UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Effective Space-time Geometry for Black Holes and Cosmology

Thesis supervisor

Prof. Sabino Matarrese

Thesis co-supervisor

Dr. Daniele Pranzetti

Candidate

Roukaya Dekhil

Academic Year 2018/2019

Contents

Abstract

Effective space-time geometries can be derived by evolving initial data sets through a modified Hamiltonian obtained within a canonical approach to quantum gravity. This problem can be formulated precisely by first selecting a *reduced* Hilbert space of the full theory Kinematical Hilbert space and then preforming a symmetry reduction in order to derive a symmetric sector of the theory. This framework has been successfully applied in the cosmological case. Recently, this setting has been extended by implementing a choice of gauge relevant for a black hole geometry, which led to the derivation of an effective Hamiltonian by means of coherent states, which are the best candidates to describe classical geometries. The classical data entering the coherent states can be seen as the initial data set to be evolved with the effective Hamiltonian. We study the algebra that the effective Hamiltonian constraint of the theory generates, as well as the equations of motion that the reduced phase space variables satisfy. With the goal to achieve closure of the algebra of the effective constraints, we extract a candidate expression for the effective diffeomorphism constraint, which is also compatible with the effective evolution equations. This is required in order to restore (a deformed version of) general covariance and to guarantee the consistency of the effective dynamics to be solved for.

Dedication

To my father who taught me how to do things right, my mother who always told me that nothing is impossible, to my older brother who pushed me to question and understand everything and to my little brother who is keeping me going forward...

Introduction

Since the time of "the father of observational astronomy", Galileo Galilei, the idea that humanity occupies a privileged position at the center of the Universe has been challenged by the regularity of the cosmos as found by observations. This very ancient idea laid out the ground for the concept of homogeneous and isotropic Cosmology, i.e. that in the Universe there are no preferred points or choices of direction for performing observations. Indeed, this idea of isotropy has been proven to be very favorable also by modern experiments with increasingly high precision.
However, observations must be explained and formulated in the framework of physical theories, that can deliver predications and able to provide explanations to them. This is one of the various applications of the General theory of Relativity (GR) that comes into play. Indeed, under the assumption of homogeneity and isotropy, General Relativity conveyed a very important lesson in Cosmology, namely, that we are living in an expanding universe that started from a *singular* state, i.e a state that ideally could be obtained by homogeneous shrinking of spacetime to a point. Singularities are generic features of cosmological solutions and gravitational collapse, which was first proven by Roger Pensrose [1].

Aside from cosmological singularities, one can reflect on more concrete situations such as black holes singularities, where according to GR spacetime ends. The theory of black holes has produced tremendous results and predictions of the nature of these physical objects. Perhaps one of the most intriguing aspects that stands out among them is the striking analogy between the mathematically rigorous laws that governs black holes and the theory of Thermodynamics. Indeed, the most important analogy between the two theories is the law of entropy mirrored in the law of the area increase in black hole physics. This strong similarity hints to the presence of an underlying microscopic structure and a quantum nature for black holes.
To what extent quantum effects can reshape the predictions acquainted in the classical formulation of singularities and black holes has been a long standing unresolved problem in theoretical physics. It is strongly believed that these very quantum effects can depict the scenarios accounting for a resolution of the singularities. If that is to be taken true, various enigmas can be smoothly washed away from the challenge-list of theoretical physics.

In the context of black hole physics, another long-lasting mystery is that of the cosmic censorship principle and the information loss paradox, highlighted by S. Hawking [2]. Hawking was the first to formulate the argument that the usual rules applied in Quantum Mechanics do not apply in the life span of a black hole, namely from its formation till evaporation. If we were to take the semi-classical approximation deployed by Hawking to be exact the unitarity principle will be violated, as black hole evaporation predicts that a pure quantum state will eventually evolve to a mixed one [3]. There has been recent proposal to solve this paradox, some argue that black hole event horizons develop into "firewalls" or "fuzzballs", while others suggest the existence of Planck-sized remnants of the evaporation process, and singularity resolution by quantum gravity effects that lead to an extension of the spacetime diagram for evaporating black holes. Beyond all speculations, a definitive fate of a classical singularity is only predicted by a detailed full Quantum Gravity calculation. Given the intimate relation between the last stages of black hole evaporation and Planck-scale physics, this paradox will likely be resolved by a better understanding of a Quantum Gravity description. Admittedly, it is true that General Relativity is a revolutionary theory on many aspects and can provide answers and prophecies for an unaccountable physical scenarios, however, it is still not reconciled with the second dominant theory in physics, that is the theory of Quantum Fields (QFT). If Quantum Field theory principles are to be applied to General Relativity, then one might convey that GR is perhaps an approximation to some more fundamental theory, from which GR emerges. The mainstream approach to approach this fundamental problem in current physics, is the theory of Quantum Gravity (QG).

Combining the main lesson of General Relativity, i.e. "spacetime and geometry are the gravitational field, and are dynamical, physical entities" [4] with the main lessons of Quantum Mechanics, i.e. "all physical systems possess quantum properties: irreducible uncertainty, probabilistic nature of physical quantities, entanglement, etc", we know that spacetime and geometry should exhibit a similar quantum nature. This very possibility raises conceptual difficulties of the most profound nature, and our picture of the world remains contradictory and incomplete.
The main apprehensive approaches to QG [4] rely on the straightforward quantization of GR. Some models apply quantization schemes directly to the full spacetime geometry, for instance canonical Loop Quantum Gravity [5, 6]; others follow path integral formulations of quantum GR and their modern evolution mostly based on lattice structures (e.g. Causal Dynamical Triangulations [7], or Group Field Theories [4, 8]). String theory [9], started off as a tentative enhancement to graviton-based formalism to Quantum Gravity, naturally resulting from considering extended string-like (and brane-like) variables, has as well mirrored striking quantum aspects of the gravitational field despite the absence of a more fundamental description of its microscopic nature. All these approaches agree on the *quantum nature* of spacetime, but more importantly, they inevitably hint towards a more *fundamental structure* constituting the very core of gravity.
Despite this vast landscape of diverse, yet complementary, approaches to the problem of QG, these models commonly indicate the crucial change in perspective towards the quantum nature of spacetime [10], namely that the fundamental nature of our usual notion of continuous spacetime is actually tightly related to the notion of building blocks "atoms of space", of no direct gravitational, spatiotemporal or geometric interpretation, and from which it has to *emerge*, whence, giving rise (in a suitable approximation) to the usual notion of geometry, gravity and fields, producing the physics we are familiar with. Black holes are often considered the true fate of quantum gravity approaches, which should provide a microscopic derivation of their thermodynamic properties, and solve the paradoxes arising from a semi-classical treatment. This generalizes the modeling of black holes inspired by loop quantum gravity (LQG) [11].

LQG [12, 13, 14] lays a preferable ground for the non-perturbative quantization of GR. The most studied case of quantum black holes and singularities [15, 16] are content to the investigation of the interior geometry and the event horizon of a Schwarzschild black hole describing homogeneous geometries captured by the so-called Kantowski-Sachs type metrics. This particular geometry can be treated as a minisuperspace for which the techniques developed in the cosmological context [17, 18, 19] of Loop Quantum Cosmology (LQC) are available and can be readily used. The study in this framework delivered results hinting towards a singularity resolution of the bouncing cosmological type.

In this work we focus on a similar approach based on the full theory of LQG, the so-called Quantum Reduced Loop Gravity (QRLG)[20] applied to spherical symmetric geometries [21] with Kerr-Schild foliation [22, 23]. The QRLG program was mainly implemented in LQC sector for cosmologies with homogeneous anisotropies [17]. For instance, in the case of Bianchi I spacetime, the program of QRLG relies on treating the partially reduced phase space endowed with the symmetry reduction at the quantum level, contrary to the mainstream approach of LQC in implementing it at the classical level.
Indeed, performing the symmetry reduction [24] at the classical level before quantization has the negative drawback that most of the full Hilbert space structure is lost, and this is the key difference that QRLG hinders, i.e. by reverting the direction in which the treatment is done. This consequently allows to work with the complete structure of the full theory, consisting of quantum states of polymeric nature decorated by graphs and $SU(2)$ representations.
Moreover, in the case of black holes, [20] the main results present in the literature are always restricted to the interior region where space is homogeneous [23, 25] and not stressing the importance of the exterior region. Hence such approaches tend to separate the two regions, and therefore, employ different respective Hilbert spaces, giving rise to some subtleties once one wishes to sew the two regions. Tentative investigation to extend the framework to include the exterior, has been little, perhaps even inexistent.
However, the foundations for a systematic treatment of spherically symmetric spacetimes in [21] set the extended treatment option to include the investigations on various sets of coordinate systems such as horizon penetrating coordinates or coordinates restricted to the *interior* or *exterior* of the event horizon of a black hole, since the obtained results are foliations-independent and therefore one can also study the exterior region of a black hole.
Indeed, to present a quantum theory of gravity that is capable of describing the full space (interior and exterior) should accommodate the ambiguities of diffeomorphisms covariance, arising in the quantum theory, that has to reconcile with the classical theory by producing the right classical predications in some suitable approximation. Therefore, there are two crucial ingredients a quantum theory of gravity has to provide for a consistent description for quantum black holes with horizon penetrating horizon, namely, expressing the effective quantum equivalent for the *diffeomorphisms* constraint and on-shell closure of the constraint algebra to avoid anomalies. Once this is achieved, solving the dynamics should describe the world as we know, and more interestingly open doors to new physics.
This work tackles the problem of diffeomorphism constraint on the effective level in the QRLG setup for spherical symmetric geometries. With the goal to achieve closure of the algebra of the effective constraints, we present a tentative strategy to extract a candidate expression for the effective diffeomorphism constraint, which is also compatible with the effective evolution equations. This is required in order to restore (perhaps even a deformed version of) general covariance and to assure the consistency of the effective dynamics to be solved for.

This work is organized as follows. We will start in chapter 1 by outlining the basic ingredients of the canonical formulation of General Relativity, where we go through the 3+1 decomposition of spacetime that would allow to formulate GR as a constrained theory. Then we discuss possible reformulations of the Einstein-Hilbert action, namely the ADM- and connection formulation respectively, where in the latter we introduce new conjugate variables based on the triad formalism. The change of variables, accounted for by introducing the triads, enables one to consider a new set of canonical conjugate variables, presented by fluxes and the Ashtekar-Barbero connection. It will lay out the ground for the LQG quantization scheme by introducing the notion of holonomies, which will be the main subject of chapter 2. In chapter 3, we outline the so-called gauge unfixing of first order gravity procedure and present the results it predicts for spherical geometries and more specifically the resulting constraints for the Kerr-Schild metric. At a second stage, we lay out the basic steps and key concepts in the QRLG quantization technique applied to spherical geometries and then compute the resulting effective Hamiltonian constraint specified to a black hole spacetime geometry described by a Kerr-Schild metric. In the last two chapters, chapter 4 and 5 are devoted to the investigation of the constraint algebra. We first explore in chapter 4 the classical constraint algebra for the spherical symmetric geometries with Kerr-Schild foliation and the on-shell consistency of the evolution equations of motion for the phase space variables that are trivially satisfied for the stationary case of the Kerr-Schild metric. In chapter 5 we derive the effective quantum equations of motion for the canonical pair concerning two scenarios, namely when the characteristic quantum parameters are taken to be constant and dynamical, i.e. as space functions. This was the preliminary set up to start studying the effective algebra with the aim to propose a candidate expression for the effective diffeomorphism constraint presented in chapter 6, where for the sake of simplicity, only the case of the constant quantum parameters will be presented. At last, we will discuss the results obtained for our proposal of the effective diffeomorphism constraint.

Chapter 1

Canonical formulation of General Relativity

Generally covariance are a fundamental technical tool to account for the fact that the choice of coordinates, which is merely a way to conveniently describe a phenomenon, should be of no physical relevance [26]. For gauge theories such as GR, with its underlying symmetry principle of general covariance, the Hamiltonian formulation has proved itself to be particularly useful in encoding important insights. The application of several different Hamiltonian formulations to the theory of GR has been developed in the existing literature, see for instance [27]. Canonical structures play a role for a general analysis of the systems of dynamical equations encountered in this setting, for the issue of observables, for the specific types of equation as they occur in cosmology or the physics of black holes, for a numerical investigation of solutions, and, last but not least, for diverse set of ambiguities forming the basis of quantum gravity (QG). This establishes the basis from which we will be able to study spherical symmetric geometries in the framework of quantum gravity [16].

The first step in a canonical formulation of a theory is the introduction of the conjugate momenta of the field theories. Consequently, the canonical scheme relies on a specific time choice in order to work with the time derivative to obtain the conjugate momenta, which should underly the manifest covariance of the theory [28]. Accordingly, canonical equations of motion are formulated for spatial tensors rather than space-time tensors. Introducing momenta after performing a space-time coordinate transformation would oftentimes result in a different set of canonical variables. The resulting setting does not have a direct action of space-time diffeomorphism on all its configurations, making the covariant feature underlying the theory a non-trivial one. Indeed, even though the space-time symmetry is no longer manifest and not obvious from the canonical equations, it must still be present; for all what has been done is just reformulating the classical theory. The mathematical basis of Hamiltonian methods is encoded in the symplectic and Poisson geometry [14].

A thorough presentation requires the prerequisites that we will sketch in the first section of this chapter regarding constrained Hamiltonian systems and covariant theories, such as GR. In the second section we will apply what we have learned about the canonical treatment to the general theory of relativity. For simplicity, we will work in vacuum GR [29, 28], and derive its Hamiltonian formulation. To this aim we will start by presenting the canonical analysis in metric variables in terms of ADM (Arnowitt, Deser and Misner) [30] variables using the *3+1 decomposition* [31] and move on to the introduction of new variables in the framework of the triad formulation and comment on the Cauchy problem in GR [31, 29, 22, 12, 13].

1.1 General Relativity as a constrained system

To point out the formulation of general relativity as a constrained theory, we can examine the Einstein tensorial equation

$$G_{ab} = 8\pi G T_{ab} \ , \tag{1.1.1}$$

where G_{ab} is the Einstein tensor defined as $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$, T_{ab} is the energy momentum tensor, G is Newton's gravitational constant, R_{ab} is the Ricci tensor and R being the Ricci scalar.
The tensorial equation (1.1.1), once split into components, comprises ten equations describing the fundamental interaction of gravity resulting from the fact that spacetime is being curved in the presence of mass and energy. These equations are of different types. They mainly constitute a system of second order partial differential equations, and thus an initial-value problem. Hence, in order to extract the underlying physics from these equations, one would have to pose the values of fields and their first-order time derivatives. As interesting and straightforward this might seem, there are two components of (1.1.1) that are of first order, namely the time parties G^0_a and G^0_0. As a matter of fact, it turns out the set of these equations contribute as constraints on the initial values of second-order ones. For instance the equation

$$G^0_a = 8\pi G T^0_a \ , \tag{1.1.2}$$

relates initial values of fields instead of determining *how* fields evolve. Another important property, derived from the Bianchi identity $\nabla_a G^a_b = 0$, is that the constraints are preserved in time; their time derivative automatically vanishes if the spatial part of Einstein's equation is satisfied and if the constraints hold at one time. Moreover, while examining explicitly how second order derivative can arise in the components of (1.1.1), another crucial property emerges. To this end, let us study the Ricci tensor, that is given by

$$R_{\mu\nu} = \partial_\nu \Gamma^\nu{}_{\mu\rho} - \partial_\mu \Gamma^\nu{}_{\nu\rho} + \Gamma^\alpha{}_{\mu\rho}\Gamma^\nu{}_{\nu\alpha} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\nu{}_{\mu\alpha} \ , \tag{1.1.3}$$

with $\Gamma^\nu{}_{\mu\rho}$ being the Christoffel symbols that read

$$\Gamma^\rho{}_{\mu\nu} = \frac{g^{\rho\sigma}}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \ . \tag{1.1.4}$$

One can notice that a second order time derivative can appear only from the first two terms, since the Christoffel symbols are first order derivative in the metric. Thus the first two terms in (1.1.3) will contribute to the second order only for time components $\mu = \nu = 0$. Computing the Christoffel symbols in this case and plugging it in the expression for the Ricci tensor gives

$$\begin{aligned} R_{00} &= g^{0\lambda}\partial_0^2 g_{0\lambda} - \frac{1}{2}g^{00}\partial_0^2 g_{00} - \frac{1}{2}g^{\kappa\lambda}\partial_0^2 g_{\kappa\lambda} + \cdots = \frac{1}{2}g^{ab}\partial_0^2 g_{ab} + \cdots \ , \\ R_{0a} &= \frac{1}{2}g^{0b}\partial_0^2 g_{ab} + \cdots \ , \\ R_{ab} &= -\frac{1}{2}g^{00}\partial_0^2 g_{ab} + \cdots \ . \end{aligned} \tag{1.1.5}$$

At this stage, we can spot a crucial property that occurs in cosmological models: only spatial components g_{ab} of the metric appear with their second order time derivatives. Concerning the other components, g_{00} which plays the role of the lapse function that we will discuss later on and g_{0a}, appear only with lower-order time derivative; pointing out the difference in the dynamical role they play when compared to the one g_{ab} does.
Moving on to computing the Ricci scalar and the Einstein tensor's components G_{00}, G_{0a} and G_{ab} and focusing on the components G^0_a and G^0_0, one can conclude that they, indeed, do not contain any second-order time derivative. The corresponding parts of Einstein's equation are hence of lower order in time derivatives than the whole system. This set of equations provides constraints on the initial values while the spatial components dictate the evolution.

Revealing the existence of constraints in (1.1.1), shows not only that initial values are not allowed to be taken arbitrarily, but also hints towards underlying symmetries. Indeed, constraints come in different types, first, second, etc, as we will show in the next section. In particular those that appear as a first class set of constraints, as it is the case in general relativity, generate gauge transformations. Classically, the gauge transformations of GR are equivalent to a change of coordinates and therefore entailing the general covariance under this transformation. In this context of constrained systems, the existence of gauge symmetries gives rise to singular systems and GR is one example of a more special class of generally covariant theories in which the local symmetries are given by coordinate transformations.

However, if we want to implement the Hamiltonian setting to this kind of constrained theories and its respective generated gauge transformations, several subtleties emerge. We know that the canonical setting relies on the time variable that will allow the reconstructed Hamiltonian to dictate the evolution of the system. This certainly requires a deeper understanding of the geometry of general relativity, and it is the origin of several characteristic and hard problems to be addressed in numerical relativity and quantum gravity. In practice, the appearance of constraints is an evidence of the redundancy the formulation of a theory has in terms of fields on a space-time. Despite the fact that what is physically important is the geometry, specific, coordinate-dependent values of fields such as the space-time metric at specific points are used in any field theoretic structure. In fact, there might exist coordinate transformations relating solutions that formally appear distinctive when expressed as fields, but evolve from the same initial values. A deterministic theory, however, cannot have a scenario where different solutions evolve starting from the same initial values. Solutions concerning the same initial values but describing field with different values in a future region must be classified and considered as two distinct representations of the same physical configuration. The number of distinguishable physical solutions is hence smaller than what is naturally expected from the number of initial values required for a set of second-order partial differential equations for a certain number of fields. Therefore, additional restrictions on the initial conditions must exist, which are inherited in constraints. This is the crucial reason why the constraints must enter into play: functionals on the phase space that do not contribute with equations for time derivatives of canonical variables but rather, non-trivial relations between them. They imply conditions to be satisfied by suitable initial values, but also point out how different parametrization of the same physical solution must be selected.

While the invariance under coordinate transformations is well known and present already in the Lagrangian formulation of general relativity, the canonical set up involving constraints has several convenient implications. Maybe the most important one is the ability to study the structure

of space-time in terms of the algebra of constraints undertaking Poisson brackets, without reference to coordinates. This algebraic simplicity is important, for instance, in approaches to quantum gravity, where a continuum manifold with coordinates may not be available but instead can be replaced by new structures. In terms of quantized constraints, one would still be able to study the underlying quantum geometry of space-time. In fact, at the classical level, a canonical formulation has indeed several important features, when it comes to handling and imposing gauge choices which is accessible and more physical to consider in terms of space-time fields rather than coordinates.
Before venturing into working out the Hamiltonian formalism for GR, let us present some mathematical tools that will turns out to be of great use to help us reconstruct the canonical version of GR.

1.2 Constrained Hamiltonian systems

The Hamiltonian formalism is the basis of any canonical treatment with the goal to quantize a theory. In order to incorporate gauge symmetries various steps and consideration should be taken into account. In this section we will discuss the case of constrained Hamiltonian systems in the absence and presence of gauge symmetries with the aim to apply this technology to general relativity.

Hamiltonian systems without gauge symmetry

The starting point of any Hamiltonian treatment for a given theory is the derivation of the Legendre transform and equations of motion. To this aim, one can proceed in two different steps, namely, defining the Hamiltonian system from scratch or in the most common way, by starting with a Lagrangian L and Legendre transform it, since it exhibits manifest invariance. If one considers a time independent Lagrangian

$$L\left(q^1,\dots,q^n,\dot{q}^1,\dots,\dot{q}^n\right) \equiv L\left(q^i,\dot{q}^i\right) \ , \tag{1.2.1}$$

then writing down the action

$$S = \int dt L \ , \tag{1.2.2}$$

in order apply the *least action principle*, namely requiring that the variation $\delta L = 0$ vanishes, then one ends up with the familiar Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i} \quad \Leftrightarrow \quad \ddot{q}^j \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = \frac{\partial L}{\partial q^i} - \dot{q}^j \frac{\partial^2 L}{\partial \dot{q}^j \partial q^i} \ . \tag{1.2.3}$$

Bear in mind for now, that in the case where we do not have to deal with gauge symmetries, the accelerations $\ddot{q}^j$ are uniquely defined which is equivalent to saying that the determinant $\det \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \neq 0$. The canonical momenta read then

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \ . \tag{1.2.4}$$

These are the basic steps into the Hamiltonian treatment of a given theory, that is consequently established on assuming that the variabels q^i and p_i are independent and setting up first order evolution equation for them. To this aim, one can treat functions that have a variation in terms of q^i and p_i only, yielding

$$\delta\left(p_i\dot{q}^i - L\right) = \dot{q}^i\delta p_i + p_i\delta\dot{q}^i - \frac{\partial L}{\partial q^i}\delta q^i - \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i = \dot{q}^i\delta p_i - \frac{\partial L}{\partial q^i}\delta q^i \ , \tag{1.2.5}$$

so that the Hamiltonian is defined as

$$H := p_i\dot{q}^i\left(q^j, p_j\right) - L = H\left(q^i, p_i\right) \ . \tag{1.2.6}$$

This Hamiltonian is uniquely defined which means that we write the $\dot{q}^i$ in terms of q^j, p_j along with the necessary condition $\det\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j} = \det\frac{\partial p_i}{\partial\dot{q}^j} \neq 0$.
Applying again the least action principal, the canonical equations of motion read

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}, \quad \dot{q}^i = \frac{\partial H}{\partial p_i} \ . \tag{1.2.7}$$

The set of variables $q^i, \cdots, q^N$ represents the set of coordinates for the configuration space and similarly the set of all the q^i and p_i coordinatise the phase space which we will denote by Γ. In this formulation, the physical observables are smooth functions of (q^i, p_i) and the set of phase space functions form an algebra. The symplectic structure for two functions yields the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} \ . \tag{1.2.8}$$

Furthermore, the canonical equations of motion can be written

$$\dot{q}^i = \left\{q^i, H\right\}, \quad \dot{p}_i = \{p_i, H\} \ . \tag{1.2.9}$$

The lesson to take from this Hamiltonian formulation in the absence of gauge symmetries is that the physical degrees of freedom of the system that coordinatise phase space are distinct points in this space, where each point corresponds to a distinct physical situation. Moreover the Hamiltonian generates a flow on the phase space that can be viewed as physical evolution. This will not be the case for gauge systems as we will see in the next paragraph, for the main reason that the distinct points in phase space can correspond to the same physical situation and hence the Hamiltonian flow is not uniquely defined for points (physical situations) in the phase space where the necessary condition

$$\det\left(\frac{\partial^2 L}{\partial\dot{q}^i\partial\dot{q}^j}\right) = 0 \ .$$

ceases to be viable .

Hamiltonian systems with gauge symmetry

For our treatment we are interested in Hamiltonian formulation of gauge theories such as general relativity. Let us start by recalling the Lagrangian equations of motion

$$\ddot{q}^j\frac{\partial^2\mathcal{L}}{\partial\dot{q}^i\partial\dot{q}^j} = \frac{\partial\mathcal{L}}{\partial\dot{q}^i} - \dot{q}^j\frac{\partial^2\mathcal{L}}{\partial\dot{q}^i\partial q^j} \ . \tag{1.2.10}$$

We saw that to get access to the evolution of the phase space variables $\ddot{q}$ as a function of $(\dot{q}^i, \dot{q}^j)$ that is unique, is equivalent to requiring the non vanishing of det $\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j}\right) \neq 0$. Since it is not the case for gauge theories, the canonical momenta $p(x) = \frac{\partial \mathcal{L}}{\partial \dot{q}(x)}$ can not be uniquely defined, allowing therefore to vary $\dot{q}^j$ without affecting their canonical pair $p(x)$. The fact that we allow the momenta to not be independent of the velocity, implies that there is a condition relating the momenta which can be expressed as

$$V_m(p(x), q(x)) = 0 \quad m = 0, \cdots, M \ . \tag{1.2.11}$$

This relation is called a *primary constraint* for the dynamical variables in the Hamiltonian formalism. Due to (1.2.11) the Hamiltonian is not uniquely determined since we can always add any linear combination of (1.2.11). Alternatively one can define a notion of total Hamiltonian, for which our theory cannot distinguish between the latter and the original one H, as

$$H_T(p(x),q(r)) = H(p(x), q(x)) + c_m V_m(p(x), q(x)) \ , \tag{1.2.12}$$

where the coefficients c_m can be arbitrary functions of the phase space variables. These primary constraints affect the Poisson structure of the theory as well. In fact, we obtain an extended Poisson bracket to the c_m that are in some way consistent with the symmetries of the bracket.

$$\dot{f} = \{f, H + c_m V_m\} = \{f, H\} + c_m \{f, V_m\} + \{f, c_m\} V_m = \{f, H\} + c_m \{f, V_m\} \ . \tag{1.2.13}$$

To fix the notation, we follow Dirac's notation for this treatment. A weak equality is denoted by $\approx$ and means equality modulo constraints. It is important to stress out that it shall be used only after all Poisson brackets have been evaluated. The consistency of $V_m \approx 0$ with the evolution equations implies

$$\dot{V}_m = \{V_m, H_T\} = \{V_m, H + c_m V_n\} \approx \{V_m, H\} + c_n \{V_m, V_n\} \approx 0 \ . \tag{1.2.14}$$

which further implies to consider four physical cases where the above equation must hold, namely, when (1.2.14) is trivially satisfied, e.g. $0 = 0$, or we are dealing with an inconsistent theory, e.g. $1 = 0$, or we could put further condition on the c_m and therefore considering new constraint $\chi_k(q,p) = 0$, independent of the c_m. The relevant and more interesting case turns out to be the last one. We call $\chi_k(q,p) = 0$ *secondary constraints*. For secondary constraints, one uses the equations of motion, as opposed to primary constraints. These constraints will reiterate the consistency algorithm generating therefore tertiary constraints. This iterative process will either come to an end at a certain point or not, and in the latter case the theory is inconsistent. Hence we end up with K new constraints and the set of all constraints is thus denoted by $\{V_1, \cdots, V_{M+K}\} := \{V_1, \cdots, V_M, \chi_1, \cdots, \chi_K\}$, where we define $V_j, j = 1, \cdots, J = M + K$..
Going back to include gauge transformations, we call a phase space function of first class if it has vanishing Poisson bracket with all constraints, i.e.

$$\{f, V_m\} \approx 0 \ . \tag{1.2.15}$$

Otherwise, it is called second class. Moreover, another property was proved in Dirac's conjecture, stating that all first class constraints generate gauge transformations. The dynamic of the theory are encoded in an "extended" Hamiltonian H_E defined as being H_T plus an arbitrary combination of first class constraints.
The generalization to field theories is straightforward once we go over to an infinite number of degrees of freedom, namely

- $q^n, n = 1,2,\ldots$ becomes $q(x), x \in \mathbb{R}^3$.

- $\sum_n$ becomes $\int d^3x$.
- $\frac{\partial L}{\partial \dot{q}^n} = p_n$ becomes $\frac{\delta L}{\delta \dot{q}(x)} = p(x)$ where $L = \int d^3x \mathcal{L}(x)$ and $p(x)$ are defined as $\delta_{\dot{q}} L = \int d^3x p(x) \delta \dot{q}(x)$.

For example the relation that holds for independent variabels of the phase space $\frac{\partial q^i}{\partial q^j} = \delta^i_j$ becomes a functional derivative for field theories denoted by

$$\frac{\delta \varphi(x)}{\delta \varphi(y)} = \delta(x - y) \ . \tag{1.2.16}$$

where the familiar derivative for discrete quantities is changed with the variational symbol δ and $\delta(x - y)$ is the Dirac delta-distribution. Furthermore, to avoid subtleties that arise when using distributions and instead work with well-defined algebraic relationships, one can smear fields or functionals of them. In the same manner one can pose a well defined Poisson bracket, that yields the algebraic relation

$$\{f, g\} = \int \mathrm{d}^n x \left(\frac{\delta f}{\delta \varphi(x)} \frac{\delta g}{\delta \pi_\varphi(x)} - \frac{\delta f}{\delta \pi_\varphi(x)} \frac{\delta g}{\delta \varphi(x)} \right) \ , \tag{1.2.17}$$

for a scalar field and its momentum π_φ.

The constraint surface geometry

It is important to point out the clear splitting of the constraints into first and second class. The first class constraints are related to the gauge transformations of the fields, whereas the second class pairs come hand in hand with the transformation generators and gauge fixing. Admittedly, solving the constraints is often times complicated and it might be that one is not able to explicitly solve all them. In this case one may work with the constrained system implicitly by using the constraints without solving them. More precisely, if one chooses to work with the dynamical flow of phase-space functions generated by the Hamiltonian, it is then important to verify that they stay contained in the constraint surface. This would be guaranteed in the case where all constraints are solved explicitly but not if some of them cannot be solved. For first class constraints, the Hamiltonian flow is tangent to the constraint surface and they do not give rise to subtleties [1]. However, what will be more important is dealing with the second-class constraints generating a flow transversal to the constraint surface. In this case, if one does not solve the second-class constraints, they may contribute to the Hamiltonian making the flow move off the constraint surface. Fortunately, instead of solving all second-class constraints, one may modify the Poisson brackets in a way to guarantee that the Hamiltonian flow generated by the old constraints with respect to the new Poisson structure is tangent to the constraint surface. This is possible by introducing the Dirac bracket,

$$\{f,g\}_* = \{f,g\} - \{f,\chi_\alpha\} C^{\alpha\beta} \{\chi_\beta,g\} \ , \tag{1.2.18}$$

where the $*$ denotes the Poisson bracket after solving the constraint, χ_i are the second class constraints and $C^{\alpha\beta} C_{\alpha\beta} = \delta^\alpha_\beta$ is defined as $\det C_{\alpha\beta} \neq 0$ everywhere on $\chi_\alpha = 0$ with C_{ij} being the matrix defined in terms of the constraints, namely $C_{ij} = \{V_i, V_j\}$ such that

$$C_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & C_{\alpha\beta} \end{pmatrix} \tag{1.2.19}$$

[1] See appendix for symplectic manifolds and Hamiltonian flow.

that can be interpreted as a subset of equal number of the original constraints to be handled as gauge conditions for the set $\{\chi_k\}$. The Dirac bracket is unaffected by choosing a different set of second class constraints on the constraint surface. Indeed, with this new Poisson bracket, the flow generated by the second-class constraints stays on the constraint surface, leaving the Poisson structure for functions generating a flow tangent to the constraint surface unchanged, while removing any flow off the constraint surface generated by functions of the second-class constraints. Even if not all second-class constraints can be solved, the Dirac bracket, which is often easier to compute, makes sure that they do not lead to spurious flows off the constraint surface. However, while proceeding with the quantization scheme, the first class constraints preserve all properties of the Poisson bracket which is particularly important, because those properties are translated by commutators upon quantization. Whereas the second class constraints reflects inconsistencies at the quantum level. Indeed, their action do not preserve the constraint surface. This is simply because they do not Poisson-commute with some of the constraints and practically one can choose the constraints as local coordinates of the constraint surface. However, in this case they can not be treated as gauge generators. One way to go around this, is to work with the above Dirac bracket.

1.2.1 The gauge unfixing procedure

The usual treatment of the GU procedure is to basically transform a second class constraint into first class ones. In our case, we will proceed with a slightly different method. We will consider some first class constraints, turn them into second class ones by imposing some set of gauge fixing conditions for some of the phase space coordinates. The new set of first class constraints we obtain after preforming GU represents a system where the original constraints were traded with the initial gauge fixing conditions we have imposed. To be concrete, let us outline briefly the program that will be applied in order to get the extended phase space. We have the set of first class constraint fulfilling the relation

$$\{V_i(\vec{x}),\, V_j(\vec{y})\} = 0 \ , \tag{1.2.20}$$

and the imposed gauge conditions denoted by

$$\chi_a \approx 0 \ , \tag{1.2.21}$$

where it is more convenient to adapt the indices ($K \to a$) according to the phase space variables for a field theory, namely Q_a and P^a. They denote the configuration and momentum fields of our field theory and obey the symplectic structure

$$\{P^a(\vec{x}), Q_b(\vec{y})\} = \gamma \delta^a_b \delta(\mathrm{X} - \vec{y}) \ , \tag{1.2.22}$$

where now $a, b, c, \cdots$, stand both for internal and tangential indices and γ is a constant depending on the theory one is working on. The set of condition (1.2.21) implies that a subset of the configuration fields $\{Q_a\}$ vanishes and hence we are left with an enlarged set of constraints, namely, $\{V_i,\, \chi_a\}$ that in return are second class. Applying the GU procedure to this system boils down to writing the $\{\chi_a\}$ as first class constraints according to the condition (1.2.15), while treating an equal number of the original constraints of $\{V_i\}$, denoted by $\{C_i\}$, as gauge conditions for the $\{\chi_a\}$. To this aim, we turn to computing the gauge invariant extensions of the corresponding momenta $\{P^a_\chi\}$. Let us write the extended momenta $\tilde{P}^a_\chi$ as

$$\tilde{P}^a_\chi(\vec{x}) = P^a_\chi(\vec{x}) + \int d\vec{y} C_i(\vec{y}) \mathrm{N}^{ia}(\vec{y},\vec{x}) + \dots \ , \tag{1.2.23}$$

where the dots stand for higher powers of the C_i's. In N^{ia} is a distributional matrix and, together with its higher power counterparts, it must be fixed by requiring the gauge invariance of $\tilde{P}^a_\chi$, namely $\{\chi_a(\vec{x}),\tilde{P}^b_\chi(\vec{y})\} \approx 0$.
Combining the above conditions and employing the expression for the $\tilde{P}^a_\chi$

$$0 \approx -\gamma\delta^a_b\delta(\vec{x}-\vec{z}) + \int d\vec{y}\{\chi_b(\vec{z}),\, C_i(\vec{y})\}\mathrm{N}^{ia}(\vec{y},\vec{x}) \ .$$

from which we see that N^{ia} is the inverse of the matrix

$$\mathrm{A}_{ai} = \gamma^{-1}\{\chi_a(\vec{z}),\, C_i(\vec{y})\} \ . \tag{1.2.24}$$

The procedure of the GU thus reduces to finding the inverse matrix $(\mathrm{A}^{-1})^{ia}$ and replacing $\mathrm{N}^{ia} = (\mathrm{A}^{-1})^{ia}$ inside the extended momenta (1.2.23). Finally, promoting P^a_χ to $\tilde{P}^a_\chi$, we end up with a theory invariant under the gauge conditions while being able to work only with the physical degrees of freedom and the eventual gauge residual ones. After computing the extended momenta $\tilde{A}^3_A$ $\tilde{A}^I_r$, one can derive the extended version for the constraints (3.1.2) by promoting the momenta to their extended version, restricting the results to the gauge surface.

1.3 The geometry of hypersurfaces and foliations

In this section we will provide the necessary definitions and geometrical concepts to rewrite GR in the 3+1 formalism. The 3+1 formalism is mainly an approach that relies on slicing the four dimensional space time by three dimensional hypersurfaces, that in turn, have to be spacelike in order to recover the Lorentzian signature of the induced metric. From the mathematical point of view and as we will show later on, this procedure enables us to formulate the problem of solving Einstein's equations as a Cauchy problem with constraints and build its Hamiltonian counterpart. This formalism is based on the notion of hypersurfaces and foliations. We will briefly explore these two blocks of the theory and provide with their help the 3+1 decomposition of Einstein equation.

In this section, we will adapt the following notation, a space-time is labeled by $(M,\boldsymbol{g})$, where M is a smooth manifold of dimension 4 and $\boldsymbol{g}$ is a Lorentzian metric of signature $(-,+,+,+)$. We denote by $\boldsymbol{\nabla}$ the affine connection associated to the metric $\boldsymbol{g}$ and name it space-time connection. $T_p(M)$ stands for the tangent space of a given point $p \in M$ while its dual, denoted by $T^*_p(M)$, is build up by all linear forms at p. As what concerns the indices, we adapt the following convention usually used in the literature: all Greek indices run in $\{0,\ 1,\ 2,\ 3\}$, we will use letters in the beginning of the alphabet $(\alpha,\beta,\gamma,\dots)$ for free indices and letters starting from $(\mu(\mu,\nu,\rho,\dots)$ as dumb indices for contraction. Lower case Latin indices starting from $i(i,j,k,\dots)$ are in $\{1,2,3\}$, whereas those starting form the beginning of the alphabet $(a,\ b,\ c,\cdots)$ run in $\{2,3\}$ only.

1.3.1 The geometry of Hypersurfaces

Hypersurface

A hypersurface Σ of M is the image of a 3-dimensional manifold $\hat{\Sigma}$ by a one-to-one embedding (homeomorphism), guaranteeing the non-intersection of Σ with itself, namely the map $\Phi : \hat{\Sigma} \to M$. This embedding induces two important notions for the subsequent analysis, the push-forward and the pull-back operations, Φ_* and Φ^* respectively.

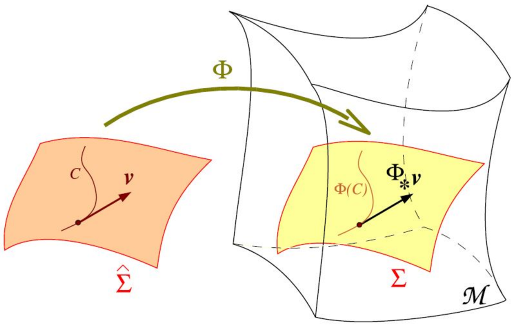


Figure 1.1: Embedding of the 3-dimensional manifold Φ into the 4-dimensional manifold M, defining the hypersurface $\Sigma = \Phi(\hat{\Sigma})$. The push-forward $\Phi_* v$ of a vector v tangent to some curve C in Σ is a vector tangent to $\Phi(C)$ in M.

First fundamental form:

A very important case of pull-back operation is the one acting on the spacetime metric $\boldsymbol{g}$, which generates the induced metric on Σ, namely

$$\boldsymbol{\gamma} := \boldsymbol{\Phi}^* \boldsymbol{g} \ . \tag{1.3.1}$$

γ is the 3-metric, called also *first fundamental form* of Σ. The pull-back and push-forward mappings allow us to define other geometric objects such as the intrinsic and extrinsic curvatures that we will shortly introduce.

Normal vector

Let us define the hypersurface Σ as level surface of scalar field $t \in M$. Then the gradient 1-form $\boldsymbol{dt}$ is normal to Σ such that the well defined scalar product of any vector $\boldsymbol{v} \in T_p(\Sigma)$ and $\boldsymbol{dt}$ vanishes identically. The metric dual to this 1-form gradient is the vector $\vec{\boldsymbol{\nabla}} t$[2]. Once this vector is normalized, the obtained unit vector, defining the *normal vector*, can be expressed as

$$n := \left(\pm \vec{\boldsymbol{\nabla}} t \cdot \vec{\boldsymbol{\nabla}} t\right)^{-1/2} \vec{\boldsymbol{\nabla}} t \ , \tag{1.3.2}$$

where the + sign stands for a time-like hypersurface and the - sign for a spacelike one.

Intrinsic curvature

In the following we restrict our attention to space-like and time-like hypersurfaces. We consider Σ, where in this case the induced metric $\boldsymbol{\gamma}$ is positive definite or Lorentzian. The 3-metric $\boldsymbol{\gamma}$ is not degenerate, hence indicating that there is a unique connection (or covariant derivative) $\boldsymbol{D}$ on the manifold Σ that is torsion-free and fulfills the condition $\boldsymbol{D}\boldsymbol{\gamma} = 0$. $\boldsymbol{D}$ here is the so-called Levi-Civita associated with the metric γ. The Riemann tensor associated with this connection represents the *intrinsic curvature* of (Σ,$\boldsymbol{\gamma}$) . We shall denote it by **Riem**, and its components

[2]The component of $\vec{\boldsymbol{\nabla}} t$ are $\nabla^\alpha t = g^{\alpha\mu} \nabla_\mu t = g^{\alpha\mu} (\boldsymbol{dt})_\mu$

by the letter R, as R^k_{lij}. [3] The corresponding Ricci tensor is denoted $\mathbf{R}$ (such that $R_{ij} = R^k_{ikj}$) and the Ricci scalar (scalar curvature) is expressed as R such that $R = \gamma^{ij} R_{ij}$. The scalar curvature is also called the Gaussian curvature of (Σ, γ). Notice that in three dimensions, the Riemann tensor can be fully determined from the knowledge of the Ricci tensor, according to the relation

$$R^i_{jkl} = \delta^i_k R_{jl} - \delta^i_l R_{jk} + \gamma_{jl} R^i_k - \gamma_{jk} R^i_l + \frac{1}{2} R(\delta^i_l \gamma_{jk} - \delta^i_k \gamma_{jl}) \tag{1.3.4}$$

Second fundamental form

Along with the intrinsic curvature defined above, we can consider an additional type of curvature characterizing hypersurfaces, namely the one that measures the "bending" of Σ in M quantifying, therefore, the change in direction of the normal vector $\boldsymbol{n}$ as one moves on the hypersurface Σ. To be more concrete, one can define the Weingarten map (also called the "shape" operator) as the endomorphism of $T_p(\Sigma)$ which identifies each vector tangent to Σ with the variation of the normal along that vector. This variation is measured by the spacetime connection $\boldsymbol{\nabla}$ and is endowed with the map

$$\begin{aligned} \boldsymbol{\chi} :& T_p(\Sigma) \to T_p(\Sigma) \\ & \boldsymbol{v} \to \boldsymbol{\nabla}_{\boldsymbol{v}} \boldsymbol{n} \end{aligned} \tag{1.3.5}$$

One can also show that the Weinberg map $\boldsymbol{\chi}$ is self-adjoint and therefore its eigenvalues are all real numbers. They are called the principal curvatures of the hypersurface Σ and the corresponding eigenvectors define the principal directions of Σ. The property of being self-adjoint of the Weingarten map implies that the bilinear form defined on Σ's tangent space by

$$\begin{aligned} \boldsymbol{K} : \quad & T_p(\Sigma) \times T_p(\Sigma) \to \mathbb{R} \\ & (\boldsymbol{u},\boldsymbol{v}) \to -\boldsymbol{u}.\boldsymbol{\chi}(\boldsymbol{v}) \ . \end{aligned} \tag{1.3.6}$$

is actually symmetric.

This is denoted by the *second fundamental form* of the hypersurface Σ. In the following, we will also call it the extrinsic curvature tensor of Σ. Given the expression of $\boldsymbol{\chi}$ in (1.3.5), the extrinsic curvature takes the form $\boldsymbol{K} = -\boldsymbol{u}\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{n}$. Its respective trace is denoted by $K := \gamma^{ij} K_{ij}$. It is important to make the distinction between the role both the Gaussian and extrinsic curvature play respectively. The extrinsic curvature reflects the properties of the Weinberg map and all its implications, including the role the principal and mean curvature play in the embedding, contrary to the intrinsic one that has no affect on how the embedding is proceeded.

1.3.2 Spacelike hypersurface and geometrical relations

Now that we have exposed some of the basic geometrical notions needed for the 3+1 treatment, we shift our focus on the geometry on spacelike hypersurfaces and the fundamental relations

[3] **Riem** can be thought as a geometrical device that measures the noncommutativity of two successive covariant derivatives $\boldsymbol{D}$, as expressed by the Ricci identity but now in three dimensions:

$$\forall v \in T(\Sigma) \ , \ (D_i D_j - D_j D_i) v^k = R^k_{lij} v^l \ . \tag{1.3.3}$$

underlying the decomposition of all the geometrical quantities we are familiar with from GR, i.e. Ricci scalar and tensor as well as the metric.

Orthogonal projector

The space of all spacetime vectors can be orthogonally decomposed in terms of the 1-dimensional subspace generated by the normal vector $\boldsymbol{n}$, in the following way

$$T_p(M) = T_p(\Sigma) \oplus \mathrm{Vect}(\boldsymbol{n}) \ . \tag{1.3.7}$$

One can associate an orthogonal projector onto Σ linked to the above decomposition and defined as

$$\begin{aligned} \vec{\gamma}: \quad & T_p(M) \to T_p(\Sigma) \\ & \boldsymbol{v} \mapsto \boldsymbol{v} + (\boldsymbol{n}.\boldsymbol{v})\boldsymbol{n} \ . \end{aligned} \tag{1.3.8}$$

In particular the components of the projection operator $\vec{\gamma}$ with respect to a basis (e_α) of $T_p(M)$ yield the equation

$$\gamma^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta \ . \tag{1.3.9}$$

The concept of the orthogonal projector naturally allows us to identify a reverse mapping between the tangent space of the point p on Σ and its respective tangent space on the manifold M, contrarily to the push-forward $(T_p(\Sigma) \to T_p(M))$ and pull-back operations $(T^*_p(M) \to \mathcal{T}^*_p(\Sigma))$ that are one directional mapping. This mapping acting on the bilinear form in Σ, i.e $\gamma : \vec{\gamma}^*_{\mathcal{M}}\boldsymbol{\gamma}$ coincide with $\boldsymbol{\gamma}$ and allows to build an extension to all vectors on $T^*_p(M)$. In what follows, this extended variable will be denoted by the same symbol for $\boldsymbol{\gamma}$. We can write this extended object in terms of the linear form $\boldsymbol{n}$ dual to the normal vector $\boldsymbol{n}$ as

$$\gamma = \mathbf{g} + \boldsymbol{n} \oplus \boldsymbol{n} \ . \tag{1.3.10}$$

Gauss-Codazzi relations

A priori, all the geometrical objects we already defined in the previous section were defined either on the manifolds M or Σ. The role of this new reverse map for the extended induced map can be expanded to act on all quantities we need to measure the change in the geometrical structure we are studying. Importantly, the relations underlying the extrinsic curvature with the tensor field $\boldsymbol{\nabla}_{\boldsymbol{n}}$, $\boldsymbol{D}$.

For the purpose of this work, we will skip all the thorough computations to derive these mathematical relations and present directly the action of the orthogonal projection operator on the curvatures and their interrelations embodied in the Gauss-Codazzi equations. These relations constitute the basis of the 3+1 formalism for general relativity. They are decompositions of the spacetime Riemann tensor, ${}^4\mathbf{Riem}$ in terms of quantities relative to the spacelike hypersurface Σ, namely the Riemann tensor associated with the induced metric $\boldsymbol{\gamma}$, $\mathbf{Riem}$ and the extrinsic curvature tensor $\boldsymbol{K}$. These relations yield the equations:

$$\gamma^\mu_\alpha \gamma^\nu_\beta \gamma^\gamma_\rho {}^4R^\rho_{\sigma\mu\nu} = R^\gamma_{\delta\alpha\beta} + K^\gamma_\alpha K_{\delta\beta} - K^\gamma_\beta K_{\alpha\delta} \qquad \text{Gauss relation} \ , \tag{1.3.11}$$

$${}^4R + 2\,{}^4R_{\mu\nu}n^\mu n^\nu = R + K^2 - K_{ij}K^{ij} \qquad \text{Scalar Gauss relation} \ . \tag{1.3.12}$$

The last equation constitutes a generalization of Gauss's famous Theorema Egregium. It relates the intrinsic curvature of Σ, represented by the Ricci scalar R, to its extrinsic curvature, written as $K^2 - K_{ij}K^{ij}$. The Codazzi relation reads

$$\gamma^{\gamma}_{\rho} n^{\sigma} \gamma^{\mu}_{\alpha} \gamma^{\nu}_{\beta} {}^4R^{\rho}{}_{\sigma\mu\nu} = D_{\beta} K^{\gamma}{}_{\alpha} - D_{\alpha} K^{\gamma}{}_{\beta} \ . \tag{1.3.13}$$

The contracted Codazzi relation reads

$$\gamma^{\mu}_{\alpha} n^{\nu} {}^4R_{\mu\nu} = D_{\alpha} K - D_{\mu} K^{\mu}{}_{\alpha} \ . \tag{1.3.14}$$

1.3.3 Globally hyperbolic space-times and foliation kinematics

Up to now, we considered the mathematical set up to describe geometrical operations on a single spacelike hypersurface. We want to be able to formulate a dynamical theory, such as GR, that allows us to introduce the notion of evolution into play. To this end, we consider a continuous set of hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$ that cover the manifold M. This is possible for a wide class of space-times, particularly, globally hyperbolic spacetimes. Actually the latter ones cover most of the space-times of astrophysical or cosmological interest.

Cauchy surface

A Cauchy surface is a spacelike hypersurface Σ in M such that each causal curve without end point intersects Σ once and only once. Equivalently, Σ is a Cauchy surface if and only if its domain of dependence is the whole spacetime M. Note that not all spacetimes admit a Cauchy surface. For instance spacetimes with closed time-like curves do not. A spacetime $(M, \boldsymbol{g})$ that admits a Cauchy surface is said to be *globally hyperbolic*. The latter admits a decomposition $\Sigma \times \mathbb{R}$.

Any globally hyperbolic spacetime $(M, \boldsymbol{g})$ can be foliated by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$. Foliation or slicing, underlines the existence of a smooth scalar field $\hat{t}$ on M, which is regular (in the sense that its gradient never vanishes), such that each hypersurface is considered as a level surface of this scalar field:

$$\forall t \in \mathbb{R},\ \Sigma_t := \left\{ p \in M, \hat{t}(p) = t \right\} \ . \tag{1.3.15}$$

Each hypersurface Σ_t is called a leaf or a slice of the foliation. We assume that all Σ_t's are spacelike and that the foliation covers M.

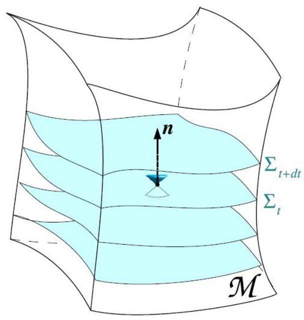


Figure 1.2: Spacetime M foliated by a family of spacelike hypersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$.

Lapse function and normal evolution vector

So far all we did is to present the basic mathematical notions and relation in the 3+1 decomposition formalism, but we still need to specify the kinematical aspect and how these geometrical quantities evolve. To this aim, we start by introducing the normal evolution vector defined as the time-like vector normal to Σ_t such that

$$\boldsymbol{m} := N\boldsymbol{n} \ . \tag{1.3.16}$$

where N is defined as the *lapse function*, that relates the normal vector $\boldsymbol{n}$ to the 1-form $\boldsymbol{d}t$

$$\boldsymbol{n} := -N\vec{\boldsymbol{\nabla}}t \tag{1.3.17}$$

$$N := (-\vec{\boldsymbol{\nabla}}t \cdot \vec{\boldsymbol{\nabla}}t)^{-1/2} = (-\langle \boldsymbol{d}t, \vec{\boldsymbol{\nabla}}t\rangle)^{-1/2} \ . \tag{1.3.18}$$

Since $\boldsymbol{n}$ is a unit vector, the scalar square of $\boldsymbol{m}$ is simply $\boldsymbol{m} \cdot \boldsymbol{m} = -N^2$. Moreover

$$\begin{aligned} \langle \boldsymbol{d}t, m\rangle &= N \langle \boldsymbol{d}t, \boldsymbol{n}\rangle = N^2 \underbrace{\left(- \langle \boldsymbol{d}t, \vec{\nabla}_t\rangle\right)}_{N^{-2}} = 1 \ , \\ &= \boldsymbol{\nabla}_{\boldsymbol{m}} t = m^\mu \nabla_\mu t = 1 \ , \end{aligned} \tag{1.3.19}$$

highlights the fact that normal vector $\boldsymbol{m}$ is most "suitable" to the scalar field t, contrarily to the normal vector $\boldsymbol{n}$. Geometrically, this fact can be pictured as the case where the hypersurface $\Sigma_{t+\delta t}$ can be obtained from the neighboring hypersurface Σ_t by the small displacement $\delta t\boldsymbol{m}$ of each point of Σ_t. Indeed let us consider some point p in Σ_t and move it by the infinitesimal vector $\delta t\boldsymbol{m}$ to the point $p' = p + \delta t\boldsymbol{m}$. From the very definition of the gradient 1-form $\boldsymbol{d}$t, the scalar field t at p' takes the value

$$t(p') = t(p + \delta t\boldsymbol{m}) = t(p) + \langle t, \delta t\boldsymbol{m}\rangle = t(p) + \delta t \underbrace{\langle \boldsymbol{d}t, \boldsymbol{m}\rangle}_{=1} = t(p) + \delta t \ . \tag{1.3.20}$$

This last equation proves that $p' \in \Sigma_{t+\delta t}$. Hence the vector $\delta t\boldsymbol{m}$ carries the hypersurface Σ_t into the neighboring one $\Sigma_{t+\delta t}$. One equivalently says that the hypersurfaces (Σ_t) are *Lie dragged* by the vector $\boldsymbol{m}$. This justifies the name normal evolution vector given to $\boldsymbol{m}$.
An immediate consequence of the Lie dragging of the hypersurfaces Σ_t by the vector $\boldsymbol{m}$ is that the Lie derivative along $\boldsymbol{m}$ of any vector tangent to Σ_t is also a vector tangent to Σ_t:

$$\forall \boldsymbol{v} \in T(\Sigma_t) \ , \ \boldsymbol{\mathcal{L}}_{\boldsymbol{m}}\boldsymbol{v} \in T(\Sigma_t) \tag{1.3.21}$$

This is obvious from the geometric definition of the Lie derivative. The reader who is not familiar with the concept of Lie derivative may consult Appendix A. This dragging induced by the Lie derivative can be applied to derive the evolution of the 3-metric and the orthogonal projector.

Evolution of the 3-metric and orthogonal projector

To be more concrete, the evolution of the induced metric $\boldsymbol{\gamma}$ of the hypersurface Σ_t is obtained by computing the Lie derivative along the evolution vector $\boldsymbol{m}$ and yields the equation

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{m}}\boldsymbol{\gamma} = -2N\boldsymbol{K} \ . \tag{1.3.22}$$

As a consequence, one can express the extrinsic curvature as $\boldsymbol{K} = -\frac{1}{2}\boldsymbol{\mathcal{L}_n}\gamma$. More interestingly, the evolution of the orthogonal projector reads

$$\boldsymbol{\mathcal{L}_m}\vec{\gamma} = 0 \ , \tag{1.3.23}$$

which implies that the Lie derivative of any tensor field $\mathbf{T}$ along $\boldsymbol{m}$ tangent to Σ_t is also a tensor field tangent to it.

Last step in the projection of the spacetime Riemann tensor

The Gauss-Codazzi equations represent the projection of the spacetime Riemann tensor that includes only fields tangent to Σ_t and their derivatives in the direction parallel to it ($\boldsymbol{\gamma}, \boldsymbol{K},$ Riem and $\boldsymbol{DK}$). However, it is exactly due to this type of decomposition that they are only defined on a single hypersurface. One way around this is to project the Riemann tensor twice onto Σ_t and twice along the normal which will result in a derivative in the direction *normal* to the hypersurface. The resulting equation depends on the spacetime Ricci tensor instead and reads

$$\gamma^{\mu}_{\alpha}\gamma^{\nu}_{\beta}{}^{4}R_{\mu\nu} = -\frac{1}{N}\mathcal{L}_{\boldsymbol{m}}K_{\alpha\beta} - \frac{1}{N}D_{\alpha}D_{\beta}N + R_{\alpha\beta} + KK_{\alpha\beta} - 2K_{\alpha\mu}K^{\mu}{}_{\beta} \ . \tag{1.3.24}$$

The scalar curvature reads ${}^{4}R$

$$R = R + K^2 + K_{ij}K^{ij} - \frac{2}{N}\mathcal{L}_m K - \frac{2}{N}D_iD^iN \ . \tag{1.3.25}$$

1.4 3+1 Einstein equations

Using (1.3.25) and (1.1.1), we can finally write Einstein's equation in the form 3+1 splitting, namely by projecting it onto the hypersurface Σ_t and along its normal. This amounts to applying the projector operator to (1.1.1). There are, however, three different possible projections to preform: a full projection onto Σ_t, a full projection perpendicular to Σ_t and a mixed projection (once onto Σ_t and once along $\mathbf{n}$). The resulting equations of these three operations build up a system of constraints that reads

$$\begin{aligned} &R + K^2 - K_{ij}K^{ij} = 0 && \text{Hamiltonian constraint} \\ &D_j k^j{}_i - D_i K = 0 && \text{Momentum constraint} \\ &\mathcal{L}_{\boldsymbol{m}} K_{ij} = -D_iD_jN + N\left\{R_{ij} + KK_{ij} - 2K_{ik}K^k_j\right\} && \end{aligned} \tag{1.4.1}$$

In this decomposed representation, the Einstein equations are summarized in the system of constraints (1.4.1).

Foliation adapted-coordinates

The system of equations we obtained in (1.4.1) is of tensorial type and in order to be able to manipulate it we need to examine its respective components and write them as a system of differential equations. To this aim, one should work in a adapted coordinate system. Therefore on each hypersurface Σ_t we define the spacial coordinate $(x^i) = (x^1, x^2, x^3)$ and we choose the set of vectors

$$\boldsymbol{\partial}_t := \frac{\partial}{\partial t} \tag{1.4.2}$$

$$\partial_i := \frac{\partial}{\partial x^i} \ , \quad i \in \{1,2,3\} \ . \tag{1.4.3}$$

as a natural basis $(\boldsymbol{\partial_\alpha})$=$(\boldsymbol{\partial}_t, \boldsymbol{\partial}_i)$ of $T_p(M)$ associated with the chosen coordinates $(x^\alpha) = (t, x^1, x^2, x^3)$. The time vector $\boldsymbol{\partial}_t$ has the same property as the vector $\boldsymbol{m}$ in the sense that, it

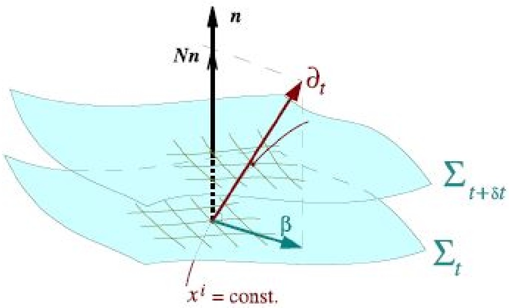


Figure 1.3: The coordinate set(x^i) on Σ_t: each constant line of the spacial coordinate x^i crosses the foliation and gives rise to the natural basis $\boldsymbol{\partial_\alpha}$.

Lie drags the hypersurface. However they differ by the *the shift vector* defined as:

$$\boldsymbol{\partial}_t := \boldsymbol{m} + \boldsymbol{\beta} \ . \tag{1.4.4}$$

It is useful to rewrite it as

$$\boldsymbol{\partial}_t := N\boldsymbol{n} + \boldsymbol{\beta} \ , \tag{1.4.5}$$

$$\boldsymbol{\partial}_t.\boldsymbol{\partial}_t = -N^2 + \boldsymbol{\beta}.\boldsymbol{\beta} \ . \tag{1.4.6}$$

In this set of coordinate and basis, the metric can be computed and it takes the form

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -N^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix} \tag{1.4.7}$$

The line element is then

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right) \tag{1.4.8}$$

The components of the inverse metric are also given by

$$g^{\alpha\beta} = \begin{pmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{pmatrix} = \begin{pmatrix} -\frac{1}{N^2} & \frac{\beta^j}{N^2} \\ \frac{\beta^2}{N^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{N^2} \end{pmatrix} \tag{1.4.9}$$

The set of relations between the metric $\boldsymbol{g}$ and the induced one $\boldsymbol{\gamma}$ yields

$$\begin{aligned} g &:= \det\left(g_{\alpha\beta}\right) \ , \\ \gamma &:= \det\left(\gamma_{\alpha\beta}\right) \ , \\ \sqrt{-g} &= N\sqrt{\gamma} \ . \end{aligned} \tag{1.4.10}$$

Finally, Einstein's equations expressed as a system of partial differential equations in these adapted coordinate basis can be written in this basis as

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)\gamma_{ij} = -2NK_{ij} \ , \tag{1.4.11}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta\right)K_{ij} = -D_iD_jN + N\left\{R_{ij} + KK_{ij} - 2K_{ik}K^k_j\right\} \ , \tag{1.4.12}$$

$$R + K^2 - K_{ij}K^{ij} = 0 \ , \tag{1.4.13}$$

$$D_jK^j_i - D_iK = 0 \ . \tag{1.4.14}$$

1.5 ADM Hamiltonian formulation of GR

In order to derive the Hamiltonian of the Einstein Hilbert action, we start by rewriting it in the 3+1 form that provides us with a natural set up to introduce the notion of time evolution and hence specify the role of time derivative needed to obtain the conjugate variables. Let us start by the standard GR action

$$S = \int_{\mathcal{V}} {}^4R\sqrt{-g}d^4x \ , \tag{1.5.1}$$

where $\mathcal{V}$ is a part of the spacetime manifold M delimited by two hypersurfaces Σ_{t_1} and Σ_{t_2} of the foliation $(\Sigma_t)_{t\in\mathbb{R}}$. We already encountered the decomposition of the Ricci scalar in (1.3.11) and thanks to the set of relations (1.4.10) we can write

$$S = \int_{\mathcal{V}} \left[N\left(R + K^2 + K_{ij}K^{ij}\right) - 2\mathcal{L}_mK - 2D_iD^iN\right]\sqrt{\gamma}d^4x \ . \tag{1.5.2}$$

We also can make use of the expression of the extrinsic curvature in terms of $\boldsymbol{\mathcal{L}_m}$ that reads

$$\boldsymbol{\mathcal{L}_m}K = m^\mu\nabla_\mu K = Nn^\mu\nabla_\mu K = N\left[\nabla_\mu\left(Kn^\mu\right) - K\underbrace{\nabla_\mu n^\mu}_{=-K}\right] \tag{1.5.3}$$

$$= N\left[\nabla_\mu\left(Kn^\mu\right) + K^2\right] \ , \tag{1.5.4}$$

plugging it in (1.5.2) and discarding divergence terms we finally obtain

$$S = \int_{t_1}^{t_2}\left\{\int_{\Sigma_t} N\left(R + K_{ij}K^{ij} - K^2\right)\sqrt{\gamma}d^3x\right\}dt \ . \tag{1.5.5}$$

This is the *3+1 form* of the Hilbert action. This action is a functional of the configurations variables $(\gamma_{ij}, N, \beta^i)$ and their time derivatives. Expressing the extrinsic curvature thanks to the system of equations (1.4.1), we can write K_{ij} as

$$K_{ij} = \frac{1}{2N}\left(\gamma_{ik}D_j\beta^k + \gamma_{jk}D_i\beta^k - \dot{\gamma}_{ij}\right) \ . \tag{1.5.6}$$

The Lagrangian density reads

$$L(\gamma, \dot{\gamma}) = N\sqrt{\gamma}\left(R + K_{ij}K^{ij} - K^2\right) = N\sqrt{\gamma}\left[R + \left(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl}\right)K_{ij}K_{kl}\right] \ . \tag{1.5.7}$$

The next step is to compute the conjugate momenta of the configuration variables according to

$$\pi^{ij} := \frac{\partial L}{\partial\dot{\gamma}_{ij}} \ , \tag{1.5.8}$$

where we get

$$\pi^{ij} = N\sqrt{\gamma}\left[\left(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl}\right)K_{kl} + \left(\gamma^{ki}\gamma^{lj} - \gamma^{kl}\gamma^{ij}\right)K_{kl}\right] \times \left(-\frac{1}{2N}\right) \tag{1.5.9}$$
$$= \sqrt{\gamma}\left(K\gamma^{ij} - K^{ij}\right) \ .$$

Now, the Hamiltonian density is derived by preforming the Legendre transform, namely

$$\mathcal{H} = \pi^{ij}\dot{\gamma}_{ij} - L \ , \tag{1.5.10}$$

and the corresponding Hamiltonian is hence

$$H = \int_{\Sigma_t} \mathcal{H} d^3x \tag{1.5.11}$$
$$= -\int_{\Sigma_t} \left(NC_0 - 2\beta^i C_i\right)\sqrt{\gamma}d^3x \ , \tag{1.5.12}$$

where, for simplicity, we have introduced the notation

$$C_0 := R + K^2 - K_{ij}K^{ij} \ , \tag{1.5.13}$$
$$C_i := D_j K_i^j - D_i K \ , \tag{1.5.14}$$

that represent the left-hand side of the constraint equations derived in (1.4.1). Notice that the Hamiltonian H is a functional of the canonical phase space variables $(\gamma_{ij}, N, \beta^i)$ and their conjugate momenta $(\pi^{ij}, \pi^N, \pi_i^{\beta})$ with

$$\pi^N := \frac{\partial L}{\partial \dot{N}} = 0 \quad \text{and} \quad \pi_i^{\beta} := \frac{\partial L}{\partial \dot{\beta}^i} = 0 \ . \tag{1.5.15}$$

and the inverse extrinsic curvature reads

$$K_{ij} = K_{ij}[\gamma, \pi] = \frac{1}{\sqrt{\gamma}}\left(\frac{1}{2}\gamma_{kl}\pi^{kl}\gamma_{ij} - \gamma_{ik}\gamma_{jl}\pi^{kl}\right) \ . \tag{1.5.16}$$

Hamiltonian equations of motion are derived using the action principle and yield

$$\frac{\delta H}{\delta \pi^{ij}} = \dot{\gamma}_{ij} \ , \tag{1.5.17}$$
$$\frac{\delta H}{\delta \gamma_{ij}} = -\dot{\pi}^{ij} \ , \tag{1.5.18}$$
$$\frac{\delta H}{\delta N} = -\dot{\pi}^N = 0 \ , \tag{1.5.19}$$
$$\frac{\delta H}{\delta \beta^i} = -\dot{\pi}_i^{\beta} = 0 \ . \tag{1.5.20}$$

Note that the lapse function and shift vector play no dynamical role in the theory and the true dynamical variabels are γ_{ij} and π^{ij}. They turn out to be Lagrange multipliers and this represents consequently the Hamiltonian constraint and the momentum constraint respectively, encountered previously.
Note that the ten components of the spacetime metric are replaced by the six components of the induced metric γ_{ij} in addition to the three components of the shift vector $\boldsymbol{\beta}$.
In the following, we will change the notation to be in accordance with the exiting literature for the ADM action of GR. We will adapt the following modification [12]

$$\gamma_{ij} \to q_{ab} \ ,$$

$$\beta \to N^a \ ,$$
$$C_i \to H_a \ ,$$
$$C_0 \to H \ .$$

Expressing the Hamiltonian and momentum constraints following this notation and in terms of the metric and its conjugate momentum π^{ab} yields

$$C_i := \nabla_a^{(3)} \left(q^{-1/2} \pi^{ab} \right) \ , \tag{1.5.21}$$
$$C_0 := \left(q^{1/2} \left[R^{(3)} - q^{-1} \pi_{cd} \pi^{cd} + \frac{1}{2} q^{-1} \pi^2 \right] \right) \ . \tag{1.5.22}$$

Writing explicitly the ADM action in this notation, it reads

$$S \left[q_{ab}, \pi^{ab}, N_a, N \right] = \int dt \int_{\Sigma} dx^3 \Bigg[\pi^{ab} \dot{q}_{ab} + 2 N_b \nabla_a^{(3)} \left(q^{-1/2} \pi^{ab} \right) + N \left(q^{1/2} \left[R^{(3)} - q^{-1} \pi_{cd} \pi^{cd} + \frac{1}{2} q^{-1} \pi^2 \right] \right) \Bigg] \ . \tag{1.5.23}$$

Variation with respect to the lapse and shift produce the four following constraints

$$-H^b(q_{ab},\pi^{ab}) = 2 \nabla^{(3)}{}_a (q^{-1/2} \pi^{ab}) = 0 \qquad \text{Vector constraint}$$
$$-H(q_{ab},\pi^{ab}) = \left(q^{(1/2)} \left[R^{(3)} - q^{-1} \pi_{cd} \pi^{cd} + \frac{1}{2} q^{-1} \pi^2 \right] \right) = 0 \qquad \text{Scalar constraint} \tag{1.5.24}$$

Hence, one can rewrite the Hamiltonian (1.5.23) as a linear combination of the first class constraints $H^\mu := (H^a, \ H)$, in this case it reads

$$S[q_{ab},\pi^{ab}, N_a, N] = \int dt \int_{\Sigma_t} dx^3 \left(\pi^{ab} \dot{q_{ab}} - N_b H^b(q_{ab}, \pi^{ab}) - N H(q_{ab},\pi^{ab}) \right) \ . \tag{1.5.25}$$

This action is quite interesting. Its respective Hamiltonian is indeed peculiar, since it is proportional to the Lagrange multipliers and hence on-shell it vanishes which means that there is no dynamics and hence no physical evolution in t. However, this is the powerful aspect of the diffeomorphism invariance of GR that tells us that time is nothing but a mere parameter.

1.6 Kerr-Schild metric

Kerr-Schild metric properties

The prescription of physically realistic initial data for spherical symmetric geometries is a crucial ingredient in the construction of quantum black hole geometry, previous work in [32] succeed in describing the interior of a Schwarzschild black hole and provided promising results depicting the quantum gravity predictions for black hole singularities.
The ADM formalism outlined above provides us with the necessary tools to describe the Hamiltonian evolution of initial data set in general relativity. Our goal is to apply spherical symmetry in this geometrical set up to simulate black holes in quantum gravity. To this aim we consider the intrinsic metric on the set of (Σ_t) that can be glued together to build up the

spacetime metric given by (1.4.8). We are interested in a Cauchy hypersurface that has the topology $\Sigma_t = \mathbb{R} \times S^2$ with a spherical symmetric spacetime. As we have seen in the previous sections, once a foliation is chosen, two independent functions are sufficient to describe an arbitrary metric in spherical symmetry. The most generic spacetime- and intrinsic metric in this case yield

$$ds^2 = -N^2 dt^2 + \Lambda^2 (dr + N^r dt)^2 + R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{1.6.1}$$

$$d\sigma^2 = \Lambda^2 dt^2 + R^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{1.6.2}$$

where N, N^r, R, Λ are functions or r and t. Note that $-\infty < t$ and $r < \infty$. The functions $\Lambda(t,r)$ and $R(t,r)$ are assumed to be positive functions that constitute the set of canonical variables together with their conjugate momenta. We can finally apply the previous set up for spherical geometries simulating black hole in GR to a specific foliation namely the Kerr-Schild one. In fact, the initial data in Kerr-Schild form, which for the Schwarzschild case correspond to ingoing Eddington-Finklestein coordinates, also extend from spatial infinity to the singularity and penetrate the horizon. Classically it encodes the outside geometry of a spinning object. The metric takes the form

$$g_{ab} = \eta_{ab} + k_a k_b \ , \tag{1.6.3}$$

where η_{ab} is the Minkowski metric and k_a is a null vector. The metric poses some intriguing properties and consequences once the Einstein equations in vacuum are imposed. This rather special characteristics are satisfied by the null vector k^a. The first property can be deduced once $R_{ab} = 0$, where in this case k^a gives a shear-free congruence of null geodesics. Moreover, the null vector k^a is a geodesic and one can write it in an affine parametrized expression, in terms of new null vector ℓ^a

$$\ell^a = \sqrt{H} k^a \ , \tag{1.6.4}$$

whence, the metric takes the form

$$g_{ab} = \eta_{ab} + H \ell_a \ell_b \tag{1.6.5}$$

where H is a smooth function on $\mathbb{R}^4$. Raising indices yields

$$\ell^a = g^{ab}\ell_b = \eta^{ab}\ell_b \ , \tag{1.6.6}$$

$$g^{ab}\ell_a\ell_b = \eta^{ab}\ell_a\ell_b = -(\ell_t)^2 + \ell^i\ell_i = 0 \tag{1.6.7}$$

Note that all Kerr-Schild metrics admit a Killing field which is also a Killing field of the flat metric η_{ab}. This killing vector field has the nice property of being translational Killing vector field, which makes the full metric g_{ab} stationary. To write down explicitly the expression for the metric we work with

$$H = \frac{2Mr^3}{r^4 + a^2 z^2} \tag{1.6.8}$$

$$\tag{1.6.9}$$

where a is the black hole's spin, m its mass. In fact it can be checked that, the null vector ℓ^a is a principal null direction of the Weyl tensor. It turns out that both the Schwarzschild and Kerr spacetimes are of type D, i.e. the Weyl tensor has another principal null direction n_a given by the usual ingoing null vector

$$n_a = \nabla_a t + \nabla_a r \ . \tag{1.6.10}$$

Choosing inertial coordinates (t,x^i) adapted to the Minkowski metric we can write

$$g_{ab}dx^a dx^b = \left(-1 + 2H\ell_t^2\right) dt^2 + 4H\ell_t\ell_i dt dx^i + \left(\delta_{ij} + 2H\ell_i\ell_j\right) dx^i dx^j \ . \tag{1.6.11}$$

A priori once we take the Schwarzschild limit (with spin zero $a \to 0$) and in the limit $r \to 0$ one can derive a solution for the system (1.6.8) and the Kerr-Schild metric yields

$$H = \frac{M}{r} \ , \tag{1.6.12}$$

$$\ell_i = \frac{x_i}{r} = \partial_i r \ , \tag{1.6.13}$$

$$r^2 = \delta^{ij} x_1 x_j \tag{1.6.14}$$

The metric is then given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{4m}{r} dt dr + \left(1 + \frac{2m}{r}\right) dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{1.6.15}$$

Solving the constraints

The lapse and shift are

$$N = -\sqrt{\frac{1}{1 + \frac{2m}{r}}}, \quad N^r = \frac{\frac{2m}{r}}{1 + \frac{2m}{r}} \ . \tag{1.6.16}$$

Expressing the constraints (1.5.24) adapted to the spherical symmetry reads

$$-{}^{(3)}R + h^{-1}\pi^{ab}\pi_{ab} - \frac{1}{2}h^{-1}\pi^2 = 0 \ , \tag{1.6.17}$$

$$D_a\left(h^{-1/2}\pi^{ar}\right) = 0 \tag{1.6.18}$$

In the following computations we assume stationarity and rewrite the above constraints in terms of the ADM variables. This is equivalent to the set of equations

$$\begin{aligned} & N^2\left(\Lambda^3 + 2RR'\Lambda' - \Lambda\left[R'^2 + 2RR''\right]\right) + \\ & + \Lambda^2\left(N^r R' - \dot{R}\right)\left[N^r\left(\Lambda R' + 2R\Lambda'\right) + \Lambda\left(2RN^{r'} - \dot{R}\right) - 2R\dot{\Lambda}\right] \quad = 0 \\ & N^r\left[R'\left(\Lambda N' + N\Lambda'\right) - N\Lambda R''\right] \\ & - NR'\dot{\Lambda} + \Lambda\left(-N'\dot{R} + N\dot{R}'\right) = 0 \end{aligned} \tag{1.6.19}$$

If we consider the simplest equation, namely the second one in (1.6.19), we can solve it for R. In fact

$$\frac{R''}{R'} = \frac{N'\Lambda + N\Lambda'}{N\Lambda} \ , \tag{1.6.20}$$

which implies that

$$\log R' = \log N + \log\Lambda + C_1 \tag{1.6.21}$$

$$, \tag{1.6.22}$$

and hence we obtain an expression for R', namely

$$R' = C_2 N \Lambda \ , \tag{1.6.23}$$

where the parameters $C_i, \quad i = 1 \cdot n$ are some integration constants. Plugging (1.6.23) in (1.6.19) yields

$$\begin{aligned} & r(2m+r)R' \left[r\left(2m + r - rC_2^2\right) + 4m^2 R^2 \right] \\ & - 2mR\left[r^2 C_2^2 + 4mR' \left((m+r)R' - r(2m+r)R''\right) \right] = 0 \end{aligned} \tag{1.6.24}$$

For $C_2 = \pm 1$ this equation is integrable. The solutions in this case read

$$R = r \ . \tag{1.6.25}$$

Plugging this solution back in (1.6.19) one obtains the solution for the metric function Λ

$$\Lambda = \pm 1/N \ . \tag{1.6.26}$$

In the following we will work with the + convention. According to (1.6.1), with these solution for Λ and R one recovers the Kerr-Schild metric in (1.6.15).

1.7 Tetrad formulation

The ADM action (1.5.25) describes general relativity as a constrained system, where the dynamics of the theory are all encoded in the set of constraints (1.5.24). This canonical formulation of GR reflects the powerful setup of constrained systems that will become more clear once one considers Dirac's quantization program.
In this section, we will highlight the main reasons that call to introduce new variables into play instead of the ADM phase space parameters. One of the most straightforward reasons to consider more suitable variabels for quantization is the attempt to apply Dirac's approach to quantize the action (1.5.25). This method relies on defining the physical states as the ones annihilated by the constraints, an explicit well defined scalar product as well as a physical interpretation of the observables [14, 13]. This program can be illustrated in three simple steps

1. Write down a presentation for the phase space variables as operators in an auxiliary kinematical Hilbert $\mathcal{H}_{kin}$ space endowed with the commutation relation $\{\cdot,\cdot\} \longrightarrow \frac{1}{i\hbar}[\cdot,\cdot]$.
2. Promote the constraints H^μ to operators in $\mathcal{H}_{kin}$.
3. Define the space of solutions of the constraints as the physical space $\mathcal{H}_{phy}$, satisfying the equations

$$\hat{H}^\mu \psi = 0 \quad \forall \psi \in \mathcal{H}_{phys} \ . \tag{1.7.1}$$

Applying this scheme to the gravitational constrained action encounters problems already at the level of defining the scalar product of the Hilbert space. Formally, one could write

$$\int \mathrm{d}g \overline{\psi[g]} \psi'[g] \equiv \langle \psi | \psi' \rangle \ . \tag{1.7.2}$$

However, the Lebesgue measure is ill-defined here, namely in the space of metrics modulo diffeomorphisms. This implies that we are unable to verify if the momentum and metric operators are hermitian or not. Moreover, even if one ignores this ambiguity and assumes the existence of a well defined scalar product, the next problem that rises is solving the constraints. Let us consider the schematic step

$$\mathcal{H}_{kin} \overset{\hat{H}^a=0}{\longrightarrow} \mathcal{H}_{Diff} \overset{\hat{H}=0}{\longrightarrow} \mathcal{H}_{phys} \ . \tag{1.7.3}$$

If we consider first the vector constraint, one can prove that

$$\psi\left[q_{ab} + 2\nabla_{(a}N_{b)}\right] \equiv \psi\left[q_{ab}\right] \ . \tag{1.7.4}$$

This implies that the solutions of the vector constraint are those functionals of the metric that are left invariant under the action of diffeomorphism. This is indeed very interesting, as it reflects at the quantum level the desired action of the classical constraints. However the space of solutions for $\mathcal{H}_{Diff}$ is again ill-defined, since it inherits from the kinematical one the absence of a well defined measure. The ambiguity of finding space of solutions becomes more drastic when trying to work out the Hamiltonian constraint, mainly due to the necessity to introduce the notion of ordering the products of operators.
A way out of this paradigm is to consider new variabels that will render the application of Dirac's quantization program more convenient for the ADM action. To get inspired of what kind of variables one can rely on, one can try to consider the issue of the coupling of fermions in general relativity. In the following, we will explore this path and discover a new set of extended variables that, as we will see later on, will play a major role in quantizing the theory.

1.7.1 Tetrad variables

Let us start with the first issue mentioned above, namely the coupling of fermions in general relativity. In order to achieve this, the set of variables where a local action of the rotation group (more generally Lorentz transformations) is defined, presents itself as a suitable choice. This is naturally achieved by describing the spacetime geometry in terms of an orthonormal frame instead of a metric where local Lorentz transformations are basically the set of transformations relating different orthonormal frames. This consequently reveals a more algebraically simple formulation of the theory in comparison to the ADM one.
Concretely, one can introduce an orthonormal frame field defined by four co-vectors e^I_a (with the index $I = 0, \ldots, 3; a$ and other Latin indices denote spacetime indices) and write the spacetime metric as a composite object

$$\begin{aligned} g_{ab} &= -e^0_a e^0_b + e^1_a e^1_b + e^2_a e^2_b + e^3_a e^3_b \\ &= e^I_a e^J_b \eta_{IJ} \ , \end{aligned} \tag{1.7.5}$$

where in the second line the internal Minkowski metric is $\eta_{IJ} = diag(-1, 1, 1, 1)$. From this definition, this quadrupole of 1-forms relate a general reference frame and an inertial one by a local isomorphism. Moreover, notice that in the familiar three dimensional space there are infinitely many frame-fields connected by local rotations; in the present four dimensional Lorentzian setting the same issue still reside, since the above equation is invariant under Lorentz transformations. Since both e and $\tilde{e}$ are solutions with $e^I \to \tilde{e}^I = \Lambda^I{}_J e^J_a$ which we will write in matrix notation as

$$e_a \to \tilde{e}_a = \Lambda e_a \ , \tag{1.7.6}$$

where $\Lambda^I{}_J$ satisfies the relation

$$\eta_{IJ} = \Lambda^I{}_J \Lambda^J{}_M \ . \tag{1.7.7}$$

Indeed, this is a new symmetry. It is an additional gauge symmetry of general relativity once we write it in terms of these variables. Concretely, let us considerλ^I (with internal index) an object that transforms covariantly under a Lorentz transformation Λ^I_J, namely

$$\lambda^I \to \tilde{\lambda^I} = \Lambda\lambda \ ,$$

then its covariant (exterior) derivative d_ω is the covariant exterior derivative

$$d_\omega \lambda^I = d\lambda^I + \omega^{IJ} \wedge \lambda_J \ ,$$

transforms covariantly as well, since ω_a^{AB} transforms inhomogeneously under internal Lorentz transformations, i.e.

$$\omega \to \tilde{\omega} = \Lambda\omega\Lambda^{-1} + \Lambda d\Lambda^{-1} \ .$$

Indeed, as in any gauge theory, derivatives of covariant fields come with the introduction of a connection $\omega^{IJ} = -\omega^{JI}$, a one-form called the Lorentz connection in this case with values in the Lorentz algebra, hence defining the covariant derivative. Hence, it is clear that the Lorentz connection ω^{IJ} is an additional filed that is necessary to have in order to work in the tetrad formulation and define derivatives within a framework where frames can be locally transformed by a local Lorentz transformation. The covariant differentiation is thus evaluated as follows

$$D_\mu v^I(x) = \partial_\mu v^I(x) + \omega^I_{\mu J}(x) v^J(x) \ . \tag{1.7.8}$$

Analogously to the Levi-Civita connection that is metric compatible, we require that the spin connection to be tetrad compatible, meaning that $D_\mu e^I_\nu = 0$. This consequently implies the symmetric and antisymmetric combinations

$$\partial_{(\mu} e^I_{\nu)} + \omega^I_{(\mu} J e^J_{\nu)} = \Gamma^\rho_{(\nu\mu)} e^I_\rho, \quad \partial_{[\mu} e^I_{\nu]} + \omega^I_{[\mu} J e^J_{\nu]} = \Gamma^\rho_{[\nu\mu]} e^I_\rho \equiv 0 \ . \tag{1.7.9}$$

From the above equations, we can derive the relation between the spin and Levi-Civita connection

$$\omega^I_{\mu J} = e^I_\nu \nabla_\mu e^\nu_J \ , \tag{1.7.10}$$

as well as the so called Cartan first structure equation

$$\mathrm{d}_\omega e^I = \mathrm{d}e^I + \omega^I_J \wedge e^J = \left(\partial_\mu e^I_\nu + \omega^I_{\mu J} e^J_\nu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = 0 \ , \tag{1.7.11}$$

We define the curvature and its components respectively as

$$F^{IJ} = \mathrm{d}\omega^{IJ} + \omega^I_K \wedge \omega^{KJ} \tag{1.7.12}$$

$$F^{IJ}_{\mu\nu} = \partial_\mu \omega^{IJ}_\nu - \partial_\nu \omega^{IJ}_\mu + \omega^I_{K\mu}\omega^{KJ}_\nu - \omega^J_{K\mu}\omega^{KI}_\nu \ . \tag{1.7.13}$$

Using the relation (1.7.10), the Riemann tensor obtained from the tetrad e^I_μ reads

$$F^{IJ}_{\mu\nu}(\omega(e)) \equiv e^{I\rho} e^{J\sigma} R_{\mu\nu\rho\sigma}(e) \ . \tag{1.7.14}$$

notice that the curvature F transforms covariantly under a local Lorentz transformation $(F \to \Lambda F \Lambda^{-1})$. This relation highlights the fact that general relativity is a gauge theory with the Lorentz group as a local gauge group and the Riemann tensor plays the role of the field's

strength.
The corresponding Einstein-Hilbert action, written in this formulation can be expressed as

$$S[e_a^{IJ},\omega_a^{AB}] = \frac{1}{2\kappa}\int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL}(\omega) \ , \tag{1.7.15}$$

where,the Levi-Civita symbol ϵ_{ABCD} is a totally anti-symmetric internal tensor such that $\epsilon_{0123} = 1$ is invariant under the simultaneous action of the Lorentz group on its four entries. The action is, in this way, invariant under the Lorentz gauge transformations (1.7.7) and (1.7.6), that define the (internal) Lorentz gauge transformations of the basic fields present in the action. Nevertheless, the gauge transformations (1.7.7) and (1.7.6) need not be listed in addition to (1.7.15); the field equations coming from the action know about these symmetries. This is especially explicit in the Hamiltonian formulation where gauge symmetries are in direct correspondence with constraints which, in turn, are the generators of gauge transformations, as discussed in the first section of this chapter. These constraints, namely generators of gauge transformations, constitute a part of the field equations. In addition to internal Lorentz transformations, the action (1.7.15) is also invariant under diffeomorphisms, underlying the general covariance aspect of the theory. At the technical level, this is reflected from the fact that the action (1.7.15) is the integral of a 4-form (a completely anti-symmetric tensor with 4 contravariant indices). Notice that under coordinate transformation $x^\mu \to y^\mu$ the fields transform as tensors

$$e_\mu^I dx^\mu = e_\mu^J \frac{\partial x^\mu}{\partial y^\alpha} dy^\alpha \ , \tag{1.7.16}$$

$$\omega_\mu^{IJ} dx^\mu = \omega_\mu^{IK} \frac{\partial x^\mu}{\partial y^\alpha} dy^\alpha \ , \tag{1.7.17}$$

while the integral remains unchanged as the 4-form transforms respectively by multiplication by the Jacobian $|\frac{\partial x^\mu}{\partial y^\alpha}|$.
Once more, such symmetry will be dictated to us by the equations of motion coming from the action if not explicitly taken into account. The equations of motion stemming from (1.7.15) are obtained from

$$\delta_e S = 0 \quad \delta_\omega S = 0 \ , \tag{1.7.18}$$

where the equations of motion read

$$\epsilon_{ABCD} e^J \wedge F^{KL}(\omega) = 0 \ , \tag{1.7.19}$$

$$d_\omega(e^I \wedge e^J) = 0 \ . \tag{1.7.20}$$

Notice their algebraic simplicity. If the tetrad field is invertible meaning that a non degenerate metric can be constructed, then the above equations are equivalent to Einstein's equation. However, the field equations, as well as the action (1.7.15) continue to make sense for degenerate tetrads. For example the no-geometry state $e = 0$ (diffeomorphism invariant vacuum) solves the equations and makes perfect sense in terms of the new variables.

1.7.2 Hamiltonian analysis

General covariance is the distinctive feature of general relativity and we have recalled how this is explicitly encoded in the action principles for gravity. The major difficulty of quantum gravity is to generalize what we have learnt about quantum field theory and apply it to understand the generally covariant physics of gravity. The notion of measurement is naturally related to the concept of localization of physical events. However, in GR, localizing spacetime events is possible only in a relational fashion where some degrees of freedom are related to others, labeling therefore, a generally covariant observable. Hence, in GR the concept of localization is always realized in a relational fashion using the notion of test observers. Test observers are crucial in the spacetime description of general relativity; the observables that follow from them are always non local in spacetime. An illustrating example is the definition of a black hole event horizon which separates those observers that can in principle escape out to infinity from those that cannot: test photons are used to define the horizon in a coordinate independent fashion. Indeed, all observables are non-local in this theory. Along these lines, the idea of extended variables might be best suited for the definition of a quantum theory of gravity. Even when the motivations are sometimes different non local objects are also central in other approaches such as strings, branes, twistor theory or causal sets. An advantage of the new variables in (1.7.15) is that they allow for the introduction of natural quantities associated to extended subsets (submanifolds) of the spacetime. These quantities are the fluxes of $e \wedge e$ and the holonomies of the Lorentz connection ω. The fluxes are

$$E(\alpha,S) = \int_S \alpha_{IJ} e^I \wedge e^J \ , \tag{1.7.21}$$

where α_{IJ} is a smearing field and S is a two-dimensional surface. The holonomy assigns an element $\Lambda(l,\omega)$ of the Lorentz group to any one dimensional path in spacetime, by the rule

$$\Lambda(l,\omega) = P\exp(-\int_l \omega) \ , \tag{1.7.22}$$

where P exp denotes the path ordered exponential. None of these extended variables are diffeomorphism invariant; however, they transform in a very simple way under coordinate transformations: the action of a diffeomorphism on them amounts to the deformation of the surface S and the ℓ by the action of the diffeomorphism on spacetime points. This behavior makes these extended variables suitable for the construction of covariant non local operators for the quantum theory. The above non-local variables are the basic building blocks in the attempts of giving a meaning to the path integral definition of quantum gravity based on action (1.7.15). Such research direction is known as the spin foam approach.

Hamiltonian formalism

To derive the Hamiltonian formulation for tetrad representation, one follows the procedure that relies on the 3+1 decomposition of spacetime [31] and the adapted coordinate (t,x). Thus we can introduce the lapse function and shift vector as in the previous section. We work in the ADM formulation of the metric. In this set up, it is straightforward to see that the tetrad for a given metric reads

$$e^I_0 = e^I_\mu \tau^\mu = Nn^I + N^a e^I_a, \quad \delta_{ij} e^i_a e^j_b = g_{ab}, i = 1,2,3 \ . \tag{1.7.23}$$

where the triad e^i_a represents the spatial counterpart of the tetrad. As usual, the next step is to identify the conjugate momentum and perform the Legendre transform. For simplicity, it is practical to work in the "time" gauge. The idea is to demand the co-vector e^0, representing the time axis of the frame field, to be perpendicular to the time slices Σ_t or equivalently to be aligned with the unit normal n^μ ($\boldsymbol{n}$). Thus, it is defined as

$$e^I_\mu n^\mu = \delta^I_0 \ , \tag{1.7.24}$$

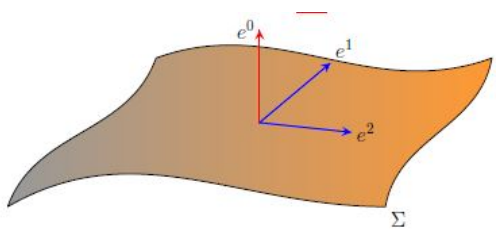


Figure 1.4: *Time gauge*: restricting the Lorentz gauge symmetry group to the $SO(3)$ subgroup. This is defined by the condition that the time e^0 component of the tetrad has to be normal to the time slice Σ_t.

$$e^0_\mu = (N, 0) \longrightarrow e^I_0 = \left(N, N^a e^i_a\right) \ . \tag{1.7.25}$$

This restricts the Lorentz gauge group to the rotation subgroup that leaves the time normal vector to the hypersurface invariant, namely $SU(2) \subset SL(2, \mathbb{C})$. Notice that such partial gauge fixing is very natural in the Hamiltonian formulation of the 3+1 decomposition of the equation (1.1.1), since it naturally provides the slicing of spacetime in terms of space-like hypersurfaces. In this sense, time-gauge amounts to adjusting the time axis in our frame field to the one that is singled out by the foliation. We furthermore define the *densitized triad*

$$E^a_i = e e^a_i = \frac{1}{2}\varepsilon_{ijk}\varepsilon^{abc}e^j_b e^k_c \ , \tag{1.7.26}$$

and the Ashtekar-Barbero connection

$$A^i_a = \gamma\omega^{0i}_a + \frac{1}{2}\varepsilon^i_{jk}\omega^{jk}_a \ . \tag{1.7.27}$$

As a matter of fact, the densitized triad and the above defined connection turn out to be conjugate variables. Now we can write the action in terms of this new set of variables as

$$S\left(A, E, N, N^a\right) = \frac{1}{\gamma}\int \mathrm{d}t \int_\Sigma \mathrm{d}^3x \left[\dot{A}^i_a E^a_i - A^i_0 D_a E^a_i - NH - N^a H_a\right] \ . \tag{1.7.28}$$

As expected, the invariance under local Lorentz transformations gives rise to new gauge symmetry in the action and hence additional constraints. We can identify the constraints as

$$\begin{aligned} G_j &\equiv D_a E^a_i = \partial_a E^a_j + \varepsilon_{jk\ell}A^j_a E^{a\ell} \\ H_a &= \frac{1}{\gamma}F^j_{ab}E^b_j - \frac{1+\gamma^2}{\gamma}K^i_a G_i \\ H &= \left[F^j_{ab} - \left(\gamma^2+1\right)\varepsilon_{jmn}K^m_a K^n_b\right]\frac{\varepsilon_{jk\ell}E^a_k E^b_\ell}{\det E} + \frac{1+\gamma^2}{\gamma}G^i\partial_a\frac{E^a_i}{\det E} \end{aligned} \tag{1.7.29}$$

The action (1.7.28) is equivalent to (1.5.2) with the difference in the canonical conjugate variables being (A,E). Lapse and shift are still Lagrange multipliers, and consistently we still refer to $H(A, E)$ and $H_a(A, E)$ as the Hamiltonian and space-diffeomorphism constraints. Furthermore, the algebra is still first class. The new formulation in terms of tetrads has brought up an extra constraint, the Gauss constraint (G_j), that is responsible for generating gauge transformations. Indeed, one can show that E^b_j and A^i_a transform respectively as an $SU(2)$ vector and as an $SU(2)$ connection under this transformation. It is true that we started with $SO(3)$ local gauge transformation of the covariant action (1.5.2) and a long the way it got swapped to an $SU(2)$ one. The change of variabels we preformed and the partial gauge fixing (the time gauge) is the reason behind this change in the gauge symmetry group since we are working with an $SU(2)$ connection (Ashtekar-Barbero connection) and not the Lorentz one as previously done. In fact the Ashtekar-Barbero configuration variable transforms as an $SU(2)$ gauge connection under the $SU(2)$ residual gauge symmetry after requiring the time-gauge. It should be seen as an auxiliary variable with the aim to recast first order constraints. The symplectic structure in this formulation yields

$$\left\{A_a^i(x), E_j^b(y)\right\} = \gamma \delta_a^b \delta_j^i \delta^3(x,y) \ . \tag{1.7.30}$$

where the new internal index i corresponds to the adjoint representation of $SU(2)$.
It will be later on useful to deal with the smeared version of these canonical variabels in the quantization procedure. The densitized triad is a 2-form and thus, it is natural to smear it on a surface, namely

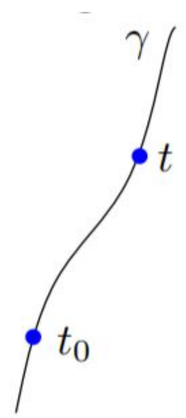
$$E_i(S) \equiv \int_S n_a E_i^a \mathrm{d}^2\sigma \quad , \tag{1.7.31}$$

with $n_a = \varepsilon_{abc} \frac{\partial x^b}{\partial \sigma_1} \frac{\partial x^c}{\partial \sigma_2}$ is the normal to the surface. With this smearing we identify the quantity $E_i(S)$ as the flux of E across a surface S. The connection is a 1-form, so we can smear it on a one dimensional path. If we consider a path γ and a corresponding parametrization

$$x^a(s) : [0,1] \to \Sigma \ , \tag{1.7.32}$$

we can associate to a given connection A_a^i an element of $SU(2)$ such that $A_a \equiv A_a^i \tau_i$, with τ_i being the generators of $SU(2)$. We can therefore integrate A_a along the path γ given as a line element, namely

$$A_a^i \longrightarrow \int_\gamma A \equiv \int_0^1 \mathrm{d}s A_a^i(x(s)) \frac{\mathrm{d}x^a(s)}{\mathrm{d}s} \tau_i \ . \tag{1.7.33}$$



We introduce the notion of holonomy[a] of A along γ as

$$h_\gamma = \mathcal{P} \exp\left(\int_\gamma A\right) \ . \tag{1.7.34}$$

with the path-ordered product $\mathcal{P}$ and the parametrization $s \in [0,1]$.

$$h_\gamma = \sum_{n=0}^{\infty} \iiint_{s_n > 0} A\left(\gamma\left(s_1\right)\right) \cdots A\left(\gamma\left(s_n\right)\right) ds_1 \cdots ds_n \tag{1.7.35}$$

[a]The reader is referred to read the appendix for more properties of the holonomies.

1.8 Spherically symmetric phase space

For the treatment of the canonical quantization program [21], we will consider the convenient formulation in terms of the Ashtekar-Barbero connection A_a^i and the densitized triad E_i^a instead of q_{ab} and π^{ab}. In the context of spherically symmetric geometries, we consider the Cauchy surface with the topology $\Sigma_t = \mathbb{R} \times S^2$. Its characteristic most generic spacetime and intrinsic metric are given by (1.6.1) and we present the derivation of its associated flux and connection, namely the set of components $\{E_i^a, A_a^i\}$. To explicitly compute the latter, we use the expression of the metric in terms of the tetrad components; $g_{\alpha\beta} = e_\alpha^I e_{I\beta}$, from which one can write down the metric components:

$$\begin{aligned} g_{tt} &= -N^2 + \Lambda^2 (N^r)^2 = -\left(e_t^0\right)^2 + \left(e_t^3\right)^2 \ , \\ g_{tr} &= \Lambda^2 N^r = -e_t^0 e_r^0 + e_t^3 e_r^3 \ , \\ g_{rr} &= \Lambda^2 = -\left(e_r^0\right)^2 + \left(e_r^3\right)^2 \ , \\ g_{\theta\theta} &= R^2 = e_\theta^1 e_{1\theta} + e_\theta^2 e_{2\theta} \ , \\ g_{\varphi\varphi} &= R^2 (\sin\theta)^2 = e_\theta^1 e_{1\varphi} + e_\varphi^2 e_{2\varphi} \ , \\ g_{\theta\varphi} &= 0 = e_\theta^1 e_{1\varphi} + e_\theta^2 e_{2\varphi} \ , \end{aligned} \tag{1.8.1}$$

from which one obtains

$$\begin{aligned} e^0 &= N dt \\ e^3 &= \Lambda N^r dt + \Lambda dr \ , \\ e^1 &= R\cos\tilde{\alpha} d\theta - R\sin\theta\sin\tilde{\alpha} d\varphi \ , \\ e^2 &= R\sin\tilde{\alpha} d\theta + R\sin\theta\cos\tilde{\alpha} d\varphi \ . \end{aligned} \tag{1.8.2}$$

where one leaves a rotation freedom in the angle $\tilde{\alpha}$ for the components e^1 and e^2.
In the time gauge $e_a^0 = n_a$, the densitized triad reads[4]

$$E = E_i^a \tau^i \partial_a \ , \tag{1.8.3}$$

$$E_i^a = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk} e_b^j e_c^k \ . \tag{1.8.4}$$

To compute the Ashtekar-Barbero connection, we make use the torsion-free condition $de^I = -\omega_J^I \wedge e^J$. The Ashtekar-Barbero connection given by

$$A^i = \Gamma^i + \gamma K^i \ , \quad \Gamma^i = -\frac{1}{2}\epsilon_{jk}^i \omega^{jk} \ , \quad K^i = \omega^{0i} \ , \tag{1.8.5}$$

takes the form $A = A_a^i \tau_i dx^a$.
The spherically symmetric Ashtekar-Barbero connection and triad are then expressed as

$$E = E^r(t,r)\sin\theta\tau_3\partial_r + \left[E^1(t,r)\tau_1 + E^2(t,r)\tau_2\right]\sin\theta\partial_\theta + \left[E^1(t,r)\tau_2 - E^2(t,r)\tau_1\right]\partial_\varphi \ , \tag{1.8.6}$$

$$A = A_r(t,r)\tau_3 dr + \left[A_1(t,r)\tau_1 + A_2(t,r)\tau_2\right] d\theta + \sin\theta\left[A_1(t,r)\tau_2 - A_2(t,r)\tau_1\right] d\varphi + \cos\theta\tau_3 d\varphi \ . \tag{1.8.7}$$

[4] τ_i denotes the anti-hermitian basis satisfying the commutation relation $[\tau_i, \tau_j] = \epsilon_{ij}^k \tau_k$ and $(\tau_i)^2 = -\frac{1}{4}\mathbb{I}$ for all i's and $\mathrm{Tr}(\tau_i\tau_j) = -\frac{1}{2}\delta_{ij}$.

For the Kerr-Schild metric given by (1.6.15), the components of the flux and connection read

$$E^r(t,r) = R^2, \quad E^1(t,r) = \Lambda R \cos\tilde{\alpha}, \quad E^2(t,r) = \Lambda R \sin\tilde{\alpha} \ , \tag{1.8.8}$$

$$A_r(t,r) = -\gamma \frac{(\Lambda' N^r + \Lambda N^{r'} - \dot{\Lambda})}{N} \ , \tag{1.8.9}$$

$$A_1(t,r) = \frac{R'}{\Lambda}\left[\gamma\left(N^r - \frac{\dot{R}}{R'}\right)\cos\tilde{\alpha} - \sin\tilde{\alpha}\right] \ , \tag{1.8.10}$$

$$A_2(t,r) = \frac{R'}{\Lambda}\left[\gamma\left(N^r - \frac{\dot{R}}{R'}\right)\sin\tilde{\alpha} + \cos\tilde{\alpha}\right] \ . \tag{1.8.11}$$

The Poisson brackets takes the form:

$$\{A_r(t,r), E^r(t,r')\} = 2G\delta(r - r') \ , \tag{1.8.12}$$

$$\left\{A_1(t,r), E^1(t,r')\right\} = G\delta(r - r') \ , \tag{1.8.13}$$

$$\left\{A_2(t,r), E^2(t,r')\right\} = G\delta(r - r') \ . \tag{1.8.14}$$

Chapter 2

Loop Quantum Gravity

Gravity is a fundamental interaction that is conceptually different from all the other known forces. The theory of general relativity tells us that the degrees of freedom of the gravitational field are at the core of the spacetime geometry and the lesson that it teaches us is that spacetime is fully dynamical [27]. Indeed, GR describes geometry as the gravitational field on top of which its own degrees of freedom and those of matter fields live. This is made clear from the framework of the initial value formulation [29, 27, 28] of GR encountered in the previous sections. Given a suitable set of initial conditions on a 3-dimensional manifold , Einstein's equations dictate the dynamics that generates the reconstruction of the spacetime geometry with all the fields propagating on it. In classical physics, general relativity is not only an outstanding description of the very nature of the gravitational interaction. Its fundamental principle of general covariance provides the basic framework to the most important lesson it delivers, namely, there is no well defined notion of absolute space and it only makes sense to describe physical entities in relation to other physical ones.
Having said this, GR is still unable to provide answers to a number of important physical situations. In particular classical general relativity predicts the existence of singularities in physically realistic situations such as the case of black hole physics and cosmology. Once one tries to present a valid consistent description of the gravitational degrees of freedom near these singularities, the theory breaks down. These are one of the reasons why a theory of quantum gravity is needed, for which Loop Quantum Gravity (LQG) presents itself as a candidate model to underlie the ambiguous relation between the principles of general relativity and quantum mechanics [4, 7, 12].
LQG is a background independent approach to quantum gravity with the challenge to define quantum field theory in the absence of any pre-defined notion of distance, namely a quantum field theory without a metric. In this section, we will rely on the tetrad formalism and the resulting smeared algebra of $h_\gamma[A]$ and $E_i(S)$, the so called holonomy-flux algebra and sketch the main aspects of LQG framework [13].

2.1 Kinematics

We have seen that GR can be written as a totally constrained theory and the tetrad formulation of it allowed us to view it as an $SU(2)$ gauge theory, which is manifest by the presence of the Gauss constraint and the Ashtekar-Barbero auxiliary connection. The set of constraints encountered read

$$G_i = 0 \qquad \text{Gauss law} \tag{2.1.1}$$
$$H_a = 0 \qquad \text{Spatial diffeomorphism invariance} \tag{2.1.2}$$
$$H = 0 \qquad \text{Hamiltonian constraint} \tag{2.1.3}$$

The usual procedure for canonical quantization of a gauge theory relies on the role played by the metric that enables the definition of a measure for the kinematical Hilbert space. However, general relativity as we have seen when we tried to implement Dirac's quantization program in section (1.7), do not have a background metric at disposal to define the integration measure. The challenge then is to define such a measure on the space of connections without relying on a background metric, which is naturally provided by *the cylindrical functions*.
A graph γ is defined to be the collection of paths $e \subset \Sigma_t$ meeting at most at their end points. Given such a graph $\gamma \subset \Sigma_t$ with N_e being the number of edges that it contains, an element $\psi_{\gamma,f} \in \text{Cyl}_\gamma$ is labeled by a smooth function f and a graph γ. This smooth function f is defined as

$$f : SU(2)^{N_e} \to \mathbb{C} \ , \tag{2.1.4}$$

and it is given by a functional of the connection defined as

$$\psi_{\gamma,f}[A] := f\left(h_{e_1}[A], h_{e_2}[A], \cdots h_{e_{N_e}}[A]\right) \ , \tag{2.1.5}$$

where e_i for $i = 1, \cdots N_e$ are the edges of the corresponding graph. Taking the union of all these functionals constitute the notion of cylindrical functions of generalized connections denoted Cyl, such that

$$\text{Cyl} = \cup_\gamma \text{Cyl}_\gamma \ . \tag{2.1.6}$$

This represents the algebra of the physical observables upon which we will define the kinematical Hilbert space $\mathcal{H}_{kin}$. This space of functionals once endowed with a proper scalar product can be

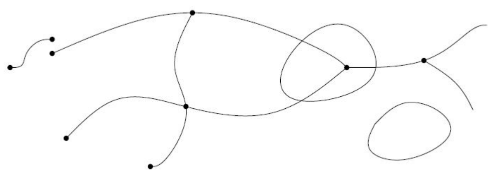


Figure 2.1: Collection of paths $\gamma = \{e_1 \cdots e_{N_e}\}$

turned into an Hilbert space. Now the switching step from the connection to the holonomy will turn out to be very convenient since the holonomy is an element of $SU(2)$, and the integration over $SU(2)$ is well-defined. Indeed, there is a unique gauge-invariant and normalized measure dh, called the de Haar measure. Whence, for N_e copies of the de Haar measure, we define on Cyl_γ the scalar product

$$\left\langle \psi_{(\gamma,f)} | \psi_{(\gamma,f')} \right\rangle \equiv \int \prod_e \mathrm{dh}_e \, \overline{f\left(h_{e_1}[A], \ldots, h_{e_{N_e}}[A]\right)} f'\left(h_{e_1}[A], \ldots, h_{e_{N_e}}[A]\right) \ . \tag{2.1.7}$$

This scalar product turns Cyl_γ into a Hilbert space $\mathcal{H}_\gamma$. The full kinematical Hilbert space is provided by the direct sum $\mathcal{H}_{kin} = \underset{\gamma \subset \Sigma_t}{\oplus} \mathcal{H}_\gamma$, where $\mathcal{H}_{kin}$ is the kinematical Hilbert space over all gauge connections A on Σ_t, thanks to the Ashtekar and Lewandowski,

$$\mathcal{H}_{kin} = L_2\left[A, \mathrm{d}\mu_{AL}\right] \ , \tag{2.1.8}$$

with $\mathrm{d}\mu_{AL}$ introduced as the Ashtekar-Lewandowski measure and defined as

$$\mu_{AL}\left(\psi_{\gamma,f}\right) = \int \prod_{e\subset\gamma} dh_e f\left(h_{e_1}, h_{e_2}, \cdots h_{e_{N_e}}\right) \ , \tag{2.1.9}$$

with the scalar product being

$$\begin{aligned} < \psi_{\gamma,f}, \psi_{\gamma',g} >: &= \mu_{AL}\left(\overline{\psi_{\gamma,f}}\psi_{\gamma',g}\right) = \\ &= \int \prod_{e\subset\Gamma_{\gamma\gamma'}} dh_e \overline{f\left(h_{e_1}, \cdots h_{e_{N_e}}\right)} g\left(h_{e_1}, \cdots h_{e_{N_e}}\right) \ . \end{aligned} \tag{2.1.10}$$

Let us turn now to find a suitable representation for the flux-holonomy algebra. We start by introducing an orthogonal basis on the space using Peter-Weyl theorem[1] This implies that any function $\psi_{(\gamma,f)}[A] \in \mathcal{H}_\gamma$ can be decomposed in the following way

$$\psi_{(\Gamma,f)}[A] = \sum_{j_e,m_e,n_e} \hat{f}^{j_1,\dots,j_n}_{m_1,\dots,m_n,n_1,\dots,n_n} D^{(j_1)}_{m_1 n_1}\left(h_{e_1}[A]\right) \dots D^{(j_n)}_{m_n n_n}\left(h_{e_n}[A]\right) \ . \tag{2.1.12}$$

In return, this allows us to move the second step in finding a Schroedinger representation for the functions. What we have accomplished with this construction is the definition of a well-behaved kinematical Hilbert space for general relativity. It carries a representation of the canonical Poisson algebra, and as a bonus, this representation is unique. Following Dirac, we now have a well-posed problem of reduction by the constraints

$$\mathcal{H}_{kin} \overset{\hat{G}_i=0}{\longrightarrow} \mathcal{H}^0_{kin} \overset{\hat{H}^a=0}{\longrightarrow} \mathcal{H}_{Diff} \overset{\hat{H}=0}{\longrightarrow} \mathcal{H}_{phys} \ . \tag{2.1.13}$$

We now proceed to finding the solutions of the quantum Gauss constraint, which are basically the state that are gauge invariant under $SU(2)$. Due to the properties of the holonomy, we have in a generic irrep j the gauge transformation

$$D^{(j)}\left(h_e\right) \longrightarrow D^{(j)}\left(h'_e\right) = D^{(j)}\left(g_{s(e)} h_e g^{-1}_{t(e)}\right) = D^{(j)}\left(g_{s(e)}\right) D^{(j)}\left(h_e\right) D^{(j)}\left(g^{-1}_{t(e)}\right) \ . \tag{2.1.14}$$

This illustrates that gauge transformations operate on the source and targets of the links, namely on the nodes of a graph. Imposing gauge-invariance then means requiring the cylindrical function to be invariant under action of the group at the nodes, which can be easily implemented via *group averaging.* This amounts to inserting on each edge a projector selecting the gauge invariant part of $\bigotimes_e V^{(j_e)}$, namely defined as

$$\mathcal{P} = \int \mathrm{d}g \prod_{e\in n} D^{(j_e)}(g) \ , \tag{2.1.15}$$

where

$$\prod_e D^{(j_e)}_{m_e n_e}\left(h_e\right) \in \bigotimes_e V^{(j_e)} \ . \tag{2.1.16}$$

[1] Peter-Weyl theorem states that a basis on the Hilbert space of functions equipped with de Haar measure on a compact group G is given by the matrix elements of the unitary irreducible representation of the group. For the interesting case of $SU(2)$, this is given by

$$f(g) = \sum_j \hat{f}^j_{mn} D^{(j)}_{mn}(g) \qquad \begin{aligned} j &= 0, \tfrac{1}{2}, 1, \dots \\ m &= -j, \dots, j \end{aligned} \tag{2.1.11}$$

with $D^{(j)}_{mn}(g)$ are the Wigner matrices.

Let us denote i_α a ket in the basis of $\mathcal{P}$, with $\alpha = 1, \dots, \dim V^{(0)}$, V^0 being the "singlet" space, and i^*_α the dual, i.e. the bra such that $\mathcal{P} = \sum_{\alpha=1}^{\dim V^{(0)}} i_\alpha i^*_\alpha$. We call these invariant object "intertwiners". A suitable method to build them amounts to add first two irreps only, then the third, and so on, that lead to a virtual decomposition of links. For the case of n=4 and n=5, we get something that is similar to Figure 2.2. This leads us to the notion of *spin network*

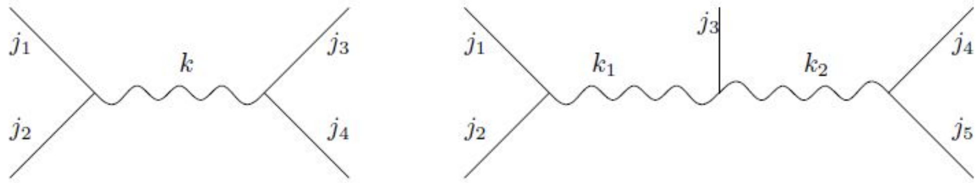


Figure 2.2: Picturing intertwiners for the case of $n = 4$ and $n = 5$: the process of adding first two irreps only, then the third and so on generates a decomposition over virtual links. The virtual spins k_i stand for the intertwiners.

states. These states are labeled with a graph γ, with an irreducible representation $D^{(j)}(h)$ of spin-j of the holonomy h along each link, and with an element i of the intertwiner space $\mathcal{H}_n \equiv \mathrm{Inv}\left[\bigotimes_{e\in n} V^{(j_e)}\right]$ and are defined as

$$\psi_{(\gamma, j_e, i_n)}\left[h_e\right] = \bigotimes_e D^{(j_e)}\left(h_e\right) \underset{n}{\otimes} i_n \ . \tag{2.1.17}$$

Imposing the gauge invariance in this manner allow us to present the solutions for the Gauss constraints, namely $\hat{G}_i\psi = 0$ where the spin network basis form a complete basis of the Hilbert space of solutions $\mathcal{H}^0_{kin}$ of it. $\mathcal{H}^0_{kin}$ decomposes as a direct sum over spaces on a fixed graph that subsequently decomposes as sum over intertwiner spaces, namely

$$\mathcal{H}^0_{kin} = \underset{\Gamma\subset\Sigma}{\oplus} \mathcal{H}^0_\Gamma \ , \tag{2.1.18}$$

$$\mathcal{H}^0_\Gamma = L_2\left[SU(2)^L/SU(2)^N, \mathrm{d}\mu_{Haar}\right] = \oplus_{j_l}\left(\otimes_n \mathcal{H}_n\right) \ . \tag{2.1.19}$$

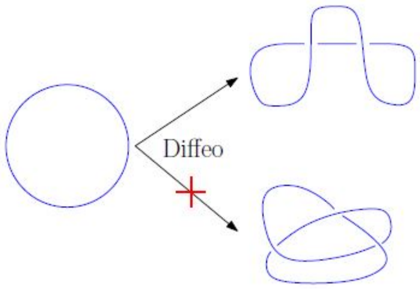
These equations are of the same nature of equations that simulate a Fock-decomposition of a Hilbert space.

2.2 Dynamics

The next step is to implement the spatial diffeomorphism quantum constraint. There are a couple of subtleties that we have to deal with during this procedure. Let us remind the reader that we are dealing at this stage with graphs that possess certain symmetries. In fact, each graph we work with is left unchanged under some diffeomorphism acting trivially on it. One can actually distinguish two cases: the diffeomorphisms that exchange the links among themselves without changing γ denoted by GS_γ , and those that preserve each link, and merely interchange the points inside the link denoted by $TDiff_\gamma$. Moreover, requiring invariance under diffeomorphism comes hand in hand with the problem of non-compactness of the group. Similarly to the Gauss law treated above and due to the notion of linear functional on $\mathcal{H}^0_{kin}$, namely $\eta(\hat{\phi}\psi) = \eta(\psi), \quad \forall\psi \in \mathcal{H}^0_{kin}$, with $\eta \in \mathcal{H}^{0*}_{kin}$ (the space of linear functionals), we can define a projector $\mathcal{P}_{Diff}$ on $\mathcal{H}_{Diff}$ in a way that we sum over all diffeomorphism except those that correspond to the trivial ones in $TDiff_\gamma$. Hence the expression for the projector yield

$$\langle\psi|\psi'\rangle_{Diff} \equiv \langle\psi\,|\mathcal{P}_{Diff}|\,\psi'\rangle = \sum_{\phi\in\mathrm{Diff}/\mathrm{TDiff}_\gamma} \left\langle\hat{\phi}\psi|\psi'\right\rangle \ . \tag{2.2.1}$$

This step amounts to ordering the spin network states into equivalence classes of graphs under diffeomorphisms which we will call *knots*.



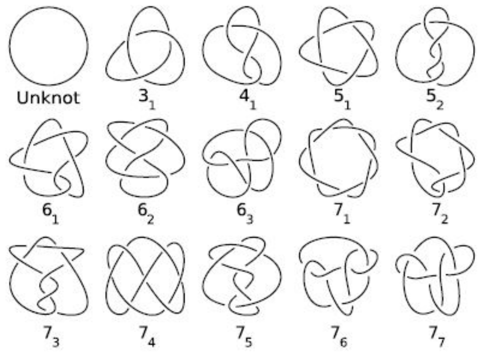


Figure 2.3: Left: Diffeomorphism acting via the embedding, leaving knots within the graph invariant. Right: examples of knots

Hamiltonian constraint

We proceed to the last step in the quantization procedure, namely the treatment of the Hamiltonian constraint. We start by rewriting it in terms of flux and holonomy. A way to do this is by using Thiemann's trick using the properties of the geometrical operator of the volume, and other established entities in section 2.3, which allows to write the classical Hamiltonian

$$\begin{aligned} H(N) &= \int \mathrm{d}^3x N\epsilon^{ij}_{k}\frac{E^a_iE^b_j}{\sqrt{\det(E)}}\left(F^k_{ab} - 2\left(1+\gamma^2\right)K^i_{[a}K^j_{b]}\right) \\ &= H^E(N) - 2\left(1+\gamma^2\right)T(N) \ . \end{aligned} \tag{2.2.2}$$

in terms of the flux and connection

$$H^E(N) = \int \mathrm{d}^3x N\epsilon^{abc}\delta_{ij}F^i_{ab}\left\{A^j_c, V\right\} \ , \tag{2.2.3}$$

$$T(N) = \int \mathrm{d}^3x\frac{N}{\gamma^3}\epsilon^{abc}\epsilon_{ik}\left\{A^i_a,\left\{H^E(1),V\right\}\right\}\left\{A^j_b,\left\{H^E(1),V\right\}\right\}\left\{\left\{A^k_c,V\right\}\right\} \ . \tag{2.2.4}$$

Exploiting the relation between the holonomy and the connection and consider a cellular decomposition C_I of Σ. We can write

$$H^E = \lim_{\epsilon\to 0}\sum_I \epsilon^N_I 3\epsilon^{abc}\,\mathrm{Tr}\left(F_{ab}\left\{A_c, V\right\}\right) \tag{2.2.5}$$

$$= \lim_{\epsilon\to 0}\sum_I N_I\epsilon^{abc}\,\mathrm{Tr}\left(\left(h_{\alpha_{ab}} - h^{-1}_{\alpha_{ab}}\right)h^{-1}_{e_c}\left\{h_{e_c}, V\right\}\right) \ . \tag{2.2.6}$$

We can specify the cellular decomposition in terms of a triangulation, namely a collection of tetrahedral cells. This can be promoted to a well defined operator

$$\widehat{H}^E = \lim_{\epsilon\to 0} \sum_I N_I \epsilon^{abc}\,\mathrm{Tr}\left(\left(\hat{h}_{\alpha_{ab}} - \hat{h}^{-1}_{\alpha_{ab}}\right)\hat{h}^{-1}_{e_c}\left[\hat{h}_{e_c}, \hat{V}\right]\right) \ . \tag{2.2.7}$$

This operator inherits the property of the volume operator of acting only on the nodes of the spin network. From the holonomies, it acts on the spin network by creating new links carrying spin $\frac{1}{2}$ around the node.

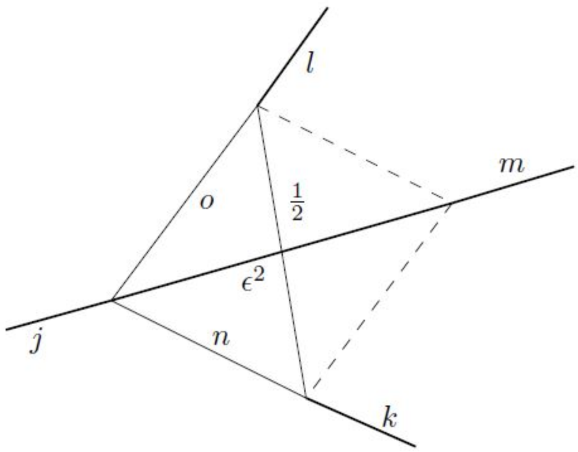


Figure 2.4: Action of the Hamiltonian operator.

2.3 Geometric operators

The main physical predication of the above outlined quantum gravity model, relies on the crucial role that some geometric operators play. In this paragraph, we will sketch their main properties. But before that, we need to specify the operator version of the fluxes and connection $(E{,}A)$ in order to specify the properties of the these geometric operators. Since the flux induces naturally a 2-form with values in the $SU(2)$ Lie algebra, we expect its operator to become an operator valued distribution. This naturally suggests that the smearing should look like

$$\begin{aligned}\widehat{E}[S,\alpha] &= \int_S d\sigma^1 d\sigma^2 \frac{\partial x^a}{\partial\sigma^1}\frac{\partial x^b}{\partial\sigma^2}\alpha^i \widehat{E^a_i}\epsilon_{abc} \\ &= -i\hbar\kappa\gamma \int_S d\sigma^1 d\sigma^2 \frac{\partial x^a}{\partial\sigma^1}\frac{\partial x^b}{\partial\sigma^2}\alpha^i \frac{\delta}{\delta A^i_c}\epsilon_{abc} \ ,\end{aligned} \tag{2.3.1}$$

where α is the smearing function. Notice that this expression presents a natural generalization of the notion of electric flux in the electromagnetism theory which sheds more light on its geometric implications. Furthermore, one can see that

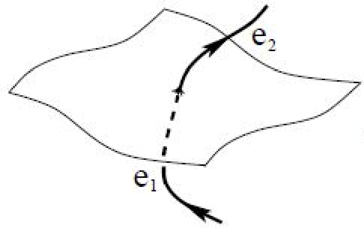
$$\begin{aligned}\frac{\delta}{\delta A^i_c} h_e[A] &= \frac{\delta}{\delta A^i_c}\left(\mathrm{P}\exp\int ds \dot{x}^d(s) A^k_d \tau_k\right) \\ &= \int ds \dot{x}^c(s)\delta^{(3)}(x(s)-x) h_{e_1}[A]\tau_i h_{e_2}[A] \ ,\end{aligned} \tag{2.3.2}$$

where $h_{e_1}[A]$ and $h_{e_2}[A]$ are the holonomies associated with the point on which the triad acts. Hence we get

$$\widehat{E}[S,\alpha]h_e[A] = -i8\pi\ell_p^2\gamma \int d\sigma^1 d\sigma^2 d\sigma^3 \frac{\partial x^a}{\partial \sigma^1}\frac{\partial x^b}{\partial \sigma^2}\frac{\partial x^c}{\partial s}\epsilon_{abc}\delta^{(3)}(x(\sigma),x(s))\alpha^i h_{e_1}[A]\tau_i h_{e_2}[A] \ . \tag{2.3.3}$$

Integrating the above expression one ends up with a very simple equation that dictates the action of the flux operator on the holonomy The flux operator is a self-adjoint operator and an

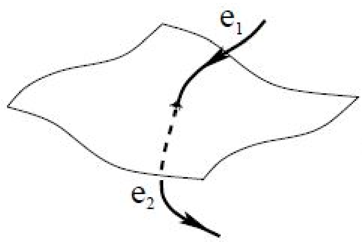
$$\widehat{E}[S,\alpha]h_e[A] = -i8\pi\ell_p^2\gamma\alpha^i h_{e_1}[A]\tau_i h_{e_2}[A]$$



,

and

$$\widehat{E}[S,\alpha]h_e[A] = i8\pi\ell_p^2\gamma\alpha^i h_{e_1}[A]\tau_i h_{e_2}[A]$$



,

$SU(2)$ gauge covariant. It also enjoys the property of encoding all the quantum Riemannian geometrical information of the space slice Σ_t and therefore it is naturally convenient to construct geometrical operators based on it.

2.3.1 The area operator

A surface S is characterized by its normal n_a and the densitized triad such that

$$A(S) = \int_S d\sigma_1 d\sigma_2 \sqrt{E_i^a E^{bi} n_a n_b} \ . \tag{2.3.4}$$

Recall that the densitized triad acts as a functional derivative (2.3.1). For a surface intersected only once by the holonomy path. We introduce a decomposition of S in N two-dimensional cells, and write the integral as the limit of a Riemann sum,

$$A(S) = \lim_{N\to\infty} A_N(S) \tag{2.3.5}$$

$$= \lim_{N\to\infty} \sum_{I=1}^{N} \sqrt{E_i(S_I)E^i(S_I)} \ , \tag{2.3.6}$$

where $E_i(S_I)$ is the flux going through the I'th cell. Promoting it to an operator amounts to defining the area operator in the following way

$$\hat{A}(S) = \lim_{N\to\infty} \hat{A}_N(S) \ , \tag{2.3.7}$$

The operator $\hat{E}^i$ now acts on a generic spin network state, labeling a generic graph γ. Therefore once the decomposition is sufficiently fine so that each S_I is punctured once and only once and hence taking a further refinement has no effect. The limit now amounts to simply sum the contributions of the finite number of punctures p of S created by the links of the graph which reads

$$\hat{A}(S) = \lim_{N\to\infty} \sum_{I=1}^{N} \sqrt{\hat{E}_i(S_I)\hat{E}^i(S_I)}\psi_{(\Gamma,f)} = \sum_{p\in S\cup\gamma} \hbar\sqrt{\gamma^2 j_p(j_p+1)}\psi_\gamma \ . \tag{2.3.8}$$

This expression presents the spectrum of the area operator. It is important to notice that this spectrum is completely known and quantized, meaning that the area can be assigned only discrete values, with minimal excitation being proportional to the squared Planck length $\ell_P^2 = \hbar G$. Moreover the operator acts diagonally on spin networks. Therefore, spin network states are eigenstates of the area operator.

2.3.2 Volume operator

Similarly to the area operator, we proceed by considering a partition of the region into cubic cells. This would allow to write the definition of the volume in terms of the flux and it reads

$$V(R) = \lim_{\epsilon\to 0} \sum_I \sqrt{\frac{1}{48}\epsilon_{ijk} \sum_{\alpha\beta\gamma} E_i(S_I^\alpha)E_j(S_I^\beta)E_k(S_I^\gamma)} \ , \tag{2.3.9}$$

where the surface of the cubic cells ∂C_I was divided into surfaces S^α in a way that $\cup_\alpha S_I^\alpha$. The size of the cell ϵ is sent to zero in the continuum limit and the cell shrinks to point x. Similarly to the area operator, one can notice that there is a convenient subdivision. This is achieved when the nodes of a graph γ can fall only in the interior of a cell and this cell contains at most one node. Moreover, notice the presence of the epsilon tensor implying that the three fluxes must be different and that the volume does not act on links. Hence the volume operator acts only on the *nodes* of the graph. In fact its matrix elements vanishes between different intertwiner spaces.

Since every intertwiner space is finite dimensional, the spectrum of the volume operator is discrete with minimal excitations proportional to the Planck length cube ℓ_P^3.

Chapter 3

Quantum reduced spherically symmetric geometries

The last step in our program to model quantum black holes [21] is the gauge fixing of the redundant degrees of freedom and finally proceed with the quantization scheme. However, it is crucial to specify which step comes ahead of the other, for all we are sure about is these two steps do not commute. The model we are adapting to sketch the quantum theory to describe spherical symmetric geometries is the so called Quantum Reduced Loop Gravity (QRLG) [33]. The basic idea behind the method of QRLG [21] is to implement the choice of symmetry compatible to the coordinate system at the quantum level rather than at the classical one. To have a better idea of this framework, it is handful to briefly go through its application in the context of homogeneous anisotropic cosmologies [19]. For Bianchi I spacetime, one can always choose a coordinate space in which the metric is diagonal and time dependent, making it a subspace of the full ADM phase space of homogeneous diagonal metrics [24].

To obtain a block diagonal metric, one usually proceeds with partial gauge fixing techniques, introducing second class constraints that has to be dealt with, and selecting a partially reduced phase space. A subspace of the latter is where coordinate independent metrics are singled out by symmetry. In this context, based on the classical picture of minisuperspaces and the procedure employed on them is to first derive the partially reduced phase space and then confine the study to its symmetric sector.
The QRLG approach to cosmology was devoted to access this sector at the quantum level as opposed to Loop Quantum Cosmology (LQC), in which the symmetry reduction is performed classically and one is then left with finite-dimensional systems. In fact, preforming the symmetry reduction beforehand of the quantization, one ends up with a Hilbert space where most of the full structure is lost. On the other hand, the advantage that the QRLG approach [21] present is to prevent this from occurring by reverting the process of symmetry reduction and quantization to derive a symmetric sector of LQG. To this end, one proceeds as follows: one selects first a Hilbert space with a diagonal metric from the full one and then performs the symmetry reduction, selecting homogeneous coherent states. This procedure provides us with the possibility to work with the complete structure of the full theory, consisting of quantum states of polymeric nature labeled by graphs and $SU(2)$ representations, as it will become clear later on. Moreover, it shows that the minisuperspace effective quantization of LQC [33] [18] can be reproduced at the level of the expectation values of quantum operators acting on the partially gauge fixed Hilbert space.

In this chapter, we will outline how this program is applied to spherical symmetric geometries. One starts by selecting the partially gauge fixed Hilbert space, namely by implementing the gauge fixing condition at the quantum level. This step amounts to having a triad with entries only in $E^r_3, E^\theta_1, E^\theta_2, E^\phi_1, E^\phi_2$ and a corresponding diagonal metric with non-vanishing components in $rr, \theta\theta, \theta\phi$ and $\phi\phi$. In the second step, the designated constraints in the full theory are projected to represent the classical gauge unfixed [34] constraints. The sequel step is then to define states belonging to this kinematical Hilbert space where the classical notion of symmetry can be inherited with help of spherically symmetric coherent states. Finally, one can introduce the notion of effective constraints by preforming the expectation value of the quantum reduced constraints on the symmetry reduced states [21].

In this chapter, we intend to go through the various step in the orthogonal gauge unfixing program for spherical symmetric geometries, to end up with an extended version of the set of constraints [34] (1.7.29). Once this is preformed, we present in a schematic manner the program of QRLG for the case of spherically symmetric geometries.

3.1 Orthogonal gauge fixing of first order gravity

In this section present the first order connection formulation of 4D general relativity in the "orthogonal" gauge. Working in the canonical formulation of general relativity amounts to dealing with a constrained system, as shown in chapter 1, where the phase space is parametrized by the symmetric 3-metric tensor and its conjugate momentum. This amounts $6 + 6 = 12$ local degrees of freedom, of which only four are physical. The extra redundant degrees of freedom are gauge degrees of freedom resulting from the four diffeomorphism constraints that arise in the canonical formulation of the Einstein-Hilbert action. In fact, an extra gauge redundancy is introduced in the first order Ashtekar connection formulation outlined in section 1.5, where the phase space configuration variables become a gauge connection and its conjugate momentum is a densitized triad giving rise to a total of 18 degrees of freedom. The additional 6 components are taken care of by 3 extra first class constraints associated to the local rotational invariance of the triad, yielding thus a total of 4 physical degrees of freedom.

These gauge symmetries are the reason underlying the difficulty to find explicit general solutions of general relativity and the reason why physical applications are usually limited to symmetry reduced cases. However, when one steps into the quantum theory, these difficulties are further amplified by the presence of ordering ambiguities in the quantization procedure and its non-anomaly free constraint algebra. Hence, also in the quantum theory one would like to implement a symmetry reduction scheme for physical applications.
The symmetry reduction strategy depends on an important choice, namely the one concerning the order in which the symmetry reduction is preformed and the quantization procedures. Indeed, the two steps in general do not commute and the relation between the quantum theories and its outcome of the two alternative choices (first reduction and then quantization or the other way around) is often difficult to assess. The classical symmetry reduction path is usually the simplest one as it is conceptually more straightforward and it makes the quantization process fairly attainable. This is oftentimes the route chosen in canonical quantum gravity in the quantization programs applied to cosmology and black hole physics. However this choice

cannot easily be used o obtain predictions, due to the ambiguities this procedure bring along the way: for instance in the cosmological case where one encounters the issue of defining a precise relation between Loop Quantum Cosmology and the full theory, as well as in in the black hole case, concerning the role of the Immirzi parameter in the recovery of the Bekenstein–Hawking entropy-area law.

In this chapter we will describe an alternative program, which interpolates between the two steps of reduction first or quantization first, namely, Quantum Reduced Loop Gravity (QRLG). Originally applied in a cosmological setting, it was also extended to the spherically symmetric sector of GR in the first order connection formulation, in order to apply it to the quantization of a Schwarzschild black hole geometry with LQG techniques. In this chapter, we will outline the first part of the analysis and show how to recast the classical phase space in a "orthogonal" gauge, that is compatible with a spherical symmetry reduction, mainly by completing the Dirac analysis. It turns out that the appearance of second class constraints in the theory is a consequence of the partial gauge fixing of the phase space canonical coordinates. The Dirac treatment of the system is successfully completed by utilizing the gauge unfixing procedure. The procedure of gauge unfixing is equivalent to the inversion of the Dirac matrix and provides the tool to work directly with the reduced phase space and the ordinary Poisson bracket. This treatment has the advantage of making calculations concerning the Hamiltonian constraint operator more easy to handle. However, since in the QRLG program we will not have to rely on point holonomies, there will be more degrees of freedom encapsulated by the reduced kinematical Hilbert space, allowing our construction to be closer to the full theory one and yielding additional quantum corrections at the effective level of the dynamics. Let us stress that, while our main motivation is to apply the results obtained here to the LQG quantization of a black hole, the classical treatment is interesting on its own, since it represents a successful study of a second class Hamiltonian system according to the Dirac program, allowing therefore to recast full 4D GR in the first order formulation in a partial gauge.

3.1.1 Constraints and spherical symmetry implementation

We are interested in spherical symmetric geometries in order to consider the physics underlying black holes. To this aim, the first step to proceed is to impose gauge conditions that best model such geometries. The starting point is to implement these gauge conditions in the (1.7.28) that is adapted to a reduction to spherical symmetry. We consider as usual a spacetime that admits a foliation by smooth 3D hypersurfaces Σ_t and work in the time gauge. Let us recall the action in this formulation

$$S = \frac{1}{16\pi G}\int dt \int_{\Sigma_t} d^3x \left(\frac{2}{\gamma}E^a_i \mathcal{L}_t A^i_a - NH - N^a V_a - \Lambda^i G_i\right) , \tag{3.1.1}$$

The action (3.1.1) defines the phase space coordinates in terms of an $SU(2)$ connection configuration variable A and its conjugate momentum E, and it describes a pure constraint theory as outlined in chapter 1. As we previously discussed in section 1.7, let us recall the constraints of GR in this formulation:

$$\begin{aligned}
G_i &= \partial_a E^a_i + \epsilon_{ij}^k A^j_a E^a_k \ , && \text{Gauss constraint}\\
H_a &= F^i_{ab} E^b_i \ , && \text{Vector constraint}\\
H &= \frac{\gamma E^a_i E^b_j}{2\sqrt{\det(E)}}\left(\epsilon^{ij}_k F^k_{ab} - 2(1+\gamma^2) K^i_{[a} K^j_{b]}\right) \ , && \text{Hamiltonian constraint}
\end{aligned} \tag{3.1.2}$$

The canonical Poisson brackets (PB) induced by the above action (3.1.1) are

$$\begin{aligned}\{A_a^i(\vec{x}), E_j(\vec{y})\} &= 8G\gamma\delta_a^b\delta_j^i\delta(\vec{x}-\vec{y}) \ , \\ \delta(\vec{x}-\vec{y}) &= \delta(r_x - r_y)\delta(\theta_x \ -\theta_y)\delta(\phi_x - \phi_y) \ .\end{aligned} \tag{3.1.3}$$

where the algebra of the constraints determined by (3.1.2) turns out to be first class. The next step is to parametrize a neighborhood of a point in a given constant time slice Σ_t. We therefore introduce a set of local spherical coordinates (r, θ, ϕ), that grants a natural way to work with radial geodesics and that can always be locally defined. For the subsequent steps, we do not need to specify the finite interval for the angular coordinates and we assume the radial coordinate to take values in the finite range $r \in [0, \overline{r}]$. Moreover, we make the further restricting requirement that the radial evolution vector has vanishing shift. This additional condition makes the simplification that r^a, the unit spacelike radial vector, is proportional to δ^{ar}. Given the above setup, the spatial index a takes values $a = r, \theta, \phi$, and the integration element in (3.1.1) becomes $d^3x = drd\theta d\phi$. The $SU(2)$ internal index i takes the values $i = 1, 2, 3$. Now, in order to fix the system in the desired gauge adapted to the foliation of Σ_t, we choose an "orthogonal" gauge defined by E_3^a being aligned with the unit spacelike radial vector r^a, which is equivalent to demanding, after decomposing along radial and tangential indices,

$$\begin{aligned} E_I^r &= 0, \quad I = 1,2 \\ E_3^A &= 0, \quad A = \theta, \phi \ . \end{aligned} \tag{3.1.4}$$

where capital letters $I, J, K\cdot$ label internal indices 1, 2 and capital letters A, B, C label tangential coordinates θ, ϕ. The set of equations in (3.1.4) can be considered as a set of four gauge conditions for our original theory (3.1.1) that provides a natural diagonal structure. This can be better interpreted once reprocessed in a matricial form with internal indices 3, I labeling rows and space indices labeling columns. Hence we can write (3.1.4) in the following form

$$\begin{bmatrix} E_3^r & 0 & 0 \\ 0 & E_1^\theta & E_1^\phi \\ 0 & E_2^\theta & E_2^\phi \end{bmatrix}$$

It is then evident the similarity with the radial gauge choice structure of the spatial metric h_{ab}, where h_{ab} is a block diagonal 3x3 matrix of the form

$$\begin{bmatrix} h_{rr} & 0 & 0 \\ 0 & h_{\theta\theta} & h_{\phi\theta} \\ 0 & h_{\phi\theta} & h_{\phi\phi} \end{bmatrix}$$

With the underlined imposition of the gauge conditions , we use the expression "orthogonal" gauge to denote the block-diagonal in the sense explained above. In order to look at the PB algebra between the gauge conditions (3.1.4) and the constraints (3.1.2), it is more convenient to work with the diffeomorphism constraint instead of the the vector constraint V_a generating spatial diffeomorphism on Σ_t, and taking the expression[1]

$$H_a = V_a - A_a^i G_i \ , \tag{3.1.5}$$

In order to proceed with the computation of the PB between the constraints and the gauge conditions, we adapt the following notation; we denote by N^a a smearing vector field having a

[1]From now on we work in units $8\pi G = 1$. Furthermore, indices in the same positions are not summed over, unless otherwise specified and we assume vanishing boundary conditions for the smearing functions.

non-vanishing component only along the a-th direction, $(N^a)^b = \eta^a \delta^{ab}$. Hence, the smearing $\vec{H}[\vec{N}^a]$ selects only the a-th component of H_b. For example, $N^\theta \equiv (0, \eta^\theta, 0)$ and $\vec{H}[\vec{N}^\theta] \equiv \int \eta^\theta H_\theta$. We also denote by $\vec{\Lambda}^i$ a vector in the internal space with the non vanishing component only along the i-th internal direction, $(\Lambda^i)^j = \lambda^i \delta^{ij}$. Hence, $\vec{G}[\vec{\Lambda^i}]$ picks up only the i-th component of G_j[2] With these conventions, $\vec{H}[\vec{N}]$ is the smeared diffeomorphisms constraint, yielding the equation

$$\vec{H}[\vec{N}] = \int d^3x N^a H_a \ . \tag{3.1.6}$$

Note that the condition $h_{rr} = 1$ implies, in terms of fluxes, $E^r = \epsilon_3^{IJ} E_I^\theta E_J$. We also assume vanishing boundary conditions for the above introduced smearing functions.
The PB brackets to be considered read

$$\{E_i^a, \vec{H}[\vec{N}]\} = \gamma \mathcal{L}_{\vec{N}} E_i^a = \gamma(N^b \partial_b E_i^a E_i \partial_a N^a + \partial_b N^b E_i^a) \ , \tag{3.1.7}$$

$$\{A_a^i, \vec{H}[\vec{N}]\} = \gamma \mathcal{L}_{\vec{N}} A_a^i = \gamma(N^b \partial_b A_a^i + A_b^i \partial_a N) \ , \tag{3.1.8}$$

With these conventions one obtains on the gauge surface selected by (3.1.4), (3.1.7)

$$\{E_I^r, \vec{H}[\vec{N}^A] \approx -\gamma E_I^B \partial_B \eta^A \delta^{Ar} = 0 \ , \tag{3.1.9}$$

$$\{E_I^r, \vec{H}[\vec{N}^r]\} \approx -\gamma E_I^A \partial_A \eta^r \ , \tag{3.1.10}$$

$$\{E_3^A, \vec{H}[\vec{N}^B]\} \approx -\gamma E_3^r \partial_r \eta^B \delta^{AB}, \tag{3.1.11}$$

$$\{E_3^A, \vec{H}[\vec{N}^r]\} \approx -\gamma E_3^r \partial_r \eta^r \delta^{Ar} = 0 \ , \tag{3.1.12}$$

where the symbol $\approx$ denotes projection of the phase space onto the gauge surface (3.1.4). It is well known the role that the Gauss constraint plays in the theory. In fact, it represents the constraint that generates the internal rotations orthogonal to the third internal direction and therefore it is not surprising that it is first class with E_I^r as well as E_3^A. On the other hand, we can see that E_I^r and E_3^A are second class with G_I, namely

$$\{E_I^r, \vec{G}[\vec{\Lambda}^J]\} \approx -\gamma \lambda^J \epsilon_I^J E_3^r \ , \tag{3.1.13}$$

$$\{E_3^A, \vec{G}[\vec{\Lambda}^J]\} \approx \gamma \lambda^J \epsilon^{JI} E_I^A \ . \tag{3.1.14}$$

Notice also that while E_3^A is second class only with H_A , E_I^r is second class only with H_r . The Dirac treatment of a second class Hamiltonian system was described in section 1.2. It is clear that, in order to preserve the number of physical degrees of freedom of the phase space, the second class constraints must be twice as many as the gauge conditions. In our case this implies that, since (3.1.4) are four conditions, four and only four out of the original seven constraints G_i, H_a and H are second class with them. Consequently, we are left with three residual first class constraints. They do not necessarily match the three constraints from the initial set, since they can be written as a linear combinations with the others. Once this splitting is done, one must invert the Dirac matrix that will enables us to implement the second class constraints through the Dirac brackets as in expressed in Eq (1.2.18). However, working out a representation of the Dirac brackets can be tedious, bringing along serious further complications at the quantization level. One way to avoid these ambiguities is to impose these second class constraints in the framework of the *gauge unfixing procedure* (GU). This alternative route provides a natural tool to directly manipulate the reduced phase space variables, while still using the ordinary Poisson

[2] In this notation, we can make the following example: $\Lambda^1 \equiv (\lambda^1,0,0)$ and $\vec{G}[\vec{\Lambda}^1] \equiv \int \lambda^1 G_1$.

brackets. Moreover, it provides a straightforward way to compute the gauge invariant residual first class constraints.
Applying the program outlined in section 1.2.1 to the case of interest of spherically symmetric geometries, one ends up with a new set of extended constraints ready to be manipulated once the quantization machinery is at play. In this setting, the GU consists of finding an extension of the phase space invariant under the flow of the gauge conditions. In the case of (3.1.4), this amounts to finding extensions of A_r^I and A_A^3. To avoid confusion, these extensions are denoted with a tilde: $\tilde{A}_r^I$ and $\tilde{A}_A^3$. They are obtained by adding to A_r^I and A_A^3 terms proportional to the original constraints

3.1.2 Constraints for the Kerr-Schild foliation

Since we are mainly interested in the Kerr-Schild metric, we can extract the extended constraints that will be useful for the subsequent work. A preliminary step is reproduce the constraints (not in the extended form yet) in the spherically reduced form. The Gauss and radial diffeomorphism in this case are computed to be

$$G_3 = \sin\theta\left(\partial_r E^r + 2A_1E^2 - 2A_2E^1\right) = 0\ , \tag{3.1.15}$$

$$\begin{aligned} H_r &= F^I_{rA}E^A_I - A^3_r G_3 & (3.1.16)\\ &= 2\sin\theta\left(\left(A_1' - A_2A_r\right)E^1 + \left(A_2' + A_1A_r\right)E^2\right) = 0\ . & (3.1.17)\end{aligned}$$

Whereas the Hamiltonian constraint reads

$$\begin{aligned} \kappa H &= -\frac{1}{\gamma^2}\left(H^E + H^L\right)\ , & (3.1.18)\\ &= -\frac{1}{\gamma^2}\left[\frac{\epsilon_k^{ij}E^a_iE^b_jF^k_{ab}}{\sqrt{\det(E)}} + \left(1+\gamma^2\right)\sqrt{\det(E)}R\right]\ . & (3.1.19)\end{aligned}$$

The Euclidean Hamiltonian in the above expression yields the equation

$$\begin{aligned} H^E &= \frac{\epsilon_k^{ij}E^a_iE^b_jF^k_{ab}}{\sqrt{\det(E)}} & (3.1.20)\\ &= \frac{\sin\theta}{\sqrt{\left((E^1)^2+(E^2)^2\right)E^r}}\Bigg[4E^rA_r\left(E^1A_1+E^2A_2\right) + 4E^r\left(E^1A_2' - E^2A_1'\right) \\ &\quad + 2\left(\left(E^1\right)^2+\left(E^2\right)^2\right)\left(\left(A_1^2+A_2^2\right)-1\right)\Bigg]\ , \end{aligned}$$

and we have for the Lorentzian part

$$\begin{aligned} H^L &= \left(1+\gamma^2\right)\sqrt{\det(E)}R & (3.1.21)\\ &= -\frac{(1+\gamma^2)}{2\sqrt{E^r}\left[(E^1(r))^2+(E^2(r))^2\right]^{3/2}} \\ &\quad\times\Bigg[-4\sin\theta\left[\left(E^2(r)\right)^2+\left(E^1(r)\right)^2\right]^2 \\ &\quad + \sin\theta\left[\left(E^1(r)\right)^2+\left(E^2(r)\right)^2\right]\left(\left(E^{r\prime}(r)\right)^2 + 4E^r(r)E^{r\prime\prime}(r)\right)\end{aligned}$$

$$
\begin{aligned}
&- 4\sin\theta E^r(r) E^{r'}(r)\left[E^1(r)E^{1'}(r) + E^2(r)E^{2'}(r)\right]\Bigg] \\
&= -\sin\theta\left(1+\gamma^2\right)\left[2\Lambda(r)\left(\frac{(R'(r))^2}{\Lambda^2(r)} - 1\right) + 4\frac{R(r)}{\Lambda(r)}\left(R''(r) - \frac{\Lambda'(r)R'(r)}{\Lambda(r)}\right)\right] .
\end{aligned}
$$

3.1.3 Extended constraints

At this point one can derive the extended version of the above considered constraints. Let us define the reduced radial diffeomorphism constraint as the constraint not containing the (3,r) component in the momenta. It then reads

$$
\mathcal{H}_r = \left(\partial_r A^I_A\right) E^A_I - A^3_r \partial_r E^r_3 \ , \tag{3.1.22}
$$

and the extended radial constraint reads

$$
\tilde{H}_r\left[N^r\right] \approx \mathcal{H}_r\left[N^r\right] + \int d\vec{x}\,\left(\partial_A N^r\right)\left[\frac{\epsilon^{IJ}E^A_I\partial_B E^B_J}{E^r_3} + \frac{\delta^{IJ}E^A_I E^B_J \mathcal{I}_B}{\left(E^r_3\right)^2}\right] , \tag{3.1.23}
$$

$$
\mathcal{I}_A \equiv \int_0^r dr'\left[D_A + E^r_3\partial_A A^3_r\right]_{r'} . \tag{3.1.24}
$$

The Hamiltonian constraint splits into its Lorentzian and Euclidean parts. In the quantum theory, the Lorentzian part is traditionally treated by rewriting it in terms of commutators of the Euclidean part with the volume operator. The reduced Euclidean Hamiltonian can be written in the form

$$
\mathcal{H}_E \approx \frac{\gamma}{\sqrt{\det(E)}}\left(E^r_3 E^A_I \epsilon^I_J \partial_r A^J_A + E^A_I E^B_J A^I_{[A}A^J_{B]} + E^r_3 E^A_I A^3_r A^I_A\right) . \tag{3.1.25}
$$

The extended version then yields

$$
\begin{aligned}
\tilde{H}_E \approx\ & \mathcal{H}_E + \frac{\gamma}{\sqrt{\det(E)}} \\
&\Bigg\{E^A_I E^B_J\left[-\delta^{IJ}\frac{\mathcal{I}_A\mathcal{I}_B}{\left(E^r_3\right)^2} + \epsilon^{IJ}\partial_A\left(\frac{\mathcal{I}_B}{E^r_3}\right)\right] \\
&- E^r_3 E^A_I\left[\frac{\epsilon^{IJ}\left(\partial_B E^B_J\right)\mathcal{I}_A}{\left(E^r_3\right)^2} + \partial_A\left(\frac{\epsilon^{IJ}E^B_J\mathcal{I}_B}{\left(E^r_3\right)^2} - \frac{\delta^{IJ}\partial_B E^B_J}{E^r_3}\right)\right]\Bigg\} ,
\end{aligned} \tag{3.1.26}
$$

where for simplicity we defined

$$
\begin{aligned}
\gamma^A =& \int_r^{\overline{r}} dr'\Bigg[\frac{\partial_B}{E^r_3}\left(\frac{N E^{[A}_I E^{B]}_J \epsilon^{IJ}}{\sqrt{\det(E)}}\right) \\
&+ \frac{E^A_I\partial_B}{\left(E^r_3\right)^2}\left(\frac{N\epsilon^{IJ}E^B_J E^r_3}{\sqrt{\det(E)}}\right) - \frac{N E^A_I}{\sqrt{\det(E)}}\left(\frac{E^B_J\delta^{IJ}A^3_B}{E^r_3} + A^I_r + \frac{\epsilon^{IJ}}{E^r_3}G_J\right) \\
&- \frac{N E^A_I E^B_J \delta^{IJ}}{\sqrt{\det(E)}\left(E^r_3\right)^2}\int_0^{r'} dr'' H_B\left(r''\right)\Bigg]_{r'}
\end{aligned} \tag{3.1.27}
$$

$$
\gamma^I = -\frac{\partial_A}{E^r_3}\left(\frac{N\delta^{IJ}E^A_J E^r_3}{\sqrt{\det(E)}}\right) - \frac{N\epsilon^{IJ}E^A_J A^3_A}{\sqrt{\det(E)}} .
$$

This analysis presents the basis for the quantum description of black holes that we will soon encounter. More precisely, the orthogonal gauge fixing outlined above will allow us to deal with the spherically symmetry reduction of a 3D spatial geometry.
The strategy is to generalize techniques introduced for cosmological applications within the framework of Quantum Reduced Loop Gravity to impose the gauge fixing conditions in terms of expectation values on kinematical quantum states of the full theory. We can then use these reduced spin networks to construct coherent states for a Kerr-Schild quantum geometry, thus activating the spherical symmetry reduction at the quantum level. The quantum dynamics will be encoded in the operatorial version of the extended Euclidean Hamiltonian constraint (and its Lorentzian contribution as well). Time evolution of the Kerr-Schild geometry initial data according to the resulting modified semi-classical Hamiltonian is expected to give rise to an effective quantum corrected metric.

3.2 Quantum reduction implemented for black holes

One can now proceed with the symmetry reduction at the quantum level. This is performed by starting with the full kinematical Hilbert space $\mathcal{H}^K$ of the full theory and then implementing the reduction on $\mathcal{H}^K$.
In this section we point out the main steps to construct a reduced kinematical Hilbert space that reflects at the quantum level the radial partial gauge fixing for spherical geometries. The strategy is the following; we start by the full theory Hilbert space, which for simplicity will be denoted in the following as $\mathcal{H}^K$. We then demand a weak version of the condition (3.1.4) restricting therefore the non-gauge-invariant spin network basis states contained in $\mathcal{H}^R$. The geometrical set up for this scheme relies on the choice of the spatial manifold triangulation convenient to describe the topology we are interested in. This can be naturally depicted by choosing cuboidal triangulations for the spherically symmetric geometry. At each vertex we have three directions, one corresponding to the radial direction and the other two to the angular directions on the 2-spheres foliating the spatial manifold.
Later on, we will have to deal with the specific choice of subclass of graphs of spin networks basis to complete the precess of reduced quantization.

3.2.1 Reduced spin network states

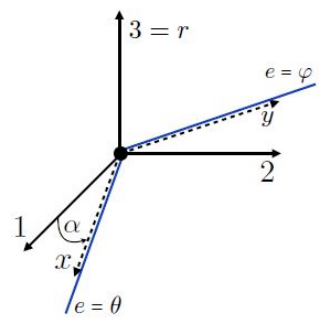


Figure 3.1: Tangent and internal directions

Let us consider our choice of the flux's smearing surface. We consider fluxes to two-dimensional surface S^a whose normal vectors are aligned with the tangent vectors (r, θ, φ). We work in two

$su(2)$ orthonormal basis, namely $\{x, y, z\}$ and $\{1, 2, 3\}$ such that the elements 3 and z coincide. Now, we take the direction r to be aligned with the internal direction 3 while the directions θ and φ are aligned with x and y. This set up basically amounts to imposing the following gauge fixing

$$E_3\left(S^\theta\right) = 0 = E_3\left(S^\varphi\right) \ , \tag{3.2.1}$$

$$E_1\left(S^r\right) = 0 = E_2\left(S^r\right) \ , \tag{3.2.2}$$

where

$$E_i\left(S^a\right) = \int E_i^a n_a d\sigma_1 d\sigma_2 \ . \tag{3.2.3}$$

At the quantum level, this gauge fixing translates to choosing the edges l_e aligned with the three directions $\{r, \theta, \varphi\}$ such that $\dot{l}_e^a \propto \delta_e^a$ and the holonomies in this configuration read

$$g_e = \mathcal{P}_e \int_{\ell_e} \tau_i A_a^i \dot{\ell}_e^a(s) ds \ . \tag{3.2.4}$$

The 2-sphere is equipped at a each value of component r with a grid of plaquettes carrying edges labeled by the tangential coordinates θ and φ, with $\epsilon_r, \epsilon_\theta,$ and ϵ_φ denote the coordinate

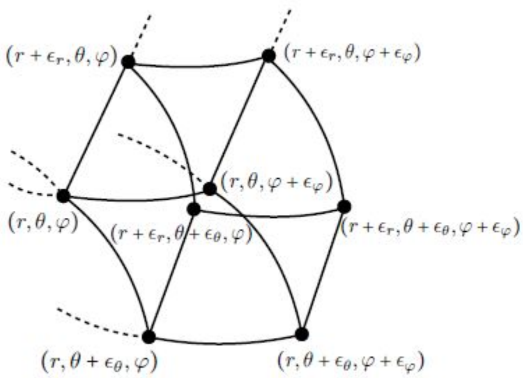


Figure 3.2: Tangent and internal directions

lengths in the tangential directions.
The projector operator acting on $\mathcal{H}^K = \oplus_\gamma \mathcal{H}_\gamma^K$ proceed in two steps. It will first act on restricted γ to Γ and then by projecting $\mathcal{H}_\Gamma^K$ to its reduced subspace $\mathcal{H}_\Gamma^R$. The kinematical Hilbert space is then

$$\mathcal{H}^R = \oplus_\Gamma \mathcal{H}_\mathrm{r}^R \ . \tag{3.2.5}$$

The second step is to implement an $SU(2)$ gauge fixing acting on the basis of $\mathcal{H}_\mathrm{r}^R$. The basis that label each link in the reduced Hilbert space in a given tangential direction can be written as

$$\begin{aligned} {}^x D^{j_x}_{m_x \overline{n}_x}\left(g_\theta\right) &= \left\langle \overline{m}_x, \vec{u}_x \left| D^{j_x}\left(g_\theta\right) \right| \overline{n}_x, \vec{u}_x \right\rangle = \left\langle g_\theta | x, j_x, \tilde{m}_x, \tilde{n}_x \right\rangle \ , \\ {}^y D^{j_y}_{m_y \overline{n}_y}\left(g_\varphi\right) &= \left\langle \overline{m}_y, \vec{u}_y \left| D^{j_y}\left(g_\varphi\right) \right| \overline{n}_y, \vec{u}_y \right\rangle = \left\langle g_\varphi | y, j_y, \tilde{m}_y, \tilde{n}_y \right\rangle \ , \\ D^{j_z}_{m_z \overline{n}_z}\left(g_r\right) &= \left\langle \overline{m}_z, j_z \left| D^{j_z}\left(g_r\right) \right| j_z, \overline{n}_z \right\rangle = \left\langle g_r | r, j_r, \tilde{m}_z, \tilde{n}_z \right\rangle \ . \end{aligned} \tag{3.2.6}$$

that can also be expressed as

$${}^I D^{j_I}_{m_I \overline{n}_I}(g) = D^{j_I -1}_{m_I m}\left(u_I\right) D^{j_I}_{mn}(g) D^{j_I}_{n \overline{n}_I}\left(u_I\right) \ , \tag{3.2.7}$$

once we introduce the orthogonal unit vector $\vec{u}_I$ (with $I \in \{x, y\}$) on the (1,2)$-$ plane

$$|\overline{n}_I, \vec{u}_I\rangle = D^{j_I}\left(\vec{u}_I\right)|j_I, \overline{n}_I\rangle = \sum_m |j_I, m\rangle\, D^{j_I}_{m\overline{n}_I}\left(u_I\right) \ . \tag{3.2.8}$$

The gauge fixing in (3.2.1) amount to choosing the parametrization

$$\begin{aligned} u_x &= R\left(\alpha, \frac{\pi}{2}, 0\right) = e^{\alpha\tau_3} e^{\frac{\pi}{2}\tau_2} \ , \\ u_y &= R\left(\alpha + \frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\right) = e^{\left(\alpha+\frac{\pi}{2}\right)\tau_3} e^{\frac{\pi}{2}\tau_2} e^{-\frac{\pi}{2}\tau_3} \ . \end{aligned} \tag{3.2.9}$$

Notice that the angle α is independent of the $\tilde{\alpha}$ entering the classical construction. Now, implementing the residual $U(1)$ gauge invariance amounts to integrating over the α in the reduced states. Meanwhile we keep $\tilde{\alpha}$, around which our semi-classical states are peaked, fixed. It is important to stress that unlike the reduced states usually built for cosmological applications, in (3.2.6) the off-diagonal terms are also taken into consideration. Those are the states peaked on maximum-minimum magnetic numbers and not just maximum-maximum or minimum-minimum. They play an important role in the case of black hole, since the symmetry reduced Ashtekar-Barbero connection contains general off-diagonal terms. It follows then, that the computation of the expectation value of the fluxes components is a straightforward task.

Quantum Gauss constraint

The projection of the kernel of the full theory Gauss constraint is done through the operation

$$P^{\dagger}\hat{G}_i P \ . \tag{3.2.10}$$

When we were studying the kinematical Hilbert space of the full theory, we encountered that the kernel of $\hat{G}_i$ is given by the well-known gauge-invariant spin network states obtained by contraction with the $SU(2)$ intertwiners at the nodes of the graphs. In this case, the gauge fixed operation (3.2.10) represents a map between the kernel of $\hat{G}_i$ to the kernel of ${}^R\hat{G}_3$, representing the classical phase space reduction. The states annihilated by ${}^R\hat{G}_3$ are now α invariant, where α is parameter of rotation around the internal 3-axis that has been aligned to the r-axis by the projector P.

Quantum vector constraint

The quantum vector constraint is imposed in the full theory by group averaging the spin network states over spatial diffeomorphisms as we have studied in section 2. In fact, solving the finite version of the vector constraint is basically the quest of finding all ψ that satisfy

$$\mathcal{U}[\phi]\psi = \psi \ . \tag{3.2.11}$$

This equation, however, has no nontrivial solutions in $\mathcal{H}^K$. Nonetheless, it can be solved for $\psi \in \text{Cyl}^*$. In principle one should try to find the kernel of $\hat{\tilde{H}}_r$. But this goal is too ambitious at the moment. Although, we know $\tilde{H}_r =^R \tilde{H}_r$ classically and in the case where the shift vector N^r depends only on the radial direction. Therefore, we may assume that the kernels of the corresponding quantized operators coincide. We expect that averaging the kinematical states constructed here over the group of radial diffeomorphisms will provide the required solutions to Eq. (3.2.11).

Semiclassical states

To derive an effective Hamiltonian for the reduced version of the constraints, one can rely on the semi-classical states. The construction of these states in $\mathcal{H}^R$ is based on the heat kernel of the Laplace operator for each edge l of the corresponding graph operating on the delta function of its respective $SU(2)$ group element g_ℓ. Coherent semi-classical states are then obtained through analytic continuation carried from $g \in SU(2)$ to $G \in SL(2,\mathbb{C})$, namely

$$\psi_G^\lambda\left(g_\ell\right) = K_\lambda\left(g_\ell, G\right) \ , \tag{3.2.12}$$

where

$$G = g\exp\left(i\frac{\lambda}{\kappa\gamma}E_i\left(S^\ell\right)\tau^i\right) \ , \tag{3.2.13}$$

plays the role of the complexifier. The semi classical states for the three tangential directions read

$$\begin{aligned}
\psi_G^\lambda\left(g_r\right) &= \sum_{j_z=0}^{\infty}\sum_{\overline{m}_z,\overline{n}_z}\left(2j_z+1\right)e^{-\frac{\lambda}{2}j_z(j_z+1)}D^{j_z}_{\overline{m}_z\overline{n}_z}\left(g_r^{-1}\right)D^{j_z}_{\overline{n}_z\overline{m}_z}\left(e^{\epsilon_r A_r\tau_3}e^{i\frac{\lambda\delta_r^2}{\kappa\gamma}E^r\sin\theta\tau_3}\right)\\
&= \sum_{j_z=0}^{\infty}\sum_{\overline{m}_z}\left(2j_z+1\right)e^{-\frac{\lambda}{2}j_z(j_z+1)}e^{\lambda\overline{m}_z\frac{\delta_r^2E^r\sin\theta}{\kappa\gamma}}D^{j_z}_{\overline{n}_z\overline{m}_z}\left(e^{\epsilon_r A_r\tau_3}\right)D^{j_z}_{\overline{m}_z\overline{n}_z}\left(g_r^{-1}\right) \ ,\\
\psi_G^\lambda\left(g_\theta\right) &= \sum_{j_x=0}^{\infty}\sum_{\overline{m}_x,\overline{n}_x}\left(2j_x+1\right)e^{-\frac{\lambda}{2}j_x(j_x+1)x}D^{j_x}_{\overline{m}_x\overline{n}_x}\left(g_\theta^{-1}\right)^xD^{j_x}_{\overline{n}_x\overline{m}_x}\left(e^{\epsilon_\theta(A_1\tau_1+A_2\tau_2)}e^{i\frac{\lambda\delta_\theta^2}{\pi\gamma}\left(E^1\tau_1+E^2\tau_2\right)\sin\theta}\right)\\
&= \sum_{j_x=0}^{\infty}\sum_{\overline{m}_x,\overline{n}_x}\left(2j_x+1\right)e^{-\frac{\lambda}{2}j_x(j_x+1)}e^{\lambda\overline{m}_x\frac{\delta_\theta^2E^x}{\kappa\gamma}x}D^{j_x}_{\overline{n}_x\overline{m}_x}\left(e^{\epsilon_\theta(A_1\tau_1+A_2\tau_2)}\right)^xD^{j_x}_{\overline{m}_x\overline{n}_x}\left(g_\theta^{-1}\right) \ ,\\
\psi_G^\lambda\left(g_\varphi\right) &= \sum_{j_y=0}^{\infty}\sum_{\overline{m}_y,\overline{n}_y}\left(2j_y+1\right)e^{-\frac{\lambda}{2}j_y(j_y+1)y}D^{j_y}_{\overline{m}_y\overline{n}_y}\left(g_\varphi^{-1}\right)^yD^{j_y}_{\overline{n}_y\overline{m}_y}\left(e^{\epsilon_\varphi((A_1\tau_2-A_2\tau_1)\sin\theta)}e^{i\frac{\lambda\delta_\varphi^2}{\kappa\gamma}\left(E^1\tau_2-E^2\tau_1\right)}\right)\\
&= \sum_{j_z=0}^{\infty}\sum_{\overline{m}_z,\overline{n}_z}\left(2j_y+1\right)e^{-\frac{\lambda}{2}j_y(j_y+1)}e^{\lambda\overline{m}_y\frac{\delta_\varphi^2E^y}{\kappa\gamma}}yD^{j_y}_{\overline{n}_y\overline{m}_y}\left(e^{\epsilon_\varphi((A_1\tau_2-A_2\tau_1)\sin\theta)}\right)^yD^{j_y}_{\overline{m}_y\overline{n}_y}\left(g_\varphi^{-1}\right) \ ,
\end{aligned} \tag{3.2.14}$$

where $\delta_x^2 = \epsilon_r\epsilon_\varphi, \delta_y^2 = \epsilon_r\epsilon_\theta, \delta_z^2 = \epsilon_\theta\epsilon_\varphi$. Notice that for $j_x, j_y, j_z \gg 1$, the coefficients appearing in the coherent states become Gaussian weights, peaking around the geometry around the classical data. The normalized reduced semi-classical states are then

$$\widetilde{\psi_G^\lambda}\left(g_\ell\right) = \frac{\psi_G^\lambda\left(g_\ell\right)}{\left|\psi_G^\lambda\left(g_\ell\right)\right|} \ . \tag{3.2.15}$$

Reduced Hamiltonian operator

The Hamiltonian constraint in (2.2.7) has a well defined action on a graph-dependent triangulation of the spacelike hypersurfaces. One can implement this construction to the cubulation we are adapting to define $\mathcal{H}^R$, by borrowing techniques developed in the cosmological sector [33]. Since the classical Hamiltonian operator in (2.2.7) inherits the action of the volume operator on nodes, one should first describe the action on a three valent nodes and define the recoupling theory. The reduced 3-valent vertex state, denoted by $\left|v_3^R(j)\right\rangle$, is obtained from the gauge

invariant spin network version of (3.2.9), after projecting them on ${}^4\mathcal{H}$. Keeping in mind that the Haar measure is confined to the $U(1)$ rotation around the z-axis, one finally derives the reduced holonomies. Now, let us consider a region containing the 3-valent node v. It is straightforward to derive the action of the reduced volume operator regularized on the cube dual to v by using the properties of the reduced fluxes, manifesting a diagonal action on the reduced 3-valent vertex states that render the computation for the Hamiltonian constraint simpler. The action of the holonomy operator on the reduced Hilbert space is prescribed by the recoupling rules for our states, derived in [21].
The reduced Hamiltonian we intend to present is an operator constructed out of the reduced fluxes and holonomies and its action is restricted to the reduced states. Using the recoupling rules and the action of the reduced volume operator, the computations greatly simplify thanks to the gauge we imposed and one can derive the action of the ${}^RH$ on the reduced states.
Taking into account that the quantization procedure acting on the full expression of the spherically symmetric Hamiltonian constraint, including the additional terms resulting from the phase space extension of the gauge unfixing procedure, one obtains an effective Hamiltonian constraint by computing the expectation value of these contributing parts[3]. Taking the continuum limit $\epsilon_r, \epsilon_\theta, \epsilon_\varphi \to 0$, one obtains an effective Hamiltonian constraint for a spherically symmetric spacetime including quantum corrections. The effective Hamiltonian for the general case of spherically has been derived in [21].

3.3 Kerr-Schild effective Hamiltonian constraint

The computation of the effective Hamiltonian for the Kerr-Schild foliation is based on the technology outlined above, applied to the gauge unfixed extended constraint in Eq(3.1.18). The effective Hamiltonian in the $\alpha = 0$ gauge is then the sum of the Euclidean and Lorentzian constraints, where the Euclidean reads

$$\begin{aligned}
\mathcal{H}^E = 4\sqrt{E^r}\Bigg\{ \epsilon_\theta \Bigg[&\cos\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\sin\theta\epsilon_\phi\right]\sin\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\sin\theta\epsilon_\phi\right] \\
&\times \frac{\left(\sin\left[\frac{A_r(r)+A_r(r+\epsilon_r)}{2}\epsilon_r\right]A_1(r+\epsilon_r)+\cos\left[\frac{A_r(r+\epsilon_r)+A_r(r)}{2}\epsilon_r\right]A_2(r+\epsilon_r)\right)}{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}} \\
&-\sin\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\sin\theta\epsilon_\phi\right]\cos\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\sin\theta\epsilon_\phi\right] \\
&\times \frac{\left(\sin\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_1(r)+\cos\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_2(r)\right)}{\sqrt{A_1^2(r)+A_2^2(r)}}\Bigg] \\
&+\epsilon_\phi\sin\theta\Bigg[\cos\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\epsilon_\theta\right]\sin\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\epsilon_\theta\right] \\
&\times \frac{\left(\sin\left[\frac{A_r(r)+A_r(r+\epsilon_r)}{2}\epsilon_r\right]A_1(r+\epsilon_r)+\cos\left[\frac{A_r(r+\epsilon_r)+A_r(r)}{2}\epsilon_r\right]A_2(r+\epsilon_r)\right)}{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}
\end{aligned}$$

[3] The quantization of the Lorentzian piece follows from using its expression in terms of the densitized scalar curvature expressed as a function of the fluxes and their derivatives.

$$
\begin{aligned}
& - \sin\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\epsilon_\theta\right]\cos\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\epsilon_\theta\right] \\
& \times \frac{\left(\sin\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_1(r)+\cos\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_2(r)\right)}{\sqrt{A_1^2(r)+A_2^2(r)}}\Bigg]\Bigg\} \\
& + 2\epsilon_r\frac{E^1}{\sqrt{E^r}}\sin\left[\sqrt{A_1^2(r)+A_2^2(r)}\epsilon_\theta\right]\sin\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\left(\sin\theta+\sin(\theta+\epsilon_\theta)\right)\epsilon_\phi\right] \\
& + 2\frac{\epsilon_r\epsilon_\phi}{\epsilon_\theta}\frac{E^1}{\sqrt{E^r}}\left(\sin(\theta+2\epsilon_\theta)-2\sin(\theta+\epsilon_\theta)+\sin\theta\right) \ , \qquad (3.3.1)
\end{aligned}
$$

and the Lorentzian term reads

$$
\begin{aligned}
\mathcal{H}^L = & -\frac{(1+\gamma^2)}{2\sqrt{E^r(r)}\left(E^1(r)\right)^2} \\
& \times \Bigg[4\frac{\epsilon_r\epsilon_\varphi}{\epsilon_\theta}\left(E^1(r)\right)^3\left(\sin\left(\theta+2\epsilon_\theta\right)-2\sin\left(\theta+\epsilon_\theta\right)+\sin\theta\right) \\
& + \frac{\epsilon_\theta\epsilon_\varphi}{\epsilon_r}\sin\theta E^1(r)\Big(\left(E^r\left(r+\epsilon_r\right)-E^r(r)\right)^2+4E^r(r)\left(E^r\left(r+2\epsilon_r\right)-2E^r\left(r+\epsilon_r\right)+E^r(r)\right)\Big) \\
& - 4\frac{\epsilon_\theta\epsilon_\varphi}{\epsilon_r}\sin\theta E^r(r)\left(E^r\left(r+\epsilon_r\right)-E^r(r)\right)\Big(E^1\left(r+\epsilon_r\right)-E^1(r)\Big)\Bigg] \ .
\end{aligned}
$$

Hence, the total effective Hamiltonian reads

$$
\begin{aligned}
-2\kappa\gamma^2\mathcal{H} = 4\sqrt{E^r}\Bigg\{ & \epsilon_\theta\Bigg[\cos\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\sin\theta\epsilon_\phi\right]\sin\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\sin\theta\epsilon_\phi\right] \\
& \times \frac{\left(\sin\left[\frac{A_r(r)+A_r(r+\epsilon_r)}{2}\epsilon_r\right]A_1(r+\epsilon_r)+\cos\left[\frac{A_r(r+\epsilon_r)+A_r(r)}{2}\epsilon_r\right]A_2(r+\epsilon_r)\right)}{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}} \\
& - \sin\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\sin\theta\epsilon_\phi\right]\cos\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\sin\theta\epsilon_\phi\right] \\
& \times \frac{\left(\sin\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_1(r)+\cos\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_2(r)\right)}{\sqrt{A_1^2(r)+A_2^2(r)}}\Bigg] \\
& + \epsilon_\phi\sin\theta\Bigg[\cos\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\epsilon_\theta\right]\sin\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\epsilon_\theta\right] \\
& \times \frac{\left(\sin\left[\frac{A_r(r)+A_r(r+\epsilon_r)}{2}\epsilon_r\right]A_1(r+\epsilon_r)+\cos\left[\frac{A_r(r+\epsilon_r)+A_r(r)}{2}\epsilon_r\right]A_2(r+\epsilon_r)\right)}{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}} \\
& - \sin\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\epsilon_\theta\right]\cos\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\epsilon_\theta\right] \\
& \times \frac{\left(\sin\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_1(r)+\cos\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_2(r)\right)}{\sqrt{A_1^2(r)+A_2^2(r)}}\Bigg]\Bigg\} \\
& + 2\epsilon_r\frac{E^1}{\sqrt{E^r}}\sin\left[\sqrt{A_1^2(r)+A_2^2(r)}\epsilon_\theta\right]\sin\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\left(\sin\theta+\sin(\theta+\epsilon_\theta)\right)\epsilon_\phi\right]
\end{aligned}
$$

$$
\begin{aligned}
&- 2\gamma^2 \frac{\epsilon_r\epsilon_\phi}{\epsilon_\theta}\frac{E^1}{\sqrt{E^r}}\left(\sin(\theta+2\epsilon_\theta) - 2\sin(\theta+\epsilon_\theta) + \sin\theta\right) \\
&- (1+\gamma^2)\frac{\sin\theta}{2(E^1(r))^2\sqrt{E^r(r)}}\frac{\epsilon_\theta\epsilon_\phi}{\epsilon_r} \\
&\times \Bigg\{ E^1(r)\left[(E^r(r+\epsilon_r)-E^r(r))^2 + 4E^r(r)\left(E^r(r+2\epsilon_r) - 2E^r(r+\epsilon_r) + Er(r)\right)\right] \\
&- 4E^r\left(E^r(r+\epsilon_r) - E^r(r)\right)\left(E^1(r+\epsilon_r) - E^1(r)\right)\Bigg\} . \qquad (3.3.2)
\end{aligned}
$$

The above expression for the Hamiltonian (3.3.2) can be further simplified, namely by summing over the angular plaquettes. To this end, it is useful to introduce a set of parameters that will set up the coordinate lengths of the plaquettes on the 2-sphere as $\epsilon_\theta = \frac{\pi}{N_\theta}, \quad \epsilon_\varphi = \frac{2\pi}{N_\varphi}$, where N_ϕ and N_θ are two integers such that $N_\phi N_\theta$ is the total number of the plaquettes on the 2-sphere. In the limit in which these numbers are large, one can see

$$\int_0^\pi d\theta = \sum_{n_\theta=0}^{N_\theta} \epsilon_\theta = \pi \ , \qquad (3.3.3)$$

$$\int_0^{2\pi} d\phi = \sum_{n_\phi=0}^{N_\phi} \epsilon_\phi = 2\pi \ , \qquad (3.3.4)$$

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta = \lim_{N_\theta,N_\phi\to\infty} \sum_{n_\phi=0}^{N_\phi}\sum_{n_\theta=0}^{N_\theta} \epsilon_\phi\epsilon_\theta \sin\left(\frac{\pi n_\theta}{N_\theta}\right) = \lim_{N_\theta\to\infty}\frac{2\pi^2}{N_\theta}\cot\left(\frac{\pi}{2N_\theta}\right) = 4\pi \ . \qquad (3.3.5)$$

Hence, in this limit and keeping in mind that the Hamiltonian is independent of the angle φ, one can write the sum over a given 2-sphere plaquettes of the effective Hamiltonian constraint as follows

$$
\begin{aligned}
\lim_{\epsilon_\theta,\epsilon_\phi\to\infty}\sum_{p\in S^2}\mathcal{H} &= \lim_{\epsilon_\theta,\epsilon_\phi\to\infty}\frac{1}{\epsilon_\phi\epsilon_\theta}\sum_{n_\phi n_\theta}\epsilon_\phi\epsilon_\theta\mathcal{H} \qquad (3.3.6) \\
&= \lim_{\epsilon_\theta,\epsilon_\phi\to\infty}\frac{1}{\epsilon_\phi\epsilon_\theta}\int_0^{2\pi} d\phi\int_0^\pi d\theta\mathcal{H} \\
&= \lim_{N_\theta,N_\phi\to\infty}\frac{N_\theta N_\phi}{\pi}\int_0^\pi d\theta\mathcal{H} \ .
\end{aligned}
$$

The sum over angular plaquettes yields the total effective Hamiltonian constraint

$$
\begin{aligned}
-\kappa\gamma^2\tilde{H} &:= -\kappa\gamma^2 \lim_{\epsilon_\theta,\epsilon_\phi\to\infty}\sum_{p\in S^2}\mathcal{H} \\
&= \lim_{N_\theta,N_\phi\to\infty}\Bigg\{\frac{4\pi}{\epsilon_\phi}\sqrt{Er(r)}\Bigg\{\Bigg[\Bigg(H_0\left[\left(\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)} + \sqrt{A_1(r)^2+A_2(r)^2}\right)\epsilon_\phi\right] \\
&+ H_0\left[\left(\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)} - \sqrt{A_1(r)^2+A_2(r)^2}\right)\epsilon_\phi\right]\Bigg) \\
&\times\left(\frac{\sin\left[\frac{A_r(r)+A_r(r+\epsilon_r)}{2}\epsilon_r\right]A_1(r+\epsilon_r) + \cos\left[\frac{A_r(r)+A_r(r+\epsilon_r)}{2}\epsilon_r\right]A_2(r+\epsilon_r)}{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}\right) +
\end{aligned}
$$

$$
\begin{aligned}
&- \Bigg[\Bigg(H_0\Bigg[\left(\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}+\sqrt{A_1^2(r)+A_2^2(r)}\right)\epsilon_\phi\Bigg] \\
&- H_0\Bigg[\left(\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}-\sqrt{A_1^2(r)+A_2^2(r)}\right)\epsilon_\phi\Bigg]\Bigg) \\
&\times \frac{\left(\sin\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_1(r)+\cos\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_2(r)\right)}{\sqrt{A_1^2(r)+A_2^2(r)}}\Bigg]\Bigg\} \\
&+ \frac{16\pi}{\epsilon_\theta}\sqrt{Er(r)}\Bigg\{\cos\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\epsilon_\theta\right]\sin\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\epsilon_\theta\right] \\
&\times \frac{\sin\left[\frac{A_r(r)+A_r(r+\epsilon_r)}{2}\epsilon_r\right]A_1(r+\epsilon_r)+\cos\left[\frac{A_r(r+\epsilon_r)+A_r(r)}{2}\epsilon_r\right]A_2(r+\epsilon_r)}{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}} \\
&- \sin\left[\frac{\sqrt{A_1^2(r)+A_2^2(r)}}{2}\epsilon_\theta\right]\cos\left[\frac{\sqrt{A_1^2(r+\epsilon_r)+A_2^2(r+\epsilon_r)}}{2}\epsilon_\theta\right] \\
&\times \frac{\left(\sin\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_1(r)+\cos\left[\frac{A_r(r+\epsilon_r)-A_r(r)}{2}\epsilon_r\right]A_2(r)\right)}{\sqrt{A_1^2(r)+A_2^2(r)}}\Bigg\} \\
&\frac{16\gamma^2E^1(r)\epsilon_r\sin^2\left(\frac{\epsilon_\theta}{2}\right)\cos\left(\epsilon_\theta\right)}{\epsilon_\theta^2} \\
&+ \frac{2\pi E^1(r)\epsilon_r H_0\left(\sqrt{A_1^2(r)+A_2^2(r)}\epsilon_\phi\right)\sin\left(\epsilon_\theta\sqrt{A_1^2(r)+A_2^2(r)}\right)}{\epsilon_\theta\epsilon_\phi} \\
&- \frac{4\pi}{\epsilon_r}\frac{(1+\gamma^2)}{(E^1(r))^2\sqrt{E^r(r)}} \\
&\times \Bigg\{E^1(r)\left[\left(E^r(r+\epsilon_r)-E^r(r)\right)^2+4E^r(r)\left(E^r(r+2\epsilon_r)-2E^r(r+\epsilon_r)+Er(r)\right)\right] \\
&- 4E^r(r)\left[E^r(r+\epsilon_r)-E^r(r)\right]\left[E^1(r+\epsilon_r)-E^1(r)\right]\Bigg\}\Bigg\}\,. \qquad (3.3.7)
\end{aligned}
$$

Chapter 4

Classical constraint algebra

In chapter 1 we emphasized the major role the constraints play in the quantization program of a theory à la Dirac and the crucial, and maybe even the only criteria, of the off-shell closure of their algebra for a quantum gravity model to be consistent [10, 28, 13].
Covariance is a central property and a fundamental concept in general relativity. Even if one proceeds with a canonical formulation of the theory [10, 14, 28], one can encounter a formal absence of covariance although the theory itself is still manifestly covariant, and is actually inherited in the Poisson or Dirac algebra of the constraint's action on the phase space variables, since at the end of the day all what was done is to write the physics in a different representation. This aspect doesn't hold anymore when stepping into the quantum representation of general relativity. Besides the ambiguities that emerge during the procedure of defining a proper Hamiltonian and diffeomorphism constraints, the closure of the quantum constraint's algebra is perhaps the crucial consistency requirement that must be met by any candidate theory of quantum gravity. In the context of RLQG [21], with the implemented spherical symmetry case, passing this test is necessary to prove its powerful treatment for the case of a quantum black hole and describing its microscopic degrees of freedom as well as recovering the classical limit.
In this chapter we will study the closure of the constraint algebra in the classical gauge unfixed case. We will rely on the results mentioned in chapter 3. More importunately, we focus on the on-shell closure of the equations of motion satisfied by the phase space variables $\left(A_a^i, E_j^b\right)$, leaving the off-shell closure investigation for future work. Concretely, we will present a possible formulation of the Hamiltonian evolution equation in terms of the constraints of the theory [35]. In a second step, we will study the Poisson algebra structure that sheds more light on the Hamiltonian and diffeomorphism constraints, that would set out the path we will follow in the next chapter to derive a candidate expression for the effective diffeomorphism constraint (Kerr-Schild foliation is considered) and its respective evolution equations.

4.1 Constraint algebra for constrained general relativity

The ADM action (1.5.2) allows us to study the phase space of general relativity parametrized by the canonical pair $(q_{ab},\ \pi^{ab})$ carrying the symplectic structure

$$\left\{\pi^{ab}(t,x), q_{cd}\left(t,x'\right)\right\} = \delta^a_{(c}\delta^b_{d)}\delta\left(x-x'\right) \ , \tag{4.1.1}$$

where we define the Poisson bracket of two functions A and B

$$\{A(y), B(z)\} = \int \mathrm{d}^3x \frac{\delta A(y)}{\delta q_{ab}(x)} \frac{\delta B(z)}{\delta \pi^{ab}(x)} - \frac{\delta B(z)}{\delta q_{ab}(x)} \frac{\delta A(y)}{\delta \pi^{ab}(x)} \ . \tag{4.1.2}$$

The constraints derived in (1.7.29) satisfy the following commutation relation

$$\begin{aligned} &\{H_a(x), H_b(y)\} = H_a(y)\partial_b\delta(x-y) - H_b(x)\partial'_a\delta(x-y) \ , \\ &\{H_a(x), H(y)\} = H(x)\partial_a\delta(x-y) \\ &, \{H(x), H(y)\} = H^a(y)\partial_a\delta(x-y) - H^a(y)\partial'_a\delta(x-y) \ . \end{aligned} \tag{4.1.3}$$

An interesting property the above constraints have is that they are first class. In fact, the right hand side vanishes on the constraints surface whence the Poison flow generated by them preserves the constraints hypersurface. As discussed in section (1.2), first class constraints generate gauge transformations on the constraints surface, which is made manifest once the respective Poisson brackets are evaluated:

$$\left\{H(\vec{N}), q_{ab}\right\} = \mathcal{L}_{\overline{N}} q_{ab}, \quad \left\{H(\vec{N}), \pi^{ab}\right\} = \mathcal{L}_{\overline{N}}\pi^{ab} \ , \tag{4.1.4}$$

where it is useful to introduce the smeared version of the constraints[1]

$$H(\vec{N}) = \int_{\Sigma_t} H^a(x)N_a(x)\mathrm{d}^3x, \quad H(N) = \int_{\Sigma_t} H(x)N(x)\mathrm{d}^3x \ . \tag{4.1.5}$$

The gauge transformations generated by the vector constraint are the ones accounted for the space-diffeomorphism. The Hamiltonian constraint commutator, however, satisfy

$$\{H(N), q_{ab}\} = \mathcal{L}_{\vec{n}N} q_{ab} \ , \tag{4.1.6}$$

$$\left\{H(N), \pi^{ab}\right\} = \mathcal{L}_{\vec{n}N}\pi^{ab} + \frac{1}{2}q^{ab}NH - 2N\sqrt{q}q^{c[a}q^{b]d}R_{cd} \ . \tag{4.1.7}$$

The first bracket stands for time diffeomorphism on the induced metric. Whereas the second bracket contains two additional terms that vanish on shell, i.e. for $H = 0$ and $R_{cd} = 0$.
In the subsequent, we will consider more general configuration treatment for the constraint algebra that is not a Lie algebra, the so called Dirac or Bargmann-Komar algebra. This algebra is peculiar mainly due the "structure constants" that appear outside the constraint's surface. Actually, considering the commutator of the smeared version of the Hamiltonian constraint:

$$\{H[N_1], H[N_2]\} = H\left(g^{ab}(N_1\partial_b N_2 - N_2\partial_b N_1)\right) \ , \tag{4.1.8}$$

one can spot the presence of the metric g_{ab} that is a dynamical variable. The mere fact of introducing a foliated spacetime to work in the canonical formalism preserved the role played by the diffeomorphism symmetry, but it gave rise to a new symmetry that is responsible for its deformations by acting on the foliation and changing them. We will present some computations that will turn out to be helpful once we move on to the treatment of spherical symmetric geometries with a Kerr-Schild foliation. Let us start by considering a smeared diffeomorphism,

$$H(\vec{N}) = \int \mathrm{d}^3x N^l H_l \ , \tag{4.1.9}$$

where we anticipate the following action of the Lie derivative

$$\mathcal{L}_{\vec{N}} q_{ab} = (\nabla_a N^c) q_{cb} + (\nabla_b N^c) q_{ac} - \underbrace{N^c \nabla_c q_{ab}}_{=0} = 2\nabla_{(a}N_{b)} \ . \tag{4.1.10}$$

[1]To avoid confusion between the Hamiltonian and diffeomorphism constraints, we refer to the defined smeared version in Eq.(4.1.5)

The Poisson bracket then can be computed

$$\left\{q_{ab}(y), H(\vec{N})\right\} = \int \mathrm{d}^3x N^l \frac{\delta H_l(x)}{\delta\pi^{ab}(y)} = 2\nabla_{(a}N_{b)} = L_{\vec{N}}q_{ab} \ . \tag{4.1.11}$$

This is a key relation for the treatment of the effective diffeomorphism constraint and the on-shell closure of the constraints. It generates infinitesimal spatial diffeomorphism. An other relation that is useful is the Poisson bracket with the Hamiltonian constraint that generates the diffeomorphism in time.
Swapping the canonical pair (q_{ab}, π^{ab}) with the connection and densitized triad $\left(A^i_a, E^b_j\right)$ to inherit their role as phase space variables, we encountered the appearance of the Gauss constraint that generates the local $SU(2)$ gauge transformations inheriting the properties of Yang Mills gauge theories. Let us then study the smeared version of the Gauss, namely

$$G(\alpha) = \int_{\Sigma_t} dx^3 \alpha^i G_i \left(A^i_a, E^a_i\right) = \int_{\Sigma} dx^3 \alpha^i D_a E^a_i \ , \tag{4.1.12}$$

then a direct computation gives

$$\delta_G A^i_a = \left\{A^i_a, G(\alpha)\right\} = -D_a\alpha^i \ , \tag{4.1.13}$$

$$\delta_G E^a_i = \left\{E^a_i, G(\alpha)\right\} = [E, \alpha]_i \ . \tag{4.1.14}$$

They also characterize the algebra's structure

$$\{G(\alpha), G(\beta)\} = G([\alpha, \beta]) \ , \quad \alpha = \alpha^i\tau_i \in su(2), \quad \beta = \beta^i\tau_i \in su(2) \ . \tag{4.1.15}$$

Hamilton's equations yield

$$\dot{A}^i_a = \left\{A^i_a, H\left(\alpha, N^a, N\right)\right\} = \left\{A^i_a, S(N)\right\} + \left\{A^i_a, G(\alpha)\right\} + \left\{A^i_a, V\left(N^a\right)\right\} \ , \tag{4.1.16}$$

$$\dot{E}^a_i = \left\{E^a_i, H\left(\alpha, N^a, N\right)\right\} = \left\{E^a_i, S(N)\right\} + \left\{E^a_i, G(\alpha)\right\} + \left\{E^a_i, V\left(N^a\right)\right\} \ . \tag{4.1.17}$$

4.2 Equations of motion of the phase space variables for spherical symmetric geometries with a Kerr-Schild foliation

It is more convenient to proceed with our program for the on-shell consistency check for the evolution equations in the Ashtekar-connection formalism. This is relevant for two main reason, namely, to show explicitly that admittedly for the stationarity property provided by the Kerr-Schild metric the evolution equations are trivially satisfied on-shell and thus we end up with time independent quantities. Moreover, the derived classical equations of motion for the flux and connection should be consistent with the effective version of them, i.e. when the effective equations of motion of the phase space variabels are expanded in to first order in the quantum parameters should reproduce the classical ones.
This is essential for the subsequent chapter, when we intend to derive a candidate expression for the effective diffeomorphism constraint to solve with help of the effective equations of motion of $(E^i,\ A_i)$ to be solved for the Kerr-Schild foliation. In this section we will present the derivation of the equations of motions for the phase space variables and as for the on-shell consistency check, we refer to the appendix for the calculations. For spherical symmetric geometries, we know that the symplectic structure satisfied by the phase space variables in the $\alpha = 0$ gauge is given by (1.8.12) with minor difference $\{A_2(t,r), E^2(t,r')\} = 0$.

For simplicity, we will use the following notation for the Hamiltonian constraint

$$\begin{aligned} H &= -\frac{1}{2\kappa\gamma^2}\left(H^E + H^L\right) \\ &= -\frac{1}{2\kappa\gamma^2}\frac{N(r)\sin\theta}{\sqrt{(E^1)^2+(E^2)^2}\sqrt{E^r}}\left(\mathcal{H}^E + \frac{(1+\gamma^2)}{2\left((E^1)^2+(E^2)^2\right)\sqrt{E^r}}\mathcal{H}^L\right) . \end{aligned} \tag{4.2.1}$$

In order to compute the equations of motion for the phase space variables, it is useful to write down explicitly the smeared constraints, namely

$$\begin{aligned} H^E &= \frac{\sin\theta}{2G}\int dr \frac{N(r)}{E^1\sqrt{E^r}}\left[4E^rE^1\left(A_rA_1 + A_2'\right) + 2(E^1)^2\left(A_1^2 + A_2^2 - 1\right)\right] \\ &\equiv \frac{\sin\theta}{2G}\int dr \frac{N(r)}{(E^1)\sqrt{E^r}}\mathcal{H}^E[N] . \end{aligned} \tag{4.2.2}$$

$$\begin{aligned} H^L &= \frac{(1+\gamma^2)\sin\theta}{2G}\int dr \frac{N(r)}{2\sqrt{E^r}(E^1)^3}\left[4(E^1)^2 - (E^1)^2\left((E^{r\prime})^2 + 4E^rE^{r\prime\prime}\right) + 4E^rE^{r\prime}E^{1\prime}E^1\right] . \\ &\equiv \frac{(1+\gamma^2)\sin\theta}{2G}\int dr \frac{N(r)}{2\sqrt{E^r}(E^1)^3}\mathcal{H}^L \end{aligned} \tag{4.2.3}$$

$$H^r = \frac{1}{G}\int dr N^r(r)\left(A_1' - \frac{1}{2}A_rE^{r\prime}\right) . \tag{4.2.4}$$

The connection and flux can be expressed in terms of the ADM variables and they read

$$\begin{aligned} E^r(r) &= R^2, \quad E^1(r) = \Lambda R, \quad E^2(r) = 0 , \\ A_r(r) &= -\gamma\frac{\Lambda' N^r + \Lambda N^{r\prime}}{N} = -\frac{\gamma}{N}\partial_r(N^r\Lambda) , \\ A_1(t,r) &= -\frac{\gamma N^r R'}{N} , \\ A_2(t,r) &= \frac{R'}{\Lambda} . \end{aligned} \tag{4.2.5}$$

Proving the closure of the constraints and more importantly the vanishing of the equations of motion on-shell boils down to computing the dynamical evolution of the flux and connection and expressing them in terms of the constraints. The previous setup allows us to write down the evolution in terms of evolution along the normal direction and orthogonal one. However we saw that these tow directional derivatives are equivalent to the gauge transformation of constrained GR, namely the and time an space diffeomorphism. Hence, in order for the equations of motion to be satisfied on-shell, the Poisson bracket of the phase space variable, say (A_i) has to vanish identically with the one computed with respect to the Hamiltonian $\{A_1,H[N]\}$. This property is also inherited from the non-vanishing shift we are considering in the stationary case of the Kerr-Schild metric. Since the computation of the Poisson brackets does not imply the angular dependence, we will drop the $\sin\theta$ and κ terms for simplicity.

4.2.1 Ashtekar-Barbero connection evolution equations

The equation of motion for the different components of the connection are given by the classical Poisson bracket of the Hamiltonian and the connection

$$\dot{A}^i = \{A_i,H\} = \{A_i,H[N]\} + \{A_i,G[\alpha]\} + \{A_i,H^r[N^r]\} , \tag{4.2.6}$$

where $i \in \{1,r\}$. We will start studying the equation of motion of $A_1(r)$ and $A_r(r)$, where we investigate the Euclidean and Lorentzian part of the Hamiltonian constraint separately and then the bracket with the diffeomorphisms. The Poisson bracket of A_1 with the Euclidean Hamiltonian can be computed in the following way

$$\begin{aligned}\dot{A}_1^E &= \left\{A_1,H^E[N]\right\} = \int d\rho \frac{\delta A_1(r)}{\delta A_1(\rho)}\frac{\delta H^E(r')}{\delta E_1(\rho)} - \frac{\delta A_1(r)}{\delta E^1(\rho)}\frac{\delta H^E(r')}{\delta A_1(\rho)} \\ &= G\int d\rho\delta(r-\rho)\frac{\delta H^E[N(r')]}{\delta E^1(\rho)} \\ &= \frac{1}{2}\int dr\frac{\delta \mathcal{H}^E[N(r')]}{\delta E^1(r)} \\ &= \frac{1}{2}\frac{N(r)}{E^1\sqrt{E^r}}\left\{\frac{-\mathcal{H}^E}{E^1} + 4E^r\left[A_rA_1 + A_2'\right] + 4E^1\left(A_1^2 + A_2^2 - 1\right)\right\} \\ &= \frac{N}{\sqrt{E^r}}\left[A_1^2 + A_2^2 - 1\right] \ .\end{aligned} \tag{4.2.7}$$

The commutator with the Lorentzian part gives:

$$\begin{aligned}\dot{A}_1^L &= \left\{A_1,H^L[N]\right\} = \int d\rho \frac{\delta A_1(r)}{\delta A_1(\rho)}\frac{\delta H^L(r')}{\delta E_1(\rho)} - \frac{\delta A_1(r)}{\delta E^1(\rho)}\frac{\delta H^L(r')}{\delta A_1(\rho)} \\ &= G\int d\rho\delta(r-\rho)\frac{\delta H^L[N(r')]}{\delta E^1(\rho)} \\ &= \frac{1}{4}\int dr\frac{\delta H^L[N(r')]}{\delta E^1(r)} \\ &= \int dr\left\{\frac{N(r)\delta(r-r')}{4(E^1)^3\sqrt{E^r}}\left[16(E^1)^3 - 2E^1\left((E'^r)^2 + 4E^rE^{r''}\right) + 4E^{r'}E^rE^{1'}\right.\right. \\ &\left.\left. - 4\partial_r(E^rE^{r'}E^1) - \frac{3\mathcal{H}^L}{E^1}\right]\right\} - \int dr 4(E^rE^{r'}E^1\delta(r-r'))\partial_r\left(\frac{N(r)}{2(E^1)^3\sqrt{E^r}}\right) \\ &= \left(N(r')\left[-\frac{(E^1)^2}{\sqrt{(E^1)^2E^r}} - \frac{(E^{r'})^2}{4\left(E^1\right)^2\sqrt{E^r}} - \frac{E^rE^{r''}}{\left(E^1\right)^2\sqrt{E^r}}\right.\right. \\ &\left.\left. + \frac{2E^{r'}E^{1'}E^r}{\left(E^1\right)^3\sqrt{E^r}} + \partial_r\left(\frac{E^rE^{r'}\sqrt{E^r}}{\left(E^1\right)^2}\right)\right] + \frac{E^rE^{r'}}{\left(E^1\right)^2\sqrt{E^r}}N'(r')\right) \ .\end{aligned} \tag{4.2.8}$$

Computing the partial derivative yields

$$\partial_r\left(\frac{E^rE^{r'}\sqrt{E^r}}{\left(E^1\right)^2}\right) = \frac{E^{r''}\sqrt{E^r}}{\left(E^1\right)^2} - 2\frac{E^{1'}E^{r'}\sqrt{E^r}}{\left(E^1\right)^3} + \frac{\left(E^{r'}\right)^2}{2\left(E^1\right)^2\sqrt{E^r}} \ . \tag{4.2.9}$$

Plugging it back into (4.2.8), and writing explicitly the prefactor for the Lorentzian term, we finally obtain

$$\dot{A}_1^L = \frac{(1+\gamma^2)}{4\left(E^1\right)^2\sqrt{E^r}}\left\{\underbrace{N\left(\left(E^{r'}\right)^2 - 4\left(E^1\right)^2\right) + 4\frac{N'}{N}E^rE^{r'}}_{:=c_1}\right\} \ . \tag{4.2.10}$$

Putting everything together, the commutator with the Hamiltonian constraint for the connection reads

$$
\begin{aligned}
\dot{A}_1^H =& \frac{(1+\gamma^2)}{4\left(E^1\right)^2\sqrt{E^r}}\left\{N\left(\left(E^{r\prime}\right)^2-4\left(E^1\right)^2\right)+4\frac{N'}{N}E^rE^{r\prime}\right\}\\
&+\frac{N}{\sqrt{E^r}}\left[\left(A_1^2+A_2^2-1\right)\right] \ .
\end{aligned}
\tag{4.2.11}
$$

The commutator with the smeared diffeomorphism gives

$$
\dot{A}_1^r = N^r\partial_r A_1 \ . \tag{4.2.12}
$$

Collecting all the contributions from (4.2.11) and (4.2.12) the equation of motion for A_1 is then

$$
\dot{A}_1 = -\frac{1}{2\gamma^2}\dot{A}_1^H + \dot{A}_1^r = 0 \ . \tag{4.2.13}
$$

The Poisson bracket of A_r with the Euclidean Hamiltonian is

$$
\begin{aligned}
\dot{A}_r^E &= \int d\rho \frac{\delta A_r(r)}{\delta A_r(\rho)}\frac{\delta H^E[N(r')]}{\delta E^r(\rho)} - \frac{\delta A_r(r)}{\delta E^r(\rho)}\frac{\delta H^E[N(r')]}{\delta A_r(\rho)}\\
&= 2G\int d\rho\delta(r-\rho)\frac{\delta H^E}{\delta E^r}\\
&= \frac{2G}{2G}\int dr\frac{N(r)}{E^1\sqrt{E^r}}\left[-\frac{\mathcal{H}^E}{2E^r}+4E^1\left(A_rA_1+A_2'\right)\right]\delta(r-r')\\
&= -\frac{2N}{\sqrt{E^r}E^1}\left\{E^1\left(A_rA_1+A_2'\right)-\frac{(E^1)^2}{E^r}\left(A_1^2+A_2^2-1\right)\right\} \ .
\end{aligned}
\tag{4.2.14}
$$

Moving to the Lorentzian part of the Hamiltonian constraint, one should expect terms including only the lapse N and additional ones including the first and second derivatives N' and N'', generated by the variation with respect to of $E^{r\prime}$ and $E^{r\prime\prime}$, namely

$$
\begin{aligned}
\dot{A}_r^L =& \int d\rho \frac{\delta A_r(r)}{\delta A_r(\rho)}\frac{\delta H^L[N(r')]}{\delta E^r(\rho)} - \frac{\delta A_r(r)}{\delta E^r(\rho)}\frac{\delta H^L[N(r')]}{\delta A_r(\rho)}\\
=& 2G\int d\rho\delta(r-\rho)\frac{\delta H^L}{\delta E^r}\\
=& (1+\gamma^2)\left\{N\left\{\frac{E^1}{(E^r)^{\frac{3}{2}}}-\frac{(E^{r\prime})^2}{4E^1\left(E^r\right)^{3/2}}-\partial_r\left(\frac{E^{r\prime}}{E^1\sqrt{E^r}}\right)+\frac{E^{r\prime}}{E^1\sqrt{E^r}}+2\partial_r^2\left(\frac{\sqrt{E^r}}{E^1}\right)-\frac{E^{1\prime}E^{r\prime}}{\left(E^1\right)^2\sqrt{E^r}}\right.\right.\\
&\left.\left.+2\partial_r\left(\frac{E^{1\prime}}{\left(E^1\right)^2}\sqrt{E^r}\right)\right\}-\left(\frac{E^{r\prime}}{E^1\sqrt{E^r}}\right)N'+2\frac{\sqrt{E^r}}{E^2}N''+4N'\partial_r\frac{\sqrt{E^r}}{E^1}+2\left(\frac{E^{1\prime}}{\left(E^1\right)^2}\sqrt{E^r}\right)N'\right\}\\
=& \left(1+\gamma^2\right)\left\{N\left[\frac{E^1}{\left(E^r\right)^{3/2}}-\frac{\left(E'\right)^2}{4E^1\left(E^r\right)^{3/2}}+\frac{E^{r\prime}}{E^1\sqrt{E^r}}-\frac{E^{1\prime}E^{r\prime}}{\left(E^1\right)^2\sqrt{E^r}}\right]\right.\\
&\left. N'\left(\frac{E^{r\prime}}{E^1\sqrt{E^r}}-2\frac{E^{1\prime}\sqrt{E^r}}{\left(E^1\right)^2}\right)+2\frac{\sqrt{E^r}}{E^2}N''\right\}\\
=& \frac{(1+\gamma^2)N}{(2(E^1)^3\sqrt{E^r}}\left(-\frac{\mathcal{H}^L}{2E^r}+4E^{r\prime}E^{1\prime}E^1-4E^{r\prime\prime}(E^1)^2\right)-\frac{(1+\gamma^2)N'(r)}{2(E^1)^3\sqrt{E^r}}\left(4E^rE^{1\prime}E^1+2E^r(E^1)^2\right)\\
&+\frac{N''}{2(E^1)^3\sqrt{E^r}}\left(4E^r(E^1)^2\right) \ .
\end{aligned}
\tag{4.2.15}
$$

The term coming form the diffeomorphism constraint can be computed in a similar way using the Poisson bracket. One obtains :

$$\dot{A}_r^r = \frac{2}{\gamma}\partial_r(N^r A_r)\ . \tag{4.2.16}$$

The total evolution equation read from (4.2.16), (4.2.15) and (4.2.14)

$$\dot{A}_r = \dot{A}_r^r - \frac{1}{2\gamma^2}\left(\dot{A}_r^L + \dot{A}_r^E\right) = 0\ . \tag{4.2.17}$$

4.2.2 Fluxes evolution equations

The equation of motion for the different components of the flux are given by

$$\dot{E}^i = \left\{E^i,H\right\} = \left\{E^i,H[N]\right\} + \left\{E^i,G[\alpha]\right\} + \left\{E^i,H^r[N^r]\right\}\ . \tag{4.2.18}$$

Let's start with E^r. The classical Poisson bracket gives

$$\begin{aligned}\dot{E}_L^r &= \left\{E^r(r),H^E[N(r')]\right\} \\ &= \int d\rho 2\left\{\frac{\delta E^r(r)}{\delta A_r(\rho)}\frac{\delta H^L[N(r')]}{\delta A_r(\rho)} - \frac{\delta E^r(r)}{\delta E^r(\rho)}\frac{\delta H^L[N(r')]}{\delta A_r(\rho)}\right\} = 0\ .\end{aligned} \tag{4.2.19}$$

Whereas the only contribution comes from the Euclidean part, namely

$$\begin{aligned}\dot{E}_H^r &= -2G\int d\rho\delta(r-\rho)\frac{\delta H^E[N(r')]}{\delta A_r(\rho)} \\ &= \frac{2G}{2G}\int dr\frac{\delta \mathcal{H}^E[N(r')]}{\delta A_r(r)} \\ &= -4N(r)\sqrt{E^r}A_1\ .\end{aligned} \tag{4.2.20}$$

Notice that the Lorentzian contribution in the Hamiltonian constraint do not contain any dependence on the Ashtekar-Barbero connection and therefore there is no contribution coming from it in the evolution equations. The commutator with the smeared diffeomorphism reads

$$\begin{aligned}\dot{E}_r^r &= -2G\frac{\delta H^r[N^r(r')]}{\delta A_r} \\ &= -\frac{2G}{2G}\int dr\frac{\delta \mathcal{H}^r[N^r(r')]}{\delta A_r(r)} \\ &= \frac{2}{\gamma}N^r(r)\left(\partial_r E^r(r)\right)\ .\end{aligned} \tag{4.2.21}$$

Hence, the full equation of motion reads

$$\dot{E}^r = -\frac{1}{2\gamma^2}\left(-4N\sqrt{E^r}A_1\right) + \frac{2}{\gamma}N^r(\partial_r E^r) = 0\ . \tag{4.2.22}$$

For the E^1 component

$$\dot{E}_H^1 = -G\frac{\delta H^E}{\delta A_1} = \frac{-2N}{\sqrt{E^r}}\left(A_r E^r + E^1 A_1\right)\ . \tag{4.2.23}$$

For the diffeomorphism PB, one ends up with the equation

$$\dot{E}^1_r = -G\frac{\delta H^r}{\delta A_1} = \frac{2}{\gamma}\partial_r(N^r E^1) \ . \tag{4.2.24}$$

Hence, the total equation of motion reads

$$\dot{E}^1 = \frac{2}{\gamma}\partial_r(N^r E^1) - \frac{1}{2\gamma^2}\left[\frac{-2N}{\sqrt{E^r}}\left(A_r E^r + E^1 A_1\right)\right] = 0 \ . \tag{4.2.25}$$

The on-shell consistency proof of the above computed equations can be found in the accompanying appendix, as well as the commutators involving the diffeomorphism constraint. The evolution equations are trivially satisfied, admittedly due to the stationarity imposed by the Kerr-Schild metric.

4.3 Diffeomorphism constraint

We want to check if the commutator (4.1.8) is reproduced at the effective unfixed reduced level of the theory. To this aim, one computes the commutator of the Hamiltonian constraints with its self. Therefore, one can split the computation in the following manner

$$\begin{aligned}\{H[N_1], H[N_2]\} &= \left\{H^L[N_1] + H^E[N_2]\, ,\, H^L[N_2] + H^E[N_1]\right\}\\ &= \left\{H^L[N_1]\, ,\, H^L[N_2]\right\} + 2\left\{H^E[N_1]\, ,\, H^L[N_2]\right\} + \left\{H^E[N_1]\, ,\, H^E[N_2]\right\}\\ &= \left\{H^E[N_1]\, ,\, H^E[N_2]\right\} + \left\{H^E[N_1]\, ,\, H^L[N_2]\right\} \ .\end{aligned} \tag{4.3.1}$$

The brackets $\left\{H^E[N_1]\, ,\, H^L[N_2]\right\}$ vanishes. The bracket between the Euclidean and Lorentzian part vanishes identically, since the tow parties commute. We will show that the latter holds and that the PB of the Euclidean contribution in the Hamiltonian is the one that contributes to the algebra.

In order to make the computation easier, it is useful to manipulate the equations of motion for the connection and fluxes in terms of the constraints.

$$\left\{H^E[N_1], H^E[N_2]\right\} \tag{4.3.2}$$

$$= G\int d\rho\left\{2\frac{\delta H^E[N_1(r_1)]}{\delta A_r(\rho)}\frac{\delta H^E[N_2(r_2)]}{\delta E^r(\rho)} + \frac{\delta H^E[N_1(r_1)]}{\delta A_1(\rho)}\frac{\delta H^E[N_2(r_2)]}{\delta E^1(\rho)} - (N_1 \leftrightarrow N_2)\right\}$$

$$= \frac{1}{4}\int d\rho 2\left(\dot{E}^r_E[N_1]\dot{A}^E_r[N_2]\right) + \left(\dot{E}^1_E[N_1]\dot{A}^E_1[N_2]\right)$$

$$- (N_1 \leftrightarrow N_2)$$

$$= \frac{1}{4}\int d\rho\left(-8N_1\sqrt{E^r}A_1\right)\left(\frac{N_2}{E^1\sqrt{E^r}}\left[-\frac{\mathcal{H}^E}{2E^r} + 4\left(\frac{E^1}{2A_2}\partial_r(A_1^2) - \frac{\gamma}{2A_2}A_1\mathcal{H}^r\right) + 4E^1A_2'\right]\right)$$

$$- (N_1 \leftrightarrow N_2) \tag{4.3.3}$$

$$+\left[\frac{N_2}{E^1\sqrt{E^r}}\left\{\frac{\mathcal{H}^E}{E^1} + 2\frac{E^r}{A_2E^1}\left(\gamma A_1\mathcal{H}^r - E^1\partial_r(A_1^2)\right)\right\} - \frac{2N_2A_2'E^r}{E^1\sqrt{E^r}}\right]\left(-\frac{2N_1}{\sqrt{E^r}}\left[A_rE^+A_1E^1\right]\right)$$

$$- (N_1 \leftrightarrow N_2)$$

$$
\begin{aligned}
&= \frac{1}{G}\int d\rho - 4N_1N_2A_1\partial_r(A_1^2) - 4N_1N_2\frac{E^rA_r\partial_r(A_1^2)}{E^1A_2} - 4N_1N_2A_1A_2' - 4N_1N_2A_2E^1\partial_r(A_1^2) \\
&- (N_1 \leftrightarrow N_2) \\
&= \frac{1}{G}\int d\rho A_2'N_1\left(\frac{E^rA_r}{E^1}A_2 - A_1A_2\right) - (N_1 \leftrightarrow N_2) \\
&= \frac{1}{G}\int d\rho A_2'N_1\frac{E^r}{(E^1)^2}\left(E^1\left[A_rA_2 - \frac{(E^1)^2}{E^r}A_1A_2\right]\right) - (N_1 \leftrightarrow N_2) \\
&= A_2'N_1\frac{E^r}{(E^1)^2}\left(E^1\left[\frac{E^{r\prime}A_1}{A_2}\right]\right) - (N_1 \leftrightarrow N_2) \\
&= \frac{1}{G}\int d\rho A_2'N_1\frac{E^r}{(E^1)^2}\left(E^1\left[A_rA_2 - A_1'\right]\right) - (N_1 \leftrightarrow N_2) \\
&= H^r\left[\frac{E^r\left(N_1'N_2 - N_2'N_1\right)}{\left(E^1\right)^2}\right] + G_3\left[\frac{A_rE^r\left(N_1'N_2 - N_2'N_1\right)}{\left(E^1\right)^2}\right] \\
&= \left(\frac{E^r\left(N_1'N_2 - N_2'N_1\right)}{\left(E^1\right)^2}\right)\mathcal{H}^r \ .
\end{aligned}
$$

Where in the last step one can perform an integration by part. Notice that all the other terms cancel each other. Furthermore, to reintroduce the derivatives appearing in the lapses, we made use of the constraint form of the equations of motion where there is an interplay between the lapse function and A_2. Indeed, in the $\alpha = 0$ gauge, we encounter the disappearance of terms contributing to the derivatives of the lapse function making the integration by parts hidden and non trivial.
The commutator of the Lorentzian and Euclidean contribution can be computed in the following way

$$
\begin{aligned}
&\left\{H^L\left[N_1\right], H^E\left[N_2\right]\right\} \\
&= G\int d\rho\ 2\left\{\left(\frac{-2N_1}{\sqrt{E^r}}\left[A_rE^r + A_1E^1\right]\right)\left((1+\gamma^2)E^{r\prime}\left[\frac{A_2}{\sqrt{E^r}E^1} + \frac{4A_2'\sqrt{E^r}}{E^{r\prime}E^1}\right]\right)\right\} \\
&+ \left(-4N_1\sqrt{E^r}A_1\right)\left(-4(1+\gamma^2)\frac{N_2A_2'}{\sqrt{E^r}}\right) \\
&= G\int d\rho\ 4(1+\gamma^2)N_1N_2A_1A_2' + (1+\gamma^2)\frac{E^{r\prime}A_2E^rA_r}{E^rE^1} + \frac{E^{r\prime}A_2'\sqrt{E^r}E^1A_1}{\sqrt{E^r}E^{r\prime}E^1} \\
&= G\int d\rho\ 4(1+\gamma^2)N_1N_2A_1A_2' + 4(1+\gamma^2)N_1N_2\left(\left(\frac{E^{r\prime}A_2A_r}{E^1}\right) + A_1A_2'\right) \\
&= G\int d\rho\ 4(1+\gamma^2)N_1N_2A_1A_2' + 4(1+\gamma^2)N_1N_2\left(\frac{E^{r\prime}A_1'}{E^1} + A_1A_2'\right) \\
&= G\int d\rho\ 4(1+\gamma^2)N_1N_2A_1A_2' + 4(1+\gamma^2)N_1N_2\left(-2A_1A_2' + A_1A_2'\right) \\
&= 0 \ ,
\end{aligned}
\tag{4.3.4}
$$

where in the last step, once once one preforms an integration by part of

$$
\int d\rho\frac{E^{r\prime}}{E_1}A_1' = -\int d\rho\partial_r\left(\underbrace{\frac{E^{r\prime}}{E_1}}_{=2A_2}\right)A_1 = -\int d\rho\ 2A_2'A_1 \ . \tag{4.3.5}
$$

Hence, the Lorentzian and Euclidean parts of the classical Hamiltonian commute and the PB reads

$$\left\{H^L\left[N_1\right], H^E\left[N_2\right]\right\} = 0 \; . \tag{4.3.6}$$

Chapter 5

Effective dynamics of a black hole spacetime

As we explored the classical theory in chapter 1 and 4 the features of the constraints and their algebra [10], we turn in this chapter to study the effective counterpart of them. The main issue of any theory of quantum gravity is to preserve the manifest covariance of general relativity during the transition to the quantum theory. Revealing a diffeomorphism constraint and solving it in quantum gravity approaches is still one the most challenging endeavors faced, along with surviving the test of off-shell closure inhibited by the constraint algebra [4, 36].
The investigation of the quantum evolution for black holes initial data [21] has set the general playground for the formulation of spherical symmetric geometries and the basis for this work to study a stationary black hole with a Kerr-Schild metric. Still, even the Quantum Reduced Loop Gravity framework inherits the same conceptual and mathematical issues from the full theory concerning the diffeomorphism invariance. One strategy to tackle some of the above mentioned ambiguities, is to put the off-shell closure property of the constraints to extract a candidate expression for the effective diffeomorphism constraint and hence restore general covariance and maintain the consistency of the effective theory at the dynamical level. The precise role of the effective diffeomorphism constraint becomes crucial once one considers the interior and exterior sewed together for black holes, which is the ultimate objective of a quantum gravity black hole.

In this chapter, we rely on the results derived chapter 3 and 4. More concretely, based on the obtained effective Hamiltonian for the Kerr-Schild metric (encoding also the inhomogeneous character of the exterior), we will derive the quantum effective evolution equations and study some features of the effective algebra involving the diffeomorphism constraint for the case of constant quantum parameters and in a second stage, in the case of phase space dependent quantum parameters. The goal is to provide a candidate expression for the diffeomorphism constraint for which the dynamics should be solved.
More precisely, the procedure to find the set of functions representing the ADM phase space variables that satisfy the derived equations of motion depends on the number of parameters to be solved for and the number of the differential equations. This relies on the mathematical condition that states: if the number of differential equations should be less than the number of the unknown variables, then the system is said to be non-solvable. In the case it exceeds or equals the set of unknown variables, it is said to be overdetermined and solvable, respectively. In our case, we intend to provide a set of evolutions equation (which are four equations obtained for the fluxes and connection components) and a candidate expression for the effective diffeomorphism

constraint that adds up to six non-local differential equations that should be solved to determine the effective metric functions. These are namely determined by Λ and R. The distinction in considering the quantum parameters, constants or phase space functions, becomes evident and of physical value once the dynamics are solved.

5.1 Constant quantum parameters

In chapter 3.2 we have expressed the effective Hamiltonian in terms of the quantum parameters ϵ (which is equivalent to expressing it in N's). There are two approaches to solve the constraints and to compute the equations of motion for the phase space variables, namely, one can consider these quantum parameters as constants or represent them as a function of the phase space variables. The approach to the two cases differ mathematically as well as on the level of the physical predication described by the dynamics. Working with coherent states where quantum numbers are constant, amounts to evaluating

$$\begin{aligned} \epsilon_r &= \sqrt{8\pi\gamma}\ell_p \frac{\sqrt{E^r}}{E^1}\sqrt{\frac{j_x j_y}{j_z}} \ , \\ \epsilon_\theta &= \sqrt{8\pi\gamma}\ell_r \frac{1}{\sqrt{E^r}}\sqrt{\frac{jzj_y}{j_x}} \ , \\ \epsilon_\varphi &= \sqrt{8\pi\gamma\ell_p}\frac{1}{\sin\theta\sqrt{E^r}}\sqrt{\frac{j_z j_x}{j_y}} \ . \end{aligned} \tag{5.1.1}$$

at a given instant of time such that they constitute constants of motion. Their explicit expression (eventually values) can be later on fixed (upon physical arguments) once the dynamics are solved. If we now set $j_1 = j_2 = j$ the relations above yield (5.1.1)

$$\begin{aligned} \epsilon_r &= \frac{\sqrt{8\pi\gamma}\ell_p}{\Lambda(r)}\frac{j}{\sqrt{j_3}} := \frac{\alpha}{\Lambda} \ , \\ \epsilon &= \frac{\sqrt{8\pi\gamma}\ell_p}{R(r)}\sqrt{j_3} := \frac{\beta}{R} \ . \end{aligned} \tag{5.1.2}$$

These are the two quantum parameters we will be working with in this section. They represent the angular and longitudinal coordinate lengths of the plaquettes, that are sewn together to build discrete geometric structure on constant time surfaces. In both cases, in the classical limit, the computations should deliver the same result for the classical equations of motion. Note that using properties the trigonometric functions exhibit, the expression for the effective Euclidean Hamiltonian yields the simpler form

$$\begin{aligned} -\kappa\gamma^2\tilde{H}^E := \frac{4\pi^2\sqrt{E^r(r)}}{\epsilon}&\Bigg(\Bigg(H_0\left[\frac{\epsilon}{2}\left(\sqrt{A_1(r)^2+A_1(r)^2}-\sqrt{A_1(r+\epsilon_r)^2+A_1(r+\epsilon_r)^2}\right)\right] \\ &+ H_0\left[\frac{\epsilon}{2}\left(\sqrt{A_1(r)^2+A_1(r)^2}+\sqrt{A_1(r+\epsilon_r)^2+A_1(r+\epsilon_r)^2}\right)\right]\Bigg) \\ &+\frac{2}{\pi}\Bigg(\sin\left[\frac{\epsilon}{2}\left(\sqrt{A_1(r)^2+A_1(r)^2}-\sqrt{A_1(\epsilon_r+r)^2+A_1(\epsilon_r+r)^2}\right)\right] \\ &+\sin\left[\frac{\epsilon}{2}\left(\sqrt{A_1(\epsilon_r+r)^2+A_1(\epsilon_r+r)^2}+\sqrt{A_1(r)^2+A_1(r)^2}\right)\right]\Bigg) \end{aligned}$$

$$
\begin{aligned}
&\times \sin\left[\tan^{-1}\left(\frac{A_1(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{\epsilon_r}{2}\left(A_r\left(\epsilon_r+r\right)+A_r(r)\right)\right]\Bigg)-\\
&\Bigg(\Bigg(H_0\left[\frac{\epsilon}{2}\left(\sqrt{A_1(r)^2+A_1(r)^2}+\sqrt{A_1\left(r+\epsilon_r\right)^2+A_1\left(r+\epsilon_r\right)^2}\right)\right]\\
&-H_0\left[\frac{\epsilon}{2}\left(\sqrt{A_1(r)^2+A_1(r)^2}-\sqrt{A_1(r+\epsilon_r)^2+A_1(r+\epsilon_r)^2}\right)\right]\Bigg)\\
&+\frac{2}{\pi}\Bigg(\sin\left[\frac{\epsilon}{2}\left(\sqrt{A_1(\epsilon_r+r)^2+A_1(\epsilon_r+r)^2}+\sqrt{A_1(r)^2+A_1(r)^2}\right)\right]\\
&-\sin\left[\frac{\epsilon}{2}\left(\sqrt{A_1(r)^2+A_1(r)^2}-\sqrt{A_1\left(\epsilon_r+r\right)^2+A_1\left(\epsilon_r+r\right)^2}\right)\right]\Bigg)\Bigg)\\
&\times \sin\left[\tan^{-1}\left(\frac{A_1(r)}{A_1(r)}\right)+\frac{\epsilon_r}{2}\left(A_r(\epsilon_r+r)-A_r(r)\right)\right]\\
&+\frac{4\pi^2E^1(r)\epsilon_r}{\epsilon^2\sqrt{E^r(r)}}\left(H_0\left[\epsilon\sqrt{A_1(r)^2+A_1(r)^2}\right]\sin\left(\epsilon\sqrt{A_1(r)^2+A_1(r)^2}\right)\right)\ ,
\end{aligned}
$$

whereas the Lorentzian Hamiltonian stays intact.

In order to study the effective equations of motion for the fluxes and connection, one should study their respective Poisson bracket with the effective Hamiltonian. However, the discrete nature of the effective constraint call for new techniques to go through the calculations, namely the calculus of discrete derivatives and integration.
Indeed, for the case of classical canonical equations of motion, functional derivatives and integrals are well defined continuous operations that are preformed to thoroughly compute equations. In the present case of the effective constraint, we are dealing with non-local equations which loses all notions of continuity and thus the calculations rely crucially on the discrete version of the mathematical tool we are familiar with. To this aim, it is important to take into consideration that the evaluation of the usual functional variation of the Hamiltonian constraint appears at two different spatial coordinate, due to the discreetness of the functional, namely once at r and once in a neighborhood of r, $r+n\epsilon_r$, where n is some integer. This motivates to introduce the following notation.
Since we are dealing with functionals evaluated in points in a small enough neighborhood ϵ_r, labeling the spatial shift for the radial variable r, we will denote it as $n\epsilon_r$. In this notation the phase space variables corresponding to those functional evaluated at different shifts or steps can be written as

$$
A_i^n(r+n\epsilon_r)\ ,\quad E_n^i(r+n\epsilon_r)\ . \tag{5.1.3}
$$

where the index $i\in\{1,2,r\}$[1] Since the classical Hamiltonian contains terms with first and second continuous derivatives, at the effective level, they are equivalent to terms including a shifts in ϵ_r and $2\epsilon_r$.

[1]Bear in mind that we are working in the $\alpha=0$, i.e $E^2=0$ gauge and therefore this phase space variable do not enter into play at the effective level as well.

5.1.1 Ashtekar-Barbero connection effective equations of motion

The effective evolution equations of the Ashtekar-Barbero connection are obtained in the usual way, namely by computing the Poisson Bracket.

$$\dot{A}_i^H(r) = \left\{A_i(r),\tilde{\mathcal{H}}(N[r])\right\} \ . \tag{5.1.4}$$

Following the notation in (5.1.3) the Ashtekar-Barbero connection component A_1 effective equation of motion obtained from $\tilde{H}(r)$ yields

$$\begin{aligned}\dot{A}_1^0(r) = \frac{2\pi}{\sqrt{E^r(r)}\epsilon_r}\Bigg\{\frac{2\epsilon_r^2}{\epsilon^2}\Bigg(\pi H_0\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\times\sin\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\\ + 8\gamma^2\sin^2\left[\frac{\epsilon}{2}\right]\cos[\epsilon]\Bigg) + \frac{(\gamma^2+1)}{E^1(r)^3}\Bigg(8E^r(r)E^1\left(\epsilon_r+r\right)\left(E^r(r)-E^r\left(\epsilon_r+r\right)\right)\\ + E^1(r)\left[E^r(r)\left(4E^r\left(2\epsilon_r+r\right)-6E^r\left(\epsilon_r+r\right)\right)+E^r\left(\epsilon_r+r\right)^2+E^r(r)^2\right]\Bigg)\Bigg\} \ .\end{aligned} \tag{5.1.5}$$

The contribution coming from $\tilde{H}(r-\epsilon_r)$

$$\dot{A}_1^1(r) = \frac{8\pi\left(\gamma^2+1\right)\left(E^r(r)-E^r\left(r-\epsilon_r\right)\right)\sqrt{E^r\left(r-\epsilon_r\right)}}{\epsilon_r E^1\left(r-\epsilon_r\right)^2} \ . \tag{5.1.6}$$

The total equation of motion then reads

$$\begin{aligned}\dot{A}_1^H = -\frac{1}{4\gamma}\sum_r\Bigg\{\frac{2\pi}{\epsilon_r\epsilon^2E^1(r)^3\sqrt{E^r(r)}}\Bigg\{2\epsilon_r^2E^1(r)^3\Bigg(\pi H_0\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\sin\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\\ + 8\gamma^2\sin^2\left[\frac{\epsilon}{2}\right]\cos(\epsilon)\Bigg) - \left(\gamma^2+1\right)\epsilon^2E^1(r)\Bigg(2E^r(r)(E^r(\epsilon_r+r)-2E^r(2\epsilon_r+r))\\ - E^r(\epsilon_r+r)^2+3E^r(r)^2\Bigg) + 8\left(\gamma^2+1\right)\epsilon^2E^r(r)E^1(\epsilon_r+r)(E^r(r)-E^r(\epsilon_r+r))\Bigg\}\Bigg\} \ .\end{aligned} \tag{5.1.7}$$

The A_r component of the connection is computed in a similar fashion and for the contribution from $\tilde{H}(r)$ it reads

$$\begin{aligned}\dot{A}_r^0 = \frac{\pi}{E^r(r)^{3/2}}\Bigg\{ - \Bigg(\frac{2\epsilon E^r(r)}{\epsilon^2}\\ \times\left[\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}-\sqrt{A_1\left(r+\epsilon_r\right)^2+A_2\left(r+\epsilon_r\right)^2}\right)\right]\right.\\ \times\left[\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)-A_r(r)\right)\right]\right.\\ \left.\left. + \sin\left[\tan^{-1}\left(\frac{A_2\left(\epsilon_r+r\right)}{A_1\left(\epsilon_r+r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)+A_r(r)\right)\right]\right]\right.\\ + \pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}+\sqrt{A_1\left(r+\epsilon_r\right)^2+A_2\left(r+\epsilon_r\right)^2}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\times \left[\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right) + \frac{1}{2}\epsilon_r \left(A_r\left(\epsilon_r + r\right) - A_r(r)\right)\right]\right. \\
&\left.- \sin\left[\tan^{-1}\left(\frac{A_2\left(\epsilon_r + r\right)}{A_1\left(\epsilon_r + r\right)}\right) + \frac{1}{2}\epsilon_r \left(A_r\left(\epsilon_r + r\right) + A_r(r)\right)\right]\right] \\
&+ 4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r) + A_2^2(r)}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1\left(\epsilon_r + r\right)^2 + A_2\left(\epsilon_r + r\right)^2}\right] \\
&\times \sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right) + \frac{1}{2}\epsilon_r \left(A_r\left(\epsilon_r + r\right) - A_r(r)\right)\right] - 4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1\left(\epsilon_r + r\right)^2 + A_2\left(\epsilon_r + r\right)^2}\right] \\
&\times \cos\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r) + A_2^2(r)}\right]\sin\left[\tan^{-1}\left(\frac{A_2\left(\epsilon_r + r\right)}{A_1\left(\epsilon_r + r\right)}\right) + \frac{1}{2}\epsilon_r \left(A_r\left(\epsilon_r + r\right) + A_r(r)\right)\right]\Bigg] \\
&+ E^1(r)\epsilon_r \left(\pi H_0\left[\epsilon\sqrt{A_1^2(r) + A_2^2(r)}\right]\sin\left[\epsilon\sqrt{A_1^2(r) + A_2^2(r)}\right] + 8\gamma^2\sin^2\left[\frac{\epsilon}{2}\right]\cos(\epsilon)\right)\Bigg) \\
&- \frac{\left(\gamma^2 + 1\right)}{E^1(r)^2\epsilon_r}\Bigg(4E^r(r)E^1\left(\epsilon_r + r\right)\left(3E^r(r) - E^r\left(\epsilon_r + r\right)\right) \\
&+ E^1(r)\left(E^r(r)\left(4E^r\left(2\epsilon_r + r\right) - 6E^r\left(\epsilon_r + r\right)\right) - E^r\left(\epsilon_r + r\right)^2 + 3E^r(r)^2\right)\Bigg)\Bigg\}\,,
\end{aligned}
\tag{5.1.8}
$$

and while considering the commutator with $\tilde{H}(r-\epsilon_r)$, one obtains

$$
\dot{A}_r^1(r) = \frac{4\pi\left(\gamma^2 + 1\right)\left(2E^1(r)E^r\left(r - \epsilon_r\right) - E^1\left(r - \epsilon_r\right)\left(E^r(r) - 3E^r\left(r - \epsilon_r\right)\right)\right)}{\epsilon_r E^1\left(r - \epsilon_r\right)^2\sqrt{E^r\left(r - \epsilon_r\right)}}\,. \tag{5.1.9}
$$

Finally, from the computation involving $\tilde{H}(r-2\epsilon_r)$ one gets

$$
A_r^2(r) = \frac{8\pi\left(\gamma^2 + 1\right)E^r\left(r - 2\epsilon_r\right)}{\epsilon_r E^1\left(r - 2\epsilon_r\right)}\,. \tag{5.1.10}
$$

Putting all the contributions from the different "shifted" Hamiltonian together yield the equation of motion

$$
\begin{aligned}
\dot{A}_r^H = \frac{1}{2\gamma}\sum_r\Bigg\{&\frac{\pi}{\epsilon_r\epsilon^2E^1(r)^2E^r(r)^{3/2}}\Bigg\{-2\epsilon_r\epsilon E^1(r)^2E^r(r)\Bigg(\\
&\times 4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r) + A_2^2(r)}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r + r)^2 + A_2(\epsilon_r + r)^2}\right] \\
&\times \sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right) + \frac{1}{2}\epsilon_r(A_r(\epsilon_r + r) - A_r(r))\right] \\
&- 4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r) + A_2^2(r)}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r + r)^2 + A_2(\epsilon_r + r)^2}\right] \\
&\times \sin\left[\tan^{-1}\left(\frac{A_2(\epsilon_r + r)}{A_1(\epsilon_r + r)}\right) + \frac{1}{2}\epsilon_r(A_r(\epsilon_r + r) + A_r(r))\right] \\
&+ \pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r) + A_2^2(r)} - \sqrt{A_1(r + \epsilon_r)^2 + A_2(r + \epsilon_r)^2}\right)\right] \\
&\times \left(\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right) + \frac{1}{2}\epsilon_r(A_r(\epsilon_r + r) - A_r(r))\right]\right. \\
&\left.+ \sin\left[\tan^{-1}\left(\frac{A_2(\epsilon_r + r)}{A_1(\epsilon_r + r)}\right) + \frac{1}{2}\epsilon_r(A_r(\epsilon_r + r) + A_r(r))\right]\right)
\end{aligned}
$$

$$+ \pi H_0 \left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)} + \sqrt{A_1(r+\epsilon_r)^2 + A_2(r+\epsilon_r)^2}\right)\right]$$
$$\times \left(\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right) + \frac{1}{2}\epsilon_r(A_r(\epsilon_r+r) - A_r(r))\right]\right.$$
$$\left.\left. - \sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right) + \frac{1}{2}\epsilon_r(A_r(\epsilon_r+r) - A_r(r))\right]\right)\right)$$
$$- 2\epsilon_r^2 E^1(r)^3 \left(\pi H_0\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\sin\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right] + 8\gamma^2\sin^2\left[\frac{\epsilon}{2}\right]\cos(\epsilon)\right)$$
$$+ \left(\gamma^2+1\right)\epsilon^2 E^1(r)\left(2E^r(r)(E^r(\epsilon_r+r) - 2E^r(2\epsilon_r+r)) + E^r(\epsilon_r+r)^2 + 8E^r(r)^{5/2} + 9E^r(r)^2\right)$$
$$\left.\left. - 4\left(\gamma^2+1\right)\epsilon^2 E^r(r) E^1(\epsilon_r+r)(E^r(r) - E^r(\epsilon_r+r))\right\}\right\} . \tag{5.1.11}$$

5.1.2 Fluxes equations of motion

The same machinery is applied to derive the fluxes equations of motion by considering the Poisson bracket

$$\dot{E}^i_H(r) = \left\{E^i(r), \tilde{\mathcal{H}}(N[r])\right\} . \tag{5.1.12}$$

In the case of the flux E^1, the contribution coming from Hamiltonian $\tilde{\mathcal{H}}(r)$ reads

$$\dot{E}^1_0 = \frac{2\pi}{\epsilon\sqrt{E^r(r)}\,(A_1(r)^2 + A_2(r)^2)}$$
$$\left\{ 2A_2(r)E^r(r)\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right) + \frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right) - A_r(r)\right)\right]\right.$$
$$\left(\pi\left[H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)} - \sqrt{A_1\left(r+\epsilon_r\right)^2 + A_2\left(r+\epsilon_r\right)^2}\right)\right]\right.\right.$$
$$\left. + H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)} + \sqrt{A_1\left(r+\epsilon_r\right)^2 + A_2\left(r+\epsilon_r\right)^2}\right)\right]\right]$$
$$\left. + 4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\cos\left(\frac{1}{2}\epsilon\sqrt{A_1\left(\epsilon_r+r\right)^2 + A_2\left(\epsilon_r+r\right)^2}\right)\right)$$
$$- A_1(r)\sqrt{A_1^2(r)+A_2^2(r)}\left(\epsilon E^r(r)\right.$$
$$\times\left\{\pi H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)} - \sqrt{A_1\left(r+\epsilon_r\right)^2 + A_2\left(r+\epsilon_r\right)^2}\right)\right]\right.$$
$$\times\left[\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right) + \frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right) - A_r(r)\right)\right]\right.$$
$$\left. + \sin\left[\tan^{-1}\left(\frac{A_2\left(\epsilon_r+r\right)}{A_1\left(\epsilon_r+r\right)}\right) + \frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right) + A_r(r)\right)\right]\right]$$
$$+ \pi H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)} + \sqrt{A_1\left(r+\epsilon_r\right)^2 + A_2\left(r+\epsilon_r\right)^2}\right)\right]$$

$$\times\left[\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)-A_r(r)\right)\right]\right.$$
$$\left.-\sin\left[\tan^{-1}\left(\frac{A_2\left(\epsilon_r+r\right)}{A_1\left(\epsilon_r+r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)+A_r(r)\right)\right]\right]$$
$$+4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1\left(\epsilon_r+r\right)^2+A_2\left(\epsilon_r+r\right)^2}\right]$$
$$\sin\left[\tan^{-1}\left(\frac{A_2\left(\epsilon_r+r\right)}{A_1\left(\epsilon_r+r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)+A_r(r)\right)\right]$$
$$+4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1\left(\epsilon_r+r\right)^2+A_2\left(\epsilon_r+r\right)^2}\right]$$
$$\left.\times\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)-A_r(r)\right)\right]\right\}$$
$$-2\pi E^1(r)\epsilon_r\left(H_{-1}\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\sin\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\right.$$
$$\left.\left.\left.+H_0\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\cos\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\right)\right)\right\}. \tag{5.1.13}$$

The second equation that denotes the shift in epsilon $n=-1$ reads

$$\dot{E}_1^1(r)=\frac{\pi\sqrt{E^r\left(r-\epsilon_r\right)}}{\epsilon\left(A_1(r)^2+A_2(r)^2\right)}\left\{2\epsilon A_1(r)\sqrt{A_1^2(r)+A_2^2(r)}\right.$$
$$\left\{\pi H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}-\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right)\right]\right.$$
$$\left[\sin\left[\tan^{-1}\left(\frac{A_2\left(r-\epsilon_r\right)}{A_1\left(r-\epsilon_r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r(r)-A_r\left(r-\epsilon_r\right)\right)\right]\right.$$
$$\left.+\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(r-\epsilon_r\right)+A_r(r)\right)\right]\right]$$
$$+\pi H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}+\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right)\right]$$
$$\left[\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(r-\epsilon_r\right)+A_r(r)\right)\right]\right.$$
$$\left.-\sin\left[\tan^{-1}\left(\frac{A_2\left(r-\epsilon_r\right)}{A_1\left(r-\epsilon_r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r(r)-A_r\left(r-\epsilon_r\right)\right)\right]\right]$$
$$+4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right]$$
$$\times\sin\left[\tan^{-1}\left(\frac{A_2\left(r-\epsilon_r\right)}{A_1\left(r-\epsilon_r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r(r)-A_r\left(r-\epsilon_r\right)\right)\right]$$
$$\sin\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right]$$
$$\times\sin\left[\tan^{-1}\left(\frac{A_2\left(r-\epsilon_r\right)}{A_1\left(r-\epsilon_r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r(r)-A_r\left(r-\epsilon_r\right)\right)\right]$$

$$\times\left(\pi\left(H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}-\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right)\right]\right.\right.$$
$$\left.+H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}+\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right)\right]\right)$$
$$\left.\left.+4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right]\right)\right\}. \tag{5.1.14}$$

Notice that since the Lorentzian part of the effective Hamiltonian constraint has no connection dependence, there is only two discrete derivatives for the fluxes. The total effective evolution equation for the flux E^1 reads

$$\dot{E}^1_H=-\frac{1}{4\gamma}\sum_r\left\{\frac{\pi}{\epsilon\sqrt{E^r(r)}}\left\{\frac{8\epsilon}{\sqrt{A_1^2(r)+A_2^2(r)}\sqrt{A_1(r+v)^2+A_2(r+\epsilon_r)^2}}A_1(r+\epsilon_r)E^r(r)\right.\right.$$
$$\times\cos\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right]$$
$$\times\sin\left[\frac{1}{2}\epsilon_r(A_r(r)+A_r(r+\epsilon_r))+\tan^{-1}\left(\frac{A_2(r+\epsilon_r)}{A_1(r+\epsilon_r)}\right)\right]\sqrt{A_1^2(r)+A_2^2(r)}$$
$$+8\epsilon A_1(r+\epsilon_r)E^r(r)\sin\left[\frac{1}{2}\epsilon_r(A_r(r+\epsilon_r)-A_r(r))+\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]$$
$$\times\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right]\sqrt{A_1^2(r)+A_2^2(r)}$$
$$-8\epsilon A_1(r)\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}E^r(r)\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]$$
$$\times\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right]\sin\left[\frac{1}{2}\epsilon_r(A_r(r)+A_r(r+\epsilon_r))+\tan^{-1}\left(\frac{A_2(r+\epsilon_r)}{A_1(r+\epsilon_r)}\right)\right]$$
$$-8\epsilon A_1(r)\cos\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right]E^r(r)$$
$$\times\sin\left[\frac{1}{2}\epsilon_r(A_r(r+\epsilon_r)-A_r(r))+\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)\right]\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}$$
$$+4\pi\epsilon_rA_1(r)E^1(r)\sin\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}$$
$$\times H_{-1}\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]+2\pi\epsilon E^r(r)$$
$$\times\left(\sin\left[\frac{1}{2}\epsilon_r(A_r(r+\epsilon_r)-A_r(r))+\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)\right]\right.$$
$$\left.+\sin\left[\frac{1}{2}\epsilon_r(A_r(r)+A_r(r+\epsilon_r))+\tan^{-1}\left(\frac{A_2(r+\epsilon_r)}{A_1(r+\epsilon_r)}\right)\right]\right)$$
$$\left(A_1(r+\epsilon_r)\sqrt{A_1^2(r)+A_2^2(r)}-A_1(r)\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)$$
$$\times H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]$$
$$-2\pi\epsilon\left(A_1(r+\epsilon_r)\sqrt{A_1^2(r)+A_2^2(r)}+A_1(r)\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)E^r(r)$$

$$
\begin{aligned}
&\times\left(\sin\left[\frac{1}{2}\epsilon_r(A_r(r+\epsilon_r)-A_r(r))+\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)\right]\right.\\
&\left.-\sin\left[\frac{1}{2}\epsilon_r(A_r(r)+A_r(r+\epsilon_r))+\tan^{-1}\left(\frac{A_2(r+\epsilon_r)}{A_1(r+\epsilon_r)}\right)\right]\right)\\
&\times H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\\
&+4\pi\epsilon_r A_1(r)\cos\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]E^1(r)\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}H_0\left[\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\\
&-\frac{4A_2(r+\epsilon_r)}{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\cos\left[\frac{1}{2}\epsilon_r(A_r(r)+A_r(r+\epsilon_r))+\tan^{-1}\left(\frac{A_2(r+\epsilon_r)}{A_1(r+\epsilon_r)}\right)\right]E^r(r)\\
&\times\left(4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right]\right.\\
&-\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\\
&\left.+\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\right)\\
&+\frac{4A_2(r)}{A_1(r)^2+A_2(r)^2}\cos\left[\frac{1}{2}\epsilon_r(A_r(r+\epsilon_r)-A_r(r))+\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)\right]E^r(r)\\
&\times\left(4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\right.\\
&+\pi\left(H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\right.\\
&\left.\left.\left.\left.+H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\right)\right)\right\}\right\}. \qquad (5.1.15)
\end{aligned}
$$

When considering the case $n=0$, the radial component of the flux E^r yields the expression

$$
\begin{aligned}
\dot{E}_0^r(r)=\frac{2\pi}{\epsilon}\sqrt{E^r(r)}\epsilon_r\Bigg\{&\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}-\sqrt{A_1\left(r+\epsilon_r\right)^2+A_2\left(r+\epsilon_r\right)^2}\right)\right]\\
&\times\left[\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)-A_r(r)\right)\right]\right.\\
&\left.-\cos\left[\tan^{-1}\left(\frac{A_2\left(\epsilon_r+r\right)}{A_1\left(\epsilon_r+r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)+A_r(r)\right)\right]\right]\\
&+\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}+\sqrt{A_1\left(r+\epsilon_r\right)^2+A_2\left(r+\epsilon_r\right)^2}\right)\right]\\
&\left[\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)-A_r(r)\right)\right]\right.\\
&\left.+\cos\left[\tan^{-1}\left(\frac{A_2\left(\epsilon_r+r\right)}{A_1\left(\epsilon_r+r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)+A_r(r)\right)\right]\right]\\
&+4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1\left(\epsilon_r+r\right)^2+A_2\left(\epsilon_r+r\right)^2}\right]
\end{aligned}
$$

$$\times\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)-A_r(r)\right)\right]$$
$$+4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1\left(\epsilon_r+r\right)^2+A_2\left(\epsilon_r+r\right)^2}\right]$$
$$\times\cos\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\cos\left[\tan^{-1}\left(\frac{A_2\left(\epsilon_r+r\right)}{A_1\left(\epsilon_r+r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(\epsilon_r+r\right)+A_r(r)\right)\right]\Bigg\}\,. \tag{5.1.16}$$

The contribution coming form the effective Hamiltonian $\tilde{\mathcal{H}}(r-\epsilon_r)$ yields

$$\dot{E}_1^r(r)=\frac{2\pi\epsilon_r\sqrt{E^r\left(r-\epsilon_r\right)}}{\epsilon}\Bigg\{\pi H_0\left(\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}-\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right)\right)$$
$$\times\Bigg[\cos\left[\tan^{-1}\left(\frac{A_2\left(r-\epsilon_r\right)}{A_1\left(r-\epsilon_r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r(r)-A_r\left(r-\epsilon_r\right)\right)\right]$$
$$+\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(r-\epsilon_r\right)+A_r(r)\right)\right]\Bigg]$$
$$+\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}+\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right)\right]$$
$$\times\Bigg[\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(r-\epsilon_r\right)+A_r(r)\right)\right]$$
$$-\cos\left[\tan^{-1}\left(\frac{A_2\left(r-\epsilon_r\right)}{A_1\left(r-\epsilon_r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r(r)-A_r\left(r-\epsilon_r\right)\right)\right]\Bigg]$$
$$+4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right]$$
$$\times\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r\left(A_r\left(r-\epsilon_r\right)+A_r(r)\right)\right]$$
$$-4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r-\epsilon_r\right)^2+A_2\left(r-\epsilon_r\right)^2}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]$$
$$\times\cos\left[\tan^{-1}\left(\frac{A_2\left(r-\epsilon_r\right)}{A_1\left(r-\epsilon_r\right)}\right)+\frac{1}{2}\epsilon_r\left(A_r(r)-A_r\left(r-\epsilon_r\right)\right)\right]\Bigg\}\,. \tag{5.1.17}$$

The total effective evolution equation of motion is then:

$$\dot{E}_H^r=-\frac{1}{2\gamma}\sum_r\Bigg\{\frac{4\pi\epsilon_r}{\epsilon}\sqrt{E^r(r)}\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right] \tag{5.1.18}$$
$$\times\Bigg\{4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1^2(r)+A_2^2(r)}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right]$$
$$-\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]$$
$$+\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1^2(r)+A_2^2(r)}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\Bigg\}Biggl\}\,.$$

Expanding (5.1.18) and (5.1.15) to first order in the pair $(\epsilon_r,\ \epsilon)$, one recovers the contribution of the Hamiltonian constraint in the classical equations of motion, namely (4.2.20) and (4.2.23).

5.2 Phase space dependent quantum parameters

Considering the quantum relation connecting the quantum parameters to the fluxes suggests that it is in fact natural to employ them in their dynamical feature rather as Dirac observables, due to (5.1.1), which is more amenable for physical predictions [32]. Therefore, in this section we will consider the evolution equations resulting from the effective Hamiltonian in term of the phase space dependent quantum parameters

$$\epsilon_r = \alpha \frac{\sqrt{E^r(r)}}{E^1(r)} \ , \tag{5.2.1}$$

$$\epsilon = \frac{\beta}{\sqrt{E^r(r)}} \ , \tag{5.2.2}$$

where the constants $\alpha = \frac{\sqrt{8\pi\gamma}\ell_p j}{\sqrt{j_3}}$ and $\sqrt{8\pi\gamma}\ell_p\sqrt{j_3}$. To this aim, we will introduce the following notation

$$\delta_r^{\pm} A_r[\alpha_n] = \frac{\alpha\sqrt{E^r}}{E^1}\left(A_r(r) \pm A_r\left(r + n\frac{\alpha\sqrt{E^r}}{E^1}\right)\right) \ , \tag{5.2.3}$$

$$\mathcal{A}_n = \sqrt{A_1^2\left(r + n\frac{\alpha\sqrt{E^r}}{E^1}\right) + A_2^2\left(n\frac{\alpha\sqrt{E^r}}{E^1}\right)} \ , \tag{5.2.4}$$

$$\Delta^{\pm}\left[\mathcal{A}_n,\mathcal{A}_{n+1}\right] = \sqrt{A_1^2(r + n\frac{\alpha\sqrt{E^r}}{E^1}) + A_2^2(r + n\frac{\alpha\sqrt{E^r}}{E^1})} \pm \tag{5.2.5}$$

$$\sqrt{A_1^2\left(r + (n+1)\frac{\alpha\sqrt{E^r}}{E^1}\right) + A_2^2\left(r + (n+1)\frac{\alpha\sqrt{E^r}}{E^1}\right)} \ . \tag{5.2.6}$$

where n is as usual the step's number. For simplicity, we will also use the short notation for the phase space variables and write them as a functional of $\alpha_n = \alpha[E^r,E^1]$ with n is the number of functional step (shifts) affecting the flux and connection components respectively, and by expressing them as $E^1[\alpha_n],A_1[\alpha_n]$. Moreover, we will also use $\Delta^{\pm}$ to denote the difference operation acting on other functions than $\mathcal{A}_n$.
The bracket we intend to compute are

$$\dot{A}_i^H = \{A_i[\alpha_n,\beta], \tilde{\mathcal{H}}[N]\} \ , \tag{5.2.7}$$

$$\dot{E}_H^i = \{E^i[\alpha_n,\beta], \tilde{\mathcal{H}}[N]\} \ . \tag{5.2.8}$$

5.2.1 Ashtekar-Barbero connection effective evolution equations

Effective Euclidean contribution

As for the euclidean part of $A_1^E$the connection

$$\dot{A}_1^E = \frac{1}{4\gamma^4}\sum_r \left\{ \frac{\pi\alpha N(r)}{\sqrt{E^r(r)}E^1(r)\left(\beta\mathcal{A}_0\left(\Delta^+[\mathcal{A}_1,\mathcal{A}_2]\right)^{3/2}\right)} \left\{ \left(2\pi E^r(r)\left(A_1(r)\cos\left[\delta^- A_r[\alpha_1]\right]\right.\right.\right.\right.$$

$$\left. - A_2(r)\sin\left[\delta^- A_r[\alpha_1]\right]\right)\left(H_0\left[\frac{\beta\Delta^-[\mathcal{A}_0,\mathcal{A}_1]}{2E^r(r)}\right] + H_0\left[\frac{\beta\left(\Delta^+[\mathcal{A}_0,\mathcal{A}_1]\right)}{2E^r(r)}\right]\right)$$

$$
\begin{aligned}
&\times \left(\delta^- A_r[\alpha_1]E^1(r) + \alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right)\Bigg) \\
&\times \left(\Delta^+[\mathcal{A}_1,\mathcal{A}_2]\right)^{3/2} - 2\pi\beta E^1(r)\left(A_2(r)\cos\left[\delta^- A_r[\alpha_1]\right] + A_1(r)\sin\left[\delta^- A_r[\alpha_1]\right]\right) \\
&\times \left(H_{-1}\left[\frac{\beta\Delta^-[\mathcal{A}_0,\mathcal{A}_1]}{2E^r(r)}\right] - H_{-1}\left[\frac{\beta\left(\Delta^+[\mathcal{A}_0,\mathcal{A}_1]\right)}{2E^r(r)}\right]\right)\left(A_1[\alpha_1]A_1'[\alpha_1] + A_2[\alpha_1]A_2'[\alpha_1]\right)\left(\Delta^+[\mathcal{A}_1,\mathcal{A}_2]\right) \\
&+ 2\pi E^r(r)\mathcal{A}_0\left(H_0\left[\frac{\beta\Delta^-[\mathcal{A}_0,\mathcal{A}_1]}{2E^r(r)}\right] - H_0\left[\frac{\beta\left(\Delta^+[\mathcal{A}_0,\mathcal{A}_1]\right)}{2E^r(r)}\right]\right) \\
&\times \Bigg(\cos\left[\delta^+ A_r[\alpha_1]\right]\left(2E^1(r)A_2'[\alpha_1] + A_1[\alpha_1]\left(\delta^+ A_r[\alpha_1]E^1(r) + \alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right)\right) \\
&- \sin\left[\delta^+ A_r[\alpha_1]\right]\left(A_2[\alpha_1]\left(\delta^+ A_r[\alpha_1]E^1(r) + \alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right) - 2E^1(r)A_1'[\alpha_1]\right)\Bigg)\left(\Delta^+[\mathcal{A}_1,\mathcal{A}_2]\right) \\
&- 2\pi\beta\mathcal{A}_0\Delta^+[\mathcal{A}_1,\mathcal{A}_2]E^1(r)\left(A_2[\alpha_1]\cos\left[\delta^+ A_r[\alpha_1]\right] + A_1[\alpha_1]\sin\left[\delta^+ A_r[\alpha_1]\right]\right) \\
&\times \left(H_{-1}\left[\frac{\beta\Delta^-[\mathcal{A}_0,\mathcal{A}_1]}{2E^r(r)}\right] + H_{-1}\left[\frac{\beta\left(\Delta^+[\mathcal{A}_0,\mathcal{A}_1]\right)}{2E^r(r)}\right]\right)\left(A_1[\alpha_1]A_1'[\alpha_1] + A_2[\alpha_1]A_2'[\alpha_1]\right) \\
&- 4\pi\mathcal{A}_0E^1(r)E^r(r)\left(A_2[\alpha_1]\cos\left[\delta^+ A_r[\alpha_1]\right] + A_1[\alpha_1]\sin\left[\delta^+ A_r[\alpha_1]\right]\right) \\
&\times \left(H_0\left[\frac{\beta\Delta^-[\mathcal{A}_0,\mathcal{A}_1]}{2E^r(r)}\right] - H_0\left[\frac{\beta\left(\Delta^+[\mathcal{A}_0,\mathcal{A}_1]\right)}{2E^r(r)}\right]\right)\left(A_1[\alpha_1]A_1'[\alpha_1] + A_2[\alpha_1]A_2'[\alpha_1]\right) \\
&- 8\left(\Delta^+[\mathcal{A}_1,\mathcal{A}_2]\right)^{3/2}\Bigg(-E^r(r)\cos\left[\frac{\beta\sqrt{\Delta^+[\mathcal{A}_1^2,\mathcal{A}_2^2]}}{2E^r(r)}\right]\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right] \\
&\times \left(A_1(r)\cos\left[\delta^- A_r[\alpha_1]\right] - A_2(r)\sin\left[\delta^- A_r[\alpha_1]\right]\right)\left[\delta^- A_r[\alpha_1]\right]E^1(r) + \alpha\sqrt{E^r(r)}A_r'[\alpha_1]\Bigg) \\
&+ \beta E^1(r)\left(A_2(r)\cos\left[\delta^- A_r[\alpha_1]\right] + A_1(r)\sin\left[\delta^- A_r[\alpha_1]\right]\right)\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\sin\left[\frac{\beta\sqrt{\Delta^+[\mathcal{A}_1,\mathcal{A}_2]}}{2E^r(r)}\right] \\
&\times \left(A_1[\alpha_1]A_1'[\alpha_1] + A_2[\alpha_1]A_2'[\alpha_1]\right)\left(\Delta^+[\mathcal{A}_1,\mathcal{A}_2]\right) \\
&+ E^r(r)\mathcal{A}_0\left(\Delta^+[\mathcal{A}_1,\mathcal{A}_2]\right)\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\sin\left[\frac{\beta\sqrt{\Delta^+[\mathcal{A}_1,\mathcal{A}_2]}}{2E^r(r)}\right] \\
&\times \Bigg\{\cos\left[\delta^+ A_r[\alpha_1]\right]\left(2E^1(r)A_2'[\alpha_1] + A_1[\alpha_1]\left(\delta^+ A_r[\alpha_1]E^1(r) + \alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right)\right) \\
&- \sin\left[\delta^+ A_r[\alpha_1]\right]\left(A_2[\alpha_1]\left(\delta^+ A_r[\alpha_1]E^1(r) + \alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right) - 2E^1(r)A_1'[\alpha_1]\right)\Bigg\} \\
&- 2\mathcal{A}_0\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]E^1(r)E^r(r)\sin\left[\frac{\beta\Delta^+[\mathcal{A}_1,\mathcal{A}_2]}{2E^r(r)}\right]\left(A_1[\alpha_1]A_1'[\alpha_1] + A_2[\alpha_1]A_2'[\alpha_1]\right) \\
&\times \left(A_2[\alpha_1]\cos\left[\delta^+ A_r[\alpha_1]\right] + A_1[\alpha_1]\sin\left[\delta^+ A_r[\alpha_1]\right]\right) \\
&+ \beta\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\cos\left[\frac{\beta\Delta^+[\mathcal{A}_1,\mathcal{A}_2]}{2E^r(r)}\right]\mathcal{A}_0\Delta^+[\mathcal{A}_1,\mathcal{A}_2]\left(A_1[\alpha_1]A_1'[\alpha_1] + A_2[\alpha_1]A_2'[\alpha_1]\right)
\end{aligned}
$$

$$\times E^1(r)\left(A_2[\alpha_1]\cos\left[\frac{\alpha\delta^+ A_r[\alpha_1]\sqrt{E^r(r)}}{2E^1(r)}\right]+A_1[\alpha_1]\sin\left[\frac{\alpha\delta^+ A_r[\alpha_1]\sqrt{E^r(r)}}{2E^1(r)}\right]\right)\Bigg\}\Bigg\}\,. \tag{5.2.9}$$

Whereas the for the A_r component the commutation follows similarly and it reads

$$
\begin{aligned}
\dot{A}_r^E = &-\frac{1}{4\gamma^4}\sum_r\Bigg\{\frac{\pi N(r)}{\beta^2}\Bigg\{\frac{8\beta^2}{\sqrt{E^r(r)}}\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\cos\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right]\left(A_2(r)\cos\left[\delta^- A_r[\alpha_1]\right]+A_1(r)\sin\left[\delta^- A_r[\alpha_1]\right]\right)\\
&+\frac{24}{\mathcal{A}_1}\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\left(\mathcal{A}_1\cos\left[\delta^+ A_r[\alpha_1]\right]+A_1[\alpha_1]\sin\left[\delta^+ A_r[\alpha_1]\right]\right)\sin\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right]\sqrt{E^r(r)}\beta\\
&-4\pi\alpha\mathcal{A}_0\sin\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]H_{-1}\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]\beta-4\pi\alpha\mathcal{A}_0\cos\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]H_0\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]\beta\\
&-\frac{24}{\mathcal{A}_0}\cos\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right]\sqrt{E^r(r)}\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\left(A_2(r)\cos\left[\delta^- A_r[\alpha_1]\right]+A_1(r)\sin\left[\delta^- A_r[\alpha_1]\right]\right)\beta\\
&+8\pi\alpha E^r(r)\sin\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]H_0\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]-\frac{6\pi\beta}{\mathcal{A}_1}\sqrt{E^r(r)}\left(\mathcal{A}_1\cos\left[\delta^+ A_r[\alpha_1]\right]+A_1[\alpha_1]\sin\left[\delta^+ A_r[\alpha_1]\right]\right)\\
&\times\left(H_0\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]-H_0\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\right)\\
&-\frac{6\pi\beta}{\mathcal{A}_0}\sqrt{E^r(r)}\left(A_2(r)\cos\left[\delta^- A_r[\alpha_1]\right]+A_1(r)\sin\left[\delta^- A_r[\alpha_1]\right]\right)\\
&\times\left(H_0\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]+H_0\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\right)\\
&-\frac{8\alpha\beta}{\mathcal{A}_1^{3/2}E^1(r)}\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]E^r(r)\sin\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right]\\
&\times\left(\mathcal{A}_1\cos\left[\delta^+ A_r[\alpha_1]\right]+A_1[\alpha_1]\sin\left[\delta^+ A_r[\alpha_1]\right]\right)\left(A_1[\alpha_1]A_1'[\alpha_1]+\mathcal{A}_1A_2'[\alpha_1]\right)\\
&+\frac{2\pi\alpha\beta}{\mathcal{A}_1^{3/2}E^1(r)}E^r(r)\left(\mathcal{A}_1\cos\left[\delta^+ A_r[\alpha_1]\right]+A_1[\alpha_1]\sin\left[\delta^+ A_r[\alpha_1]\right]\right)\\
&\times\left(H_0\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]-H_0\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\right)\\
&\times\left(A_1[\alpha_1]A_1'[\alpha_1]+\mathcal{A}_1A_2'[\alpha_1]\right)-\frac{4\beta^2}{\mathcal{A}_1E^1(r)\sqrt{E^r(r)}}\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\cos\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right]\\
&\times\left(\mathcal{A}_1\cos\left[\delta^+ A_r[\alpha_1]\right]+A_1[\alpha_1]\sin\left[\delta^+ A_r[\alpha_1]\right]\right)\left(2\mathcal{A}_1E^1(r)-\alpha\sqrt{E^r(r)}\left(A_1[\alpha_1]A_1'[\alpha_1]+\mathcal{A}_1A_2'[\alpha_1]\right)\right)\\
&+\frac{\pi\beta^2}{\mathcal{A}_1E^1(r)\sqrt{E^r(r)}}\left(\mathcal{A}_1\cos\left[\delta^+ A_r[\alpha_1]\right]+A_1[\alpha_1]\sin\left[\delta^+ A_r[\alpha_1]\right]\right)\\
&\times\Bigg(H_{-1}\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\left(-2\mathcal{A}_1E^1(r)-2\mathcal{A}_0\mathcal{A}_1E^1(r)+\alpha\sqrt{E^r(r)}\left(A_1[\alpha_1]A_1'[\alpha_1]+\mathcal{A}_1A_2'[\alpha_1]\right)\right)\\
&+H_{-1}\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]\left(-2\mathcal{A}_1E^1(r)+2\mathcal{A}_0\mathcal{A}_1E^1(r)+\alpha\sqrt{E^r(r)}\left(A_1[\alpha_1]A_1'[\alpha_1]+\mathcal{A}_1A_2'[\alpha_1]\right)\right)\Bigg)\\
&+\frac{\pi\beta^2}{\mathcal{A}_0\mathcal{A}_1E^1(r)\sqrt{E^r(r)}}\left(A_2(r)\cos\left[\delta^- A_r[\alpha_1]\right]+A_1(r)\sin\left[\delta^- A_r[\alpha_1]\right]\right)\\
&\times\Bigg(H_{-1}\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]\left(-2\mathcal{A}_1E^1(r)+2\mathcal{A}_0\mathcal{A}_1E^1(r)+\alpha\sqrt{E^r(r)}\left(A_1[\alpha_1]A_1'[\alpha_1]+\mathcal{A}_1A_2'[\alpha_1]\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ H_{-1}\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\left(2\left(A_1[\alpha_1]^2+\mathcal{A}_1^2+\mathcal{A}_0\mathcal{A}_1\right)E^1(r)-\alpha\sqrt{E^r(r)}\left(A_1[\alpha_1]A_1'[\alpha_1]+\mathcal{A}_1A_2'[\alpha_1]\right)\right)\Bigg) \\
&- \frac{4\alpha\beta}{\mathcal{A}_0E^1(r)^2}\cos\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right]E^r(r)\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right] \\
&\times\left(A_1(r)\cos\left[\delta^-A_r[\alpha_1]\right]-A_2(r)\sin\left[\delta^-A_r[\alpha_1]\right]\right)\left(\delta^-A_r[\alpha_1]E^1(r)+\alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right) \\
&- \frac{\pi\alpha\beta}{\mathcal{A}_0E^1(r)^2}E^r(r)\left(A_1(r)\cos\left[\delta^-A_r[\alpha_1]\right]-A_2(r)\sin\left[\delta^-A_r[\alpha_1]\right]\right) \\
&\times\left(H_0\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]+H_0\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\right)\left(\left(\Delta A_r[\alpha_1]\right)E^1(r)+\alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right) \\
&+ \frac{\pi\alpha\beta}{\mathcal{A}_1E^1(r)^2}E^r(r)\left(H_0\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]-H_0\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\right) \\
&\times\Bigg(\sin\left[\delta^+A_r[\alpha_1]\right]\left(\mathcal{A}_1\left(\left(A_r(r)+A_r[\alpha_1]\right)E^1(r)+\alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right)-2E^1(r)A_1'[\alpha_1]\right) \\
&- \cos\left[\delta^+A_r[\alpha_1]\right]\left(2E^1(r)A_2'[\alpha_1]+A_1[\alpha_1]\left(\left(A_r(r)+A_r[\alpha_1]\right)E^1(r)+\alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right)\right)\Bigg) \\
&+ \frac{4\alpha\beta}{\mathcal{A}_1E^1(r)^2}\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]E^r(r)\sin\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right] \\
&\times\Bigg(\cos\left[\delta^+A_r[\alpha_1]\right]\left(2E^1(r)A_2'[\alpha_1]+A_1[\alpha_1]\left(\left(A_r(r)+A_r[\alpha_1]\right)E^1(r)+\alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right)\right) \\
&- \sin\left[\delta^+A_r[\alpha_1]\right]\left(\mathcal{A}_1\left(\left(A_r(r)+A_r[\alpha_1]\right)E^1(r)+\alpha\sqrt{E^r(r)}A_r'[\alpha_1]\right)-2E^1(r)A_1'[\alpha_1]\right)\Bigg) \\
&- \frac{4\beta^2}{\mathcal{A}_0\mathcal{A}_1E^1(r)\sqrt{E^r(r)}}\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\sin\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right] \\
&\times\left(A_2(r)\cos\left[\delta^-A_r[\alpha_1]\right]+A_1(r)\sin\left[\delta^-A_r[\alpha_1]\right]\right)\left(2\mathcal{A}_1E^1(r)-\alpha\sqrt{E^r(r)}\left(A_1[\alpha_1]A_1'[\alpha_1]+\mathcal{A}_1A_2'[\alpha_1]\right)\right) \\
&+ \frac{8\beta^2}{\mathcal{A}_1\sqrt{E^r(r)}}\mathcal{A}_0\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\sin\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right]\left(\mathcal{A}_1\cos\left[\delta^+A_r[\alpha_1]\right]+A_1[\alpha_1]\sin\left[\delta^+A_r[\alpha_1]\right]\right)\Bigg\}\Bigg\}\,.
\end{aligned}
\tag{5.2.10}
$$

Analogously to the fluxes, expanding the effective equation for the Ashtekar-Barbero connection, namely (5.2.10) and (5.2.9) to first order in the quantum parameters α and β to first order, one recovers the classical equations of motion of the Euclidean contribution in the classical Hamiltonian derived in chapter 4, namely the equations (4.2.14) and (4.2.7).

Effective Lorentzian contribution

The Lorentzian contribution reads for the A_1 connection yields the equation

$$
\begin{aligned}
\dot{A}_1^L &= -\frac{1}{4\gamma^4}\sum_r\Bigg\{\frac{2\pi N(r)}{\alpha\beta^2(E^1(r))^2E^1[\alpha_2]^3E^r(r)E^r[\alpha_2]^{3/2}} \\
&\times\Bigg\{-16\alpha^3\gamma^2\cos\left[\frac{\beta}{E^r(r)}\right]E^1[\alpha_2]^3\sin^2\left[\frac{\beta}{2E^r(r)}\right]\left(E^1[\alpha_2]E^{r'}[\alpha_2]-2E^r[\alpha_2]E^{1'}[\alpha_2]\right)E^r(r)^{7/2}
\end{aligned}
$$

$$
\begin{aligned}
&+4\alpha^2\gamma^2\left(-2\cos\left[\frac{\beta}{E^r(r)}\right]+\cos\left[\frac{2\beta}{E^r(r)}\right]+1\right)E^1(r)E^1[\alpha_2]^4E^r[\alpha_2]E^r(r)^3\\
&-\beta^2\left(\gamma^2+1\right)E^1(r)^3E^1[\alpha_2]E^r[\alpha_2]\Bigg(E^1[\alpha_2]\left(E^r[\alpha_1]^2-6E^r[\alpha_2]E^r[\alpha_1]+E^r[\alpha_2]^2+4E^r(r)E^r[\alpha_2]\right)\\
&+4E^1[\alpha_1]\left(E^r[\alpha_2]-E^r[\alpha_1]\right)E^r[\alpha_2]\Bigg)-\alpha\beta^2\left(\gamma^2+1\right)E^1(r)^2\sqrt{E^r(r)}\Bigg\{\\
&\Bigg(2\left(E^r[\alpha_1]-3E^r[\alpha_2]\right)E^r[\alpha_2]E^{r'}[\alpha_1]+\left(-E^r[\alpha_1]^2-6E^r[\alpha_2]E^r[\alpha_1]+3E^r[\alpha_2]^2+4E^r(r)E^r[\alpha_2]\right)\\
&\times E^{r'}[\alpha_2]\Bigg)E^1[\alpha_2]^2-2E^r[\alpha_2]\Bigg(\left(E^{1'}[\alpha_2]-2E^1[\alpha_1]\right)E^r[\alpha_2]^2+4E^r(r)E^{1'}[\alpha_2]E^r[\alpha_2]\\
&+2\left(E^r[\alpha_1]\left(E^1[\alpha_1]-3E^{1'}[\alpha_2]\right)+E^1[\alpha_1]\left(E^{r'}[\alpha_1]-3E^{r'}[\alpha_2]\right)\right)E^r[\alpha_2]\\
&+E^r[\alpha_1]\left(E^r[\alpha_1]E^{1'}[\alpha_2]+2E^1[\alpha_1]E^{r'}[\alpha_2]\right)\Bigg)E^1[\alpha_2]+16E^1[\alpha_1]\left(E^r[\alpha_1]-E^r[\alpha_2]\right)E^r[\alpha_2]^2E^{1'}[\alpha_2]\Bigg\}\Bigg\}\Bigg\}\,.
\end{aligned}
\tag{5.2.11}
$$

For the case of the radial component of the connection A_r one obtains

$$
\begin{aligned}
\dot{A}_r^L=&-\frac{1}{4\gamma^4}\sum_r\Bigg\{-\frac{2\pi}{\alpha^2E^1(r)^2E^r(r)^2}\Bigg\{2\alpha^2\left(\gamma^2+1\right)N(r)\left(\left(E^r(r)-E^r[\alpha_1]\right)E^{1\prime}[\alpha_1]-E^1[\alpha_1]E^{r\prime}[\alpha_1]\right)\\
&\times E^r(r)^{3/2}+E^1(r)^2\Bigg(4\alpha\beta\gamma^2E^r(r)^{3/2}\left(\sin\left[\frac{2\beta}{\sqrt{E^r(r)}}\right]-\sin\left[\frac{\beta}{\sqrt{E^r(r)}}\right]\right)-\alpha\left(\gamma^2+1\right)E^r[\alpha_1]^2N(r)\\
&+E^r(r)^2\left(4N(r)\beta\left(-2\cos\left[\frac{\beta}{\sqrt{E^r(r)}}\right]+\cos\left[\frac{2\beta}{\sqrt{E^r(r)}}\right]+1\right)\gamma^2+\beta\left(\gamma^2+1\right)\right)\Bigg)\\
&+\alpha\left(\gamma^2+1\right)E^1(r)N(r)\sqrt{E^r(r)}\Bigg[4E^1[\alpha_1]E^r(r)^{3/2}+4\alpha E^{r\prime}[\alpha_2]E^r(r)\\
&+\alpha\left(E^r[\alpha_1]-3E^r(r)\right)E^{r\prime}[\alpha_1]\Bigg]\Bigg\}+\frac{\pi}{2E^r[\alpha_1]^{3/2}}\Bigg\{\frac{8\gamma^2\beta}{\alpha}\Bigg(2E^r(r)\left(\sin\left[\frac{\alpha}{\sqrt{E^r(r)}}\right]-\sin\left[\frac{2\alpha}{\sqrt{E^r(r)}}\right]\right)\\
&+\frac{E^r[\alpha_1]E^{r\prime}(r)}{E^1[\alpha_1]}\left(-2\cos\left[\frac{\alpha}{\sqrt{E^r(r)}}\right]+\cos\left[\frac{2\alpha}{\sqrt{E^r(r)}}\right]+1\right)\Bigg)+\frac{4\left(\gamma^2+1\right)N(r)}{\alpha E^1(r)^2\sqrt{E^r(r)}}\sqrt{E^r[\alpha_1]}\\
&\times\Bigg(4E^r(r)\sqrt{E^r[\alpha_1]}E^1[\alpha_1]^2+\alpha\left(3E^r(r)-E^r[\alpha_1]\right)E^{r\prime}(r)E^1[\alpha_1]+2\alpha E^r(r)\Bigg(E^r(r)\\
&-E^r[\alpha_2]\Bigg)E^{1\prime}(r)\Bigg)+\frac{\left(\gamma^2+1\right)N(r)}{\alpha E^1(r)E^r(r)^{3/2}}\Bigg(2E^1[\alpha_1]E^r(r)\left(5E^r(r)^2+6E^r[\alpha_1]E^r(r)-3E^r[\alpha_1]^2\right)\\
&-\alpha\sqrt{E^r[\alpha_1]}\Bigg(8E^{r\prime}[\alpha_2]E^r(r)^2+\Bigg[-3E^r(r)^2+6E^r[\alpha_1]E^r(r)-4E^r[\alpha_2]E^r(r)+E^r[\alpha_1]^2\Bigg]E^{r\prime}(r)\Bigg)\Bigg)\\
&-\frac{16\left(\gamma^2+1\right)}{E^1(r)^3}E^1[\alpha_1]\sqrt{E^r(r)}\left(E^r(r)-E^r[\alpha_1]\right)\sqrt{E^r[\alpha_1]}N(r)E^{1\prime}(r)\Bigg\}\\
&+\frac{\pi N(r)}{E^r[\alpha_2]^{3/2}}\Bigg\{\frac{8\beta}{\alpha}\Bigg(E^r(r)\left(\sin\left[\frac{\alpha}{\sqrt{E^r(r)}}\right]-\sin\left[\frac{2\alpha}{\sqrt{E^r(r)}}\right]\right)
\end{aligned}
$$

$$+\frac{\gamma^2 E^r[\alpha_2]E^{r\prime}(r)}{E^1[\alpha_2]}\left(-2\cos\left[\frac{\beta}{\sqrt{E^r(r)}}\right]+\cos\left[\frac{2\alpha}{\sqrt{E^r(r)}}\right]+1\right)\Bigg)$$

$$+\frac{(\gamma^2+1)}{\alpha E^1(r)E^r(r)^{3/2}}\Bigg(E^1[\alpha_2]E^r(r)\left(E^r(r)^2-6E^r[\alpha_1]E^r(r)-4E^r[\alpha_2]E^r(r)+E^r[\alpha_1]^2\right)$$

$$+\alpha\sqrt{E^r[\alpha_2]}\Bigg[\left(3E^r(r)^2-6E^r[\alpha_1]E^r(r)+4E^r[\alpha_2]E^r(r)-E^r[\alpha_1]^2\right)E^{r\prime}(r)$$

$$+2E^r(r)\left(E^r[\alpha_1]-3E^r(r)\right)E^{r\prime}[\alpha_1]\Bigg]\Bigg)\Bigg\}$$

$$-\frac{2(\gamma^2+1)}{\alpha E^1(r)^2\sqrt{E^r(r)}}\Bigg\{\alpha\sqrt{E^r[\alpha_2]}\Bigg(\left(E^{1\prime}(r)-2E^{1\prime}[\alpha_2]\right)E^r(r)^2$$

$$+\left(4E^r[\alpha_2]E^{1\prime}(r)+2E^r[\alpha_1]\left(E^{1\prime}\left(r+\frac{\alpha\sqrt{E^r(r)}}{E^1(r)}\right)-3E^{1\prime}(r)\right)\right)E^r(r)+E^r[\alpha_1]^2E^{1\prime}(r)\Bigg)$$

$$+2E^1[\alpha_1]\Bigg(E^1[\alpha_2]E^r(r)\left(E^r[\alpha_1]-E^r(r)\right)+\alpha\sqrt{E^r[\alpha_2]}\left(E^r(r)\left(E^{r\prime}[\alpha_1]-3E^{r\prime}(r)\right)+E^r[\alpha_1]E^{r\prime}[\alpha_2]\right)\Bigg)\Bigg\}$$

$$-\frac{16(\gamma^2+1)}{E^1(r)^3}E^1[\alpha_1]\sqrt{E^r(r)}\left(E^r(r)-E^r[\alpha_1]\right)\sqrt{E^r[\alpha_2]}E^{1\prime}(r)\Bigg\} \tag{5.2.12}$$

Expanding the effective equation for the Ashtekar-Barbero connection, namely (5.2.11) and (5.2.12) to first order in the quantum parameters α and β to first order, one recovers the classical equations of motion of the Lorentzian contribution in the classical Hamiltonian derived in chapter 4, namely the equations (4.2.15) and (4.2.8).
Thus, the total effective of motion yields

$$\dot{A}_1^H=\dot{A}_1^L+\dot{A}_1^E\ , \tag{5.2.13}$$

$$\dot{A}_r^H=\dot{A}_r^L+\dot{A}_r^E\ . \tag{5.2.14}$$

5.2.2 Fluxes effective evolution equations

The evolution equation of the flux E^r reads

$$E^r_H=-\frac{1}{4\gamma^4}\sum_r\Bigg\{\frac{2\pi\alpha E^r(r)^{3/2}N(r)}{\beta}\Bigg\{4\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\cos\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right]\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\delta_r^-A_r[\alpha_1]\right]$$

$$+4\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\sin\left[\frac{\beta\mathcal{A}_1}{2E^r(r)}\right]\cos\left[\tan^{-1}\left(\frac{A_2[\alpha_2]}{A_r[\alpha_1]}\right)+\delta_r^+A_r[\alpha_1]\right]$$

$$+\pi H_0\left[\frac{\beta(\mathcal{A}_0-\mathcal{A}_1)}{2E^r(r)}\right]\left(\cos\left[\tan^{-1}\left(\frac{A_2[\alpha_1]}{A_2[\alpha_1]}\right)+\delta_r^+A_r[\alpha_1]\right]-\cos\left[\tan^{-1}\left(\frac{A_2[\alpha_0]}{A_2[\alpha_0]}\right)\delta^-A_r[\alpha_1]\right]\right) \tag{5.2.15}$$

$$+\pi H_0\left[\frac{\beta(\mathcal{A}_0+\mathcal{A}_1)}{2E^r(r)}\right]\left(\cos\left[\tan^{-1}\left(\frac{A_2[\alpha_0]}{A_2[\alpha_0]}\right)+\delta^-A_r[\alpha_1]\right]+\cos\left[\tan^{-1}\left(\frac{A_2[\alpha_1]}{A_2[\alpha_1]}\right)+\delta^+A_r[\alpha_1]\right]\right)\Bigg\}\Bigg\}\ . \tag{5.2.16}$$

For the E^1 component of the flux, one can find that the equation of motion reads

$$\dot{E}^1_H = -\frac{1}{4\gamma^4}\sum_r\Bigg\{\frac{4\pi E^r(r)N(r)\sqrt{E^r(r)}}{\beta}\Bigg\{4\Bigg\{\frac{A_1(r)}{(\mathcal{A}_0)^{3/2}}\cos\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right] \tag{5.2.17}$$

$$\begin{aligned}
&\times\left(A_2(r)\cos\left[\delta_r^- A_r[\alpha_1]\right]+A_1(r)\sin\left[\delta_r^- A_r[\alpha_1]\right]\right)\\
&-\frac{\beta A_1(r)}{2\left(\mathcal{A}_0\right)^{1/2}E^r(r)}\cos\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\cos\left[\frac{\beta\mathcal{A}_0+\mathcal{A}_1}{2E^r(r)}\right]\\
&\times\left(A_2(r)\cos\left[\delta_r^- A_r[\alpha_1]\right]+A_1(r)\sin\left[\delta_r^- A_r[\alpha_1]\right]\right)\\
&-\frac{\beta\sqrt{E^r(r)}}{2\mathcal{A}_0+\mathcal{A}_1E^r(r)}A_1(r)\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\sin\left[\frac{\beta\mathcal{A}_0+\mathcal{A}_1}{2E^r(r)}\right]\\
&\times\left(A_2[\alpha_1]\cos\left[\delta^+A_r[\alpha_1]\right]+A_1[\alpha_1]\sin\left[\delta^+A_r[\alpha_1]\right]\right)\\
&-\frac{1}{\mathcal{A}_0}\cos\left[\frac{\beta\mathcal{A}_0+\mathcal{A}_1}{2E^r(r)}\right]\sin\left[\frac{\beta\mathcal{A}_0}{2E^r(r)}\right]\sin\left[\delta_r^- A_r[\alpha_1]\right]\Bigg\}\\
&+\pi\Bigg\{-\frac{\beta A_1(r)}{2\left(A_1(r)^2+A_2(r)^2\right)E^r(r)}\left(A_2(r)\cos\left[\delta_r^- A_r[\alpha_1]\right]+A_1(r)\sin\left[\delta_r^- A_r[\alpha_1]\right]\right)\\
&\times\left(H_{-1}\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]+H_{-1}\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\right)\\
&+\frac{A_1(r)}{\left(A_1(r)^2+A_2(r)^2\right)^{3/2}}\left(A_2(r)\cos\left[\delta_r^- A_r[\alpha_1]\right]+A_1(r)\sin\left[\delta_r^- A_r[\alpha_1]\right]\right)\\
&\times\left(H_0\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]+H_0\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\right)\Bigg\}\\
&-\frac{\beta}{2\mathcal{A}_0+\mathcal{A}_1E^r(r)}A_1(r)\left(A_2[\alpha_1]\cos\left[\delta^+A_r[\alpha_1]\right]+A_1[\alpha_1]\sin\left[\delta^+A_r[\alpha_1]\right]\right)\\
&\times\left(H_{-1}\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]-H_{-1}\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\right)\\
&-\frac{\sqrt{E^r(r)}}{\mathcal{A}_0}\sin\left[\delta_r^- A_r[\alpha_1]\right]\left(H_0\left[\frac{\beta\left(\mathcal{A}_0-\mathcal{A}_1\right)}{2E^r(r)}\right]+H_0\left[\frac{\beta\left(\mathcal{A}_0+\mathcal{A}_1\right)}{2E^r(r)}\right]\right)\\
&+\frac{\pi\alpha A_1(r)\sin\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]H_{-1}\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]}{\mathcal{A}_0}+\frac{\pi\alpha A_1(r)\cos\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]H_0\left[\frac{\beta\mathcal{A}_0}{E^r(r)}\right]}{\mathcal{A}_0}\Bigg\}\Bigg\}\,.
\end{aligned}\tag{5.2.18}$$

Expanding the effective equation for the fluxes, namely (5.2.15) and (5.2.17) to first order in the quantum parameters α and β to first order, one recovers the classical equations of motion derived in chapter 4, namely the equations (4.2.20) and (4.2.23).

The effective evolution equations

$$\dot{A}_1^H = \dot{A}_1^L + \dot{A}_1^E\,, \tag{5.2.19}$$

$$\dot{A}_r^H = \dot{A}_r^L + \dot{A}_r^E\,, \tag{5.2.20}$$

and

$$\dot{E}^1_H\,,\quad \dot{E}^r_H\,, \tag{5.2.21}$$

are part of the set of equations that should account for the dynamics of the theory once solved. Indeed, these effective equations obtained from computing the Poisson bracket of the phase

space variables with the effective Hamiltonian should presumably cancel the ones obtained from a proposed diffeomorphism constraint, as it should follow analogously from the classical studied case in (B). This is a key step in ensuring the consistency and reliability of the present framework. The effective radial evolution equations will be discussed in the next chapter.
Moreover, once we extract a proposal for the effective diffeomorphism constraint we will end up with six as the total number of nonlocal differential equations to be solved for, namely four coming from the evolution equations added up to the Hamiltonian and diffeomorphism constraints. Once one makes a choice for the lapse function and shift vector two of the constraints become redundant and one may conveniently choose four out of the six equations to solve for the four phase space variables.

Chapter 6

Effective diffeomorphism constraint

In this chapter we will consider a proposal for the effective diffeomorphism constraint in the case of constant quantum numbers, leaving the scenario of the quantum parameters expressed as phase space functions for future work.
At the effective level, the only available constraint that we will be able to work with and extract a candidate expression for the diffeomorphism constraint, is the effective Hamiltonian $\tilde{H}$. To this end, a convenient program to write down a solvable set of equations for the candidate effective counterpart of the diffeomorphism and the effective equations of motion, is to make use of the constraint algebra closure. Though this should include commutators involving the Hamiltonian constraint, which is mainly provided by

$$\{H\left[N_1\right], H\left[N_2\right]\} = H\left(g^{ab}\left(N_1\partial_b N_2 - N_2\partial_b N_1\right)\right) \ , \tag{6.0.1}$$

In our case this is translated in

$$\{H\left[N_1\right], H\left[N_2\right]\} = H\left(g^{rr}\left(N_1\partial_r N_2 - N_2\partial_r N_1\right)\right) \ , \tag{6.0.2}$$

Notice the metric component g^{rr}, that should factor out during the computations. Although at the classical level only the Euclidean term in the Hamiltonian gives rise to the diffeomorphism expression smeared by the structure functions, it might not hold anymore at the effective level. This is mainly due to the fact that, the quantum corrections at that stage may dominate and one is not able anymore to prove the vanishing of the bracket of the Lorentzian and Euclidean terms. Therefore they shall be included in the following computations.
The strategy to extract a candidate expression for the effective diffeomorphism constraint will put into play the derived effective equations of motion in the previous chapter. With help of discrete calculus techniques, one can factorize the discrete version of the structure functions that multiplies the "would-be" diffeomorphism constraint. We will present the arguments supporting a specific choice for the expression of the diffeomorphism, highlighting the reason supporting it. Once this is done, we will start the mechanism of consistency checking for the selected effective diffeomorphism constraint, presenting the first one as the derivation of its respective radial evolution equations for the phase space variables.
The obtained radial equations should in principal cancel the ones derived from the effective Hamiltonian constraint in the previous chapter (this would be the second consistency test) building up a system of six nonlocal differential equations to be solved for, namely four vanishing evolution equations in addition to the candidate diffeomorphism expression and the effective Hamiltonian constraint. Once a specific choice of lapse and shift is made, two become redundant and one is contented with four equations out of the initial six to solve for the four phase space variables.

6.1 Effective Hamiltonian bracket

However, the ordinary continuous derivative present in the classical Hamiltonian expression is replaced with discrete derivative. Indeed, this quantum nature inherited in the equations is the source of some technical subtleties. More concretely, in order to arrive at the smeared diffeomorphism in (6.0.2), integration by parts is preformed to obtain first oder derivative on the lapses (N,M), which consequently requires terms including derivatives in the phase space variables to serve as the basis for such smearing. To implement this strategy at the quantum level, one should make use of the spatial shifts defined in the previous chapter to work with new discrete derivative and integration operations. To this aim, we will first study the bracket written in the following way

$$\begin{aligned}\left\{\tilde{\mathcal{H}}[N_1(r)],\tilde{\mathcal{H}}[N_2(r')]\right\} &= \frac{\kappa\gamma}{4\kappa^2\gamma^2}\sum_r\left\{\tilde{\mathcal{H}}[N_1(r)],\tilde{\mathcal{H}}[N_2(r')]\right\}\\ &= \frac{1}{4\kappa\gamma^3}\sum_r\left[\frac{\delta\tilde{\mathcal{H}}[N_1(r)]}{\delta A_1(\rho)}\frac{\delta\tilde{\mathcal{H}}[N_2(r')]}{\delta E^1(\rho)}-\frac{\delta\tilde{\mathcal{H}}[N_1(r)]}{\delta E^1(\rho)}\frac{\delta\tilde{\mathcal{H}}[N_2(r')]}{\delta A_1(\rho)}\right.\\ &\left.+2\left(\frac{\delta\tilde{\mathcal{H}}[N_1(r)]}{\delta A_r(\rho)}\frac{\delta\tilde{\mathcal{H}}[N_2(r')]}{\delta E^r(\rho)}-\frac{\delta\tilde{\mathcal{H}}[N_1(r)]}{\delta E^(\rho)}\frac{\delta\tilde{\mathcal{H}}[N_2(r')]}{\delta A^r(\rho)}\right)\right],\end{aligned}$$

where the effective Hamiltonian entails the contribution of similar terms, yet shifted by ϵ_r, namely

$$\tilde{\mathcal{H}}[N(r)] = \sum_{n=0}\tilde{\mathcal{H}}[N(r+n\epsilon_r)]\ , \tag{6.1.1}$$

where for the case of the (A_1,A_r) equations of motion, it would involves only up to $n = 2$ whereas for the fluxes (E^1, E^r) it comprises terms up to $n = 1$, since the Lorentzian term responsible for the presence of second derivatives is connection-independent.
The direct analytic computation of the above Poisson bracket is a lengthy one and it is not straight forward to extract the correct expression of the desired constraint. The scheme we adapted is to go through the computation in a symbolic numerical one. Before moving on to that point, let us consider a simple manipulation in the spatial shift using the back and forward derivation operations defined as

$$\begin{aligned}\mathcal{F}^+ &:= f(x+\epsilon)-f(x) = \Delta^+_\epsilon[f]\ ,\\ \mathcal{F}^- &:= f(x)-f(x-\epsilon) = \Delta^-_\epsilon[f]\ .\end{aligned} \tag{6.1.2}$$

Hence, the classical equivalent of, for instance, the first derivative of a function $N(r)$

$$\mathcal{F}^+N(r) = N(r+\epsilon)-N(r) \tag{6.1.3}$$
$$= N(r)-N(r-\epsilon) \tag{6.1.4}$$
$$\approx N'(r)\ . \tag{6.1.5}$$

These are the equivalent discrete derivatives for left and right continuous ones, and it hold $\mathcal{F}^+ = \mathcal{F}^-$ for $\epsilon \to 0$. In fact we will use both of these operations, since there is no information loss while doing it, due to the fact that the forward operation of f at r equals the backward operation of f at $r+\epsilon$.
In an analogous way, one can obtain the finite difference to higher order derivatives and differential operators, and this yields for the second order derivative

$$\Delta^+_{2\epsilon}[f] = f(r+2\epsilon)-2f(r+\epsilon)+f(r) \tag{6.1.6}$$

$$\approx f''(r) \ , \tag{6.1.7}$$

$$\Delta^-_{2\epsilon}[f] = f(r) - 2f(r-\epsilon) + f(r-2\epsilon) \tag{6.1.8}$$

$$\approx f''(r) \ , \tag{6.1.9}$$

where the symbol $\approx$ is used to denote taking the limit $\epsilon \to 0$ and dividing by ϵ to switch from discrete to continuous.
The forward difference can be considered as an operator that maps the function f to $\Delta^+_{n\epsilon}[f]$, we will call it the shift operator functioning in steps ϵ, and denote it by $\mathcal{D}_\epsilon$. This operator satisfies a particular Leibniz rule and linearity property respectively

$$\begin{aligned} \mathcal{D}_\epsilon(f(r)g(r)) &= (\mathcal{D}_\epsilon f(r))\, g(r+\epsilon) + f(r)\,(\mathcal{D}_\epsilon g(r)) && \text{Leibniz rule} \\ \mathcal{D}_\epsilon[\alpha f + \beta g](r) &= \alpha \mathcal{D}_\epsilon[f](r) + \beta \mathcal{D}_\epsilon[g](r) \ , && \text{Linearity} \\ \mathcal{D}_\epsilon(f \cdot g)(r) &= f(r+\epsilon)\mathcal{D}_\epsilon g(r) + \mathcal{D}_\epsilon f(r) g(r) \ . && \text{Product rule} \end{aligned} \tag{6.1.10}$$

When it comes to integration, and more relevantly to this work integration by parts, the usual notion of integral is formally translated into a Riemannian sum to reproduce it in the continuum limit. Hence, to derive the analogous of the integration by part, we play with the product rule in (6.1.10). Thus in order to preform this "summation by parts" in the subsequent computations, we make use of the following relations in discrete calculus

$$\mathcal{D}_\epsilon(f \cdot g)(r) - f(r+\epsilon)\mathcal{D}_\epsilon g(r) = g(r)\mathcal{D}_\epsilon f(r) \ . \tag{6.1.11}$$

Summing in both sides of the above equation yields

$$\sum_r g(r)\mathcal{D}_\epsilon f(n) = -\sum_r f(r+\epsilon)\mathcal{D}_\epsilon g(r) \ . \tag{6.1.12}$$

We apply this technique to the case of interest, namely terms in the computations that will appear including terms in first- and second derivatives of the lapse function multiplying functionals of the phase space variables

$$\begin{aligned} \sum_r N''(g'f) &\approx -\sum_r N'\mathcal{D}_\epsilon(g'f) \\ &\approx -\sum_r \Delta^-_\epsilon[N]\mathcal{D}_\epsilon(\Delta^-_\epsilon[g]f) \\ &\approx -\sum_r \Delta^-_\epsilon[N]\left((\Delta^-_\epsilon[g](r+\epsilon)f(r+\epsilon) - \Delta^-_\epsilon[g](r)f(r))\right) \\ &\approx -\sum_r \Delta^-_\epsilon[N]\left(\Delta^+_\epsilon[g]f(r+\epsilon) - \Delta^-_\epsilon[g]f(r)\right) \ , \\ \sum_r N''(r)(fg) &\approx -\sum_r N'\mathcal{D}_\epsilon(fg) \\ &\approx -\sum_r \Delta^-_{n\epsilon}[N]\left(f(r+\epsilon)\mathcal{D}_\epsilon(g) + g\mathcal{D}_\epsilon(f)\right) \ . \end{aligned} \tag{6.1.13}$$

Since the classical Hamiltonian contains terms with first and second continuous derivative, at the effective level they are respectively equivalent to a shift ϵ_r and $2\epsilon_r$. To be more precise, in the case of the Poisson Bracket we are examining this is translated in the following expression

$$
\begin{aligned}
&\left\{\tilde{\mathcal{H}}[N_1(r)],\tilde{\mathcal{H}}[N_2(r')]\right\} \approx \frac{1}{4\kappa\gamma^3}\sum_r\Bigg\{\left(N_1(r)\frac{\delta\tilde{\mathcal{H}}[N_1(r)]}{\delta A_1(\rho)}+N_1(r-\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_1(r-\epsilon_r)]}{\delta A_1(\rho)}\right)\\
&\underbrace{\times\left(N_2(r)\frac{\delta\tilde{\mathcal{H}}[N_2(r)]}{\delta E^1(\rho)}+N_2(r-\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_2(r-\epsilon_r)]}{\delta E^1(\rho)}\right)-(N_1\Longleftrightarrow N_2)}_{PB_1}\\
&+2\left(N_1(r)\frac{\delta\tilde{\mathcal{H}}[N_1(r)]}{\delta A_r(\rho)}+N_1(r-\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_1(r-\epsilon_r)]}{\delta A_r(\rho)}\right)\\
&\underbrace{\times\left(N_2(r)\frac{\delta\tilde{\mathcal{H}}[N_2(r)]}{\delta E^r(\rho)}+N_1(r-\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_2(r-\epsilon_r)]}{\delta E^r(\rho)}+N_1(r-2\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_1(r-2\epsilon_r)]}{\delta E^r(\rho)}\right)-(N_1\Longleftrightarrow N_2)}_{PB_r}\Bigg\}\,.
\end{aligned}
\tag{6.1.14}
$$

We discarded terms that has the same shift in the lapse functions such as $N_1(r)N_2(r)$ since they cancel each other during the computation. The indices (1,r) stand for the commutation equations involving variation with respect (E^1, A_1) or (E^r, A_r). The next step is to simplify this expression and massage the tow terms PB_1 and PB_r with the aim to make the presence of the structure functions explicit and thus recover an analytic relation similar to (6.0.2).

$$
\begin{aligned}
PB_r = 2\Bigg\{&\left(N_1(r)\frac{\delta\tilde{\mathcal{H}}[N_1(r)]}{\delta A_r(\rho)}+N_1(r-\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_1(r-\epsilon_r)]}{\delta A_r(\rho)}\right)\\
&\times\left(N_2(r)\frac{\delta\tilde{\mathcal{H}}[N_2(r)]}{\delta E^r(\rho)}+N_1(r-\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_2(r-\epsilon_r)]}{\delta E^r(\rho)}+N_1(r-2\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_1(r-2\epsilon_r)]}{\delta E^r(\rho)}\right)\\
&-(N_1\Longleftrightarrow N_2)\Bigg\}\\
&=2\Bigg\{N_1(r)N_2(r-\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_1(r)]}{\delta A_r(\rho)}\frac{\delta\tilde{\mathcal{H}}[N_2(r)]}{\delta E^r(\rho)}+N_1(r-\epsilon_r)N_2(r)\frac{\delta\tilde{\mathcal{H}}[N_1(r-\epsilon_r)]}{\delta A_r(\rho)}\frac{\delta\tilde{H}[N_1(r)]}{\delta E^r(\rho)}\\
&+N_1(r)N_2(r-2\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_1(r)]}{\delta A_r(\rho)}\frac{\delta\tilde{H}[N_2(r-2\epsilon_r)]}{\delta E^r(\rho)}\\
&+N_1(r-\epsilon_r)N_1(r-2\epsilon_r)\frac{\delta\tilde{\mathcal{H}}[N_1(r-\epsilon_r)]}{\delta A_r(\rho)}\frac{\delta\tilde{\mathcal{H}}[N_2(r-2\epsilon_r)]}{\delta E^r(\rho)}-(N_1\Longleftrightarrow N_2)\Bigg\}\\
&=2\Bigg\{\frac{N_1(r)N_2(r-\epsilon_r)}{2}\dot{E}^r_0\dot{A}^1_r+\frac{N_1(r)N_2(r-2\epsilon_r)}{2}\dot{E}^r_0\dot{A}^2_r+\frac{N_1(r-\epsilon_r)N_2(r-\epsilon_r)}{2}\dot{E}^r_1(r)\dot{A}^0_r\\
&+\frac{N_1(r-\epsilon_r)N_2(r-2\epsilon_r)}{2}\dot{E}^r_1\dot{A}^2_r\Bigg\}\\
&=\frac{1}{2}\Bigg\{\left(N_1(r-\epsilon_r)N_2(r)-N_1(r)N_2(r-\epsilon_r)\right)\dot{E}^r_0\dot{A}^1_r\\
&+\underbrace{\left(N_1(r)N_2(r-\epsilon_r)-N_1(r-\epsilon_r)N_2(r)\right)}_{:=a_1}\dot{E}^r_1\dot{A}^0_r\\
&+\left(N_1(r-2\epsilon_r)N_2(r)-N_1(r)N_2(r-2\epsilon_r)\right)\dot{E}^r_0\dot{A}^2_r
\end{aligned}
$$

$$+\underbrace{\left(N_1(r-2\epsilon_r)N_2(r-\epsilon_r)-N_1(r-\epsilon_r)N_2(r-2\epsilon_r)\right)}_{:=a_2}\dot{E}_1^r\dot{A}_r^2\Big\}\ .$$

The key idea to reveal the smearing structure functions in terms of lapses and their derivatives, (namely a terms similar to $(N_1N_2'-N_2N_1')$) is to manipulate the terms (a_i) to bring them to a form similar to the forward and backward derivatives by using (6.1.10). Now, since we intend to preform the integration by part for the a second derivative, it follows in two steps to get rid of the second order derivative by making use of (6.1.13). In the same spirit, the discrete derivative in $2\epsilon_r$ would take different terms to translate its analog continuous one. For instance, applying (6.1.6) in the case of the term a_1, one can rewrite it as

$$\begin{aligned}a_1 &= N_1(r-2\epsilon_r)N_2(r)-N_1(r)N_2(r-2\epsilon_r)\\ &= \left[N_1(r-2\epsilon_r)-2N_1(r-\epsilon_r)+N_1(r)\right]N_2(r)-\left[N_2(r-2\epsilon_r)-2N_2(r-\epsilon_r)+N_2(r)\right]N_1(r)+\\ &+2\left[N_1(r-\epsilon_r)N_2(r)-N_1(r)N_2(r-\epsilon_r)\right]\\ &\approx N_1''(r)N_2(r)-N_2''(r)N_1(r)+2\left[N_1(r)N_2'(r)-N_2(r)N_1('r)\right]\ ,\end{aligned}\tag{6.1.15}$$

and for the a_2 term, it reads

$$\begin{aligned}a_2 &= N_1(r-2\epsilon_r)N_2(r-\epsilon_r)-N_1(r-\epsilon_r)N_2(r-2\epsilon_r)\\ &= \left[N_1(r-2\epsilon_r)-2N_1(r-\epsilon_r)+N_1(r)\right]\left[N_2(r-\epsilon_r)-N_2(r)\right]\\ &-\left[N_1(r-\epsilon_r)-N_1(r)\right]\left[N_2(r-2\epsilon_r)-2N_2(r-\epsilon_r)+N_2(r)\right]\\ &-N_1(r-2\epsilon_r)N_2(r)+2N_1(r-\epsilon_r)N_2(r-\epsilon_r)-2N_1(r-2\epsilon_r)N_2(r)\\ &-2N_1(r-\epsilon_r)N_2(r-\epsilon_r)+N_1(r)N_2(r)-N_1(r)N_2(r)-N_1(r)N_2(r-\epsilon_r)-N_1(r)N_2(r-2\epsilon_r)\\ &\approx (N_1(r)N_2'(r)-N_1'(r)N_2(r))+(N_1''(r)N_2(r)-N_1(r)N_2''(r))+(N_1'(r)N_2''(r)-N_1''(r)N_2'(r))\ .\end{aligned}\tag{6.1.16}$$

The PB_1 term can be written as

$$\begin{aligned}PB_1 &= \left(N_1(r)\frac{\delta\tilde{H}[N_1(r)]}{\delta A_1(\rho)}+N_1(r-\epsilon_r)\frac{\delta\tilde{H}[N_1(r-\epsilon_r)]}{\delta A_1(\rho)}\right)\\ &\times\left(N_2(r)\frac{\delta\tilde{H}[N_2(r)]}{\delta E^1(\rho)}+N_2(r-\epsilon_r)\frac{\delta\tilde{H}[N_2(r-\epsilon_r)]}{\delta E^1(\rho)}\right)-(N_1\Longleftrightarrow N_2)\\ &= \left[N_1(r)-N_2(r-\epsilon_r)\right]\dot{E}_1^1\dot{A}_1^1+\left[N_1(r-\epsilon_r)N_2(r)\right]\dot{E}_1^1\dot{A}_1^0\\ &\approx \left[N_1(r)N_2'-N_2N_1'\right]\left(\dot{E}_1^1\dot{A}_1^1-\dot{E}_1^1\dot{A}_1^0\right)\ .\end{aligned}\tag{6.1.17}$$

Hence the bracket including the radial and the -1 contributions yields

$$\begin{aligned}\left\{\tilde{\mathcal{H}}[N_1(r)],\tilde{\mathcal{H}}[N_2(r')]\right\} &\approx \frac{1}{4\kappa\gamma^3}\sum_r\left[N_1(r)N_2'-N_2N_1'\right]\left(\dot{E}_1^1\dot{A}_1^1-\dot{E}_1^1\dot{A}_1^0\right)\\ &+\frac{1}{2}\Big\{\left(N_1(r)N_2'(r)-N_1'(r)N_2(r)\right)\dot{E}_0^r\dot{A}_r^1+2\left(N_1(r)N_2'(r)-N_1'(r)N_2(r)\right)\dot{E}_0^r\dot{A}_r^2\\ &+\left(N_1(r)N_2'(r)-N_1'(r)N_2(r)\right)\dot{E}_1^r\dot{A}_r^0+\left(N_1(r)N_2'(r)-N_1'(r)N_2(r)\right)\dot{E}_0^r\dot{A}_r^2\\ &+\left(N_1(r)N_2''(r)-N_1''(r)N_2(r)\right)\dot{E}_0^r\dot{A}_r^2+\left(N_1(r)N_2''(r)-N_1''(r)N_2(r)\right)\dot{E}_1^r\dot{A}_r^2\\ &+\left(N_1'(r)N_2''(r)-N_1''(r)N_2'(r)\right)\dot{E}_1^r\dot{A}_r^2\Big\}\end{aligned}\tag{6.1.18}$$

$$
\begin{aligned}
&\approx \frac{1}{4\kappa\gamma^3}\sum_r \Bigg\{ (N_1(r)N_2'(r) - N_1'(r)N_2(r)) \\
&\times \Bigg(\underbrace{\dot{E}_0^1\dot{A}_1^1 - \dot{E}_1^1\dot{A}_1^0 + \frac{E_0^r}{2}\left[2\dot{A}_r^2 + \dot{A}_r^1\right] + \frac{E_1^r}{2}\left[2\dot{A}_r^2 - \dot{A}_r^0\right]}_{b_1} \Bigg) \\
&+ (N_1(r)N_2''(r) - N_1''(r)N_2(r)) \Bigg(\underbrace{\frac{\dot{A}_r^2}{2}\left[-\dot{E}_0^r - E_1^r\right]}_{b_2} \Bigg) \\
&+ (N_1(r)'N_2''(r) - N_1''(r)N_2'(r)) \Bigg(\underbrace{\frac{\dot{A}_r^2}{2}E_1^r}_{b_3} \Bigg) \Bigg\}
\end{aligned} \tag{6.1.19}
$$

The next step is to implement the summation by parts to the above expression to get rid of the second derivative in the lapse with the goal to obtain a factorized structure constant factorizing the "supposed-to-be" effective diffeomorphism constraint. This is achieved by employing the calculus techniques derived in (6.1.13) to the terms multiplying b_2 and b_3. The Poisson bracket (6.1.18) reads

$$
\begin{aligned}
\left\{\tilde{\mathcal{H}}[N_1(r)], \tilde{\mathcal{H}}[N_2(r')]\right\} = \frac{1}{4\kappa\gamma^3}\sum_r \Bigg\{ &\Bigg(\Delta^-_{\epsilon_r}[N_1(r)]\Delta^+_{\epsilon_r}[N_2(r)] - \Delta^+_{\epsilon_r}[N_1(r)]\Delta^-_{\epsilon_r}[N_1(r)] \Bigg) \\
&\times \Bigg(b_2(r+\epsilon_r) - b_3(r+\epsilon_r) \Bigg) \\
&+ \Bigg(N_1(r)\Delta^-_{\epsilon_r}[N_2(r)] - \Delta^-_{\epsilon_r}[N_1(r)]N_2(r) \Bigg) \Big(\Delta^+_{\epsilon_r} b_3(r) + b_1 \Big) \Bigg\} \, .
\end{aligned} \tag{6.1.20}
$$

Notice that the above Hamiltonian bracket reproduce two smeared terms with different structure functions and thus admittedly deviate from the classical Hamiltonian bracket in (4.3.1). In order to distinguish the candidate expression for the effective diffeomorphism constraint, one can expand the terms $\Delta^+_{\epsilon_r} b_3(r) + b_1$ and $b_2(r+\epsilon_r) - b_3(r+\epsilon_r)$ separately and check if the leading order of the expansion actually reproduces the classical familiar expression of the diffeomorphism constraint. An other selection criterion is: once one elects which term delivers the right classical expression, one can investigate if it contains higher order terms in ϵ_r since it subsidize to the power-counting of the Barbero-Immirzi parameter γ. Therefore this higher order contribution should not ideally reproduce the classical diffeomorphism constraint, since it will generate the right proportionality to γ for the constraint algebra to close. In view of these consideration, the residual term can be reabsorbed. In light of these considerations, the terms $b_2(r+\epsilon_r) - b_3(r+\epsilon_r)$ stands out as an initiatory proposal for the effective diffeomorphism constraint and it reads

$$
\begin{aligned}
\tilde{D} = -\frac{1}{4\kappa\gamma^3}\sum_r \frac{\pi(E^1)^2(r)}{\epsilon E^r(r)} \Bigg\{ &- \frac{16\left(\gamma^2+1\right)}{E^1\left(r-\epsilon_r\right)} E^r\left(r-\epsilon_r\right) \Bigg(\sin\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r\right)^2 + A_2\left(r\right)^2}\right] \\
&\times -4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r+\epsilon_r\right)^2 + A_2\left(r+\epsilon_r\right)^2}\right] \cos\left[\tan^{-1}\left(\frac{A_2\left(r\right)}{A_1\left(r\right)}\right) + \frac{1}{2}\left(A_r(r-\epsilon_r) - A_r\left(r\right)\right)\epsilon_r\right] \\
&- \pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2 + A_2(r)^2} - \sqrt{A_1\left(r+\epsilon_r\right)^2 + A_2\left(r+\epsilon_r\right)^2}\right)\right] \\
&\times \Bigg\{ \cos\left[\tan^{-1}\left(\frac{A_2\left(r\right)}{A_1\left(r\right)}\right) + \frac{1}{2}\left(A_r(r-\epsilon_r) - A_r\left(r\right)\right)\epsilon_r\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\cos\left[\tan^{-1}\left(\frac{A_2\left(r+\epsilon_r\right)}{A_1\left(r+\epsilon_r\right)}\right)+\frac{1}{2}\left(A_r(r-\epsilon_r)-A_r\left(r\right)\right)\epsilon_r\right]\Bigg\}\\
&+\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}+\sqrt{A_1\left(r+\epsilon_r\right)^2+A_2\left(r+\epsilon_r\right)^2}\right)\right]\\
&\times\Bigg\{-\cos\left[\tan^{-1}\left(\frac{A_2\left(r\right)}{A_1\left(r\right)}\right)+\frac{1}{2}\left(A_r(r-\epsilon_r)-A_r\left(r\right)\right)\epsilon_r\right]\\
&+\cos\left[\tan^{-1}\left(\frac{A_2\left(r+\epsilon_r\right)}{A_1\left(r+\epsilon_r\right)}\right)+\frac{1}{2}\left(A_r(r-\epsilon_r)-A_r\left(r\right)\right)\epsilon_r\right]\Bigg\}\Bigg)\\
&+\frac{8\left(\gamma^2+1\right)E^r(r-\epsilon_r)}{\epsilon E^1\left(r-\epsilon_r\right)}\Bigg\{\sin\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r+\epsilon_r\right)^2+A_2\left(r+\epsilon_r\right)^2}\right]\\
&\times\Bigg(4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r+2\epsilon_r\right)^2+A_2\left(r+2\epsilon_r\right)^2}\right]\\
&\times\cos\left[\tan^{-1}\left(\frac{A_2\left(r+\epsilon_r\right)}{A_1\left(r+\epsilon_r\right)}\right)+\frac{1}{2}\left(A_r(r+2\epsilon_r)-A_r\left(r+\epsilon_r\right)\right)\epsilon_r\right]\\
&+4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r+\epsilon_r\right)^2+A_2\left(r+\epsilon_r\right)^2}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1\left(r+2\epsilon_r\right)^2+A_2\left(r+2\epsilon_r\right)^2}\right]\\
&\times\cos\left[\tan^{-1}\left(\frac{A_2\left(r+2\epsilon_r\right)}{A_1\left(r+2\epsilon_r\right)}\right)+\frac{1}{2}\left(A_r\left(r+\epsilon_r\right)+A_r\left(r+2\epsilon_r\right)\right)\epsilon_r\right]\Bigg)\\
&+\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1\left(r+\epsilon_r\right)^2+A_2\left(r+\epsilon_r\right)^2}-\sqrt{A_1\left(r+2\epsilon_r\right)^2+A_2\left(r+2\epsilon_r\right)^2}\right)\right]\\
&\times\Bigg\{\cos\left[\tan^{-1}\left(\frac{A_2\left(r+\epsilon_r\right)}{A_1\left(r+\epsilon_r\right)}\right)+\frac{1}{2}\left(A_r\left(r+2\epsilon_r\right)-A_r\left(r+\epsilon_r\right)\right)\epsilon_r\right]\\
&+\cos\left[\tan^{-1}\left(\frac{A_2\left(r+2\epsilon_r\right)}{A_1\left(r+2\epsilon_r\right)}\right)+\frac{1}{2}\left(A_r\left(r+\epsilon_r\right)+A_r\left(r+2\epsilon_r\right)\right)\epsilon_r\right]\Bigg\}\\
&+\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1\left(r+\epsilon_r\right)^2+A_2\left(r+\epsilon_r\right)^2}+\sqrt{A_1\left(r+2\epsilon_r\right)^2+A_2\left(r+2\epsilon_r\right)^2}\right)\right]\\
&\times\Bigg\{\cos\left[\tan^{-1}\left(\frac{A_2\left(r+\epsilon_r\right)}{A_1\left(r+\epsilon_r\right)}\right)+\frac{1}{2}\left(A_r\left(r+2\epsilon_r\right)-A_r\left(r+\epsilon_r\right)\right)\epsilon_r\right]\\
&-\cos\left[\tan^{-1}\left(\frac{A_2\left(r+2\epsilon_r\right)}{A_1\left(r+2\epsilon_r\right)}\right)+\frac{1}{2}\left(A_r\left(r+\epsilon_r\right)+A_r\left(r+2\epsilon_r\right)\right)\epsilon_r\right]\Bigg\}\Bigg\}\Bigg\}\,.
\end{aligned}
\tag{6.1.21}
$$

Expanding in the quantum parameters ϵ_r and ϵ and taking into account only the leading order, one recovers the classical expression of the diffeomorphism constraint.

6.2 Effective radial evolution equations

The proposal for the effective diffeomorphism constraint should undergo further consistency tests, to ensure it truly represents the right effective constraint which establishes the closure of algebra and classical predictions. To this aim, one can study whether the vanishing of the total evolution equations (including the Hamiltonian- and diffeomorphism brackets) of the phase space variables is achieved. This is similar to the classical case where the radial equations of motion cancels out the Hamiltonian ones derived in (B). A preliminary consistency check is to review if the extracted evolution equations from (6.1.21) actually reproduces its respective classical expressions. As a last test, the proposal for the diffeomorphism constraint should ensure the closure of the algebra and hence it ought to satisfy the commutation relations derived in (4.1.3). The first step is then to derive the contribution effective equation of motion coming form the above constraint that will add up to the ones obtained from the effective Hamiltonian. This will eventually provide a solvable set of non-local differential equations for the fluxes and Ashtekar-Barbero connection components.

6.2.1 Fluxes equation of motion

The effective evolution equations are computed in similar manner as in the previous ones, namely by considering the Poisson bracket of the phase space variables and the smeared diffeomorphism constraint, which in this case the candidate expression we derived above. Note that we are dealing as usual with discrete computations, hence a sum over the plaquettes replacing the integration that appears again as in the previous computations.

$$
\begin{aligned}
\dot{E}^1_D = {} & \frac{1}{4\gamma^2}\sum_r -\frac{8\pi^2\left(\gamma^2+1\right)E^1(r)^2N^r(r)E^r(r-\epsilon_r)}{\epsilon\sqrt{E^r(r)}\left(A_1(r)^2+A_2(r)^2\right)E^1(r-\epsilon_r)}\Bigg\{ \\
& \times \epsilon A_1(r)\sqrt{A_1(r)^2+A_2(r)^2}\Bigg(4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right] \\
& \times\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right] \\
& +4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right] \\
& \times\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right] \\
& +\pi H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right] \\
& \times\left(\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\right. \\
& \left.+\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\right) \\
& +\pi H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right]\right) \\
& \times\left(\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\right. \\
& \left.\left.-\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\right)\right) \\
& +2A_2(r)\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right] \\
& \times\Bigg(4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\pi\Bigg(H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\\
&+H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right]\right]\Bigg)\Bigg)\Bigg\}\,. \qquad (6.2.1)
\end{aligned}
$$

As for the flux E^1, it yields

$$
\begin{aligned}
\dot{E}^r_D &= \frac{1}{2\gamma^2}\sum_r\Bigg\{-\frac{8\pi^2\left(\gamma^2+1\right)N^r(r)\epsilon_r E^1(r)^2 E^r(r-\epsilon_r)}{\epsilon\sqrt{E^r(r)}E^1(r-\epsilon_r)}\Bigg\{\\
&\times 4\Bigg(\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right]\\
&\times\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\\
&+\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right]\\
&\times\sin\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\Bigg)\\
&+\pi H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\\
&\times\Bigg(\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\\
&-\sin\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\Bigg)\\
&+\pi\left[\frac{1}{2}\epsilon\left[\sqrt{A_1(r)^2+A_2(r)^2}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right]\right]\\
&\times\Bigg(\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\\
&+\sin\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\Bigg)\Bigg\}\Bigg\}\,. \qquad (6.2.2)
\end{aligned}
$$

6.2.2 Connection components equations of motion

The The evolution equation of the connection A_1 obtained from the Poisson bracket with the effective diffeomorphism constraint reads

$$
\begin{aligned}
\dot{A}^D_1 &= -\frac{1}{4\gamma^2}\sum_r\Bigg\{-\frac{8\pi^2\left(\gamma^2+1\right)N^r(r)E^1(r)^2E^r(r-\epsilon_r)}{\epsilon\sqrt{E^r(r)}\left(A_1(r)^2+A_2(r)^2\right)E^1(r-\epsilon_r)}\Bigg\{\\
&\times\epsilon A_1(r)\sqrt{A_1(r)^2+A_2(r)^2}\Bigg\{4\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\cos\left(\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right)\\
&\times\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\\
&+4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\\
&+\pi H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\\
&\times\left(\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\right.\\
&\left.+\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\right)\\
&+\pi H_{-1}\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\\
&\times\left(\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\right.\\
&\left.\left.-\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\right)\right\}\\
&+2A_2(r)\sin\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\\
&\left(4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right]\right.\\
&+\pi\left(H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\right.\\
&\left.\left.\left.+H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\right)\right)\right\}
\end{aligned}
\tag{6.2.3}
$$

For the radial component of the connection, the evolution equation yields

$$
\begin{aligned}
\dot{A}_r^D &= -\frac{1}{2\gamma^2}\sum_r\left\{\frac{16\pi^2\left(\gamma^2+1\right)N^r(r)E^1(r)^4E^r(r-\epsilon_r)}{\epsilon E^r(r)^3E^1(r-\epsilon_r)}\right\{
\end{aligned}
\tag{6.2.4}
$$

$$
\begin{aligned}
&-4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right]\\
&\times\left(2\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\right.\\
&\left.+\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(2\epsilon_r+r)^2+A_2(2\epsilon_r+r)^2}\right]\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(2\epsilon_r+r)-A_r(\epsilon_r+r))\right]\right)\\
&+\cos\left[\frac{1}{2}\epsilon\sqrt{A_1(\epsilon_r+r)^2+A_2(\epsilon_r+r)^2}\right]\left(8\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(r)^2+A_2(r)^2}\right]\right.\\
&\times\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\\
&-4\sin\left[\frac{1}{2}\epsilon\sqrt{A_1(2\epsilon_r+r)^2+A_2(2\epsilon_r+r)^2}\right]\\
&\left.\times\cos\left[\tan^{-1}\left(\frac{A_2(2\epsilon_r+r)}{A_1(2\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(2\epsilon_r+r))\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\pi\Bigg\{H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}-\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\\
&\times\Bigg(\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\\
&+\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\Bigg)\Bigg\}\\
&+2H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r)^2+A_2(r)^2}+\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}\right)\right]\\
&\times\Bigg(\cos\left[\tan^{-1}\left(\frac{A_2(r)}{A_1(r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)-A_r(r))\right]\\
&-\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(r))\right]\Bigg)\\
&+H_0\left(\frac{1}{2}\epsilon\left(\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}-\sqrt{A_1(r+2\epsilon_r)^2+A_2(r+2\epsilon_r)^2}\right)\right)\\
&\Bigg(\cos\left[\tan^{-1}\left(\frac{A_2(2\epsilon_r+r)}{A_1(2\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(2\epsilon_r+r))\right]\\
&-\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(2\epsilon_r+r)-A_r(\epsilon_r+r))\right]\Bigg)\\
&-H_0\left[\frac{1}{2}\epsilon\left(\sqrt{A_1(r+\epsilon_r)^2+A_2(r+\epsilon_r)^2}+\sqrt{A_1(r+2\epsilon_r)^2+A_2(r+2\epsilon_r)^2}\right)\right]\\
&\times\Bigg(\cos\left[\tan^{-1}\left(\frac{A_2(\epsilon_r+r)}{A_1(\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(2\epsilon_r+r)-A_r(\epsilon_r+r))\right]\\
&+\cos\left[\tan^{-1}\left(\frac{A_2(2\epsilon_r+r)}{A_1(2\epsilon_r+r)}\right)+\frac{1}{2}\epsilon_r(A_r(\epsilon_r+r)+A_r(2\epsilon_r+r))\right]\Bigg)\Bigg\}\Bigg\}\ .
\end{aligned}
$$

The total effective evolution equations of the connection and flux are provided once we add up the Hamiltonian and diffeomorphism contributions, formally it is given by

$$ {}^{\text{eff}}\dot{A}_1 = \dot{A}_1^H + \dot{A}_1^D\ , \tag{6.2.5} $$
$$ {}^{\text{eff}}\dot{A}_r = \dot{A}_r^H + \dot{A}_r^D\ , \tag{6.2.6} $$
$$ {}^{\text{eff}}\dot{E}^1 = \dot{E}^{1H} + \dot{E}_D^1\ , \tag{6.2.7} $$
$$ {}^{\text{eff}}\dot{E}^r = \dot{E}^{rH} + \dot{E}_D^r\ , \tag{6.2.8} $$

where the left superscript index refers to the effective equations.
These are the effective equations of motion based on our candidate expression for the effective diffeomorphism constraint. The first consistency check regarding the classical radial equations of motion established the classical equations, namely (4.2.12,4.2.16) and(4.2.21,4.2.24).
In order to support this proposal and ensure self-consistency of this work, one must prove that the obtained equations of motion do vanish. Ideally, one could just add the two contribution obtained from the Hamiltonian and diffeomorphism and see if they actually give zero. However, due to their non-trivial and formally complicated aspect, the set of equations (6.2.5) should inherit the same behavior as the classical one. As it is indicated in (B), the use of constraints and the interplay between the phase space variables was sort of necessary to prove the vanishing

of the equations of motion. At the effective level one should expect the same to happen, i.e the use of the only effective constraint at hand (Hamiltonian constraint) and the emergence of some effective on-shell closure relations to appear. This is still work in progress.
Moreover one has to show that the rest of the algebra employing the proposed diffeomorphism constraint closes. Adding the candidate diffeomorphism expression in (6.1.21) after surviving all the consistency checks, we end up with a set of five non-local differential equations to be solved for. Their solutions will ensure the explicit description of the metric functions Λ and R, which will generate the beginning of physical prediction and reliability of this framework.

Discussion and outlook

In this work, we studied the formulation of black holes in a quantum gravity model. We started by describing the classical theory of the canonical treatment of general relativity. The latter is based on decomposing spacetime à la 3+1 decomposition, mainly splitting space to globally hyperbolic hypersurface endowed with familiar notion of time evolution that is encoded in the directional Lie dragging operation. This was the mathematical setup to build the Hamiltonian formulation for the Einstein-Hilbert action, where in this reformulation Einstein equations were recast in a set of constraints that accounts for the gauge transformations of the theory. This alternative presentation of GR can also be written in a triad formulation that, along with the introduction of auxiliary parameters for the ADM phase space variables, prepared the framework to go through the process of quantization. We moved on to introduce loop quantum gravity program using the point holonomy entity and the generated flux-holonomy algebra and sketched the construction of LQG Hilbert space.
To specify the work to black holes, we have provided the canonical coordinates of GR phase space parametrized by the Ashtekar-Barbero $SU(2)$ connection and its conjugate momentum formulation some partial gauge fixing conditions from which second class constraints appeared. The latter was taken into consideration in the framework of gauge unfixing. This procedure, even though is still equivalent to working with the inverse matrix of Dirac, provided an easier method based on the standard Poisson bracket between the remaining (reduced) phase space coordinates. This has the advantage of discarding the difficulties accommodated on the quantum Dirac bracket at the price of introducing some nonlocal extra terms in the remaining first class constraints.
This formalism was the basis to present the foundations for a systematic investigation of spherically symmetric geometries in the QRLG framework. More concretely, this is put into craft by the implementation of a quantization program that identifies a symmetric sector at the quantum level, emphasizing the reverting process of symmetry reduction and quantization, the mainstreamed approach to quantum black holes in the existing literature. The main results in [21] set up the basic playground for the consideration of horizon penetrating foliation to eventually be able to provide predictions and a consistent description of the interior and exterior regions of a quantum black hole in QG, delivering the result of an effective Hamiltonian for the subsequent work.
The first step in this program with the aim to glue both geometries (dedicated for future work) is to make sure that the theory is anomaly free, namely, but studying the constraint algebra classically and in second step, to administer a candidate expression for the effective counterpart of the diffeomorphism constraint that will enable the exposition of the gluing procedure. This was mainly done, with the goal to achieve the off-shell closure the constraint algebra.

This investigation concerning the effective dynamics for the Kerr-Schild foliation is an initial step to attain the goal of gluing the interior and exterior of a "quantum gravity" black hole,

which has not been done so far in quantum gravity approaches. The main results of this work concerns the proposal for the effective diffeomorphism constraint. Even though the extracted equations of option matches the classical ones (obtained from the commutator with the classical constraint), the proposed expression (6.1.21) still have to surpass the challenges of the remaining consistency checks. Indeed, showing that the total effective equations of motion in (6.2.5) vanish for the promoted effective diffeomorphism constraint and that it also closes the algebra are the "acid" test it has to survive.
Furthermore, in the scenario where the quantum parameters are represented as phase space functions, a proposal for an effective diffeomorphism constraint is still work in progress and hints towards a different approach to extract a candidate expression for the effective diffeomorphism constraint. Ideally one can proceed as in the constant case by using discrete calculus techniques but the functional-nature of the parameters suggests that one should consider discrete functional derivatives as well. At the point where this is achieved, the same consistency algorithm would be applied to the extracted would-be effective diffeomorphism constraint. As in the constant quantum parameters, one will end up with five equations to be solved for.
The remaining consistency checks and the study of the case of phase-space quantum parameters is still work in progress.

One should also stress that the derived effective evolution equations for the fluxes and Ashtekar-Barbero connection are nonlocal differential equations encoding the non-homogeneity of the metric and its discrete nature. Solving these equations is therefore highly non-trivial. There are very few methods to solve non-local differential equations exactly and the common approach is the numerical one that appears in the context of fluid mechanics. Therefore this framework provides the motivation to study nonlocal equations in the context of a quantum gravity model, which is (more generally) a powerful tool to investigate the discreetness of geometry.
The ultimate fate of the obtained nonlocal differential evolution equations for the fluxes and Ashtekar-Barbero connection is to provide a numerical solutions encoding quantum gravity corrections inherited in an effective metric. Once this is achieved, one can start investigating if the derived results are able to answer the longstanding unanswered questions concerning black holes and perhaps even the opportunity to link them to the observational sector.
Once the evolution equations are solved, one can study the dynamics the current setup predicts, namely investigate if one obtains the classical limit and what is the impact of the quantum fluctuations on the physical predictions. It is indeed a very prosperous field to search for answers concerning all the puzzles black holes still face.

Appendix A

Differential geometry and topology

A.1 Lie derivative

Let M, N be manifolds and ϕ_t a one-parameter group of diffeomorphisms that is generated by a vector field v^a. We define the map that "carries along" tangent vectors at $p \in M$ to tangent vectors at $\phi(p) \in N$, such that

$$(\phi^* v)(f) = v(f \circ \phi) \ . \tag{A.1.1}$$

ϕ^* can be viewed as the derivative of ϕ at p. The Lie derivative $\mathcal{L}$ is defined as the operation that compares $T^{a_1 \cdots a_k}{}_{b_1 \ldots b_l}$ and $\phi^*_{-t} T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$, for some tensor $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$. Formally, it is denoted by $\mathcal{L}_v$ and it can be expressed as

$$\mathcal{L}_v T^{a_1} \cdots a_{k_b} \ldots b_l = \lim_{\leftrightarrow \to 0} \left\{ \frac{\phi^*_{-t} T^{a_1 \cdots a_{k_b}} \ldots b_l - T^{a_1 \cdots a_k}_{b_1 \cdots b_l}}{t} \right\} \ . \tag{A.1.2}$$

More explicitly, the Lie derivative $\mathcal{L}_v$ is a linear map from smooth tensor fields of type $(k{,}l)$ to smooth tensor fields of the same type. To study the action of its operation on an arbitrary tensor field, is it useful to work in a set of coordinates on M, so that $v^a = \left(\frac{\partial}{\partial x^1}\right)^a$. Hence the action the one-parameter group of diffeomorphism ϕ_{-t} is $x \to x + t$, while the other coordinates are held fixed. Consequently, the components of the Lie derivative of $T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l}$ reads

$$\mathcal{L}_v T^{\mu_1 \cdots \mu_k}{}_{v_1 \ldots v_l} = \frac{\partial T^{\mu_1 \cdots \mu_k}{}_{v_1 \ldots v_l}}{\partial x^1} \ . \tag{A.1.3}$$

A coordinate independent formulation for the Lie derivative of a vector field w^a can be derived as follows. In an adapted coordinate system we have

$$\mathcal{L}_v w^\mu = \frac{\partial w^\mu}{\partial x^1} \ , \tag{A.1.4}$$

$$w^a = \sum_\mu w^\mu \left(\frac{\partial}{\partial x^\mu}\right)^a \ . \tag{A.1.5}$$

Using (A.1.3), the commutator of the two vector fields v^a,w^a yields the equation

$$\begin{aligned} {}^\mu &= \sum_\nu \left(v^\nu \frac{\partial w^\mu}{\partial x^\nu} - w^\nu \frac{\partial v^\mu}{\partial x^\nu} \right) \\ &= \frac{\partial w^\mu}{\partial x^1} \ , \end{aligned} \tag{A.1.6}$$

which corresponds to the components of $\mathcal{L}_v w^\mu$. Since they are defined in a coordinate independent manner, the following equality holds

$$\mathcal{L}_v w^a = [v,w]^a \ . \tag{A.1.7}$$

The action of the Lie derivative on a dual vector reads

$$\mathcal{L}_v \left(\mu_a w^a\right) = w^a \mathcal{L}_v \mu_a + \mu_a [v,w]^a \ . \tag{A.1.8}$$

For an arbitrary derivative operator ∇_a, one can express the Lie derivative as

$$\mathcal{L}_v \mu_a = v^b \nabla_b \mu_a + \mu_b \nabla_a v^b \ . \tag{A.1.9}$$

More generally, we have the relation that holds for any derivative operator

$$\begin{aligned}\mathcal{L}_v T^{a_1 \cdots a_k}{}_{b_1 \cdot b_k} &= v^c \nabla_c T^{a_1 \cdots a_k}{}_{b_1 \cdot b_l} \\ &+ \sum_{i=1}^{k} T^{a_1 \cdots a_k}{}_{b_1 \cdot b_k} \nabla_c v^{ai} + \sum_{j=1}^{l} T^{a_1 \cdots a_k}{}_{b_1 \cdot b_l} \nabla_j v^c \ .\end{aligned} \tag{A.1.10}$$

A.2 Holonomy

Some important properties of the holonomy are listed in this section. Given a unique solution of the differential equation

$$\frac{d}{ds} h_e[A,s] + \dot{x}^\mu(s) A_\mu h_e[A,s] = 0 \ , \tag{A.2.1}$$

where the one dimensional path e reads

$$e : [0,1] \subset \mathbb{R} \to \Sigma \ , \tag{A.2.2}$$

sending the variable $s \in [0,1] \to x^\mu(s)$. The unique solution $h_e[A,s]$ exists for the initial value $h_e[A,0] = \mathbb{1}$ and the holonomy along the path e is defined as

$$h_e[A] = h_e[A,1] \ , \tag{A.2.3}$$

satisfying the properties:

- $h_e[A] = h_e[A,1]$ is independent of the parametrization of e.
- It satisfies:

  $$h_e[A] = h_{e_1}[A] h_{e_2}[A] \ , \tag{A.2.4}$$

  where the multiplication on the right is the $SU(2)$ multiplication. the above property holds for the holonomy of a path given by a single point is the identity and given two oriented paths e_1 and e_2 such that the end of one of them coincide with the other i.e. $e = e_1 e_2$.
-

  $$h_{e^{-1}}[A] = h_e^{-1}[A] \ . \tag{A.2.5}$$

- Gauge transformations generated by the Gauss constraint act on the holonomy in the following way

$$h'_e[A] = g(x(0))h_e[A]g^{-1}(x(1)) \ . \tag{A.2.6}$$

Gauge transformations generated by the diffeomorphism constraint acts on the holonomy in the following way

$$h_e\left[\phi^* A\right] = h_{\phi^{-1}(e)}[A] \ . \tag{A.2.7}$$

where ϕ acts on the connection A. Transforming the connection with a diffeomorphism can be pictured as moving the path with ϕ^{-1}.

The holonomy is a device that dictates the rule of parallel transporting spinors along a path e.

Appendix B

On-shell vanishing evolution equations

B.1 The equations of motion for the connection

A_1 connection component

The evolution equation of the connection A_1 coming from the Euclidean contribution reads

$$\dot{A}_1^E = \frac{N}{\sqrt{E^r}}\left[\left(A_1^2 + A_2^2 - 1\right)\right] \tag{B.1.1}$$
$$= \frac{N}{E^1\sqrt{E^r}}\left\{\underbrace{\frac{\mathcal{H}^E}{E^1} - 4E^r\left(A_r A_1 + A_2'(r')\right)}_{=\frac{2\mathcal{H}^E}{E^1}-\frac{\mathcal{H}^E}{E^1}-4E^r\left(A_rA_1+A_2'\right)}\right\} .$$

Note that one can write A_rA_1 as follows

$$A_rA_1 = \frac{1}{2E^1A_2}\left(2A_1A_1'E^1 - \gamma A_1\mathcal{H}^r\right) , \tag{B.1.2}$$
$$= \frac{1}{2E^1A_2}\left(E^1\partial_r(A_1^2) - \gamma A_1\mathcal{H}^r\right) .$$

Plugging it back into the equation for $\dot{A}_1^E$

$$\dot{A}_1^E = \frac{N}{E^1\sqrt{E^r}}\left\{\frac{\mathcal{H}^E}{E^1} + 2\frac{E^r}{A_2E^1}\left(\gamma A_1\mathcal{H}^r - E^1\partial_r(A_1^2)\right)\right\} \tag{B.1.3}$$
$$- \frac{2NA_2'E^r}{E^1\sqrt{E^r}}$$

For the Lorentzian contribution, we will be working on (4.2.8) and using the constraint before arriving at the final result in Eq(4.2.10). A preliminary computation is needed, namely

$$4(E^rE^{r\prime}E^1)\partial_r\left(\frac{N(r)}{4(E^1)^3\sqrt{E^r}}\right) = \frac{(E^rE^{r\prime}E^1)}{(E^1)^3\sqrt{E^r}}\left(\frac{4N'(E^1)^3\sqrt{E^r}}{4(E^1)^3\sqrt{E^r}} - N\frac{12(E^1)^2\sqrt{E^r}E^{1\prime} + \frac{2(E^1)^3E^{r\prime}}{\sqrt{E^r}}}{4(E^1)^3\sqrt{E^r}}\right)$$

$$= \frac{(E^r E^{r\prime} E^1)}{(E^1)^3\sqrt{E^r}} \left[N' - \frac{12(E^1)^2 E^r (E^1)' + 2(E^1)^3 (E^r)'}{4(E^1)^3} \right]$$

$$= \frac{N}{4(E^1)^3\sqrt{E^r}} \left[\frac{N'}{N} 12 E^r (E^r)'(E^1)' + 2(E^{r\prime})^2 E^1 \right] . \tag{B.1.4}$$

One can write $\dot{A}_1^L$ as

$$\dot{A}_1^L = \frac{N(r)}{2(E^1)^3\sqrt{E^r}} \Bigg[-\frac{3\mathcal{H}^L}{E^1} - 2E^1 \left((E^{r\prime})^2 + 4E^r E^{r\prime\prime} \right) + 4E^{r'} E^r E^{1'} - 4(E^{r\prime})^2 E^1 - 4E^1 E^r E^{r\prime\prime}$$

$$- 4E^{r'} E^r E^{1'} + 12 E^{r'} E^r E^{1'} + 2(E^{r\prime})^2 E^1 - 4\frac{N'}{N} E^{r'} E^r E^1 \Bigg]$$

$$= \frac{N(r)}{2(E^1)^3\sqrt{E^r}} \left[\underbrace{\frac{4\mathcal{H}^L}{E^1} - \frac{3\mathcal{H}^L}{E^1}}_{\frac{\mathcal{H}^L}{E^1}} + \underbrace{4E^r E^{r\prime\prime} E^1 + 4E^r E^{r\prime} E^1 - 4\frac{N'}{N} E^{r'} E^r E^1}_{:=a} \right] \tag{B.1.5}$$

Notice that on-shell one can make use of the following relations:

$$A_2 = N \quad G_3 = 0 \tag{B.1.6}$$

$$E^{r\prime} = \gamma G_3 + 2A_2 E^1 \Rightarrow E^{r\prime\prime} = \gamma G_3' + 2\left(E^{1\prime} A_2 + A_2' E^1 \right) \tag{B.1.7}$$

The term denoted by a becomes

$$a = 8\left(E^r (E^1)^2 A_2' + E^r E^1 E^{1\prime} \right) - 4\left(A_2' E^r E^1 \frac{E^1 A_2}{A_2} + A_2 E^{1\prime} E^1 E^r \right) - 2\gamma G_3 + 4\gamma \partial_r G_3 \tag{B.1.8}$$

$$= 2E^r E^{r\prime\prime} E^1 - 2\gamma G_3 + 4\gamma \partial_r G_3 \ .$$

In terms of the constraints, the equation of motion for A_1 coming from the Hamiltonian constraint read

$$\dot{A}_1^H = -\frac{1}{\gamma^2} \Bigg\{ \frac{H}{E_1} + \frac{N}{4(E^1)^3\sqrt{E^r}} \left((1+\gamma^2) \left[\underbrace{2E^r E^{r\prime\prime} E^1}_{c_2} - 2\gamma G_3 + 4\gamma \partial_r G_3 \right] \right) \tag{B.1.9}$$

$$+ \frac{N}{4(E^1)\sqrt{E^r}} \left\{ 2\frac{E^r}{A_2 E^1} \left(\gamma A_1 \mathcal{H}^r - E^1 \partial_r (A_1^2) \right) \right\} + \frac{2N A_2' E^r}{E^1 \sqrt{E^r}} \Bigg\} \ .$$

The c_2 term is equivalent to the previous term c_1, which can be written as

$$c_1 = c_2 = N(E^{r\prime})^2 - 4N \left(\frac{E^{r\prime}}{2A_2} \right)^2 + 4\frac{N'}{N} E^{r\prime} E^r$$

$$= N(E^{r\prime})^2 - A_2 (E^{r\prime})^2 + 8\frac{N'}{N} E^1 A_2 E^r$$

$$= 8\frac{N'}{N} E^1 A_2 E^r \ . \tag{B.1.10}$$

If we equate all the constraints to zero, replacing the shift function with A_2 and we use the contributing term computed above we are left with

$$\begin{aligned}\dot{A}_1^H &= -\frac{1}{\gamma^2}\left(-4\frac{\sqrt{E^r}A_1'A_1}{E^1} - 2\frac{\sqrt{E^r}A_2A_2'}{E^1} + (1+\gamma^2)\frac{2A_2'A_2\sqrt{E^r}}{E^1}\right) \\ &= -\gamma^2\frac{A_2'A_2\sqrt{E^r}}{E^1} + \frac{2}{\gamma^2}\frac{\sqrt{E^r}A_1'A_1}{E^1} \ .\end{aligned} \tag{B.1.11}$$

Some useful equation that will be used in the following computations read:

$$\begin{aligned} N^r &= -\frac{NA_1}{\gamma R'} = -\frac{A_2A_1}{\gamma R'} = -\frac{2\sqrt{E^r}A_2A_1}{\gamma(E^r)'} \ , \\ \Lambda &= \frac{E^1}{\sqrt{E^r}} \ , \\ A_1 &= -\frac{\gamma N^r R'}{N} = -\gamma N^r\Lambda \ , \\ A_r &= -\gamma\frac{\partial_r(\Lambda N^r)}{A_2} = -\gamma\frac{\partial_r(\frac{N^rE^1}{\sqrt{E^r}})}{A_2} = \frac{A'}{A_2} \ . \end{aligned} \tag{B.1.12}$$

where in the expression for A_r, we used the relation $\Lambda = \frac{E^1}{\sqrt{E^r}}$ and we trade $E^{r\prime}$ every time it seems convenient to, in order to simplify terms including A_2's and E^1's. To prove closure of the evolution equation at this stage, one should show that all the terms that are written in terms of constraints cancel the term coming from the diffeomorphism commutator. The PB with the diffeomorphism gives

$$\dot{A}_1^r = \frac{2N^r}{\gamma}\partial_rA_1 = \frac{2N^r}{\gamma}(A_2A_r) = N^rA_2\frac{\partial_r(\frac{N^rE^1}{\sqrt{E^r}})}{A_2} \tag{B.1.13}$$

$$= -\frac{2\sqrt{E^r}A_2A_1}{\gamma(E^r)'}\left[\frac{E^1}{\sqrt{E^r}}N^{r\prime} + N^r\left(\frac{E^1}{\sqrt{E^r}}\right)'\right] \tag{B.1.14}$$

$$= \gamma^2\frac{A_2'A_2\sqrt{E^r}}{E^1} - 2\frac{2}{\gamma^2}\frac{\sqrt{E^r}A_1'A_1}{E^1} \ . \tag{B.1.15}$$

where we replaces the expression for the shift N^r, A^r (that follows from the radial diffeomorphism constraint), $E^{r\prime}$ with the expressions in (B.1.12). This exactly the contribution of the Hamiltonian constraint in the evolution equation for A_1. Hence we obtains the desired result

$$\dot{A}_1 = \dot{A}_1^r + \dot{A}_1^H = 0 \ . \tag{B.1.16}$$

A_r connection component

For A_r evolution equations, while considering the Euclidean contribution to the total equation of motion one can notice that the expression in (4.2.14) can be written in the following way

$$\begin{aligned}\dot{A}_r^E &= \frac{N(r)}{E^1\sqrt{E^r}}\left[-\frac{\mathcal{H}^E}{2E^r} + 4\left(\frac{E^1}{2A_2}\partial_r(A_1^2) - \frac{\gamma}{2A_2}A_1\mathcal{H}^r\right) + 4E^1A_2'\right] \\ &= \frac{N(r)}{E^1\sqrt{E^r}}\left(-\frac{\mathcal{H}^E}{2E^r} - \frac{\gamma A_1\mathcal{H}^r}{2A_2}\right) + \left(2\frac{A_2\partial_r(A_1^2)}{\sqrt{E^r}} + 4\underbrace{\frac{A_2'A_2}{\sqrt{E^r}}}_{a}\right) \ .\end{aligned} \tag{B.1.17}$$

where in the last equation we substitute $N(r)$ with A_2. Note that there are two extra terms that will cancel the diffeomorphism contribution and a part from the Lorentzian PB. The equation in (4.2.15) for $\dot{A}_r^L$ can also be computed in an alternative way that would make writing it in terms of the constraints easier to manipulate, namely by using the relation $E^{r\prime} = \gamma G + 2A_2E^1$ and $E^{r\prime\prime} = (\gamma G + 2A_2E^1)'$. We eventually obtain

$$\begin{aligned}
\dot{A}_r^L &= \int dr \frac{(1+\gamma^2)N(r)}{2(E^1)^3\sqrt{E^r}}\Bigg(\frac{\delta}{\delta E^r}\left(-(E^1)^2\left((2A_2E^1)^2 + 4E^r(\gamma G + 2A_2E^1)'\right)\right) \\
&+ \frac{\delta}{\delta E^r}\left[4(\gamma G + 2A_2E^1)E^1E^{1\prime}E^r\right] - \frac{\mathcal{H}^L}{2E^r}\Bigg) \\
&= (1+\gamma^2)\frac{N(r)}{2(E^1)^3\sqrt{E^r}}\Bigg(-\frac{\mathcal{H}^L}{2E^r} - 4(E^1)^2\left[2A_2'E^1 + 2E^{1\prime}A_2\right] - (E^1)^2\gamma G' \\
&+ 8A_2(E^1)^2E^{1\prime} + 2\gamma G\partial_r(E^1)^2\Bigg) \\
&= (1+\gamma^2)\frac{N(r)}{2(E^1)^3\sqrt{E^r}}\left(-\frac{\mathcal{H}^L}{2E^r} - 4(E^1)^2\gamma G' + 2\gamma G\partial_r(E^1)^2\right) + (1+\gamma^2)\frac{-8N(E^1)^3A_2'}{2(E^1)^3\sqrt{E^r}} \\
&= (1+\gamma^2)\frac{N(r)}{2(E^1)^3\sqrt{E^r}}\left(-\frac{\mathcal{H}^L}{2E^r} - 4(E^1)^2\gamma G' + 2\gamma G\partial_r(E^1)^2\right) - 4(1+\gamma^2)\underbrace{\frac{A_2A_2'}{\sqrt{E^r}}}_{b}
\end{aligned} \tag{B.1.18}$$

The b term in the above equation will cancel the second term in a. Hence the PB with the Hamiltonian reads

$$\begin{aligned}
\dot{A}_r^H = \frac{-1}{2\gamma^2}\Bigg\{&\underbrace{\frac{N(r)}{E^1\sqrt{E^r}}\left(-\frac{\mathcal{H}^E}{2E^r} - \frac{(1+\gamma^2)}{2E^1}\frac{\mathcal{H}^L}{2E^r}\right)}_{=H} - \frac{\gamma A_1\mathcal{H}^r}{2E^1\sqrt{E^r}} \\
&+ (1+\gamma^2)\frac{N(r)}{2(E^1)^3\sqrt{E^r}}\left(-4(E^1)^2\gamma G' + 2\gamma G\partial_r(E^1)^2\right)\Bigg\} \\
&+ \frac{-1}{2\gamma^2}\left\{\left(2\frac{A_2\partial_r(A_1^2)}{\sqrt{E^r}}\right) - 4\gamma^2\frac{A_2A_2'}{\sqrt{E^r}}\right\} .
\end{aligned} \tag{B.1.19}$$

As for the contribution from the diffeomorphism constraint in (4.2.16), it can be manipulated in the same fashion as the case of A_1 in (B.1.13). This time one should consider

$$\frac{2}{\gamma}\partial_r(N^rA_r) = \frac{2}{\gamma}\partial_r\left(-\gamma N^r\frac{\partial_r(\Lambda N^r)}{N}\right) = \frac{2}{\gamma^2}\partial_r\left(\frac{\sqrt{E^r}}{E^1A_2}\partial_r(A_1^2)\right) \tag{B.1.20}$$

which brings us to the similar calculation of (B.1.13). Computing it explicitly will produce the canceling term coming from the Hamiltonian contribution. Notice that there is a relation between the evolution equations attributed to A_r and A_1, namely the ones coming from the Hamiltonian contribution

$$\dot{A}_r^H = \frac{E^1}{E^r}\dot{A}_1^H = -\frac{2E^1}{E^r}\dot{A}_1^r \tag{B.1.21}$$

$$= -\frac{2E^1}{\gamma E^r}N^r\partial_rA_1 . \tag{B.1.22}$$

If we consider the computation before evaluating the delta function present in the bracket (this is useful to preform integration by parts), one can show that

$$\begin{aligned}
\dot{A}_r^H &= \frac{E^1}{E^r}\dot{A}_1^H = -\frac{2E^1}{E^r}\dot{A}_1^r \\
&= \int d\rho \left\{ \delta(r-\rho) \left(-\frac{4E^1}{\gamma E^r} N^r \partial_r A_1 \right) \right\} \\
&= \int d\rho \left\{ \delta(r-\rho) \left(-\frac{4E^1}{\gamma E^r} N^r A_r A_2 \right) \right\} \\
&= \int d\rho \left\{ \delta(r-\rho) \left(-2N^r A_r \frac{E^{r\prime}}{\gamma E^r} \right) \right\} \\
&= \frac{1}{\gamma} \int d\rho \left\{ \delta(r-\rho) \left(2N^r A_r \right) \right\} \\
&= \frac{2}{\gamma} \partial_r \left(N^r A_r \right) \\
&= -\dot{A}_r^r \ .
\end{aligned} \tag{B.1.23}$$

Therefore the evolution equation for A_r vanishes

$$\dot{A}_r = \dot{A}_r^H + \dot{A}_r^r = 0 \ .$$

B.2 Fluxes evolution equations

E^r flux component

In order to show explicitly that the evolution equations are satisfied on-shell, a crucial feature, as we saw for the connection, is to express the shift in terms of the connection and fluxes component. Hence, an accessible way to do so is to rely on the connection component A_1, which will turn out to be important since it will grant the right proportionality for the γ factors contributing from the Hamiltonian and diffeomorphism constraint.[1] Using the fact that $N = A_2$ and that $R' = \frac{(E^r)'}{2\sqrt{E^r}}$, the shift N^r can be expressed as

$$N^r = -\frac{NA_1}{\gamma R'} = -\frac{A_2 A_1}{\gamma R'} = -\frac{2\sqrt{E^r} A_2 A_1}{\gamma (E^r)'} \ . \tag{B.2.1}$$

Furthermore, expressing $E^{r\prime}$ in (4.2.22) in terms of the Gauss constraint, the evolution equation yields

$$\begin{aligned}
\dot{E}^r &= \frac{1}{\gamma^2} 4N\sqrt{E^r} A_1 + \frac{2N^r}{\gamma} \left(\gamma G_3 + 2A_2 E^1 \right) \\
&= \frac{1}{\gamma^2} 4N\sqrt{E^r} A_1 + 2N^r A_2 E^1 \\
&= \frac{1}{\gamma^2} 4N\sqrt{E^r} A_1 - 4\frac{1}{\gamma^2} \left(\frac{\sqrt{E^r} A_2 A_1}{(E^r)'} A_2 E^1 \right) \\
&= 0 \ .
\end{aligned}$$

[1] We reintroduced the γ^2 in all the calculations

where one can set the constraint G_3 to zero and use the relation $2A_2E^1 = E^{r\prime}$ to arrive at the last equality. Therefore the total evolution equation for the flux E^r vanishes, i.e.

$$\dot{E}^r = 0 \ .$$

E^1 flux component

For the E^1 component

$$\dot{E}^1_H = \frac{-2N}{\sqrt{E^r}} \left(A_r E^r + E^1 A_1 \right) \ . \tag{B.2.2}$$

Notice the combination

$$\begin{aligned} \frac{E^r}{2}\mathcal{H}^1_r + A_1 G_3 &= \frac{\sin\theta}{\gamma}\left[E^r A_1' + A_1 E^{r\prime} - 2A_2(E^1 A_1 + E^r A_r) \right] \\ &= \frac{\sin\theta}{\gamma}\left(\partial_r(E^r A_1) - 2A_2(E^1 A_1 + E^r A_r) \right) = 0 \ . \end{aligned} \tag{B.2.3}$$

Therefore the full equation of motion for the flux E^1 reads

$$\begin{aligned} \dot{E}^1 &= \frac{-2N}{\sqrt{E^r}} \left(A_r E^r + E^1 A_1 \right) + \frac{2}{\gamma}\partial_r(N^r E^1) \\ &= \frac{-N(r)}{A_2\sqrt{E^r}} \left\{ \frac{\gamma E^r}{2}\mathcal{H}^1_r + \gamma G_3 A_r - \partial_r(E^r A_1) \right\} + \frac{2}{\gamma}\partial_r(N^r E^1) \ . \end{aligned} \tag{B.2.4}$$

Now all what one should prove is that the third term in the curly bracket cancels the contribution coming the PB with $\mathcal{H}^r$. To this aim let us examine the term coming from the commutator with the diffeomorphism constraint. One writes the useful relation

$$E^1 = \frac{1}{2A_2}\left(E^{r\prime} - \gamma G_3 \right) \ , \tag{B.2.5}$$

and notice that $\dot{E}^1_r$ can be written as:

$$\begin{aligned} \partial_r(N^r E^1) &= \frac{2}{\gamma}\partial_r\left(\frac{N^r}{2A_2}(E^{\prime r} - \gamma G_3) \right) \\ &= \frac{2}{\gamma}\partial_r\left(\frac{N^r}{2A_2}E^{\prime r} \right) \\ &= -\frac{2}{\gamma}\partial_r\left(\frac{N A_1\sqrt{E^r}}{2A_2\gamma} \right) \\ &= -\frac{1}{\gamma^2}\partial_r\left(A_1\sqrt{E^r} \right) \ , \\ &= -\frac{1}{4}\partial_r\left(\dot{E}^r_E \right) \end{aligned}$$

As we did for the connection in the previous paragraph, we evaluate the equation of motion of $\dot{E}^1_E$ before integrating over the delta function. Putting everything together

$$\dot{E}^1 = \frac{2N}{\gamma A_2}\partial_r(\sqrt{E^r}A_1) - \frac{2}{\gamma}\partial_r(A_1\sqrt{E^r}) = 0 \ . \tag{B.2.6}$$

Acknowledgment

I would like to thank my supervisors Daniele Pranzetti and Prof. Sabino Matarrese for their support and instructions to conduct this work.
I would also like to express my gratitude to Silke Rosemarie Pranzetti for her constant advice and encouragement. I am grateful to Daniele Oriti, Prof. Luca Martucci, Prof. Marco Matone, Alice Di Tucci, Alba Kalaja, Riccardo Ciccone and Giovanni Frezzato, Nicholas Turetta, Victor David Ortega and Matteo Laudonio not only for the useful and very helpful discussions but also for their support.

This research was supported in part by Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through the Department of Innovation, Science and Economic Development and by the Province of Ontario through the Ministry of Research and Innovation.

Bibliography

[1] Roger Penrose. Gravitational collapse and space-time singularities. *Phys. Rev. Lett.*, 14:57–59, Jan 1965.

[2] S. W. Hawking. Particle creation by black holes. *Communications in Mathematical Physics*, 43(3):199–220, Aug 1975.

[3] John Preskill. Do black holes destroy information? In *International Symposium on Black holes, Membranes, Wormholes and Superstrings Woodlands, Texas, January 16-18, 1992*, pages 22–39, 1992.

[4] Daniele Oriti. *Approaches to Quantum Gravity.* Cambridge University Press, 2009.

[5] Alejandro Perez. The Spin Foam Approach to Quantum Gravity. *Living Rev. Rel.*, 16:3, 2013.

[6] Kristina Giesel and Hanno Sahlmann. From Classical To Quantum Gravity: Introduction to Loop Quantum Gravity. *PoS*, QGQGS2011:002, 2011.

[7] J. Ambjorn, A. Goerlich, J. Jurkiewicz, and R. Loll. Nonperturbative Quantum Gravity. *Phys. Rept.*, 519:127–210, 2012.

[8] Daniele Oriti. Group Field Theory and Loop Quantum Gravity. 2014.

[9] Olaf Hohm, Dieter Lüst, and Barton Zwiebach. The Spacetime of Double Field Theory: Review, Remarks, and Outlook. *Fortsch. Phys.*, 61:926–966, 2013.

[10] Hermann Nicolai, Kasper Peeters, and Marija Zamaklar. Loop quantum gravity: An Outside view. *Class. Quant. Grav.*, 22:R193, 2005.

[11] Daniele Pranzetti and Hanno Sahlmann. Horizon entropy with loop quantum gravity methods. *Phys. Lett.*, B746:209–216, 2015.

[12] Pietro Dona and Simone Speziale. Introductory lectures to loop quantum gravity. In *Gravitation Théorie et Expérience.Proceedings, Troisième école de physique théorique de Jijel: Jijel, Algeria, September 26?October 03, 2009*, pages 89–140, 2013.

[13] Alejandro Perez. Introduction to loop quantum gravity and spin foams. 2004.

[14] Thomas Thiemann. *Modern Canonical Quantum General Relativity.* Cambridge University Press, Cambridge, UK, 2007.

[15] Abhay Ashtekar and Martin Bojowald. Quantum geometry and the schwarzschild singularity. *Classical and Quantum Gravity*, 23(2):391–411, dec 2005.

[16] Miguel Campiglia, Rodolfo Gambini, and Jorge Pullin. Loop quantization of spherically symmetric midi-superspaces : The Interior problem. *AIP Conf. Proc.*, 977(1):52–63, 2008.

[17] Abhay Ashtekar, Tomasz Pawlowski, and Parampreet Singh. Quantum Nature of the Big Bang: An Analytical and Numerical Investigation. I. *Phys. Rev.*, D73:124038, 2006.

[18] Abhay Ashtekar and Parampreet Singh. Loop Quantum Cosmology: A Status Report. *Class. Quant. Grav.*, 28:213001, 2011.

[19] Abhay Ashtekar, Martin Bojowald, and Jerzy Lewandowski. Mathematical structure of loop quantum cosmology. *Adv. Theor. Math. Phys.*, 7(2):233–268, 2003.

[20] Emanuele Alesci, Gioele Botta, Francesco Cianfrani, and Stefano Liberati. Cosmological singularity resolution from quantum gravity: the emergent-bouncing universe. *Phys. Rev.*, D96(4):046008, 2017.

[21] Emanuele Alesci, Sina Bahrami, and Daniele Pranzetti. Quantum evolution of black hole initial data sets: Foundations. *Phys. Rev.*, D98(4):046014, 2018.

[22] István Rácz and Jeffrey Winicour. Black hole initial data without elliptic equations. *Phys. Rev.*, D91(12):124013, 2015.

[23] Abhay Ashtekar and Badri Krishnan. Dynamical horizons and their properties. *Phys. Rev.*, D68:104030, 2003.

[24] Norbert Bodendorfer, Jerzy Lewandowski, and Jedrzej Świeżewski. General relativity in the radial gauge: Reduced phase space and canonical structure. *Phys. Rev.*, D92(8):084041, 2015.

[25] Abhay Ashtekar, Javier Olmedo, and Parampreet Singh. Quantum extension of the Kruskal spacetime. *Phys. Rev.*, D98(12):126003, 2018.

[26] Abhay Ashtekar. Loop quantum cosmology: an overview. *General Relativity and Gravitation*, 41(4):707–741, Apr 2009.

[27] Robert M Wald. *General relativity.* Chicago Univ. Press, Chicago, IL, 1984.

[28] Martin Bojowald. *Canonical Gravity and Applications: Cosmology, Black Holes, and Quantum Gravity.* Cambridge University Press, 2010.

[29] Claus Kiefer. *Quantum Gravity.* Oxford University Press, 2004.

[30] Richard L. Arnowitt, Stanley Deser, and Charles W. Misner. The Dynamics of general relativity. *Gen. Rel. Grav.*, 40:1997–2027, 2008.

[31] Eric Gourgoulhon. 3+1 formalism and bases of numerical relativity. 2007.

[32] Emanuele Alesci, Sina Bahrami, and Daniele Pranzetti. Quantum gravity predictions for black hole interior geometry.

[33] Emanuele Alesci and Francesco Cianfrani. Quantum reduced loop gravity: Universe on a lattice. *Phys. Rev.*, D92:084065, 2015.

[34] Emanuele Alesci, Costantino Pacilio, and Daniele Pranzetti. Orthogonal gauge fixing of first order gravity. *Phys. Rev.*, D98(4):044052, 2018.

[35] Alejandro Perez. Black Holes in Loop Quantum Gravity. *Rept.Prog.Phys.*, 80(12):126901, 2017.

[36] Kinjal Banerjee, Gianluca Calcagni, and Mercedes Martin-Benito. Introduction to loop quantum cosmology. *SIGMA*, 8:016, 2012.