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BCS-BEC Crossover of Superfluid Fermions with Rabi Coupling

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Preface

The aim of this thesis is the study of an ultracold gas of Rabi coupled atoms interacting via a two body contact potential. Apart from Rabi interaction, such is the model describing the familiar BCS-BEC crossover, in which the gas goes from being superconductive when the coupling constant is small in modulus to being superfluid as the attraction gets more intense. The BCS-BEC crossover enriched with Rabi coupling has almost no counterpart in the literature, so that many of the results presented are a first.

This work is divided in three main parts. Chapter 1 introduces a key quantity to our treatment: the scattering length a_F, through which we will express the contact potential thereby renormalizing our theory; chapters 2 and 3 are review sections, in which some of the main techniques used for the study of the BCS-BEC crossover are studied and the behaviour of various physical quantities will be displayed along the whole crossover in absence of Rabi coupling. In particular, in chapter 2 we review the mean field treatment of the model both at the critical temperature and at zero temperature, while in chapter 3 we try to go beyond mean field, studying specifically the critical temperature of the system along the crossover with different approaches. In chapters 4 and 5 we introduce Rabi coupling in the model and try to replicate the results obtained in the Rabiless case both at the mean field level in chapter 4 and beyond mean field in chapter 5.

To appreciate the work fully and not get lost in the calculations, it is strongly suggested not to read this thesis section by section in the order given. The work was developed by writing a section about the Rabiless BCS-BEC crossover and then by producing the Rabi coupled counterpart. This is why chapter 2 should be read together with 4, while chapter 3 should be studied parallely to chapter 5.

In all the thesis, much attention was payed to making calculations accessible, at the cost of sometimes being heavy. The idea is that if one reads this work, he or she should be able to replicate all the results without filling pages and pages of cumbersome calculations on his or her own. It is not always beautiful, but I hope it will at least be clear and helpful to the ones who will work in the vast and dense world of the BCS-BEC crossover. Moreover, many calculations were performed but not fully exploited: for example, in the Gaussian fluctuations sections, the explicit expressions for the propagators were derived for any temperatures, even though I only used them at the critical temperature. I hope that these calculations will be used in the future to explore the zero temperature regime, too.

Finally, before getting into the realms of this work, I would like to thank Prof. L. Salasnich for his useful suggestions, often asked for and provided promptly way after midnight; I would like to thank my parents, who supported me and inspired me for all these years; my closest friend Roberta deserves all my gratefulness for keeping me strong and in touch with the real world even during the toughest times; and finally Lisa, who made me realise what really is important in life.

And now, enough with the jibber jabber. In the words of N. David Mermin:

shut up and calculate!

Contents

1 Scattering Theory

The objective of this section is to investigate the behaviour of a very dilute ultracold gas of atoms which is interacting via a two body potential. In particular, since we want to write the action functional of the gas in terms of a contact potential, we investigate how the interaction constant g is derived and in what regimes such approximation is valid. The treatment will be taken mainly from [1] and [2], but also from [3].

1.1 Two Body Scattering

The Schrodinger equation for two particles interacting via a potential $V(\mathbf{q}_1 - \mathbf{q}_2)$ is

$$i\hbar\frac{\partial\Psi(\mathbf{q}_1, \mathbf{q}_2, t)}{\partial t} = \Big[-\frac{\hbar^2}{2m_1}\nabla^2 - \frac{\hbar^2}{2m_2}\nabla^2 + V(\mathbf{q}_1 - \mathbf{q}_2)\Big]\Psi(\mathbf{q}_1, \mathbf{q}_2, t). \tag{1.1}$$

In order to solve it, one may get in the center of mass frame of reference by introducing the canonical change of variables

$$\begin{cases} \mathbf{q} &= \mathbf{q}_1 - \mathbf{q}_2 \\ \mathbf{Q} &= \frac{m_1\mathbf{q}_1 + m_2\mathbf{q}_2}{m_1 + m_2}, \end{cases} \tag{1.2}$$

meaning that

$$\begin{cases} \nabla_{\mathbf{q}_1} = \nabla_{\mathbf{q}} + \frac{m_1}{M}\nabla_{\mathbf{Q}} \\ \nabla_{\mathbf{q}_2} = -\nabla_{\mathbf{q}} + \frac{m_2}{M}\nabla_{\mathbf{Q}}. \end{cases} \tag{1.3}$$

By substituting (1.3) in (1.1) we find that

$$i\hbar\frac{\partial\Psi(\mathbf{q}, \mathbf{Q}, t)}{\partial t} = \Big[-\frac{\hbar^2}{2M}\nabla^2_{\mathbf{Q}} - \frac{\hbar^2}{2\mu}\nabla^2_{\mathbf{q}} + V(\mathbf{q})\Big]\Psi(\mathbf{q}, \mathbf{Q}, t), \tag{1.4}$$

which means that the wave function may factorize in a part dependent on the center of mass coordinate, which behaves like a free particle, and one depending on the relative coordinate. We then write $\Psi(\mathbf{q}, \mathbf{Q}, t) = \phi(\mathbf{Q}, t)\psi(\mathbf{q}, t)$ and consider the equation for $\psi(\mathbf{q}, t)$

$$i\hbar\frac{\partial\psi(\mathbf{q}, t)}{\partial t} = \Big[-\frac{\hbar^2}{2\mu}\nabla^2 + V(\mathbf{q})\Big]\psi(\mathbf{q}, t). \tag{1.5}$$

We have reduced the problem to a single particle one.

Now, we want to find the solution of the stationary Schrodinger equation

$$\Big[-\frac{\hbar^2}{2\mu}\nabla^2 + V(\mathbf{q})\Big]\psi(\mathbf{q}) = E\psi(\mathbf{q}). \tag{1.6}$$

1.1.1 Solution of the Equation

In the case of an incoming stream of particles scattering via the potential $V(\mathbf{q})$, we show that the wave function solving (1.6) is a superposition of an incoming plane wave and an outgoing spherical wave at long distance [3]. To do so, we solve the equation by using the Green's function method. In particular, by defining the Green's function as $G^{(+)}(\mathbf{q}-\mathbf{q}')$ satisfying the equation

$$\Big[\nabla^2_{\mathbf{q}} + \frac{2\mu}{\hbar^2}E\Big]G^{(+)}(\mathbf{q}-\mathbf{q}') = -\delta^3(\mathbf{q}-\mathbf{q}'), \tag{1.7}$$

we find that

$$\psi(\mathbf{q}) = e^{i\mathbf{k}\cdot\mathbf{q}} - \frac{2\mu}{\hbar^2}\int_V d^3\mathbf{q}' G^{(+)}(\mathbf{q}-\mathbf{q}')V(\mathbf{q}')\psi(\mathbf{q}') \tag{1.8}$$

is the exact solution of (1.6), given that $\mathbf{k}^2 = \frac{2\mu}{\hbar^2}E$. To solve (1.7) one goes in Fourier space to find the algebraic equation

$$\tilde{G}(\mathbf{k}) = \frac{1}{\mathbf{k}^2 - \frac{2\mu}{\hbar^2}E}, \tag{1.9}$$

and goes back to coordinate space to find that

$$G(\mathbf{q}) = \frac{e^{ikq}}{4\pi q}, \tag{1.10}$$

with $k = \sqrt{2\mu E/\hbar^2}$.

At this point one may substitute (1.10) into (1.8) and compute its asymptotic value for large distances by imposing that $|\mathbf{q}-\mathbf{q}'| \approx q - \frac{\mathbf{q}}{q}\cdot\mathbf{q}'$, which is true for $q \gg q'$, to obtain the result we wanted,

$$\psi(\mathbf{q}) = e^{i\mathbf{k}\cdot\mathbf{q}} - \frac{2\mu}{\hbar^2}\frac{e^{ikq}}{4\pi q}\int_V d^3q' e^{-ik\frac{\mathbf{q}}{q}\cdot\mathbf{q}'}V(\mathbf{q}')\psi(\mathbf{q}'). \tag{1.11}$$

In particular, by calling $k\frac{\mathbf{q}}{q}\cdot\mathbf{q}' = \mathbf{k}'$, which is a wavevector of modulus k and the direction of $\mathbf{q}$, one realizes that the integral is a Fourier transform, so that the overall result may be written as

$$\psi(\mathbf{q}) = e^{i\mathbf{k}\cdot\mathbf{q}} - \frac{2\mu}{\hbar^2}\frac{e^{ikq}}{4\pi q}\langle\mathbf{k}'|V|\psi\rangle\,. \tag{1.12}$$

In particular the solution of the stationary Schrodinger equation (1.6) is a superposition of an incoming plane wave and an outgoing spherical wave in the large distance limit.

Conventionally, the scattering amplitude $f(\mathbf{k}',\mathbf{k})$ is then introduced to write such wavefunction in a more meaningful way:

$$\psi(\mathbf{q}) = e^{i\mathbf{k}\cdot\mathbf{q}} + f(\mathbf{k}',\mathbf{k})\frac{e^{ikq}}{q}. \tag{1.13}$$

Notice that the scattering amplitude only depends on the modulus of the incoming plane wave $k = \sqrt{2\mu E/\hbar^2}$ and the angle that $\mathbf{k}$ makes with $\mathbf{q}$, as its explicit expression is

$$f(\mathbf{k}',\mathbf{k}) = -\frac{2\mu}{4\pi\hbar^2}\langle\mathbf{k}'|V|\psi\rangle\,. \tag{1.14}$$

1.1.2 The Scattering Amplitude

The scattering amplitude contains a lot of physical information [2]. For example, one may want to calculate the scattering cross section of the system following equation (1.5), i.e. the ratio between the number of particles scattered on a given solid angle per unit time and the number of incident particles. The probability per unit time that a scattered particle will go through the surface element $dS = q^2 d\Omega$, where $d\Omega$ is the solid angle, can be calculated using the continuity equation

$$\frac{d|\psi(\mathbf{q},t)|^2}{dt} = -\nabla \cdot \mathbf{j}(\mathbf{q},t), \tag{1.15}$$

with

$$\mathbf{j}(\mathbf{q},t) = \frac{1}{2mi}\Big[\psi^*(\mathbf{q},t)\nabla\psi(\mathbf{q},t) - \psi(\mathbf{q},t)\nabla\psi^*(\mathbf{q},t)\Big]. \tag{1.16}$$

By calculating $\nabla \cdot \mathbf{j}$ explicitly with $\psi(\mathbf{q}) = f(\mathbf{k}',\mathbf{k})\frac{e^{ikq}}{q}$, one finds that the flux over dS of equation (1.15) is

$$\left(\frac{k}{m}\right)|f(\mathbf{k}',\mathbf{k})|^2 d\Omega. \tag{1.17}$$

The number of incident particles, instead, can be calculated simply as $\mathbf{j} \cdot d\mathbf{S}$ substituting $\psi(\mathbf{q}) = e^{i\mathbf{k}\cdot\mathbf{q}}$ in (1.16), and reads $\left(\frac{k}{m}\right)$. The cross section at large distances, then, depends only on the scattering amplitude:

$$d\sigma = |f(\mathbf{k}',\mathbf{k})|^2 d\Omega. \tag{1.18}$$

Another interesting relation can be found by taking (1.8) in momentum space [3], where it reads

$$\psi(\mathbf{p}) = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k}) - \frac{2\mu}{\hbar^2}\frac{1}{\mathbf{p}^2 - \frac{2\mu}{\hbar^2}E}\int \frac{d^3p'}{(2\pi)^3}V(\mathbf{p}')\psi(\mathbf{p}-\mathbf{p}'). \tag{1.19}$$

The inverse of the Green's function (1.9) has come out of the integral naturally this time, without the need for any large distance approximation. One may call the modified scattering amplitude

$$\tilde{f}(\mathbf{p},\mathbf{k}) = -\frac{2\mu}{\hbar^2}\int \frac{d^3p'}{(2\pi)^3}V(\mathbf{p}')\psi(\mathbf{p}-\mathbf{p}'), \tag{1.20}$$

where the dependence on $\mathbf{k}$ is hidden in ψ. At this point one may multiply both sides of (1.19) by $V(\mathbf{p}'-\mathbf{p})$ and integrate over $\mathbf{p}$ to obtain an integral equation for f in terms of V:

$$\tilde{f}(\mathbf{p}',\mathbf{k}) = V(\mathbf{p}'-\mathbf{k}) + \int \frac{d^3p}{(2\pi)^3}\frac{\tilde{f}(\mathbf{p},\mathbf{k})V(\mathbf{p}'-\mathbf{p})}{\mathbf{p}^2 - \frac{2\mu}{\hbar^2}E}. \tag{1.21}$$

1.1.3 Partial Wave Expansion

We start the analysis by solving the Schrodinger equation (1.6) in polar coordinates, exploiting the fact that the system is invariant under rotations, so that the angular part and the radial part of the wave function factorize, leaving the radial equation

$$\Big[\frac{1}{q^2}\frac{d}{dq}q^2\frac{d}{dq} - \frac{l(l+1)}{q^2} - \frac{2\mu}{\hbar^2}V(q) + \frac{2\mu}{\hbar^2}E\Big]R_l(q) = 0, \tag{1.22}$$

which has as an asymptotic solution for $q \to +\infty$

$$R_l(q) \approx \frac{a_l \sin(kq - l\pi/2 + \delta_l)}{q}. \tag{1.23}$$

The wave function (1.13) only depends on the modulus of $\mathbf{k}$ and the angle between $\mathbf{k}$ and $\mathbf{q}$, which in polar coordinates is $\theta \in (0, \pi)$. This means that $\psi(\mathbf{q})$ does not depend on the azimuthal angle ϕ, so that its angular part can be written as an expansion over Legendre polynomials $P(\cos\theta)$:

$$\psi(\mathbf{q}) = \sum_{l=0}^{+\infty}(2l+1)A_l P_l(\cos\theta)\frac{\sin(kq - l\pi/2 + \delta_l)}{kq}. \tag{1.24}$$

The unknowns in such expression are the coefficients A_l and the phases δ_l. To write $A_l(\delta_l)$, we first write

$$e^{i\mathbf{k}\cdot\mathbf{q}} = \sum_{l=0}^{+\infty}(2l+1)i^l P_l(\cos\theta)\frac{\sin(kq - \pi l)}{kq}, \tag{1.25}$$

and then calculate

$$\psi(\mathbf{q}) - e^{i\mathbf{k}\cdot\mathbf{q}} =$$

$$= \sum_{l=0}^{+\infty}(2l+1)\frac{P_l(\cos\theta)}{2ikq}\Big[A_l\Big(e^{i(kq - l\pi/2+\delta_l)} - e^{-i(kq-l\pi/2+\delta_l)}\Big) - i^l\Big(e^{i(kq-l\pi/2)} - e^{-i(kq-l\pi/2)}\Big)\Big]. \tag{1.26}$$

Such is the spherical wave going out of the scattering center, meaning that it cannot have any term proportional to e^{-ikq}. For this condition to be satisfied

$$A_l = i^l e^{i\delta_l}, \tag{1.27}$$

so that

$$\psi(\mathbf{q}) = \sum_{l=0}^{+\infty}(2l+1)\frac{P_l(\cos\theta)}{2ikq}((-1)^l e^{-ikq} - e^{2i\delta_l}e^{ikq}). \tag{1.28}$$

Moreover, plugging (1.27) into (1.26) it becomes clear that

$$f(\mathbf{k}', \mathbf{k}) = \sum_{l=0}^{+\infty}(2l+1)\frac{P_l(\cos\theta)}{2ik}(e^{2i\delta_l} - 1). \tag{1.29}$$

The scattering amplitude is completely determined based on the form of δ_l, and the total cross section, meaning the integration over $d\Omega$ of (1.18) takes the form

$$\int d\sigma = 2\pi \int \sin(\theta)|f(\theta)|^2 = \frac{4\pi}{k^2}\sum_{l=0}^{+\infty}(2l+1)\sin^2(\delta_l), \tag{1.30}$$

knowing that the integral of $P_l^2(\cos\theta)$ over $d\theta\sin\theta$ is $\frac{2}{2l+1}$, that Legendre polynomials are orthogonal and that for the derivation of δ_l it is clear that it only depends on the modulus of $\mathbf{k}$. Each term in the sum is a partial effective cross section, as it regards particles with angular momentum l.

Given these expressions, the scattering amplitude is often written with a different notation [1],

$$f(\mathbf{k}', \mathbf{k}) = \sum_{l=0}^{+\infty}(2l+1)f_l(k)P_l(\cos\theta), \tag{1.31}$$

with

$$f_l(k) = \frac{1}{2ik}(e^{2i\delta_l(k)} - 1). \tag{1.32}$$

1.1.4 Born's Formula

Suppose the potential in equation (1.6) is a small contribution, so that one may use perturbation theory up to first order to get a reasonable solution to the problem. In order to do so, we write the wave function $\psi(\mathbf{q}) = \sum_{n=0}^{+\infty}\psi^{(n)}(\mathbf{q})$, with $|\psi^{n+1}| \ll |\psi^{(n)}|$. The biggest contribution, by adiabatic continuity, will come from the plane wave, so that $\psi^{(0)}(\mathbf{q}) = e^{i\mathbf{k}\cdot\mathbf{q}}$. Up to first order, then, the Schrodinger equation is

$$\left[\frac{\hbar^2}{2m} + E\right]\psi^{(1)}(\mathbf{q}) = V(\mathbf{q})\psi^{(0)}(\mathbf{q}), \tag{1.33}$$

whose solution can be calculated using the Green's function method, so that

$$\psi^{(1)}(\mathbf{q}) = -\int d^3q' G(\mathbf{q}-\mathbf{q}')V(\mathbf{q}')\psi^{(0)}(\mathbf{q}'), \tag{1.34}$$

with

$$G(\mathbf{q}) = \frac{m}{2\pi\hbar^2}\frac{e^{ikq}}{q}. \tag{1.35}$$

Let a be the characteristic range of interaction of the potential and let the energy of the particle be small, so that ka is of the order of unity. In this case the oscillating term of the Green's function in the integral can be neglected and we see that

$$\psi^{(1)}(\mathbf{q}) \approx \psi^{(0)}(\mathbf{q})|V|a^2\frac{m}{\hbar^2}. \tag{1.36}$$

Such expression satisfies $\psi^{(1)} \ll \psi^{(0)}$ iff

$$|V| \ll \frac{\hbar^2}{ma^2}, \tag{1.37}$$

which is of the order of the kinetic energy of a particle enclosed in a volume of linear dimension a.

At this point, considering the wave function to be the sum of the plane wave and of $\psi^{(1)}(\mathbf{q})$, one may deduce the approximate shape of the scattering amplitude by calculating explicitly

$$\psi^{(1)}(\mathbf{q}) = -\frac{m}{2\pi\hbar^2}\frac{e^{ikq}}{q}\int d^3q' e^{-ik\frac{\mathbf{q}}{q}\cdot\mathbf{q}'}e^{i\mathbf{k}\cdot\mathbf{q}'}V(\mathbf{q}'), \tag{1.38}$$

so that the scattering amplitude is actually the Fourier transform of the potential:

$$f(\mathbf{k}', \mathbf{k}) = -\frac{m}{2\pi\hbar^2}\int d^3q' e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{q}'}V(\mathbf{q}') = -\frac{m}{2\pi\hbar^2}V(\mathbf{k}' - \mathbf{k}). \tag{1.39}$$

1.1.5 Analytical Properties of the Scattering Amplitude

One may consider the scattering amplitude as a function of the energy E of the system in the center of mass reference frame [2]. We consider $V(q)$ vanishing rapidly at infinity as usual and we suppose that the angular momentum of the particle is $l = 0$. Then, the solution to (1.22) is asymptotically

$$u(q) = rR_0(q) = A(E)e^{-\frac{\sqrt{-2\mu E}}{\hbar}q} + B(E)e^{\frac{\sqrt{-2mE}}{\hbar}q}, \tag{1.40}$$

which we can analytically continue by thinking of E as a complex variable z. If $E < 0$, the exponentials are real, and so have to be $A(E)$ and $B(E)$. This means that

$$A(z^*) = A^*(z), \qquad B(z^*) = B^*(z). \tag{1.41}$$

The function $\sqrt{-z}$ is not single valued, but the function $u(q)$ has to be. To solve the problem we introduce a branch cut on the energy complex plane, starting at the origin and running through the positive real axis up to infinity, making $\sqrt{-z}$ single valued. This way on the upper half of the plane

$$u(q) = A(z)e^{ikq} + B(z)e^{-ikq}, \tag{1.42}$$

meaning that we impose $\sqrt{-z} = -i\sqrt{z}$, while on the lower half

$$u(q) = A^*(z)e^{-ikq} + B^*(z)e^{ikq}, \tag{1.43}$$

so that $\sqrt{-z} = i\sqrt{z}$. In this way we have that everywhere in the Riemann sheet $Re\{\sqrt{-z}\} > 0$ and that the square root of $-z$ is indeed single valued. For $u(q)$ to be single valued, though, we also need the condition

$$A(z) = B^*(z), \tag{1.44}$$

making $u(q)$ real, too, as it should be. The complex plane cut in this way is referred as the physical Riemann sheet. The functions $A(z)$ and $B(z)$ are regular everywhere on the physical sheet except on the branch point $z = 0$, since $u(q)$ is a solution of an equation with finite coefficients. Since everywhere on the physical sheet $Re\{\sqrt{-z}\} > 0$, the first term in (1.40) decreases expronentially for $Re\{z\} < 0$, while the second term increases exponentially, meaning that the two terms have different orders of magnitude and such expression would not be legitimate as the asymptotic form of the wave function, as the small term is clearly neglegible with respect to the second one. The ratio of the small and large terms must not be less than the relative order of magnitude of the potential energy V/E, which is actually neglected in the asymptotic region. In other words the potential must descease faster than $e^{-\frac{2\sqrt{2m}}{\hbar}qRe\{\sqrt{-z}\}}$ as $q \to +\infty$. In such situation (1.40) is valid everywhere on the physical sheet, otherwise it is only for $Re\{z\} > 0$.

An important point is that the bound states of a particle in the field $V(q)$ are wavefunctions that vanish at $q \to +\infty$, so that $B(E) = 0$ in those cases. This means that $B(z)$ has zeroes on the discrete energy levels of the system, and that all the zeros of $B(z)$ are for real z.

Now, if $E > 0$, recalling (1.23),

$$u(q) = constant\big(e^{i(kq+\delta_0)} - e^{-i(kq+\delta_0)}\big), \tag{1.45}$$

which compared to (1.43) yields the relation

$$-\frac{A(z)}{B(z)} = e^{2i\delta_0}, \tag{1.46}$$

so that according to (1.32)

$$f_0 = \frac{\hbar}{2\sqrt{-2\mu E}}\Big(\frac{A}{B} + 1\Big). \tag{1.47}$$

Now, by analytically continuing f_0 to the physical sheet, one realizes that its poles correspond to the discrete energy levels of the system, which correrspond to bound states, and that it has no other singular points.

1.1.6 Scattering of Slow Particles

We consider particles with small velocities, so that their wavelength is large compared to the range of interaction a of the potential, meaning that $ka \ll 1$, and that the kinetic energy of the particle is small compared to the field within the radius a, as in the case of ultracold atomic gases.

It can be shown that the phases δ_l of the solution for such a problem are proportional to k^2l, so that the partial amplitudes with $l \neq 0$ can be neglected in the sum (1.31), leading to the so called *s-wave scattering*. By using equation (1.32) one finds that

$$f(\mathbf{k}', \mathbf{k}) \approx f_0 = \frac{1}{kcot(\delta_0(k)) - ik}, \tag{1.48}$$

which is spherically symmetric, meaning that scattering at low energies is isotropic. By defining

$$\lim_{k\to 0} \frac{\delta_0(k)}{k} = -\frac{1}{a_F}, \tag{1.49}$$

one may expand the cotangent at the denominator of (1.40) and obtain

$$f(\mathbf{k}', \mathbf{k}) \approx -\frac{a_F}{1 - \frac{a_F}{2} q_{eff} k^2 + i a_F k}, \tag{1.50}$$

which can be written in terms of small energies by recalling that $k^2 = \frac{2\mu}{\hbar^2} E$. By analytical continuation, letting the energy E be a complex number z, one may see that

$$f(\mathbf{k}', \mathbf{k}) \approx -\frac{a_F}{1 - a_F q_{eff} \frac{\mu}{\hbar^2} z + i a_F \sqrt{\frac{2\mu}{\hbar^2} z}}, \tag{1.51}$$

meaning that the scattering amplitude has a pole at the energy

$$E_m = -\frac{\hbar^2}{m a_F^2}. \tag{1.52}$$

Such pole signals the presence of a two body bound state with a small binding energy E_m according to the last point of the discussion made in section 1.1.5.

1.1.7 Scattering from a Square Well Potential

Consider low energy particles, so that the s-wave scattering approximation can be used, in a square well potential of the form

$$V(q) = \begin{cases} V_0 \text{ if } q < a \\ 0 \text{ otherwise} \end{cases}. \tag{1.53}$$

The potential is spherically symmetric and we only consider solutions of the Schrodinger equation with $l = 0$, so that the general solution of the problem for $q \to +\infty$ is of the kind (1.40). The equation of motion is

$$\left[\frac{d^2}{dq^2} - \frac{2\mu}{\hbar^2} V(q) + \frac{2\mu}{\hbar^2} E\right] u(q) = 0, \tag{1.54}$$

where $u(q) = q R_0(q)$ as usual. The general solution can be written as

$$u(q) = \begin{cases} A e^{ikq} + B e^{-ikq} \text{ for } q > a \\ C e^{iKq} + D e^{-iKq} \text{ for } q < a \end{cases}, \tag{1.55}$$

with $k = \frac{\sqrt{2\mu E}}{\hbar}$ and $K = \frac{\sqrt{2\mu}}{\hbar} \sqrt{E - V_0}$. We impose the boundary condition that $u(0) = 0$, so that $C = -D$ and use equation (1.46) to show that

$$e^{2i\delta_0} = -\frac{C}{D}. \tag{1.56}$$

Finally, we impose boundary conditions, letting the two functions join smoothly at a, meaning that they must have the same value and the same derivative at a. From these conditions we find that

$$\delta_0(k) = -ka + \tan^{-1}\left[\frac{k}{K}\tan(Ka)\right]. \tag{1.57}$$

If one lets $V_0 \to +\infty$, the scattering length just takes the form

$$a_F = a, \tag{1.58}$$

the range of interaction of the potential. For low energies and long wavelengths, the details of a short ranged potential are not probed, and are unimportant to the treatment. One is allowed to model any short ranged potential as a hard core potential of range a_F.

2 Mean Field Analysis of the BCS-BEC Crossover

In this chapter we seek a mean field description of a gas of ultracold fermionic atoms whose interaction is modeled with an attractive contact potential of value $-g$, with $g > 0$. Since the gas is very dilute, the most relevant contributions come from two body interactions, so that the second quantized Hamilton operator can be written in the form of the BCS Hamiltonian

$$\hat{H} = \int_V d^3q \Big[\hat{a}^\dagger_\sigma(\mathbf{q})\Big(-\frac{\nabla^2}{2m}\Big)\hat{a}_\sigma(\mathbf{q}) - g\hat{a}^\dagger_\uparrow(\mathbf{q})\hat{a}^\dagger_\downarrow(\mathbf{q})\hat{a}_\downarrow(\mathbf{q})\hat{a}_\uparrow(\mathbf{q})\Big], \tag{2.1}$$

where V is the volume of the system, $\hbar = 1$, $\mathbf{q}$ is the coordinate of the particles, σ is the spin index; the operators $\hat{a}_\sigma(\mathbf{q})$ and $\hat{a}^\dagger_\sigma(\mathbf{q})$ are respectively the fermionic annihilation and creation operators; the sum over repeated indeces is implicit.

The partition function at equilibrium, using the coherent state path integral formalism [5], then, is of the form

$$Z = \int D[\bar{\psi}_\sigma(\mathbf{q},\tau)\psi_\sigma(\mathbf{q},\tau)]e^{-S[\bar{\psi}_\sigma(\mathbf{q},\tau)\psi_\sigma(\mathbf{q},\tau)]}, \tag{2.2}$$

in which the action $S[\bar{\psi}_\sigma(\mathbf{q},\tau)\psi_\sigma(\mathbf{q},\tau)]$ reads

$$\begin{gathered} S[\bar{\psi}_\sigma\psi_\sigma] = \\ \int_0^\beta d\tau \int_V d^3q \Big[\bar{\psi}_\sigma(\mathbf{q},\tau)\Big(\partial_\tau - \frac{\nabla^2}{2m} - \mu\Big)\psi_\sigma(\mathbf{q}\tau) - g\bar{\psi}_\uparrow(\mathbf{q},\tau)\bar{\psi}_\downarrow(\mathbf{q},\tau)\psi_\downarrow(\mathbf{q},\tau)\psi_\uparrow(\mathbf{q},\tau)\Big], \end{gathered} \tag{2.3}$$

where $\beta = \frac{1}{k_B T}$, with k_B the Boltzmann constant, T the absolute temperature, $\psi_\sigma(\mathbf{q},\tau)$ is a time, position and spin dependent Grassmann field and μ is the chemical potential of the system. Our objective is to study this system while varying the value of the potential g, which can be tuned by changing the scattering length a_F of the two body scattering events. Such quantity can be controlled continuously and arbitrarily thanks to the use of Feshback resonances [1].

2.1 Gap and Number Equations

The aim of this section is to derive the mean field gap and number equations in order to relate the macroscopic quantities of the system. Such equations contain most of the Physics of interest, and already at the mean field level allow a glimpse of the superconducting and superfluid nature of our gas in the different regimes of the crossover.

2.1.1 Hubbard-Stratonovich Transformation

The treatment of the partition function (2.2) will not differ much from the standard BCS one, as what we want to investigate is the behaviour of the Cooper pairs formed below the critical temperature T_c, which, as the interaction gets stronger, will have a shorter and shorter correlation length, until the formation of actual bosonic molecules occurs [4]. We know, then, that to investigate the behaviour of Cooper pairs, we may introduce the spinless bosonic field $\Delta(\mathbf{q},\tau)$ in the partition function via a Hubbard-Stratonovich transformation. We then separate the interaction using the Cooper channel and obtain the new action

$$\begin{aligned} S[\bar{\Delta}\Delta\bar{\psi}\psi] = \\ = \int_0^\beta d\tau \int_V d^3q \Big[\frac{|\Delta(\mathbf{q},\tau)|^2}{g} - \Delta(\mathbf{q},\tau)\bar{\psi}_\uparrow(\mathbf{q},\tau)\bar{\psi}_\downarrow(\mathbf{q},\tau) - \bar{\Delta}(\mathbf{q},\tau)\psi_\downarrow(\mathbf{q},\tau)\psi_\uparrow(\mathbf{q},\tau) + \\ + \bar{\psi}_\sigma(\mathbf{q},\tau)\Big(\partial_\tau - \frac{\nabla^2}{2m} - \mu\Big)\psi_\sigma(\mathbf{q},\tau)\Big]. \end{aligned} \tag{2.4}$$

Now that the quartic term in the fermionic fields is decoupled one can introduce the Nambu spinors

$$\bar{\Psi}(\mathbf{q},\tau) = \begin{pmatrix}\bar{\psi}_\uparrow(\mathbf{q},\tau) & \psi_\downarrow(\mathbf{q},\tau)\end{pmatrix}, \qquad \Psi(\mathbf{q},\tau) = \begin{pmatrix}\psi_\uparrow(\mathbf{q},\tau) \\ \bar{\psi}_\downarrow(\mathbf{q},\tau)\end{pmatrix}, \tag{2.5}$$

and rewrite the action as

$$S[\bar{\Delta}\Delta\bar{\psi}\psi] = \int_0^\beta d\tau \int_V d^3q \Big[\frac{|\Delta(\mathbf{q},\tau)|^2}{g} - \bar{\Psi}(\mathbf{q},\tau)G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta)\Psi(\mathbf{q},\tau)\Big] + \beta\sum_{\mathbf{p}}\xi_{\mathbf{p}}, \tag{2.6}$$

with $\xi_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \mu$ being the free particle energy, while the inverse fermionic propagator is

$$G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta) = \begin{pmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu & \Delta(\mathbf{q},\tau) \\ \bar{\Delta}(\mathbf{q},\tau) & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{pmatrix}. \tag{2.7}$$

The last term comes from the fact that the fermionic fields anticommute. Such term yields a constant, and can then be taken out of the path integral. Its role will come into play in the calculation of the number equation.

The path integral over the fermionic variables can then be solved, yielding the determinant of $G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta)$. We then managed to write a new effective theory for the field $\Delta(\mathbf{q},\tau)$, whose action reads

$$\boxed{S^{eff}[\bar{\Delta}\Delta] = \int_0^\beta d\tau \int_V d^3q \frac{|\Delta(\mathbf{q},\tau)|^2}{g} - Tr[\ln(G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta))] + \beta\sum_{\mathbf{p}}\xi_{\mathbf{p}}}\,. \tag{2.8}$$

2.1.2 Derivation of the Gap and Number Equations

In the spirit of mean field, to treat the path integral we use the saddle point approximation imposing $\Delta(\mathbf{q},\tau) = \Delta_0$ to be a homogeneous field in space and time. To find the value that minimizes (2.8) one may compute S_0, the effective action calculated at Δ_0, and derive it with respect to Δ_0, imposing the condition

$$\frac{\partial S_0}{\partial \Delta_0} = 0. \tag{2.9}$$

First of all, then, we consider the action (2.8) calculated at $\Delta(\mathbf{q},\tau)=\Delta_0$,

$$S_0=\beta V\frac{|\Delta_0|^2}{g}-Tr[\ln G^{-1}_{KP}(\bar{\Delta}_0,\Delta_0)], \tag{2.10}$$

where $G^{-1}_{KP}(\bar{\Delta}_0,\Delta_0)$ is the inverse fermionic propagator written in Matzubara representation,

$$G^{-1}_{KP}(\bar{\Delta}_0,\Delta_0)=\begin{pmatrix}(i\Omega^F_n-\xi_{\mathbf{p}})\delta^{(4)}_{P,K} & \Delta_0\delta^{(4)}_{K,-P}\\ \Delta_0\delta^{(4)}_{K,-P} & (i\Omega^F_n+\xi_{\mathbf{p}})\delta^{(4)}_{P,K}\end{pmatrix}, \tag{2.11}$$

where by P we mean the four vector $P=\big(i\Omega^F_n,\mathbf{p}\big)$, with

$$\Omega^F_n=\frac{(2n+1)\pi}{\beta},\quad n\in\mathbb{Z}, \tag{2.12}$$

the fermionic Matzubara frequencies, and $\xi_{\mathbf{p}}=\frac{\mathbf{p}^2}{2m}-\mu$ is the free particle energy.

The isolated zeros of the inverse propagator correspond to the discrete energy levels of the fermions of the theory. In particular, by imposing that the determinant of (2.11) is null when the field is homogeneous and by performing a Wick rotation to real time $t=-i\tau$ corresponding to $\omega=i\Omega^F_n$, we find that the excitation energies of the fermions are

$$\boxed{\omega=\sqrt{\xi^2_{\mathbf{p}}+|\Delta_0|^2},} \tag{2.13}$$

which is the spectrum of a free particle plus an energy gap $|\Delta_0|$, interpreted as the energy necessary to break a Cooper pair.

By deriving (2.10), then,

$$\frac{\partial S_0}{\partial\Delta_0}=\beta V\frac{\bar{\Delta}_0}{g}-Tr\Big[G_{KP}(\bar{\Delta}_0,\Delta_0)\begin{pmatrix}0&1\\0&0\end{pmatrix}\Big], \tag{2.14}$$

where in Matzubara representation

$$G_{KP}(\bar{\Delta}_0,\Delta_0)=-\frac{1}{(\Omega^F_n)^2+\xi^2_{\mathbf{p}}+|\Delta_0|^2}\begin{pmatrix}(i\Omega^F_n+\xi_{\mathbf{p}})\delta^{(4)}_{P,K} & -\Delta_0\delta^{(4)}_{P,-K}\\ -\bar{\Delta}_0\delta^{(4)}_{P,-K} & (i\Omega^F_n-\xi_{\mathbf{p}})\delta^{(4)}_{P,K}\end{pmatrix}. \tag{2.15}$$

Given these considerations, equation (2.9), which is the familiar *gap equation*, reads

$$\frac{1}{g}=\frac{k_BT}{V}\sum_p\frac{1}{(\Omega^F_n)^2+\xi^2_{\mathbf{p}}+|\Delta_0|^2}. \tag{2.16}$$

Such relation depends on both $|\Delta_0|$ and μ, which are unknowns for the interacting system, since for $g\neq 0$ the chemical potential $\mu\neq\omega_F$, the Fermi energy $\omega_F=\frac{(3\pi^2n)^{\frac{2}{3}}}{2m}$.

Despite the fact that we are working in the grand canonical ensamble for convenience, we want to provide a model for an experiment in which the number of particles N is fixed.

Since we will be working in the thermodynamic limit, though, the number of particles and the volume V will take infinite values, while the density of particles $n = \frac{N}{V}$ will be kept fixed, yielding results analogous to the ones obtained in the canonical ensemble. Given the mean field partition function that we found in our treatment,

$$Z_{MF} = e^{-S_0}, \tag{2.17}$$

the average number of particles can be calculated as

$$n = \frac{k_B T}{V} \partial_\mu \ln(Z_{MF}) = \sum_{\mathbf{p}} 1 - \partial_\mu \frac{|\Delta_0|^2}{g} + \frac{k_B T}{V} Tr\Big[G_{KP} \begin{pmatrix} 1 & \partial_\mu \Delta_0 \\ \partial_\mu \bar{\Delta}_0 & -1 \end{pmatrix} \Big], \tag{2.18}$$

which recalling (2.15) reads

$$\begin{aligned} n = \sum_{\mathbf{p}} 1 - \frac{\partial_\mu |\Delta_0|^2}{g} - \frac{k_B T}{V} \sum_P \frac{2\xi_{\mathbf{p}} - \partial_\mu |\Delta_0|^2}{(\Omega_n^F)^2 + \xi_{\mathbf{p}}^2 + |\Delta_0|^2} = \\ = \sum_{\mathbf{p}} 1 - \frac{k_B T}{V} \sum_P \frac{2\xi_{\mathbf{p}}}{(\Omega_n^F)^2 + \xi_{\mathbf{p}}^2 + |\Delta_0|^2}, \end{aligned} \tag{2.19}$$

in which the last equality holds thanks to the gap equation (2.16). It is indeed very lucky that the terms proportional to the derivative of the energy gap cancel out at equilibrium.

By solving (2.16) and (2.19) in a consistent way, one may get the mean field values of the gap Δ_0 and of the chemical potential of the system μ. Actually, the gap equation written in such a way diverges in the ultraviolet and needs regularization. The divergence comes from the contact potential approximation, and can be fixed by improving it with the introduction of its definition in terms of the scattering length a_F, as will be shown. Indeed, by performing the sum over Matzubara frequencies of the gap equation and the number equation one gets the system

$$\boxed{\begin{cases} \frac{1}{g} = \frac{1}{2V} \sum_{\mathbf{p}} \frac{\tanh\left(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\right)}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} \\ n = \frac{1}{V} \sum_{\mathbf{p}} \left[1 - \frac{\xi_{\mathbf{p}}}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} \tanh\left(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\right)\right] \end{cases}}\,. \tag{2.20}$$

In appendix A one can find the calculation of the mean field grand canonical potential, from which one may derive the number and gap equations directly from its minimization and its derivative with respect to μ. We now investigate the behaviour of the system at $T \to 0^+$ and $T = T_c$, the critical temperature.

2.2 Critical Temperature

The critical temperature of the system T_c is the lowest temperature at which the energy gap becomes $|\Delta_0(T_c)| = 0$. In fact, $|\Delta_0| = 0$ is always a solution of (2.9), meaning that it always extremizes the grand canonical potential, but it corresponds to a maximum only for temperatures below T_c. Below the critical temperature the spontaneous symmetry breaking of the $U(1)$ gauge symmetry of the theory occurs, and the field Δ_0 that minimizes the grand potential becomes non zero, solving the gap equation. In this section we investigate the behaviour of the critical temperature of the system in the different regimes that depend on the strength of g.

2.2.1 Analytical Results

To find an expression for the critical temperature one may simply impose the condition $|\Delta_0| = 0$ in the system of equations (2.20). First, though, one would like to get rid of the ultraviolet divergence present in the gap equation. The divergence comes from the contact potential approximation, and may be cured by regularizing it, defining it in terms of the scattering length a_F of the system:

$$\boxed{\frac{1}{g} = -\frac{m}{4\pi a_F} + \frac{1}{V}\sum_{\mathbf{p}} \frac{m}{\mathbf{p}^2},} \tag{2.21}$$

where the second term is indeed divergent and happens to cancel exactly the divergence of the gap equation.

The system of equations then becomes

$$\begin{cases} -\frac{m}{4\pi a_F} = \frac{1}{V}\sum_{\mathbf{p}}\left[\frac{\tanh\left(\frac{\beta_c}{2}\xi_{\mathbf{p}}\right)}{2\xi_{\mathbf{p}}} - \frac{m}{\mathbf{p}^2}\right] \\ n = \frac{1}{V}\sum_{\mathbf{p}}\left[1 - \tanh\left(\frac{\beta_c}{2}\xi_{\mathbf{p}}\right)\right] \end{cases}. \tag{2.22}$$

From these equations one may obtain analytical results with respect to the expressions of the critical temperature and of the chemical potential in the BCS and deep BEC limit.

First, we turn the sums into integrals in polar coordinates with the usual prescription $\frac{1}{V}\sum_{\mathbf{p}} \to \frac{1}{(2\pi)^3}\int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta \int_0^{+\infty} dpp^2$, then introduce a unit integral $\int_{-\infty}^{+\infty} d\varepsilon\delta(\frac{p^2}{2m} - \varepsilon)$ and integrate over the momentum p, with the overall effect of a change of variables yielding

$$\begin{cases} -\frac{m}{4\pi a_F} = \frac{m^{\frac{3}{2}}}{\sqrt{2}\pi^2}\int_0^{+\infty} d\varepsilon\sqrt{\varepsilon}\left[\frac{\tanh\left(\frac{\beta_c}{2}(\varepsilon-\mu)\right)}{2(\varepsilon-\mu)} - \frac{1}{2\varepsilon}\right] \\ n = \frac{m^{\frac{3}{2}}}{\sqrt{2}\pi^2}\int_0^{+\infty} d\varepsilon\sqrt{\varepsilon}\left[1 - \tanh\left(\frac{\beta_c}{2}(\varepsilon-\mu)\right)\right] \end{cases}. \tag{2.23}$$

The first integral has an analytical solution when $\mu > 0$, and in fact by substituting $\varepsilon/\mu = z$ one gets the equation

$$-\frac{1}{a_F} = \frac{\sqrt{8m\mu}}{\pi}\int_0^{+\infty} dz\sqrt{z}\left[\frac{\tanh\left(\frac{\beta_c\mu}{2}(z-1)\right)}{2(z-1)} - \frac{1}{2z}\right], \tag{2.24}$$

so that

$$-\frac{1}{a_F} = \frac{\sqrt{8m\mu}}{\pi}\ln\left(\frac{8\gamma}{\pi e^2 K_B T_c}\right), \tag{2.25}$$

with γ the Euler-Mascheroni constant and e the Euler constant [4]. The critical temperature can then be extracted and reads

$$\boxed{k_B T_c = \frac{8\gamma}{\pi e^2}\mu e^{\frac{\pi}{\sqrt{8m\mu}a_F}}}\,. \tag{2.26}$$

We have a relation between the critical temperature and the chemical potential in all the cases in which $\mu > 0$.

In particular we have that in the BCS regime, in which $\frac{1}{a_F} \to -\infty$ and the interaction is very weak, the critical temperature is much smaller than the chemical potential, meaning that we can approximate μ with the Fermi energy ω_F, as predicted in the plot in figure 4.

In the BEC regime, instead, we expect a bound state to develop and the chemical potential to change sign, with the realization of a bosonic system. Considering the critical temperature to be $T_c \ll |\mu|$ the hyperbolic tangent can be set to unity, so that one may exploit the fact that

$$\int_0^{+\infty} dz\sqrt{z}\Big[\frac{1}{2(z-1)} - \frac{1}{z}\Big] = -\frac{\pi}{2}, \tag{2.27}$$

to rewrite the gap equation as

$$-\frac{m}{4\pi a_F} = -\frac{m^{\frac{3}{2}}\sqrt{\mu}}{2\sqrt{2}\pi}, \tag{2.28}$$

yielding

$$\boxed{\mu = \frac{1}{2ma_F^2}}, \tag{2.29}$$

which corresponds to half of the predicted energy of the new molecular bound state, as stated in equation (1.52). By substituting (2.29) in the number equation in (2.23) while expanding the hyperbolic tangent around $T_c \to 0^+$ one gets

$$n \approx 2\frac{m^{\frac{3}{2}}}{\sqrt{2}\pi^2}\int_0^{+\infty} d\varepsilon\sqrt{\varepsilon}e^{-\frac{\varepsilon-\mu}{k_B T_c}}, \tag{2.30}$$

so that the critical temperature should be

$$\boxed{T_c \approx \frac{1}{4ma_F^2 \ln\left(\frac{1}{2ma_F^2\omega_F}\right)^{\frac{3}{2}}}}\,. \tag{2.31}$$

This result is not consistent with the previous approximations we made, as the critical temperature turns out to be divergent in the strong coupling limit, but can still be interpreted physically as the dissociation temperature of the molecules. To fix this inconsistency one should go beyond the mean field analysis and consider quantum fluctuations around the saddle point of the theory [10].

2.2.2 Numerical Treatment

The unknowns in the system of equations (2.22) are the critical temperature and the chemical potential. We are interested in studying the behaviour of the critical temperature with respect to the variation of the scattering length a_F along the whole crossover. In order to do so, we procede in a similar fashion as done in [9] for the $T \to 0^+$ case by introducing the dimensionless variables

$$x^2 = \frac{\beta_c p^2}{2m}, \qquad z_0 = \beta_c \mu, \tag{2.32}$$

and getting rid of the divergences by integrating by parts, thereby obtaining the expressions

$$\begin{cases} \frac{1}{a_F} = \frac{4}{\pi}\sqrt{2mk_BT_c}\int_0^{+\infty} dx \left[\frac{x^4}{2(x^2-z_0)\cosh^2\left[\frac{1}{2}(x^2-z_0)\right]} - z_0\frac{x^2\tanh\left(\frac{1}{2}(x^2-z_0)\right)}{(x^2-z_0)^2}\right] \\ n = \frac{(2mk_BT_c)^{\frac{3}{2}}}{6\pi^2}\int_0^{+\infty} dx \frac{x^4}{\cosh^2\left[\frac{1}{2}(x^2-z_0)\right]} \end{cases}, \tag{2.33}$$

which we recast by writing

$$\begin{cases} \frac{1}{a_F} = \frac{4}{\pi}\sqrt{2mk_BT_c} I_3(z_0) \\ n = \frac{(2mk_BT_c)^{\frac{3}{2}}}{6\pi^2} I_4(z_0) \end{cases}. \tag{2.34}$$

Both integrals depend on only one parameter. In particular, from the number equation we can recover the expression for the Fermi energy $\omega_F = \frac{(3\pi^2 n)^{\frac{2}{3}}}{2m}$, from which one gets that

$$\boxed{\frac{k_BT_c}{\omega_F} = \left(\frac{2}{I_4(z_0)}\right)^{\frac{2}{3}}, \quad \frac{\mu}{\omega_F} = \frac{\mu}{k_BT_c}\frac{k_BT_c}{\omega_F} = z_0\left(\frac{2}{I_4(z_0)}\right)^{\frac{2}{3}}, \quad \frac{1}{k_Fa_F} = \frac{4}{\pi}\frac{2^{\frac{1}{3}} I_3(z_0)}{I_4(z_0)^{\frac{1}{3}}}.} \tag{2.35}$$

The plot we obtain is shown below, and shows how the crcitical temperature increases as the scattering length goes to $\frac{1}{a_F} \to +\infty$ as predicted by (2.31).

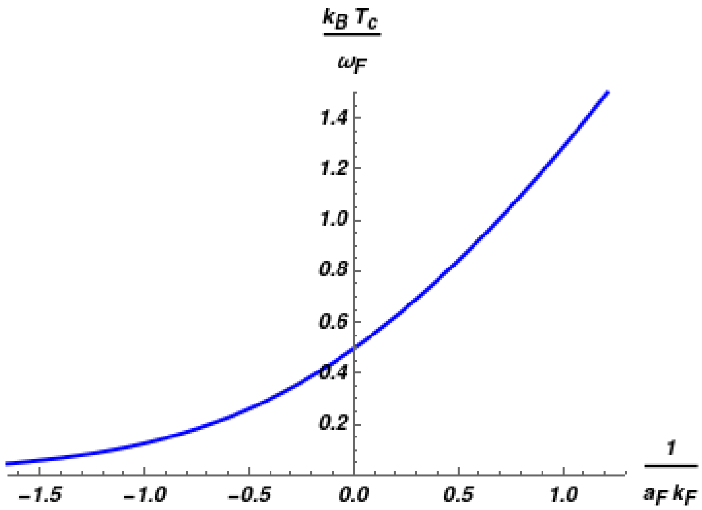

Figure 1: Mean field critical temperature *vs.* inverse scattering length over the whole crossover.

2.3 $T \to 0^+$ Limit

In this section we show the behaviour of some of the most significant physical quantities of the system at zero temperature, among which the variation of the energy gap, the chemical potential and the condensed fraction of Cooper pairs along the whole crossover. Most of the results will be achieved numerically, even though some analytical results can be extracted.

2.3.1 Numerical Solutions for the Gap and Number Equations

At zero temperature the system of equations (2.20) can be reduced to two one dimensional integrals that can be written as linear combinations of elliptic integrals as shown in [9], thanks to the fact that the hyperbolic tangent at the numerator becomes unity. In order to reproduce such result, though, we have to eliminate the ultraviolet divergence present in the gap equation, which using (2.21) will read

$$-\frac{m}{4\pi a_F} = \frac{1}{V}\sum_{\mathbf{p}}\Big[\frac{1}{2\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} - \frac{m}{\mathbf{p}^2}\Big]. \tag{2.36}$$

One may start the treatment of the equation by transforming the sum over $\mathbf{p}$ into an integral in polar coordinates with the usual prescription. Then, it is useful to make the integral dimensionless by introducing the new variables

$$x^2 = \frac{p^2}{2m}\frac{1}{|\Delta_0|}, \qquad x_0 = \frac{\mu}{|\Delta_0|}, \tag{2.37}$$

so that one may get the expression for the scattering length

$$-\frac{1}{a_F} = \frac{2}{\pi}(2m|\Delta_0|)^{\frac{1}{2}}\int_0^{+\infty} dx x^2\Big[\frac{1}{[(x^2-x_0)^2+1]^{\frac{1}{2}}} - \frac{1}{x^2}\Big] = \frac{4(2m|\Delta_0|)^{\frac{1}{2}}}{\pi}I_1(x_0). \tag{2.38}$$

In the meantime, using the same substitutions, the number equation can be written as

$$n = \frac{(2m|\Delta_0|)^{\frac{3}{2}}}{4\pi^2}\int_0^{+\infty} dx x^2\Big[1 - \frac{x^2-x_0}{[(x^2-x_0)^2+1]^{\frac{1}{2}}}\Big] = \frac{(2m|\Delta_0|)^{\frac{3}{2}}}{3\pi^2}I_2(x_0). \tag{2.39}$$

One may explicitly eliminate the divergences by integrating by parts $I_1(x_0)$ and $I_2(x_0)$. As far as the first integral is concerned, indeed, the term emerging from parts integration is

$$\lim_{x\to+\infty}\Big[\frac{x^3}{[(x^2-x_0)^2+1]^{\frac{1}{2}}} - x\Big] = 0, \tag{2.40}$$

while for the latter

$$\lim_{x\to+\infty}\Big[\frac{x^3}{3} - \frac{x^3(x^2-x_0)}{3[(x^2-x_0)^2+1]}\Big] = 0, \tag{2.41}$$

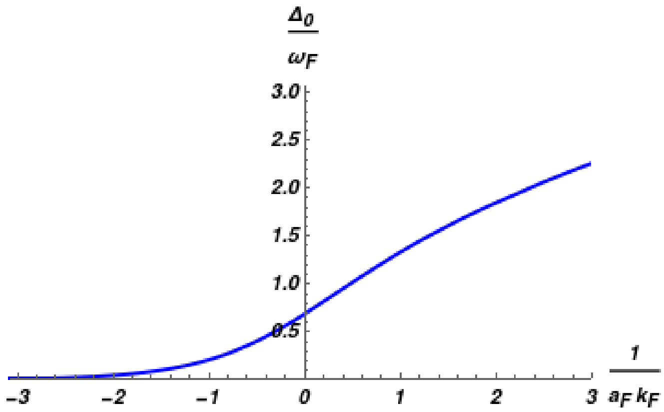

Figure 2: Energy gap *vs.* inverse scattering length along the whole crossover

leaving the system of equations as

$$\begin{cases} -\frac{1}{a_F} = \frac{4(2m|\Delta_0|)^{\frac{1}{2}}}{\pi} \int_0^{+\infty} dx \frac{x_0 x^2 (x^2 - x_0) - x^2}{[(x^2 - x_0)^2 + 1]^{\frac{3}{2}}} \\ n = \frac{(2m|\Delta_0|)^{\frac{3}{2}}}{3\pi^2} \int_0^{+\infty} dx \frac{x^4}{[(x^2 - x_0)^2 + 1]^{\frac{3}{2}}} \end{cases}, \tag{2.42}$$

which is explicitly convergent and treatable numerically. These calculations are shown explicitly, since for the treatment of the Rabi coupled diluted fermion gas the steps will be very similar.

From (2.42) one can write the physical quantities of interest in terms of the integrals $I_1(x_0)$ and $I_2(x_0)$ as

$$\boxed{\frac{|\Delta_0|}{\omega_F} = \frac{1}{I_2(x_0)^{\frac{2}{3}}}, \qquad \frac{\mu}{\omega_F} = \frac{\mu}{|\Delta_0|}\frac{|\Delta_0|}{\omega_F} = \frac{x_0}{I_2(x_0)^{\frac{2}{3}}}, \qquad \frac{1}{k_F a_F} = -\frac{4}{\pi}\frac{I_1(x_0)}{I_2(x_0)^{\frac{1}{3}}},} \tag{2.43}$$

with $\omega_F = \frac{(3\pi^2 n)^{\frac{2}{3}}}{2m}$ the Fermi energy of the system and $k_F = (3\pi^2 n)^{\frac{1}{3}}$ its Fermi momentum. Using these expressions one may plot how the energy gap and the chemical potential vary with the scattering length, keeping the density of particles n fixed. In choosing a given ratio x_0 one may deduce how the energy gap or the chemical potential depend on the free parameter a_F. In other words, by fixing a value of the scatttering length a_F and x_0, both values of $|\Delta_0|$ and μ are automatically fixed by (2.43). The plots describing the behaviour of $|\Delta_0|$ and μ at thermodynamic equilibrium with respect to the change of the scattering length are shown in figures 2 and 3 respectively. It is also interesting to see how the energy gap changes together with the chemical potential, which is shown in figure 4.

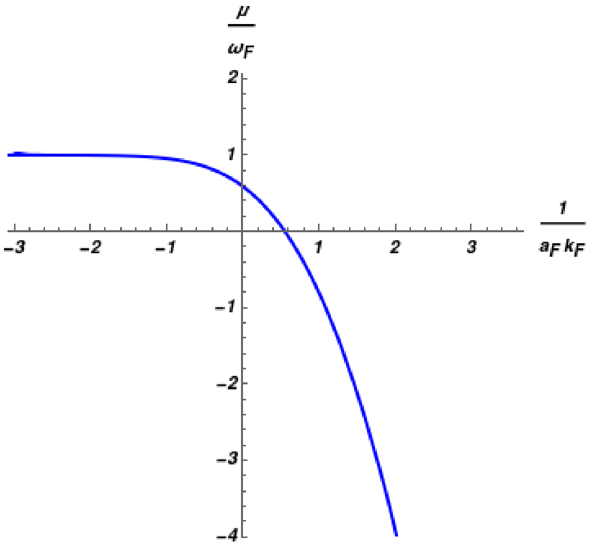

Figure 3: Chemical potential *vs.* inverse scattering length along the whole crossover

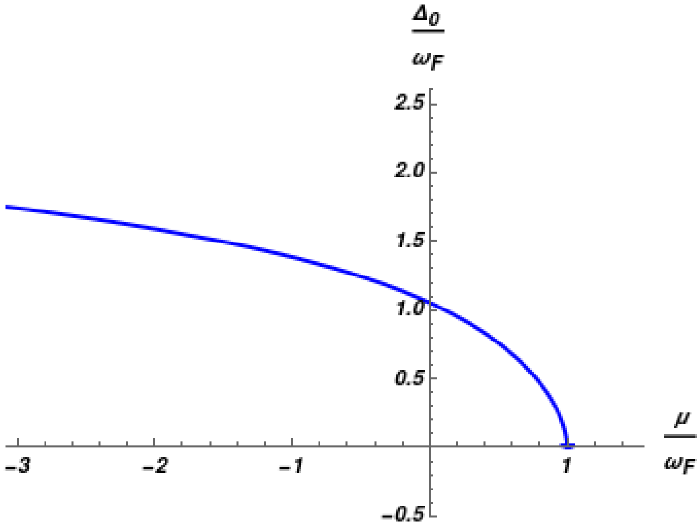

Figure 4: The energy gap as a function of the chemical potential. Notice how as the critical temperature approaches zero the chemical potential converges to the Fermi energy.

2.3.2 Reduced Density Matrices and BEC

The fact that below a critical temperature T_c the value of $|\Delta_0|$ takes non vanishing values signifies the spontaneous symmetry breaking of the $U(1)$ gauge symmetry of the model. The solution $|\Delta_0| = 0$ indeed always satisfies (2.9), minimizing the grand potential above T_c and locally maximizing it below it. Such phenomenon translates to a phase transition with the formation of Cooper pairs, which behave like spinless bosonic particles. Such pairs may undergo Bose-Einstein condensation, and an accessible quantity one may calculate is their condensate fraction. Before tackling the problem of the calculation of the condensate fraction for an interacting Fermi gas, we briefly review the arguments made by Penrose and Onsager in [12] and by Yang in [13]. In particular we will see the connection between BEC and off diagonal long range order (OLDRO).

Let $\hat{\rho}$ be the density matrix of an N particle fermionic system, meaning

$$\hat{\rho} = \sum_{j_1,...,j_N} p_{j_1,...,j_N} \left| j_1, ..., j_N \right\rangle \left\langle j_1, ..., j_N \right|, \tag{2.44}$$

where the index j_k stands for the quantum numbers identifying the $k-th$ particle and $p_{j_1,...,j_N}$ is the probability for the system to be in the pure state $|j_1, ..., j_N\rangle$. In particular, for a system in thermal equilibrium like the one we are studying,

$$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}. \tag{2.45}$$

In this work, we are always working in thermal equilibrium and in the thermodynamic limit, meaning that the particle density $n = \frac{N}{V}$ will be kept fixed, while the number of particles N and the volume V will be considered to be infinite.

We may define the elements of the reduced density matrices $\hat{\rho}_1$ and $\hat{\rho}_2$ as

$$\langle j|\hat{\rho}_1|k\rangle = Tr[\hat{a}_j \hat{\rho} \hat{a}_k^\dagger], \quad \langle jk|\hat{\rho}_2|lm\rangle = Tr[\hat{a}_j \hat{a}_k \hat{\rho} \hat{a}_m^\dagger \hat{a}_l^\dagger], \tag{2.46}$$

where $\hat{a}_j$ and $\hat{a}_j^\dagger$ are respectively the annihilation and creation operators for a single particle with quantum numbers j and by $Tr[\cdot]$ we mean the trace operator. Both the reduced density matrices are positive semidefinite, since they are products of positive semidefinite operators. Moreover, since $Tr[\hat{\rho}] = 1$, $Tr[\hat{\rho}_1] = N$ and $Tr[\hat{\rho}_2] = N(N-1)$, the eigenvalues of $\hat{\rho}_1$ must be smaller than N, while the ones of $\hat{\rho}_2$ have to be smaller than $N(N-1)$. The physical meaning of the elements of such operators can be understood by noticing that

$$\langle j|\hat{\rho}_1|k\rangle = \langle \hat{a}_k^\dagger \hat{a}_j \rangle, \quad \langle jk|\hat{\rho}_2|lm\rangle = \langle \hat{a}_m^\dagger \hat{a}_l^\dagger \hat{a}_j \hat{a}_k \rangle, \tag{2.47}$$

so that they are strictly related to the single and two particle thermal Green's functions of the system.

Both $\hat{\rho}_1$ and $\hat{\rho}_2$ are hermitian, and can thus be diagonalized as

$$\hat{\rho}_1 = \sum_j n_j \left| j \right\rangle \left\langle j \right|, \quad \hat{\rho}_2 = \sum_{j,k} n_{jk} \left| jk \right\rangle \left\langle jk \right|, \tag{2.48}$$

where $|j\rangle$ and $|lm\rangle$ are the eigenstates of $\hat{\rho}_1$ and $\hat{\rho}_2$ respectively, not necessarily corresponding to quantities related to the eigenstates of $\hat{H}$. It is natural, knowing the trace of such operators, to interpret n_j, the eigenvalues of $\hat{\rho}_1$, as the numbers of particles in the single particle state $|j\rangle$ and the eigenvalues of $\hat{\rho}_2$, n_{jk}, as the numbers of pairs in the two particle state $|jk\rangle$. The criterion to have Bose Einstein condensation in the case of a fermionic gas as the one we are studying is that there must be at least an eigenvalue of $\hat{\rho}_2$, which we will call λ_2, of order N, so that the condensate fraction

$$\boxed{n_0 = \frac{\lambda_2}{V}}, \tag{2.49}$$

is finite in the thermodynamic limit, meaning that there is a macroscopic occupation of a state $|jk\rangle$ by Cooper pairs [14]. It is clear that the eigenvalues of $\hat{\rho}_1$ will all be smaller than one because of the Pauli exclusion principle in the case of a Fermi gas, while for a Bose system one may calculate the condensate fraction directly as the largest eigenvalue of $\hat{\rho}_1$. In particular, it is possible to show that for a fermionic system the maximum eigenvalue of $\hat{\rho}_2$ can be at most of order N, and not N^2 [13]. The physical interpretation of the formation of bosonic particles made of fermionic pairs, in this picture, holds.

2.3.3 OLDRO

A different way of facing the question of whether BEC may occur in a Fermi gas is to look at the behaviour of the elements of the reduced density matrix $\hat{\rho}_2$ in coordinate representation

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = Tr[\hat{a}_\downarrow(\mathbf{q}_1')\hat{a}_\uparrow(\mathbf{q}_2')\hat{\rho}\hat{a}_\uparrow^\dagger(\mathbf{q}_2)\hat{a}_\downarrow^\dagger(\mathbf{q}_1)]. \tag{2.50}$$

What Yang shows [13] is that if such elements do not vanish at infinite interparticle distances,

$$\boxed{\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle \neq 0 \text{ for } \begin{cases} |\mathbf{q}_1' - \mathbf{q}_1| \to +\infty \\ |\mathbf{q}_2' - \mathbf{q}_2| \to +\infty \end{cases},} \tag{2.51}$$

the reduced density matrix $\hat{\rho}_2$ will certainly have an eigenvalue of order N in the fermionic case.

The element of $\hat{\rho}_2$ in coordinate representation, using its diagonal form (2.48), reads

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = \sum_{j,k} n_{jk} \langle \mathbf{q}_1'\mathbf{q}_2'|jk\rangle \langle jk|\mathbf{q}_1\mathbf{q}_2\rangle = \sum_{j,k} n_{jk}\psi_{jk}(\mathbf{q}_1', \mathbf{q}_2')\psi_{jk}^*(\mathbf{q}_1, \mathbf{q}_2), \tag{2.52}$$

where $\psi_{jk}(\mathbf{q}_1\mathbf{q}_2)$ is the eigenvector of $\hat{\rho}_2$ relative to the eigenvalue n_{jk} written in coordinate representation.

We now show explicitly that equation (2.51) holds at the mean field level for the theory of action (2.6), taking into account that $|\Delta_0|$ is kept fixed at its saddle point value. The path integration formalism comes in handy in this task, since we are dealing now with a Gaussian theory for the fermionic degrees of freedom. In particular, then, when calculating the elements of $\hat{\rho}_2$ in coordinate representation, one may exploit Wick's theorem as

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = \langle\bar{\psi}_\uparrow(\mathbf{q}_1)\bar{\psi}_\downarrow(\mathbf{q}_2)\rangle\langle\psi_\downarrow(\mathbf{q}_1')\psi_\uparrow(\mathbf{q}_2')\rangle - \langle\psi_\uparrow(\mathbf{q}_2')\bar{\psi}_\uparrow(\mathbf{q}_1)\rangle\langle\bar{\psi}_\downarrow(\mathbf{q}_2)\psi_\downarrow(\mathbf{q}_1')\rangle, \tag{2.53}$$

where the single terms can be calculated using the explicit expression of the fermionic propagator in (2.15), which is

$$G(\bar{\Delta}_0, \Delta_0) = \begin{pmatrix} \langle\psi_\uparrow\bar{\psi}_\uparrow\rangle & \langle\psi_\uparrow\psi_\downarrow\rangle \\ \langle\bar{\psi}_\downarrow\bar{\psi}_\uparrow\rangle & \langle\bar{\psi}_\downarrow\psi_\downarrow\rangle \end{pmatrix}. \tag{2.54}$$

By working in Matzubara representation, then, after the sum over Matzubara frequencies, one gets that

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = \frac{1}{4V^2}\sum_{\mathbf{p},\mathbf{k}}\Big[\frac{|\Delta_0|^2\tanh(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2})\tanh(\frac{\beta}{2}\sqrt{\xi_{\mathbf{k}}^2+|\Delta_0|^2})}{\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\sqrt{\xi_{\mathbf{k}}^2+|\Delta_0|^2}}e^{i\mathbf{p}\cdot(\mathbf{q}_1'-\mathbf{q}_2')}e^{-i\mathbf{k}\cdot(\mathbf{q}_1-\mathbf{q}_2)}+$$
$$+\Big(\frac{\xi_{\mathbf{p}}\xi_{\mathbf{k}}\tanh(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2})\tanh(\frac{\beta}{2}\sqrt{\xi_{\mathbf{k}}^2+|\Delta_0|^2}}{\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\sqrt{\xi_{\mathbf{k}}^2+|\Delta_0|^2}}-1\Big)e^{i\mathbf{p}\cdot(\mathbf{q}_2'-\mathbf{q}_2)}e^{i\mathbf{k}\cdot(\mathbf{q}_1'-\mathbf{q}_1)}\Big], \tag{2.55}$$

where the first term corresponds to $\langle\bar{\psi}_\uparrow(\mathbf{q}_1)\bar{\psi}_\downarrow(\mathbf{q}_2)\rangle\langle\psi_\downarrow(\mathbf{q}_1')\psi_\uparrow(\mathbf{q}_2')\rangle$, while the second to the other term in (2.53). What is interesting about this formula is that it makes it explicit that the second term in the sum in the thermodynamic limit, in which the sum over momenta can be transformed into an integral, will vanish in the limit $\begin{cases}|\mathbf{q}_1'-\mathbf{q}_1|\to+\infty\\|\mathbf{q}_2'-\mathbf{q}_2|\to+\infty\end{cases}$ due to the Riemann-Lebesgue lemma. In the meantime, the other term will survive only below the critical temperature, since it is proportional to $|\Delta_0|^2$. Then, OLDRO will hold below T_c, and the two particle reduced density matrix will factorize as

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = \langle\bar{\psi}_\uparrow(\mathbf{q}_1)\bar{\psi}_\downarrow(\mathbf{q}_2)\rangle\langle\psi_\downarrow(\mathbf{q}_1')\psi_\uparrow(\mathbf{q}_2')\rangle \text{ for } \begin{cases}|\mathbf{q}_1'-\mathbf{q}_1|\to+\infty\\|\mathbf{q}_2'-\mathbf{q}_2|\to+\infty\end{cases}. \tag{2.56}$$

Since OLDRO is present, we can be certain that there will be a non-vanishing condensate fraction below the critical temperature T_c. For completeness, we write (2.56) explicitly, as

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = \frac{1}{4V^2}\sum_{\mathbf{p},\mathbf{k}}\frac{|\Delta_0|^2\tanh(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2})\tanh(\frac{\beta}{2}\sqrt{\xi_{\mathbf{k}}^2+|\Delta_0|^2})}{\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\sqrt{\xi_{\mathbf{k}}^2+|\Delta_0|^2}}e^{i\mathbf{p}\cdot(\mathbf{q}_1'-\mathbf{q}_2')}e^{-i\mathbf{k}\cdot(\mathbf{q}_1-\mathbf{q}_2)}. \tag{2.57}$$

2.3.4 Condensate Fraction in the $T\to 0^+$ Limit

In this section we want to calculate the fraction of condensed Cooper pairs along the whole crossover, giving particular attention to the $T\to 0^+$ case. In particular we can recover an analytical result depending on the chemical potential and the energy gap [15], whose values along the whole crossover are reported in figure 4. We will then be able to construct the plot for the condensed fraction *vs* the scattering length a_F.

As stated in the previous sections, we want to calculate the highest eigenvalue of (2.57) below the critical temperature. Such quantity can be calculated as

$$\boxed{n_0=\frac{1}{V}\int_V d^3q_1\int_V d^3q_2|\langle\psi_\downarrow(\mathbf{q}_1)\psi_\uparrow(\mathbf{q}_2)\rangle|^2}\,. \tag{2.58}$$

By working in Matzubara representation and recalling (2.57), thanks to the orthonormality of plane waves, we see that such expression corresponds to

$$n_0 = \frac{|\Delta_0|^2}{4V} \sum_{\mathbf{p}} \frac{\tanh^2(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2})}{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}. \tag{2.59}$$

In particular, we want to focus on the $T \to 0^+$ limit, where n_0 will take the highest values. In such regime (2.59) reads

$$n_0 = \frac{|\Delta_0|^2}{4V} \sum_{\mathbf{p}} \frac{1}{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} = \frac{|\Delta_0|^2}{8\pi^2} \int_0^{+\infty} dp \frac{p^2}{\xi_p^2 + |\Delta_0|^2}. \tag{2.60}$$

By making the change of variables

$$x^2 = \frac{p^2}{2m} \frac{1}{|\Delta_0|}, \qquad x_0 = \frac{\mu}{|\Delta_0|}, \tag{2.61}$$

one gets that

$$n_0 = \frac{(2m|\Delta_0|^{\frac{3}{2}})}{8\pi^2} \int_0^{+\infty} dx \frac{x^2}{(x^2 - x_0)^2 + 1}, \tag{2.62}$$

which is solvable analytically, and yields

$$\boxed{n_0 = \frac{(m|\Delta_0|)^{\frac{3}{2}}}{8\pi} \sqrt{x_0 + \sqrt{x_0^2 + 1}}}\,. \tag{2.63}$$

Written in a more convenient way, using the number equation in (2.42), we see that

$$\boxed{\frac{n_0}{n/2} = \frac{3\pi}{4\sqrt{8} I_2(x_0)} \sqrt{x_0 + \sqrt{x_0^2 + 1}}}\,, \tag{2.64}$$

meaning that we can use the data obtained in figure 4 to plot such normalized condensed fraction *vs* the scattering length, as shown in figure 5.

Notice that in the BEC regime the fraction approaches unity, meaning that all fermions will be involved in Cooper pairing. In the BCS regime, instead, the fraction will be lower, but still non vanishing for finite falues of $\frac{1}{k_F a_F}$.

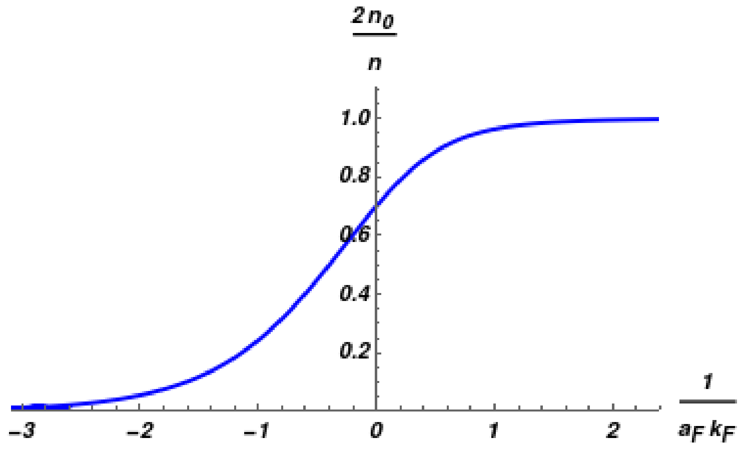

Figure 5: Condensate fraction *vs* inverse scattering length over the whole crossover at $T \to 0^+$.

3 Gaussian Fluctuations in the BCS-BEC Crossover

We now want to improve our approximation with the introuction of Gaussian fluctuations in the partition function. The results we found so far, in fact, may describe well the BCS regime, in which the attraction is very weak, but fail to reproduce the wanted results in the BEC regime, as we saw for example in the calculation of the critical temperature in equation (2.31). Our objective is the calculation of a more refined grand canonical potential in order to establish a more meaningful relation between the chemical potential μ and the number of particles n, by obtaining a new number equation $n = -\partial_\mu \Omega$ as done in [10]. Such relation will enable us to calculate again the physical values of interest.

3.1 General Form of the Grand Canonical Potential

We start this section with the derivation of the grand canonical potential with the inclusion of Gaussian fluctuations. From the effective action

$$S^{eff}[\bar{\Delta}\Delta] = \int_0^\beta d\tau \int_V d^3q \frac{|\Delta(\mathbf{q},\tau)|^2}{g} - Tr[\ln(G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta))] + \beta \sum_{\mathbf{p}} \xi_{\mathbf{p}}, \tag{3.1}$$

with

$$G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta) = \begin{pmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu & \Delta(\mathbf{q},\tau) \\ \bar{\Delta}(\mathbf{q},\tau) & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{pmatrix}, \tag{3.2}$$

one may introduce fluctuations by separating the bosonic field into the homogeneous classical part Δ_0 satisfying the gap equation

$$\frac{1}{g} = \frac{1}{2V} \sum_{\mathbf{p}} \frac{\tanh\left(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\right)}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}}, \tag{3.3}$$

and its fluctuations around it, which will behave as a complex field that we will call $\eta(\mathbf{q},\tau)$,

$$\Delta(\mathbf{q},\tau) = \Delta_0 + \eta(\mathbf{q},\tau). \tag{3.4}$$

Then, the task at hand is to expand $Tr[\ln(G^{-1}_{\mathbf{q},\tau})]$ about $\eta(\mathbf{q},\tau) \to 0$. To do so, one may write the inverse fermionic propagator in Matzubara representation as

$$\begin{aligned} G^{-1}_{KP} = \begin{pmatrix} (i\Omega_n^F - \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & \Delta_0 \delta^{(4)}_{K,-P} \\ \bar{\Delta}_0 \delta^{(4)}_{K,-P} & (i\Omega_n^F + \xi_{\mathbf{p}})\delta^{(4)}_{K,P} \end{pmatrix} + \begin{pmatrix} 0 & \eta_{P+K} \\ \bar{\eta}_{P+K} & 0 \end{pmatrix} = \\ = \tilde{G}^{-1}_{KP} + \eta_{KP}, \end{aligned} \tag{3.5}$$

and express the trace of the logarithm of G^{-1}_{KP} as

$$Tr[\ln G^{-1}_{KP}] \approx Tr[\ln \tilde{G}^{-1}_{kp}] + Tr[\tilde{G}_{KP}\eta_{PK}] - \frac{1}{2} Tr[\tilde{G}_{KP}\eta_{PL}\tilde{G}_{LM}\eta_{MK}]. \tag{3.6}$$

The first term in the expansion contributes to the action at the saddle point, yielding

$$S_0 = \beta V \frac{|\Delta_0|^2}{g} - Tr[\ln G^{-1}_{KP}(\bar{\Delta}_0, \Delta_0)] + \beta \sum_{\mathbf{p}} \xi_{\mathbf{p}}, \tag{3.7}$$

which goes outside the path integral, since it depends only on Δ_0 and corresponds to the mean field theory. The second term ends up cancelling the linear terms in $\eta(\mathbf{q}, \tau)$, leaving us with a purely Gaussian theory. The quadratic term, instead, reads

$$\begin{gathered} \frac{1}{2} Tr[\tilde{G}_{KP} \eta_{PL} \tilde{G}_{LM} \eta_{MK}] = \\ \frac{k_B T}{2V} \sum_{P,K} \frac{\Delta_0^2 \bar{\eta}_{K-P} \bar{\eta}_{K-P} + \bar{\Delta}_0^2 \eta_{K-P} \eta_{P-K} + 2(i\Omega_n^F + \xi_{\mathbf{p}})(i\tilde{\Omega}_{n^F} - \xi_{\mathbf{k}}) \bar{\eta}_{P+K} \eta_{P+K}}{((\Omega_n^F)^2 + \xi_{\mathbf{p}}^2 + |\Delta_0|^2)(\tilde{\Omega}_{n^F}^2 + \xi_{\mathbf{k}}^2 + |\Delta_0|^2)}, \end{gathered} \tag{3.8}$$

with both

$$\Omega_n^F = \frac{(2n+1)\pi}{\beta}, \quad \tilde{\Omega}_{m^F} = \frac{(2m+1)\pi}{\beta}, \quad n, m \in \mathbb{Z}, \tag{3.9}$$

being fermionic Matzubara frequencies. Denoting by K the four vector $(i\Omega_m^B, \mathbf{k})$, with

$$\Omega_m^B = \frac{2m\pi}{\beta}, \quad m \in \mathbb{Z} \tag{3.10}$$

bosonic Matzubara frequencies, the new effective action can be written as

$$\boxed{S_G^{eff}[\bar{\eta}\eta] = S_0 + \sum_K \begin{pmatrix} \bar{\eta}_K & \eta_{-K} \end{pmatrix} M_K \begin{pmatrix} \eta_K \\ \bar{\eta}_{-K} \end{pmatrix}}, \tag{3.11}$$

with the inverse propagator of the bosonic fluctuations being

$$\boxed{M_K = \frac{1}{2g} \mathbb{I}d + \chi_K,} \tag{3.12}$$

where $\mathbb{I}d$ is the 2×2 identity matrix in Nambu space and

$$\chi_K = \frac{k_B T}{2V} \sum_P \begin{pmatrix} \frac{(i\Omega_n^F - \xi_{\mathbf{p}})(i(\Omega_m^B + \Omega_n^F) + \xi_{\mathbf{k}+\mathbf{p}})}{((\Omega_n^F)^2 + \xi_{\mathbf{p}}^2 + |\Delta_0|^2)((\Omega_m^B + \Omega_n^F)^2 + \xi_{\mathbf{k}+\mathbf{p}}^2 + |\Delta_0|^2)} & \frac{\Delta_0^2}{((\Omega_n^F)^2 + \xi_{\mathbf{p}}^2 + |\Delta_0|^2)((\Omega_m^B + \Omega_n^F)^2 + \xi_{\mathbf{k}+\mathbf{p}}^2 + |\Delta_0|^2)} \\ \frac{\bar{\Delta}_0^2}{((\Omega_n^F)^2 + \xi_{\mathbf{p}}^2 + |\Delta_0|^2)((\Omega_m^B + \Omega_n^F)^2 + \xi_{\mathbf{k}+\mathbf{p}}^2 + |\Delta_0|^2)} & \frac{(i\Omega_n^F - \xi_{\mathbf{p}})(i(-\Omega_m^B + \Omega_n^F) + \xi_{-\mathbf{k}+\mathbf{p}})}{((\Omega_n^F)^2 + \xi_{\mathbf{p}}^2 + |\Delta_0|^2)((-\Omega_m^B + \Omega_n^F)^2 + \xi_{-\mathbf{k}+\mathbf{p}}^2 + |\Delta_0|^2)} \end{pmatrix}. \tag{3.13}$$

More explicitly, one may perform the sums over Matzubara frequencies for the elements of χ_K. The calculations are reported in appendix B, and the results are presented below, noticing that $(\chi_K)_{11} = (\chi_{-K})_{22}$ and that $(\chi_K)_{12}$ is the complex conjugate of $(\chi_K)_{21}$,

$$\begin{aligned} (\chi_K)_{11} = \frac{1}{2V} \sum_{\mathbf{p}} \Big[& \frac{(\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} - \xi_{\mathbf{p}})(i\Omega_m^B + \sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} + \xi_{\mathbf{p}+\mathbf{k}}) \tanh\left(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\right)}{2\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\left((\Omega_m^B)^2 + \xi_{\mathbf{p}+\mathbf{k}}^2 - \xi_{\mathbf{p}}^2 - 2i\Omega_m^B\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\right)} + \\ & + \frac{(\sqrt{\xi_{\mathbf{p}+\mathbf{k}}^2 + |\Delta_0|^2} + \xi_{\mathbf{p}+\mathbf{k}})(\sqrt{\xi_{\mathbf{p}+\mathbf{k}}^2 + |\Delta_0|^2} - \xi_{\mathbf{p}} - i\Omega_m^B) \tanh\left(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\right)}{2\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\left((\Omega_m^B)^2 + \xi_{\mathbf{p}+\mathbf{k}}^2 - \xi_{\mathbf{p}}^2 + 2i\Omega_m^B\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\right)} \Big], \end{aligned} \tag{3.14}$$

while

$$(\chi_K)_{12} = \frac{1}{2V}\sum_{\mathbf{p}} \frac{\tanh\left(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}\right)}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}((\Omega_m^B)^2 + \xi_{\mathbf{p}+\mathbf{k}}^2 - \xi_{\mathbf{p}}^2 - 2i\Omega_m^B\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2})}. \tag{3.15}$$

The partition function of the system, since our new theory is Gaussian, reads

$$\boxed{Z_G = \int D[\bar{\eta}\eta] e^{-S_G^{eff}[\bar{\eta}\eta]} = e^{-S_0}\Big[\det\Big(M\Big)\Big]^{-1}}. \tag{3.16}$$

and the corresponding grand canonical potential is

$$\boxed{\frac{\Omega_G}{V} = \frac{k_B T}{V} S_0 + \frac{k_B T}{V} Tr[\ln\Big(M\Big)] = \frac{\Omega_{MF}}{V} + \frac{k_B T}{V} Tr[\ln\Big(M\Big)]}, \tag{3.17}$$

where by M we mean the infinite dimensional matrix whose blocks are M_K and $\frac{k_B T}{V} S_0 = \frac{\Omega_{MF}}{V}$ corresponds to the mean field grand canonical potential. Since M is a block matrix, its determinant may be computed as the product of the determinants of all the M_K.

3.2 Critical Temperature

Now that we have an expression for the effective action with the inclusion of Gaussian fluctuations and a general expression for the partition function and the grand canonical potential, we can restrict ourselves to the critical temperature regime. Our objective is to improve the results obtained in the mean field case, and in particular to understand the behaviour of T_c along the crossover.

3.2.1 M_K at the Critical Temperature

We want to investigate the properties of M_K before deriving a feasible expression for the number equation at the critical temperature. First, we start by noticing that by imposing $\Delta_0 = 0$ the off-diagonal terms vanish since they are proportional to Δ_0^2, so that it is no longer useful to consider M_K in Nambu space. Then, one may write

$$M_K = \frac{1}{g} - \frac{1}{V}\sum_{\mathbf{p}} \frac{\tanh\left(\frac{\beta_c}{2}\xi_{\mathbf{p}}\right)}{\xi_{\mathbf{p}+\mathbf{k}} + \xi_{\mathbf{p}} - i\Omega_m^B}. \tag{3.18}$$

First, we want to eliminate the dependence on mixed $\mathbf{p}$ and $\mathbf{k}$ terms from the denominator, since we want to transform the sum over $\mathbf{p}$ into an integral in polar coordinates. To do so, we substitute $\mathbf{p} \to \mathbf{p} - \frac{\mathbf{k}}{2}$ and $\mathbf{k} \to -\mathbf{k}$, so that

$$M_K = \frac{1}{g} - \frac{1}{V}\sum_{\mathbf{p}} \frac{\tanh\left(\frac{\beta_c}{2}\xi_{\mathbf{p}+\mathbf{k}/2}\right)}{\xi_{\mathbf{p}+\mathbf{k}/2} + \xi_{\mathbf{p}-\mathbf{k}/2} - i\Omega_m^B}, \tag{3.19}$$

meaning that the dependence on the angle is present only in the argument of the hyperbolic tangent: explicitly, by using the usual regularization for the potential,

$$M_K = -\frac{m}{4\pi a_F} - \frac{m}{(2\pi)^2}\int_{-1}^{1} d\cos\theta \int_0^{+\infty} dp p^2 \Big[\frac{\tanh\left(\frac{\beta_c}{2}(\frac{p^2}{2m} + \frac{k^2}{8m} + \frac{pk\cos\theta}{2m} - \mu)\right)}{p^2 + \frac{k^2}{4} - 2m\mu - im\Omega_m^B} - \frac{1}{p^2}\Big], \quad (3.20)$$

where we mean $k = |\mathbf{k}|$.

The integration over $d\cos\theta$ is straight forward, since only the hyperbolic tangent will be affected by it, and

$$\int_{-1}^{1} d\cos\theta \tanh\Big[\frac{\beta_c}{2}(\frac{p^2}{2m} + \frac{k^2}{8m} + \frac{pk\cos\theta}{2m} - \mu)\Big] =$$
$$\frac{4mk_BT_c}{kp}\Big[\ln\Big(\cosh\Big(\frac{\beta_c}{2}\Big[\frac{(p+k/2)^2}{2m} - \mu\Big]\Big)\Big) - \ln\Big(\cosh\Big(\frac{\beta_c}{2}\Big[\frac{(p-k/2)^2}{2m} - \mu\Big]\Big)\Big)\Big]. \quad (3.21)$$

We may make the integral adimensional by performing the changes of variables

$$\boxed{\beta_c\Omega_n^B = \omega, \quad z = \beta_c\frac{\mathbf{k}^2}{2m}, \quad z_0 = \beta_c\mu.} \quad (3.22)$$

so that finally, calling

$$\boxed{A(x,z) = \ln\Big[\cosh\Big(\frac{1}{2}\Big[(x+\frac{z}{2})^2 - z_0\Big]\Big)\Big],} \quad (3.23)$$

one gets that the inverse propagator M_K has the form

$$\boxed{M_K = -\frac{m}{4\pi}\Big[\frac{1}{a_F} + \frac{2\sqrt{2mk_BT_c}}{z\pi}\int_0^{+\infty} dx\Big[\frac{x[A(x,z) - A(x,-z)]}{x^2 + \frac{z^2}{4} - z_0 - i\frac{\omega}{2}} - z\Big].} \quad (3.24)$$

It is straight forward to check that the integrand vanishes at infinity, since

$$\lim_{x\to+\infty}\Big[\ln\Big(\cosh\Big[\frac{1}{2}\Big((x+\frac{z}{2})^2 - z_0\Big)\Big]\Big) - \ln\Big(\cosh\Big[\frac{1}{2}\Big((x-\frac{z}{2})^2 - z_0\Big)\Big]\Big)\Big] = \lim_{x\to+\infty} xz. \quad (3.25)$$

We make the remark that for $\mathbf{k} = 0$ and $\Omega_n^B = 0$ the matrix element M_K vanishes, as it takes exactly the form of the gap equation (3.3) calculated at the critical temperature. This means that the theory is actually divergent and that the grand potential will experience a divergence at the critical temperature, as expected. This problem can be avoided by considering $\eta_0 = 0$ as a classical field, so that it can be taken out of the path integral, making the theory convergent. Such procedure is physically significant, similar to the one carried out in the treatment of a non interacting Bose gas experiencing BEC [4]. Notice also that if $\Omega_n^B = 0$, the imaginary part of M_K vanishes exactly for any value of $\mathbf{k}$.

Moreover, one can see that $M_K = M_{-K}^*$ from (3.24), ensuring that the partition function and the grand potential will be real.

3.2.2 Number and Gap Equations at the Critical Temperature

At the critical temperature, given the above considerations, the grand canonical potential (3.17) reduces to

$$\Omega_G - \Omega_{MF} = \frac{k_B T_c}{V} \sum_{K \neq 0} \ln(M_K), \tag{3.26}$$

with M_K being the one defined in (3.24). The sum over Matzubara frequencies can be carried out in a similar fashion as the one used for the calculation of the grand potential in appendix A. First, we write the sum over frequencies as

$$\Omega_G - \Omega_{MF} = -\frac{1}{2\pi i}\frac{1}{V} \sum_{\mathbf{k}} \int_C d\tilde{\omega} n_B(\tilde{\omega}) \ln(M_{\tilde{\omega},\mathbf{k}}), \tag{3.27}$$

with C being a contour containing the imaginary axis of the complex $\tilde{\omega}$ plane and $n_B(\tilde{\omega}) = (e^{\beta\tilde{\omega}} - 1)^{-1}$ being the Bose-Einstein distribution; with the notation $M_{\tilde{\omega},\mathbf{k}}$ we mean that the dependence on the four vector $K = (i\Omega_n^B, \mathbf{k})$ of M_K changes to the one on $(\tilde{\omega}, \mathbf{k})$, with the overall result of a change of variables

$$i\Omega_n^B \to \tilde{\omega} \in \mathbb{C} \tag{3.28}$$

in (3.24), yielding

$$M_{\omega,z} = -\frac{m}{4\pi}\Big[\frac{1}{a_F} + \frac{2\sqrt{2mk_BT_c}}{z\pi}\int_0^{+\infty} dx \Big[\frac{x[A(x,z) - A(x,-z)]}{x^2 + \frac{z^2}{4} - z_0 - \frac{\omega}{2}} - z\Big], \tag{3.29}$$

with

$$\boxed{\beta_c\tilde{\omega} = \omega, \quad z = \beta_c\frac{\mathbf{k}^2}{2m}, \quad z_0 = \beta_c\mu.} \tag{3.30}$$

Given such form for $M_{\tilde{\omega},\mathbf{k}}$ one realizes that $\ln(M_{\tilde{\omega},\mathbf{k}})$ has a branch cut over the whole real axis in the complex $\tilde{\omega}$ plane and has no isolated poles, so that the integration contour C can be modified to one containing the real axis. The result, then, is that

$$\Omega_G - \Omega_{MF} = -\frac{1}{2\pi i}\frac{1}{V}\sum_{\mathbf{k}} \int_{-\infty}^{+\infty} d\tilde{\omega} n_B(\tilde{\omega}) \Big[\ln(M_{\tilde{\omega}+i\varepsilon,\mathbf{k}}) - \ln(M_{\tilde{\omega}-i\varepsilon,\mathbf{k}})\Big]. \tag{3.31}$$

with $\varepsilon \to 0$. Now, as done in [11], we may write $M_{\tilde{\omega}\pm i\varepsilon,\mathbf{k}}$ in the Euler representation as

$$M_{\tilde{\omega}\pm i\varepsilon,\mathbf{k}} = |M_{\tilde{\omega}\pm i\varepsilon,\mathbf{k}}| e^{\pm i\delta(\tilde{\omega}+i\varepsilon,\mathbf{k})}, \tag{3.32}$$

where for convenience we exploited the fact that by changing the sign of $i\varepsilon$ the phases of the two quantities have to be opposite in sign. The phase can be written as

$$\boxed{\delta(\tilde{\omega},\mathbf{k}) = \arctan\Big(\frac{Im[M_{\tilde{\omega},\mathbf{k}}]}{Re[M_{\tilde{\omega},\mathbf{k}}]}\Big)}, \tag{3.33}$$

where $Re[M_{\tilde{\omega},\mathbf{k}}]$ and $Im[M_{\tilde{\omega},\mathbf{k}}]$ can be calculated using (3.29).

Notice in fact that the integrand in (3.29) has two poles at

$$\pm p_0(\omega, z_0) = \pm\sqrt{\frac{\omega}{2} + z_0 - \frac{z^2}{4}}, \tag{3.34}$$

so that if $p_0(\omega, z_0) \in \mathbb{R}$, one has to use residue calculus to get a finite result, since $p_0(\omega, z_0)$ will lie on the integration domain. In such case, then,

$$\boxed{\begin{aligned} M_{\omega,z} = -\frac{m}{4\pi}\Big[\frac{1}{a_F} + \frac{2\sqrt{2mk_BT_c}}{z\pi}P\int_0^{+\infty} dx\Big(\frac{x[A(x,z) - A(x,-z)]}{x^2 + \frac{z^2}{4} - z_0 - \frac{\omega}{2}} - z\Big) + \\ +i\frac{\sqrt{2mk_BT_c}}{z}\Big(A(p_0(\omega,z),z) - A(p_0(\omega,z),-z)\Big)\Big], \end{aligned}} \tag{3.35}$$

where by $P\cdot$ we mean the principal value, showing that for $p_0^2(\omega, z) > 0$, $M_{\tilde{\omega},z}$ actually has a non vanishing imaginary part.

Then one may split the logarithms and get the identity

$$\Omega_G - \Omega_{MF} = -\frac{1}{\pi}\frac{1}{V}\sum_{\mathbf{k}}\int_{-\infty}^{+\infty} d\tilde{\omega} n_B(\tilde{\omega})\delta(\tilde{\omega}, \mathbf{k}). \tag{3.36}$$

The number equation $n = -\partial_\mu \Omega_G$ at the critical temperature, recalling the form of the number equation at the mean field level in (2.20) can then be written as

$$n = \frac{1}{V}\sum_{\mathbf{p}}\Big[1 - \tanh\Big(\frac{\beta}{2}\xi_{\mathbf{p}}\Big)\Big] + \frac{1}{\pi V}\sum_{\mathbf{k}}\int_{-\infty}^{+\infty} d\tilde{\omega} n_B(\tilde{\omega})\frac{\partial\delta(\tilde{\omega}, \mathbf{k})}{\partial\mu}, \tag{3.37}$$

where the first contribution comes from the mean field grand potential, so that the system of equations to be solved is

$$\boxed{\begin{cases} -\frac{m}{4\pi a_F} = \frac{1}{2V}\sum_{\mathbf{p}}\Big[\frac{\tanh\left(\frac{\beta}{2}\xi_{\mathbf{p}}\right)}{\xi_{\mathbf{p}}} - \frac{m}{\mathbf{p}^2}\Big] \\ n = \frac{1}{V}\sum_{\mathbf{p}}\Big[1 - \tanh\Big(\frac{\beta}{2}\xi_{\mathbf{p}}\Big)\Big] + \frac{1}{\pi V}\sum_{\mathbf{k}}\int_{-\infty}^{+\infty} d\tilde{\omega} n_B(\tilde{\omega})\frac{\partial\delta(\tilde{\omega},\mathbf{k})}{\partial\mu}, \end{cases}} \tag{3.38}$$

We make the remark that the gap equation does not distinguish itself from the one derived in the mean field case. Once fixed the scattering length a_F one can establish the relation between μ and the critical temperature T_c, which will be the same as in the mean field case and does not depend on n. The difference from the mean field analysis lies in the relation between the chemical potential μ and the number of particles n which, once fixed the scattering length, can be extracted from the number equation. Another big difference is the dependence of the number equation from the scattering length, which is implicit in the definition of $\delta(\omega, \mathbf{k})$ due to the form of $M_{\tilde{\omega},\mathbf{k}}$ in (3.35).

As in the mean field analysis, from such relations we will be able to construct the plot $\frac{k_BT_c}{\omega_F}$ *vs* $\frac{1}{k_Fa_F}$. In order to do so, though, it will be useful to make the integrals dimensionless via the changes of variables in equation (3.30), since we are going to transform the sum over **k** into an integral in polar coordinates. The number equation, recalling the expression for the particle density at the mean field level given in (2.34), will become

$$n = \frac{(2mk_BT_c)^{\frac{3}{2}}}{3\pi^2}\Big[\frac{1}{2}\int_0^{+\infty} dx \frac{x^4}{\cosh^2\left[\frac{1}{2}(x^2-z_0)\right]} + \frac{3}{2\pi}\int_0^{+\infty} dz z^2 \int_{-\infty}^{+\infty} d\omega \frac{1}{e^{\omega}-1}\frac{\partial\delta(\omega,z)}{\partial z_0}\Big] = \\ = \frac{(2mk_BT_c)^{\frac{3}{2}}}{3\pi^2}\Big[\frac{1}{2}I_4(z_0) + \frac{3}{2\pi}I_5(z_0)\Big]. \tag{3.39}$$

Such can be recast in the form

$$\boxed{\frac{k_BT_c}{\omega_F} = \Big(\frac{1}{\frac{1}{2}I_4(z_0)+\frac{3}{2\pi}I_5(z_0)}\Big)^{\frac{2}{3}}}\,, \tag{3.40}$$

Working on the gap equation, which can be rewritten as a one dimensional integral as in (2.34), instead, one gets that

$$\boxed{\frac{1}{k_Fa_F} = \frac{4}{\pi}\Big(\frac{1}{\frac{1}{2}I_4(z_0)+\frac{3}{2\pi}I_5(z_0)}\Big)^{\frac{1}{3}} I_3(z_0)}\,, \tag{3.41}$$

with

$$I_3(z_0) = \int_0^{+\infty} dx \Big[\frac{x^4}{2(x^2-z_0)\cosh^2\left[\frac{1}{2}(x^2-z_0)\right]} - z_0\frac{x^2\tanh\left(\frac{1}{2}(x^2-z_0)\right)}{(x^2-z_0)^2}\Big]. \tag{3.42}$$

In the meantime, to express $\frac{1}{a_F}$, present in the definition of $M_{\tilde{\omega},\mathbf{k}}$ (3.35), we can use directly the expression in equation (2.34),

$$\frac{1}{a_F} = \frac{4}{\pi}\sqrt{2mk_BT_c}\,I_3(z_0). \tag{3.43}$$

We were not able to implement such equations to retrieve the plot of the critical temperature *vs.* the scattering length, due to the computational difficulty of the problem and the absence of time. Despite that, results with this method (NSR), are reported in [11].

3.3 Beyond Mean Field Critical Temperature from the Phase Stiffness

In this section we follow [18] in a computationally simpler approach to the calculation of the critical temperature in the BCS-BEC crossover, exploiting only mean field quantities. This approach is certainly effective in two dimensions, but we try to generalize it to three, in a somewhat heuristic way. We will start with the calculation of the superfluid density of the system, which we will use to perform our calculation of the critical temperature.

3.3.1 Mean Field Superfluid Fraction

As explained by Tisza and Landau in [6] and [7], the condensate formed below T_c is formed by a mixture of two interpenetrating components: the superfluid one and the normal one. The superfluid part has no viscosity, while the normal part is viscous.

The number density of the fluid, then, can be written as

$$n = n_s(T) + n_n(T), \tag{3.44}$$

where $n_s(T)$ is the superfluid density and $n_n(T)$ is the density of the viscous fluid. As proven below, an explicit expression for $n_n(T)$ can be derived.

Following [8], we consider the liquid at a finite temperature close to $T = 0$. In this situation the fluid is not in its ground state and contains excitations. We think of the excitations as a gas of quasi particles moving with respect to the liquid at velocity $\mathbf{v}$. Let us take a coordinate system in which the gas is at rest as a whole, so that the liquid is moving at velocity $-\mathbf{v}$. The total energy E of the liquid in such reference frame is given by

$$E = E_0 - \mathbf{P}_0 \cdot \mathbf{v} + \frac{1}{2}Mv^2, \tag{3.45}$$

where M is the mass of the gas while E_0 and P_0 are the energy and momentum of the liquid in its rest frame. If we consider an excitation of energy $\omega(\mathbf{p})$ arising, the additional energy of the liquid will be $\omega - \mathbf{p} \cdot \mathbf{v}$ in such frame. Then, the distribution of the gas moving as a whole is $n(\omega - \mathbf{p} \cdot \mathbf{v})$, where $\mathbf{p}$ is the momentum of the particle and $n(\omega)$ is the Bose-Einstein distribution $n(\omega) = (e^{-\beta\omega} - 1)^{-1}$.

The total momentum per unit volume of the quasi particle gas, then, is

$$\mathbf{P} = \int d^3q \mathbf{p} n(\omega - \mathbf{p} \cdot \mathbf{v}). \tag{3.46}$$

If the velocity is small and the system is homogeneous and isotropic, as in our case, one can rewrite such equation as

$$\mathbf{P} = \frac{1}{3}\mathbf{v} \int d^3q \Big(- \frac{dn(\omega)}{d\omega} \Big) p^2, \tag{3.47}$$

so that one can express the density of the normal fluid as

$$n_n(T) = -\frac{1}{3} \int d^3q \frac{dn(\omega)}{d\omega} \frac{p^2}{m} = -\frac{1}{3} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{m} \frac{dn(\omega(\mathbf{p}))}{d\omega}. \tag{3.48}$$

Finally, one can express the superfluid density of the system as

$$\frac{n_s(T)}{n} = 1 - \frac{\beta}{6\pi^2 n} \int_0^{+\infty} dp \frac{p^4}{m} \frac{e^{\beta\omega(p)}}{(e^{\beta\omega(p)} - 1)^2}, \tag{3.49}$$

with

$$\omega(\mathbf{p}) = \sqrt{\Big(\frac{p^2}{2m} - \mu\Big)^2 + |\Delta_0|^2}, \tag{3.50}$$

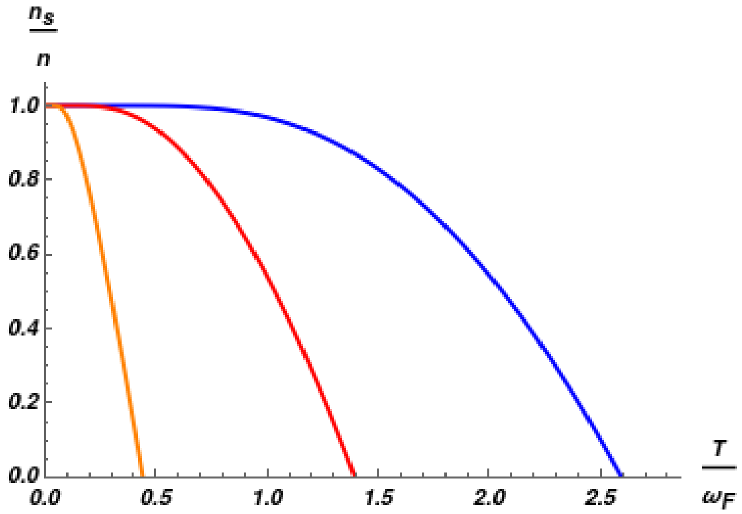

Figure 6: Superfluid fraction $\frac{n_s(T)}{n}$ *vs.* temperature. Blue line: $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}) = (-2.78, 1.71)$; red line: $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}) = (-0.26, 1.16)$; orange line: $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}) = (0.83, 0.44)$.

the fermionic excitation energies. In this treatment we are only considering fermionic excitation energies, neglecting the fluctuations of the Hubbard Stratonovich bosonic field $|\Delta(\mathbf{q}, \tau)|$.

To study the behaviour of $n_s(T)$ one can perform the change of variables $x = \frac{p}{k_F}$ in the integral (3.49), with $k_F = (3\pi n)^{\frac{1}{3}}$ the Fermi momentum, thereby obtaining

$$\frac{n_s(T)}{n} = 1 - \frac{\omega_F}{k_B T} \int_0^{+\infty} dx x^4 \frac{e^{\frac{\omega_F}{k_B T}\sqrt{(x^2 - \frac{\mu}{\omega_F})^2 + \frac{|\Delta_0|^2}{\omega_F^2}}}}{\left(e^{\frac{\omega_F}{k_B T}\sqrt{(x^2 - \frac{\mu}{\omega_F})^2 + \frac{|\Delta_0|^2}{\omega_F^2}}} - 1\right)^2}. \tag{3.51}$$

Since the equation is valid for small temperatures, it makes sense to consider the pairs $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F})$ we calculated at the mean field level from the solution of the number and gap equations at zero temperature, shown in figure 4.

In figure 6 we show the behaviour of the superfluid density varying with temperature for fixed $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F})$. From the plot we can see the behaviour of the temperature at which the superfluid fraction becomes null: the more negative the chemical potential, the higher the temperature.

Notice also how the superfluid fraction is unity at $T \to 0^+$ in all regimes, contrary to the condensate fraction, whose behaviour is reported in figure 5. The superfluid density and the condensate density indeed differ, as pointed out at the beginning of the section.

3.3.2 Beyond Mean Field Critical Temperature with the Kleinert method

As reported in [19], the effective action we derived in 2.1.1

$$S[\bar{\Delta}\Delta\bar{\psi}\psi] = \int_0^\beta d\tau \int_V d^3q \Big[\frac{|\Delta(\mathbf{q},\tau)|^2}{g} - \bar{\Psi}(\mathbf{q},\tau)G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta)\Psi(\mathbf{q},\tau)\Big] + \beta\sum_{\mathbf{p}}\xi_{\mathbf{p}}, \tag{3.52}$$

with $\xi_{\mathbf{p}} = \frac{\mathbf{p}^2}{2m} - \mu$ being the free particle energy and the inverse fermionic propagator being

$$G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta) = \begin{pmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu & \Delta(\mathbf{q},\tau) \\ \bar{\Delta}(\mathbf{q},\tau) & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{pmatrix}, \tag{3.53}$$

can be mapped to an XY model one, recalling that

$$\bar{\Psi}(\mathbf{q},\tau) = \begin{pmatrix} \bar{\psi}_\uparrow(\mathbf{q},\tau) & \psi_\downarrow(\mathbf{q},\tau) \end{pmatrix}, \qquad \Psi(\mathbf{q},\tau) = \begin{pmatrix} \psi_\uparrow(\mathbf{q},\tau) \\ \bar{\psi}_\downarrow(\mathbf{q},\tau) \end{pmatrix}. \tag{3.54}$$

This is achieved by imposing that $\Delta(\mathbf{q},\tau) \to \Delta(\mathbf{q},\tau)e^{i\theta(\mathbf{q},\tau)}$, so that also $\psi_\sigma(\mathbf{q},\tau) \to \psi_\sigma(\mathbf{q},\tau)e^{i\frac{\theta(\mathbf{q},\tau)}{2}}$ and by assuming that the phase gradients are small.

The effective XY model obtained has the Hamiltian

$$H = \frac{J}{2}\int d^3q[\nabla\theta(\mathbf{q})]^2, \tag{3.55}$$

with stiffness parameter

$$J = \frac{n_s(T_c)}{4m}, \tag{3.56}$$

where in our approximation $n_s(T)$ will be the superfluid density calculated in equation (3.49).

The critical temperature of the model can be heuristically calculated with the equation

$$k_BT_c = 3J\Big(\frac{2}{n}\Big)^{\frac{1}{3}} = 3\frac{n_s(T_c)}{4m}\Big(\frac{2}{n}\Big)^{\frac{1}{3}}, \tag{3.57}$$

Such formula is a generalization of the exact one obtained in the 2 dimensional XY model, and is numerically consistent with Monte Carlo simulations of the 3D XY model [20]. By dividing both sides of the equation by the Fermi energy ω_F we manage to get an expression independent of n: explicitly

$$\frac{k_BT_c}{\omega_F} = \Big(\frac{3}{4\pi^4}\Big)^{\frac{1}{3}}\frac{n_s(T_c)}{n}. \tag{3.58}$$

Once again, to solve the equation, we plug in the values of the pairs $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F})$ taken from figure 4 in (3.58). To calculate the inverse scattering length corresponding to such temperature, instead, we use the gap equation at the critical temperature derived in the mean field treatment in section 2.2.2

$$\frac{1}{k_Fa_F} = \frac{4}{\pi}\frac{2^{\frac{1}{3}}I_3(z_0)}{I_4(z_0)^{\frac{1}{3}}}, \tag{3.59}$$

with

$$I_3(z_0) = \int_0^{+\infty} dx \Big[\frac{x^4}{2(x^2 - z_0)\cosh^2\left[\frac{1}{2}(x^2 - z_0)\right]} - z_0 \frac{x^2 \tanh\left(\frac{1}{2}(x^2 - z_0)\right)}{(x^2 - z_0)^2}\Big],$$
$$I_4(z_0) = \int_0^{+\infty} dx \frac{x^4}{\cosh^2\left[\frac{1}{2}(x^2 - z_0)\right]}, \tag{3.60}$$

where $z_0 = \frac{\mu}{k_B T_c}$; the critical temperatures are the ones derived from solving (3.58), and the chemical potentials are the corresponding values used to solve the same equation.

The so obtained beyond mean field critical temperature is is shown in figure 7 along the whole crossover, for varying inverse scattering length. Despite our approximations, this heuristic approach yields a critical temperature whose behaviour resembles the one obtained with the introduction of Gaussian fluctuations, derived in [10], with the difference that in this case $k_B T_c$ does not have a maximum in the intermediate regime. The curve we obtained is more similar to the one displayed in fig. 3 of [21] through the application of the Thouless criterion.

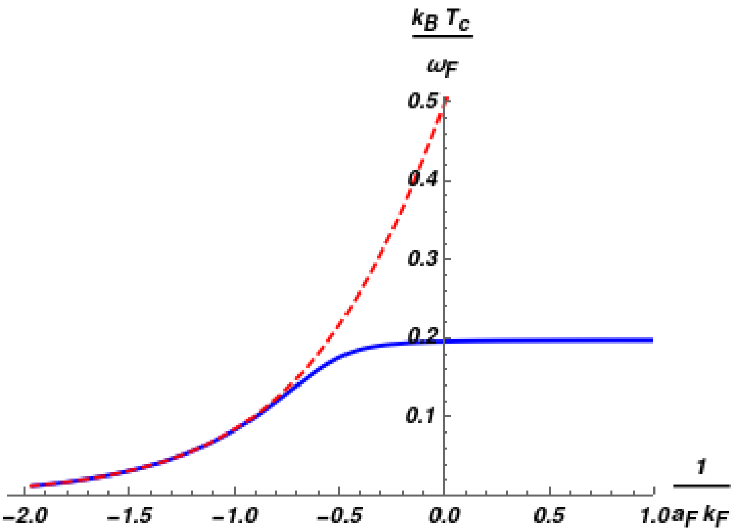

Figure 7: Critical temperature calculated with the implicit equation (3.58) *vs.* inverse scattering length along the whole crossover.

3.3.3 A Small Step Further

In the deep BEC limit, as the chemical potential μ approaches negative infinity, the superfluid fraction (3.51) clearly approaches unity, meaning that the critical temperature stabilizes at

$$\frac{k_B T_c}{\omega_F} \xrightarrow[\mu \to -\infty]{} \left(\frac{3}{4\pi^4}\right)^{\frac{1}{3}} \approx 0.197, \tag{3.61}$$

as also clear from figure 7. If we consider a gas of free bosons of mass $2m$ and density $\frac{n}{2}$ undergoing Bose Einstein condensation, given $\zeta(x) = \sum_{n=1}^{+\infty} \frac{1}{n^x}$, we find the familiar formula

[4] for the critical temperature

$$k_B T_c = \frac{2\pi}{2m}\left(\frac{n}{2\zeta(\frac{3}{2})}\right)^{\frac{2}{3}} \leftrightarrow \frac{k_B T_c}{\omega_F} = \frac{2}{(6\sqrt{\pi}\zeta(\frac{3}{2}))^{\frac{2}{3}}} \approx 0.218. \tag{3.62}$$

The values of (3.61) and (3.62) are actually similar, meaning that our system in the BEC limit has a similar behaviour as a bosonic gas of free particles with mass $2m$ and density $\frac{n}{2}$.

The main difference of our result and the one in [21] is in the BEC limit, where we would like the critical temperature to stabilize at the value given by (3.62). To get to such a result one may improve equation (3.57) by promoting the factor 3 to an arbitrary α

$$k_B T_c = 3\frac{n_s(T_c)}{4m}\left(\frac{2}{n}\right)^{\frac{1}{3}} \to k_B T_c = \alpha\frac{n_s(T_c)}{4m}\left(\frac{2}{n}\right)^{\frac{1}{3}}, \tag{3.63}$$

that will yield

$$\frac{k_B T_c}{\omega_F} \xrightarrow[\mu\to-\infty]{} \frac{2}{(6\sqrt{\pi}\zeta(\frac{3}{2}))^{\frac{2}{3}}}. \tag{3.64}$$

In order for the limit (3.64) to be achieved, we have that

$$\alpha = 2\left(\frac{\sqrt{\pi}}{\zeta(\frac{3}{2})}\right)^{\frac{2}{3}} \approx 1.55 \longrightarrow \frac{k_B T_c}{\omega_F} = \frac{2}{(6\sqrt{\pi}\zeta(\frac{3}{2}))^{\frac{2}{3}}}\frac{n_s(T_c)}{n}. \tag{3.65}$$

The results obtained with the same approach as the one used in the previous section are displayed in figure 8.

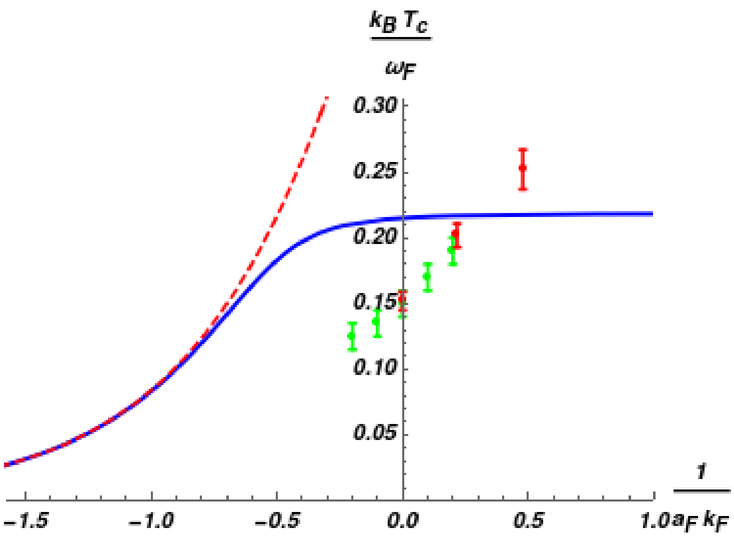

Figure 8: Critical temperature *vs.* inverse scattering length along the whole crossover. Red dashed line: mean field result; thick blue line: data obtained from (3.65); red points: results from diagrammatic Monte Carlo approach [22]; green points: results from Monte Carlo simulations [23].

One may be tempted to look for a more accurate approach to the solution of the implicit equation (3.65), in which we use values of $\frac{\mu}{\omega_F}$ and $\frac{|\Delta_0|}{\omega_F}$ consistent with the mean field number and gap equations calculated at a generic temperature T, which were derived in (2.20). In fact, by taking (2.20), transforming the sums over momenta in integrals with the usual prescription $\frac{1}{V}\sum_{\mathbf{p}} \to \frac{1}{(2\pi)^3}\int d^3p$, working in polar coordinates and performing the changes of variables $\frac{p^2}{k_F^2} = x^2$, one obtains the system of equations

$$\begin{cases} -\frac{1}{k_F a_F} = \frac{2}{\pi}\int_0^{+\infty} dx x^2 \left[\frac{\tanh\left(\frac{\omega_F}{2k_BT}\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}\right)}{\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}} - \frac{1}{x^2}\right] \\ 1 = \frac{3}{2}\int_0^{+\infty} dx x^2 \left[1 - \frac{x^2-\frac{\mu}{\omega_F}}{\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}}\tanh\left(\frac{\omega_F}{2k_BT}\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}\right)\right] \end{cases}, \quad (3.66)$$

which can be coupled to equation (3.65) to get the triplets $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}, \frac{k_BT}{\omega_F})$ for different values of $\frac{1}{k_F a_F}$. The pairs made of the chemical potential and the temperature will be used to calculate the inverse scattering length with equation (3.59), since the temperatures we found will be considered to be the critical ones.

This approach improves only a little the results we obtained in figure 8, as one can see from figure 9, although being computationally more expensive. The deep BEC limit is reached thanks to our improvement of the Kleinert equation and the BCS regime resembles the mean field one, as expected.

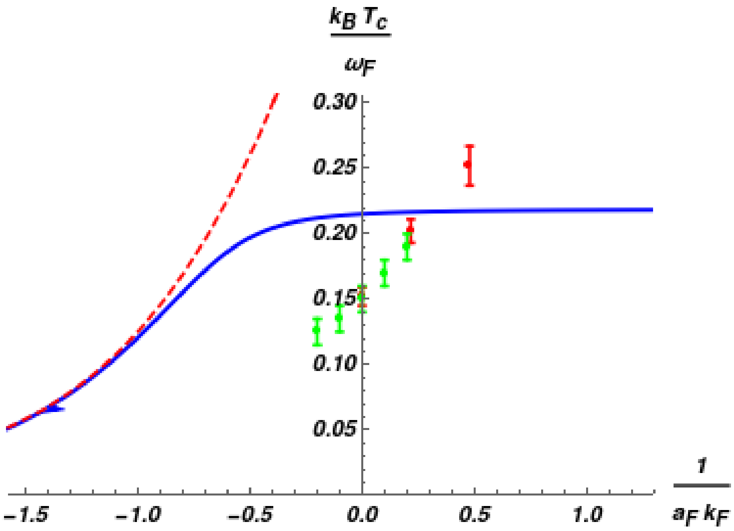

Figure 9: Critical temperature *vs.* inverse scattering length along the whole crossover. Red dashed line: mean field result; thick blue line: plot obtained from solving the system of equations made of (3.66) and (3.65); red points: results from diagrammatic Monte Carlo approach [22]; green points: results from Monte Carlo simulations [23].

4 Mean Field Treatment of the BCS-BEC Crossover with Rabi Coupling

In this chapter the ultra cold Fermi gas model treated previously will be enriched with the addition of Rabi coupling, which enables the spin of the particles involved to flip. The action, omitting the explicit dependence of the fields on space and time, then, reads

$$S[\bar{\psi},\psi]=\int_0^\beta d\tau\int d^3q\Big[\bar{\psi}_\sigma\Big(\partial_\tau-\frac{\nabla^2}{2m}-\mu\Big)\psi_\sigma-g\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow+\omega_R(\bar{\psi}_\uparrow\psi_\downarrow+\bar{\psi}_\downarrow\psi_\uparrow)\Big]. \qquad (4.1)$$

To recover the physical degrees of freedom of interest, we may perform the same Hubbard Stratonovich transformation made in section 2.1.1, so that the model can be rewritten in terms of a new action depending also on a new spinless complex field $\Delta(\mathbf{q},\tau)$:

$$S[\bar{\Delta}\Delta\bar{\psi}\psi]=\int_0^\beta d\tau\int_V d^3q\Big[\frac{|\Delta(\mathbf{q},\tau)|^2}{g}-\frac{1}{2}\bar{\Psi}(\mathbf{q},\tau)G^{-1}\Psi(\mathbf{q},\tau)\Big]+\beta\sum_{\mathbf{p}}\xi_{\mathbf{p}}, \qquad (4.2)$$

where the modified Nambu spinors have to be four dimensional and take the form

$$\bar{\Psi}(\mathbf{q},\tau)=\begin{pmatrix}\bar{\psi}_\uparrow(\mathbf{q},\tau) & \psi_\downarrow(\mathbf{q},\tau) & \bar{\psi}_\downarrow(\mathbf{q},\tau) & \psi_\uparrow(\mathbf{q},\tau)\end{pmatrix}, \quad \Psi(\mathbf{q},\tau)=\begin{pmatrix}\psi_\uparrow(\mathbf{q},\tau)\\ \bar{\psi}_\downarrow(\mathbf{q},\tau)\\ \psi_\downarrow(\mathbf{q},\tau)\\ \bar{\psi}_\uparrow(\mathbf{q},\tau)\end{pmatrix} \qquad (4.3)$$

while the inverse fermionic propagator in coordinate representation is

$$G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta)=\begin{pmatrix}-\partial_\tau+\frac{\nabla^2}{2m}+\mu & \Delta(\mathbf{q},\tau) & -\omega_R & 0\\ \bar{\Delta}(\mathbf{q},\tau) & -\partial_\tau-\frac{\nabla^2}{2m}-\mu & 0 & \omega_R\\ -\omega_R & 0 & -\partial_\tau+\frac{\nabla^2}{2m}+\mu & -\Delta(\mathbf{q},\tau)\\ 0 & \omega_R & -\bar{\Delta}(\mathbf{q},\tau) & -\partial_\tau-\frac{\nabla^2}{2m}-\mu\end{pmatrix}. \qquad (4.4)$$

The new theory is Gaussian in the fermionic degrees of freedom, meaning that their integration in the path integral can be performed to obtain an effective theory for the complex field $\Delta(\mathbf{q},\tau)$, whose action reads

$$S^{eff}[\bar{\Delta},\Delta]=\int_0^\beta d\tau\int_V d^3q\frac{|\Delta(\mathbf{q},\tau)|^2}{g}-\frac{1}{2}Tr[\ln G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta)]+\beta\sum_{\mathbf{p}}\xi_{\mathbf{p}}. \qquad (4.5)$$

With such theory in mind, we may procede in its mean field analysis and in the study of its Gaussian fluctuations as we did for the Rabiless case.

4.1 Gap and Number Equations

The mean field analysis of this model will be carried out on a similar ground with respect to the standard one, but some differences will arise in the BCS regime, both at $T\to 0^+$ and at the critical temperature $T=T_c$. In particular, the calculation of the grand potential will be of great importance in the understaing of how the gap and number equations differ from the ones in the Rabiless case at low temperatures.

4.1.1 Derivation of the Gap and Number Equations

Given the action (4.5), we want to calculate its saddle point value by imposing the complex field $\Delta(\mathbf{q}, \tau)$ to be space and time homogeneous. The mean field action, then, reads

$$\boxed{S_{MF} = \beta V \frac{|\Delta_0|^2}{g} - \frac{1}{2} Tr[\ln G^{-1}_{KP}(\bar{\Delta}_0, \Delta_0)] + \beta \sum_{\mathbf{p}} \xi_{\mathbf{p}},} \tag{4.6}$$

where again to calculate the trace of the logarithm of the mean field fermionic inverse propagator we will work in the Matzubara representation. The four momenta will be denoted by a capital letter such as $K = (i\Omega^F_n, \mathbf{p})$, with the fermionic Matzubara frequencies

$$\Omega^F_n = \frac{(2n+1)\pi}{\beta}, \quad n \in \mathbb{Z}. \tag{4.7}$$

To minimize (4.6) we derive the mean field action with respect to Δ_0, getting the equation

$$\beta V \frac{\bar{\Delta}_0}{g} - \frac{1}{2} Tr\left[G_{KP}(\bar{\Delta}_0 \Delta_0) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right] = 0. \tag{4.8}$$

Since it has not yet been written explicitly and it will be useful both in this mean field treatment and in the calculation of Gaussian fluctuations, the explicit expression for $G^{-1}_{KP}(\bar{\Delta}, \Delta)$ for an arbitrary field Δ_K is reported below

$$G^{-1}_{KP}(\bar{\Delta}, \Delta) = \begin{pmatrix} (i\Omega^F_n - \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & \Delta_{K+P} & -\omega_R \delta^{(4)}_{K,-P} & 0 \\ \bar{\Delta}_{K+P} & (i\Omega^F_n + \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & 0 & \omega_R \delta_{k,-p} \\ -\omega_R \delta^{(k,-p)} & 0 & (i\Omega^F_n - \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & -\Delta_{K+P} \\ 0 & \omega_R \delta^{(4)}_{K,-P} & -\bar{\Delta}_{K+P} & (i\Omega^F_n + \xi_{\mathbf{p}})\delta^{(4)}_{K,P}; \end{pmatrix}. \tag{4.9}$$

By imposing that the complex field is homogeneous in space and time in (4.9) we may calculate its determinant

$$\det(G^{-1}_{KP}(\bar{\Delta}, \Delta)) = ((\Omega^F_n)^2 + \omega^2_+(\mathbf{p}, \omega_R))((\Omega^F_n)^2 + \omega^2_-(\mathbf{p}, \omega_R)), \tag{4.10}$$

with

$$\boxed{\omega_+(\mathbf{p}, \omega_R) = \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2} + \omega_R, \quad \omega_-(\mathbf{p}, \omega_R) = \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2} - \omega_R}. \tag{4.11}$$

These two energies correspond to the poles of the fermionic propagator in real time, meaning that they are the single particle excitation energies of the theory, and they differ from the expression (2.13) from the Rabiless case only by a constant shift of ω_R. The presence of Rabi coupling splits the excitation energies calculated in (2.13) into two different energy levels separated by a shift of $2\omega_R$. It is immediately clear that $\omega_-(\mathbf{p},\omega_R)$ may take negative values, which is somewhat unexpected. This may happen for $|\Delta_0|<\omega_R$, a regime which will be proven to be unphysical, unless $|\Delta_0|=0$.

In order to write (4.8) in a compact form, firstly one has to compute the elements of the propagator $G_{KP}(\bar{\Delta},\Delta)$, which read

$$
\begin{aligned}
G_{11}=G_{33}=&\frac{(i\Omega_n^F+\xi_{\mathbf{p}})^2(i\Omega_n^F-\xi_{\mathbf{p}})-\omega_R^2(i\Omega_n^F-\xi_{\mathbf{p}})-|\Delta_{K+P}|^2(i\Omega_n^F+\xi_{\mathbf{p}})}{((\Omega_n^F)^2+\omega_+^2(\mathbf{p},\omega_R))((\Omega_n^F)^2+\omega_-^2(\mathbf{p},\omega_R))},\\
G_{22}=G_{44}=&\frac{(i\Omega_n^F-\xi_{\mathbf{p}})^2(i\Omega_n^F+\xi_{\mathbf{p}})-\omega_R^2(i\Omega_n^F+\xi_{\mathbf{p}})-|\Delta_{K+P}|^2(i\Omega_n^F-\xi_{\mathbf{p}})}{((\Omega_n^F)^2-\omega_+^2(\mathbf{p},\omega_R))((\Omega_n^F)^2+\omega_-^2(\mathbf{p},\omega_R))},\\
G_{21}=-G_{34}=\bar{G}_{12}=-\bar{G}_{43}=&\frac{\Delta_{K+P}\Big[((\Omega_n^F)^2+\xi_{\mathbf{p}}^2-\omega_R^2+|\Delta_{K+P}|^2\Big]}{((\Omega_n^F)^2+\omega_+^2(\mathbf{p},\omega_R))((\Omega_n^F)^2+\omega_-^2(\mathbf{p},\omega_R))},\\
G_{13}=G_{31}=-\bar{G}_{24}=-\bar{G}_{42}=&\frac{|\Delta_{K+P}|^2\omega_R-\omega_R^3+\omega_R(i\Omega_n^F+\xi_{\mathbf{p}})^2}{((\Omega_n^F)^2+\omega_+^2(\mathbf{p},\omega_R))((\Omega_n^F)^2+\omega_-^2(\mathbf{p},\omega_R))},\\
G_{14}=-G_{32}=\bar{G}_{41}=-\bar{G}_{23}=&\frac{2i\Omega_n^F\omega_R\Delta_{K+P}}{((\Omega_n^F)^2+\omega_+^2(\mathbf{p},\omega_R))((\Omega_n^F)^2+\omega_-^2(\mathbf{p},\omega_R))}.
\end{aligned}
\tag{4.12}
$$

Given these results, the gap equation in the presence of Rabi interaction can be written as

$$
\frac{1}{g}=\frac{k_BT}{V}\sum_P\frac{(\Omega_n^F)^2+\xi_{\mathbf{p}}^2+|\Delta_0|^2-\omega_R^2}{((\Omega_n^F)^2+\omega_+^2(\mathbf{p},\omega_R))((\Omega_n^F)^2+\omega_-^2(\mathbf{p},\omega_R))},\tag{4.13}
$$

which after the Matzubara frequency sum yields

$$
\frac{1}{g}=\frac{1}{4V}\sum_{\mathbf{p}}\Big[\frac{\tanh\big(\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\big)}{\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}}+\frac{\tanh\big(\frac{\beta}{2}\omega_-(\mathbf{p},\omega_R)\big)}{\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}}\Big].\tag{4.14}
$$

As expected, for $\omega_R=0$ one recovers exactly the result obtained in equation (2.16). Once again the gap equation (4.14) is divergent in the ultraviolet and requires the same regularization as the one used for the case of the previous section:

$$
\frac{1}{g}=-\frac{m}{4\pi a_F}+\frac{1}{V}\sum_{\mathbf{p}}\frac{m}{\mathbf{p}^2},\tag{4.15}
$$

with a_F being the scattering length of the system.

As far as the number equation is concerned, instead, one has that

$$
n=\frac{k_BT}{V}\partial_\mu\ln Z_{MF}=-\partial_\mu\frac{|\Delta_0|^2}{g}+\frac{k_BT}{2V}Tr\Big[G_{KP}\begin{pmatrix}1&\partial_\mu\Delta_0&0&0\\ \partial_\mu\bar{\Delta}_0&-1&0&0\\ 0&0&1&-\partial_\mu\Delta_0\\ 0&0&-\partial_\mu\bar{\Delta}_0&-1\end{pmatrix}\Big],\tag{4.16}
$$

which expanded reads

$$n = -\partial_\mu \frac{|\Delta_0|^2}{g} + \frac{k_B T}{2V} \sum_P \Big[G_{11} + G_{33} - G_{22} - G_{44} + (G_{12} - G_{34}) \partial_\mu \bar{\Delta}_0 + (G_{21} - G_{43}) \partial_\mu \Delta_0 \Big]. \quad (4.17)$$

Recalling the relations (4.12) one may rewrite such equation as

$$n = -\partial_\mu \frac{|\Delta_0|^2}{g} + \frac{k_B T}{2V} \sum_P \Big[2G_{11} - 2G_{22} + 2Re\{G_{12}\}(\partial_\mu \Delta_0 + \partial_\mu \bar{\Delta}_0) \Big], \quad (4.18)$$

where by $Re\{\cdot\}$ we mean the real part. Explicitly, substituting the results gotten in (4.12), one gets that

$$\begin{aligned} n = -\partial_\mu \frac{|\Delta_0|^2}{g} + \sum_{\mathbf{p}} 1 + \\ + \frac{k_B T}{V} \sum_P \Big[\frac{((\Omega_n^F)^2 + \xi_{\mathbf{p}}^2 + |\Delta_0|^2 - \omega_R^2) \partial_\mu |\Delta_0|^2 - 2\xi_{\mathbf{p}}((\Omega_n^F)^2 + \xi_{\mathbf{p}}^2 + |\Delta_0|^2 - \omega_R^2)}{((\Omega_n^F)^2 + \omega_+^2(\mathbf{p}, \omega_R))((\Omega_n^F)^2 + \omega_-^2(\mathbf{p}, \omega_R))} \Big]. \end{aligned} \quad (4.19)$$

The sum over Matzubara frequencies has the same form as the one of the gap equation, so that the result is analogous apart from factors, meaning

$$n = -\partial_\mu \frac{|\Delta_0|^2}{g} + \sum_{\mathbf{p}} 1 + \frac{1}{4V} \sum_{\mathbf{p}} (\partial_\mu \Delta_0^2 - 2\xi_{\mathbf{p}}) \Big[\frac{\tanh\left(\frac{\beta}{2}\omega_+(\mathbf{p}, \omega_R)\right)}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} + \frac{\tanh\left(\frac{\beta}{2}\omega_-(\mathbf{p}, \omega_R)\right)}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} \Big]. \quad (4.20)$$

The expression for $\omega_R = 0$ goes back to the original one (2.19). Moreover, exploiting (4.14) one can see once again that the terms proportional to $\partial_\mu |\Delta_0|$ exactly cancel out.

Including the regularization of the contact potential, then, the system of equations with Rabi coupling reads

$$\boxed{\begin{cases} -\frac{m}{4\pi a_F} = \frac{1}{V} \sum_{\mathbf{p}} \left[\frac{\tanh\left(\frac{\beta}{2}\omega_{\mathbf{p}}^+\right)}{4\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} + \frac{\tanh\left(\frac{\beta}{2}\omega_{\mathbf{p}}^-\right)}{4\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} - \frac{m}{\mathbf{p}^2} \right] \\ n = \frac{1}{V} \sum_{\mathbf{p}} \left[1 - \frac{\xi_{\mathbf{p}}}{2} \frac{\tanh\left(\frac{\beta}{2}\omega_{\mathbf{p}}^+\right)}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} - \frac{\xi_{\mathbf{p}}}{2} \frac{\tanh\left(\frac{\beta}{2}\omega_{\mathbf{p}}^-\right)}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} \right] \end{cases}}, \quad (4.21)$$

The difference with respect to the Rabiless case is a shift of $\pm\omega_R$ in the arguments of the hyperbolic tangents, which makes the derivation of analytic results more demanding.

4.1.2 Calculation of the Gran Potential

The procedure we will follow to calculate the grand potential is analogous to the one carried out in appendix A, with the complication that the determinant of $G_{KP}^{-1}(\bar{\Delta}_0, \Delta_0)$ is now quartic in the Matzubara frequencies instead of quadratic, meaning that there will be two integrals of the kind (A.5) instead of one, because of the splitting of the excitation energies due to the presence of Rabi coupling.

The determinant of the inverse fermionic propagator (4.9) calculated at a homogeneous value Δ_0 is given by (4.10), and can be recast conviniently as

$$\det(G^{-1}_{KP}(\bar{\Delta},\Delta)) = (-(\Omega^F_n)^2 - \omega^2_+(\mathbf{p},\omega_R))(-(\Omega^F_n)^2 - \omega^2_-(\mathbf{p},\omega_R)), \tag{4.22}$$

in such a way that when taking its logarithm all the frequencies $i\Omega^F_n$ may be written with a positive sign. The grand potential, given the mean field action (4.6), is given by

$$\Omega_{MF} = k_B T S_{MF} = V\frac{|\Delta_0|^2}{g} - \frac{1}{2}\ln\Big(\prod_p (-(\Omega^F_n)^2 - \omega^2_+(\mathbf{p},\omega_R))(-(\Omega^F_n)^2 - \omega^2_-(\mathbf{p},\omega_R))\Big). \tag{4.23}$$

By looking at the shape of the argument of the logarithm it is clear that the reasoning made in appendix A follows step by step, so that we can safely say that the grand potential for the Rabi coupled fermi gas at the mean field level has the form

$$\boxed{\frac{\Omega_{MF}}{V} = \frac{|\Delta_0|^2}{g} - \frac{k_B T}{2V}\sum_{\mathbf{p}}\Big[\ln\Big(2\big(1+\cosh[\beta\omega_+(\mathbf{p},\omega_R)]\big)\Big) + \ln\Big(2\big(1+\cosh[\beta\omega_-(\mathbf{p},\omega_R)]\big)\Big)\Big]}\,. \tag{4.24}$$

From this expression one could derive the number and gap equations again. The real practical use of this expression, though, will be manifest in the following, where we investigate the behaviour of the system at $T \to 0^+$.

4.2 Critical Temperature

We now investigate the behaviour of the system at the critical temperature T_c, at which the energy gap $|\Delta_0(T_c)| = 0$. In particular, following a similar procedure to the one used for the Rabiless case, we can obtain the plot for the critical temperature varying with the scattering length.

4.2.1 Numerical Results

To obtain such result, we start from the gap and number equations at the critical temperature, which read

$$\begin{cases} -\frac{m}{4\pi a_F} = \frac{1}{V}\sum_{\mathbf{p}}\Big[\frac{\tanh\left(\frac{\beta_c}{2}(\xi_{\mathbf{p}}+\omega_R\right)}{4\xi_{\mathbf{p}}} + \frac{\tanh\left(\frac{\beta_c}{2}(\xi_{\mathbf{p}}-\omega_R)\right)}{4\xi_{\mathbf{p}}} - \frac{m}{\mathbf{p}^2}\Big] \\ n = \frac{1}{V}\sum_{\mathbf{p}}\Big[1 - \tanh\left(\frac{\beta_c}{2}(\xi_{\mathbf{p}}+\omega_R)\right) - \tanh\left(\frac{\beta_c}{2}(\xi_{\mathbf{p}}-\omega_R)\right)\Big] \end{cases}, \tag{4.25}$$

and can be manipulated as usual, turning the sums into integrals over momenta and by making the substitution

$$\frac{\beta_c p^2}{2m} = x^2, \tag{4.26}$$

making the integrals adimensional. In the following we use parts integration on the number equation in order to have an expression quickly converging at infinity, which was found to be computed faster numerically: namely, we rewrite the system of equations as

$$\begin{cases} \frac{1}{a_F} = \frac{2}{\pi}(2mk_BT_c)^{\frac{1}{2}} J_3\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big) \\ n = \frac{1}{12\pi^2}(2mk_BT_c)^{\frac{3}{2}} J_4\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big) \end{cases}, \tag{4.27}$$

with

$$J_3\Big(\frac{\mu}{\omega_F},\omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big) = \int_0^{+\infty} dx x^2 \Big[\frac{\tanh\Big(\frac{1}{2}(x^2-\beta_c\mu+\beta_c\omega_R)\Big)}{2(x^2-\beta_c\mu)} + \frac{\tanh\Big(\frac{1}{2}(x^2-\beta_c\mu-\beta_c\omega_R)\Big)}{2(x^2-\beta_c\mu)} - \frac{1}{x^2}\Big]. \tag{4.28}$$

and

$$J_4\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c,\frac{\omega_R}{\omega_F}\Big) = \int_0^{+\infty} dx \Big[\frac{x^4}{\cosh^2\big[\frac{1}{2}(x^2-\beta_c\mu+\beta_c\omega_R)\big]} + \frac{x^4}{\cosh^2\big[\frac{1}{2}(x^2-\beta_c\mu-\beta_c\omega_R)\big]}\Big]. \tag{4.29}$$

The complication with respect to the case with no Rabi interaction is that one cannot let the integrals J_3 and J_4 depend only on one parameter $z_0 = \beta_c\mu$, since the dependence on the Rabi frequency ω_R does not allow it. To solve the problem, then, one has to first find the level curves for k_BT_c in function of μ. This can be done numerically by exploiting the number equation in (4.27), which can be recast in the form

$$\boxed{\frac{k_BT_c}{\omega_F} = \Big(\frac{4}{J_4\big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\big)}\Big)^{\frac{2}{3}}}. \tag{4.30}$$

Such equation depends only on the two variables $\frac{\mu}{\omega_F}$ and $\omega_F\beta_c$ and the plot obtained is shown below:

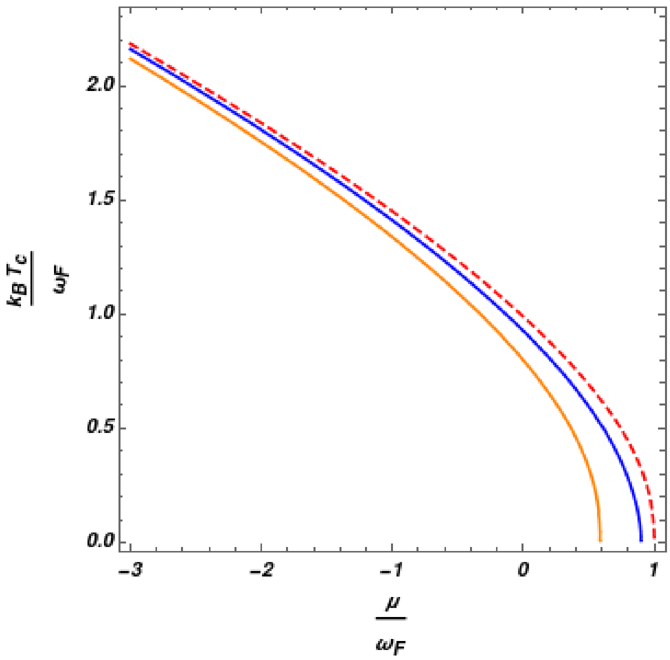

Figure 10: Level curves of (4.30). Red dashed line: Rabiless case, i.e. $\frac{\omega_R}{\omega_F} = 0$. Blue line: fixed value of $\frac{\omega_R}{\omega_F} = 0.6$. Orange Line: fixed value of $\frac{\omega_R}{\omega_F} = 1$.

It is interesting to notice that the value of the chemical potential at very low temperatures is suppressed by the Rabi coupling, as will also be shown analytically in the discussion made in the zero temperature section. As already seen in section 2.2, in the Rabiless case the chemical potential at low critical temperature approaches the Fermi energy; in the presence of Rabi interaction, instead, such value decreases and, as will be discussed, using equation (4.42) its value can be predicted. Such solution to the equation, though, has proven to be unphysical, as is clear also from the plot in figure 11. In the BEC regime, instead, as the critical temperature increases, the effect of Rabi coupling tends to vanish, as clear from the raw expressions of (4.28) and (4.29), which depend on $\beta_c \omega_R$.

Working with the gap equation in (4.27), then, one may obtain the expression for the scattering length

$$\boxed{\frac{1}{k_F a_F} = \frac{2}{\pi}\Big(\frac{4}{J_4\big(\frac{\mu}{\omega_F}, \omega_F \beta_c, \frac{\omega_R}{\omega_F}\big)}\Big)^{\frac{1}{3}} J_3\Big(\frac{\mu}{\omega_F}, \omega_F \beta_c, \frac{\omega_R}{\omega_F}\Big)}, \tag{4.31}$$

so that using the data in figure 10 one can produce a plot of the critical temperature varying with the scattering length as the one in figure 11.

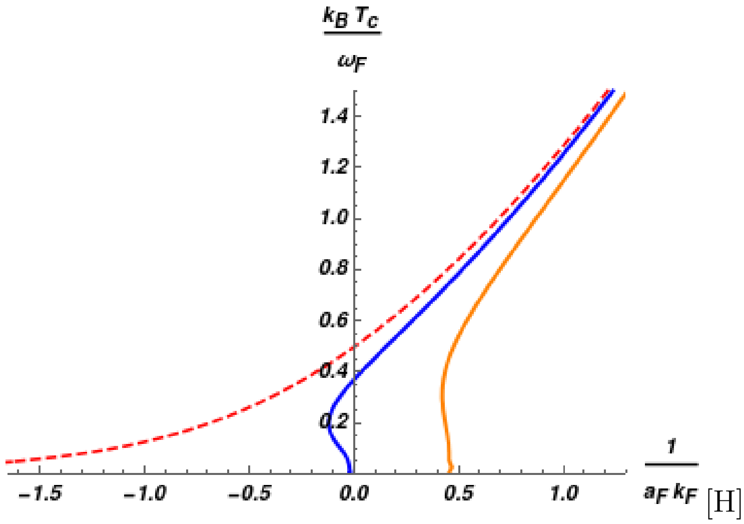

Figure 11: Red dashed line: critical temperature *vs* scattering length with no Rabi interaction; thick blue line: critical temperature *vs* scattering length over the whole crossover with fixed $\frac{\omega_R}{\omega_F} = 0.5$; thick orange line: critical temperature *vs* scattering length over the whole crossover with fixed $\frac{\omega_R}{\omega_F} = 1$.

The net effect of the Rabi coupling manifests itself in the BCS regime, in which there is no critical temperature. Rabi coupling inhibits the formation of Cooper pairs. The stronger the Rabi coupling, the higher the lowest possible critical temperature and the higher the lowest scattering length at which the existence of a critical temperature is possible at the mean field level. From the plot in figure 11 it would seem like there exists a region of scattering lengths for which two different critical temperatures exist. Actually, we will prove in the zero temperature section that the solution for $T_c = 0$ is unphysical, and so will be all the ones in the lower part of the curve.

4.3 $T \to 0^+$ Limit

In this section we investigate the behaviour of the energy gap, chemical potential and of the condensate fraction with respect to the variation of the scattering length of the system a_F along the whole crossover at zero temperature.

4.3.1 Gap and Number Equations

As we did in the Rabiless case, we may reduce (4.21) to a system of two expressions written in terms of one dimensional integrals exploiting the fact that the hyperbolic tangent becomes unity at $T \to 0^+$ thereby obtaining relations similar to the ones in (2.43). In doing so, one has to be careful in the study of the sign of $\omega_-(\mathbf{p}, \omega_R)$, which affects the form of the equations. In fact, if $\omega_-(\mathbf{p}, \omega_R) > 0$ for any value of the momentum $\mathbf{p}$, the number and gap equations (4.21) will take the same form as the ones with no Rabi interaction (2.20), while if for some values of $\mathbf{p}$ the energy $\omega_-(\mathbf{p}, \omega_R) < 0$, the equations will take a different form, as we will show below.

To start, we see from the form of the excitation energies in equation (4.11) that

$$\omega_-(\mathbf{p}, \omega_R) > 0 \longleftrightarrow \mathbf{p}^4 - 4m\mu\mathbf{p}^2 + 4m^2(\mu^2 + |\Delta_0|^2 - \omega_R^2) > 0. \tag{4.32}$$

It is clear that for $|\Delta_0| > \omega_R$ such inequality is satisfied for any value of the momentum $\mathbf{p}$, since $\mathbf{p}^4 - 4m\mu\mathbf{p}^2 + 4m^2\mu^2 = (\mathbf{p}^2 - 2m\mu)^2 > 0$. Actually, one finds that the minimum value of $\omega_-(\mathbf{p}, \omega_R)$ with respect to $\mathbf{p}$ is taken for $|\mathbf{p}| = \sqrt{2m\mu}$. Then, by imposing that $\omega_-(\sqrt{2m\mu}) > 0$ one obtains the inequality

$$\omega_-(\mathbf{p}, \omega_R) > 0 \ \forall \ \mathbf{p} \longleftrightarrow \ |\Delta_0| > \omega_R. \tag{4.33}$$

In the case $\omega_R > |\Delta_0|$, the solutions for (4.32) are

$$\frac{\mathbf{p}^2}{2m} < \mu - \sqrt{\omega_R^2 - |\Delta_0|^2} \text{ or } \frac{\mathbf{p}^2}{2m} > \mu + \sqrt{\omega_R^2 - |\Delta_0|^2}, \tag{4.34}$$

in which the first inequality may be satisfied if

$$\mu^2 > \omega_R^2 - |\Delta_0|^2. \tag{4.35}$$

We then conclude that

$$\omega_-(\mathbf{p}, \omega_R) < 0 \longleftrightarrow \begin{cases} \mu - \sqrt{\omega_R^2 - |\Delta_0|^2} < \frac{\mathbf{p}^2}{2m} < \mu + \sqrt{\omega_R^2 - |\Delta_0|^2}, \\ |\Delta_0| < \omega_R \end{cases} . \tag{4.36}$$

The terms in (4.21) that satisfy (4.36) cancel out, since

$$\lim_{x \to +\infty} \tanh(x) = 1 \text{ and } \lim_{x \to -\infty} \tanh(x) = -1, \tag{4.37}$$

meaning that the effect of Rabi interaction on the gap and number equations is null if $|\Delta_0| > \omega_R$ and to change the momenta domain of integration in the complementary case.

To be more precise, one may rewrite the gap equation as

$$\boxed{\begin{aligned} \frac{1}{g} = \frac{1}{2V} \sum_{\mathbf{p}} \Big[\frac{\Theta(|\Delta_0| - \omega_R)}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} + \\ \frac{\Theta(\xi_{\mathbf{p}} - \sqrt{\omega_R^2 - |\Delta_0|^2}) + \Theta(-\xi_{\mathbf{p}} - \sqrt{\omega_R^2 - |\Delta_0|^2})}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} \Theta(\omega_R - |\Delta_0|) \Big], \end{aligned}} , \tag{4.38}$$

where $\Theta(\cdot)$ is the Heaviside step function. This expression is valid for any value of the chemical potential, energy gap and Rabi frequency. The number equation suffers the same fate as (4.38) and may be rearranged similarly as

$$\boxed{\begin{aligned} n = \frac{1}{V}\sum_{\mathbf{p}}\Big[1 - \frac{\xi_{\mathbf{p}}}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}}\Theta(|\Delta_0| - \omega_R) + \\ -\frac{\xi_{\mathbf{p}}[\Theta(\xi_{\mathbf{p}} - \sqrt{\omega_R^2 - |\Delta_0|^2}) + \Theta(-\xi_{\mathbf{p}} - \sqrt{\omega_R^2 - |\Delta_0|^2})]}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}}\Theta(\omega_R - |\Delta_0|)\Big]. \end{aligned}} \tag{4.39}$$

Both the number and gap equations are obtained as derivatives of the mean field grand potential Ω_{MF} written in (4.24). One can in fact obtain them in the same form by deriving the grand potential calculated at zero temperature, which reads

$$\frac{1}{V}\lim_{T\to 0^+}\Omega_{MF} = \frac{|\Delta_0|^2}{g} - \frac{1}{2V}\sum_{\mathbf{p}}\Big[\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} + \omega_R + \Big|\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} - \omega_R\Big|\Big]. \tag{4.40}$$

From the shape of the grand potential one can conclude that for $|\Delta_0| > \omega_R$ the Physics of the system will not at all be affected by Rabi coupling. It is also interesting to notice that the right hand sides of both (4.38) and (4.39) are continuous at $|\Delta_0| = \omega_R$, even though the grand potential is not smooth.

The objective is now to find the solutions for the system made of equations (4.38) and (4.39). By considering the case in which (4.33) holds, we know that the solutions will be the same as the ones reported in the plots in figures 2, 3 and 4 of the Rabiless case, since we know from the previous discussion that Rabi coupling will not have any effect on the system. The presence of Rabi coupling, though, will affect the physics of the system, since for $|\Delta_0| < \omega_R$ the solutions obtained in the system with no Rabi coupling will no longer be solutions of (4.21). We then want to understand the nature of the new kind of solutions, obtained in the regime $|\Delta_0| < \omega_R$.

We want to show through an analytical example that the solutions for $|\Delta_0| < \omega_R$ are unphysical. The case in which $|\Delta_0| = 0$, corresponding to $T_c = 0$, is particularly easy to study, and it contains useful information about the validity of the solutions of the gap equation. In that case the number equation takes the form

$$n = \frac{1}{V}\sum_{\mathbf{p}}\Big[1 - \Theta(\frac{\mathbf{p}^2}{2m} - \mu - \omega_R) - \Theta(-\frac{\mathbf{p}^2}{2m} + \mu - \omega_R)\Big], \tag{4.41}$$

which can easily be solved and yields the equation

$$\frac{(\mu + \omega_R)^{\frac{3}{2}}}{2\omega_F^{\frac{3}{2}}} - \frac{(\mu - \omega_R)^{\frac{3}{2}}}{2\omega_F^{\frac{3}{2}}}\Theta(\mu - \omega_R) = 1. \tag{4.42}$$

From this we may extract an explicit expression for the case in which $\mu < \omega_R$, which reads

$$\frac{\mu}{\omega_F} = 2^{2/3} - \frac{\omega_R}{\omega_F}. \tag{4.43}$$

By imposing again that $\omega_R > \mu$ we get the relation

$$\frac{\omega_R}{\omega_F} > \frac{1}{2^{\frac{1}{3}}} \approx 0.7937. \tag{4.44}$$

With this solution of the chemical potential and energy gap in mind, we calculate the second derivative of the grand potential, which in this case for $\omega_R > \mu$ and $\omega_R > 0.7937$ reads

$$\frac{1}{V}\frac{\partial^2\Omega_{MF}}{\partial|\Delta_0|^2} = \frac{|\Delta_0|^2}{2\pi^2}\Big[\int_{\sqrt{2m(\mu+\sqrt{\omega_R^2-|\Delta_0|^2})}}^{+\infty} dp\frac{p^2}{2[\xi_p^2+|\Delta_0|^2]^{\frac{3}{2}}} - \frac{(2m)^{\frac{3}{2}}\sqrt{\mu+\sqrt{\omega_R^2-|\Delta_0|^2}}}{4\omega_R\sqrt{\omega_R^2-|\Delta_0|^2}}\Big]. \tag{4.45}$$

In the $|\Delta_0| = 0$ case we see that such expression is identically zero, giving no useful information. We can explore the region for which $|\Delta_0| \approx 0^+$, from which we may understand if it corresponds to a maximum or a minimum of the grand potential. To do this, we expand (4.45) up to second order in $|\Delta_0|$, get rid of the constant factors and impose equation (4.43). Then one has that (4.45), after the change of variables

$$\frac{p^2}{2m\omega_F} = z^2$$

and after getting rid of unimportant global positive factors becomes

$$f(\omega_R) = \frac{1}{2^{\frac{4}{3}}}\Big(\frac{\omega_F}{\omega_R}\Big)^2 - \int_{2^{1/3}}^{+\infty} dz\frac{z^2}{2[z^2-2^{\frac{2}{3}}+\frac{\omega_R}{\omega_F}]^3}. \tag{4.46}$$

Such function can be plotted with varying ω_R, showing that it is actually always negative for $\omega_R > 0.7937$:

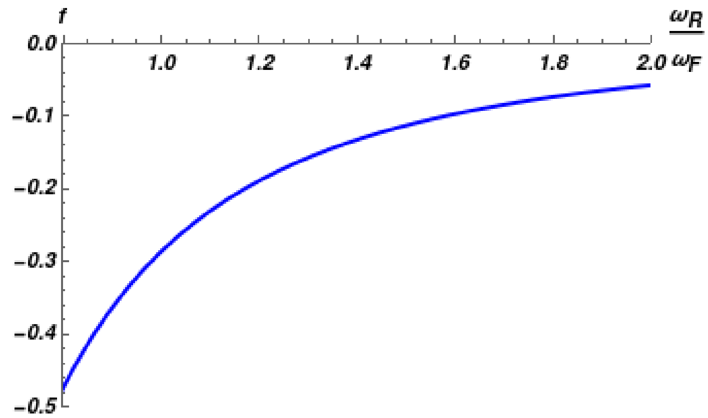

Figure 12: Second derivative of the grand potential near vanishing critical temperature (apart from positive factors) *vs* Rabi frequency.

The solution of the system of equations (4.38) and (4.39) that we found in this regime is then proven to be unphysical, since it corresponds to a maximum of the grand potential. With this result, we may go on assuming that all the solutions of (4.21) in the regime not satisfying (4.33) are maxima of Ω_{MF}.

4.3.2 Numerical Results

To solve the system of equations made of (4.38) and (4.39) in the general case, the procedure is analogous to the one followed in the Rabiless case, with the obvious complication that the expressions are more convoluted. For instance, here it is not possible to reduce such equations to one dimensional integrals depending only on one parameter because of the presence of the Rabi frequency in the Heaviside step functions, so that for a fixed Rabi frequency one will first have to find the level curves of the energy gap *vs* the chemical potential. We start from the number equation by turning the sum into an integral in spherical coordinates and by performing the change of variables

$$\frac{p^2}{2m|\Delta_0|} = x^2, \quad x_0 = \frac{\mu}{|\Delta_0|}, \quad \eta = \frac{\omega_R}{|\Delta_0|}. \tag{4.47}$$

Then we may use the Heaviside step functions to change the domains of integration of the integrals and by integrating by parts we can eliminate the ultraviolet divergences. Finally, the number equation translates to

$$n = \frac{(2m|\Delta_0|)^{\frac{3}{2}}}{3\pi^2}\Big[\int_0^{+\infty} dx \frac{x^4\Theta(|\Delta_0| - \omega_R)}{[(x^2 - x_0) + 1]^{\frac{3}{2}}} + \Big(\frac{1}{2\eta}(x_0 + \sqrt{\eta^2 - 1})^{\frac{3}{2}}\sqrt{\eta^2 - 1} + \\ + \int_{\sqrt{x_0 + \sqrt{\eta^2-1}}}^{+\infty} dx \frac{x^4}{[(x^2 - x_0)^2 + 1]^{\frac{3}{2}}} - \frac{3}{2}\int_0^{\sqrt{x_0 - \sqrt{\eta^2-1}}} dx \frac{x^2}{[(x^2 - x_0)^2 + 1]^{\frac{1}{2}}}\Big)\Theta(\omega_R - |\Delta_0|)\Big], \tag{4.48}$$

which can be recast in the form

$$n = \frac{(2m|\Delta_0|)^{\frac{3}{2}}}{3\pi^2} J_2\Big(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}, \frac{\omega_R}{\omega_F}\Big), \tag{4.49}$$

from which one obtains the relation

$$\frac{|\Delta_0|}{\omega_F} = \Big(\frac{1}{J_2\Big(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}, \frac{\omega_R}{\omega_F}\Big)}\Big)^{\frac{2}{3}}. \tag{4.50}$$

The plot for the solutions of such equation is reported in figure 13, and it shows both the physical and unphysical solutions.

To carry on the treatment we take the gap equation (4.38), substitute the contact potential with its regularized counterpart and follow the same procedure carried out for the number equation, obtaining the equation

$$-\frac{1}{a_F} = \frac{4}{\pi}(2m|\Delta_0|)^{\frac{1}{2}}\Big[\int_0^{+\infty} dx \frac{x_0 x^2(x^2 - x_0) - x^2}{[(x^2 - x_0)^2 + 1]^{\frac{3}{2}}}\Theta(|\Delta_0| - \omega_R) + \Big(-\frac{(x_0 + \sqrt{\eta^2 - 1})^{\frac{3}{2}}}{2\eta} + \\ + \int_{\sqrt{x_0 + \sqrt{\eta^2-1}}}^{+\infty} dx \frac{x_0 x^2(x^2 - x_0) - x^2}{[(x^2 - x_0)^2 + 1]^{\frac{3}{2}}} + \frac{1}{2}\int_0^{\sqrt{x_0 - \sqrt{\eta^2-1}}} dx \frac{x^2}{[(x^2 - x_0)^2 + 1]^{\frac{1}{2}}}\Big)\Theta(\omega_R - |\Delta_0|)\Big]. \tag{4.51}$$

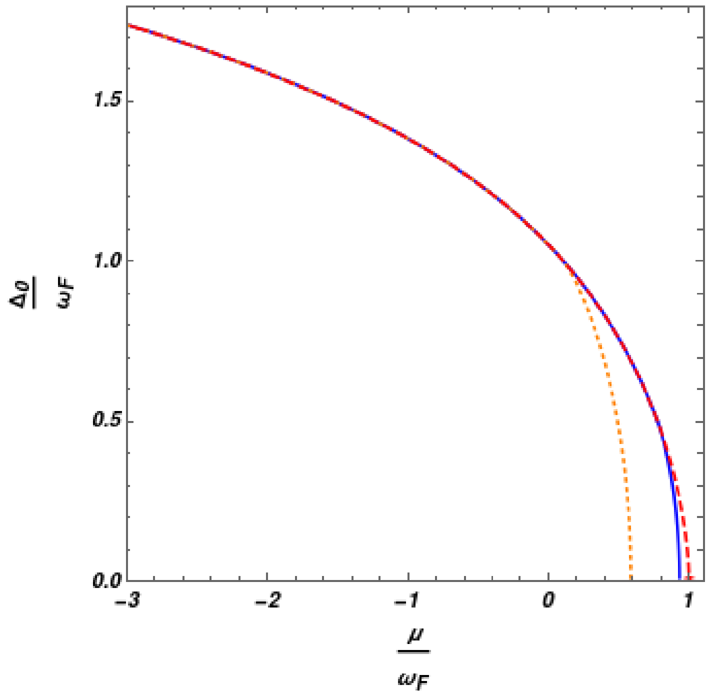

Figure 13: Values of the energy gap *vs* the chemical potential solving equation (4.50) over the whole crossover. Red dashed line: null Rabi frequency $\frac{\omega_R}{\omega_F} = 0$. Thick blue line: fixed value of $\frac{\omega_R}{\omega_F} = 0.5$. Orange dotted line: dixed value of $\frac{\omega_R}{\omega_F} = 1$.

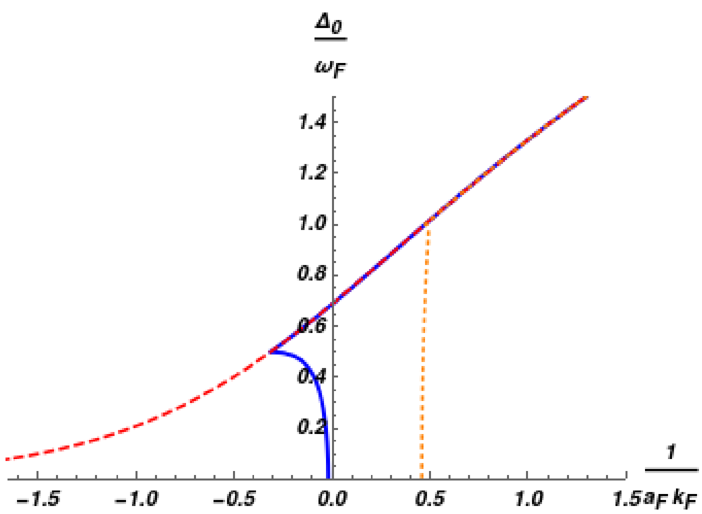

Figure 14: Energy gap *vs* inverse scattering length solving the system of equations (4.21) in the $T \to 0^+$ limit in the presence of Rabi Coupling. Red dashed line: null Rabi frequency $\frac{\omega_R}{\omega_F} = 0$. Thick blue line: fixed value of $\frac{\omega_R}{\omega_F} = 0.5$. Orange dotted line: fixed value of $\frac{\omega_R}{\omega_F} = 1$.

The corresponding plots for the energy gap and chemical potential varying with the scattering length are then analogous to the ones in the case with no Rabi coupling for $|\Delta_0| > \omega_R$, but exhibit a different behaviour below such threshold. These results are reported in figure 14, in which it is clear that for some given scattering lengths a_F, the system of equations (4.21) may have two solutions. As we showed, only the ones above $|\Delta_0| \geqslant \omega_R$ are physical, though.

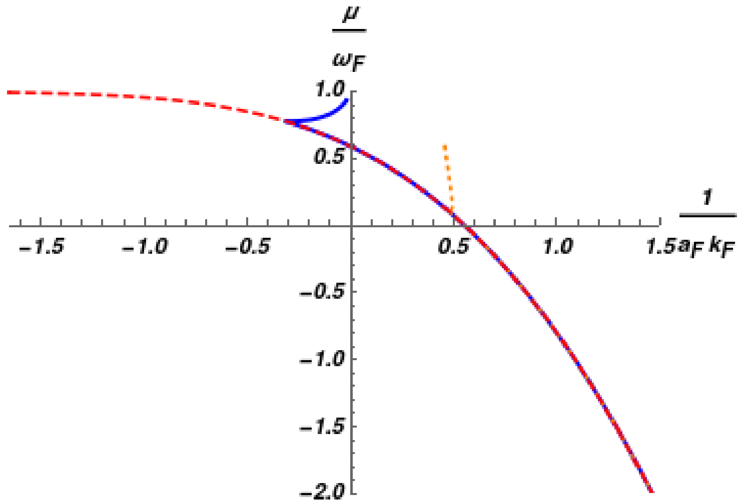

Figure 15: Chemical potential *vs* inverse scattering length solving the system of equations (4.21) in the $T \to 0^+$ limit in the presence of Rabi Coupling. Red dashed line: null Rabi frequency $\frac{\omega_R}{\omega_F} = 0$. Thick blue line: fixed value of $\frac{\omega_R}{\omega_F} = 0.5$. Orange dotted line: fixed value of $\frac{\omega_R}{\omega_F} = 1$.

4.3.3 Condensate Fraction

We now investigate how the Rabi coupling affects the condensate fraction of the system at all temperatures, focusing in particular in the $T \to 0^+$ case as we did in the Rabiless case. First, we want to investigate when OLDRO is present, as we did in section 2.3.3 and then find an explicit expression for the condensate fraction n_0.

The fermionic propagator of the theory at the mean field level, whose elements are listed in equation (4.12), contains the necessary information for the calculation of the two particle reduced density matrices

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = Tr[\hat{a}_\downarrow(\mathbf{q}_1')\hat{a}_\uparrow(\mathbf{q}_2')\hat{\rho}\hat{a}_\uparrow^\dagger(\mathbf{q}_1)\hat{a}_\downarrow^\dagger(\mathbf{q}_2)] \tag{4.52}$$

and

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\tilde{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = Tr[\hat{a}_\uparrow(\mathbf{q}_1')\hat{a}_\uparrow(\mathbf{q}_2')\hat{\rho}\hat{a}_\uparrow^\dagger(\mathbf{q}_1)\hat{a}_\uparrow^\dagger(\mathbf{q}_2)]. \tag{4.53}$$

In particular,

$$G(\bar{\Delta}_0, \Delta_0) = \begin{pmatrix} \langle\psi_\uparrow\bar{\psi}_\uparrow\rangle & \langle\psi_\uparrow\psi_\downarrow\rangle & \langle\psi_\uparrow\bar{\psi}_\downarrow\rangle & \langle\psi_\uparrow\psi_\uparrow\rangle \\ \langle\bar{\psi}_\downarrow\bar{\psi}_\uparrow\rangle & \langle\bar{\psi}_\downarrow\psi_\downarrow\rangle & \langle\bar{\psi}_\downarrow\bar{\psi}_\downarrow\rangle & \langle\bar{\psi}_\downarrow\psi_\uparrow\rangle \\ \langle\psi_\downarrow\bar{\psi}_\uparrow\rangle & \langle\psi_\downarrow\psi_\downarrow\rangle & \langle\psi_\downarrow\bar{\psi}_\downarrow\rangle & \langle\psi_\downarrow\psi_\uparrow\rangle \\ \langle\bar{\psi}_\uparrow\bar{\psi}_\uparrow\rangle & \langle\bar{\psi}_\uparrow\psi_\downarrow\rangle & \langle\bar{\psi}_\uparrow\bar{\psi}_\downarrow\rangle & \langle\bar{\psi}_\uparrow\psi_\uparrow\rangle \end{pmatrix}, \tag{4.54}$$

so that by using Wick's theorem we may rewrite the elements of the reduced density matrix $\hat{\rho}_2$, for example, as

$$\begin{aligned}\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = \langle\bar{\psi}_\uparrow(\mathbf{q}_1)\bar{\psi}_\downarrow(\mathbf{q}_2)\rangle\langle\psi_\downarrow(\mathbf{q}_1')\psi_\uparrow(\mathbf{q}_2')\rangle + \langle\psi_\downarrow(\mathbf{q}_1')\bar{\psi}_\downarrow(\mathbf{q}_2)\rangle\langle\psi_\uparrow(\mathbf{q}_2')\bar{\psi}_\uparrow(\mathbf{q}_1)\rangle + \\ -\langle\psi_\downarrow(\mathbf{q}_1')\bar{\psi}_\uparrow(\mathbf{q}_1)\rangle\langle\psi_\uparrow(\mathbf{q}_2')\bar{\psi}_\downarrow(\mathbf{q}_2)\rangle\end{aligned}. \tag{4.55}$$

By working in Matzubara representation and taking the limit $\begin{cases}|\mathbf{q}_1' - \mathbf{q}_1| \to +\infty \\ |\mathbf{q}_2' - \mathbf{q}_2| \to +\infty\end{cases}$ we see again that the second and third terms vanish due to the Riemann-Lebesgue lemma, so that the matrix factorizes again as

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\hat{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = \langle\bar{\psi}_\uparrow(\mathbf{q}_1)\bar{\psi}_\downarrow(\mathbf{q}_2)\rangle\langle\psi_\downarrow(\mathbf{q}_1')\psi_\uparrow(\mathbf{q}_2')\rangle. \tag{4.56}$$

In the same way,

$$\langle \mathbf{q}_1'\mathbf{q}_2'|\tilde{\rho}_2|\mathbf{q}_1\mathbf{q}_2\rangle = \langle\bar{\psi}_\uparrow(\mathbf{q}_1)\bar{\psi}_\uparrow(\mathbf{q}_2)\rangle\langle\psi_\uparrow(\mathbf{q}_1')\psi_\uparrow(\mathbf{q}_2')\rangle. \tag{4.57}$$

The condensate fraction may split into two contributions, then: the singlet one coming from (4.56) and the triplet one from (4.57),

$$n_0 = n_s + n_t, \tag{4.58}$$

with

$$\boxed{n_s = \frac{1}{V}\int_V d^3q_1 \int_V d^3q_2 |\langle\psi_\downarrow(\mathbf{q}_1)\psi_\uparrow(\mathbf{q}_2)\rangle|^2} \tag{4.59}$$

and

$$\boxed{n_t = \frac{1}{V}\int_V d^3q_1 \int_V d^3q_2 |\langle\psi_\uparrow(\mathbf{q}_1)\psi_\uparrow(\mathbf{q}_2)\rangle|^2}. \tag{4.60}$$

It is straightforward to prove that after the Matzubara frequency summation

$$\langle\bar{\psi}_\uparrow(\mathbf{q}_1)\bar{\psi}_\downarrow(\mathbf{q}_2)\rangle = \frac{\bar{\Delta}_0}{4V}\sum_{\mathbf{p}}\Big(\frac{\tanh[\frac{\beta}{2}(\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} + \omega_R)]}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} + \frac{\tanh[\frac{\beta}{2}(\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} - \omega_R)]}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}}\Big)e^{-i\mathbf{p}\cdot(\mathbf{q}_1 - \mathbf{q}_2)}, \tag{4.61}$$

so that one may write the singlet contribution of the condensate fraction as

$$n_s = \frac{|\Delta_0|^2}{16V}\sum_{\mathbf{p}}\Big(\frac{\tanh[\frac{\beta}{2}(\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} + \omega_R)]}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}} + \frac{\tanh[\frac{\beta}{2}(\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} - \omega_R)]}{\sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2}}\Big)^2. \tag{4.62}$$

The calculation of $\langle\bar{\psi}_\uparrow(\mathbf{q}_1)\bar{\psi}_\uparrow(\mathbf{q}_2)\rangle$, instead, yields exactly zero after the sum over Matzubara frequencies, since that element of the propagator is odd in $i\omega$. Rabi coupling, then, does not enable the formation of spinful Cooper pairs, contrary to what happens in the similar case of spin-orbit coupling [16], meaning that

$$n_t = 0. \tag{4.63}$$

Once again we proved that OLDRO may occur only below the critical temperature T_c, when $|\Delta_0| \neq 0$, since our formula for n_0 is proportional to $|\Delta_0|^2$.

As it happened for the gap and number equations at zero temperature, in the $T \to 0^+$ limit, the expression for the condensate fraction reduces to the one of the Rabiless case. The only difference that we have to keep in mind is the behaviour of the energy gap, which cannot take values below the Rabi frequency ω_R and instead has to abruptly go to zero at that point, meaning that

$$\boxed{n_0 = \begin{cases} \frac{(m|\Delta_0|)^{\frac{3}{2}}}{8\pi}\sqrt{\frac{\mu}{|\Delta_0|} + \sqrt{\left(\frac{\mu}{|\Delta_0|}\right)^2 + 1}} \text{ for } |\Delta_0| > \omega_R \\ 0 \text{ for } |\Delta_0| \leqslant \omega_R. \end{cases}} \tag{4.64}$$

The plot in figure 16 shows the values of the condensate fraction corresponding to the values of the chemical potential and energy gap solving (4.21) reported in figure 13. The parts of the curves getting away from the one of the $\frac{\omega_R}{\omega_F} = 0$ case, then, are unphysical.

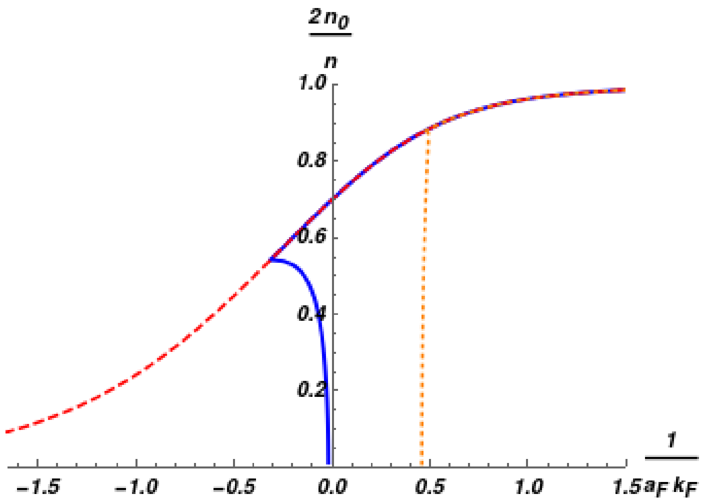

Figure 16: Condensate fraction *vs* inverse scattering length over the whole crossover in the $T \to 0^+$ limit for different values of $\frac{\omega_R}{\omega_F}$. Red dashed line: curve in the absence of Rabi coupling, $\frac{\omega_R}{\omega_F} = 0$. Thick blue line: fixed value of $\frac{\omega_R}{\omega_F} = 0.5$. Orange dotted line: fixed value of $\frac{\omega_R}{\omega_F} = 1$.

5 Gaussian Fluctuations in the Rabi Coupled BCS-BEC Crossover

We now introduce Gaussian fluctuations in the partition function of the system. The objective is analogous to the one of the Rabiless case: derive a more precise form for the number equation in order to understand the role of quantum fluctuations on the relation between the chemical potential μ and the density of particles n.

Moreover, in this section we are going to show also another approach to go beyond mean field at the critical temperature over the whole crossover.

5.1 General Form of the Grand Canonical Potential

In this section we derive the expression for the grand canonical potential at a generic inverse temperature β. This will be the starting point of the treatment at the critical temperature.

5.1.1 Expansion of the Action

We start from the effective action that we obtained from performing the Hubbard-Stratonovich transformation and by integrating out the fermionic degrees of freedom

$$S^{eff}[\bar{\Delta},\Delta] = \int_0^\beta d\tau \int_V d^3q \frac{|\Delta(\mathbf{q},\tau)|^2}{g} - \frac{1}{2} Tr[\ln G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta)] + \beta \sum_{\mathbf{p}} \xi_{\mathbf{p}}, \tag{5.1}$$

where in Matzubara representation

$$G^{-1}_{KP}(\bar{\Delta},\Delta) = \begin{pmatrix} (i\Omega^F_n - \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & \Delta_{K+P} & -\omega_R \delta^{(4)}_{K,-P} & 0 \\ \bar{\Delta}_{K+P} & (i\Omega^F_n + \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & 0 & \omega_R \delta_{k,-p} \\ -\omega_R \delta^{(k,-p)} & 0 & (i\Omega^F_n - \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & -\Delta_{K+P} \\ 0 & \omega_R \delta^{(4)}_{K,-P} & -\bar{\Delta}_{K+P} & (i\Omega^F_n + \xi_{\mathbf{p}})\delta^{(4)}_{K,P}; \end{pmatrix}. \tag{5.2}$$

In order to introduce fluctuations we separate the field $\Delta(\mathbf{q},\tau)$ in its homogeneous part Δ_0 minimizing the grand potential at the mean field level and its fluctuations $\eta(\mathbf{q},\tau)$ around it so that

$$\Delta(\mathbf{q},\tau) = \Delta_0 + \eta(\mathbf{q},\tau). \tag{5.3}$$

Recall that $|\Delta_0|$ has to solve the gap equation

$$-\frac{m}{4\pi a_F} = \frac{1}{V} \sum_{\mathbf{p}} \Big[\frac{\tanh\left(\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right)}{4\sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2}} + \frac{\tanh\left(\frac{\beta}{2}\omega_-(\mathbf{p},\omega_R)\right)}{4\sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2}} - \frac{m}{\mathbf{p}^2} \Big], \tag{5.4}$$

with

$$\omega_+(\mathbf{p},\omega_R) = \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2} + \omega_R, \quad \omega_-(\mathbf{p},\omega_R) = \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2} - \omega_R, \tag{5.5}$$

below the critical temperature. This way one may write

$$G^{-1}_{KP}(\bar{\Delta},\Delta) = \tilde{G}^{-1}_{KP} + \eta_{KP}, \tag{5.6}$$

with

$$\tilde{G}^{-1}_{KP} = \begin{pmatrix} (i\Omega^F_n - \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & \Delta_0\delta^{(4)}_{K,-P} & -\omega_R\delta^{(4)}_{K,-P} & 0 \\ \bar{\Delta}_0\delta^{(4)}_{K,-P} & (i\Omega^F_n + \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & 0 & \omega_R\delta_{k,-p} \\ -\omega_R\delta^{(k,-p)} & 0 & (i\Omega^F_n - \xi_{\mathbf{p}})\delta^{(4)}_{K,P} & -\Delta_0\delta^{(4)}_{K,-P} \\ 0 & \omega_R\delta^{(4)}_{K,-P} & -\bar{\Delta}_0\delta^{(4)}_{K,-P} & (i\Omega^F_n + \xi_{\mathbf{p}})\delta^{(4)}_{K,P}; \end{pmatrix} \tag{5.7}$$

and

$$\eta_{PK} = \begin{pmatrix} 0 & \eta_{K+P} & 0 & 0 \\ \bar{\eta}_{K+P} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta_{K+P} \\ 0 & 0 & -\bar{\eta}_{K+P} & 0 \end{pmatrix}. \tag{5.8}$$

Then, the expansion of the trace of the logarithm of G^{-1}_{KP} is

$$Tr[\ln G^{-1}_{KP}] \approx Tr[\ln \tilde{G}^{-1}_{kp}] + Tr[\tilde{G}_{KP}\eta_{PK}] - \frac{1}{2}Tr[\tilde{G}_{KP}\eta_{PL}\tilde{G}_{LM}\eta_{MK}], \tag{5.9}$$

as in the Rabiless case, yielding the mean field action

$$\boxed{S_{MF} = \beta V\frac{|\Delta_0|^2}{g} - \frac{1}{2}Tr[\ln \tilde{G}^{-1}_{KP}]} \tag{5.10}$$

from the first term, the cancellation of linear terms in η_K from the second one and a Gaussian term from the last one. For the calculation of the last term, we start by stating that

$$\begin{aligned} \frac{1}{2}Tr[\tilde{G}_{KP}\eta_{PL}\tilde{G}_{LM}\eta_{MK}] = & \frac{k_BT}{V}\sum_{P,K}\bar{\eta}_{-K}\Big[(\tilde{G}_{12})_P(\tilde{G}_{12})_{P+K} + (\tilde{G}_{14})_P(\tilde{G}_{14})_{P+K}\Big]\bar{\eta}_K + \\ & +\frac{k_BT}{V}\sum_{P,K}\eta_{-K}\Big[(\tilde{G}_{12})_P(\tilde{G}_{12})_{P+K} + (\tilde{G}_{14})_P(\tilde{G}_{14})_{P+K}\Big]\eta_K + \\ & +2\frac{k_BT}{V}\sum_{P,K}\bar{\eta}_K\Big[(\tilde{G}_{11})_P(\tilde{G}_{22})_{K-P} + (\tilde{G}_{13})_P(\bar{\tilde{G}}_{13})_{K-P}\Big]\eta_K, \end{aligned} \tag{5.11}$$

so that such term in the action may be written in Nambu space as

$$\boxed{S^{eff}_G = S_{MF} + \frac{1}{2}\sum_K \begin{pmatrix}\bar{\eta}_K & \eta_{-K}\end{pmatrix} M_K \begin{pmatrix}\eta_K \\ \bar{\eta}_{-K}\end{pmatrix}}, \tag{5.12}$$

with

$$\boxed{M_K = \frac{1}{g}\mathbb{I} + \chi_K}, \tag{5.13}$$

where χ_K is the contribution coming from the trace of the logarithm and $\mathbb{I}$ denotes the 2×2 identity matrix in Nambu space. The components of χ_K read

$$\begin{aligned}(\chi_K)_{11} = (\chi_{-K})_{22} = \frac{k_B T}{V} \sum_P \frac{[(i\Omega_n^F - \xi_{\mathbf{p}})((i\Omega_n^F)^2 - \xi_{\mathbf{p}}^2 - |\Delta_0|^2) - \omega_R^2(i\Omega_n^F + \xi_{\mathbf{p}})]}{((\Omega_n^F)^2 + \omega_+^2(\mathbf{p}, \omega_R))((\Omega_n^F)^2 + \omega_-^2(\mathbf{p}, \omega_R))} \times \\ \times \frac{[(i\Omega_m^B + i\Omega_n^F + \xi_{\mathbf{k}+\mathbf{p}})((i\Omega_m^B + i\Omega_n^F)^2 - \xi_{\mathbf{k}+\mathbf{p}}^2 - |\Delta_0|^2) - \omega_R^2(i\Omega_m^B + i\Omega_n^F - \xi_{\mathbf{k}+\mathbf{p}})]}{((\Omega_m^B + \Omega_n^F)^2 + \omega_+^2(\mathbf{k}+\mathbf{p}, \omega_R))((\Omega_m^B + \Omega_n^F)^2 + \omega_-^2(\mathbf{k}+\mathbf{p}, \omega_R)} + \\ + \omega_R \frac{k_B T}{V} \sum_P \frac{[|\Delta_0|^2 - \omega_R^2 + (i\Omega_n^F - \xi_{\mathbf{p}})^2]}{((\Omega_n^F)^2 + \omega_+^2(\mathbf{p}, \omega_R))((\Omega_n^F)^2 + \omega_-^2(\mathbf{p}, \omega_R))} \times \\ \times \frac{[|\Delta_0|^2 - \omega_R^2 + (i\Omega_m^B + i\Omega_n^F + \xi_{\mathbf{p}+\mathbf{k}})^2]}{((\Omega_m^B + \Omega_n^F)^2 + \omega_+^2(\mathbf{p}+\mathbf{k}, \omega_R))((\Omega_m^B + \Omega_n^F)^2 + \omega_-^2(\mathbf{p}+\mathbf{k}, \omega_R))}\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}(\chi_K)_{12} = (\chi_K)_{21} = |\Delta_0|^2 \frac{k_B T}{V} \sum_P \frac{\left[((\Omega_n^F)^2 + \xi_{\mathbf{p}}^2) - \omega_R^2 + |\Delta_0|^2\right]}{((\Omega_n^F)^2 + \omega_+^2(\mathbf{p}, \omega_R))((\Omega_n^F)^2 + \omega_-^2(\mathbf{p}, \omega_R))} \times \\ \times \frac{\left[((\Omega_m^B + \Omega_{n_F})^2 + \xi_{\mathbf{p}+\mathbf{k}}^2) - \omega_R^2 + |\Delta_0|^2\right]}{((\Omega_m^B + \Omega_n^F)^2 + \omega_+^2(\mathbf{p}+\mathbf{k}))((\Omega_m^B + \Omega_n^F)^2 + \omega_-^2(\mathbf{p}+\mathbf{k}))} + \\ -4\omega_R^2 |\Delta_0|^2 \frac{k_B T}{V} \sum_P \frac{\Omega_n^F}{((\Omega_n^F)^2 + \omega_+^2(\mathbf{p}, \omega_R))((\Omega_n^F)^2 + \omega_-^2(\mathbf{p}, \omega_R))} \times \\ \times \frac{(\Omega_m^B + \Omega_n^F)}{((\Omega_m^B + \Omega_n^F)^2 + \omega_+^2(\mathbf{p}+\mathbf{k}, \omega_R))((\Omega_m^B + \Omega_n^F)^2 + \omega_-^2(\mathbf{p}+\mathbf{k}, \omega_R))}.\end{aligned} \tag{5.15}$$

5.1.2 The Grand Canonical Potential

Our objective is to find an expression for the grand canonical potential from which we can recover a treatable expression for the contribution of the Gaussian fluctuations to the number equation as the one in [17]. The theory we obtained is Gaussian, meaning that it can be integrated explicitly, giving

$$Z_G = e^{-S_{MF}} \prod_K{}' \left[\det(M_K) \right]^{-1}, \tag{5.16}$$

where by $\prod'_K$ we mean the product of all K except for the ones such that $\det(M_K) = 0$. For example, as in the Rabiless case at the critical temperature, in fact, for $K = 0$ the determinant of M_0 yields zero, since it takes the form of the gap equation. To avoid the divergence of the grand potential and the degeneration of the theory, then, we may consider the fluctuation field η_0 to be classical, meaning that it will not be involved in the path integration.

The grand potential will simply be

$$\frac{\Omega_G}{V} = \frac{k_B T}{V} S_{MF} + \frac{k_B T}{V} \sum_K{}' \ln[\det(M_K)], \tag{5.17}$$

with the prime in $\sum_K'$ taking the same meaning as the one in the product in (5.16). To compute the sum over Matzubara frequencies in (5.16) one may analytically continue the argument of the sum by promoting

$$i\Omega_n^B \to \tilde{\omega} \tag{5.18}$$

as we did in the Rabiless case and transform the sum in an integral so that

$$\frac{\Omega_G}{V} - \frac{k_B T}{V} S_{MF} = -\frac{1}{V 2\pi i} \sum_{\mathbf{k}} \int_C d\tilde{\omega} n_B(\tilde{\omega}) \ln \Big[\det(M_{\tilde{\omega},\mathbf{k}}) \Big], \tag{5.19}$$

with

$$n_B(\tilde{\omega}) = (e^{\beta\tilde{\omega}} - 1)^{-1} \tag{5.20}$$

the Bose-Einstein distribution and C is a closed integration contour containing the whole imaginary axis of the complex $\tilde{\omega}$ plane.

Supposing that $\ln \Big[\det(M_{\tilde{\omega},\mathbf{k}}) \Big]$ has no poles in the complex $\tilde{\omega}$ plane and a branch cut on the real axis, as will be shown, we can modify the integration contour C to one containing the real axis, so that one gets

$$\frac{\Omega_G}{V} - \frac{k_B T}{V} S_{MF} = -\frac{1}{V 2\pi i} \sum_{\mathbf{k}} \int_{-\infty}^{+\infty} d\tilde{\omega} n_B(\tilde{\omega}) \Big(\ln \Big[\det(M_{\tilde{\omega}+i\varepsilon,\mathbf{k}}) \Big] - \ln \Big[\det(M_{\tilde{\omega}-i\varepsilon,\mathbf{k}}) \Big] \Big). \tag{5.21}$$

In doing so, one has to be careful not to take into account the contribution coming from $\tilde{\omega} = 0$, which is a pole of the Bose Einstein distribution. By using the Euler representation for the determinant of $M_{\tilde{\omega}\pm i\varepsilon,\mathbf{k}}$ one may write

$$\det(M_{\tilde{\omega}\pm i\varepsilon,\mathbf{k}}) = |\det(M_{\tilde{\omega}\pm i\varepsilon,\mathbf{k}})| e^{\pm i\delta(\tilde{\omega}+i\varepsilon,\mathbf{k})}, \tag{5.22}$$

with the phase

$$\boxed{\delta(\tilde{\omega},\mathbf{k}) = \arctan\Big(\frac{Im[\det(M_{\tilde{\omega},\mathbf{k}}]}{Re[\det(M_{\tilde{\omega},\mathbf{k}}]} \Big).} \tag{5.23}$$

Finally, then, one can write the grand potential in terms of an integral of the phase $\delta(\tilde{\omega},\mathbf{k})$ as

$$\boxed{\frac{\Omega_G}{V} = \frac{k_B T}{V} S_{MF} - \frac{1}{\pi V} \sum_{\mathbf{k}} \int_{-\infty}^{+\infty} d\tilde{\omega} n_B(\tilde{\omega}) \Big[\delta(\tilde{\omega},\mathbf{k}) - \delta(0,\mathbf{k}) \Big].} \tag{5.24}$$

The issue for future calculations, then, will be to identify the real and imaginary parts of the determinant of $M_{\tilde{\omega},\mathbf{k}}$ in order to compute the phase $\delta(\tilde{\omega},\mathbf{k})$.

5.1.3 Number and Gap Equations

Now that we have an explicit form of the grand canonical potential, we may write the number equation for the system as

$$n = -\partial_\mu \frac{\Omega_G}{V}. \tag{5.25}$$

Such will consist of two contributions, then, one coming from the mean field action, which will be identical to the one found in (4.21) and one coming from Gaussian fluctuations. Namely,

$$\boxed{n = \frac{1}{V}\sum_{\mathbf{p}}\Big[1 - \frac{\xi_{\mathbf{p}}}{2}\frac{\tanh\left(\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right)}{\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}} - \frac{\xi_{\mathbf{p}}}{2}\frac{\tanh\left(\frac{\beta}{2}\omega_-(\mathbf{p},\omega_R)\right)}{\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}}\Big] + \frac{1}{\pi}\sum_{\mathbf{k}}\int_{-\infty}^{+\infty} d\tilde{\omega} n_B(\tilde{\omega})\frac{\partial\delta(\tilde{\omega},\mathbf{k})}{\partial\mu}.} \tag{5.26}$$

To determine the physical quantities of the system we still have to take the gap equation into consideration. Such is the one that determines the value of the energy gap $|\Delta_0|$ around which the bosonic field $\eta(\mathbf{q},\tau)$ fluctuates, meaning that it will not change its form with respect to the one in the mean field treatment. In particular the equation will read

$$\boxed{-\frac{m}{4\pi a_F} = \frac{1}{V}\sum_{\mathbf{p}}\Big[\frac{\tanh\left(\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right)}{4\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}} + \frac{\tanh\left(\frac{\beta}{2}\omega_-(\mathbf{p},\omega_R)\right)}{4\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}} - \frac{m}{\mathbf{p}^2}\Big],} \tag{5.27}$$

always taking into account the same renormalization of the contact potential

$$\frac{1}{g} = -\frac{m}{4\pi a_F} + \frac{1}{V}\sum_{\mathbf{p}}\frac{m}{\mathbf{p}^2}. \tag{5.28}$$

Solving the system made of (5.27) and (5.26) one can retrieve information about the main physical quantities of the system, as in the mean field case. The main difference is provided by the modified form of the number equation, which leads to a different dependence of the chemical potential μ on the fixed number of particles $n = \frac{N}{V}$. The difficult part in the solution of such equations is to find a treatable expression for the phase $\delta(\tilde{\omega},\mathbf{k})$ that appears in the number equation. Our main efforts in the following will be dedicated to this task.

5.1.4 Sum over Matzubara Frequencies for χ_k

The sum over Matzubara frequencies of the elements of χ_K (5.14) and (5.15) can be performed explicitly. The procedure is all in all the same one used in appendix B for the Rabiless case, with the difference that more terms will be involved. In particular we have completely new contributions which come from the Rabi coupling, proportional to ω_R, both in the diagonal and offdiagonal terms.

In the following we will separate the contributions proportional to ω_R from the other ones in order to have clearly in mind the effects coming purely from the Rabi coupling.

Moreover, in order to write such elements in a compact form, we will introduce some auxiliary functions of the single particle energy $\omega_+(\mathbf{p},\omega_R)$ defined in (5.5), of the bosonic four vector $K=(i\Omega_n^B,\mathbf{k})$ and of the Rabi frequency ω_R. All the information regarding $(\chi_K)_{11}$ can be put inside these three functions

$$f_1(\omega_+(\mathbf{p},\omega_R),K)=\frac{(\omega_+(\mathbf{p},\omega_R)-\xi_{\mathbf{p}})(\omega_+^2(\mathbf{p},\omega_R)-\xi_{\mathbf{p}}^2-|\Delta_0|^2)}{8\omega_R\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\omega_+(\mathbf{p},\omega_R)}\times$$
$$\times\frac{(\omega_+(\mathbf{p},\omega_R)+i\Omega_n^B+\xi_{\mathbf{p+k}})((\omega_+(\mathbf{p},\omega_R)+i\Omega_n^B)^2-\xi_{\mathbf{p+k}}-|\Delta_0|^2)\tanh\left[\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right]}{(\Omega_n^{B2}+\omega_+^2(\mathbf{p+k})-\omega_+^2(\mathbf{p},\omega_R)-2i\Omega_n^B\omega_+(\mathbf{p},\omega_R))(\Omega_n^{B2}+\omega_-^2(\mathbf{p+k})-\omega_+^2(\mathbf{p},\omega_R)-2i\Omega_n^B\omega_+(\mathbf{p},\omega_R))};$$
$$f_2(\omega_+(\mathbf{p},\omega_R),K)=\frac{[(\omega_+(\mathbf{p},\omega_R)+\xi_{\mathbf{p}})(\omega_+(\mathbf{p},\omega_R)+i\Omega_n^B+\xi_{\mathbf{p+k}})((\omega_+(\mathbf{p},\omega_R)+i\Omega_n^B)^2-\xi_{\mathbf{p+k}}-|\Delta_0|^2)+}{8\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\omega_+(\mathbf{p},\omega_R)}\times$$
$$\times\frac{+(\omega_+(\mathbf{p},\omega_R)+i\Omega_n^B-\xi_{\mathbf{p+k}})(\omega_+(\mathbf{p},\omega_R)-\xi_{\mathbf{p}})(\omega_+^2(\mathbf{p},\omega_R)-\xi_{\mathbf{p}}^2-|\Delta_0|^2)]\tanh\left[\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right]}{(\Omega_n^{B2}+\omega_+^2(\mathbf{p+k})-\omega_+^2(\mathbf{p},\omega_R)-2i\Omega_n^B\omega_+(\mathbf{p},\omega_R))(\Omega_n^{B2}+\omega_-^2(\mathbf{p+k})-\omega_+^2(\mathbf{p},\omega_R)-2i\Omega_n^B\omega_+(\mathbf{p},\omega_R))}$$
$$f_3(\omega_+(\mathbf{p},\omega_R),K)=\frac{(\omega_+(\mathbf{p},\omega_R)+\xi_{\mathbf{p}})}{8\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\omega_+(\mathbf{p},\omega_R)}\times$$
$$\times\frac{(\omega_+(\mathbf{p},\omega_R)+i\Omega_n^B-\xi_{\mathbf{p+k}})\tanh\left[\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right]}{(\Omega_n^{B2}+\omega_+^2(\mathbf{p+k})-\omega_+^2(\mathbf{p},\omega_R)-2i\Omega_n^B\omega_+(\mathbf{p},\omega_R))(\Omega_n^{B2}+\omega_-^2(\mathbf{p+k})-\omega_+^2(\mathbf{p},\omega_R)-2i\Omega_n^B\omega_+(\mathbf{p},\omega_R))},$$
$$f_4(\omega_+(\mathbf{p},\omega_R),K)=\frac{((\omega_+(\mathbf{p},\omega_R)-\xi_{\mathbf{p}})^2+|\Delta_0|^2-\omega_R^2)}{8\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\omega_+(\mathbf{p},\omega_R)}\times$$
$$\times\frac{((\omega_+(\mathbf{p},\omega_R)+i\Omega_n^B+\xi_{\mathbf{p+k}})^2+|\Delta_0|^2-\omega_R^2)\tanh\left[\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right]}{(\Omega_n^{B2}+\omega_+^2(\mathbf{p+k})-\omega_+^2(\mathbf{p},\omega_R)-2i\Omega_n^B\omega_+(\mathbf{p},\omega_R))(\Omega_n^{B2}+\omega_-^2(\mathbf{p+k})-\omega_+^2(\mathbf{p},\omega_R)-2i\Omega_n^B\omega_+(\mathbf{p},\omega_R))} \tag{5.29}$$

keeping in mind that

$$\omega_+(\mathbf{p},-\omega_R)=\omega_-(\mathbf{p},\omega_R). \tag{5.30}$$

The diagonal element $(\chi_K)_{11}$, then, has the form

$$\boxed{\begin{aligned}(\chi_k)_{11}=&\\ \frac{1}{V}\sum_{\mathbf{p}}\Big[&(-f_1+\omega_R f_2-\omega_R^3 f_3-\omega_R f_4)(\omega_+(\mathbf{p},\omega_R),K)+\\ &+(-f_1+\omega_R f_2-\omega_R^3 f_3-\omega_R f_4)(-\omega_+(\mathbf{p},\omega_R),K)+\\ &+(f_1-\omega_R f_2+\omega_R^3 f_3+\omega_R f_4)(\omega_+(\mathbf{p},-\omega_R),K)+\\ &+(f_1-\omega_R f_2+\omega_R^3 f_3-\omega_R f_4)(-\omega_+(\mathbf{p},-\omega_R),K)\Big],\end{aligned}} \tag{5.31}$$

where we placed the arguments of the functions at the end to make the equation more compact. It is useful to have this expression written in such a way to highlight the symmetry in the sum, since it will make further calculations more straight forward.

To express the off-diagonal elements of χ_K, instead, we define the three functions

$$\begin{aligned}
g_1(\omega_+(\mathbf{p},\omega_R),K) &= \frac{(\omega_+^2(\mathbf{p},\omega_R)+\xi_{\mathbf{p}}^2+|\Delta_0|^2)}{8\omega_R\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\omega_+(\mathbf{p},\omega_R)}\times \\
&\times\frac{[(\omega_+(\mathbf{p},\omega_R)+\Omega_n^B)^2+\xi_{\mathbf{p+k}}^2+|\Delta_0|^2]\tanh\left[\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right]}{((\Omega_m^B+\Omega_n^F)^2+\omega_+^2(\mathbf{p+k},\omega_R))((\Omega_m^B+\Omega_n^F)^2+\omega_-^2(\mathbf{p+k},\omega_R))};\\
g_2(\omega_+(\mathbf{p},\omega_R),K) &= \frac{[\omega_+^2(\mathbf{p},\omega_R)+\xi_{\mathbf{p}}^2+|\Delta_0|^2+}{8\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\omega_+(\mathbf{p},\omega_R)}\times \\
&\times\frac{+(\omega_+(\mathbf{p},\omega_R)+\Omega_n^B)^2+\xi_{\mathbf{p+k}}^2+|\Delta_0|^2]\tanh\left[\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right]}{((\Omega_m^B+\Omega_n^F)^2+\omega_+^2(\mathbf{p+k},\omega_R))((\Omega_m^B+\Omega_n^F)^2+\omega_-^2(\mathbf{p+k},\omega_R))};\\
g_3(\omega_+(\mathbf{p},\omega_R),K) &= \frac{1}{8\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\omega_+(\mathbf{p},\omega_R)}\times \\
&\times\frac{\tanh\left[\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right]}{((\Omega_m^B+\Omega_n^F)^2+\omega_+^2(\mathbf{p+k}))((\Omega_m^B+\Omega_n^F)^2+\omega_-^2(\mathbf{p+k}))}\\
g_4(\omega_+(\mathbf{p},\omega_R),K) &= \frac{\omega_+(\mathbf{p},\omega_R)}{8\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\omega_+(\mathbf{p},\omega_R)}\times \\
&\times\frac{(\omega_+(\mathbf{p},\omega_R)+\Omega_n^B)\tanh\left[\frac{\beta}{2}\omega_+(\mathbf{p},\omega_R)\right]}{((\Omega_m^B+\Omega_n^F)^2+\omega_+^2(\mathbf{p+k},\omega_R))((\Omega_m^B+\Omega_n^F)^2+\omega_-^2(\mathbf{p+k},\omega_R))}.
\end{aligned} \tag{5.32}$$

Then, one can express $(\chi_K)_{12}$ in a similar way with respect to $(\chi_K)_{11}$, as

$$\boxed{\begin{aligned}
(\chi_K)_{12} = \frac{|\Delta_0|^2}{V}\sum_{\mathbf{p}}\Big[&(-g_1+\omega_R g_2-\omega_R^3 g_3+4\omega_R g_4)(\omega_+(\mathbf{p},\omega_R),K)+\\
&+(-g_1+\omega_R g_2-\omega_R^3 g_3+4\omega_R g_4)(-\omega_+(\mathbf{p},\omega_R),K)+\\
&+(g_1-\omega_R g_2+\omega_R^3 g_3-4\omega_R g_4)(\omega_+(\mathbf{p},-\omega_R),K)+\\
&+(g_1-\omega_R g_2+\omega_R^3 g_3-4\omega_R g_4)(-\omega_+(\mathbf{p},-\omega_R),K)\Big].
\end{aligned}} \tag{5.33}$$

With these expressions in mind, we can procede with the treatment of the system at the critical temperature.

5.2 Critical Temperature

We now procede with the treatment of the system at the critical temperature with the inclusion of Gaussian fluctuations. The aim is to find a reasonable expression for χ_K in order to rewrite the number equation in a treatable way numerically. In doing so, one may obtain the plot of the behaviour of the critical temperature with respect to the varying of the scattering length along the whole crossover.

5.2.1 M_K at the Critical Temperature

The off-diagonal terms of M_K vanish at the critical temperature, as they are proportional to $|\Delta_0|^2$. As a result, one can consider M_K as a scalar instead of a matrix in Nambu space. The technical point is the calculation of χ_K, which is twice (5.31) in this case. The situation will be quite different from the Rabiless case, as some completely new contributions will appear due to Rabi coupling. Anyway, we will be able to write it in a quite compact form.

The first thing to notice is that by imposing that the energy gap $|\Delta_0| = 0$ the first three terms in the first row of (5.31) cancel each other, implying that the same happens to the first three in the third row. Equivalently, the fourth term on the first row of (5.31) vanishes, and consequently the last term in the third row will, too. Then, we are left with eight contributions, six of which come purely from Rabi coupling, since they are proportional to ω_R, and two that correspond to the one in the Rabiless case, meaning that they reduce to the Rabiless inverse propagator of the fluctuations in the case in which $\omega_R = 0$. All in all the eight terms coming from Rabi coupling can be put together to obtain an even more compact form of χ_K with some simple algebraic calculations, so that calling

$$D_1(\mathbf{p}, \omega_R, K) = (\xi_{\mathbf{p}+\mathbf{k}} - \xi_{\mathbf{p}} - 2\omega_R + i\Omega_n^B)(\xi_{\mathbf{p}+\mathbf{k}} + \xi_{\mathbf{p}} + 2\omega_R - i\Omega_n^B), \tag{5.34}$$

one gets that

$$\begin{aligned}\chi_K = \frac{1}{2V}\sum_{\mathbf{p}}\Big[-\frac{[\xi_{\mathbf{p}+\mathbf{k}} - \xi_{\mathbf{p}} + i\Omega_n^B]}{D_1(\mathbf{p}, K)} \tanh\Big[\frac{\beta_c}{2}(\xi_{\mathbf{p}} + \omega_R)\Big] + \omega_R\Big(\frac{(\xi_{\mathbf{p}+\mathbf{k}} + 3\xi_{\mathbf{p}} + 2\omega_R - 3i\Omega_n^B)}{D_1(\mathbf{p}, K)(\xi_{\mathbf{p}+\mathbf{k}} + \xi_{\mathbf{p}} - i\Omega_n^B)} + \\ - \frac{1}{(\xi_{\mathbf{p}+\mathbf{k}} + \xi_{\mathbf{p}} + 2\omega_R - i\Omega_n^B)(\xi_{\mathbf{p}+\mathbf{k}} + \xi_{\mathbf{p}} - i\Omega_n^B)}\Big) \tanh\Big[\frac{\beta_c}{2}(\xi_{\mathbf{p}} + \omega_R)\Big]\Big] + \\ + (\omega_R \to -\omega_R),\end{aligned} \tag{5.35}$$

where with the notation $(\omega_R \to -\omega_R)$ we mean the sum of all of the same terms with the substitution $\omega_R \to -\omega_R$. Notice that only the first term is ultraviolet divergent, while the purely Rabi contributions are all convergent.

To further simplify the expression, we add $\frac{2\omega_R}{D_1(\mathbf{p},K)}$ to the first term and subtract it to the second one. This way the expression will become much more feasible:

$$
\begin{aligned}
\chi_K = \frac{1}{2V}\sum_{\mathbf{p}}\Big[&- \frac{\tanh\left[\frac{\beta_c}{2}(\xi_{\mathbf{p}}+\omega_R)\right]}{(\xi_{\mathbf{p+k}}+\xi_{\mathbf{p}}+2\omega_R-i\Omega_n^B)}+ \\
&- \frac{2\omega_R}{(\xi_{\mathbf{p+k}}+\xi_{\mathbf{p}}+2\omega_R-i\Omega_n^B)(\xi_{\mathbf{p+k}}+\xi_{\mathbf{p}}-i\Omega_n^B)}\tanh\left[\frac{\beta_c}{2}(\xi_{\mathbf{p}}+\omega_R)\right]\Big]+ \\
&+(\omega_R\to-\omega_R).
\end{aligned}
\tag{5.36}
$$

The great simplification lies in the fact that now there are no more terms of the kind $\xi_{\mathbf{p+k}}-\xi_{\mathbf{p}}$, but only of the kind $\xi_{\mathbf{p+k}}+\xi_{\mathbf{p}}$, meaning that by shifting the argument of the sum $\mathbf{p}\to\mathbf{p}-\frac{\mathbf{k}}{2}$ we can move the dependence on the angle between $\mathbf{p}$ and $\mathbf{k}$ to the argument of the hyperbolic tangent alone. Thanks to this fact one can transform the sum over $\mathbf{p}$ into an integral in polar coordinates $\sum_{\mathbf{p}}\to\int_0^{2\pi}d\varphi\int_{-1}^{1}d\cos\theta\int_0^{+\infty}dpp^2$ and solve analytically the integral over $d\cos\theta$, leaving a one dimensional integral that can be solved numerically.

All in all, recalling the definition (5.13), the inverse propagator of the fluctuations can be written as

$$
\boxed{
\begin{aligned}
M_K = -\frac{m}{4\pi a_F} - \frac{1}{2V}\sum_{\mathbf{p}}\Big[&\frac{\tanh\left[\frac{\beta_c}{2}(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\omega_R)\right]}{(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\xi_{\mathbf{p}-\frac{\mathbf{k}}{2}}+2\omega_R-i\Omega_n^B)}-\frac{m}{\mathbf{p}^2}\Big]+ \\
+\frac{\omega_R}{V}\sum_{\mathbf{p}}&\frac{\tanh\left[\frac{\beta_c}{2}(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\omega_R)\right]}{(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\xi_{\mathbf{p}-\frac{\mathbf{k}}{2}}+2\omega_R-i\Omega_n^B)(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\xi_{\mathbf{p}-\frac{\mathbf{k}}{2}}-i\Omega_n^B)}+(\omega_R\to-\omega_R).
\end{aligned}
}
\tag{5.37}
$$

5.2.2 Real and Imaginary parts of $M_{\tilde{\omega},\mathbf{k}}$

To treat the number equation, one has to continue analytically M_K in (5.37) by promoting

$$
i\Omega_n^B\to\tilde{\omega}\in\mathbb{C}
\tag{5.38}
$$

and calculate the real and imaginary parts of

$$
\begin{aligned}
M_{\tilde{\omega},\mathbf{k}} = -\frac{m}{4\pi a_F} - \frac{1}{2V}\sum_{\mathbf{p}}\Big[&\frac{\tanh\left[\frac{\beta_c}{2}(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\omega_R)\right]}{(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\xi_{\mathbf{p}-\frac{\mathbf{k}}{2}}+2\omega_R-\tilde{\omega})}-\frac{m}{\mathbf{p}^2}\Big]+ \\
-\frac{\omega_R}{V}\sum_{\mathbf{p}}&\frac{\tanh\left[\frac{\beta_c}{2}(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\omega_R)\right]}{(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\xi_{\mathbf{p}-\frac{\mathbf{k}}{2}}+2\omega_R-\tilde{\omega})(\xi_{\mathbf{p}+\frac{\mathbf{k}}{2}}+\xi_{\mathbf{p}-\frac{\mathbf{k}}{2}}-\tilde{\omega})}+(\omega_R\to-\omega_R).
\end{aligned}
\tag{5.39}
$$

In order to do so we transform the sum over $\mathbf{p}$ into an integral in polar coordinates and define the quantities

$$
\boxed{x^2=\beta_c\frac{p^2}{2m},\quad z^2=\beta_c\frac{k^2}{2m},\quad \omega=\beta_c\tilde{\omega},\quad z_0=\beta_c\mu,\quad f=\beta_c\omega_R,}
\tag{5.40}
$$

so that one can render the integral dimensionless, getting

$$\begin{aligned} M_{\omega,z} = -\frac{m}{4\pi}\Big[\frac{1}{a_F}+ \\ +\frac{\sqrt{2mk_BT_c}}{2\pi}\int_{-1}^{1} d\cos\theta\int_0^{+\infty} dx x^2\Big(\frac{\tanh\left(\frac{1}{2}\left[x^2+\frac{z^2}{4}+xz\cos\theta-z_0+f\right]\right)}{x^2+\frac{z^2}{4}+f-z_0-\frac{\omega}{2}}-\frac{1}{x^2}\Big)+ \\ +\frac{f\sqrt{2mk_BT_c}}{2\pi}\int_{-1}^{1} d\cos\theta\int_0^{+\infty} dx\frac{x^2\tanh\left(\frac{1}{2}\left[x^2+\frac{z^2}{4}+xz\cos\theta-z_0+f\right]\right)}{(x^2+\frac{z^2}{4}+f-z_0-\frac{\omega}{2})(x^2+\frac{z^2}{4}-z_0-\frac{\omega}{2})}+ \\ +(f\rightarrow -f). \end{aligned} \tag{5.41}$$

The integral over $d\cos\theta$ is readily performed since

$$\begin{aligned} \int_{-1}^{1} d\cos\theta\tanh\left(\frac{1}{2}\left[x^2+\frac{z^2}{4}+xz\cos\theta-z_0+f\right]\right) = \\ \frac{2}{xz}\Big(\ln\left[\cosh\left(\frac{1}{2}\left[\left(x+\frac{z}{2}\right)^2-z_0+f\right]\right)\right]-\ln\left[\cosh\left(\frac{1}{2}\left[\left(x-\frac{z}{2}\right)^2-z_0+f\right]\right)\right]\Big), \end{aligned} \tag{5.42}$$

so that calling

$$\boxed{A(x,z,f)=\ln\left[\cosh\left(\frac{1}{2}\left[(x+\frac{z}{2})^2-z_0+f\right]\right)\right],} \tag{5.43}$$

one gets that

$$\begin{aligned} M_{\omega,z} = -\frac{m}{4\pi}\Big[\frac{1}{a_F}++\frac{\sqrt{2mk_BT_c}}{\pi z}\int_0^{+\infty} dx x\Big(\frac{A(x,z,f)-A(x,-z,f)}{x^2+\frac{z^2}{4}+f-z_0-\frac{\omega}{2}}-\frac{z}{x}\Big)+ \\ +\frac{f\sqrt{2mk_BT_c}}{\pi z}\int_0^{+\infty} dx\frac{x[A(x,z,f)-A(x,-z,f)]}{(x^2+\frac{z^2}{4}+f-z_0-\frac{\omega}{2})(x^2+\frac{z^2}{4}-z_0-\frac{\omega}{2})}+ \\ +(f\rightarrow -f). \end{aligned} \tag{5.44}$$

We now define the poles of the integrands as

$$\boxed{p_0^{\pm}=\sqrt{z_0+\frac{\omega}{2}\mp f-\frac{z^2}{4}},\quad p_1=\sqrt{z_0+\frac{\omega}{2}-\frac{z^2}{4}},} \tag{5.45}$$

since we will need to use residue calculus to get finite results from the integrals in the regimes in which such quantities are real.

In the end, then, we are able to separate the real and imaginary parts of $M_{\omega,z}$ as

$$\boxed{\begin{aligned} Re[M_{\omega,z}] = -\frac{m}{4\pi}\Big[\frac{1}{a_F}+\frac{\sqrt{2mk_BT_c}}{\pi z}P\int_0^{+\infty} dx x\Big(\frac{A(x,z,f)-A(x,-z,f)}{x^2+\frac{z^2}{4}+f-z_0-\frac{\omega}{2}}-\frac{z}{x}\Big)+ \\ +\frac{f\sqrt{2mk_BT_c}}{\pi z}P\int_0^{+\infty} dx\frac{x[A(x,z,f)-A(x,-z,f)]}{(x^2+\frac{z^2}{4}+f-z_0-\frac{\omega}{2})(x^2+\frac{z^2}{4}-z_0-\frac{\omega}{2})}+ \\ +(f\rightarrow -f), \end{aligned}} \tag{5.46}$$

where by $P\cdot$ we mean the principal part of the integral, while the imaginary part will read

$$\boxed{Im[M_{\omega,z}] = -\frac{m}{4\pi}\frac{\sqrt{2mk_BT_c}}{z}\frac{A(p_1,z,f) - A(p_1,-z,f)}{2}\Theta(\frac{\omega}{2} + z_0 - \frac{z^2}{4}) + (f \to -f),} \tag{5.47}$$

since all the other contributions cancel and where by $\Theta(\cdot)$ we denote the Heavyside step function. Notice that there is a lower threshold for ω, namely

$$\boxed{\omega_t(z) = \frac{z^2}{2} - 2z_0,} \tag{5.48}$$

below which the imaginary part of $M_{\omega,z}$ vanishes exactly. Such is the same as in the Rabiless case, contrary to the similar spin orbit coupling studied, for example, in [17], for which $\omega_t(z)$ changes.

5.2.3 Numerical Treatment

As already pointed out in section 5.1.3, the gap equation remains unchanged in the treatment including Gaussian fluctuations, meaning that it will read

$$\frac{1}{a_F} = \frac{2}{\pi}(2mk_BT_c)^{\frac{1}{2}} J_3\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big), \tag{5.49}$$

with

$$\begin{gathered} J_3\Big(\frac{\mu}{\omega_F},\omega_F\beta_c,\frac{\omega_R}{\omega_F}\Big) = \\ \int_0^{+\infty} dx x^2 \Big[\frac{\tanh\left(\frac{1}{2}(x^2 - \beta_c\mu + \beta_c\omega_R)\right)}{2(x^2 - \beta_c\mu)} + \frac{\tanh\left(\frac{1}{2}(x^2 - \beta_c\mu - \beta_c\omega_R)\right)}{2(x^2 - \beta_c\mu)} - \frac{1}{x^2}\Big]. \end{gathered} \tag{5.50}$$

By fixing the inverse scattering length, then, one may find the relation between the critical temperature T_c and the chemical potential μ, which can then be used in the number equation.

The number equation (5.26), instead, can be written in terms of the changes of variables (5.40) as

$$\begin{aligned} n = \frac{(2mk_BT_c)^{\frac{3}{2}}}{3\pi^2}\Big[\frac{1}{4}J_4\Big(\frac{\mu}{\omega_F},\omega_F\beta_c\Big) + \frac{3}{2\pi}\int_0^{+\infty} dz z^2 \int_{\omega_t(z)}^{+\infty} d\omega n_B(\omega)\frac{\partial\delta(\omega,k)}{\partial z_0}\Big] = \\ = \frac{(2mk_BT_c)^{\frac{3}{2}}}{3\pi^2}\Big(\frac{1}{4}J_4\Big(\frac{\mu}{\omega_F},\omega_F\beta_c\Big) + \frac{3}{2\pi}J_5\Big(\frac{\mu}{\omega_F},\omega_F\beta_c\Big)\Big), \end{aligned} \tag{5.51}$$

with $\omega_t(z)$ given in (5.48) and

$$\begin{gathered} J_4\Big(\frac{\mu}{\omega_F},\omega_F\beta_c,\frac{\omega_R}{\omega_F}\Big) = \\ \int_0^{+\infty} dx \Big[\frac{x^4}{\cosh^2\left[\frac{1}{2}(x^2 - \beta_c\mu + \beta_c\omega_R)\right]} + \frac{x^4}{\cosh^2\left[\frac{1}{2}(x^2 - \beta_c\mu - \beta_c\omega_R)\right]}\Big], \end{gathered} \tag{5.52}$$

meaning that one can recover the usual relation

$$\boxed{\frac{k_B T_c}{\omega_F} = \Big(\frac{1}{\frac{1}{2}J_4\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big) + \frac{3}{2\pi}J_5\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big)}\Big)^{\frac{2}{3}}.} \tag{5.53}$$

Then, by dividing both members of (5.49) by the Fermi momentum k_F and using (5.53) one gets that

$$\boxed{\frac{1}{k_F a_F} = \frac{2}{\pi}\Big(\frac{1}{\frac{1}{2}J_4\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big) + \frac{3}{2\pi}J_5\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big)}\Big)^{\frac{1}{3}} J_3\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big).} \tag{5.54}$$

5.3 Beyond Mean Field Critical Temperature From The Phase Stiffness

We now take a little step back from the cumbersome calculations of the past section and briefly go back to the use of mean field quantities. The objective of this section is the generalization of the calculations made in section 3.3 in the case of the presence of Rabi interaction. First, then, the superfluid density of the system will be calculated, using the arguments of [8], and the beyond mean field critical temperature will be obtained with the method proposed in [18].

5.3.1 Mean Field Superfluid Density

For the calculation of the mean field superfluid density we proceed as in section 3.3.1 in the Rabiless case. Generalizing formula (3.48) taken from [8] to the Rabi case, we get that

$$n_s(T) = n + \frac{1}{3}\int \frac{d^3p}{(2\pi)^3}\frac{p^2}{m}\frac{1}{2}\Big[\frac{dn(\omega_+)}{d\omega_+} + \frac{dn(\omega_-)}{\omega_-}\Big], \tag{5.55}$$

where

$$\omega_+(\mathbf{p}, \omega_R) = \sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} + \omega_R, \quad \omega_-(\mathbf{p}, \omega_R) = \sqrt{\xi_{\mathbf{p}}^2 + |\Delta_0|^2} - \omega_R, \tag{5.56}$$

are the single particle fermionic excitation energies derived in equation (4.11), $n(\omega) = (e^{\beta\omega} - 1)^{-1}$ is the Bose-Einstein distribution and n is the total number density. Once again, it is due to point out that this is an approximation, since we are only considering the fermionic excitation energies, and neglecting all the corrections coming from the fluctuations of the bosonic field $\Delta(\mathbf{q}, \tau)$.

Calculating explicitly the derivatives, integrating over the angles and dividing both sides by $\frac{n}{2}$ one gets the equation

$$\frac{n_s(T)}{n} = 1 - \frac{\beta}{12\pi^2 n}\int_0^{+\infty} dp\frac{p^4}{m}\Big[\frac{e^{\beta\omega_+(p)}}{(e^{\beta\omega_+(p)} - 1)^2} + \frac{e^{\beta\omega_-(p)}}{(e^{\beta\omega_+(p)} - 1)^2}\Big]. \tag{5.57}$$

Once again, since this equation is valid for small temperatures, we will use the pairs $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F})$ that satisfy the gap and number equations at $T \to 0^+$, reported in figure 13.

To work with dimensionless quantities, we perform the change of variables $x = \frac{p}{k_F}$, where $k_F = (3\pi^2 n)^{\frac{1}{3}}$ is the Fermi momentum, obtaining the expression

$$\boxed{\begin{aligned}\frac{n_s(T)}{n} = 1 - \frac{\omega_F}{2k_BT}\int_0^{+\infty} dx x^4 \Bigg[&\frac{e^{\frac{\omega_F}{k_BT}\left(\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}+\frac{\omega_R}{\omega_F}\right)}}{\left(e^{\frac{\omega_F}{k_BT}\left(\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}+\frac{\omega_R}{\omega_F}\right)}-1\right)^2} + \\ &+ \frac{e^{\frac{\omega_F}{k_BT}\left(\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}-\frac{\omega_R}{\omega_F}\right)}}{\left(e^{\frac{\omega_F}{k_BT}\left(\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}-\frac{\omega_R}{\omega_F}\right)}-1\right)^2}\Bigg].\end{aligned}} \tag{5.58}$$

In figure 17 we show the behaviour of the superfluid density in the BEC regime at different values of $\frac{\omega_R}{\omega_F}$. The effect of Rabi coupling is to lower the temperature T^* at which $n_s(T^*) = 0$. At very low temperatures, as expected, the behaviour is the same as the one taken by the Rabi less system.

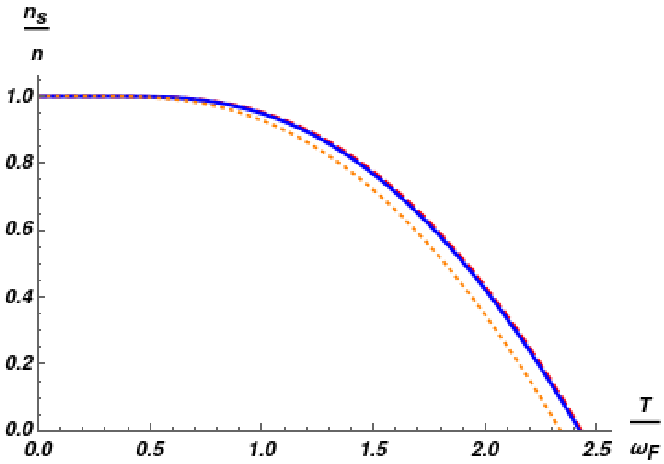

Figure 17: Superfluid fraction *vs.* temperature at different values of the Rabi frequency ω_R at fixed $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}) = (-2.42, 1.65)$. Red dashed line: Rabiless case, i.e. $\frac{\omega_R}{\omega_F} = 0$; blue line: $\frac{\omega_R}{\omega_F} = 0.5$; orange dotted line: $\frac{\omega_R}{\omega_F} = 1$. Here we have set $k_B = 1$

Instead, in figure 18, we show the superfluid fraction changing with the temperature in different regimes of the crossover at fixed $\frac{\omega_R}{\omega_F} = 1$. In particular this plot shows, compared to the one in the Rabiless case in figure 6, that the temperature T^* decreases much more drammatically with the change of $\frac{\mu}{\omega_F}$ in the presence of Rabi coupling. In particular, it gives more evidence of the fact that in the deep BEC regime the system resembles the Rabi less one, while in the BCS regime, after a threshold, the values different from zero solving the

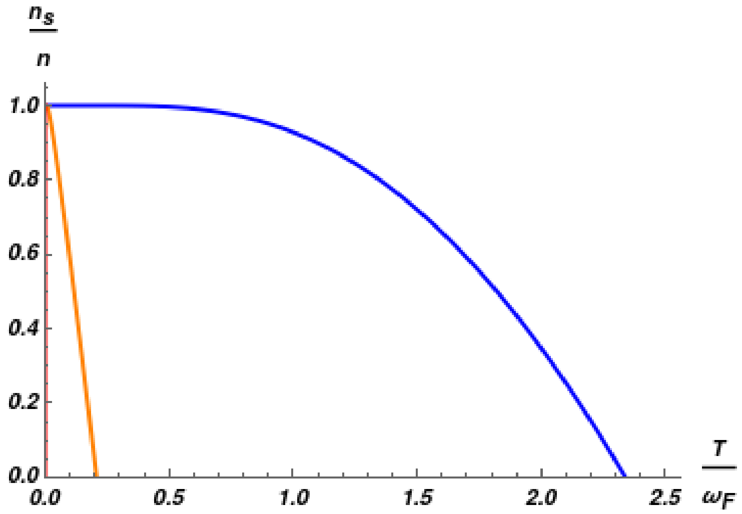

Figure 18: Superfluid fraction *vs.* temperature for $\frac{\omega_R=1}{\omega_F}$ at fixed $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F})$. Blue line: $\frac{\omega_R}{\omega_F} = 1$ at fixed $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}) = (-2.43, 1.66)$; orange line: $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}) = (0.035, 1.04)$; red line: $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}) = (0.30, 0.82)$. Here we have set $k_B = 1$.

gap equation are unphysical, i.e. they maximize, not minimize, the grand potential. This is evident, since there is no superfluid fraction for $T > 0$ in such regime, meaning that no phase transition occurs. In particular, equation (5.58) develops a pole when $|\Delta_0| < \omega_R$, making the expression unphysical.

5.3.2 Calculation of the Beyond Mean Field Critical Temperature

As already reported in section 3.3.2 for the Rabiless case, by following the calculations of [19] we can map the effective action

$$S[\bar{\Delta}\Delta\bar{\psi}\psi] = \int_0^\beta d\tau \int_V d^3q \Big[\frac{|\Delta(\mathbf{q},\tau)|^2}{g} - \frac{1}{2}\bar{\Psi}(\mathbf{q},\tau)G^{-1}\Psi(\mathbf{q},\tau)\Big] + \beta\sum_{\mathbf{p}}\xi_{\mathbf{p}}, \tag{5.59}$$

where the modified Nambu spinors are

$$\bar{\Psi}(\mathbf{q},\tau) = \begin{pmatrix}\bar{\psi}_\uparrow(\mathbf{q},\tau) & \psi_\downarrow(\mathbf{q},\tau) & \bar{\psi}_\downarrow(\mathbf{q},\tau) & \psi_\uparrow(\mathbf{q},\tau)\end{pmatrix}, \quad \Psi(\mathbf{q},\tau) = \begin{pmatrix}\psi_\uparrow(\mathbf{q},\tau)\\ \bar{\psi}_\downarrow(\mathbf{q},\tau)\\ \psi_\downarrow(\mathbf{q},\tau)\\ \bar{\psi}_\uparrow(\mathbf{q},\tau)\end{pmatrix} \tag{5.60}$$

and

$$G^{-1}_{\mathbf{q},\tau}(\bar{\Delta},\Delta) = \begin{pmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu & \Delta(\mathbf{q},\tau) & -\omega_R & 0 \\ \bar{\Delta}(\mathbf{q},\tau) & -\partial_\tau - \frac{\nabla^2}{2m} - \mu & 0 & \omega_R \\ -\omega_R & 0 & -\partial_\tau + \frac{\nabla^2}{2m} + \mu & -\Delta(\mathbf{q},\tau) \\ 0 & \omega_R & -\bar{\Delta}(\mathbf{q},\tau) & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{pmatrix} \tag{5.61}$$

to the one of a three dimensional XY model by making the changes of variables $\psi_\sigma(\mathbf{q},\tau) \rightarrow \psi_\sigma(\mathbf{q},\tau)e^{i\frac{\theta(\mathbf{q},\tau)}{2}}$ and $\Delta(\mathbf{q},\tau) \rightarrow \Delta(\mathbf{q},\tau)e^{i\theta(\mathbf{q},\tau)}$ and making a gradient expansion, considering long wavelength contributions of the phase as the most significant.

The Hamiltonian of the effective XY model obtained has the form

$$H = \frac{J}{2}\int d^3q[\nabla\theta(\mathbf{q})]^2, \tag{5.62}$$

where the stiff parameter J is related to the superfluid density as

$$J = \frac{n_s(T)}{4m}, \tag{5.63}$$

with $n_s(T)$ given by equation (5.58) in our approximation. The critical temperature of such model, as derived through Montecarlo simulations in [20], can be approximated by the implicit equation

$$k_B T_c = 3\frac{n_s(T_c)}{4m}\Big(\frac{2}{n}\Big)^{\frac{1}{3}}, \tag{5.64}$$

which we will improve to

$$k_B T_c = 2\Big(\frac{\sqrt{\pi}}{\zeta(\frac{3}{2})}\Big)^{\frac{2}{3}}\frac{n_s(T_c)}{4m}\Big(\frac{2}{n}\Big)^{\frac{1}{3}}, \tag{5.65}$$

with $\zeta(x) = \sum_{n=1}^{+\infty}\frac{1}{n^x}$, as already done in the Rabiless case in section 3.3.3 in order to get a critical temperature analogue to the one of a gas of free bosons of mass $2m$ and density $\frac{n}{2}$ in the deep BEC regime. Since the expression for $n_s(T)$ is valid at small temperatures, such expression will be solved by using the pairs $(\frac{\mu}{\omega_F}, \frac{|\Delta|}{\omega_F})$ solving the number and gap equations at zero temperature, reported in figure 13, as done in the previous section.

By dividing both sides of (5.65) by the Fermi energy $\omega_F = \frac{(3\pi^2 n)^{\frac{2}{3}}}{2m}$ we finally get the expression that, coupled with (5.58), is used to obtain our numerical results:

$$\boxed{\frac{k_B T_c}{\omega_F} = \frac{2}{(6\sqrt{\pi}\zeta(\frac{3}{2}))^{\frac{2}{3}}}\frac{n_s(T_c)}{n}.} \tag{5.66}$$

To solve such problem along the whole crossover, given that the expression for $n_s(T)$ is valid only for small temperatures, we first find the temperatures T_c that satisfy (5.66) coupled with the pairs $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F})$ that solve the mean field number equation at zero temperature found in section 4.3.2, whose values are reported in figure 13. Then, to calculate the inverse scattering length, we use the equation

$$\frac{1}{k_F a_F} = \frac{2}{\pi}\Big(\frac{4}{J_4\big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\big)}\Big)^{\frac{1}{3}} J_3\Big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\Big), \tag{5.67}$$

from the mean field treatment at the critical temperature derived in section 4.2, in which the integrals $J_3\big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\big)$ and $J_4\big(\frac{\mu}{\omega_F}, \omega_F\beta_c, \frac{\omega_R}{\omega_F}\big)$ are given in equations (4.28) and (4.29).

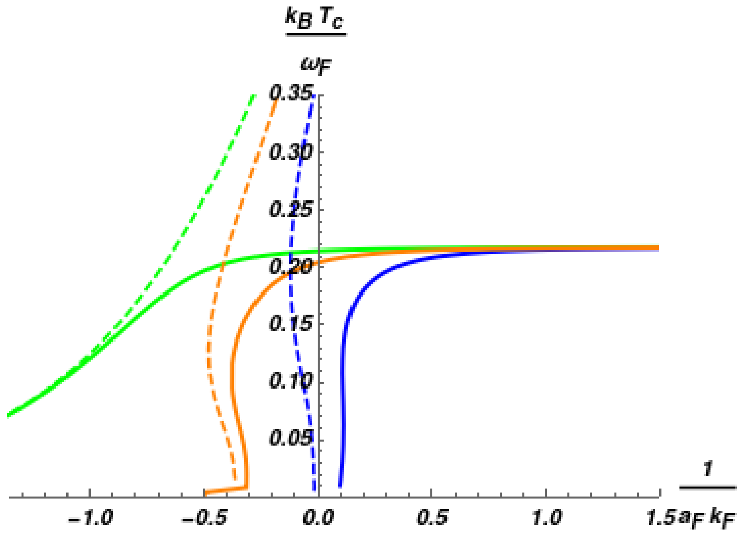

Figure 19: Critical temperature *vs.* inverse scattering length along the whole crossover for different values of the Rabi frequency $\frac{\omega_R}{\omega_F}$. Green dashed line: Rabiless case, i.e. $\omega_R = 0$ at the mean field level; blue dashed line: $\frac{\omega_R}{\omega_F} = 0.5$ at the mean field level; orange dashed line: $\frac{\omega_R}{\omega_F} = 0.3$ at the mean field level; thick green line: Rabiless case beyond mean field; thick blue line: $\frac{\omega_R}{\omega_F} = 0.5$ beyond mean field; thick orange line: $\frac{\omega_R}{\omega_F} = 0.3$ beyond mean field.

The pairs $(\frac{\mu}{\omega_F}, \frac{k_B T_c}{\omega_F})$ plugged in such equation are the critical temperature calculated from solving (5.66) and the corresponding value of the chemical potential used to solve the same equation.

In figure 19 we show the results obtained along the whole crossover for different values of the Rabi frequency. It is clear that Rabi interaction does not affect the deep BEC regime, where the behaviour of the beyond mean field critical temperature resembles perfectly the one of the Rabiless case.

5.3.3 A Small Step Further

We replicate the procedure followed in the treatment of the Kleinert equation (5.66) carried out in the Rabiless case in the last part of section 3.3.3. In order to do so, we manipulate the mean field gap and number equations for a generic temperature obtained in equation (4.21), transforming them in one dimensional integrals. To do so, we transform the sums over momenta in integrals in polar coordinates with the usual prescription $\frac{1}{V}\sum_{\mathbf{p}} \to \frac{1}{(2\pi)^3}\int d^3p$ and perform the change of variables $\frac{p^2}{k_F^2} = x^2$.

The gap equation can then be rewritten as

$$
\begin{aligned}
-\frac{1}{k_F a_F} = \frac{2}{\pi}\int_0^{+\infty} dx x^2 \Big[&\frac{\tanh\left(\frac{\omega_F}{2k_B T}\left[\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}+\frac{\omega_R}{\omega_F}\right]\right)}{2\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}}+ \\
&+\frac{\tanh\left(\frac{\omega_F}{2k_B T}\left[\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}-\frac{\omega_R}{\omega_F}\right]\right)}{2\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}} - \frac{1}{x^2}\Big]
\end{aligned}, \tag{5.68}
$$

while the number equation will read

$$
\begin{aligned}
1 = \frac{3}{2}\int_0^{+\infty} dx x^2 \Big[1 + & \\
-\frac{x^2-\frac{\mu}{\omega_F}}{2\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}} &\tanh\left(\frac{\omega_F}{2k_B T}\left[\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}+\frac{\omega_R}{\omega_F}\right]\right)\Big]+ \\
-\frac{x^2-\frac{\mu}{\omega_F}}{2\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}} &\tanh\left(\frac{\omega_F}{2k_B T}\left[\sqrt{(x^2-\frac{\mu}{\omega_F})^2+\frac{|\Delta_0|^2}{\omega_F^2}}-\frac{\omega_R}{\omega_F}\right]\right)\Big]
\end{aligned}. \tag{5.69}
$$

The prescription is to solve the system of equations made of (5.68), (5.69) and (5.66) to get triplets of $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F}, \frac{k_B T}{\omega_F})$ for different values of $\frac{1}{k_F a_F}$. The pairs $(\frac{\mu}{\omega_F}, \frac{k_B T}{\omega_F})$ will then be used to calculate the inverse scattering length obtained with the mean field equation at the critical temperature, (5.67), as such temperatures will be considered the critical ones. The results we obtained are reported in figure 20.

As expected, the only values for $(\frac{\mu}{\omega_F}, \frac{|\Delta_0|}{\omega_F})$ that yield a non null critical temperature from equation (5.64) are the ones that we labelled as physical, or in other words that minimize the mean field grand potential at zero temperature. The unphysical values yield a null superfluid density, since no phase transition occurs in that regime, as shown in figure 18. In fact, while in the Rabiless case we see that the critical temperature decreases as $\frac{1}{k_F a_F}$ decreases, but never reaches the exact zero, in the Rabi case we have an actual point at which the critical temperature becomes null, for a finite value of $\frac{1}{k_F a_F}$. The main takeaway is that in the BEC regime Rabi interaction does not affect the physics of our system, which will behave as a bosonic field of free Cooper pairs as discussed in section 3.3.2, while as the chemical potential μ grows, it inhibits the bosonic nature of the system, until a point is reached at which the second order phase transition becomes a first order one.

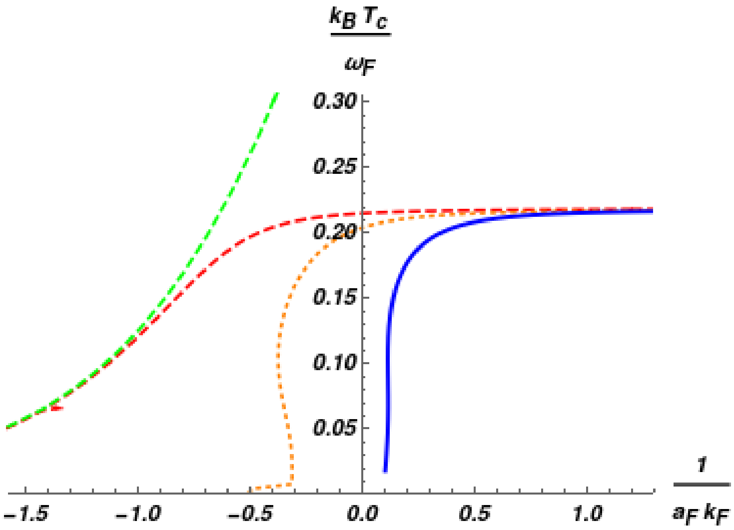

Figure 20: Critical temperature along the whole crossover for different values of Rabi frequency. Green dashed line: mean field critical temperature in absence of Rabi coupling; red dashed line: beyond mean field critical temperature with no Rabi coupling; orange dotted line: beyond mean field critical temperature at $\frac{\omega_F}{\omega_F} = 0.3$; thick blue line: beyond mean field critical temperature at $\frac{\omega_F}{\omega_F} = 0.5$.

6 Conclusions

The BCS-BEC crossover has been studied thoroughly both in the standard Rabiless case and in the presence of Rabi coupling using the coherent state path integral formalism. The behaviour of many physical quantities has been studied along the whole crossover, including the mean field critical temperature, the beyond mean field critical temperature, the mean field energy gap at zero temperature, the condensate fraction and the superfluid fraction. The results we obtained were compared, enabling us to understand the effect of Rabi coupling in the crossover.

In particular, we found that both at zero temperature and at the critical temperature the system is not affected by Rabi interaction in the deep BEC regime ($\mu \to -\infty$) (figures 14, 16, 19 as a few examples). In the BCS regime, instead, the phase transition that characterizes superconductivity and has a second order nature in the Rabiless case, is inhibited by Rabi coupling, so much that at a given inverse scattering length depending on the strength of Rabi interaction the transition becomes a first order one (figures 11 and 19 as examples); below such scattering length, no phase transition is possible. Such scattering length does not have to be smaller than zero necessarily, so that Rabi interaction, if strong enough, will affect the BEC regime, too.

The beyond mean field treatment was carried out following two different and independent procedures: the NSR [11] and the Kleinert [18] ones, the first being more formal and the second being more heuristic, but much less computationally costly. In particular the Kleinert approach was improved, getting the desired result in the deep BEC limit.

Explicit calculations for the NSR approach were given in section 5.2, even though we were not able to obtain the plots of the critical temperature due to the computational difficulty of the problem. Despite that, one can use the cumbersome calculations we performed to try and implement them: no calculation is to be thrown away!

A Calculation of the Gran Potential at the Mean Field Level

The calculation of the grand potential in this case will be of no pratical use, but the procedure to evaluate it will be the same in the case of the Rabi-coupled gas, where the form of the mean field grand potential will play an important role in the interpretation of the gap and number equations, in particular in the $T \to 0^+$ limit.

The grand potential of the system at mean field level will be denoted by Ω_{MF} and is defined as

$$\Omega_{MF} = -k_B T \ln(Z_{MF}) = k_B T S_0. \tag{A.1}$$

The non obvious part of this calculation is the evaluation of the trace of the logarithm of the inverse fermionic propagator $G^{-1}_{KP}(\bar{\Delta}_0, \Delta_0)$. To find its compact form, we use the identity

$$Tr[\ln(G^{-1}_{KP}(\bar{\Delta}_0, \Delta_0))] = \ln\Big(\det(G^{-1}_{KP}(\bar{\Delta}_0, \Delta_0))\Big), \tag{A.2}$$

which is easier to calculate thanks to the fact that in the mean field approximation $G^{-1}_{KP}(\bar{\Delta}_0, \Delta_0)$ is diagonal in its Matzubara representation, as clear from (2.11), in which $\Delta_{K+P} \to \Delta_0$. Then one has that

$$\Omega_{MF} = V\frac{|\Delta_0|^2}{g} - k_B T \ln\Big(\prod_P (i\Omega^F_n + \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2})(i\Omega^F_n - \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2})\Big), \tag{A.3}$$

which can be simplified by taking the product out of the logarithm and by performing the sum over Matzubara frequencies in the usual way.

The sum over the frequencies is delicate because of the presence of a branch cut in the complex plane of frequencies due to the multivalued nature of the logarithm. The integration contour will have to be modified with respect to the usual case in order to avoid the branch cut [4]. To be more precise, the integral to calculate is

$$\int_{\Gamma_1} \frac{dz}{2\pi i}\frac{1}{e^{\beta z}+1}\ln\big(z + \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2}\big) + \int_{\Gamma_2} \frac{dz}{2\pi i}\frac{1}{e^{\beta z}+1}\ln\big(z - \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2}\big), \tag{A.4}$$

where Γ_1 is a circle covering the whole complex plane and curved in order to avoid the branch cut on the real axis starting at $z = -\infty$ and ending at $z = -\sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2}$, while Γ_2 is the same contour, but avoiding the branch cut starting at $z = -\infty$ and ending at $\sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2}$. Since the integrands have no poles on the real axis, the two contours can be modified by extending the deformation around the branch cuts up to $z = +\infty$, so that the grand potential can be written as

$$\Omega_{MF} = V\frac{|\Delta_0|^2}{g} - \sum_{\mathbf{p}} \int_{\Gamma} \frac{dz}{2\pi i}\frac{1}{e^{\beta z}+1}\Big[\ln\big(z + \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2}\big) + \ln\big(z - \sqrt{\xi^2_{\mathbf{p}} + |\Delta_0|^2}\big)\Big], \tag{A.5}$$

where Γ is the new contour. The integrals vanish at infinity over the whole complex plane, so that what remains of them is

$$\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\frac{1}{e^{\beta z}+1}\Big[\ln(z+i\eta+\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2})-\ln(z-i\eta+\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2})+ \\ +\ln(z+i\eta-\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2})-\ln(z-i\eta-\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2})\Big], \tag{A.6}$$

where $\eta\to 0^+$. Such expression can be calculated explicitly by noticing that $(e^{\beta z}+1)^{-1}=-\frac{1}{\beta}\partial_z\ln(1+e^{-\beta z})$ and by integrating by parts. Explicitly,

$$k_BT\sum_{\mathbf{p}}\int_{-\infty}^{+\infty}\frac{dz}{2\pi i}\ln(1+e^{-\beta z})\Big[\frac{1}{z+i\eta+\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}}-\frac{1}{z-i\eta+\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}}+ \\ +\frac{1}{z+i\eta-\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}}-\frac{1}{z-i\eta-\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}}\Big]. \tag{A.7}$$

Now, by using the fact that

$$\lim_{\eta\to 0^+}\frac{1}{z+i\eta}=-i\pi\delta(z)+P\Big[\frac{1}{z}\Big], \tag{A.8}$$

where by $P\Big[\cdot\Big]$ we denote the principal part, we get that

$$\Omega_{MF}=V\frac{|\Delta_0|^2}{g}-k_BT\sum_{\mathbf{p}}\ln\Big([1+e^{-\beta\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0^2|}}][1+e^{\beta\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0^2|}}]\Big), \tag{A.9}$$

which in a more familiar form reads

$$\Omega_{MF}=V\frac{|\Delta_0|^2}{g}-k_BT\sum_{\mathbf{p}}\ln\Big[2\Big(1+\cosh[\beta\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}]\Big)\Big]. \tag{A.10}$$

From such equation one may rederive the gap equation and the number equation.

B Sum over Matzubara Frequencies For The Calculation of M_K

We compute the sums over Matzubara frequencies of χ_K for a generic temperature T, by first noticing that $(\chi_K)_{11}=(\chi_{-K})_{22}$ and that $(\chi_K)_{12}$ is the complex conjugate of $(\chi_K)_{21}$, meaning that χ_K only has two independent elements. For the diagonal element we obtain four different contributions

$$(\chi_K)_{11}=\frac{k_BT}{2V}\sum_P\frac{(i\Omega_n^F-\xi_{\mathbf{p}})(i(\Omega_m^B+\Omega_n^F)+\xi_{\mathbf{k+p}})}{((\Omega_n^F)^2+\xi_{\mathbf{p}}^2+|\Delta_0|^2)((\Omega_m^B+\Omega_n^F)^2+\xi_{\mathbf{k-p}}^2+|\Delta_0|^2)}=\frac{1}{2V}\sum_{\mathbf{p}}[f_1+f_2+f_3+f_4], \tag{B.1}$$

with, given $n_F(z) = (e^{\beta z}+1)^{-1}$ the Fermi distribution,

$$
\begin{aligned}
f_1 &= -\frac{(\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}-\xi_{\mathbf{p}})(i\Omega_m^B+\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}+\xi_{\mathbf{p+k}})n_F\Big(\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\Big)}{2\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\big((\Omega_m^B)^2+\xi_{\mathbf{p+k}}^2-\xi_{\mathbf{p}}^2-2i\Omega_m^B\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\big)},\\
f_2 &= -\frac{(\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}+\xi_{\mathbf{p+k}})(\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}-\xi_{\mathbf{p}}-i\Omega_m^B)n_F\Big(\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}\Big)}{2\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}\big((\Omega_m^B)^2-\xi_{\mathbf{p+k}}^2+\xi_{\mathbf{p}}^2+2i\Omega_m^B\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}\big)},\\
f_3 &= \frac{(\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}+\xi_{\mathbf{p}})(\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}-\xi_{\mathbf{p+k}}-i\Omega_m^B)n_F\Big(-\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\Big)}{2\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\big((\Omega_m^B)^2+\xi_{\mathbf{p+k}}^2-\xi_{\mathbf{p}}^2+2i\Omega_m^B\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\big)},\\
f_4 &= \frac{(\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}-\xi_{\mathbf{p+k}})(i\Omega_m^B+\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}+\xi_{\mathbf{p}})n_F\Big(-\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}\Big)}{2\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}\big((\Omega_m^B)^2-\xi_{\mathbf{p+k}}^2+\xi_{\mathbf{p}}^2-2i\Omega_m^B\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}\big)}.
\end{aligned}
\tag{B.2}
$$

One may substitute the sum parameter $\mathbf{p}$ with $\mathbf{l} = \mathbf{p}+\mathbf{k}$ in the terms involving f_2 and f_4, rename $\mathbf{l} \to \mathbf{p}$ and finally, in the action, change $\mathbf{k} \to -\mathbf{k}$. This way the factor of the Fermi distribution in f_1 and f_4 will become equal, and the same will happen for f_2 and f_3. After these manipulations, then, $(\chi_K)_{11}$ can be rewritten as

$$
\begin{aligned}
(\chi_K)_{11} = \frac{1}{2V}\sum_{\mathbf{p}}\Big[&\frac{(\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}-\xi_{\mathbf{p}})(i\Omega_m^B+\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}+\xi_{\mathbf{p+k}})\tanh\Big(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\Big)}{2\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\big((\Omega_m^B)^2+\xi_{\mathbf{p+k}}^2-\xi_{\mathbf{p}}^2-2i\Omega_m^B\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\big)}+\\
&+\frac{(\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}+\xi_{\mathbf{p+k}})(\sqrt{\xi_{\mathbf{p+k}}^2+|\Delta_0|^2}-\xi_{\mathbf{p}}-i\Omega_m^B)\tanh\Big(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\Big)}{2\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\big((\Omega_m^B)^2+\xi_{\mathbf{p+k}}^2-\xi_{\mathbf{p}}^2+2i\Omega_m^B\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\big)}\Big]
\end{aligned}
\tag{B.3}
$$

The calculation is the same for the off-diagonal term, with the difference that there is no frequency dependent term at the numerator. The four terms emerging from the sum, then, will give the same contribution and yield

$$
(\chi_K)_{12} = \frac{1}{2V}\sum_{\mathbf{p}}\frac{\tanh\Big(\frac{\beta}{2}\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}\Big)}{\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2}((\Omega_m^B)^2+\xi_{\mathbf{p+k}}^2-\xi_{\mathbf{p}}^2-2i\Omega_m^B\sqrt{\xi_{\mathbf{p}}^2+|\Delta_0|^2})}. \tag{B.4}
$$

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