda(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\sigma(\tau_2,\,\mathbf{k}_2,\,-\mathbf{q}_1) & \mathcal{Q}_\lambda(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\sigma(\tau_2,\,\mathbf{k}_2,\,\mathbf{q}_1-\mathbf{k}_1) \end{array} \tag{B.93}$$

or, replacing $\mathbf{k}_2 = -\mathbf{k}_1$,

$$\begin{array}{cc} \mathcal{Q}_\sigma(\tau_1,\,-\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\lambda(\tau_2,\,\mathbf{k}_1,\,-\mathbf{q}_1) & \mathcal{Q}_\sigma(\tau_1,\,-\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\lambda(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1+\mathbf{k}_1) \\ \mathcal{Q}_\lambda(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\sigma(\tau_2,\,-\mathbf{k}_1,\,-\mathbf{q}_1) & \mathcal{Q}_\lambda(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\sigma(\tau_2,\,-\mathbf{k}_1,\,\mathbf{q}_1-\mathbf{k}_1) \end{array} \tag{B.94}$$

Using the property eq. (B.92), and in half of them changing the dummy variable $\mathbf{q}_1 \to \mathbf{p}_1 - \mathbf{q}_1$, one can show that the bottom terms are equal to those in the top line of eq. (B.93) with the change $\tau_1 \leftrightarrow \tau_2$. Again, recall that this holds also for the other factors in the integrand. In conclusion, this means that the integrand is symmetric around the axis $\tau_2 = \tau_1$. We can exploit the symmetry property to change the integration from the original domain (left fig. B.2) to the square $[0,\tau]\times[0,\tau]$ (right) and account of this with a factor $\frac{1}{2}$.

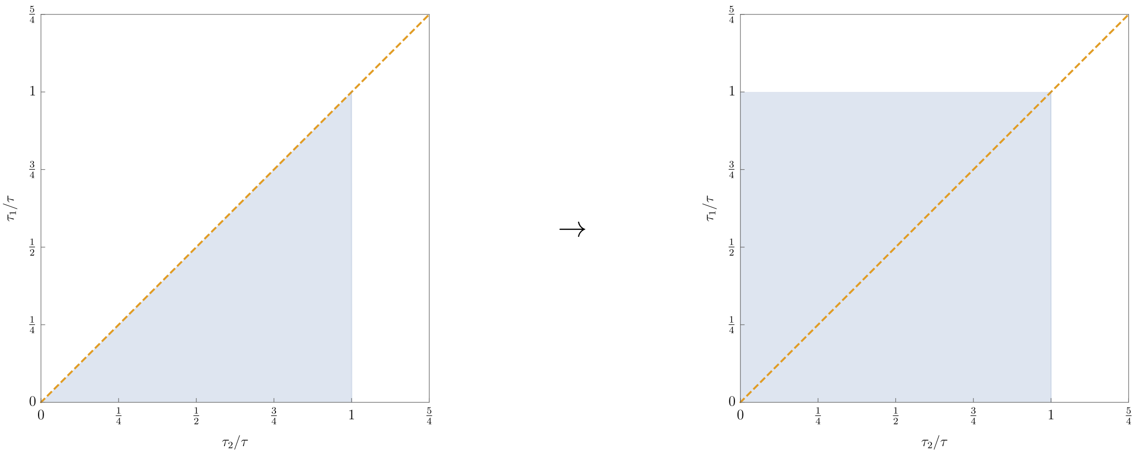


Figure B.2: The time-integrations $\iint d\tau_2\, d\tau_1$ in eq. (B.90) are performed over the domain shown in the left picture as the shadowed region. The integrand is symmetric over the axis $\tau_1 = \tau_2$ (orange). We can therefore integrate over the full square (right) and put a prefactor 1/2.

We obtain the result

$$\begin{aligned} \delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\, v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= (2\pi)^3\delta^{(3)}(k_1+k_2)\,\frac{2\pi^2}{k_1^3}\,\left(\frac{4}{9}M_p\right)^2 k_1\int_0^\tau d\tau_2\int_0^\tau d\tau_1\; a(\tau_1)\,a(\tau_2) \\ &\times\sum_{\alpha\beta}\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\,\frac{2\pi^2}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\,\sin\left[k_1(\tau_2-\tau)\right]\,\sin\left[k_1(\tau_1-\tau)\right] \\ &\times\Big\{\mathcal{Q}_\sigma(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\lambda(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1)+\mathcal{Q}_\lambda(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\,\mathcal{Q}_\sigma(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big\} \end{aligned} \tag{B.95}$$

This integral is exactly the one in eq. (B.51). Let us demonstrate this.
To this end, it is convenient to extract the polarisation vector from $\mathcal{Q}_\lambda$. Recalling the definition

eq. (5.45) (reported below)

$$\mathcal{Q}_\lambda(\tau,\,\mathbf{k},\,\mathbf{q}) \equiv \mathrm{e}^*_\lambda(\hat{k},\,\mathbf{q})\,\Big\{3\,T(\tau,\,q)\,T(\tau,\,|\mathbf{k}-\mathbf{q}|) + \frac{1}{\mathcal{H}^2}\,T'(\tau,\,q)\,T'(\tau,\,|\mathbf{k}-\mathbf{q}|) + \frac{1}{\mathcal{H}}\,\partial_\tau\left[\,T(\tau,\,q)\,T(\tau,\,|\mathbf{k}-\mathbf{q}|)\,\right]\Big\}$$

We introduce the quantity Q as

$$\mathcal{Q}_\lambda(\tau,\,\mathbf{k},\,\mathbf{q}) \equiv \mathrm{e}^*_\lambda(\hat{k},\,\mathbf{q})\;Q(\tau,\,\mathbf{k},\,\mathbf{q}) \tag{B.96}$$

It is convenient to rewrite the integral eq. (B.95) in the following form

$$\begin{aligned}\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= (2\pi)^3\delta^{(3)}(k_1+k_2)\,\frac{2\pi^2}{k_1^3}\,\left(\frac{4}{9}M_p\right)^2\,k_1\int^\tau d\tau_2\int^\tau d\tau_1\\ &\times\sum_{\alpha\beta}\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\frac{2\pi^2}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\,\mathrm{e}^*_\sigma(\hat{k}_1,\,\mathbf{q}_1)\,\mathrm{e}^*_\lambda(\hat{k}_1,\,\mathbf{q}_1)\\ &\times\Big\{\Big(a(\tau_1)\sin\left[k_1(\tau_1-\tau)\right]\,Q(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)\,\Big(a(\tau_2)\,\sin\left[k_1(\tau_2-\tau)\right]\,Q(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)\\ &+\Big(a(\tau_1)\sin\left[k_1(\tau_1-\tau)\right]\,Q(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)\,\Big(a(\tau_2)\sin\left[k_1(\tau_2-\tau)\right]\,Q(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)\Big\}\end{aligned} \tag{B.97}$$

From sec. A.5 we know that for different polarisations $\lambda\neq\sigma$ the integral vanishes. Therefore we factor out a $\delta_{\lambda\sigma}$ and we go ahead in the case $\lambda=\sigma$. Using the identity eq. (B.50), we realize that

$$\mathrm{e}^*_\lambda(\hat{k}_1,\,\mathbf{q}_1)\,\mathrm{e}^*_\lambda(\hat{k}_1,\,\mathbf{q}_1) = \frac{1}{4}\,q_1^4\,\sin^4\theta \tag{B.98}$$

where θ is again the angle between the wavevector $\vec{k}_1$ and the dummy vector $\vec{q}_1$. With this simplification the expression becomes

$$\begin{aligned}\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\,\frac{2\pi^2}{k_1^3}\,\left(\frac{4}{9}M_p\right)^2\,k_1\int^\tau d\tau_2\int^\tau d\tau_1\\ &\times\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\frac{2\pi^2}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\,\frac{1}{4}\,q_1^4\,\sin^4\theta\\ &\times\Big\{2\Big(a(\tau_1)\sin\left[k_1(\tau_1-\tau)\right]\,Q(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)\,\Big(a(\tau_2)\,\sin\left[k_1(\tau_2-\tau)\right]\,Q(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)\Big\}\end{aligned} \tag{B.99}$$

To demonstrate that this is indeed eq. (B.51), we must work out the last line of eq. (B.99). We use the trigonometric identity $\sin(a-b)=\sin(a)\cos(b)-\cos(a)\sin(b)$ and we simplify the product. Moreover, the second line of eq. (B.99) does not depend on time, therefore we can move the (independent) integrations over time in the round brackets. Focussing on the terms in curly brackets, we have

$$\begin{aligned}&\Big[\frac{\cos(k_1\tau)}{2k_1}\Big(-4k_1\int d\tau_1\,a(\tau_1)\,\sin(k_1\tau_1)\,Q(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)+\frac{\sin(k_1\tau)}{2k_1}\Big(4k_1\int d\tau_1\,a(\tau_1)\,\cos(k_1\tau_1)\,Q(\tau_1,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)\Big]\\ &\Big[\frac{\cos(k_1\tau)}{2k_1}\Big(-4k_1\int d\tau_2\,a(\tau_2)\,\sin(k_1\tau_2)\,Q(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)+\frac{\sin(k_1\tau)}{2k_1}\Big(4k_1\int d\tau_2\,a(\tau_2)\,\cos(k_1\tau_2)\,Q(\tau_2,\,\mathbf{k}_1,\,\mathbf{q}_1)\Big)\Big]\end{aligned} \tag{B.100}$$

The functions in the round brackets are exactly eq. (B.43) and eq. (B.42). The curly bracket of eq. (B.99) is therefore

$$\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\,\frac{2\pi^2}{k_1^3}\,\left(\frac{4}{9}M_p\right)^2\,k_1 \\ \times\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\,\frac{2\pi^2}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\,\mathcal{P}_\zeta(|\mathbf{k}_1-\mathbf{q}_1|)\,\frac{1}{4}\,q_1^4\,\sin^4\theta \\ \times\left\{\frac{1}{(2k_1)^2}\,\Big[\cos(k_1\tau)\,\mathcal{I}_c\,(x,y)+\sin(k_1\tau)\,\mathcal{I}_s\,(x,y)\Big]^2\right\} \tag{B.101}$$

As one can verify by comparison, indeed the two expressions eq. (B.51) and eq. (B.101) are equal.

It is needed a comment about the time integration in the functions $\mathcal{I}_{c,s}$. In the definition eq.s(B.43)-(B.42) the time integration begins at $\tau_0 = 0$. This because with the Green function method, where we first defined them, the conformal time was normalized such that the RD era began at $\tau = 0$. This is a normalization convention. In our case, technically, the integration starts at $\tau = \tau_* < 0$. The proper accounting of *this* convention requires the prescription, in this case, of using the transfer function $T(\tau) \to T(\tau-\tau_*)$. This consideration was straightforward, but it is worth to be noted in that a sloppy application of the correspondence above may lead to errors.

If would change variables, accordingly to (B.54), then the (B.101) would match exactly with eq. (B.57). For completeness, we report it (below)

$$\delta_2\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\left(\frac{M_p}{2}\right)^2\,\frac{1}{81} \\ \iint_{\mathcal{D}} dx\,dy\,\mathcal{P}_\zeta(x\,k_1)\,\mathcal{P}_\zeta(y\,k_1)\,\left(\frac{x}{y}\right)^2\Big[1-\Big(\frac{1+x^2-y^2}{2x}\Big)^2\Big]^2 \\ \times\Big[\sin(k_1\tau)\,\mathcal{I}_s\,(x,y)+\cos(k_1\tau)\,\mathcal{I}_c\,(x,y)\Big]^2 \tag{B.102}$$

B.7 The GW power spectrum from enhanced scalar perturbations

The power spectrum of the RD-era second-order sourced GWs from primordial enhanced scalar perturbations is obtained inserting (5.35) in eq. (B.102). The possible terms appearing in the r.h.s. depends from the magnitude of the enhancement A_s (see eq. (5.35)) in the three possible ways

$$\mathcal{P}_{h_\lambda}(\tau,k_1) = \mathcal{P}^{(0)}_{h_\lambda}(\tau,k_1)\;+\;A_s\,\mathcal{P}^{(1)}_{h_\lambda}(\tau,k_1)\;+\;A_s^2\,\mathcal{P}^{(2)}_{h_\lambda}(\tau,k_1) \tag{B.103}$$

for example $\mathcal{P}^{(0)}_{h_\lambda}$ is the GW power spectrum due to the standard flat primordial scalar power spectrum. The biggest effect of the scalar enhancement is contained in the term $\sim A_s^2$. It is the one resulting from eq. (B.102) when one uses as $\mathcal{P}_\zeta \sim A_s$ the bump of eq. (5.35). If one focusses on this term ($x_\star \equiv \frac{p_*}{k_1}$)

$$\Big\langle v_\lambda(\tau, \mathbf{k}_1)\, v_\sigma(\tau, \mathbf{k}_2) \Big\rangle = (2\pi)^3\, \delta_{\lambda\sigma} \delta^{(3)}(k_1 + k_2) \frac{2\pi^2}{k_1^3} \left(\frac{M_p}{2}\right)^2 \frac{1}{81}$$
$$\left(A_s\, \frac{p_*}{k_1}\right)^2 \iint_{\mathcal{D}} dx\, dy\; \delta(x - x_\star)\, \delta(y - x_\star) \left(\frac{x}{y}\right)^2 \left[1 - \left(\frac{1 + x^2 - y^2}{2x}\right)^2\right]^2$$
$$\times \Big[\sin(k_1 \tau)\, \mathcal{I}_s\,(x, y) + \cos(k_1 \tau)\, \mathcal{I}_c\,(x, y)\Big]^2 \quad \text{(B.104)}$$

In the equation above, we made use of the identity

$$\delta(x\, k_1 - p_*) = \frac{1}{k_1}\, \delta\Big(x - x_\star\Big) \quad \text{(B.105)}$$

The δ-functions make the integral over $dx\, dy$ trivial: it gives the integrand in the point $(x, y) = (x_\star,\, x_\star)$ provided that this point lies inside the domain of integration, otherwise it gives zero. The condition $(x_\star,\, x_\star) \in \mathcal{D}$ is simply $x_\star \geqslant 1/2$, see fig. B.1. One can write the result in a compact form using a θ-function, that is

$$\Big\langle v_\lambda(\tau, \mathbf{k}_1)\, v_\sigma(\tau, \mathbf{k}_2) \Big\rangle = (2\pi)^3\, \delta_{\lambda\sigma} \delta^{(3)}(k_1 + k_2) \frac{2\pi^2}{k_1^3}$$
$$\times \left\{\theta\left(x_\star - \frac{1}{2}\right) \left(\frac{M_p}{2}\right)^2 \left(\frac{A_s\, x_\star}{9}\right)^2 \left[1 - \frac{1}{(2x_\star)^2}\right]^2 \Big[\sin(k_1 \tau)\, \mathcal{I}_s\,(x_\star, x_\star) + \cos(k_1 \tau)\, \mathcal{I}_c\,(x_\star, x_\star)\Big]^2\right\}$$
$$\text{(B.106)}$$

The second line of the expression above is the sought-after dominant contribution to the GW power spectrum coming from the enhanced scalar perturbations. For future convenience, let us express this in terms of the tensor field h_{ij}. Recalling $v^\lambda(\tau, \mathbf{k}) = \frac{M_p}{2}\, a(\tau)\, h^\lambda(\tau, \mathbf{k})$, one has

$$\Big\langle h_\lambda(\tau, \mathbf{k}_1)\, h_\sigma(\tau, \mathbf{k}_2) \Big\rangle = (2\pi)^3\, \delta_{\lambda\sigma} \delta^{(3)}(k_1 + k_2) \frac{2\pi^2}{k_1^3}$$
$$\times \left\{\theta\left(x_\star - \frac{1}{2}\right) \frac{1}{a^2(\tau)} \left(\frac{A_s\, x_\star}{9}\right)^2 \left[1 - \frac{1}{(2x_\star)^2}\right]^2 \Big[\sin(k_1 \tau)\, \mathcal{I}_s\,(x_\star, x_\star) + \cos(k_1 \tau)\, \mathcal{I}_c\,(x_\star, x_\star)\Big]^2\right\}$$
$$\text{(B.107)}$$

or

$$\begin{aligned} \mathcal{P}_h(\tau,\, k_1) \;&=\; \theta\left(x_\star - \frac{1}{2}\right) \frac{1}{a^2(\tau)} \left(\frac{A_s\, x_\star}{9}\right)^2 \left[1 - \frac{1}{(2x_\star)^2}\right]^2 \\ &\times \Big[\sin(k_1 \tau)\, \mathcal{I}_s\,(x_\star, x_\star) + \cos(k_1 \tau)\, \mathcal{I}_c\,(x_\star, x_\star)\Big]^2 \end{aligned} \quad \text{(B.108)}$$

B.8 The energy density and 2-point GW correlator

Let us recall the discussion of sec. 5.4 about the stress-energy tensor of the gravitational waves. The GWs energy density is to be found in the 00-component of the stress-energy tensor. Also, remember that for a stochastic GW signal, one may express the energy density as a suitable time average of the GW power spectrum. In fact, the 00-component of the GW stress-energy is

$$
\begin{aligned}
\rho^{(\mathrm{GW})}(\tau,\vec{x}) &= -\frac{M_p^2}{4}\overline{\left\langle a^{-2}\partial_k h_{ij}(\tau,\vec{x})\,\partial_k h_{ij}(\tau,\vec{x})\right\rangle} \\
&= -\frac{M_p^2}{4}a^{-2}\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\mathrm{e}^{i\vec{x}\cdot(\vec{p_1}+\vec{p_2})}\,e^{\lambda}_{ij}(\hat{p_1})e^{\sigma}_{ij}(\hat{p_2})\,\vec{p_1}\cdot\vec{p_2}\overline{\left\langle h_\lambda(\tau,\vec{p_1})h_\sigma(\tau,\vec{p_2})\right\rangle} \\
&= -\frac{M_p^2}{4}a^{-2}\int\frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\mathrm{e}^{i\vec{x}\cdot(\vec{p_1}+\vec{p_2})}\,e^{\lambda}_{ij}(\hat{p_1})e^{\sigma}_{ij}(\hat{p_2})\,\vec{p_1}\cdot\vec{p_2}\,(2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}\left(\vec{p_1}+\vec{p_2}\right)\frac{2\pi^2}{p_1^3}\overline{\mathcal{P}_h(\tau,p_1)} \\
&= \frac{M_p^2}{4}a^{-2}\int\frac{d^3p_1}{(2\pi)^3}\,e^{\lambda}_{ij}(\hat{p_1})e^{\lambda}_{ij}(-\hat{p_1})\,\frac{2\pi^2}{p_1}\overline{\mathcal{P}_h(\tau,p_1)} \\
&= 2\frac{M_p^2}{4}a^{-2}\int_0^\infty dp_1\,p_1\,\overline{\mathcal{P}_h(\tau,p_1)} = \frac{M_p^2}{2}\int d\left(\log p\right)\,\left(\frac{p}{a}\right)^2\overline{\mathcal{P}_h(\tau,p)}
\end{aligned}
\tag{B.109}
$$

where we used the definition of the power spectrum and the identity (A.23). The factor 2 which appeared in the last line derives from the sum over the polarisations, where we assumed unpolarised gravitational waves (e.g. the 2-point GW function is independent from λ). The line over the GW power spectrum denotes the time average over several oscillation periods, see sec. 5.4.1.
It is customary to measure the ratio between the energy density of the stochastic gravitational waves and the critical energy density $\rho_c \equiv \frac{3H^2}{8\pi G}$. This quantity measures the relative contribution to the universe energetic budget coming from the gravitational waves. It is denoted by Ω_{GW} and in momentum space it is defined as (i.e. see [2])

$$
\Omega_{(\mathrm{GW})}(\tau,p) \equiv \frac{1}{\rho_c}\frac{d\rho_{\mathrm{GW}}}{d\ln p} = \frac{1}{6}\left(\frac{p}{aH}\right)^2\overline{\mathcal{P}_h(\tau,p)}
\tag{B.110}
$$

where $\rho_c = 3M_p^2H^2$ is the critical density at the time τ.

B.8.1 The density parameter of the first-order unsourced gravitational waves

Let us evaluate the $\Omega^{(\mathrm{GW})}$ contribution from the first-order GWs propagating freely in the radiation dominated era - that is, in the case where no enhancement is present and second-order sourced GWs are smaller than first-order ones. We start from eq. (5.15) and recall the free mode functions (B.70) in the RD era. The temporal average is performed on scales much smaller than the cosmological ones, but much larger than the GW period. We have

$$
\begin{aligned}
\rho^{(\mathrm{GW})} &= \frac{M_p^2}{4a^2}\frac{4}{a^2M_p^2}\sum_{\lambda,\sigma}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\left\langle e^{i\,\vec{p}\cdot\vec{x}}\,v'_\lambda(\tau,\,\vec{p})\mathrm{e}_{ij,\lambda}(\hat{p})\;\cdot\; e^{i\,\vec{q}\cdot\vec{x}}\,v'_\sigma(\tau,\,\vec{q})\mathrm{e}_{ij,\sigma}(\hat{q})\right\rangle \\
&= \frac{1}{a^4}\sum_{\lambda,\sigma}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\,e^{i\,\vec{p}\cdot\vec{x}}\,e^{i\,\vec{q}\cdot\vec{x}}\,\left\langle v'_\lambda\,(\tau,\,\vec{p})\;\,v'_\sigma\,(\tau,\,\vec{q})\right\rangle\,\mathrm{e}_{ij,\lambda}\,(\hat{p})\,\mathrm{e}_{ij,\sigma}\,(\hat{q}) \\
&= \frac{1}{a^4}\sum_{\lambda,\sigma}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\,e^{i\,(\vec{p}+\vec{q})\cdot\vec{x}}\,\left\langle v'_\lambda\,(\tau,\,\vec{p})\;\,v'_\sigma\,(\tau,\,\vec{q})\right\rangle\,\mathrm{e}_{ij,\lambda}\,(\hat{p})\,\mathrm{e}_{ij,\sigma}\,(\hat{q}) \\
&= \frac{1}{a^4}\sum_{\lambda,\sigma}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\,e^{i\,(\vec{p}+\vec{q})\cdot\vec{x}}\,\,(2\pi)^3\,\delta_{\lambda\sigma}\delta^{(3)}\,(\vec{p}+\vec{q})\,\,v'_{p,\lambda}(\tau)v'^{*}_{p,\lambda}(\tau)\,\,\mathrm{e}_{ij,\lambda}\,(\hat{p})\,\mathrm{e}_{ij,\sigma}\,(\hat{q}) \\
&= \frac{1}{a^4}\sum_{\lambda}\int\frac{d^3p}{(2\pi)^3}\,v'_{p,\lambda}(\tau)v'^{*}_{p,\lambda}(\tau)\,\,\mathrm{e}_{ij,\lambda}\,(\hat{p})\,\mathrm{e}_{ij,\lambda}\,(-\hat{p}) \\
&= \frac{2}{a^4}\int\frac{d^3p}{(2\pi)^3}\left[H^2_{\mathrm{inf}}\frac{a(\tau)^2}{\tau^2}\frac{\cos^2(p\tau)}{2p^3}+\dots\right]
\end{aligned}
\tag{B.111}
$$

where we have used $\sum_{ij}\mathrm{e}_{ij,\lambda}\,(\hat{p})\,\mathrm{e}_{ij,\lambda}\,(\hat{p}) = 1$ and the independence of the free-field mode functions from the polarisation. The dots are again terms suppressed by $\mathrm{O}\,(k\tau_*)$ and higher.
Performing the time average one obtains

$$
\rho_{\mathrm{GW}} = \frac{1}{2\pi^2}\frac{1}{a^4}\frac{a^2H^2_{\mathrm{inf}}}{\tau^2}\int\frac{dk}{k}\left(\frac{1}{2}\right)+\dots = \frac{1}{4\pi^2}\frac{H^2_{\mathrm{inf}}}{a^2\tau^2}\int\frac{dk}{k}+\dots \tag{B.112}
$$

The radiation energy density is given by

$$
\rho_\gamma = 3M_p^2H^2 \tag{B.113}
$$

so that an observer, who inhabit the universe during the radiation dominated era, would measure the contribution to the density parameter from GWs given by

$$
\Omega^{(0)}_{(\mathrm{GW})} \equiv \frac{1}{\rho_\gamma}\frac{d\rho_{\mathrm{GW}}}{d\ln k} = \frac{1}{12\,\pi^2}\frac{H^2_{\mathrm{inf}}}{M_p^2}\frac{1}{a^2\tau^2H} = \frac{1}{12\,\pi^2}\frac{H^2_{\mathrm{inf}}}{M_p^2} \tag{B.114}
$$

where we used the fact that in a radiation-dominated universe, at late time $\tau \gg \tau_*$, it holds $aH\tau = 1$. The reader is invited to notice that the energy density of the GWs scales as radiation (e.g. $\sim a^{-4}$), so that for a radiation dominated universe the GW density parameter is conserved.

B.8.2 The density parameter of the second-order sourced gravitational waves: dominant contribution

Let us now evaluate the contribution to the GW density parameter from the second-order GWs produced by the enhanced scalar perturbations. Here we are concerned with the dominant contribution (5.37) only, the sub-dominant one will be studied in Appendix C.5. From the definition (B.110), first we need the time averaged power spectrum.

Using eq. (B.108) we have

$$\begin{aligned}\overline{\delta_2\mathcal{P}_h(\tau,\,k_1)}\;&\simeq\;\theta\left(x_\star-\frac{1}{2}\right)\,\frac{1}{a^2(\tau)}\,\left(\frac{A_s\,x_\star}{9}\right)^2\left[1-\frac{1}{(2x_\star)^2}\right]^2\\&\quad\times\left[\overline{\sin^2(k_1\tau)}\,\mathcal{I}_s^2+\overline{\cos^2(k_1\tau)}\,\mathcal{I}_c^2-\overline{\sin(2k_1\tau)}\,\mathcal{I}_c\,\mathcal{I}_s\right]^2_{x_\star=p_*/k_1}\\&=\;\frac{1}{2}\theta\left(x_\star-\frac{1}{2}\right)\,\frac{1}{a^2(\tau)}\,\left(\frac{A_s\,x_\star}{9}\right)^2\left[1-\frac{1}{(2x_\star)^2}\right]^2\left[\,\mathcal{I}_s^2+\mathcal{I}_c^2\,\right]_{x_\star=p_*/k_1}\end{aligned}\tag{B.115}$$

where we used (6.16). Using the definition eq. (B.110), the late-time density parameter of the sourced second-order GWs from the dominant interaction (5.37) is ($aH\tau\simeq 1$)

$$\begin{aligned}\Omega^{(2)}_{(\mathrm{GW})}(k_1)&=\frac{1}{6}\left(\frac{k_1}{aH}\right)^2\,\overline{\mathcal{P}_h(\tau,k_1)}\\&=\theta\left(x_\star-\frac{1}{2}\right)\,\frac{(A_s\,x_\star)^2}{3\cdot 18^2}\,\frac{k_1^2}{a^4(\tau)\,H^2}\left[1-\frac{1}{(2x_\star)^2}\right]^2\left[\,\mathcal{I}_s^2+\mathcal{I}_c^2\,\right]_{x_\star=p_*/k_1}\\&=\theta\left(x_\star-\frac{1}{2}\right)\,\frac{(A_s\,x_\star)^2}{3\cdot 18^2}\,\frac{a^{-4}}{H^2}\left[1-\frac{1}{(2x_\star)^2}\right]^2\left[\,(k_1\mathcal{I}_s)^2+(k_1\mathcal{I}_c)^2\,\right]_{x_\star=p_*/k_1}\\&\simeq\theta\left(x_\star-\frac{1}{2}\right)\,\frac{(A_s\,x_\star)^2}{3\cdot 18^2}\left[1-\frac{1}{(2x_\star)^2}\right]^2\left[\,(\alpha_i\,k_1\,\mathcal{I}_s)^2+(\alpha_i\,k_1\,\mathcal{I}_c)^2\,\right]_{x_\star=p_*/k_1}\end{aligned}\tag{B.116}$$

where in the last line we used the late-time relation

$$\alpha_i\equiv H_{\mathrm{inf}}\tau_*^2\,,\qquad\qquad \frac{a^{-2}}{(aH)^2}=a^{-2}\frac{1}{\mathcal{H}^2}=a^{-2}\,\tau^2\;\;\simeq\;\;\left(\frac{\tau}{H_{\mathrm{inf}}\,\tau_*^2}\right)^{-2}\tau^2\;\;=\;\;\alpha_i^2\tag{B.117}$$

Notice that the quantities $(\alpha_i\,k_1\,\mathcal{I}_{c,s})$ are adimensional functions of the adimensional variable $x_\star$ only, e.g. see the closed relations in page 95. Explicitly

$$\Big(\alpha_i\,k_1\,\mathcal{I}_c(x_\star\,x_\star)\Big)\;=\;\left[-36\pi\frac{\left(d^2+s^2-2\right)^2}{\left(s^2-d^2\right)^3}\theta(s-1)\right]_{s=\frac{2}{\sqrt{3}}x_\star\;\;,\;\;d=0}\tag{B.118}$$

$$\Big(\alpha_i\,k_1\,\mathcal{I}_s(x_\star\,x_\star)\Big)\;=\;\left[-36\frac{\left(d^2+s^2-2\right)}{\left(s^2-d^2\right)^2}\left(\frac{\left(d^2+s^2-2\right)}{\left(s^2-d^2\right)}\log\frac{\left(1-d^2\right)}{|s^2-1|}+2\right)\right]_{s=\frac{2}{\sqrt{3}}x_\star\;\;,\;\;d=0}\tag{B.119}$$

This result is discussed in page 69.

Appendix C

The sourced GWB from primordial scalar perturbations with the operatorial CTP formalism: a subdominant contribution

This appendix has the purpose of supporting the contents of chapter 6. It provides additional details for the calculations which were first introduced there but, for the sake of readability, were omitted.

C.1 The source. Expanding the action |Act 2

In this section we expand the action eq. (B.13) in the perturbations to find the interactions of the form $h_{ij}\, h_{kl}\, \Phi$, see (5.38). The indices here contract each others or with derivative operators, which we have omitted, in such a way to obtain scalar quantities. We use the notation of Appendix B.
For later convenience, it is useful to define the quantity $\hat{\mathcal{L}}$

$$S \equiv \int d^4x \, \sqrt{-g}\, \hat{\mathcal{L}} = \int d^4x \, \sqrt{-g} \left[\frac{M_p^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V \right] \tag{C.1}$$

notice the difference with the lagrangian density $\mathcal{L}$. The two are related each other by $\mathcal{L} = \sqrt{-g}\hat{\mathcal{L}}$.

Once we insert the explicit expansion (B.4) for the perturbed metric determinant, the sought-after interactions are conveniently expressed

$$\mathcal{L} \supset \left[\sqrt{-g} \left(\frac{M_p^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V \right) \right]_{h^2\Phi} = a^4 \left[\hat{\mathcal{L}}|_{h^2\Phi} - 2\Phi\, \hat{\mathcal{L}}|_{h^2} - \frac{1}{4} h_{ij} h_{ij}\, \hat{\mathcal{L}}|_{\Phi} - \frac{1}{2} h^{ij}\, h_{ij}\, \Phi\, \hat{\mathcal{L}}|_0 \right] \tag{C.2}$$

where we have

$$\hat{\mathcal{L}}|_{h^2\Phi} = \frac{M_p^2}{2} R|_{h^2\Phi} \tag{C.3}$$

$$\hat{\mathcal{L}}|_{h^2} = \frac{M_p^2}{2} R|_{h^2} \tag{C.4}$$

$$\hat{\mathcal{L}}|_{\Phi} = \frac{M_p^2}{2} R|_{\Phi} - a^{-2} \Phi \left(\varphi_0' \right)^2 + a^{-2} \varphi_0' \delta\varphi' - \frac{\partial V}{\partial \varphi_0} \delta\varphi \tag{C.5}$$

$$\hat{\mathcal{L}}|_0 = \frac{M_p^2}{2} R|_0 + \frac{1}{2a^2} {\varphi_0'}^2 - V(\varphi_0) \tag{C.6}$$

Notice that $\mathcal{L}|_0$ denotes a function of background quantities only.
The calculation is divided in two contributions: the one coming from the geometry and the remainder from the inflaton field.

1/2 | Let us start from the geometry: we need the Ricci scalar. After a little amount of algebra, we get the quantities

$$R|_0 = \frac{6a''}{a^3} \tag{C.7}$$

$$R|_\Phi = -\frac{6}{a^2}\left(\Phi'' + 4\mathcal{H}\Phi' + 2\frac{a''}{a}\Phi - \frac{1}{3}\nabla^2\Phi\right) \tag{C.8}$$

$$R|_{h^2} = -\frac{1}{a^2}\left[h_{ij}\left(h''_{ij} - \nabla^2 h_{ij} + 3\mathcal{H}h'_{ij}\right) + \frac{3}{4}\left(h'_{ij}h'_{ij} - \partial_k h_{ij}\partial_k h_{ij}\right) + \frac{1}{2}\partial_i h_{jk}\partial_j h_{ik}\right] \tag{C.9}$$

$$\begin{aligned} R|_{h^2\Phi} = -\frac{2}{a^2}\Bigg\{ h_{ij}\Bigg[&\left(h''_{ij} + 3\mathcal{H}h'_{ij} - 3\nabla^2 h_{ij}\right)\Phi + h'_{ij}\Phi' - \left(3\partial_k h_{ij} - 2\partial_j h_{ik}\right)\partial_k\Phi + h_{jk}\partial_i\partial_k\Phi \Bigg] \\ &+ h^2_{ij}\Big(\Phi'' + 3\mathcal{H}\Phi' - \nabla^2\Phi\Big) + \frac{3}{4}\Phi\Big(h'_{ij}h'_{ij} + 2\partial_i h_{jk}\partial_j h_{ik} - 3\partial_k h_{ij}\partial_k h_{ij}\Big)\Bigg\} \end{aligned} \tag{C.10}$$

The contribution from the geometry is easily obtained with eq.s(C.3)-(C.6) and the expressions above, it is

$$\begin{aligned} \mathcal{L} \supset \frac{M_p^2}{2}a^4 &\left[R|_{h^2\Phi} - 2\Phi\, R|_{h^2} - \frac{1}{4}h_{ij}h_{ij}\, R|_\Phi - \frac{1}{2}h_{ij}h_{ij}\Phi\, R|_0\right] \\ &= \frac{M_p^2}{2}a^2\left\{\frac{1}{2}h_{ij}h_{ij}\left(\Phi'' + 4\mathcal{H}\Phi' + \nabla^2\Phi\right) - \left(\partial_k h_{ij}\right)^2\Phi\right\} \end{aligned} \tag{C.11}$$

2/2 | The contribution from the inflaton field is

$$\begin{aligned} \mathcal{L} \supset -a^2\frac{1}{2}h_{ij}h_{ij}&\left[\frac{1}{2}\varphi'_0\delta\varphi' - a^2\left(\frac{1}{2}\frac{\partial V}{\partial\varphi_0}\delta\varphi + \Phi V(\varphi_0)\right)\right] \\ &= -a^2\frac{M_p^2}{2}h_{ij}h_{ij}\left(\Phi'' + 3\mathcal{H}\Phi'\right) \end{aligned} \tag{C.12}$$

Summing the two contributions, we obtain the desired interaction lagrangian

$$\mathcal{L}^{\text{int}}|_{h^2\Phi} = -\frac{M_p^2}{2}a^2\left[\frac{1}{2}h_{ij}h_{ij}\left(\Phi'' + 2\mathcal{H}\Phi' - \nabla^2\Phi\right) + \left(\partial_k h_{ij}\right)^2\Phi\right] \tag{C.13}$$

Let us summarize the procedure we followed. We started from the action eq. (C.1) and we expanded in the perturbations. We were interested in the dynamics of the gravitational waves. From the expansion, first we have got the bilinear (free) kinetic term for the GWs eq. (5.16). We then focussed on the interactions, so we expanded the action in higher terms n-linear ($n > 2$) in the fields. We were interested in the interactions involving the GW field. Therefore we were searching for n-linear terms containing at least one h_{ij} field. We first found the interaction eq. (B.20), which is linear in the tensor perturbation and bilinear in the scalar perturbation. In the case when there is an enhancement in the primordial scalar spectrum, we had claimed - and eventually we demonstrated it - that the dominant contribution to the 2-point GW correlator came from the interaction eq. (B.20). However, the interaction lagrangian to third-order in perturbation theory contains another term, the one we have just found here. The full interaction lagrangian to third order in the perturbations is therefore given by the sum eq. (B.20) and eq. (C.13), where the former is responsible for the dominant contribution to the 2-point enhanced GW correlator and the latter is A_s^{-1} suppressed.

The complete interaction lagrangian for the sources of the second-order GWs is

$$\begin{aligned}\mathcal{L}^{\text{int}} \equiv \mathcal{L}^{\text{int}}|_{h\Phi\Phi} + \mathcal{L}^{\text{int}}|_{h^2\Phi} &= -\frac{1}{2}\, M_p^2\, a^2 h_{ij} \left[-2\partial_i\Phi\partial_j\Phi - \partial_i\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\partial_j\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)\right] \\ &\quad -\frac{1}{2}\, M_p^2\, a^2 \left[\frac{1}{2}h_{ij}h_{ij}\left(\Phi''+2\mathcal{H}\Phi'-\nabla^2\Phi\right)+\left(\partial_k h_{ij}\right)^2\Phi\right]\end{aligned} \tag{C.14}$$

C.2 Equation of motion for the sourced GW |Act 2

The evolution equation for the second-order sourced gravitational waves follows from the interaction lagrangian we reported above. We have already calculated the Euler-Lagrange equation for the GW in the case where only the dominant source terms (5.37) were considered, see eq. (B.25).

This time, we perform the variation of the action, accounting also for the contributions above. In particular, they enter the GW Euler-Lagrange equation with

$$\partial_\mu \frac{\partial \mathcal{L}|_{h^2\Phi}}{\partial\left(\partial_\mu h_{ij}\right)} = -M_p^2\, a^2\left(\nabla^2 h_{ij}\Phi + \partial_k h_{ij}\partial_k\Phi\right) \tag{C.15}$$

$$-\frac{\partial \mathcal{L}|_{h^2\Phi}}{\partial h_{ij}} = \frac{M_p^2}{2}\, a^2\, h_{ij}\Big[\Phi''+2\mathcal{H}\Phi'-\nabla^2\Phi\Big] \tag{C.16}$$

The Euler-Lagrange equation for the second-order sourced GW, comprehensive of all the second-order sources from primordial perturbations, is

$$\begin{aligned}h''_{ij}+2\mathcal{H}h'_{ij}-\nabla^2 h_{ij} &= -2\left[4\,\Phi\,\Phi_{,ij}+2\,\Phi_{,i}\Phi_{,j}-\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)_{,i}\left(\frac{\Phi'}{\mathcal{H}}+\Phi\right)_{,j}\right]^{TT} \\ &\quad + 4\Big[\partial_k\left(\Phi\partial_k h_{ij}\right)\Big]^{TT} - \,2\,h_{ij}\Big[\Phi''+2\mathcal{H}\Phi'-\nabla^2\Phi\Big]\end{aligned} \tag{C.17}$$

where the tensor perturbation h_{ij} in the second line is the linear GW field from the inflation eq. (5.20). One should not confuse it with the second-order GW field in the l.h.s., we used the same symbol because in fact no ambiguity may ever occur if one counts the perturbative order of each term.

For example, to remove any doubts one may express the fields in terms of the perturbative series

$$h_{ij} = \sum_{n=1}^{\infty} h_{ij}^{(n)}\ , \qquad\qquad \Phi = \sum_{n=1}^{\infty}\Phi^{(n)} \tag{C.18}$$

here $h_{ij}^{(1)}$ is the first-order GW field, $h_{ij}^{(2)}$ the second-order GW field and so on. The Euler-Lagrange equation, order-by-order in the perturbations, is obtained inserting this expansion in eq. (C.17). At the first/second-order we have

$$h''^{(1)}_{ij} + 2\,\mathcal{H}\,h'^{(1)}_{ij} - \nabla^2\,h^{(1)}_{ij} = 0 \tag{C.19}$$

$$\begin{aligned} h''^{(2)}_{ij} + 2\,\mathcal{H}\,h'^{(2)}_{ij} - \nabla^2\,h^{(2)}_{ij} = -2\Big[4\,\Phi^{(1)}\,\Phi^{(1)}_{,ij} + 2\,\Phi^{(1)}_{,i}\Phi^{(1)}_{,j} - \left(\frac{\Phi'^{(1)}}{\mathcal{H}} + \Phi^{(1)}\right)_{,i}\left(\frac{\Phi'^{(1)}}{\mathcal{H}} + \Phi^{(1)}\right)_{,j}\Big]^{TT} \\ +4\Big[\partial_k\left(\Phi^{(1)}\partial_k h^{(1)}_{ij}\right)\Big]^{TT} - \;2\,h^{(1)}_{ij}\Big[\Phi''^{(1)} + 2\mathcal{H}\Phi'^{(1)} - \nabla^2\Phi^{(1)}\Big] \end{aligned} \tag{C.20}$$

The solution for the first-order GWs in the radiation dominated era was given in eq. (B.70). The key point here is the enhancement in the primordial scalar perturbation $\sim \Phi^{(1)}$. The enhancement is the feature we are interested. For convenience, one may think of the enhancement $\sim A_s$ as parametrically much bigger than the conventional $\mathcal{P}_\zeta^{(0)}$. Therefore the source in second-line is small in comparison with those in the first-line, because it is A_s-times smaller. Also, the second-line contains the primordial tensor perturbation, which is typically smaller than $\mathcal{P}_\zeta^{(0)}$.

C.3 The sourced GWB from primordial scalar perturbations with the operatorial CTP formalism |Act 2

This section is in support of sec. 6.2, where we present synthetically the application of the in-in formalism to the calculation of the sub-dominant contribution to the second-order GWs production in RD era from primordial tensor and enhanced-scalar perturbations. Again, the endpoint is the 2-point GW function.

We take advantage of the previous discussion about the classicality of the primordial first-order scalar/tensor perturbation to consider the fields ζ, h_λ from the very beginning as classical functions. That is, they are $\mathbb{C}$-valued functions times the identity operator acting on the quantum states. This greatly reduces the amount of calculations.

C.3.1 The interaction lagrangian

It is convenient to perform the calculations using the field v_λ, in terms of which the action is canonically normalized. The interaction lagrangian, in the RD era, must be a functional of the fields v_λ, ζ. We use the transfer function eq. (5.41) to express the Φ field in terms of the curvature perturbation ζ. We are concerned with the sub-dominant sources only.

The interaction lagrangian is

$$L^{(\mathrm{GW})}_{int}|_{h^2\Phi} \;\equiv\; \int d^3x\,\mathcal{L}^{(\mathrm{GW})}_{int}|_{h^2\Phi} = -\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\,v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\,\zeta^*(\mathbf{k}_1+\mathbf{k}_2)\,\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) \tag{C.21}$$

here we used the reality condition in momentum-space $\zeta(-\vec{p}) = \zeta^*(\vec{p})$, and we introduced the function

$$\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) \equiv \frac{2}{3}\,|k_1+k_2|^2\,\mathrm{e}^{\lambda}_{ij}(\hat{k}_1)\,\mathrm{e}^{\sigma}_{ij}(\hat{k}_2) \left[T''\left(|k_1+k_2|\tau\right) + \frac{2\mathcal{H}}{|k_1+k_2|}T'\left(|k_1+k_2|\tau\right) + \left(1-\frac{2\,k_1\cdot k_2}{(k_1+k_2)^2}\right)T\left(|k_1+k_2|\tau\right)\right] \tag{C.22}$$

The promotion of the classical theory to the quantum realm does not present any difficulties. Treating the first-order scalar/tensor fields as classical external sources corresponds, in practise, in the consideration of the following quantum hamiltonian

$$\hat{H}^I_{int}(\tau) = -\hat{L}^{(\mathrm{GW})}_{int}|_{h\,h\Phi} \;=\; 2\int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\;\hat{v}_\lambda(\tau,\,\mathbf{k}_1)\;v_\sigma(\tau,\,\mathbf{k}_2)\;\zeta^*(\mathbf{k}_1+\mathbf{k}_2)\;\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k}_1,\,\mathbf{k}_2) \tag{C.23}$$

The reader is invited to note that the only quantum operator in this interaction comes from the sourced GW field, as we emphasized by momentarily reintroducing the hats. One may argue in disfavour of using the same notation for the external classical tensor source and the quantum tensor perturbation. However, we note that no possible ambiguities ever occur. For example, if we would have associated the second-order GW quantum field with $v_\sigma(\tau,\,\mathbf{k}_2)$ then a straightforward relabelling of the dummy variables/indices and the proper accounting of the properties of the $\mathcal{R}_{\lambda\sigma}$-function would be sufficient to recover the form above. This explains the factor 2 appearing in (C.23).

C.3.2 Subdominant contribution

The implementation of the in-in master equation follows closely the one we pursued in the previous. The contribution to the 2-point GW function from (C.23) is contained in the N=2 expansion. It is denoted by $\delta_2^{(\mathrm{b})}$ and it is given by

$$\begin{aligned}\delta_2^{(\mathrm{b})}\Big\langle\hat{v}_\lambda(\tau,\,\mathbf{k}_1)\,\hat{v}_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle \equiv -4\int^{\tau}d\tau_2\int^{\tau_2}d\tau_1\sum_{\alpha\beta\gamma\delta}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3}&\\ \times\,\mathcal{R}_{\alpha\beta}(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{R}_{\gamma\delta}(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\,\Big\langle\zeta^*(\mathbf{p}_1+\mathbf{q}_1)\,\zeta^*(\mathbf{p}_2+\mathbf{q}_2)\Big\rangle&\\ \times\Big\langle\Big[\hat{v}_\alpha(\tau_1,\,\mathbf{p}_1)\,v_\beta(\tau_1,\,\mathbf{q}_1)\,,\,\Big[\hat{v}_\gamma(\tau_2,\,\mathbf{p}_2)\,v_\delta(\tau_2,\,\mathbf{q}_2)\,,\,\hat{v}_\lambda(\tau,\,\mathbf{k}_1)\,\hat{v}_\sigma(\tau,\,\mathbf{k}_2)\Big]\Big]\Big\rangle&\end{aligned} \tag{C.24}$$

This expression is much more clean if we move the classical scalar & tensor external sources and leave between the vacuum expectation value only the second-order GW quantum operator

$$\begin{aligned}\delta_2^{(\mathrm{b})}\Big\langle\hat{v}_\lambda(\tau,\,\mathbf{k}_1)\,\hat{v}_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = -4\int^{\tau}d\tau_2\int^{\tau_2}d\tau_1\sum_{\alpha\beta\gamma\delta}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3}&\\ \times\,\mathcal{R}_{\alpha\beta}(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{R}_{\gamma\delta}(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\,\Big\langle\zeta^*(\mathbf{p}_1+\mathbf{q}_1)\,\zeta^*(\mathbf{p}_2+\mathbf{q}_2)\Big\rangle\,\Big\langle v_\beta(\tau_1,\,\mathbf{q}_1)\,v_\delta(\tau_2,\,\mathbf{q}_2)\Big\rangle&\\ \times\Big\langle\Big[\hat{v}_\alpha(\tau_1,\,\mathbf{p}_1)\,,\,\Big[\hat{v}_\gamma(\tau_2,\,\mathbf{p}_2)\,,\,\hat{v}_\lambda(\tau,\,\mathbf{k}_1)\,\hat{v}_\sigma(\tau,\,\mathbf{k}_2)\Big]\Big]\Big\rangle&\end{aligned} \tag{C.25}$$

The meaning of the angled brackets containing the external scalar and tensor sources (second line) is that of the ensemble average. The angled brackets containing the quantum fields (third line) denote the vacuum ket/bra states. We have conventionally associated the external sources as the fields with dummy variables $\vec{q}_i\;\;i=1,2$, so that the hats on quantum operators is no longer necessary.

Let us simplify this expression starting from the innermost commutator. This is analogous to eq. (B.73), and the identity eq. (B.72) must be used. This way, the last line of eq. (C.25) is written as the sum of products of two GW commutators

$$\begin{aligned}\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= -4\int^\tau d\tau_2\int^{\tau_2} d\tau_1\sum_{\alpha\beta\gamma\delta}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3}\\ &\times\mathcal{R}_{\alpha\beta}(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{R}_{\gamma\delta}(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\,\Big\langle\zeta^*(\mathbf{p}_1+\mathbf{q}_1)\,\zeta^*(\mathbf{p}_2+\mathbf{q}_2)\Big\rangle\Big\langle v_\beta(\tau_1,\,\mathbf{q}_1)\,v_\delta(\tau_2,\,\mathbf{q}_2)\Big\rangle\\ \times\Bigg\{&\Big[v_\gamma(\tau_2,\,\mathbf{p}_2),\,v_\lambda(\tau,\,\mathbf{k}_1)\Big]\Big[v_\alpha(\tau_1,\,\mathbf{p}_1)\,,\;v_\sigma(\tau,\,\mathbf{k}_2)\Big]+\Big[v_\gamma(\tau_2,\,\mathbf{p}_2),\,v_\sigma(\tau,\,\mathbf{k}_2)\Big]\Big[v_\alpha(\tau_1,\,\mathbf{p}_1)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big]\Bigg\}\end{aligned} \tag{C.26}$$

Notice that the second term in curly brackets can be obtained from the first with $(\lambda, \mathbf{k_1}) \leftrightarrow (\sigma, \mathbf{k}_2)$.

$$\begin{aligned}\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= -4\int^\tau d\tau_2\int^{\tau_2} d\tau_1\sum_{\alpha\beta\gamma\delta}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3}\\ &\times\mathcal{R}_{\alpha\beta}(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{R}_{\gamma\delta}(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\,\Big\langle\zeta^*(\mathbf{p}_1+\mathbf{q}_1)\,\zeta^*(\mathbf{p}_2+\mathbf{q}_2)\Big\rangle\Big\langle v_\beta(\tau_1,\,\mathbf{q}_1)\,v_\delta(\tau_2,\,\mathbf{q}_2)\Big\rangle\\ &\times\Bigg\{\Big[v_\gamma(\tau_2,\,\mathbf{p}_2),\,v_\lambda(\tau,\,\mathbf{k}_1)\Big]\Big[v_\alpha(\tau_1,\,\mathbf{p}_1)\,,\;v_\sigma(\tau,\,\mathbf{k}_2)\Big]+\Big((\lambda,\mathbf{k}_1)\leftrightarrow(\sigma,\mathbf{k}_2)\Big)\Bigg\}\end{aligned} \tag{C.27}$$

The second term in the curly bracket renders the integrand symmetric over the $\tau_2 = \tau_1$ axis. For example, one can see this by performing the variable relabelings $(\alpha,\, p_1) \leftrightarrow (\gamma,\, p_2)$, $(\beta,\, q_1) \leftrightarrow (\delta,\, q_2)$ and by rearranging the factors. This let us trade the original time-domain for the square $[0,\, \tau] \times [0,\, \tau]$ by accounting of the doubling of the original integral by dividing by two. This is analogous to the case of page 101 for the dominant contribution.

$$\begin{aligned}\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle &= -2\int_{\tau_0}^\tau d\tau_2\int_{\tau_0}^{\tau} d\tau_1\sum_{\alpha\beta\gamma\delta}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3}\\ &\times\Big\langle\zeta^*(\mathbf{p}_1+\mathbf{q}_1)\,\zeta^*(\mathbf{p}_2+\mathbf{q}_2)\Big\rangle\Bigg\{\mathcal{R}_{\alpha\beta}(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\,\mathcal{R}_{\gamma\delta}(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\,\Big\langle v_\beta(\tau_1,\,\mathbf{q}_1)\,v_\delta(\tau_2,\,\mathbf{q}_2)\Big\rangle\\ &\times\Big[v_\gamma(\tau_2,\,\mathbf{p}_2)\,,\;v_\lambda(\tau,\,\mathbf{k}_1)\Big]\Big[v_\alpha(\tau_1,\,\mathbf{p}_1)\,,\;v_\sigma(\tau,\,\mathbf{k}_2)\Big]+\Big(\tau_1\leftrightarrow\tau_2\Big)\Bigg\}\end{aligned} \tag{C.28}$$

The reader is invited to note that the integrand is proportional to $\mathcal{P}_\zeta^{(0)}\,\mathcal{P}_h^{(0)}$. We replace the 2-point scalar/tensor function with the expressions (5.35)/(5.61).

The raw expression is very clumsy. It is reported below

$$\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = 2\left(\frac{H_{\rm inf}}{\sqrt{2}}\right)^2\frac{1}{k_1\,k_2}\int_{\tau_0}^{\tau}d\tau_2\int_{\tau_0}^{\tau}d\tau_1\sum_{\alpha\beta\gamma\delta}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{q}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d^3\vec{q}_2}{(2\pi)^3}$$
$$\Bigg\{(2\pi)^{12}\delta_{\beta\delta}\,\delta^{(3)}(p_1+p_2+q_1+q_2)\delta^{(3)}(q_1+q_2)\;\delta_{\gamma\lambda}\delta_{\alpha\sigma}\delta^{(3)}(k_1+p_2)\delta^{(3)}(p_1+k_2)\;\frac{1}{q_1^3}\frac{2\pi^2}{|\mathbf{p}_1+\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)$$
$$\Big(\frac{a(\tau_1)}{k_1\tau_1}\,\sin(q_1\tau_1)\,\sin\left[k_1(\tau_1-\tau)\right]\,\mathcal{R}_{\alpha\beta}(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1)\Big)\,\Big(\frac{a(\tau_2)}{k_2\tau_2}\,\sin(q_2\tau_2)\,\sin\left[k_1(\tau_2-\tau)\right]\,\mathcal{R}_{\gamma\delta}(\tau_2,\,\mathbf{p}_2,\,\mathbf{q}_2)\Big)$$
$$+\Big(\tau_1\leftrightarrow\tau_2\Big)\Bigg\} \tag{C.29}$$

There are four δ-functions. One factorizes the total momentum conservation $\delta^{(3)}(k_1+k_2)$. There remains three independent δ-functions. We perform the trivial integrations over $d^3\vec{q}_2$, $d^3\vec{p}_1$, $d^3\vec{p}_2$. The resulting expression is (we change $\mathbf{q}_1 \;\rightarrow\; \textbf{-}\mathbf{q}_1$)

$$\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\;\left(\frac{H_{\rm inf}}{\sqrt{2}}\right)^2\;2k_1\sum_\beta\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\frac{1}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)$$
$$\times\int_{\tau_0}^{\tau}d\tau_2\int_{\tau_0}^{\tau}d\tau_1\;\Bigg\{\Big(\frac{a(\tau_1)}{k_1\tau_1}\,\sin(q_1\tau_1)\,\sin\left[k_1(\tau_1-\tau)\right]\,\mathcal{R}_{\sigma\beta}(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\Big)$$
$$\times\Big(\frac{a(\tau_2)}{k_1\tau_2}\,\sin(q_1\tau_2)\,\sin\left[k_1(\tau_2-\tau)\right]\,\mathcal{R}_{\lambda\beta}(\tau_2,\,-\mathbf{k}_1,\,\mathbf{q}_1)\Big)+\Big(\tau_1\leftrightarrow\tau_2\Big)\Bigg\} \tag{C.30}$$

It is most convenient to extract the polarisation vectors out of the $\mathcal{R}_{\alpha\beta}$ function. This is analogous to what we have already done in page 102 for $\mathcal{Q}_\lambda$. We introduce the quantity R

$$\mathcal{R}_{\lambda\sigma}(\tau,\,\mathbf{k},\,\mathbf{q})\;\equiv\;\mathrm{e}_{ij}^{\lambda}(\hat{k})\,\mathrm{e}_{ij}^{\sigma}(\hat{q})\;R(\tau,\,\mathbf{k},\,\mathbf{q}) \tag{C.31}$$

Notice the similarity between the scalar function $R(\tau,\,\mathbf{k},\,\mathbf{q})$ and $Q(\tau,\,\mathbf{k},\,\mathbf{q})$ of eq. (B.96). In fact, the corresponding property holds

$$R(\tau_1,\,\mathbf{p}_1,\,\mathbf{q}_1) = R(\tau_1,\,-\mathbf{p}_1,\,-\mathbf{q}_1) \tag{C.32}$$

This way the expression (C.30) becomes

$$\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\;\left(\frac{H_{\rm inf}}{\sqrt{2}}\right)^2\;2\,k_1\sum_\beta\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\frac{1}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)$$
$$\mathrm{e}_{ij}^{\sigma}(\hat{k}_1)\,\mathrm{e}_{kl}^{\lambda}(-\hat{k}_1)\sum_\beta\mathrm{e}_{ij}^{\beta}(-\hat{q}_1)\,\mathrm{e}_{kl}^{\beta}(\hat{q}_1)\times\int_{\tau_0}^{\tau}d\tau_2\int_{\tau_0}^{\tau}d\tau_1\;\Bigg\{\Big(\frac{a(\tau_1)}{k_1\tau_1}\,\sin(q_1\tau_1)\,\sin\left[k_1(\tau_1-\tau)\right]\,R(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\Big)$$
$$\times\Big(\frac{a(\tau_2)}{k_1\tau_2}\,\sin(q_1\tau_2)\,\sin\left[k_1(\tau_2-\tau)\right]\,R(\tau_2,\,-\mathbf{k}_1,\,\mathbf{q}_1)\Big)+\Big(\tau_1\leftrightarrow\tau_2\Big)\Bigg\} \tag{C.33}$$

The expression above can be further simplified. Let us organize the following in blocks.

(a) The polarisation sum eq. (C.34)

(b) The content of the round brackets. The functions $\mathcal{I}_{c,s}$

(c) Final result

Point (a) - Polarisation vectors

In this subsection we simplify the product

$$\mathrm{e}^{\sigma}_{ij}(\hat{k}_1)\,\mathrm{e}^{\lambda}_{kl}(-\hat{k}_1)\sum_{\beta}\mathrm{e}^{\beta}_{ij}(-\hat{q}_1)\,\mathrm{e}^{\beta}_{kl}(\hat{q}_1) \tag{C.34}$$

appearing in eq. (C.33). The integral for different external $\lambda \neq \sigma$ polarisations vanishes. It is a consequence of the fact that the first-order GWs are unpolarised and the interaction sources do not create GWs with a privileged polarisation.

As a preliminary, we manipulate the identity (A.23) summing over the polarisations, we have

$$\begin{aligned}
\sum_{\beta}\mathrm{e}^{\beta}_{ab}(-\hat{q})\,\mathrm{e}^{\beta}_{cd}(\hat{q}) &= \sum_{\beta=\pm1}\frac{1}{4}\Big[\delta_{ac}-\hat{q}_a\hat{q}_c+i\beta\epsilon_{ace}\hat{q}_e\Big]\Big[\delta_{bd}-\hat{q}_b\hat{q}_d+i\beta\epsilon_{bdf}\hat{q}_f\Big]\\
&= \sum_{\beta=\pm1}\frac{1}{4}\Big[\delta_{ac}\delta_{bd}-(\delta_{ac}\hat{q}_b\hat{q}_d+\delta_{bd}\hat{q}_a\hat{q}_c)+i\alpha(\cdots)-\alpha^2\epsilon_{ace}\epsilon_{bdf}\hat{q}_e\hat{q}_f+\hat{q}_a\hat{q}_b\hat{q}_c\hat{q}_d\Big]\\
&= \frac{1}{2}\left[\delta_{ac}\delta_{bd}-(\delta_{ac}\hat{q}_b\hat{q}_d+\delta_{bd}\hat{q}_a\hat{q}_c)-\epsilon_{ace}\epsilon_{bdf}\hat{q}_e\hat{q}_f+\hat{q}_a\hat{q}_b\hat{q}_c\hat{q}_d\right]
\end{aligned} \tag{C.35}$$

A straightforward but long computation let us express the product (C.34) as (subscripts '$_1$' are omitted)

$$\begin{aligned}
&\delta_{\lambda\sigma}\left(\sum_{i,j,k,l,\beta}\mathrm{e}^{\lambda}_{ij}(\hat{k})\,\mathrm{e}^{\lambda}_{kl}(-\hat{k})\,\mathrm{e}^{\beta}_{ij}(-\hat{q})\,\mathrm{e}^{\beta}_{kl}(\hat{q})\right)\\
&\quad= \delta_{\lambda\sigma}\sum_{i,j,k,l,\beta}\frac{1}{4}\Big[\delta_{ik}\delta_{jl}-\Big(\delta_{ik}\hat{k}_j\hat{k}_l+\delta_{jl}\hat{k}_i\hat{k}_k\Big)-i\lambda\left(\delta_{ik}\epsilon_{jlh}+\delta_{jl}\epsilon_{ikh}\right)\hat{k}_h\\
&\qquad\qquad -\lambda^2\epsilon_{ikg}\epsilon_{jlh}\hat{k}_g\hat{k}_h+i\lambda\hat{k}_h\Big(\hat{k}_i\hat{k}_k\epsilon_{jlh}+\hat{k}_j\hat{k}_l\epsilon_{ikh}\Big)+\hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l\Big]\\
&\qquad\qquad\times\frac{1}{2}\Big[\delta_{ik}\delta_{jl}-(\delta_{ik}\hat{q}_j\hat{q}_l+\delta_{jl}\hat{q}_i\hat{q}_k)-\epsilon_{ike}\epsilon_{jlf}\hat{q}_e\hat{q}_f+\hat{q}_i\hat{q}_j\hat{q}_k\hat{q}_l\Big]
\end{aligned} \tag{C.36}$$

Using the relations

$$\lambda^2 = 1\ , \qquad\qquad \epsilon_{ikg}\epsilon_{jlh}\epsilon_{ike}\epsilon_{jlf} = 4\delta_{ig}\delta_{kh} \tag{C.37}$$

and also expressing $\hat{k}_1\cdot\hat{q}_1 = \cos\theta$, one finally obtains the neat expression

$$\delta_{\lambda\sigma}\Big(\sum_{i,j,k,l,\beta}\mathrm{e}^{\lambda}_{ij}(\hat{k}_1)\,\mathrm{e}^{\lambda}_{kl}(-\hat{k}_1)\,\mathrm{e}^{\beta}_{ij}(-\hat{q}_1)\,\mathrm{e}^{\beta}_{kl}(\hat{q}_1)\Big) = \delta_{\lambda\sigma}\,\frac{1}{8}\Big(1+6\cos^2\theta+\cos^4\theta\Big) \tag{C.38}$$

Point (b) - The $\mathcal{I\!I}_{c,s}$ functions

Let us introduce the functions

$$\begin{aligned}\mathcal{I\!I}_c(x,\,y) \equiv\; & \frac{k_1}{|k_1-q_1|^2}\int_0^{\tau} d\tau_1\, \left(\frac{a(\tau_1)}{k_1\tau_1}\,\sin(q_1\tau_1)\,\sin(k_1\tau_1)\,R(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\right)\Big|_{(x,y)} \\ =\; & \frac{k_1}{(y\,k_1)^2}\int_0^{\tau} d\tau_1\,\frac{a(\tau_1)}{k_1\tau_1}\,\sin(x\,k_1\tau_1)\,\sin(k_1\tau_1)\,R(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\Big|_{(x,y)}\end{aligned} \tag{C.39}$$

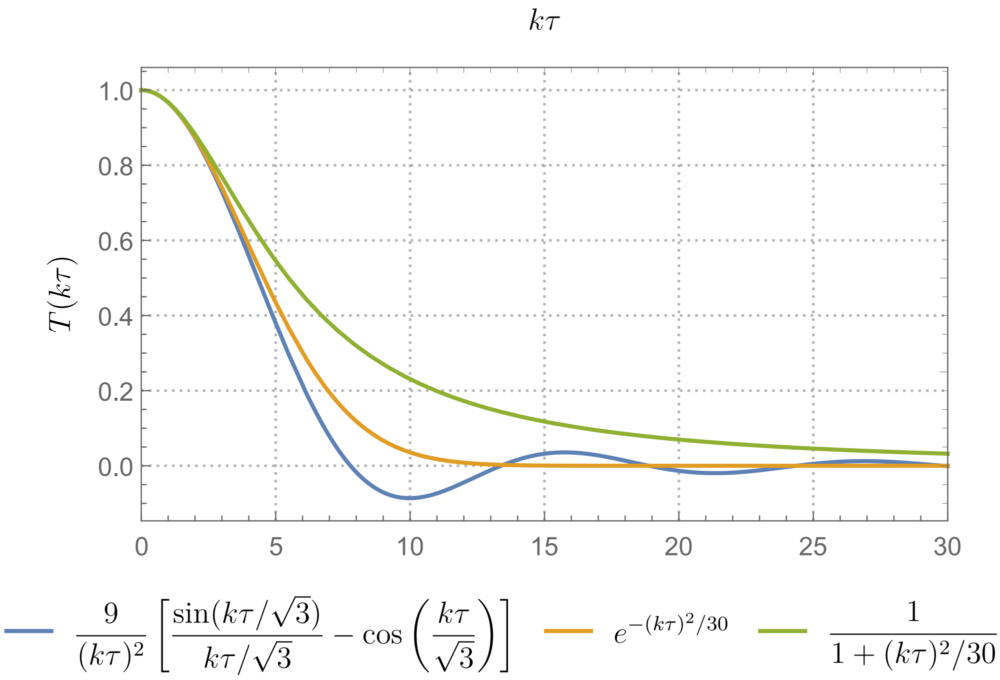
$$\begin{aligned}\mathcal{I\!I}_s(x,\,y) \equiv\; & -\frac{k_1}{|k_1-q_1|^2}\int_0^{\tau} d\tau_1\, \left(\frac{a(\tau_1)}{k_1\tau_1}\,\sin(q_1\tau_1)\,\cos(k_1\tau_1)\,R(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\right) \\ =\; & -\frac{k_1}{(y\,k_1)^2}\int_0^{\tau} d\tau_1\,\frac{a(\tau_1)}{k_1\tau_1}\,\sin(x\,k_1\tau_1)\,\cos(k_1\tau_1)\,R(\tau_1,\,\mathbf{k}_1,\,-\mathbf{q}_1)\Big|_{(x,y)}\end{aligned} \tag{C.40}$$

where the $(x,y) = \left(\frac{q_1}{k_1},\,\frac{|\mathbf{q}_1\text{-}\mathbf{k}_1|}{k_1}\right)$. These quantities are adimensional functions of the adimensional variables (x,y), defined accordingly to (B.54). The functions $\mathcal{I\!I}_{c,s}$ in eq.s (C.39)-(C.40) are similar to eq.s (B.42)-(B.43). Analogously to the earlier case, these quantities depends on τ though the integration domain. Again, one may 'eliminate' the τ-dependence with the physical insight that the late-time observables ($\tau \gg\; \tau_*$) are insensitive to what happened in the far past. The validity of this approximation is ultimately supported by the structure of the integrand, which contains the transfer functions. In fact, it decays $\sim (k\tau)^{-3}$ for large times and the corresponding contributions to the total integral are small.
For instance, in the case of $\mathcal{I\!I}_c$ the 'late-time approximation' leads to

$$\begin{aligned}\mathcal{I\!I}_c(x,\,y) = k_1 & \int_0^{\tau} d\tau_1\,\frac{a(\tau_1)}{k_1\tau_1}\,\sin(x\,k_1\tau_1)\,\sin(k_1\tau_1)\,\frac{2}{3}\left[T''(k_1\,y\,\tau_1)+\frac{2\mathcal{H}}{k_1\,y}\,T'(k_1\,y\,\tau_1)+\frac{1+x^2}{y^2}\,T(k_1\,y\,\tau_1)\right] \\ = & \int_0^{k_1\,\tau} dz\,\left(\frac{a(z/k_1)}{z}\right)\,\sin(xz)\,\sin(z)\,\frac{2}{3}\left[T''(y\,z)+\frac{2\mathcal{H}(z/k_1)}{k_1\,y}\,T'(y\,z)+\frac{1+x^2}{y^2}\,T(y\,z)\right] \\ \simeq\; & \frac{2}{3}\frac{1}{H_{\text{inf}}\,k_1\,\tau_*^2}\int_0^{\infty} dz\,\sin(xz)\,\sin(z)\,\left[T''(y\,z)+\frac{2}{y\,z}\,T'(y\,z)+\frac{1+x^2}{y^2}\,T(y\,z)\right]\end{aligned} \tag{C.41}$$

An analogous expression holds for $\mathcal{I\!I}_s$, with $\sin(z) \to$ - $\cos(z)$. In the last line one has approximated the scale factor (and the Hubble rate) for late-times. It appears the adimensional normalization $\alpha_i\,k_1 = H_{\text{inf}}\tau_*^2\,k_1$, analogously to the case of eq.s (B.60)-(B.61). However we will see later that, as we expect, the α_i does not appear in the final result (e.g. the power spectrum) because it is compensated once one returns to the tensor perturbation $v_\lambda \to h_\lambda$.
The reader is invited to notice that these approximations are consistent with those one makes to find explicitly the functions $\mathcal{I}_{c,s}$ for the dominant contribution. The oscillatory behaviour of the transfer function makes the numerical evaluation of these functions difficult. Despite it exists at least the closed form for $\mathcal{I\!I}_c$, it is not particularly illuminating. Contrariwise, to evaluate these functions numerically/analytically we will use the further approximation of replacing the oscillatory transfer

function with a resummed exponential series (orange line below)



(C.42)

The goodness of this approximation is to be found in the comparison between the numerical integration (using eq. (5.42)) and the approximated one (with $T(k\tau) = e^{-x^2/30}$). Also, we notice that is appears also in literature, e.g. see [33, App.B][1] where the transfer function (in blue) is approximated with the green one.

Final steps

Using the trigonometric identity $\sin[k_1(\tau_1 - \tau)] = \sin(k_1\tau_1)\cos(k_1\tau) - \cos(k_1\tau_1)\sin(k_1\tau)$, the expression (C.33) assumes the neat form

$$\begin{aligned}
\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\, v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\left(\frac{H_{\mathrm{inf}}}{\sqrt{2}}\right)^2\,4\,k_1\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\frac{1}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)\\
\frac{1}{8}\Big(1+6\cos^2\theta+\cos^4\theta\Big)\;(k_1\,y^2)^2\Big[\cos(k_1\tau)\,\mathcal{I}_c(x,\,y)+\sin(k_1\tau)\,\mathcal{I}_s(x,\,y)\Big]\\
\times\,\Big[\cos(k_1\tau)\,\mathcal{I}_c(x,\,y)+\sin(k_1\tau)\,\mathcal{I}_s(x,\,y)\Big]
\end{aligned} \tag{C.43}$$

[1] Be cautious, some typographical errors are present.

or

$$\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\left(\frac{H_{\rm inf}}{\sqrt{2}}\right)^2\,4\,k_1^3\int\frac{d^3\vec{q}_1}{(2\pi)^3}\,\frac{1}{q_1^3}\frac{1}{|\mathbf{k}_1-\mathbf{q}_1|^3}\,\mathcal{P}_\zeta(q_1)$$
$$\frac{1}{8}\Big(1+6\cos^2\theta+\cos^4\theta\Big)\times\left[y^2\bigg(\cos(k_1\tau)\,\mathit{I\!I}_c(x,\,y)+\sin(k_1\tau)\,\mathit{I\!I}_s(x,\,y)\bigg)\right]^2_{(x,y)=\left(\frac{q_1}{k_1},\,\frac{|\mathbf{q}_1-\mathbf{k}_1|}{k_1}\right)} \tag{C.44}$$

Let us change variables, accordingly to (B.54). It will help us in comparing this result with the former for the dominant contribution. We have

$$\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\left(\frac{H_{\rm inf}}{\sqrt{2}}\right)^2\,\frac{1}{2(2\pi)^2}\iint_{\mathcal{D}}dxdy\,\frac{\mathcal{P}_\zeta(x\,k_1)}{(xy)^2}\,y^4$$
$$\times\left[1+6\Big(\frac{1+x^2-y^2}{2x}\Big)^2+\Big(\frac{1+x^2-y^2}{2x}\Big)^4\right]\left[\cos(k_1\tau)\,\mathit{I\!I}_c(x,\,y)+\sin(k_1\tau)\,\mathit{I\!I}_s(x,\,y)\right]^2 \tag{C.45}$$

The domain of integration is the one we have already encountered in eq. (B.57), namely $\mathcal{D}=\Big\{(x,y)\in\mathbb{R}^+\times\mathbb{R}^+\;|\;|1-x|<y<1+x\Big\}$. It is illustrated in fig. B.1.

C.4 The GW power spectrum from enhanced scalar perturbations: subdominant contribution

Finally, let us specialize this result to case-study of enhanced primordial scalar perturbations. Using eq. (5.35), it is easy to prove that the sub-dominant contribution to the second-order GWs sourced from the primordial first-order enhanced scalar perturbation and the first-order tensor perturbation, is given by ($x_\star = p_*/k_1$)

$$\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\left(\frac{H_{\rm inf}}{\sqrt{2}}\right)^2\,\frac{1}{2(2\pi)^2}\iint_{\mathcal{D}}dxdy\,\frac{(A_s\,p_*)}{(xy)^2}\,\frac{1}{k_1}\delta(x-x_\star)\,y^4$$
$$\left[1+6\Big(\frac{1+x^2-y^2}{2x}\Big)^2+\Big(\frac{1+x^2-y^2}{2x}\Big)^4\right]\left[\cos(k_1\tau)\,\mathit{I\!I}_c(x,\,y)+\sin(k_1\tau)\,\mathit{I\!I}_s(x,\,y)\right]^2 \tag{C.46}$$

The δ-function of the x-variable turns the integration from the original domain $\mathcal{D}$ to the $y-$segment such that $|1-x_\star|<y<1+x_\star$. Therefore, we have

$$\delta_2^{(\mathrm{b})}\Big\langle v_\lambda(\tau,\,\mathbf{k}_1)\,v_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\left(\frac{H_{\rm inf}}{\sqrt{2}}\right)^2\,\frac{1}{2(2\pi)^2}\,\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star}dy\,y^2$$
$$\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right]\left[\cos(k_1\tau)\,\mathit{I\!I}_c(x_\star,\,y)+\sin(k_1\tau)\,\mathit{I\!I}_s(x_\star,\,y)\right]^2 \tag{C.47}$$

or, in terms of the tensor field h_{ij}, we have $(v^\lambda(\tau,\mathbf{k}) = \frac{M_p}{2}\, a(\tau)\, h^\lambda(\tau,\mathbf{k}))$

$$\delta_2^{(\mathrm{b})}\Big\langle h_\lambda(\tau,\,\mathbf{k}_1)\, h_\sigma(\tau,\,\mathbf{k}_2)\Big\rangle = (2\pi)^3\,\delta^{(3)}(k_1+k_2)\frac{2\pi^2}{k_1^3}\,\frac{2}{\pi^2}\left(\frac{H_{\rm inf}}{M_p}\right)^2\frac{1}{a^2(\tau)}\frac{1}{8}\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star} dy\; y^2$$
$$\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right]\left[\cos(k_1\tau)\,\mathcal{I}_c(x_\star,\,y)+\sin(k_1\tau)\,\mathcal{I}_s(x_\star,\,y)\right]^2 \quad \text{(C.48)}$$

The GW power spectrum in terms of the tensor perturbation h_{ij} is given by

$$\delta_2^{(b)}\mathcal{P}_h(\tau,k_1) = \frac{2}{\pi^2}\left(\frac{H_{\rm inf}}{M_p}\right)^2\frac{1}{a^2(\tau)}\frac{1}{8}\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star} dy\; y^2\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right]$$
$$\times\left[\cos(k_1\tau)\,\mathcal{I}_c(x_\star,\,y)+\sin(k_1\tau)\,\mathcal{I}_s(x_\star,\,y)\right]^2_{x_\star=\frac{p_*}{k_1}} \quad \text{(C.49)}$$

C.5 The density parameter of the second-order sourced gravitational waves: subdominant contribution

The contribution to the GW power spectrum from the sub-dominant diagram (6.1b) of fig. 6.1 was given in eq. (C.49). Now we are interested in the density parameter of the GWs with power spectrum above. First we need the time-averaged power spectrum, we have

$$\overline{\delta_2^{(b)}\mathcal{P}_h(\tau,k_1)} = \frac{2}{\pi^2}\left(\frac{H_{\rm inf}}{M_p}\right)^2\frac{1}{a^2(\tau)}\frac{1}{8}\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star} dy\; y^2\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right]$$
$$\times\left[\overline{\cos^2(k_1\tau)}\,\mathcal{I}_c^2(x_\star,\,y)+\overline{\sin^2(k_1\tau)}\,\mathcal{I}_s^2(x_\star,\,y)+\overline{\sin(2k_1\tau)}\,\mathcal{I}_c^2(x_\star,\,y)\mathcal{I}_s^2(x_\star,\,y)\right]_{x_\star=\frac{p_*}{k_1}} \quad \text{(C.50)}$$

or

$$\overline{\delta_2^{(b)}\mathcal{P}_h(\tau,k_1)} = \frac{2}{\pi^2}\left(\frac{H_{\rm inf}}{M_p}\right)^2\frac{1}{a^2(\tau)}\frac{1}{8}\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star} dy\; y^2\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right]$$
$$\times\frac{1}{2}\left[\mathcal{I}_c^2(x_\star,\,y)+\mathcal{I}_s^2(x_\star,\,y)\right]_{x_\star=\frac{p_*}{k_1}} \quad \text{(C.51)}$$

Apart from the integration, this expression resembles the eq. (B.115) for the dominant contribution. We indicate the late-time density parameter of the sourced second-order GWs from the sub-dominant

(5.38) as $\Omega^{(2b)}_{(\mathrm{GW})}$. Using the definition eq. (B.110), at late-times one has

$$\Omega^{(2b)}_{(\mathrm{GW})}(k_1) = \frac{1}{48\pi^2}\left(\frac{H_{\inf}}{M_p}\right)^2 \frac{k_1^2}{a^2(\tau)\mathcal{H}^2}\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star} dy\, y^2\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right] \times \left[\mathcal{I\!I}_c^2(x_\star,\, y)+\mathcal{I\!I}_s^2(x_\star,\, y)\right]_{x_\star=\frac{p_*}{k_1}} \tag{C.52}$$

$$\simeq \frac{\Omega^{(0)}_{(\mathrm{GW})}}{4}\frac{A_s}{x_\star}\int_{|1-x_\star|}^{1+x_\star} dy\, y^2\left[1+6\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^2+\Big(\frac{1+x_\star^2-y^2}{2x_\star}\Big)^4\right] \times \left[(\alpha_i\, k_1\, \mathcal{I\!I}_c)^2+(\alpha_i\, k_1\, \mathcal{I\!I}_s^2)\right] \tag{C.53}$$

The quantities $(\alpha_i\, k_1\, \mathcal{I\!I}_{c,s})$ are analogous to the $(\alpha_i\, \mathcal{I}_c)_{c,s}$ encountered in page 107. Again, the quantities $(\alpha_i\, k_1\, \mathcal{I\!I}_{c,s})$ are adimensional functions of the adimensional variables $(x_\star,\, y)$. Moreover, the normalization α_i does not appear in the final result since it is contained in the $\mathcal{I\!I}_{c,s}$ functions, e.g. $\mathcal{I\!I}_{c,s} \propto \alpha_i^{-1}$.
This result is discussed in page 70.

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