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Extension to $\mathbb{Q}_p$ of p-adic Γ and L functions and p-adic measures on $\mathbb{Q}_p$

Candidato:
Alessandro Ariu [1174499]

Relatore:
prof. Francesco Baldassarri

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Contents

Introduction

This thesis project is dedicated to the study of some p-adic locally analytic objects which are classically defined on a neighborhood of $\mathbb{Z}_p$ in $\mathbb{C}_p$, such as Morita's Γ-function, the p-adic Riemann zeta function ζ_p, and Kubota-Leopoldt's L-functions, and to the possibility of extending them to a neighborhood of all of $\mathbb{Q}_p$ in $\mathbb{C}_p$. We only mention in passing the interest of shedding any light whatsoever on the values of Riemann's zeta function ζ at rational non-integral points, values for which no closed formula is available and no conjectures seem to have been formulated. The relation of Morita's p-adic Γ-function Γ_p to Gauss sums is the content of the Gross-Koblitz formula. Maurizio Boyarsky ([9]) following Dwork ([16]) explained the appearance of Γ_p in the Frobenius matrix of Fermat curves with p-adically good reduction. The variable of Γ_p was then naturally interpreted as parametrizing and analytically interpolating characters of the geometric fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$, while the value of Γ_p at the character χ was giving the action of Frobenius on the (1-dimensional) χ-isotypical component of cohomology. This is the essence the so-called *Boyarsky Principle*.

It was Coleman [11] who performed the calculation of the Frobenius matrix in the Hyodo-Kato cohomology of potentially semistable Fermat curves (which admit a semistable model over $\mathbb{Z}_p[\mu_{p^n}]$, for some n). He showed that the action of Frobenius on 1-dimensional isotypical components of that cohomology, when expressed in dependence of characters of the geometric fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$, could naturally be expressed in terms of an analytic extension of Γ_p to $\mathbb{Q}_p$, with a number of remarkable properties.

In a different vein, Jack Diamond in [13] defined a p-adic $\log \Gamma$ on $\mathbb{C}_p \setminus \mathbb{Z}_p$, and correspondingly introduced, using the distribution formula, certain Gamma measures. In terms of such measures, $L_p(s, \chi)$ admits an integral expression; Diamond then obtained new formulas for the values $L_p(r, \chi)$ for a Dirichlet character modulo p^m and r a *positive* integer.

The classical Amice transform (see [1] and [12]) defines an isomorphism between the algebra of $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p$ and the algebra of $\mathbb{Q}_p$-analytic functions defined and bounded by 1 on the open unit disc. Colmez himself has extended that construction to a certain algebra of $\mathbb{Z}_p$-valued measures on $\mathbb{Q}_p$; values are taken in a ring of projective systems of the previous type of analytic functions. On the other hand, in ancient thesis work, [3] F. Baldassarri had proposed the same type of extension of the Amice transform which applies to $\mathbb{Z}_p$-valued *uniform* measures on $\mathbb{Q}_p$ and takes values in the ring of functions on the formal perfectoid open disc defined over $\mathbb{Z}_p$. That construction, still unpublished, will appear in [2].

In view of the previous facts, the goal of this thesis is to propose a possible way to generalise the definition of Kubota-Leopoldt's p-adic L-functions.

The thesis is organised as follows. The first chapter is dedicated to introduce the p-adic analytic tools that we need, with particular emphasis on the definition and properties of two special functions, namely the Morita Γ_p-function and Baldassarri's $\Psi = \Psi_p$ function. We define the former in the style of Dwork, using in particular the construction of Γ_p as Frobenius matrix in a certain exponential analytic cohomology. One obtains a p-adic analytic function defined in a neighborhood of $\mathbb{Z}_p$ such that

$$\begin{aligned} \Gamma_p(s+1) &= -s\Gamma_p(s) \qquad && |s| = 1 \\ \Gamma_p(s+1) &= -\Gamma_p(s) \qquad && |s| < 1 \end{aligned}$$

and $\Gamma_p(0) = 1$.
The latter special function Ψ was introduced by F. Baldassarri in [4]and [6] via the functional equation:

$$\sum_{j=0}^{\infty} p^{-j} \Psi(p^j T)^{p^j} = T \ .$$

The role of Ψ is the one of analytically trivializing the addition law of Barsotti's Witt covectors and it has a relation, via the Artin-Hasse isomorphism, with a system of exponential-like test functions for a Fourier theory on $\mathbb{Q}_p$.

In the second chapter we introduce the classical theory of p-adic measures over $\mathbb{Z}_p$ and, in particular, we give the definition of the Kubota-Leopoldt p-adic L functions L_p via Mellin transform $\mathscr{M}$ and Bernoulli measures $\nu_{1,c}$:

$$L_p(1-s,\chi) = \frac{-1}{1-\chi(c)\langle c\rangle^s} \mathscr{M}(\chi\nu_{1,c})(s)$$

Here we get our first result: in fact, after a reformulation of this definition we calculated the Mahler series associated to L_p i.e. :

$$L_p(-s,\chi) = \frac{1-p}{1-\chi(c)\langle c\rangle^{s+1}} \sum_{n=0}^{\infty} p^n \frac{B_{n+1,\chi}}{n+1} \binom{s}{n}$$

After that we define, following Diamond, the p-adic logΓ function G_p and the related Γ-measures. As an application, we can:

- Reformulate the definition of L_p using the Diamond's Gamma measure $\mu_{1,c}$ (observe that the left hand side of the equation does not depend by the constant c):

$$L_p(1-s,\chi) = \frac{-1}{1-\chi(c)\langle c\rangle^s}\int_{\mathbb{Z}_p^\times}\langle x\rangle^s\chi(x)d\mu_{1,c}(x)$$

- Prove the following formulas about special values of p-adic L functions:

$$L_p(0,\omega) = \frac{c}{(c-1)\log_p(c)}\int_{\mathbb{Z}_p^\times}\log_p(x)d\nu_{1,c} = 0$$

$$L_p(1+k,\omega^{-k}) = \frac{(-1)^k(1-c^{-(k+1)})}{(c^{-k}-1)k!}(\log_p\Gamma_p)^{(k+1)}(0)$$

with $k \geq 1$

- In particular, we can produce the Amice transform of the measure $\mu_{1,c}$:

$$\mu_{1,c} = \log_p(c)\left(1-\frac{1}{p}\right)+\sum_{n=2}^{\infty}\left(\log_p(c)\left(1-\frac{1}{p}\right)+\sum_{k=2}^{n}\frac{s(n,k)}{n!}\frac{B_{k,\omega^k}(1-c^k)}{k}\right)(\Delta_1-\Delta_0)^n$$

In the third chapter, we address the study of the Coleman's Γ-function denoted Γ_C and its geometric meaning linked with the action of the Frobenius automorphism in the de Rham cohomology of some Fermat curves. A similar interpretation exists also for the classical p-adic Γ function of Morita as explained in [5]. First of all we prove the crucial (for our applications) arithmetic properties of Γ_C. Namely, for $s = \frac{i}{p^n}+m$ with $0 < \frac{i}{p^n} < 1$ and $m \in \mathbb{N}$ we check that

$$\Gamma_C\left(\frac{i}{p^n}+m\right) = \frac{\prod_{k=0}^{m-1}\exp_p\log_p\left(\frac{i}{p^n}+k\right)}{\exp_p\log_p\left(\frac{i}{p^n}\right)\exp_p\log_p\left(\frac{i}{p^n}-1\right)},$$

in which $\log_p$ is the Iwasawa logarithm, which says in particular that for $s \in \mathbb{Q}_p\setminus\mathbb{Z}_p$ the value $\Gamma_C(s) \equiv 1 \bmod p\mathbb{Z}_p$. Then we compare that to G_p which, a priori, is a quite different function. We prove that there is a formula which expresses Γ_C in terms of G_p and the p-adic exponential:

$$\Gamma_C(s) = \exp_p(G_p(s)-G_p(s-[s]_p)) \qquad s \in \mathbb{Q}_p\setminus\mathbb{Z}_p$$

This allows us to remark that Γ_C is a locally analytic function and that it can be easily linked to the Diamond's measure and so to p-adic L functions, in fact the above formula say to us that:

$$\log_p\Gamma_C(s) = G_p(s)-G_p(s-[s]_p)$$

where $G_p(s-[s]_p)$ is a locally costant function and in particular:

$$(\log_p \Gamma_C(s))' = G_p'(s)$$

but $\mu_{1,c}$ is defined by G_p'. This fact could open the possibility of a geometric interpretation of these p-adic measures.

The last part of this thesis is dedicated to motivate a possible extension of L_p to $\mathbb{Q}_p$. We first review Huber's adic spaces and introduce Scholze's perfectoid spaces. We recall from [2] the definitions of the "formal perfectoid open unit discs" $\mathbb{D}(0)$ and $\mathbb{D}(1)$ (centered at 0 and 1, respectively), and a special choice of coordinates on them. We also review the notions of uniform and bounded measures over $\mathbb{Q}_p$ following [2]. Since the measure $\mu_{1,c}$ is so versatile, we propose the following extension to $\mathbb{Q}_p$ of $\mu_{1,c}$:

$$\tilde{\mu}_{1,c}(a+p^N\mathbb{Z}_p) = \begin{cases} p^{-v_p(a)} \cdot \mu_{1,c}(u + p^{N-v_p(a)}\mathbb{Z}_p) & a+p^N\mathbb{Z}_p \subset \mathbb{Q}_p \setminus p\mathbb{Z}_p \\ 0 \qquad a+p^N\mathbb{Z}_p \subset p\mathbb{Z}_p \end{cases} \tag{1}$$

with $a \in \mathbb{Q}_p^\times$ and $a = p^{v_p(a)}u$, in particular $\tilde{\mu}_{1,c}$is a uniform measure.

The main point is that $\mathbb{D}(0)$ and $\mathbb{D}(1)$ are isomorphic and that their perfectoid algebra is isomorphic to the ring $\mathscr{D}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$ of $\mathbb{Z}_p$-valued uniform measures on $\mathbb{Q}_p$. Moreover,

$$\mathscr{D}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p) = \text{ the } (p,T)\text{-adic compl. of } \quad \mathbb{Z}_p[T^{1/p^\infty}] \ .$$

. This is Baldassarri's extension of the Amice transform. The choice of coordinates on $\mathbb{D}(0)$ and $\mathbb{D}(1)$ is as follows. We change the original choice of parameter $T = \Delta_1 - \Delta_0$, where Δ_a is the Dirac mass at $a \in \mathbb{Q}_p$ into

$$\mu_{\text{can}} := \lim_{n \to +\infty} E_p(\Delta_{p^{-n}} - \Delta_0)^{p^n} \in \mathscr{D}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p) \ ,$$

where $E_p(x) \in 1 + x\mathbb{Z}_p[[x]]$ is the Artin-Hasse logarithm. Then a "coordinate" on $\mathbb{D}(0)$ (resp. on $\mathbb{D}(1)$) is the monoid morphism

$$(\mathbb{Z}[1/p]_{>0}, +) \to (\mathscr{D}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p), \cdot) \ \ , \ \ q \longmapsto \mu_{\text{can}}^q$$

(resp.

$$(\mathbb{Z}[1/p]_{\geq 0}, +) \to (\mathscr{D}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p), \cdot) \ \ , \ \ q \longmapsto \Delta_q \ ) \ .$$

Then, for any perfectoid extension field K of $\mathbb{Q}_p$, the K-valued analytic points of $\mathbb{D}(0)$ (resp. of $\mathbb{D}(1)$) are monoid morphisms

$$\chi : (\mathbb{Z}[1/p]_{>0}, +) \to (K^{\circ\circ}, \cdot) \ \ , \ \ (\mu_{\text{can}}^q)_q \longmapsto (\chi(q))_q$$

(resp.

$$\psi : (\mathbb{Z}[1/p]_{\geq 0}, +) \to (1 + K^{\circ\circ}, \cdot) \ \ , \ \ (\Delta_q)_q \longmapsto (\psi(q))_q \ .$$

which in fact extends to $(\mathbb{Q}_p, +) \to (1+K^{\circ\circ}, \cdot)$). We denote by $\mathbb{D}_K(0)$ (resp. on $\mathbb{D}_K(1)$) this set of K-valued analytic points.

Notice that the classical open analytic discs of radius $r_p := p^{-\frac{1}{p-1}}$ are isomorphic through the action of $\exp_p$ and $\log_p$:

$$D(0, r_p) \xrightarrow{\exp_p} D(1, r_p) \ .$$

Let $F_p(x) \in 1 + x\mathbb{Z}_p[[x]]$ be the Artin-Hasse analytic function $D(0,1) \to D(1,1)$ and let $E_p(x) \in x\mathbb{Z}_p[[x]]$ be its inverse. We denote by $\overline{F}_p(x) \in 1 + x\mathbb{F}_p[[x]]$ the Artin-Hasse series reduced modulo p. The perfectoid analog, following from Scholze's theory and spelled-out in [2], is the isomorphism $\mathbb{D}(0) \xrightarrow{F_p^\sharp} \mathbb{D}(1)$ obtained from the untilting $F_p^\sharp$ of F_p. For every $\chi \in \mathbb{D}_K(0)$, the image $F_p^\sharp(\chi)$ is then a continuous group homomorphism $(\mathbb{Q}_p, +) \to (1 + K^{\circ\circ}, \cdot)$.

Chapter 1

Preliminaries

1.1 p-adic analytic tools

We fix a prime $p > 2$ and denote by $\mathbb{Q}_p$ the field of p-adic numbers, $\mathbb{Z}_p$ the ring of p-adic integers with standard valuation v_p such that $v_p(p) = 1$, $\overline{\mathbb{Q}}_p$ will be an algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ its completion with absolute value denoted by $|\cdot|$.
For this part we will follow [28],[27],[12] and [18].

Definition 1. Let $x \in \mathbb{C}_p$ the closed disc centered in x of radius r is defined as:

$$D(x,r) = \{a \in \mathbb{C}_p | x - a| \leq r\}$$

the open disc centered in x of radius r is defined as:

$$D(x,r^-) = \{a \in \mathbb{C}_p | x - a| < r\}$$

Proposition 1. *A power series in* $\mathbb{C}_p$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

converges in the open disc of radius ρ (i.e. in $D(0,\rho)$) given by:

$$\rho = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}$$

and diverges for $|x| > \rho$.

Definition 2. A function

$$f : D(a,r) \to \mathbb{C}_p$$

is called analytic if could be written as a power series converging in $D(a,r)$.

Definition 3. Let $X \subset \mathbb{C}_p$, a function :

$$f : X \to \mathbb{C}_p$$

is called locally analytic if for every $x \in X$ exists $r_x > 0$ such that:

$$f_{|D(x,r_x)} : D(x, r_x) \subset X \to \mathbb{C}_p$$

is analytic.

Definition 4. A p-adic Banach space is a $\mathbb{Q}_p$-vector space with a lattice B^0 wich is a $\mathbb{Z}_p$-lattice such that:

$$B^0 = \varprojlim_{n \in \mathbb{N}} B^0/p^n B^0$$

and it is endowed by a valuation v_B satisfying:

1. $v_B(x + y) \geq \min(v_B(x), v_B(y))$;
2. for $\lambda \in \mathbb{Q}_p$, $v_B(\lambda x) = v_p(\lambda) + v_B(x)$

s.t. B is complete under the norm induced by the valuation.

Definition 5. A Banach basis for a p-adic Banach space B is a family $(e_i)_i$ such that:

1. For every $x \in B$ we have $x = \sum_i x_i e_i$ and $x_i \to 0$ for $i \to \infty$ for $x_i \in \mathbb{Q}_p$;
2. $v_B(x) = \inf_i v_p(x_i)$

An example of p-adic Banach space the set of continuous functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ denoted $C(\mathbb{Z}_p, \mathbb{Q}_p)$ A well know result is:

Theorem 1. *A Banach basis for $C(\mathbb{Z}_p, \mathbb{Q}_p)$ is given by $\binom{x}{n}$ i.e. if $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ then:*

$$f = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$$

this basis is called Mahler basis and the coefficients are given recoursively by:

$$f^{[0]} = f \qquad f^{[k-1]}(x) = f^{[k]}(x+1) - f^{[x]}(x)$$

and setting $a_n(f) = f^{[n]}(0)$ More explicitely:

$$f^{[x]}(x) = \sum_{i=1}^{n} (-1)^i \binom{n}{i} f(x + n - i)$$

$$a_n(f) = \sum_{i=1}^{n} (-1)^i \binom{n}{i} f(n - i)$$

Example 1. Another example is given by the space of analytic functions on the disc $D(x, r^{-})$, where the norm is induced by the valuation :

$$v_{D(x,r^{-})}(\phi) = \inf_{x \in D(x,r^{-})} (\phi(x))$$

with the banach basis given by the functions:

$$\chi_{D(x,r^{-})} \frac{(a-x)^k}{p^{\lfloor kr \rfloor}} \qquad k \in \mathbb{N}$$

and χ the characteristic function.

Definition 6. Let K be a NA field then a topological vector space X over K is a Fréchet space if:

- X is an Hausdorff space;
- exists a countable family of valuations $\{w_i\}_{i \in \mathbb{N}}$ which defines the topology of X i.e. a base of neighborhoods of $x \in X$ is given by:

$$\{y \in X |\, w_i(x-y) > N,\, N > 0\}$$

- X is complete with respect to this topology.

Definition 7. (p-adic exponential) We define the p-adic exponential as the power series:

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so the region of convergence of this power series is: $D(0, p^{1/1-p^{-}})$ and we observe that $p^{1/(1-p)} < 1$ so we don't have an anologous of the real e.

Observation 1. In $\mathbb{Q}_p$ we have that the disk of convergence of $\exp_p$ is $p\mathbb{Z}_p$.

Definition 8. (p-adic logarithm) We define the p-adic logarithm as the power series:

$$\log_p(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$$

We recall that this power series converges in the disc $D\left(1, 1^{-}\right)$.

Observation 2. In $\mathbb{Q}_p$ we have that the disk of convergence of $\log_p$ is $1 + p\mathbb{Z}_p$.

Observation 3. In the disc of convergence these two functions have the same proprierties of their real analogous.

Proposition 2. *The map* $\log_p : (1 + p\mathbb{Z}_p, \cdot) \to (p\mathbb{Z}_p, +)$ *is an isomorphism with inverse the* p*-adic exponential* $\exp_p$.

Unfortunately these two function can't be extended to entire functions in all $\mathbb{C}_p$, for us it will be enough to give a definition of an extension of this logarithm which will be locally analytic and the restriction it will coincide with the power series above.

Definition 9. (Iwasawa logarithm) Exists a unique continuos and locally analytic function $f : \mathbb{C}_p^\times \to \mathbb{C}_p$ such that:

1. $f(xy) = f(x) + f(y)$
2. $f_{|D(1,1^-)} = \log_p$
3. $f(p) = 0$

This logarithm is called "Iwasawa logarithm" and it will be still denoted as $\log_p$.

Definition 10. (Artin-Hasse exponential) The Artin-Hasse exponential is defined as:

$$F_p(x) = \exp_p\left(\sum_{n=0}^{\infty} \frac{x^{p^n}}{p^n}\right)$$

It converges on the disc $D(0, 1^-)$

Definition 11. Let $\pi \in \mathbb{C}_p$ such that $\pi^{p-1} = p$, we define the following function

$$\Theta(x) = \exp_p(\pi(x - x^p))$$

which is the Dwork-exponential.

Definition 12. [28, p.77] Let $f : X \to \mathbb{Q}_p$, $X \subset \mathbb{Q}_p$ without isolated points, f is strictly differentiable at $a \in X$ iff the following limit exists for every $(x, y), x \neq y$:

$$\lim_{(x,y)\to(a,a)} \frac{f(x) - f(y)}{x - y}$$

In [28, p. 167] is defined the integral of a (strictly differentiable function) as:

$$\int_{\mathbb{Z}_p} f(t)dt = \lim_{n\to\infty} p^{-n} \sum_{i=0}^{p^n-1} f(i)$$

So if the function is strictly differentiable it converges and it has the follwing propriertie**s**:

Proposition 3. *Let* f *strictly differentiable,* $s \in \mathbb{Z}_|$ *then:*

1. $\int_{\mathbb{Z}_p} f(t + x + 1)dx = \int_{\mathbb{Z}_p} f(t + x)dx + f'(x)$

2. $\frac{d}{dt}\int_{\mathbb{Z}_p} f(x+t)dx = \int_{\mathbb{Z}_p} f'(x+t)dx$

Proof. See [28] $\square$

Proposition 4. *Let f strictly differentiable then:*

$$\int_{i+p^n\mathbb{Z}_p} f(t)dt = \int_{p^n\mathbb{Z}_p} f(i+t)dt = p^{-n}\int_{\mathbb{Z}_p} f(i+p^n t)dt$$

Proof. Consider that by definition:

$$\int_{i+p^n\mathbb{Z}_p} f(t)dt = \int_{p^n\mathbb{Z}_p} f(i+t)dt$$

Now again by definition:

$$\int_{p^n\mathbb{Z}_p} f(i+t)dt = \lim_{s\to\infty} p^{-s}(f(j)+\cdots+f(j+(p^{s-n}-1)p^n))$$

Set now $h(x) = f(j+p^n x)$ then the above limits could be written as:

$$\lim_{s\to\infty} p^{-s}(h(0)+h(1)+\cdots+h(p^{s-n}-1)) = p^{-n}\int_{\mathbb{Z}_p} h(x)dx$$

$\square$

1.2 Dwork's approach on Gamma functions

First of all we give a brief review of the properties of the p-adic Gamma function from[17] and [18]. We recall also the definition and basic proprierties of the classic Γ from [30]:

Definition 13. The Γ function is defined as:

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$$

The main properties are:

- The functional equation: $\Gamma(s+1) = s\Gamma(s)$
- In particular for $n \in \mathbb{N}$: $\Gamma(n+1) = n!$
- The reflection formula:
$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$
- The distribution formula:
$$\prod_{i=1}^{n-1}\Gamma\left(\frac{s+i}{n}\right) = n^{1-s}\Gamma(s)\prod_{i=0}^{n-1}\Gamma\left(\frac{i}{n}\right)$$

We will see that there are a lot of analogous formulas in the p-adic case, so let's introduce the p-adic Gamma function following Dwork:

Definition 14. [17, p.2] The Morita's Gamma function is defined by the functional equation:

$$\Gamma_M(s+1) = -s\Gamma_M(s) \qquad |s| = 1$$

$$\Gamma_M(s+1) = -\Gamma_M(s) \qquad |s| < 1$$

and $\Gamma_M(0) = 1$

Observation 4. For $n \in \mathbb{N}$ this function works like the factorial in fact:

$$\Gamma_M(n) = (-1)^n \prod_{0<i<n, p\nmid n} i$$

Definition 15. Let $a \in U = \mathbb{Q} \cap \mathbb{Z}_p \setminus \mathbb{Z}$ consider X^a a symbol, $L_{0,\infty}$ the set of Laurent series converging in some $\{r_1 < |x| < r_2\}$ with $r_1 < 1 < r_2$ then :

$$\Omega_a^0 = \{X^a \xi | \xi \in L_{0,\infty}\}$$

We can define a derivation operator D setting:

$$D(X^a\xi) = X^a \left(X\frac{d}{dX} + a + \pi X \right) \xi$$

and set $\overline{\Omega_a} = \Omega_a^0 / D\Omega_a^0$ which is a vector space of dimension 1 with base the image of X^a. The base depends only on a mod $\mathbb{Z}$ and the relation for the change of the basis is:

$$X^{a+m} \equiv \frac{\Gamma(a+m)}{\Gamma(a)}(-\pi)^{-m} X^a$$

Consider the map $\alpha : \Omega^a \to \Omega^b$ defined as $\alpha X^a \xi = X^b \psi(\xi X^{a-pb}\Theta)$ where

$$\psi(\xi) = p^{-1} \sum_{X^p = y} \xi(X)$$

then $\gamma_p(a,b)$ is the unique number given by $\alpha X^a \equiv \gamma_p(a,b) X^b$ mod $D\Omega_a^0$ and is given explicitely by :

$$\gamma_p(a,b) = (-\pi)^{-i} \sum_{pi+t \geq 0} c_{pi+t} \frac{\Gamma(b+i)}{\Gamma(b)}$$

where c_i are the coefficient of the series expansion of Θ.

Theorem 2. *[18, p.245] The function $\gamma(a,b)$ satisfies for every integers m and n the following relation:*

$$\gamma_p(a+m, b+n) = \gamma_p(a,b) \frac{\Gamma(a+m)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+n)} (-\pi)^{n-m}$$

where π is such that $\pi^{p-1} = -p$

Observation 5. We can define the Boyarsky's Gamma function using γ_p by setting $a = pb-t$ with $0 < t = Rep(-a) < p-1$ ($Rep(-a) \in \{0, \dots, p-1\}$ and $|a + Rep(-a)| < 1$)

$$\Gamma^B(a) = \gamma_p(a,b)\pi^{-Rep(-a)}$$

in fact if $t \neq 0$:

$$\Gamma^B(a+1) = \gamma_p(pb-t+1,b)\pi^{-Rep(-a)-1} = -\gamma_p(a,b)\frac{\Gamma(pb-t+1)}{\Gamma(pb-t)}\pi^{-Rep(-a)} = -a\Gamma^B(a)$$

otherwise we can write $a = pb$ then $a+1 = pb+1-p+p = p(b+1)-(p-1)$ then

$$\Gamma^B(a+1) = -\gamma_p(a,b)\pi^{-Rep(-a)}\frac{py}{y} = -\Gamma^B(a)$$

so this function satisfies the functional equation of he Morita's gamma function.

From now on we will denote the Morita Gamma function as Γ_p

1.3 The function Ψ

We introduce now a new special function after Γ_p, we follow [2].

Definition 16. The function Ψ is defined by the following functional equation:

$$\sum_{j=0}^{\infty} p^{-j}\Psi(p^jT)^{p^j} = T$$

We have that $\Psi(T) \in T + T^2\mathbb{Z}[[T]]$ i.e.

$$\Psi(T) = T + \sum_{j=2}^{\infty} a_jT^j$$

We list now some proprierties of this functions discussed in [2]:

Proposition 5. *The function Ψ is entire i.e.*

$$\limsup_{n\to\infty} |a_i|^{1/i} = 0$$

Proposition 6. *The function Ψ has the property that for $x \in \mathbb{Q}_p$ then $\Psi(x) \in \mathbb{Z}_p$, and in particular if:*

$$x = \sum_{i>>-\infty}^{+\infty} x_ip^i$$

then $\Psi(p^ix) \equiv x_{-i}$ mod $p\mathbb{Z}_p$ i.e.

$$x = \sum_{i>>-\infty}^{+\infty} [\Psi(p^ix)]p^i$$

where [.] is the Teichmuller character.

Observation 6. Using these remarkable properties of Ψ we can define now a new function Ψ_i, consider $x \in \mathbb{Q}_p$ we define the following:

$$\Psi_i(x) = \frac{\Psi(p^i x)}{\exp_p \log_p \Psi(p^i x)}$$

where $\log_p$ is the Iwasawa logarithm.

If $x \in \mathbb{Q}_p$ and x_{-i} is a non-zero $-i$-th p-adic component then:

$$x_{-i} = \frac{\Psi(p^i x)}{\exp_p \log_p \Psi(p^i x)} = \Psi_i(x)$$

Proof. Consider $\Psi(p^i x)$, since we have the property that

$$\Psi(p^i x) \equiv x_{-i}$$

then $|\Psi(p^i x)| = 1$. Now $\Psi(p^i x) \in \mathbb{Z}_p^\times$ then by the fact that $\mathbb{Z}_p^\times \cong \mu_{p-1} \times (1 + p\mathbb{Z}_p)$, $\Psi(p^i x) = \zeta u$ where $\zeta = [\Psi(p^i x)]$ and u it will be denoted $\langle \Psi(p^i x) \rangle$. Now the logarithm kills roots of 1 i.e. $\log_p \Psi(p^i x) = \log_p \langle \Psi(p^i x) \rangle$ and since $\langle \Psi(p^i x) \rangle \in 1 + p\mathbb{Z}_p$ we have $\langle \Psi(p^i x) \rangle = \exp_p \log_p \langle \Psi(p^i x) \rangle = \exp_p \log_p \Psi(p^i x)$. So, by $\Psi(p^i x) = [\Psi(p^i x)] \langle \Psi(p^i x) \rangle$

$$x_{-i} = [\Psi(p^i x)] = \frac{\Psi(p^i x)}{\langle \Psi(p^i x) \rangle} = \frac{\Psi(p^i x)}{\exp_p \log_p \langle \Psi(p^i x) \rangle} = \frac{\Psi(p^i x)}{\exp_p \log_p \Psi(p^i x)}$$

□

In general it does not work for zero p-adic components, in fact, if $x_{-i} = 0$ we have that

$$\Psi(p^i x) = p^n \alpha \qquad \alpha \in Z_p$$

and so

$$\Psi_i(x) = \frac{p^n \alpha}{\alpha / \omega(\alpha)} = p^n \omega(\alpha)$$

which is 0 is and only is $\Psi(p^{-i} x) = 0$. So in general using Ψ we know that $x_i = \lim_{n \to \infty} \Psi(p^{-i} x)^{p^n}$ and we obtain the i-th p-adic component, but this is not an analytic process. If $x_i \neq 0$ using this trick we can revover it in an 'analytical way'.

Chapter 2

p-adic measures on $\mathbb{Z}_p$

2.1 p-adic distributions and measures

Definition 17. We define the Bernoulli polynomials $B_k(x)$ by the following generaing function [23, p.35]):

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} t^k$$

and we call $B_k(0)$ (denoted by B_k) k-th Bernoulli number.

By this definition we can immediately deduce some useful properties like $B_k'(x) = kB_{k-1}(x)$ (where $'$ means differeniation), $B_k(x+1) = B_k(x) + kx^{k-1}$ and the distribution formula for $N > 0$:

$$\sum_{j=0}^{N-1} B_k\left(\frac{x+j}{N}\right) = \frac{B_k(x)}{N^{k-1}}$$

Proof.

$$\sum_{i=0}^{N-1} \frac{te^{(x+a)t}}{e^{Nt} - 1} = \frac{1}{N} \sum_{i=0}^{N-1} \frac{Nte^{\frac{(x+a)}{N}Nt}}{e^{Nt} - 1}$$

Using now the series expansion of the exponential and the definition of Bernoulli polynomial

$$\frac{1}{N} \sum_{i=0}^{N-1} \sum_{k=0}^{\infty} B_k\left(\frac{x+i}{N}\right) \frac{(Nt)^k}{k!} = \sum_{k=0}^{\infty} \sum_{i=0}^{N-1} N^{k-1} B_k\left(\frac{x+i}{N}\right) \frac{t^k}{k!}$$

On the other hand $\sum_{i=0}^{N-1} e^{it} = \dfrac{1 - e^{Nt}}{1 - e^t}$ and so:

$$\sum_{i=0}^{N-1} \frac{te^{(x+a)t}}{e^{Nt} - 1} = \frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

□

Definition 18. We can define more generally the χ-Bernoulli Polinomials and [23, p. 37]) as:

$$te^{tx}\frac{\sum_{r=0}^{m}\chi(r)e^{rt}}{e^{mt}-1}=\sum_{k=0}^{\infty}\frac{B_{k,\chi}(x)}{k!}t^k$$

(with χ a Dirichlet character mod m).

Proposition 7. *There is an integral formula for B_k which is:*

$$B_k=\int_{\mathbb{Z}_p}x^k dx$$

Proof. Let $a \in p\mathbb{Z}_p$ then :

$$\int_{\mathbb{Z}_p}\exp_p(ax)dx=\frac{a}{\exp_p(a)-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}a^k$$

(see [28] pag.171) on the other hand

$$\int_{\mathbb{Z}_p}\exp_p(ax)dx=\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p}x^n dx\right)\frac{a^n}{n!}$$

So comparing the terms of the 2 series above:

$$\int_{\mathbb{Z}_p}x^k dx=B_k$$

which is the result. $\square$

Definition 19. (we follow now[23] , [22]) Consider $\mathbb{Z}_p \subset \mathbb{Q}_p$ then a distribution on $\mathbb{Z}_p$ is a map ν from the set $\{x+p^N\mathbb{Z}_p |\, x\in\mathbb{Z}_p, N\geq 0\}$ to $\mathbb{Q}_p$ such that, for every $x+p^N\mathbb{Z}_p\subset\mathbb{Z}_p$

$$\nu(x+p^N\mathbb{Z}_p)=\sum_{i=0}^{p-1}\nu(x+ip^N+p^{N+1}\mathbb{Z}_p)$$

If the distribution is bounded then is called measure.

Observation 7. We can define the set of distributions as a continuous linear map to the set of locally costant functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ and given a distribution ν then we define $\int_{\mathbb{Z}_p} f d\nu=\nu(f)$ where f is locally costant. We recover the old definition setting:

$$\nu(x+p^N\mathbb{Z}_p)=\int_{\mathbb{Z}_p}\chi_{x+p^N\mathbb{Z}_p}d\nu$$

with $\chi_{x+p^N\mathbb{Z}_p}$ the characteristic function. We recall also that locally costant functions are countinuous because $\mathbb{Z}_p$ is totally disconnected then clearly if f is locally costant is costant in different connected components which does not intersect. So we have clearly a continuous function.

Lemma 1. *The set of locally costant functions is dense in the set of continuous functions* $C(\mathbb{Z}_p, \mathbb{Q}_p)$.

Proof. Let f continuos functions, define the sequence of functions $g_n(x) = f(i)$ such that $0 \leq i \leq p^n - 1$ and $i \equiv x \mod p^n\mathbb{Z}_p$. By the continuity of f we have that $g_n \rightrightarrows f$. To see this we observe that $x = i + p^n a$ then $|f(x) - g_n(x)| = |f(i + p^n a) - f(i)| \to 0$ for $n \to \infty$. $\square$

Definition 20. Let f be a locally costant function (assume costant in discs of radius p^{-n}) then we define the integral of f against a distribution ν (thought as a locally costant map on the balls) as:

$$\int_{\mathbb{Z}_p} f d\nu = \sum_{i=0}^{p^n-1} f(i)\nu(i + p^n\mathbb{Z}_p)$$

We can think a measure as a map from the set of continuous functionals to $\mathbb{Q}_p$, in fact we have an isomorphism:

$$Meas(\mathbb{Z}_p, \mathbb{Q}_p) \cong (C(\mathbb{Z}_p, \mathbb{Q}_p))'_{strong}$$

where $C(\mathbb{Z}_p, \mathbb{Q}_p)$ is the set of continuous functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ and $Meas(\mathbb{Z}_p, \mathbb{Q}_p)$ is the set of measures on $\mathbb{Z}_p$ with values in $\mathbb{Q}_p$. So, given a measure ν, we can define the integral of a continuous function as:

$$\int_{\mathbb{Z}_p} f(x)d\nu(x) = \lim_{N\to\infty} \sum_{i=0}^{p^N-1} f(i)\nu(i + p^N\mathbb{Z}_p)$$

(i.e. extending by continuity the definition above, this definition does not depend on the choice of the sequence converging to f) now we can see the duality stated before, in fact given an addittive map μ on balls in $\mathbb{Z}_p$ we get the continuous functional:

$$\mu \mapsto \left(f \mapsto \int_{\mathbb{Z}_p} f d\mu \right)$$

Conversely given a continuous functional ϕ we define the measure μ_ϕ :

$$\mu_\phi(x + p^n\mathbb{Z}_p) = \phi(\chi_{x+p^n\mathbb{Z}_p})$$

where χ is the characteristic function.

Observation 8. In general for distributions is not well defined the Integral defined using "Riemann sums" like measures.

Definition 21. Consider ν a measure and f a continuous function we define the measure $f\nu$ as:

$$\int_{\mathbb{Z}_p} h(x)d(f\nu)(x) = \int_{\mathbb{Z}_p} (fh)(x)d\nu(x)$$

Definition 22. (Convolution product)
Given two measures μ and ν we define the convolution product $\mu * \nu$ in the following way:

$$\int_{\mathbb{Z}_p} f\, d(\mu * \nu) = \int_{\mathbb{Z}_p} \left(\int_{\mathbb{Z}_p} f(x+y)\, d\mu(x) \right) d\nu(y)$$

Definition 23. (Amice transform) For a measure ν we define the Amice transform as the power series:

$$A_\nu(T) = \int_{\mathbb{Z}_p} (1+T)^x d\nu(x) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\nu(x) \right) T^n$$

Theorem 3. *The map $\mu \to A_\mu$ defines an isomorphism :*

$$(C(\mathbb{Z}_p, \mathbb{Q}_p))'_{strong} \to \mathbb{Z}_p[[T]]$$

from the space of measures and power series with coefficients in $\mathbb{Z}_p$.

Observation 9. This theorem tells us an important fact: the topological isomorphism stated in the theorem is given by

$$T \mapsto \Delta_1 - \Delta_0$$

Where Δ_a is the dirac measure i.e. the measure which acts on the functions in the following way:

$$\int_{\mathbb{Z}_p} f\, d\Delta_a = \Delta_a(f) = f(a)$$

for a fixed $a \in \mathbb{Z}_p$. So as remarked in [10] the Amice transform of a measure μ gives a way to write this measure with respect to the basis $\{T^n\}_{n\in\mathbb{N}}$ (which is in fact the dual basis of the Mahler basis) i.e.

$$\mu = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) \right) T^n = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) \right) (\Delta_1 - \Delta_0)^n$$

in fact if f is continuous by Mahler's theorem :

$$f = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$$

then :

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) \right) T^n(f) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) \right) (\Delta_1 - \Delta_0)^n (f) =$$

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x) \right) a_n(f) = \int_{\mathbb{Z}_p} \left(\sum_{n=0}^{\infty} a_n(f) \binom{x}{n} \right) d\mu(x) = \int_{\mathbb{Z}_p} f d\mu(x) = \mu(f)$$

so these measures coincides.

Example 2. We can define the Haar "measures" as :

$$\mu_{haar}(x + p^N \mathbb{Z}_p) = \frac{1}{p^N}$$

We stress that in our context the Haar measure is not bounded and so it is not properly a measure but a distribution. We remark also that in this case the definition of the integral of a function against a distribution using Riemann sums make sense, in fact, in general we have this definition:

$$\int_{\mathbb{Z}_p} f(x) d\mu(x) = \lim_{n\to\infty} \sum_{i=0}^{p^n-1} f(i)\mu(i + p^n \mathbb{Z}_p)$$

but in this case we clearly have:

$$\lim_{n\to\infty} \sum_{i=0}^{p^n-1} f(i)\mu_{haar}(i + p^n \mathbb{Z}_p) = \lim_{n\to\infty} p^{-n} \sum_{i=0}^{p^n-1} f(i)$$

so we get:

$$\int_{\mathbb{Z}_p} f(x) dx = \int_{\mathbb{Z}_p} f(x) d\mu_{haar}(x)$$

which make sense for every strictly differentiable functions.

2.2 Bernoulli Measures and p-adic L functions

Definition 24. (Bernoulli Distributions) see [23, p.36] and [22, p. 34] Using Bernoulli polynomials we can define a distribution on $\mathbb{Z}_p$ as:

$$\nu_k(x + p^N \mathbb{Z}_p) = p^{N(k-1)} B_k\left(\frac{x}{p^N}\right)$$

where $0 \le x \le p^N - 1$ thank to the distribution formula this is a well defined distribution on the p-adic integers:

$$\nu_k(x + p^N \mathbb{Z}_p) = p^{N(k-1)} \left(p^{k-1} \sum_{j=0}^{p-1} B_k\left(\frac{x + p^N j}{p^{N+1}}\right)\right) = \sum_{j=0}^{p-1} \nu_k(x + jp^N + p^{N+1}\mathbb{Z}_p)$$

since: $0 \le x + p^N j \le x + p^{N+1} - p^N \le p^N - 1 + p^{N+1} - p^N = p^{N+1} - 1$ In general this is not a measure since $B_k\left(\frac{x}{p^N}\right) \sim \left(\frac{x}{p^N}\right)^k$ which is not bounded.

Observation 10. We can easily build a measure with a standard technique: fix c rational integer prime to p then

$$\nu_{k,c} = \nu_k(x + p^N\mathbb{Z}_p) - c^{-k}\nu_k(cx + p^N\mathbb{Z}_p)$$

these are called Bernoulli-Measures. We can observe also this interesting relation (since the function 1 is constant):

$$\int_{\mathbb{Z}_p} d\nu_k = \nu_k(\mathbb{Z}_p) = p^0 B_k(0) = B_k$$

(all the relations with Bernoulli polynomials can be translated with similar properties involving χ-Bernoulli polynomials for example the expression above becames:$B_{k,\chi} = \int \chi d\nu_k$).

Proposition 8. *([22, p.37 Th. 5] and [23, p.39 Thm 2.1])*
Take D_k the least common denominator of $B_k(x)$ and $d_k = v_p(D_k)$.

- *$\nu_{1,c}$ takes values in $\mathbb{Z}_p$*
- *$D_k\nu_{k,c}(x + p^N\mathbb{Z}_p) \equiv kD_k x^{k-1}\nu_{1,c}(x + p^N\mathbb{Z}_p)$ mod p^N*
- *$\nu_{k,c}$ takes values in $\mathbb{Z}_p$*

Proof. For the first assertion, first of all, we observe that $B_1(x) = x - \frac{1}{2}$ then let's compute explicitely $\nu_{1,c}(a + p^N\mathbb{Z}_p)$ (we denote by $[cx]$ the reduction mod p^N of cx):

$$\nu_{1,c}(x + p^N\mathbb{Z}_p) = \frac{x}{p^N} - \frac{1}{2} - \frac{1}{c}\left(\frac{[cx]}{p^N} - \frac{1}{2}\right) = \frac{1/c - 1}{2} + \frac{x}{p^N} - \frac{1}{c}\left(\frac{cx}{p^N} - \left[\frac{cx}{p^N}\right]\right) =$$

and so

$$\mu_{1,c}(x + p^N\mathbb{Z}_p) = \frac{1}{c}\left[\frac{cx}{p^N}\right] + \frac{1/c - 1}{2}$$

For the second statement we have to make a similar calculation. We observe that since the congruence mod p^N and the presence $p^{N(k-1)}$ we can only consider the first two terms of $D_k B_K(x)$ which are $D_k x^k + \frac{k}{2}D_k x^{k-1}$ and so:

$$D_k\nu_{k,c}(x + p^N\mathbb{Z}_p) \equiv D_k p^{N(k-1)}\left[\left(\frac{x}{p^N}\right)^k - \left(\frac{1}{2}\right)^k - \left(\frac{c[c^{-1}x]_{p^N}}{p^N}\right)^k + c\frac{k}{2}\left(\frac{c[c^{-1}x]}{p^N}\right)^{k-1}\right]$$

using now the same argument as above we find $D_k\nu_{k,c}(x + p^N\mathbb{Z}_p) \equiv D_k x^{k-1}\nu_{1,c}(x + p^N\mathbb{Z}_p)$ mod p^N . For the last part, if $N - d_k > 0$ then it is clear for the firsts two points. Otherwise by the distribution law we have:

$$\nu_{k,c}(x + p^N\mathbb{Z}_p) = \sum_{i \equiv x \, mod p^N} \nu_{k,c}(i + p^M\mathbb{Z}_p)$$

with $M - d_k > 0$. $\square$

Observation 11. $\nu_{k,c} = kx^{k-1}\nu_{1,c}$ in fact by the definition of integral e by the proposition above:

$$\int_{\mathbb{Z}_p} f(x) d\nu_{k,c}(x) = \lim_{N\to\infty} \sum_{i=0}^{p^N-1} f(i)\nu_{k,c}(i + p^N\mathbb{Z}_p) = \lim_{N\to\infty} \sum_{i=0}^{p^N-1} f(i)ki^{k-1}\nu_{1,c}(i + p^N\mathbb{Z}_p) + p^{2N-d_k}$$

but the last term on the right goes to zero, this proves the equality $\int_{\mathbb{Z}_p} f(x)d\nu_{k,c}(x) = k\int_{\mathbb{Z}_p} f(x)x^{k-1}d\nu_{1,c}(x)$.

These measures are important for p-adic L-functions, in fact we can introduce the Mellin transorm ([23, p.105]):

Definition 25. (Mellin transform) Take μ a measure on $\mathbb{Z}_p$ then the Mellin transform of μ is:

$$\mathscr{M}(\mu)(s) = \int_{\mathbb{Z}_p^\times} \langle a\rangle^s a^{-1} d\mu(a)$$

Considering now s a variable in $\mathbb{Z}_p$ we can define the p-adic L-function as:

Definition 26. (Mazur, Kubota-Leopoldt) Taking χ a Dirichlet character mod p^N, and $c \in \mathbb{Z}_p^\times$ s.t. $\chi(c)\langle c\rangle \neq 1$ we define (see [23, p.107]):

$$L_p(1-s,\chi) = \frac{-1}{1-\chi(c)\langle c\rangle^s}\mathscr{M}(\chi\nu_{1,c})(s)$$

Lemma 2. *Take $k > 0$, using the above assumptions we get:*

$$\frac{B_{k,\chi}}{k} = \frac{1}{1-\chi(c)c^{-k}}\int_{\mathbb{Z}_p} x^{k-1}\chi(x)d\nu_{1,c}$$

Proof. By the integral formula for Bernoulli numbers seen before:

$$\frac{B_{k,\chi}}{k} = \int_{\mathbb{Z}_p} \chi d\nu_{k,c} + \int_{\mathbb{Z}_p} \chi c^k d\nu_k(c^{-1}x)$$

Then from the relation $\nu_{k,c} = kx^{k-1}\nu_{1,c}$ and sending $x \mapsto cx$ we find:

$$B_{k,\chi} = k\int_{\mathbb{Z}_p} x^{k-1}\chi d\nu_{1,c} + \chi(c)c^{-k}B_{k,\chi}$$

and so the result. $\square$

Theorem 4. *For every $k > 0$ we have:*

$$L_p(1-k,\chi) = -\frac{B_{k,\chi\omega^{-k}}}{k}$$

Proof. Using the lemma above with a direct calculation:

$$\int_{\mathbb{Z}_p} \langle x\rangle^{k-1}\chi(x)\omega(x)^{-1}d\nu_{1,c} = \int_{\mathbb{Z}_p} x^{k-1}\chi(x)\omega(x)^{1-k}\omega(x)^{-1}d\nu_{1,c} = \frac{1}{k}(1-\chi\omega^{-k}(c)c^k)B_{k,\chi\omega^{-k}}$$

$\square$

Theorem 5. *(Mahler expansion of L_p) The Kubota-leopoldt p-adic L function satisfies the following formula:*

$$L_p(-s,\chi) = \frac{1-p}{1-\chi(c)\langle c\rangle^{s+1}} \sum_{n=0}^{\infty} p^n \frac{B_{n+1,\chi}}{n+1}\binom{s}{n}$$

Proof. Recall that the p-adic Mellin trasform is defined as:

$$\mathscr{M}(\mu)(s) = \int_{\mathbb{Z}_p^\times} \langle x\rangle^s x^{-1}\, d\mu(x)$$

With a change of variables we get:

$$\mathscr{M}(\mu)(s) = (p-1)\int_{1+p\mathbb{Z}_p} x^{s-1}\, d\mu(x)$$

Now the last change is : $x \to 1+px$, so we get an integral on $\mathbb{Z}_p$:

$$\mathscr{M}(\mu)(s) = (p-1)\int_{\mathbb{Z}_p} (1+px)^{s-1}\, d\mu(x)$$

Now we want to make all the calculations i.e.:

$$(1+px)^{s-1} = \sum_{n=0}^{\infty}\binom{s-1}{n}(px)^n$$

and so:

$$\mathscr{M}(\mu)(s+1) = \sum_{n=0}^{\infty}\left(p^n\int_{\mathbb{Z}_p} x^n\, d\mu\right)\binom{s}{n}$$

So this is a continuous function defined by the above Mahler series. Recall that if $f \in C(\mathbb{Z}_p,\mathbb{Z}_p)$ then:

$$f(x) = \sum_{n=0}^{\infty} a_n(f)\binom{x}{n}$$

where:

$$a_n(f) = \int_{\mathbb{Z}_p} f(x)\, d(\Delta_1 - 1)^n$$

So we get (let $T = \Delta_1 - 1$) :

$$\int_{\mathbb{Z}_p} \mathscr{M}(\mu)(s+1)\, dT^n = a_n(\mathscr{M}(\mu)(s+1)) = (p-1)p^n\int_{\mathbb{Z}_p} x^n\, d\mu(x)$$

By definition the Kubota-Leopoldt p-adic L function is defined as:

$$L_p(-s,\chi) = \frac{-1}{1-\chi(c)\langle c\rangle^{s+1}}\mathscr{M}(\chi\nu_{1,c})(s+1)$$

with χ a Dirichlet character and $\nu_{1,c}$ the Bernoulli measure. Eventually:

$$a_n(\mathscr{M}(\chi\nu_{1,c})(s+1)) = (p-1)p^n\int_{\mathbb{Z}_p} x^n\chi(x)\,d\nu_{1,c}(x) = (p-1)p^n\frac{B_{n+1,\chi}}{n+1}$$

□

2.3 Diamond's LogGamma function

Definition 27. (We follow in particular [13] but also[28, p.182]) The Diamond's LogGamma function is defined in $\mathbb{C}_p \setminus \mathbb{Z}_p$ by:

$$G_p(x) = \lim_{k\to\infty}\frac{1}{p^k}\sum_{n=0}^{p^k-1}(x+n)\log(x+n)-(x+n)$$

We can see this definition using an inegral form on $\mathbb{Z}_p$:

$$G_p(x) = \int_{\mathbb{Z}_p}(x+t)(\log_p(x+t)-1)dt$$

Observation 12. We observe that in this case we have an action of the Haar distribution on the function $f(x) = x\log_p(x) - x$, similar to the action of measures to continuous functions defined in [10] by 'convolution'

$$(\mu f)(x) = \int_{\mathbb{Z}_p} f(x+t)d\mu(t)$$

The same argument could be done for Bernoulli polynomials :

$$B_k(x) = \int_{\mathbb{Z}_p}(x+t)^k d\mu_{haar}(t)$$

see [28].

Theorem 6. *This function G_p is locally analytic in $\mathbb{C}_p \setminus \mathbb{Z}_p$ and for every $x \in \mathbb{C}_p$ the disc of holomorphicity is the largest disc such that it does not intersect $\mathbb{Z}_p$.*

Proof. Let $f(x,t) = (x+t)\log_p(x+t) - (x+t)$ fix $a \in \mathbb{C}_p$ and $\rho = d(a,\mathbb{Z}_p)$ take $x \in \mathbb{C}_p$ s.t. $|x| < \rho$ then $G_p(a+x) = \int_{\mathbb{Z}_p} f(x+a,t)dt$. Now :

$$f(a+x,t) = (a+x+t)\left(\log_p\left(1+\frac{x}{a+t}\right)+\log_p(a+t)\right)-(a+t+x) =$$

$$(a+x+t)\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{x^n}{(a+t)^n}\right)+(a+t+x)(\log_p(a+t)-1)$$

Now making calculations and taking the integral we obtain:

$$G_p(a+x)=G_p(a)+x\int_{\mathbb{Z}_p}\log_p(x+t)dt+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}\left(\int_{\mathbb{Z}_p}(x+t)^{-n}dt\right)x^{n+1}$$

□

Proposition 9. *The function G_p satisfies the functional equation [28, p.182, Thm 60]:*

$$G_p(x+1)-G_p(x)=\log_p(x)$$

Proof. let $f(x,t)=(x+t)\log_p(x+t)-(x+t)$ then $f(x+1,t)-f(x,t)=f(x,t+1)-f(x,t)$ then by the properties of the Volkenborn integral $G_p(x+1)-G_p(x)=\frac{\partial}{\partial t}f(x,0)=\log_p(x)$ □

For us one of the most important proprierties is:

Theorem 7. *We have a distribution law:*

$$G_p(x)=\sum_{a=0}^{p^r-1}G_p\left(\frac{x+a}{p^r}\right)$$

Proof. Let $f(x,t)=(x+n)\log_p(x+n)-(x+n)$, using the additivity of the integral we have:

$$\int_{\mathbb{Z}_p}f(x,t)d\mu_{haar}(t)=\sum_{i=0}^{p^N-1}\int_{i+p^N\mathbb{Z}_p}f(x,t)d\mu_{haar}(t)$$

now using the properties of Haar measures:

$$\int_{i+p^N\mathbb{Z}_p}f(x,t)d\mu_{haar}(t)=\frac{1}{p^N}\int_{\mathbb{Z}_p}f(x,i+p^Nt)d\mu_{haar}(t)$$

using the definition of $f(x,t)$:

$$\int_{\mathbb{Z}_p}f(x,i+p^Nt)d\mu_{haar}(t)=p^N\int_{\mathbb{Z}_p}f\left(\frac{x+i}{p^N},t\right)d\mu_{haar}(t)$$

and so:

$$G_p(x)=\sum_{a=0}^{p^r-1}G_p\left(\frac{x+a}{p^r}\right)$$

□

Observation 13. In the paper of Diamond and in [28, p.175] is proved a stronger version of this distribution law:

$$G_p(x) = \left(x - \frac{1}{2}\right)\log_p(m) + \sum_{a=0}^{m-1} G_p\left(\frac{x+a}{m}\right)$$

But for our purpose is enought the case in which $m = p^r$.

Theorem 8. *G_p satisfies the following "Stirling Formula" for $|x| > 1$:*

$$G_p(x) = \left(x - \frac{1}{2}\right)\log(x) - x + \sum_{r=1}^{\infty} \frac{B_{r+1}}{r(r+1)x^r}$$

Proof. Recall hat by definition $G_p(x) = \int_{\mathbb{Z}_p} f(x,t)d\mu_{haar}(t)$ and $\int_{\mathbb{Z}_p}(x+t)^k d\mu_{haar}(t) = B_k(x)$ so:

$$G_p(x) = \int_{\mathbb{Z}_p}(x+t)(\log_p(x+t)-1)d\mu_{haar}(t) = \int_{\mathbb{Z}_p}(x+t)(\log_p(x(1+t/x))d\mu_{haar}(t) - B_1(x)$$

Using then the series expansion of logarithm (since $|t/x| < 1$)

$$\int_{\mathbb{Z}_p}(x+t)(\log_p(x(1+t/x))d\mu_{haar}(t) = \log_p x B_1(x) + \int_{\mathbb{Z}_p}(x+t)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{t^n}{x^n}d\mu_{haar}(t)$$

then, in the last integral, separating the sum and interchanging the integral sign with the series sign we obtain the following two series:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{nx^{n+1}}\int_{\mathbb{Z}_p} t^n d\mu_{haar}(t) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{nx^n}\int_{\mathbb{Z}_p} t^{n+1}d\mu_{haar}(t)$$

Now changing the index in the first series $n \to n+1$ we obtain that the term for $n = 0$ is B_1 and so:

$$\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{(n+1)x^n}\int_{\mathbb{Z}_p} t^{n+1}d\mu_{haar}(t) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{nx^n}\int_{\mathbb{Z}_p} t^{n+1}d\mu_{haar}(t)$$

Then comparing term by term the two series and adding this to the first part of the calculation we obtain the result:

$$G_p(x) = \log_p(x)B_1(x) - x + \sum_{r=1}^{\infty}\frac{B_{r+1}}{r(r+1)x^r}$$

□

Proposition 10. *The function G_p has the following reflection formula:*

$$G_p(x) + G_p(1-x) = 0$$

Proof. By the Stirling formula proved above we have $G_p(x) - G_p(-x) = \log_p(x)$, then for $|x| > 1$ we have also the functional equation $G_p(x+1) = G_p(x) + \log_p(x)$ and so changing $x \to -x$ in the last formula we have:

$$G_p(1-x) = G_p(-x) + \log_p(x) = -G_p(x) - \log_p(x) + \log_p(x) = -G_p(x)$$

For $|x| \leq 1$ we have :

$$G_p(x) + G_p(1-x) = \sum_{a=0}^{p^r-1} \left(G_p\left(\frac{x+a}{p^r}\right) + G_p\left(\frac{1-x+a}{p^r}\right)\right) =$$

$$\sum_{a=0}^{p^r-1} \left(G_p\left(\frac{x+a}{p^r}\right) + G_p\left(\frac{1-x+a}{p^r}\right)\right) = \sum_{a=0}^{p^r-1} \left(G_p\left(\frac{x+a}{p^r}\right) - G_p\left(1 - \frac{1-x+a}{p^r}\right)\right)$$

then :

$$G_p(x) + G_p(1-x) = \sum_{a=0}^{p^r-1} \left(G_p\left(\frac{x+a}{p^r}\right) - G_p\left(\frac{p^r+1-x+a}{p^r}\right)\right) = 0$$

□

Theorem 9. *We have a relation with Morita's Gamma functions and the G_p, let $x \in \mathbb{Z}_p$ then:*

$$\log_p \Gamma_p(x) = \sum_{j=0, |x+j|=1}^{p-1} G_p\left(\frac{x+j}{p}\right)$$

2.4 Gamma measures and special values of p-adic L functions

Thank to the distribution property we can define a distribution and then a measure using G_p (see [14]) and its derivatives in fact we have:

$$G_p^{(k)}(x) = p^{-k} \sum_{j=0}^{p-1} G_p^{(k)}\left(\frac{x+j}{p}\right)$$

where the (k) at he exponent means k-th derivative.

Definition 28. We define the p-adic $k-th$ Gamma distribution as:

$$\mu_{G,k}(a + p^N \mathbb{Z}_p) = p^{-kN} G_p^{(k)}\left(\frac{a}{p^N}\right)$$

where $a \in \mathbb{Z}$ s.t. $0 < a < p^N$ and $(a,p) = 1$
Diamond's in his paper defines measures in a similar way in which are defined Bernoulli measures:

Definition 29. Fix $c \in \mathbb{Z}$ with $(c, p) = 1$ in such a way to obtain a p-adic unit then:

- $\mu_{0,c}(X) := \mu_{G,0}(X) - c^{-1}\mu_{G,0}(cX) + \frac{\log_p(c)}{c}\nu_{1,c}(cX)$
- $\mu_{1,c}(X) := \mu_{G,1}(X) - \mu_{G,1}(cX) + \log_p(c)\mu_{haar}(cX)$
- For $k > 1$ $\mu_{k,c}(X) := \dfrac{(-1)^k}{(k-2)!}(\mu_{G,k}(X) - c^{-k}\mu_{G,k}(cX))$

Theorem 10. *In the same hypothesis as above we have the following equalities of measures in* $\mathbb{Z}_p^\times$ *:*

1. $\mu_{0,c}(x) = \log_p(x)\nu_{1,c}(x)$
2. $\mu_{1,c}(x) = \frac{1}{x}\nu_{1,c}(x)$
3. $k > 1$ $\mu_{k,c}(x) = (1-k)x^{-k}\nu_{1,c}(x)$

Proof. By definition we have the following:

$$\mu_{1,c}(x + p^m\mathbb{Z}_p) = p^{-m}G_p'\left(\frac{[cx]}{p^m}\right) - p^{-m}G_p'\left(\frac{x}{p^m}\right) + p^{-m}\log_p(x)$$

Now we use the "Stirling formula" proved above to obtain (all he congruences are meant mod p^m):

$$\mu_{1,c}(x + p^m\mathbb{Z}_p) \equiv p^{-m}\left(\log_p\left(\frac{cx}{[cx]}\right) - \frac{p^m}{2x} + \frac{p^m}{2[cx]}\right) \equiv$$

$$p^{-m}log_p\left(1 + \frac{p^m(cx/p^m - [cx/p^m])}{[cx]}\right) + \frac{1}{2}\frac{1}{x}\left(\frac{1}{c}\right) - 1\right) \equiv \frac{1}{c}\left(\frac{1}{c}\left[\frac{cx}{p^n}\right] + \frac{1/c - 1}{2}\right)$$

finally:

$$\mu_{1,c}(x + p^m\mathbb{Z}_p) \equiv \frac{1}{x}\nu_{1,c}(x + p^m\mathbb{Z}_p)$$

Then by the definition of integral (using Riemann sums) we have the equality of the integrals

$$\int_{X\subset\mathbb{Z}_p^\times} f(x)d\mu_{1,c} = \int_{X\subset\mathbb{Z}_p^\times} f(x)x^{-1}d\nu_{1,c}$$

The other cases are similar. $\square$

Application 1. Recall the definition of p-adic L function given in the section before:

$$L_p(1-s, \chi) = \frac{-1}{1 - \chi(c)\langle c\rangle^s}\int_{\mathbb{Z}_p^\times} \langle x\rangle^s\chi(x)x^{-1}d\nu_{1,c}(x)$$

we can trivially reformulate these definition in terms of Diamond's Gamma measures:

$$L_p(1-s, \chi) = \frac{-1}{1 - \chi(c)\langle c\rangle^s}\int_{\mathbb{Z}_p^\times} \langle x\rangle^s\chi(x)d\mu_{1,c}(x)$$

This type of integral is called in [23] Gamma transform, so the p-adic L function can be seen as the Gamma transform of the measure $\chi\mu_{1,c}$

Application 2. We can compute $L(0,\omega)$ in fact:

$$\int_{\mathbb{Z}_p^\times} \log_p(x) d\nu_{1,c} = \frac{\log_p(c)}{c} \int_{\mathbb{Z}_p^\times} d\nu_{1,c} = 0$$

and so

$$L_p(0,\omega) = \frac{c}{(c-1)\log_p(c)} \int_{\mathbb{Z}_p^\times} \log_p(x) d\nu_{1,c} = 0$$

Proof. For the Diamond's distribution we have:

$$\int_{\mathbb{Z}_p^\times} d\mu_{G,0} = \sum_{i=1}^{p-1} \int_{i+p\mathbb{Z}_p} d\mu_{G,0} = \sum_{i=1}^{p-1} G_p\left(\frac{i}{p}\right)$$

By theorem 9 we get:

$$\sum_{i=1}^{p-1} G_p\left(\frac{i}{p}\right) = \sum_{i=1}^{p-1} G_p\left(\frac{0+i}{p}\right) = \log_p \Gamma_p(0) = 0$$

by the relation above we have:

$$\int_{\mathbb{Z}_p^\times} d\mu_{G,0} = 0$$

Now by definition:

$$\int_{\mathbb{Z}_p^\times} d\mu_{0,c}(x) = \int_{\mathbb{Z}_p^\times} d\mu_{G,0}(x) - c^{-1} \int_{\mathbb{Z}_p^\times} d\mu_{G,0}(cx) + \frac{\log_p(c)}{c} \int_{\mathbb{Z}_p^\times} d\nu_{1,c}(cx)$$

with the change $x \to c^{-1}x$ (which is an isomorphism of $\mathbb{Z}_p^\times$)in the second and third integral in the right hand side we obtain:

$$\int_{\mathbb{Z}_p^\times} d\mu_{0,c}(x) = (1-c^{-1}) \int_{\mathbb{Z}_p^\times} d\mu_{G,0}(x) + \frac{\log_p(c)}{c} \nu_{1,c}(\mathbb{Z}_p^\times)$$

eventually

$$\int_{\mathbb{Z}_p^\times} \log_p(x)\, d\nu_{1,c} = \frac{\log_p c}{c} \nu_{1,c}(\mathbb{Z}_p^\times) = 0$$

because:

$$\nu_{1,c}(x + p^N\mathbb{Z}_p) = \frac{1}{c}\left[\frac{cx}{p^N}\right] + \frac{1/c - 1}{2}$$

Then:

$$\nu_{1,c}(\mathbb{Z}_p^\times) = \nu_{1,c}(\mathbb{Z}_p) - \nu_{1,c}(p\mathbb{Z}_p) = \nu_{1,c}(0 + p^0\mathbb{Z}_p^\times) - \nu_{1,c}(0 + p\mathbb{Z}_p) = 0$$

The link with p-adic L-functions is given now by:

$$\int_{\mathbb{Z}_p^\times} d\nu_{1,c} = \int_{\mathbb{Z}_p^\times} xx^{-1} d\nu_{1,c} = \int_{\mathbb{Z}_p^\times} \langle x\rangle\omega(x)x^{-1} d\nu_{1,c} = (\langle c\rangle\omega(c) - 1)L_p(0,\omega)$$

$$L_p(0,\omega) = \frac{c}{(c-1)\log_p(c)} \int_{\mathbb{Z}_p^\times} \log_p(x) d\nu_{1,c}$$

$\square$

Application 3. We can compute $\int_{\mathbb{Z}_p^\times} x^{-1} d\nu_{1,c}(x)$ in fact:

$$\int_{\mathbb{Z}_p^\times} d\mu_{1,c}(x) = \int_{\mathbb{Z}_p^\times} d\mu_{G,1}(x) - \int_{\mathbb{Z}_p^\times} d\mu_{G,1}(cx) + \log_p(c)\mu_{haar}(\mathbb{Z}_p^\times)$$

as usual $x \to c^{-1}x$ gives to us

$$\int_{\mathbb{Z}_p^\times} x^{-1} d\nu_{1,c}(x) = \log_p(c)\left(1 - \frac{1}{p}\right)$$

Application 4. (Values of L-functions at positive integers and p-adic logGamma functions) Consider $L_p(1+k,\omega^{-k})$, where as usual ω is the Teichmuller character and $k \geq 1$ we can give an interesting link from G_p and these special values of p-adic L functions. Consider :

$$L_p(1-(-k),\omega^{-k}) = \frac{-1}{1-\omega^{-k}(c)\langle c\rangle^{-k}} \int_{\mathbb{Z}_p^\times} \langle x\rangle^{-k}\omega^{-k}(x)x^{-1} d\nu_{1,c}$$

Now we have:

$$\int_{\mathbb{Z}_p^\times} \langle x\rangle^{-k}\omega^{-k}(x)x^{-1} d\nu_{1,c} = \int_{\mathbb{Z}_p^\times} x^{-k}x^{-1} d\nu_{1,c} = \int_{\mathbb{Z}_p^\times} x^{-(k+1)} d\nu_{1,c}$$

By the equalities of measures given by Diamond:

$$\int_{\mathbb{Z}_p^\times} x^{-(k+1)} d\nu_{1,c} = -\frac{1}{k}\int_{\mathbb{Z}_p^\times} d\mu_{k+1,c} = \frac{(-1)^k(1-c^{-(k+1)})}{k!} p^{-(k+1)} \sum_{i=1}^{p-1} G_p^{(k+1)}\left(\frac{i}{p}\right)$$

Again by theorem 9 we get:

$$(\log_p \Gamma_p)^{(k)}(x) = p^{-k} \sum_{j=0,|x+j|=1}^{p-1} G_p^{(k)}\left(\frac{x+j}{p}\right)$$

Then:

$$L_p(1+k,\omega^{-k}) = \frac{(-1)^k(1-c^{-(k+1)})}{(c^{-k}-1)k!}(\log_p \Gamma_p)^{(k+1)}(0)$$

Theorem 11. *The expansion in base $\Delta_1 - \Delta_0$ of μ_D is given by:*

$$\mu_D = \log_p(c)\left(1-\frac{1}{p}\right) + \sum_{n=2}^{\infty}\left(\log_p(c)\left(1-\frac{1}{p}\right) + \sum_{k=2}^{n} \frac{s(n,k)}{n!}\frac{B_{k,\omega^k}(1-c^k)}{k}\right)(\Delta_1-\Delta_0)^n$$

Proof. To compute the Amice transform of a measure it's necessary to compute the integrals:

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu$$

for every $n \in \mathbb{N}$, in particular using the fact that:

$$\binom{x}{n} = \sum_{k=0}^{n} \frac{s(n,k)}{n!} x^k$$

where $s(n,k)$ are the Stirling numbers, it is enough to know the integral only over the monomials x^k, so let's compute it (we think μ_D as extended by 0 in $p\mathbb{Z}_p$). The degree zero of the series is given by the integral:

$$\int_{\mathbb{Z}_p^\times} d\mu_{1,c}(x) = \int_{\mathbb{Z}_p^\times} x^{-1} d\nu_{1,c}(x) = \log_p(c)\left(1 - \frac{1}{p}\right)$$

The higher degrees are:

$$\int_{\mathbb{Z}_p^\times} x^k d\mu_{1,c} = \int_{\mathbb{Z}_p^\times} x^{k-1} d\nu_{1,c} = \frac{B_{k,\omega^k}}{k}(1 - c^k)$$

for $k \geq 2$, which gives:

$$\mu_{1,c} = \log_p(c)\left(1 - \frac{1}{p}\right) + \sum_{n=2}^{\infty}\left(\log_p(c)\left(1 - \frac{1}{p}\right) + \sum_{k=2}^{n} \frac{s(n,k)}{n!} \frac{B_{k,\omega^k}(1 - c^k)}{k}\right)(\Delta_1 - \Delta_0)^n$$

In practice this is the Amice transorm of the measure μ_D which is given in fact setting $T = \Delta_1 - \Delta_0$ i.e.

$$A_{\mu_{1,c}}(T) = \log_p(c)\left(1 - \frac{1}{p}\right) + \sum_{n=2}^{\infty}\left(\log_p(c)\left(1 - \frac{1}{p}\right) + \sum_{k=2}^{n} \frac{s(n,k)}{n!} \frac{B_{k,\omega^k}(1 - c^k)}{k}\right) T^n$$

We observe that there is no term of degree one because:

$$\int_{\mathbb{Z}_p^\times} \binom{x}{1} d\mu_{1,c} = \int_{\mathbb{Z}_p^\times} x \, d\mu_{1,c} = \int_{\mathbb{Z}_p^\times} d\nu_{1,c} = 0$$

□

Chapter 3

Coleman's Gamma function

3.1 Relations between G_p and Γ_C

Definition 30. We follow the notations of Coleman introduced in [11], the Coleman's Γ function is defined by the following functional equation for $s \in \mathbb{Q}_p$:

$$\Gamma_C(s+1) = s^* \Gamma_C(s) \quad s \in \mathbb{Q}_p - \mathbb{Z}_p$$

$$\Gamma_C(s+1) = -\{s\} \Gamma_C(s) \qquad s \in \mathbb{Z}_p$$

with the normalization $\Gamma_C(s) = 1$ if $s \in \mathbb{Z}[1/p]$ and $0 \leq s < 1$.

We recall that $s^* = p^{-v(s)} \zeta_s^{-1} s$. In this case by ζ_s we mean the unique root of 1 (with order prime with p) in $\overline{\mathbb{Q}_p}$ such that $|\zeta_s - sp^{-v(s)}| < 1$. So for $s \in \mathbb{Z}_p^\times$ we define $\{s\} = 1$ for $s \in \mathbb{Z}_p^\times$ and $\{s\} = s$ for $s \in p\mathbb{Z}_p$. This implies that in $\mathbb{Z}_p$, Γ_C coincides with the classical Morita's Γ function.
In general the properties of the Morita's Γ function are well known so we will focus our study of Γ_C outside $\mathbb{Z}_p$ in particular our purpose is to understand some crucial (for us) arithmetic properties, if this is an analytic function and if there are relations with p-adic L functions. Our approach will be to study the properties of Γ_C and to compare them with the ones of G_p, the advantage is that Diamond gives to us precise analytic properties for its function and it will hallow us to deduce a lot of informations for Γ_C.

Proposition 11. *Let $s \in \mathbb{Q}_p \setminus \mathbb{Z}_p$ such that $s = \frac{i}{p^n} + m$ with $0 < \frac{i}{p^n} < 1$and $m \in \mathbb{N}$, then*

$$\Gamma_C\left(\frac{i}{p^n} + m\right) = \frac{\prod_{k=0}^{m-1} \exp_p \log_p \left(\frac{i}{p^n} + k\right)}{\exp_p \log_p \left(\frac{i}{p^n}\right) \exp_p \log_p \left(\frac{i}{p^n} - 1\right)}$$

and extends by continuity to all $\mathbb{Q}_p \setminus \mathbb{Z}_p$, in particular $\Gamma_C(s) \in 1 + p\mathbb{Z}_p$ for $s \in \mathbb{Q}_p \setminus \mathbb{Z}_p$.

Proof. By the definition of Coleman, we recall again that :

$$s^* = \frac{s}{p^{v_p(s)} \zeta_s}$$

where ζ_s is the unique root of 1 of order prime to p such that $|\zeta_s - sp^{-v(s)}| < 1$, it's easy now to determine it in fact, if we denote $[\cdot]$ the Teichmuller character, the $(p-1)$-root of 1 $[p^{-v_p(s)}s]$ satisfies clearly the property $|[p^{-v_p(s)}s] - p^{-v_p(s)}s| < 1$. In particular this implies that:

$$s^* = \frac{p^{-v_p(s)}s}{[p^{-v_p(s)}s]} \in 1 + p\mathbb{Z}_p$$

The function s^* for $s \neq 0$ could be expressed in a more 'analytic way' in fact we claim that:

$$s^* = \exp_p \log_p(s)$$

with $\log_p$ the Iwasawa logarithm. This beacuse if $s \in \mathbb{Q}_p^\times$ then $s = p^{v_p(s)}u$ with $u \in \mathbb{Z}_p^\times$ because $|u| = 1$. Then $u = [u]\langle u \rangle$ so evaluating that we get:

$$\exp_p \log_p(s) = \exp_p \log_p(p^{v_p(s)}[u]\langle u \rangle) = \exp_p \log_p(\langle u \rangle)$$

we observe that in fact $\langle u \rangle = s^* \in 1 + p\mathbb{Z}_p$ but we already mentioned the isomorphism:

$$\log_p : (1 + p\mathbb{Z}_p, \cdot) \to (p\mathbb{Z}_p, +)$$

and so:

$$\exp_p \log_p(s) = \exp_p \log_p(\langle u \rangle) = \langle u \rangle = s^*$$

Considering numbers on the form $\frac{i}{p^n} + m$ with $0 < \frac{i}{p^n} < 1$ and $m \in \mathbb{N}$, which are dense in $\mathbb{Q}_p \setminus \mathbb{Z}_p$, Γ_C could be written as the extension by continuity of:

$$\Gamma_C\left(\frac{i}{p^n} + m\right) = \frac{\prod_{k=0}^{m-1} \exp_p \log_p\left(\frac{i}{p^n} + k\right)}{\exp_p \log_p\left(\frac{i}{p^n}\right) \exp_p \log_p\left(\frac{i}{p^n} - 1\right)}$$

and in particular

$$\Gamma_C\left(\frac{i}{p^n} + m\right) \in 1 + p\mathbb{Z}_p$$

It's easy to check that it satisfies the functional equation:

$$\Gamma_C\left(\frac{i}{p^n} + m + 1\right) = \frac{\prod_{k=0}^{m} \exp_p \log_p\left(\frac{i}{p^n} + k\right)}{\exp_p \log_p\left(\frac{i}{p^n}\right) \exp_p \log_p\left(\frac{i}{p^n} - 1\right)} =$$

$$= \exp_p \log_p\left(\frac{i}{p^n} + m\right) \frac{\prod_{k=0}^{m-1} \exp_p \log_p\left(\frac{i}{p^n} + k\right)}{\exp_p \log_p\left(\frac{i}{p^n}\right) \exp_p \log_p\left(\frac{i}{p^n} - 1\right)}$$

which says that:

$$\Gamma_C\left(\frac{i}{p^n} + m + 1\right) = \left(\frac{i}{p^n} + m\right)^* \Gamma_C\left(\frac{i}{p^n} + m\right)$$

Finally:

$$\Gamma_C\left(\frac{i}{p^n}\right) = \frac{\exp_p \log_p\left(\frac{i}{p^n}\right)\exp_p \log_p\left(\frac{i}{p^n}-1\right)}{\exp_p \log_p\left(\frac{i}{p^n}\right)\exp_p \log_p\left(\frac{i}{p^n}-1\right)} = 1$$

Later we will give a more conctrete formula which will enphasize the analyticness of Γ_C. $\square$

We can see this formula as an anlogue of the classical product formula for the Gamma function which is:

$$\Gamma(s) = \frac{1}{s}\prod_{n=1}^{\infty}\frac{\left(1+\frac{s}{n}\right)^s}{1+\frac{s}{n}} = \lim_{n\to\infty}\frac{n!(n+1)^s}{s(s+1)\dots(s+n)}$$

for more details we suggest [30, Pg. 237]

Corollary 1. *For $s \in \mathbb{Q}_p \setminus \mathbb{Z}_p$ we have that $\Gamma_C(s) \neq 0$.*

Proof. It's an easy consequence of the proposition in fact $|\Gamma_C(s)| = 1$ so it can't vanish. $\square$

Definition 31. Consider now $\log_p$ the Iwasawa logarithm, we define the Coleman's log-Gamma function as $\log_p \Gamma_C$, this definition make sense since $\Gamma_C(s)$ does not vanish (the Iwasawa logarithm is defined in all $\mathbb{C}_p^\times$). We start studying the analytical properties of Γ_C by comparing G_p and $\log_p \Gamma_C$ (we denote our ζ_s as $\omega(s)$):

Proposition 12. *The function $\log_p \Gamma_C$ is locally analytic, it is given explicitely by :*

$$\log_p \Gamma_C(s) = G_p(s) - G_p(s - [s]_p)$$

in particular $(\log_p \Gamma_C(s))' = G_p'(s)$

Proof. By applying the logarithm to the Gamma function we obtain that by the functional equation for $|s| > 1$:

$$\log_p \Gamma_C(s+1) - \log_p \Gamma_C(s) = \log_p(s) + \log_p(p^{-v(s)}\omega^{-1}(s))$$

but by the properties of this logarithm $\log_p(p^{-v(s)}) = 0$ and $\log_p \omega^{-1}(s) = 0$ since $\omega^{-1}(s)$ is a root of 1 (if x is a root of 1 then $0 = \log_p(1) = \log_p(x^n) = n\log_p(x)$ which implies $\log_p(x) = 0$)so the Coleman's logGamma function satisfies the same functional equation of the Diamond's logGamma function (i.e. $f(x+1) - f(x) = \log_p(x)$). This implies that: call $y(s) = \log_p \Gamma_C(s) - G_p(s)$, where $G_p(s)$ is the Diamond's logGamma function, then:

$$y(s+1) - y(s) = 0$$

by the observation above. We can conclude that $\log_p \Gamma_C(s) = G_p(s) + y(s)$ where y is a locally costant function on discs of radius 1 in $\mathbb{Q}_p$.
This because if $y(s+1) - y(s) = 0$ then we clearly have $y(s+n) - y(s) = 0$ for $n \in \mathbb{Z}$ and so by continuity:

$$y(s) = y(s+\mathbb{Z}) = y(s+\mathbb{Z}_p)$$

Now we want describe more explicitely the Coleman's gamma function using the information above: we know that $\Gamma_C(s) = 1$ if $s \in \mathbb{Z}[1/p]$ such that $0 < s = i/p^n < 1$ by the above relation:

$$0 = \log_p 1 = \log_p \Gamma_C(i/p^n) = G_p(i/p^n) + y(i/p^n)$$

so $y(i/p^n) = -G_p(i/p^n)$. For every $s \in \mathbb{Q}_p - \mathbb{Z}_p$ we have to find i/p^n such that $|s - i/p^n| \leq 1$. This is easy in fact, for example, we can write $s = \dfrac{a_{-i}}{p^i} + \cdots + \frac{a_{-1}}{p} + a_0 + a_1 p + \dots$ and take the fractional part which is in fact $x - [x]_p$, so we obtain:

$$\log_p \Gamma_C(s) = G_p(s) - G_p(s - [s]_p)$$

By the equation $\log_p \Gamma_C(s) = G_p(s) + y(s)$ we deduce also that

$$(\log_p \Gamma_C(s))' = G_p'(s)$$

We know already that G_p is locally analytic and the discs of convergence are given by the largest discs which does not intersect $\mathbb{Z}_p$. Since the radius of convergence of the function is the same of its derivative we get that $\log_p \Gamma_C$ (thought as a function with domain $\mathbb{Q}_p - \mathbb{Z}_p$)is locally analytic with radius of convergence of the discs equal of the one of the G_p.

□

Observation 14. We have that, since $(\log_p \Gamma_C(s))' = G_p'(s)$, then $(\log_p \Gamma_C(s))'$ can be used to define distributions. We know in fact that the p-adic L function could be written as the Gamma transorm of the measure $\mu_{1,c}$ as:

$$L_p(1-s,\chi) = \frac{-1}{1-\chi(c)\langle c\rangle^s} \int_{\mathbb{Z}_p^\times} \langle x\rangle^s \chi(x) d\mu_{1,c}(x)$$

but we have already seen that the derivative of the Coleman's logGamma function and the Diamond's one are equal. So the L-function could be described using a measure linked with the Coleman's gamma function which could be interesting in view of the geometrical meaning of this function.

Proposition 13. *Consider $s \in \mathbb{Q}_p \setminus \mathbb{Z}_p$, then $\Gamma_C(s)$ is a locally analytic function satisfying the following formula which relates it to G_p:*

$$\Gamma_C(s) = \exp_p(G_p(s) - G_p(s - [s]_p))$$

Proof. Consider in fact $\exp_p(G_p(s) - G_p(s - [s]_p)$ this because we proved $\Gamma_C(s) \in 1 + p\mathbb{Z}_p$ where the logarithm is invertible by the p-adic exponential but let's check precisely that this function satisfies the functional equation given at the beginning of the chapter.

Recall that $G_p(s+1) - G_p(s) = \log_p(s)$ if $s \in \mathbb{Q}_p - \mathbb{Z}_p$ and so by definition : $[s+1]_p = [s]_p + 1$ which gives:

$$\exp_p(G_p(s+1) - G_p(s+1-[s+1]_p) = \exp_p(G_p(s) - G_p(s-[s]_p) + \log_p(s))$$

Now by the properties of the Iwasawa logarithm $\log_p s = \log_p(s^*)$ hence:

$\exp_p(G_p(s) - G_p(s-[s]_p) + \log_p(s)) = \exp_p(G_p(s) - G(s-[s]_p) + \log_p(s^*)) = \exp_p(G_p(s) - G(s-[s]_p)s^*$

morover if $a \in \mathbb{Z}[1/p]$ and $0 < a < 1$ we get $a - [a]_p = a$ because $[a] = 0$ hence:

$$\exp_p(G_p(a) - G_p(a - [a]_p)) = \exp_p(G_p(a) - G_p(a)) = 1$$

This function is a solution for the functional equation of the Coleman Gamma function so we get:

$$\Gamma_C(s) = \exp_p(G_p(s) - G_p(s - [s]_p))$$

In this way we see that it is a composition of locally analytic functions, by this definition we get in fact that Γ_C is locallyt analytic in a neighborhood of $\mathbb{Q}_p$ in $\mathbb{C}_p$. □

We recall that if we consider Γ_p, the classical Morita's Gamma function, it is well known that is analytic (see [18], [5]) in the uninion of the discs:

$$D = \bigcup_{t=0}^{p-1} D(-t, \rho)$$

where $\rho = p^{-\frac{1}{p} - \frac{-1}{p-1}}$ so globally Γ_C is locally analytic. From now on we denote $\langle x \rangle_p = x - [x]_p$

For G_p we have already studied the distribution and reflection formula, these reflects directly on Γ_C in fact:

Theorem 12. *We give for Γ_C the distribution formula ($|s| > 1$) :*

$$\Gamma_C(s) = \prod_{j=0}^{p-1} \Gamma_C\left(\frac{s+j}{p}\right)$$

Proof. Recall that in this situation:

$$\Gamma_C(s) = \exp_p(G_p(s) - G_p(\langle s \rangle_p))$$

By the distribution properties of $G_p(s)$:

$$\Gamma_C(s) = \exp_p\left(\sum_{j=0}^{p-1}\left(G_p\left(\frac{s+j}{p}\right) - G_p\left(\frac{\langle s \rangle_p + j}{p}\right)\right)\right)$$

Clearly now we have :

$$\frac{\langle s \rangle_p + j}{p} = \left\langle \frac{s+j}{p} \right\rangle_p$$

And finally:

$$\Gamma_C(s) = \exp_p\left(\sum_{j=0}^{p-1}\left(G_p\left(\frac{s+j}{p}\right) - G_p\left\langle \frac{s+j}{p} \right\rangle_p\right)\right) = \prod_{j=0}^{p-1} \Gamma_C\left(\frac{s+j}{p}\right)$$

□

Theorem 13. *For $|s| > 1$ we prove the reflection formula:*

$$\Gamma_C(s)\Gamma_C(1-s) = 1$$

Proof.

$$\Gamma_C(s)\Gamma_C(1-s) = \exp_p(G_p(s) - G_p(\langle s\rangle_p)) \exp_p(G_p(1-s) - G_p(\langle 1-s\rangle_p))$$

$$\Gamma_C(s)\Gamma_C(1-s) = \exp_p(G_p(s) + G_p(1-s) - (G_p(\langle s\rangle_p) + G_p(\langle 1-s\rangle_p)))$$

Since $G_p(s) + G_p(1-s) = 0$

$$\Gamma_C(s)\Gamma_C(1-s) = \exp_p(0) = 1$$

□

3.2 Overview of the De Rham cohomology

We follow [21], let R an S-algebra (where S is a ring), we define the R-module of Kahler differential $\Omega_{R/S}$ as the free algebra generated by symbols dr with $r \in R$ modulo the relations:

- $ds = 0$ for every $s \in S$
- $d(a+b) = da + db$
- $d(ab) = d(a)b + ad(b)$

We denote now $\Omega^i_{R/S} = \bigwedge_{n=0}^{i} \Omega_{R/S}$, now we have a linear derivation $d : R \to \Omega_{R/S}$ with the universal property that if $f : R \to M$ is S linear then f factors throught d. Thank to this universal property we have maps $\Omega^i_{R/S} \to \Omega^{i+1}_{R/S}$ in this way we obtain a complex $\Omega^\bullet_{R/S}$:

Proposition 14. *It exists a unique map $d : \Omega^i_{R/S} \to \Omega^{i+1}_{R/S}$ such that:*

1. $d^2 = 0$
2. *At degree 0 coincides with the dirrerential $d : R \to \Omega_{R/S}$*

Proof. The definition is the one of the classical exterior differential i.e.:

$$d(y\, df_1 \wedge \cdots \wedge df_n) = dy \wedge df_1 \cdots \wedge df_n$$

and then we extend by linearity. □

Observation 15. If ω_p is a p-form and ω_q is a q-form then:

$$d(\omega_p \wedge \omega_q) = d\omega_p \wedge \omega_q + (-1)^p \omega_p \wedge d\omega_q$$

Let $f : X \to S$ be an S-scheme with $X = SpecR$ and $S = SpecA$ then we define $\Omega_{X/S} = (\Omega_{R/A})\tilde{}$. With $\tilde{}$ we mean the $\mathscr{O}_X$-module associated to $\Omega_{R/S}$ (if we have any scheme we can take an open affine cover and glue the sheaves) so now we have an induced complex of sheaves:

$$\mathscr{O}_X \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to$$

which we call the "algebraic De Rham complex".

Now we want to introduce the " Hypercohomology ", take C and D abelian categories, C with enought injectives and $F : C \to D$ a left exact functor, consider $A^\bullet$ a bounded below complex in C, let $A^\bullet \to I^\bullet$ be a quasi isomorphism then we define the right hyper-derived functor of F as:

$$R^n F(A^\bullet) = H^n(F(I^\bullet))$$

Now if C is the category of Sheaves on abelian groups of X we define the Hypercohomology groups as:

$$\mathbb{H}^n(X, \mathscr{F}^\bullet) = R^n(\Gamma(X, \mathscr{F}^\bullet))$$

We are ready to define De Rham cohomology, in fact consider the De Rham complex $\Omega^\bullet_{R/S}$ then the n-th de Rham cohomology group of X is :

$$H^n_{dR}(X) = \mathbb{H}^n(X, \Omega^\bullet_{R/S})$$

Observation 16. If X is affine then we clearly have (recall that $\Omega_{X/S}$ is quasi-coherent):

$$H^n_{dR}(X) = H^n(\Gamma(X, \Omega^\bullet_{X/S})) = H^n(\Omega^\bullet_{R/A})$$

where $(\Omega_{R/S})\tilde{} = \Omega_{X/S}$

As an example we compute the de Rham cohomology of $X = \mathbb{G}_m(K) = SpecK[x, x^{-1}]$, Let $R = K[x, x^{-1}]$ we clearly have:

$$\Omega^0_{R/K} = K[x, x^{-1}] \longrightarrow \Omega^1_{R/K}$$

Since $K[x, x^{-1}] = K[x]_x$ (the localization of $K[x]$at x) we can compute : $\Omega^1_{K[x]/K} = K[x]dx$ because the exterior differential acts in the way $f(x) \mapsto f'(x)dx$. A well known property ofthe module of Kahler differential is :

$$\Omega_{S^{-1}R/K} \cong S^{-1}\Omega_{R/K}$$

i.e. $\Omega^1_{R/K} = (K[x]dx)_x = K[x, x^{-1}]dx$, so we have only the 0 differential:

$$d^0 : K[x, x^{-1}] \to K[x, x^{-1}]dx$$

($\Omega^i = 0$ for $i \geq 2$ because the wedge product vanishes if there are repetitions i.e. $\cdots \wedge dx \wedge dx \wedge \cdots = 0$). So if $\frac{f(x)}{x^k} \in K[x, x^{-1}]$ then $d\left(\frac{f(x)}{x^k}\right) = \frac{f'(x)}{x^k}dx - kx^{-k-1}$.

Eventually :

$$H^0_{dR}(\mathbb{G}_m) \cong K \qquad H^1_{dR}(\mathbb{G}_m) \cong \Omega^1_{R/K}/dK[x, x^{-1}] \cong K\frac{dx}{x}$$

3.3 Connections and Frobenius action in De Rham cohomology

Consider X a scheme over $\mathbb{F}_p$ (see [26]), so $\mathscr{O}_X$ is a sheaf of rings of characteristic p. The Frobenius endomorphism induces a map of sheaves : $\mathscr{O}_X \to \mathscr{O}_X$ and clearly it induces on X the following map:

$$F_X : X \to X$$

called the absolute Frobenius. Let now S be a scheme over $\mathbb{F}_p$ and consider the S-scheme $X \to S$ then we call $X^{(p)}$ the fibered product $X \times_S S$ via $F_S : S \to S$. Then consider $F_X : X \to X$ so by the pullback property we have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[dr, dotted, "F_{X/S}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\
& X^{(p)} \arrow[r, "p"] \arrow[d] & X \arrow[d] \\
& S \arrow[r, "F_S"] & S
\end{tikzcd}$$

Definition 32. The map of S-schemes $F_{X/S}$ is called relative Frobenius.

It is in general an important problem to understand the action in cohomology induced by the Frobenius automorphism, in the next section we will sketch the ideas of Dwork and Coleman. They were able to compute explicitely the matrix of the endomorpshim in cohomology associated, in fact, to the Frobenius.

Definition 33. Let X be a scheme, a vector bundle $\mathscr{E}$ on X is an $\mathscr{O}_X$-module locally free of finite rank.

Definition 34. Let X a smooth algebraic variety over a field of characteristic 0, let Ω_X its sheaf of differentials, consider $\mathscr{E}$ a vector bundle on X then a connection is an $\mathscr{O}_X$-linear map :

$$\nabla : \mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_X} \Omega_X$$

s.t. for every $U \subset X$ open and $f \in \Gamma(U, \mathscr{O}_U)$ and every $s \in \Gamma(U, \mathscr{E})$ it satisfies the Leibniez rule:

$$\nabla(fs) = f\nabla(s) + s \otimes df$$

A section s is horizontal if $\nabla(s) = 0$. ∇ is called integrable if $\nabla^2 = 0$. Now given a pair ($\mathscr{E}$, ∇) we can define the complex:

$$\mathscr{E} \to \mathscr{E} \otimes_{\mathscr{O}_X} \Omega^1_X \to \mathscr{E} \otimes_{\mathscr{O}_X} \Omega^2_X \to \dots$$

taking the hypercohomology of this complex we can define the de Rham cohomology with coefficients in ($\mathscr{E}$, ∇):

$$H^n_{dR}(X, (\mathscr{E}, \nabla)) = \mathbb{H}^n(X, \mathscr{E} \otimes_{\mathscr{O}_X} \Omega^\bullet_X)$$

Now we can review the part of the first section regarding p-adic Gamma functions, we follow [5]: Consider $X = Spec\, \mathbb{C}_p[x, x^{-1}] =: \mathbb{G}_m$ for $a \in \mathbb{Q}_p \cap \mathbb{Z}_p - \mathbb{Z}$, the connection:

$$\nabla_a : \mathbb{C}_p[x, x^{-1}] \longrightarrow \mathbb{C}_p[x, x^{-1}]dx$$
$$f \mapsto \left(x\frac{d}{dx} + \pi x + a \right)(f)\frac{dx}{x}$$

recall that π is such that $\pi^{p-1} - p = 0$. Completing p-adically $\mathbb{C}_p[x, x^{-1}]$ we obtain the algebra $L_{0,\infty}$ so we have that (we still denote the connection ∇_a):

$$\nabla_a : L_{0,\infty} \longrightarrow L_{0,\infty}\frac{dx}{x}$$

Finally we have:

$$H^1_{dR}(\mathbb{G}_m, (\mathscr{O}_{\mathbb{G}_m}, \nabla_a)) = L_{0,\infty}\frac{dx}{x} / \nabla_a L_{0,\infty}$$

. This cohomology group is spanned by classes $\left[\frac{dx}{x}\right]_a$ satisfying the following:

$$\left[x^m\frac{dx}{x}\right]_a = \left[\frac{dx}{x}\right]_{a+m} = \frac{\Gamma(a+m)(-\pi)^{-m}}{\Gamma(a)}\left[\frac{dx}{x}\right]_a$$

Consider now a, b like above and s.t. $a - pb = t \in \mathbb{Z}$ we define the map

$$F(a, b) : (L_{0,\infty}, \nabla_a) \longrightarrow (L_{0,\infty}, \nabla_b)$$
$$f(x) \mapsto \frac{f(x^p)x^t}{\Theta(x)}$$

and the left inverse:

$$D(a, b) : (L_{0,\infty}, \nabla_b) \longrightarrow (L_{0,\infty}, \nabla_a)$$
$$f(x) \mapsto \psi(f(x)x^t\Theta(x))$$

Recall that here ψ is:

$$\psi(f) = \sum_{x^p = y} f(y)$$

These two maps induces two isomorphism (one the inverse of the other) in cohomology

$$Frob(a, b) : H^1_{dR}(\mathbb{G}_m, (\mathscr{O}_{\mathbb{G}_m}, \nabla_b)) \to H^1_{dR}(\mathbb{G}_m, (\mathscr{O}_{\mathbb{G}_m}, \nabla_a))$$

and

$$Dw(a, b) : H^1_{dR}(\mathbb{G}_m, (\mathscr{O}_{\mathbb{G}_m}, \nabla_a)) \to H^1_{dR}(\mathbb{G}_m, (\mathscr{O}_{\mathbb{G}_m}, \nabla_b))$$

such that:

$$Dw(a, b)\left(\left[\frac{dx}{x}\right]\right)_a = \frac{1}{p}\gamma_p(a, b)\left[\frac{dx}{x}\right]_b$$

so the function $\gamma_p(a, b)$ gives the 'matrix' associated to the morphism $Dw(a, b)$. We remark that this function was introduced by Dwork and is more flexible than Γ_p in fact it is also a meromorphic function in the set :

$$\mathscr{D}^{(t)}(p\rho) = \{(x, y) \in \mathbb{C}_p^2 \,|py - x = t,\ |y| = p\rho\}$$

3.4 The geometric meaning of Γ_C

Now we recall some facts of [7], as usual denote by $\overline{\mathbb{Q}}_p$ an algebraic closure of the field of p-adic numbers and denote by $\mathbb{Q}_p^{ur}$ the maximal unramified extension of $\mathbb{Q}_p$.

Definition 35. We define the 'Cristalline' Weil group as:

$$W_{cris}(\overline{\mathbb{Q}}_p) = \{\phi \in Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) : \phi_{|\mathbb{Q}_p^{ur}} = Frob^n, n \in \mathbb{Z}\}$$

i.e. the elements of this group are automorphisms of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ which restricted to $\mathbb{Q}_p^{ur}$ are integral powers of the Frobenius automorphism.

Given the map $deg : W_{cris}(\overline{\mathbb{Q}}_p) \to \mathbb{Z}$ defined as $deg(\phi) = n$, we recall that it exists an exact sequence:

$$1 \to I_{cris}(\overline{\mathbb{Q}}_p) \to W_{cris}(\overline{\mathbb{Q}}_p) \to \mathbb{Z} \to 1$$

Here: $I_{cris}(\overline{\mathbb{Q}}_p) := \mathrm{Ker}(deg)$ is, in fact, the Galois group $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{ur})$.

Definition 36. Let V be a $\overline{\mathbb{Q}}_p$-vector space, we define a semi-linear action $W_{cris}(\overline{\mathbb{Q}}_p)$ on V as a map

$$\rho : W_{cris}(\overline{\mathbb{Q}}_p) \to End_{\overline{\mathbb{Q}}_p}(V)$$

such that $\rho(\phi)(ax) = \phi(a)\rho(\phi(x))$ for every $a \in \overline{\mathbb{Q}}_p$ and $x \in V$.

Definition 37. Let X a smooth proper $\overline{\mathbb{Q}}_p$-scheme, we say that X has good reduction iff exists a finite extension K of $\mathbb{Q}_p$ and a smooth proper O_K-scheme X' toghether with an isomorphism $\alpha : X'_{\overline{\mathbb{Q}}_p} \to X$ where $X'_{\overline{\mathbb{Q}}_p} = X' \times_{Spec\, O_K} Spec\, \overline{\mathbb{Q}}_p$.

Definition 38. Let X a smooth proper K-scheme with K finite extension of $\mathbb{Q}_p$, we say that X has potentially good reduction if $X_{\overline{K}_p}$ has good reduction.

Theorem 14. *Let X a smooth proper $\overline{\mathbb{Q}}_p$-scheme with good reduction, then exists a unique action ρ_{cris} of $W_{cris}(\overline{\mathbb{Q}}_p)$ on $H^n_{dR}(X/\overline{\mathbb{Q}}_p)$ which is functorial in X.*

Proof. See [7] theorem (4.2). □

Observation 17. If X/K has potentially good reduction we have an action of the Weil group on $H^n_{dR}(X_{\overline{\mathbb{Q}}_p}/\overline{\mathbb{Q}}_p)$.

Before to explain the geometric meaning of Γ_C, we need to introduce the notion of the Jacobian of a curve. We will only sketch the main results that we need, more details are given in [24]. To a non singular curve C it is in fact possible to associate an abelian variety J called the jacobian of C.

Let C a complete non-singular curve over a field k then:

Definition 39. Consider T a connected k-scheme, let $\mathscr{L}$ an invertible sheaf on $C \times_{Spec\, k} T$ we define the abelian group:

$$P_C^0(T) = \{\mathscr{L} \in Pic(C \times_{Spec\, k} T) | deg(\mathscr{L}) = 0\}/q^* Pic(T)$$

we recall that $Pic(X) = \{Invertible\ \mathscr{O}_X\ modules\}/\cong$, i.e. the abelian group under $\otimes_{\mathscr{O}_X}$ of invertible sheaves up to isomorphisms.

Theorem 15. *The functor :*

$$P_C^0 : Sch(k) \longrightarrow Ab$$

is representable, its representant is an abelian variety J called the Jacobian of C.

Proof. See [24] Theorem 1.1 □

Proposition 15. *The tangent space of the Jacobian variety J at 0 is isomorphic with $H^1(C, \mathscr{O}_C)$.*

Proof. See [24] Proposition 2.1 □

Theorem 16. *It exists an isomorphism :*

$$\Gamma(J, \Omega_J) \to \Gamma(C, \Omega_C)$$

Proof. For details see [24] proposition 2.2 . □

Observation 18. This theorem hallow us to identify the first de Rham cohomology group of the curve with the one of its jacobian variety.

Let F_m be the Fermat curve (affine) with equation $x^m + y^m = 1$, if $(m,p) = 1$ then F_m has good reduction over $\mathbb{Q}_p$ so we can study the action of the Frobenius endomorphism in cohomology using techniques (briefly) described in section 3.3. Now, if $p|m$, the situation is much more complicated (all he calculations are due to Coleman) but we have that the Jacobian J_m of F_m has still potentially good reduction i.e. the Weil group $W_{cris}(\overline{\mathbb{Q}}_p)$ acts on $H^1_{dR}((J_m)_{\overline{\mathbb{Q}}_p})$. Now since we have:

$$H^1_{dR}((J_m)_{\overline{\mathbb{Q}}_p}) \cong H^1_{dR}(J_m/\mathbb{Q}_p) \otimes \overline{\mathbb{Q}}_p$$

(this is true for every proper smooth scheme [7] section (4.7)). By our remarks on the jacobian we have also :

$$H^1_{dR}(J_m/\mathbb{Q}_p) \otimes \overline{\mathbb{Q}}_p \cong H^1_{dR}(F_m/\mathbb{Q}_p) \otimes \overline{\mathbb{Q}}_p$$

So ρ_{cris} acts in the Cohomology of F_m, the work [11] of Coleman describes the matrix of this action using the extension of the classical p-adic Gamma function which we have denoted Γ_C. Let now $r, s \in \mathbb{Q}/\mathbb{Z} - 0$ such that $r + s$ is not 0 then for a fixed Fermat curve F_m we denote by $\langle r \rangle$ the smallest representative of r mod $\mathbb{Z}$.

Observation 19. We recall (see [11]) that a base of differentials (of the second kind) for F_m is given by:

$$\omega_{i,j} = x^i y^j (x/y) d(x/y)$$

with $0 < i, j < m$ and $i + j$ different from m.

Let $\varepsilon(r, s) = \langle r + s \rangle - \langle r \rangle - \langle s \rangle$ and $L(r, s) = \langle r + s \rangle^{\varepsilon(r,s)}$. $v_{r,s}$ will denote the class of the differential $mL(r, s)\omega_{i,j}$ where $i = m\langle r \rangle$ and $j = m\langle s \rangle$.

Proposition 16. *Denote by $q = (r, s)$, let $mq = 0$, $(m, p) = 1$ then :*

$$\rho_{cris}(\sigma)(v_{\sigma^{-1}q}) = \beta_\sigma(q) v_q$$

where $\beta_\sigma(q)$ is the matrix of the action and in this case is given explicitely by :

$$\beta_\sigma(q) = (-1)^{\varepsilon(q)} p^{\varepsilon(\sigma^{-1}q)} \frac{\Gamma_C \langle r + s \rangle}{\Gamma_C \langle r \rangle \Gamma_C \langle s \rangle}$$

Proof. [11] Theorem 1.7 □

Consider now the case in which $p|m$, let q ad above denote by η_q the following differential form:

$$\eta_q = L(q) x^i y^j (x/y) d(x/y)$$

(where i, j are as above) then define $\gamma_\sigma(q)$ as the matrix of the action:

$$\rho_{cris}(\sigma)\eta_{\sigma^{-1}q} = \gamma_\sigma(q)\eta_q$$

then

Theorem 17. *Let q as always, we have that :*

$$\gamma_\sigma(q) = (-1)^{\mu(\sigma,r)+\mu(\sigma,s)} \beta_\sigma(q)$$

(here $\mu(\sigma, r) = \langle \sigma^{-1} r \sigma(-\langle r \rangle) \rangle$) Morover if $mq = 0$ and $deg\sigma = 1$ by the theorem above:

$$\gamma_\sigma(q) = \frac{(-p)^{\varepsilon(\sigma^{-1}q)}}{\Gamma_C \langle -(r + s) \rangle \Gamma_C \langle r \rangle \Gamma_C \langle s \rangle}$$

Proof. [11] Proposition 1.9 □

Chapter 4

p-adic measures on $\mathbb{Q}_p$

In the first part of this section we recall some definitions and results of the theory of perfectoid and adic spaces the main references are : [25], [8] and [29] but most of the proofs for adic spaces are in [20] and [19].

4.1 Perfectoid rings

Definition 40. Let R be a topological ring, R is said to be integral perfectoid if :

- exists a non zero divisor π such that $R \cong \varprojlim_n R/\pi^n R$;
- $p \in \pi^p R$;
- The map $R/\pi R \to R/\pi^p R$, $x \to x^p$ is an isomorphism.

Such an element π is called (perfectoid) pseudo-uniformizer (p.u.).

Example 3. Consider the field:

$$\mathbb{Q}_p(p^{1/p^\infty}) = \varinjlim_n \mathbb{Q}_p(p^{1/p^n})$$

then the p-adic completion of its ring of integers $\mathbb{Z}_p[p^{1/p^\infty}]$ is integral perfectoid, the same for $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Z}_p[\zeta_{p^\infty}]$.

Definition 41. (Tilting functor) For an integral perfectoid ring R we define the tilting functor $(-)^\flat$ as :

$$R^\flat := \varprojlim_{x \to x^p} (R/pR)$$

In this way $R^\flat$ is a perfect ring of characteristic p equipped with the inverse limit topology. The elements of $R^\flat$ are in fact sequences of compatiple p-power roots i.e. if $a \in R^\flat$ then $a = (a_i)_{i \in \mathbb{N}}$ such that:

$$(a_{i+1})^p = a_i$$

Proposition 17. *If R is π-adically complete and $\pi|p$ then, we have the following isomorphism:*

$$\varprojlim_{x\to x^p} R \cong \varprojlim_{x\to x^p} (R/\pi R)$$

the isomorphism is clearly only of multiplicative monoids.

Proof. Let $\{a_n\},\{b_n\}$ two sequences such that $a_n \equiv b_n \mod \pi$ in particular by the properties of the projective limit $a_{n+k}^{p^k} = a_n$. We can observe also that by the above equality $\lim_{k\to\infty} a_{n+k}^{p^k} = a_n$ then by assumpion $\lim_{k\to\infty}(b_{n+k}+\pi x)^{p^k} = a_n$, but

$$\lim_{k\to\infty}(b_{n+k}+\pi x)^{p^k} = \lim_{k\to\infty} b_{n+k}^{p^k} = b_n$$

and finally $a_n = b_n$.
For the surjectivity: let $\{\tilde{a_n}\} \in \varprojlim(R/pR)$ for every $\tilde{a_n}$ let a_n be a lift in R. This implies in particular $a_{n+k+1}^p \equiv a_{n+k} \mod \pi$, a well known fact is that: $a \equiv b \mod \pi \Rightarrow a^{p^n} \equiv b^{p^n} \mod \pi^{n+1}$. The last implies that the sequence $k \to a_{n+k}^{p^k}$ is Cauchy then it has a limit x_n.
Now $(x_{n+1})^p = (\lim_{k\to\infty} a_{n+k+1}^{p^k})^p = \lim_{k\to\infty} a_{n+k+1}^{p^{k+1}} = \lim_{k\to\infty} a_{n+k}^{p^k} = x_n$. Finally:

$$\overline{x_n} = \lim_{k\to\infty} \overline{a}_{n+k}^{p^k} = \lim_{k\to\infty} \tilde{a}_{n+k}^{p^k} = \lim_{k\to\infty} \tilde{a}_n = \tilde{a}_n$$

□

Lemma 3. *Let R integral perfectoid and π a p.u. then, up to a unit, π has in R a compatible system of p-power roots : $\pi^{1/p}, \pi^{1/p^2}, \ldots$.*

Proof. Since the Frobenius $R/\pi R \to R/\pi^p R$ is an isomorphism we have an induced isomorphism $\varprojlim_{x\to x^p}(R/\pi R) \cong \varprojlim_{x\to x^p}(R/\pi^p R)$. By the above lemma we get the isomorphism: $\varprojlim_{x\to x^p}(R/\pi^p R) \cong \varprojlim_{x\to x^p} R$ so it exists $a = (a_0, a_1, \dots) \in \varprojlim_{x\to x^p} R$ such that $a_0 \equiv \pi \mod \pi^p R$ and $a_0 = u\pi$ with $u \in 1 + \pi^{p-1}R \subset R^\times$ which is a unit. □

Definition 42. Consider a ring R and consider its tilting $R^\flat$, we define the 'untilting' map as:

$$\sharp : R^\flat \to R$$

$$b = (b_0, b_1, \ldots) \mapsto \lim_{i\to\infty} \tilde{b_i}^{p^i} = b^\sharp$$

where $\tilde{b_i}$ is any lift of $b_i \in R/pR$ in R.

Observation 20. This is in general the projection to the zero factor.

Observation 21. In general $\sharp$ is multiplicative but not additive in fact:

$$(a+b)^\sharp = \lim_{i\to\infty}\left((a^{\frac{1}{p^i}})^\sharp + (b^{\frac{1}{p^i}})^\sharp\right)^{p^i}$$

$\sharp$ in particular we can compose it with the reduction mod p :

$$R^\flat = \varprojlim_{x \to x^p} (R/pR) \xrightarrow{red \circ \sharp} R/pR$$

obtaining in fact a ring homomorphism. Moreover if R is in characteristic p the map $\sharp$ is an isomorphism.

Example 4. (see [25]) Let R an integral perfectoid ring and consider π a p.u., define $R\langle T^{1/p^\infty}\rangle$ the π-adic completion of $\bigcup_{n\geq 0} R[T^{1/p^n}]$. Then this is an integral perfectoid ring, moreover we remark that $R\langle T^{1/p^\infty}\rangle^\flat$ contains the element $T^\flat = (T, T^{1/p}, \dots)$ which provides the following topological isomorphism:

$$R^\flat\langle U^{1/p^\infty}\rangle \xrightarrow{\cong} R\langle T^{1/p^\infty}\rangle^\flat$$

$$U \mapsto T^\flat$$

Definition 43. We define a perfectoid field as a complete NA field K with norm $|\cdot| : K \to \mathbb{R}$ such that:

- $|K^*| \subset \mathbb{R}$ is dense;
- exists $\pi \in \mathbb{M}_K$ s.t. $p \in \pi^p O_K$;
- the Frobenius map $O_K/p \to O_K/p$ is an isomorphism.

In this assumption we have in fact that O_K is integral perfectoid, so we get the following definition:

Definition 44. (Tilting of a perfectoid field) Let K be a perfectoid field then the tilting of K is defined as:

$$K^\flat := Frac(O_K^\flat)$$

we observe that: $Frac(O_K^\flat) = O_K^\flat\left[\frac{1}{\pi^b}\right]$ where π is a pseudo-uniformizer.

4.2 Adic spaces

Definition 45. (Huber ring)

An Huber ring is a topological ring R such that it exists an open subring $R_0 \subset R$ with a finitely generated ideal $I \leq R_0$ which induces the I-adic topology on R (i.e. the family $\{I^n\}_{n\geq 0}$ is a base of open neighborhoods of 0). R_0 is usually called ring of definition and I ideal of definition.

Example 5. An example is $\mathbb{Q}_p$ with ring of definition $\mathbb{Z}_p$ and ideal of definition $p\mathbb{Z}_p$.

Definition 46. A subset S of a topological ring R is called bounded if for all open neighborhoods U of 0 exists an open neighborhood V of 0 such that $VS \subset U$.

Definition 47. (Tate ring)

An Huber ring is called Tate ring if it contains a unit $\pi \in R^\times$ such that $\pi^n \to 0$ as $n \to \infty$ (An element satisfying this condition is called topological nilpotent).

Definition 48. An element $f \in R$ with R a Huber ring is called power bounded if the set $\{f^n | n \geq 0\}$ is bounded. The set of all power bounded elements is denoted R°.

We denote also by $R^{\circ\circ}$ the set of topological nilpotent elements and so:

Proposition 18. *Let R a Huber ring then :*

- *R° is open and integrally closed;*
- *R° is the union of all rings of definition:*
- *$R^{\circ\circ}$ is an open ideal of R°.*

Proof. [25, Lemma 2.13] □

Observation 22. In the case of a NA field K we have that K° is the ring of integers O_K and $K^{\circ\circ}$ is the maximal ideal $\mathbb{M}_K$.

Definition 49. (Huber pair) An Huber pair (R, R^+) is given by a Huber ring R and an open and integrally closed subring $R^+ \subset R_0$ (R_0 is the ring of definition). In particular we can choose $R^+ = R^\circ$.

Definition 50. A morphism of Huber pairs $(A, A^+) \to (B, B^+)$ is a continuous ring homomorphism $f : A \to B$ such that $f(A) \subset B$.

Definition 51. (Affinoid Tate ring) An affinoid Tate ring is given by a pair (R, R^+) of a Tate ring R and a ring of integral elements R^+ of R.

Definition 52. A continuous absolute value (or valuation) on a topological ring R is a map:

$$|\cdot| : R \to \Gamma \cup \{0\}$$

with Γ a totally ordered (multiplicative) abelian group, such that:

- $|0| = 0$, $|1| = 1$;
- $|ab| = |a||b|$;
- $|a + b| \leq \max(|a|, |b|)$;
- for every $\gamma \in \Gamma$, the set $\{x \in R \mid |x| < \gamma\}$ is open.

The usual convention is that for evert $\gamma \in \Gamma$ then $0\gamma = 0$ and $0, \gamma$.

Definition 53. (Adic spectrum) The adic spectrum of an Huber pair (R, R^+) is defined as: $Spa(R, R^+) :=$ the set of equivalence classes of continuous valuations such that $|R^+| \leq 1$. It is endowed with the coarsest topology given by the sets:

$$Spa(R, R^+)\left(\frac{f}{g}\right) = \{x \in Spa(R, R^+) | \, |f(x)| \leq |g(x)| \neq 0\}$$

for any $f, g \in R$ (for $x \in Spa(R, R^+)$ we denote $x(f)$ as $|f(x)|$).

Definition 54. (Rational subset) Given a finite family of $Spa(R, R^+)\left(\frac{f_i}{g}\right)$ with $i = 1, \ldots, n$ we define :

$$Spa(R, R^+)\left(\frac{f_1, \ldots, f_n}{g}\right) = \bigcap_{i=1}^{n} Spa(R, R^+)\left(\frac{f_i}{g}\right)$$

which are called rational subsets.

Theorem 18. *Given a Huber pair (R, R^+) we have the following facts:*

- *$Spa(R, R^+)$ is spectral;*
- *the rational subsets form a basis;*
- *the rational subsets are quasi-compact;*
- $R^\circ = \{f \in R | \forall x \in Spa(R, R^+) \, |f(x)| \leq 1\}$;
- $R^{\circ\circ} = \{f \in R | \forall x \in Spa(R, R^+) \, |f(x)| < 1\}$;
- $R^\times = \{f \in R | \forall x \in Spa(R, R^+) \, |f(x)| \neq 0\}$.

Proof. All the he proofs are in [19]. □

Observation 23. The map $(R, R^+) \to Spa(R, R^+)$ gives a contravariant functor from the category of Huber pairs to the one of topological spaces.

Definition 55. Let K a perfectoid field with tilt $K^\flat$ and pseudo uniformiser t such that, letting $\pi = t^\sharp$, $|p| \leq |\pi| < 1$ then: a K-algebra R is perfectoid if R° is bounded and the Frobenius $R^\circ/p \to R^\circ/p$ is surjective.

Observation 24. Let K be a perfectoid field, and let R be a perfectoid K-algebra. Then (R, R°) is a complete affinoid Tate ring, and it admits a natural structure map from (K, K°).

Definition 56. An affinoid Tate K-algebra (R, R^+) is perfectoid if R is perfectoid.

Theorem 19. *(Tilting of adic spaces) Let K a perfectoid field and (R, R^+) a perfectoid affinoid K-algebra then for any continuous valuation $x \in Spa(R, R^+)$ the composition:*

$$R^\flat \xrightarrow{\sharp} R \xrightarrow{x} \Gamma \cup \{0\}$$

gives an element $x^\flat \in Spa(R^\flat, R^{\flat+})$ which induces an isomorphism of adic-spaces:

$$Spa(R, R^+) \cong Spa(R^\flat, R^{\flat+})$$

preserving rational subsets i.e. if $U \subset Spa(R, R^+)$ is a rational subset then $U^\flat \subset Spa(R^\flat, R^{\flat+})$ is a rational subset.

Definition 57. (Perfectoid spaces). The adic space $Spa(R, R^+)$ attached to a perfectoid affinoid K-algebra (R, R^+) is called an affinoid perfectoid space over K. More generally, a perfectoid space over K is an adic space over $Spa(K, K^\circ)$ that is locally isomorphic to an affinoid perfectoid space.

Proposition 19. *The category of perfectoid spaces over K admits fiber products.*

Proof. (Sketch)
Consider (A, A^+), (B, B^+) and (C, C^+) three perfectoid affinoid K-algebras such that we have the diagram:

$$\begin{array}{ccc} (A, A^+) & \longrightarrow & (B, B^+) \\ \big\downarrow & & \\ (C, C^+) & & \end{array}$$

Let $D_0 = B \otimes_A C$ and D_0^+ the integral closure of the image of the map $B^+ \otimes_{A^+} C^+ \to D_0$. Finally consider (D, D^+), the t-adic completion of (D_0, D_0^+). Locally we define $Spa(A, A^+) \times_K Spa(B, B^+) := Spa(D, D^+)$, this gives the required pullback because the pair (D, D^+) is a pushout in the category of (K, K°)-perfectoid affinoid algebras. $\square$

4.3 The modules of Measures on $\mathbb{Q}_p$

The goal of this section is to develop what we need about the theory of measures on $\mathbb{Q}_p$(the main reference is [2]), for this purpose we need some notions of adic spaces and perfectoid rings that we introduced in the previous sections. We will see that is possible to extend the notions of measures in $\mathbb{Z}_p$ to $\mathbb{Q}_p$ (which is clearly non-compact) using in fact the theory of perfectoids and adic spaces. We will give also a definition of a possible extension of the measure of Diamond $\mu_{1,c}$.

Definition 58. Consider R a complete and separated linearly topologized ring and denote by $P(R)$ the basis of open ideals, let M a R-linearly topologized separated and complete topological R-module with basis of submodules $\{IM\}_{I\in P(R)}$. For every set of indices A we say that the series $\sum_{a\in A} m_a$ is unconditionally convergent iff for every IM open sumbodule $m_a \in IM$ for almost all $a \in A$.

Definition 59. We define the set $\Sigma_{unif}(\mathbb{Q}_p)$ as the set of uniformly open subsets i.e. those subsets which are union of balls of the same radius (for balls of radius p^{-n} in $\mathbb{Q}_p$ we mean sets on the form $a + p^n\mathbb{Z}_p$ with $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$).

Definition 60. We denote by $\Sigma_n(\mathbb{Q}_p)$ the family of open subset which are union of balls of the same radius p^{-n} for every $n \in \mathbb{Z}$.

Definition 61. We define $\Sigma(\mathbb{Q}_p)$ as the family of clopen subset of $\mathbb{Q}_p$.

Definition 62. (Uniform measure)

A uniform measure in $\mathbb{Q}_p$ with values in k (separated and complete linearly topologized ring) is an additive map

$$\mu : \Sigma_{unif}(\mathbb{Q}_p) \to k$$

such that: for every $n \in \mathbb{Z}$ and any family $\{U_\alpha\} \subset \Sigma_n(\mathbb{Q}_p)$ then if $U = \bigcup_\alpha U_\alpha$ we have that $\sum_\alpha \mu(U_\alpha)$ converges unconditionally to $\mu(U)$.

Definition 63. (Bounded measure)

A bounded measure in $\mathbb{Q}_p$ with values in k(separated and complete linearly topologized ring) is an additive map

$$\mu : \Sigma(\mathbb{Q}_p) \to k$$

such that: for any $I \in P(k)$ exists an open compact subset $Z_I \subset \mathbb{Q}_p$ such that $\mu(V) \in I$ for any $V \in \Sigma(\mathbb{Q}_p - Z_I)$.

Definition 64. We denote the algebra of p-adic uniform measures over $\mathbb{Q}_p$ with values in k as $\mathscr{D}_{unif}(\mathbb{Q}_p, k)$ and we give to it the topology induced by the family of submodules:

$$\mathscr{U}_{n,J} = \{\mu \in \mathscr{D}_{unif}(\mathbb{Q}_p, k) | \mu(U) \in J,\ \forall U \in \Sigma_n(\mathbb{Q}_p)\}$$

with $J \in P(k)$

Definition 65. We denote the algebra of p-adic bounded measures over $\mathbb{Q}_p$ with values in k as $\mathscr{D}_{bd}(\mathbb{Q}_p, k)$ and we give to it the topology induced by the family of submodules:

$$\mathscr{U}_J = \{\mu \in \mathscr{D}_{bd}(\mathbb{Q}_p, k) | \mu(U) \in J,\ \forall U \in \Sigma(\mathbb{Q}_p)\}$$

with $J \in P(k)$

Definition 66. Consider now the the direct system of $\mathbb{Z}_p[[T^{1/p^n}]]$ induced by the inclusion maps $\mathbb{Z}_p[[T^{1/p^n}]] \hookrightarrow \mathbb{Z}_p[[T^{1/p^m}]]$ for $n \leq m$, so we define $\mathscr{D}$ as :

$$\mathscr{D} = \varinjlim_n \mathbb{Z}_p[[T^{1/p^n}]]$$

i.e. $\mathscr{D}$ could be identified with the completion of the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[T^{1/p^\infty}] = \bigcup_{n\geq 0} \mathbb{Z}_p[T^{1/p^n}]$ in the (p,T)-adic topology. Moreover we define $\mathscr{D}_{bd}$ as he p-adic completion of $\bigcup_{n\geq 0} \mathbb{Z}_p[[T^{1/p^n}]]$, so clearly we have the embedding: $\mathscr{D}_{bd} \hookrightarrow \mathscr{D}$.

Definition 67. Eventually we introduce also the ring $\tilde{\mathscr{D}} = \mathscr{D}/p\mathscr{D}$ i.e. the (t)-adic completion of $\bigcup_{n\geq 0} \mathbb{F}_p[[t^{1/p^n}]]$ where t is the reduction mod p of T.

Theorem 20. *(see [2])*
Consider the two rings $\mathscr{D}$ and $\mathscr{D}_{unif}(\mathbb{Q}_p, k)$ then the identification given by :

$$\mathscr{D}_{unif}(\mathbb{Q}_p, k) \longrightarrow \mathscr{D}$$

$$\Delta_{p^i} \mapsto \lim_{n\to\infty} F_p\left(T^{p^{i-n}}\right)^{p^n}$$

gives a topological isomorphism.

Observation 25. We are indentifying in this case :

$$T \mapsto \lim_{n\to\infty} E_p(\Delta_{p^{-n}} - \Delta_0)^{p^n}$$

where $E_p(T)$ is the Artin-Hasse logarithm i.e. the series such that:

$$E_p(F_p(T) - 1) = T \qquad F_p(E_p(T)) = E(T)$$

Observation 26. This is the $\mathbb{Q}_p$-analog of the classical statement in the case of $\mathbb{Z}_p$ in which we have a topological isomorphism between the algebra of measures and $\mathbb{Z}_p[[T]]$ given by $\Delta_1 - \Delta_0 \mapsto T$.

Theorem 21. *(Functional interpretation [10, Thm. 5.4])*
Let $\mathscr{D}_{unif}(\mathbb{Q}_p, \mathbb{Z}_p)$ and $\mathscr{D}_{bd}(\mathbb{Q}_p, \mathbb{Z}_p)$ as above we have the following two strong perfect duality pairings: The first one:

$$\int_{\mathbb{Q}_p} : \mathscr{D}_{bd}(\mathbb{Q}_p, \mathbb{Z}_p) \times C(\mathbb{Q}_p, \mathbb{Z}_p) \to \mathbb{Z}_p$$

which identifies $\mathscr{D}_{bd}(\mathbb{Q}_p, \mathbb{Z}_p) \cong (C(\mathbb{Q}_p, \mathbb{Z}_p))'_{strong}$. The second one:

$$\int_{\mathbb{Q}_p} : \mathscr{D}_{unif}(\mathbb{Q}_p, \mathbb{Z}_p) \times C_{unif}(\mathbb{Q}_p, \mathbb{Z}_p) \to \mathbb{Z}_p$$

which identifies $(\mathscr{D}_{unif}(\mathbb{Q}_p,\mathbb{Z}_p))'_{strong} \cong C_{unif}(\mathbb{Q}_p,\mathbb{Z}_p)$ *compatible with the canonical inclusion:*

$$C_{unif}(\mathbb{Q}_p,\mathbb{Z}_p) \hookrightarrow C(\mathbb{Q}_p,\mathbb{Z}_p)$$

and the map given functorially:

$$(-)_{unif} : \mathscr{D}_{bd}(\mathbb{Q}_p,\mathbb{Z}_p) \to \mathscr{D}_{unif}(\mathbb{Q}_p,\mathbb{Z}_p)$$

in the sense that: if $f \in C_{unif}(\mathbb{Q}_p,\mathbb{Z}_p)$ *and* $\mu \in \mathscr{D}_{bd}(\mathbb{Q}_p,\mathbb{Z}_p)$ *then:*

$$\int_{\mathbb{Q}_p} f d\mu = \int_{\mathbb{Q}_p} f d\mu_{unif}$$

Observation 27. This theorem is true in a more general situation, instead of $\mathbb{Z}_p$ one can take any linearly topologized topological ring k and instead of $\mathbb{Q}_p$ a so called STS-space. For STS-space we mean a 0-dimensional locally compact paracompact space.

Example 6. (An extension of the Diamond's measure)
Let $a \in \mathbb{Q}_p^\times$, then $a = p^{v_p(a)}u$ where $u \in \mathbb{Z}_p^\times$ (this follows from the structure of $\mathbb{Q}_p^\times$ in fact we have $\mathbb{Q}_p^\times \cong p^{\mathbb{Z}} \times \mathbb{Z}_p^\times$). Consider now an open compact in $\mathbb{Q}_p^\times$ on the form $a + p^N\mathbb{Z}_p$ then we have $a = p^{v_p(a)}u$. From now on we fix $c \neq 1$ an integer coprime with p and we denote μ_D the measure of Diamond $\mu_{1,c}$.

Definition 68. We define the extension of the Diamond's measure $\mu_{1,c}$ as the following:

$$\tilde{\mu}_{1,c}(a+p^N\mathbb{Z}_p) = \begin{cases} p^{-v_p(a)} \cdot \mu_{1,c}(u + p^{N-v_p(a)}\mathbb{Z}_p) & a+p^N\mathbb{Z}_p \subset \mathbb{Q}_p \setminus p\mathbb{Z}_p \\ \\ 0 \qquad a+p^N\mathbb{Z}_p \subset p\mathbb{Z}_p \end{cases} \tag{4.1}$$

where $0 < u < p^{N-v_p(a)}$ s.t. $(u,p)=1$, $N \in \mathbb{Z}$.
Consider $a + p^N\mathbb{Z}_p \subset \mathbb{Q}_p \setminus p\mathbb{Z}_p$ where $a = p^{v_p(a)}u$ then since $p^{v_p(a)}u + p^N\mathbb{Z}_p \subset \mathbb{Q}_p \setminus p\mathbb{Z}_p$ we have $N - v_p(a) > 0$.

Observation 28. If $a + p^n\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ then $v_p(a) = 0$ (and a satisfies the above properties) this implies that this measure coincides with the measure of Diamond $\mu_{1,c}$, in fact in $\mathbb{Z}_p^\times$:

$$\tilde{\mu}_{1,c}(a+p^n\mathbb{Z}_p) = p^{-v_p(a)}\mu_{1,c}(a+p^n\mathbb{Z}_p) = \mu_{1,c}(a+p^n\mathbb{Z}_p)$$

Proposition 20. *This function* $\tilde{\mu}_{1,c}$ *extends to a uniform measure*

Proof. Denote by $\mu := \tilde{\mu}_{1,c}$ and by $\mu_D := \mu_{1,c}$, it's easy to see that it is additive in open compact sets on the form $a + p^n\mathbb{Z}_p$, in fact if $a = p^{v_p(a)}u$:

$$\mu(a+p^n\mathbb{Z}_p) = p^{-v_p(a)}\mu_D(u + p^{n-v_p(a)}\mathbb{Z}_p)$$

and, using the fact that μ_D is a measure, we get:

$$p^{-v_p(a)}\mu_D(u+p^{n-v_p(a)}) = p^{-v_p(a)}\sum_{j=0}^{p-1}\mu_D(u+p^{n-v_p(a)}j+p^{n+1-v_p(a)}\mathbb{Z}_p)$$

with $n+1 > v_p(a)$ since $n > v_p(a)$, moreover $|a| > |p^n j|$ this implies $|a| = |a+p^n j|$ i.e. $v_p(a+p^n j) = v_p(a)$ $(0 \le j \le p-1)$.

$$\sum_{j=0}^{p-1} p^{-v_p(a)}\mu_D(u+p^n j+p^{n+1-v_p(a)}\mathbb{Z}_p) = \sum_{j=0}^{p-1} p^{-v_p(a+p^n j)}\mu_D(u+p^{n-v_p(a)}j+p^{n+1-v_p(a)}\mathbb{Z}_p)$$

finally:

$$\mu(a+p^n\mathbb{Z}_p) = \sum_{j=0}^{p-1}\mu(a+p^n j+p^{n+1}\mathbb{Z}_p)$$

Let $V \in \Sigma_{unif}(\mathbb{Q}_p)$ and $V = \bigcup_{i\in I}(x_i + p^r\mathbb{Z}_p)$ with r the smallest integer such that $r - v_p(x_i) \ge 0$ for every $x_i \ne 0$ and $0 < x_i \le p$, so by definition we set:

$$\mu(V) = \sum_{i\in I}\mu(x_i + p^r\mathbb{Z}_p)$$

and we get a map :

$$\mu : \Sigma_{unif}(\mathbb{Q}_p) \longrightarrow \mathbb{Z}_p$$

but now we have to check that this series converges unconditionally for every index set I. Clearly if I is finite then μ acts like a classical measure. If I is infinite then V is not bounded, so if we fix a ball $p^N\mathbb{Z}_p \in P(\mathbb{Z}_p)$, then up to a finite number of balls $x_i + p^r\mathbb{Z}_p \subset V$ we have $v_p(x) < -N$ for arbitrary N. We observe that by definition:

$$|\mu(x_i + p^r\mathbb{Z}_p)| = \left|p^{-v_p(x_i)}\right| \cdot |\mu_D(u + p^{r-v_p(x_i)}\mathbb{Z}_p)| \le p^{-N}$$

for $v_p(x_i) \le -N$ because μ is a measure on $\mathbb{Z}_p$ and so $|\mu| \le 1$.
By the argument above we get:

$$\mu(x_i + p^r\mathbb{Z}_p) \in p^N\mathbb{Z}_p$$

for almost all $i \in I$. This implies the unconditionally convergence of the above series. $\square$

Observation 29. We justify the fact of extending μ by zero in $p\mathbb{Z}_p$ by the fact that, in general, μ_D is only a measure in $\mathbb{Z}_p^\times$ and since the definition of Mellin transform is given integrating in $\mathbb{Z}_p^\times$. So extending by 0 this measure does not influences the classical definition. Outside $p\mathbb{Z}_p$ we defined it as :

$$\mu(a + p^N\mathbb{Z}_p) = p^{-v_p(a)} \cdot \mu_D(u + p^{N-v_p(a)}\mathbb{Z}_p)$$

Also this does not influences the classical definition of μ_D because we're asking that:

$$\mu_D(p^n U) = p^{-n}\mu_D(U)$$

with U compact/open in $\mathbb{Z}_p^\times$, but $p^n U$ is not contained in $\mathbb{Z}_p^\times$ so this situation can't happen.

4.4 Artin-Hasse isomorphism and formal perfectoid discs

Consider again the Artin-Hasse series:

$$F_p(T) = \exp_p\left(\sum_{i=0}^{\infty} \frac{T^{p^i}}{p^i}\right)$$

we remark that $F_p(T) \in 1+T+T^2\mathbb{Z}_{(p)}[[T]]$. The Dieudonné formula (proved in [15]) provides to the following equation:

$$\prod_{i=0}^{\infty} F_p(x_i T^{p^i}) = \exp_p\left(\sum_{i=0}^{\infty} x^{(i)} T^{p^i}\right) = 1 + \sum_{i=1}^{\infty} g_i(x_0, \ldots, x_{\lfloor \log(i) \rfloor}) T^{p^i}$$

with $\log(i)$, the real logarithm, and

$$x^{(i)} = \sum_{n=0}^{i} p^{n-i} x_n^{p^{i-n}}$$

After the change $x_i \to x_{i-n}$ and $T^i \to T^{i/p^n}$ we get the formula:

$$\prod_{i=-n}^{\infty} F_p(x_i T^{p^i}) = \exp_p\left(\sum_{i=0}^{\infty} x^{(i)} T^{p^i}\right) = 1 + \sum_{i=1}^{\infty} g_i(x_{-n}, \ldots, x_{\lfloor \log(i/p^n) \rfloor}) T^{p^{i/p^n}}$$

We give now a definition wich will be useful from now on:

Definition 69. We define S as the set of indices given by:

$$S = \mathbb{Z}\left[\frac{1}{p}\right] \cap \mathbb{R}_{\geq 0}$$

we observe that is clearly countable.

So in [2, Sect. 4] is defined a new topological algebra $\widehat{\mathscr{P}}$ in which it is possible to compute the limit $n \to \infty$ in the formula above obtaining the following:

$$\prod_{i=-\infty}^{\infty} F_p(x_i T^{p^i}) = \sum_{q \in S} g_q(x) T^q$$

Moreover, again in [2], is given an analytic specialization of the latter. In fact are introduced other topological algebras in wich it is possible to compute those formulas using the change $x_i \to \Psi(p^{-i}x)$. So we obtain the new formula :

$$\prod_{i=-\infty}^{\infty} F_p(\Psi(p^i x) T^{p^i}) = \sum_{q \in S} G_q(x) T^q$$

where the polynomials $G_q(x)$ are obtained by g_q after the change $x_i = \Psi(p^{-i}x)$.

Remark 1. We remark that this definition gives to the polynomials G_q the following proprierties:

$$G_q(x+y) = \sum_{q_1+q_2=q} G_{q_1}(x)G_{q_2}(y) \qquad G_{qp}(px) = G_q(x) \qquad G_q(\mathbb{Q}_p) \subset \mathbb{Z}_p$$

The binomial coefficients $\binom{x}{n}$ has similar proprierties, in fact the polynomials $\{G_q\}_{q\in S}$ are a Banach basis for $C(\mathbb{Q}_p, \mathbb{Z}_p)$.

The latter algebra is denoted $\mathscr{E}^{\frac{p}{p-1},1}$ in [2] and defined in the following way:
Consider $\mathscr{E}$: the algebra of entire functions in $\mathbb{C}_p$, such that restricted to $\mathbb{Q}_p$ are uniformly continuous and if $f \in \mathscr{E}$ then $f(\mathbb{Q}_p) \subset \mathbb{Z}_p$. This algebra is complete under the Fréchet topology induced by the family of semivaluations $\{w_r\}_{r\in\mathbb{Z}}$ given by:

$$w_r(f) = \inf_{x\in p^{-r}\mathbb{C}_p^\circ} v_p(f(x))$$

Example 7. Consider the function $\Psi(x)$, defined also in [2] and briefly introduced Section 1, then by definition $\Psi(x) \in \mathscr{E}$.

Then $\mathscr{E}^{\frac{p}{p-1},1}$ is the algebra of series on the form $\sum_{q\in S} a_q T^q$ such that for every $C \in \mathbb{R}$:

$$w_r(a_q) + \ell(q) + \frac{p}{p-1}(\max(qp^r, 1) - 1) > C$$

form almost all $q \in S$,

Remark 2. More precisely $\mathscr{E}^{\frac{p}{p-1},1}$ is the completion of $\mathscr{E}[T^{1/p^\infty}]$ under the Frechet topology induced by the semivaluations:

$$w_r^{\frac{p}{p-1},1}\left(\sum_{q\in S} a_q T^q\right) = \inf_{q\in S}(w_r(a_q) + \ell(q) + \frac{p}{p-1}(\max(qp^r, 1) - 1))$$

introduced in [2].

Definition 70. (Formal perfectoid disc over $\mathbb{Z}_p$) We define the 'formal perfetoid' open unit disc over $\mathbb{Z}_p$ as the adic space $\mathbb{D} = Spa(\mathscr{D}, \mathscr{D})$.

Definition 71. (Formal perfectoid disc over $\mathbb{F}_p$) We define the 'formal perfetoid' open unit disc over $\mathbb{F}_p$ as the adic space $\mathbb{D}_{\mathbb{F}_p} = Spa(\tilde{\mathscr{D}}, \tilde{\mathscr{D}})$.

So consider now two characters Θ_0 and Θ_1, the first given by:

$$\Theta_0 : (S, +) \to \mathscr{D}(\mathbb{Q}_p, \mathbb{Z}_p)$$
$$q \mapsto T^q$$

we remark that $(S, +)$ is only a monoid. The second is:

$$\Theta_1 : \mathbb{Q}_p \to \mathscr{D}(\mathbb{Q}_p, \mathbb{Z}_p)$$
$$q \mapsto \Delta_q$$

Definition 72. The formal perfectoid open unit disc centered in 0 (resp. 1) over $\mathbb{Z}_p$ is the couple $(\mathbb{D}, \Theta_0)$ (resp. $(\mathbb{D}, \Theta_1)$) and it will be denoted $\mathbb{D}(0)$ (resp. $\mathbb{D}(1)$).

Definition 73. The formal perfectoid open unit disc centered in 0 (resp. 1) over $\mathbb{F}_p$ is the couple $(\mathbb{D}_{\mathbb{F}_p}, \Theta_0)$ (resp. $(\mathbb{D}_{\mathbb{F}_p}, \Theta_1)$) and it will be denoted $\mathbb{D}_{\mathbb{F}_p}(0)$ (resp. $\mathbb{D}_{\mathbb{F}_p}(1)$).

Observation 30. So using $\mathscr{D}$ is possible to introduce this notion of 'formal perfectoid disc', we can compare the situation of the classical analytic discs. Take $D(0, r_p)$ and $D(1, r_p)$ the classical exponential gives an insomorphism :

$$\exp_p : D(0, r_p) \xrightarrow{\sim} D(1, r_p)$$

in this more abstract situation the role of the $\exp_p$ is played by F_p i.e. the Artin-Hasse exponential.

So for any perfectoid extension $K/\mathbb{Q}_p$ the Artin-Hasse series $F_p(T)$ give to rise to an isomoprhism between the two perfectoid formal discs:

$$\overline{F_p} : \mathbb{D}_{\mathbb{F}_p}(0) \xrightarrow{\sim} \mathbb{D}_{\mathbb{F}_p}(1)$$

this because we've already remarked that $t \mapsto \overline{F_p}(t)$ gives an isomorphism at the level of algebras of measures, in fact, $t \to \lim_{n\to\infty} \overline{F_p}(t^{p^{-n}})^{p^n} = \overline{F_p}(t)$ because we are in characteristic p. Since we have functor given by $(R, R^+) \mapsto Spa(R, R^+)$, the isomorphism carries at level of adic spaces. As a consequence of the tilting correspondence for adic spaces this isomorphism lifts also in characteristic 0

$$F_p^\sharp : \mathbb{D}_K(0) \xrightarrow{\sim} \mathbb{D}_K(1)$$

Here $\mathbb{D}_K$ is obtained by 'base change' i.e.:

$$\mathbb{D}_K = Spa(\mathscr{D}\hat{\otimes}_{\mathbb{Z}_p} K, \mathscr{D}\hat{\otimes}_{\mathbb{Z}_p} K)$$

for more details see [2]. So for any perfectoid extension K of $\mathbb{Q}_p$ and any choice of pseudo-uniformizer $\varpi = (\varpi^{(i)})_{i\in\mathbb{N}} \in (K^\flat)^{\circ\circ}$ we can interpret a point of $\mathbb{D}(0)$ as a character $\chi := \chi_\varpi$ depending on ϖ which acts in the following way:

$$T^q \to \chi(q) := \lim_{j\to\infty} (\varpi^{(j)})^{qp^j}$$

So in practice it can be considered a character $\chi : (S \setminus 0, +) \to (K^{\circ\circ}, \cdot)$.
Again by the Artin-Hasse isomorphism between the two perfectoid discs we have :

$$F_p^\sharp : \mathbb{D}_K(0) \to \mathbb{D}_K(1)$$

$$\chi \longmapsto F_\chi^\sharp(x)$$

Our main interest in this work is now to study the property of convergence in $\mathbb{Q}_p$ and $\mathbb{C}_p$ the formal equation given by the Dieudonné formula after the last specialization given by $T^q \to \chi(q)$.

$$\prod_{i=-\infty}^{+\infty} F_p(\Psi(p^i x)\chi(p^i)) = \sum_{q\in S} \chi(q) G_q(x)$$

We are mainly interested in prove that the series $\sum_{q\in S}\chi(q)G_q(x)$ converges in some sense to:

$$\psi : (\mathbb{Q}_p,+) \to (1+K^{\circ\circ},\cdot)$$
$$x \mapsto F_\chi{}^\sharp(x) = \sum_{q\in S}\chi(q)G_q(x)$$

which it would be a continuous group homomorphism.

4.5 Extension of characters and convergence properties

So we need from [6] and [2] the following estimates for Ψ and G_q:

Theorem 22. *(Estimates for G_q)*
Consider the family of valuations $\{w_r\}_{r\in\mathbb{Z}}$ introduced in the last section and let $\ell(q) = \lfloor log(q)\rfloor$ (log is the real logarithm in base p here) , then for $G_q(x)$ we have the following estimates for any $c\in\mathbb{R}$ and $N\in\mathbb{Z}$:

- $r \leq -\ell(q) \Rightarrow w_r(G_q(x)) \geq -v(q)+\ell(q)$
- $r \geq -\ell(q) \Rightarrow w_r(G_q(x)) \geq -v(q)+\ell(q) - c((p^{r+\ell(q)})^N - 1)$
  Moreover, these two conditions can be put togheter in the following one:

$$w_r(G_q(x)) \geq -v(q) + \max(\ell(q),-r) - p\frac{p^{\max(p^r+\ell(q),0)}-1}{p-1}$$

Proof. [2] Remark 5.4 and remark 5.7 . □

Theorem 23. *(Estimates for Ψ)*
For $i = 1,2,\dots$ and $v_p(x) \geq -i$ (resp. $v_p(x) > -i$) we have $v_p(\Psi(x)) \geq -\frac{p^i-1}{p-1}$ (resp. $v_p(\Psi(x)) > -\frac{p^i-1}{p-1}$). If $v_p(x) > -1$ then $v_p(\Psi(x)) = v_p(x)$.

Proof. [6] Corollary 4.6. □

Lemma 4. *The valuation $v(\chi(q)) = qv(\varpi^{(0)})$ where $\chi = \chi_\varpi$, in particular we have the estimates: $v(\chi(q)) > q$.*

Proof. We gave explicitely the character χ i.e.

$$\chi(q) = \lim_{j\to\infty}(\varpi^{(j)})^{qp^j}$$

and so:

$$v(\chi(q)) = v(\lim_{j\to\infty}(\varpi^{(j)})^{qp^j}) = qv(\lim_{j\to\infty}(\varpi^{(j)})^{p^j})$$

since ϖ is an element of the tilt of $K^{\circ\circ}$ we have by definition $(\varpi^{(j+1)})^p = \varpi^{(j)}$.

$$v(\chi(q)) = qv(\lim_{j\to\infty}(\varpi^{(0)})) = qv(\varpi^{(0)})$$

and since $\varpi^{(0)} \in K^{\circ\circ}$ we get $v(\chi(q)) > q$. □

Lemma 5. *Given the sequence $\{a_i\}_{i\in\mathbb{N}}$ in $\mathbb{C}_p$ and the infinite product :*

$$\prod_{i=1}^{\infty}(1+a_i)$$

if $a_i \neq -1$ and $a_i \to 0$, then the infinite product converges

Proof. [27] Page 279. □

Lemma 6. *Consider the infinite product:*

$$\prod_{i=0}^{\infty} F_p(\Psi(p^i x)\chi(p^{-i}))$$

Then it converges as a product of functions in $C(\mathbb{Q}_p, K^\circ)$ with the topology of uniform convergence on compact subsets

Proof. Using Lemma 1, the infinite product $\prod(1+a_n)$ converges if the sequence a_n in $\mathbb{C}_p$ is such that: every $a_n \neq -1$ and $a_n \to 0$. So consider the product :

$$\prod_{i=0}^{\infty}((F_p(\Psi(p^i x)\chi(p^{-i})) - 1) + 1)$$

then since $\chi(q) \in K^{\circ\circ}$ for every $q \in S$ we have $|\chi(q)| < 1$ moreover if $x \in \mathbb{Q}_p$ then clearly $p^i x \in \mathbb{Q}_p$ and so $\Psi(p^i x) \in \mathbb{Z}_p$, this means that:

$$|\Psi(p^i x)\chi(p^{-i}))| < 1$$

Since $F_p(x) \in 1 + x\mathbb{Z}_{(p)}[[x]]$ we have: $|F_p(\Psi(p^i x)\chi(p^{-i}))| = 1$ and so it is not 0 in particular $F_p(\Psi(p^i x)\chi(p^i)) - 1 \neq -1$.
Now by the properties of the Artin-Hasse exponential :

$$|F_p(\Psi(p^i x)\chi(p^{-i})) - 1| = |\Psi(p^i x)\chi(p^{-i})| < |\Psi(p^i x)|$$

and now:

$$|\Psi(p^i x)| \to 0$$

as $i \to \infty$, because $\Psi(x) \in x\mathbb{Z}_p[[x]]$. □

Lemma 7. *Consider the infinite product:*

$$\prod_{i=0}^{\infty} F_p(\Psi(p^{-i}x)\chi(p^i))$$

It converges uniformly as a product of functions in $C(\mathbb{Q}_p, K^\circ)$ with the sup-norm

Proof. The same argument of above works until the fact that:

$$|F_p(\Psi(p^{-i}x)\chi(p^i)) - 1| = |\Psi(p^{-i}x)\chi(p^i)|$$

now we use a different estimate:

$$|\Psi(p^{-i}x)\chi(p^i)| \leq |\chi(p^i)|$$

Now since χ is an additive character $\chi(p^i \cdot 1) = \chi(1)^{p^i}$ then since $|\chi(1)| < 1$:

$$|\chi(p^i)| = |\chi(1)|^{p^i} \to 0$$

as $i \to \infty$.
In fact $v(\chi(p^i)) = p^i\, v(\varpi^{(0)}) > p^i \to \infty$ as $i \to \infty$. ☐

Proposition 21. *Consider the infinite product:*

$$\prod_{i=1}^{\infty} F_p(\Psi(p^i x)\chi(p^{-i}))$$

It converges as a product of functions in $An(D\left(0, p^+\right), v_{D(0,p^+)})$.

Proof. Again, we know that $|\chi(q)| < 1$ for every $q \in S$, let $x \in \mathbb{C}_p$, we have to check that $|\Psi(p^i x)\chi(p^{-i})| < 1$ for every $i \in \mathbb{N}_{\geq 1}$ in order to have the convergence of the Artin-Hasse exponential. We have to impose $v_p(\Psi(p^i x)) + v_p(\chi(p^{-i})) > 0$, we know $v_p(\chi(p^{-i})) > p^{-i} \Longrightarrow v_p(\chi(p^{-i})) + v_p(\Psi(p^i x)) > v_p(\Psi(p^i x)) + p^{-i}$ for $i \in \mathbb{N}_{\geq 1}$. So we need to ask:

$$v_p(\Psi(p^i x)) \geq 0$$

in this case the Artin-Hasse exponential converges for all $i \in \mathbb{N}_{\geq 1}$. By the estimates of Ψ, for $v_p(p^i x) \geq 0$, we have $v_p(\Psi(p^i x)) = v_p(p^i x) \geq 0$

$$v_p(p^i x) \geq 0 \Longleftrightarrow v_p(x) \geq -i$$

so we need at least $v_p(x) \geq -1$, at this point we can use the same argument of the lemmas above because:

$$|F_p(\Psi(p^i x)\chi(p^{-i})) - 1| = |\Psi(p^i x)\chi(p^{-i})|$$

☐

Proposition 22. *Consider the infinite product:*

$$\prod_{i=0}^{\infty} F_p(\Psi(p^{-i}x)\chi(p^i))$$

It converges as a product of functions in $An(D\,(0,1^+)\,, v_{D(0,1^+)})$.

Proof. Again we need to check the convergence of the Artin-Hasse exponential so we need:

$$v_p(\Psi(p^{-i}x)\chi(p^i)) > 0$$

What we know is that for $i >> 0$:

$$v_p(\Psi(p^{-i}x)\chi(p^i)) > v_p(\Psi(p^{-i}x)) + p^i \geq p^i - \frac{p^{i-\lfloor v_p(x)\rfloor} - 1}{p-1}$$

if $v_p(x) \geq 0$ then $v_p(\Psi(p^{-i}x)\chi(p^i)) > 0$ which implies the convergence of F_p. Again if $v_p(x) \geq 0 \Longrightarrow \lfloor v_p(x) \rfloor \geq 0$ so:

$$p^i - \frac{p^{i-\lfloor v_p(x)\rfloor} - 1}{p-1} \to \infty$$

as $i \to \infty$ which implies the convergence of the product beacause if $v_p(\Psi(p^{-i}x)\chi(p^i)) > 0$ then

$$v_p(F_p(\Psi(p^{-i}x)\chi(p^i)) - 1) = v_p(\Psi(p^{-i}x)\chi(p^i)) \to \infty$$

as $i \to \infty$ $\square$

Theorem 24. *The map* $\psi(x) = F_p^\sharp(\chi)$ *given by the untilting of the Artin-Hasse isomorphism:*

$$\psi : (\mathbb{Q}_p, +) \to (1 + K^{\circ\circ}, \cdot\,)$$

$$x \mapsto \sum_{q\in S} \chi(q) G_q(x)$$

is a group homomorphism.

Proof. It follows from the additive properties of $G_q(x)$ in fact:

$$\psi(x)\psi(y) = \left(\sum_{q\in S}\chi(q)G_q(x)\right)\cdot\left(\sum_{q\in S}\chi(q)G_q(y)\right) = \sum_{q\in S}\left(\sum_{q_1+q_2=q} G_{q_1}(x)G_{q_1}(y)\right)\chi(q) =$$

$$\sum_{q\in S}\chi(q)G_q(x+y) = \psi(x+y)$$

$\square$

These characters are the analogous for the continuous characters on $\mathbb{Z}_p$, we recall in fact that every additive (to multiplicative) character from $\mathbb{Z}_p$ to $\mathbb{C}_p^\times$ is on the form :

$$\psi : (\mathbb{Z}_p, +) \to (1 + \mathbb{C}_p^{\circ\circ}, \cdot\,)$$

$$s \mapsto x^s$$

with $|x-1| < 1$. We give now a different interpretation of this fact, we stress that in the classical case we have the following identification:

$$D(0,1^-) \xrightarrow{1+} D(1,1^-) \xrightarrow{\cong} Hom_{cts}(\mathbb{Z}_p, \mathbb{C}_p^\times)$$

$$x \longmapsto (1+x) \longmapsto (s \to (1+x)^s)$$

for every $s \in \mathbb{Z}_p$. It is well known the series expansion in terms of the variable x of this character, which is:

$$x^s = \sum_{n=0}^{\infty} \binom{s}{n} (x-1)^n$$

this says that, with respect to x, it is analytic in $D(1,1^-)$.
So we can see two types of characters: we get an additive (to multiplicative) map

$$\phi : (\mathbb{Z}_{>0}, +) \to (\mathbb{C}_p^{\circ\circ}, \cdot)$$

$$n \to x^n$$

for every $x \in D(0,1^-)$, i.e. every such a character is parametrised by x. Now, as already mentioned, we obtain another character:

$$\psi : (\mathbb{Z}_p, +) \to (1 + \mathbb{C}_p^{\circ\circ}, \cdot\,)$$

$$s \mapsto (1+x)^s$$

parametrised by $(1+x) \in D(1,1^-)$.
So in the case of $\mathbb{Z}_p$, the correspondence given by $F_p^\sharp$ could be seen as :

$$(\phi : n \mapsto x^n) \xrightarrow{F_p^\sharp} (F_p^\sharp(\phi) : s \to (1+x)^s)$$

Now, in the case of $\mathbb{Q}_p$, we have a point χ in the disc $\mathbb{D}_K(0)$ i.e.

$$\chi : (S \setminus \{0\}, +) \to (K^{\circ\circ}, \cdot)$$

$$q \to \chi(q)$$

this time S plays the role of $\mathbb{Z}$. Its image $F_p^\sharp(\chi) \in \mathbb{D}_K(1)$ through the Artin-Hasse series is a group homomorphism

$$F_p^\sharp(\chi) : (\mathbb{Q}_p, +) \to (1 + K^{\circ\circ}, \cdot)$$

given by the series:

$$x \mapsto \sum_{q \in S} \chi(q) G_q(x)$$

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