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Generation and evolution of cosmological second-order gravitational waves and density perturbations

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1 Introduction

The study of cosmological perturbations is a very powerful tool to get information about the origin of the Universe. In fact, the seeds of the present observable structures have all been produced in the very early phases of the Universe, from quantum fluctuations during inflation. They have been stretched by the accelerated expansion of the space-time during this phase, exiting the cosmological horizon and remaining constant until they could re-enter it during the successive radiation- or matter-dominated phases of the Universe. Then, once re-entered the horizon and having been affected by the causal physics, some of those perturbations (e.g. the density perturbations) undergo gravitational instability and grow, at a pace which depends on the phase in which the Universe is, i.e. on the component dominating its energy density at that time. So, eventually those perturbations generate the structures that we observe.
Studying the generation of those perturbations in the early Universe and their evolution up to now, we may compare the predictions of our models with the observations (from the Cosmic Microwave Background or the Large Scale Structure). This way, one could give constraints on different models of the early Universe, proving or ruling out inflation.

The study of cosmological perturbations is usually performed with a perturbative approach -as exact solutions of the Einstein equations are very difficult to obtain- and in regimes where those perturbations are not so relevant with respect to the background. In fact, while on the smallest scales, non-linear processes have prevailed in the structure formation, so it is not possible to describe them with a perturbative approach, it is still possible to do that on the largest scales, where the Universe appears to be a homogeneous and isotropic background with small perturbations superimposed. So, the line element of the Friedmann-Robertson-Walker metric is expanded adding to it three kinds of perturbations: scalars, vectors and tensors. Also, a general relativistic approach is considered, to account for perturbations scales bigger than the cosmological horizon at some time.

At linear order, the scalar, vector and tensor perturbations of the metric are separated: a source for each perturbation can be only of the same type. So, for example, scalar linear metric perturbations are generated by scalar perturbations of the stress-energy tensor (like density perturbations of a fluid), while the tensor perturbations predicted from a single field inflationary model would have no source at first-order, being just vacuum fluctuations of the gravitational field itself. From second-order on, there is a mixing between the different kind of modes: we can obtain density perturbations generated by combinations of first-order tensor modes, as well as gravitational waves from first-order density perturbations.

The study of those second-order perturbations is becoming more and more important in Cosmology, in the perspective of reaching higher observational resolutions and to make more accurate predictions about observables, like non-Gaussianity, which could give important information to discriminate between different inflationary models. In fact, they predict small (but different) levels of non-Gaussianity, so a three-point correlation function (or its Fourier transform, the bispectrum) of the fluctuations from inflation which is different from zero. But also the non-linear, post-inflationary evolution of those fluctuations generates a substantial level of non-Gaussianity, more relevant than the small amount predicted by inflation itself [9]. Thus, the second-order contribution to the bispectrum has to be considered if we want to constrain the primordial one from the observations.

Another fundamental information coming from cosmological perturbations would be the amplitude of the stochastic background of primordial gravitational waves, that would set the energy scale of inflation. This way, it represents an important knowledge for Theoretical and Particle Physics, too. Unfortunately, the amplitude of those linear gravitational waves seems to be very small, such that we have not observed them yet (for example, from the polarization of the CMB).
Also scalar perturbations could be a source of gravitational waves at second-order, once they re-enter the horizon after inflation. In particular, scalar perturbations re-entering the horizon during matter domination could lead to a non-negligible contribution to the spectrum of gravitational waves: there can be an enhancement mechanism to this spectrum during an early matter dominated era after inflation (such as reheating) depending on the duration of this era, and the amplitude of those GW could possibly reach future observational limits (advanced LIGO or LISA). Furthermore, second-order GW generated by first-order perturbations could affect the CMB polarization and limit the possibility of estimating the first-order tensor modes, reducing the constraints on the energy scale of inflation [8, 10].

All of these (and many other) reasons justify the effort of going beyond the linear approach to cosmological perturbations, that anyway helped us so much in defining our picture of the Universe. In this work we will focus mainly on second-order scalar and tensor perturbations, with an original contribution on the first ones. In fact, in the literature it has been extensively studied the contribution of first-order scalars (which are the most relevant linear perturbation) to the second-order perturbations, neglecting linear vectors (which should not be produced during inflation, and would anyway decay) and linear gravitational waves (for their very small amplitude). To our knowledge, only few works (like [20, 21, 24]) have taken into account tensors in the source terms. So, it can be asked whether the contribution to second-order scalar perturbations coming from linear gravitational waves would really be totally negligible, or not. In other words, whether or not those perturbations could undergo gravitational instability and grow in time, such that, even if starting from very small amplitudes, they could have some feature making them non-negligible.

This work is structured as follows: in the first part (sections 2), there is an introduction about the production of perturbations during inflation and their evolution in the Newtonian theory; in sections 3-6 there is all the relativistic derivation of second-order density and metric perturbations in two gauges and in an Einstein-de Sitter Universe, following the procedure described in [1]. Then, in section 7, a possible enhancement mechanism for second-order gravitational waves during an early matter-dominated era [8] is mentioned. In section 8, there is the derivation of the time evolution of second-order perturbations in EdS (using the results obtained in the first part), followed by the conclusions, in section 9. At the end, the Appendices present all the (sometimes lengthy) expressions which have been useful for the derivations of our results.

2 Generation and evolution of cosmological perturbations

2.1 Perturbations from inflation

The inflation is a phase at the very beginning of our Universe that has been modeled mainly to answer the questions about its homogeneity and isotropy on large scales, its flatness and the absence of "strange" relics (like magnetic monopoles, topological defects...) that one would expect from the possible phase transitions in its primordial phases.

Defining the cosmological horizon as the Hubble radius:

$$r_H = \frac{1}{aH},$$

(with $a(t)$ the scale factor, $H = \frac{da/dt}{a} = \frac{\dot{a}}{a}$ the Hubble parameter), all those problems are solved requiring that r_H is decresing during this phase. It is easy to see that this requirement is equivalent to having an accelerated expansion of the scale factor:

$$\dot{r}_H < 0 \leftrightarrow -\frac{\ddot{a}}{(\dot{a})^2} < 0 \leftrightarrow \ddot{a} > 0,$$

which, for the third Friedmann equation, is equivalent to having a species with equation of state $\rho + 3p < 0 \leftrightarrow w = \frac{p}{\rho} < -\frac{1}{3}$. The simplest model of inflation assumes that the species driving this phase is a scalar field φ, with $w \simeq -1$, so a quasi de-Sitter Universe. We cannot have exactly $w = -1$ as this would be equivalent to a constant H, so to an eternal inflation.
To achieve this accelerated expansion, the inflaton field φ should satisfy the *slow-roll condition*: it should have a very flat (but not constant) potential $V(\varphi)$, such that it would reach the minimum of its potential very slowly (so, the kinetic term should satisfy $\frac{1}{2}\dot{\varphi}^2 \ll V$). This requirement is connected to the constraint on the equation of state: in fact, explicitating the pressure and the energy density for a scalar field,

$$w_\varphi = \frac{p_\varphi}{\rho_\varphi} = \frac{\frac{1}{2}\dot{\varphi}^2 - V}{\frac{1}{2}\dot{\varphi}^2 + V} \simeq -1.$$

It is also useful to define the slow-roll parameters ϵ and η, which quantify respectively how flat the potential is and for how long it will remain sufficiently flat. Their expressions with respect to the scalar field and the potential are, respectively:

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi}{M_{Pl}^2}\frac{\dot{\varphi}^2}{H^2}; \quad \eta = -\frac{\ddot{\varphi}}{3H\dot{\varphi}}; \quad \epsilon_V = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2; \quad \eta_V = \frac{1}{8\pi G}\left(\frac{V''}{V}\right). \tag{2.1}$$

The slow-roll conditions are fulfilled as long as the slow-roll parameters are $\epsilon \ll 1, \eta \ll 1$. As soon as they get close to 1, it means that the inflaton is approaching the minimum of its potential, where it will start oscillating getting the inflation to an end. During these oscillations, it will decay in relativistic particles that will constitute the radiation fluid dominating the successive phase of the classical Hot Big Bang Universe. This last phase at the end of inflation is called *reheating*.

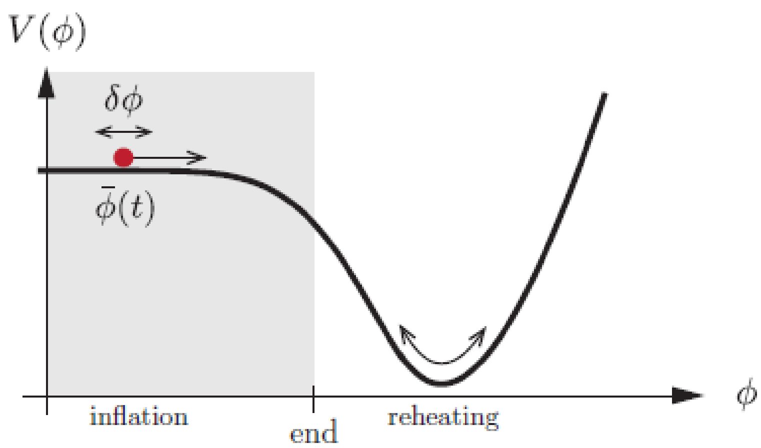

Figure 1: Representation of a possible inflationary potential, with the inflaton field slowly-rolling during inflation and reaching the end of this phase once the minimum has been approached. On its path down the potential, the scalar field has perturbations depending on the position which kick it a little bit up or down, so a little further or closer to the end of inflation. This way, at different positions we would have slightly different durations of inflation. Figure from [13].

2.1.1 Density perturbations from inflation

The inflationary models do not just explain the homogeneity, isotropy and flatness of the Universe, they also predict the formation of density perturbations which are adiabatic, almost scale-invariant and almost Gaussian; this is found to be in perfect agreement with the current observations. Those perturbations are generated as quantum fluctuations of the inflaton field during inflation: $\varphi(\mathbf{x}, \tau) = \varphi_0(\tau) + \delta\varphi(\mathbf{x}, \tau)$, where τ is the conformal time (given by $d\tau = dt/a(t)$) and $\varphi_0(\tau)$ is the background contribution, which does not depend on space but just on times (as it has to evolve, to let the inflation end). The perturbation $\delta\varphi(\mathbf{x},\tau)$ can be also redefined as $\delta\tilde{\varphi}(\mathbf{x},\tau) = a\,\delta\varphi(\mathbf{x},\tau)$, which can be promoted to a quantum operator:

$$\delta\tilde{\varphi}(\mathbf{x},\tau) = \int \frac{d\mathbf{k}^3}{(2\pi)^3}[u_k(\tau)\,\hat{a}_{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}} + u_k^*(\tau)\,\hat{a}_{\mathbf{k}}^+\,e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{2.2}$$

where $u_k(\tau), u_k^*(\tau)$ satisfy the commutation relation $u_k^*(\tau)u_k'(\tau) - u_k(\tau)u_k'^*(\tau) = -i$ (the prime indicates $' = \frac{d}{d\tau}$). The creation and annihilation operators $\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}^+$ satisfy:

$$\hat{a}_{\mathbf{k}}|0\rangle = 0, \quad \langle 0|\hat{a}_{\mathbf{k}}^+ = 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}] = 0, \quad [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^+] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}'),$$

where $|0\rangle$ is the vacuum state. The action of the inflaton field, minimally coupled to gravity, would be:

$$S = \int d^4x\,\sqrt{-g}\left[\frac{1}{2}M_{Pl}^2 R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - V(\varphi)\right], \tag{2.3}$$

with R the Ricci scalar, $g = \det(g_{\mu\nu})$. The equation of motion (Klein-Gordon equation) for $\delta\varphi$ is obtained from the variation of the action S, perturbed up to first-order in the scalar field (we neglect the perturbations of the metric for simplicity). For the eigenfunctions $u_k(\tau)$ the equation of motion reads:

$$u_k'' + \left(k^2 - \frac{a''}{a} + m_\varphi^2 a^2\right)u_k = 0, \tag{2.4}$$

where $m_\varphi^2 = \frac{\partial^2 V}{\partial\varphi^2}$ is the effective mass. This equation can be recast in this form:

$$u_k'' + \left(k^2 - \frac{\nu_\varphi^2 - \frac{1}{4}}{\tau^2}\right)u_k = 0, \tag{2.5}$$

where $\nu_\varphi^2 = \left(\frac{9}{4} - \frac{m_\varphi^2}{H^2}\right) \simeq \frac{9}{4} + 3\epsilon - 3\eta_V$. For constant m_φ^2 and real ν_φ this is a Bessel equation, whose solution can be expressed in terms of the Hankel functions of first and second kind:

$$u_k(\tau) = \sqrt{-\tau}\,[c_1(k)H_{\nu_\varphi}^{(1)}(-k\tau) + c_2(k)H_{\nu_\varphi}^{(2)}(-k\tau)]. \tag{2.6}$$

The constants $c_1(k), c_2(k)$ can be fixed with the requirement that on sub-horizon scales (i.e. $k \gg aH$, corresponding in quasi de-Sitter to $-k\tau \gg 1$, as $\tau \simeq -\frac{1}{aH(1-\epsilon)}$) the solution matches the one we would expect in a flat space-time, so plane waves: $u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}$. Applying this limit to the Hankel functions:

$$H_{\nu_\varphi}^{(1)}(x \gg 1) \simeq \sqrt{-\frac{2}{\pi x}}e^{-i\left(k\tau + \frac{\pi}{4} + \frac{\pi}{2}\nu_\varphi\right)}; \quad H_{\nu_\varphi}^{(2)}(x \gg 1) \simeq \sqrt{-\frac{2}{\pi x}}e^{-i\left(-k\tau + \frac{\pi}{4} + \frac{\pi}{2}\nu_\varphi\right)},$$

so we need to impose $c_2(k) = 0, c_1(k) = \frac{\sqrt{\pi}}{2} e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\nu_\varphi\right)}$. This way, the exact solution becomes:

$$u_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i\left(\frac{\pi}{4}+\frac{\pi}{2}\nu_\varphi\right)} \sqrt{-\tau}\, H^{(1)}_{\nu_\varphi}(-k\tau).$$

To obtain the super-horizon limit of this solution, we can also consider the limit $k \ll aH \leftrightarrow -k\tau \ll 1$, that for the Hankel function of first kind corresponds to:

$$H^{(1)}_{\nu_\varphi}(x \ll 1) \simeq \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\nu_\varphi - \frac{3}{2}} \frac{\Gamma(\nu_\varphi)}{\Gamma(3/2)} (x)^{-\nu_\varphi},$$

such that $u_k(\tau)$ becomes:

$$u_k(\tau) = e^{i\left(\nu_\varphi - \frac{1}{2}\right)\frac{\pi}{2}} 2^{\nu_\varphi - \frac{3}{2}} \frac{\Gamma(\nu_\varphi)}{\Gamma(3/2)} \frac{1}{\sqrt{2k}} (-k\tau)^{\frac{1}{2}-\nu_\varphi}. \tag{2.7}$$

So, the super-horizon limit of the inflaton perturbation $\delta\varphi = \frac{u_k}{a}$ would be:

$$|\delta\varphi|(-k\tau \ll 1) \simeq \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\varphi}, \tag{2.8}$$

to lowest order in the slow-roll parameters.

If we would consider also the perturbations of the metric (as we are not in an unperturbed FRW background), the term with the effective mass in equation (2.4) would become $Ma^2 \simeq \frac{M}{H^2\tau^2} \simeq \frac{3\eta_V - 6\epsilon}{\tau^2}$, so $\nu_\varphi^2 = \frac{9}{4} + 9\epsilon - 3\eta_V$, and the wavefunction u_k would be associated not simply with $\delta\varphi$, but with the gauge-invariant Sasaki-Mukhanov variable Q_φ:

$$Q_\varphi \equiv \delta\varphi + \frac{\varphi'}{\mathcal{H}} \phi, \tag{2.9}$$

where $\mathcal{H} = \frac{a'}{a}$ is the conformal Hubble parameter and ϕ is the spatial curvature perturbation (defined in (4.1)). So, at the end one would obtain, on super-horizon scales:

$$|Q_\varphi(k)|(-k\tau \ll 1) \simeq \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\varphi \simeq \eta_V - 3\epsilon}. \tag{2.10}$$

It is useful to define other quantities to connect Q_φ with the observations. First of all, the curvature perturbation, which is not gauge-invariant:

$$\hat{\Phi} \equiv \phi + \frac{1}{6}\nabla^2 \chi^{\parallel}, \tag{2.11}$$

where ϕ and $\chi^{\parallel}$ are scalar perturbations of the space-space part of the metric (defined in (4.1)). This is strictly related to the linear intrinsic spatial curvature on hypersurfaces of constant conformal time:

$$^{(3)}R = \frac{4}{a^2}\nabla^2\hat{\Phi}. \tag{2.12}$$

The corresponding gauge-invariant quantity is the curvature perturbation on uniform density hypersurfaces:

$$-\zeta \equiv \hat{\Phi} + \frac{\delta\rho}{\rho'}\mathcal{H}, \tag{2.13}$$

which reduces to the curvature perturbation in the uniform density gauge ($\delta\rho = 0$). The advantage of using ζ is not only its gauge invariance, but mostly the fact that it remains constant on super-horizon scales and in the absence of non-adiabatic perturbations (whose pressure perturbations have a δp_{nad} which does not depend on ρ). This means that, once having been produced, the inflationary expansion stretches ζ out of the horizon, where it does not evolve: it encodes all the informations from the primordial Universe until its horizon re-entry.

To connect ζ with Q_φ, it can also be defined the gauge-invariant curvature perturbation on comoving hypersurfaces:

$$\mathcal{R} \equiv \hat{\Phi} + \frac{\delta\varphi}{\varphi'}\mathcal{H}, \tag{2.14}$$

which is directly connected to the Sasaki-Mukhanov variable: $\mathcal{R} = \frac{\mathcal{H}}{\varphi'}Q_\varphi$. It is also related to ζ through:

$$-\zeta = \mathcal{R} + \frac{2\rho}{9(\rho+p)}\left(\frac{k}{aH}\right)^2\psi, \tag{2.15}$$

where ψ is the scalar perturbation of the time-time part of the metric (defined in (4.1)). So, on super-horizon scales $-\zeta \simeq \mathcal{R}$ and its amplitude would become:

$$|\zeta(k)|(-k\tau \ll 1) \simeq \frac{H^2}{\sqrt{2k^3}\dot{\varphi}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu_\varphi \simeq \eta_V - 3\epsilon}. \tag{2.16}$$

Depending on the gauge choice, ζ at linear order is connected either with the curvature perturbation (in a uniform density gauge) or with the linear density contrast $\delta = \frac{\delta\rho}{\rho}$ (in the uniform curvature gauge). At the first horizon crossing (instant $t_{\mathrm{H}}^{(1)}(k)$), the density perturbations of the inflaton are frozen out, so they remain constant until they re-enter the horizon (instant $t_{\mathrm{H}}^{(2)}(k)$), when they generate density perturbations in the fluid dominating the Universe. This can be shown, using the slow-roll equation $3H\dot{\varphi} \simeq -\frac{\partial V}{\partial \varphi}$ and the third Friedmann equation:

$$\zeta\Big|_{t_{\rm H}^{(1)}(k)} = -H\frac{\delta\varphi}{\dot{\varphi}} \simeq -H\frac{\delta\rho_\varphi}{\dot{\rho}_\varphi} = \frac{1}{3H(1+w)}\frac{\delta\rho}{\rho} = \zeta\Big|_{t_{\rm H}^{(2)}(k)}. \tag{2.17}$$

So, in principle, on the largest scales we would have access to the information coming directly from the primordial phases of the Universe.

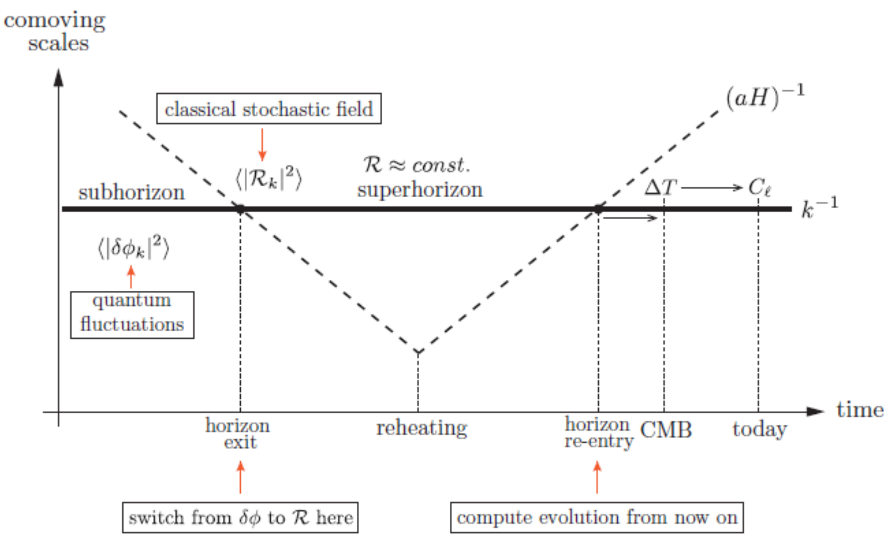

Figure 2: This figure shows the development of the Hubble radius during and after inflation, which leads to the exit and re-entering of a mode with a certain k. It is also emphasized how the quantum fluctuations from the inflaton field on sub-horizon scales are translated into classical fluctuations in the curvature perturbation $\mathcal{R}$, which stay constant on super-horizon scales. Once they re-enter the horizon, they affect the post-inflationary Universe, evolving in the temperature perturbations that we can observe from the CMB. Figure from [13].

The quantum to classical transition of the perturbations passing from sub- to super-horizon scales can be explained evaluating the number of particles n_k which are produced on super-horizon scales. As it would be much bigger than unity, the perturbations can be considered classical, such that their energy can be estimated "classically": $H_k = \omega_k(n_k + \frac{1}{2}) \simeq \omega_k n_k$ [14].

To evaluate the amplitude and the statistical properties of the fluctuations we can compute their correlation functions, or, classically, their ensamble average.

For quantum fluctuations, like $\delta\varphi$, in Fourier space it is defined as:

$$\langle 0|\,\delta\varphi(\mathbf{k}_1)\,\delta\varphi^*(\mathbf{k}_2)\,|0\rangle = \frac{2\pi^2}{k_1^3}\delta^3(\mathbf{k}_2 - \mathbf{k}_2)P_\varphi(k_1), \tag{2.18}$$

such that:

$$\begin{aligned}\langle 0|\,\delta\varphi(\mathbf{x},t)\,\delta\varphi^*(\mathbf{x+r},t)\,|0\rangle &= \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\int \frac{d^3\mathbf{k}_2}{(2\pi)^3} e^{i\mathbf{k}_1\cdot\mathbf{x}} e^{-i\mathbf{k}_2\cdot(\mathbf{x+r})}\langle 0|\,\delta\varphi(\mathbf{k}_1)\,\delta\varphi(\mathbf{k}_2)\,|0\rangle = \\ &= \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} e^{-i\mathbf{k}_1\cdot\mathbf{r}}\frac{2\pi^2}{k_1^3}P_\varphi(k_1).\end{aligned} \tag{2.19}$$

So, the variance will be:

$$\langle 0|\,\delta\varphi(\mathbf{x},t)\,\delta\varphi^*(\mathbf{x},t)\,|0\rangle = \langle 0|\,\delta\varphi(\mathbf{x},t)^2\,|0\rangle = \int \frac{dk}{k}P_\varphi(k). \tag{2.20}$$

For classical perturbations, their evaluation on the vacuum state is replaced by their ensemble average $\langle\cdot\rangle$, and the complex perturbations become $\delta\varphi^*(\mathbf{k}) = \delta\varphi(-\mathbf{k})$. Comparing with the amplitudes which has beed found for the curvature perturbation ζ, its power spectrum would be:

$$P_\zeta(k) = \frac{k^3}{2\pi^2}\,|\zeta(k)|^2 = \left(\frac{H^2}{2\pi\dot\varphi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu_\varphi \simeq 2\eta_V - 6\epsilon}, \tag{2.21}$$

where the spectral index is defined as $n_s - 1 = 3 - 2\nu_\varphi$.
At the horizon crossing: $P_\zeta(k) = (\frac{H^2}{2\pi\dot\varphi})^2_*$.

2.1.2 Gravitational waves from inflation

The generation of linear gravitational waves during inflation can be explained with a mechanism completely analogous to the one for scalar perturbations. Perturbing the action (2.3) with respect to tensors at first-order, one obtains the evolution equation:

$$\ddot h_{ij} + 3\frac{\dot a}{a}\,\dot h_{ij} - \frac{\nabla^2 h_{ij}}{a^2} = 0\,, \tag{2.22}$$

which is a wave equation, solved by

$$h_{ij}(\mathbf{x},t) = \sum_{\lambda=+,\times} h^{(\lambda)}(t)\,e^{(\lambda)}_{ij}(\mathbf{x})\,, \tag{2.23}$$

where the polarization tensor e_{ij} is symmetric, transverse and traceless ($e_{ij} = e_{ji}, k^i e_{ij} = 0, e^i_i = 0$) and $\lambda = +, \times$ are the two polarization states.

The following transformation can be performed:

$$v_{ij}\,(\mathbf{x}, t) = \frac{aM_{Pl}}{\sqrt{2}}\, h_{ij}(\mathbf{x}, t)\,, \tag{2.24}$$

where this new variable can be expanded in Fourier space this way:

$$v_{ij}\,(\mathbf{x}, t) = \int \frac{d^3(k)}{2\pi^3} \sum_{\lambda=+,\times} e^{i\mathbf{k}\cdot\mathbf{x}}\, v^{(\lambda)}_{\mathbf{k}}(t)\, e^{(\lambda)}_{ij}(\mathbf{x})\,, \tag{2.25}$$

quantizing the field $v_{\mathbf{k}}$ as in (2.2). The evolution equation for the linear tensor modes becomes:

$$v''^{(\lambda)}_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right) v^{(\lambda)}_{\mathbf{k}} = 0\,, \tag{2.26}$$

which can be recasted in the same form as equation (2.5), with a parameter $\nu_{\mathrm{T}} \simeq \frac{3}{2} + \epsilon$. So, the solution for this equation is completely analogous to the one for the scalar perturbations (2.6). The amplitude of the variable $v_{\mathbf{k}}$ is:

$$|v_{\mathbf{k}}|^2 = \frac{H^2}{2k^3}\left(\frac{k}{aH}\right)^{3-2\nu_{\mathrm{T}}}; \tag{2.27}$$

this way, the power spectrum for linear gravitational waves (defined as the one of scalar perturbations, in (2.18)), becomes:

$$P_{\mathrm{T}} = \frac{k^3}{2\pi^2} \sum_{\lambda=+,\times} |h^{(\lambda)}_{\mathbf{k}}|^2 = \frac{8}{M^2_{\mathrm{pl}}}\left(\frac{H}{2\pi}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu_{\mathrm{T}}}, \tag{2.28}$$

where the spectral index for tensors is: $n_{\mathrm{T}} = 3 - 2\nu_{\mathrm{T}} = -2\epsilon$.

The power spectra for scalars and tensors present the form: $P(k) = \Delta(k_0)(\frac{k}{k_0})^n$, so the ratio between their amplitudes can be evaluated:

$$r = \frac{\Delta_{\mathrm{T}}}{\Delta_\zeta} = \frac{8}{M^2_{\mathrm{pl}}}\left(\frac{\dot\varphi}{H}\right)^2 = 16\epsilon\,. \tag{2.29}$$

The amplitude of the spectrum of scalar perturbations from inflation has been evaluated from the measurements of the temperature fluctuations of the CMB: $\Delta_\zeta \simeq 2 \times 10^{-9}$ [18]. Measuring also the amplitude of the stochastic background of primordial gravitational waves Δ_T would set the energy scale of the parameter ϵ, so it would allow to distinguish between different scenarios of inflation (for example, between large-field or small-field models).

Furthermore, as the spectral index for tensors is $n_T = -2\epsilon$ in single field inflation, the consistency relation $r = -8n_T$ would allow to put constraints on the possibility of having single- or multi-field models.

Up to now, only the spectral index of scalar perturbations has been probed: $n_s \simeq 0.96$ (from CMB measurements [18]), whose deviations from 1 are due to the small contribution from the slow-roll parameters ϵ, η. So, we have an almost scale-invariant scalar power spectrum.

2.2 Evolution of perturbations in a matter dominated Universe

After inflation, the perturbations re-enter the horizon and are affected by the causal physics. To study the evolution of perturbations of non-relativistic matter well inside the cosmological horizon, a very classical approach is the Newtonian one. Even though this work is focused on the complete, general relativistic treatment, a parenthesis can be opened on the Newtonian procedure, which easily gives insights on the physical processes in action. Then, in the next sections, the relativistic treatment is followed again, but some comparisons with the Newtonian one can still be drawn (section 6.3).

On large scales, we can make the assumption that matter behaves like a perfect fluid, with energy density $\rho(\mathbf{x}, t)$, 3-velocity $\mathbf{v}(\mathbf{x}, t)$, pressure $p \ll \rho$ and entropy $s(\mathbf{x}, t)$. A perfect fluid element satisfies the following equations:

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{v}\rho) = 0\,; \tag{2.30}$$

Euler equation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho}\nabla p + \nabla \phi = 0\,; \tag{2.31}$$

Poisson equation:

$$\nabla^2 \phi - 4\pi G\rho = 0\,; \tag{2.32}$$

Entropy conservation:

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0\,, \tag{2.33}$$

where ϕ is the gravitational potential.
The easiest possible solution is the static one: $\rho = \rho_0$, $\mathbf{v} = 0$, $s = s_0$, $p = p_0$ and $\nabla\phi = 0$, where this last constraint requires $\rho_0 = 0$. So, a dinamical Universe is a natural consequence of having an average energy density, naturally arising also in Newtonian theory. Its expansion can be described by the scale factor $a(t)$, relating the physical, Eulerian coordinate $\mathbf{r}$ to the Lagrangian one $\mathbf{x}$, comoving with the fluid element, through:

$$\mathbf{r} = a(t)\mathbf{x}\,.$$

If we assume as a dinamical background a homogeneous and isotropic Universe, the background energy density will be a function of time only: $\rho_0 = \rho_0(t)$. Plugging it into the continuity equation, it evolves like:

$$\dot{\rho}_0 + 3H\rho_0 = 0\,,$$

where the dot indicates a derivative with respect to time. The velocity of the fluid element is:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{\dot{a}}{a}\mathbf{r} + a\,\dot{\mathbf{x}} = \mathbf{v}_0 + \delta\mathbf{v}\,.$$

It is convenient to consider derivatives with respect to the comoving coordinate $\mathbf{x}$. The derivative in space simply becomes:

$$\nabla_{\mathbf{r}} = \frac{\nabla_{\mathbf{x}}}{a}\,; \tag{2.34}$$

the time derivative is derived comparing the partial derivative at constant $\mathbf{x}$ to the one at constant $\mathbf{r}$:

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)_{\mathbf{r}} &= \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{r}} \cdot \nabla_{\mathbf{x}} = \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial a^{-1}(t)\mathbf{r}}{\partial t}\right)_{\mathbf{r}} \cdot \nabla_{\mathbf{x}} \\ &= \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} - H\mathbf{x}\cdot\nabla_{\mathbf{x}} = \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} - \dot{a}\mathbf{x}\cdot\nabla_{\mathbf{r}} = \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} - \mathbf{v}_0\cdot\nabla_{\mathbf{r}}\,.\end{aligned} \tag{2.35}$$

We can consider small perturbations with respect to the background (stopping at linear order):

$$\rho = \rho_0 + \delta\rho; \quad \mathbf{v} = \mathbf{v}_0 + \delta\mathbf{v}; \quad \phi = \phi_0 + \varphi; \quad s = s_0; \quad p = p_0 + \delta p = p_0 + c_s^2\delta\rho;$$

where the pressure perturbation $\delta p = c_s^2\delta\rho + \frac{\partial p}{\partial s}\delta s = c_s^2\delta\rho$ reduces just to the first term, as we neglect entropy perturbations. Inserting the perturbed variables into the equations (2.30) - (2.32), we obtain a set of linearized equations:

Continuity equation:

$$\left(\frac{\partial \delta\rho}{\partial t}\right)_{\mathbf{r}} + \rho_0 \, \nabla_{\mathbf{r}} \cdot \delta\mathbf{v} + \nabla_{\mathbf{r}}(\delta\rho \cdot \mathbf{v}_0) = 0\,; \tag{2.36}$$

Euler equation:

$$\left(\frac{\partial \delta\mathbf{v}}{\partial t}\right)_{\mathbf{r}} + (\mathbf{v}_0 \cdot \nabla_{\mathbf{r}})\,\delta\mathbf{v} + (\delta\mathbf{v} \cdot \nabla_{\mathbf{r}})\mathbf{v}_0 + \frac{c_s^2}{\rho_0}\,\nabla_{\mathbf{r}}\,\delta\rho + \nabla_{\mathbf{r}}\,\varphi = 0\,; \tag{2.37}$$

Poisson equation:

$$\nabla^2_{\mathbf{r}}\,\varphi - 4\pi G\delta\rho = 0\,. \tag{2.38}$$

Then, inserting the definitions (2.34) and (2.35) to have spatial derivatives with respect to $\mathbf{x}$ and time derivatives at constant $\mathbf{x}$, after few passages [13, 22] one gets:

Continuity equation:

$$\left(\frac{\partial \delta}{\partial t}\right)_{\mathbf{x}} + \frac{1}{a}\,\nabla_{\mathbf{x}} \cdot \delta\mathbf{v} = 0\,; \tag{2.39}$$

Euler equation:

$$\left(\frac{\partial \delta\mathbf{v}}{\partial t}\right)_{\mathbf{x}} + H\delta\mathbf{v} + \frac{c_s^2}{a}\,\nabla_{\mathbf{x}}\,\delta + \frac{\nabla_{\mathbf{x}}}{a}\,\varphi = 0\,; \tag{2.40}$$

Poisson equation:

$$\nabla^2_{\mathbf{x}}\,\varphi - 4\pi G a^2 \rho_0 \delta = 0\,; \tag{2.41}$$

where δ is the density contrast, defined as $\delta = \frac{\delta\rho}{\rho_0}$.

Taking the divergence of the Euler equation and combining it with the continuity and the Poisson equation, we get an evolution equation for the linear density contrast:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{c_s^2}{a^2}\,\nabla^2_{\mathbf{x}}\,\delta - 4\pi G\rho_0\delta = 0\,, \tag{2.42}$$

that, considering a pressureless fluid $p = 0 \rightarrow c_s^2 = 0$, becomes:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_0\delta = 0\,. \tag{2.43}$$

Expressing equation (2.42) in the Fourier space:

$$\ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} + \left(\frac{c_s^2}{a^2}\,k^2 - 4\pi G\rho_0\right)\delta_{\mathbf{k}} = 0\,, \tag{2.44}$$

we can define as a critical lengthscale the Jeans length:

$$\lambda_J = \frac{2\pi}{k_J} = \frac{c_s}{a}\sqrt{\frac{\pi}{G\rho_0}}\,,$$

where the physical wavelength would be $\lambda_J^{phys} = a\lambda_J$. As in a flat, matter dominated Universe $\rho_0 = (6\pi G t^2)^{-1}$, the physical Jeans length would be:

$$\lambda_J^{phys} \simeq c_s t\,,$$

approximately the sound horizon. If the lengthscale of the perturbation is smaller than the Jeans scale: $\lambda \ll \lambda_J^{phys} \to k \gg k_J$, the gravitational term can be neglected and the solutions are oscillating functions: the pressure opposes to the gravitational instability process.

If the lengthscale is larger $\lambda \gg \lambda_J^{phys} \to k \ll k_J$, the pressure term is the one to be neglected. Considering the case of a flat, matter-dominated Universe (our model for the following sections), where $a \propto t^{2/3}$ and $H = \frac{2}{3t}$, equation (2.43) becomes:

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0\,. \tag{2.45}$$

The solution of this differential equation is a power-law $\delta \propto t^\alpha$, that, plugged into (2.45), gives the straightforward result:

$$\delta = c_1 t^{2/3} + c_2 t^{-1}\,. \tag{2.46}$$

The growing mode $\delta \propto t^{2/3}$ gives the rate of growth of density perturbations. It is not the most efficient growth rate: in the case of a static Universe, this rate would be exponential [13, 22, 23]. It is easy to understand that this weaker rate of perturbation growth is due to the expansion of the Universe, which reduces the accretion mechanism.

The fact that vector perturbations are negligible can be understood by looking at the Euler equation (2.40): in the absence of gravitational and pressure perturbations, their evolution becomes:

$$\dot{\delta\mathbf{v}} + H\delta\mathbf{v} \simeq 0\,,$$

from which $\delta\mathbf{v} \propto a^{-1}$. So, linear vector perturbations are decreasing with time and can be neglected.

3 Gauge choice and gauge transformations

In a complete relativistic setup, dealing with spacetime perturbations means considering small deviations with respect to an unperturbed background, which in our case is the flat FRW Universe. Any perturbation ΔT of the quantity T (e.g. a tensor field) is defined as the difference between the background quantity T_0 and the value T assumed on the perturbed, physical spacetime. Since T and T_0 are defined in two different spacetimes, for differential geometry we cannot compare them: the difference between the two quantities has to be evaluated at the same point. So we need a one-to-one map between the background and the physical spacetime, to "transport" the physical value T to the unperturbed spacetime and compare it with T_0. The choice of a particular map is the "gauge choice". The problem about the gauge choice is that the value of T at any spacetime point in general changes with the change of gauge: thus, also the perturbation ΔT is not (in general) a gauge invariant quantity. So, it can sometimes be useful to define a particular gauge simplifying our problem, or otherwise to define some gauge invariant quantities. For our purposes, it is certainly convenient to deepen the discussion about the gauge issue and gauge transformations.

We can see the background and the physical, perturbed spacetime as distinct manifolds $\mathcal{M}_0$ and $\mathcal{M}_\lambda$ (a one-parameter family of manifolds), where λ is the order of the perturbation.
The choice of a one-to-one correspondence (a one-parameter function of λ) between points of $\mathcal{M}_0$ and points of $\mathcal{M}_\lambda$ is a gauge choice. For example, assigning the coordinates x^μ to the background $\mathcal{M}_0$, those can be transported over $\mathcal{M}_\lambda$ through a map ψ_λ, defining the gauge. This map connects e.g. the point p on $\mathcal{M}_0$, with coordinates $x^\mu(p)$, to the point $O = \psi_\lambda(p)$ on $\mathcal{M}_\lambda$; however, there could be a different map φ_λ connecting the point O to another point q on the background, with coordinates $\tilde{x}^\mu(q)$: $O = \psi_\lambda(p) = \varphi_\lambda(q)$. The gauge transformation is this change of correspondence, where a point on $\mathcal{M}_\lambda$ is associated to different points on $\mathcal{M}_0$ (keeping the point on $\mathcal{M}_\lambda$ fixed): so we can see it as a *one-to-one correspondence between different points in the background*. In fact, as $q = \varphi_\lambda^{-1}(O)$ and $O = \psi_\lambda(p)$, we end up with $q = \varphi_\lambda^{-1}(\psi_\lambda(p)) := \Phi_\lambda(p)$. Thus, we have that the coordinates of q are a one-parameter function of those of p: $\tilde{x}^\mu(q) = \Phi_\lambda^\mu(x^\alpha(p))$.

This transformation can be seen as an "active coordinate transformation", in which a coordinate system moves one point to another, or as a "passive coordinate transformation", a simple relabeling of coordinates to each point.
Suppose a coordinate system x^μ on a manifold $\mathcal{M}$ and a vector field ξ such that $\xi^\mu = dx^\mu/d\lambda$. ξ generates a congruence of curves $x^\mu(\lambda)$, where λ is the parameter

along the congruence. It can be defined a point p lying on one of these curves, associated to $\lambda = 0$, and a point q at a distance λ from p. So, the coordinate of the point q is (eliminating the dependence on the specific points):

$$\tilde{x}^{\mu}(\lambda) = x^{\mu} + \lambda\,\xi^{\mu} + ... \tag{3.1}$$

This is the "active" approach. The "passive" approach consists in defining a new coordinate system y^{μ} on $\mathcal{M}$ such that:

$$y^{\mu}(q) := x^{\mu}(p) = x^{\mu}(q) - \lambda\,\xi^{\mu}(x(p)) + ... \simeq x^{\mu}(q) - \lambda\,\xi^{\mu}(x(q)) + ... \tag{3.2}$$

at first-order in λ.

We now consider a vector field Z on $\mathcal{M}$ with components Z^{μ} in the x-coordinate system. Once the relation (3.1) between points has been established, a new vector field with components $\tilde{Z}^{\mu}$ in the x-coordinates can be defined, such that, at the point $x^{\mu}(p)$, they are equal to the components Z'^{μ} that Z has in the y-coordinates, at the point $y^{\mu}(q)$:

$$\tilde{Z}^{\mu}(x(p)) := Z'^{\mu}(y(q)) = \left(\frac{\partial y^{\mu}}{\partial x^{\nu}}\right)_{x(q)} Z^{\nu}(x(q)). \tag{3.3}$$

Substituting the equation (3.2) into (3.3) and then expanding the RHS at first-order in λ about $x(p)$ one obtains:

$$\begin{aligned} \tilde{Z}^{\mu}(\lambda) &= Z^{\mu} + \lambda\,\mathcal{L}_{\xi}Z^{\mu} + ... \\ \mathcal{L}_{\xi}Z^{\mu} &:= Z^{\mu}_{,\nu}\,\xi^{\nu} - \xi^{\mu}_{,\nu}\,Z^{\nu} \end{aligned} \tag{3.4}$$

where ", ν" $= \frac{\partial}{\partial x^{\nu}}$, the dependence on the point p has been omitted and $\mathcal{L}_{\xi}$ can be defined as the Lie derivative[1] with respect to the vector field ξ^{μ}, in the limit $\lambda \to 0$. So the vector $\tilde{Z}^{\mu}(\lambda)$ (the pullback of Z from q to p) is defined at the same point as the original vector Z^{μ} and they can be compared.

This can also be extended to higher order, as (3.1) is the first-order solution of the differential equation $\xi^{\mu} = dx^{\mu}/d\lambda$. Its exact solution at second-order would be (eliminating the dependence on the points):

$$\tilde{x}^{\mu}(\lambda) = x^{\mu} + \lambda\,\xi^{\mu} + \frac{\lambda^2}{2}\,\xi^{\mu}_{,\nu}\,\xi^{\nu} + ... = \exp[\lambda\mathcal{L}_{\xi}]x^{\mu}, \tag{3.5}$$

[1]The Lie derivative of a scalar is: $\mathcal{L}_{\xi}f = f_{,\mu}\,\xi^{\mu}$, for a contravariant vector: $\mathcal{L}_{\xi}Z^{\mu} = Z^{\mu}_{,\nu}\,\xi^{\nu} - \xi^{\mu}_{,\nu}\,Z^{\nu}$, for a covariant tensor: $\mathcal{L}_{\xi}T_{\mu\nu} = T_{\mu\nu,\sigma}\,\xi^{\sigma} + \xi^{\sigma}_{,\mu}\,T_{\sigma\nu} + \xi^{\sigma}_{,\nu}\,T_{\mu\sigma}$.

where $d^2x^\mu/d\lambda^2 = \xi^\mu_{,\nu}\xi^\nu$. From the "passive" approach, using the definition $y^\mu(q) = x^\mu(p)$, the equation (3.5), expanding all terms about $x(q)$ and omitting the $x(q)$ dependence, one obtains:

$$y^\mu(\lambda) = x^\mu - \lambda\,\xi^\mu + \frac{\lambda^2}{2}\,\xi^\mu_{,\nu}\xi^\nu + \dots . \tag{3.6}$$

Using equation (3.6) into equation (3.3), expanding all terms about $x(p)$ and omitting the dependence on the point, we get the pullback $\tilde{Z}^\mu(\lambda)$ at second-order :

$$\tilde{Z}^\mu(\lambda) = \exp[\lambda\mathcal{L}_\xi]Z^\mu = Z^\mu + \lambda\,\mathcal{L}_\xi Z^\mu + \frac{\lambda^2}{2}\,\mathcal{L}^2_\xi Z^\mu + \dots . \tag{3.7}$$

Finally, one can generalize the equation (3.3) to a generic tensor of type (p, q) using the right number of transformation matrices, to obtain the pullback $\tilde{T}$:

$$\tilde{T}^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q}(x(p)) := T'^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q}(y(q)) = \left[\frac{\partial y^{\mu_1}}{\partial x^{\rho_1}}\dots\frac{\partial y^{\mu_p}}{\partial x^{\rho_p}}\frac{\partial x^{\sigma_1}}{\partial y^{\nu_1}}\dots\frac{\partial x^{\sigma_q}}{\partial y^{\nu_q}}\right]_{x(q)} T^{\rho_1\dots\rho_p}_{\sigma_1\dots\sigma_q}(x(q)). \tag{3.8}$$

Using equation (3.5) like before:

$$\tilde{T}(\lambda) = T + \lambda\,\mathcal{L}_\xi T + \frac{\lambda^2}{2}\,\mathcal{L}^2_\xi T + \dots . \tag{3.9}$$

This represents the expression for the pullback $\tilde{T}(\lambda)$ for a one-parameter group of transformations. Through this, it is possible to generalize the definition of Lie derivative:

$$\mathcal{L}_\xi T := \left[\frac{d}{d\lambda}\right]_{\lambda=0}\tilde{T}(\lambda) = \lim_{\lambda\to 0}\frac{1}{\lambda}[\tilde{T}(\lambda) - T]. \tag{3.10}$$

Gauge transformations do not form a one-parameter group, but a one-parameter family of transformations. The action of a one-parameter family of transformations can be given by the successive action of one-parameter groups, becoming evident at non-linear order. This means that a vector field $\xi_{(k)}$ has to be introduced, with parameter $\lambda_{(k)}$, associated to the kth one-parameter group of transformations. At a given order n of the expansion, a number $k = 1, .., n$ of such vector fields has to be introduced. At second order the tranformation becomes:

$$\tilde{x}^\mu(\lambda) = x^\mu + \lambda\,\xi^\mu_{(1)} + \frac{\lambda^2}{2}\,(\xi^\mu_{(1),\nu}\xi^\nu_{(1)} + \xi^\mu_{(2)})\dots \tag{3.11}$$

where $\lambda_{(1)} = \lambda$, $\lambda_{(2)} = \lambda^2/2$. Given this transformation, equation (3.11) can be used to define the y coordinates:

$$y^\mu(q) := x^\mu(p) = x^\mu(q) - \lambda\,\xi^\mu_{(1)}(x(p)) - \frac{\lambda^2}{2}\,[\xi^\mu_{(1),\nu}(x(p))\,\xi^\nu_{(1)}(x(p)) + \xi^\mu_{(2)}(x(p))] + \ldots\,. \tag{3.12}$$

Expanding as usual the RHS about q and neglecting the specific point p, we get:

$$y^\mu(\lambda) = x^\mu - \lambda\,\xi^\mu_{(1)} + \frac{\lambda^2}{2}\,(\xi^\mu_{(1),\nu}\,\xi^\nu_{(1)} + \xi^\mu_{(2)})\ldots\,. \tag{3.13}$$

Finally, the pullback for a generic tensor T can be derived substituting equation (3.13) into equation (3.8):

$$\tilde{T}(\lambda) = T + \lambda\,\mathcal{L}_{\xi_{(1)}}T + \frac{\lambda^2}{2}\,(\mathcal{L}^2_{\xi_{(1)}} + \mathcal{L}_{\xi_{(2)}})T + \ldots\,. \tag{3.14}$$

Coming back to the gauge problem for perturbations: consider the tensor field T_λ on each $\mathcal{M}_\lambda$ (where T_0 is the unperturbed tensor field, T_λ is the perturbed one, at order λ). We can choose two different gauges ψ_λ and φ_λ to represent T_λ on $\mathcal{M}_0$: they can be called $T(\lambda)$ and $\tilde{T}(\lambda)$ respectively. They are defined in such a way to have the same components of T_λ in their particular gauge. On the other hand, we have a relation between $T(\lambda)$ and $\tilde{T}(\lambda)$ given by Φ_λ, relating their components through the transformation (3.8). Now, in each gauge we have a field representing T_λ on $\mathcal{M}_0$, so those fields can be compared to the unperturbed one T_0 in order to define the perturbation. We would have $\Delta T_\lambda = T_\lambda - T_0$ in the first gauge and $\Delta\tilde{T}_\lambda = \tilde{T}_\lambda - T_0$ in the second one: in general, they are not equal.

Since we can expand the fields in the new gauges as:

$$T(\lambda) = T_0 + \lambda\,\delta T + \frac{\lambda^2}{2}\,\delta^2 T + \mathcal{O}(\lambda^3), \tag{3.15}$$

$$\tilde{T}(\lambda) = T_0 + \lambda\,\delta\tilde{T} + \frac{\lambda^2}{2}\,\delta^2\tilde{T} + \mathcal{O}(\lambda^3), \tag{3.16}$$

substituting those expansions in equation (3.14), one finally obtains the gauge transformations for the perturbations at first and second-order:

$$\delta\tilde{T} = \delta T + \mathcal{L}_{\xi_{(1)}}T_0, \tag{3.17}$$

$$\delta^2\tilde{T} = \delta^2 T + 2\mathcal{L}_{\xi_{(1)}}\delta T + \mathcal{L}^2_{\xi_{(1)}}T_0 + \mathcal{L}_{\xi_{(2)}}T_0. \tag{3.18}$$

Equation (3.18) shows that there can be special second-order transformations only due to the second-order generator ξ_2, in the case $\xi_1 = 0$. On the other hand, a non-vanishing ξ_1 always affects both first-order and second-order transformations, inducing an effect of δT on $\delta^2 \tilde{T}$.

4 Perturbed flat FRW Universe

We consider perturbations with respect to a homogeneous and isotropic background, in a flat, matter dominated Universe, described by the FRW metric: $ds^2 = a^2(\tau)(-d\tau^2 + dx^2)$, where $a(\tau)$ is the scale factor and τ is the conformal time. Those metric perturbations can be defined separating them in scalars, vectors and tensors (STV-decomposition [12]):

$$
\begin{aligned}
g_{00} &= -a^2(\tau)\left(1 + 2\sum_{r=1}^{+\infty}\frac{1}{r!}\psi^{(r)}\right) \\
g_{0i} &= a^2(\tau)\sum_{r=1}^{+\infty}\frac{1}{r!}\omega_i^{(r)} \\
g_{ij} &= a^2(\tau)\left\{\left[1 - 2\left(\sum_{r=1}^{+\infty}\frac{1}{r!}\phi^{(r)}\right)\right]\delta_{ij} + \sum_{r=1}^{+\infty}\frac{1}{r!}\chi_{ij}^{(r)}\right\},
\end{aligned}
\tag{4.1}
$$

where $\chi_i^{(r)\,i} = 0$, so $\chi_{ij}^{(r)}$ is the traceless part of the spatial metric perturbation. The index (r) represents the rth-order of the perturbation.
The scalar (or longitudinal) parts are related to a scalar potential, the vector parts to transverse (divergence-free or solenoidal) vector fields and the tensor parts to transverse, trace-free tensors. The shift $\omega_i^{(r)}$ can be decomposed, as any vector, in the sum of an irrotational and a divergence-free vector:

$$\omega_i^{(r)} = \partial_i\,\omega^{(r)\parallel} + \omega_i^{(r)\perp}$$

where $\partial^i\,\omega_i^{(r)\perp} = 0$. Similarly, the traceless part of the spatial metric can be decomposed in the following way:

$$\chi_{ij}^{(r)} = \mathrm{D}_{ij}\chi^{(r)\parallel} + \partial_i\chi_j^{(r)\perp} + \partial_j\chi_i^{(r)\perp} + \chi_{ij}^{(r)\mathrm{T}},$$

where $\chi^{(r)\parallel}$ is a scalar, $\chi_i^{(r)\perp}$ is a solenoidal vector field, $\partial^i\chi_{ij}^{(r)\mathrm{T}} = 0$, and $\mathrm{D}_{ij} = \partial_i\partial_j - \frac{1}{3}\,\delta_{ij}\nabla^2$.

Also the energy density ρ and the four velocity u^μ can be expanded:

$$
\begin{aligned}
\rho &= \rho_{(0)} + \sum_{r=1}^{+\infty}\frac{1}{r!}\delta^r\rho \\
u^\mu &= \frac{1}{a}\left(\delta_0^\mu + \sum_{r=1}^{+\infty}\frac{1}{r!}v_{(r)}^\mu\right),
\end{aligned}
$$

where u^μ is subjected to the normalization $u^\mu u^\nu g_{\mu\nu} = -1$. From this normalization condition, at any order the time component $v^0_{(r)}$ is related to the lapse perturbation $\psi^{(r)}$. Also the velocity perturbation can be split into a scalar and a solenoidal vector part:

$$v^i_{(r)} = \partial^i v^{\|}_{(r)} + v^\mu_{(r)\perp}.$$

The generators of the gauge transformations are the vectors $\xi_{(r)}$, which can be split into a time and a space part:

$$\begin{aligned} \xi^0_{(r)} &= \alpha^{(r)} \\ \xi^i_{(r)} &= \partial^i \beta^{(r)} + d^{(r)\,i}, \end{aligned}$$

with $\partial_i d^{(r)\,i} = 0$.
As this generator is determined by two scalars and one vector, a gauge is defined by the constraints on e.g. two scalars and one vector metric perturbations, or on the density, on the velocity...

We know that General Relativity is invariant under coordinate transformations, but not under gauge transformations. To solve the Einstein equation one can choose a particular gauge: two very useful gauges in Cosmology are the comoving synchronous gauge and the Poisson gauge.

5 Evolution in the synchronous gauge

The synchronous gauge is defined by the conditions $g_{00} = -a^2(\tau)$, $g_{0i} = 0$ (corresponding to the two scalars and one vector constraints: $\psi_{(r)} = 0$, $\omega^{(r)\parallel} = 0$ and $\omega_i^{(r)\perp} = 0$). It is called this way as the proper time of an observer at fixed spatial coordinates is equal to the cosmic time in FRW: $-ds^2 = a^2(\tau)\, d\tau^2 = dt^2$. We consider an Einstein-de Sitter Universe dominated by a perfect fluid of irrotational dust, in synchronous and comoving coordinates. The line element can be written as:

$$ds^2 = a^2(\tau)[-d\tau^2 + \gamma_{ij}(\mathbf{x}, \tau)\, dx^i dx^j],$$

where $\mathbf{x}$ are the Lagrangian coordinates of the fluid elements. We indicate spacetime indices with greek letters, spatial indices with latin letters, and $' = \frac{\partial}{\partial \tau}$. The scale factor in this case evolves like $a(\tau) \propto \tau^2$.

It is useful to introduce the concept of extrinsic curvature. While the Riemann tensor measures the intrinsic curvature of space, the extrinsic curvature depends on how the space is embedded in a larger one. In our case, we consider a spatial hypersurface Σ, its normal vector n^μ and define the projection tensor $P_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$, projecting any vector of the space cotangent to Σ onto its tangent space. It also acts like the "spatial metric" for vectors tangent to the hypersurface [2].
If we think to extend the normal vector field n^μ, the extrinsic curvature is defined as the Lie derivative of the projection tensor (the spatial metric if Σ is spacelike) along the normal vector field, expressing the rate of change of the hypersurface metric as we move orthogonally away from Σ: $\theta_{\mu\nu} = \frac{1}{2}\mathcal{L}_n P_{\mu\nu}$.
In our case, the normal vector field would be timelike $n^\mu = \delta^\mu_0$, and the spatial metric is γ_{ij}, so the extrinsic curvature is

$$\theta_{ij} = \frac{1}{2}\mathcal{L}_n \gamma_{ij} = \frac{1}{2}(\gamma_{ij,\sigma}\, n^\sigma + 2n^\sigma_{,i}\gamma_{\sigma j}) = \frac{1}{2}(\gamma_{ij,\sigma}\, n^\sigma) = \frac{1}{2}\gamma'_{ij},$$

and

$$\theta^i_j = \frac{1}{2}\gamma^{ik}\,\gamma'_{kj},$$

as the spatial metric γ_{ij} would raise and lower the indices in the spatial hypersurface.

An advantage of this gauge is that the Einstien equations can be expressed in terms of geometric quantities only. Expressing the Christoffel symbols and the conformal Ricci tensor of the spatial hypersurface $\mathcal{R}^i_j$ with respect to the spatial metric γ_{ij} (appendix B in [1]), one can express the Einstein equations with respect to the extrinsic curvature, after having subtracted the background contribution. The energy constraint is:

$$\theta^2 - \theta^i_j \theta^j_i + \frac{8}{\tau}\theta + \mathcal{R} = \frac{24}{\tau^2}\delta \tag{5.1}$$

where $\mathcal{R} = \mathcal{R}^i_i$ is the conformal Ricci scalar and $\delta \equiv \frac{\rho - \rho_{(0)}}{\rho_{(0)}}$ is the density contrast, with $\rho(\mathbf{x}, \tau)$ the mass density of the fluid and $\rho_{(0)}(\tau) = \frac{3}{2\pi G a^2(\tau)\tau^2}$ its background mean value.
The momentum constraints is:

$$\theta^i_{j;i} = \theta_{,j}, \tag{5.2}$$

where "; i" represents the covariant derivative with respect to i.
Replacing the density from the energy constraints and subtracting the background contribution, the evolution equation becomes:

$$\theta^{i\prime}_{\ j} + \frac{4}{\tau}\theta^i_j + \theta^i_j\theta + \frac{1}{4}(\theta^k_l\theta^l_k - \theta^2)\,\delta^i_j + \mathcal{R}^i_j - \frac{1}{4}\mathcal{R}\delta^i_j = 0 \tag{5.3}$$

Considering the vector n^μ normal to the spatial hypersurfaces as the tangent to a timelike geodesic, the extrinsic curvature $\theta_{\mu\nu}$ would be also equal to the covariant derivative with respect to n^μ: $\theta_{\mu\nu} = D_\mu n_\nu$, describing the extent to which neighbouring geodesics deviate from remaning parallel [2]. So, the scalar θ expresses the peculiar volume expansion, whose evolution is described through Raychaudhuri equation:

$$\theta' + \theta^i_j\theta^j_i + \frac{2}{\tau}\theta + \frac{6}{\tau^2}\delta = 0. \tag{5.4}$$

Also the density contrast can be expressed in terms of the extrinsic curvature by solving the continuity equation $\dot{\rho} = -\theta\rho$, which is equivalent to $\delta' + (1+\delta)\,\theta = 0$ [19]. The result is:

$$\delta(\mathbf{x}, \tau) = (1 + \delta_0(\mathbf{x}))[\gamma(\mathbf{x}, \tau)/\gamma_0(\mathbf{x})]^{-1/2} - 1, \tag{5.5}$$

where $\gamma = \det\gamma_{ij}$ and the subscript 0 represents the quantity at the initial condition.

5.1 First-order perturbations

From the previous equations we can get the evolution of the first-order perturbations. Expanding the conformal spatial metric tensor we get at linear order $\gamma_{ij} = \delta_{ij} + \gamma^{(1)}_{Sij}$, where, according to the general definition

$$\gamma^{(1)}_{Sij} = -2\phi^{(1)}_S\delta_{ij} + D_{ij}\chi^{(1)\parallel}_S + \partial_i\chi^{(1)\perp}_{Sj} + \partial_j\chi^{(1)\perp}_{Si} + \chi^{(1)T}_{ij},$$

with $\partial^i \chi_{Si}^{(1)\perp} = \chi_i^{(1)\mathrm{T}i} = \partial^i \chi_{ij}^{(1)\mathrm{T}} = 0$ and the subscrit S indicates the gauge (not present for the first-order tensor mode, which is gauge invariant).

Linearizing the traceless part of the evolution equation (5.3), we get the equation of motion for the first-order tensor mode:

$$\chi_{ij}^{(1)\mathrm{T}\,\prime\prime} + \frac{4}{\tau}\chi_{ij}^{(1)\mathrm{T}\,\prime} - \nabla^2\chi_{ij}^{(1)\mathrm{T}} = 0, \tag{5.6}$$

that is the equation for the propagation of gravitational waves in the Einstein-de Sitter Universe (where $2\mathcal{H} = 4/\tau$, $\mathcal{H}$ being the Hubble parameter in conformal time). The general solution, expanded in Fourier space, is

$$\chi_{ij}^{(1)\mathrm{T}}(\mathbf{x},\tau) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,\exp(i\mathbf{k}\cdot\mathbf{x})\,\chi_\sigma^{(1)}(\mathbf{k},\tau)\,\epsilon_{ij}^\sigma(\hat{\mathbf{k}}), \tag{5.7}$$

where $\epsilon_{ij}^\sigma(\hat{\mathbf{k}})$ is the polarization tensor ($\sigma = +,\times$) and $\chi_\sigma^{(1)}(\mathbf{k},\tau)$ is the amplitude of the polarization states, which is an oscillating function:

$$\chi_\sigma^{(1)}(\mathbf{k},\tau) = A(k)a_\sigma(\mathbf{k})\left(\frac{3j_1(k\tau)}{k\tau}\right), \tag{5.8}$$

with j_1 spherical Bessel function of order 1 and $a_\sigma(\mathbf{k})$ a random variable with autocorrelation function $\langle a_\sigma(\mathbf{k})a_{\sigma'}(\mathbf{k}')\rangle = (2\pi)^3\,k^{-3}\delta^3(\mathbf{k}+\mathbf{k}')\delta_{\sigma\sigma'}$. The spectrum of gravitational waves background $A(k)$ depends on the model for its production: in most inflationary models, it is a nearly scale invariant spectrum, depending on $\mathcal{H}$ during inflation. It would also give an important information about the energy scale of inflation, allowing to discriminate among different models.

As we are dealing with an irrotational fluid, at linear order the gauge modes for vector perturbations can be set to zero: $\chi_i^{(1)\perp} = 0$.

The remaining two scalar modes are related by the momentum constraint (5.2), giving:

$$\phi_\mathrm{S}^{(1)} + \frac{1}{6}\nabla^2\chi_\mathrm{S}^{(1)\parallel} = \phi_\mathrm{S0}^{(1)} + \frac{1}{6}\nabla^2\chi_\mathrm{S0}^{(1)\parallel}. \tag{5.9}$$

From the energy constraint (5.1):

$$\nabla^2\left[\frac{2}{\tau}\chi_\mathrm{S}^{(1)\parallel\,\prime} + \frac{6}{\tau^2}\left(\chi_\mathrm{S}^{(1)\parallel} - \chi_\mathrm{S0}^{(1)\parallel}\right) + 2\phi_\mathrm{S0}^{(1)} + \frac{1}{3}\nabla^2\chi_\mathrm{S0}^{(1)\parallel}\right] = \frac{12}{\tau^2}\,\delta_0, \tag{5.10}$$

and from the trace part of the evolution equation:

$$\chi_S^{(1)\parallel\,\prime\prime} + \frac{4}{\tau}\chi_S^{(1)\parallel\,\prime} + \frac{1}{3}\nabla^2\chi_S^{(1)\parallel} = -2\phi_S^{(1)}. \tag{5.11}$$

Combining the equations (5.10) and (5.11), we obtain an equation only for the scalar mode $\chi_S^{(1)\parallel}$:

$$\nabla^2\left[\chi_S^{(1)\parallel\,\prime\prime} + \frac{2}{\tau}\chi_S^{(1)\parallel\,\prime} - \frac{6}{\tau^2}\left(\chi_S^{(1)\parallel} - \chi_{S0}^{(1)\parallel}\right)\right] = -\frac{12}{\tau^2}\,\delta_0. \tag{5.12}$$

To obtain an equation for the density contrast, one linearizes the solution of the continuity equation:

$$\delta_S^{(1)} = \delta_0 - \frac{1}{2}\,\nabla^2(\chi_S^{(1)\parallel} - \chi_{S0}^{(1)\parallel}); \tag{5.13}$$

deriving this expression and inserting it in equation (5.12) we obtain:

$$\delta_S^{(1)\,\prime\prime} + \frac{2}{\tau}\,\delta_S^{(1)\,\prime} - \frac{6}{\tau^2}\,\delta_S^{(1)} = 0. \tag{5.14}$$

The solution to this equation is a simple power-law. The residual gauge ambiguity of the synchronous gauge can be used to simplify the previous equations: fixing $\nabla^2\chi_{S0}^{(1)\parallel} = -2\delta_0$, the equation (5.12) for $\chi_S^{(1)\parallel}$ assumes the same form of (5.14), and its solution would be:

$$\chi_S^{(1)\parallel}(\mathbf{x},\tau) = \chi_+(\mathbf{x})\,\tau^2 + \chi_-(\mathbf{x})\,\tau^{-3},$$

where $\chi_\pm$ are the growing/decaying modes. From $\nabla^2\chi_{S0}^{(1)\parallel} = -2\delta_0$ and the Poisson equation $\nabla^2\varphi(\mathbf{x}) = 4\pi G a^2\rho_0\delta_0 = \frac{6}{\tau_0^2}\,\delta_0$ (where $\varphi(\mathbf{x})$ is the peculiar gravitational potential), considering only the growing mode: $\nabla^2\chi_{S0}^{(1)\parallel} = (\nabla^2\chi_+)\,\tau_0^2 = -2\delta_0 = -\frac{1}{3}\,\tau_0^2\,\nabla^2\varphi \;\rightarrow\; \chi_+ = -\frac{1}{3}\varphi$. Therefore,

$$\mathrm{D}_{ij}\chi_S^{(1)\parallel} = -\frac{\tau^2}{3}\left(\varphi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\varphi\right), \tag{5.15}$$

and using equation (5.11) we obtain the remaining scalar mode:

$$\phi_S^{(1)}(\mathbf{x},\tau) = \frac{5}{3}\,\varphi(\mathbf{x}) + \frac{\tau^2}{18}\nabla^2\varphi(\mathbf{x}). \tag{5.16}$$

Collecting those results, the linear metric perturbation becomes:

$$\gamma_{Sij}^{(1)} = -\frac{10}{3}\,\varphi\,\delta_{ij} - \frac{\tau^2}{3}\,\varphi_{,ij} + \chi_{ij}^{(1)\mathrm{T}}; \tag{5.17}$$

with growing mode only, the linear density contrast is

$$\delta_S^{(1)} = \frac{\tau^2}{6}\nabla^2\varphi. \tag{5.18}$$

5.2 Second-order perturbations

The results obtained at first-order are used to solve the second-order case. The conformal spatial metric tensor up to second-order is:

$$\gamma_{ij} = \delta_{ij} + \gamma^{(1)}_{Sij} + \frac{1}{2}\gamma^{(2)}_{Sij},$$

with

$$\gamma^{(2)}_{Sij} = -2\phi^{(2)}_{S}\,\delta_{ij} + \chi^{(2)}_{Sij}$$

and $\chi^{(2)\,i}_{Si} = 0$, traceless part. This expansion of the metric tensor is inserted in equations (5.1) - (5.4), to obtain expressions for $\phi^{(2)}_{S}$ and $\chi^{(2)}_{Sij}$ in terms of the initial peculiar gravitational potential φ and the linear tensor modes $\chi^{(1)\mathrm{T}}_{ij}$. These equations are reported in the Appendix A. They can be solved using for the initial conditions the simplifying assumption $\tau_0 = 0$, implying $\delta_0 = 0$. The trace part of the second-order metric tensor can be obtained from the Raychaudhuri equation, using also the energy constraint to obtain the subleading mode. The result is:

$$\phi^{(2)}_{S} = \frac{\tau^4}{252}\left(-\frac{10}{3}\varphi^{,ki}\varphi_{,kj} + (\nabla^2\varphi)^2 \right) + \frac{5\tau^2}{18}\left(\varphi^{,k}\varphi_{,k} + \frac{4}{3}\varphi\nabla^2\varphi \right) + \phi^{(2)}_{S(t)}, \qquad (5.19)$$

where $\phi^{(2)}_{S(t)}$ is the part of $\phi^{(2)}_{S}$ generated by combinations of linear tensor modes. It satisfies equation (B.1) and it can be solved through the method of variation of arbitrary constants: given the two solutions of the homogeneous part of the equation (B.1), $y_1 = \tau^{-3}$ and $y_2 = \tau^2$, and the Wronskian, defined as $W(\tau) = y_1 y_2' - y_2 y_1' = 5\tau^{-2}$, the general solution of the inhomogeneous equation is given by $y(\tau) = c_1 y_1(\tau) + c_2 y_2(\tau) + y_P(\tau)$, where $y_P(\tau)$ is:

$$y_P(\tau) = y_2(\tau)\int^{\tau} \frac{y_1(s)\mathcal{Q}(s)\,ds}{W(s)} - y_1(\tau)\int^{\tau} \frac{y_2(s)\mathcal{Q}(s)\,ds}{W(s)}.$$

$\mathcal{Q}(\mathbf{x},\tau)$ is the source term of the inhomogeneous differential equation (explicitly written in Appendix B). The solution of the differential equation, in our case, becomes:

$$\phi^{(2)}_{S(t)} = \frac{\tau^2}{5}\int_0^{\tau} \frac{d\tau'}{\tau'}\,\mathcal{Q}(\tau') - \frac{1}{5\tau^3}\int_0^{\tau} d\tau'\,\tau'^4\,\mathcal{Q}(\tau'), \qquad (5.20)$$

(where the constants c_1 and c_2 have been set to zero).

The result for $\chi^{(2)}_{Sij}$ is obtained replacing the expression for $\phi^{(2)}_{S}$ in the equations (A.2) - (A.4) and solving them in the order: energy constraint → momentum constraint

→ traceless part of the evolution equation. We get:

$$\chi^{(2)}_{Sij} = \frac{\tau^4}{126}\left(19\,\varphi^{,k}_{,i}\varphi_{,kj} - 12\,\varphi_{,ij}\,\nabla^2\varphi + 4\,(\nabla^2\varphi)^2\,\delta_{ij} - \frac{19}{3}\,\varphi^{,kl}\varphi_{,kl}\,\delta_{ij}\right) + \frac{5\tau^2}{9}\left(-6\,\varphi_{,i}\varphi_{,j} - 4\,\varphi\varphi_{,ij} + 2\,\varphi^{,k}\varphi_{,k}\,\delta_{ij} + \frac{4}{3}\,\varphi\nabla^2\varphi\,\delta_{ij}\right) + \pi_{Sij} + \chi^{(2)}_{S(t)ij}, \tag{5.21}$$

where $\chi^{(2)}_{S(t)ij}$ is the part of $\chi^{(2)}_{Sij}$ generated by combinations of linear tensor modes (it can be obtained solving the equations in Appendix B); π_{Sij} is the second-order transverse and traceless part, generated by scalar perturbations. It is determined by the wave equation:

$$\pi''_{Sij} + \frac{4}{\tau}\,\pi'_{Sij} - \nabla^2\,\pi_{Sij} = -\frac{\tau^4}{21}\nabla^2\,\mathcal{S}_{ij}, \tag{5.22}$$

whose source term is:

$$\mathcal{S}_{ij} = \nabla^2\,\Psi_0\,\delta_{ij} + \Psi_{0,ij} + 2(\varphi_{,ij}\nabla^2\,\varphi - \varphi^{,k}_{,i}\varphi_{,kj}), \tag{5.23}$$

with

$$\nabla^2\,\Psi_0 = -\frac{1}{2}[(\nabla^2\varphi)^2 - \varphi^{,kl}\varphi_{,kl}]. \tag{5.24}$$

The solution for π_{Sij} can be found through the Green method:

$$\pi_{Sij}(\mathrm{x},\tau) = \frac{\tau^4}{21}\mathcal{S}_{ij}(\mathrm{x}) + \frac{4\tau^2}{3}\mathcal{T}_{ij} + \tilde{\pi}_{ij}(\mathrm{x},\tau), \tag{5.25}$$

where $\nabla^2\mathcal{T}_{ij} = \mathcal{S}_{ij}$ and $\tilde{\pi}_{ij}$ satisfies the evolution equation (6.8). Its solution can be found with the method of variation of arbitrary constants described before, and its expression is:

$$\tilde{\pi}_{ij}(\mathrm{x},\tau) = \frac{1}{(2\pi)^3}\int d^3\mathbf{k}\,\exp(i\mathbf{k}\cdot\mathbf{x})\frac{40}{k^4}\mathcal{S}_{ij}(\mathbf{k})\left(\frac{1}{3} - \frac{j_1(k\tau)}{k\tau}\right), \tag{5.26}$$

with $S_{ij}(\mathbf{k}) = \int d^3\mathbf{x}\,\exp(-i\mathbf{k}\cdot\mathbf{x})\,\mathcal{S}_{ij}(\mathbf{x})$.
It represents the real gravitational wave contribution generated by linear scalar modes, with a term constant in time and another one oscillating with decreasing amplitude (like in the first order tensor mode (5.8)).

The synchronous gauge tensor mode (5.25) contains four terms: the first one, $\propto\ \tau^4$, represents a Newtonian contribution, describing the dynamical tidal induction acting from the environment on the fluid element; the term $\propto\ \tau^2$ is a post-Newtonian term; then there is a constant post-post-Newtonian term, required by the vanishing initial conditions and having no obvious observational effects, and, finally, a wave-like piece, which has the usual form as free cosmological gravitational waves [3].

The second-order density contrast in synchronous and comoving gauge is found from equation (5.5), expanding up to second-order the determinant of the metric (Appendix B in [1]). The result is [2]:

$$\begin{aligned}\delta_{\mathrm{S}}^{(2)} =& \frac{\tau^4}{252}[4\,\varphi^{,ij}\varphi_{,ij} + 10\,(\nabla^2\varphi)^2] + \frac{\tau^2}{18}(15\,\varphi^{,k}\varphi_{,k} + 40\,\varphi\nabla^2\varphi - 6\,\varphi^{,ij}\chi_{ij}^{(1)\mathrm{T}}) \\ &+ \frac{1}{2}(\chi^{(1)\mathrm{T}ij}\chi_{ij}^{(1)\mathrm{T}} - \chi_0^{(1)\mathrm{T}ij}\chi_{0ij}^{(1)\mathrm{T}}) + 3\,\phi_{\mathrm{S(t)}}^{(2)}.\end{aligned} \tag{5.27}$$

From equations (5.26) and (5.27) the consequence of the mixing between scalars and tensors (here we are neglecting vectors) at non-linear order is evident: we have a second-order wave-like tensor generated by combinations of scalars built up from the peculiar gravitational potential and a second-order density contrast which would be generated by combinations of linear tensor modes even in the absence of initial density fluctuations.

[2] This density contrast is twice the one found in equation (4.39) in [1], where a factor $\frac{1}{2}$ has been forgotten in the derivation (from $\delta = \delta^{(1)} + \frac{1}{2}\delta^{(2)}$, the whole $\frac{1}{2}\delta^{(2)}$ has been extracted, without removing the factor).

6 Evolution in the Poisson gauge

The Poisson gauge is defined by the two scalar conditions $\omega^{(r)\parallel} = 0$ and $\chi^{(r)\parallel} = 0$ and the vector condition $\chi_i^{(r)\perp} = 0$ (equivalent to the conditions $\omega_i^{(r),i} = \chi_{ij}^{(r),j} = 0$). This generalizes the longitudinal gauge ($\omega_i^{(r)} = \chi_{ij}^{(r)} = 0$), where only scalar modes are present. As vector and tensor fields are put to zero by hand, the longitudinal gauge is not useful to describe second-order evolution.
The Poisson gauge has an important physical interpretation, as the scalar perturbations of the metric are equivalent, in this gauge, to the gauge-invariant Bardeen potentials [4]. So, it is useful to transfer all the first- and second-order perturbations that have been solved in the synchronous and comoving gauge to this one, using the general gauge transformations (3.17) and (3.18). All the equations for the gauge transformation are presented in Appendix C.

6.1 First-order perturbations

First of all, one can obtain the parameters of the transformation at first-order, replacing the definition of $\chi_S^{(1)\parallel}$ (equation (5.15)) in (C.6), to obtain $\beta^{(1)}$; then using (C.3) to get $\alpha^{(1)}$:

$$\begin{aligned} \alpha^{(1)} &= \frac{\tau}{3}\varphi, \\ \beta^{(1)} &= \frac{\tau^2}{6}\varphi; \end{aligned} \tag{6.1}$$

we also have $d^{(1)i} = 0$ in the absence of initial vector modes.

The metric perturbations are obtained from equations (C.2), (C.4), (C.5) and (C.8):

$$\begin{aligned} \psi_P^{(1)} &= \phi_P^{(1)} = \varphi, \\ \chi_{P\,ij}^{(1)} &= \chi_{ij}^{(1)T}. \end{aligned} \tag{6.2}$$

This result shows the well-known equivalence of the scalar perturbations in the longitudinal gauge with the peculiar gravitational potential and the gauge invariance of the first-order tensor mode.

For the linear density contrast, from equation (C.10):

$$\delta_P^{(1)} = -2\varphi + \frac{\tau^2}{6}\nabla^2\varphi. \tag{6.3}$$

The linear four-velocity is given by equations (C.12), (C.13):

$$v_P^{(1)0} = -\varphi, \tag{6.4}$$

$$v_{\mathrm{P}}^{(1)\,i} = -\frac{\tau}{3}\,\varphi^{,i}. \tag{6.5}$$

6.2 Second-order perturbations

From the results just obtained at linear order and replacing the second-order metric perturbations in synchronous gauge (equations (5.19), (5.21)) in the expressions for the second-order gauge parameters (equations (C.25) - (C.27)), we can solve those equations to get these parameters:

$$\begin{aligned}
\alpha^{(2)} &= -\frac{2}{21}\,\tau^3\Psi_0 + \tau\left(\frac{10}{9}\,\varphi^2 + 4\,\Theta_0\right) + \alpha^{(2)}_{(\mathrm{t})},\\
\beta^{(2)} &= \frac{\tau^4}{6}\left(\frac{1}{12}\,\varphi^{,i}\,\varphi_{,i} - \frac{1}{7}\,\Psi_0\right) + \frac{\tau^2}{3}\left(\frac{7}{2}\,\varphi^2 + 6\,\Theta_0\right) + \beta^{(2)}_{(\mathrm{t})},\\
\nabla^2 d^{(2)}_j &= \frac{4\,\tau^2}{3}\,(-\varphi_{,j}\,\nabla^2\varphi + \varphi^{,i}\,\varphi_{,ij} - 2\,\Psi_{0,j}) + \nabla^2 d^{(2)}_{(\mathrm{t})\,j},
\end{aligned} \tag{6.6}$$

where $\nabla^2\,\Theta_0 = \Psi_0 - \frac{1}{3}\,\varphi^{,i}\,\varphi_{,i}$, and the subscript (t) represents the piece generated by combinations of linear tensor modes (Appendix D).

From equations (C.15) - (C.18) we obtain the second-order metric perturbations in the Poisson gauge:

$$\begin{aligned}
\psi^{(2)}_{\mathrm{P}} &= \tau^2\left(\frac{1}{6}\,\varphi^{,i}\,\varphi_{,i} - \frac{10}{21}\,\Psi_0\right) + \frac{16}{3}\,\varphi^2 + 12\,\Theta_0 + \psi^{(2)}_{\mathrm{P(t)}},\\
\phi^{(2)}_{\mathrm{P}} &= \tau^2\left(\frac{1}{6}\,\varphi^{,i}\,\varphi_{,i} - \frac{10}{21}\,\Psi_0\right) + \frac{4}{3}\,\varphi^2 - 8\,\Theta_0 + \phi^{(2)}_{\mathrm{P(t)}},\\
\nabla^2\,\omega^{(2)\,i}_{\mathrm{P}} &= -\frac{8\,\tau}{3}\,(\varphi^{,i}\,\nabla^2\varphi - \varphi^{,ij}\,\varphi_{,j} + 2\,\Psi_0^{,i}) + \nabla^2\,\omega^{(2)\,i}_{\mathrm{P(t)}},\\
\chi^{(2)}_{\mathrm{P}\,ij} &= \tilde{\pi}_{ij} + \chi^{(2)}_{\mathrm{P(t)}\,ij}.
\end{aligned} \tag{6.7}$$

In the Poisson gauge, the second-order tensor mode contains only the scalar-generated, gravitational wave-like tensor mode $\tilde{\pi}_{ij}$ and the piece generated by first-order tensor modes $\chi^{(2)}_{\mathrm{P(t)}\,ij}$, without any other Newtonian and post-Newtonian scalar-generated term (present in the synchronous gauge, eq. (5.21)). This makes its interpretation much easier than in the synchronous and comoving gauge. The tensor mode has an evolution equation (from (5.22)):

$$\tilde{\pi}''_{ij} + \frac{4}{\tau}\,\tilde{\pi}'_{ij} - \nabla^2\,\tilde{\pi}_{ij} = -\frac{40}{3}\,\mathcal{T}_{ij}, \tag{6.8}$$

and its solution is given in (5.26), containing a constant and a wave-like piece.

Then, we can obtain the second-order density contrast in the Poisson gauge, from the gauge transformation (C.21) and the synchronous gauge analogous (5.27):

$$\delta_{\mathrm{P}}^{(2)} = \frac{\tau^4}{252}[10\,(\nabla^2\varphi)^2 + 4\,\varphi_{,ij}\,\varphi^{,ij} + 14\,\varphi_{,i}\nabla^2\varphi^{,i}] + \frac{\tau^2}{18}\left(9\,\varphi_{,i}\varphi^{,i} + 32\,\varphi\nabla^2\varphi - 6\,\varphi^{,ij}\chi_{ij}^{(1)\mathrm{T}}\right)$$
$$+\frac{4\,\tau^2}{7}\Psi_0 + \frac{1}{2}\left(\chi_{ij}^{(1)\mathrm{T}}\chi^{(1)\mathrm{T}ij} - \chi_{0ij}^{(1)\mathrm{T}}\chi_0^{(1)\mathrm{T}ij}\right) - \frac{8}{3}\varphi^2 - 24\,\Theta_0 + 3\phi_{\mathrm{S}(t)}^{(2)} - \frac{6}{\tau}\alpha_{(\mathrm{t})}^{(2)}; \tag{6.9}$$

also in this gauge there are terms generated by a combination of linear tensor modes. Note that this result is different from the one found in [1] (equation (6.10)) because of the previous error in the synchronous gauge expression; it is instead in agreement (in the scalar sector) with two independent and more general results found for a ΛCDM Universe -with no tensor and vector modes- (equation (5.54) in [5], equation (29) in [6]), in the limit of vanishing cosmological constant and vanishing primordial non-Gaussianity (what is defined as a_{NL} in these references is set to zero).
The second-order terms generated by linear tensor modes (those with the subscript (t)) are presented in Appendices B and D.

The second-order velocity perturbation is obtained from the equations (C.23) - (C.24):

$$v_{\mathrm{P}}^{(2)\,0} = \frac{\tau^2}{3}\left(-\frac{1}{6}\,\varphi_{,i}\varphi^{,i} + \frac{10}{7}\Psi_0\right) - \frac{7}{3}\varphi^2 - 12\,\Theta_0 - \psi_{\mathrm{P(t)}}^{(2)}; \tag{6.10}$$

$$v_{\mathrm{P}}^{(2)\,i} = \frac{\tau^3}{9}\left(-\,\varphi_{,j}\,\varphi^{,ij} + \frac{6}{7}\Psi_0^{,i}\right) - 2\tau\left(\frac{16}{9}\,\varphi\,\varphi^{,i} + 2\,\Theta_0^{,i}\right) - d^{(2)\,i\,\prime} - \beta_{(\mathrm{t})}^{(2)\prime\,,i}. \tag{6.11}$$

6.3 Relation between Newtonian and relativistic treatment

To compare the Newtonian treatment of the problem of cosmological perturbations to the relativistic one developed so far, one can re-express the fluid equations (2.39) - (2.41) using comoving time [19] and considering always a pressureless, perfect irrotational fluid. Redefining the Newtonian density perturbation as δ_{N} and the velocity perturbation as $\mathbf{v}$, the continuity equation becomes:

$$\frac{\partial\delta_{\mathrm{N}}}{\partial\tau} + \nabla\cdot[(1+\delta_{\mathrm{N}})\mathbf{v}] = 0\,; \tag{6.12}$$

the Euler equation is:

$$\frac{\partial\,\mathbf{v}}{\partial\tau} + \mathcal{H}\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} + \nabla\,\varphi = 0\,; \tag{6.13}$$

and the Poisson equation is always:

$$\nabla^2 \varphi - 4\pi G a^2 \rho_{(0)} \delta_{\rm N} = 0\,. \tag{6.14}$$

Combining the divergence of the Euler equation with the Poisson equation, one gets:

$$\frac{\partial(\nabla \cdot \mathbf{v})}{\partial \tau} + \mathcal{H}\, \nabla \cdot \mathbf{v} + \nabla \cdot (\mathbf{v} \cdot \nabla)\, \mathbf{v} + 4\pi G a^2 \rho_{(0)} \delta_{\rm N} = 0\,. \tag{6.15}$$

We can define an analogous of the relativistic deformation tensor in the Newtonian case $\theta_{\rm N}^{ij} = \partial^i v^j$ and, using the fact that for an irrotational fluid $v^i = \partial^i v$, we can rewrite:

$$\begin{aligned} \nabla \cdot (\mathbf{v} \cdot \nabla)\, \mathbf{v} &= \partial_i (v^j\, \partial_j) v^i = \partial_i (\partial^j v\, \partial_j)\, \partial^i v = \\ &= \theta^i_{{\rm N}\,j} \theta^j_{{\rm N}\,i} + \partial^j v\, \partial_j \theta_{\rm N}\,. \end{aligned}$$

Defining a Lagrangian time derivative:

$$\frac{d}{d\tau} = \frac{\partial}{\partial \tau} + \mathbf{v} \cdot \nabla\,,$$

we can rewrite the continuity equation and equation (6.15):

$$\frac{d\delta_{\rm N}}{d\tau} + (1 + \delta_{\rm N})\, \theta_{\rm N} = 0\,; \tag{6.16}$$

$$\frac{d\theta_{\rm N}}{d\tau} + \mathcal{H}\theta_{\rm N} + \theta^i_{{\rm N}\,j} \theta^j_{{\rm N}\,i} + 4\pi G a^2 \rho_{(0)} \delta_{\rm N} = 0\,, \tag{6.17}$$

obtaining, respectively, an analogous of the relativistic continuity equation and the Raychaudhuri equation (5.4) in Lagrangian coordinates (i.e. the comoving sinchronous gauge). The equivalence is evident, if one considers derivatives with respect to τ in the synchronous comoving gauge for the relativistic case and the convective Lagrangian time derivative in the Newtonian case.
The difference remains between the energy and momentum constraint (equations (5.1), (5.2)) versus the Poisson equation (6.14): the relativistic equations combine in a Poisson equation analogous to the Newtonian one at first-order only.

This equivalence can be summarized this way:

$$\begin{aligned} \text{Newtonian Lagrangian} &\leftrightarrow \text{relativistic comoving} \\ \frac{d}{d\tau} &\leftrightarrow \frac{\partial}{\partial \tau} \\ \partial^i v_j &\leftrightarrow \theta^i_j \\ \delta_{\rm N} &\leftrightarrow \delta_{\rm S}\,, \end{aligned}$$

where the equivalence between the density constrast and the deformation tensors in the two gauges is exact, while the gravitational potential φ would correspond to the gravitational potentials $\phi_{\rm P}^{(1)}, \psi_{\rm P}^{(1)}$ in the Poisson gauge at first order. It has to be noted that the definition of the deformation tensor is gauge dependent, so in gauges other that the comoving synchronous one there would also be terms depending on v (so, it would be different from the extrinsic curvature).

7 Second-order gravitational waves generated during an early matter era

In the context of the study of second-order perturbations, it is interesting to mention in this section the results of reference [8], which studies the production of gravitational waves generated by linear scalars during an early matter-dominated era after inflation. In most inflationary models, the end of inflation is provided by a period in which the energy density of the Universe is dominated by an oscillating scalar field, behaving like a massive field. During the oscillations, the field decays into radiation, that will dominate the successive epoch of the Universe.
Using the previous notation, the second-order tensor perturbation in the Poisson gauge generated by scalars at linear order is given by the evolution equation:

$$\tilde{\pi}''_{ij} + \frac{4}{\tau}\tilde{\pi}'_{ij} - \nabla^2\tilde{\pi}_{ij} = S^{\mathrm{TT}}_{ij} \tag{7.1}$$

where S^{TT}_{ij} is the tranverse and traceless source term. Passing to the Fourier space, we have:

$$\tilde{\pi}''_{\mathbf{k}} + \frac{4}{\tau}\tilde{\pi}'_{\mathbf{k}} + k^2\,\tilde{\pi}_{\mathbf{k}} = S_{\mathbf{k}} \tag{7.2}$$

with

$$S_{\mathbf{k}} = \frac{40}{3}\int \frac{d^3\mathbf{k}'}{(2\pi)^3} e(\mathbf{k},\mathbf{k}')\,\psi^{(1)}_{\mathrm{P}}(\mathbf{k}-\mathbf{k}')\,\psi^{(1)}_{\mathrm{P}}(\mathbf{k}'), \tag{7.3}$$

where $e(\mathbf{k},\mathbf{k}') = \epsilon^{+\,ij}(\mathbf{k})k'_i k'_j$ (for a definition of the polarization modes, (8.5)). As in the Poisson gauge at first order $\psi^{(1)}_{\mathrm{P}} = \phi^{(1)}_{\mathrm{P}} = \varphi$, we can also substitute those perturbations with the peculiar gravitational potential.
In a matter dominated epoch, the solution for the tensor mode in (7.2) is:

$$\tilde{\pi}_{\mathbf{k}} = \frac{S_{\mathbf{k}}}{k^2} + A_{\mathbf{k}}\left(\frac{\sin(k\tau) - k\tau\cos(k\tau)}{k^3\tau^3}\right) - B_{\mathbf{k}}\left(\frac{\cos(k\tau) + k\tau\sin(k\tau)}{k^3\tau^3}\right) = \frac{S_{\mathbf{k}}}{k^2} + A_{\mathbf{k}}\frac{j_1(k\tau)}{k\tau} + B_{\mathbf{k}}\frac{y_1(k\tau)}{k\tau}, \tag{7.4}$$

where the last two terms are the solutions for the first-order gravitational waves (considering also the second mode, singular for $\tau \to 0$), while the first term is the one generated by first-order scalars. This solution is analogous to the one already found in (5.26). As φ is constant in an EdS Universe, the source term $S_{\mathbf{k}}$ is constant, supporting a part of the tensor perturbation at late times, while $\frac{j_1(k\tau)}{k\tau}$ oscillates with an amplitude decaying like a^{-1} and $\frac{y_1(k\tau)}{k\tau}$ rapidly decays. Choosing the initial conditions $\tilde{\pi} = \tilde{\pi}' = 0$ for $\tau = 0$, so $B_{\mathbf{k}} = 0$ (to remove the singularity at early times) and $A_{\mathbf{k}} = -3\frac{S_{\mathbf{k}}}{k^2}$, we got:

$$\tilde{\pi}_{\mathbf{k}} = \frac{S_{\mathbf{k}}}{k^2}\left[1 + 3\frac{k\tau\cos(k\tau) - \sin(k\tau)}{k^3\tau^3}\right], \tag{7.5}$$

which, for large scales, in the super-Hubble limit $k \ll \mathcal{H}$, gives a growing tensor perturbation

$$\tilde{\pi}_{\mathbf{k}} = \frac{S_{\mathbf{k}}}{10}\tau^2; \tag{7.6}$$

and on sub-Hubble scales ($k \gg \mathcal{H}$), the amplitude is constant in time

$$\tilde{\pi}_{\mathbf{k}} = \frac{S_{\mathbf{k}}}{k^2}. \tag{7.7}$$

So a growth function can be defined:

$$g(k\tau) = 1 + 3\frac{k\tau\cos(k\tau) - \sin(k\tau)}{k^3\tau^3}, \tag{7.8}$$

such to rewrite the solution for the tensor mode:

$$\tilde{\pi}_{\mathbf{k}} = \frac{40\, g(k\tau)}{3\, k^2}\int_0^{k_{\rm dom}} \frac{d^3\mathbf{k}'}{(2\pi)^3} e(\mathbf{k},\mathbf{k}')\, \varphi(\mathbf{k}-\mathbf{k}')\, \varphi(\mathbf{k}'), \tag{7.9}$$

where $k_{\rm dom}$ is the comoving Hubble scale at the beginning of this matter-dominated era.

The two-point correlation function of this second-order tensor modes can be considered :

$$\langle \tilde{\pi}_{\mathbf{k}}(\tau)\tilde{\pi}_{\tilde{\mathbf{k}}}(\tau)\rangle = \left(\frac{40\, g(k\tau)}{3\, k^2}\right)^2 \int_0^{k_{\rm dom}} \frac{d^3\mathbf{k}'}{(2\pi)^3}\frac{d^3\tilde{\mathbf{k}}'}{(2\pi)^3} e(\mathbf{k},\mathbf{k}')\, e(\tilde{\mathbf{k}},\tilde{\mathbf{k}}')\, \langle \varphi(\mathbf{k}-\mathbf{k}')\, \varphi(\mathbf{k}')\varphi(\tilde{\mathbf{k}}-\tilde{\mathbf{k}}')\, \varphi(\tilde{\mathbf{k}}')\rangle, \tag{7.10}$$

where

$$\langle \varphi(\mathbf{k}-\mathbf{k}')\, \varphi(\mathbf{k}')\varphi(\tilde{\mathbf{k}}-\tilde{\mathbf{k}}')\, \varphi(\tilde{\mathbf{k}}')\rangle = \langle \varphi(\mathbf{k}-\mathbf{k}')\, \varphi(\tilde{\mathbf{k}}-\tilde{\mathbf{k}}')\rangle\langle \varphi(\mathbf{k}')\, \varphi(\tilde{\mathbf{k}}')\rangle + \\ + \langle \varphi(\mathbf{k}-\mathbf{k}')\, \varphi(\tilde{\mathbf{k}}')\rangle\langle \varphi(\tilde{\mathbf{k}}-\tilde{\mathbf{k}}')\, \varphi(\mathbf{k}')\rangle.$$

The two point function for tensor and scalar modes are defined here as:

$$\langle \tilde{\pi}_{\mathbf{k}}(\tau)\tilde{\pi}_{\tilde{\mathbf{k}}}(\tau)\rangle = \frac{1}{2}\frac{2\pi^2}{k^3}\delta^3(\mathbf{k}+\tilde{\mathbf{k}})\mathcal{P}_{\tilde{\pi}}(k,\tau)$$

$$\langle \varphi_{\mathbf{k}}(\tau)\varphi_{\tilde{\mathbf{k}}}(\tau)\rangle = \frac{2\pi^2}{k^3}\delta^3(\mathbf{k}+\tilde{\mathbf{k}})\mathcal{P}(k)$$

where $\mathcal{P}_{\tilde{\pi}}(k,\tau), \mathcal{P}(k)$ are the respective power spectra. On very large scales, for primordial scalar perturbations[3] $\mathcal{P}(k) = \frac{9}{25}\Delta^2_{\mathcal{R}}(\frac{k}{k_*})^{n_s-1}$, where the numerical factor is given by the relation between scalar curvature perturbation in the Poisson and comoving gauges on large scales in a matter dominated era. From observations of the CMB, for density perturbations on scales $\approx$ 100 Mpc today, the amplitude is $\Delta^2_{\mathcal{R}} \approx 2.4\times10^{-9}$, while the spectral index is $n_s \approx 0.96$ (those values, anyway, can be different for perturbations on much smaller scales, relevant for the detection of gravitational waves).
Substituting the definition of the two-point function into (7.10):

$$\langle\tilde{\pi}_{\mathbf{k}}(\tau)\tilde{\pi}_{\tilde{\mathbf{k}}}(\tau)\rangle =$$
$$= \left(\frac{40\,g(k\tau)}{3\,k^2}\right)^2 \frac{\pi\,\delta^3(\mathbf{k}+\tilde{\mathbf{k}})}{2\,k^4}\int_0^{k_{\text{dom}}} d^3\mathbf{k}'\,e(\mathbf{k},\mathbf{k}')[e(\tilde{\mathbf{k}},\mathbf{k}') + e(\tilde{\mathbf{k}},\mathbf{k}-\mathbf{k}')]\mathcal{P}(k-k')\mathcal{P}(k'). \tag{7.11}$$

Assuming for simplicity that the power spectrum of primordial scalar perturbations is scale invariant ($n_s = 1$), one obtains:

$$\mathcal{P}_{\tilde{\pi}}(k,\tau) = 2\left(\frac{24\,g(k\tau)}{5}\right)^2 \Delta^4_{\mathcal{R}}\left(\frac{k_{\text{dom}}}{k}\right) I_1(k/k_{\text{dom}}), \tag{7.12}$$

where the integral

$$I_1(k/k_{\text{dom}}) = \frac{1}{2\pi k_{\text{dom}}}\int d^3\mathbf{k}'\,\frac{[e(\mathbf{k},\mathbf{k}')]^2}{k'^3|\mathbf{k}-\mathbf{k}'|^3}\,\Theta(k_{\text{dom}}-k')\,\Theta(k_{\text{dom}}-|\mathbf{k}-\mathbf{k}'|) \tag{7.13}$$

can be approximated as

$$I_1(x) \approx \frac{16}{15} - \frac{4}{3}x + \frac{16}{35}x^2.$$

The step functions in (7.13) have been introduced to cut off the smallest scales $k > k_{\text{dom}}$, which could enter a non-linear regime if the matter era lasts long enough ($k_{\text{dom}} > k_{\text{NL}}$, where k_{NL} is the scale where non-linear effects become relevant). That is why a cut-off on the power spectrum of density perturbations is imposed, such that $\mathcal{P}(k) = 0$ for $k > k_{\text{cut}}$, where $k_{\text{cut}} = \min[k_{\text{NL}}(\tau), k_{\text{dom}}]$. This provides a lower bound on the amplitude of second-order gravitational waves.

[3]The relation between the scalar metric perturbation in Poisson gauge $\phi_P = \varphi = \hat{\phi}$ (where $\hat{\phi}$ is the curvature perturbation) and the primordial, inflationary scalar perturbation δ_φ is given by the gauge invariant curvature perturbation on uniform density hypersurfaces $\zeta = -\hat{\phi} - H\frac{\delta_\rho}{\dot{\rho}}$, plus the relation $\delta_\varphi = \dot{\varphi}\frac{\delta_\rho}{\dot{\rho}}$ (where $\dot{} = \frac{d}{dt}$).

The end of the early matter-dominated era corresponds to $k = k_{\rm dec} = \tau_{\rm dec}^{-1}$; the power spectrum for this scale is given by substituting $g(k/k_{\rm dec})$ in equation (7.12). The behavior of the power spectrum on all scales corresponding to the dimension of the Hubble size before, during and after the early matter dominated epoch is shown in Figure (3):

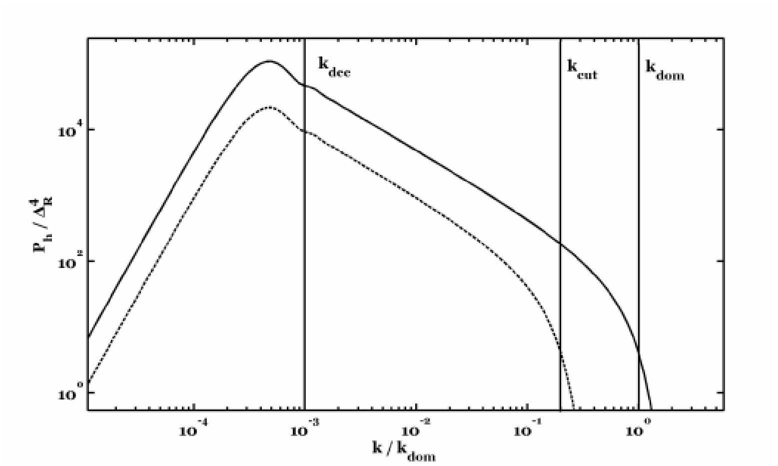

Figure 3: This is the power spectrum of second-order gravitational waves generated during the early matter-dominated era $\mathcal{P}_{\tilde{\pi}}(k, \tau)$, with respect to the wavenumber k. The comoving Hubble scale at the beginnig of this epoch is $k_{\rm dom}$, at its end is $k_{\rm dec}$ (signing the start of the classic radiation-dominated era). The thick line shows the power spectrum for $k_{\rm dom} = 10^3\, k_{\rm dec}$; the dotted line shows the result for the matter power spectrum truncated at $k_{\rm cut} = 200\, k_{\rm dec}$. Figure from [8].

In the super-Hubble limit, the growth function becomes $g(k\tau) = \frac{k^2}{10}\tau^2$, such that on scales larger than the Hubble size at the end of the matter-dominated era $k < k_{\rm dec} = \tau_{\rm dec}^{-1}$, we obtain:

$$\mathcal{P}_{\tilde{\pi}}(k, \tau_{\rm dec}) \simeq 0.5\, \Delta_{\mathcal{R}}^4 \left(\frac{k^3 k_{\rm dom}}{k_{\rm dec}^4} \right), \tag{7.14}$$

using $I_1(k/k_{\rm dom}) \approx 16/15$ and $g(k/k_{\rm dec}) = \frac{k^2}{10\, k_{\rm dec}^2}$. This way, as we can also see in Figure (3), on large scales we have a steep blue spectrum $\mathcal{P}_{\tilde{\pi}} \propto k^3$, whose amplitude decreases for increasing scales (and decreasing k).
For scales entering the horizon between the start and the end of the matter era $k_{\rm dec} < k < k_{\rm dom}$, the growth function has reached the constant value $g(k\tau_{\rm dec}) \simeq 1$ (corresponding to smaller scales, see equation (7.7)) by the end of the matter era, so we have:

$$\mathcal{P}_{\tilde{\pi}}(k, \tau_{\rm dec}) \simeq 46\, \Delta_{\mathcal{R}}^4 \left(\frac{k_{\rm dom}}{k} \right) I_1(k/k_{\rm dom}). \tag{7.15}$$

Assuming $I_1(k/k_{\rm dom})$ to be constant, we have $\mathcal{P}_{\tilde{\pi}} \propto k^{-1}$, a power spectrum decreasing on the smallest scales (actually, $I_1(x)$ decreases for $x \to 1$, so it would suppress even more the spectrum for increasing k). This way, the highest amplitude it can be reached corresponds to scales entering the Hubble size at the end of the matter-dominated era (as it can be seen in Figure (3)):

$$\mathcal{P}_{\tilde{\pi}}^{\rm max} = \mathcal{P}_{\tilde{\pi}}(k_{\rm dec}, \tau_{\rm dec}) \simeq 50\, \Delta_{\mathcal{R}}^4 \left(\frac{k_{\rm dom}}{k_{\rm dec}}\right), \tag{7.16}$$

where $I_1(k_{\rm dec}/k_{\rm dom}) \simeq 16/15$ for $k_{\rm dom} \gg k_{\rm dec}$. This results shows that for sub-Hubble scales, for which the gravitational wave amplitude is supported by the constant scalar source term $S_{\mathbf{k}}$ (equation (7.7)), the power spectrum has an additional factor $\frac{k_{\rm dom}}{k_{\rm dec}}$ with respect to the predictable square of the amplitude of the first-order scalar power spectrum, $\Delta_{\mathcal{R}}^4$. This factor increases the amplitude of second-order tensor perturbations, depending on the duration of the matter-dominated era (how much $k_{\rm dom} \gg k_{\rm dec}$).

Actually, this estimation has to take into account the (already mentioned) non-linear cutoff. For a very long matter-dominated era, the Hubble scale becomes much larger than the scale corresponding to $k_{\rm dom}$. On sub-Hubble scales the constant gravitational potential, for gravitational instability, leads to the increase of the density contrast $\delta\rho/\rho$, giving a breakdown of the results valid in linear theory. Below the non-linear scale $k > k_{\rm NL}(\tau)$, a perturbative analysis shows that the gravitational potential φ decreases and thus the source term $S_{\mathbf{k}}$. For simplicity, one can assume a cutoff such that the power spectrum for the scalar perturbations vanishes on scales smaller than the non-linear one $k > k_{\rm NL}(\tau) > k_{\rm dom}$, providing a lower bound for the gravitational waves power spectrum. As the extreme of integration becomes the time dependent $k_{\rm NL}(\tau)$ in the place of $k_{\rm dom}$, also the source term becomes time dependent. Anyway, as its rate of change is slow with respect to the decay time of the most relevant part of the solution (7.4) in sub-Hubble limit, one can take the quasi-static generalization of equation (7.7):

$$\tilde{\pi}_{\mathbf{k}} \approx \frac{S_{\mathbf{k}}(k_{\rm NL}(\tau))}{k^2}. \tag{7.17}$$

This way, the generalization of the power spectrum for tensor perturbations on sub-Hubble scales at the end of the matter-dominated era ($k_{\rm cut}(\tau) > k > k_{\rm dec}$, with $k_{\rm cut} = \min[k_{\rm NL}(\tau), k_{\rm dom}]$ and $k_{\rm NL}(\tau) \simeq 200\, k_{\rm dec}$) is:

$$\mathcal{P}_{\tilde{\pi}}(k, \tau_{\rm dec}) \simeq 46\, \Delta_{\mathcal{R}}^4 \left(\frac{k_{\rm cut}}{k}\right) I_1(k/k_{\rm cut}). \tag{7.18}$$

For very small scales $k \gg k_{\rm NL}$, as already said, the potential decays and the source term $S_{\mathbf{k}}$ tends to zero. The solution to the second-order tensor mode (equation (7.4)) thus propagates as a free gravitational wave, oscillating and with an amplitude decaying as $\tilde{\pi} \propto a^{-1}$ on sub-Hubble scales. The power spectrum in this case becomes:

$$\mathcal{P}_{\tilde{\pi}}(k, \tau_{\rm dec}) \simeq 46\, \Delta_{\mathcal{R}}^4 \left(\frac{k_{\rm NL}(\tau)}{k}\right)^4, \tag{7.19}$$

which, being $\propto k^{-4}$, is suppressed on the scales smaller than the non-linear ones $k \gg k_{\rm NL}$. The fact that scalar perturbations, on the smallest scales, decay because of non-linear effects is due to the velocity of matter, which becomes non-negligible.

7.1 Present density of gravitational waves

The energy density of gravitational waves on sub-Hubble scales is given by [11]:

$$\rho_{\rm GW} = \frac{1}{32\pi G}\langle \dot{h}_{ij}\, \dot{h}^{ij}\rangle = \frac{k^2}{32\pi G a^2}\int d(\ln k)\, \mathcal{P}_{\tilde{\pi}}(k, \tau). \tag{7.20}$$

This gives the density parameter:

$$\Omega_{\rm GW}(k,\tau) = \frac{d\rho_{\rm GW}/d\ln k}{\rho_{\rm critical}} = \frac{1}{12}\left(\frac{k}{\mathcal{H}}\right)^2 \mathcal{P}_{\tilde{\pi}}(k,\tau), \tag{7.21}$$

scaling like non-interacting relativistic particles during and after the radiation-dominated era. So, the present density of gravitational waves is:

$$\Omega_{\rm GW,0}(k) \simeq \frac{\Omega_{\gamma,0}}{12}\left(\frac{k}{k_{\rm dec}}\right)^2 \mathcal{P}_{\tilde{\pi}}(k, \tau_{\rm dec}) \tag{7.22}$$

where $\Omega_{\gamma,0} \simeq 1.2 \times 10^{-5}$ is the present energy density of radiation.
Considering the amplitude of the power spectrum of tensor perturbations at the beginning of the radiation-dominated era $k < k_{\rm dec}$ (equation (7.15)), their density is:

$$\Omega_{\rm GW}(k,\tau) \simeq \frac{23}{12}\,\Delta_{\mathcal{R}}^4\left(\frac{k_{\rm dom}k}{k_{\rm dec}^2}\right) I_1(k/k_{\rm dom}), \tag{7.23}$$

remaining constant in time during the radiation era, if there is no further production of gravitational waves. Substituting in (7.22), the present density is:

$$\Omega_{\rm GW,0}(k) \simeq \frac{23}{12}\,\Omega_{\gamma,0}\Delta_{\mathcal{R}}^4\left(\frac{k_{\rm dom}k}{k_{\rm dec}^2}\right) I_1(k/k_{\rm dom}), \tag{7.24}$$

which is shown in Figure (4) as a function of k.
The power spectrum has a maximum at the end of the early matter era, as shown in equation (7.16); in the case of the density of gravitational waves, the maximum is provided for the scale corresponding to the beginning of the matter era $k_{\rm dom}$:

$$\Omega_{\rm GW,0}(k) \simeq \frac{23}{12}\,\Omega_{\gamma,0}\Delta^4_{\mathcal{R}}\left(\frac{k_{\rm dom}}{k_{\rm dec}}\right)^2 I_1(k/k_{\rm dom}). \tag{7.25}$$

Just as the case of the power spectrum, there is an enhancement factor with respect to the simple expectation for the present density $\Omega_{\rm GW,0}(k) \simeq \Omega_{\gamma,0}\Delta^4_{\mathcal{R}} \simeq 3\times 10^{-22}$, which is

$$F^2 = \left(\frac{k_{\rm dom}}{k_{\rm dec}}\right)^2. \tag{7.26}$$

This represents the duration of the early matter-dominated era, as $k \propto \mathcal{H} \propto aH \propto t^{-1/3} \propto H^{1/3}$. So, $F^2 = \left(\frac{H_{\rm dom}}{H_{\rm dec}}\right)^{2/3}$.

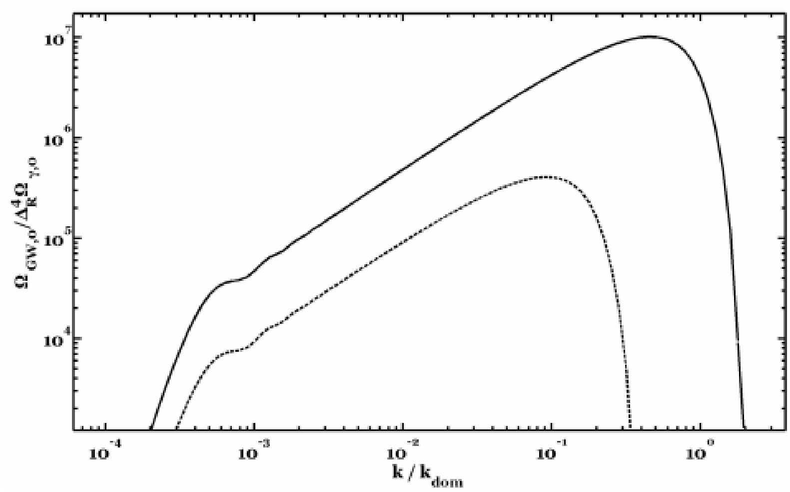

Figure 4: This is the present energy density of gravitational waves generated during the early matter-dominated era $\Omega_{\rm GW,0}(k)$, with respect to the wavenumber k. In this case, $F^2 = (k_{\rm dom}/k_{\rm dec})^2 = 10^6$ has been taken. The solid line refers to the linear matter power spectrum in the case $k < k_{\rm dom}$, the dotted line is for the spectrum truncated at $k > k_{\rm cut} = 200\,k_{\rm dec}$. Figure from [8].

If we assume that the early matter-dominated era is due to the oscillation of a scalar field around the minimum of its potential, leading to its decay in radiation, it has to be required that $H_{\rm dom} < m$, so that the expansion rate at the beginning to be smaller than the mass of the field (the oscillation rate). The end will be provided by $H_{\rm dec} \simeq \Gamma$, so by an expansion rate comparable with the decay rate of the species. This way:

$$F^2 < \left(\frac{m}{\Gamma}\right)^{2/3}.$$

An extended matter era occurs for $F \gg 1$, so in the case of scalar fields weakly coupled with radiation ($\Gamma \ll m$).

Also in this case we have to consider the effects of non-linearity, due to the fact that for $k > k_{\rm NL}$ the gravitational potential decays, and so does the amplitude of the power spectrum for these smallest scales. This way, there is an upper bound also for the enhancement factor F^2, whose estimation using linear scalar perturbations is valid up to $k_{\rm dom} = k_{\rm NL}(\tau_{\rm dec})$:

$$F^2 = \left(\frac{k_{\rm NL}(\tau_{\rm dec})}{k_{\rm dec}}\right)^2 \simeq \mathcal{P}^{-1/2} \simeq 3 \times 10^4. \tag{7.27}$$

Considering the scales under the Hubble size during matter-domination $k_{\rm dec} < k < k_{\rm NL}$ (with $k_{\rm dom} > k_{\rm NL}$), using equations (7.18) and (7.22) (plus the fact that $\left(\frac{k_{\rm NL}(\tau_{\rm dec})}{k_{\rm dec}}\right) \simeq \mathcal{P}^{-1/4} \propto \Delta_{\mathcal{R}}^{-1/2}$), we obtain:

$$\Omega_{\rm GW,0}(k) \simeq \Omega_{\gamma,0}\Delta_{\mathcal{R}}^4\left(\frac{k_{\rm NL}k}{k_{\rm dec}^2}\right) \simeq \Delta_{\mathcal{R}}^{3.5}\,\Omega_{\gamma,0}\left(\frac{k}{k_{\rm dec}}\right). \tag{7.28}$$

So, the maximum value for the present density of gravitational waves is reached for $k \simeq k_{\rm NL}$:

$$\Omega_{\rm GW,0}(k_{\rm NL}) \simeq \Omega_{\gamma,0}\Delta_{\mathcal{R}}^4\left(\frac{k_{\rm NL}}{k_{\rm dec}}\right)^2 \simeq \Delta_{\mathcal{R}}^3\,\Omega_{\gamma,0}. \tag{7.29}$$

If we consider that the amplitude of the power spectrum of primordial density perturbations is the same as the one seen on CMB scales today, $\Delta_{\mathcal{R}}^2 \simeq 2 \times 10^{-9}$, we obtain $\Omega_{\rm GW,0}(k_{\rm NL}) \simeq 10^{-17}$, which would be detectable only by future experiment (like the Big Bang Observer). This is a lower bound for the density of gravitational waves from an early matter era, as we neglected all effects coming from non-linear scales.

During radiation-dominated era, the Hubble scale is $k = aH$, where the Hubble parameter is

$$H^2 = \frac{8\pi G}{3}\left(g_*\,\frac{\pi^2}{30}T^4\right), \tag{7.30}$$

with g_* number of degrees of freedom of the relativistic species at the temperature T. The wavelenght that, today, corresponds to the wavenumber k is:

$$\lambda_0 = \frac{2\pi a_0}{k} \simeq 2 \times 10^{16} g_*^{-1/6} \left(\frac{\text{GeV}}{T} \right) \text{m}, \tag{7.31}$$

and the corresponding frequency is:

$$\nu = \frac{c}{\lambda_0} \simeq 1.2 \times 10^{-8} g_*^{1/6} \left(\frac{T}{\text{GeV}} \right) \text{Hz}. \tag{7.32}$$

From that, we have that the enhancement factor can be expressed also with respect to the frequencies:

$$F = \frac{\nu_{\text{dom}}}{\nu_{\text{dec}}}, \tag{7.33}$$

so, a long matter-dominated era corresponds to gravitational waves produced on a large range of frequencies.
From the primordial nucleosynthesis, we have a constraint on the temperature at the end of the early matter-dominated era ($T_{\text{dec}} > 1$ MeV), which is reflected on a constraint on the frequency at the end of matter-dominated era:

$$\nu_{\text{dec}} > 10^{-11} \text{ Hz}.$$

Comparing with the sensitivity of present and future detectors, gravitational waves generated during an early matter era could be detected on LIGO frequencies $\nu_{\text{LIGO}} \simeq 100$ Hz, if $T_{\text{dec}} < 10^{10}$ GeV and on LISA frequencies $\nu_{\text{LISA}} \simeq 10^{-3}$ Hz, if $T_{\text{dec}} < 10^5$ GeV. From the current limits from LIGO, the density of these gravitational waves has the bound $\Omega_{\text{GW},0} < 6 \times 10^{-5}$ in LIGO frequencies range, but in the future Advanced LIGO will be sensitive down to $\Omega_{\text{GW},0} \simeq 10^{-9}$. Instead, LISA could detect a background with densities $\Omega_{\text{GW},0} \simeq 10^{-11}$ at ν_{LISA} frequencies, while future experiments like Big Bang Observer could detect a background with $\Omega_{\text{GW},0} \simeq 10^{-17}$ at the frequencies $\nu_{\text{BBO}} \simeq 1$ Hz.

8 Evolution of second order scalar perturbations generated by tensor modes

We know that the most relevant perturbations at linear order are the scalar ones, and that is why the studies of second-order perturbations has mostly considered the contribution from the linear scalars, neglecting the linear tensors and vectors. It can be asked whether the second-order perturbations coming from first-order gravitational waves are really negligible, so, whether they are subjected to a relevant time evolution, or not. This way, it can be studied whether the second-order density perturbation generated only by linear tensor modes can grow in time, i.e. undergo gravitational collapse. We would also like to see the growth rate for this $\delta^{(2)}_{(\mathrm{t})}$, as well as for the other two gravitational potentials.

In the previous sections, by solving the Einstein equations in EdS Universe up to second-order, the expressions for the second-order scalar perturbations and gravitational potentials have been derived. Now, we are interested in getting the evolution in time of those quantities generated by tensor modes. To do that, we can arbitrarily set to zero the scalar modes in the source to consider just the contributions from tensor-tensor terms, and we can substitute the definition of these linear tensors in the source terms. Recalling the properties of the first-order tensor (equations (5.6)- (5.8)), its general solution is

$$\chi^{(1)\,\mathrm{T}}_{ij}(\mathbf{x},\tau) = \frac{1}{(2\pi)^3}\sum_{\sigma=+,\times}\int d^3\mathbf{k}\,\exp(i\,\mathbf{k}\cdot\mathbf{x})\,\chi^{(1)}_{\sigma}(\mathbf{k},\tau)\,\epsilon^{\sigma}_{ij}(\hat{\mathbf{k}}),$$

where the amplitude of the polarization states $\chi^{(1)}_{\sigma}$ has, in full generality, this form:

$$\chi^{(1)}_{\sigma}(\mathbf{k},\tau) = A_{\sigma}(\mathbf{k})\left(\frac{3j_1(k\tau)}{k\tau}\right) + B_{\sigma}(\mathbf{k})\left(\frac{y_1(k\tau)}{k\tau}\right). \tag{8.1}$$

with j_1, y_1 that are the spherical Bessel functions of the first and second kind. From now on we set $B_{\sigma}(\mathbf{k}) = 0$ to neglect the decreasing mode in y_1, as it is divergent for our initial condition $\tau = 0$. So the tensor mode in Fourier space is:

$$\chi^{(1)}_{\sigma}(\mathbf{k},\tau) = A_{\sigma}(\mathbf{k})\left(\frac{3j_1(k\tau)}{k\tau}\right). \tag{8.2}$$

To make χ_{σ} real, $\chi^{*}_{\sigma}(\mathbf{k},\tau) = \chi_{\sigma}(-\mathbf{k},\tau)$, and $\epsilon^{\sigma}_{ij}(\mathbf{k},\tau) = \epsilon^{\sigma}_{ij}(-\mathbf{k},\tau)$. The polarization tensor obeys also symmetry, transverse and traceless conditions:

$$\epsilon^{\sigma}_{ij}(\hat{\mathbf{k}}) = \epsilon^{\sigma}_{ji}(\hat{\mathbf{k}}) \qquad \hat{\mathbf{k}}^{i}\,\epsilon^{\sigma}_{ij}(\hat{\mathbf{k}}) = 0 \qquad \epsilon^{\sigma\,i}_{i}(\hat{\mathbf{k}}) = 0$$

and there are the orthonormal and completeness relations [7]:

$$\epsilon^{\sigma}_{ij}(\hat{\mathbf{k}})\,\epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}) = 2\delta^{\sigma\sigma'} \tag{8.3}$$

$$\sum_{\sigma=+,\times} \epsilon^{\sigma}_{ij}(\hat{\mathbf{k}})\,\epsilon^{\sigma}_{lm}(\hat{\mathbf{k}}) = P_{il}P_{jm} + P_{im}P_{jl} - P_{ij}P_{lm} \tag{8.4}$$

with the projection tensor $P_{ij} = \delta_{ij} - \frac{\hat{\mathbf{k}}_i \hat{\mathbf{k}}_j}{k^2}$. The polarization tensor can be expressed with respect to the two unit vectors normal to $\hat{\mathbf{k}}$, in each polarization:

$$\begin{aligned} \epsilon^{+}_{ij}(\hat{\mathbf{k}}) &= \hat{m}_i\hat{m}_j - \hat{n}_i\hat{n}_j \\ \epsilon^{\times}_{ij}(\hat{\mathbf{k}}) &= \hat{m}_i\hat{n}_j + \hat{n}_i\hat{m}_j; \end{aligned} \tag{8.5}$$

while the projection tensor is $P_{ij} = \hat{m}_i\hat{m}_j + \hat{n}_i\hat{n}_j$.

This study is performed both in the comoving synchronous and in the Poisson gauge: in the Poisson gauge the two linear scalar perturbations of the metric can be connected to the gauge invariant Bardeen potentials $\varphi = \Phi_A = \Psi_A$, while the linear density perturbation in the comoving synchronous gauge satisfies the generalized Poisson equation at first-order:[4]

$$\nabla^2\Psi_A = 4\pi G a^2 \rho_0 \delta^{(1)}_{\mathrm{S}}.$$

8.1 Derivation of the second-order scalar perturbations starting from the comoving synchronous gauge

In the previous sections we have already derived the second-order density contrast in the comoving synchronous gauge (5.27), but just as a solution of the continuity equation, not as a solution of an evolution equation, which would give a better physical insight.

In the comoving synchronous gauge a very simple and immediate evolution equation for the second-order density perturbation $\delta^{(2)}_{\mathrm{S}}$ can be derived. To do that, the second-order continuity equation [19]:

$$\delta'^{(2)}_{\mathrm{S}} + \theta^{(2)} = -2\delta^{(1)}_{\mathrm{S}}\theta^{(1)} \tag{8.6}$$

[4] $\delta^{(1)}_{\mathrm{S}}$ is the same as the gauge invariant density perturbation in the comoving orthogonal gauge (actually entering the Poisson equation) [15]:

$$\delta^{(1)}_{\mathrm{com}} = \delta^{(1)} + \frac{\rho'^{(0)}}{\rho^{(0)}}(v^{\parallel(1)} + \omega^{\parallel(1)}),$$

as in the comoving synchronous gauge $v^{\parallel(1)} + \omega^{\parallel(1)} = 0$.

can be combined to the Raychaudhuri equation at second-order:

$$\theta'^{(2)} + \frac{2}{\tau}\,\theta^{(2)} + \theta^{(1)\,i}_{j}\theta^{(1)\,j}_{i} + \frac{6}{\tau^2}\delta^{(2)}_{\mathrm{S}} = 0, \tag{8.7}$$

where θ^i_j is the deformation tensor. The evolution equation obtained is:

$$\begin{aligned}\delta''^{(2)}_{\mathrm{S}} + \frac{2}{\tau}\,\delta'^{(2)}_{\mathrm{S}} - \frac{6}{\tau^2}\,\delta^{(2)}_{\mathrm{S}} &= -\,2\delta'^{(1)}\theta^{(1)} - 2\delta^{(1)}\theta'^{(1)} - \frac{4}{\tau}\,\delta^{(1)}\theta^{(1)} + 2\theta^{(1)\,i}_{j}\theta^{(1)\,j}_{i} = \\ &= \frac{2}{9}\tau^2\varphi^{ij}\varphi_{ij} + \frac{5}{9}\tau^2(\nabla^2\varphi)^2 - \frac{2\tau}{3}\,\varphi^{ij}\chi^{(1)\mathrm{T}}_{ij} + \frac{1}{2}\chi'^{(1)\mathrm{T}\,ij}\chi'^{(1)\mathrm{T}}_{ij}\,,\end{aligned} \tag{8.8}$$

and it can be verified that this expression for $\delta^{(2)}_{\mathrm{S}}$:

$$\begin{aligned}\delta^{(2)}_{\mathrm{S}} =& \frac{\tau^4}{252}[4\,\varphi^{,ij}\varphi_{,ij} + 10\,(\nabla^2\varphi)^2] + \frac{\tau^2}{18}(15\,\varphi^{,k}\varphi_{,k} + 40\,\varphi\nabla^2\varphi - 6\,\varphi^{,ij}\chi^{(1)\mathrm{T}}_{ij}) \\ &+ \frac{1}{2}(\chi^{(1)\mathrm{T}\,ij}\chi^{(1)\mathrm{T}}_{ij} - \chi^{(1)\mathrm{T}\,ij}_{0}\chi^{(1)\mathrm{T}}_{0ij}) + 3\,\phi^{(2)}_{\mathrm{S(t)}},\end{aligned}$$

already obtained simply from the second-order continuity equation, satisfies our evolution equation. $\phi^{(2)}_{\mathrm{S(t)}}$ is the second-order scalar perturbation of the metric generated by linear tensors, whose evolution equation and solution are recalled in the following (equations (8.11), (8.12)).

We are interested in evaluating the second-order density perturbation generated only by first-order gravitational waves, so we can neglect the first-order scalars to isolate this contribution. This way, the evolution equation (8.8) becomes:

$$\delta''^{(2)}_{\mathrm{S(t)}} + \frac{2}{\tau}\,\delta'^{(2)}_{\mathrm{S(t)}} - \frac{6}{\tau^2}\,\delta^{(2)}_{\mathrm{S(t)}} = \frac{1}{2}\chi'^{(1)\mathrm{T}\,ij}\chi'^{(1)\mathrm{T}}_{ij}\,. \tag{8.9}$$

The structure of the homogeneous equation is the same as the evolution equation for the linear density contrast; of course in the second-order case we have also a source term different from zero. It can also be checked that this homogeneous equation expressed with respect to the conformal time matches the one expressed with respect to the physical time t (2.43):

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\rho_{(0)}\delta = 0\,.$$

Solving the inhomogeneous differential equation, the second-order density perturbation in the comoving synchronous gauge generated only by tensors is:

$$\delta^{(2)}_{\mathrm{S(t)}}(\mathbf{x},\tau) = \frac{\tau^2}{10}\int_0^\tau d\tilde{\tau}\frac{\chi'^{(1)\mathrm{T}\,ij}\chi'^{(1)\mathrm{T}}_{ij}}{\tilde{\tau}} - \frac{1}{10\tau^3}\int_0^\tau d\tilde{\tau}\,\tilde{\tau}^4\chi'^{(1)\mathrm{T}\,ij}\chi'^{(1)\mathrm{T}}_{ij} + c^{\delta}_1(\mathbf{x})\tau^2 + c^{\delta}_2(\mathbf{x})\tau^{-3}\,, \tag{8.10}$$

where the last two terms are the solution of the homogeneous part of equation (8.9) and they can be set by choosing the initial conditions for $\delta^{(2)}_{\mathrm{S(t)}}$. This result agrees with the analogous expression found in [20].

The expression of $\delta^{(2)}_{\mathrm{S(t)}}$ previously derived in the comoving synchronous gauge:

$$\delta^{(2)}_{\mathrm{S}} = \frac{1}{2}(\chi^{(1)\mathrm{T}\,ij}\chi^{(1)\mathrm{T}}_{ij} - \chi^{(1)\mathrm{T}\,ij}_{0}\chi^{(1)\mathrm{T}}_{0ij}) + 3\,\phi^{(2)}_{\mathrm{S(t)}}\,,$$

can be recasted in the form of the solution (8.10). To do that, it is necessary to substitute the expression for $\phi^{(2)}_{\mathrm{S(t)}}$, which satisfies the evolution equation:

$$\begin{aligned}\phi^{(2)\,\prime\prime}_{\mathrm{S(t)}} + \frac{2}{\tau}\,\phi^{(2)\,\prime}_{\mathrm{S(t)}} - \frac{6}{\tau^2}\,\phi^{(2)}_{\mathrm{S(t)}} &= -\frac{1}{6}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime} + \frac{2}{3\tau}\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}\\ &\quad - \frac{1}{3}\chi^{(1)\mathrm{T}\,ij}\,\nabla^2\chi^{(1)\mathrm{T}}_{ij} + \frac{1}{\tau^2}(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij} - \chi^{(1)\mathrm{T}}_{0\,ij}\chi^{(1)\mathrm{T}\,ij}_{0}) \equiv \mathcal{Q}(\mathbf{x},\tau),\end{aligned} \tag{8.11}$$

whose solution is:

$$\begin{aligned}\phi^{(2)}_{\mathrm{S(t)}}(\mathbf{x},\tau) &= c_1(\mathbf{x})\tau^2 + c_2(\mathbf{x})\tau^{-3} + \frac{\tau^2}{5}\int_0^\tau \frac{d\tilde\tau}{\tilde\tau}\,\mathcal{Q}(\mathbf{x},\tilde\tau) - \frac{1}{5\tau^3}\int_0^\tau d\tilde\tau\,\tilde\tau^4\,\mathcal{Q}(\mathbf{x},\tilde\tau) =\\ &= c_1(\mathbf{x})\tau^2 + c_2(\mathbf{x})\tau^{-3} + \frac{\tau^2}{5}\int_0^\tau \frac{1}{6\tilde\tau}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde\tau - \frac{1}{5\tau^3}\int_0^\tau \frac{\tilde\tau^4}{6}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde\tau +\\ &\quad + \frac{\tau^2}{5}\left[-\frac{1}{2\tau^2}(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}ij} - \chi^{(1)\mathrm{T}}_{0ij}\chi^{(1)\mathrm{T}ij}_{0}) - \frac{\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}ij}}{3\tau}\right|_0^\tau\\ &\quad - \frac{1}{5\tau^3}\left[\frac{\tau^3}{3}(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}ij} - \chi^{(1)\mathrm{T}}_{0ij}\chi^{(1)\mathrm{T}ij}_{0}) - \frac{\tau^4}{3}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}ij}\right|_0^\tau,\end{aligned} \tag{8.12}$$

where, as before, we leave also the terms of the homogeneous solution.

So, the expression for $\delta^{(2)}_{\mathrm{S(t)}}$ becomes:

$$\begin{aligned}\delta^{(2)}_{\mathrm{S(t)}} &= 3\phi^{(2)}_{\mathrm{S}(t)} + \frac{1}{2}\left(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}ij} - \chi^{(1)\mathrm{T}}_{0ij}\chi^{(1)\mathrm{T}ij}_{0}\right) =\\ &= \frac{\tau^2}{10}\int_0^\tau \frac{1}{\tilde\tau}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde\tau - \frac{1}{10\tau^3}\int_0^\tau \tilde\tau^4\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde\tau\\ &\quad - \frac{3\tau^2}{5}\left[-\frac{1}{2\tau^2}(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}ij} - \chi^{(1)\mathrm{T}}_{0ij}\chi^{(1)\mathrm{T}ij}_{0}) - 5c_1 - \frac{\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}ij}}{3\tau}\right|_{\tau=0}\\ &\quad + \frac{3}{5\tau^3}\left[\frac{\tau^3}{3}(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}ij} - \chi^{(1)\mathrm{T}}_{0ij}\chi^{(1)\mathrm{T}ij}_{0}) - \frac{\tau^4}{3}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}ij} + 5c_2\right|_{\tau=0},\end{aligned} \tag{8.13}$$

where the terms in parenthesis are evaluated at the initial condition $\tau = 0$, for which they are constant[5] and not divergent. If we choose to set the constants in (8.10) to zero $c_1^\delta(\mathbf{x}) = 0, c_2^\delta(\mathbf{x}) = 0$ (as, for $\tau = 0$, the decreasing mode would become divergent and the growing mode would anyway be sent to zero), the constants in (8.13) can be set to give the same result:

$$c_1^\delta = \left[\frac{3}{10\tau^2}(\chi_{ij}^{(1)\mathrm{T}}\chi^{(1)\mathrm{T}ij} - \chi_{0ij}^{(1)\mathrm{T}}\chi_0^{(1)\mathrm{T}ij}) + 3c_1 + \frac{\chi_{ij}^{(1)\mathrm{T}'}\chi^{(1)\mathrm{T}ij}}{5\tau}\right|_{\tau=0} = 0 \rightarrow$$
$$\rightarrow c_1 = -\left[\frac{1}{10\tau^2}(\chi_{ij}^{(1)\mathrm{T}}\chi^{(1)\mathrm{T}ij} - \chi_{0ij}^{(1)\mathrm{T}}\chi_0^{(1)\mathrm{T}ij}) + \frac{\chi_{ij}^{(1)\mathrm{T}'}\chi^{(1)\mathrm{T}ij}}{15\tau}\right|_{\tau=0};$$
$$c_2^\delta = \left[\frac{\tau^3}{5}(\chi_{ij}^{(1)\mathrm{T}}\chi^{(1)\mathrm{T}ij} - \chi_{0ij}^{(1)\mathrm{T}}\chi_0^{(1)\mathrm{T}ij}) + \frac{\tau^4}{5}\chi_{ij}^{(1)\mathrm{T}'}\chi^{(1)\mathrm{T}ij} + 3c_2\right|_{\tau=0} = 0 + 3c_2 = 0 \rightarrow c_2 = 0\,.$$

The final expressions for $\delta^{(2)}_{\mathrm{S(t)}}$ and $\phi^{(2)}_{\mathrm{S(t)}}$ become:

$$\delta^{(2)}_{\mathrm{S(t)}}(\mathbf{x},\tau) = \frac{\tau^2}{10}\int_0^\tau d\tilde{\tau}\frac{\chi'^{(1)\mathrm{T}ij}\chi'^{(1)\mathrm{T}}_{ij}}{\tilde{\tau}} - \frac{1}{10\tau^3}\int_0^\tau d\tilde{\tau}\,\tilde{\tau}^4\chi'^{(1)\mathrm{T}ij}\chi'^{(1)\mathrm{T}}_{ij}, \tag{8.14}$$

$$\begin{aligned}\phi^{(2)}_{\mathrm{S(t)}}(\mathbf{x},\tau) = {} & \frac{\tau^2}{5}\int_0^\tau \frac{1}{6\tilde{\tau}}\chi_{ij}^{(1)\mathrm{T}'}\chi^{(1)\mathrm{T}ij'}d\tilde{\tau} - \frac{1}{5\tau^3}\int_0^\tau \frac{\tilde{\tau}^4}{6}\chi_{ij}^{(1)\mathrm{T}'}\chi^{(1)\mathrm{T}ij'}d\tilde{\tau} \\ & - \frac{1}{6}(\chi_{ij}^{(1)\mathrm{T}}\chi^{(1)\mathrm{T}ij} - \chi_{0ij}^{(1)\mathrm{T}}\chi_0^{(1)\mathrm{T}ij}),\end{aligned} \tag{8.15}$$

which are both null[6] for $\tau = 0$: $\delta^{(2)}_{\mathrm{S(t)}}(0) = 0, \phi^{(2)}_{\mathrm{S(t)}}(0) = 0$.

[5] For this estimation, we have simply considered a limit for $\tau \rightarrow 0$ of the Fourier transform of the terms in parenthesis, leaving the other variables constant (the wavemode k and the variable k' coming from the convolution appearing in Fourier space, see (8.30)). A complete treatment should take into account also the integration over k'.

[6] Also the term $-\frac{1}{10\tau^3}\int_0^\tau \tilde{\tau}^4\chi_{ij}^{(1)\mathrm{T}'}\chi^{(1)\mathrm{T}ij'}d\tilde{\tau}$ is null for $\tau \rightarrow 0$: in fact, as we would have an indeterminate form $\frac{0}{0}$, we can use the de l'Hôpital theorem, for which

$$\lim_{\tau\rightarrow 0}\frac{\int_0^\tau \tilde{\tau}^4\chi_{ij}^{(1)\mathrm{T}'}\chi^{(1)\mathrm{T}ij'}d\tilde{\tau}}{\tau^3} \simeq \lim_{\tau\rightarrow 0}\frac{\tau^4\chi_{ij}^{(1)\mathrm{T}'}\chi^{(1)\mathrm{T}ij'}}{\tau^2} \rightarrow 0.$$

This estimation relies only on limits in τ, keeping the other variables fixed. Also here, a complete procedure should take into account the integration over the variable k' of the convolution in Fourier space.

The second-order density contrast in the Poisson gauge has been derived by a gauge transformation from the result obtained in the synchronous gauge (6.9):

$$\begin{aligned}\delta^{(2)}_{P(t)} &= \delta^{(2)}_{S(t)} - \frac{6}{\tau}\alpha^{(2)}_{(t)} = \frac{1}{2}\left(\chi^{(1)T}_{ij}\chi^{(1)Tij} - \chi^{(1)T}_{0ij}\chi^{(1)Tij}_{0}\right) + 3\phi^{(2)}_{S(t)} - \frac{6}{\tau}\alpha^{(2)}_{(t)} = \\ &= \frac{\tau^2}{10}\int_0^\tau d\tilde{\tau}\frac{\chi'^{(1)T\,ij}\chi'^{(1)T}_{ij}}{\tilde{\tau}} - \frac{1}{10\tau^3}\int_0^\tau d\tilde{\tau}\,\tilde{\tau}^4\chi'^{(1)T\,ij}\chi'^{(1)T}_{ij} - \frac{6}{\tau}\alpha^{(2)}_{(t)},\end{aligned} \tag{8.16}$$

where the term coming from the gauge transformation is:

$$-\frac{6}{\tau}\,\alpha^{(2)}_{(t)} = \frac{9}{2\tau}\nabla^{-2}\nabla^{-2}\chi'^{(2)\,i,j}_{S(t)\,j,i}\,.$$

An expression for $\chi'^{(2)\,ij}_{S(t)\,,ij}$ can be obtained deriving in space the momentum constraint found in the synchronous gauge (considering only tensors in the source term):

$$\begin{aligned}2\nabla^2\phi'^{(2)}_{S(t)} + \frac{1}{2}\chi'^{(2)\,i,j}_{S(t)\,j,i} &= \chi^{(1)T\,ik,j}\chi'^{(1)T}_{kj,i} - \frac{3}{2}\chi^{(1)T\,ik,j}\chi'^{(1)T}_{ik,j} - \chi^{(1)T\,ik}\nabla^2\chi'^{(1)T}_{ik} - \frac{1}{2}\chi'^{(1)T\,ik}\nabla^2\chi^{(1)T}_{ik} \equiv \\ &\equiv \mathcal{W}(\mathbf{x},\tau),\end{aligned} \tag{8.17}$$

from which we obtain:

$$\begin{aligned}-\frac{6}{\tau}\,\alpha^{(2)}_{(t)} &= -\frac{18}{\tau}\nabla^{-2}\phi'^{(2)}_{S(t)} + \frac{9}{\tau}\nabla^{-2}\nabla^{-2}\mathcal{W}(\mathbf{x},\tau) = \\ &= -\frac{18}{\tau}\nabla^{-2}\left(\frac{2\tau}{5}\int_0^\tau \frac{1}{6\tilde{\tau}}\chi^{(1)T'}_{ij}\chi^{(1)T\,ij'}d\tilde{\tau} + \frac{3}{5\tau^4}\int_0^\tau \frac{\tilde{\tau}^4}{6}\chi^{(1)T'}_{ij}\chi^{(1)T\,ij'}d\tilde{\tau} - \frac{1}{3}\chi^{(1)T}_{ij}\chi'^{(1)Tij}\right) \\ &+ \frac{9}{\tau}\nabla^{-2}\nabla^{-2}\left(\chi^{(1)T\,ik,j}\chi'^{(1)T}_{kj,i} - \frac{3}{2}\chi^{(1)T\,ik,j}\chi'^{(1)T}_{ik,j} - \chi^{(1)T\,ik}\nabla^2\chi'^{(1)T}_{ik} - \frac{1}{2}\chi'^{(1)T\,ik}\nabla^2\chi^{(1)T}_{ik}\right).\end{aligned} \tag{8.18}$$

For completeness, we recall also the second order gravitational potentials in the Poisson gauge derived by gauge transformation from the comoving synchronous gauge (where we have already neglected all the scalar terms):

$$\begin{aligned}\nabla^2\nabla^2\psi^{(2)}_{P(t)} &= \frac{18}{\tau^2}\nabla^2\phi^{(2)}_{S(t)} + \frac{5}{8}\nabla^2(\chi^{(1)T'}_{ij}\chi^{(1)T\,ij'}) - \frac{1}{\tau}\chi^{(1)T}_{ij}\nabla^2\chi^{(1)T\,ij'} - \frac{3}{2}\nabla^2\chi^{(1)T}_{kj,i}\chi^{(1)T\,ij,k} \\ &+ \frac{1}{2\tau}\chi^{(1)T\,ij'}\nabla^2\chi^{(1)T}_{ij} - \frac{1}{2\tau}\chi^{(1)T\,ij,k'}\chi^{(1)T}_{ij,k} + \frac{3}{\tau}\chi^{(1)T'}_{ij,k}\chi^{(1)T\,kj,i} - \frac{1}{4}\nabla^2\chi^{(1)T}_{ij}\nabla^2\chi^{(1)T\,ij} \\ &+ \frac{1}{4}\nabla^2\chi^{(1)T}_{ij,k}\chi^{(1)T\,ij,k} - \frac{3}{2}\chi^{(1)T'}_{ij,k}\chi^{(1)T\,kj,i'} + \frac{1}{2}\chi^{(1)T}_{ij}\nabla^2\nabla^2\chi^{(1)T\,ij} \\ &+ \frac{3}{\tau^2}\nabla^2\left(\chi^{(1)T\,ij}\chi^{(1)T}_{ij} - \chi^{(1)T\,ij}_{0}\chi^{(1)T}_{0ij}\right);\end{aligned} \tag{8.19}$$

$$
\begin{aligned}
\nabla^2\nabla^2\phi^{(2)}_{\mathrm{P(t)}} = {} & \frac{18}{\tau^2}\nabla^2\phi^{(2)}_{\mathrm{S(t)}} + \frac{1}{8}\nabla^2\left(\chi^{(1)\mathrm{T}\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\prime}\right) - \frac{1}{\tau}\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij\prime} + \frac{1}{2}\nabla^2\chi^{(1)\mathrm{T}}_{kj,i}\chi^{(1)\mathrm{T}\,ij,k} \\
& + \frac{1}{2\tau}\chi^{(1)\mathrm{T}\,ij\prime}\nabla^2\chi^{(1)\mathrm{T}}_{ij} - \frac{1}{2\tau}\chi^{(1)\mathrm{T}\prime}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k} + \frac{3}{\tau}\chi^{(1)\mathrm{T}\prime}_{kj,i}\chi^{(1)\mathrm{T}\,ij,k} - \frac{1}{2}\nabla^2\left(\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij}\right) \\
& - \frac{3}{4}\nabla^2\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k} - \frac{3}{4}\chi^{(1)\mathrm{T}}_{ij,kl}\chi^{(1)\mathrm{T}\,ij,kl} + \frac{1}{2}\chi^{(1)\mathrm{T}}_{ij,kl}\chi^{(1)\mathrm{T}\,kj,il} \\
& + \frac{3}{\tau^2}\nabla^2\left(\chi^{(1)\mathrm{T}\,ij}\chi^{(1)\mathrm{T}}_{ij} - \chi_0^{(1)\mathrm{T}\,ij}\chi^{(1)\mathrm{T}}_{0\,ij}\right),
\end{aligned}
\tag{8.20}
$$

in which one simply needs to substitute the expression (8.15) for $\phi^{(2)}_{\mathrm{S(t)}}$.

8.2 Derivation of the second-order scalar perturbations starting from the Poisson gauge

In this section we are going to obtain an evolution equation for $\delta^{(2)}_{\mathrm{P(t)}}$ directly in the Poisson gauge and to compare the results obtained with the ones previously found.

To do that, one has to solve the Einstein equations in the chosen gauge. We recall the line element in the Poisson gauge:

$$
ds^2 = a^2(\tau)\left\{-(1+\psi^{(2)})\,d\tau^2 + \left(\frac{1}{2}\omega^{(2)}_i\right)d\tau dx^i + \left[(1-\phi^{(2)})\,\delta_{ij} + \chi^{(1)\mathrm{T}}_{ij} + \frac{1}{2}\chi^{(2)\mathrm{T}}_{ij}\right]dx^i dx^j\right\},
$$

where we used the gauge conditions $\omega^{(r)\parallel} = 0$, $\chi^{(r)\parallel} = 0$ and $\chi^{(r)\perp}_i = 0$, plus the conditions coming from the fact we are dealing with a perfect, irrotational fluid: $v^{(1)\perp}_{\mathrm{P}\,i} = \omega^{(1)}_{\mathrm{P}\,i} = \chi^{(1)\perp}_{\mathrm{P}\,i} = 0$. To simplify this derivation and making it as concise as possible, we have already put ourselves in the situation in which no first order scalar exists: $\psi^{(1)} = \phi^{(1)} = \varphi = 0$, $v^{(1)\parallel}_{\mathrm{P}} = -\frac{\tau}{3}\varphi = 0$.

Using the results for the Einstein tensors listed in the Appendix E, the equations needed to solve our variables are:

Energy constraint (00 Einstein equation):

$$\frac{12}{\tau^2}\psi^{(2)}_{\mathrm{P(t)}} + \frac{6}{\tau}\phi'^{(2)}_{\mathrm{P(t)}} - \nabla^2\phi^{(2)}_{\mathrm{P(t)}} =$$
$$= -\frac{6}{\tau^2}\delta^{(2)}_{\mathrm{P(t)}} - \frac{1}{8}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime} + \frac{1}{2}\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij} - \frac{2}{\tau}\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime} + \frac{3}{8}\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k}$$
$$- \frac{1}{4}\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ik,j}; \tag{8.21}$$

Spatial derivative of momentum constraint (0i Einstein equation):

$$-\frac{2}{\tau}\nabla^2\psi^{(2)}_{\mathrm{P(t)}} - \nabla^2\phi'^{(2)}_{\mathrm{P(t)}} =$$
$$= \frac{6}{\tau^2}v^{(2)\,,i}_{\mathrm{P(t)}\,i} - \frac{1}{2}\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ik,j\,\prime} + \frac{1}{4}\chi^{(1)\mathrm{T}\,\prime}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij} + \frac{3}{4}\chi^{(1)\mathrm{T}\,\prime}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k} + \frac{1}{2}\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij\,\prime}; \tag{8.22}$$

Trace of the ij Einstein equation:

$$\begin{aligned}&\frac{6}{\tau}\psi'^{(2)}_{\mathrm{P(t)}} + \nabla^2\psi^{(2)}_{\mathrm{P(t)}} + 3\phi''^{(2)}_{\mathrm{P(t)}} + \frac{12}{\tau}\phi'^{(2)}_{\mathrm{P(t)}} - \nabla^2\phi^{(2)}_{\mathrm{P(t)}} =\\ &= -\frac{1}{2}\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij} + \frac{3}{8}\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k} - \frac{1}{4}\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ik,j} - \frac{5}{8}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime};\end{aligned} \tag{8.23}$$

Traceless part of the ij Einstein equation:

$$\nabla^2\nabla^2\phi^{(2)}_{\mathrm{P(t)}} - \nabla^2\nabla^2\psi^{(2)}_{\mathrm{P(t)}} =$$
$$= -\nabla^2\left[\frac{1}{2}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime} + \chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij} + \frac{3}{8}\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k} - \frac{1}{4}\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,jk,i}\right]$$
$$+ \frac{3}{4}\chi^{(1)\mathrm{T}}_{ij,k}\nabla^2\chi^{(1)\mathrm{T}\,ij,k} + \frac{3}{4}\nabla^2\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij} + \frac{3}{2}\chi^{(1)\mathrm{T}\,\prime}_{ij,k}\chi^{(1)\mathrm{T}\,jk,i\,\prime} + \frac{3}{2}\chi^{(1)\mathrm{T}}_{ij,k}\nabla^2\chi^{(1)\mathrm{T}\,kj,i}\,. \tag{8.24}$$

with this last equation deriving from the traceless construction of the ij Einstein equation: $\delta G^{(2)\,i,j}_{j,i} - \frac{\nabla^2}{3}\delta G^{(2)\,i}_{i}$.

Two other useful relations come from the conservation of the stress-energy tensor $D_\mu T^{\mu\nu} = 0$ [15], which, adapted to our case, become:

$D_\mu T^\mu_0 = 0$:

$$\delta'^{(2)}_{\mathrm{P(t)}} = 3\phi'^{(2)}_{\mathrm{P(t)}} + \frac{1}{2}(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij})' - v^{(2)\,,i}_{\mathrm{P(t)}\,i}; \tag{8.25}$$

$D_\mu T^\mu_i = 0$:

$$v'^{(2),i}_{\mathrm{P(t)}\,i} = -\frac{2}{\tau}\, v^{(2),i}_{\mathrm{P(t)}\,i} - \nabla^2 \psi^{(2)}_{\mathrm{P(t)}} \,. \tag{8.26}$$

Combining the definition of the second-order velocity perturbation in the Poisson gauge (equation (6.10), considering only terms generated by tensors) and the expressions of the gauge parameters (D.1), we can find that:

$$v^{(2),i}_{\mathrm{P(t)}\,i} = -\nabla^2 \alpha^{(2)}_{\mathrm{(t)}}, \tag{8.27}$$

holding when we consider linear tensor modes only, neglecting all linear scalars. This way, we can find again the expression for the gauge transformation of $\psi^{(2)}_{\mathrm{P(t)}}$ (D.2):

$$\nabla^2 \psi^{(2)}_{\mathrm{P(t)}} = \nabla^2 \alpha'^{(2)}_{\mathrm{(t)}} + \frac{2}{\tau}\, \nabla^2 \alpha^{(2)}_{\mathrm{(t)}} \,,$$

from which (8.19) is obtained.

To simplify our derivation and the comparison with what already found from the comoving synchronous gauge, we can verify that the expressions for $\psi^{(2)}_{\mathrm{P(t)}}$, $\phi^{(2)}_{\mathrm{P(t)}}$ already obtained (equations (8.19), (8.20)) satisfy the trace and traceless part of the ij Einstein equations in the Poisson gauge (equations (8.23), (8.24)). They are clearly the solutions we need to obtain the expressions of $v^{(2),i}_{\mathrm{P(t)}\,i}$ and $\delta^{(2)}_{\mathrm{P(t)}}$ (and to check they agree with what already found by gauge transformation).

So, it can be verified that the relation (8.27) holds by deriving the expression for $\frac{6}{\tau}\, \nabla^{-2} v^{(2),i}_{\mathrm{P(t)}\,i}$ through the momentum constraint in the Poisson gauge (8.22) and equations (8.19), (8.20): we would obtain exactly $\frac{6}{\tau}\, \nabla^{-2} v^{(2),i}_{\mathrm{P(t)}\,i}$= (8.18) = $-\frac{6}{\tau}\, \alpha^{(2)}_{\mathrm{(t)}}$, which is the term coming from gauge transformation in the expression of $\delta^{(2)}_{\mathrm{P(t)}}$.

The most interesting expression to obtain is the evolution equation for the second-order density contrast in this gauge. It can be obtained deriving in time the continuity equation (8.25) and replacing in it the equation (8.26) and the difference between the 00 and the traced ij Einstein equations ((8.21) - (8.23)):

$$\delta''^{(2)}_{\mathrm{P(t)}} + \frac{2}{\tau}\, \delta'^{(2)}_{\mathrm{P(t)}} - \frac{6}{\tau^2}\, \delta^{(2)}_{\mathrm{P(t)}} = \frac{1}{2} \chi^{(1)\mathrm{T}\prime}_{ij} \chi^{(1)\mathrm{T}\,ij\prime} - \frac{6}{\tau}\, \psi'^{(2)}_{\mathrm{P(t)}} + \frac{12}{\tau^2}\, \psi^{(2)}_{\mathrm{P(t)}} \,. \tag{8.28}$$

It is straightforward to see that the first term in the source of the inhomogeneous equation (8.28) is the same as the source of the evolution equation found in the synchronous gauge (8.9), while the other two terms depend on the choice of gauge. In fact, it can be verified that they come from the gauge transformation term $-\frac{6}{\tau}\alpha^{(2)}_{(t)} = \frac{6}{\tau}\nabla^{-2}v^{(2),i}_{P(t)\,i}$:

$$\left(-\frac{6}{\tau}\alpha^{(2)}_{(t)}\right)'' + \frac{2}{\tau}\left(-\frac{6}{\tau}\alpha^{(2)}_{(t)}\right)' - \frac{6}{\tau^2}\left(-\frac{6}{\tau}\alpha^{(2)}_{(t)}\right) = -\frac{6}{\tau}\psi'^{(2)}_{P(t)} + \frac{12}{\tau^2}\psi^{(2)}_{P(t)}\,,$$

where we have used also (8.26) to check that. Because of these additional gauge terms, the evolution equation in the Poisson gauge appears to be less transparent than the one found in the comoving synchronous gauge.

So, the expression (8.16) for $\delta^{(2)}_{P(t)}$ found by gauge transformation is the solution for the evolution equation (8.28), as it could also be checked by actually solving it and setting the right initial conditions (as already done before for the solution in the comoving synchronous gauge):

$$\begin{aligned}
\delta^{(2)}_{P(t)} &= \tilde{c}^{\delta}_1(\mathbf{x})\tau^2 + \tilde{c}^{\delta}_2(\mathbf{x})\tau^{-3} + \frac{\tau^2}{5}\int_0^{\tau}\frac{1}{2\tilde{\tau}}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde{\tau} + \frac{\tau^2}{5}\left(-\frac{6}{\tau^2}\psi^{(2)}_{P(t)}\right|_0^{\tau} + \frac{1}{5\tau^3}(6\tau^3\psi^{(2)}_{P(t)}\big|_0^{\tau} \\
&- \frac{1}{5\tau^3}\int_0^{\tau}\frac{\tilde{\tau}^4}{2}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde{\tau} + \frac{6}{\tau^3}(\tau^2\nabla^{-2}v^{(2),i}_{P(t)\,i}\big|_0^{\tau} = \\
&= \frac{\tau^2}{10}\int_0^{\tau}\frac{1}{\tilde{\tau}}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde{\tau} - \frac{1}{10\tau^3}\int_0^{\tau}\tilde{\tau}^4\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde{\tau} + \frac{6}{\tau}\nabla^{-2}v^{(2),i}_{P(t)\,i} \\
&- \frac{3\tau^2}{5}\left(-\frac{5}{3}\tilde{c}^{\delta}_1 - \frac{2}{\tau^2}\psi^{(2)}_{P(t)}\right|_{\tau=0} - \frac{1}{5\tau^3}(-5\tilde{c}^{\delta}_2 + 6\tau^3\psi^{(2)}_{P(t)} + 30\nabla^{-2}v^{(2),i}_{P(t)\,i}\big|_{\tau=0} = \\
&= \frac{\tau^2}{10}\int_0^{\tau}\frac{1}{\tilde{\tau}}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde{\tau} - \frac{1}{10\tau^3}\int_0^{\tau}\tilde{\tau}^4\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}d\tilde{\tau} + \frac{6}{\tau}\nabla^{-2}v^{(2),i}_{P(t)\,i}\,.
\end{aligned} \tag{8.29}$$

8.3 Time evolution of the second-order density contrast generated by tensors

Our final goal is to study the behaviour in time time of the second-order density contrast, considering only the contribution from linear gravitational waves. We want to see in particular whether it can grow in time and its possible growth rate.

It is more useful to pass to the Fourier space (8.1) in this case, where we can isolate the amplitudes $\chi^{(1)}_{\sigma}(\mathbf{k},\tau)$ (8.2) from the polarization tensors and inside $\chi^{(1)}_{\sigma}(\mathbf{k},\tau)$

we can work with the functions governing the time evolution of the linear gravitational waves in the Einstein-de Sitter Universe:

$$\chi_\sigma(\mathbf{k},\tau) = A_\sigma(\mathbf{k})\left(3\frac{j_1(k\tau)}{k\tau}\right), \quad \chi'_\sigma(\mathbf{k},\tau) = A_\sigma(\mathbf{k})\,3\left(\frac{\sin(k\tau)}{k\tau^2} - 3\frac{j_1(k\tau)}{k\tau^2}\right).$$

When passing to the Fourier space, we have to deal with the Fourier transform of products of functions. The procedure can be shown for a single term, like $\chi_{ij}^{(1)\mathrm{T}}\chi^{(1)\mathrm{T}\,ij}$:

$$\begin{aligned}
&\chi_{ij}^{(1)\mathrm{T}}(\mathbf{x},\tau)\chi^{(1)\mathrm{T}\,ij}(\mathbf{x},\tau) = \\
&= \sum_{\sigma,\sigma'=+,\times} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\, e^{i\mathbf{k}_1\cdot\mathbf{x}} \chi_\sigma^{(1)}(\mathbf{k}_1,\tau)\, \epsilon_{ij}^{\sigma}(\hat{\mathbf{k}}_1) \int \frac{d^3\mathbf{k}_2}{(2\pi)^3}\, e^{i\mathbf{k}_2\cdot\mathbf{x}} \chi_{\sigma'}^{(1)}(\mathbf{k}_2,\tau)\, \epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}_2) = \\
&= \sum_{\sigma,\sigma'=+,\times} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3}\, e^{i\mathbf{k}_1\cdot\mathbf{x}} \chi_\sigma^{(1)}(\mathbf{k}_1,\tau)\, \epsilon_{ij}^{\sigma}(\hat{\mathbf{k}}_1)\, e^{i\mathbf{k}_2\cdot\mathbf{x}} \chi_{\sigma'}^{(1)}(\mathbf{k}_2,\tau)\, \epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}_2) = \\
&= \sum_{\sigma,\sigma'=+,\times} \int \frac{d^3\mathbf{k}_1}{(2\pi)^3} \int \frac{d^3\mathbf{k}_2}{(2\pi)^3}\, e^{i(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{x}} \chi_\sigma^{(1)}(\mathbf{k}_1,\tau)\, \epsilon_{ij}^{\sigma}(\hat{\mathbf{k}}_1) \chi_{\sigma'}^{(1)}(\mathbf{k}_2,\tau)\, \epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}_2)
\end{aligned}$$

and redefining the variables $\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$, $\mathbf{k}' = \mathbf{k}_2$, $\mathbf{k}_1 = \mathbf{k} - \mathbf{k}'$, we can rewrite it:

$$\begin{aligned}
&\chi_{ij}^{(1)\mathrm{T}}\chi^{(1)\mathrm{T}\,ij} = \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{\sigma,\sigma'=+,\times} \int \frac{d^3\mathbf{k}'}{(2\pi)^3} \chi_\sigma^{(1)}(\mathbf{k}-\mathbf{k}',\tau)\chi_{\sigma'}^{(1)}(\mathbf{k}',\tau)\, \epsilon_{ij}^{\sigma}(\widehat{\mathbf{k}-\mathbf{k}'})\, \epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}') = \\
&= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} \sum_{\sigma,\sigma'=+,\times} \int \frac{d^3\mathbf{k}'}{(2\pi)^3}\, \epsilon_{ij}^{\sigma}(\widehat{\mathbf{k}-\mathbf{k}'})\, \epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}')\times \\
&\times\left[A_\sigma(\mathbf{k}-\mathbf{k}')\, A_{\sigma'}(\mathbf{k}')\left(\frac{3j_1[|\mathbf{k}-\mathbf{k}'|\tau]}{|\mathbf{k}-\mathbf{k}'|\tau}\right)\left(\frac{3j_1(k'\tau)}{k'\tau}\right)\right].
\end{aligned} \tag{8.30}$$

The Fourier transform of products of functions like $\chi_{ij}^{(1)\mathrm{T}}\chi^{(1)\mathrm{T}\,ij}$ gives the convolution of their Fourier transforms. A complete estimation of those products should pass from the integral in the mute variable of the convolution $\mathbf{k}'$, that in full generality should span from 0 to an arbitrarily big wavemode. In general, divergencies corresponding to particular values of k' (depending also on the choice of the physical scale k) could appear, such that this integration should be performed with infrared or ultraviolet cutoffs. Also, the expressions for the contractions of the polarization tensors should be found: they are functions of the angles between the vectors $\widehat{\mathbf{k}-\mathbf{k}'}$ and $\hat{\mathbf{k}}'$, which would be integrated through $\int d^3\mathbf{k}' =$

$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_{k'_{\min}}^{k'_{\max}} dk' (k')^2 \sin\theta.$

To evaluate the evolution in time of the second-order density contrast we can restrict ourselves to the easiest expression (the one in the comoving synchronous gauge, where we have no additional gauge term) and we can perform the complete procedure described before. Passing to the Fourier space, $\delta^{(2)}_{\mathrm{S(t)}}$ becomes:

$$\delta^{(2)}_{\mathrm{S(t)}}(\mathbf{k},\tau) = \sum_{\sigma,\sigma'=+,\times} \int \frac{d^3\mathbf{k}'}{(2\pi)^3}\, \epsilon^{\sigma}_{ij}(\widehat{\mathbf{k}-\mathbf{k}'})\, \epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}')\, A_\sigma(\mathbf{k}-\mathbf{k}')\, A_{\sigma'}(\mathbf{k}')\times$$
$$\times\left[\frac{\tau^2}{10}\int_0^{\tau}\frac{d\tilde\tau}{\tilde\tau}\left(3\frac{j_1[|\mathbf{k}-\mathbf{k}'|\tau]}{|\mathbf{k}-\mathbf{k}'|\tau}\right)'\left(3\frac{j_1(k'\tau)}{k'\tau}\right)' - \frac{1}{10\tau^3}\int_0^{\tau} d\tilde\tau\,\tilde\tau^4\left(3\frac{j_1[|\mathbf{k}-\mathbf{k}'|\tau]}{|\mathbf{k}-\mathbf{k}'|\tau}\right)'\left(3\frac{j_1(k'\tau)}{k'\tau}\right)'\right]. \tag{8.31}$$

A first thing to notice is that a valid perturbation, also at second-order, should satisfy $\langle\delta^{(2)}_{\mathrm{S(t)}}(\mathbf{k},\tau)\rangle = 0$, so we should consider $\delta^{(2)}_{\mathrm{S(t)}}(\mathbf{k},\tau) - \langle\delta^{(2)}_{\mathrm{S(t)}}(\mathbf{k},\tau)\rangle$ instead of (8.31). In the Fourier space, the mean value corresponds to $\langle\delta^{(2)}_{\mathrm{S(t)}}(\mathbf{k},\tau)\rangle = \delta^{(2)}_{\mathrm{S(t)}}(\mathbf{k}=0,\tau)$: the problem can be simplified considering only the modes with $\mathbf{k}\neq 0$, such that the previous expression can already represent a perturbation.

We can assume that the amplitudes of linear gravitational waves have a power-law dependence on $\mathbf{k}$, as predicted by inflation: $A_\sigma(\mathbf{k}) = A_\sigma(\mathbf{k}_*)(\frac{k}{k_*})^{n_{\mathrm{T}}}$, where $\mathbf{k}_*$ would be a particular scale. As a further, simplifying assumption, we can take this amplitude to be exactly scale-invariant (as n_{T} would be very small) and equal for the two polarizations, such that $A_\sigma(\mathbf{k}-\mathbf{k}') = A_{\sigma'}(\mathbf{k}') = A(k_*)$. Of course, we have not measured this amplitude yet, so, we are going to factor that out and consider eventually $\frac{\delta^{(2)}_{\mathrm{S(t)}}}{A^2(k_*)}$.

Before performing the integration in $\mathbf{k}'$, one has to obtain the expression for

$$\sum_{\sigma,\sigma'=+,\times} \epsilon^{\sigma}_{ij}(\widehat{\mathbf{k}-\mathbf{k}'})\, \epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}').$$

To do that, we recall the definitions of those polarization tensors (8.5), that depend on combinations of unit vectors orthogonal to the wavevector in their argument. So, we assume $\mathbf{k}$ oriented along the z-axis and $\hat{\mathbf{k}}'$ with generic angles with respect to it:

$$\hat{\mathbf{k}}' = \cos\varphi\sin\theta\,\hat{\mathbf{x}} + \sin\varphi\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}\,,$$

such that $\widehat{\mathbf{k}-\mathbf{k}'}$ has components:

$$\widehat{\mathbf{k}-\mathbf{k}'} = \frac{-k'\cos\varphi\sin\theta\,\hat{\mathbf{x}} - k'\sin\varphi\sin\theta\,\hat{\mathbf{y}} + (k - k'\cos\theta)\,\hat{\mathbf{z}}}{\sqrt{k^2 + (k')^2 - 2kk'\cos\theta}}\,.$$

The unit vectors orthogonal to $\widehat{\mathbf{k}-\mathbf{k}'}$ are:

$$\begin{aligned} \hat{u} &= (\sin\varphi, -\cos\varphi, 0)\,; \\ \hat{v} &= \left(\cos\varphi\cos\theta, \sin\varphi\cos\theta, \frac{k'\cos\theta\sin\theta}{k - k'\cos\theta}\right); \end{aligned} \tag{8.32}$$

while the ones orthogonal to $\hat{\mathbf{k}}'$ are:

$$\begin{aligned} \hat{m} &= (\sin\varphi, -\cos\varphi, 0)\,; \\ \hat{n} &= (\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta)\,. \end{aligned} \tag{8.33}$$

From that, one can compute the polarization tensors for both $\widehat{\mathbf{k}-\mathbf{k}'}$ and $\hat{\mathbf{k}}'$:

$$\begin{aligned} \epsilon^{\times}_{ij}(\widehat{\mathbf{k}-\mathbf{k}'}) &= \hat{u}_i\hat{v}_j + \hat{u}_j\hat{v}_i = \\ &= \begin{pmatrix} 2\cos\theta\cos\varphi\sin\varphi & \cos\theta\,(\sin^2\varphi - \cos^2\varphi) & \frac{k'\cos\theta\sin\theta\sin\varphi}{k-k'\cos\theta} \\ \cos\theta\,(\sin^2\varphi - \cos^2\varphi) & -2\cos\theta\cos\varphi\sin\varphi & -\frac{k'\cos\theta\sin\theta\cos\varphi}{k-k'\cos\theta} \\ \frac{k'\cos\theta\sin\theta\sin\varphi}{k-k'\cos\theta} & -\frac{k'\cos\theta\sin\theta\cos\varphi}{k-k'\cos\theta} & 0 \end{pmatrix}; \end{aligned}$$

$$\begin{aligned} \epsilon^{+}_{ij}(\widehat{\mathbf{k}-\mathbf{k}'}) &= \hat{u}_i\hat{u}_j - \hat{v}_i\hat{v}_j = \\ &= \begin{pmatrix} -\cos^2\theta\cos^2\varphi + \sin^2\varphi & -\cos\varphi\sin\varphi\,(1+\cos^2\theta) & -\frac{k'\cos^2\theta\sin\theta\cos\varphi}{k-k'\cos\theta} \\ -\cos\varphi\sin\varphi\,(1+\cos^2\theta) & \cos^2\varphi - \cos^2\theta\sin^2\varphi & -\frac{k'\cos^2\theta\sin\theta\sin\varphi}{k-k'\cos\theta} \\ -\frac{k'\cos^2\theta\sin\theta\cos\varphi}{k-k'\cos\theta} & -\frac{k'\cos^2\theta\sin\theta\sin\varphi}{k-k'\cos\theta} & -\frac{(k')^2\cos^2\theta\sin^2\theta}{(k-k'\cos\theta)^2} \end{pmatrix}; \end{aligned}$$

$$\begin{aligned} \epsilon^{\times}_{ij}(\hat{\mathbf{k}}') &= \hat{m}_i\hat{n}_j + \hat{m}_j\hat{n}_i = \\ &= \begin{pmatrix} 2\cos\theta\cos\varphi\sin\varphi & \cos\theta\,(\sin^2\varphi - \cos^2\varphi) & -\sin\theta\sin\varphi \\ \cos\theta\,(\sin^2\varphi - \cos^2\varphi) & -2\cos\theta\cos\varphi\sin\varphi & \sin\theta\cos\varphi \\ -\sin\theta\sin\varphi & \sin\theta\cos\varphi & 0 \end{pmatrix}; \end{aligned}$$

$$\epsilon^{+}_{ij}(\hat{\mathbf{k}}') = \hat{m}_i\hat{m}_j - \hat{n}_i\hat{n}_j =$$

$$= \begin{pmatrix} -\cos^2\theta\cos^2\varphi + \sin^2\varphi & -\cos\varphi\sin\varphi\,(1+\cos^2\theta) & \cos\theta\sin\theta\cos\varphi \\ -\cos\varphi\sin\varphi\,(1+\cos^2\theta) & \cos^2\varphi - \cos^2\theta\sin^2\varphi & \cos\theta\sin\theta\sin\varphi \\ \cos\theta\sin\theta\cos\varphi & \cos\theta\sin\theta\sin\varphi & -\sin^2\theta \end{pmatrix},$$

from which the sum $\sum_{\sigma,\sigma'=+,\times} \epsilon^{\sigma}_{ij}(\widehat{\mathbf{k}-\mathbf{k}'})\,\epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}') = \frac{[2k'\cos\theta - \frac{1}{2}k(3+\cos(2\theta))]^2}{(k-k'\cos\theta)^2}$ can be evaluated .

Plugging everything into (8.31):

$$\delta^{(2)}_{S(t)}(\mathbf{k},\tau) = A^2(k_*)\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_{k'_{\min}}^{k'_{\max}} \frac{dk'}{(2\pi)^3}(k')^2\sin\theta\,\frac{[2k'\cos\theta - \frac{1}{2}k(3+\cos(2\theta))]^2}{(k-k'\cos\theta)^2}\times$$
$$\times\left[\frac{\tau^2}{10}\int_0^{\tau}\frac{d\tilde{\tau}}{\tilde{\tau}}\left(3\frac{j_1[|\mathbf{k}-\mathbf{k}'|\tau]}{|\mathbf{k}-\mathbf{k}'|\tau}\right)'\left(3\frac{j_1(k'\tau)}{k'\tau}\right)' - \frac{1}{10\tau^3}\int_0^{\tau} d\tilde{\tau}\,\tilde{\tau}^4\left(3\frac{j_1[|\mathbf{k}-\mathbf{k}'|\tau]}{|\mathbf{k}-\mathbf{k}'|\tau}\right)'\left(3\frac{j_1(k'\tau)}{k'\tau}\right)'\right], \tag{8.34}$$

which is now complicated by the $\cos\theta$ dependence in the modulus $|\mathbf{k}-\mathbf{k}'|$, present in our functions of time.
To avoid this dependence and have a first estimate of this effect, we can make a further, simplifying assumption: we can assume a toy model in which gravitational waves propagate in just one direction. This way, $\mathbf{k}$ and $\mathbf{k}'$ would propagate along the same line, such that $|\mathbf{k}-\mathbf{k}'| = |k-k'|$, the 3-dim integral in $\mathbf{k}'$ would collapse in a 1-dim one:

$$\int \frac{d^3\mathbf{k}'}{(2\pi)^3} \longrightarrow \int_{-\bar{k}'}^{\bar{k}'} \frac{dk'}{(2\pi)},$$

and the contraction of the polarization tensors, according to (8.3), would become:

$$\sum_{\sigma,\sigma'=+,\times} \epsilon^{\sigma}_{ij}(\widehat{\mathbf{k}-\mathbf{k}'})\,\epsilon^{\sigma'\,ij}(\hat{\mathbf{k}}') = \frac{[2k'\cos\theta - \frac{1}{2}k(3+\cos(2\theta))]^2}{(k-k'\cos\theta)^2} \underset{\theta\to 0}{\longrightarrow} 4.$$

The new, simplified version of $\delta^{(2)}_{S(t)}(k,\tau)$ reads:

$$\delta^{(2)}_{S(t)}(k,\tau) = 4\,A^2(k_*)\int_{-\bar{k}'}^{\bar{k}'}\frac{dk'}{(2\pi)}\times$$
$$\times\left[\frac{\tau^2}{10}\int_0^{\tau}\frac{d\tilde{\tau}}{\tilde{\tau}}\left(3\frac{j_1[(k-k')\tau]}{(k-k')\tau}\right)'\left(3\frac{j_1(k'\tau)}{k'\tau}\right)' - \frac{1}{10\tau^3}\int_0^{\tau} d\tilde{\tau}\,\tilde{\tau}^4\left(3\frac{j_1[(k-k')\tau]}{(k-k')\tau}\right)'\left(3\frac{j_1(k'\tau)}{k'\tau}\right)'\right]. \tag{8.35}$$

This can be integrated numerically in Python (Appendix F), choosing an integration window for k' (which, in principle, should be from -∞ to ∞), a particular scale k and evaluating the whole expression up to a certain τ. We set $\bar{k}'$ not too big, not to make the program computationally heavy. The same also for τ: we consider only the interval $\tau = [0-2]$, which should anyway be enough to present the evolution of $\delta^{(2)}_{\mathrm{S(t)}}$.

The results of this integration procedure are shown in the following Figures: Fig. (5) and Fig. (6) show the trend for the integrals in the first and second term, respectively, and the whole terms themselves; Fig. (7) shows $\delta^{(2)}_{\mathrm{S(t)}}$, compared in particular with a generic function $\propto \tau^2$.

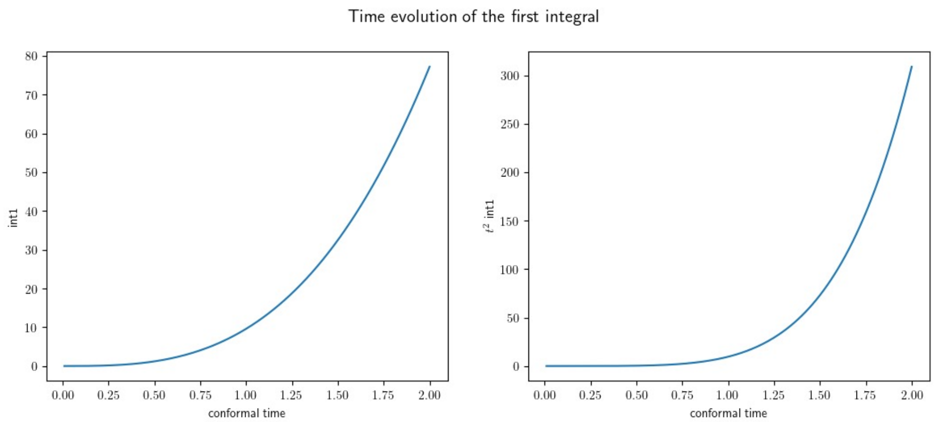

Figure 5: This Figure shows on the left the plot of the integral in the first term of $\delta^{(2)}_{\mathrm{S(t)}}$: int1 $= \int_{-\bar{k}'}^{\bar{k}'} \frac{dk'}{(2\pi)} 4 \frac{d\tilde{\tau}}{\tilde{\tau}} (3\frac{j_1[(k-k')\tau]}{(k-k')\tau})'(3\frac{j_1(k'\tau)}{k'\tau})'$, on the right, the complete first term $\tau^2 \int_{-\bar{k}'}^{\bar{k}'} \frac{dk'}{(2\pi)} 4 \int_0^{\tau} \frac{d\tilde{\tau}}{\tilde{\tau}} (3\frac{j_1[(k-k')\tau]}{(k-k')\tau})'(3\frac{j_1(k'\tau)}{k'\tau})'$. In the integration procedure, we have chosen $\bar{k}' = 10$ and $k = 10$; also the time window is quite small, but enough to appreciate the behaviour of our expressions. If we consider a dimensional scale factor, we can assume dimensionless comoving coordinates and conformal time.

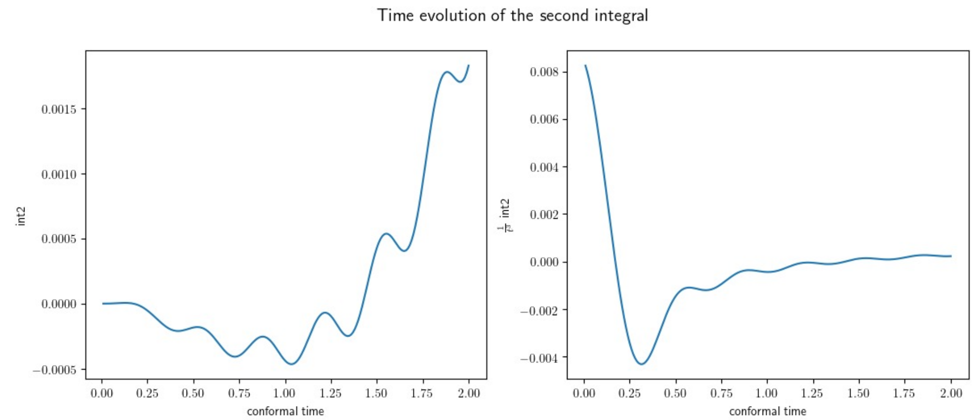

Figure 6: This Figure shows on the left the plot of the integral in the second term of $\delta^{(2)}_{\mathrm{S(t)}}$: int2 $= \int_{-\bar{k}'}^{\bar{k}'} \frac{dk'}{(2\pi)} 4 \int_0^\tau d\tilde{\tau}\, \tilde{\tau}^4 \,(3\frac{j_1[(k-k')\tau]}{(k-k')\tau})'(3\frac{j_1(k'\tau)}{k'\tau})'$, on the right, the (module of the) complete second term $\frac{1}{\tau^3}\int_{-\bar{k}'}^{\bar{k}'} \frac{dk'}{(2\pi)} 4 \int_0^\tau d\tilde{\tau}\, \tilde{\tau}^4 \,(3\frac{j_1[(k-k')\tau]}{(k-k')\tau})'(3\frac{j_1(k'\tau)}{k'\tau})'$. We have the same choice of parameters as before: $\bar{k}' = 10$, $k = 10$ and τ from 0 to 2.

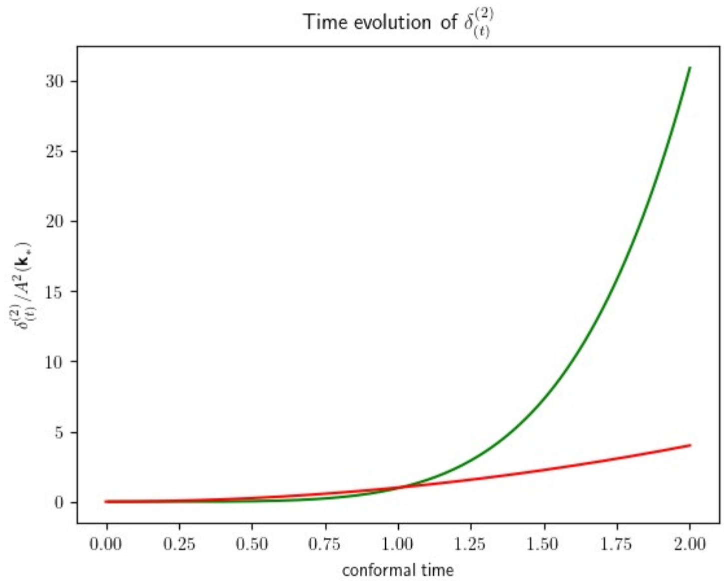

Figure 7: This Figure shows the behaviour of $\delta^{(2)}_{\mathrm{S(t)}}$ (in green), compared with a simple function $\propto \tau^2$ (in red). For this particular choice of k and of integration window for k', our result appears to have a growth higher than τ^2, provided by the contribution of the integral 1 (Fig. (6)). In particular, as we have not measured the amplitude of linear gravitational waves yet, we have estimated the fractional quantity $\frac{\delta^{(2)}_{\mathrm{S(t)}}}{A^2(k_*)}$. We have the same choice of parameters as before: $\bar{k}' = 10$, $k = 10$ and τ from 0 to 2.

It is not straightforward to always have a density contrast growing with a rate higher than τ^2: this will depend on the integration in k' (so, on the extremes of integration chosen) and on the choice of k, as well. For some scales k, the integral in the first term appears to be negative, representing an underdensity. This can be explained by the presence of k in oscillating functions, resulting in an oscillating dependence not only on time, but also on the scale itself. This eventually results in perturbations highly suppressed for those particular scales.
Of course, the simplifying assumption of gravitational waves propagating in just one direction could have oversimplified our estimation and made the growth rate too strong. In any case, there should anyway be a growth in time.

We can also quickly discuss the evolution of the second-order gravitational potentials in the Poisson gauge $\psi^{(2)}_{\mathrm{P(t)}}, \phi^{(2)}_{\mathrm{P(t)}}$. Their most relevant term is:

$$\psi^{(2)}_{\mathrm{P(t)}}, \phi^{(2)}_{\mathrm{P(t)}} \propto \frac{6}{\tau^2}\nabla^{-2}\left(3\phi^{(2)}_{\mathrm{S}(t)} + \frac{1}{2}\left(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}ij} - \chi^{(1)\mathrm{T}}_{0ij}\chi^{(1)\mathrm{T}ij}_{0}\right)\right) = \frac{6}{\tau^2}\nabla^{-2}\delta^{(2)}_{\mathrm{S(t)}}$$

which means that, even though they are affected by a factor τ^{-2}, they could still grow in time if the integral in the first term of $\delta^{(2)}_{\mathrm{S(t)}}$ is increasing in time. The other terms in the expressions of $\psi^{(2)}_{\mathrm{P(t)}}, \phi^{(2)}_{\mathrm{P(t)}}$ are all decreasing functions of τ and they are not integrated, so they would give no relevant contribution at late times.
The presence of the inverse laplacian ∇^{-2} provides a factor k^{-2}, making the contribution of the smallest scales (highest k) the most negligible.

9 Conclusions

In this work the second-order cosmological perturbations have been studied in the context of a matter dominated Universe, with perfect irrotational fluid and in two gauges. In general, one or more of these approximations can be dropped, considering the effect of a cosmological constant (like in [5,6,19]), of non-irrotational fluids or more than one fluid (e.g. [15]). Of course, the same procedure can be repeated also for different phases, like a radiation-dominated phase: this would be necessary to extrapolate results about observable quantities, that would have to experience this phase before entering the matter-dominated one.
Also, we have arbitrarily neglected the first-order scalars in the source terms of our second-order expressions, to simplify our derivation and to highlight the pure tensor contribution. Of course, this way we have neglected also the mixed scalar-tensor terms in the source of the second-order quantities, which could provide other modes.

Considering only tensor modes is interesting also in the perspective of a different model of the early Universe, where there is no substantial production of scalar perturbations. Those could be subdominant with respect to the tensor ones and more relevant scalars would be produced at second-order. Those second-order scalar perturbations sourced by tensors could in principle undergo gravitational instability, for what we have seen; of course, they would not be the main seeds of the structures that we observe. In fact, second-order perturbations provide a source of non-Gaussianity that would not agree with the observed statistics of density perturbations, which is almost Gaussian.
Models where the inflaton field does not provide the scalar perturbations that we observe already exist: for example, in the curvaton scenario [9, 14] the curvature perturbations are not produced by the inflaton field, but by another light scalar field, whose energy density is subdominant during inflation. The perturbations from the curvaton become then adiabatic when it decays into radiation much after the end of inflation.

Sticking to the classic scenario described in section 2, the approximations we have assumed could allow a description of an early matter-dominated era after inflation, like reheating, where the effects of a cosmological constant would not be that important. The perturbations produced from inflation would then enter the horizon after its end, which would be indicated by our initial condition $\tau = 0$ (while the inflation would conventionally start from $\tau = -\infty$). Also the approximation of negligible linear vector perturbations is fine, as they are not produced during an inflation driven by scalar fields (and anyway, they would decay in time

once produced).
If one would consider the production during the "classical" matter-dominated phase, one has to account for the passage through the preceding radiation-dominated one, so the initial conditions should be adjusted to have a smooth passage from one phase to another.
Our choice of the initial conditions for the expressions of the second-order density contrast has been dictated more by the necessity of having a "clean", simple expression, which would highlight more the dependence on the first two terms than on other constants coming from the initial time. Of course, it is straightforward to set these constants different from zero (in (8.10)), to set the initial conditions e.g. also for the derivatives of our expressions.

Of course, studying the propagation of second-order perturbations produced by tensor modes could be important only in the case in which they can effectively grow in time and become non-negligible in the passage from one era to another. As we have seen in the last section, it can be possible to have a growth rate for the second-order density contrast which is even higher than a trend $\propto \tau^2$. This is due to the fact that the growing mode $\propto \tau^2$ is multiplied by an integral which is an increasing function of τ. So, the growth rate would be higher that the one of the classical, linear scalar perturbations, which is $\propto \tau^2 \propto t^{2/3}$ in a matter-dominated Universe. Actually, from Figure (7), it seems to be even higher than the time dependence of the part of $\delta^{(2)}$ generated by linear scalars, which grows like $\propto \tau^4$. This is true at least for the very small time window that we have considered, for our particular choice of parameters and of integration in k' (the mute variable of the convolution in Fourier space). We expect this to change, even only for the dependence on k of the oscillating functions that are integrated: for some different choice of k (we have verified that for $k = 100$) the integral in that growing mode is negative, so $\delta^{(2)}_{\mathrm{S(t)}}$ stays negative and becomes smaller and smaller. This would indicate that, for this particular scale, the density perturbations would represent underdensities.
Some changes are expected also by the choice of a different UV cutoff for the integration in k'.

What we have actually plotted in the previous section is $\frac{\delta^{(2)}_{\mathrm{S(t)}}}{A^2(k_*)}$, to reabsorb the still unknown amplitudes of the first-order gravitational waves. To factor them out, we have made the simplifying assumption of a perfectly scale-invariant power spectrum of gravitational waves (which would anyway be not so far from what predicted by inflation) and of equal amplitudes for the two polarization modes.
The amplitude squared would provide a very small factor to the expression of $\delta^{(2)}_{\mathrm{S(t)}}$: not only because we are considering small perturbations at second-order,

but mostly because even at linear order the amplitude of tensor modes would be much smaller than the one of scalars. In fact, assuming that the consistency relation holds, the tensor-to-scalar ratio is estimated to be $r < 0.06$ [18], providing an amplitude for the tensor power spectrum $\Delta_{\mathrm{T}} \simeq A^2(k_*) < 1.2 \times 10^{-10}$.

Only future observations could have the sensitivity to probe the cosmological background of gravitational waves, maybe also through the second-order effects they can induce.

Appendix A

Evolution equations for second-order perturbations

The second-order Raychaudhuri equation:

$$
\begin{aligned}
&\phi_{\mathrm{S}}^{(2)\,\prime\prime} + \frac{2}{\tau}\,\phi_{\mathrm{S}}^{(2)\,\prime} - \frac{6}{\tau^2}\,\phi_{\mathrm{S}}^{(2)} = \\
&= -\frac{1}{6}\,\gamma_{\mathrm{S}}^{(1)\,ij\,\prime}\left(\gamma_{\mathrm{S}\,ij}^{(1)}{}' - \frac{4}{\tau}\,\gamma_{\mathrm{S}\,ij}^{(1)}\right) + \frac{1}{6}[2\gamma_{\mathrm{S}}^{(1)\,ij}(2\gamma_{\mathrm{S}\,i,jk}^{(1)\,k} - \nabla^2\gamma_{\mathrm{S}\,ij}^{(1)} - \gamma_{\mathrm{S}\,k,ij}^{(1)\,k}) - \gamma_{\mathrm{S}\,k}^{(1)\,k}(\gamma_{\mathrm{S}\,,ij}^{(1)\,ij} - \nabla^2\gamma_{\mathrm{S}\,i}^{(1)\,i})] \\
&- \frac{2}{\tau^2}\left[-\frac{1}{4}(\gamma_{\mathrm{S}\,i}^{(1)\,i} - \gamma_{\mathrm{S}0\,i}^{(1)\,i})^2 - \frac{1}{2}(\gamma_{\mathrm{S}}^{(1)\,ij}\gamma_{\mathrm{S}\,ij}^{(1)} - \gamma_{\mathrm{S}0}^{(1)\,ij}\gamma_{\mathrm{S}0\,ij}^{(1)}) + \delta_0(\gamma_{\mathrm{S}\,i}^{(1)\,i} - \gamma_{\mathrm{S}0\,i}^{(1)\,i})\right];
\end{aligned}
\tag{A.1}
$$

Energy constraint:

$$
\begin{aligned}
&\frac{2}{\tau}\,\phi_{\mathrm{S}}^{(2)\prime} - \frac{1}{3}\nabla^2\phi_{\mathrm{S}}^{(2)} + \frac{6}{\tau^2}\,\phi_{\mathrm{S}}^{(2)} - \frac{1}{12}\chi_{\mathrm{S}\,,ij}^{(2)\,ij} = \\
&= -\frac{2}{3\tau}\gamma_{\mathrm{S}}^{(1)\,ij}\gamma_{\mathrm{S}\,ij}^{(1)\prime} - \frac{1}{24}(\gamma_{\mathrm{S}}^{(1)\,ij\prime}\gamma_{\mathrm{S}\,ij}^{(1)\prime} - \gamma_{\mathrm{S}\,i}^{(1)\,i\prime}\gamma_{\mathrm{S}\,j}^{(1)\,j\prime}) + \frac{1}{6}\Big[\gamma_{\mathrm{S}}^{(1)\,ij}(-2\gamma_{\mathrm{S}\,i,jk}^{(1)\,k} + \nabla^2\gamma_{\mathrm{S}\,ij}^{(1)} + \gamma_{\mathrm{S}\,k,ij}^{(1)\,k}) \\
&+ \gamma_{\mathrm{S}\,,k}^{(1)\,ki}(\gamma_{\mathrm{S}\,j,i}^{(1)\,i} - \gamma_{\mathrm{S}\,i,j}^{(1)\,i}) + \frac{3}{4}\gamma_{\mathrm{S}\,,k}^{(1)\,ij}\gamma_{\mathrm{S}\,ij}^{(1)\,,k} - \frac{1}{2}\gamma_{\mathrm{S}\,,k}^{(1)\,ij}\gamma_{\mathrm{S}\,i,j}^{(1)\,k} - \frac{1}{4}\gamma_{\mathrm{S}\,i}^{(1)\,i,k}\gamma_{\mathrm{S}\,j,k}^{(1)\,j}\Big] \\
&+ \frac{2}{\tau^2}\left[-\frac{1}{4}(\gamma_{\mathrm{S}\,i}^{(1)\,i} - \gamma_{\mathrm{S}0\,i}^{(1)\,i})^2 - \frac{1}{2}(\gamma_{\mathrm{S}}^{(1)\,ij}\gamma_{\mathrm{S}\,ij}^{(1)} - \gamma_{\mathrm{S}0}^{(1)\,ij}\gamma_{\mathrm{S}0\,ij}^{(1)}) + \delta_0(\gamma_{\mathrm{S}\,i}^{(1)\,i} - \gamma_{\mathrm{S}0\,i}^{(1)\,i})\right];
\end{aligned}
\tag{A.2}
$$

Momentum constraint:

$$
2\phi_{\mathrm{S},j}^{(2)\prime} + \frac{1}{2}\chi_{\mathrm{S}\,j,i}^{(2)\,i\prime} = \gamma_{\mathrm{S}}^{(1)\,ik}(\gamma_{\mathrm{S}\,kj,i}^{(1)\prime} - \gamma_{\mathrm{S}\,ik,j}^{(1)\prime}) + \gamma_{\mathrm{S},i}^{(1)\,ik}\gamma_{\mathrm{S}\,kj}^{(1)\prime} - \frac{1}{2}\gamma_{\mathrm{S},j}^{(1)\,ik}\gamma_{\mathrm{S}\,ki}^{(1)\prime} - \frac{1}{2}\gamma_{\mathrm{S}\,i,k}^{(1)\,i}\gamma_{\mathrm{S}\,j}^{(1)k\prime}; \tag{A.3}
$$

Evolution equation:

$$
\begin{aligned}
&-\left(\phi_{\mathrm{S}}^{(2)\,\prime\prime}+\frac{4}{\tau}\,\phi_{\mathrm{S}}^{(2)\,\prime}\right)\delta^i_j+\frac{1}{2}\left(\chi_{\mathrm{S}\,j}^{(2)\,i\,\prime\prime}+\frac{4}{\tau}\chi_{\mathrm{S}\,j}^{(2)\,i\,\prime}\right)+\phi_{\mathrm{S},j}^{(2)\,,i}-\frac{1}{4}\,\chi_{\mathrm{S}\,kl}^{(2)\,,kl}\delta^i_j+\frac{1}{2}\chi_{\mathrm{S}\,,kj}^{(2)\,ki}+\frac{1}{2}\chi_{\mathrm{S}\,j,k}^{(2)\,k,i}\\
&-\frac{1}{2}\nabla^2\chi_{\mathrm{S}\,j}^{(2)\,i}=\\
&=\gamma_{\mathrm{S}}^{(1)\,ik\,\prime}\gamma_{\mathrm{S}\,kj}^{(1)\,\prime}-\frac{1}{2}\gamma_{\mathrm{S}\,k}^{(1)\,k\,\prime}\gamma_{\mathrm{S}\,j}^{(1)\,i\,\prime}+\frac{1}{8}[(\gamma_{\mathrm{S}\,k}^{(1)\,k\,\prime})^2-\gamma_{\mathrm{S}\,l}^{(1)\,k\,\prime}\gamma_{\mathrm{S}\,k}^{(1)\,l\,\prime}]\delta^i_j-\frac{1}{2}\bigg[-\gamma_{\mathrm{S}\,j}^{(1)\,i}(\gamma_{\mathrm{S}\,l,k}^{(1)\,k,l}-\nabla^2\gamma_{\mathrm{S}\,k}^{(1)\,k})\\
&+2\gamma_{\mathrm{S}}^{(1)\,kl}(\gamma_{\mathrm{S}\,j,kl}^{(1)\,i}+\gamma_{\mathrm{S}\,kl,j}^{(1)\,,i}-\gamma_{\mathrm{S}\,l,jk}^{(1)\,i}-\gamma_{\mathrm{S}\,lj,k}^{(1)\,,i})+2\gamma_{\mathrm{S},k}^{(1)\,kl}(\gamma_{\mathrm{S}\,j,l}^{(1)\,i}-\gamma_{\mathrm{S}\,l,j}^{(1)\,i}-\gamma_{\mathrm{S}\,jl}^{(1)\,,i})+2\gamma_{\mathrm{S}\,,l}^{(1)\,ki}\gamma_{\mathrm{S}\,kj}^{(1)\,,l}\\
&-2\gamma_{\mathrm{S}\,,l}^{(1)\,ki}\gamma_{\mathrm{S}\,j,k}^{(1)\,l}+\gamma_{\mathrm{S}\,,j}^{(1)\,kl}\gamma_{\mathrm{S}\,kl}^{(1)\,,i}+\gamma_{\mathrm{S}\,l,k}^{(1)\,l}(\gamma_{\mathrm{S}\,,j}^{(1)\,ki}+\gamma_{\mathrm{S}\,j}^{(1)\,k,i}-\gamma_{\mathrm{S}\,j}^{(1)\,i,k})\\
&-\gamma_{\mathrm{S}}^{(1)\,kl}(\nabla^2\gamma_{\mathrm{S}\,kl}^{(1)}+\gamma_{\mathrm{S}\,m,kl}^{(1)\,m}-2\gamma_{\mathrm{S}\,k,ml}^{(1)\,m})\delta^i_j-\gamma_{\mathrm{S},l}^{(1)\,kl}(\gamma_{\mathrm{S}\,m,k}^{(1)\,m}-2\gamma_{\mathrm{S}\,k,m}^{(1)\,m})\delta^i_j-\frac{3}{4}\,\gamma_{\mathrm{S}\,,m}^{(1)\,kl}\gamma_{\mathrm{S}\,kl}^{(1)\,,m}\delta^i_j\\
&+\frac{1}{2}\,\gamma_{\mathrm{S}\,,m}^{(1)\,kl}\gamma_{\mathrm{S}\,k,l}^{(1)\,m}\delta^i_j+\frac{1}{4}\,\gamma_{\mathrm{S}\,k}^{(1)\,k,m}\gamma_{\mathrm{S}\,l,m}^{(1)\,l}\delta^i_j\bigg].
\end{aligned}
\tag{A.4}
$$

Appendix B

Equations for second-order perturbations arising from linear tensor modes

Those equations present the part of the equations in Appendix A involving combinations of first-order tensor modes. They can be used to derive the parts of second-order metric perturbations generated by linear tensor modes. The calculations are still performed in synchronous and comoving gauge, in an Einstein-de Sitter Universe.

Raychaudhuri equation:

$$\phi^{(2)\prime\prime}_{\mathrm{S(t)}} + \frac{2}{\tau}\,\phi^{(2)\prime}_{\mathrm{S(t)}} - \frac{6}{\tau^2}\,\phi^{(2)}_{\mathrm{S(t)}} = \frac{\tau^2}{9}\,\varphi^{,ij}\,\nabla^2\chi^{(1)\mathrm{T}}_{ij} - \frac{1}{6}\chi^{(1)\mathrm{T}\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\prime} + \frac{2}{3\tau}\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij\prime}$$
$$- \frac{1}{3}\chi^{(1)\mathrm{T}\,ij}\,\nabla^2\chi^{(1)\mathrm{T}}_{ij} + \frac{1}{\tau^2}\left(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij} - \chi^{(1)\mathrm{T}}_{0\,ij}\chi^{(1)\mathrm{T}\,ij}_{0}\right) \equiv \mathcal{Q}(\mathbf{x},\tau). \tag{B.1}$$

Energy constraint:

$$\frac{2}{\tau}\,\phi^{(2)\prime}_{\mathrm{S(t)}} - \frac{1}{3}\nabla^2\phi^{(2)}_{\mathrm{S(t)}} + \frac{6}{\tau^2}\,\phi^{(2)}_{\mathrm{S(t)}} - \frac{1}{12}\chi^{(2)\,ij}_{\mathrm{S(t)}\,,ij} =$$
$$= \frac{5\tau}{18}\chi^{(1)\mathrm{T}\,ij\prime}\,\varphi_{,ij} + \frac{5}{9}\chi^{(1)\mathrm{T}\,ij}\,\varphi_{,ij} - \frac{\tau^2}{18}\,\nabla^2\chi^{(1)\mathrm{T}\,ij}\,\varphi_{,ij} - \frac{\tau^2}{36}\chi^{(1)\mathrm{T}\,ij,k}\,\varphi_{,ijk} - \frac{1}{24}\chi^{(1)\mathrm{T}\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\prime}$$
$$- \frac{2}{3\tau}\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij\prime} + \frac{1}{6}\chi^{(1)\mathrm{T}\,ij}\,\nabla^2\chi^{(1)\mathrm{T}}_{ij} + \frac{1}{8}\chi^{(1)\mathrm{T}\,ij,k}\,\chi^{(1)\mathrm{T}}_{ij,k} - \frac{1}{12}\chi^{(1)\mathrm{T}\,ij,k}\,\chi^{(1)\mathrm{T}}_{kj,i}$$
$$- \frac{1}{\tau^2}\left(\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij} - \chi^{(1)\mathrm{T}}_{0\,ij}\chi^{(1)\mathrm{T}\,ij}_{0}\right). \tag{B.2}$$

Momentum constraint:

$$2\phi^{(2)\prime}_{\mathrm{S(t)},j} + \frac{1}{2}\chi^{(2)\,i\prime}_{\mathrm{S(t)}\,j,i} = \frac{\tau^2}{3}\left[\left(\chi^{(1)\mathrm{T}\,ik\prime}_{,j} - \chi^{(1)\mathrm{T}\,k,i\prime}_{j}\right)\varphi_{,ik} + \frac{1}{2}\chi^{(1)\mathrm{T}\,ik\prime}\,\varphi_{,ijk} - \frac{1}{2}\chi^{(1)\mathrm{T}\,k\prime}_{j}\,\nabla^2\varphi_{,k}\right]$$
$$+ \frac{\tau}{3}\chi^{(1)\mathrm{T}\,ik}_{,j}\,\varphi_{,ik} + \frac{5}{3}\chi^{(1)\mathrm{T}\,i\prime}_{j}\,\varphi_{,i} + \chi^{(1)\mathrm{T}\,ik}\left(\chi^{(1)\mathrm{T}\prime}_{kj,i} - \chi^{(1)\mathrm{T}\prime}_{ki,j}\right) - \frac{1}{2}\chi^{(1)\mathrm{T}\,ik\prime}\chi^{(1)\mathrm{T}}_{ik,j}. \tag{B.3}$$

Evolution equation:

$$
\begin{aligned}
&-\left(\phi^{(2)\,\prime\prime}_{\mathrm{S(t)}}+\frac{4}{\tau}\,\phi^{(2)\,\prime}_{\mathrm{S(t)}}\right)\delta^i_j+\frac{1}{2}\left(\chi^{(2)\,i\,\prime\prime}_{\mathrm{S(t)}\,j}+\frac{4}{\tau}\chi^{(2)\,i\,\prime}_{\mathrm{S(t)}\,j}\right)+\phi^{(2)\,,i}_{\mathrm{S(t)},j}-\frac{1}{4}\chi^{(2)\,,kl}_{\mathrm{S(t)}\,kl}\delta^i_j+\frac{1}{2}\chi^{(2)\,ki}_{\mathrm{S(t)}\,,kj}\\
&+\frac{1}{2}\chi^{(2)\,k,i}_{\mathrm{S(t)}\,j,k}-\frac{1}{2}\nabla^2\chi^{(2)\,i}_{\mathrm{S(t)}\,j}=\\
=&-\frac{2\tau}{3}\chi^{(1)\mathrm{T}\,\prime}_{kj}\,\varphi^{,ik}-\frac{2\tau}{3}\chi^{(1)\mathrm{T}\,ik\,\prime}\,\varphi_{,kj}+\frac{\tau}{3}\chi^{(1)\mathrm{T}\,i}_{j}\,\nabla^2\varphi+\frac{\tau}{6}\chi^{(1)\mathrm{T}\,kl\,\prime}\,\varphi_{,kl}\,\delta^i_j+\frac{10}{3}\chi^{(1)\mathrm{T}\,i}_{j}\,\nabla^2\varphi\\
&+\frac{25}{3}\chi^{(1)\mathrm{T}\,kl}\,\varphi_{,kl}\,\delta^i_j-\frac{10}{3}\chi^{(1)\mathrm{T}\,ik}\,\varphi_{,kj}-\frac{10}{3}\chi^{(1)\mathrm{T}}_{kj}\,\varphi^{,ik}+\frac{10}{3}\nabla^2\chi^{(1)\mathrm{T}\,i}_{j}\,\varphi\\
&+\frac{\tau^2}{3}(\chi^{(1)\mathrm{T}\,i,kl}_{j}+\chi^{(1)\mathrm{T}\,kl,i}_{,j}-\chi^{(1)\mathrm{T}\,k,il}_{j}-\chi^{(1)\mathrm{T}\,ki,l}_{,j})\,\varphi_{,lk}+(\chi^{(1)\mathrm{T}\,i,k}_{j}+\chi^{(1)\mathrm{T}\,ki}_{,j}-\chi^{(1)\mathrm{T}\,k,i}_{j})\left(\frac{5}{3}\,\varphi_{,k}+\frac{\tau^2}{6}\nabla^2\,\varphi_{,k}\right)\\
&+\frac{\tau^2}{6}(\chi^{(1)\mathrm{T}\,kl,i}\,\varphi_{,lkj}+\chi^{(1)\mathrm{T}\,kl}_{,j}\,\varphi^{,i}_{,lk}-\nabla^2\chi^{(1)\mathrm{T}\,kl}\,\varphi_{,kl}\,\delta^i_j)-\frac{\tau^2}{12}\chi^{(1)\mathrm{T}\,kl,m}\,\varphi_{,klm}\delta^i_j+\chi^{(1)\mathrm{T}\,\prime}_{kj}\,\chi^{(1)\mathrm{T}\,ik\,\prime}\\
&-\frac{1}{8}\chi^{(1)\mathrm{T}\,\prime}_{kl}\,\chi^{(1)\mathrm{T}\,lk\,\prime}\,\delta^i_j+\chi^{(1)\mathrm{T}\,ik}_{,l}(\chi^{(1)\mathrm{T}\,l}_{j,k}-\chi^{(1)\mathrm{T}\,,l}_{kj})-\frac{1}{2}\chi^{(1)\mathrm{T}\,kl}_{,j}\chi^{(1)\mathrm{T}\,,i}_{kl}-\frac{1}{4}\chi^{(1)\mathrm{T}\,km}_{,l}\chi^{(1)\mathrm{T}\,l}_{m,k}\,\delta^i_j\\
&+\frac{1}{2}\chi^{(1)\mathrm{T}\,kl}\nabla^2\chi^{(1)\mathrm{T}}_{lk}\,\delta^i_j+\frac{3}{8}\chi^{(1)\mathrm{T}\,km}_{,l}\chi^{(1)\mathrm{T}\,,l}_{km}\,\delta^i_j+\chi^{(1)\mathrm{T}\,kl}(\chi^{(1)\mathrm{T}\,i}_{k,jl}+\chi^{(1)\mathrm{T}\,,i}_{kj,l}-\chi^{(1)\mathrm{T}\,i}_{j,kl}-\chi^{(1)\mathrm{T}\,,i}_{kl,j}).
\end{aligned}
\tag{B.4}
$$

Appendix C

From the synchronous to the Poisson gauge

We express the most general equations (3.17) and (3.18) for gauge transformation referring to the particular case of the metric, the density perturbation and the velocity perturbation, in both first- and second-order case.

C.1 First-order transformations

The first-order transformation for the metric is:

$$\delta\tilde{g}_{\mu\nu} = \delta g_{\mu\nu} + \mathcal{L}_{\xi_{(1)}} g^{(0)}_{\mu\nu}, \tag{C.1}$$

from which we obtain, in each perturbation and from synchronous to Poisson gauge:

Lapse perturbation:

$$\psi^{(1)}_{\mathrm{P}} = \alpha^{(1)\prime} + \frac{a'}{a}\,\alpha^{(1)}, \tag{C.2}$$

Shift perturbation, scalar:

$$\alpha^{(1)} = \beta^{(1)\prime}, \tag{C.3}$$

Shift perturbation, vector:

$$\omega^{(1)}_{\mathrm{P}\,i} = d^{(1)\prime}_{i}, \tag{C.4}$$

Spatial metric, trace:

$$\phi^{(1)}_{\mathrm{P}} = \phi^{(1)}_{\mathrm{S}} - \frac{1}{3}\,\nabla^2\beta^{(1)} - \frac{a'}{a}\,\alpha^{(1)}, \tag{C.5}$$

Spatial metric, traceless:

$$\mathrm{D}_{ij}\,(\chi^{(1)\parallel}_{\mathrm{S}} + 2\beta^{(1)}) = 0, \tag{C.6}$$

$$\chi^{(1)\perp}_{\mathrm{S}\,(i,j)} + d^{(1)}_{(i,j)} = 0, \tag{C.7}$$

$$\chi^{(1)\mathrm{T}}_{\mathrm{P}\,ij} = \chi^{(1)\mathrm{T}}_{\mathrm{S}\,ij}. \tag{C.8}$$

For a scalar like the energy density, we would have:

$$\delta\tilde{\rho} = \delta\rho + \mathcal{L}_{\xi_{(1)}}\rho_{(0)} \quad\longrightarrow\quad \delta\tilde{\rho} = \delta\rho + \rho'_{(0)}\,\alpha_{(1)}. \tag{C.9}$$

The density perturbation is defined as $\delta = \frac{\rho-\rho_{(0)}}{\rho_{(0)}} = \frac{\delta^{(1)}\rho+\frac{1}{2}\delta^{(2)}\rho+\dots}{\rho_{(0)}} = \delta^{(1)} + \frac{1}{2}\delta^{(2)} + \dots$, so the first-order term is $\delta^{(1)} = \frac{\delta^{(1)}\rho}{\rho_{(0)}}$ and its gauge transformation becomes:

$$\delta_{\mathrm{P}}^{(1)} = \delta_{\mathrm{S}}^{(1)} + \frac{\rho'_{(0)}}{\rho_{(0)}}\,\alpha_{(1)}. \tag{C.10}$$

For the four-velocity we have:

$$\delta\tilde{u}^{\mu} = \delta u^{\mu} + \mathcal{L}_{\xi_{(1)}} u^{\mu}_{(0)}, \tag{C.11}$$

and dividing between time and space components:

$$\tilde{v}_{\mathrm{P}}^{(1)\,0} = -\alpha^{(1)\,\prime} - \frac{a'}{a}\,\alpha^{(1)}, \tag{C.12}$$

$$\tilde{v}_{\mathrm{P}}^{(1)\,i} = -\beta'^{,i}_{(1)} - d^{i\,\prime}_{(1)}. \tag{C.13}$$

As we are in the irrotational case, $\chi^{(1)\perp}_{\mathrm{S}\,i} = v^{(1)\perp}_{i} = 0$, so $d^{(1)}_{i} = \omega^{(1)}_{\mathrm{P}\,i} = \chi^{(1)\perp}_{\mathrm{P}\,i} = 0$.

C.2 Second-order transformartions

The second-order transformation for the metric is:

$$\delta^2\tilde{g}_{\mu\nu} = \delta^2 g_{\mu\nu} + 2\mathcal{L}_{\xi_{(1)}}\delta g_{\mu\nu} + \mathcal{L}^2_{\xi_{(1)}} g^{(0)}_{\mu\nu} + \mathcal{L}_{\xi_{(2)}} g^{(0)}_{\mu\nu}. \tag{C.14}$$

Using the results of the first-order case, like equation (C.3) and $d^{(1)}_{i} = 0$, we obtain for each perturbation:

Lapse perturbation:

$$\psi^{(2)}_{\mathrm{P}} = \beta'_{(1)}\left[\beta'''_{(1)}+5\frac{a'}{a}\beta''_{(1)}+\left(\frac{a''}{a}+\frac{a'^{2}}{a^{2}}\right)\beta'_{(1)}\right]+\beta^{,i}_{(1)}\left(\beta^{(1)\,\prime\prime}_{,i}+\frac{a'}{a}\beta^{(1)\,\prime}_{,i}\right)+2\beta''^{2}_{(1)}+\alpha^{(2)\,\prime}+\frac{a'}{a}\,\alpha^{(2)}, \tag{C.15}$$

Shift perturbation, vector:

$$\omega^{(2)}_{\mathrm{P}\,i} = -2\left(2\phi^{(1)}_{\mathrm{S}}+\beta''_{(1)}-\frac{2}{3}\nabla^2\beta_{(1)}\right)\beta^{(1)\,\prime}_{,i}-2\beta^{(1)\,\prime}_{,j}\,\beta^{(1),j}_{,i}+2\chi^{(1)\,\mathrm{T}}_{ij}\beta'^{,i}_{(1)}-\alpha^{(2)}_{,i}+\beta^{(2)\,\prime}_{,i}+d^{(2)\,\prime}_{i}, \tag{C.16}$$

Spatial metric, trace:

$$\begin{aligned}\phi^{(2)}_{\rm P} &= \phi^{(2)}_{\rm S} + \beta'_{(1)}\left[2\left(\phi^{(1)\,\prime}_{\rm S} + 2\,\frac{a'}{a}\,\phi^{(1)}_{\rm S}\right) - \left(\frac{a''}{a} + \frac{a^{'\,2}}{a^2}\right)\beta'_{(1)} - \frac{a'}{a}\,\beta''_{(1)}\right] \\ &\quad - \frac{1}{3}\left(- 4\,\phi^{(1)}_{\rm S} + \beta'_{(1)}\partial_0 + \beta^{,\,i}_{(1)}\partial_i + 4\frac{a'}{a}\,\beta'_{(1)} + \frac{4}{3}\,\nabla^2\beta_{(1)}\right)\nabla^2\beta_{(1)} \\ &\quad + \beta^{,\,i}_{(1)}\left(2\,\phi^{(1)}_{{\rm S},i} - \frac{a'}{a}\,\beta^{(1)\,\prime}_{,i}\right) + \frac{2}{3}\,\beta^{,\,ij}_{(1)}\beta^{(1)}_{,\,ij} - \frac{2}{3}\,\chi^{(1)\,\rm T}_{ij}\,\beta^{,ij}_{(1)} - \frac{1}{3}\,\nabla^2\beta^{(2)} - \frac{a'}{a}\,\alpha^{(2)},\end{aligned} \tag{C.17}$$

Spatial metric, traceless:

$$\begin{aligned}\chi^{(2)}_{{\rm P}\,ij} &= \chi^{(2)}_{{\rm S}\,ij} + 2\left(- 4\,\phi^{(1)}_{\rm S} - \beta'_{(1)}\partial_0 - \beta^{,\,k}_{(1)}\partial_k + \frac{4}{3}\,\nabla^2\beta_{(1)}\right){\rm D}_{ij}\beta^{(1)} - 4\left(\beta^{(1)}_{,\,ik}\beta^{,k}_{(1)\,,j} - \frac{1}{3}\,\delta_{ij}\beta^{,lk}_{(1)}\beta^{(1)}_{,lk}\right) \\ &\quad + 2\left(\chi^{(1)\,\rm T\,\prime}_{ij} + 2\,\frac{a'}{a}\,\chi^{(1)\,\rm T}_{ij}\right)\beta'_{(1)} + 2\,\chi^{(1)\,\rm T}_{ij,k}\,\beta^{,k}_{(1)} + 2\,\chi^{(1)\,\rm T}_{ik}\,\beta^{,k}_{(1)\,,j} + 2\,\chi^{(1)\,\rm T}_{jk}\,\beta^{,k}_{(1)\,,i} - \frac{4}{3}\,\chi^{(1)\,\rm T}_{lk}\,\beta^{,lk}_{(1)}\,\delta_{ij} \\ &\quad + 2\,(d^{(2)}_{(i,j)} + {\rm D}_{ij}\beta^{(2)}).\end{aligned} \tag{C.18}$$

Then, the general second-order gauge transformation for the energy density is:

$$\delta^{(2)}\tilde{\rho} = \delta^{(2)}\rho + 2\mathcal{L}_{\xi_{(1)}}\delta\rho + (\mathcal{L}^2_{\xi_{(1)}} + \mathcal{L}_{\xi_{(2)}})\rho^{(0)}. \tag{C.19}$$

from which we derive:

$$\delta^{(2)}\tilde{\rho} = \delta^{(2)}\rho + \rho'_{(0)}\alpha_{(2)} + \rho''_{(0)}\alpha^2_{(1)} + \rho'_{(0)}\alpha_{(1)}\alpha'_{(1)} + 2\delta\rho'\alpha_{(1)} + \beta^{,i}_{(1)}\rho'_{(0)}\alpha_{(1)\,,i} + 2\beta^{,i}_{(1)}\delta\rho_{,i}, \tag{C.20}$$

and substituting the expression for $\delta^{(2)}\rho$ with respect to the second-order density contrast $\delta^{(2)}\rho = \rho_{(0)}\cdot\delta^{(2)}$, we obtain:

$$\delta^{(2)}_{\rm P} = \delta^{(2)}_{\rm S} + \frac{\rho'_{(0)}}{\rho_{(0)}}\alpha_{(2)} + \frac{\rho''_{(0)}}{\rho_{(0)}}\alpha^2_{(1)} + \frac{\rho'_{(0)}}{\rho_{(0)}}\alpha_{(1)}\alpha'_{(1)} + 2\delta^{(1)\,\prime}\alpha_{(1)} + 2\frac{\rho'_{(0)}}{\rho_{(0)}}\delta^{(1)}\alpha_{(1)} + \beta^{,i}_{(1)}\frac{\rho'_{(0)}}{\rho_{(0)}}\alpha_{(1)\,,i} + 2\beta^{,i}_{(1)}\delta^{(1)}_{,i}. \tag{C.21}$$

The general transformation for the four-velocity at second-order is:

$$\delta^2\tilde{u}^\mu = \delta^2 u^\mu + 2\mathcal{L}_{\xi_{(1)}}\delta u^\mu + (\mathcal{L}^2_{\xi_{(1)}} + \mathcal{L}_{\xi_{(2)}})u^\mu_{(0)}. \tag{C.22}$$

and dividing between time and space components, from comoving synchronous to Poisson gauge:

$$v^0_{\rm P(2)} = -\frac{a'}{a}\,\alpha_{(2)} - \alpha'_{(2)} + \beta'_{(1)}\left[\left(2\frac{a^{'2}}{a} - \frac{a''}{a}\right)\beta'_{(1)} + \frac{a'}{a}\beta''_{(1)} - \beta'''_{(1)}\right] - \beta^{,i}_{(1)}\left(\frac{a'}{a}\,\beta'_{(1)\,,i} + \beta''_{(1)\,,i}\right) + \beta''^{2}_{(1)} + \beta'_{(1)\,,i}\beta^{'\,,i}_{(1)}, \tag{C.23}$$

$$v^i_{\mathrm{P}(2)} = -\beta'^{,i}_{(2)} - d'^{\,i}_{(2)} + \beta'_{(1)}\left[2\frac{a}{a}\beta'^{,i}_{(1)} - \beta''^{,i}_{(1)}\right] + \beta'^{,i}_{(1)}\beta''_{(1)}. \tag{C.24}$$

Equations (C.15) - (C.18) couple the second-order metric perturbations in the Poisson gauge and the second-order parameters of the gauge transformation $\alpha^{(2)}, \beta^{(2)}$ and $d_i^{(2)}$. We can obtain expressions for those parameters using the following conditions:
the conditions $\partial^i \chi^{(2)}_{\mathrm{P}ij} = 0$, $\partial^i \partial^j \chi^{(2)}_{\mathrm{P}ij} = 0$ and $\partial^i d_i^{(1)} = 0$ with equation (C.18), to get:

$$\begin{aligned}
\nabla^2\nabla^2\beta^{(2)} = &-\frac{3}{4}\chi^{(2),ij}_{\mathrm{S}ij} + 6\,\phi^{(1),ij}_{\mathrm{S}}\beta^{(1)}_{,ij} - 2\nabla^2\phi^{(1)}_{\mathrm{S}}\nabla^2\beta^{(1)} + 8\phi^{(1),i}_{\mathrm{S}}\nabla^2\beta^{(1)}_{,i} + 4\,\phi^{(1)}_{\mathrm{S}}\nabla^2\nabla^2\beta^{(1)} \\
&+ 4\,\nabla^2\beta^{(1)}_{,ij}\beta^{,ij}_{(1)} - \frac{1}{6}\nabla^2\beta^{(1)}_{,i}\nabla^2\beta^{,i}_{(1)} + \frac{5}{2}\beta^{,ijk}_{(1)}\beta^{(1)}_{,ijk} - \frac{2}{3}\nabla^2\beta^{(1)}\nabla^2\nabla^2\beta^{(1)} + \frac{3}{2}\beta^{(1)\prime}_{,ij}\beta^{,ij\prime}_{(1)} \\
&- \frac{1}{2}\nabla^2\beta'_{(1)}\nabla^2\beta'_{(1)} + 2\beta^{,i\prime}_{(1)}\nabla^2\beta^{(1)\prime}_{,i} + \beta^{(1)\prime}\nabla^2\nabla^2\beta^{(1)\prime} + \beta^{(1)}_{,i}\nabla^2\nabla^2\beta^{,i}_{(1)} \\
&- \frac{3}{2}\left(\chi^{(1)\mathrm{T}\prime}_{ij} + 2\,\frac{a'}{a}\chi^{(1)\mathrm{T}}_{ij}\right)\beta'^{,ij}_{(1)} - \frac{5}{2}\chi^{(1)\mathrm{T}}_{ij,k}\beta^{,ijk}_{(1)} - 2\chi^{(1)\mathrm{T}}_{ij}\nabla^2\beta^{,ij}_{(1)} + \nabla^2\chi^{(1)\mathrm{T}}_{ij}\beta^{,ij}_{(1)};
\end{aligned} \tag{C.25}$$

the condition $\partial^i \chi^{(2)}_{\mathrm{P}ij} = 0$ and the expression found for $\beta_{(2)}$, to get:

$$\begin{aligned}
\nabla^2 d_i^{(2)} = &-\frac{4}{3}\nabla^2\beta^{(2)}_{,i} - \chi^{(2),j}_{\mathrm{S}ij} + 8\,\phi^{(1),j}_{\mathrm{S}}\mathrm{D}_{ij}\beta^{(1)} + \frac{16}{3}\phi^{(1)}_{\mathrm{S}}\nabla^2\beta^{(1)}_{,i} + \frac{2}{3}\beta^{(1)}_{,ij}\nabla^2\beta^{(1),j} + \frac{10}{3}\beta^{(1)}_{,ijk}\beta^{,jk}_{(1)} \\
&- \frac{8}{9}\nabla^2\beta^{(1)}_{,i}\nabla^2\beta_{(1)} + 2\beta^{,j\prime}_{(1)}\mathrm{D}_{ij}\beta^{(1)\prime} + \frac{4}{3}\beta^{(1)\prime}\nabla^2\beta^{(1)\prime}_{,i} + \frac{4}{3}\beta^{(1),j}\nabla^2\beta^{(1)}_{,ij} - 4\chi^{(1)\mathrm{T}}_{ij,k}\beta^{,jk}_{(1)} \\
&- 2\left(\chi^{(1)\mathrm{T}\prime}_{ij} + 2\,\frac{a'}{a}\chi^{(1)\mathrm{T}}_{ij}\right)\beta'^{,j}_{(1)} - \frac{2}{3}\chi^{(1)\mathrm{T}}_{ik}\beta^{,jk}_{(1),i} - 2\chi^{(1)\mathrm{T}}_{ij}\nabla^2\beta^{,j}_{(1)} + \frac{4}{3}\chi^{(1)\mathrm{T}}_{jk,i}\beta^{,jk}_{(1)};
\end{aligned} \tag{C.26}$$

the condition $\partial^i \omega^{(2)}_{\mathrm{P}i} = 0$ and the expression for $\beta_{(2)}$, to get:

$$\begin{aligned}
\nabla^2\alpha^{(2)} = &\nabla^2\beta^{(2)\prime} - 2\left(2\,\phi^{(1),i}_{\mathrm{S}} + \beta''^{,i}_{(1)} + \frac{1}{3}\nabla^2\beta^{,i}_{(1)}\right)\beta^{(1)\prime}_{,i} - 2\beta^{(1)\prime}_{,ij}\beta^{(1),ij} \\
&- 2\left(2\,\phi^{(1)}_{\mathrm{S}} + \beta''_{(1)} - \frac{2}{3}\nabla^2\beta_{(1)}\right)\nabla^2\beta^{(1)\prime} + 2\chi^{(1)\mathrm{T}}_{ij}\beta'^{,ij}_{(1)}.
\end{aligned} \tag{C.27}$$

Appendix D

Second-order parameters and perturbations generated by tensor modes

First, we consider the second-order gauge parameters generated by first-order tensor modes. They can be obtained from equations (C.25) - (C.27), replacing the first-order gauge parameters:

$$\begin{aligned}
\nabla^2\nabla^2\beta^{(2)}_{(\mathrm{t})} &= -\frac{3}{4}\chi^{(2),ij}_{\mathrm{S(t)}\,ij} - \frac{\tau}{2}\left(\chi^{(1)\,\mathrm{T}\,\prime}_{ij} + \frac{4}{\tau}\chi^{(1)\,\mathrm{T}}_{ij}\right)\varphi^{,ij} - \frac{5\tau^2}{12}\chi^{(1)\,\mathrm{T}}_{ij,k}\,\varphi^{,ijk} - \frac{\tau^2}{3}\chi^{(1)\,\mathrm{T}}_{ij}\,\nabla^2\varphi^{,ij} \\
&\quad + \frac{\tau^2}{6}\nabla^2\chi^{(1)\,\mathrm{T}}_{ij}\,\varphi^{,ij}, \\
\nabla^2 d^{(2)}_{(t)\,i} &= -\frac{4}{3}\nabla^2\beta^{(2)}_{(\mathrm{t}),i} - \chi^{(2)\,j}_{\mathrm{S(t)}\,i,j} - \frac{2\tau}{3}\left(\chi^{(1)\,\mathrm{T}\,\prime}_{ij} + \frac{4}{\tau}\chi^{(1)\,\mathrm{T}}_{ij}\right)\varphi^{,j} + \frac{2\tau^2}{3}\chi^{(1)\,\mathrm{T}}_{ij,k}\,\varphi^{,jk} \\
&\quad - \frac{\tau^2}{3}\chi^{(1)\,\mathrm{T}}_{ij}\,\nabla^2\varphi^{,j} - \frac{\tau^2}{9}\chi^{(1)\,\mathrm{T}}_{jk}\,\varphi^{,jk}_{,i} + \frac{2\tau^2}{9}\chi^{(1)\,\mathrm{T}}_{jk,i}\,\varphi^{,jk}, \\
\nabla^2\alpha^{(2)}_{(\mathrm{t})} &= \nabla^2\beta^{(2)\,\prime}_{(\mathrm{t})} + \frac{2\tau}{3}\chi^{(1)\,\mathrm{T}}_{ij}\,\varphi^{,ij}.
\end{aligned} \tag{D.1}$$

From equations (C.15) - (C.18) we obtain the expressions for the second-order metric perturbations:

$$\psi^{(2)}_{\mathrm{P(t)}} = \alpha^{(2)\,\prime}_{(\mathrm{t})} + \frac{2}{\tau}\alpha^{(2)}_{(\mathrm{t})}, \tag{D.2}$$

$$\omega^{(2)}_{\mathrm{P(t)}\,i} = \frac{2\tau}{3}\chi^{(1)\,\mathrm{T}}_{ij}\,\varphi^{,j} + \beta^{(2)\,\prime}_{(\mathrm{t}),i} - \alpha^{(2)}_{(\mathrm{t}),i} + d^{(2)\,\prime}_{(\mathrm{t})\,i}, \tag{D.3}$$

$$\phi^{(2)}_{\mathrm{P(t)}} = \phi^{(2)}_{\mathrm{S(t)}} - \frac{\tau^2}{9}\chi^{(1)\,\mathrm{T}}_{ij}\,\varphi^{,ij} - \frac{1}{3}\nabla^2\beta^{(2)}_{(\mathrm{t})} - \frac{2}{\tau}\alpha^{(2)}_{(\mathrm{t})}, \tag{D.4}$$

$$\begin{aligned}
\chi^{(2)}_{\mathrm{P(t)}\,ij} &= \chi^{(2)}_{\mathrm{S(t)}\,ij} + \frac{2\tau}{3}\varphi\left(\chi^{(1)\,\mathrm{T}\,\prime}_{ij} + \frac{4}{\tau}\chi^{(1)\,\mathrm{T}}_{ij}\right) + \frac{\tau^2}{3}\left(\chi^{(1)\,\mathrm{T}}_{ij,k}\,\varphi^{,k} + \chi^{(1)\,\mathrm{T}}_{ik}\,\varphi^{,k}_{,j} + \chi^{(1)\,\mathrm{T}}_{jk}\,\varphi^{,k}_{,i}\right) \\
&\quad - \frac{2\tau^2}{9}\delta_{ij}\chi^{(1)\,\mathrm{T}}_{lk}\,\varphi^{,lk} + 2\left(d^{(2)}_{(\mathrm{t})\,(i,j)} + \mathrm{D}_{ij}\beta^{(2)}_{(\mathrm{t})}\right).
\end{aligned} \tag{D.5}$$

We express the second-order metric perturbations in the Poisson gauge with respect to the synchronous gauge ones and to combinations of first-order scalar-tensor terms and tensor-tensor terms, substituting the second-order gauge parameters in equation (D.1).

Lapse perturbation:
This is found from the gauge transformation (D.2), using the momentum constraint (B.3) and the Raychaudhuri equation (B.1)[7]:

$$
\begin{aligned}
\nabla^2\nabla^2\psi^{(2)}_{\mathrm{P(t)}} = {}& \frac{18}{\tau^2}\nabla^2\phi^{(2)}_{\mathrm{S(t)}} + \frac{6}{\tau}\chi^{(1)\mathrm{T}\prime}_{ij}\varphi^{,ij} - 3\nabla^2\chi^{(1)\mathrm{T}}_{ij}\varphi^{,ij} + \frac{5}{8}\nabla^2(\chi^{(1)\mathrm{T}\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\prime}) - \frac{1}{\tau}\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij\prime} \\
& + \frac{1}{2\tau}\chi^{(1)\mathrm{T}\,ij\prime}\nabla^2\chi^{(1)\mathrm{T}}_{ij} - \frac{1}{2\tau}\chi^{(1)\mathrm{T}\,ij,k\prime}\chi^{(1)\mathrm{T}}_{ij,k} + \frac{3}{\tau}\chi^{(1)\mathrm{T}\prime}_{ij,k}\chi^{(1)\mathrm{T}\,kj,i} - \frac{1}{4}\nabla^2\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij} \\
& + \frac{1}{4}\nabla^2\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k} - \frac{3}{2}\chi^{(1)\mathrm{T}\prime}_{ij,k}\chi^{(1)\mathrm{T}\,kj,i\prime} + \frac{1}{2}\chi^{(1)\mathrm{T}}_{ij}\nabla^2\nabla^2\chi^{(1)\mathrm{T}\,ij} - \frac{3}{2}\nabla^2\chi^{(1)\mathrm{T}}_{kj,i}\chi^{(1)\mathrm{T}\,ij,k} \\
& + \frac{3}{\tau^2}\nabla^2\Big(\chi^{(1)\mathrm{T}\,ij}\chi^{(1)\mathrm{T}}_{ij} - \chi^{(1)\mathrm{T}\,ij}_{0}\chi^{(1)\mathrm{T}}_{0\,ij}\Big);
\end{aligned}
\tag{D.6}
$$

Shift perturbation:
From equation (D.3), using the momentum constraint (B.3) and the Raychaudhuri equation (B.1):

$$
\begin{aligned}
\nabla^2\nabla^2\omega^{(2)}_{\mathrm{P(t)}\,,i} = {}& (-4\chi^{(1)\mathrm{T}\prime}_{ij}\varphi^{,j} - 2\chi^{(1)\mathrm{T}\prime}_{ik,j}\chi^{(1)\mathrm{T}\,jk} + 2\chi^{(1)\mathrm{T}\prime}_{jk,i}\chi^{(1)\mathrm{T}\,jk} + \chi^{(1)\mathrm{T}\,jk}_{,i}\chi^{(1)\mathrm{T}\prime}_{jk}) \\
& + (4\chi^{(1)\mathrm{T}\prime}_{jk}\varphi^{,kj} - \chi^{(1)\mathrm{T}\prime}_{kj}\nabla^2\chi^{(1)\mathrm{T}\,kj} - 2\chi^{(1)\mathrm{T}}_{kj}\nabla^2\chi^{(1)\mathrm{T}\,kj\prime} - 3\chi^{(1)\mathrm{T}\,kj,l\prime}\chi^{(1)\mathrm{T}}_{kj,l} \\
& + 2\chi^{(1)\mathrm{T}\,kj,l\prime}\chi^{(1)\mathrm{T}}_{lj,k})_{,i}\,;
\end{aligned}
\tag{D.7}
$$

Spatial metric, trace:
It is found from the gauge transformation (D.4), momentum constraint (B.3) and the Raychaudhuri equation (B.1):

$$
\begin{aligned}
\nabla^2\nabla^2\phi^{(2)}_{\mathrm{P(t)}} = {}& \frac{18}{\tau^2}\nabla^2\phi^{(2)}_{\mathrm{S(t)}} - \nabla^2(\chi^{(1)\mathrm{T}}_{ij}\varphi^{,ij}) + \frac{1}{8}\nabla^2(\chi^{(1)\mathrm{T}\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\prime}) - \frac{1}{\tau}\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij\prime} \\
& + \frac{1}{2\tau}\nabla^2\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij\prime} - \frac{1}{2\tau}\chi^{(1)\mathrm{T}\prime}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k} + \frac{3}{\tau}\chi^{(1)\mathrm{T}\prime}_{kj,i}\chi^{(1)\mathrm{T}\,ij,k} - \frac{1}{2}\nabla^2(\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij}) \\
& - \frac{3}{4}\nabla^2\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k} - \frac{3}{4}\chi^{(1)\mathrm{T}}_{ij,kl}\chi^{(1)\mathrm{T}\,ij,kl} + \frac{1}{2}\chi^{(1)\mathrm{T}}_{ij,kl}\chi^{(1)\mathrm{T}\,kj,il} + \frac{1}{2}\nabla^2\chi^{(1)\mathrm{T}}_{kj,i}\chi^{(1)\mathrm{T}\,ij,k} \\
& + \frac{6}{\tau}\chi^{(1)\mathrm{T}\prime}_{ij}\varphi^{,ij} + \frac{3}{\tau^2}\nabla^2\Big(\chi^{(1)\mathrm{T}\,ij}\chi^{(1)\mathrm{T}}_{ij} - \chi^{(1)\mathrm{T}\,ij}_{0}\chi^{(1)\mathrm{T}}_{0\,ij}\Big);
\end{aligned}
\tag{D.8}
$$

[7] Note that the following expressions for the Lapse perturbation, the trace and the traceless part of the spatial metric are slightly different from the ones found in [1]: for the Lapse perturbation, only the factors of two terms (the first and the tenth ones in the RHS) are different; for the trace part, there are less terms with respect to [1] (for a possible error of sign during the computation); for the traceless part, there are two more terms.

Spatial metric, traceless part:
It comes from the gauge transformation (D.5) and from equations (D.1):

$$
\begin{aligned}
\nabla^2\nabla^2\chi^{(2)}_{\mathrm{P(t)}\,ij} = {} & \nabla^2\nabla^2\chi^{(2)}_{\mathrm{S(t)}\,ij} - 2\,\nabla^2\chi^{(2)\,,k}_{\mathrm{S(t)}\,k(i,j)} + \frac{1}{2}\chi^{(2)\,,kl}_{\mathrm{S(t)}\,,klij} + \frac{1}{2}\delta_{ij}\,\nabla^2\chi^{(2)\,kl}_{\mathrm{S(t)}\,,kl} - \nabla^2\Bigg\{\frac{4\tau}{3}\Bigg[\varphi^{,k}\Bigg(\chi^{(1)\mathrm{T}\,\prime}_{k(i} \\
& + \frac{4}{\tau}\chi^{(1)\mathrm{T}}_{k(i}\Bigg)\Bigg]_{,j)} + \frac{4\tau^2}{3}\,\varphi^{,kl}\chi^{(1)\mathrm{T}}_{k(i,j)l} + \frac{2\tau^2}{3}\nabla^2\,\varphi^{,k}\chi^{(1)\mathrm{T}}_{k(i,j)} - \frac{4\tau^2}{9}\Bigg(\varphi^{,kl}\,\chi^{(1)\mathrm{T}}_{lk,ij} - \frac{1}{2}\varphi^{,kl}_{,ij}\,\chi^{(1)\mathrm{T}}_{lk} \\
& + \frac{1}{2}\chi^{(1)\mathrm{T}}_{lk,(i}\,\varphi^{,kl}_{,j)}\Bigg) - \frac{2\tau^2}{3}\nabla^2\,\chi^{(1)\mathrm{T}}_{l(i}\varphi^{,l}_{,j)} - \nabla^2\Bigg[\frac{2\tau}{3}\varphi\Bigg(\chi^{(1)\mathrm{T}\,\prime}_{ij} + \frac{4}{\tau}\chi^{(1)\mathrm{T}}_{ij}\Bigg) + \frac{\tau^2}{3}\,\varphi^{,k}\chi^{(1)\mathrm{T}}_{ij,k}\Bigg] \\
& - \frac{1}{2}\,\delta_{ij}\Bigg[\frac{2\tau}{3}\,\varphi^{,kl}\Bigg(\chi^{(1)\mathrm{T}\,\prime}_{kl} + \frac{4}{\tau}\chi^{(1)\mathrm{T}}_{kl}\Bigg) - \frac{\tau^2}{3}\,\varphi^{,klm}\chi^{(1)\mathrm{T}}_{kl,m} - \frac{2\tau^2}{3}\,\varphi^{,kl}\nabla^2\chi^{(1)\mathrm{T}}_{kl}\Bigg]\Bigg\} \\
& + \frac{1}{2}\Bigg[\frac{2\tau}{3}\,\varphi^{,kl}\Bigg(\chi^{(1)\mathrm{T}\,\prime}_{kl} + \frac{4}{\tau}\chi^{(1)\mathrm{T}}_{kl}\Bigg) + \frac{5\tau^2}{9}\,\varphi^{,klm}\chi^{(1)\mathrm{T}}_{kl,m} + \frac{4\tau^2}{9}\,\chi^{(1)\mathrm{T}}_{kl}\nabla^2\varphi^{,kl} \\
& - \frac{2\tau^2}{9}\,\varphi^{,kl}\nabla^2\chi^{(1)\mathrm{T}}_{kl}\Bigg]_{,ij} - \frac{4\tau^2}{3}\,\nabla^2\Bigg(\chi^{(1)\mathrm{T}}_{k(i,(l)}\,\varphi^{,l}_{,j)}\Bigg) - \frac{2\tau^2}{3}\,\nabla^2\Bigg(\chi^{(1)\mathrm{T}}_{k(i}\nabla^2\varphi^{,k}_{,j)}\Bigg).
\end{aligned}
\tag{D.9}
$$

Appendix E

Einstein equations perturbed at second-order in the Poisson gauge

In this Appendix we show the Ricci tensors, the Einstein tensors and the stress-energy tensors obtained in the Poisson gauge and in an Einstein-de Sitter Universe. Contrary to what can be found in the literature [1–3, 8], here we have eliminated the first-order scalars (that can be obtained in those references) and we have preserved the first-order tensors, while it is usually done the other way around. Their terms are ordered from the background contribution to the second-order one. The inverse metric tensor and the Christoffel symbols are also presented in this gauge, already neglecting first-order scalars and vectors in the source.

Inverse metric tensor up to second order:

$$
\begin{aligned}
g^{00} &= -a^{-2}(1 - \psi^{(2)}_{\mathrm{P(t)}})\,; \\
g^{0i} &= a^{-2}\frac{1}{2}\omega^{(2)\,i}_{\mathrm{P(t)}}\,; \\
g^{ij} &= a^{-2}(\delta^{ij} - \chi^{(1)\mathrm{T}\,ij} - \frac{1}{2}\chi^{(2)\mathrm{T}\,ij}_{\mathrm{P(t)}} - \chi^{(1)\mathrm{T}\,ik}\chi^{(1)\mathrm{T}\,j}_{k} + \phi^{(2)}_{\mathrm{P(t)}}\,\delta^{ij})\,.
\end{aligned}
$$

Christoffel symbols up to second order:

$$
\begin{aligned}
\Gamma^0_{00} &= \frac{2}{\tau} + \frac{1}{2}\,\psi'^{(2)}_{\mathrm{P(t)}}\,; \\
\Gamma^0_{0i} &= \frac{1}{\tau}\,\omega^{(2)}_{\mathrm{P(t)}\,i} + \frac{1}{2}\,\psi^{(2)}_{\mathrm{P(t)}\,,i}\,; \\
\Gamma^i_{00} &= \frac{1}{\tau}\,\omega^{(2)\,i}_{\mathrm{P(t)}} + \frac{1}{2}\,\omega'^{(2)\,i}_{\mathrm{P(t)}} + \frac{1}{2}\,\psi^{(2)\,,i}_{\mathrm{P(t)}}\,; \\
\Gamma^i_{0j} &= \frac{2}{\tau}\,\delta^i_j + \frac{1}{2}\chi'^{(1)\mathrm{T}\,i}_{j} + \frac{1}{4}\chi'^{(2)\mathrm{T}\,i}_{\mathrm{P(t)}\,j} - \frac{1}{2}\,\phi'^{(2)}_{\mathrm{P(t)}}\delta^i_j - \frac{1}{2}\chi^{(1)\mathrm{T}\,ik}\chi'^{(1)\mathrm{T}}_{kj} + \frac{1}{4}(\omega^{(2)\,i}_{\mathrm{P(t)}\,,j} - \omega^{(2)\,,i}_{\mathrm{P(t)}\,j})\,; \\
\Gamma^0_{ij} &= \left(\frac{2}{\tau} - \frac{2}{\tau}\,\psi^{(2)}_{\mathrm{P(t)}}\right)\delta_{ij} + \frac{1}{2}\chi'^{(1)\mathrm{T}}_{ij} + \frac{1}{4}\chi'^{(2)\mathrm{T}}_{\mathrm{P(t)}\,ij} - \frac{1}{2}\,\phi'^{(2)}_{\mathrm{P(t)}}\delta_{ij} + \frac{2}{\tau}\chi^{(1)\mathrm{T}}_{ij} - \frac{2}{\tau}\,\phi^{(2)}_{\mathrm{P(t)}}\delta_{ij} + \frac{1}{\tau}\chi^{(2)\mathrm{T}}_{\mathrm{P(t)}\,ij} \\
&\quad - \frac{1}{4}(\omega^{(2)}_{\mathrm{P(t)}\,i,j} - \omega^{(2)}_{\mathrm{P(t)}\,j,i})\,; \\
\Gamma^i_{jk} &= \frac{1}{2}\left(\chi^{(1)\mathrm{T}\,i}_{k,j} + \chi^{(1)\mathrm{T}\,i}_{j,k} - \chi^{(1)\mathrm{T}\,,i}_{jk}\right) + \frac{1}{4}\left(\chi^{(2)\mathrm{T}\,i}_{\mathrm{P(t)}\,k,j} + \chi^{(2)\mathrm{T}\,i}_{\mathrm{P(t)}\,j,k} - \chi^{(2)\mathrm{T}\,,i}_{\mathrm{P(t)}\,jk}\right) - \frac{1}{\tau}\,\delta_{jk}\omega^{(2)\,i}_{\mathrm{P(t)}} \\
&\quad - \frac{1}{2}\left(\phi^{(2)}_{\mathrm{P(t)}\,,j}\,\delta^i_k + \phi^{(2)}_{\mathrm{P(t)}\,,k}\,\delta^i_j - \phi^{(2)\,,i}_{\mathrm{P(t)}}\,\delta_{jk}\right) - \frac{1}{2}\chi^{(1)\mathrm{T}\,il}\left(\chi^{(1)\mathrm{T}}_{kl,j} + \chi^{(1)\mathrm{T}}_{jl,k} - \chi^{(1)\mathrm{T}}_{jk,l}\right).
\end{aligned}
$$

Ricci tensors up to second order:

$$R_{00} = \frac{6}{\tau^2} + \frac{3}{\tau}\,\psi'^{(2)}_{\mathrm{P(t)}} + \frac{1}{2}\nabla^2\psi^{(2)}_{\mathrm{P(t)}} + \frac{3}{2}\,\phi''^{(2)}_{\mathrm{P(t)}} + \frac{3}{\tau}\,\phi'^{(2)}_{\mathrm{P(t)}} + \frac{1}{4}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime} + \frac{1}{2}\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij}$$
$$- \frac{1}{\tau}\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime}; \tag{E.1}$$

$$R_{0i} = \phi'^{(2)}_{\mathrm{P(t)},i} + \frac{2}{\tau}\,\psi^{(2)}_{\mathrm{P(t)},i} - \frac{1}{4}\nabla^2\omega^{(2)}_{\mathrm{P(t)}\,i} + \frac{3}{\tau^2}\,\omega^{(2)}_{\mathrm{P(t)}\,i} - \frac{1}{2}\chi^{(1)\mathrm{T}\,jk}\chi^{(1)\mathrm{T}\,\prime}_{ik,j} + \frac{1}{4}\chi^{(1)\mathrm{T}\,jk\,\prime}\chi^{(1)\mathrm{T}}_{jk,i} + \frac{1}{2}\chi^{(1)\mathrm{T}\,jk}\chi^{(1)\mathrm{T}\,\prime}_{jk,i}\;; \tag{E.2}$$

$$R_{ij} = \frac{6}{\tau^2}\,\delta^i_j + \frac{6}{\tau^2}\chi^{(1)\mathrm{T}}_{ij} - \frac{1}{2}\,\psi^{(2)}_{\mathrm{P(t)},ij} + \frac{1}{2}\,\phi^{(2)}_{\mathrm{P(t)},ij} + \frac{1}{4}\chi''^{(2)}_{\mathrm{P(t)}\,ij} + \frac{1}{\tau}\chi'^{(2)}_{\mathrm{P(t)}\,ij} + \frac{3}{\tau^2}\chi^{(2)}_{\mathrm{P(t)}\,ij}$$
$$+ \delta^i_j\left(- \frac{1}{\tau}\,\psi'^{(2)}_{\mathrm{P(t)}} - \frac{6}{\tau^2}\,\psi^{(2)}_{\mathrm{P(t)}} - \frac{1}{2}\,\phi''^{(2)}_{\mathrm{P(t)}} - \frac{5}{\tau}\,\phi'^{(2)}_{\mathrm{P(t)}} - \frac{6}{\tau^2}\,\phi^{(2)}_{\mathrm{P(t)}} + \frac{1}{2}\nabla^2\phi^{(2)}_{\mathrm{P(t)}} - \frac{1}{\tau}\chi^{(1)\mathrm{T}\,kl}\chi^{(1)\mathrm{T}\,\prime}_{kl}\right)$$
$$- \frac{1}{2}\chi^{(1)\mathrm{T}\,kl}\chi^{(1)\mathrm{T}}_{jl,ki} + \frac{1}{2}\chi^{(1)\mathrm{T}\,kl}\chi^{(1)\mathrm{T}}_{ij,kl} + \frac{1}{2}\chi^{(1)\mathrm{T}\,kl}\chi^{(1)\mathrm{T}}_{kl,ij} + \frac{1}{4}\chi^{(1)\mathrm{T}\,kl}_{,i}\chi^{(1)\mathrm{T}}_{kl,j} - \frac{1}{2}\chi^{(1)\mathrm{T}\,\prime}_{ik}\chi^{(1)\mathrm{T}\,k\,\prime}_{j}$$
$$+ \frac{1}{2}\chi^{(1)\mathrm{T}\,l}_{i,k}\chi^{(1)\mathrm{T}\,,k}_{lj} - \frac{1}{2}\chi^{(1)\mathrm{T}\,kl}\chi^{(1)\mathrm{T}}_{il,kj} - \frac{1}{4}\nabla^2\chi^{(2)}_{\mathrm{P(t)}\,ij} - \frac{1}{2}\,\omega'^{(2)}_{\mathrm{P(t)}\,(i,j)} - \frac{2}{\tau}\omega^{(2)}_{\mathrm{P(t)}\,(i,j)} - \frac{1}{2}\chi^{(1)\mathrm{T}\,l}_{i,k}\chi^{(1)\mathrm{T}\,i}_{j,l}\,. \tag{E.3}$$

Second-order Einstein tensors:

$$\frac{1}{2}\delta G^{(2)\,0}_{0} = \frac{1}{a^2}\left(\frac{12}{\tau^2}\,\psi^{(2)}_{\mathrm{P(t)}} + \frac{6}{\tau}\,\phi'^{(2)}_{\mathrm{P(t)}} - \nabla^2\phi^{(2)}_{\mathrm{P(t)}} + \frac{1}{8}\chi^{(1)\mathrm{T}\,\prime}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime} - \frac{1}{2}\chi^{(1)\mathrm{T}}_{ij}\nabla^2\chi^{(1)\mathrm{T}\,ij}\right.$$
$$\left. + \frac{2}{\tau}\chi^{(1)\mathrm{T}}_{ij}\chi^{(1)\mathrm{T}\,ij\,\prime} - \frac{3}{8}\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ij,k} + \frac{1}{4}\chi^{(1)\mathrm{T}}_{ij,k}\chi^{(1)\mathrm{T}\,ik,j}\right); \tag{E.4}$$

$$\frac{1}{2}\delta G^{(2)\,0}_{i} =$$
$$= \frac{1}{a^2}\left(- \frac{2}{\tau}\psi^{(2)}_{\mathrm{P(t)},i} - \phi'^{(2)}_{\mathrm{P(t)},i} + \frac{1}{4}\nabla^2\omega^{(2)}_{\mathrm{P(t)}\,i} + \frac{1}{2}\chi^{(1)\mathrm{T}\,jk}\chi^{(1)\mathrm{T}\,\prime}_{ik,j} - \frac{1}{4}\chi^{(1)\mathrm{T}\,jk\,\prime}\chi^{(1)\mathrm{T}}_{jk,i} - \frac{1}{2}\chi^{(1)\mathrm{T}\,jk}\chi^{(1)\mathrm{T}\,\prime}_{jk,i}\right); \tag{E.5}$$

$$
\begin{aligned}
&\frac{1}{2}\delta G^{(2)\,i}_{j} = \\
&= \frac{\delta^i_j}{a^2}\bigg(\frac{2}{\tau}\,\psi'^{(2)}_{\mathrm{P(t)}} + \frac{\nabla^2\psi^{(2)}_{\mathrm{P(t)}}}{2} + \phi''^{(2)}_{\mathrm{P(t)}} + \frac{4}{\tau}\phi'^{(2)}_{\mathrm{P(t)}} - \frac{\nabla^2\phi^{(2)}_{\mathrm{P(t)}}}{2} + \frac{3}{8}\chi^{(1)\mathrm{T}\,\prime}_{kl}\chi^{(1)\mathrm{T}\,kl\,\prime} - \frac{3}{8}\chi^{(1)\mathrm{T}}_{kl,m}\chi^{(1)\mathrm{T}\,kl,m} \\
&+ \frac{1}{4}\chi^{(1)\mathrm{T}}_{kl,m}\chi^{(1)\mathrm{T}\,mk,l}\bigg) + \frac{1}{a^2}\bigg(- \frac{1}{2}\,\psi^{(2),i}_{\mathrm{P(t)}\,,j} + \frac{1}{2}\,\phi^{(2)\,,i}_{\mathrm{P(t)}\,,j} + \frac{1}{4}\chi''^{(2)i}_{\mathrm{P(t)}\,j} + \frac{1}{\tau}\chi'^{(2)\,i}_{\mathrm{P(t)}\,j} - \frac{1}{4}\nabla^2\chi^{(2)i}_{\mathrm{P(t)}\,j} \\
&- \frac{1}{2}\,\omega'^{(2)\,(i}_{\mathrm{P(t)}\,,j)} - \frac{2}{\tau}\omega^{(2)\,(i}_{\mathrm{P(t)}\,,j)} - \frac{1}{2}\chi^{(1)\mathrm{T}\,kl}\chi^{(1)\mathrm{T}\,i}_{l,kj} - \frac{1}{2}\chi^{(1)\mathrm{T}\,kl}\chi^{(1)\mathrm{T}\,,i}_{jl,k} + \frac{1}{2}\chi^{(1)\mathrm{T}\,kl}\chi^{(1)\mathrm{T}\,i}_{j,kl} \\
&+ \frac{1}{2}\chi^{(1)\mathrm{T}\,kl}\chi^{(1)\mathrm{T}\,,i}_{kl,j} + \frac{1}{4}\chi^{(1)\mathrm{T}\,kl,i}\chi^{(1)\mathrm{T}}_{kl,j} - \frac{1}{2}\chi^{(1)\mathrm{T}\,ik\,\prime}\chi^{(1)\mathrm{T}\,\prime}_{kj} - \frac{1}{2}\chi^{(1)\mathrm{T}\,il,k}\chi^{(1)\mathrm{T}}_{jk,l} + \frac{1}{2}\chi^{(1)\mathrm{T}\,il,k}\chi^{(1)\mathrm{T}}_{jl,k}\bigg).
\end{aligned}
\tag{E.6}
$$

Second-order stress-energy tensors:

$$\frac{1}{2}\delta T^{(2)\,0}_{0} = -\frac{6}{a^2\tau^2}\,\delta^{(2)}_{\mathrm{P(t)}}; \tag{E.7}$$

$$\frac{1}{2}\delta T^{(2)\,0}_{i} = \frac{6}{a^2\tau^2}\,(v^{(2)}_{\mathrm{P(t)}\,i} + \omega^{(2)}_{\mathrm{P(t)}\,i}); \tag{E.8}$$

$$\frac{1}{2}\delta T^{(2)\,i}_{j} = 0\,. \tag{E.9}$$

Appendix F

Python program to plot $\delta^{(2)}_{\mathbf{S(t)}}$

```
import scipy.integrate as integrate
import scipy.special as special
import numpy as np
import matplotlib.pyplot as plt
plt.rcParams['text.usetex'] = True


k=10.
ttopvalues = np.linspace(0., 2, 201)


inttvalues1 = []
intvalues1 = []
inttvalues2 = []
intvalues2 = []
deltavalues = []


for i in range(0, 201):
ttopvalue = ttopvalues[i]


Area1 = integrate.dblquad(lambda t, k1:
(4./(2.*np.pi))*(9./t)*((np.sin((k-k1)*t)
- 3.*special.jn(1,(k-k1)*t))/((k-k1)*t**2.))*((np.sin(k1*t)
-3.*special.jn(1,k1*t))/(k1*t**2.)), 0, ttopvalue, -10, 10)
Area2 = integrate.dblquad(lambda t, k1:
(4./(2.*np.pi))*(9.*t**4.)*((np.sin((k-k1)*t)
- 3.*special.jn(1,(k-k1)*t))/ ((k-k1)*t**2.))*((np.sin(k1*t)
-3.*special.jn(1,k1*t))/(k1*t**2.)), 0, ttopvalue, -10, 10)
delta =((ttopvalue**2)/10.)*Area1[0]-(1/(10.*ttopvalue**3))*Area2[0]


intvalues1.append(Area1[0])
intvalues2.append(Area2[0])
inttvalues1.append((ttopvalue**2)*Area1[0])
inttvalues2.append((1/(ttopvalue**3))*Area2[0])
deltavalues.append(delta)
```

```
print "ciclo ", i, "int1 ", Area1[0], "int2 ", Area2[0], "delta",
delta
i+=1

G = plt.figure(1)
st = G.suptitle("Time evolution of the first integral", fontsize=
"x-large")
axes1 = G.add_subplot(121)
axes1.plot(ttopvalues, intvalues1)
axes1.set_ylabel('int1')
axes1.set_xlabel('conformal time')

axes2 = G.add_subplot(122)
axes2.plot(ttopvalues, inttvalues1)
axes2.set_ylabel(r'$\tau^2$ int1')
axes2.set_xlabel('conformal time')

plt.show()

G = plt.figure(1)
st = G.suptitle("Time evolution of the second integral", fontsize=
"x-large")
axes2 = G.add_subplot(121)
axes2.plot(ttopvalues, intvalues2)
axes2.set_ylabel('int2')
axes2.set_xlabel('conformal time')

axes4 = G.add_subplot(122)
axes4.plot(ttopvalues, inttvalues2)
axes4.set_ylabel(r'$\frac{1}{\tau^3}$ int2')
axes4.set_xlabel('conformal time')

plt.show()

b = ttopvalues**2.
```

```
plt.plot(ttopvalues, deltavalues,'g', ttopvalues, b,'r')
plt.ylabel(r'$\delta^{(2)}_{(t)}/A^2(k_*)$')
plt.xlabel('conformal time')
plt.title(r'Time evolution of $\delta^{(2)}_{(t)}$')

plt.show()
```

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