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Matrici di scattering sul worldsheet della stringa

Scattering matrices on the string worldsheet

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Introduction

The fundamental forces in our universe are well described by Standard Model, which is a quantum field theory (QFT) with gauge group: $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ and General Relativity, which is the classical theory of gravity. Quantum field theory, being the natural framework for the quantum description of interacting many-particle systems, found a fertile field of applications also in the description of condensed matter systems. Most notably, the framework of QFT can describe superconductivity, superfluidity and critical phenomena.

Although the standard model is well-established and its predictions match perfectly the outcome of scattering events performed at the Large Hadron Collider at CERN, there are some phenomena that call for a better description. In particular our understanding of QFT is mostly perturbative, in the sense that most of the computations are obtained from an expansion of some parameters around a free theory. For instance in the context of low energy quantum chromodynamics (QCD) when the strong coupling $\alpha_s \approx 1$, the usual perturbative analysis breaks down. *Confinement* is a phenomena that manifests itself at this energy scale and indeed it is still not understood. Moreover, General Relativity is only a good description of the classical theory of gravity. In particular one expects that at very short lengths scales, of the order of the Planck's length $l_p \approx 10^{-33}cm$ it breaks down. The gravitational force at these scales should be described through a quantum theory and the best candidate as a theory of quantum gravity is the *String Theory* which also provides a description of strongly interacting QFTs through the AdS/CFT duality.

AdS/CFT. One of the most spectacular results in string theory is the so-called AdS/CFT correspondence [1]. It states that a string theory formulated in a space $AdS_{d+1} \times X$ is dual to a gauge theory with conformal invariance (CFT side) which lives on the boundary of AdS_{d+1}. Here AdS is anti-de Sitter space, which is defined as the $d+1$ dimensional space of constant negative curvature, with isometries $SO(2,d)$. The previous result allows to connect gravity with gauge theory indeed, $SO(2,d)$ is also the symmetry group of a d-dimensional conformal field theory. Moreover it is a special case of the holographic principle [2] that, roughly speaking, states that the informations on the gravitational degrees of freedom in the bulk are encoded in the boundary. The most famous example is the correspondence between $\mathcal{N}=4$ super-Yang-Mills (SYM) theory in four dimensions and type IIB superstring theory on $AdS_5 \times S^5$ which is the product of five dimensional anti-de Sitter space and five dimensional sphere.

Let us spend a few more words about the correspondence, in fact they will be useful in order to appreciate how the concept of *integrability* enters in game. In a gauge theory with a gauge group $SU(N_c)$, there are two relevant parameters: the number of colors N_c which is related to the size of the gauge group and the 't Hooft coupling $\lambda = g_{YM}^2 N_c$, where g_{YM} is the Yang-Mills coupling. While in a string theory, there are the string tension T and the string coupling g_{str}.

The equivalence is supposed to hold at any point in the parameter space, but it is more testable in the *planar* or *large-N* limit, since the gauge theory simplifies a lot, as it is pointed out by 't Hooft [2]. The large-N limit is an approximation in a different direction with respect to the usual perturbation theory, which is an expansion in small g_{YM} at fixed N_c. In the usual perturbative expansions, Feynman diagrams have a higher and higher order in loops at higher order in g_{YM}. By contrast, at large N_c, diagrams naturally ordanise in a genus expansion, where the leading diagrams are planar (*i.e.* can be drawn on a sphere without self interaction) and the subleading have higher genus [3]. At a given genus, diagrams may have an arbitrary high number of loops.

This genus expansion suggests to take $N_c \to \infty$ while λ remains finite (which means taking $g_{YM} \to 0$). Correspondingly, on the string theory side of the duality, one can take an analogous planar limit. This corresponds to taking $g_{str} \to 0$, while the tension T remains fixed. The correspondence identifies the couplings on either side as

$$\lambda \propto T^2 \, , \quad \frac{1}{N_c} \propto \frac{g_{str}}{T^2} \propto \frac{g_{str}}{\lambda} . \tag{1}$$

In particular if $\lambda \gg 1$ we are in the non perturbative regime of the gauge theory, while, on the other side, $T \gg 1$ and hence we are in the perturbative regime of the string theory. This consideration allows us to clarify that the AdS/CFT correspondence is a strong/weak duality. This provides a powerful tool in order to do explicit calculation, in fact depending on the validity regime, one can move from the gauge theory to the string and vice versa.

More precisely, the correspondence provides a way to map observables from one side to the other of the duality in some particular regimes and *integrability* allows to connect the two sides also in the intermediate ones. In particular, since in the large N-limit we are dealing with free strings, natural observables are the string energy levels. Clearly also in the CFT side it is in principle possible to compute the energy spectrum which, in particular, is related to the eigenvalues of the generator of dilatations. And so, as we said before, the aim of integrability is to give a description of the spectrum valid at any intermediate coupling.

Integrability in AdS_5/CFT_4. The first hint about integrability in the large-N limit on the gauge theory side, was discovered in [4]. In particular Minahan and Zarembo showed that the problem of finding the spectrum of a gauge theory can be rephrased in terms of integrable spin chain, indeed intuitively the various gauge fields play the role of the spins pointing in different directions, at leading order in λ, in the space of flavors.

It turns out that, if we take into account loop effects in the computation of the energy levels of the CFT, then we are forced to add interaction more and more complicated to the spin chain Hamiltonian. In particular, at one loop the CFT problem is related to the Heisenberg spin chain (nearest-neighbor interactions), while at two loops the spin chain Hamiltonian has also a next to nearest-neighbors interactions [5]. It turns out that the key object on which one should focus to obtain all-loop results is not the Hamiltonian, which becomes more and more complicated at higher orders, but rather the S-matrix. In fact, this can be fixed by imposing compatibility with the symmetries of the model. In particular, in this context, the S-matrix describes the scattering of collective excitations of the spins of definite momentum, called *spin waves* or *magnons*. For instance, a one-*magnon* state in a simple two-state system would look like

$$|\Psi(p)\rangle = \sum_n e^{ipn} \, |{\downarrow\downarrow} \cdots {\downarrow\uparrow_n\downarrow} \cdots \downarrow\rangle \, , \tag{2}$$

while a two-*magnon* state would feature an incoming and a reflected wave

$$|\Psi(p_1, p_2)\rangle = \sum_{n_1 < n_2} (e^{ip_1 n_1 + ip_2 n_2} + S(p_1, p_2) e^{ip_2 n_1 + ip_1 n_2}) \, |{\downarrow} \cdots {\downarrow\uparrow_{n_1}\downarrow} \cdots {\downarrow\uparrow_{n_2}\downarrow\downarrow}\rangle \tag{3}$$

where $S(p_1, p_2)$ is the S-matrix (which in this simple example is just a compelx number, rather than a matrix). A crucial property, that makes this system exactly solvable, is that the scattering of two magnons is uniquely fixed, and the S-matrix factorizes [6] i.e. the M-body S- matrix can be rewritten as a product of two-body S-matrices. This is a typical feature of integrable theories, which amounts to satisfying the *Yang-Baxter equation*:

$$S_{23} \cdot S_{13} \cdot S_{12} = S_{12} \cdot S_{13} \cdot S_{23} \tag{4}$$

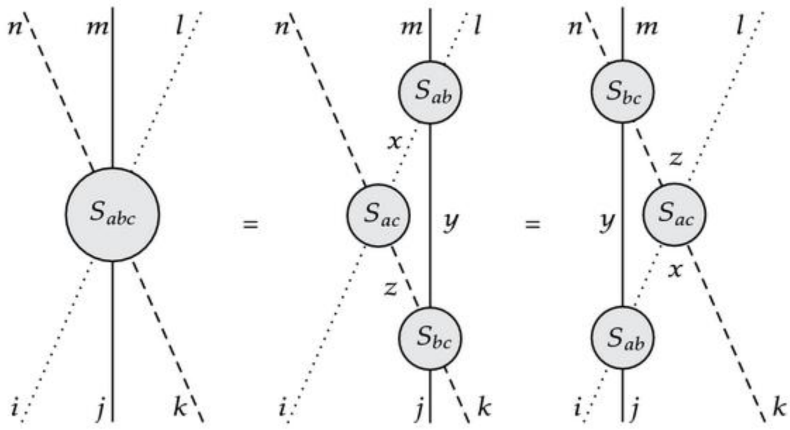

Figure 1: This figure represents the Yang-Baxter equations for the three particle S-matrix. In particular, for consistency, the two different factorizations of the three particles S-matrix must be equals. The equation is non trivial as long there is more than one particle in the spectrum, so that the difference two-body S-matrices do not trivially commute. Here i, j, k etc label the entries of the S-matrix

The integrability structure was studied also from the string point of view. The string is described as a non linear σ-model (NLSM) on the $AdS_5 \times S^5$ background. It is possible to exploit the coset description [7] of the Green-Schwarz action [8] in order to show that the theory is integrable as a classical field theory [9]. In fact the equation of motion can be recast as the flatness condition for a Lax connection $L_\alpha(z, \tau, \sigma)$ [1]

$$\partial_\tau L_\sigma(z,\tau,\sigma) - \partial_\sigma L_\tau(z,\tau,\sigma) + [L_\tau(z,\tau,\sigma), L_\sigma(z,\tau,\sigma)] = 0. \tag{5}$$

If we want to use the same approach used in the previous case, *i.e.* to derive the scattering matrix, it is crucial to take into account that the integrable model is the one arising on the worldsheet after the gauge-fixing procedure, hence in this case the excitations live on the worldsheet and they correspond to the bosonic and fermionic coordinate that parametrize the ambient space time. It turns out that the Hamiltonian of the gauge-fixed string theory is highly non-linear. However, in the large string tension limit $T \to \infty$, it is possible to implement the usual perturbation theory. In particular it is possible to extract the quartic expansion (in fields) of the string Hamiltonian and, from this, the tree-level two-body process, as in [10],[11]. In order to improve the precision also loop contributions can be taken into account such as two loops effects [12]. Eventually, also in this case, the S-matrix respects the *Yang-Baxter* equation.

The program of finding the S-matrix is so important because its knowledge allows one to construct the *Bethe-Yang* equations by imposing periodicity of the wave-function. These are the equations that one should solve to compute the spectrum of the theory.

There are several references in which the integrability structure of a string in $AdS_5 \times S^5$ background is largely treated such as [13],[14].

Deformations. One can wonder whether it is possible to deform the $AdS_5 \times S^5$ background preserving the integrable structure. The answer to the previous question could, in fact, highlight the necessary conditions in order to preserve integrability, in a less constrained system.

A possible approach in order to obtain new integrable theory consists in directly deforming the symmetry algebra by a continuous parameter q. In order to clarify this deformation procedure, let us consider a bosonic string that propagates in a sphere S^2. We can write the action of this non linear σ-model, using the coset formulation in which the target space is described as

$$S^2 \equiv \frac{SO(3)}{SO(2)} \approx \frac{SU(2)}{SO(2)} = \frac{SU(2)}{U(1)}, \tag{6}$$

[1]In general the Lax formalism is the one uses to prove classical integrability in the case of quantum field theory

where $SU(2)$ is the group of the isometries of the sphere. The q-deformation consists in deforming the algebra $\mathfrak{su}(2)$ and this implies a deformation of the sphere S^2. Geometrically, in this simple case, it corresponds to squashing the sphere to obtain an ellipsoid. While, at the Lie algebra level, $\mathfrak{su}(2)$ invariance is lost and only an $\mathfrak{u}(1)$ subalgebra is preserved, it is intuitively clear that an ellipsoid has "more symmetry" than just $\mathfrak{u}(1)$. The quantum group $\mathfrak{su}(2)_q$ is precisely what econdes such deformed symmetries. The action, for a real q-deformation, of the $AdS_5 \times S^5$ superstring was obtained by Delduc, Magro and Vicedo in [15]. It is a generalization of deformations valid for bosonic cosets [16] and it is of the type of the Yang-Baxter σ-model of Klimcik. It is sometimes referred to as η-deformation, where η is the deformation parameter related to q. In particular the classical integrability, from the S-matrix point of view, for η-deformation of the $AdS_5 \times S^5$ background was proved in [17]. The remarkable fact is that by construction the deformation procedure preserves the classical integrability of the original model.

AdS_3. Together with the study of the deformed $AdS_5 \times S^5$ model, it is interesting to analyze whether the integrable structure is preserved also at less symmetric and lower dimensional backgrounds. In particular, one could focus on AdS_3 spaces, indeed the AdS_3 gravity was the first example of the holographic duality, and it can be seen as an easier set-up to investigate the gravitational models. Moreover, AdS_3 gravity is not trivial at all, indeed as shown in [18], if there is a negative cosmological constant term, then it admits black hole solutions.

So, it is interesting to study the integrability of strings moving in $AdS_3 \times M$ background and it turns out that the case of $AdS_3 \times S^3$ is particularly rich also when it comes to deformations. An intuitive reason for this is that, only in the case of $AdS_3 \times S^3$, the isometries factorise between a "left" and "right" isometry group. In particular $SO(2,2) = SU(1,1) \times SU(1,1)$, where we note the further local (algebra) isomorphism $SU(1,1) = SL(2,\mathbb{R})$, and $SO(4) = SU(2) \times SU(2)$. This factorisation fits with the existence of a left and right sector of the dual CFT_2. We remark that for superstrings, it is possible to prove that the theory is consistent if the space time dimensions are $D = 10$. In the case of $AdS_5 \times S^5$ this constraint is respected, but not in the lower dimensional one. Indeed the $AdS_3 \times S^3$ background is always accompanied by another manifold M, for instance $M = T^4$ or $M = S^3 \times S^1$, in order to reach $D = 10$. These two choices for M are also very special in that they solve the supergravity equations and they have the maximal possible amount of (super)symmetry. Also in this case there are several reviews on the AdS_3/CFT_2 topic for instance [19].

About this thesis. This thesis in particular is devoted to confirming that some deformations of the $AdS_3 \times S^3$ background produce still integrable string models. In particular we will verify that the tree-level S-matrices of non linear σ-model on $AdS_3 \times S^3$ deformed as in [20] and as in [32], respect the Yang-Baxter equation. It is important to remark that in our study we will focus on the bosonic strings which will provide an important input for the comprehensive study of this background namely, including the fermionic degrees of freedom, and so considering the two parameter deformation of the $AdS_3 \times S^3$ superstring model.

As we said before, the S-matrix governs the scattering processes on the worldsheet. In order to obtain it, it is necessary to compute the Hamiltonian, which we will obtain using the first order formalism. This will turn out to have a very complicated dependence on the fields, and therefore we should use an expansion in the limit of large string tension to obtain its quadratic expansion. From the latter we could then obtain the S-matrix at the tree-level simply by including the expression of the fields in terms of creation and annihilation operators. Finally, to prove integrability, we should check whether this will respect the *Yang-Baxter* equation. With the same spirit, we will analyze also the, well known, integrability of the string σ-model in $AdS_3 \times S^3$ and in the η-deformation of $AdS_3 \times S^3$ backgrounds. However the document will be divide as follows:

In *chapter* 1 we will review some basic concepts in flat space bosonic string such as the classical bosonic string and the light cone quantization.

In *chapter* 2 we will discuss about the formulation of bosonic string in a less banal background. In particular we will introduce the *first order* formalism, which will be crucial in order to perform perturbative computations. In this chapter we will keep the discussion as general as possible, but with particular attention to the $AdS_3 \times S^3$ background. Moreover we will recall the basic notions in scattering theory, and finally we will present the S-matrix computation in the case of bosinc string propagates in a flat spacetime.

In *chapter* 3 we will review the concept of *integrability* both from a classical and quantum point of views. In particular our attention will be on *integrability* for field theory. Indeed in the case a system with an infinite number of degrees of freedom, will be necessary to introduce the *Lax* formalism. Moreover, in this chapter we will give some examples of classically integrable field models such as the *principal chiral model* and the *symmetric space σ-model*.

In *chapter* 4 we will concretize the previous concepts to the bosonic string in $AdS_3 \times S^3$ background. In particular first we will study the classical integrability of this model using the Lax formalism and then we will move to the perturbative computations using the first order formalism. Eventually we will present the tree-level S-matrix for this model.

In *chapter* 5 we will analyze the bosonic string in the case of one and two parameters deformations of $AdS_3 \times S^3$ background. Also in this case we will recover the previous analysis. In particular first we will study the classical integrability of these models using the Lax formalism and then we will move to the perturbative computations using the first order formalism. Eventually we will present the tree-level S-matrix for both the models.

Chapter 6 contains the conclusions.

The original results of this thesis have been published in [21].

Chapter 1

Bosonic String in flat space

The following chapter summarizes some concepts in bosonic strings, which are the building block of the thesis. The literature on bosonic strings is very vast and covers all the aspects on this topic, but for our purposes light-cone quantization and the strings spectrum are key concepts. All the material is taken, mostly, from [22], [23], [24], [25].

1.1 Action

A natural way to introduce the action for bosonic strings is to start with the description of a point-like particle moving in a background gravitational field *i.e.* in a Riemannian geometry described by a metric tensor $G_{\mu\nu}(X)$. The metric is assumed to have $D-1$ positive eigenvalues and one negative eigenvalue, moreover we set always $\hbar = c = 1$ in order to work with natural units.

Thus, with this assumptions, the action is

$$S = \frac{1}{2}\int d\tau \, \Big(e^{-1}(\tau)\dot{x}^2 - e(\tau)m^2\Big), \tag{1.1}$$

where $e(\tau)$ is identified as the world-line metric and $\dot{x}^2 = G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$.

The previous expression, called Polyakov action, is invariant under reparametrisation of $\tau \to \tilde{\tau}(\tau)$. At infinitesimal level this reads as: $\tau \to \tilde{\tau}(\tau) = \tau + \epsilon(\tau)$ with $\epsilon(\tau)$ small, which corresponds to local diffeomorphisms invariance on the world-line:

$$x^\mu(\tau) \to x^\mu(\tau) + \frac{dx^\mu}{d\tau}\epsilon(\tau) \quad , \; e(\tau) \to e(\tau) + \frac{d}{d\tau}(\epsilon(\tau)e(\tau)). \tag{1.2}$$

Moreover (1.1) is valid both for massless and massive particles. In fact if we solve the e equation of motion:

$$\frac{\delta S}{\delta e} = 0 \iff \dot{x}^2 + e^2(\tau)m^2 = 0, \tag{1.3}$$

and we substitute the $e(\tau)$ solution back in (1.1), then we find the action valid for a massive particle

$$S = -m\int d\tau\sqrt{-\dot{x}^2}. \tag{1.4}$$

The general lesson is that we can use local diffeomorphisms invariance to fix the gauge by setting the equation of motion for $e(\tau)$ to a constraint. Now, the point-like action can be extended to objects of higher dimensionality n. Hence their motions describe a $n+1$ dimensional manifold with a metric $h_{\alpha\beta}(\xi)$.

In the specific case of $n = 1$, we are dealing with one-dimensional objects, called *strings* , and in particular the space-time surface described is called a world-sheet. It is clear that, in order to obtain a dimensionless string action, there must be a multiplicative factor T with dimension of $(mass)^2$, which can be identified as the *string tension*. So the string action is

$$S = -\frac{T}{2}\int d^2\xi\ \sqrt{-h}h^{\alpha\beta}(\xi)G_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu, \tag{1.5}$$

where $\xi = (\sigma^0, \sigma) \equiv (\tau, \sigma)$ and $\sqrt{-h}$ is the square root of minus the determinant of the world-sheet metric.

It is useful to spend some more words to comment on the objects in the game. $h_{\alpha\beta}$ is the metric that describes the geometry of the world-sheet, on which there is a system of coordinates ξ. The string map: X^μ, provides an embedding of the world-sheet into the D-dimensional spacetime, which geometry is described by $G_{\mu\nu}(X)$. It is clear, for consistency, that $D \geq 2$.

Let us now analyze the symmetries of (1.5) paying particular attention to their distinction into two different types

The first are *spacetime symmetries* which, for instance, in the case of Minkowski metric $G_{\mu\nu} = \eta_{\mu\nu}$ are just reduced to the Poincaré invariance:

$$X^\mu \to \Lambda^\mu_\nu X^\nu + b^\mu, \tag{1.6}$$

with $\Lambda^\mu_\nu \in SO(1, D-1)$ and $b^\mu \in R^D$. They can be interpreted as global symmetries from the perspective of (1+1)-dimensional field theory.

The second are the *world-sheet symmetries*, that correspond to reparametrisation invariance: $\xi^\alpha \to \xi^\alpha + \epsilon^\alpha(\xi)$, under which the fields transform according to their tensorial natures:

$$X^\mu(\xi) \to X^\mu(\xi) + \epsilon^\rho\partial_\rho X^\mu, \quad h_{\alpha\beta} \to h_{\alpha\beta} + \epsilon^\rho\partial_\rho h_{\alpha\beta} + (\partial_\alpha\epsilon^\rho)h_{\rho\beta} + (\partial_\beta\epsilon^\rho)h_{\alpha\rho}. \tag{1.7}$$

Moreover, there is also a Weyl invariance [1] which is the following transformation:

$$h_{\alpha\beta} \to \Lambda(\xi)h_{\alpha\beta}, \quad \sqrt{-h}h^{\alpha\beta} \to \sqrt{-h}h^{\alpha\beta}. \tag{1.8}$$

It is crucial to appreciate that in string theory, the coordinates of the spacetime X^μ are promoted to dynamical scalar fields (scalar for the transformation property (1.7) under local diffeomorphisms) in the 2-dimensional field theory on the world-sheet defined by action (1.5). Studying bosonic string theory is equivalent to studying 2-dimensional gravity coupled to scalars. At this point it is important to compute the stress-energy tensor, which in general measures the variation of the matter part of the action with respect to the metric. In this case the equations of motion for the world-sheet metric are proportional to the stress-energy tensor as we can find considering the variations of (1.5)

$$\frac{\delta S}{\delta h^{\alpha\beta}} = 0 \iff T_{\alpha\beta} \equiv -\frac{2}{T\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}h^{\alpha'\beta'}\partial_{\alpha'}X^\mu\partial_{\beta'}X_\mu = 0 \tag{1.9}$$

Let us considering a string propagating in flat ambient spacetime $G_{\mu\nu} = \eta_{\mu\nu}$

$$S = -\frac{T}{2}\int d\sigma d\tau\ \sqrt{-h}h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu. \tag{1.10}$$

The presence of the Weyl invariance identifies string theory as a very special generalization of the point-like particle theory. Indeed, in general, $h_{\alpha\beta}$ is a symmetric tensor so it has $\frac{1}{2}n(n+1)$ independent components but in the particular case of $n = 1$, it is possible to gauge-away the $h_{\alpha\beta}$ dependence (using reparametrisation invariance and Weyl symmetry) setting $h_{\alpha\beta} = \eta_{\alpha\beta}$. This procedure leads to

$$S_{gf} = \frac{T}{2}\int_\Sigma d\tau d\sigma(\partial_\tau X^\mu\partial_\tau X_\mu - \partial_\sigma X^\mu\partial_\sigma X_\mu) = -\frac{T}{2}\int_\Sigma d\sigma d\tau\eta^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu. \tag{1.11}$$

[1]In particular the Weyl invariance is present only in the case of strings

The gauge fixing condition implies that the stress-energy tensor is automatically traceless. On the other hand, in order to find the fields equations of motion, we can also consider the general variation of (1.11):

$$\begin{aligned}\delta S_{gf} &= -T\int d\sigma d\tau\ \eta^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta\delta X\\ &= -T\int d\sigma d\tau\ \eta^{\alpha\beta}\partial_\beta(\partial_\alpha X\cdot\delta X)+T\int d\sigma d\tau\ \eta^{\alpha\beta}\partial_\beta\partial_\alpha X\cdot\delta X.\end{aligned}\tag{1.12}$$

The last, in the second line, is a volume term that gives the 2-dimensional wave equation:

$$\Box X^\mu = \Big(\frac{\partial^2}{\partial\sigma^2}\ -\ \frac{\partial^2}{\partial\tau^2}\Big)X^\mu = 0,\tag{1.13}$$

while the first is a surface term, indeed for finite strings the spatial coordinate of the world-sheet is restricted to a finite interval $\sigma\in[-l,l]$ while $\tau\in(-\infty,\infty)$. Hence we are able to rewrite

$$-T\int d\sigma d\tau\ \eta^{\alpha\beta}\partial_\beta(\partial_\alpha X\cdot\delta X) = -T\int d\tau\ \partial_\sigma X\cdot\delta X|^{\sigma=l}_{\sigma=-l}+T\int d\sigma\ \partial_\tau X\cdot\delta X|^{\tau=\infty}_{\tau=-\infty},\tag{1.14}$$

which, clearly, manifests the difference between open and closed strings, indeed the first ones are given by the vanishing of the surface term, while for the second ones the periodicity of X is sufficient to ensure the stationarity of the action. Moreover, in order to solve the equation of motion also the constraints $T_{\alpha\beta}=0$ are needed

$$T_{10}=T_{01}=\partial_\tau X\cdot\partial_\sigma X=0,\ \ T_{00}=T_{11}=\frac{1}{2}\Big((\partial_\tau X)^2+(\partial_\sigma X)^2\Big)=0.\tag{1.15}$$

A convenient set of coordinates are the light-cone coordinates defined as: $\sigma^+=\tau+\sigma$ and $\sigma^-=\tau-\sigma$. The world-sheet metric as well as the derivatives change in

$$\eta_{+-}=\eta_{-+}=-\frac{1}{2},\ \ \eta_{++}=\eta_{--}=0\to\partial_\pm=\frac{1}{2}(\partial_\tau\pm\partial_\sigma).\tag{1.16}$$

So the equation of motion for the scalars fields become: $\partial_+\partial_-X^\mu=0$ which implies that its solutions take the forms: $X^\mu(\sigma,\tau)=X^\mu_L(\sigma^+)+X^\mu_R(\sigma^-)$. Also (1.15) becomes [2]:

$$\begin{aligned}T_{++}&=\frac{1}{2}(T_{00}+T_{01})=\frac{1}{4}(\dot X^2+X'^2+2\dot X\cdot X')=\partial_+X\cdot\partial_+X\\ T_{--}&=\frac{1}{2}(T_{00}-T_{01})=\frac{1}{4}(\dot X^2+X'^2-2\dot X\cdot X')=\partial_-X\cdot\partial_-X,\end{aligned}\tag{1.17}$$

which are not automatically satisfied, while the traceless of $T_{\alpha\beta}$ is immediate since $T_{+-}=T_{-+}=0$. So the conditions that must be implemented as constraints in flat coordinates take the form: $T_{++}=T_{--}=0$.

1.2 Oscillators expansion

In the previous section we mentioned that the difference between open and closed string arises when one tries to ensure the stationarity of the action. In the following we will look closely at these solutions and it will be useful to define

$$T=\frac{1}{2\pi\alpha'}.\tag{1.18}$$

The *closed strings* are loops with no free ends, topologically equivalent to circles, thus the periodicity condition is just:

$$X^\mu(\tau,\sigma)=X^\mu(\tau,\sigma+2\pi),\tag{1.19}$$

[2] In (1.17) we have used the notation $\partial_\tau X=\dot X$ and $\partial_\sigma X=X'$.

while the *open strings* are obtained imposing certain boundary conditions, for instance the Neumann-Neumann ones reads as

$$X'^{\mu}(\tau,0) = X'^{\mu}(\tau,\pi) = 0. \tag{1.20}$$

However, in order to write down the closed string solution of the wave equation, it is useful to note that:

$$\partial_+ X^\mu = \partial_+ X^\mu_L, \quad \partial_- X^\mu = \partial_- X^\mu_R. \tag{1.21}$$

In this way (1.19) can be extended, independently, for right and left moving simply because, in the light-cone coordinates, the derivative selects only one of the modes. Thus starting from the previous, we can express X^μ_R and X^μ_L by their Fourier expansions

$$\begin{aligned} X^\mu_R(\tau-\sigma) &= \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-in(\tau-\sigma)}, \\ X^\mu_L(\tau+\sigma) &= \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau+\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-in(\tau+\sigma)}. \end{aligned} \tag{1.22}$$

It is also immediate to note that in the case of open strings, the left and right modes are not present and in fact the solution is given by

$$X^\mu(\tau,\sigma) = x^\mu + \frac{1}{\pi T}p^\mu\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-in\tau}cos(n\sigma). \tag{1.23}$$

Some observations are useful: first of all, in (1.22), $\tilde{\alpha}^\mu_n$, α^μ_n represent independent right and left Fourier modes, moreover in the sum $X^\mu = X^\mu_L + X^\mu_R$ the linear term in σ vanishes implying the consistency of the periodicity condition. Finally, in both the cases, reality of scalar fields $X^\mu = (X^\mu)^*$ implies that:

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^*, \quad \tilde{\alpha}^\mu_{-n} = (\tilde{\alpha}^\mu_n)^*. \tag{1.24}$$

The closed string zero mode are fixed conventionally to: $\tilde{\alpha}^\mu_0 = \alpha^\mu_0 = \frac{1}{\sqrt{4\pi T}}p^\mu$, while in the open case $a^\mu_0 = \frac{1}{\sqrt{\pi T}}p^\mu$. This fact implies an important difference between open and closed strings mass spectrum, but apart from this the following analysis is very similar. So let us focus in the case of closed string.

All the discussion is actually based on a flat world-sheet metric, hence also the constraints $T_{++} = T_{--} = 0$ must be imposed. It is possible to Fourier expands also (1.17)

$$\begin{aligned} T_{++} &= (\partial_+ X^\mu_L)^2 = \frac{1}{4\pi T}\sum_{m=-\infty}^{\infty}\tilde{L}_m e^{-im\sigma^+} \to \tilde{L}_m = 2T\int_0^{2\pi} d\sigma e^{im\sigma^+}T_{++} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n, \\ T_{--} &= (\partial_- X^\mu_R)^2 = \frac{1}{4\pi T}\sum_{m=-\infty}^{\infty}L_m e^{-im\sigma^-} \to L_m = 2T\int_0^{2\pi} d\sigma e^{im\sigma^-}T_{--} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n, \end{aligned} \tag{1.25}$$

hence the constraints can be rewrite as: $L_m = 0, \quad \tilde{L}_m = 0 \quad \forall m \in \mathbb{Z}$ where L_m and $\tilde{L}_m$ are called the Virasoro generators and classically respect the Witt algebra

$$\{L_m, L_n\}_{PB} = -i(m-n)L_{m+n}, \quad \{\tilde{L}_m, \tilde{L}_n\}_{PB} = -i(m-n)\tilde{L}_{m+n}. \tag{1.26}$$

It is not difficult to see that in the case of open string one obtains only the first algebraic relations of (1.26). Now, since the zero Fourier mode is related to the momentum of the string, it is interesting to analyze L_0, $\tilde{L}_0$, indeed we can extract the mass of the mode. This can be done starting from (1.25), setting $m = 0$, and finally considering that $\alpha_0^2 = \tilde{\alpha}_0^2 = -\frac{1}{2}\alpha' M^2_{closed}$:

$$M^2_{closed} = \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n). \tag{1.27}$$

In the case of opens string, following basically the same steps one can arrives at

$$M^2_{open} = \frac{1}{\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n. \tag{1.28}$$

1.3 Quantization

All the previous discussion is valid at the classical level. There are several ways to quantize a theory, for instance: the old covariant quantization (OCQ) which is a procedure that is manifest Lorentz invariant, but unitarity holds only for certain dimensions of spacetime. The light-cone quantization (LCQ) which is a procedure that leads to a unitary theory, but Lorentz invariance holds for certain dimensions of spacetime. Finally the path-integral quantization which is a method uses the Faddeev-Popov procedure. For the purpose of this thesis, the most convenient choice is the second.

1.3.1 Quantization of the oscillator algebra

In the canonical quantization the classical fields $X^\mu(\tau,\sigma)$ and their canonical momenta, defined by $p_\mu(\tau,\sigma) \equiv \frac{\delta L}{\delta \partial_\tau X^\mu}$, are promoted to operators, by replacing:

$$\{\ ,\ \}_{PB} \to -i[\ ,\ ]. \tag{1.29}$$

So the classical Poisson structures, between conjugate variables are promoted to equal time commutation relations between operators that in the case of closed string (1.22) reads as

$$\begin{aligned} \{\alpha^\mu_m, \alpha^\nu_n\}_{P.B} &= im\delta_{m+n,0}\eta^{\mu\nu} \to [a^\mu_m, a^\nu_n] = \delta_{m+n,0}\eta^{\mu\nu} \\ \{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\}_{P.B} &= im\delta_{m+n,0}\eta^{\mu\nu} \to [\tilde{a}^\mu_m, \tilde{a}^\nu_n] = \delta_{m+n,0}\eta^{\mu\nu} \\ \{\tilde{\alpha}^\mu_m, \alpha^\nu_n\}_{P.B} &= 0 \to [\tilde{a}^\mu_m, a^\nu_n] = 0, \end{aligned} \tag{1.30}$$

where, we have rescaled the modes in order to obtain the algebra of harmonic oscillators with the standard normalizations, and also we assumed to have real fields, for which $a^{\mu\dagger}_n = a^\mu_{-n}$. However, this procedure yields a non unitary quantum theory, namely there are some states, into the Hilbert space, with negative norm. For instance, if one considers a state defined as $|\psi\rangle = a^0_{-n}|0\rangle$, then

$$\langle\psi|\psi\rangle = \langle 0|\, a^0_n a^0_{-n}\,|0\rangle = \langle 0|\,[a^0_n, a^0_{-n}]\,|0\rangle = \langle 0|0\rangle\,\eta^{00} = -\langle 0|0\rangle\ < 0. \tag{1.31}$$

This is actually only an apparent issue, indeed we have still to impose the constraints $T_{\alpha\beta} = 0$, which can be rephrased using their modes expansions. The further advantage is that, in this way, we are also able to construct their quantum operators versions, which could be defined as

$$L_m = \frac{1}{2}\sum_n a_{m-n}\cdot a_n, \quad \tilde{L}_m = \frac{1}{2}\sum_n \tilde{a}_{m-n}\cdot \tilde{a}_n. \tag{1.32}$$

Clearly, the vanishing of the stress energy tensor components should be replaced by $L_m|\psi\rangle = \tilde{L}_m|\psi\rangle = 0$, but once again we find a problem, indeed the quantum Virasoro operators L_0 and $\tilde{L}_0$ manifest the *ordering problem* which is, colloquially, an ambiguity in the definition of a quantum operator starting from its classical counterpart. The root of the problem basically stays in the quantization procedure because is not clear, a priori, what is the correct order of products of non commuting operators. In order to solve this ambiguities one usually defines the *normal-ordering* as:

$$: a^\mu_m a^\nu_n := a^\mu_m a^\nu_n \quad if\ m \le n, \quad : a^\mu_m a^\nu_n := a^\nu_n a^\mu_m \quad if\ m > n. \tag{1.33}$$

With this, the problematic Virasoro operators for closed strings read as

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} a_{-n}\cdot a_n, \quad \tilde{L}_0 = \frac{1}{2}\alpha_0^2 + \sum_{n>0} \tilde{a}_{-n}\cdot\tilde{a}_n. \tag{1.34}$$

Because of the normal-ordering ambiguity, we include an undetermined constant a every time we will use L_0 and $\tilde{L}_0$, and we are allowed to call *physical* a state (for closed strings) such that

$$(L_m - a\delta_{m,0})\,|\psi\rangle = 0,\ (\tilde{L}_m - a\delta_{m,0})\,|\psi\rangle = 0\ \ \forall m \ge 0. \tag{1.35}$$

Taking the difference between the zero components of the previous equations, we obtained the so called *level matching condition*

$$(L_0 - \tilde{L}_0)\,|\psi\rangle = 0. \tag{1.36}$$

Moreover, this is not the end, indeed the ordering ambiguity also affects the quantum Virasoro algebra. Let us look at the first relation in (1.26) with the substitution of the Poisson brackets with the commutators and with the further adjoint of the normal-ordering in the definition of the Virasoro operators. Initially we also set $a = 0$, and we start by considering the following commutator, which will be the building block for all the computations

$$[a_m^\mu, L_n] = \frac{1}{2}\sum_{p=-\infty}^{\infty}[a_m^\mu, : a_p^\nu a_{n-p,\nu} :] = \frac{1}{2}\sum_{p=-\infty}^{\infty}\Big(m\delta_{m+p,0}a_{n-p}^\mu + m a_p^\mu \delta_{m+n-p,0}\Big) = m a_{n+m}^\mu. \quad (1.37)$$

In the previous, we used only the algebra (1.30) and the previous definition of normal-ordering. With this it is then possible to compute

$$\begin{aligned}[L_m, L_n] &= \frac{1}{2}\sum_{p=-\infty}^{0}[a_p^\mu a_{m-p,\mu}, L_n] + \frac{1}{2}\sum_{p=1}^{\infty}[a_{m-p}^\mu a_{p,\mu}, L_n] \\ &= \frac{1}{2}\sum_{p=-\infty}^{0}\Big(p a_{p+n}^\mu a_{m-p,\mu} + (m-p)a_p^\mu a_{m-p+n,\mu}\Big) \\ &+ \frac{1}{2}\sum_{p=1}^{\infty}\Big((m-p)a_{m-p+n}^\mu a_{p,\mu} + p a_{m-p}^\mu a_{n+p,\mu}\Big).\end{aligned} \quad (1.38)$$

Now, in order to reconstruct the harmonic oscillators algebra, it is useful to change the index to $p = q - n$ in the first and in the last terms of the previous summation

$$\begin{aligned}[L_m, L_n] = \frac{1}{2}\Big(&\sum_{q=-\infty}^{n}(q-n)a_q^\mu a_{m+n-q,\mu} + \sum_{p=-\infty}^{0}(m-p)a_p^\mu a_{m+n-p,\mu} \\ &+ \sum_{p=1}^{\infty}(m-p)a_{m+n-p}^\mu a_{p,\mu} + \sum_{q=n+1}^{\infty}(q-n)a_{m+n-q}^\mu a_{q,\mu}\Big).\end{aligned} \quad (1.39)$$

Without loss of generality we consider $n > 0$, moreover using the harmonic oscillators algebra

$$a_q^\mu a_{m+m-q,\mu} = a_{m+n-q}^\mu a_{q,\mu} + q\delta_{m+n,0}\eta_\mu^\mu = a_{m+n-q}^\mu a_{q,\mu} + qD\delta_{m+n,0}, \quad (1.40)$$

then we obtain

$$[L_m, L_n] = \frac{1}{2}\sum_{p=-\infty}^{\infty}(m-n) : a_p^\mu a_{m+n-p,\mu} : + \frac{D}{2}\delta_{m+n,0}\sum_{q=1}^{n}(q^2 - nq). \quad (1.41)$$

D is the dimension of the spacetime where the string propagates. Finally, it is possible to write down explicitly the Virasoro algebra once that the following relations are included

$$\sum_{q=1}^{n} q^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{q=1}^{n} q = \frac{1}{2}n(n+1). \quad (1.42)$$

In particular the Virasoro algebra with $a = 0$ is

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+m,0}. \quad (1.43)$$

while, if $a \neq 0$, then $L_m \to L_m - a\delta_{m,0}$ and

$$[L_m, L_n] = (m-n)L_{m+n} + \Big(\frac{D}{12}m^3 + \Big(2a - \frac{D}{12}\Big)m\Big)\delta_{m+n,0}. \quad (1.44)$$

It is also possible to prove that only for $a = 1$ and for $D = 26$ there are no ghosts, namely states with negative norm. Instead of doing this derivation, we will see how these conditions arise in light-cone gauge which will be more useful for what we want to do. At this point, before proceeding with the light-cone quantization, we can build the strings spectrum.

1.3.2 String spectrum

In the previous section we mentioned that for some values of a and D, the spectrum is ghost-free. So let us set the normal-ordering constant $a = 1$, and the dimension of the spacetime to $D = 26$. Furthermore, we have also said that the Hilbert space is obtained by acting on the vacuum with the creation operators $a^{\mu}_{-n}|0\rangle$, where $\mu = 0, \dots, 25$, but a generic state

$$|\psi\rangle = a^{\mu_1}_{n_1} a^{\mu_1}_{n_1} \cdots a^{\mu_N}_{n_N} |0\rangle, \tag{1.45}$$

is *physical* if and only if it respects the Virasoro constraints (1.35). Moreover, the masses of the excitations are modified with respect to the classical counterparts by the normal-ordering constant and in particular we must distinguish between closed and open strings. However for both the cases, the mass shell condition implements the first Virasoro condition.

Open string spectrum The mass shell condition is

$$M^2_{open} = \frac{1}{\alpha'}\Big(\sum_{n=1}^{\infty} a_{-n}\cdot a_n - 1\Big) = \frac{1}{\alpha'}(N-1), \tag{1.46}$$

where we introduced the *number operator* $N = L_0 - \frac{1}{2}a_0^2$ that satisfies

$$[N, a^{\mu}_{-n}] = n a^{\mu}_{-n}. \tag{1.47}$$

The first level is then a *Tachyon*, indeed for this state the eigenvalue of the number operator is $N = 0$ and so $M^2_{closed} = -\frac{1}{\alpha'}$. Moreover it is a physical state since all the other Virasoro constraints are satisfied. For the second level, $N = 1$, we find D-states with $M^2_{closed} = 0$

$$|A(\zeta, k)\rangle = \eta_{\mu\nu}\zeta^{\mu} a^{\nu}_{-1}|0\rangle\,, \tag{1.48}$$

where ζ^{μ} is the polarization vector of the state and k is the momentum. The other Virasoro constraints are satisfied if $\zeta \cdot k = 0$. Thus the polarization vector ζ^{μ} describes $D - 2 = 24$ transverse degrees of freedom, and the longitudinal ones are decoupled. This basically is a massless vector boson.

Closed string spectrum Following the same strategy, the mass shell condition is

$$M^2_{closed} = \frac{2}{\alpha'}\Big(\sum_{n=1}^{\infty} a_{-n}\cdot a_n + \sum_{n=1}^{\infty} \tilde{a}_{-n}\cdot\tilde{a}_n - 2\Big) = \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{1.49}$$

where $N = L_0 - \frac{1}{2}a_0^2,\ \tilde{N} = \tilde{L}_0 - \frac{1}{2}a_0^2$ that satisfies

$$[N, a^{\mu}_{-n}] = n a^{\mu}_{-n}, \quad [N, \tilde{a}^{\mu}_{-n}] = n\tilde{a}^{\mu}_{-n}. \tag{1.50}$$

Moreover, taking the difference between $L_0 - \tilde{L}_0$ we obtain the *level matching condition* $N - \tilde{N} = 0$ which tells us that in any physical state the eigenvalue of the number operator in the left and right sectors must be the same. However, also in this case the level $N = \tilde{N} = 0$ is a *Tachyon*, with a mass $M^2_{open} = -\frac{4}{\alpha'}$. For $N = \tilde{N} = 1$ we have a tensor state with $M^2_{closed} = 0$

$$|A(\zeta, k)\rangle = \zeta_{\mu\nu} a^{\mu}_{-1}\tilde{a}^{\nu}_{-1}|0\rangle\,, \tag{1.51}$$

where $\zeta_{\mu\nu}$ is the polarization tensor. This state describes various fields depending on the polarization tensor, in particular the symmetric part $\zeta_{(\mu\nu)}$ describes a massless spin 2 particle which can be treats as *Graviton*, and so its field is the metric $G_{\mu\nu}(X)$. The anti-symmetric part $\zeta_{[\mu\nu]}$ describes the degrees of freedom of an anti-symmetric tensor $B_{\mu\nu}(X)$ called *Klab-Ramond field.* Finally the trace $\zeta^{(0)}$ represents a scalar field $\Phi(X)$ called *Dilaton.* However, all the other Virasoro constraints are satisfied imposing the transversality conditions

$$\zeta_{\mu\nu}k^{\mu} = \zeta_{\mu\nu}k^{\nu} = 0, \tag{1.52}$$

which in particular identifies as physical only the transverse degrees of freedom.

1.3.3 Quantization in light-cone gauge

In the light-cone formalism, $T_{\alpha\beta}=0$ are imposed before quantization. The fundamental observation is that $h_{\alpha\beta}=\eta_{\alpha\beta}$ does not completely fix the gauge. In order to see this, it is convenient to write (1.11) in light cone coordinates:

$$S = T\int d\sigma^+ d\sigma^- \partial_+ X^\mu \partial_- X_\mu. \tag{1.53}$$

It is evident that any reparametrisation like $\sigma^+ \to \tilde{\sigma}^+ = f(\sigma^+)$ and $\sigma^- \to \tilde{\sigma}^- = f(\sigma^-)$, with $f(\sigma^\pm)$ generic functions, leaves the action invariant. Clearly, for closed strings σ^+ and σ^- are reparamerized independently, while for open strings they are connected by the boundary conditions. This implies that the light-cone coordinates transform as:

$$\begin{aligned} \tau &= \frac{1}{2}\Big(\sigma^+ + \sigma^-\Big) \to \tilde{\tau} = \frac{1}{2}\Big(\tilde{\sigma}^+(\tau+\sigma) + \tilde{\sigma}^-(\tau-\sigma)\Big) \\ \sigma &= \frac{1}{2}\Big(\sigma^+ - \sigma^-\Big) \to \tilde{\sigma} = \frac{1}{2}\Big(\tilde{\sigma}^+(\tau+\sigma) - \tilde{\sigma}^-(\tau-\sigma)\Big). \end{aligned} \tag{1.54}$$

Hence we deduce that $\partial_+\partial_-\tilde{\tau}=0$, and on the other hand, once $\tilde{\tau}$ is chosen, $\tilde{\sigma}$ is completely determined. In order to appreciate the advantage of the previous assertion, it is useful to introduce the spacetime light-cone coordinate:

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}), \quad X^i, \ i = 1,\dots,D-2\ , \tag{1.55}$$

so the spacetime metric and, consequently, the scalar product between spacetime vectors become:

$$\eta_{+-} = \eta_{-+} = -1, \ \eta_{ij} = \delta_{ij} \implies X^\mu X^\nu \eta_{\mu\nu} = -2X^+X^- + X^iX^i. \tag{1.56}$$

The fact that $\tilde{\tau}$ satisfies the same equation of X^μ implies that we can chose a solution for $\tilde{\tau}$ such that

$$X^+(\tau,\sigma) = x^+ + \frac{1}{2\pi T}p^+\tilde{\tau}, \tag{1.57}$$

which does not depend directly on σ. Having fixed $X^+(\tau,\sigma)$, then, it also possible to deduce $X^-(\tau,\sigma)$ in terms of $X^i(\tau,\sigma)$.

First order formalism We will see how to do this gauge fixing in a slightly different way, which will tour out to be more suitable for curved backgrounds which we will see later in the thesis. In particular while for flat space the light-cone gauge fixing and the choice of $h^{\alpha\beta}=\eta^{\alpha\beta}$ are compatible, this is not so for curved backgrounds as we will see later. Therefore let us start from the full form of the action before any gauge fixing [3]

$$S = -\frac{T}{2}\int d\sigma d\tau\ \gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu. \tag{1.58}$$

We introduce the light cone coordinates as in (1.55) and the light-cone momenta

$$P_\pm = \frac{\partial L}{\partial \dot{X}^\pm}, \quad P_i = \frac{\partial L}{\partial \dot{X}^i}. \tag{1.59}$$

Expressing the velocities via the momenta

$$\dot{X}^\pm = \frac{1}{T}(P_\mp - T\gamma^{\tau\sigma}X'^\pm), \quad \dot{X}^i = \frac{1}{T\gamma^{\tau\tau}}(-P^i - T\gamma^{\tau\sigma}X'^i), \tag{1.60}$$

it is then possible to write down the phase space Lagrangian density in light-cone coordinates as

$$\begin{aligned} \mathcal{L} = {} & P_i\dot{X}^i + P_-\dot{X}^- + P_+\dot{X}^+ + \frac{1}{2T\gamma^{\tau\tau}}\Big(-2P_-P_+ + P_iP^i + T^2X'_iX'^i - 2T^2X'^-X'^+\Big) \\ & + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}\Big(P_iX'^i + P_-X'^- + P_+X'^+\Big). \end{aligned} \tag{1.61}$$

[3]In the action we identified $\gamma^{\alpha\beta} = \sqrt{-h}h^{\alpha\beta}$

The phase space light-cone gauge consists in imposing the following conditions [4]

$$X^+ = \frac{p^+}{2\pi T}\tilde{\tau}, \quad P^+ = \frac{p^+}{2\pi} = const. \tag{1.62}$$

This choice allows us to completely remove the gauge degrees of freedom present in the Lagrangian that we wrote above. Moreover in (1.61) we can identified also the Virasoro constraints formulated in the phase space

$$C_1 = P_i X'^i + P_- X'^- + P_+ X'^+, \quad C_2 = -2P_- P_+ + P_i P^i + T^2 X'_i X'^i - 2T^2 X'^- X'^+. \tag{1.63}$$

Let us consider the closed string case, and let us solve some constraints followed from the Lagrangian (1.61). In particular varying the Lagrangian with respect to $\gamma^{\tau\tau}$ and imposing the phase space gauge-fixing condition, we find

$$P_+ = -P^- = -\frac{\pi}{p^+}\Big(P_i P^i + T^2 X'_i X'^i\Big). \tag{1.64}$$

Now, the following equations are the results of the variations with respect to P_- and $\gamma^{\tau\sigma}$ respectively

$$\begin{aligned} &\frac{\delta L}{\delta P_-} = \dot{X}^- - \frac{P_+}{T\gamma^{\tau\tau}} = 0 \implies \dot{X}^- = \frac{\pi}{Tp^+}(P_i P^i + T^2 X'_i X'^i), \\ &\frac{\delta L}{\delta \gamma^{\tau\sigma}} = 0 \implies X'^- = \frac{2\pi}{p^+} P_i X'^i. \end{aligned} \tag{1.65}$$

Integrating the last line in the previous formula we obtain that

$$X^-(\sigma = \pi) - X^-(\sigma = 0) = \frac{2\pi}{p^+}\int_0^{2\pi} d\sigma P_i X'^i, \tag{1.66}$$

which for a closed string must be zero. This basically is the only constraint which remains unsolved and it is the *level matching* condition. Moreover, substituting the solution of all the constraints and the gauge-fixing conditions in the Lagrangian, it is then possible to deduce the Hamiltonian

$$H = \frac{1}{2T}\int_0^{2\pi} d\sigma (P_i P^i + T^2 X'_i X'^i). \tag{1.67}$$

Thus from (1.65) we find that the zero mode of X^- evolves as

$$\dot{x}^- = \frac{H}{p^+} \implies p^- = \frac{2\pi T}{p^+} H. \tag{1.68}$$

Finally, with the following oscillations expansion we give a complete solution for the fields X^-

$$\alpha^-_n = \frac{\sqrt{\pi T}}{p^+}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{D-2} \alpha^i_{n-m}\alpha^i_m, \quad \tilde{\alpha}^-_n = \frac{\sqrt{\pi T}}{p^+}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{D-2} \tilde{\alpha}^i_{n-m}\tilde{\alpha}^i_m. \tag{1.69}$$

We can then check the mass of the string, in particular since we have found that $p^- = \frac{2\pi T}{p^+}H$ we get for the mass

$$\begin{aligned} M^2_{closed} &= -p_\mu p^\mu = 2p^+ p^- - p_i p^i = -p_i p^i + 4\pi T H \\ &= \frac{2}{\alpha'}\sum_{n=1}^{\infty}\sum_{i=1}^{D-1}(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n). \end{aligned} \tag{1.70}$$

Moreover, also the *level matching*, in terms of transverse oscillators, reads as

$$\sum_{n\neq 0}(\alpha^i_{-n}\alpha^i_n - \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n) = 0. \tag{1.71}$$

[4] For closed string

Thus, the level-matching condition tells that the left- and right-moving oscillators contribute the same amount of energy.

However at this level the objects are still classical. The Poisson structure and, after the quantization procedure, the commutation relations are present only for the transverse oscillators, just because X^+ and X^- can be expressed in terms of X^i.

$$\begin{aligned}&[X^i(\sigma,\tau),X^j(\sigma',\tau)]=[P^i(\sigma,\tau),P^j(\sigma',\tau)]=0\\&[X^i(\sigma,\tau),P^j(\sigma',\tau)]=i\delta^{ij}\delta(\sigma-\sigma'),\quad [p^+,x^-]=i.\end{aligned}\tag{1.72}$$

In particular for closed strings, the previous leads to

$$[a^i_m,a^j_n]=m\delta_{m+n,0}\delta^{ij},\quad [\tilde{a}^i_m,\tilde{a}^j_n]=m\delta_{m+n,0}\delta^{ij}.\tag{1.73}$$

It is then possible to construct the Hilbert space, just acting on the vacuum $|0\rangle$ with the creation operators:

$$a^i_m(\tilde{a}^i_m)\,|0\rangle=0,\quad a^{i\dagger}_m(\tilde{a}^{i\dagger}_m)\,|0\rangle=a^i_{-m}(\tilde{a}^i_{-m})\,|0\rangle=|\psi\rangle\,,\quad \forall m>0.\tag{1.74}$$

Since the Virasoro constraints are implemented before the quantization, one has only $D-2$ degrees of freedom, and the quantum theory is automatically unitary, indeed the problem was the presence of $\eta_{\mu\nu}$ in the commutation relations while in this case there is a δ_{ij}.

As we mentioned before, the problem in this case is the Lorentz invariance. This issue can be intuitively understood simply looking at the Hilbert space, since it is generated acting on the vacuum only with the transverse creation operators. In order to understand more rigorously how to restore the Lorentz invariance, it is necessary to check if the commutation relations, that define its algebra, are valid even at quantum level. It turns out that for $a=1$ and $D=26$, the Lorentz algebra is not anomalous.

1.3.4 Poincaré invariance and critical dimension

In this subsection we will compute explicitly the critical dimension.

In particular, for closed strings, by using the Noether theorem we can compute the conserved quantities connected to classical Lorentz invariance

$$J^{\mu\nu}=T\int_0^{2\pi}d\sigma(X^\mu\dot{X}^\nu-X^\nu\dot{X}^\mu).\tag{1.75}$$

Moreover they become the generators of the Lorentz transformations, indeed they respect the following algebra:

$$\{J^{\mu\nu},J^{\rho,\sigma}\}_{PB}=i\eta^{\mu\rho}J^{\nu\sigma}+i\eta^{\nu\sigma}J^{\mu\rho}-i\eta^{\mu\sigma}J^{\nu\rho}-i\eta^{\nu\rho}J^{\mu\sigma}.\tag{1.76}$$

Since $X^\pm$ are singled out in LCQ, a breakdown of Lorentz invariance might show up in an anomaly of the algebra satisfied by J^{i-} namely $[J^{i-},J^{j-}]\neq 0$. Let us sketch the explicit computation, that can be found in chapter 4 of [25]. First of all, the mode expansion of the problematic Lorentz generator is

$$\begin{aligned}J^{i-}&=\frac{1}{2}(x^ip^-+p^-x^i)-x^-p^i-i\sum_{n=1}^\infty\frac{1}{n}(a^i_{-n}a^-_n-a^-_{-n}a^i_n)-i\sum_{n=1}^\infty\frac{1}{n}(\tilde{a}^i_{-n}\tilde{a}^-_n-\tilde{a}^-_{-n}\tilde{a}^i_n)\\&=l^{i-}-i\sum_{n=1}^\infty\frac{1}{n}(a^{[i}_{-n}a^{-]}_n)-i\sum_{n=1}^\infty\frac{1}{n}(\tilde{a}^{[i}_{-n}\tilde{a}^{-]}_n)\\&=l^{i-}+S^{i-}+\tilde{S}^{i-}.\end{aligned}\tag{1.77}$$

Clearly, because of the intrinsic structure of the Lorentz generator, the computation of the commutation relation can be divided into separate parts whose building blocks are basically summarized in the following table

[,]	p^+	p^-	p^j	x^j	x^-	a_m^j	a_m^-
p^+	0	0	0	0	i	0	0
p^-	0	0	0	$-i\frac{p^i}{p^+}$	$-i\frac{p^-}{p^+}$	$-\frac{2\pi T}{p^+}ma_m^j$	$-\frac{2\pi T}{p^+}ma_m^-$
p^i	0	0	0	$-i\delta^{ij}$	0	0	0
x^i	0	$i\frac{p^i}{p^+}$	$i\delta^{ij}$	0	0	$i\frac{\delta^{ij}\delta_m}{4\pi T}$	$i\frac{a_m^i}{p^+}$
x^-	-i	$i\frac{p^-}{p^+}$	0	0	0	0	$i\frac{a_m^-}{p^+}$
a_n^i	0	$\frac{2\pi T}{p^+}na_n^i$	0	$-i\frac{\delta^{ij}\delta_n}{4\pi T}$	0	$n\delta^{ij}\delta_{n+m}$	$\frac{\sqrt{4\pi T}}{p^+}na_{n+m}^i$
a_n^-	0	$\frac{2\pi T}{p^+}na_n^-$	0	$-i\frac{a_n^i}{p^+}$	$-i\frac{a_n^-}{p^+}$	$-i\frac{\sqrt{4\pi T}}{p^+}ma_{n+m}^i$	$[a_n^-, a_m^-]$

Table 1.1: Important commutation relations. In particular $[a_m^-, a_n^-]$ can be computed as $[L_m, L_n]$

We report also the following important commutation relation in the general case of $a \neq 0$

$$[a_m^-, a_n^-] = \frac{\sqrt{4\pi T}}{p^+}(m-n)a_{m+n}^- + \frac{4\pi T}{p^{+2}}\Big(\frac{D-2}{12}m^3 + 2am - \frac{D-2}{12}m\Big)\delta_{m+n,0} \tag{1.78}$$

With this it then possible to evaluate the following commutations relations

$$\begin{aligned}
[l^{i-}, l^{j-}] &= \frac{i}{4}[p^i, x^j]\frac{p^-}{p^+} + \frac{p^-}{p^+}\frac{i}{4}[p^i, x^j] + \frac{i}{2}[x^-, p^-]\delta^{ij} = \frac{1}{2}\frac{p^-}{p^+}\delta^{ij} - \frac{1}{2}\frac{p^-}{p^+}\delta^{ij} = 0,\\
[l^{i-}, S^{j-}] + [S^{i-}, l^{j-}] &= -2\frac{p^-}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}a_{-n}^{[i}a_n^{j]} + \frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(-a_{-n}^{[j}a_n^{-]}p^i + a_{-n}^{[i}a_n^{-]}p^j),\\
[S^{i-}, S^{j-}] &= \frac{4\sqrt{\pi T}a_0^-}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}a_{-n}^{[i}a_n^{j]} + \frac{1}{p^+}\sum_{n=1}^{\infty}\frac{1}{n}(a_{-n}^{[j}a_n^{-]}p^i - a_{-n}^{[i}a_n^{-]}p^j)\\
&\quad - \sum_{n=1}^{\infty}\Big(\frac{4\pi T}{p^{+2}}2n - \frac{f(n)}{n^2}\Big)a_{-n}^{[i}a_n^{j]}.
\end{aligned} \tag{1.79}$$

Finally, it is now possible to compute

$$\begin{aligned}
[J^{i-}, J^{j-}] &= [S^{i-}, S^{j-}] + [l^{i-}, S^{j-}] + [S^{i-}, l^{j-}] + [l^{i-}, l^{j-}] + [\tilde{S}^{i-}, \tilde{S}^{j-}] + [l^{i-}, \tilde{S}^{j-}] + [\tilde{S}^{i-}, l^{j-}]\\
&= \frac{4\pi T}{(p^+)^2}\sum_{n=1}^{\infty}\Delta_n(a_{-n}^{[i}a_n^{j]} + \tilde{a}_{-n}^{[i}\tilde{a}_n^{j]}),\ \ \Delta_n = n\Big(\frac{D-2}{12} - 2\Big) + \frac{1}{n}\Big(2a - \frac{D-2}{12}\Big)
\end{aligned} \tag{1.80}$$

Thus the commutator matches the classical Poisson structure if and only if $D = 26$ and $a = 1$, which is the same result that we will obtain in the case of closed strings.

Finally it is possible to prove that the bosonic quantum strings are ghost free and are Lorentz invariant if the normal order constant $a = 1$ and the spacetime dimension is $D = 26$.

Chapter 2

Non linear σ-model

In this chapter we will review the basic tools that will be useful in order to approach the perturbative analysis of bosonic strings in a background described by the metric $G_{\mu\nu}(X)$. The final purpose is to evaluate the S-matrix for $AdS_3 \times S^3$ deformed background. First of all, AdS_n and S^n are maximally symmetric spaces. Focusing on the anti de Sitter space, it is a maximally symmetric solution of the free Einstein equations. This means that AdS_n (as well as S^n) has the maximal numbers of Killing vectors which are vector fields that satisfy the following equations:

$$\nabla_{(\mu} K_{\nu)} = 0, \tag{2.1}$$

where ∇ is the covariant derivative

$$\nabla_\mu K_\nu = \partial_\mu K_\nu - \Gamma^{\xi}_{\mu\nu} K_\xi, \tag{2.2}$$

and where $\Gamma^{\xi}_{\mu\nu}$ are the Christoffel symbols.

(2.1) is symmetric in the indexes μ, ν, and is called *Killing equation*. Its solutions are in one-to-one correspondence with continuous symmetries of the metric. Physically they can be understood as the directions along which the metric does not change [26]. Let us provide some more generalities about the two previous spaces [27].

Sphere. The n-dimensional sphere is a maximally symmetric space with positive curvature, which can be defined through its embedding in flat $\mathbb{R}^{n+1}$ dimensional space with metric

$$ds^2 = \sum_{i=1}^{n+1} dX_i^2, \tag{2.3}$$

as the submanifold described by

$$\sum_{i=1}^{n+1} X_i^2 = R^2. \tag{2.4}$$

The constant R is the radius of the sphere and its isotropy group is $SO(n)$. From this it is possible to define the sphere also by the coset

$$S^n \equiv \frac{SO(n+1)}{SO(n)}. \tag{2.5}$$

Anti-de Sitter. The n-dimensional anti-de Sitter is a maximally symmetric Lorentzian space with negative curvature, which can be defined through its embedding in flat $\mathbb{R}^{2,n-1}$ dimensional space with metric

$$ds^2 = \sum_{i=1}^{n-1} dX_i^2 - dX_0^2 - dX_n^2, \tag{2.6}$$

as the submanifold described by

$$\sum_{i=1}^{n-1} X_i^2 - X_0^2 - X_n^2 = -R^2. \tag{2.7}$$

The constant R is the radius of the AdS space and its isotropy group is $SO(1, n-1)$. From this it is possible to define the AdS space also by the coset

$$AdS_n \equiv \frac{SO(2, n-1)}{SO(1, n-1)}. \tag{2.8}$$

In order to evaluate the S-matrix one has to quantize the theory and, as we pointed out, the quantization scheme which is convenient to adopt in this context is the *LCQ*. However the light-cone quantization approach is problematic in the case of *AdS* background, indeed in flat space, one starts by fixing the conformal gauge

$$h^{\alpha\beta} = \eta^{\alpha\beta}, \tag{2.9}$$

and then one fixes the residual conformal diffeomorphism symmetry on the worldsheet by choosing (1.57). The requirement for being able to choose the light-cone gauge in a curved space is the existence of a null Killing vector. It turns out that the above conditions do not apply in the case of *AdS* metric as it is pointed out in [7].

The way to avoid the issue is to use first-order formalism, which has the advantage naturally producing the Hamiltonian.

2.1 $AdS_3 \times S^3$ metric

For future convenience let us spend some words on the metric of $AdS_3 \times S^3$, in particular this will be useful in order to understand how to introduce light-cone coordinates and also to appreciate intuitively what is the *decompactification* limit that we will use.

We stress one more time that our aim is to find the S-matrix for some deformation of the previous background, but once again the starting point will be the evaluation of the metric and the parametrization that we expose in this subsection will act as a starting tool. In the following we will use basically the parametrization given in [13] for the $AdS_5 \times S^5$ string, with the further simplification that in our case two of the angles in *AdS* and *S* are set to zero.

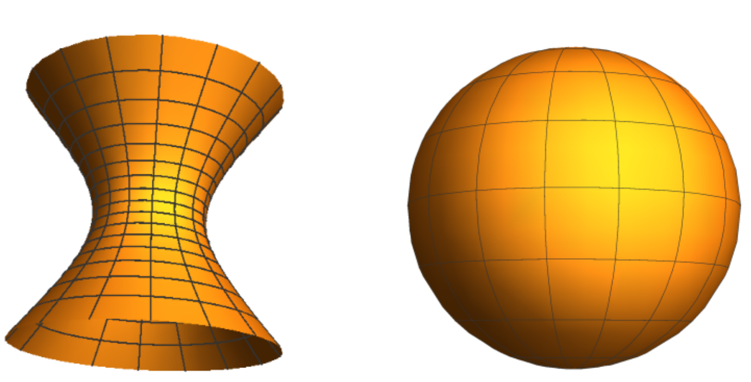

Figure 2.1: The left plot is the surface of AdS_2 embedded in $\mathcal{R}^3$ while the right one is the surface of S^2 embedded in $\mathcal{R}^3$

For the three-sphere S^3, we use the coordinate $Y_a,\ a = 1, .., 4$ to parametrize the Euclidean space $\mathbb{R}^4$. The unitary three dimensional sphere, embedded in this space, is defined by the constraint $Y_a Y_b \delta^{ab} = 1$, which is respected by the parametrization:

$$Y_1 + iY_2 = re^{i\theta}, \quad Y_3 + iY_4 = \sqrt{1 - r^2} e^{i\phi}, \tag{2.10}$$

with $0 < r < 1$, and $0 < \phi, \theta < 2\pi$. Hence the metric which is induced on the sphere is

$$ds^2_{S^3} = (1 - r^2)d\phi^2 + \frac{dr^2}{(1 - r^2)} + r^2 d\theta^2. \quad (2.11)$$

Despite the previous is the most intuitive parametrization, in this context there is a more convenient set of coordinates. Indeed the constraint can be satisfied also by

$$Y_1 + iY_2 = \frac{y_1 + iy_2}{1 + \frac{|y|^2}{4}}, \quad Y_3 + iY_4 = \frac{1 - \frac{|y|^2}{4}}{1 + \frac{|y|^2}{4}} e^{i\phi}, \quad (2.12)$$

where $|y|^2 \equiv y_1^2 + y_2^2$. With this coordinates the metric becomes

$$d^2_{S^3} = \left(\frac{1 - \frac{|y|^2}{4}}{1 + \frac{|y|^2}{4}} \right)^2 d\phi^2 + \frac{dy_1^2 + dy_2^2}{\left(1 + \frac{|y|^2}{4}\right)}. \quad (2.13)$$

In the case of AdS_3 we proceed in the same way, so the embedding space $\mathbb{R}^{2,2}$ is parametrized by the coordinates $Z_a,\ a = 0,..,3$. The three dimensional unitary anti-de Sitter space is defined by $Z_a Z_b \eta^{ab} = -1$ where $\eta^{ab} = diag(-1, 1, 1, -1)$. Also in this case there are a naive parametrization for which the new constraint is satisfied:

$$Z_1 + iZ_2 = \rho e^{i\psi}, \quad Z_0 + iZ_3 = \sqrt{1 + \rho^2} e^{it}, \quad (2.14)$$

where $0 < \rho < \infty$ and the angle $0 < \psi < 2\pi$, moreover $-\infty < t < \infty$ is the non compact time. Using this local coordinates, the metric induced on the AdS space is

$$ds^2_{AdS_3} = -(1 - \rho^2)dt^2 + \frac{d\rho^2}{(1 + \rho^2)} + \rho^2 d\psi^2. \quad (2.15)$$

Also in this case a most convenient parametrization is

$$Z_1 + iZ_2 = \frac{z_1 + iz_2}{1 - \frac{|z|^2}{4}}, \quad Z_0 + iZ_3 = \frac{1 + \frac{|z|^2}{4}}{1 - \frac{|z|^2}{4}} e^{it}, \quad (2.16)$$

where $|z|^2 = z_1^2 + z_2^2$. Thus the metric in this coordinates reads as:

$$ds^2_{AdS_3} = -\left(\frac{1 + \frac{|z|^2}{4}}{1 - \frac{|z|^2}{4}} \right)^2 dt^2 + \frac{dz_1^2 + dz_2^2}{\left(1 - \frac{|z|^2}{4}\right)^2}. \quad (2.17)$$

Finally, merging the two metrics, we can give the expression of the induced metric on $AdS_3 \times S^3$ which is

$$ds^2 = -G_{tt}dt^2 + G_{\phi\phi}d\phi^2 + G_{zz}dz^2 + G_{yy}dy^2, \quad (2.18)$$

where

$$G_{tt} = \left(\frac{1 + \frac{|z|^2}{4}}{1 - \frac{|z|^2}{4}} \right)^2, \ G_{\phi\phi} = \left(\frac{1 - \frac{|y|^2}{4}}{1 + \frac{|y|^2}{4}} \right)^2, \ G_{zz} = \frac{1}{\left(1 - \frac{|z|^2}{4}\right)^2}, \ G_{yy} = \frac{1}{\left(1 + \frac{|y|^2}{4}\right)^2}. \quad (2.19)$$

A crucial consideration is that the previous form highlights two isometries which are the shifts of the time t and space coordinate ϕ. Indeed the metric does not depends on the time t and on the angle ϕ, so the shift

$$t \to t + t_0, \quad \phi \to \phi + \phi_0, \quad (2.20)$$

with t_0 and ϕ_0 two constants, leaves ds^2 invariant. This fact has deep implications since, as we will see, the two previous translations leave also the action invariant and so, via the Noether theorem, one can define two conserved charges.

2.2 Bosonic string in light-cone-gauge

Let us consider the action of non a linear σ-model for a closed string. In this section we will slightly change the notation that we used in chapter 1. In particular $\sqrt{h}\ h^{\alpha\beta} \to \gamma^{\alpha\beta}$ (as we used, when we introduced the first order formalism for flat space), and we set $\sigma \in [-l/2, l/2]$. Moreover we also change the background metric indexes to $G_{MN}(X)$ following the notation used in [28]. This will be useful in order to appreciate the distinction between the light-cone coordinate and the transverse ones which will be denoted by the indexes $\mu\ \nu$, or in other words the difference between the longitudinal and the transverse degrees of freedom. Moreover in the previous section we note that the t and ϕ coordinates played a special role in the parametrization of the metric. So, having in mind the example of $AdS_3 \times S^3$, we denote $X^M = (t, X^\mu, \phi)$.

In the previous chapter we reviewed the flat space bosonic string, but here we want to consider a more general situation where the spacetime metric is not banal. The curved metric can be understood as a state of the graviton describing the fluctuations of the metric around $\eta_{\mu\nu}$.

Hence the general lesson is that a closed string propagates in a background described by states of its own massless fluctuations. In other words the propagation in a curved spacetime corresponds to coupling the string fields to the gravitational sector of its massless excitations. But with the same spirit, we are led to including also the other fields corresponding to the other components of the polarization tensor.

A natural generalization of (1.5) is then an action that describes a system which is known as non linear σ-model:

$$S = -\frac{T}{2}\int_\Sigma d^2\xi\ \sqrt{-h}\ [\ (h^{\alpha\beta}G_{\mu\nu}(X(\xi)) - \epsilon^{\alpha\beta}B_{\mu\nu})\partial_\alpha X^\mu \partial_\beta X^\nu + \alpha' R^{(2)}(h)\Phi(X(\xi))\ ], \tag{2.21}$$

where $\epsilon^{\alpha\beta}$ is completely anti-symmetric. As we said before, we will slightly change the notation with respect to the 1, thus (2.21) can be rewrite as

$$S = -\frac{T}{2}\int_{-\infty}^{\infty} d\tau \int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma \Big(\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN}(X) - \epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N B_{MN}(X)\Big). \tag{2.22}$$

where we set to zero the *Fradkin-Tseytlin* $R^{(2)}(h)$ term [1]. At this level we treat the target space metric G_{MN} as general as possible with special attention to $AdS_3 \times S^3$. In order to rewrite the action in the first order formalism it is necessary to evaluate the conjugate momenta to X^M

$$\begin{aligned} p_M = \frac{\delta S}{\delta \dot{X}^M} &= -T\gamma^{\tau\beta}\partial_\beta X^N G_{MN} + T\partial_\sigma X^N B_{MN} \\ &= -T\Big(\gamma^{\tau\tau}\partial_\tau X^N G_{MN} + \gamma^{\tau\sigma}\partial_\sigma X^N G_{MN} - \partial_\sigma X^N B_{MN}\Big) \end{aligned} \tag{2.23}$$

hence, denoting as usual $\partial_\tau X^M = \dot{X}^M$ and $\partial_\sigma X^M = X'^M$, it is possible to extract

$$\dot{X}^Q = \Big(-p_M + \gamma^{\tau\sigma}X'^N G_{MN} + X'^N B_{MN}\Big)\frac{G^{MQ}}{T\gamma^{\tau\tau}}. \tag{2.24}$$

Thus, with the substitution of the previous into (2.22) we obtain

$$S = \int_{-\infty}^{\infty} d\tau \int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma \Big(p_M \dot{X}^M + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}}C_1 + \frac{1}{2T\gamma^{\tau\tau}}C_2\Big), \tag{2.25}$$

where

$$\begin{aligned} &C_1 = p_M X'^M, \\ &C_2 = G^{MN}p_M p_N + T^2 G_{MN}X'^M X'^N - 2Tp_M X'^Q G^{MN}B_{NQ} + T^2 X'^P X'^Q B_{MP}B_{NQ}G^{MN}. \end{aligned} \tag{2.26}$$

[1] In the case of $AdS_3 \times S^3$ it turns out that $\Phi(X) = const$. Moreover, in two dimensions, $\int d^2\xi\ \sqrt{-h}R^{(2)}(h)$, is a topological invariant. So for our purposes the *Fradkin-Tseytlin* is a topological invariant

The previous action is a clear generalization of the one obtained integrating the Lagrangian (1.61) introduced in the fist chapter. It is evident that the action (2.25) does not longer manifest the Poincaré invariance, but this fact is not very surprising since the Hamiltonian formalism is not covariant. In chapter 1 we have pointed out also that imposing the equation of motion for the worldsheet metric was equivalent to imposing $T_{\alpha\beta} = 0$ which were called the Virasoro constraints. This was linked to the fact that exploiting the Weyl and the reparametrization invariances one can reabsorbs the worldsheet metric dependences in the action. In this context this is performs by imposing

$$\mathcal{C}_1 = 0, \quad \mathcal{C}_2 = 0, \tag{2.27}$$

so we will call these conditions the Virasoro constraints.

Let us focus on the case of $AdS_3 \times S^3$. As we mentioned in Section 2.1, the metric element of the form (2.19) leads immediate to two conserved charges

$$E = -\int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma\ p_t, \qquad J = \int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma\ p_\phi, \tag{2.28}$$

where p_t and p_ϕ are the conjugate momenta to the coordinate t and ϕ. It is clear that E is the target space energy since it is associated to time translations, while J is the total angular momentum of the string in the ϕ direction, since it is associated to ϕ translations.

With this two special coordinate, we can construct the light-cone ones

$$x^- \equiv \phi - t, \qquad x^+ \equiv (1-a)t + a\phi, \tag{2.29}$$

where a is a generic parameter. The conjugate momenta to this variables are defined in such a way that will be a liner combinations of p_t and p_ϕ

$$p_- = \frac{\delta S}{\delta \dot{x}^-} = (1-a)p_\phi - a p_t\ , \quad p_+ = \frac{\delta S}{\delta \dot{x}^+} = p_\phi + p_t. \tag{2.30}$$

In terms of the general light-cone coordinates and momenta the action (2.25) takes the form

$$\begin{aligned} S &= \int_{-\infty}^{\infty} d\tau \int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma \Big(p_- \dot{x}_+ + p_+ \dot{x}_- + p_y \dot{Y} + p_z \dot{Z} + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2T\gamma^{\tau\tau}} C_2 \Big) \\ &= \int_{-\infty}^{\infty} d\tau \int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma \Big(p_- \dot{x}_+ + p_+ \dot{x}_- + p_\mu \dot{X}^\mu + \frac{\gamma^{\tau\sigma}}{\gamma^{\tau\tau}} C_1 + \frac{1}{2T\gamma^{\tau\tau}} C_2 \Big)\ , \end{aligned} \tag{2.31}$$

where in the last line we collect the transverse fields in the expression $p_\mu \dot{X}^\mu$. The first Virasoro constraint is linear in p_+

$$\mathcal{C}_1 = p_+ x'^- + p_- x'^+ + p_\mu X'^\mu, \tag{2.32}$$

while the second is quadratic in p_+

$$\begin{aligned} \mathcal{C}_2 =& G^{++} p_+^2 + 2G^{+-} p_+ p_- + G^{--} p_-^2 + T^2 G_{--} (x'^-)^2 \\ &+ 2T^2 G_{+-} x'^+ x'^- + T^2 G_{++} (x'^+)^2 + \mathcal{H}_x. \end{aligned} \tag{2.33}$$

In both the previous expressions we have assumed that the B-field vanishes in the light-cone directions, moreover in the last one, $\mathcal{H}_x$ contains only the transverse fields

$$\mathcal{H}_x = G^{\mu\nu} p_\mu p_\nu + T^2 X'^\mu X'^\nu G_{\mu\nu} - 2T p_\mu X'^\rho B_{\nu\rho} G^{\mu\nu} + T^2 X'^\lambda X'^\rho B_{\mu\lambda} B_{\nu\rho} G^{\mu\nu}, \tag{2.34}$$

and

$$\begin{aligned} G^{++} &= a^2 G^{\phi\phi} + (a-1)^2 G^{tt},\ G^{+-} = a G^{\phi\phi} + (a-1) G^{tt},\ G^{--} = G^{\phi\phi} + G^{tt} \\ G_{--} &= (a-1)^2 G_{\phi\phi} + a^2 G_{tt},\ G_{+-} = -(a-1) G_{\phi\phi} - a G_{tt},\ G_{++} = G_{\phi\phi} + G_{tt}. \end{aligned} \tag{2.35}$$

The uniform light-cone gauge-fixing conditions are archived by setting

$$x^+ = \tau + am\sigma, \quad p_- = 1. \tag{2.36}$$

The word *uniform* is associated with the condition $p_- = 1$ which means that the momentum is distributed uniformly along the string. Moreover the integer m is the winding number of the string along the circle parametrized by ϕ. From now on, we set $m = 0$.

The advantage of this gauge choice stays in the fact that the term

$$p_-\dot{x}^+ + p_+\dot{x}^- + p_\mu \dot{X}^\mu \tag{2.37}$$

in the action (2.31) is simplified, in fact $\dot{x}^+ = 1,\ x'^+ = 0$, and so, dropping the constant and the total time derivative term $\dot{x}^-$, we end up with the gauge-fixed action

$$S_{gf} = \int_{-\infty}^{\infty} d\tau \int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma \Big(p_\mu \dot{X}^\mu - \mathcal{H}\Big), \tag{2.38}$$

where

$$\mathcal{H} = -p_+\Big(p_\mu, X^\mu, X'^\mu\Big) \tag{2.39}$$

is the Hamiltonian density of the worldsheet. A crucial observation is that it depends only on the transverse fields.

Hence, in order to find the Hamiltonian *i.e.* $-p_+$ it is necessary to solve first the linear constraint

$$\mathcal{C}_1 = x'^- + p_\mu X'^\mu = 0 \implies x'^- = -p_\mu X'^\mu, \tag{2.40}$$

and after that, we solve $\mathcal{C}_2 = 0$ using the previous solution. Since the second constraint is quadratic, we are dealing with two solutions, and we keep those that lead to a positive defined Hamiltonian.

The most general solution for the second Virasoro constraint is the following:

$$\mathcal{H} = -p_+ = \frac{G^{+-} + \sqrt{\left(G^{+-}\right)^2 - G^{--}(G^{++} + T^2 G_{++}(x'^-)^2 + \mathcal{H}_x)}}{G^{--}}. \tag{2.41}$$

If, in the previous expression, we consider $a = \frac{1}{2}$ and we include the metric elements in the case of flat space, then the square root vanishes, and we recovered the Hamiltonian found in the previous chapter, which basically is the free Hamiltonian. The situation is different if we consider $a \neq \frac{1}{2}$, indeed also in flat space the square root does not vanish. Moreover, it is clear that the worldsheet light-cone Hamiltonian has a complicated non liner dependence on the fields *i.e.* coordinates and momenta, hence it is very hard that it can be used in order to perform direct canonical quantization.

However it is possible to consider some limits, in which the Hamiltonian can be used for perturbative calculations. In order to see this, it is interesting to recast (2.28) using light-cone coordinates and momenta

$$P_+ = \int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma p_+ = J - E, \quad P_- = \int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma p_- = (1-a)J + aE, \tag{2.42}$$

where the capital momenta is related to the conserved charges while the lower case momenta represents the corresponding density charge. The gauge-fixing condition leads to $p_- = 1$ and so the second integral becomes trivial

$$P_- = l, \tag{2.43}$$

which means that the light-cone string sigma model is defined on a cylinder of radius equal to the momentum P_-. With this fact, we analyzed some limit cases in the following.

The Berenstein-Maldacena-Nastase (*BMN*) limit is performed by taking $T \to \infty$ (large string tension limit) and $P_- \to \infty$ while keeping $\frac{T}{P_-}$ fixed. The gauge-fixed action then admits a well defined expansion in power of T^{-1}, in which the leading contribution is quadratic in the fields. The subsequent contributions correspond to interaction terms.

The limit in which $P_- \to \infty$ with T fixed is called *decompactification limit*. From the observation that the momentum is related to the radius of a closed string (2.43), we can deduce that in this limit, the non linear σ model is defined on a plane instead of a cylinder. A crucial fact is that in this way the asymptotic states and the S-matrix are well defined. Then, in order to perform

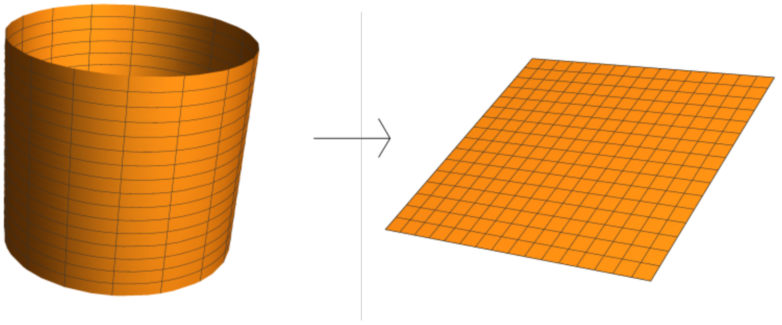

Figure 2.2: Intuitive visualization of *decompactification limit*

perturbative calculations we can take a large tension expansion in T^{-1}. The advantage of this approach is that it allows to distinguish between decompactification of the worldsheet and the expansion of the Hamiltonian.

Let us mention also that there are convenient choices for the parameter a [13]. If $a = 0$, then we obtain $t = \tau$, $P_- = J$, which is the temporal gauge. Since the Hamiltonian is a function of P_-, with this choice of the parameter the Hamiltonian depends only on J. If $a = \frac{1}{2}$, then we obtain the usual light-cone gauge

$$x^+ = \frac{1}{2}(t + \phi) = \tau, \quad P_- = \frac{1}{2}(E + J), \tag{2.44}$$

which indeed coincide with the (1.55). The last choice simplifies drastically the perturbative computations in the large string tension limit. Finally, the last interesting choice is to set $a = 1$, that leads to

$$x^+ = \phi = \tau, \quad P_- = E, \tag{2.45}$$

which means that the angle variable ϕ is identified as the worldsheet time τ and that the energy is distributed uniformly along the string.

2.3 Perturbative expansion and Quantization

As we mention in the previous section, the S-matrix as well as the asymptotic states are well defined in the *decompactification limit*. Since our aim is to calculate the S-matrix for $AdS_3 \times S^3$ deformed background it is convenient to adopt this limit, and it is also crucial to quantize the Hamiltonian (2.41) which is highly non linear. In order to proceed, we need to systematically expand it in the large-tension limit. A way to archived this is by rescaling the coordinate $\sigma \to T\sigma$ and the fields

$$X_\mu \to \frac{X_\mu}{\sqrt{T}}, \qquad p_\mu \to \frac{p_\mu}{\sqrt{T}}. \tag{2.46}$$

By the rescaling we have introduced an explicit dependence on the coupling in the string Lagrangian and Hamiltonian. In this way, sending $T \to \infty$ it is possible to use perturbative techniques. This leads

$$\begin{aligned} S_{gf} &= \int d\sigma d\tau \Big(\mathcal{L}_2 + \frac{\mathcal{L}_4}{T} + \dots\Big), \\ H &= \int d\sigma d\tau \Big(\mathcal{H}_2 + \frac{\mathcal{H}_4}{T} + \dots\Big). \end{aligned} \tag{2.47}$$

The explicit expressions for $\mathcal{L}_n$, $\mathcal{H}_n$ depend on the specific model, however the root is quite general and basically the subscript n indicates the power of the fields that compose the specific terms.

As we mentioned in the first chapter, the quantization is archived promoting the classical Poisson structures to commutations relations. In this specific case, the action depends only on the transverse fields, and so

$$[X^\mu(\sigma,\tau), p_\nu(\sigma',\tau)] = i\delta^\mu_\nu \delta(\sigma-\sigma'). \tag{2.48}$$

The commutation relations between creation and annihilation operators can be obtained once the oscillators expansion of the conjugate variables are given. In particular for real scalar fields

$$X^\mu(\sigma,\tau) = \int dp \frac{1}{\sqrt{2\omega(p)}}\Big(e^{ip\sigma}a^\mu(p,\tau) + e^{-ip\sigma}a^{\mu\dagger}(p,\tau)\Big), \tag{2.49}$$

while the analytic expression for the conjugate momenta depend on the model. For the sake of concreteness let us considering the action of a free theory composed by the previous fields

$$S = -\frac{1}{2}\int d\sigma d\tau\ \gamma^{\alpha\beta}\partial_\alpha X_\mu(\sigma,\tau)\partial_\beta X^\mu(\sigma,\tau). \tag{2.50}$$

The expression for the conjugate momenta (actually the density) are

$$\begin{aligned} p_\mu(\sigma,\tau) \equiv \frac{\delta\mathcal{L}}{\delta\partial_\tau X^\mu} &= \partial_\tau X_\mu(\sigma,\tau) \\ &= \int dp\ i\frac{\sqrt{\omega(p)}}{\sqrt{2}}\Big(e^{-ip\sigma}a^\dagger_\mu(p,\tau) - e^{ip\sigma}a_\mu(p,\tau)\Big), \end{aligned} \tag{2.51}$$

where we used

$$a^\mu(p,\tau) = e^{-i\omega(p)\tau}a^\mu(p),\ \ a^{\mu\dagger}(p,\tau) = e^{+i\omega(p)\tau}a^{\mu\dagger}(p), \tag{2.52}$$

and where $\omega(p)$ is the dispersion relation. Hence, the creation and annihilation operators satisfy the canonical commutation relations

$$[a^\mu(p,\tau), a^\dagger_\nu(p',\tau)] = \delta^\mu_\nu\delta(p-p'). \tag{2.53}$$

Let us point out that, as we exposed in chapter 1, the Hilbert space of the theory, quantized in the light-cone approach, does not has states with negative norm, *i.e.* negative probability. Indeed, the index μ and ν are associated to the transverse fields, which means that in this case δ^μ_ν has only positive eigenvalues.

2.4 Perturbative world-sheet S-matrix

In this section we will review some generalities about the scattering theory [13]. Clearly, the explicit expression for the S-matrix strongly depends on the model, but the basic notions that govern this huge topic are in common for all the models that we will analyze.

In scattering theory the S-matrix $\mathbb{S}$ is a unitary operator mapping free particle out-states to free particles in-states in the Heisenberg pictures. Both the states belong to the same Hilbert space of the model and are eigenvectors of the full Hamiltonian $\mathbb{H}$ with eigenvalues E

$$\mathbb{H}\,|p_1,p_2,..,p_n\rangle^{in/out}_{i_2,..,i_n} = E\,|p_1,p_2,..,p_n\rangle^{in/out}_{i_2,..,i_n}, \tag{2.54}$$

where $i_1,..,i_n$ are the flavor indices that label different kinds of particles, and p_k is the momentum carried by the particle of flavor i_k. In order to describe the in- and out-states we introduce creation and annihilation operators that act on the same Hilbert space and that satisfy the commutation relation (2.53). The Hilbert space has a vacuum $|\Omega\rangle$ that is annihilate by the operators $a_{in}(p) \equiv a_{in}(p,0)$, and $a_{out}(p) \equiv a_{out}(p,0)$. So the in- out-states are created as

$$\begin{aligned} |p_1,\ldots,p_n\rangle^{(in)}_{i_1,\ldots,i_n} &= a^\dagger_{in,i_1}(p_1)\ldots a^\dagger_{in,i_n}(p_n)\,|\Omega\rangle\,, \\ |p_1,\ldots,p_n\rangle^{(out)}_{i_1,\ldots,i_n} &= a^\dagger_{out,i_1}(p_1)\ldots a^\dagger_{out,i_n}(p_n)\,|\Omega\rangle\,. \end{aligned} \tag{2.55}$$

Since the in- and out-operators satisfy the canonical commutation relations, by virtue of the Stone - Von Neumann theorem, they are connected together by a unitary operator $\mathbb{S}$ which is the S-matrix

$$a^\dagger_{in}(p,\tau) = \mathbb{S} a^\dagger_{out}(p,\tau)\mathbb{S}^\dagger, \quad a_{in}(p,\tau) = \mathbb{S} a_{out}(p,\tau)\mathbb{S}^\dagger. \tag{2.56}$$

Thus, from (2.55), we deduce that in- out- states are related by the S-matrix, in fact

$$|p_1,\dots,p_n\rangle^{(in)}_{i_1,\dots,i_n} = \mathbb{S}\, |p_1,\dots,p_n\rangle^{(out)}_{i_1,\dots,i_n}, \tag{2.57}$$

which also implies that in absence of interactions we have $\mathbb{S} = \mathbb{I}$. Clearly the in- and out- operators are not the creation and annihilation operators of the interacting theory, in fact they are free operators that coincide with the operators of the interacting theory at $\tau = \infty$, $\tau = -\infty$ as we will point out in (2.64). Moreover the fact that they evolve freely implies that they respect the Heisenberg equations of motion with the free Hamiltonian (which in our case is the quadratic part $\mathcal{H}_2$ in (2.47))

$$\dot a_{in,k}(p,\tau) = i[H_2(a^\dagger_{in}, a_{in}), a_{in,k}(p,\tau)], \;\; \dot a_{out,k}(p,\tau) = i[H_2(a^\dagger_{out}, a_{out}), a_{out,k}(p,\tau)] \tag{2.58}$$

In order to compute the S-matrix in the perturbation theory usually one splits the full Hamiltonian into the free and interaction parts

$$\mathbb{H} = \mathbb{H}_0 + \mathbb{H}_{int}, \tag{2.59}$$

and one introduces creation and annihilation operators satisfying the canonical commutation relations (2.53) under which the free Hamiltonian reads as

$$\mathbb{H}_0 = \int dp \sum_k \Big(\omega_k(p) a^\dagger_k(p,\tau) a_k(p,\tau)\Big). \tag{2.60}$$

The operators $a_k(p,\tau)$, $a^\dagger_k(p,\tau)$, as well as $\mathbb{H}_0$, are interacting Heisenberg fields which respect the following equation of motion

$$\begin{aligned} \dot a_k(p,\tau) &= i[\mathbb{H}, a_k(p,\tau)] = i[\mathbb{H}_0 + \mathbb{H}_{int}, a_k(p,\tau)] \\ &= -i\omega_k(p) a_k(p,\tau) + i[\mathbb{H}_{int}, a_k(p,\tau)]. \end{aligned} \tag{2.61}$$

Also in this case it is possible to relate the two sets of creation and annihilation operators by unitary transformations, since they respect the canonical commutation relations. Moreover, for the same reason, one can connects this set of operators with the old in- out- creation and annihilation operators

$$\begin{aligned} a^\dagger(p,\tau) &= \mathbb{U}^\dagger_{in}(\tau) a^\dagger_{in}(p,\tau)\mathbb{U}_{in}(\tau), \qquad a(p,\tau) = \mathbb{U}^\dagger_{in}(\tau) a_{in}(p,\tau)\mathbb{U}_{in}(\tau) \\ a^\dagger(p,\tau) &= \mathbb{U}_{out}(\tau) a^\dagger_{out}(p,\tau)\mathbb{U}^\dagger_{out}(\tau), \quad a(p,\tau) = \mathbb{U}_{out}(\tau) a_{out}(p,\tau)\mathbb{U}^\dagger_{out}(\tau). \end{aligned} \tag{2.62}$$

If we require that

$$\mathbb{U}_{in}(-\infty) = \mathbb{I}, \quad \mathbb{U}_{out}(\infty) = \mathbb{I}, \tag{2.63}$$

then this implies that the free creation and annihilation operators tend to the interacting ones in the the limit of infinite time, in particular

$$\lim_{\tau\to\infty} a_{out}(p,\tau) \to a(p,\tau), \qquad \lim_{\tau\to-\infty} a_{in}(p,\tau) \to a(p,\tau). \tag{2.64}$$

It is also clear that the operator $\mathbb{U}$ and $\mathbb{S}$ are connected, indeed using (2.56)

$$\begin{aligned} \mathbb{U}_{out}(\tau) a^\dagger_{out}(p,\tau)\mathbb{U}^\dagger_{out}(\tau) = a^\dagger(p,\tau) &= \mathbb{U}^\dagger_{in}(\tau) a^\dagger_{in}(p,\tau)\mathbb{U}_{in}(\tau) \\ &= \mathbb{U}^\dagger_{in}(\tau)\;\mathbb{S}\; a^\dagger_{out}(p,\tau)\;\mathbb{S}^\dagger\mathbb{U}_{in}(\tau), \end{aligned} \tag{2.65}$$

which leads to

$$\mathbb{S} = \mathbb{U}_{in}(\tau)\mathbb{U}_{out}(\tau). \tag{2.66}$$

The dependence on time is only apparent in the previous formula since the in- and out- operators evolve as free operators and thus their time dependence cancels. So one can choose the most convenient instant of time τ. Clearly a possible convenient choice is $\tau = \infty$ since the boundary conditions (2.63) are valid

$$\mathbb{S} = \mathbb{U}_{in}(\infty) = \mathbb{U}_{out}(-\infty), \tag{2.67}$$

hence if we find $\mathbb{U}$, then we find an expression for the S-matrix. Let us start taking the time derivative of the first line of (2.62)

$$\begin{aligned} \dot{a}^\dagger &= \dot{\mathbb{U}}^\dagger_{in} a^\dagger_{in} \mathbb{U}_{in} + \mathbb{U}^\dagger_{in} \dot{a}^\dagger_{in} \mathbb{U}_{in} + \mathbb{U}^\dagger_{in} a^\dagger_{in} \dot{\mathbb{U}}_{in}, \\ \dot{a} &= \dot{\mathbb{U}}^\dagger_{in} a_{in} \mathbb{U}_{in} + \mathbb{U}^\dagger_{in} \dot{a}_{in} \mathbb{U}_{in} + \mathbb{U}^\dagger_{in} a_{in} \dot{\mathbb{U}}_{in}, \end{aligned} \tag{2.68}$$

now, using the equation of motion for the operators involved, we obtain

$$[\dot{\mathbb{U}}_{in}\mathbb{U}^\dagger_{in} + i\mathbb{H}_{int}, a^\dagger_{in}(p,\tau)] = 0, \quad [\dot{\mathbb{U}}_{in}\mathbb{U}^\dagger_{in} + i\mathbb{H}_{int}, a_{in}(p,\tau)] = 0. \tag{2.69}$$

It is remarkable to noticed that the interacting Hamiltonian is now a function of the in-operators instead of $a(p,\tau)$ and $a^\dagger(p,\tau)$, and the change is archived using (2.62)

$$\mathbb{H}_{int}(a_{in}, a^\dagger_{in}) = \mathbb{H}(a^\dagger_{in}, a_{in}) - \mathbb{H}_{in,0} = \mathbb{U}_{in}\mathbb{H}(a^\dagger_k, a_k)\mathbb{U}^\dagger_{in} - \mathbb{H}_{in,0}. \tag{2.70}$$

Equation (2.68) implies that

$$\dot{\mathbb{U}}_{in}\mathbb{U}^\dagger_{in} + i\mathbb{H}_{int}(a^\dagger_{in}, a_{in}) = 0. \tag{2.71}$$

The last can be solved by the following *time-order* operator

$$\mathbb{U}_{in}(\tilde{\tau}) = \mathcal{T}exp\Big(-i \int_{-\infty}^{\tilde{\tau}} d\tau \mathbb{H}_{int}(a^\dagger_{in}(\tau), a_{in}(\tau))\Big). \tag{2.72}$$

Moreover, the same consideration is valid for $\mathbb{U}_{out}$

$$\mathbb{U}_{out}(\tilde{\tau}) = \mathcal{T}exp\Big(-i \int_{\tilde{\tau}}^{\infty} d\tau \mathbb{H}_{int}(a^\dagger_{out}(\tau), a_{out}(\tau))\Big), \tag{2.73}$$

where in the two expression for $\mathbb{U}_{in}(\tau)$ and $\mathbb{U}_{out}(\tau)$ we have used the *time-ordered* exponential $\mathcal{T}exp$. Thus, it is possible to derive the explicit expression for the S-matrix using the expression found above and (2.66)

$$\begin{aligned} \mathbb{S} = \mathbb{U}_{in}(\tilde{\tau} = \infty) &= Texp\Big(-i \int_{-\infty}^{\infty} d\tau \mathbb{H}_{int}(a^\dagger_{in}(\tau), a_{in}(\tau))\Big) \\ = \mathbb{U}_{out}(\tilde{\tau} = -\infty) &= Texp\Big(-i \int_{-\infty}^{\infty} d\tau \mathbb{H}_{int}(a^\dagger_{out}(\tau), a_{out}(\tau))\Big). \end{aligned} \tag{2.74}$$

Clearly, perturbation theory requires the expansions for the previous *time-order* exponentials, which are archived by a series in the coupling constant. We will need only the leading order term

$$\mathbb{S} = \mathbb{I} + \frac{i}{T}\mathbb{T}, \quad \mathbb{T} = -T \int_{-\infty}^{\infty} d\tau \mathbb{H}_{int}(\tau) + \dots, \tag{2.75}$$

where T is, in general, the coupling constant that in this case corresponds to the *string tension.* As we said before, in the *decompactification limit* the scattering matrix is well defined, and the perturbation theory is performed in the large string tension limit. Thus we can compute the worldsheet two-particles S-matrix for the light-cone σ-model simply finding the expansion in terms of creation and annihilation operators, for the quartic Hamiltonian that we have denoted by $\mathcal{H}_4$ in (2.47). The choice of the creation and annihilation operators should be the corrected one if the quadratic Hamiltonian $\mathcal{H}_2$ has the same expression of (2.60).

In order to complete the discussion, we note that from (2.62) and setting $a(p) = a(p,0)$ and $a^\dagger(p) = a^\dagger(p,0)$, then we connect naturally

$$a^\dagger(p) = \mathbb{U}^\dagger_{in}(0) a^\dagger_{in} \mathbb{U}_{in}(0), \quad a(p) = \mathbb{U}^\dagger_{in}(0) a_{in}(p) \mathbb{U}_{in}(0), \tag{2.76}$$

as well as for out-operators. This implies that (2.62) can be rewritten as

$$|p_1,\dots,p_n\rangle^{(in)}_{i_1,\dots,i_n} = \mathbb{U}_{in}(0)a^\dagger_{i_1}(p_1)\dots a^\dagger_{i_n}(p_n)\,|0\rangle = \mathbb{U}_{in}(0)\,|\Phi_\alpha\rangle\,, \tag{2.77}$$

where $|0\rangle = \mathbb{U}^\dagger_{in}(0)\,|\Omega\rangle$ is the state such that $a_k(p)\,|0\rangle = 0$, moreover in the last formula we used the unitarity condition $\mathbb{U}^\dagger\mathbb{U} = \mathbb{I}$. In order to find the explicit expression for $\mathbb{U}_{in}(0)$ and for $\mathbb{U}_{out}(0)$ we introduce free time dependent operators

$$a^\dagger(p,\tau)_{free,k} = e^{i\omega_k(p)\tau}a^\dagger_k(p), \quad a(p,\tau)_{free,k} = e^{-i\omega_k(p)\tau}a_k(p), \tag{2.78}$$

which basically justify the expression (2.52). However since in this way they have the same time dependence of the in- out- operators, we are then able to connect

$$a^\dagger(p,\tau)_{free} = \mathbb{U}^\dagger_{in}(0)a^\dagger_{in}(p,\tau)\mathbb{U}_{in}(0), \quad a(p,\tau)_{free} = \mathbb{U}^\dagger_{in}(0)a_{in}(p,\tau)\mathbb{U}_{in}(0). \tag{2.79}$$

Thus, it is manifest that

$$\begin{aligned}\mathbb{U}_{in}(\tilde\tau) &= \mathbb{U}_{in}(0)\mathcal{T}exp\Big(-i\int_{-\infty}^{\tilde\tau} d\tau\mathbb{H}_{int}(a^\dagger(p,t)_{free}, a(p,t)_{free})\Big)\mathbb{U}^\dagger_{in}(0),\\ \mathbb{U}_{out}(\tilde\tau) &= \mathbb{U}^\dagger_{out}(0)\mathcal{T}exp\Big(-i\int_{\tilde\tau}^{\infty} d\tau\mathbb{H}_{int}(a^\dagger(p,t)_{free}, a(p,t)_{free})\Big)\mathbb{U}_{out}(0).\end{aligned} \tag{2.80}$$

Then it is possible to extract the S-matrix elements as

$$\langle out|in\rangle_{\beta,\alpha} = \langle\Phi_\beta|\,\mathbb{U}_{out}(0)\mathbb{U}_{in}(0)\,|\Phi_\alpha\rangle = \langle\Phi_\beta|\,\tilde{\mathbb{S}}\,|\Phi_\alpha\rangle\,, \tag{2.81}$$

where

$$\tilde{\mathbb{S}} = \mathbb{U}_{out}(0)\mathbb{U}_{in}(0). \tag{2.82}$$

It is important to noticed that the operators $\tilde{\mathbb{S}}$ differs from $\mathbb{S}$ present in (2.66) by the opposite order of the operators $\mathbb{U}_{in}$, $\mathbb{U}_{out}$.

2.5 S-matrix in flat space

In this section we will expose the S-matrix computation in the case of flat space in order to familiarize with the future computation using the first order formalism. Basically the starting action is (1.11), and we further set the dimension of the spacetime to $D = 4$. The line element does not depends on the fields

$$ds^2 = -dt^2 + dX_1^2 + dX_2^2 + dX_3^2, \tag{2.83}$$

and the Virasoro constraints reads as

$$C_1 = p_M X'^\mu = 0, \quad C_2 = \eta^{MN}p_M p_N + T^2\eta_{MN}X'^M X'^N = 0. \tag{2.84}$$

In the case of $AdS_3 \times S^3$ we have identified two special coordinates, because the metric did not depend explicitly on them. This fact reflects directly on the symmetry (traslations) of the model. In this case there are no preferred coordinates. Hence we can choose the time t and another generic coordinate and, from them, construct the light-cone coordinates. For instance we solve the Virasoro constraints introducing the light-cone coordinates

$$\begin{aligned}&x^- \equiv X_3 - t, \quad x^+ \equiv (1-a)t + aX_3,\\ &p_- = \frac{\delta S}{\delta\dot x^-} = (1-a)p_{x_3} - ap_t\ , \quad p_+ = \frac{\delta S}{\delta\dot x^+} = p_{x_3} + p_t.\end{aligned} \tag{2.85}$$

and the gauge-fixing condition

$$x^+ = \tau, \quad p_- = 1. \tag{2.86}$$

With this, we can solve the first equation in (2.84) obtaining

$$x'^{-} = -P_{x_1}X_1' - P_{x_2}X_2'. \tag{2.87}$$

While the second equation in (2.84) becomes

$$\begin{aligned} C_2 = P_{x_1}^2 + P_{x_2}^2 - ((1-a)p_- - p_+)^2 + (ap_- + p_+)^2 + T^2\Big(X_1'^2 + X_2'^2 \\ + ((1-a)x'_- + x'_+)^2 - (-ax'_- + x'_+)^2\Big) = 0. \end{aligned} \tag{2.88}$$

Now, we have to use (2.87) and the gauge fixing condition to extract the Hamiltonian from the previous equation

$$H = -\int d\sigma p_+(P_x, X, X'). \tag{2.89}$$

Of course we have a similar expression to (2.41), thus we need also to rescale the fields as in (2.46) in order to get the perturbative expansion of the Hamiltonian. In particular, taking the large string tension limit, we obtain the quadratic expression in the fields

$$H^{(2)} = \frac{1}{2}(P_{x_1}^2 + P_{x_2}^2 + X_1'^2 + X_2'^2). \tag{2.90}$$

It is convenient to introduce the complex variables

$$\begin{aligned} X_1 &= \frac{-i(X-\bar{X})}{\sqrt{2}},\ X_2 = \frac{-(X+\bar{X})}{\sqrt{2}}, \\ X_1' &= \frac{-i(X'-\bar{X}')}{\sqrt{2}},\ X_2' = \frac{-(X'+\bar{X}')}{\sqrt{2}}, \\ P_{x_1} &= \frac{-i(P_x-\bar{P})}{\sqrt{2}},\ P_{x_2} = \frac{-(P_x+\bar{P})}{\sqrt{2}}. \end{aligned} \tag{2.91}$$

With this $H^{(2)}$ becomes

$$H^{(2)} = P_x\bar{P}_x + X'\bar{X}'. \tag{2.92}$$

The previous corresponds to the free Hamiltonian that describes the motion of a massless complex scalar field indeed the two transverse real degrees of freedom are now encoded in the real and complex components of the field X. Now, it is important to notice that the free wave equation that come from the last quadratic Hamiltonian is solved by the following modes expansion

$$\begin{aligned} X(\sigma,\tau) &= \int dp\Big(\frac{e^{-i\omega(p)\tau+ip\sigma}}{\sqrt{2g(p)}}a_x(p) + \frac{e^{i\bar{\omega}(p)\tau-ip\sigma}}{\sqrt{2\bar{g}(p)}}a_{\bar{x}}^\dagger(p)\Big), \\ \bar{X}(\sigma,\tau) &= \int dp\Big(\frac{e^{-i\bar{\omega}(p)\tau+ip\sigma}}{\sqrt{2\bar{g}(p)}}a_{\bar{x}}(p) + \frac{e^{i\omega(p)\tau-ip\sigma}}{\sqrt{2g(p)}}a_x^\dagger(p)\Big), \\ X'(\sigma,\tau) &= \int dp\Big(\frac{e^{-i\omega(p)\tau+ip\sigma}}{\sqrt{2g(p)}}ipa_x(p) - \frac{e^{i\bar{\omega}(p)\tau-ip\sigma}}{\sqrt{2\bar{g}(p)}}ipa_{\bar{x}}^\dagger(p)\Big), \\ \bar{X}'(\sigma,\tau) &= \int dp\Big(\frac{e^{-i\bar{\omega}(p)\tau+ip\sigma}}{\sqrt{2\bar{g}(p)}}ipa_{\bar{x}}(p) - \frac{e^{i\omega(p)\tau-ip\sigma}}{\sqrt{2g(p)}}ipa_x^\dagger(p)\Big), \\ P_x(\sigma,\tau) &= \int dp\Big(-i\frac{e^{-i\omega(p)\tau+ip\sigma}}{\sqrt{2g(p)}}\omega(p)a_x(p) + i\frac{e^{i\bar{\omega}(p)\tau-ip\sigma}}{\sqrt{2\bar{g}(p)}}\bar{\omega}(p)a_{\bar{x}}^\dagger(p)\Big), \\ \bar{P}_x(\sigma,\tau) &= \int dp\Big(-i\frac{e^{-i\bar{\omega}(p)\tau+ip\sigma}}{\sqrt{2\bar{g}(p)}}\bar{\omega}(p)a_{\bar{x}}(p) + i\frac{e^{i\omega(p)\tau-ip\sigma}}{\sqrt{2g(p)}}\omega(p)a_x^\dagger(p)\Big). \end{aligned} \tag{2.93}$$

In this relations we can identify that the operators $a_x(p)$ and $a_{\bar{x}}(p)$ destroy a particle and an antiparticle respectively, while the operators $a_x^\dagger(p)$ and $a_{\bar{x}}^\dagger(p)$ create a particle and an antiparticle.

Hence, if we include the modes expansion in the quadratic Hamiltonian we then should find the structure exposed in the section 2.4.

In order to explicit write down $H^{(2)}$ in terms of creation and annihilation operators, we also need the dispersion relation $\omega(p) = \sqrt{p^2}$ and the normalization condition $g(p) = \omega(p)$. Moreover, in this case we have also that $\omega(p) = \bar{\omega}(p)$ and hence $g(p) = \bar{g}(p)$. With this considerations it is immediate to obtain

$$H^{(2)} = \int dp\ \omega(p)\Big(a_x^\dagger(p)a_x(p) + a_{\bar{x}}^\dagger(p)a_{\bar{x}}(p)\Big). \tag{2.94}$$

We are now in the correct position to start with the computation of the S-matrix. For this aim, we start by writing down the quartic expansion which follows from the rescaling and the large string tension limit of (2.41)

$$\begin{aligned} H^{(4)} = \frac{(1-2a)}{8}\Big(&(P_{x_1}^2 + P_{x_2}^2)^2 - 8P_{x_1}P_{x_2}X_1'X_2' + 2P_{x_2}^2X_1'^2 - 2P_{x_2}^2X_2'^2 \\ &- 2P_{x_1}^2X_1'^2 + 2P_{x_1}^2X_2'^2 + (X_1'^2 + X_2'^2)^2\Big). \end{aligned} \tag{2.95}$$

The expression is again more readable if one uses the change of variables (2.91), in fact

$$H^{(4)} = \frac{(-1+2a)}{2}(P_x^2 - X'^2)(\bar{P}_x^2 - \bar{X}'^2). \tag{2.96}$$

It is crucial to point out that in the case of $a = \frac{1}{2}$ the quartic Hamiltonian *i.e.* the interacting part, vanishes. This fact is actually what we expected, indeed we have to deal with flat spacetime that yields a free σ model. However, in the case of $a \neq \frac{1}{2}$ the interacting Hamiltonian is present. The tree level S-matrix computation basically consists in the rewriting of the quartic Hamiltonian in terms of oscillators. So, including (2.93), in (2.96) we obtain

$$\begin{aligned} H^{(4)} = \int dp_1 dp_2\Big(&T_{XX}(p_1,p_2)a_x^\dagger(p_1)a_x^\dagger(p_2)a_x(p_1)a_x(p_2) + \\ &T_{\bar{X}\bar{X}}(p_1,p_2)a_{\bar{x}}^\dagger(p_1)a_{\bar{x}}^\dagger(p_2)a_{\bar{x}}(p_1)a_{\bar{x}}(p_2) + \dots\Big). \end{aligned} \tag{2.97}$$

Let us spend a few more words about the computations. In particular, when products of four fields expressed as in (2.93) are integrated, continuous products of Dirac delta are encountered. These, once integrated, provide the following Jacobian

$$\Omega(p_1,p_2) = \frac{1}{\omega'(p_1) - \omega'(p_2)}, \tag{2.98}$$

which can be computed once the dispersion relationship is known. So, using $\omega(p) = \sqrt{p^2}$, it turns out that

$$\Omega(p_1,p_2) = \frac{\omega(p_1)\omega(p_2)}{p_1\omega(p_2) - p_2\omega(p_1)} = \frac{\sqrt{p_1^2}\sqrt{p_2^2}}{p_1\sqrt{p_2^2} - p_2\sqrt{p_1^2}}. \tag{2.99}$$

The previous Jacobian is a factor that must be taken into account for the calculation of each matrix element. For instance

$$T_{XX} = (1-2a)\frac{(p_1p_2 - \omega(p_1)\omega(p_2))^2}{2\omega(p_1)\omega(p_2)}\Omega(p_1,p_2), \tag{2.100}$$

that it can be rewritten, explicitly using the dispersion relation and multiplying the numerator and denominator by $p_1\sqrt{p_2^2} - p_2\sqrt{p_1^2}$, in a more readable way as

$$T_{XX} = (-1+2a)\frac{p_1\omega(p_2) - p_2\omega(p_1)}{2}. \tag{2.101}$$

It is also manifest that we can distinguish between four different situation

$$\begin{aligned} &T_{XX} = (-1+2a)p_1p_2 \iff p_1 > 0,\ p_2 < 0, \quad T_{XX} = 0 \iff p_1 > 0,\ p_2 > 0, \\ &T_{XX} = (1-2a)p_1p_2 \iff p_1 < 0,\ p_2 > 0, \quad T_{XX} = 0 \iff p_1 < 0,\ p_2 < 0, \end{aligned} \tag{2.102}$$

which tells us that, in two dimensions, two massless relativistic particles with momenta oriented in the same direction, never meet, while if the momenta are oriented in opposite directions, they can scatter. However, in the following we report all the non vanishing T-matrix elements

$$\begin{aligned} T_{XX}(p_1,p_2) &= (-1+2a)\frac{p_1\omega(p_2)-p_2\omega(p_1)}{2},\ T_{\bar{X}\bar{X}}(p_1,p_2) = (-1+2a)\frac{p_1\omega(p_2)-p_2\omega(p_1)}{2}, \\ T_{\bar{X}X}(p_1,p_2) &= (-1+2a)\frac{p_1\omega(p_2)-p_2\omega(p_1)}{2},\ T_{X\bar{X}}(p_1,p_2) = (-1+2a)\frac{p_1\omega(p_2)-p_2\omega(p_1)}{2}. \end{aligned} \tag{2.103}$$

We can then extract directly the S-matrix elements remembering that

$$\mathbb{S} = \mathbb{I} + \frac{i}{T}\mathbb{T}. \tag{2.104}$$

We remark that in the previous expressions we denoted with T_{XX} the tree-level T-matrix elements which corresponds to the scattering of two particles into two particles as well as $T_{X\bar{X}}$ corresponds to the scattering of a particle and an antiparticle into a particle and an antiparticle.

Finally, we mention that if the complete two-body S-matrix and the dispersion relation are known *i.e.* by including all the loops contributions, then it is possible to compute the spectrum of the closed string using the *Bethe-Yang* equations. From the point of view of the ZF algebra we can define an asymptotic wave function as

$$|\psi(p_1,\dots,p_M)\rangle = \sum_{\pi\in s_M} S_\pi \left|p_{\pi(1)},\dots,p_{\pi(M)}\right\rangle, \tag{2.105}$$

where $\pi \in s_M$ is a permutation and each asymptotic state reads as

$$|p_1,\dots,p_M\rangle = \int d\sigma_1\dots d\sigma_M e^{i(p_1\sigma_1+\dots+p_M\sigma_M)} A^\dagger(\sigma_1)\dots A^\dagger(\sigma_M)\,|0\rangle\,. \tag{2.106}$$

If we require the periodicity under the shift $\sigma_i \to \sigma_i + l$ [2], then we obtain the *Bethe-Yang* equations

$$e^{ip_k l}\prod_{j\neq k}^{M} S(p_k,p_j) = 1, \quad k = 1,\dots,M. \tag{2.107}$$

In particular, the spectrum matches with the one given in chapter 1.

[2] Where l is the length of the string

Chapter 3

Integrability

Integrability is a very broad concept that emerges both from classical Hamiltonian and quantum systems. Colloquially, an integrable system enjoys so many symmetries that its dynamic is completely constrained and it turns out that it is solvable exactly. The last statement means that it is possible to find some important information, for instance the spectrum, of the theory solving some system of algebraic equations. In the following we will analyze this concept, from a classical point of view as well as from the quantum prospective.

3.1 Classical integrability

Integrability in classical mechanics is a quite simple concept, which for instance is exposed in [29]. Let us focus in the case of Hamiltonian systems. In that systems we describe the motion of the n-degrees of freedom with a trajectory in a $2n$-dimensional phase-space M with local coordinates (q_i, p_j) $i, j = 1, \dots, n$. The evolution of a generic function $f \equiv f(q(t), p(t), t)$ of the phase space is given by

$$\dot{f} = \frac{\partial f(q(t), p(t), t)}{\partial t} + \Big\{ f(q(t), p(t), t), H(q(t), p(t)) \Big\}_{P.B.}. \tag{3.1}$$

A function $f(q(t), p(t))$ is called a *first integral* or a *constant of motion* if $\dot{f} = 0$, which reads as

$$\Big\{ f(q(t), p(t)), H(q(t), p(t)) \Big\}_{P.B.} = 0. \tag{3.2}$$

Two first integral are in *involution* if

$$\{ f_1(q(t), p(t)), f_2(q(t), p(t)) \}_{P.B.} = 0. \tag{3.3}$$

Moreover a set of k first-integrals are independent functions if their gradients are linearly independent

$$rk(\nabla f_1, \dots, \nabla f_k) = k. \tag{3.4}$$

A system will be solvable if it admits sufficient many first integrals. In a more formal way, integrability can be archived via the Liouville-Arnol'd theorem. It states that an integrable system consist in a $2n$ dimensional phase space together with n independents functions $f_1, \dots, f_n$ in involution

$$\{f_j,\ f_k\} = 0, \quad j, k = 1, \dots, n. \tag{3.5}$$

Let be

$$M_f \equiv \{(q, p) \in M : f_k(q, p) = c_k\}, \quad c_k = const, \quad k = 1, \dots, n, \tag{3.6}$$

a n-dimensional surface of first-integrals. If M_f is compact and connected then it is diffeomorphic to a torus

$$T^n = S^1 \times S^1 \times \cdots \times S^1, \tag{3.7}$$

and we can introduce the *action-angle* coordinates

$$I_1, \dots, I_n, \phi_1, \dots, \phi_n, \quad 0 \le \phi_k \le 2\pi. \tag{3.8}$$

Moreover the equations of motion become

$$\dot{I}_k = 0, \quad \dot{\phi}_k = \omega_k(I_1, \dots, I_n), \quad k = 1, \dots, n. \tag{3.9}$$

Typically one of the first-integrals is the Hamiltonian while the others can be found geometrically.

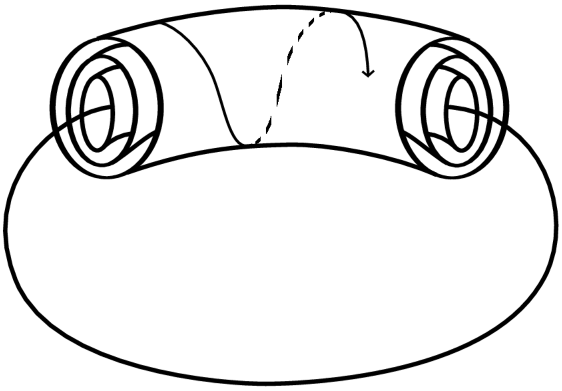

Figure 3.1: This picture, reported from [19], represents the tori foliation of the phase space. The motion, which is sketched by the arrow, is periodic and in particular it winds around one of the tori .

The crucial idea is therefore that if we have enough symmetries then the dynamic is constrained such that the model is solvable. Since, in this thesis, we are dealing with field theory, it is crucial to extend the previous concepts in the case of infinity degrees of freedom. Naturally this is not the most convenient approach in the case of fields since we will find an infinite set of conserved quantities in involution. Thus, in the case of field theory, the so-called *Lax formalism* is the most convenient formalism, as it is pointed out in [19]. Let us consider two dimensional classical field theory since the sting-NLSM is formulated on the two dimensional woldsheet. Moreover we assume that the equation of motion can be recast as

$$\partial_\tau U(\tau, \sigma, z) - \partial_\sigma V(\tau, \sigma, z) = [V(\tau, \sigma, z), U(\tau, \sigma, z)], \tag{3.10}$$

where U, V are matrices depending on the fields and a complex parameter z. We may think that the previous equation arises from the following system

$$\begin{aligned} \partial_\tau \Psi(\tau, \sigma, z) &= V(\tau, \sigma, z)\Psi(\tau, \sigma, z), \\ \partial_\sigma \Psi(\tau, \sigma, z) &= U(\tau, \sigma, z)\Psi(\tau, \sigma, z), \end{aligned} \tag{3.11}$$

since

$$\begin{aligned} \partial_\sigma \partial_\tau \Psi(\tau, \sigma, z) &= \partial_\sigma V(\tau, \sigma, z)\Psi(\tau, \sigma, z) + V(\tau, \sigma, z)\partial_\sigma \Psi(\tau, \sigma, z) \\ &= \Big(\partial_\sigma V(\tau, \sigma, z) + V(\tau, \sigma, z)U(\tau, \sigma, z)\Big)\Psi(\tau, \sigma, z), \end{aligned} \tag{3.12}$$

$$\begin{aligned} \partial_\tau \partial_\sigma \Psi(\tau, \sigma, z) &= \partial_\tau U(\tau, \sigma, z)\Psi(\tau, \sigma, z) + U(\tau, \sigma, z)\partial_\tau \Psi(\tau, \sigma, z) \\ &= \Big(\partial_\tau U(\tau, \sigma, z) + U(\tau, \sigma, z)V(\tau, \sigma, z)\Big)\Psi(\tau, \sigma, z) \end{aligned} \tag{3.13}$$

and thus this implies (3.10), by requiring that $(\partial_\sigma \partial_\tau - \partial_\tau \partial_\sigma)\Psi = 0$.

This procedure, called the *inverse scattering method*, allows to solve the systems of linear equations instead of the non linear ones (3.10). Now, if we introduce a 2-dimensional connection L_α with components

$$L_\sigma(\tau, \sigma, z) = U(\tau, \sigma, z), \qquad L_\tau(\tau, \sigma, z) = V(\tau, \sigma, z), \tag{3.14}$$

then, the equation of motion (3.10) can be rewritten as the zero-curvature equation [13]

$$\partial_\beta L_\alpha - \partial_\alpha L_\beta + [L_\alpha, L_\beta] = 0. \tag{3.15}$$

A connection L_α with this property is called *Lax connection* and the equation (3.15) is called the zero-curvature (Lax) representation of an integrable partial differential equation. The usefulness of the Lax connection lies in the fact that for an integrable model it provides a way to exhibit the conservation laws and hence the first integrals. In order to see this we define the *monodromy matrix* $T(\tau, z)$ by the path-ordered exponential

$$T(\tau, z) = \overleftarrow{exp}\Big(\int_{-\frac{l}{2}}^{\frac{l}{2}} d\sigma U(\tau, \sigma, z)\Big). \tag{3.16}$$

The evolution in time is archived once we compute $\partial_\tau T(\tau, z)$, and for the sake of simplicity we consider a closed string with $\sigma \in [0, 2\pi]$. In this way the time derivative of (3.16) is

$$\begin{aligned}\partial_\tau T(\tau, z) &= \int_0^{2\pi} d\sigma \Big(\overleftarrow{exp}\int_0^{2\pi} U(\tau, \tilde{\sigma}, z)\Big)\partial_\tau U(\tau, \sigma, z) \\ &= \int_0^{2\pi} d\sigma \Big(\overleftarrow{exp}\int_\sigma^{2\pi} U(\tau, \tilde{\sigma}, z)\Big)\partial_\tau U(\tau, \sigma, z)\Big(\overleftarrow{exp}\int_0^{\sigma} U(\tau, \tilde{\sigma}, z)\Big).\end{aligned} \tag{3.17}$$

Now, using 3.10

$$\begin{aligned}\partial_\tau T(\tau, z) &= \int_0^{2\pi} d\sigma \Big(\overleftarrow{exp}\int_\sigma^{2\pi} U(\tau, \tilde{\sigma}, z)\Big)(\partial_\sigma V + [V, U])\Big(\overleftarrow{exp}\int_0^{\sigma} U(\tau, \tilde{\sigma}, z)\Big) \\ &= \int_0^{2\pi} d\sigma \partial_\sigma \Big[\Big(\overleftarrow{exp}\int_\sigma^{2\pi} U(\tau, \tilde{\sigma}, z)\Big)V(\tau, \sigma, z)\Big(\overleftarrow{exp}\int_0^{\sigma} U(\tau, \tilde{\sigma}, z)\Big)\Big],\end{aligned} \tag{3.18}$$

which can be rewrite in a more compact and useful way as

$$\partial_\tau T(\tau, z) = [V(\tau, 0, z), T(\tau, z)]. \tag{3.19}$$

The last equation is crucial because the eigenvalues of $T(\tau, z)$, defined by

$$det(T(\tau, z) - \mu \mathbb{I}) = 0, \tag{3.20}$$

actually do not depend on τ, and hence they are integrals of motion. So, in the Lax formalism, the integrability of a field theory is archived once it is possible to recast the equation of motion using the components of the Lax connection as in (3.10). In this way it is possible to define the monodromy matrix whose eigenvalues basically encode the (infinite) conserved charges.

3.2 Principal chiral model

In the following section we will concretize the Lax formalism for the principal chiral model using the same notation of [13]. This toy model is the most simple example of integrable 2-dimensional field theory and basically it is a non linear σ-model with base space $\mathbb{R}^{1,1}$ and target space a Lie group G, hence the fields is the group elements $g \equiv g(\sigma, \tau) \in G$, and their dynamic is governed by the action

$$S = -\frac{1}{2}\int d\tau d\sigma \gamma^{\alpha\beta} Tr(\partial_\alpha g g^{-1} \partial_\beta g g^{-1}). \tag{3.21}$$

where we use the notation $\gamma^{\alpha\beta} \equiv \sqrt{-\gamma}\gamma^{\alpha\beta}$. The equations of motion

$$\begin{aligned}\partial_\alpha(\gamma^{\alpha\beta}\partial_\beta g g^{-1}) = 0, \\ \partial_\alpha(\gamma^{\alpha\beta} g^{-1}\partial_\beta g) = 0,\end{aligned} \tag{3.22}$$

can be rewritten in terms of

$$A_l^\alpha = -\gamma^{\alpha\beta}\partial_\beta g g^{-1}, \quad A_r^\alpha = -\gamma^{\alpha\beta} g^{-1}\partial_\beta g, \tag{3.23}$$

as

$$\partial_\alpha A_l^\alpha = \partial_\alpha A_r^\alpha = 0. \tag{3.24}$$

It is useful to introduce a connection that will depends on A^r, A^l and also on two parameters l_1, l_2, which basically depend on the spectral parameter z introduced in the previous section

$$L_\alpha = l_1 A_\alpha + l_2 \gamma_{\alpha\beta}\epsilon^{\beta\rho} A_\rho. \tag{3.25}$$

In this context, namely in two dimensions, it is useful to rewrite the zero curvature equation as

$$2\epsilon^{\alpha\beta}\partial_\alpha L_\beta - \epsilon^{\alpha\beta}[L_\alpha, L_\beta] = 0, \tag{3.26}$$

where $\epsilon^{\alpha\beta}$ is a completely antisymmetric tensor, and so expanding explicitly the previous contractions one obtains

$$\begin{aligned} &2\epsilon^{\tau\sigma}\partial_\tau L_\sigma + 2\epsilon^{\sigma\tau}\partial_\sigma L_\tau - \epsilon^{\tau\sigma}[L_\tau, L_\sigma] + \epsilon^{\sigma\tau}[L_\sigma, L_\tau] = 0,\\ &2\partial_\tau L_\sigma - 2\partial_\sigma L_\tau - [L_\tau, L_\sigma] - [L_\sigma, L_\tau] = 0, \end{aligned} \tag{3.27}$$

which is (3.10). Now, using the identity $\epsilon^{\alpha\beta}\gamma_{\beta\rho}\epsilon^{\rho\delta} = \gamma^{\alpha\delta}$, we can write down the zero curvature equation for the connection (3.25)

$$2l_1\epsilon^{\alpha\beta}\partial_\alpha A_\beta - (l_1^2 - l_2^2)\epsilon^{\alpha\beta}[A_\alpha, A_\beta] + 2l_2\partial_\alpha A^\alpha = 0. \tag{3.28}$$

The term

$$2l_2\partial_\alpha A^\alpha \tag{3.29}$$

vanishes due to the equations of motion. On the other hand, both A^r, A^l are flat

$$\partial_\alpha A_\beta^l - \partial_\beta A_\alpha^l + [A_\alpha^l, A_\beta^l] = 0, \quad \partial_\alpha A_\beta^r - \partial_\beta A_\alpha^r - [A_\alpha^r, A_\beta^r] = 0, \tag{3.30}$$

hence, the first terms in (3.28) become

$$2l_1 - 2(l_1^2 - l_2^2)[A_\alpha^r, A_\beta^r] = 0, \quad 2l_1 + 2(l_1^2 - l_2^2)[A_\alpha^l, A_\beta^l] = 0. \tag{3.31}$$

We deduce that the zero curvature equation (on-shell) is respected if

$$\begin{aligned} &l_1^2 - l_2^2 - l_1 = 0, \quad A = A^r,\\ &l_1^2 - l_2^2 + l_1 = 0, \quad A = A^l. \end{aligned} \tag{3.32}$$

Finally we can also give the expression for the Lax connection for the principal chiral model, when the previous system is solved in terms of a spectral parameter z

$$\begin{aligned} L_\alpha^r &= -\frac{z^2}{1-z^2}A_\alpha^r + \frac{z}{1-z^2}\gamma_{\alpha\beta}\epsilon^{\beta\rho}A_\rho^r,\\ L_\alpha^l &= \frac{z^2}{1-z^2}A_\alpha^l + \frac{z}{1-z^2}\gamma_{\alpha\beta}\epsilon^{\beta\rho}A_\rho^l. \end{aligned} \tag{3.33}$$

Eventually we rephrased the equation of motion for the principal chiral model as the zero curvature condition for the previous Lax connection.

3.3 Symmetric space σ model

The symmetric space σ model is another classically integrable field theory [27], but this example highlights also the classical integrability of $AdS_3 \times S^3$ string NLSM since its Lagrangian can be recast in a similar fashion to the one of the symmetric space σ model. This is a two dimensional filed theory with base manifold $\mathbb{R}^{1,1}$ and with target space $M = G/H$, where G is a connected semi-simple Lie group and H a particular subgroup of G. Moreover, the Lie algebra admits a $\mathbb{Z}_2$-graded decomposition

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(2)}, \quad [\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j \ mod 4)}, \tag{3.34}$$

where $\mathfrak{g}^{(0)}$ label the algebra of H. It it then useful to introduce the projectors into the single subspace in which the algebra is decomposed $P^{(j)}X = X^{(j)}$, where $X \in \mathfrak{g}^{(j)}$ and $j = 1, 2$. In the light-cone coordinates the action can be written as

$$\begin{aligned} S &= T \int d^2\sigma Tr\Big(g^{-1}\partial_- g P^{(2)} g^{-1}\partial_+ g\Big) \\ &= T \int d^2\sigma Tr\Big((g^{-1}\partial_- g)^{(2)} (g^{-1}\partial_+ g)^{(2)}\Big), \end{aligned} \tag{3.35}$$

where the Tr is the ad-invariant Killing form on $\mathfrak{g} = Lie(G)$. In particular, the ad-invariance is archived as

$$Tr[Ad_g^{-1}(X)Ad_g^{-1}(Y)] = Tr[Ad_g(X)Ad_g(Y)] = Tr[XY], \quad \forall g \in G, \quad \forall X,\ Y \in \mathfrak{g}, \tag{3.36}$$

and where we used Ad_g to label the adjoint representations $Ad_g(X) = gXg^{-1}$. In particular the expression in the first line of the equation (3.35) is clearly invariant under the global transformations $g \to g_0 g$ with $g_0 \in G$. Thus, it is possible to extract the conserved current associated to the previous group transformations

$$J_\pm = Ad_g(g^{-1}\partial_\pm g)^{(2)}, \tag{3.37}$$

which, as in the case of the principal chiral model, satisfies the flatness condition

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0. \tag{3.38}$$

Moreover the equation of motion is the conservation equation

$$\partial_- J_+^{(2)} + \partial_- J_-^{(2)} + [J_-^{(0)}, J_+^{(2)}] + [J_+^{(0)}, J_-^{(2)}] = 0. \tag{3.39}$$

One can notice that the last equation decomposes according to the $\mathbb{Z}_2$ grading. Also the flatness condition can be decompose in the same way

$$\begin{aligned} \partial_+ J_-^{(0)} - \partial_- J_+^{(0)} + [J_+^{(0)}, J_-^{(0)}] + [J_+^{(2)}, J_-^{(2)}] = 0, \\ \partial_+ J_-^{(2)} - \partial_- J_+^{(2)} + [J_+^{(0)}, J_-^{(0)}] + [J_+^{(2)}, J_-^{(2)}] = 0. \end{aligned} \tag{3.40}$$

As we performed in the previous section, we can collect all this information on a Lax connection that respect a flatness condition

$$L_\pm(z) = J_\pm^{(0)} + z^{\pm 1} J_\pm^{(2)}, \tag{3.41}$$

where z is again the spectral parameter.

3.4 Quantum integrability

In this section we will analyze the integrability of a 2-dimensional QFT (following [19]) since the string nonlinear σ-model turns out to be of this type. Let us assume that we have quantized a classical integrable theory. Let be

$$\mathcal{Q}_1, \dots, \mathcal{Q}_n, \dots, \qquad [\mathcal{Q}_i, \mathcal{Q}_j] = 0, \tag{3.42}$$

the infinite conserved charges. Since they commute between each other, we can simultaneously diagonalize them. We identify a basis of the Hilbert space labeled with the momentum and flavors α. The charges act on this basis as

$$\mathcal{Q}_n \left|p\right\rangle_\alpha^{(in,out)} = Q_n(p;\alpha) \left|p\right\rangle_\alpha^{(in,out)}. \tag{3.43}$$

Generically, these charges are built up as an invariant tensor constructed out of the labels α and many functions of particle momenta. For instance in some theories it is possible to have $Q_n(p;\alpha) \propto p^n$. Now, if we consider the action of such charges on the M-particle state

$$\mathcal{Q}_n \left|p_1, \dots, p_M\right\rangle_{\alpha_1,\dots,\alpha_M}^{(in)} = \Big(Q_n(p_1;\alpha_1) + \dots + Q_n(p_M;\alpha_M)\Big) \left|p_1, \dots, p_M\right\rangle_{\alpha_1,\dots,\alpha_M}^{(in)}, \tag{3.44}$$

and we consequently evolve the in-state into the out-state $|\tilde{p}_1, \ldots, \tilde{p}_M\rangle^{(out)}_{\tilde{\alpha}_1,\ldots,\tilde{\alpha}_M}$, then considering that the charges are conserved, we obtain the following identity

$$\sum_{j=1}^{M} Q_n(p_j;\alpha_j) = \sum_{k=1}^{\tilde{M}} Q_n(\tilde{p}_k;\tilde{\alpha}_k). \tag{3.45}$$

The only way for the previous equality to be worth $\forall Q_n$ is that the incoming momenta correspond to the outcoming momenta *i.e.* $M = \tilde{M}$. This fact in 2-dimensional QFT has deeper implications for the scattering events.

Let us consider the 2-particles scattering event. As we said in the chapter 2, the in-state $|p_1, p_2\rangle^{(in)}_{\alpha_1,\alpha_2}$ is defined at $\tau = -\infty$, moreover we order the momenta as $p_1 > p_2$. Since there is only one spatial dimension, the particles can move on a line, and at some point they scatter. For the conservation laws (3.45), the resulting state is of the form $|p_2, p_1\rangle_{\tilde{\alpha}_2,\tilde{\alpha}_1}$ where the momenta are exactly the same, while the flavor indexes may change.

In the case of M-particles events the in-state is $|p_1, \ldots, p_M\rangle^{(in)}_{\alpha_1,\ldots,\alpha_M}$, where we assume $p_1 > p_2 > \cdots > p_M$. At some point, two of the particles that compose this state undergo a $2 \to 2$ scattering event, whose dynamic is described above. After a sequence of $\frac{1}{2}M(M-1)$ scattering, the particles are spatially order following their momenta $p_M, \ldots, p_1$. We then deduce that the M-particles scattering, *factorizes* in a series of 2-particles events. This special property was pointed out by the Zamolodchikov brothers in their seminal paper [30]. Since the property of the scattering process is encoded in the S-matrix, the fundamental object that we need to determinate is the 2-particles S-matrix $\mathcal{S}(p_1, p_2)$.

Clearly, the S-matrix must respect some conditions. The first is that there are no $2 \to M$ scatterings unless $M = 2$, namely there are no particle production. Another one is that, apparently, there are several ways to factorize a $M \to M$ process into $2 \to 2$ events. Indeed we can group together the particles in several ways. It is clear that, for consistency, all the factorizations must bring to the same S-matrix. For instance in the case of a $3 \to 3$ event, we can have two different factorizations. Each of the two give a cubic expression in terms of the S-matrix, and their equality gives the famous *Yang-Baxter* equation. The physical reason for the Yang-Baxter equation is that, as we mentioned before, the charges $\mathcal{Q}_n$ are in general functions of particle momenta. They are unitary operators in the quantum theory that implement the transformations, under which the classical action is invariant, into the Hilbert space of the quantum theory. These transformations in particular shift the momentum p_j of the particles. Clearly, the amount of the shift may change from one particle to the others.

3.4.1 The Zamolodchikov-Faddeev algebra

The Zamolodchikov-Faddeev (ZF) algebra consists in a series of relations which are respected by a particular representation of the asymptotic Hilbert space. In this description we will follow [19] as well as [13]. However, the ZF algebra is defined in terms of abstract creation and annihilation operators $A^\dagger_\alpha(p)$, $A_\alpha(p)$ which act on the vacuum as

$$A^\dagger_\alpha |0\rangle = |p\rangle^{(in)}_\alpha = |p\rangle^{(out)}_\alpha, \quad A^\alpha(p)\,|0\rangle = 0. \tag{3.46}$$

These are related to the in- out-states, introduced in the chapter 2, by the following relations

$$\begin{aligned} |p_1, \ldots, p_n\rangle^{(in)}_{i_1,\ldots,i_n} &= A^\dagger_{i_1}(p_1) \ldots A^\dagger_{i_n}(p_n)\,|0\rangle, \\ |p_1, \ldots, p_n\rangle^{(out)}_{i_1,\ldots,i_n} &= (-1)^{\sum_{k<l}\epsilon_{i_k}\epsilon_{i_l}} A^\dagger_{i_n}(p_n) \ldots A^\dagger_{i_1}(p_1)\,|0\rangle, \end{aligned} \tag{3.47}$$

where we included the fact that when two fermions are exchanged, then we must take into account an overall minus sign, which in the previous formula is given by $(-1)^\epsilon$. So in the case of fermions $\epsilon = -1$ while for bosons $\epsilon = 0$. The in- out-states are the eigenstates of the Hamiltonian of the model $\mathbb{H}$ and the order of momenta are $p_1 > \ldots, p_n$. It is important to notice that the ZF creation and annihilation operators are not the Heisenberg in- out- operators $a_{in}(p)$, $a_{out}(p)$, in fact they do not satisfy the canonical commutation relations in the interacting theory, so they are not related to the in- out- operators by a unitary transformation. The final advantage is that in this new point of view for the asymptotic states, a scattering process, are understood as a reordering of particles in the momentum space.

Let us focus in the case of 2-body collisions, where we set $p_1 > p_2$,

$$\begin{aligned} |p_1, p_2\rangle^{(in)}_{\alpha_1,\alpha_2} &= A^\dagger_{\alpha_1}(p_1)A^\dagger_{\alpha_2}(p_2)\,|0\rangle \\ |p_1, p_2\rangle^{(out)}_{\alpha_3,\alpha_4} &= (-1)^{\epsilon_{\alpha_3}\epsilon_{\alpha_4}} A^\dagger_{\alpha_3}(p_2)A^\dagger_{\alpha_4}(p_1)\,|0\rangle\,. \end{aligned} \tag{3.48}$$

As it is pointed out before, in the case of fermions $\epsilon_\alpha = 1$, which allows the presence of a minus sign due to the exchange of the two fermionic particles. Hence, in the case of the $2 \to 2$ process, there are two possibilities. In the first ones, known as forward scattering, particles keep their initial momenta. In the second case, known as backward scattering, particles exchange their initial momenta. From the discussion of the previous chapter we can write

$$|p_1, p_2\rangle^{(in)}_{\alpha_1,\alpha_2} = \mathbb{S}^{\alpha_3,\alpha_4}_{\alpha_1,\alpha_2}\,|p_1, p_2\rangle^{(out)}_{\alpha_3,\alpha_4}\,, \tag{3.49}$$

and so we are able to find the commutations relation for the ZF operators

$$A^\dagger_{\alpha_1}(p_1)A^\dagger_{\alpha_2}(p_2) = (-1)^{\epsilon_{\alpha_3}\epsilon_{\alpha_4}}\mathbb{S}^{\alpha_3,\alpha_4}_{\alpha_1,\alpha_2}(p_1,p_2)A^\dagger_{\alpha_3}(p_2)A^\dagger_{\alpha_4}(p_1), \tag{3.50}$$

which manifest the fact that the ZF operators satisfy a different algebra respect to the in- out- operators.

In order to simplify the notation, it is useful to adopt matrix formalism. For this aim, let E_j and E^j be rows and column vectors with all vanishing entries except the one in the i-th position which is equal to the identity. These are the basis respectively in the vector space $\mathcal{V}$ and in the dual vector space $\mathcal{V}^*$. A basis of matrices that act on $\mathcal{V}$ is given by $E^i_j = E_j \otimes E^i$. Then we can express the ZF operators as rows and columns

$$A^\dagger = A^\dagger_\alpha(p)E^\alpha, \quad A = A^\alpha(p)E_\alpha. \tag{3.51}$$

We can then reabsorbing the permutation sign into the R matrix, which is defined as

$$R(p,q) = R^{\alpha_3,\alpha_4}_{\alpha_1,\alpha_2}(p,q)E^{\alpha_1}_{\alpha_3}\otimes E^{\alpha_2}_{\alpha_4} = (-1)^{\epsilon_{\alpha_3}\epsilon_{\alpha_4}}\mathbb{S}^{\alpha_3,\alpha_4}_{\alpha_1,\alpha_2}(p,q)E^{\alpha_1}_{\alpha_3}\otimes E^{\alpha_2}_{\alpha_4}, \tag{3.52}$$

hence the (3.50) takes the form

$$A^\dagger_{(1)}A^\dagger_{(2)} = A^\dagger_{(2)}A^\dagger_{(1)}R_{(12)}, \tag{3.53}$$

where the subscript indices specify the subspace of the tensor product $\mathcal{V}\otimes\mathcal{V}$ on which the operators act *i.e.*

$$A^\dagger_{(1)}A^\dagger_{(2)} = A^\dagger_i(p_1)A^\dagger_j(p_2)E^i\otimes E^j. \tag{3.54}$$

In order to complete the algebra it is necessary to manifest also the relations between the annihilation operators. So the complete ZF algebra is

$$\begin{aligned} &A^\dagger_{(1)}A^\dagger_{(2)} = A^\dagger_{(2)}A^\dagger_{(1)}R_{(12)}, \quad A_{(1)}A_{(2)} = R_{(12)}A_{(2)}A_{(1)}, \\ &A_{(1)}A^\dagger_{(2)} = A_{(2)}R_{(21)}A^\dagger_{(1)} + \delta(p_1 - p_2)\mathbb{I}. \end{aligned} \tag{3.55}$$

3.4.2 Yang-Baxter equation

From (3.50) we immediately deduce that

$$A^\dagger_{(1)}A^\dagger_{(2)} = A^\dagger_{(2)}A^\dagger_{(1)}R_{(12)} = A^\dagger_{(1)}A^\dagger_{(2)}R_{(21)}R_{(12)}, \tag{3.56}$$

and so the consistency condition

$$R_{(21)}R_{(12)} = R_{(12)}R_{(21)} = \mathbb{I}, \tag{3.57}$$

called *braiding unitarity*. The previous condition supplements the usual *unitarity* of the S-matrix as well as for the R-matrix, which reads as

$$R_{(12)}R^\dagger_{(12)} = R^\dagger_{(12)}R_{(12)} = \mathbb{I}. \tag{3.58}$$

Since our aim is to highlight the factorization of the S-matrix, we should extend our argument to 3-particle states like

$$\begin{aligned} |p_1, p_2, p_3\rangle^{(in)}_{\alpha_1,\alpha_2,\alpha_3} &= A^\dagger_{\alpha_1}(p_1)A^\dagger_{\alpha_2}(p_2)A^\dagger_{\alpha_3}(p_3)\,|0\rangle\,, \\ |p_1, p_2, p_3\rangle^{(out)}_{\alpha_1,\alpha_2,\alpha_3} &= (-1)^{\sum_{k<l}\epsilon_{\alpha_k}\epsilon_{\alpha_l}} A^\dagger_{\alpha_1}(p_3)A^\dagger_{\alpha_2}(p_2)A^\dagger_{\alpha_3}(p_1)\,|0\rangle\,. \end{aligned} \tag{3.59}$$

Let us denote, as in the previous cases, $A^\dagger_{(1)}A^\dagger_{(2)}A^\dagger_{(3)}$ a combination that act on $\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{V}$.

If we connect the creation operators with the annihilation ones then we find two different relations

$$\begin{aligned} A^\dagger_{(1)}A^\dagger_{(2)}A^\dagger_{(3)} &= A^\dagger_{(3)}A^\dagger_{(2)}A^\dagger_{(1)}R_{(12)}R_{(13)}R_{(23)}, \\ A^\dagger_{(1)}A^\dagger_{(2)}A^\dagger_{(3)} &= A^\dagger_{(3)}A^\dagger_{(2)}A^\dagger_{(1)}R_{(23)}R_{(13)}R_{(12)}, \end{aligned} \tag{3.60}$$

where we denoted

$$\begin{gathered} R_{(12)} = R^{kl}_{ij}E^i_k\otimes E^j_l\otimes\mathbb{I}_2, \quad R_{(13)} = R^{kl}_{ij}E^i_k\otimes\mathbb{I}_2\otimes E^j_l, \\ R_{(23)} = R^{kl}_{ij}\ \mathbb{I}_2\otimes E^i_k\otimes E^j_l. \end{gathered} \tag{3.61}$$

Hence the equality of the products in (3.60) leads to the famous *Yang-Baxter* equation

$$R_{(12)}R_{(13)}R_{(23)} = R_{(23)}R_{(13)}R_{(12)}, \tag{3.62}$$

which can be rewritten in terms of S-matrix as

$$\mathbb{S}_{(12)}(p_1,p_2)\mathbb{S}_{(13)}(p_1,p_3)\mathbb{S}_{(23)}(p_2,p_3) = \mathbb{S}_{(23)}(p_2,p_3)\mathbb{S}_{(13)}(p_1,p_3)\mathbb{S}_{(12)}(p_1,p_2). \tag{3.63}$$

It is crucial to recognize that both the left and the right side represent the 3-particles S-matrix and the equation is just an equality for its *factorization*. So we end up with the crucial observation that a necessary condition for the factorization of the S-matrix is that it must respect the Yang-Baxter equation.

We conclude this chapter with some further observations. The first one is that the S-matrix in (3.63) can be expanded in power of the coupling. This feature, typically happen in perturbation theory, and as me mentioned in the chapter 2 we will consider only

$$\mathbb{S} = \mathbb{I} + \frac{i}{T}\mathbb{T}. \tag{3.64}$$

Using this expansion, (3.63) simply reads as

$$[\mathbb{T}_{(12)}, \mathbb{T}_{(13)}] + [\mathbb{T}_{(12)}, \mathbb{T}_{(23)}] + [\mathbb{T}_{(13)}, \mathbb{T}_{(23)}] = 0, \tag{3.65}$$

which basically is the classical version of the Yang-Baxter equation. The second is a summary of what we had exposed, that will turn useful in future. If one has to deal with a classically integrable field theory, the most convenient formalism in order to describe this property is the Lax one. The infinite many conserved charges, once the theory is quantized, force the factorization of the S-matrix. Hence the integrable structure is encoded into the structure of the scattering process. In particular, a 2-dimensional quantum field theory is integrable if its scattering matrix respects the Yang-Baxter equation. So, our final purpose is basically to extract the S-matrix and verify if it respects the previous equation.

Chapter 4

$AdS_3 \times S^3$

In this chapter we will focus on the integrability structure underlying the non linear σ-model with target space $AdS_3 \times S^3$. First of all we will review the classical integrability of this model, and then we will move to the computation of the tree-level S-matrix. Finally we will check if the Yang-Baxter equations are respected.

4.1 Classical integrability

In this section we will analyze the classical integrability structure of the non linear σ-model with target space $AdS_3 \times S^3$ according to [20]. First of all we point out that it is possible to use the coset parametrization for this space that has the advantage of highlighting its integrable structure

$$AdS_3 \times S^3 \equiv \frac{SO(2,2) \times SO(4)}{SO(1,2) \times SO(3)} \cong \frac{SU(1,1) \times SU(2)}{SO(1,2) \times SO(3)}. \tag{4.1}$$

In this formulation we have linked the group $SO(2,2)$ in (2.8) to $SU(1,1)$ as well as the group $SO(4)$ to $SU(2)$.

We can think that $SU(1,1)$ are the isometries of AdS_3 while $SU(2)$ are the isometries of S^3. In this way the action of the string can be written as a non linear σ-model with base space the world-sheet and with target space $AdS_3 \times S^3$. This was first realized on the $AdS_5 \times S^5$ background and then specialized in our case. We also mention that this formalism work also in the case of superstring, in which the coset are

$$AdS_3 \times S^3 \equiv \frac{PSU(1,1|2) \times PSU(1,1|2)}{SO(1,2) \times SO(3)}. \tag{4.2}$$

In particular $PSU(1,1|2)$ contains the bosonic subgroup $SU(1,1) \times SU(2)$. The integrability structure becomes evident in the case of superstring and clearly it also implies the integrability of their bosonic part.

For the sake of simplicity, let us focus only on the S^3 space that forms the full spacetime, indeed all the following discussion can be extended also to the AdS_3 part.

The S^3 σ-model can be written as a principal chiral model, that was mentioned in chapter 3, for the group $SU(2)$. The action, in the light-cone coordinates, can be written as

$$S = -\frac{1}{2} \int d^2\sigma Tr(J_+ J_-), \quad J = g^{-1} dg \in \mathfrak{su}(2). \tag{4.3}$$

J is the conserved current under the global left group transformation, and $g \in SU(2)$. This action is equivalent to the one of the symmetric space σ-model for

$$\frac{F}{F_0} = \frac{SU(2) \times SU(2)}{SU(2)_{diag}}. \tag{4.4}$$

In order to see this, let us analyze in more detail the last model. In particular the algebra $\mathfrak{f} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ admits a $\mathbb{Z}_2$-decomposition in

$$\mathfrak{f} = \mathfrak{f}_0 \oplus \mathfrak{f}_2, \quad [\mathfrak{f}_i, \mathfrak{f}_j] \subset f_{i+j,\ mod\ 4}. \tag{4.5}$$

where $\mathfrak{f}_0$ is the algebra corresponding to the diagonal subalgebra of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ while $\mathfrak{f}_2$ is the orthogonal complement. Using a block diagonal matrix realization for the group F, then its algebra elements read as

$$A = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \bar{\mathcal{A}} \end{pmatrix} \in \mathfrak{f}, \ \mathcal{A}, \bar{\mathcal{A}} \in \mathfrak{su}(2). \tag{4.6}$$

In particular, using the projectors into the graded subspaces, we can then express the decomposition as

$$\begin{aligned} P_0 \mathcal{A} &= \begin{pmatrix} \mathcal{A}_0 & 0 \\ 0 & \mathcal{A}_0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathcal{A} + \tilde{\mathcal{A}} & 0 \\ 0 & \mathcal{A} + \tilde{\mathcal{A}} \end{pmatrix}, \\ P_2 \mathcal{A} &= \begin{pmatrix} \mathcal{A}_2 & 0 \\ 0 & -\mathcal{A}_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathcal{A} - \tilde{\mathcal{A}} & 0 \\ 0 & \tilde{\mathcal{A}} - \mathcal{A} \end{pmatrix}, \end{aligned} \tag{4.7}$$

with P_0, P_2 two projectors that project into $\mathfrak{f}_0$ and $\mathfrak{f}_2$ respectively. From the previous, it is manifest that

$$Tr(f_i f_j) = 0, \quad i + j \neq 0 \ mod\ 4, \tag{4.8}$$

since, for instance

$$Tr(f_0 f_2) = \frac{1}{4} Tr \left(\begin{pmatrix} \mathcal{A} + \tilde{\mathcal{A}} & 0 \\ 0 & \mathcal{A} + \tilde{\mathcal{A}} \end{pmatrix} \begin{pmatrix} \mathcal{A} - \tilde{\mathcal{A}} & 0 \\ 0 & \tilde{\mathcal{A}} - \mathcal{A} \end{pmatrix} \right) = \mathcal{A}^2 - \tilde{\mathcal{A}}^2 - \mathcal{A}^2 + \tilde{\mathcal{A}}^2 = 0. \tag{4.9}$$

The advantage is that the action can be rewritten as

$$S = - \int d^2\sigma Tr(\mathcal{J}_+ (P_2 \mathcal{J}_-)), \tag{4.10}$$

where

$$\begin{gathered} \mathcal{J} = f^{-1} df = \begin{pmatrix} J & 0 \\ 0 & \tilde{J} \end{pmatrix} = \begin{pmatrix} g^{-1} dg & 0 \\ 0 & \tilde{g}^{-1} d\tilde{g} \end{pmatrix}, \\ g, \tilde{g} \in SU(2), \quad f \in SU(2) \times SU(2), \quad J,\ \tilde{J} \in \mathfrak{su}(2). \end{gathered} \tag{4.11}$$

Then it is possible to recover (4.3), setting $\tilde{J} = 0$ and using the previous expressions for the projectors and for $\mathcal{J}$ in (4.10).

Now, the equation of motion following from the action (4.10) is

$$\partial_+ (P_2 \mathcal{J}_-) + \partial_- (P_2 \mathcal{J}_+) + [\mathcal{J}_+, P_2 \mathcal{J}_-] + [\mathcal{J}_-, P_2 \mathcal{J}_+] = 0, \tag{4.12}$$

where $\mathcal{J}$ is the left invariant current and so it respects the flatness condition

$$\partial_- \mathcal{J}_+ - \partial_+ \mathcal{J}_- + [\mathcal{J}_-, \mathcal{J}_+] = 0. \tag{4.13}$$

It now possible to project the flatness condition and the equation of motion into $\mathfrak{f}_0$ and $\mathfrak{f}_2$ using the projectors, and then eventually to prove that the resulting equations follow from the flatness condition of the following Lax connection

$$L_\pm = \mathcal{J}_{0\pm} + z^{\pm 2} \mathcal{J}_{2\pm}. \tag{4.14}$$

This basically proves the integrability of the non linear S^3 sigma model. Moreover, this formulation will be the starting point for the deformed background analysis. We also pointed out that the above construction can be analytically continued from S^3 to AdS_3 or equivalently from $SU(2)$ to $SU(1,1)$. Indeed all the formulas written in terms of algebra-value fields, will be the same except for some minus signs that take into account the correct signature of the metric.

The next step, is to give an explicit expression for the metric as well as for the action. We will use the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.15}$$

With this, the following parametrization of $f \in SU(2) \times SU(2)$

$$f = \begin{pmatrix} exp\Big(i\frac{\sigma_3}{2}(\phi+\theta) + i\frac{\sigma_1}{2}arcsin(r)\Big) & 0 \\ 0 & exp\Big(i\frac{\sigma_3}{2}(\theta-\phi) - i\frac{\sigma_1}{2}arcsin(r)\Big) \\ , & \end{pmatrix} \tag{4.16}$$

gives the non linear σ-model with target space described by the metric

$$ds^2_{S^3} = \frac{dr^2}{1-r^2} + (1-r^2)d\phi^2 + r^2 d\theta^2, \tag{4.17}$$

which matches (2.11). Moreover, if we parametrize $f \in SU(1,1) \times SU(1,1)$ as

$$f = \begin{pmatrix} exp\Big(i\frac{\sigma_3}{2}(\psi+t) + i\frac{\sigma_1}{2}arcsinh(\rho)\Big) & 0 \\ 0 & exp\Big(i\frac{\sigma_3}{2}(\psi-t) - i\frac{\sigma_1}{2}arcsinh(\rho)\Big) \end{pmatrix} \tag{4.18}$$

then, this gives the non linear σ-model with target space described by the metric

$$ds^2_{AdS_3} = \frac{d\rho^2}{1+\rho^2} - (1+\rho^2)dt^2 + \rho^2 d\psi^2, \tag{4.19}$$

which matches (2.15). Then we can deduce that the non linear σ-model in $AdS_3 \times S^3$, with the metric

$$ds^2 = ds^2_{S^3} + ds^2_{AdS_3}, \tag{4.20}$$

is a classical integrable field theory, since the sphere can be connected with the anti de Sitter space and we had proven that the equation of motion for S^3 σ-model can be recast as the flatness condition for a Lax connection. At this point we can also clarify the meaning of the fact that the AdS_3 non linear σ-model can be obtained by analytically continuing the S^3 one. Indeed the analytic continuation from S^3 to AdS_3 can be implemented at the level of the coordinates as

$$r \to -i\rho, \quad \phi \to t, \quad \theta \to \psi. \tag{4.21}$$

4.2 Quadratic Hamiltonian

The starting point is the metric

$$ds^2 = \frac{d\rho^2}{1+\rho^2} - (1+\rho^2)dt^2 + \rho^2 d\psi^2 + \frac{dr^2}{1-r^2} + (1-r^2)d\phi^2 + r^2 d\theta^2, \tag{4.22}$$

where

$$\begin{aligned} &r \in [0,1], \quad \theta \in (-\pi,\pi], \quad \phi \in (-\pi,\pi], \\ &\rho \in [0,\infty), \quad \psi \in [-\pi,\pi], \quad t \in (-\infty,\infty). \end{aligned} \tag{4.23}$$

We further used the following change of coordinates

$$\rho \to \sqrt{Z_1^2 + Z_2^2}, \ r \to \sqrt{Y_1^2 + Y_2^2}, \ \psi \to -\arctan\Big(\frac{Z_2}{Z_1}\Big), \ \theta \to \arctan\Big(\frac{Y_2}{Y_1}\Big). \tag{4.24}$$

With this, the metric tensor becomes

$$\begin{pmatrix} -1-Z_1^2-Z_2^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1+Z_2^2}{1+Z_1^2+Z_2^2} & -\frac{Z_1Z_2}{1+Z_1^2+Z_2^2} & 0 & 0 & 0 \\ 0 & -\frac{Z_1Z_2}{1+Z_1^2+Z_2^2} & \frac{1+Z_1^2}{1+Z_1^2+Z_2^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-Y_1^2-Y_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{-1+Y_2^2}{-1+Y_1^2+Y_2^2} & -\frac{Y_1Y_2}{-1+Y_1^2+Y_2^2} \\ 0 & 0 & 0 & 0 & -\frac{Y_1Y_2}{-1+Y_1^2+Y_2^2} & -\frac{-1+Y_1^2}{-1+Y_1^2+Y_2^2} \end{pmatrix}. \tag{4.25}$$

In the previous we have identified as transverse fields $Y_1,\ Y_2$ in the case of S^3 and $Z_1,\ Z_2$ in the case of AdS_3.

Now, since in the first order formalism it is necessary to find the momenta conjugate to the canonical variables, let us write the Lagrangian in the action (2.22) exploiting the parametrization given above and expanding it up to second order in the fields and in their derivatives. Moreover, we further set the B-field to zero

$$L^{(2)} = -\frac{T}{2}\gamma^{\alpha\beta}\Big(-(1+Z^2)\partial_\alpha t\partial_\beta t + \partial_\alpha Z_i\partial_\beta Z_i + (1-Y^2)\partial_\alpha\phi\partial_\beta\phi + \partial_\alpha Y_i\partial_\beta Y_i\Big). \tag{4.26}$$

In the previous expression we have implicit used the Einstein notation for the Y and Z fields, namely $\partial_\alpha Z_i\partial_\beta Z_i = \partial_\alpha Z_1\partial_\beta Z_1 + \partial_\alpha Z_2\partial_\beta Z_2$ and $Z^2 = Z_1^2 + Z_2^2$. The conjugate momenta are defined as

$$P_{y_i} = \frac{\delta L^{(2)}}{\delta\partial_\tau Y^i} = \dot{Y}_i, \quad P_{z_i} = \frac{\delta L^{(2)}}{\delta\partial_\tau Z^i} = \dot{Z}_i. \tag{4.27}$$

With this, it is then possible to introduce the first order formalism as we exposed in the section 2.2. Hence rescaling the fields on the worldsheet as in (2.46), it is then possible to expand (2.41) taking the large string tension limit. In particular the quadratic term is

$$H^{(2)} = \frac{1}{2}\Big(P_{z_1}^2 + P_{z_2}^2 + P_{y_1}^2 + P_{y_2}^2 + Y_1^2 + Y_2^2 + Z_1^2 + Z_2^2 + Y_1'^2 + Y_2'^2 + Z_1'^2 + Z_2'^2\Big), \tag{4.28}$$

where

$$Y_i' = \partial_\sigma Y_i, \quad Z_i' = \partial_\sigma Z_i. \tag{4.29}$$

As we expected, $H^{(2)}$ depends only on the transverse fields, and it is the free Hamiltonian of four massive bosons with unitary mass. As we mentioned before, the Y_i coordinates describe the fields associated to the sphere, while the Z_i coordinates describe the fields related to the anti-de Sitter space.

It is convenient to introduce two complex fields, which have the advantage of leading to a more readable perturbative Hamiltonian. Clearly, also the derivatives of the fields change, in particular

$$\begin{aligned} &Z = \frac{-Z_2 + i\ Z_1}{\sqrt{2}},\ \bar{Z} = \frac{-Z_2 - i\ Z_1}{\sqrt{2}},\ \bar{Y} = \frac{-Y_1 - i\ Y_2}{\sqrt{2}},\ Y = \frac{-Y_1 + i\ Y_2}{\sqrt{2}}, \\ &P_z = \frac{-P_{z_2} + i\ P_{z_1}}{\sqrt{2}},\ \bar{P}_z = \frac{-P_{z_2} - i\ P_{z_1}}{\sqrt{2}},\ \bar{P}_y = \frac{-P_{y_1} - i\ P_{y_2}}{\sqrt{2}},\ P_y = \frac{-P_{y_1} + i\ P_{y_2}}{\sqrt{2}}, \\ &Z' = \frac{-Z_2' + i\ Z_1'}{\sqrt{2}},\ \bar{Z}' = \frac{-Z_2' - i\ Z_1'}{\sqrt{2}},\ \bar{Y}' = \frac{-Y_1' - i\ Y_2'}{\sqrt{2}},\ Y' = \frac{-Y_1' + i\ Y_2'}{\sqrt{2}}. \end{aligned} \tag{4.30}$$

Whit this redefinitions, the quadratic part becomes

$$H^{(2)} = P_y\bar{P}_y + P_z\bar{P}_z + Y\bar{Y} + Z\bar{Z} + Y'\bar{Y}' + Z'\bar{Z}'. \tag{4.31}$$

This is the free Hamiltonian of two complex fields, whose real and imaginary components encode the previous real degrees of freedoms. In order to diagonalized $H^{(2)}$, we can introduce the following oscillators expansion

$$
\begin{aligned}
Z(\sigma,\tau) &= \int dp \Bigg(\frac{e^{-i\omega(p)\tau+ip\sigma}}{\sqrt{2g(p)}} a_z(p) + \frac{e^{i\bar{\omega}(p)\tau-ip\sigma}}{\sqrt{2\bar{g}(p)}} a^\dagger_{\bar{z}}(p) \Bigg), \\
Y(\sigma,\tau) &= \int dp \Bigg(\frac{e^{-i\omega(p)\tau+ip\sigma}}{\sqrt{2g(p)}} a_y(p) + \frac{e^{i\bar{\omega}(p)\tau-ip\sigma}}{\sqrt{2\bar{g}(p)}} a^\dagger_{\bar{y}}(p) \Bigg), \\
\bar{Z}(\sigma,\tau) &= \int dp \Bigg(\frac{e^{-i\bar{\omega}(p)\tau+ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{z}}(p) + \frac{e^{i\omega(p)\tau-ip\sigma}}{\sqrt{2g(p)}} a^\dagger_{z}(p) \Bigg), \\
\bar{Y}(\sigma,\tau) &= \int dp \Bigg(\frac{e^{-i\bar{\omega}(p)\tau+ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{y}}(p) + \frac{e^{i\omega(p)\tau-ip\sigma}}{\sqrt{2g(p)}} a^\dagger_{y}(p) \Bigg).
\end{aligned}
\tag{4.32}
$$

Moreover, the previous ansatz implies also that

$$
\begin{aligned}
P_y(\sigma,\tau) &= \int dp \Bigg(- i\frac{e^{-i\omega(p)\tau+ip\sigma}}{\sqrt{2g(p)}} \omega(p) a_y(p) + i\frac{e^{i\bar{\omega}(p)\tau-ip\sigma}}{\sqrt{2\bar{g}(p)}} \bar{\omega}(p) a^\dagger_{\bar{y}}(p) \Bigg), \\
\bar{P}_y(\sigma,\tau) &= \int dp \Bigg(- i\frac{e^{-i\bar{\omega}(p)\tau+ip\sigma}}{\sqrt{2\bar{g}(p)}} \bar{\omega}(p) a_{\bar{y}}(p) + i\frac{e^{i\omega(p)\tau-ip\sigma}}{\sqrt{2g(p)}} \omega(p) a^\dagger_{y}(p) \Bigg), \\
Y'(\sigma,\tau) &= \int dp \Bigg(ip\ \frac{e^{-i\omega(p)\tau+ip\sigma}}{\sqrt{2g(p)}} a_y(p) - ip\frac{e^{i\omega(p)\tau-ip\sigma}}{\sqrt{2\bar{g}(p)}} a^\dagger_{\bar{y}}(p) \Bigg), \\
\bar{Y}'(\sigma,\tau) &= \int dp \Bigg(ip\frac{e^{-i\bar{\omega}(p)\tau+ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{y}}(p) - ip\frac{e^{i\omega(p)\tau-ip\sigma}}{\sqrt{2g(p)}} a^\dagger_{y}(p) \Bigg),
\end{aligned}
\tag{4.33}
$$

and similar for the Z', $\bar{Z}'$, P_z, $\bar{P}_z$. Using the equation of motion, we find that

$$
\omega(p) = \sqrt{1+p^2}, \quad \bar{\omega}(p) = \sqrt{1+p^2}. \tag{4.34}
$$

Moreover, if

$$
g(p) = \omega(p), \quad \bar{g}(p) = \bar{\omega}(p) = \omega(p), \tag{4.35}
$$

then, the fields and their canonical conjugate momenta satisfy the canonical commutation relations

$$
\begin{aligned}
&[a_z(p), a^\dagger_z(p')] = \delta(p-p'), \quad [a_y(p), a^\dagger_y(p')] = \delta(p-p'), \\
&[a_{\bar{z}}(p), a^\dagger_{\bar{z}}(p')] = \delta(p-p'), \quad [a_{\bar{y}}(p), a^\dagger_{\bar{y}}(p')] = \delta(p-p'),
\end{aligned}
\tag{4.36}
$$

From the previous one can deduce that the creation operators are $a^\dagger_i$ while the annihilation ones are the a_i. In particular

oscillator	*particle*
$a^\dagger_z(p)$	$\vert Z(p)\rangle$
$a^\dagger_{\bar{z}}(p)$	$\vert \bar{Z}(p)\rangle$
$a^\dagger_y(p)$	$\vert Y(p)\rangle$
$a^\dagger_{\bar{y}}(p)$	$\vert \bar{Y}(p)\rangle$

Table 4.1: Particles created by the corresponding creation operators

We are now able to find the expansion of the quadratic Hamiltonian which is

$$
H^{(2)} = \int dp\ \omega(p) \Big(a^\dagger_z(p) a_z(p) + a^\dagger_y(p) a_y(p) + a^\dagger_{\bar{z}}(p) a_{\bar{z}}(p) + a^\dagger_{\bar{y}}(p) a_{\bar{y}}(p) \Big). \tag{4.37}
$$

4.3 S-matrix for $AdS_3 \times S^3$

With the previous considerations it is then possible to extract the S-matrix. This process essentially boils down to finding the interaction Hamiltonian and rewriting it in terms of oscillators. All the conceptual steps will be valid also in the deformed case. Thus, the expansion of (2.41), with the substitution (4.30) leads to the following *a-gauge* quartic Hamiltonian

$$\begin{aligned}H^{(4)} = \frac{1}{2}\Big(& - P_y^2\bar{P}_y^2 - 2P_y\bar{P}_yP_z\bar{P}_z - P_z^2\bar{P}_z^2 - \bar{P}_y^2Y^2 + \bar{P}_y^2Y'^2 - 4P_y\bar{P}_yY\bar{Y} - 2P_z\bar{P}_zY\bar{Y} - P_y^2\bar{Y}^2 + 3Y^2\bar{Y}^2 \\ & + Y'^2\bar{Y}^2 - 2P_z\bar{P}_zY'\bar{Y}' + P_y^2\bar{Y}'^2 + Y^2\bar{Y}'^2 - Y'^2\bar{Y}'^2 + \bar{P}_z^2Z^2 + 2\bar{P}_y\bar{P}_zY'Z' + 2P_y\bar{P}_z\bar{Y}'Z' \\ & + \bar{P}_z^2Z'^2 + 2P_y\bar{P}_yZ\bar{Z} + 4P_z\bar{P}_zZ\bar{Z} + 2Y\bar{Y}Z\bar{Z} + 2Y'\bar{Y}'Z\bar{Z} + P_z^2\bar{Z}^2 - Z^2\bar{Z}^2 - Z'^2\bar{Z}^2 \\ & + 2P_yP_z\bar{Y}'\bar{Z}' - 2P_y\bar{P}_yZ'\bar{Z}' - 2Y\bar{Y}Z'\bar{Z}' - 2Y'\bar{Y}'Z'\bar{Z}' + P_z^2\bar{Z}'^2 - Z^2\bar{Z}'^2 + 2\bar{P}_yP_zY'\bar{Z}' \\ & - Z'^2\bar{Z}'^2 + 2\Big(- (Y\bar{Y} + Z\bar{Z})^2 + ((P_y - Y')(\bar{P}_y - \bar{Y}') + (P_z - Z')(\bar{P}_z - \bar{Z}')) \\ & ((P_y + Y')(\bar{P}_y + \bar{Y}') + (P_z + Z')(\bar{P}_z + \bar{Z}'))\Big)a\Big). \end{aligned} \tag{4.38}$$

The inclusion of the modes decomposition into the previous expression leaves several integrals in the variables $\sigma,\ \tau,\ p_1,\ p_2,\ p_3,\ p_4$. The integrals in τ and σ imply momentum and energy conservations since their integrands are like

$$\int d\tau d\sigma e^{-i\tau(\omega(p_1)+\omega(p_2)-\omega(p_3)-\omega(p_4))}e^{i\sigma(p_1+p_2-p_3-p_4)} \propto \delta(\omega(p_1)+\omega(p_2)-\omega(p_3)-\omega(p_4))\delta(p_1+p_2-p_3-p_4). \tag{4.39}$$

Clearly, the Dirac delta must be then integrated in the momenta, and this operation yields a Jacobian

$$\Omega(p_1,p_2) = \frac{1}{\omega(p_1)' - \omega(p_2)'} \tag{4.40}$$

which depends on the derivative of the dispersion relation. In particular, in this case we have $\omega(p) = \sqrt{1+p^2}$, which basically gives

$$\Omega(p_1,p_2) = \frac{\omega(p_1)\omega(p_2)}{\omega(p_1)p_2 - \omega(p_2)p_1}. \tag{4.41}$$

Eventually, we are left with a sum of expressions of the form

$$H^{(4)} = \cdots + \int dp_1dp_2\ T_{ij}^{kl}\ a_k^\dagger(p_1)a_l^\dagger(p_2)a^i(p_1)a^j(p_2) + \ldots, \tag{4.42}$$

where the flavors $\{i,\ j,\ k,\ l\}$ ranging in $\{y,\ \bar{y},\ z,\ \bar{z}\}$. As in the flat case, T_{ij}^{kl} is an elements of the tree-level T-matrix and it turns out that it is diagonal *i.e.* takes the form

$$T_{ij}^{kl} = \delta_i^k\delta_j^lT_{ij}. \tag{4.43}$$

Clearly, the explicit structure of the non vanishing elements defines the scattering processes, but apart from that, the crucial property is the diagonal form of the scattering matrix. Indeed the Yang-Baxter equation is automatically satisfied, since one of the elements in which the the 3-body scattering matrix is factorized can be obtain via the following tensor product

$$S_{12} = S(p_1,p_2) \otimes \mathbb{I}_4. \tag{4.44}$$

For this reason the Yang-Baxter equation can be written in terms of the S-matrix as

$$S(p_2,p_3) \otimes \mathbb{I}_4 \cdot \mathbb{I}_4 \otimes S(p_1,p_3) \cdot S(p_1,p_2) \otimes \mathbb{I}_4 = \mathbb{I}_4 \otimes S(p_1,p_2) \cdot S(p_1,p_3) \otimes \mathbb{I}_4 \cdot \mathbb{I}_4 \otimes S(p_2,p_3). \tag{4.45}$$

From the T-matrix, whose non vanishing elements are (4.50), it is then easy to construct the $\mathbb{S}$ matrix as

$$\mathbb{S} = \mathbb{I} + \frac{i}{T}\mathbb{T}, \tag{4.46}$$

which satisfies the (4.45). We conclude this chapter by mentioning also that in the limit $a \to \frac{1}{2}$ we obtain the same results that can be found in [13] for the bosonic sector. Moreover in the literature there are also the matrix elements in the case of generic a as in [10]. In this paper T.Klose et al found that the matrix elements involved in the scatterings are

$$\begin{aligned}
&Y(p_1)Y(p_2) \to Y(p_1)Y(p_2)\\
&T_{YY\to YY} = \frac{1}{2}\Big((1-2a)(\omega(p_2)p_1 - \omega(p_1)p_2) + \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) + \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
&Z(p_1)Z(p_2) \to Z(p_1)Z(p_2)\\
&T_{ZZ\to ZZ} = \frac{1}{2}\Big((1-2a)(\omega(p_2)p_1 - \omega(p_1)p_2) - \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) - \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
&Z(p_1)Y(p_2) \to Z(p_1)Y(p_2)\\
&T_{ZY\to ZY} = \frac{1}{2}\Big((1-2a)(\omega(p_2)p_1 - \omega(p_1)p_2) + \frac{p_1^2-p_2^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big).
\end{aligned} \tag{4.47}$$

It is also crucial to notice that the elements of the scattering matrix (4.50) can be related together. Indeed in the case of $a \to \frac{1}{2}$, we can connected

$$T_{YY} = -T_{\bar{Z}\bar{Z}}, \;\; T_{\bar{Y}\bar{Y}} = -T_{ZZ}, \;\; T_{\bar{Y}Y} = -T_{Z\bar{Z}}. \tag{4.48}$$

This fact basically is related to the analytic continuation (4.21) between AdS_3 and S^3. Moreover, another crucial consideration is related to the dependence of (4.50) on the a parameter. Indeed, the previous dependence is quite general, and in particular one can obtains the S-matrix in the general a-gauge, simply shifting the computation performed at $a = \frac{1}{2}$

$$T_{ij}(p_1,p_2) = T_{ij}(p_1,p_2)|_{a=\frac{1}{2}} - (a - \frac{1}{2})(p_1\omega(p_2) - p_2\omega(p_1)). \tag{4.49}$$

The result of our computations for the matrix elements are presented in the following

$$
\begin{aligned}
T_{ZZ} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) - \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) - \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
T_{YY} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) + \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
T_{ZY} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{p_1^2-p_2^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big),\\
T_{YZ} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{p_2^2-p_1^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big),\\
T_{\bar{Z}\bar{Z}} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) - \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) - \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
T_{\bar{Y}\bar{Y}} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) + \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
T_{\bar{Z}\bar{Y}} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{p_1^2-p_2^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big),\\
T_{\bar{Y}\bar{Z}} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{p_2^2-p_1^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big),\\
T_{Z\bar{Z}} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) - \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) - \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
T_{Y\bar{Y}} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) + \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
T_{Z\bar{Y}} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{p_1^2-p_2^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big),\\
T_{Y\bar{Z}} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{p_2^2-p_1^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big),\\
T_{\bar{Z}Z} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) - \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) - \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
T_{\bar{Y}Y} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{(p_1-p_2)^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big) + \frac{2p_1p_2}{\omega(p_2)p_1 - \omega(p_1)p_2},\\
Y_{\bar{Z}Y} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{p_1^2-p_2^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big),\\
T_{\bar{Y}Z} &= \frac{1}{2}\Big((1-2a)\Big(\omega(p_2)p_1 - \omega(p_1)p_2\Big) + \frac{p_2^2-p_1^2}{\omega(p_2)p_1 - \omega(p_1)p_2}\Big).
\end{aligned}
\tag{4.50}
$$

Chapter 5

Deformation of $AdS_3 \times S^3$

In this chapter we will analyze the integrability structure of the bosonic string non linear σ-model in the $AdS_3 \times S^3$ deformed background. First of all we will review the formulation of the deformed model using the same structure exposed in the previous chapter and then we will move to the perturbative analysis.

5.1 Classical Integrability

All this discussion is based on [20]. Our starting point is the coset realization presented in section 4.1. Also in this case the target spacetime is the product of two maximally symmetric spaces, but with some parameters that take into account the deformations. Thus, we can still focus only on the deformation of the σ-model formulated on the sphere and study its possible deformations. Then we can still perform an analytic continuation from S^3 to AdS_3. The bi-Yang-Baxter deformation of the σ-model formulated on $SU(2)$ is described by the action

$$S = -\frac{1}{2}\int d^2\sigma Tr\Big(J_+ \frac{1}{1-\alpha R_g - \beta R_{\tilde{g}}} J_-\Big) \tag{5.1}$$

where $R_g = Ad_g^{-1} R Ad_g$ and R is a solution of the modified classical Yang-Baxter equation

$$[RM, RN] + R([RM, N] + [M, RN]) = [M, N], \tag{5.2}$$

where $M,\ N$ are elements of the algebra. Moreover, in (5.1), α and β are two real parameters and J is the left-invariant current

$$J = g^{-1}dg, \quad g \in SU(2). \tag{5.3}$$

Also in this case the action can be recast as the action of the symmetric space coset sigma model. Let us consider the group-valued field $f \in SU(2) \times SU(2)$, and a solution R of the modified classical Yang-Baxter equation for the algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. The action for this model reads as

$$S = -\int d^2\sigma Tr\Big(\mathcal{J}_+ \Big(P_2 \frac{1}{1 - I_{\chi_{L,R}} R_f P_2} \mathcal{J}_-\Big)\Big), \tag{5.4}$$

where

$$I_{\chi_{L,R}} = \begin{pmatrix} \chi_L I & 0 \\ 0 & \chi_R I \end{pmatrix}. \tag{5.5}$$

Now we want to connect the deformed action for the symmetric space σ-model with the one of the bi-Yang-Baxter deformation. In order to do this, it is useful to consider

$$R = \begin{pmatrix} R & 0 \\ 0 & \pm R \end{pmatrix}, \ \mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & \tilde{J} \end{pmatrix}, \ J = g^{-1}dg, \quad \tilde{J} = \tilde{g}^{-1}d\tilde{g}. \tag{5.6}$$

Then including this expressions in (5.4), with the further identification $\chi_L = 2\alpha$, $\chi_R = \pm 2\beta$, it is possible to obtain the following action

$$S = -\frac{1}{2}\int d^2\sigma Tr\Big((J_+ - \tilde{J}_+)\frac{1}{1-\alpha R_g - \beta R_{\tilde{g}}}(J_- - \tilde{J}_-)\Big). \tag{5.7}$$

In particular it turn out that the previous action is obtained promoting $SU(2)$ to a gauge group in (5.1) and so using the gauge fixing condition $\tilde{J} = 0$ in (5.7) it is possible to recover (5.1). At this point we can investigate the classical integrability structure using the same procedure used in 4.1. The equation of motion which follows from the variation of the action (5.4) is

$$\begin{aligned} &\partial_+(P_2 O_-^{-1}\mathcal{J}_-) + [\mathcal{J}_+, P_2 O_-^{-1}\mathcal{J}_-] + \partial_-(P_2 O_+^{-1}\mathcal{J}_+) + [\mathcal{J}_-, P_2 O_+^{-1}\mathcal{J}_+] \\ &\quad + I_{\chi_{L,R}}([R_f P_2 O_-^{-1}\mathcal{J}_-, P_2 O_+^{-1}\mathcal{J}_+] + [P_2 O_-^{-1}\mathcal{J}_-, R_f P_2 O_+^{-1}\mathcal{J}_+]) = 0, \end{aligned} \tag{5.8}$$

where

$$O_\pm \equiv 1 \pm I_{\chi_{L,R}} R_f P_2. \tag{5.9}$$

$\mathcal{J}$ satisfies the flatness condition

$$\partial_-\mathcal{J}_+ - \partial_+\mathcal{J}_- + [\mathcal{J}_-, \mathcal{J}_+] = 0. \tag{5.10}$$

The next step should be the projection into the $\mathbb{Z}_2$ decomposition of the algebra using the projectors. For this aim it is useful to define

$$K_\pm = O_\pm^{-1}\mathcal{J}_\pm, \quad K = \begin{pmatrix} K & 0 \\ 0 & \tilde{K} \end{pmatrix}. \tag{5.11}$$

With this, the equation of motion an the flatness condition become

$$\begin{aligned} &\Xi = \partial_+(P_2 K_-) + [K_+, P_2 K_-] + \partial_-(P_2 K_+) + [K_-, P_2 K_+] = 0, \\ &\partial_- K_+ - \partial_+ K_- + [K_+, K_-] + I^2_{\chi_{L,R}}[P_2 K_-, P_2 K_+] + I_{\chi_{L,R}} R_f \Xi = 0. \end{aligned} \tag{5.12}$$

It is also useful to define

$$\tilde{K}_0 = K_0 + \chi_+\chi_- K_2, \quad \tilde{K}_2 = \sqrt{1+\chi_+^2}\sqrt{1+\chi_-^2}K_2, \tag{5.13}$$

where

$$\chi_\pm = \frac{1}{2}(\chi_L \pm \chi_R). \tag{5.14}$$

It is then possible to prove that $\tilde{K}_0$ and $\tilde{K}_2$ satisfy the equation of motion and the flatness condition that, in the case of the undeformed model, were satisfied by J_0 and J_2, namely

$$\begin{aligned} &\partial_+\tilde{K}_{2-} + [\tilde{K}_{0+}, \tilde{K}_{2-}] + \partial_-\tilde{K}_{2+} + [\tilde{K}_{0-}, \tilde{K}_{2+}] = 0, \\ &\partial_-\tilde{K}_{0+} - \partial_+\tilde{K}_{0-} + [\tilde{K}_{0-}, \tilde{K}_{0+}] + [\tilde{K}_{2-}, \tilde{K}_{2+}] = 0, \\ &\partial_-\tilde{K}_{2+} + [\tilde{K}_{0-}, \tilde{K}_{2+}] - \partial_+\tilde{K}_{2-} - [\tilde{K}_{0+}, \tilde{K}_{2-}] = 0. \end{aligned} \tag{5.15}$$

The previous equations follow from the flatness condition of a Lax connection defined as

$$L_\pm = \tilde{K}_{0\pm} + z^{\pm 2}\tilde{K}_{2\pm}, \tag{5.16}$$

or equivalently

$$L_\pm = K_{0\pm} + \chi_+\chi_- K_{2\pm} + z^{\pm 2}\sqrt{1+\chi_+^2}\sqrt{1+\chi_-^2}K_{2\pm}. \tag{5.17}$$

Thus, at the end, we are able to recast the equation of motion as the flatness condition for a Lax connection. This basically is sufficient to ensure the classical integrability.

As in the previous chapter, in order to explicitly perform perturbative computations we need to have the expression of the action. For this aim, it is crucial to have the spacetime metric. If we use

(4.18) for the AdS_3 part and (4.16) for the S^3 one and we insert this expressions in (5.4) then we obtain the non linear σ-model with target space described by the metrics

$$\begin{aligned} ds^2_{S^3} &= \frac{1}{1+\chi_-^2(1-r^2)+\chi_+^2 r^2}\left(\frac{dr^2}{1-r^2}+(1-r^2)\Big(1+\chi_-^2(1-r^2)\Big)d\phi^2\right.\\ &\qquad\left.+r^2(1+\chi_+^2 r^2)d\theta^2+2\chi_-\chi_+ r^2(1-r^2)d\phi d\theta\right),\\ ds^2_{AdS_3} &= \frac{1}{1+\chi_-^2(1+\rho^2)-\chi_-^2\rho^2}\left(\frac{d\rho^2}{1+\rho^2}-(1+\rho^2)\Big(1+\chi_-^2(1+\rho^2)\Big)dt^2\right.\\ &\qquad\left.+\rho^2(1-\chi_+^2\rho^2)d\psi^2+2\chi_-\chi_+\rho^2(1+\rho^2)dtd\psi\right). \end{aligned} \tag{5.18}$$

5.2 One parameter deformation

Here, we will set one the previous deformation parameters to zero in order to recover the so called η-deformation of $AdS_3 \times S^3$. The S-matrix of string in η-deformed $AdS_5 \times S^5$ can be found in [17]. In this section we will treat the lower dimensional case.

The starting point is to write down the metric. In particular we recover the line element used in the previous reference by sending to zero one of the deformation parameters present in (5.18). We can note that $\chi_- \to 0$, reproduces the line element of the η-deformed background

$$\begin{aligned} ds^2 =& \frac{1}{1+\chi^2 r^2}\left(\frac{dr^2}{1-r^2}+(1-r^2)d\phi^2+r^2(1+\chi^2 r^2)d\theta^2\right)\\ &+\frac{1}{1-\chi^2\rho^2}\left(\frac{d\rho^2}{1+\rho^2}-(1+\rho^2)dt^2+\rho^2(1-\chi^2\rho^2)d\psi^2\right), \end{aligned} \tag{5.19}$$

where we also set $\chi_+ = \chi$.

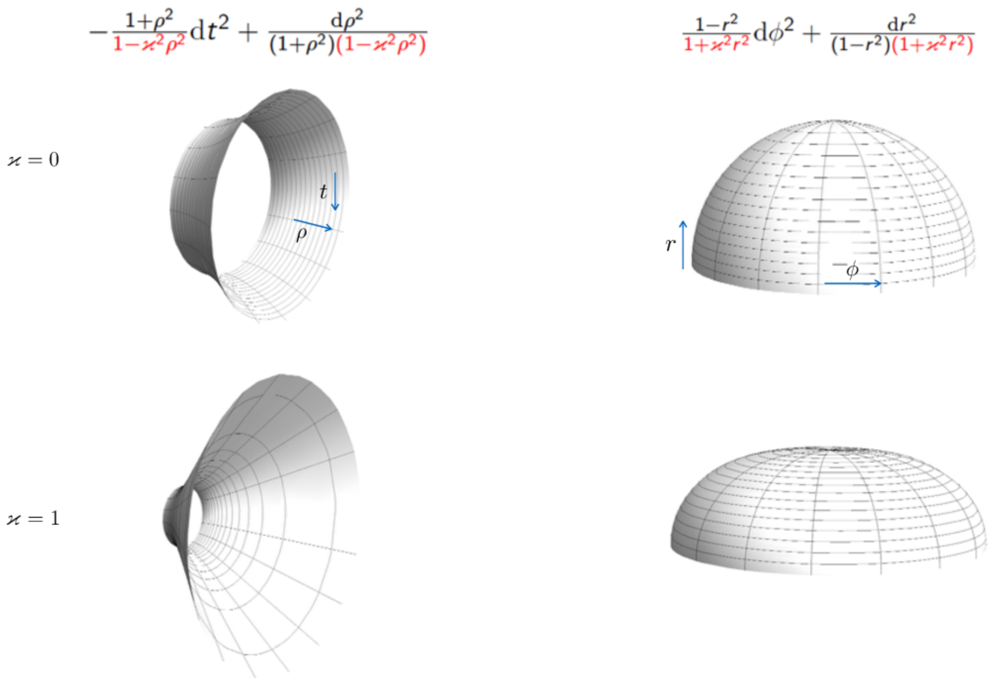

Figure 5.1: [31]. η deformation for the AdS_2 space (left) and for the S^2 space (right). In particular, the first two figures are obtained considering $\chi = 0$ while the last two are obtained with $\chi = 1$

As in the undeformed case, it is convenient to use the following further change of coordinates

$$\rho \to \sqrt{Z_1^2+Z_2^2},\ r \to \sqrt{Y_1^2+Y_2^2},\ \psi \to -\arctan\Big(\frac{Z_2}{Z_1}\Big),\ \theta \to \arctan\Big(\frac{Y_2}{Y_1}\Big). \tag{5.20}$$

This produce the following non vanishing elements of the metric tensor

$$\begin{aligned}
G_{tt} &= \frac{1+Z_1^2+Z_2^2}{-1+(Z_1^2+Z_2^2)\chi^2},\quad G_{z_1z_1} = \frac{-1+Z_2^4\chi^2+Z_2^2(-1+\chi^2+Z_1^2\chi^2)}{(1+Z_1^2+Z_2^2)(-1+Z_1^2\chi^2+Z_2^2\chi^2)},\\
G_{z_1z_2} &= \frac{-Z_1^3Z_2\chi^2+Z_1Z_2(1-(1+Z_2^2)\chi^2)}{(1+Z_1^2+Z_2^2)(-1+Z_1^2\chi^2+Z_2^2\chi^2)},\\
G_{z_2z_2} &= \frac{-1+Z_1^4\chi^2+Z_1^2(-1+\chi^2+Z_2^2\chi^2)}{(1+Z_1^2+Z_2^2)(-1+Z_1^2\chi^2+Z_2^2\chi^2)},\\
G_{\phi\phi} &= -\frac{-1+Y_1^2+Y_2^2}{1+(Y_1^2+Y_2^2)\chi^2},\quad G_{y_1y_1} = \frac{-1+Y_2^4\chi^2+Y_2^2(1-\chi^2+Y_1^2\chi^2)}{(-1+Y_1^2+Y_2^2)(1+Y_1^2\chi^2+Y_2^2\chi^2)},\\
G_{y_1y_2} &= -\frac{Y_1^3Y_2\chi^2+Y_1Y_2(1+(-1+Y_2^2)\chi^2)}{(-1+Y_1^2+Y_2^2)(1+Y_1^2\chi^2+Y_2^2\chi^2)},\\
G_{y_2y_2} &= \frac{-1+Y_1^4\chi^2+Y_1^2(1-\chi^2+Y_2^2\chi^2)}{(-1+Y_1^2+Y_2^2)(1+Y_1^2\chi^2+Y_2^2\chi^2)}.
\end{aligned} \tag{5.21}$$

In the previous expressions we have identified as transverse fields $Y_1,\ Y_2$ in the case of S^3 and $Z_1,\ Z_2$ in the case of AdS_3.

5.2.1 Quadratic Hamiltonian

The first order formalism allows us to extract the perturbative Hamiltonian, but as a preliminary step it is necessary to deduce the conjugate momenta. We start our analysis using the usual string action in a generic spacetime with the B-field set to zero

$$S = -\frac{T}{2}\int d\sigma d\tau\gamma^{\alpha\beta}\Big(\partial_\alpha X^M\partial_\beta X^N G_{MN}(X)\Big). \tag{5.22}$$

Let us first expand the metric elements (5.21) up to quartic order in fields, and insert the resulting expressions in the previous action, taking into account only the quadratic terms in the fields and in their derivatives. This procedure leads to the following quadratic Lagrangian

$$\begin{aligned}
L^{(2)} = -\frac{T}{2}\gamma^{\alpha\beta}\Big(&-\Big(1+Z^2(1+\chi^2)\Big)\partial_\alpha t\partial_\beta t+\partial_\alpha Z_i\partial_\beta Z_i+\Big(1-Y^2(1+\chi^2)\Big)\partial_\alpha\phi\partial_\beta\phi\\
&+\partial_\alpha Y_i\partial_\beta Y_i\Big),
\end{aligned} \tag{5.23}$$

where, as in the undeformed case, we used $\partial_\alpha Y_i\partial_\beta Y_i = \partial_\alpha Y_1\partial_\beta Y_1+\partial_\alpha Y_2\partial_\beta Y_2$ and also $Y^2 = Y_1^2+Y_2^2$. From this we can extract the conjugate momenta to the canonical variables

$$P_{y_i} \equiv \frac{\partial L^{(2)}}{\partial\partial_\tau Y_i} = \dot{Y}_i,\qquad P_{z_i} \equiv \frac{\partial L^{(2)}}{\partial\partial_\tau Z_i} = \dot{Z}_i. \tag{5.24}$$

Thus, we can use the first order formalism introduced in chapter 2, and extract for instance the quadratic Hamiltonian in the general light-cone gauge

$$H^{(2)} = \frac{1}{2}\Big(P_{y_1}^2+P_{y_2}^2+P_{z_1}^2+P_{z_2}^2+(Y_1^2+Y_2^2)(1+\chi^2)+(Z_1^1+Z_2^2)(1+\chi^2)+Y_1'^2+Y_2'^2+Z_1'^2+Z_2'^2\Big). \tag{5.25}$$

This is the free Hamiltonian of 2 massive bosons described by the fields $Y_1,\ Y_2$ coming from the sphere, and 2 massive bosons described by the fields $Z_1,\ Z_2$ coming from the anti-de Sitter space. Before moving to a further change of coordinates let us consider the equations of motion that follows from the (5.23), in particular let us set the worldsheet metric to be diagonal with $\gamma_{\tau\tau} = -1$ and

$\gamma_{\sigma\sigma} = 1$ and also $t = \phi = \tau$ which is satisfied in the uniform light-cone gauge (2.44). With this the Euler-Lagrange equations , that follow from (5.23), are

$$\begin{aligned} \partial_\alpha \frac{\partial L^{(2)}}{\partial \partial_\alpha Y^i} - \frac{\partial L^{(2)}}{\partial Y^i} &= \partial_\alpha \partial^\alpha Y_i + (1+\chi^2) Y_i = 0, \\ \partial_\alpha \frac{\partial L^{(2)}}{\partial \partial_\alpha Z^i} - \frac{\partial L^{(2)}}{\partial Z^i} &= \partial_\alpha \partial^\alpha Z_i + (1+\chi^2) Z_i = 0. \end{aligned} \tag{5.26}$$

If we consider the plane wave solution as $Y_i \approx e^{-i\omega(p)t+ip\sigma}$ and $Z_i \approx e^{-i\omega(p)t+ip\sigma}$, then we find that the dispersion relation is

$$\omega(p) = \sqrt{p^2 + (1+\chi^2)}. \tag{5.27}$$

So, as we can deduce from the Hamiltonian, the mass of the bosons is $m^2 = (1+\chi^2)$.

Let us consider the same change of coordinates that we have introduced in the previous chapter in (4.30)

$$\begin{aligned} &Z = \frac{-Z_2 + i\ Z_1}{\sqrt{2}},\ \bar{Z} = \frac{-Z_2 - i\ Z_1}{\sqrt{2}},\ \bar{Y} = \frac{-Y_1 - i\ Y_2}{\sqrt{2}},\ Y = \frac{-Y_1 + i\ Y_2}{\sqrt{2}}, \\ &P_z = \frac{-P_{z_2} + i\ P_{z_1}}{\sqrt{2}},\ \bar{P}_z = \frac{-P_{z_2} - i\ P_{z_1}}{\sqrt{2}},\ \bar{P}_y = \frac{-P_{y_1} - i\ P_{y_2}}{\sqrt{2}},\ P_y = \frac{-P_{y_1} + i\ P_{y_2}}{\sqrt{2}}, \\ &Z' = \frac{-Z'_2 + i\ Z'_1}{\sqrt{2}},\ \bar{Z}' = \frac{-Z'_2 - i\ Z'_1}{\sqrt{2}},\ \bar{Y}' = \frac{-Y'_1 - i\ Y'_2}{\sqrt{2}},\ Y' = \frac{-Y'_1 + i\ Y'_2}{\sqrt{2}}, \end{aligned} \tag{5.28}$$

This complex coordinates have the advantage to lead a more convenient form for the future quartic Hamiltonian. So, $H^{(2)}$ becomes

$$H^{(2)} = P_y \bar{P}_y + P_z \bar{P}_z + Y\bar{Y}(1+\chi^2) + Z\bar{Z}(1+\chi^2) + Y'\bar{Y}' + Z'\bar{Z}'. \tag{5.29}$$

The diagonalizaton of (5.29) as well as the solution of the equations of motion (once the complex fields are introduced) are archived via the same ansatz on the oscillators expansion used in the undeformed case 4.32. Also in this case, from the equations of motion for the complex fields, we deduce that $\omega(p) = \bar{\omega}(p)$.

Since the momenta is simply the time derivative of the corresponding conjugate variables, we have also that the momenta and the *sigma* derivative ($\partial_\sigma Y = Y'$) of the fields are the same with respect to the undeformed case. Moreover the commutation relations between the fields are transferred to the creation and annihilation operators, once 4.32 are included. In particular the following canonical commutation relations are obtained requiring the equality between the normalization coefficients $\bar{g}(p) = g(p)$

$$\begin{aligned} &[a_{\bar{z}}(p), a^\dagger_{\bar{z}}(p')] = \delta(p-p'), \quad [a_{\bar{y}}(p), a^\dagger_{\bar{y}}(p')] = \delta(p-p'), \\ &[a_z(p), a^\dagger_z(p')] = \delta(p-p'), \quad [a_y(p), a^\dagger_y(p')] = \delta(p-p'). \end{aligned} \tag{5.30}$$

Once the oscillators expansion is included in the (5.29) we obtain

$$H^{(2)} = \int dp\ \omega(p) \Big(a^\dagger_{\bar{z}}(p) a_{\bar{z}}(p) + a^\dagger_{\bar{y}}(p) a_{\bar{y}}(p) + a^\dagger_z(p) a_z(p) + a^\dagger_y(p) a_y(p) \Big), \tag{5.31}$$

which is the correct form of the free Hamiltonian of two complex bosons.

5.2.2 Quartic Hamiltonian

Now, it is possible to extract the quartic Hamiltonian from the general (2.41) which in the η-deformed case and in the general a-gauge is

$$\begin{aligned} H^{(4)} = \frac{1}{8}\Big(& -16P_y\bar{P}_y Y\bar{Y} + 16P_z\bar{P}_z Z\bar{Z} + m^2\Big(8P_y\bar{P}_y Z\bar{Z} - 8P_z\bar{P}_z Y\bar{Y} + 8Y'\bar{Y}'Z\bar{Z} - 8Y\bar{Y}Z'\bar{Z}' \\ & + 8m^2 Y\bar{Y}Z\bar{Z}(1-2a) - 4Y^2\bar{Y}^2(-3+2m^2a+\chi^2) - 4Z^2\bar{Z}^2(1+2a+(-3+2a)\chi^2)\Big) \\ & + (1-2a)\Big(4\bar{P}_z^2 Z'^2 + 4\bar{P}_y^2 Y'^2 + 4P_y^2\bar{Y}'^2 + 8\bar{P}_y\bar{P}_z Y'Z' + 8P_y\bar{P}_z\bar{Y}'Z' + 8\bar{P}_y P_z Y'\bar{Z}' \\ & + 8P_yP_z\bar{Y}'\bar{Z}' + 4P_z^2\bar{Z}'^2 - 4P_y^2\bar{P}_y^2 - 8P_y\bar{P}_yP_z\bar{P}_z - 4P_z^2\bar{P}_z^2 - 8P_z\bar{P}_zY'\bar{Y}' \\ & - 8Y'\bar{Y}'Z'\bar{Z}' - 4Z'^2\bar{Z}'^2 - 4Y'^2\bar{Y}'^2 - 8P_y\bar{P}_yZ'\bar{Z}'\Big) - 16\chi^2\Big(YY'\bar{Y}\bar{Y}' - ZZ'\bar{Z}\bar{Z}'\Big) \\ & + 4(\chi^2-1)\Big(\bar{P}_y^2Y^2 + P_y^2\bar{Y}^2 - Y'^2\bar{Y}^2 - Y^2\bar{Y}'^2 - \bar{P}_z^2Z^2 - P_z^2\bar{Z}^2 + Z'^2\bar{Z}^2 + Z^2\bar{Z}'^2\Big)\Big). \end{aligned} \tag{5.32}$$

Hence, in order to find the S-matrix we must include the oscillators expansions in the previous formula. As we found for the undeformed model, this process yields several integrals in the variables $\sigma,\ \tau,\ p_1,\ p_2,\ p_3,\ p_4$.

The integrations on τ and σ give a product of Dirac delta, that still gives the same Jacobian obtained in the undeformed case

$$\Omega(p_1, p_2) = \frac{\omega(p_1)\omega(p_2)}{p_1\omega(p_2) - p_2\omega(p_1)}, \tag{5.33}$$

with a different dispersion relation. Eventually, we obtained the following expression

$$H^{(4)} = \cdots + \int dp_1 dp_2 T_{ij}^{kl}(p_1,p_2) a_k^\dagger(p_1) a_l^\dagger(p_2) a^i(p_1) a^j(p_2) + \ldots, \tag{5.34}$$

from which we deduced the scattering matrix elements. In particular, in the case of $a = \frac{1}{2}$

$$\begin{aligned} T_{ZZ} &= -\frac{1}{2}\frac{(p_1-p_2)^2 + \chi^2(\omega(p_1)-\omega(p_2))^2 + 4p_1p_2 + 4\chi^2\omega(p_1)\omega(p_2)}{\omega(p_2)p_1 - \omega(p_1)p_2}, \\ T_{\bar{Z}\bar{Z}} &= -\frac{1}{2}\frac{(p_1-p_2)^2 + \chi^2(\omega(p_1)-\omega(p_2))^2 + 4p_1p_2 + 4\chi^2\omega(p_1)\omega(p_2)}{\omega(p_2)p_1 - \omega(p_1)p_2}, \\ T_{\bar{Z}Z} &= -\frac{1}{2}\frac{(p_1-p_2)^2 + \chi^2(\omega(p_1)-\omega(p_2))^2}{\omega(p_2)p_1 - \omega(p_1)p_2}, \\ T_{Z\bar{Y}} &= \frac{1}{2}\frac{m^2(p_2^2-p_1^2)}{\omega(p_2)p_1 - \omega(p_1)p_2}, \\ T_{ZY} &= \frac{1}{2}\frac{m^2(p_2^2-p_1^2)}{\omega(p_2)p_1 - \omega(p_1)p_2}, \\ T_{\bar{Z}Y} &= \frac{1}{2}\frac{m^2(p_2^2-p_1^2)}{\omega(p_2)p_1 - \omega(p_1)p_2}, \\ T_{\bar{Z}\bar{Y}} &= \frac{1}{2}\frac{m^2(p_2^2-p_1^2)}{\omega(p_2)p_1 - \omega(p_1)p_2}. \end{aligned} \tag{5.35}$$

In the previous chapter we pointed out that it is sufficient to compute the the S-matrix elements with $a = \frac{1}{2}$ and then obtain the general ones simply by the shift (4.49), moreover all the other matrix elements can be obtained by the analytic continuation (4.21). However, also in this case the S-matrix is diagonal, and hence it respects the Yang-Baxter equation 4.45.

5.3 Two parameter deformation

Let us move to the most general case of $AdS_3 \times S^3$ with two deformation parameters. The line element is the sum of the one that comes from the sphere and the one that comes from the anti-de Sitter space

(5.18). Using the same strategy used before, the first operation that we performed is the change of coordinates (5.20). This leads to a quite complicated expression for the metric tensor

$$
\begin{aligned}
G_{tt} &= \frac{(1+Z_1^2+Z_2^2)(1+(1+Z_1^2+Z_2^2)\chi_-^2)}{-1-\chi_-^2+(Z_1^2+Z_2^2)(-\chi_-^2+\chi_+^2)} \\
G_{Z_1Z_1} &= \frac{-1+Z_2^4\chi_+^2+Z_2^2(-1+Z_1^2\chi_+^2+\chi_+^2)}{(1+Z_1^2+Z_2^2)(-1-\chi_-^2+Z_1^2(-\chi_-^2+\chi_+^2)+Z_2^2(-\chi_-^2+\chi_+^2))} \\
G_{Z_2Z_2} &= \frac{-1+Z_1^4\chi_+^2+Z_1^2(-1+Z_1^2\chi_+^2+\chi_+^2)}{(1+Z_1^2+Z_2^2)(-1-\chi_-^2+Z_1^2(-\chi_-^2+\chi_+^2)+Z_2^2(-\chi_-^2+\chi_+^2))} \\
G_{tZ_1} &= -\frac{Z_2(1+Z_1^2+Z_2^2)\chi_+\chi_-}{-1-\chi_-^2+(Z_1^2+Z_2^2)(-\chi_-^2+\chi_+^2)} \\
G_{tZ_2} &= \frac{Z_1(1+Z_1^2+Z_2^2)\chi_+\chi_-}{-1-\chi_-^2+(Z_1^2+Z_2^2)(-\chi_-^2+\chi_+^2)} \\
G_{Z_1Z_2} &= \frac{-Z_1^3Z_2\chi_+^2+Z_1Z_2(1-\chi_+^2(1+Z_2^2))}{(1+Z_1^2+Z_2^2)(-1-\chi_-^2+Z_1^2(-\chi_-^2+\chi_+^2)+Z_2^2(-\chi_-^2+\chi_+^2))} \\
G_{\phi\phi} &= \frac{(-1+Y_1^2+Y_2^2)(-1+(-1+Y_1^2+Y_2^2)\chi_-^2)}{1+\chi_-^2-(Y_1^2+Y_2^2)(\chi_-^2-\chi_+^2)} \\
G_{Y_1Y_1} &= \frac{-1+Y_2^4\chi_+^2+Y_2^2(1+Y_1^2\chi_+^2-\chi_+^2)}{(-1+Y_1^2+Y_2^2)(1+\chi_-^2-Y_1^2(+\chi_-^2-\chi_+^2)-Y_2^2(-\chi_-^2+\chi_+^2))} \\
G_{Y_2Y_2} &= \frac{-1+Y_1^4\chi_+^2+Y_1^2(1+Y_2^2\chi_+^2-\chi_+^2)}{(-1+Y_1^2+Y_2^2)(1+\chi_-^2-Y_1^2(+\chi_-^2-\chi_+^2)-Y_2^2(-\chi_-^2+\chi_+^2))} \\
G_{\phi Y_1} &= \frac{Y_2(-1+Y_1^2+Y_2^2)\chi_+\chi_-}{1+\chi_-^2-(Y_1^2+Y_2^2)(+\chi_-^2-\chi_+^2)} \\
G_{\phi Y_2} &= -\frac{Y_1(-1+Y_1^2+Y_2^2)\chi_+\chi_-}{1+\chi_-^2-(Y_1^2+Y_2^2)(+\chi_-^2-\chi_+^2)} \\
G_{Y_1Y_2} &= -\frac{Y_1^3Y_2\chi_+^2+Y_1Y_2(1+\chi_+^2(-1+Y_2^2))}{(-1+Y_1^2+Y_2^2)(1+\chi_-^2-Y_1^2(\chi_-^2-\chi_+^2)-Y_2^2(\chi_-^2-\chi_+^2))}
\end{aligned}
\tag{5.36}
$$

We also implicitly used the fact that the metric tensor is symmetric, and hence the terms $G_{Y_1\phi} = G_{\phi Y_1}$, and so on. Moreover as in the previous cases, we have identified $Y_1,\ Y_2$ as the transverse fields coming from sphere, while $Z_1,\ Z_2$ the transverse fields coming from the anti-de Sitter space.

5.3.1 Quadratic Hamiltonian

With the previous metric, it is then possible to write down the gauge-fixed action and so we introduce the light-cone coordinates $x^\pm$ and their canonical momenta $p_\pm$ by setting

$$t = x^+ - ax^-, \quad \phi = x^+ + (1-a)x^-, \quad p_t = (1-a)p_+ - p_-, \quad p_\phi = ap_+ + p_-, \tag{5.37}$$

where a is the same real parameter introduced in the previous cases. Then the uniform light-cone gauge is fixed by

$$x^+ = \tau, \quad p_- = 1. \tag{5.38}$$

Thus following the procedure explained in chapter 2 we arrive at the light-cone Hamiltonian

$$H(X^\mu, X'^\mu, P_\mu) = -\int d\sigma\ p_+((X^\mu, X'^\mu, P_\mu). \tag{5.39}$$

We stress again that at this level the Hamiltonian is still highly non linear, hence is not useful in order to directly quantize the model. As we said before, once the fields are rescaled as in (2.46), then it is possible to perform the large-tension expansion $T \to \infty$ and extract the perturbative Hamiltonian. In particular, the explicit computation gives the following expression for $H^{(2)}$

$$
\begin{aligned}
H^{(2)} =& \frac{1}{2(1+\chi_-^2)}\Big(P_{y_1}^2(1+\chi_-^2)^2 + P_{y_2}^2(1+\chi_-^2)^2 + P_{z_1}^2(1+\chi_-^2)^2 + P_{z_2}^2(1+\chi_-^2)^2 \\
&+ (1+\chi_-^2)(1+\chi_+^2)\Big(Y_1^2 + Z_1^2 + Z_2^2 + Y_2^2\Big) + Z_1'^2 + Z_2'^2 + Y_1'^2 + Y_2'^2 \\
&+ \chi_-(1+\chi_-^2)\chi_+\Big(2P_{z_2}Z_1 - 2P_{z_1}Z_2 - 2P_{y_2}Y_1 + 2P_{y_1}Y_2\Big)\Big).
\end{aligned}
\tag{5.40}
$$

It is immediate to notice that the quadratic Hamiltonian is not canonically normalized, hence it is convenient to perform the following rescaling

$$
\begin{aligned}
&P_{y_i} \to \frac{P_{y_i}}{\sqrt{1+\chi_-^2}}, \quad P_{z_i} \to \frac{P_{z_i}}{\sqrt{1+\chi_-^2}}, \\
&Y_i \to \sqrt{1+\chi_-^2}\,Y_i, \quad Z_i \to \sqrt{1+\chi_-^2}\,Z_i, \\
&Y_i' \to \sqrt{1+\chi_-^2}\,Y_i', \quad Z_i' \to \sqrt{1+\chi_-^2}\,Z_i',
\end{aligned}
\tag{5.41}
$$

where $i = 1, 2$. Moreover, the expression is further simplified by introducing the same complex fields introduced in the η-deformed model and also in the undeformed model (4.30). Eventually, the quadratic Hamiltonian reads as

$$
H^{(2)} = P_y\bar{P}_y + P_z\bar{P}_z + Y'\bar{Y}' + Z'\bar{Z}' + m^2(Y\bar{Y} + Z\bar{Z}) + i\chi_+\chi_-(\bar{P}_yY + \bar{P}_zZ - P_y\bar{Y} - P_z\bar{Z}), \tag{5.42}
$$

with

$$
m^2 = (1+\chi_+^2)(1+\chi_-^2). \tag{5.43}
$$

This is the free Hamiltonian of two complex massive bosons, with quite surprising behaviours, indeed the dispersion relations for particles and antiparticles are different.

In order to see this, let us write down the quadratic Lagrangian and then let us find and solve the equation of motion. Moreover this has another further advantage indeed we can find the conjugate momenta that will be useful when we will introduce the oscillators expansion. Clearly the momenta could be obtained also from the Hamiltonian, using the Hamilton equations.

So we included the expansion of the metric (5.36) in the general action of non linear σ-model with the B-fields set to zero

$$
S = \int d\sigma d\tau \gamma^{\alpha\beta}(\partial_\alpha X^M \partial_\beta X^N G_{MN}(X)). \tag{5.44}
$$

Moreover, in the first order formalism the world-sheet metric is imposed to be diagonal, with $\gamma^{\tau\tau} = -1$ and $\gamma^{\sigma\sigma} = 1$, by the Virasoro constraints, so let us consider this metric in the previous action. Let us also set $\phi = t = \tau$, which basically is realized in the uniform light-cone gauge (2.44). With this considerations, and after a further expansion of the metric in order to take into account the quadratic terms in the fields and in their derivatives, then the quadratic Lagrangian reads as

$$
\begin{aligned}
L^{(2)} =& \frac{1}{2(1+\chi_-^2)}\Big((\partial_\sigma Y_1)^2 + (\partial_\sigma Y_2)^2 + (\partial_\sigma Z_1)^2 + (\partial_\sigma Z_2)^2 - (\partial_\tau Y_1)^2 - (\partial_\tau Y_2)^2 - (\partial_\tau Z_1)^2 - (\partial_\tau Z_2)^2 \\
&+ (1+\chi_-^2+\chi_+^2)(Y_1^2 + Y_2^2 + Z_1^2 + Z_2^2) + 2\chi_+\chi_-(\partial_\tau Y_2 Y_1 - \partial_\tau Y_1 Y_2 - \partial_\tau Z_2 Z_1 + \partial_\tau Z_1 Z_2)\Big).
\end{aligned}
\tag{5.45}
$$

As in the Hamiltonian case, we introduce a similar expression to (4.30), but we must take into account that the momenta in the Lagrangian formalism is not present, and so we must use the time derivative of the complex fields. However, dropping the overall constant, we find

$$
\begin{aligned}
L^{(2)} =& \partial_\sigma Y \partial_\sigma \bar{Y} + \partial_\sigma Z \partial_\sigma \bar{Z} - \partial_\tau Y \partial_\tau \bar{Y} - \partial_\tau Z \partial_\tau \bar{Z} + (1+\chi_+^2+\chi_-^2)(Y\bar{Y} + Z\bar{Z}) \\
&+ i\chi_+\chi_-(\partial_\tau \bar{Y} Y - \partial_\tau Y \bar{Y} + \partial_\tau \bar{Z} Z - \partial_\tau Z \bar{Z}).
\end{aligned}
\tag{5.46}
$$

At this point we find, for instance, the equations of motion for the Y and $\bar{Y}$ fields as

$$\begin{aligned} \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha Y} - \frac{\partial \mathcal{L}}{\partial Y} &= \partial_\sigma^2 \bar{Y} - \partial_\tau^2 \bar{Y} - 2i\chi_+\chi_- \partial_\tau \bar{Y} - (1+\chi_+^2+\chi_-^2)\bar{Y} = 0, \\ \partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha \bar{Y}} - \frac{\partial \mathcal{L}}{\partial \bar{Y}} &= \partial_\sigma^2 Y - \partial_\tau^2 Y + 2i\chi_+\chi_- \partial_\tau Y - (1+\chi_+^2+\chi_-^2)Y = 0. \end{aligned} \tag{5.47}$$

Hence, considering solutions like

$$Y \propto e^{-i\omega(p)\tau + ip\sigma}, \quad \bar{Y} \propto e^{-i\bar{\omega}(p)\tau + ip\sigma}, \tag{5.48}$$

we find that

$$\begin{aligned} \bar{\omega}(p)^2 - 2\chi_+\chi_-\bar{\omega}(p) - p^2 - (1+\chi_-^2+\chi_+^2) = 0, \\ \omega(p)^2 + 2\chi_+\chi_-\omega(p) - p^2 - (1+\chi_-^2+\chi_+^2) = 0. \end{aligned} \tag{5.49}$$

Finally, the solution of the previous quadratic equations are

$$\omega(p) = -\chi_+\chi_- + \sqrt{p^2 + (1+\chi_+^2)(1+\chi_-^2)}, \quad \bar{\omega}(p) = \chi_+\chi_- + \sqrt{p^2 + (1+\chi_+^2)(1+\chi_-^2)}, \tag{5.50}$$

which confirms the previous considerations. The same results are valid also for the Z and $\bar{Z}$ fields. Moreover the conjugate momenta are defined as

$$\begin{aligned} \bar{P}_y \equiv \frac{\delta \mathcal{L}}{\delta \partial_\tau Y}, \quad P_y \equiv \frac{\delta \mathcal{L}}{\delta \partial_\tau \bar{Y}} \\ \bar{P}_z \equiv \frac{\delta \mathcal{L}}{\delta \partial_\tau Z}, \quad P_z \equiv \frac{\delta \mathcal{L}}{\delta \partial_\tau \bar{Z}}. \end{aligned} \tag{5.51}$$

Hence, in this case we find that

$$\begin{aligned} \bar{P}_y = \partial_\tau \bar{Y} + i\bar{Y}\chi_+\chi_-, \quad P_y = \partial_\tau Y - iY\chi_+\chi_-, \\ \bar{P}_z = \partial_\tau \bar{Z} + i\bar{Z}\chi_+\chi_-, \quad P_z = \partial_\tau Z - iZ\chi_+\chi_-. \end{aligned} \tag{5.52}$$

In order to diagonalize the quadratic Hamiltonian we introduce the oscillators expansion of the fields, which in this case are more insidious with respect to the undeformed and to the η-deformed models.

For instance we must take into account that there are two different dispersion relations, but apart from this, the equation of motion can be solved by the same ansatz used in the previous cases that we report in the following

$$\begin{aligned} Z(\sigma,\tau) &= \int dp \left(\frac{e^{-i\omega(p)\tau + ip\sigma}}{\sqrt{2g(p)}} a_z(p) + \frac{e^{i\bar{\omega}(p) - ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{z}}^\dagger(p) \right), \\ \bar{Z}(\sigma,\tau) &= \int dp \left(\frac{e^{-i\bar{\omega}(p)\tau + ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{z}}(p) + \frac{e^{i\omega(p)\tau - ip\sigma}}{\sqrt{2g(p)}} a_z^\dagger(p) \right), \\ Y(\sigma,\tau) &= \int dp \left(\frac{e^{-i\omega(p)\tau + ip\sigma}}{\sqrt{2g(p)}} a_y(p) + \frac{e^{i\bar{\omega}(p) - ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{y}}^\dagger(p) \right), \\ \bar{Y}(\sigma,\tau) &= \int dp \left(\frac{e^{-i\bar{\omega}(p)\tau + ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{y}}(p) + \frac{e^{i\omega(p)\tau - ip\sigma}}{\sqrt{2g(p)}} a_y^\dagger(p) \right), \end{aligned} \tag{5.53}$$

where $\omega(p)$ and $\bar{\omega}(p)$ are the dispersion relation given in (5.50), moreover we have still to find the correct normalization namely we have to find $g(p)$ and $\bar{g}(p)$. Since the normalizations are obtained by imposing the canonical commutation relations between the fields, in this case, it is useful to present also the σ-derivative of the fields and the conjugate momenta. In particular

$$\begin{aligned} Z' &= \int dp \left(ip \frac{e^{-i\omega(p)\tau + ip\sigma}}{\sqrt{2g(p)}} a_z(p) - ip \frac{e^{i\bar{\omega}(p) - ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{z}}^\dagger(p) \right), \\ \bar{Z}' &= \int dp \left(ip \frac{e^{-i\bar{\omega}(p)\tau + ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{z}}(p) - ip \frac{e^{i\omega(p)\tau - ip\sigma}}{\sqrt{2g(p)}} a_z^\dagger(p) \right), \end{aligned} \tag{5.54}$$

and the same consideration is valid also for Y' and $\bar{Y}'$

$$\begin{aligned}
Y' &= \int dp \left(ip \frac{e^{-i\omega(p)\tau + ip\sigma}}{\sqrt{2g(p)}} a_y(p) - ip \frac{e^{i\bar{\omega}(p) - ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{y}}^{\dagger}(p) \right), \\
\bar{Y}' &= \int dp \left(ip \frac{e^{-i\bar{\omega}(p)\tau + ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{y}}(p) - ip \frac{e^{i\omega(p)\tau - ip\sigma}}{\sqrt{2g(p)}} a_y^{\dagger}(p) \right).
\end{aligned} \tag{5.55}$$

For the conjugate momenta we have to consider (5.52), in particular

$$\begin{aligned}
P_y &= \partial_\tau Y - iY\chi_+\chi_- \\
&= \int dp \left(-i(\omega(p) + \chi_+\chi_-) \frac{e^{-i\omega(p)\tau + ip\sigma}}{\sqrt{2g(p)}} a_y(p) + i(\bar{\omega}(p) - \chi_+\chi_-) \frac{e^{i\bar{\omega}(p) - ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{y}}^{\dagger}(p) \right), \\
\bar{P}_y &= \partial_\tau \bar{Y} + i\bar{Y}\chi_+\chi_- \\
&= \int dp \left(-i(\bar{\omega}(p) - \chi_-\chi_+) \frac{e^{-i\bar{\omega}(p)\tau + ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{y}}(p) + i(\omega(p) + \chi_-\chi_+) \frac{e^{i\omega(p)\tau - ip\sigma}}{\sqrt{2g(p)}} a_y^{\dagger}(p) \right), \\
P_z &= \partial_\tau Z - iZ\chi_+\chi_- \\
&= \int dp \left(-i(\omega(p) + \chi_+\chi_-) \frac{e^{-i\omega(p)\tau + ip\sigma}}{\sqrt{2g(p)}} a_z(p) + i(\bar{\omega}(p) - \chi_+\chi_-) \frac{e^{i\bar{\omega}(p) - ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{z}}^{\dagger}(p) \right), \\
\bar{P}_z &= \partial_\tau \bar{Z} + i\bar{Z}\chi_+\chi_- \\
&= \int dp \left(-i(\bar{\omega}(p) - \chi_-\chi_+) \frac{e^{-i\bar{\omega}(p)\tau + ip\sigma}}{\sqrt{2\bar{g}(p)}} a_{\bar{z}}(p) + i(\omega(p) + \chi_-\chi_+) \frac{e^{i\omega(p)\tau - ip\sigma}}{\sqrt{2g(p)}} a_z^{\dagger}(p) \right).
\end{aligned} \tag{5.56}$$

At the classical level the sympletic structures underlying the Hamiltonian formalism, induces the Poisson parenthesis between the variables and their conjugate momenta. Let us focus on the Y and $\bar{Y}$ fields. We have that

$$\begin{aligned}
[Y(\sigma,\tau), \bar{P}_y(\tilde{\sigma},\tau)] = \int dp_1 dp_2 \Bigg(& -i(\bar{\omega}(p_2) - \chi_+\chi_-) \frac{e^{-i\tau(\bar{\omega}(p_2) - \bar{\omega}(p_1)) + i\tilde{\sigma}p_2 - i\sigma p_1}}{\sqrt{2\bar{g}(p_1)}\,\sqrt{2\bar{g}(p_2)}} [a_{\bar{y}}^{\dagger}(p_1,\tau), a_{\bar{y}}(p_2,\tau)] \\
& + i(\omega(p_2) + \chi_+\chi_-) \frac{e^{+i\tau(\omega(p_2) - \omega(p_1)) - i\tilde{\sigma}p_2 + i\sigma p_1}}{\sqrt{2g(p_1)}\,\sqrt{2g(p_2)}} [a_y(p_1,\tau), a_y^{\dagger}(p_2,\tau)] \Bigg),
\end{aligned} \tag{5.57}$$

where we used the second line in the following canonical commutation relations between creation and annihilation operators

$$\begin{aligned}
&[a_{\bar{y}}(p_1,\tau), a_{\bar{y}}^{\dagger}(p_2,\tau)] = \delta(p_1 - p_2), \quad [a_y(p_1,\tau), a_y^{\dagger}(p_2,\tau)] = \delta(p_1 - p_2) \\
&[a_{\bar{y}}(p_1,\tau), a_y^{\dagger}(p_2,\tau)] = 0, \quad [a_y(p_1,\tau), a_{\bar{y}}^{\dagger}(p_2,\tau)] = 0.
\end{aligned} \tag{5.58}$$

Hence, using the first line of the previous, and integrating along p_2, we can obtain

$$\begin{aligned}
[Y(\sigma,\tau), \bar{P}_y(\tilde{\sigma},\tau)] &= \int dp_1 \left(i(\bar{\omega}(p_1) - \chi_+\chi_-) \frac{e^{ip_1(-\tilde{\sigma}+\sigma)}}{2\bar{g}(p_1)} + i(\omega(p_1) + \chi_+\chi_-) \frac{e^{ip_1(\sigma-\tilde{\sigma})}}{2g(p_1)} \right) \\
&= \int dp_1 i \left(\frac{\sqrt{p_1^2 + (1+\chi_+^2)(1+\chi_-^2)}}{2\bar{g}(p_1)} + \frac{\sqrt{p_1^2 + (1+\chi_+^2)(1+\chi_-^2)}}{2g(p_1)} \right) e^{ip_1(\sigma-\tilde{\sigma})}.
\end{aligned} \tag{5.59}$$

In the second line we used the explicit expression for the dispersion relations. Moreover it is immediate to notice that we obtain the canonical commutation relation between the fields and their canonical momenta if and only if

$$\bar{g}(p_1) = \bar{\omega}(p_1) - \chi_+\chi_-, \quad g(p_1) = \omega(p_1) + \chi_+\chi_-, \tag{5.60}$$

indeed

$$[Y(\sigma,\tau),\bar{P}_y(\tilde{\sigma},\tau)] = \int dp_1 i\left(\frac{\sqrt{p_1^2+(1+\chi_+^2)(1+\chi_-^2)}}{2\sqrt{p_1^2+(1+\chi_+^2)(1+\chi_-^2)}}+\frac{\sqrt{p_1^2+(1+\chi_+^2)(1+\chi_-^2)}}{2\sqrt{p_1^2+(1+\chi_+^2)(1+\chi_-^2)}}\right)e^{ip_1(\sigma-\tilde{\sigma})}$$
$$= i\delta(\sigma-\tilde{\sigma}). \tag{5.61}$$

The same considerations are valid also for the other fields, and thus commutation relations between creation and annihilation operators linked to the Z and $\bar{Z}$ fields are

$$\begin{aligned}&[a_{\bar{z}}(p_1,\tau),a^\dagger_{\bar{z}}(p_2,\tau)] = \delta(p_1-p_2), \quad [a_z(p_1,\tau),a^\dagger_z(p_2,\tau)] = \delta(p_1-p_2)\\&[a_{\bar{z}}(p_1,\tau),a^\dagger_z(p_2,\tau)] = 0, \quad [a_z(p_1,\tau),a^\dagger_{\bar{z}}(p_2,\tau)] = 0.\end{aligned} \tag{5.62}$$

With (5.62) and (5.58) we also deduce that $a^\dagger_{\bar{z}}$ creates an antiparticle with flavors Z while $a^\dagger_z$ creates a particle of flavors Z as well as $a^\dagger_{\bar{y}}$ creates a antiparticle of flavors Y while $a^\dagger_y$ creates a particle of flavors Y. Clarified this, then we include the oscillators expansions of the fields in the expression of (5.42)

$$H^{(2)} = \int dp\Big(\omega(p)\Big(a^\dagger_z(p)a_z(p)+a^\dagger_y(p)a_y(p)\Big)+\bar{\omega}(p)\Big(a^\dagger_{\bar{z}}(p)a_{\bar{z}}(p)+a^\dagger_{\bar{y}}(p)a_{\bar{y}}(p)\Big)\Big). \tag{5.63}$$

5.3.2 Quartic Hamiltonian

The quartic Hamilton is crucial in order to compute the S-matrix. The following expression is obtained considering $a=\frac{1}{2}$

$$\begin{aligned}H^{(4)} =&\frac{1}{2}\Big(m^2\Big(2P_y\bar{P}_yZ\bar{Z}-2P_z\bar{P}_zY\bar{Y}+2Y\bar{Y}'Z\bar{Z}-2Y\bar{Y}Z'\bar{Z}'-3i\bar{P}_yY^2\bar{Y}\chi_-\chi_++3iP_yY\bar{Y}^2\chi_-\chi_+\\&-i\bar{P}_zY\bar{Y}Z\chi_-\chi_+-iP_y\bar{Y}Z\bar{Z}\chi_-\chi_++3i\bar{P}_zZ^2\bar{Z}\chi_-\chi_+-3iP_zZ\bar{Z}^2\chi_-\chi_++iP_zY\bar{Y}\bar{Z}\chi_-\chi_+\\&+i\bar{P}_yYZ\bar{Z}\chi_-\chi_+\Big)+\chi_-\chi_+\Big(iP_y^2\bar{P}_y\bar{Y}-iP_y\bar{P}_y^2Y-i\bar{P}_yP_z\bar{P}_zY+iP_yP_z\bar{P}_z\bar{Y}-i\bar{P}_yY'^2\bar{Y}\\&+iP_yY\bar{Y}'^2+iP_y\bar{P}_y\bar{P}_zZ+iP_z\bar{P}_z^2Z+i\bar{P}_zY'\bar{Y}'Z-i\bar{P}_zY'\bar{Y}Z'+i\bar{P}_zY\bar{Y}'Z'-iP_y\bar{P}_yP_z\bar{Z}\\&-iP_zY'\bar{Y}'\bar{Z}-iP_z^2\bar{P}_z\bar{Z}+i\bar{P}_yY'Z'\bar{Z}+iP_y\bar{Y}'Z'\bar{Z}+i\bar{P}_zZ'^2\bar{Z}-iP_zY'\bar{Y}\bar{Z}'+iP_zY\bar{Y}'\bar{Z}'\\&-i\bar{P}_yY'Z\bar{Z}'-iP_y\bar{Y}'Z'\bar{Z}'-i\bar{P}_yYZ'\bar{Z}'+iP_y\bar{Y}Z'\bar{Z}'-iP_zZ\bar{Z}'^2\Big)\\&+4(\chi_--\chi_+)(\chi_-+\chi_+)\Big(YY'\bar{Y}\bar{Y}'-ZZ'\bar{Z}\bar{Z}'\Big)+(1+\chi_-^2)(-1+\chi_+^2)\Big(\bar{P}_y^2Y^2+P_y^2\bar{Y}^2\\&-Y'^2\bar{Y}^2-Y^2\bar{Y}'^2-\bar{P}_z^2Z^2-P_z^2\bar{Z}^2+Z'^2\bar{Z}^2+Z^2\bar{Z}'^2\Big)\\&+2(1+\chi_-^2)^2(-1+\chi_+^2)(1+\chi_+^2)\Big(Z^2\bar{Z}^2-Y^2\bar{Y}^2\Big)\\&+4(1+\chi_-^2(2+\chi_+^2))\Big(+P_z\bar{P}_zZ\bar{Z}-P_y\bar{P}_yY\bar{Y}\Big)\Big).\end{aligned} \tag{5.64}$$

We then plug the oscillators expansions in the (5.64), as we made for the quadratic Hamiltonian and then, as for the undeformed and for the η-deformed models, we must compute several integrals. In particular, the Jacobian is the same with respect to the one parameter case, indeed in the products of the exponentials

$$e^{i\tau(\omega(p_1)+\omega(p_2)-\bar{\omega}(p_3)-\bar{\omega}(p_4))} \tag{5.65}$$

the shifting factor cancels out in the sum $\omega(p_i)-\bar{\omega}(p_j)$.

In the following we will present the tree-level T-matrix in the case of $a = \frac{1}{2}$.

$$
\begin{aligned}
T_{ZZ} =& \frac{1}{2D}\Big((p_1+p_2)^2 + (\chi_-^2+\chi_+^2)(\omega(p_1)+\omega(p_2))^2 - \chi_-\chi_+\Big(4(\omega(p_1)+\omega(p_2)) \\
&+ 4\chi_-\chi_+(1-\omega(p_1)\omega(p_2)) + (p_2\omega(p_1)-p_1\omega(p_2))(p_2-p_1)\Big)\Big), \\
T_{\bar{Z}\bar{Z}} =& \frac{1}{2D}\Big((p_1+p_2)^2 + (\chi_-^2+\chi_+^2)(\bar{\omega}(p_1)+\bar{\omega}(p_2))^2 + \chi_-\chi_+\Big(4(\bar{\omega}(p_1)+\bar{\omega}(p_2)) \\
&- 4\chi_-\chi_+(1-\bar{\omega}(p_1)\bar{\omega}(p_2)) + (p_2\bar{\omega}(p_1)-p_1\bar{\omega}(p_2))(p_2-p_1)\Big)\Big), \\
T_{\bar{Z}Z} =& \frac{1}{2D}\Big(-(p_1-p_2)^2 - (\chi_-^2+\chi_+^2)(\bar{\omega}(p_1)-\omega(p_2))^2 \\
&+ \chi_-\chi_+\Big(\omega(p_2)(4+p_1^2+p_1p_2) - \bar{\omega}(p_1)(4+p_2^2+p_1p_2) + 4\chi_+\chi_-(1+\omega(p_2)\bar{\omega}(p_1))\Big)\Big), \\
T_{Z\bar{Y}} =& \frac{1}{2D}\Big(m^2(p_2^2-p_1^2) + \chi_-\chi_+(p_1-p_2)\Big(-p_1(\bar{\omega}(p_2)-\chi_-\chi_+) + p_2(\omega(p_1)+\chi_+\chi_-)\Big)\Big), \\
T_{ZY} =& \frac{1}{2D}\Big(m^2(p_2^2-p_1^2) + \chi_-\chi_+(p_1+p_2)\Big(p_1(\omega(p_2)+\chi_-\chi_+) - p_2(\omega(p_1)+\chi_+\chi_-)\Big)\Big), \\
T_{\bar{Z}Y} =& \frac{1}{2D}\Big(m^2(p_2^2-p_1^2) + \chi_-\chi_+(p_2-p_1)\Big(-p_1(\omega(p_2)+\chi_-\chi_+) + p_2(\bar{\omega}(p_1)-\chi_+\chi_-)\Big)\Big), \\
T_{\bar{Z}\bar{Y}} =& \frac{1}{2D}\Big(m^2(p_2^2-p_1^2) + \chi_-\chi_+(p_1+p_2)\Big(-p_1(\bar{\omega}(p_2)-\chi_-\chi_+) + p_2(\bar{\omega}(p_1)-\chi_+\chi_-)\Big)\Big),
\end{aligned}
\tag{5.66}
$$

where we introduced the short-hand,

$$
\begin{aligned}
D &= \left(p_1\sqrt{p_2^2+m^2} - p_2\sqrt{p_1^2+m^2}\right) \\
&= \left(p_1\sqrt{p_2^2+(1+\chi_+^2)(1+\chi_-^2)} - p_2\sqrt{p_1^2+(1+\chi_+^2)(1+\chi_-^2)}\right).
\end{aligned}
\tag{5.67}
$$

As it is pointed out in the case of undeformed and in the η-deformed $AdS_3 \times S^3$ backgrounds, the others non vanishing elements of the T-matrix are obtained exploiting the analytic continuation from S^3 to AdS_3 (4.21). Moreover, another equivalent approach in order to deduce the other matrix elements, is to exploit the unitarity of the S-matrix, that at the tree-level reads as

$$
T_{ij}(p_1,p_2) = -T_{ji}(p_2,p_1). \tag{5.68}
$$

Finally, it is clear that the also in this case the S-matrix is diagonal, and so it respects the *Yang-Baxter* equations (4.45). This fact basically proves the classical integrability of the model.

5.4 Explicit results for the three parameter deformation

In this section we will report the bosonic tree-level S-matrix computed in [21], for the most general three-parameter deformation of ref.[32]. In particular the line element and B-field take the form

$$ds^2 = ds^2_{(1)} + ds^2_{(2)}\,, \qquad B = B_{(1)} + B_{(2)}\,, \tag{5.69}$$

where the subscript labels (1) and (2) refer to AdS_3 and S^3, respectively, and

$$\begin{aligned} ds^2_{(1)} =& \frac{1}{F_{(1)}}\Big[\frac{1-q^2\rho^2(1+\rho^2)}{1+\rho^2}\,d\rho^2 - 2q\chi_-\rho(1+\rho^2)\,d\rho\,dt + 2q\chi_+\rho^3\,d\rho\,d\psi \\ &-(1+\chi_-^2(1+\rho^2))(1+\rho^2)\,dt^2 + 2\chi_+\chi_-\rho^2(1+\rho^2)\,dt\,d\psi + \rho^2(1-\rho^2\chi_+^2)\,d\,\psi^2\Big], \\ ds^2_{(2)} =& \frac{1}{F_{(2)}}\Big[\frac{1+q^2r^2(1-r^2)}{1-r^2}\,dr^2 - 2q\chi_-r(1-r^2)\,dr\,d\omega - 2q\chi_+r^3\,dr\,d\phi \\ &+(1+\chi_-^2(1-r^2))(1-r^2)\,d\omega^2 + 2\chi_+\chi_-r^2(1-r^2)\,d\omega\,d\phi + r^2(1+\chi_+^2r^2)\,d\phi^2\Big], \end{aligned} \tag{5.70}$$

and

$$\begin{aligned} B_{(1)} =& \frac{a\,q}{F_{(1)}}\,\rho^2\Big[2+(1+\rho^2)q^2+(1+\rho^2)\chi_-^2+(1-\rho^2)\chi_+^2\Big]\,dt\wedge d\psi\,, \\ B_{(2)} =& \frac{a\,q}{F_{(2)}}\,r^2\Big[2+(1-r^2)q^2+(1-r^2)\chi_-^2+(1+r^2)\chi_+^2\Big]\,d\omega\wedge d\phi\,, \end{aligned} \tag{5.71}$$

with

$$\begin{aligned} F_{(1)} =& 1-\chi_+^2\rho^2+\chi_-^2(1+\rho^2)-q^2\rho^2(1+\rho^2)\,, \\ F_{(2)} =& 1+\chi_+^2r^2+\chi_-^2(1-r^2)+q^2r^2(1-r^2)\,, \end{aligned} \tag{5.72}$$

and

$$a = \frac{1}{\sqrt{(q^2+\chi_+^2+\chi_-^2)^2+4(q^2-\chi_+^2\chi_-^2)}}\,. \tag{5.73}$$

Before proceeding with the expansion of the Hamiltonian we use the change of coordinate (5.20) and then, in this case, it is convenient to perform also the following canonical transformation:

$$P_j = \frac{\widetilde{P}_j}{\sqrt{1+\chi_-^2}} - \frac{q\,\chi_-}{\sqrt{1+\chi_-^2}}\widetilde{X}^j\,, \qquad X^j = \sqrt{1+\chi_-^2}\,\widetilde{X}^j\,. \tag{5.74}$$

Then, the quadratic Hamiltonian reads

$$\begin{aligned} H^{(2)} =& P_z\bar{P}_z + \acute{Z}\acute{\bar{Z}} + m^2 Z\bar{Z} + i\chi_+\chi_-(Z\bar{P}_z - \bar{Z}P_z) - i\lambda(Z\acute{\bar{Z}} - \bar{Z}\acute{Z}) \\ &+ P_y\bar{P}_y + \acute{Y}\acute{\bar{Y}} + m^2 Y\bar{Y} + i\chi_+\chi_-(Y\bar{P}_y - \bar{Y}P_y) - i\lambda(Y\acute{\bar{Y}} - \bar{Y}\acute{Y})\,, \end{aligned} \tag{5.75}$$

with

$$m^2 = q^2 + (1+\chi_+^2)(1+\chi_-^2)\,, \qquad \lambda = a\,q\,(2+q^2+\chi_-^2+\chi_+^2)\,. \tag{5.76}$$

In this case the three deformation parameters are $\chi_+,\ \chi_-,\ q$. Now, using the same expansions given for the previous cases (4.32), we obtain

$$H^{(2)} = \int dp\Big[\omega(p)\Big(a_z^\dagger(p)a_z(p) + a_y^\dagger(p)a_y(p)\Big) + \bar{\omega}(p)\Big(a_{\bar{z}}^\dagger(p)a_{\bar{z}}(p) + a_{\bar{y}}^\dagger(p)a_{\bar{y}}(p)\Big)\Big]\,, \tag{5.77}$$

where

$$\omega(p) = \sqrt{p^2-2\lambda p+m^2} - \chi_+\chi_-\,, \qquad \bar{\omega}(p) = \sqrt{p^2+2\lambda p+m^2} + \chi_+\chi_-\,, \tag{5.78}$$

where m and λ are given by eq. (5.76). Below we write the quartic Hamiltonian $H^{(4)}$ for $a = 1/2$

$$
\begin{aligned}
H^{(4)} = \;& \frac{1}{2}\Big\{ m^2\Big(-2P_z\bar{P}_zY\bar{Y} + 2P_y\bar{P}_yZ\bar{Z} + 2\acute{Y}\acute{\bar{Y}}Z\bar{Z} - 2Y\bar{Y}\acute{Z}\acute{\bar{Z}} \\
& + \chi_-(q-i\chi_+)\Big(+\bar{P}_zY\bar{Y}Z - \bar{P}_yYZ\bar{Z}\Big) + \chi_-(q+i\chi_+)\Big(P_zY\bar{Y}\bar{Z} - P_y\bar{Y}Z\bar{Z}\Big)\Big) \\
& + \lambda\Big(-i\bar{P}_y^2Y\acute{Y} - iP_z\bar{P}_z\acute{Y}\bar{Y} + iP_z\bar{P}_zY\acute{\bar{Y}} + iP_y^2\bar{Y}\acute{\bar{Y}} - i\acute{Y}^2\bar{Y}\acute{\bar{Y}} + iY\acute{Y}\acute{\bar{Y}}^2 \\
& + i\bar{P}_y\bar{P}_z\acute{Y}Z + iP_y\bar{P}_z\acute{\bar{Y}}Z - i\bar{P}_y\bar{P}_zY\acute{Z} + iP_y\bar{P}_z\bar{Y}\acute{Z} + i\bar{P}_z^2Z\acute{Z} - i\bar{P}_yP_z\acute{Y}\bar{Z} \\
& - iP_yP_z\acute{\bar{Y}}\bar{Z} + iP_y\bar{P}_y\acute{Z}\bar{Z} + i\acute{Y}\acute{\bar{Y}}\acute{Z}\bar{Z} - i\bar{P}_yP_zY\acute{\bar{Z}} + iP_yP_z\bar{Y}\acute{\bar{Z}} - iP_y\bar{P}_yZ\acute{\bar{Z}} \\
& - i\acute{Y}\acute{\bar{Y}}Z\acute{\bar{Z}} - i\acute{Y}\bar{Y}\acute{Z}\acute{\bar{Z}} + iY\acute{\bar{Y}}\acute{Z}\acute{\bar{Z}} - iP_z^2\bar{Z}\acute{\bar{Z}} + i\acute{Z}^2\bar{Z}\acute{\bar{Z}} - iZ\acute{Z}\acute{\bar{Z}}^2 \\
& + (m^2 - 2iq\chi_-^2\chi_+)\Big(-iY\acute{\bar{Y}}Z\bar{Z} + iY\bar{Y}Z\acute{\bar{Z}}\Big) - \chi_-(iq-\chi_+)\Big(2\bar{P}_zZ\acute{Z}\bar{Z} - 2\bar{P}_yY\acute{Y}\bar{Y}\Big) \\
& + (m^2 + 2iq\chi_-^2\chi_+)\Big(+i\acute{Y}\bar{Y}Z\bar{Z} - iY\bar{Y}\acute{Z}\bar{Z}\Big) + \chi_-(iq+\chi_+)\Big(2P_zZ\bar{Z}\acute{\bar{Z}} - 2P_yY\bar{Y}\acute{\bar{Y}}\Big) \\
& + (\chi_-\chi_+)\Big(-2\bar{P}_zY\bar{Y}\acute{Z} + 2\bar{P}_y\acute{Y}Z\bar{Z} + 2P_y\acute{\bar{Y}}Z\bar{Z} - 2P_zY\bar{Y}\acute{\bar{Z}}\Big)\Big) \\
& + \chi_-(q-i\chi_+)\Big(-P_y^2\bar{P}_y\bar{Y} - P_yP_z\bar{P}_z\bar{Y} + \bar{P}_y\acute{Y}^2\bar{Y} + \bar{P}_z\acute{Y}\bar{Y}\acute{Z} + P_y\bar{P}_yP_z\bar{Z} \\
& + P_z^2\bar{P}_z\bar{Z} + P_z\acute{Y}\acute{\bar{Y}}\bar{Z} - \bar{P}_y\acute{Y}\acute{Z}\bar{Z} - P_y\acute{\bar{Y}}\acute{Z}\bar{Z} - \bar{P}_z\acute{Z}^2\bar{Z} + P_z\acute{Y}\bar{Y}\acute{\bar{Z}} - P_y\bar{Y}\acute{Z}\acute{\bar{Z}}\Big) \\
& + 4(1+\chi_-^2(2+\chi_+^2))\Big(P_z\bar{P}_zZ\bar{Z} - P_y\bar{P}_yY\bar{Y}\Big) + \chi_-(q+i\chi_+)\Big(-P_y\bar{P}_y^2Y - \bar{P}_yP_z\bar{P}_zY \\
& + P_yY\acute{\bar{Y}}^2 + P_y\bar{P}_y\bar{P}_zZ + P_z\bar{P}_z^2Z + \bar{P}_z\acute{Y}\acute{\bar{Y}}Z + \bar{P}_zY\acute{\bar{Y}}\acute{Z} + P_zY\acute{\bar{Y}}\acute{\bar{Z}} - \bar{P}_y\acute{Y}Z\acute{\bar{Z}} \\
& - P_y\acute{\bar{Y}}Z\acute{\bar{Z}} - \bar{P}_yY\acute{Z}\acute{\bar{Z}} - P_zZ\acute{\bar{Z}}^2\Big) + (q^2 - \chi_-^2 + \chi_+^2)\Big(-4Y\acute{Y}\bar{Y}\acute{\bar{Y}} + 4Z\acute{Z}\bar{Z}\acute{\bar{Z}}\Big) \\
& + q\chi_-^2\chi_+\Big(-2iP_y\bar{P}_z\bar{Y}Z + 2i\bar{P}_yP_zY\bar{Z} - 2iY\acute{\bar{Y}}\acute{Z}\bar{Z} + 2i\acute{Y}\bar{Y}Z\acute{\bar{Z}}\Big) \\
& + (q\lambda\chi_-)\Big(-2i\bar{P}_zY\acute{\bar{Y}}Z + 2iP_z\acute{Y}\bar{Y}\bar{Z} - 2iP_y\bar{Y}\acute{Z}\bar{Z} + 2i\bar{P}_yYZ\acute{\bar{Z}}\Big) \\
& + (1+\chi_-^2 + (q-i\chi_+)(q-i(1+\chi_-^2)\chi_+))\Big(\acute{Y}^2\bar{Y}^2 + P_z^2\bar{Z}^2 - P_y^2\bar{Y}^2 - \acute{Z}^2\bar{Z}^2\Big) \\
& + (1+\chi_-^2 + (q+i\chi_+)(q+i(1+\chi_-^2)\chi_+))\Big(-\bar{P}_y^2Y^2 + Y^2\acute{\bar{Y}}^2 + \bar{P}_z^2Z^2 - Z^2\acute{\bar{Z}}^2\Big) \\
& + (q^4 + 2q^2(1+\chi_+^2) + (1+\chi_-^2)^2(-1+\chi_+^4))\Big(-2Y^2\bar{Y}^2 + 2Z^2\bar{Z}^2\Big) \\
& + \chi_-(q(q^2 + 4iq\chi_+ + 1 + 5\chi_+^2 + \chi_-^2(-3+\chi_+^2)) - 3i\chi_+m^2)\Big(P_zZ\bar{Z}^2 - P_yY\bar{Y}^2\Big) \\
& + \chi_-(q(q^2 - 4iq\chi_+ + 1 + 5\chi_+^2 + \chi_-^2(-3+\chi_+^2)) + 3i\chi_+m^2)\Big(\bar{P}_zZ^2\bar{Z} - \bar{P}_yY^2\bar{Y}\Big) \\
& + aq(3q^4 - 2iq^3\chi_-^2\chi_+ - 2iq\chi_-^2\chi_+(2+\chi_-^2+\chi_+^2) + q^2(9+6\chi_+^2 - \chi_-^2(-2+\chi_+^2)) \\
& - (1+\chi_+^2)(2+\chi_-^4 - 3\chi_+^2 + \chi_-^2(7+\chi_+^2)))\Big(+iY^2\bar{Y}\acute{\bar{Y}} - iZ^2\bar{Z}\acute{\bar{Z}}\Big) \\
& + aq(3q^4 + 2iq^3\chi_-^2\chi_+ + 2iq\chi_-^2\chi_+(2+\chi_-^2+\chi_+^2) + q^2(9+6\chi_+^2 - \chi_-^2(-2+\chi_+^2)) \\
& - (1+\chi_+^2)(2+\chi_-^4 - 3\chi_+^2 + \chi_-^2(7+\chi_+^2)))\Big(-iY\acute{Y}\bar{Y}^2 + iZ\acute{Z}\bar{Z}^2\Big)\Big\}.
\end{aligned}
\tag{5.79}
$$

Finally, including the expansion in creation and annihilation operators in the previous expression we find the following tree-level S-matrix elements

$$
\begin{aligned}
T_{ZY} = \frac{1}{2D}\Big[& m^2(p_2^2 - p_1^2) + \lambda(p_1 - p_2)\Big(m^2 + \chi_-^2\chi_+^2 + p_1p_2 - \omega_1\omega_2\Big) \\
& + \chi_-\chi_+\Big(p_2 + p_1 - 2\lambda\Big)\Big(p_1(\omega_2 + \chi_-\chi_+) - p_2(\omega_1 + \chi_-\chi_+)\Big)\Big],
\end{aligned}
\tag{5.80}
$$

$$
\begin{aligned}
T_{Z\bar{Y}} = \frac{1}{2D}\Big[& m^2(p_2^2 - p_1^2) + \lambda(p_1+p_2)\Big(m^2 + \chi_-^2\chi_+^2 - p_1p_2 + \omega_1\bar{\omega}_2\Big) \\
& + \chi_-\chi_+\Big(p_1 - p_2 - 2\lambda\Big)\Big(- p_1(\bar{\omega}_2 - \chi_-\chi_+) + p_2(\omega_1 + \chi_-\chi_+)\Big)\Big],
\end{aligned} \tag{5.81}
$$

$$
\begin{aligned}
T_{\bar{Z}Y} = \frac{1}{2D}\Big[& m^2(p_2^2 - p_1^2) - \lambda(p_1+p_2)\Big(m^2 + \chi_-^2\chi_+^2 - p_1p_2 + \bar{\omega}_1\omega_2\Big) \\
& + \chi_-\chi_+\Big(p_2 - p_1 - 2\lambda\Big)\Big(- p_1(\omega_2 + \chi_-\chi_+) + p_2(\bar{\omega}_1 - \chi_-\chi_+)\Big)\Big],
\end{aligned} \tag{5.82}
$$

$$
\begin{aligned}
T_{\bar{Z}\bar{Y}} = \frac{1}{2D}\Big[& m^2(p_2^2 - p_1^2) - \lambda(p_1-p_2)\Big(m^2 + \chi_-^2\chi_+^2 + p_1p_2 - \bar{\omega}_1\bar{\omega}_2\Big) \\
& + \chi_-\chi_+\Big(p_1 + p_2 + 2\lambda\Big)\Big(- p_1(\bar{\omega}_2 - \chi_-\chi_+) + p_2(\bar{\omega}_1 - \chi_-\chi_+)\Big)\Big],
\end{aligned} \tag{5.83}
$$

$$
\begin{aligned}
T_{ZZ} =& \frac{1}{2D}\Big[(p_1+p_2)^2 + (q^2 + \chi_-^2 + \chi_+^2)(\omega_1+\omega_2)^2 - 4q^2(\omega_1\omega_2 - p_1p_2 - 1) \\
& - \chi_-\chi_+(4(\omega_1+\omega_2) + 4\chi_-\chi_+(1-\omega_1\omega_2) + (p_2\omega_1 - p_1\omega_2)(p_2-p_1)) \\
& - \lambda\Big(p_1(p_2^2 - \omega_2(\omega_1 + 2\chi_-\chi_+) + p_2(p_1^2 - \omega_1(\omega_2 + 2\chi_-\chi_+)))\Big) \\
& - aq(p_1+p_2)C\Big],
\end{aligned} \tag{5.84}
$$

$$
\begin{aligned}
T_{\bar{Z}\bar{Z}} =& \frac{1}{2D}\Big[(p_1+p_2)^2 + (q^2 + \chi_-^2 + \chi_+^2)(\bar{\omega}_1+\bar{\omega}_2)^2 - 4q^2(\bar{\omega}_1\bar{\omega}_2 - p_1p_2 - 1) \\
& + \chi_-\chi_+(4(\bar{\omega}_1+\bar{\omega}_2) - 4\chi_-\chi_+(1-\bar{\omega}_1\bar{\omega}_2) + (p_2\bar{\omega}_1 - p_1\bar{\omega}_2)(p_2-p_1)) \\
& + \lambda\Big(p_1(p_2^2 - \bar{\omega}_2(\bar{\omega}_1 - 2\chi_-\chi_+)) + p_2(p_1^2 - \bar{\omega}_1(\bar{\omega}_2 - 2\chi_-\chi_+))\Big) \\
& + aq(p_1+p_2)C\Big],
\end{aligned} \tag{5.85}
$$

$$
\begin{aligned}
T_{\bar{Z}Z} =& \frac{1}{2D}\Big[- (p_1-p_2)^2 - (q^2 + \chi_-^2 + \chi_+^2)(\bar{\omega}_1-\omega_2)^2 - 4q^2(\bar{\omega}_1\omega_2 - p_1p_2 + 1) \\
& + \lambda\Big(- p_1(p_2^2 + \omega_2(\bar{\omega}_1 - 2\chi_-\chi_+)) + p_2(p_1^2 + \bar{\omega}_1(\omega_2 + 2\chi_-\chi_+))\Big) \\
& + \chi_-\chi_+\Big(\omega_2(4 + p_1^2 + p_1p_2) - \bar{\omega}_1(4 + p_2^2 + p_1p_2) + 4\chi_-\chi_+(1 + \omega_2\bar{\omega}_1)\Big) \\
& + aq(-p_1+p_2)C\Big],
\end{aligned} \tag{5.86}
$$

where we introduced the short-hand,

$$
D = ((p_2 \pm \lambda)(\omega_1 \pm \chi_-\chi_+) - (p_1 \pm \lambda)(\omega_2 \pm \chi_-\chi_+)), \tag{5.87}
$$

where the sign of the shift $\pm\lambda$ is positive for Z and Y and negative for $\bar{Z}$ and $\bar{Y}$ and ω_j is either $\omega(p_j)$ or $\bar{\omega}(p_j)$ depending on the particle's flavour. Similarly, we introduce the constant C

$$
C = q^2(7 + q^2 + 2(\chi_-^2 + \chi_+^2)) + (2 - \chi_-^2 - \chi_+^2 + \chi_-^4 + \chi_+^4 - 6\chi_-^2\chi_+^2)\,. \tag{5.88}
$$

Also in this case the the remaining S-matrix elements can be computed either by braising unitarity or by observing the relation between AdS_3 and S^3 fields,

$$
S_{YY} = -S_{\bar{Z}\bar{Z}}\,, \qquad S_{\bar{Y}\bar{Y}} = -S_{ZZ}\,, \qquad S_{\bar{Y}Y} = -S_{Z\bar{Z}}\,. \tag{5.89}
$$

Moreover the expression for general α can be obtained by

$$
T_{ij}(p_1,p_2) = T_{ij}(p_1,p_2)\Big|_{a=1/2} + (a - \tfrac{1}{2})\Big(p_2\omega(p_1) - p_1\omega(p_2)\Big)\,. \tag{5.90}
$$

We deduce that also in this case the matrix is diagonal and consequently the Yang-Baxter equation is trivially satisfied

Chapter 6

Conclusions

In our work we have computed the tree-level bosonic S-matrix for various models all related to the underlying background $AdS_3 \times S^3$.

In particular we first analyzed the string in the undeformed $AdS_3 \times S^3$ background. The integrability structure of this model follows from the well known integrability of the superstrings in $AdS_5 \times S^5$. However, we first pointed out that the equation of motion for this model can be rephrased in terms of the a Lax connection, implying that the non linear σ-model is classically integrable. Then we moved to a perturbative analysis, using the first order formalism, in order to deduce the light-cone gauge-fixed Hamiltonian of the model. For our purposes it was particularly important to find the quartic expansion (in fields) of the above Hamiltonian, since the computation of the tree-level S-matrix essentially boils down to rewriting the quartic Hamiltonian in terms of oscillators. Eventually we recovered the S-matrix elements, which can be found in literature. Moreover, from (4.50), we can immediately recognize the factorization of S-matrix, which indeed is a crucial property for an integrable system.

Then we moved to the deformed background. In particular we considered the η-deformation , the two-parameters deformation $(\chi_-,\ \chi_+)$ and the three-parameters deformation $(\chi_-,\ \chi_+,\ q)$ of the $AdS_3 \times S^3$ background. Also in this case we first pointed out that the equation of motion, for general two-parameters deformed string σ-model, can be recast as the zero curvature condition for a Lax connection. Afterwards, we used the first order formalism in order to deduce the perturbative Hamiltonian of the mode. As in the undeformed model, the computations of the S-matrix consist in the substitution of the oscillators expansion of the fields into the quartic (in fields) Hamiltonian of the model. Eventually, for the η-deformed model, we recovered the matrix elements that can be found in literature, while, in the two-parameters and three-parameters deformations cases the matrix elements are not present in literature. However, we find that, in all the cases, the S-matrix is diagonal, and therefore it respects the Yang-Baxter equation as expected.

A consistency check for the matrix elements in the two-parameters and in the three-parameters cases is given by the limit to the η-deformed and to the undeformed models. In particular, for the first deformations, sending $\chi_- \to 0$, it is straightforward to check that (5.66) become (5.35), while the limit $\chi_+ \to 0,\ \chi_- \to 0$ of the matrix elements (5.66) recover the undeformed ones (4.50). For the three-parameters in addition to the previous limits it must also be added $q \to 0$, and it is evident that all the previous models are recovered.

We stress also that we found that the bosonic model is only classically integrable. The quantum validity of the previous statement does not follow immediately, as we limited ourselves to tree-level computations.

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