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The model of Solid Inflation and its outcomes

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Introduction

The theory of General Relativity provides the theoretical framework for the study of the Universe on large scales. This theory, however, consists of a complex system of non-linear differential equations. Therefore, the addition of the *Cosmological Principle* allows for a simple and effective description of the Universe dynamics. The Cosmological Principle states that, for comoving observers, the Universe must look the same in every direction and at every point of observation, which corresponds to imposing strong symmetry requirements to the metric tensor. The Einstein equations combined with the Cosmological Principle are the basic ingredients of modern cosmology.

A generic prediction of this framework is that the Universe is not static but rather it is expanding. This implies that, by proceeding back in time, the Universe must have had in its past a hotter and denser phase than nowadays. This description of the history of the Universe is provided by the standard model of *Hot Big Bang* and its predictions have been confirmed on observational grounds since the discovery of distant galaxies moving away from the Earth made by Hubble in 1929. Another important prediction is the existence of a relic radiation from an early epoch of the Universe, which has been observed by Penzias and Wilson in 1965. This further prediction was fundamental in proving that the Universe was denser and hotter in the past. This relic radiation was named *Cosmic Microwave Background* (CMB) and represents the "Rosetta stone" of modern cosmology. Indeed it is one of the most important subjects of cosmological observation as it provides information about the composition, geometry and evolution of the Universe.

With the increasing amount of information derived from observational data, cosmologists eventually realized that the description of the Universe given by the Hot Big Bang model has two important issues. Firstly, from the observation of the CMB, which originated when photons decoupled from matter, we can infer that in this phase the Universe was in thermal equilbrium with a isotropic black-body distribution. Within the Hot Big Bang model it is not possible to explain such a degree of thermal homogeneity. One is then forced to assume that regions that were never in causal connection were in the same condition before the last scattering. This is defined as the *Horizon* problem. Secondly, CMB observations indicate that the total density of the Universe is such to make the spatial curvature of the Universe negligible today [4]. According to the Hot Big Bang model, we must infer that this was true also in the past by orders of magnitude. This issue is known as the *Flatness* problem. The above issues highlight how using the Hot Big Bang model to explain the initial stage of the Universe is contrived. Nonetheless, if one assumes an early period of accelerated expansion of the Universe, which is known as "inflation", the horizon and flatness problems are solved dynamically, namely we are not forced to make such strict initial assumptions.

This theory was first proposed by A. Guth [36] and then revised by A. Linde [47], A. Albrecht and P. Steinhardt [12]. The essential idea is that inflation was triggered by a scalar field slowly rolling down its potential, whose energy density was dominating the Universe. The inflationary stage ended when the above field (dubbed "the inflaton") started to reach the minimum of its potential, thus exiting the slow-roll phase. The field then ceased its energy by decaying into relativistic particles, thus setting the beginning of the standard radiation dominated epoch of the Hot Big Bang picture.

Most importantly, the theory of inflation intrinsically provides an explanation for the production of the first density perturbations in the early Universe, which are at the origin of the small temperature anisotropies impressed in the CMB and the subsequential formation of large scale structures through gravitational instability. We can explain this phenomenon exploiting the quantum fluctuations of the inflaton field occurred during inflation as they were stretched to macroscopic scales through the exponential expansion of space. In this scenario, such quantum fluctuations induced fluctuations in the space-time metric, which can be classified in *scalar* (*e.g.* density), *vector* and *tensor* (*i.e.* gravitational waves) ones. This fact left specific signatures on the CMB, which are amenable to be tested on observational grounds in the form of statistical correlations between temperatures anisotropies in the CMB maps. The analysis of the CMB can thus reveal precious information on the primordial inflationary stage of the Universe. The first type of information that is possible to obtain is the power spectrum of the above primordial fluctuations, namely the Fourier transform of the two-point correlation function. In particular, the characteristics of the scalar power spectrum predicted by the single field slow-roll model of inflation have been confirmed by the *Plank* satellite data with unprecedent accuracy [6]. Despite this success, a vast class of inflationary models with different fields interactions have been proposed on theoretical grounds, which can lead to very similar predictions for the scalar power spectrum. This fact makes it difficult to select a single candidate for the description of the early Universe.

Given the abundance of possible inflationary scenarios, it is necessary to have available observables quantities whose values encode distinctive features of specific classes of models. This way, it is possible to operate a selective investigation through which one can confirm or rule out specific inflationary scenarios. Such observables are provided by the primordial *non-Gaussianities* of the CMB anisotropies, *i.e.* their deviation from pure Gaussian statistics due to the existence of higher-order connected correlation functions. These correlation functions are linked to the n-point expectation values of primordial quantum fluctuations. The above expectation values result from the interactions experienced during inflation by the quantum field(s) that characterized the physics of the very early Universe. Specific classes of inflationary models leaves therefore their specific non-Gaussian signature in the CMB. The leading order non-Gaussian signal that we consider is the three-point (connected) correlation function, or its Fourier tranform, the *bispectrum*. At present the single-field slow-roll model is still valid, since its primordial bispectrum of scalar fluctuations lays in the confidence region provided by recent observations [5]. Leaving aside all the observational issues, on a purely theoretical level it is *per se* important to study the features and the type of bispectra predicted within a specific model. By "type" of bispectra we mean every possible combination of bispectrum involving scalar, vector and tensor fluctuations that specific models can predict.

The model that will be studied in this thesis is the one of *Solid Inflation* [32]. The name derives from the fact that this model can be understood as a basic effective field theory for a solid, constructed therefore with three scalar field $\{\Phi^I\}$ which represent the internal degrees of freedom of the medium. The configuration that triggered inflation is the one of a solid at rest, namely

$$\{\Phi^I\} = x^I \,, \quad I = 1,\, 2,\, 3 \,. \tag{1}$$

The three scalars individually break isotropy and homogeneity. Nevertheless, they can be combined in such a way that they can produce a homogeneous and isotropic expansion. However, this peculiar symmetry breaking leads to a phenomenology which is rich, being characterized by uncommon and interesting features.

We will immediately mention that vector perturbations within Solid Inflation are sustained during the inflationary expansion, while in many other theories this type of metric fluctuations are not considered because rapidly decaying. Another important feature is that tensor fluctuations acquire a mass term. As far as primordial non-Gaussianities are concerned, it has been proved in [32] that the primordial bispectrum of scalar fluctuations shows a non-trivial angular dependence, which is another uncommon outcome among inflationary models (see also [16, 19]).

This thesis therefore is structured as follows:

- Chapter 1 will start by presenting a brief review of the inflationary paradigm. More precisely, the chapter is focused on the model of single-field slow-roll inflation. The discussion will then proceed with the evolution of the quantum fluctuations of the scalar field in an simplified version, *i.e.* without considering the dynamics of gravity perturbations.

- Chapter 2 will commence with the explanation of the theory of cosmological perturbations in General Relativity. More specifically, the perturbative theory will be studied to the linear order. The inclusion of the metric perturbation dynamics is the following and necessary step in order to have a more comprehensive picture of the evolution of primordial perturbation during inflation than the one presented in Chapter 1. Subsequently, we will apply this approach to the single-field slow-roll model in order to derive the scalar and tensor power spectra predicted by this scenario.
- Chapter 3 will continue developing the study of primordial perturbation in order to address the subject of primordial non-Gaussianities. After having introduced some of the primordial bispectra analyzed by the *Planck* team, we will focus on the theoretical methodology by which the primordial bispectra are computed. The approach that will be followed has its base on the parametrization of the metric proposed by Arnowitt, Deser and Misner (ADM) in conjunction with the *in-in* formalism.
- Chapter 4 reviews the model of Solid Inflation. The discussion will begin with the construction of the effective field theory for the medium under consideration, following with the analysis of the perturbation dynamics to linear order predicted by this model. Moreover, the mode functions of the above perturbations will be explicitely computed.
- Chapter 5 contains the original contribution of this thesis. The methodology exposed in Chapter 3 is applied to the model of Solid Inflation. The computation of six bispectra involving scalar, vector and tensor perturbations predicted within this model will be extensively reported.

The convention we will use in this thesis are the following:

- we will utilize natural units: $\hbar = c = 1$;
- our metric signature is $(-,+,+,+)$;
- our convention for the Fourier transform is

$$f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})\,. \tag{2}$$

- the reduced Plank mass is defined as $M_P = (8\pi G)^{-1/2}$, where G is Newton gravitational constant.

Chapter 1

The paradigm of inflation

Before proceeding towards the subject of the thesis, it is necessary to introduce the basic theoretical framework in which the inflationary theory of cosmological perturbations is developed. We will present a model involving a single scalar real field that is coupled to gravity, which is the simplest way for implementing an inflationary expansion of the Universe and for discussing what happens to its quantum fluctuations.

Such fluctuations are in fact believed to be at the origin of the temperature anisotropies impressed in the Cosmic Microwave Background (CMB) radiation and, ultimately, the seeds of the cosmic structures we observe in the current Universe. The above idea allows the theory to be successfully predictive: whithin the inflationary paradigm quantum fluctuations are naturally provived and leave specific signature in the sky that are amenable to be tested on observational grounds. As we will see later, especially in discussing primordial non-Gaussianity in Chapter 3, different models of inflation give quite specific predictions. Given the above considerations, the single field model turns out not only to be a tool for a pedagogical introduction but it will allow to emphasize the peculiarities of the Solid Inflation model.

1.1 Inflation as a solution for the Horizon and Flatness problems

Cosmology describes the structure and evolution of the Universe on large scales. The Einstein equation provides such a description linking the geometry of space-time on such large scales with the content of the Universe:

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{1.1}$$

where $R_{\mu\nu}$ and R are respectively the Ricci tensor and the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, G is the Newtonian constant and $T_{\mu\nu}$ is the energy-momentum tensor associated with particles and fields filling the Universe. We can adopt an additional assumption on the symmetries that the Universe obeys: by imposing it to be homogeneous and isotropic for comoving observers, we get a remarkable semplification in dealing with an otherwise very complicated task. This is the statement of *The Cosmological Principle*, which is the basis of modern cosmology. This assumption is also consistent with the observation of both Large-Scale Structures and temperature anisotropies of CMB [4].

Three distinctive geometry of the Universe results by adopting this symmetry argument [59], described by the Robertson-Walker (RW) metric whose line element is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + sin^2(\theta) d\phi^2) \right], \tag{1.2}$$

where t is the cosmic time, r, θ, ϕ are comoving polar coordinates, $a(t)$ is the scale-factor and K is the spatial curvature parameter of 3-dimensional hypersurfaces, which can assume the vales

$K = 0, +1, -1$ according to a flat, closed or open geometry respectively. The scale-factor has a dimension of length. As for the energy-momentum tensor, it should be emphasized that the symmetry requirements of homogeneity and isotropy can be satisfied if it assumes the following form[59]:

$$T_{\mu\nu} = \Big(\rho(t) + p(t)\Big) u_\mu u_\nu + p(t) g_{\mu\nu} \tag{1.3}$$

where ρ and p are the energy density and the isotropic pressure, while $u_\mu \equiv dx_\mu/d\lambda$, being λ some affine parameter. This is the form of an energy-momentum tensor of a perfect fluid. If more matter species are present ρ and p are undestood as sum of all components

$$\rho = \sum_i \rho_i\,, \qquad p = \sum_i p_i\,. \tag{1.4}$$

The Einstein equations will therefore result in two coupled, non linear differential equations, the so-called *Friedmann Equations*,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}\,, \tag{1.5}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\,. \tag{1.6}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter which has unit of inverse time and overdots denote derivatives with respect the cosmic time t. In addition to the Friedmann equations, the covariant conservation of the energy-momentum tensor $D_\mu T^{\mu\nu} = 0$, implied by the *Bianchi Identity* [59], yields the continuity equation

$$\dot{\rho} + 3H(\rho + p) = 0\,, \tag{1.7}$$

which is usually combined with (1.5) and (1.6) for convenience, although it is not an independent equation with respect to the other two. It should be noticed that in an expanding Universe ($\dot{a} > 0$) filled with ordinary matter, *i.e.* matter satisfying the Strong Energy Condition (SEC) $\rho + 3p \geq 0$, equation (1.6), it is implied both that $\ddot{a} < 0$ and the existence of a time value t_* such that $a(t_*) = 0$. In order to determine the time dependence of the scale factor, the additional information provided by the equation of state of the fluid filling the Universe is necessary, namely

$$p = w\rho\,. \tag{1.8}$$

We report below the three solutions obtained for the cases of a flat Universe ($K = 0$) dominated by non-relativistic matter ($w = 0$, MD), radiation or relativistic matter ($w = \frac{1}{3}$, RD) and a vacuum energy ($w = -1$, Λ):

$$a(t) \propto \begin{cases} t^{\frac{2}{3}} & \text{MD} \\ t^{\frac{1}{2}} & \text{RD} \\ e^{Ht} & \Lambda \end{cases}\,. \tag{1.9}$$

We can then express the scale factor and the Hubble parameter for $w \geq -1$ as

$$a(t) \propto t^{\frac{2}{3(1+w)}}\,, \qquad H(t) = \frac{3}{2}(1+w)\frac{1}{t}\,. \tag{1.10}$$

1.1.1 The Horizon problem

In order to appreciate the importance of inflation in solving the Horizon problem that was mentioned in the Introduction, we have to present some crucial quantities that describe the causal structures of the Universe.

Given that the characteristic time-scale of expansion of the Universe is determined by the Hubble time $\tau_H \equiv H^{-1}$, we can define the distance that light covers during the time τ_H by

$$R_H(t) = \tau_H = \frac{a}{\dot{a}} \qquad (c = 1 \text{ in our convention}), \tag{1.11}$$

which is the so-called *Hubble radius* or *Hubble horizon* size [22, 44]. The respective comoving scale is the *comoving Hubble horizon*, namely

$$r_H(t) \equiv \frac{R_H}{a} = \frac{1}{\dot{a}} = (aH)^{-1} \,, \tag{1.12}$$

which corresponds to the distance travelled by light during one expansion time. The Hubble horizon is the scale that measures the causal connection between two separate regions of the Universe: if two regions of the Universe are separated by a distance greater than R_H, light cannot cover this distance and therefore no information can travel from a region to the other. Such regions are therefore *causally disconnected* at that time.

It should be highlighted that the Hubble horizon is a time-dependent quantity: according to the type of evolution of the scale factor $a(t)$ listed in (1.9), R_H increases or decreases, therefore regions of the Universe can pass from the outside to inside of the sphere of ray R_H, and viceversa. In particular, for values $w > -1$, which include the cases of a dominated Universe by radiation and matter, R_H is monothonically increasing as

$$R_H = \frac{3}{2}(1+w)t \,. \tag{1.13}$$

As for the comoving counterpart r_H, its time dependence is given by

$$r_H(t) = \frac{3}{2}(1+w)t^{\frac{1+3w}{3(1+w)}} \,. \tag{1.14}$$

The second scale we introduce is the *particle horizon*, which is the maximum distance a photon can cover from the time of the origin of the Universe ($t = 0$) to the generic time t:

$$d_p(t) \equiv a(t)\int_0^t \frac{dt'}{a(t')} \,. \tag{1.15}$$

The particle horizon defines a sphere of the ray $d_p(t)$, centered at the origin $\mathbf{x} = 0$, that includes all the points that have been in causal connection with the origin until the time t. If two regions of the Universe at a given time are separated by distance greater than $d_p(t)$, they have never been causally connected at any time before t. The comoving counterpart of d_p is the *comoving particle horizon*, given by

$$\chi_p \equiv \int_0^t \frac{dt'}{a(t')} = \int_0^a d\ln(a')\frac{1}{a'H(a')} \,. \tag{1.16}$$

It should be noted that the comoving particle horizon is defined as the integral of the comoving Hubble horizon. For a Universe dominated by radiation or matter, *i.e.* $w \geq 0$, we have that the comoving Hubble horizon grows monotonically [22] and the fraction of the Universe in causal contact increases in time

$$\chi_p \simeq a^{\frac{1}{2}(1+3w)} \begin{cases} a & \text{RD} \\ a^{\frac{1}{2}} & \text{MD} \end{cases} \,. \tag{1.17}$$

The Horizon problem can therefore be recast as follows: if we assume that the Universe was dominated first by radiation and then by matter, as in the picture of the model of Hot Big Bang, this assumption implies that at the time of the last scattering[1] the regions of the Universe that we observe in the CMB must have never been causally connected, despite the uniformity that the CMB reveals.

It is nonetheless possible to justify this uniformity by assuming an early period of time during which r_H decreased, allowing for regions that were previously causally connected to exit the comoving Hubble horizon. The decreasing of r_H corresponds to

$$\frac{d}{dt}\left(\frac{1}{aH}\right) < 0 \,, \tag{1.18}$$

[1] The CMB originated at a time correspondig to a redshift $z \sim 1100$ [22, 59].

which readily implies $\ddot{a} > 0$. A generic stage of accelerated increasing of the scale factor is dubbed "inflation". From equation (1.6) then follows that the content of the Universe during this stage must be characterized by

$$p < -\frac{\rho}{3} \,. \tag{1.19}$$

Therefore, an inflationary stage was triggered by a violation of the SEC. We will see in the next section that a scalar field rolling down its potential can potentially satisfies the above condition.

1.1.2 The Flatness problem

It is possible to quantity how the Universe is well approximated by a flat space ($K = 0$). Defining the ratio $\Omega(a)$ of the energy density ρ relative to the critical energy value

$$\rho_c(a) \equiv \frac{3H^2(a)}{(8\pi G)} \,, \tag{1.20}$$

the Friedmann equation (1.5) can be reformulated as follows,

$$1 - \Omega(a) = \frac{-K}{(aH)^2} = -K r_H^2(t) \,. \tag{1.21}$$

As explained in the previous paragraph, since r_H grows with time when considering matter or radiation dominated stages, from (1.21) we see that the value of $|\Omega - 1|$ must diverge accordingly. The critical value $\Omega = 1$ is therefore an unstable fixed point, unless K is fine-tuned at the value $K = 0$. By defining $\Omega_K \equiv -K/(aH)^2$, its current value is confirmed to be very close to zero by observations [4], namely $|1 - \Omega(t_0)| \lesssim 10^{-2}$. Within the model of Hot Big Bang, this was true also in the past by orders of magnitude. For example, at the Planck time ($t_P \sim 10^{-43}\, s$) it results in [22, 59]

$$|1 - \Omega(t_P)| = \mathcal{O}(10^{-64})\, |1 - \Omega(t_0)| \,. \tag{1.22}$$

From (1.21) we can readily see that the introduction of inflation solves the problem: the decreasing of $r_H(t)$ during that stage resulted in a suppression of $|1 - \Omega|$, potentially providing a negligible number that can justify the current observations.

1.2 Background dynamics of the single scalar field model

As we mentioned in the beginning of this chapter, the easiest way to implement inflation is by introducing a single scalar real field ϕ, the so-called *inflaton*, whose energy density dominated the Universe at early times. The subject is thouroughly discussed, for example, in [59, 44]. The inclusion of gravity in the dynamics is implemented by the *minimal coupling.* The whole dynamics of the system under consideration is encoded in the following action:

$$S = S_{EH} + S_\phi = \int dt d^3x \sqrt{-g} \left[\frac{M_p^2}{2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \,. \tag{1.23}$$

Here, $M_p^2 = (8\pi G)^{-1}$ is the reduced Planck mass, g is the determinant of the metric tensor, while the $V(\phi)$ term describes the self-interactions of the inflaton. It will not be necessary to specify the form of $V(\phi)$ in this context. The energy-momentum tensor of the scalar field is obtained in the usual fashion,

$$T_{\mu\nu}^{(\phi)} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial^\alpha \phi \partial_\alpha \phi + V(\phi) \right) \,. \tag{1.24}$$

The equation of motion of ϕ are obtained through the minimization of S_ϕ, namely

$$\frac{\delta S_\phi}{\delta \phi} = 0 \qquad \Longrightarrow \qquad \Box \phi = V'(\phi) \,. \tag{1.25}$$

The meaning of the symbols introduced in the above equation is given by

$$V'(\phi) = \frac{dV(\phi)}{d\phi}\,, \tag{1.26}$$

while $\Box$ is the covariant D'Alembert operator

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\right)\,. \tag{1.27}$$

By adopting the metric in (1.2) with $K = 0$ the equation of motion becomes

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2\phi}{a^2} = -V'(\phi)\,. \tag{1.28}$$

It should be noted that the above equation differs from the Klein-Gordon equation in Minkowski space. More specifically, equation (1.28) is characterized by a "friction" term given by $3H\dot{\phi}$ and by the scale factor under the gradient term. The latter will play an important role in the dynamics of perturbations, redshifting away the waveleghts of the field. Such differences are due to the interplay of the inflaton with the evolving background space in which it lives. In order to see how inflation is achieved, the inflaton field has to be split into its background value and its perturbation,

$$\phi(t, \mathbf{x}) = \bar{\phi}(t) + \delta\phi(t, \mathbf{x})\,. \tag{1.29}$$

Here, $\bar{\phi}$ is the classical vacuum expectation value of the ϕ, *i.e.*

$$\langle 0|\phi(t, \mathbf{x})|0\rangle = \bar{\phi}\,. \tag{1.30}$$

The dependence of $\bar{\phi}$ only on time is due to the symmetry of the space-time background. The perturbation $\delta\phi$ is understood as quantum fluctuations, whose amplitude is much smaller compared to the value of $\bar{\phi}$. The latter assumption is motivated in general by the size of the temperature fluctuations in the CMB ($\Delta T/T \sim 10^{-5}$) and it allows to separate the respective evolutions of $\bar{\phi}$ and $\delta\phi$. The analysis of the dynamics is therefore perfomed order by order in perturbations. As for equation (1.28), the expression at zeroth order in perturbation results in

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} = -V'(\bar{\phi})\,, \tag{1.31}$$

while the respective equation to the first-order in $\delta\phi$ is given by

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{\nabla^2}{a^2}\delta\phi = -m_\phi^2\delta\phi\,. \tag{1.32}$$

In the above expression the effective mass of the inflaton has been introduced, which is defined as $m_\phi^2 = V''(\bar{\phi})$. The two equations above have been obtained by expanding the potential V around $\bar{\phi}$ and truncating the expansion up to first order in $\delta\phi$ as follows

$$V'(\phi) = V'(\bar{\phi} + \delta\phi) \approx V'(\bar{\phi}) + V''(\bar{\phi})\delta\phi\,. \tag{1.33}$$

In this section only the dynamics of the background $\bar{\phi}$, described by equation (1.31), will be considered. By applying the same expansion to the energy-momentum tensor, its zeroth order term in $\delta\phi$ is characterized by a perfect fluid form, where

$$\bar{\rho} \equiv \rho_{\bar{\phi}} = \frac{\dot{\bar{\phi}}^2}{2} + V(\bar{\phi})\,, \tag{1.34}$$

$$\bar{p} \equiv p_{\bar{\phi}} = \frac{\dot{\bar{\phi}}^2}{2} - V(\bar{\phi})\,. \tag{1.35}$$

Equation (1.6) dictates the necessary condition to have a period of accelerated expansion: in order to have $\ddot{a} > 0$, the pressure must be subjected to the upper bound

$$\bar{p} < -\frac{\bar{\rho}}{3}\,, \tag{1.36}$$

which corresponds to $w < -\frac{1}{3}$. In particular, the condition for a near exponential expansion must be $\bar{p} \simeq -\bar{\rho}$, as shown in equation (1.9). Therefore the following condition should be satisfied,

$$V(\bar{\phi}) \gg \dot{\bar{\phi}}^2 \,, \tag{1.37}$$

namely, the energy density of the scalar field must be dominated by the potential V. The above requirement can be achieved if the field $\bar{\phi}$ is in a nearly flat region of its potential, such that $\bar{\phi}$ is slowly rolling down V. The condition (1.37) is also named *first slow-roll condition.* Moreover, if the friction effect of $3H\dot{\bar{\phi}}$ in (1.31) is large, the acceleration $\ddot{\phi}$ can be neglected when compared to the friction term, leading to

$$\ddot{\bar{\phi}} \ll 3H\dot{\bar{\phi}} \,. \tag{1.38}$$

The last condition enforces the regime of slow rolling of $\bar{\phi}$ down its potential and it is known as the *second slow-roll condition.* By combining (1.37) and (1.38), the result is a condition on the shape of the potential, more precisely on its first derivative:

$$\left(\frac{V'}{3H}\right)^2 \ll V \,. \tag{1.39}$$

When applied to (1.5), the first slow-roll condition leads to

$$H^2 \approx \frac{8\pi G}{3} V(\bar{\phi}) \,, \tag{1.40}$$

so that equation (1.39) becomes

$$\epsilon_V \equiv \frac{M_P^2}{3} \left(\frac{V'}{V}(\bar{\phi})\right)^2 \ll 1 \,. \tag{1.41}$$

The newly introduced parameter ϵ_V allows to recast the condition on $V'(\bar{\phi})$: in order to have cosmic inflation, the *first slow-roll parameter* ϵ_V must be small. It is then possible to reformulate it in a similar manner the second slow-roll condition. By starting with

$$\ddot{\bar{\phi}} = \frac{\partial \dot{\bar{\phi}}}{\partial t} \approx \frac{\partial}{\partial t}\left(\frac{V'(\bar{\phi})}{3H}\right) = \frac{V''(\bar{\phi})}{2H} \dot{\bar{\phi}} \,, \tag{1.42}$$

the condition (1.38) then becomes

$$\eta_V \equiv \frac{M_P^2}{3} \left(\frac{V''}{V}(\bar{\phi})\right) \ll 1 \,, \tag{1.43}$$

where equation (1.5) has also been used. The symbol η_V is named the *second slow-roll parameter.* The conditions to obtain the slow-roll regime are now simply

$$\epsilon_V \ll 1 \,, \quad |\eta_V| \ll 1 \,. \tag{1.44}$$

The smallness of the slow-roll parameters forces the potential V to be nearly flat for all the time of cosmic inflation. As long as the ϵ_V and η_V are small, the scale factor has a nearly exponential growth with time, so that the background space-time is approximately a de-Sitter space. The end of the inflationary stage is triggered when the inflaton field stop slowly rolling down its potential, laying on a minimum of V. The last condition is reached when the slow-roll parameters cease to be small.

It is possible to define alternative slow-roll parameters in the following way [44]:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \,, \quad \eta \equiv \frac{\dot{\epsilon}}{\epsilon H} \,. \tag{1.45}$$

The above definition of ϵ quantifies the rate of change of H during inflation, therefore how much the background space-time differs from de-Sitter, where $\dot{H} = 0$. It should be noticed that

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = H^2(1 - \epsilon)\,, \tag{1.46}$$

so that the end of inflation is signalled by $\epsilon \geq 1$. It can be shown [44] that the two types of slow-roll parameters are related by

$$\epsilon = \epsilon_V + \mathcal{O}(\epsilon_V^2, \eta_V^2) \qquad \eta = 4\epsilon_V - 2\eta_V + \mathcal{O}(\epsilon_V^2, \eta_V^2)\,. \tag{1.47}$$

However, it should be emphasized that the necessity of a nearly flat potential V is a feature of the model of the standard simplest model of inflation we are discussing here. As we will see in Chapter 4, in Solid Inflation there is no necessity of a slow-roll motion towards a local minimum point: η and ϵ still parametrize the inflationary stage but their values are controlled by another type of mechanism.
In order to fully appreciate the physical processes at the base of inflation we will need introduce the parameter *Number of e-folding* N, which is an estimate of the duration of the inflationary epoch:

$$N = \ln\left(\frac{a(t_e)}{a(t_{in})}\right) = \int_{t_{in}}^{t_e} dt\, H(t)\,, \tag{1.48}$$

where t_e and t_{in} represent the time of the end and of the beginning of inflation respectively. Another useful parameter is given by

$$N_k = \int_{t(k)}^{t_e} dt\, H(t) = \ln\left(\frac{a(t_e)}{a(t(k))}\right)\,, \tag{1.49}$$

where $t(k)$ is the time when the scale corresponding to k leaves the horizon during inflation before the end of inflation. For the scales of CMB anisotropies, $N_k \simeq 40 - 60$ [17].

1.3 Quantum fluctuations during inflation

So far, we have summarized the background evolution of the Universe in its inflationary expansion. We have discarded the quantum fluctuations of the inflaton field and of the metric tensor, assuming implicitly that they do not affect the evolution of the background fields. This approximation does not provide a satisfatory description of the Universe as it really is: all the structures we see today must have had a beginning, a primordial density inhomogeneity that has grown through gravitational instability. The same local deviations from the mean behaviour of the Universe can be seen as CMB anisotropies, revealing their presence at the time of the decoupling. If perturbations of the inflaton field are considered in the description in conjunction with metric perturbations, the inflation can naturally provide an explanation for the origin of the primordial density inhomogeneities. A mandatory outcome of this picture is also the generation of gravitational waves.

We will now present a brief qualitative description of the generation of primordial fluctuations during the inflationary epoch [17, 51]. It is believed that quantum fluctuations arose on scales much smaller than the comoving Hubble radius $r_H = (aH)^{-1}$. During the time of inflation r_H decreases with time and, consequently, the wavelenghs of the scalar fluctuations soon exceed the Hubble radius. On such scales the fluctuations are not causally connected anymore: since no microscopic physics affects their evolution, they maintain an almost constant amplitude, while their waveleghts grow exponentially. In the jargon of cosmologists, they are "frozen-in" after the horizon crossing. Such frozen fluctuations appear as a realization of a classical random field. In order to understand how the inflaton fluctuations are related to the fluctuations of space-time, it is necessary to take into account also the dynamics of metric perturbation. A thorough analysis of the last subject will be given in the next chapter.

In the following paragraphs, the discussion will be restricted only to the quantum fluctuations of the field ϕ, equivalent to considering a quantum scalar field living in a quasi de-Sitter space-time with no gravitational backreaction. The starting point of the discussion is the evolution equation (1.32) of the quantum fluctuation $\delta\phi$ to first order. For convenience of notation the subscript ϕ will be avoided in the mass term:

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - \frac{\nabla^2}{a^2}\delta\phi = -m^2\delta\phi\,. \tag{1.50}$$

In order to study the above equation it is convenient to use the conformal time $\tau \equiv \int dt/a(t)$. The derivative operators with respect to the cosmic time have therefore to be transformed in accordance with the above choice,

$$\frac{\partial}{\partial t} = \frac{1}{a}\frac{\partial}{\partial\tau}\,, \qquad \frac{\partial^2}{\partial t^2} = \frac{1}{a}\frac{\partial}{\partial\tau}\left(\frac{1}{a}\frac{\partial}{\partial\tau}\right)\,. \tag{1.51}$$

Whithin few passages, equation (1.50) results in

$$(a\delta\phi)'' - \nabla^2\,(a\delta\phi) + \left(a^2m^2 - \frac{a''}{a}\right)(a\delta\phi) = 0\,, \tag{1.52}$$

where the primes denote derivatives with respect to the conformal time. the disappearance of the viscosity friction term should be noted, which means that in the conformal coordinate system the scalar field $\delta\phi$ perceives the evolving background differently. The expanding background space is in fact accounted by a time dependent frequency in the equation of a free harmonic oscillator, as it is evident after the field redefinition $a\delta\phi = \psi$:

$$\psi'' - \nabla^2\psi + \left(a^2m^2 - \frac{a''}{a}\right)\psi = 0\,. \tag{1.53}$$

Since the geometry of space is assumed to be flat ($K = 0$), we are allowed to use the plane wave expansion

$$\psi(\tau,\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\mathbf{k}\cdot\mathbf{x}}\,\hat{\psi}(\tau,\mathbf{k})\,, \tag{1.54}$$

where the reality condition requires $\hat{\psi}(\tau,-\mathbf{k}) = \hat{\psi}^*(\tau,\mathbf{k})$.

In order to proceed with the quantization of the theory it is necessary to provide an algebra for the operators involved. It can be shown directly from (1.23) that the field ψ has the following action

$$S_\psi = \int d^3x d\tau\left[\frac{1}{2}\,(\psi')^2 - \frac{1}{2}\,(\partial_i\psi)^2 - a^2m^2\psi^2 + \frac{a''}{a}\phi^2\right]\,, \tag{1.55}$$

by which we can define the conjugate momentum of ψ, namely

$$\pi(\tau,\mathbf{x}) \equiv \frac{\partial\mathcal{L}}{\partial\psi'} \qquad \Longrightarrow \qquad \pi(\tau,\mathbf{x}) = \psi'(\tau,\mathbf{x})\,. \tag{1.56}$$

The field ψ is then promoted to a field operator decomposed in the usual second quantization fashion:

$$\psi(\tau,\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int d^3k\left[u(\tau,k)\,a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + u^*(\tau,k)\,a^\dagger_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right]\,, \tag{1.57}$$

where $u(\tau,k)$ is the *mode function*, which encodes the time evolution of the single k-mode. We can provide a time evolution equation for the mode function by inserting the expression above in (1.53):

$$u''(\tau,\mathbf{k}) + \left(k^2 - \frac{a''}{a} + a^2m^2\right)u(\tau,\mathbf{k}) = 0\,. \tag{1.58}$$

The fields ψ is then quantized according to the canonical *equal time commutation relations* between ψ and its conjugate momentum π

$$[\psi(\tau,\mathbf{x})\,,\,\pi(\tau,\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{1.59}$$

$$[\psi(\tau,\mathbf{x})\,,\,\psi(t,\mathbf{y})] = 0\,, \qquad [\pi(\tau,\mathbf{x})\,,\,\pi(\tau,\mathbf{y})] = 0\,. \tag{1.60}$$

Adopting the decomposition (1.57), one can infer that the operators $a_{\mathbf{k}}$ and $a^\dagger_{\mathbf{k}}$ obey the usual annihilation/creation algebra, that is

$$[a_{\mathbf{k}}\,,\,a_{\mathbf{p}}] = 0 \qquad \left[a^\dagger_{\mathbf{k}}\,,\,a^\dagger_{\mathbf{p}}\right] = 0 \qquad \left[a_{\mathbf{k}}\,,\,a^\dagger_{\mathbf{p}}\right] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})\,, \tag{1.61}$$

if and only if the following requirement on the mode funtion is imposed:

$$u'(\tau,k)u^*(\tau,k) - \,(c.c.) = -i\,. \tag{1.62}$$

The above equation fixes the normalization of $u(\tau,\mathbf{k})$. In order to determine the mode function completely, it is also necessary to specify the initial condition of $u(\tau,\mathbf{k})$, which is equivalent to fixing the vacuum state.

Within inflationary perturbation theories the so-called *Buch-Davies vacuum* [17, 51, 22, 29] represents a preferential choice of the vacuum state. It consists in demanding that, when the mode is deep inside the horizon, *i.e.* $k/aH \longrightarrow \infty$, it should approach a plane wave solution of the form

$$u(\tau,k) \longrightarrow \frac{e^{-ik\tau}}{\sqrt{2k}}\,. \tag{1.63}$$

Since the latter is a typical solution of an ordinary quantum field theory in Minkowski space, the physical interpretation of the Bunch-Davies prescription is that the mode function does not perceive any gravitational effect at scales considered small when compared with the cosmological ones.

1.3.1 Massless scalar field in de-Sitter

In order to illustrate the implications of these quantization prescriptions, we will consider the case of a massless scalar field in an exact de-Sitter background. Within this comfortable approximation many calculation subtleties will be avoided. When considering the pure de-Sitter space-time the Hubble parameter is constant and the scale factor has a simple form:

$$\tau = \int \frac{dt}{a(t)} = \int \frac{da}{a^2 H} = -\frac{1}{aH} + \int \frac{da}{a}\frac{d}{da}\underbrace{\left[\frac{1}{H}\right]}_{=0}\,, \tag{1.64}$$

so that a''/a is given by

$$\frac{a''}{a} = -H\tau\frac{d^2}{d\tau^2} = \frac{2}{\tau^2} = 2r_H^2\,, \tag{1.65}$$

and the equation (1.58) becomes

$$u''(\tau,k) + \underbrace{\left(k^2 - \frac{2}{\tau^2}\right)}_{k^2-2r_H^2} u(\tau,k) = 0\,, \tag{1.66}$$

where we emphasize the comparison with the comoving Hubble radius. The exact solution of (1.66) is given by

$$u(\tau,\mathbf{k}) = C_1\left(1-\frac{i}{k\tau}\right)\frac{e^{-ik\tau}}{\sqrt{2k}} + C_2\left(1+\frac{i}{k\tau}\right)\frac{e^{ik\tau}}{\sqrt{2k}}\,, \tag{1.67}$$

where $C_{1,2}$ are two unconstrained coefficient. The imposition of the normalization condition (1.62) leads to the following constraint on $C_{1,2}$,

$$|C_1|^2 - |C_2|^2 = 1\,. \tag{1.68}$$

In order to match with the asymptotic behaviour of the type required in (1.63), the coefficients are fixed to the values $C_2 = 0$ and $C_1 = 1$. The mode function results therefore in

$$u(\tau,k) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1-\frac{i}{k\tau}\right) = i\frac{aH}{\sqrt{2k^3}}(1+ik\tau)e^{-ik\tau}\,. \tag{1.69}$$

Turning our attention to the field $\delta\phi(\tau, \mathbf{x}) = \psi(\tau, \mathbf{x})/a$, it is possible to show what was previously anticipated: after the horizon crossing, the amplitude of a quantum fluctuation becomes frozen at the constant value

$$|\delta\phi(\tau, k)| = \left|\frac{u(\tau, k)}{a}\right| \xrightarrow{-k\tau\to 0} \frac{H}{\sqrt{2k^3}} \,. \tag{1.70}$$

1.3.2 Light scalar field in quasi-de-Sitter

In a proper inflationary scenario, the background space-time is not purely de-Sitter, nor the scalar field should be considered massless. The rate of the Hubble expansion is therefore not exactly constant during inflation, but rather slowly changing, namely

$$\dot{H} = -\epsilon H \,. \tag{1.71}$$

Here, the dot means a derivative with respect to the cosmic time t, while ϵ is the first slow-roll parameter defined in (1.45). In addition, it can be proved that the time derivatives of the slow-roll parameters are $\mathcal{O}(\epsilon^2, \eta^2)$ [44], therefore we will consider them as constant in time. Consequently, the expression of the scale factor in conformal time is more complex than in equation (1.64):

$$\tau = \int \frac{dt}{a(t)} = -\frac{1}{aH} + \int \frac{da}{a}\frac{d}{da}\left[\frac{1}{H}\right] = -\frac{1}{aH} + \int \frac{da}{a}\frac{1}{\dot{a}}\frac{d}{dt}\left[\frac{1}{H}\right] = -\frac{1}{aH} - \epsilon\int\frac{da}{a^2 H} \,, \tag{1.72}$$

so that

$$a(\tau) = -\frac{1}{H}\frac{1}{\tau(1-\epsilon)} \,, \tag{1.73}$$

where the dependence on time of the slow-roll parameter have been neglected. By exploiting the smallness of ϵ, we then get

$$\begin{aligned}\frac{a''}{a} &= \frac{1}{a}a\frac{d}{dt}\left(a\frac{da}{dt}\right) = a^2\left[2\left(\frac{\dot{a}}{a}\right)^2 - \underbrace{\left(\frac{\dot{a}}{a}\right)^2 + \left(\frac{\ddot{a}}{a}\right)^2}_{\dot{H}^2}\right] \\ &= a^2H^2\left(2 + \frac{\dot{H}}{H^2}\right) = \frac{1}{\tau^2}\frac{1}{(1-\epsilon)^2}(2-\epsilon) \approx \frac{2}{\tau^2}(1+\frac{3}{2}\epsilon) \,. \end{aligned} \tag{1.74}$$

The effective mass of the scalar field is considered small, being related to η_V through (1.43). We thus recast the effective mass as $\eta_V \equiv (m_\phi^2/3H^2) \ll 1$. The mass term in (1.58) results therefore in

$$a^2m^2 = \frac{m_\phi^2}{H^2\tau^2}\frac{1}{(1-\epsilon)^2} \approx \frac{m_\phi^2}{H^2\tau^2}(1+2\epsilon) \approx \frac{3\eta_V}{\tau^2} \,, \tag{1.75}$$

where we have kept the expansion in η_V and ϵ at the lowest order. By inserting equations (1.74) and (1.75) in (1.58), we can reformulate the equation of motion of the mode function as

$$u_k'' + \left[k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right]u_k = 0 \,, \tag{1.76}$$

where

$$\nu^2 = \frac{9}{4} + 3\epsilon - 3\eta_V \,, \quad \nu \simeq \frac{3}{2} + \epsilon - \eta_V \,. \tag{1.77}$$

Since the time derivatives of the slow-roll parameters are $\mathcal{O}(\epsilon^2)$, ν can be safely considered as constant in time within the approximation that has been adopted so far. Equation (1.76) can then be recast in the form of the so-called *Bessel equation* [1] by performing the field redefinition $y_k(\tau) = u_k(\tau)/\sqrt{-\tau}$:

$$\tau^2 y_k^2 + \tau y_k + \left(\tau^2k^2 - \nu^2\right)y_k = 0 \,. \tag{1.78}$$

The solution of the above equation is given by a combination of the *Bessel functions* of the first and second kind, respectively J_ν and Y_ν, whose properties are extensively reviewed in [1] and [34]. We are now able to provide the general solution of (1.76) by expressing u_k in terms of y_k:

$$u_k(\tau) = \sqrt{-\tau}\left[C_+(k)J_\nu(-k\tau) + C_-(k)Y_\nu(-k\tau)\right] . \tag{1.79}$$

For real value of ν the above solution can be expressed in the alternative form

$$u_k(\tau) = \sqrt{-\tau}\left[C_1(k)H_\nu^{(1)}(-k\tau) + C_2(k)H_\nu^{(2)}(-k\tau)\right] , \tag{1.80}$$

where $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the so-called *Hankel functions* of first and second kind, which are defined as

$$H_\nu^{(1)} = J_\nu + iY_\nu\, , \tag{1.81}$$
$$H_\nu^{(2)} = J_\nu - iY_\nu\, . \tag{1.82}$$

The properties of the Hankel functions that concerns the scopes of this thesis are reported in Appendix B. In order to impose the Bunch-Davies condition, the asymptotic behaviour of $H_\nu^{(1)}$ and $H_\nu^{(2)}$ deep inside the horizon should be analyzed. Introducing the variable $x = -k\tau$, we know that

$$H_\nu^{(1)}(x) \underset{x\to+\infty}{\sim} \sqrt{\frac{2}{\pi x}}e^{i(x-\frac{\pi}{2}\nu-\frac{\pi}{4})}\, , \qquad H_\nu^{(2)}(x) \underset{x\to+\infty}{\sim} \sqrt{\frac{2}{\pi x}}e^{-i(x-\frac{\pi}{2}\nu-\frac{\pi}{4})}\, . \tag{1.83}$$

If we impose that for $k \gg aH$ ($-k\tau \gg 1$) the solution $u_k(\tau)$ matches the behaviour (1.63), it follows that $C_2(k) = 0$, while $C_1(k)$ must be constrained by the normalization condition (1.62):

$$u_k^*(\tau)u_k'(\tau) - (c.c)\Big|_{-k\tau\to+\infty} = -i\frac{4}{\pi}|C_1(k)|^2 = -i\, ,$$
$$|C_1(k)| = \frac{\sqrt{\pi}}{2}\, . \tag{1.84}$$

Finally, in order to satisfy the condition (1.63), the phase of C_1 have to be $(\nu+\frac{1}{2})\frac{\pi}{2}$. The mode function $u_k(\tau)$ results therefore in

$$u_k(\tau) = \frac{\sqrt{\pi}}{2}e^{i(\nu+\frac{1}{2})\frac{\pi}{2}}\sqrt{-\tau}H_\nu^{(1)}(-k\tau)\, . \tag{1.85}$$

The superhorizon behaviour of $u_k(\tau)$ is obtained by exploiting the asymptotic behaviour of $H_\nu^{(1)}(x)$ reported in Appendix B:

$$H_\nu^{(1)}(x) \underset{x\to 0}{\sim} \sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}2^{(\nu-\frac{3}{2})}\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}x^{-\nu}\, , \tag{1.86}$$

where Γ stands for the Gamma function. The amplitude of $u_k(\tau)$ on scales beyond the horizon therefore results in

$$|u_k(\tau)| \underset{-k\tau\to 0}{\sim} 2^{(\nu-\frac{3}{2})}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k}}(-k\tau)^{\frac{1}{2}-\nu} \tag{1.87}$$
$$= 2^{(\nu-\frac{3}{2})}\frac{\Gamma(\nu)}{\Gamma(3/2)}\frac{1}{\sqrt{2k^3}}\left(\frac{k}{aH(1-\epsilon)}\right)^{\frac{3}{2}-\nu}aH(1-\epsilon) \simeq \frac{aH}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}\, . \tag{1.88}$$

Here, τ has been rewritten exploiting its relation with the scale factor, while in the last step the numerical coefficient has been given to the leading order in ϵ. It is now possible to analyze the amplitude of the physical fluctuation $\delta\phi_k = u_k/a$ on superhorizon scales, expressed below at the lowest-order in the slow-roll parameters:

$$|\delta\phi_k(\tau)|\Big|_{-k\tau\to 0} = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu}\, . \tag{1.89}$$

1.4 The power spectrum

The discussion so far has been limited to the linear evolution of the quantum fluctuations: we have in fact discarded both possible self-couplings of the field $\delta\phi$ beyond the quadratic order and the coupling with other fields. Thanks to this choice, it has been possible to quantize the field because we were dealing with a superposition of harmonic oscillators (although with time-dependent frequencies) and to track its linear evolution. If this level of approximation is kept, it leads inevitably to Gaussian statistics of realizations of the perturbation field. To describe the statistics of a random field (and in particular its statistics beyond a Gaussian approximation) we can turn to the n-point correlation functions $\langle \delta\phi(x_1) \cdots \delta\phi(x_n) \rangle$. For a Gaussian field, only the 2-point correlation function is relevant, being the higher-order functions just products of the 2-point correlation functions if n is even, or equal to zero if n is odd [17, 59]. This is what happens if non-linear interactions are discarded, as we have done so far. It is evident that, if one considers an odd product of the field ψ as given in (1.57). The result vanishes because an odd product of operators a and $a^\dagger$ between two vacuum states equates to zero. As we shall see, the *non-Gaussian* statistics of the fields (*i.e.* the non-Gaussianity) is generated only by taking into account the non-linear interactions.

At this point of the discussion, we can introduce anyway an important statistical tool that measures the amplitude of the fluctuations at a given scale: *the power spectrum* $P(k)$. It is defined through the usage of the two-point correlation function in Fourier space:

$$\langle \delta\phi(\mathbf{k}) \delta\phi^*(\mathbf{k}') \rangle = \delta^{(3)}(\mathbf{k} - \mathbf{k}') P_{\delta\phi}(k) \,. \tag{1.90}$$

We can also define the *dimensionless* power spectrum as

$$\mathcal{P}(k) = \frac{2\pi^2}{k^3} P(k) \,. \tag{1.91}$$

The latter can be interpreted as the logarithmic scale contribution to the variance of the fluctuations, indeed:

$$\begin{aligned} \sigma^2_{\delta\phi} \equiv \langle \delta\phi^2(\mathbf{x}) \rangle &= \frac{1}{(2\pi)^3} \int d^3k d^3k' e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} \langle \delta\phi(\mathbf{k}) \delta\phi^*(\mathbf{k}') \rangle \\ &= \frac{1}{(2\pi)^3} \int d^3k d^3k' \delta^{(3)}(\mathbf{k} - \mathbf{k}') P_{\delta\phi}(k) = \int \frac{dk}{k} \mathcal{P}_{\delta\phi}(k) \,. \end{aligned} \tag{1.92}$$

To describe the slope of the power spectrum it is standard practice to define a *spectral index* $n_{\delta\phi}(k)$, through

$$n_{\delta\phi}(k) - 1 \equiv \frac{d \ln \mathcal{P}_{\delta\phi}}{d \ln k} \,. \tag{1.93}$$

In the case of a scalar field $\delta\phi = \psi/a$ we can combine (1.57) together with the algebra (1.61) to evaluate $\mathcal{P}_\delta\phi$ spectrum:

$$\langle 0 | \delta\phi(\tau, \mathbf{k}) \delta\phi^*(\tau, \mathbf{k}') | 0 \rangle = \frac{1}{a^2} \langle 0 | u(\tau, k) u^*(\tau, k') a_{\mathbf{k}} a^*_{\mathbf{k}'} | 0 \rangle \tag{1.94}$$

$$= \frac{H^2}{2k^3}(1 + k^2\tau^2) \delta^{(3)}(\mathbf{k} - \mathbf{k}') \Longrightarrow \mathcal{P}_{\delta\phi}(k) = \frac{k^3}{2\pi^2} |\delta\phi(\tau, k)|^2 = \frac{H^2}{(2\pi)^2}(1 + k^2\tau^2) \,, \tag{1.95}$$

where we considered the case (1.69) of a massless scalar field in de Sitter space-time. Evaluating the spectrum on superhorizon scales we get a scale-invariant dimensionless power spectrum with a spectral index equal to one:

$$\mathcal{P}_{\delta\phi}(k) = \frac{H^2}{(2\pi)^2} \,. \tag{1.96}$$

As for the other analized case of a light scalar field in a quasi de-Sitter Universe (1.89), on superhorizon scales the corresponding results are

$$\mathcal{P}_{\delta\phi}(k) = \frac{H^2}{(2\pi)^2} \left(\frac{k}{aH} \right)^{3-2\nu} \,, \qquad n_{\delta\phi}(k) - 1 = 3 - 2\nu \,. \tag{1.97}$$

Chapter 2

Cosmological perturbations

Like any perturbative approach, cosmological perturbation theory is based on searching solutions of some field equation via successive approximations, as long as we consider the perturbation field as a small deviation from a known background solution. The basic difference arising in General Relativity is that we have to consider the perturbation of the space-time geometry itself.

The main hypothesis consists in considering two distinct space-time manifolds: a "background" $\mathcal{M}_0$, equipped with the usual FRW metric $g^{(0)}_{\mu\nu}$, and a "physical" $\mathcal{M}_{\rm phys}$, that represents the real Universe; the latter does not have the same space-time symmetries of the background. Each of them is considered as the solution of the Einstein field equation, respectively

$$R^{(0)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} R = 8\pi G T^{(0)}_{\mu\nu} \qquad , \qquad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \,. \tag{2.1}$$

The second equation is solved via successive approximations, that have as zeroth order solution the FRW metric and the energy-momentum tensor of a perfect fluid.

Within the perturbative approach, the "physical" metric and energy-momentum tensor are structrured as follows, where the perturbed quantities δ are intended to be small:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} \qquad , \qquad T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu} \,. \tag{2.2}$$

It is now of fundamental importance to analyze the conceptual set-up of the perturbative theory in General Relativity; as we anticipated, we are dealing with a perturbed *geometry*: since the theory of gravity has Differetial Geometry in its core construction, this involves the discussion of some important geometrical subtleties that have to be clarified. Being the procedure of splitting tensors in background plus perturbation not a covariant procedure (in the sense of General Relativity), we will face an ambiguity in defining perturbations that is a direct consequence of a changing in the coordinate frame: this is called *the gauge problem.* We will keep the discussion on as concise as possible since we are interested in presenting the *first order* perturbations of Einstein equations. More on this subject is contained in [26], where a detailed geometrical analysis for perturbations of higher order is presented.

2.1 Perturbations in General Relativity

We start assuming the existence of family of solutions of the field equation parametrized by $\lambda \in \mathbb{R}$; in other words, we consider a family of metric tensors g_λ and matter fields collectively denoted as ϕ_λ that depend smoothly on λ and satisfy the Einstein equations

$$\mathcal{F}(g_\lambda, \phi_\lambda) \equiv R_{\mu\nu,\lambda} - \frac{1}{2} g_{\mu\nu,\lambda} R_\lambda = 8\pi G T_{\mu\nu,\lambda} \,. \tag{2.3}$$

The fields that solve (2.3) with $\lambda = 0$ are the background solutions. We rethink the situation introducing the manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$; in this way we have a foliation $(\mathcal{M}_\lambda)_{\lambda\in\mathbb{R}}$ of $\mathcal{N}$. It is thus possible to consider a generic tensor field T_λ living in $\mathcal{M}_\lambda$ with the field T in $\mathcal{N}$ for a fixed value of λ: $T_\lambda(\mathcal{O}) \equiv T(\lambda, \mathcal{O})$, with $\mathcal{O} \in \mathcal{M}_\lambda$. In this sense g_λ and ϕ_λ are metric and fields living on the submanifold $\mathcal{M}_\lambda$. We can then identify the physical Universe with the manifold $\mathcal{M}_\lambda$, while the physical quantities we are interested in T_λ.

The perturbation δT of a any tensor field is naively written as the following difference, where the subscript means the quantity related to the background space-time:

$$\delta T_\lambda \equiv T_\lambda - T_0 \,. \tag{2.4}$$

It should be emphasized that, if we want that the expression above makes sense at all, T_λ and T_0 *must be evaluated at the same point in space-time*; tensor fields live in fact in curved space-time and because of this we need to 'drag' them at the same point in spacetime. More over, T_λ and T_0 live in different space-time manifolds, respectevily $\mathcal{M}_\lambda$ and $\mathcal{M}_0$. We then need a prescription that relates the points of this two manifolds. A diffeomorphism $\varphi_\lambda : \mathcal{N} \longrightarrow \mathcal{N}$ can provide such a one-to-one corrispondence, such that

$$\varphi_\lambda|_{\mathcal{M}_0} : \mathcal{M}_0 \longrightarrow \mathcal{M}_\lambda \,. \tag{2.5}$$

Such a map gives the identity for $\lambda = 0$: for $p \in \mathcal{M}_0$ we have $\varphi_0(p) \equiv \varphi(0,p) = p$.

We anticipate from now that φ_λ is not the only existing diffeomorphic map for the two varieries above and we'll make use of this fact; our second choice is labeled with ψ_λ. The difference between these two maps stands in the way they connect singular points: more specifically, if $p, q \in \mathcal{M}_0$, with $p \neq q$, and $\mathcal{O} \in \mathcal{M}_\lambda$ we have two distinct points of the background mapped in same one of the "physical" variety (note that we don't give a local chart):

$$\varphi_\lambda(p) = \psi_\lambda(q) = \mathcal{O} \,. \tag{2.6}$$

It is known from Differential Geometry that the map φ_λ induces a differentiable map between tangent spaces whose name is *push-forward*

$$\varphi_{*\lambda}|_p : T_p\mathcal{M}_0 \longrightarrow T_{\varphi_\lambda(p)}\mathcal{M}_\lambda \,, \tag{2.7}$$

and a map between cotangent spaces called *pull-back*

$$\varphi^*_\lambda|_{\varphi_\lambda(p)} : T^*_{\varphi_\lambda(p)}\mathcal{M}_\lambda \longrightarrow T^*_p\mathcal{M}_0 \,. \tag{2.8}$$

For example, the action of the pull-back φ^*_λ on a tensor field $T_\lambda \in \mathcal{T}^{0,2}_{\varphi_\lambda(p)}(\mathcal{M}_\lambda)$ is given by pushing-forward the vectors of $T_p\mathcal{M}_0$

$$(\varphi^*_\lambda T)(v_1, v_2) = T(\varphi_{\lambda *}v_1, \varphi_{\lambda *}v_2) \,, \tag{2.9}$$

while the push-forward acts on tensors $T_0 \in \mathcal{T}^{2,0}_p(\mathcal{M}_0)$ by pulling-back the covectors of $T^*_{\varphi_\lambda(p)}M$

$$(\varphi_{*\lambda}T_0)(\omega_1, \omega_2) = T_0(\varphi^*_\lambda\omega_1, \varphi^*_\lambda\omega_2) \,. \tag{2.10}$$

Thanks to the existence of the inverse map $\varphi^{-1}_\lambda : \mathcal{M}_\lambda \to \mathcal{M}_0$, we can also define the corresponding inverse ones; in summary:

$$(\varphi_{*\lambda})^{-1} : T_{\varphi_\lambda(p)}\mathcal{M}_\lambda \longrightarrow T_p\mathcal{M}_0 \,, \tag{2.11}$$

$$(\varphi^*_\lambda)^{-1} : T^*_p\mathcal{M}_0 \longrightarrow T^*_{\varphi_\lambda(p)}\mathcal{M}_\lambda \,. \tag{2.12}$$

It is then possible to define the transformation property of a mixed-type tensor; for example, we consider the pulled-back tensor $T \in \mathcal{T}^{1,1}_{\varphi_\lambda(p)}(\mathcal{M}_\lambda)$:

$$(\varphi^*_\lambda T) = T(\varphi^*_\lambda\omega, (\varphi_{*\lambda})^{-1}w) \,, \qquad \omega \in T^*_{\varphi_\lambda(p)}\mathcal{M}_\lambda \,, \, w \in T_{\varphi_\lambda(p)}\mathcal{M}_\lambda \,. \tag{2.13}$$

We are finally able to make a proper definition tensor perturbations in General Relativity:

$$\Delta T_\lambda(p) \equiv (\varphi^*_\lambda T)(p) - T_0(p) \,, \tag{2.14}$$

here, we remberer, p is a point of the background variety; in this way is formalized the common statement in literature that "perturbations are fields living in the background". We can now make a Taylor expanion in λ of the pulled-back tensor field $(\varphi_\lambda^* T)$:

$$(\varphi_\lambda^* T)(p) = \sum_{k=0}^{+\infty} \frac{\lambda^k}{k!} \left[\frac{d^k}{d\lambda^k} (\varphi_\lambda^* T)(p) \right]_{\lambda=0} = T_0(p) + \underbrace{\sum_{k=1}^{+\infty} \frac{\lambda^k}{k!} \left[\frac{d^k}{d\lambda^k} (\varphi_\lambda^* T)(p) \right]_{\lambda=0}}_{\lambda \Delta T_\lambda(p)} . \tag{2.15}$$

The above equation includes, at least formally, every order in perturbation; at the first order, we have that the term is the *Lie derivative* of the tensor T_0:

$$\frac{d}{d\lambda} \left[(\varphi_\lambda^* T)(p) \right]_{\lambda=0} \equiv \mathcal{L}_{\xi_\varphi} T_0(p) . \tag{2.16}$$

We now turn our attention on the possibility of making a second choice ψ_λ to connect $\mathcal{M}_0$ and $\mathcal{M}_\lambda$; using the inverse φ_λ^{-1} it is then construncted a map that moves p in q, that is

$$q = \Phi_\lambda(p) \equiv \phi_\lambda^{-1}(\psi_\lambda(p)) . \tag{2.17}$$

Putting a chart on $\mathcal{M}_0$, it is rewritten at first order in λ as follows:

$$x_q^\mu = x_p^\mu + \lambda \xi^\mu + \mathcal{O}(\lambda^2) . \tag{2.18}$$

The map Φ_λ is called *gauge transformation.* More specifically, a map that moves each point to another is often called *active* coordinate transformation. We now want to show how tensor perturbation ΔT transforms under Φ at first order. We start considering the pulled-back tensor $(\psi_\lambda^* T)(p)$ and inserting the identity map $(\varphi_\lambda^*)^{-1} \varphi_\lambda^*$; it gives us simply

$$(\psi_\lambda^* T)(p) = (\psi_\lambda^* (\varphi_\lambda^*)^{-1} \varphi_\lambda^* T)(p) = \Phi_\lambda^* (\varphi_\lambda^* T)(p) . \tag{2.19}$$

At this point it is possible to perform an expansion in λ like in (2.15). When considering just the first order, we get

$$\begin{aligned} \Phi_\lambda^* (\varphi_\lambda^* T)(p) &= T_0(p) + \lambda \Delta T(p) + \lambda (\mathcal{L}_\xi (T_0(p) + \lambda \Delta T(p)) + \mathcal{O}(\lambda^2) \\ &= T_(p) + \lambda \left(\Delta T(p) + \mathcal{L}_\xi (T_0(p) \right) \mathcal{O}(\lambda^2) , \end{aligned} \tag{2.20}$$

and from the equation above we get the trasformation property of the perturbation ΔT under gauge transformation at first order:

$$\widetilde{\Delta T(p)} = \Delta T(p) + \mathcal{L}_\xi (T_0(p) . \tag{2.21}$$

As a result. a crucial issue concerning perturbation theory in General Relativity consists in distinguishing between the physical perturbation and gauge artifacts. Two ways are available to erase the gauge dependence: we can fix the coordinate system (*i.e.* choosing a gauge) or we can use a gauge invariant approach.

In concluding this section, we give the formulas of the Lie derivative that must be used to explicitly compute the transformation of fields (that, until now, we have generically dubbed as "tensor") according to their covariant nature. We thus have scalars (S), vectors (V) and tensors (t) under diffeomorphisms:

$$\mathcal{L}_\xi S = \xi^\alpha D_\alpha S , \tag{2.22}$$
$$\mathcal{L}_\xi V_\mu = \xi^\alpha D_\alpha V_\mu + V_\alpha D_\mu \xi^\alpha , \tag{2.23}$$
$$\mathcal{L}_\xi V^\mu = \xi^\alpha D_\alpha V^\mu - V^\alpha D_\alpha \xi^\mu , \tag{2.24}$$
$$\mathcal{L}_\xi t_{\mu\nu} = \xi^\alpha D_\alpha t_{\mu\nu} + t_{\mu\alpha} D_\nu \xi^\alpha + t_{\alpha\nu} D_\mu \xi^\alpha . \tag{2.25}$$

where D_μ denotes the covariant derivative.

We will discuss the metric perturbations first, as part of the left hand side of the Einstein equations. Only then, we will move to consider the perturbation of the right-hand side, *i.e.* the perturbation of the energy momentum tensor.

2.2 Metric perturbations

Whether within a quantum or classical physical theory, the time-evolution of a field is determined by the solution of a dynamical differetial equation and the imposition of initial conditions. General Relativity makes no exception. It is thus necessary to realize some kind of distinction in the equation, in order to identify a time flow and the object whose evolution we want to track, which is, as our intuition suggests, a three-dimensional hypersurface. Perturbative cosmology makes no exception: the study of the metric perturbations at cosmological level needs in fact such formulation to be addressed.

2.2.1 The (3+1) decomposition and the ADM variables

In order to discuss the theory of cosmological perturbations on a FRW background we need to introduce an important parametrization of the metric tensor, commonly known in literature as $(3 + 1)$ decomposition. This formalism is applicable to a broad class of problems including, but not limited to the perturbative ones. What makes this formalism an important achivement in the context of General Relativity is that, with the (3+1) decomposition, the Einstein equations assume manifestly the form of a time-evolution equation [55]. In summary, it is possible to demonstrate that a wide class of space-time manifold $\mathcal{M}$ can be foliated into hypersurfaces $\{\Sigma_t\}_{t\in\mathbb{R}}$ labeled by a time parameter t [1] that, in our case, is the cosmic time. The manifold $\mathcal{M}$ is thus given by the product $\mathbb{R} \times \Sigma$; here, Σ is an evolving structure in time, called *Cauchy surface*. It is customary in literature to refer to such a foliation as *slicing*, while the choice of the spatial coordinates on Σ is named *threading*. We can state that a gauge transformation changes the slicing and threading.

According to the decomposition we described above, the metric tensor is expressed in the so called 'ADM form'. The acronym refers to Arnowitt, Deser and Misner, who introduced such a parametrization for the first time [13]:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + N^k N_k & N_j \\ N_i & h_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^j}{N^2} \\ \frac{N^i}{N^2} & h^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}, \tag{2.26}$$

where N, N^i and h_{ij} depend on $\mathbf{x}$ and t. We will refer to them as the *ADM variables*. The line element is therefore expressed in the ADM variables as follows,

$$\begin{aligned} ds^2 &= (-N^2 + N_k N^k)dt^2 + 2N_i dt dx^i + h_{ij} dx^i dx^j \\ &= -N^2 dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)\,. \end{aligned} \tag{2.27}$$

The variable $N(t, \mathbf{x})$ is named *lapse function*, while $N^i(t, \mathbf{x})$ is named *shift vector* [56]. The quantity $h_{ij}(t, \mathbf{x})$ is the metric on the spatial hypersurface. It lowers and raises the spatial indices as follows,

$$N^i = h^{ij} N_j\,, \qquad N_i = h_{ij} N^j\,. \tag{2.28}$$

We have now to apply the ADM formalism to perturbative cosmology. The ADM variables provide in fact a way to decribe a slightly anisotropic and inhomogeneous space-time, which consists of a FRW background $\bar{g}_{\mu\nu}$ plus the perturbation part $\delta g_{\mu\nu}$, as in (2.2). All the following symbols that will be labeled with δ have to be considered as first order perturbations, so that every term of higher order is implicitly disregarded.

We first have to establish the rules by which we lower and raise the indices of the perturbed metric $\delta g_{\mu\nu}$. We first have to consider the identity $\delta^\mu_\nu = g^{\mu\alpha} g_{\alpha\nu}$, which is valid at every order in perturbations. Therefore, by splitting the metric as $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, the above identity gives

$$\delta^\mu_\nu = (\bar{g}^{\mu\alpha} + \delta g^{\mu\alpha})(\bar{g}_{\alpha\nu} + \delta g_{\alpha\nu})\,. \tag{2.29}$$

[1]The space-time $\mathcal{M}$ must be *globally hyperbolic* for the foliation to be possible. A thorough discussion on this subject is contained, for example, in [55]. For almost all the cases of cosmological and astrophysical interest, such a condition is satisfied.

By neglecting higher order terms and knowing that $\bar{g}^{\mu\alpha}\bar{g}_{\alpha\nu} = \delta^\mu_\nu$, we have

$$\delta^\mu_\nu = \bar{g}^{\mu\alpha}\bar{g}_{\alpha\nu} + \delta g^{\mu\alpha}\bar{g}_{\alpha\nu} + \bar{g}^{\mu\alpha}\delta g_{\alpha\nu} \tag{2.30}$$
$$\delta g^{\mu\alpha}\bar{g}_{\alpha\nu} = -\bar{g}^{\mu\alpha}\delta g_{\alpha\nu}\,. \tag{2.31}$$

The rules to raise and lower indices are thus the following,

$$\delta g_{\mu\nu} = -\bar{g}_{\mu\alpha}\bar{g}_{\nu\beta}\delta g^{\alpha\beta}\,, \qquad \delta g^{\mu\nu} = -\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\delta g_{\alpha\beta}\,. \tag{2.32}$$

The FRW metric tensor in cosmic time is given, togheter with its inverse, by

$$\bar{g}_{\mu\nu} = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & a^2\delta_{ij} \end{pmatrix}\,, \qquad \bar{g}^{\mu\nu} = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & \frac{1}{a^2}\delta^{ij} \end{pmatrix}\,, \tag{2.33}$$

The ADM variables must then be expanded in the same manner as $g_{\mu\nu}$. As for the lapse function N, we have

$$N = N_0 + N_1 + N_2 + \cdots = N_0 + \delta N\,. \tag{2.34}$$

In regard to the shift vector, given that the $(i0)$ components in the FRW metric are null, the expansion of N^i starts at first order:

$$N^i = N^i_1 + N^i_2 + \cdots\,. \tag{2.35}$$

As we are considering first order analysis, we will identify the perturbation just with N^i. Finally, for the three-metric h_{ij}, we have

$$h_{ij} = h_{0,ij} + h_{1,ij} + \cdots = \bar{h}_{ij} + \delta h_{ij}\,. \tag{2.36}$$

Thus, by comparing the expansion of the metric in the ADM form and the FRW background, we are able to establish the following correspondences:

$$N_0 = 1\,, \qquad \bar{h}_{ij} = a^2\delta_{ij}\,, \qquad \bar{h}^{ij} = \frac{1}{a^2}\delta^{ij}\,, \tag{2.37}$$
$$\delta g_{00} = -2\delta N \equiv -2\phi\,,\,, \tag{2.38}$$
$$\delta g_{i0} = N_i \equiv aB_i\,, \qquad \delta g^{i0} = N^i \equiv \frac{B^i}{a}\,, \tag{2.39}$$
$$\delta g_{ij} = \delta h_{ij} \equiv 2a^2\hat{h}_{ij}\,, \qquad \delta g^{ij} = \delta h^{ij} = -\frac{2}{a^2}\hat{h}^{ij}\,. \tag{2.40}$$

The newly introduced quantities B^i, $\hat{h}_{ij}$ are defined in such a way that their indices are raised and lowered by the Kronecker delta symbol; this is directly proved by using (2.32),

$$B^i = a\delta g^{i0} = \frac{a}{a^2}\delta^{ij}\delta g_{j0} = \delta^{ij}B_j\,, \qquad \hat{h}^{ij} = -\frac{a^2}{2}\delta g^{ij} = \delta^{ik}\delta^{jm}\hat{h}_{km}\,. \tag{2.41}$$

The above property shows that ϕ, B^i and $\hat{h}^{ij}$ are, respectively, a scalar, a vector and a tensor of the 3D Euclidean space. The resulting line element ds^2 is thus given by

$$ds^2 = -(1+2\phi)dt^2 + 2a(t)B_i dt dx^i + a^2(t)(\delta_{ij} + 2\hat{h}_{ij})dx^i dx^j\,. \tag{2.42}$$

In the following, we may refer to ϕ as “lapse function” and to B^i as “shift vector” [49].

2.2.2 The (S,V,T) decomposition

As it is possible to do so, it is convenient to split the metric components according to the irreducible representations of the Euclidean rotations group, *i.e* scalars, vectors and tensors of $SO(3)$. It should be pointed out the reason why such a splitting is convenient: as it has been proved [59], the linear Einstein equations do not mix scalar, vectors and tensors dynamically. This means, for example, that no scalar will enter in the dynamics of a vector or a tensor.

Vector fields can be decomposed in a *longitudinal* and a *tranverse* part which are, respectively, curl-free and divergence-free; as for the shift vector B^i, we have

$$B^i = B^i_L + B^i_T\,, \qquad \partial_i B^i_T = 0\,, \qquad \varepsilon^{ijk}\partial^k B^k_L = 0. \tag{2.43}$$

The significance of the longitudinal and transverse parts is more apparent in Fourier space, where $\tilde{B}^i$ refers to the Fourier transformed of B^i. If we consider the vector k^i in the momentum space, two projectors are constructed with it, that is,

$$P^{ij}_T = (\delta^{ij} - \frac{k^i k^j}{k^2})\,, \qquad P^{ij}_L = \frac{k^i k^j}{k^2}\,. \tag{2.44}$$

The operator P^{ij}_T projects in the tranverse direction with respect to $\mathbf{k}$, while P^{ij}_L projects along it. In real space, the resulting projected components of $\tilde{B}^i$ are indeed B^i_L and B^i_T. As for B^i_L, in real space we can adopt two defintions:

$$B^i_L \equiv \partial_i \hat{B}\,, \text{ or } B^i_L \equiv \frac{\partial_i}{\sqrt{-\nabla^2}} B\,; \tag{2.45}$$

the difference beetween the two choices above laying simply in a rescaling $1/k$ of the scalar function $\hat{B}$ in Fourier space:

$$\tilde{\hat{B}} = \frac{\tilde{B}}{k}\,. \tag{2.46}$$

In order to emphasize the different roles of the longitudinal and transverse components in the perturbations dynamics, we opt to give them different names; therefore, we set $B^i_T = S^i$, so that we have

$$B^i = \partial_i B - S^i\,, \qquad \partial_i S^i = 0\,; \tag{2.47}$$

or,

$$B^i = \frac{\partial_i B}{\sqrt{-\nabla^2}} - S^i\,, \qquad \partial_i S^i = 0\,. \tag{2.48}$$

We will adopt (2.48) when discussing the model of Solid Inflation, as in [32]. The splitting of B^i into a scalar B plus a vector S^i have thus been completed.

Tensors can be decomposed in irreducible $SO(3)$ representations as well; $\hat{h}_{ij}$ can in fact be splitted as follows [49],

$$\hat{h}_{ij} = -\psi\delta_{ij} + \partial_i\partial_j E + \partial_{(i}F_{j)} + \frac{1}{2}\gamma_{ij}\,, \tag{2.49}$$

where

$$\partial_{(i}F_{j)} = \frac{1}{2}\left(\partial_i F_j + \partial_j F_i\right)\,, \qquad \partial_i F^i = 0\,. \tag{2.50}$$

As for γ_{ij}, it is a symmetric, transverse and traceless tensor, that is

$$\partial_i \gamma_{ij} = 0\,, \qquad \gamma^i_i = \gamma_{ii} = 0\,. \tag{2.51}$$

All the perturbations of the metric can thus be grouped by type:

(S) four scalars $\{\,\phi\,,\,\psi\,,\,E\,,\,B\,\}$;

(V) two vectors $\{\,S_i\,,\,F_i\,\}$;

(T) one tensor γ_{ij} .

There are 10 degrees of freedom in total, which are indeed the independent components of $\delta g_{\mu\nu}$.

It should be now discussed how a gauge transformation acts on each metric perturbation. The gauge transformation rules are obtained by making explicit the Lie derivative in (2.25). Such a computation involves the four-vector ξ^μ of (2.18), which we now express in components as

$$\xi^\mu = \left(\alpha, \beta^i\right)\,, \tag{2.52}$$

the vector β^i being splitted into the transverse and longitudinal parts,

$$\beta^i = \partial^i \beta + \gamma^i , \qquad \partial_i \gamma^i = 0 . \tag{2.53}$$

The transformation (2.25), when applied to the perturbed metric, gives

$$\widetilde{\delta g_{\mu\nu}} = \delta g_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu} . \tag{2.54}$$

Given that in a FRW background the only non-vanishing components of the Christoffel symbol are $\bar{\Gamma}^i_{0j} = H\delta^i_j$, $\bar{\Gamma}^0_{ij} = H\delta_{ij}$, and $D_\alpha \bar{g}_{\mu\nu}$=0, the Lie derivative along ξ^μ results in

$$\begin{aligned} \mathcal{L}_\xi \bar{g}_{\mu\nu} &= \bar{g}_{\mu\alpha} D_\nu \xi^\alpha + \bar{g}_{\nu\alpha} D_\mu \xi^\alpha , \\ &= \bar{g}_{\mu\alpha} \left(\partial_\mu \xi^\alpha + \bar{\Gamma}^\alpha_{\nu\rho} \xi^\rho \right) + (\mu \leftrightarrow \nu) . \end{aligned} \tag{2.55}$$

The resulting components of $\mathcal{L}_\xi \bar{g}_{\mu\nu}$ are thus

$$\mathcal{L}_\xi \bar{g}_{00} = -2\dot{\alpha} , \tag{2.56}$$
$$\mathcal{L}_\xi \bar{g}_{i0} = a^2(\dot{\beta}_i) - \partial_i \alpha , \tag{2.57}$$
$$\mathcal{L}_\xi \bar{g}_{ij} = 2a^2 \alpha H \delta_{ij} + 2a^2 \partial_{(i} \beta_{j)} . \tag{2.58}$$

The scalar perturbations will finally look as follows:

$$\widetilde{\phi} = \phi + \dot{\alpha} , \tag{2.59}$$
$$\widetilde{\psi} = \psi - H\alpha , \tag{2.60}$$
$$\widetilde{B} = B - \frac{1}{a}\alpha + a\dot{\beta} , \tag{2.61}$$
$$\widetilde{E} = E + \beta . \tag{2.62}$$

As for the vector perturbations, we will have

$$\widetilde{S}_i = S_i - a\dot{\gamma}_i , \tag{2.63}$$
$$\widetilde{F}_i = F_i + \gamma_i . \tag{2.64}$$

The tensor γ_{ij} is invariant at first order under gauge trasformations, as it is apparent in (2.58).

2.3 Energy-momentum tensor

It is important to stress that the symmetry requirements demanded by the Cosmological Principle shape the energy-momentum tensor into a *perfect fluid form*, even if the content of the Universe is not a proper fluid one is familiar with. The space could in fact be filled with a single scalar field, or even with a solid; in both cases, the zeroth order $\bar{T}_{\mu\nu}$ must have the following form:

$$\bar{T}_{\mu\nu} = (\bar{\rho}(t) + \bar{p}(t))\, \bar{u}_\mu \bar{u}_\nu + \bar{p}(t) \bar{g}_{\mu\nu} . \tag{2.65}$$

Here, ρ and p represent the energy density and the isotropic pressure. As for $\bar{u}^\mu$, it is the zeroth order four-velocity of the fluid, a time-like unit vector, thus obeying to the condition

$$\bar{g}_{\mu\nu} \bar{u}^\mu \bar{u}^\nu = \bar{u}^\mu \bar{u}_\mu = -1 . \tag{2.66}$$

In the fluid comoving frame we choose $u^\mu = (1, \mathbf{0})$.

The Universe that we are going to describe as part of the perturbative theory has to evade from the Cosmological Principle since we need to include, despite their small size, inhomogeneities and anisotropies on cosmological scales. While the tensor form (2.65) represents a very good approximation, it is nonetheless inadequate in order to provide a more realistic and detailed description of the Universe. It is therefore necessary to introduce a type of energy-momentum tensor that efficiently includes the system anisotropies and inhomogeneities.

2.3.1 Imperfect fluid form

In this paragraph we will follow a method reported in [25, 22] in order to construct an energy-momentum tensor of *imperfect fluid* form, which can take into account the deviations from perfect homoneity and isotropy.

Let us consider a unit future-directed time-like vector field U^μ, that can be regarded as the four-velocityof the observer:

$$U^\mu U_\mu = -1\,,\, U^0 \geq 0\,. \tag{2.67}$$

It is then possible to construct a tensor which projects along the orthogonal direction of U^μ, namely

$$P_{\mu\nu} = g_{\mu\nu} + U_\nu U_\mu\,. \tag{2.68}$$

The projector $P_{\mu\nu}$ is defined as the metric of the three-dimensional spatial section orthogonal to U_μ [25, 22]. We can now use $P_{\mu\nu}$ to project quantities orthogonal to U^μ into the rest frame of the observer:

$$\begin{aligned} T_{\mu\nu} = \delta^\alpha_\mu \delta^\beta_\nu T_{\alpha\beta} &= \left(P^\alpha_\mu - U^\alpha U_\mu\right)\left(P^\beta_\nu - U^\beta U_\nu\right) T_{\alpha\beta} \\ &= \left(P^\alpha_\mu P^\beta_\nu - U_\mu U^\alpha P^\beta_\nu - U_\nu U^\beta P^\alpha_\mu + U_\mu U_\nu U^\alpha U^\beta\right) T_{\alpha\beta} \\ &= \rho\, U_\mu U_\nu + U_{(\mu} q_{\nu)} + p\left(g_{\mu\nu} + U_\mu U_\nu\right) + \pi_{\mu\nu}\,. \end{aligned} \tag{2.69}$$

The variables introduced have been built through the following projections of the energy momentum tensor:

$$\rho = T_{\mu\nu} U_\mu U_\nu\,, \tag{2.70}$$

$$3p = T_{\mu\nu} P^{\mu\nu}\,, \tag{2.71}$$

$$q_\mu = -P^\alpha_\mu T_{\alpha\beta} U^\beta\,, \tag{2.72}$$

$$\pi_{\mu\nu} = P^\alpha_\mu P^\beta_\nu T_{\alpha\beta} - \frac{1}{3} P_{\mu\nu} P_{\alpha\beta} T^{\alpha\beta}. \tag{2.73}$$

In addition to the energy density ρ and the isotropic pressure p, this time $T_{\mu\nu}$ includes two new terms with respect to the perfect fluid form, namely

1. the *anisotropic stress* $\pi_{\mu\nu}$;
2. the energy flux q^μ, which is, by contruction, orthogonal to the observer's four-velocity U^μ ($q_\mu U^\mu = 0$).

A proper choice of U^μ can simplify the expression (2.69), making disappear the term q_μ. We can see that the requirement on such four-velocity must be

$$q_\mu = 0 \qquad \Longrightarrow \qquad T_{\mu\beta} U^\beta = -\left(T_{\alpha\beta} U^\alpha U^\beta\right) U_\mu = -\rho\, U_\mu\,, \tag{2.74}$$

therefore U^μ must be the unit eigen-vector of the energy-momentum tensor. This four-vector is called *energy frame* [25] and it will be labeled by u^μ_E. The four-vector u^μ_E is the eigen-vector of the energy-momentum tensor with $-\rho$ as the corresponding eigen-value:

$$T^\mu_\nu u^\nu_{(E)} = -\rho u^\mu_{(E)}\,. \tag{2.75}$$

Once we have adopted the energy frame of the matter $u^\mu_{(E)}$, the tensor $T_{\mu\nu}$ results in

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu + p g_{\mu\nu} + \pi_{\mu\nu}\,, \tag{2.76}$$

where for simplicity we avoided to write the subsript (E).

At this point, one can see that the anisotropic stress must obey the constraints

$$u^\mu \pi_{\mu\nu} = 0\,, \qquad u^\mu u^\nu \pi_{\mu\nu} = 0\,, \qquad \pi^\mu_\mu = 0 \tag{2.77}$$

as implied by the definitions of p and ρ in equation (2.70) and by the fact that u^μ is the energy density eigen-vector. These results are valid at every order in perturbation theory. In particular, the second equation above implies that the (00) component of the anisotropic stress is null: being an invariant quantity under diffeomorphism, the above mentioned statement is veryfied by choosing $u^\mu = (1, \mathbf{0})$.

2.3.2 Energy-momentum tensor perturbation

Our analysis of first order perturbations of the energy-momentum tensor begins with expressing u^μ in the coordinates of the background; its spatial components are defined as

$$u^i = \delta u^i = \frac{dx^i}{ds} = \frac{a}{a}\frac{dx^i}{ds} = \frac{v^i}{a}\,. \tag{2.78}$$

Here, v^i is the peculiar velocity of the matter, and has to be regarded as a first order quantity. To derive the temporal component we need to use the normalization condition

$$u^\mu u_\mu = g_{00}(u^0)^2 + 2g_{0i}u^0u^i + g_{ij}u^iu^j = -1\,. \tag{2.79}$$

By taking the perturbed quantities of both sides of the above equation, we obtain

$$\begin{aligned} \delta g_{00}(\bar{u}^0)^2 + 2\bar{g}_{00}\bar{u}^0\delta u^0 &= 0\,, \\ \delta u^0 &= -\phi\,. \end{aligned}$$

Finally, we will have

$$u^\mu = \bar{u}^\mu + \delta u^\mu = \left(1-\phi, \frac{v^i}{a}\right)\,. \tag{2.80}$$

By lowering the indices in the usual manner and neglecting higher order quantities, we obtain as well u_μ:

$$u_\mu = \left(-1-\phi, a(v_i + B_i)\right)\,. \tag{2.81}$$

In the same manner as (2.47), the peculiar velocity can be split into a longitudinal and a transverse part:

$$v^i = \partial_i v + v^i_T\,, \qquad \partial_i v^i_T = 0\,. \tag{2.82}$$

We will now consider the anisotropic stress (2.73). As the background space-time is isotropic and homogeneous, $\pi_{\mu\nu}$ cannot be part of the background energy-momentum tensor. Therefore, the anisotropic stress has to be regarded at least as a first order perturbation of the background energy-momentum tensor. We already know that $\pi_{00} = 0$; moreover, the equation (2.77) tells us that π_{i0} is null to first order in perturbations:

$$u^\mu \pi_{\mu j} = u^0\pi_{0j} + \underbrace{u^i\pi_{ij}}_{II order} = 0\,. \tag{2.83}$$

We are therefore considering only the spatial part of the anisotropic stress, which we split into a trace-free scalar part Π^S, a vector part Π^V_i and a tensor part Π^T_{ij}, as previously done for $\hat{h}_{ij}$ in (2.49). We thus have

$$\pi_{ij} = a^2\left[\left(\partial_i\partial_j - \delta_{ij}\frac{1}{3}\nabla^2\right)\Pi^S + \partial_{(i}\Pi^V_{j)} + \Pi^T_{ij}\right]\,. \tag{2.84}$$

From the analysis of the above results, we are finally able to provide the expression of the perturbed quantity $\delta T_{\mu\nu}$; by denoting with $\delta\rho$ and δp the energy density and pressure perturbation, the perturbed components of $T_{\mu\nu}$ are

$$\delta T^0_0 = \delta\left[(\rho+p)u^0u_0 + p\delta^0_0\right] = \bar{u}^0\bar{u}_0(\delta\rho+\delta p) + \delta p = -\delta\rho\,, \tag{2.85}$$

$$\delta T^0_i = \delta\left[(\rho+p)u^0u_i + p\delta^0_i\right] = (\bar{\rho}+\bar{p})\bar{u}^0\delta u_i = a(\bar{\rho}+\bar{p})\left(v_i+B_i\right), \tag{2.86}$$

$$\delta T^i_0 = \delta\left[(\rho+p)u_0u^i + p\delta^i_0\right] = (\bar{\rho}+\bar{p})\bar{u}_0\delta u^i = -(\bar{\rho}+\bar{p})\,\frac{v^i}{a}\,, \tag{2.87}$$

$$\delta T^i_j = \delta\left[(\rho+p)u^iu_j + p\delta^i_j\right] + \bar{g}^{ik}\pi_{kj} = \delta p\delta^i_j + \frac{1}{a^2}\pi^i_j\,. \tag{2.88}$$

In order to complete the analysis of the subject, what is left to illustrate are the gauge trasformations of $\delta\rho$, δp and v^i.

As scalars under the action of diffeomorphisms, ρ and p have Lie derivative given in (2.22); therefore, under a gauge transformation they change into

$$\widetilde{\delta\rho} = \delta\rho + \dot{\bar{\rho}}\alpha\,, \qquad \widetilde{\delta p} = \delta p + \dot{\bar{p}}\alpha\,. \tag{2.89}$$

As for the four-velocity perturbation, according to (2.24), it transforms as

$$\widetilde{\delta u^\mu} = \delta u^\mu + \mathcal{L}_\xi\bar{u}^\mu = \delta u^\mu - \bar{u}^0\partial_0\xi^\mu\,. \tag{2.90}$$

From the above, we infer how v^i changes under a gauge transformation:

$$v^i \longrightarrow v^i - a\dot{\beta}^i\,. \tag{2.91}$$

We can also introduce an additional variable in the energy-momentum tensor, the *momentum density* δV^i, defined by

$$\delta V^i \equiv a(\bar{\rho}+\bar{p})\left(v^i+B^i\right) = \partial_i q + q^i\,, \quad \partial_i q^i = 0\,; \tag{2.92}$$

its longitudinal and transverse components transform as follows:

$$\delta q \equiv a(\bar{\rho}+\bar{p})(v+B) \longrightarrow \delta q - (\bar{\rho}+\bar{p})\alpha\,, \tag{2.93}$$

$$\delta q^i \equiv a(\bar{\rho}+\bar{p})(v^i_T - S^i) = \widetilde{\delta q^i}\,. \tag{2.94}$$

Finally, in order to find the gauge transformation law of $\pi_{\mu\nu}$, we consider the general transformation (2.25) applied to $T_{\mu\nu}$. The computation reveals that δT_{ij} changes in the same way as $\delta p\delta_{ij}$, namely

$$\widetilde{\delta T_{ij}} = \delta T_{ij} + \dot{\bar{p}}\alpha\delta_{ij}\,. \tag{2.95}$$

This fact therefore entails that the anisotropic stress is gauge invariant.

2.3.3 A simple example: the energy-momentum tensor of a single scalar field

We will now compare the above results with the case of a single scalar real field φ. The aim is to provide an expression of the quantities of interest in terms of the field φ and its perturbation $\delta\varphi$. The energy-momentum tensor of such a field is

$$T^\varphi_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi + V(\phi)\right)\,. \tag{2.96}$$

This is a tensor in a perfect fluid form, as one can demonstrate by identifying ρ, p and u_μ as follows [22, 49]:

$$u_\mu = \frac{\partial_\mu \varphi}{\sqrt{\partial_\alpha \varphi \partial^\alpha \varphi}} \,, \tag{2.97}$$

$$\rho = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + V(\varphi) \,, \tag{2.98}$$

$$p = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \,. \tag{2.99}$$

From the above, one can see that no anisotropic stress term appears in $T^\varphi_{\mu\nu}$.

By splitting φ and the metric into

$$\varphi(\tau, \mathbf{x}) = \bar{\varphi}(\tau) + \delta\varphi(\tau, \mathbf{x}) \,, \qquad g_{\mu\nu} = \bar{g}_{\mu\nu}(\tau) + \delta g_{\mu\nu}(\tau.\mathbf{x}) \,, \tag{2.100}$$

we obtain the first order perturbation of $T^\mu_{\nu,\varphi}$:

$$\begin{aligned} \delta T^\mu_{\nu,\varphi} =& \delta g^{\mu\alpha} \partial_\alpha \bar{\varphi} \partial_\nu \bar{\varphi} + \bar{g}^{\mu\alpha} \partial_\alpha \bar{\varphi} \partial_\nu \delta\varphi + \bar{g}^{\mu\alpha} \partial_\alpha \delta\varphi \partial_\nu \bar{\varphi} \\ & - \frac{\delta^\mu_\nu}{2} \left(\delta g^{\alpha\beta} \partial_\alpha \bar{\varphi} \partial_\beta \bar{\varphi} + \bar{g}^{\alpha\beta} \partial_\alpha \bar{\varphi} \partial_\beta \delta\varphi + \bar{g}^{\alpha\beta} \partial_\alpha \delta\varphi \partial_\beta \bar{\varphi} \right) - \delta^\mu_\nu \frac{\partial V}{\partial \varphi}(\bar{\varphi}) \delta\varphi \,. \end{aligned} \tag{2.101}$$

In conformal time, it results in

$$\delta T^0_0 = \frac{1}{2} \delta g^{00} \partial_0 \bar{\varphi} \partial_0 \bar{\varphi} + \bar{g}^{00} \partial_0 \bar{\varphi} \partial_0 \delta\varphi - \frac{\partial V}{\partial \varphi} \delta\varphi = \frac{1}{a^2} \bar{\varphi}' (\phi \bar{\varphi}' - \delta\varphi') - \frac{\partial V}{\partial \varphi} \delta\varphi \tag{2.102}$$

$$\delta T^0_i = \delta g^{0\alpha} \partial_\alpha \bar{\varphi} \partial_i \bar{\varphi} + \bar{g}^{0\alpha} \partial_\alpha \bar{\varphi} \partial_i \delta\varphi + \bar{g}^{0\alpha} \partial_\alpha \delta\varphi \partial_i \bar{\varphi} = -\frac{1}{a^2} \bar{\varphi}' \partial_i \delta\varphi \tag{2.103}$$

$$\begin{aligned} \delta T^i_j =& -\frac{\delta^i_j}{2} \left(2 \frac{\partial V}{\partial \varphi} \delta\varphi + \delta g^{\alpha\beta} \partial_\alpha \bar{\varphi} \partial_\beta \bar{\varphi} + \bar{g}^{\alpha\beta} \partial_\alpha \bar{\varphi} \partial_\beta \delta\varphi + \bar{g}^{\alpha\beta} \partial_\alpha \delta\varphi \partial_\beta \bar{\varphi} \right) \\ =& \delta^i_j \left[-\frac{\partial V}{\partial \varphi} \delta\varphi - \frac{1}{a^2} \bar{\varphi}' \left(\delta\varphi' - \phi \bar{\varphi}' \right) \right] \,. \end{aligned} \tag{2.104}$$

Through the comparison of the above equations with (2.85), (2.86) and (2.88), we can notice the absence of any tensor and vector perturbation.

2.4 The dynamics of the first order perturbations

Within the theory of General Relativity, the metric and energy-momentum tensor perturbations are dynamically coupled by the Einstein equations (2.1), namely

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \,. \tag{2.105}$$

The term $\delta G_{\mu\nu}$ encodes a set of dynamical equations for each type of the metric perturbations previously defined, while $\delta T_{\mu\nu}$ contains the relative source terms.

We will now briefly summarize the steps necessary to formulate the set of dynamical equations of the cosmological perturbations. The right hand side of (2.105) is obtained by expanding the full Einstein equations (2.1) to first order in $\delta g_{\mu\nu}$. One has to calculate progressively the Christoffel symbol $\Gamma^\sigma_{\mu\nu}$ to first order in $\delta g_{\mu\nu}$, and then the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R as well. Since this is quite a long task, we prefer to report here just the final result: a set of differential equations for the scalar, vector and tensor perturbations. The following analisys is based on comprehensive study of subject reported in [49, 22, 41]. As anticipated, the Einstein equations do not couple dynamically the different types, therefore they can safely be treated separately. It is also important to notice that this is true only within the approximation we have adopted (first order), but no longer true at higher orders.

At this point of the discussion we will see how a proper choice of the coordinate system, *i.e.* a gauge fixing, can simplify the form of the equations and, more importantly, erase the non-physical degrees of freedom. In the following analysis we will introduce also *gauge independent* quantities; their importance stands in the fact that they are good candidates for physical observables, by virtue of their independency on the coordinate system. They also turn out to be valuable instruments to shed the light on the physical meaning of the perturbed Einstein equations.

2.4.1 Scalar perturbations

The (00) and (0i) components of (2.105) relative to scalar perturbations are

$$3a^2H(\dot{\psi}+H\phi)-\nabla^2\left[\psi+H(a^2\dot{E}-aB)\right]=-4\pi a^2G\delta\rho\,, \tag{2.106}$$

$$\dot{\psi}+H\phi=-4\pi G\delta q\,. \tag{2.107}$$

It should be noted that the equations (2.106) and (2.107) do not represent proper dynamical equations, because that would require a second time derivative. Instead, they stand as *constraint equations*, which are commonly known as "energy constraint" and "momentum constraint" respectively. Once they have been solved, their solution is meant to be substituted into the dynamical equations. As we shall soon see in section 2.5.2, the constraint equations can be solved straightforwardly in the spatially flat gauge.

The trace of (2.1) results in

$$\frac{\partial}{\partial t}(a^2\dot{E}-aB)+3H(a^2\dot{E}-aB)+(\psi-\phi)=8\pi Ga^2\Pi^S\,; \tag{2.108}$$

while the off-trace components give

$$\ddot{\psi}+3H\dot{\psi}+H\dot{\phi}+(3H^2+2\dot{H})\phi=4\pi G\left(\delta p+\frac{2}{3}\nabla^2\Pi^S\right)\,. \tag{2.109}$$

The covariant conservation of the energy momentum tensor, $D_\mu T^{\mu\nu}=0$, produces the *continuity equation* and the *Euler equation*; they are, respectively, the density energy and scalar momentum conservation equation:

$$\delta\dot{\rho}+3H(\delta\rho+\delta p)=(\bar{\rho}+\bar{p})\left[3\dot{\psi}-\nabla^2(\dot{E}-\frac{B}{a})\right]-\frac{\nabla^2}{a^2}\delta q\,, \tag{2.110}$$

$$\delta\dot{q}+3H\delta q=-(\bar{\rho}+\bar{p})\phi-\delta p-\frac{2}{3}\nabla^2\Pi^S=0\,. \tag{2.111}$$

The equations (2.106), (2.107), (2.108) and (2.110) look rather cumbersome; however, by fixing a convenient gauge, they can become simpler. For that reason, we will discuss the physical implications of the scalar perturbation equations only after we have introduced some popular gauge choices and the gauge-invariant variables.

2.4.2 Vector perturbations

It is possible to extrapolate from (2.105) two equations associated to vector perturbations. The (0i) components of (2.105), when restricting to vectors, give the following constraint equation:

$$\nabla^2\left(\dot{F}_i+\frac{S_i}{a}\right)=-16\pi G\delta q_i\,. \tag{2.112}$$

A second equation relative to perturbations of vector type originates from the covariant conservation of $T_{\mu\nu}$:

$$\delta\dot{q}_i+3H\delta q_i=-\nabla^2\Pi_i^V\,. \tag{2.113}$$

The latter is interpreted as the conservation of the vector part of the momentum density δV_i, previously defined in (2.92).

It should be noticed that it is possible to combine the vectors of $\delta g_{\mu\nu}$ in such a way that the combination is unaffected by the tranformations (2.63), namely

$$G_i\equiv\left(\dot{F}_i+\frac{S_i}{a}\right),\qquad \tilde{G}_i=G_i\,. \tag{2.114}$$

The vector G_i is therefore a gauge-invariant variable representing, by virtue of such a property, a truly observable perturbation. Since, as illustrated in (2.93) and (2.95), δq_i and $\pi_{\mu\nu}$ are gauge-invariant too, the equations (2.112) and (2.113) can be reformulated in terms of physical observables:

$$\nabla^2 G_i = -16\pi G \delta q_i \,, \tag{2.115}$$

$$\nabla^2 \left(\dot{G}_i + 3HG_i\right) = -16\pi G \nabla^2 \Pi_i^V \,. \tag{2.116}$$

Vector perturbations are often not discussed in the context of inflation, and in many cases not even mentioned. The reason of such a lack of interest on the subject is explained by the fact that, for a wide class of models, vector perturbations do not produce sizeable effects on cosmological scales; they have in fact an amplitude decaying in time. In order to prove this statement, it is sufficient to consider the case in which the vector part of the anisotropic stress Π_i^V is null. The equation (2.116) therefore takes the following form:

$$\nabla^2 \left(\dot{G}_i + 3HG_i\right) = 0 \,. \tag{2.117}$$

By solving (2.117) in Fourier space, it gives

$$\dot{G}(t)_{\mathbf{k}} = -3\frac{\dot{a}}{a}(t) G_{\mathbf{k}}(t) \,, \tag{2.118}$$

which has $G_{\mathbf{k}}(t) \propto a(t)^{-3}$ as solution.

In conclusion, vector perturbations are not sustained, unless a non-zero vector part of the anisotropic stress is present. We anticipate that such a condition is instead satisfied in the model of Solid Inflation, which will be discussed in chapter 4. A for the single scalar field case, vector perturbations are null at any given time: as it can be seen in (2.102), no δq_i is contained in the perturbed energy momentum-tensor $T^{\varphi}_{\mu\nu}$.

2.4.3 Tensor perturbations

The (ij) components of (2.105) encode the following dynamical equation for the tensor γ_{ij}:

$$\ddot{\gamma}_{ij} + 3H\dot{\gamma}_{ij} - \frac{\nabla^2}{a^2}\gamma_{ij} = 16\pi G \Pi^T_{ij} \,. \tag{2.119}$$

As it has been mentioned in section 2.2.2, tensor perturbations are gauge-invariant at first order; therefore the equation (2.119) has the same solution in any coordinate system. By mapping (2.119) in momentum space, and by using the conformal time τ, the amplitudes $\gamma_{\mathbf{k}}(\tau)$ and $\Pi^T_{\mathbf{k}}(\tau)$ of the k-modes obey the equation

$${\gamma^s_{\mathbf{k}}}''(\tau) + 2\mathcal{H}{\gamma^s_{\mathbf{k}}}'(\tau) + k^2\gamma^s_{\mathbf{k}}(\tau) = 16\pi G a^2 \Pi^T_{\mathbf{k}}(\tau). \tag{2.120}$$

If no tensor part Π^T_{ij} is included in the anisotropic stress, as per the single scalar field case or a perfect fluid, the equation (2.120) is the one of a massless scalar field in an unperturbed FRW metric, that is equation (1.76) with $\eta_\chi = 0$. However, it should be pointed out that $\Pi^T_{\mathbf{k}}$ can potentially result in a mass term for gravitons. We will see in chapter 4 that this is indeed the case in Solid Inflation.

2.5 Gauge fixing and gauge independent variables

We will now discuss four popular gauge choices in literature [49, 59, 22, 51] that will be employed in this thesis: the *spatially flat*, the *conformal newtonian*, the *uniform density* and the *comoving gauge*. The first one will be extensively used for calculations beyond the linear order in Solid Inflation, while the others have mainly an illustrative purpose.

2.5.1 Conformal newtonian gauge

The functions α, β and γ^i in the four-vector ξ^μ that defines the gauge transformation, can be chosen to eliminate two of the four scalar perturbations $\{\phi\,,\,\phi\,,\,E\,,\,B\}$ and one of the two vector perturbations $\{\,S_i\,,\,F_i\,\}$ of the metric. We first define the following two variables, named *Bardeen potentials* [49, 59, 22]:

$$\Phi \equiv \phi - \frac{\partial}{\partial t}\left[a^2\left(\dot{E} - \frac{B}{a}\right)\right]\,, \tag{2.121}$$

$$\Psi \equiv \psi + a^2 H(\dot{E} - \frac{B}{a})\,. \tag{2.122}$$

Such two functions are constructed in a way that makes them invariant under the tranformations (2.59), *i.e.* they are gauge invariant. Moreover, with a suitable choice of α it is possible to make them coincide with the scalars ϕ and ψ. The latter occurs when it is required that:

$$\alpha_{c.n.} = -a^2\Big(\dot{E} - \frac{B}{a}\Big)\,, \tag{2.123}$$

where the subscript *c.n.* stands for 'conformal newtonian' gauge, which is the name of the coordinate system defined by the condition (2.123). In order to fully specify the conformal newtonian coordinate system, it is also necessary to set the values of β and γ_i; we then set $\beta_{c.n.} = -E$ and $a\dot{\gamma}_{i,c.n.} = S_i$, so that the line element ds^2 becomes

$$ds^2 = -(1+2\Phi)dt^2 + a^2[(1-2\Psi)\delta_{ij} + 2\partial_{(i}F_{j)} + \gamma_{ij}]dx^i dx^j\,. \tag{2.124}$$

It is interesting to notice how the Einstein equations for scalar perturbations look within the conformal newtonian gauge:

$$3H(\dot{\Psi} + H\Phi) - \frac{\nabla^2}{a^2}\Psi = -4\pi G\delta\rho_{c.n.}\,, \tag{2.125}$$

$$\dot{\Psi} + H\Phi = -4\pi G\delta q_{c.n.}\,. \tag{2.126}$$

Since $E = B = 0$, the trace and the trace-off components are

$$\ddot{\Psi} + 3H\dot{\Psi} + H\dot{\Phi}(3H^2 + 2\dot{H})\Phi = 4\pi G\left(\delta p_{c.n.} + \frac{2}{3}\nabla^2\Pi^S\right)\,, \tag{2.127}$$

$$\Psi - \Phi = 8\pi G a^2 \Pi^S\,. \tag{2.128}$$

The last equation highlights an interesting result: $\Phi = \Psi$ results only if the anisotropic stress is absent; in this case the scalar perturbations are fully encoded in just one function.

2.5.2 Spatially flat gauge

The spatially flat gauge is defined by the absence of both scalar and vector perturbations in the 3-metric; such a condition is obtained through

$$\begin{cases} \psi_{flat} = \psi - H\alpha_{flat} = 0 \\ E_{flat} = E + \beta_{flat} = 0 \\ F_{i,flat} = F_i + \gamma_{i,flat} = 0 \end{cases} \implies \begin{cases} \alpha_{flat} = \frac{\psi}{H} \\ \beta_{flat} = -E \\ \gamma_{i,flat} = -F_i \end{cases}\,. \tag{2.129}$$

With the above conditions, one can notice that the gauge is completely fixed. It is possible construct three gauge invariant variables, given by

$$\hat{\phi} \equiv \phi_{flat} = \phi + \psi + a\frac{\partial}{\partial t}\left(\frac{\psi}{aH}\right) , \tag{2.130}$$

$$\hat{B} \equiv B_{flat} = B - a\dot{E} - \frac{\psi}{aH} , \tag{2.131}$$

which in the spatially flat gauge correspond to the lapse function and the scalar part of the shift. Therefore, the line element results in

$$ds^2 = -(1+2\hat{\phi})dt^2 + a(\partial_i \hat{B} - \hat{S}_i)dtdx^i + a^2(\delta_{ij} + \gamma_{ij})dx^i dx^j , \tag{2.132}$$

where $\hat{S}$ denotes the following gauge-invariant combination

$$\hat{S} = S_i + a\dot{F}_i = S_{i,flat} . \tag{2.133}$$

The constraints (2.106) and (2.107) are now simplier in respect to their form, making it easier to solve them and express the lapse function in terms of matter perturbations:

$$3H^2\hat{\phi} + H\nabla^2\frac{\hat{B}}{a} = -4\pi G\delta\rho_{flat} , \tag{2.134}$$

$$H\hat{\phi} = -4\pi G\delta q_{flat} . \tag{2.135}$$

2.5.3 Uniform density gauge

The uniform density gauge (*u.d.*) is identified by the condition $\delta\rho = 0$. From (2.89) the above condition is equivalent to

$$\alpha_{u.d.} = -\frac{\delta\rho}{\dot{\rho}}. \tag{2.136}$$

We can define an important gauge-invariant variable, that within this gauge corresponds to $-\psi_{u.d.}$:

$$\zeta \equiv -\psi_{u.d.} = -\left(\psi + H\frac{\delta\rho}{\dot{\rho}}\right) . \tag{2.137}$$

We then name ζ the *curvature perturbation on uniform density hypersurfaces.* It should be noticed, however, that the gauge is not completely fixed, since β and γ_i are still unspecified.

The relation that links ζ and the density perturbation in flat gauge, where $\psi = 0$ holds, is apparent:

$$\zeta = -\frac{H}{\dot{\rho}}\delta\rho_{flat} = \frac{1}{3}\frac{\delta\rho_{flat}}{\bar{\rho} + \bar{p}} ; \tag{2.138}$$

here, the second equality follows from the background continuity equation.

In the context of cosmological perturbations ζ represents a key quantity, since it has the remarkable feature of being conserved on super-horizon scales ($k << aH$) in a large variety of cases. Such a property makes ζ a convenient variable when studying primordial density perturbations: if it is conserved, in fact, cosmologists have at their disposal a quantity that relates the primordial inflationary stage with observations on a more recent Universe.

2.5.4 Comoving gauge

The last gauge we will rewiew is the comoving gauge: as the name suggests, it consists in choosing the coordinate system in such a way that an observer from within sees the matter at rest. By setting $\tilde{v}^i = 0$ and $\delta\tilde{q} = 0$, we obtain the following conditions on the gauge functions:

$$\alpha_{c.g} = (v + B) , \qquad \dot{\beta} = \frac{v}{a} , \qquad \dot{\gamma}^i = \frac{v_T^i}{a} . \tag{2.139}$$

As already done when discussing the previous gauges, we define as well a gauge invariant variable that within the comoving gauge corresponds to the curvature perturbation:

$$\mathcal{R} \equiv \psi - aH(v + B) = \psi - H\frac{\delta q}{\bar{\rho} + \bar{p}}\,. \tag{2.140}$$

From the conservation equations (2.110), an interesting relation connecting ζ with $\mathcal{R}$ is infered:

$$-\zeta = \mathcal{R} - \frac{2\bar{\rho}}{3(\bar{\rho} + \bar{p})}\frac{\nabla^2}{(aH)^2}\Psi\,, \tag{2.141}$$

where Ψ is the Bardeen potential. From the above equation we can infer that on super horizon scales ($k \ll aH$) ζ and $\mathcal{R}$ coincide if Ψ stays constant in that limit.

2.6 The outcomes of single scalar field inflation at linear order

In section 1.3 we derived the power spectrum in the 'spectator field' case; such a case corresponds to neglecting the coupling of the scalar field with the perturbed metric. Given the conceptual tools we have developed in the current chapter, it is now possible to include the effects of a perturbed metric in the evolution of the scalar field. This is indeed the case of the inflaton, the field generating inflation and sourcing the scalar perturbations of the metric.

We can summarize the features we established for the inflaton case as follows:

- $T^{\varphi}_{\mu\nu}$ is the energy-momentum tensor of a perfect fluid;
- no 3-momentum δq_i are included in $\delta T^{\varphi}_{\mu\nu}$, implying that vector perturbations are identically null;
- no anisotropic stress is included $\delta T^{\varphi}_{\mu\nu}$, therefore no source term is present for tensor perturbations;
- ζ is thee only scalar perturbation (adiabaticity) .

2.6.1 Curvature perturbation: power spectrum and spectral index n_s

Since in the single-field slow-roll model the energy density of the Universe is dominated by the potential $V(\varphi)$, the following applies

$$\rho \approx V(\varphi) \implies \delta\rho \approx \frac{dV}{d\varphi}(\bar{\varphi})\delta\varphi = -3H\dot{\bar{\varphi}}\delta\varphi\,, \tag{2.142}$$

where the second slow-roll condition (1.38) has been used. The background continuity equation gives $\dot{\bar{\rho}} = -3H\dot{\bar{\varphi}}$, so that the variable ζ in the spatially flat gauge is given by

$$\zeta = -\mathcal{H}\frac{\delta\rho_{flat}}{\bar{\rho}'} = -H\frac{\delta\rho_{flat}}{\dot{\bar{\rho}}} = -H\left(\frac{\delta\varphi}{\dot{\bar{\varphi}}}\right)_{flat}\,. \tag{2.143}$$

We now want to compute the evolution equation of $\delta\varphi$, this time including the perturbation of the metric tensor. By combining the first order Einstein equations (2.105), or equivalently by minimazing the action $S[\delta\varphi, \delta g; t]$ in section 1.23 in respect to $\delta\phi$, it is possible to extrapolate the Klein-Gordon equation, which in spatially flat gauge results in

$$\delta\varphi'' + 2\mathcal{H}\delta\varphi' - \nabla^2\delta\varphi + a^2\frac{\partial^2 V}{\partial\varphi^2}\delta\varphi + 2a^2\hat{\phi}\frac{\partial V}{\partial\varphi} - \bar{\varphi}'\hat{\phi}' = 0\,. \tag{2.144}$$

We then solve the constraint (2.135) in order to make $\hat{\phi}$ explicit:

$$\hat{\phi} = -\frac{4\pi G}{\mathcal{H}}\bar{\varphi}'\delta\varphi, \tag{2.145}$$

at the same time exploiting the second slow-roll condition (1.38). The dynamical equation (2.144) will be expressed in terms of the so-called *Mukanhov-Sasaki* variable Q_φ [51, 17, 59], a gauge-invariant quantity corresponding to $\delta\varphi$ in the spatially flat gauge:

$$Q_\varphi \equiv \delta\varphi + \frac{\bar{\varphi}'}{\mathcal{H}}\psi\,. \tag{2.146}$$

With few additional steps, the equation (2.144) then becomes

$$(aQ_\varphi)'' + \left(k^2 - \frac{a''}{a} + a^2\mathcal{M}_\varphi^2\right)(aQ_\varphi) = 0\,, \tag{2.147}$$

with the mass term M^2 given in terms of the slow-roll parameters (1.41) and (1.43) as follows

$$\mathcal{M}_\varphi^2 = \frac{\partial^2 V}{\partial\varphi^2} - \frac{8\pi G}{a\mathcal{H}}(\bar{\varphi}')^2 \simeq H^2(3\eta_V - 6\epsilon_V)\,. \tag{2.148}$$

The equation (2.147) is known as the *Mukhanov-Sasaki equation*; its solution can be obtained and examined by following the methodology explained in section 1.3, since the equation of motion of Q_φ has the same form of (1.76). The gauge-invariant variable Q_φ has therefore the following amplitude on super-horizon scales:

$$|Q_\varphi(\tau,k)| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{3/2-\nu_\varphi}\,, \tag{2.149}$$

with ν_φ given by

$$\nu_\varphi \simeq \frac{3}{2} - \eta_V + 3\epsilon_V\,. \tag{2.150}$$

Given that the relation between ζ and Q_φ is $\zeta = -H\left(\frac{Q_\varphi}{\dot{\bar{\phi}}}\right)$, we obtain the dimensionless power spectrum $\mathcal{P}_\zeta$ on super-horizon scales by exploiting (2.149):

$$\mathcal{P}_\zeta(k) = \left(\frac{H^2}{2\pi\dot{\bar{\varphi}}}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu_\varphi}\,. \tag{2.151}$$

According to the definition (1.93), the spectral scalar index relative to the curvature perturbation, n_s, is thus given as the following combination of slow-roll parameters:

$$n_s - 1 = -6\epsilon_V + 2\eta_V\,. \tag{2.152}$$

The dimensionless scalar power spectrum is parametrized by its amplitude A_s and its spectral index n_s [6]:

$$P_\zeta(k) = A_s(k_0)\left(\frac{k}{k_0}\right)^{n_s(k_0)}\,, \tag{2.153}$$

where k_0 is a pivot scale.

2.6.2 Gravitational waves: power spectrum and tensor spectral index

We saw in section 2.4.3 that the amplitude of the gravitational waves $\gamma_{\mathbf{k}}^s(\tau)$ obey the same equation (2.120) as massless scalar fields , up to a numerical factor. It is thus possible to state that, after being generated on sub-horizon scales, the mode function $\gamma^s(\tau,k)$ freezes at horizon crossing at the value

$$|\gamma^s(\tau,k)| \underset{-k\tau\to 0}{\simeq} \sqrt{32\pi G}\frac{H}{\sqrt{2k^3}}\,. \tag{2.154}$$

As done previously for curvature perturbation, the dimensionless power spectrum $\mathcal{P}_\gamma$ for the gravitational waves is defined as follows:

$$\langle \gamma^s_{\mathbf{k}}(\tau)\gamma^{s'}_{\mathbf{k}'}(\tau)\rangle = (2\pi)^3\delta^{ss'}\delta^{(3)}(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}\mathcal{P}_\gamma(\tau,k)\,, \tag{2.155}$$

which, by using (2.154), on super-horizon scales results in

$$\mathcal{P}_\gamma(k) = \frac{H^2}{M_p^2\pi^2}\,. \tag{2.156}$$

As per scalar perturbations, the tensor power spectrum is parametrized by an amplitude A_T and a spectral index n_T:

$$\mathcal{P}_\gamma(k) = A_T(k_0)\left(\frac{k}{k_0}\right)^{n_T(k_0)}\,, \qquad n_T = \frac{d\ln\left(\mathcal{P}_\gamma(k)\right)}{d\ln(k)}\,, \tag{2.157}$$

being k_0 a pivot scale. According to the previous analysis, the tensor spectral index predicted by the single field model of inflation is

$$n_T = -2\epsilon\,. \tag{2.158}$$

The measurement of the four parameters $\{A_s\,,\,A_T\,,\,n_s\,,\,n_T\,\}$ can in principle allow to recontruct the shape of the potential $V(\phi)$; H is in fact a measure of the scale of the potential, ϵ_V of its first derivative V' and η_v of its second derivative V''. However, since the amplitude of scalar perturbations is observed at $A_s \sim 10^{-9}$ [2], tensor fluctuations can be normalized in respect to A_s; in this way, it is defined another important cosmological parameter, the so called *tensor-to-scalar ratio* r:

$$r \equiv \frac{2\mathcal{P}_h(k)}{\mathcal{P}_\zeta(k)}\,. \tag{2.159}$$

In the above, the factor 2 takes into account the two polarized states of tensor perturbations. By combining (2.151) and (2.156), one obtains the value of r within the single scalar field scenario, that is

$$r = 16\epsilon\,. \tag{2.160}$$

The value of the ratio r is therefore related to the tensor spectral index by the following relation, also known as *consistency relation*:

$$r = -8n_T\,. \tag{2.161}$$

It should be noticed that within the paradigm of single field slow-roll inflation the consistency relation holds regardless the shape of the potential $V(\varphi)$, standing therefore as a general prediction of the model. By taking into account the consistency relation, it is thus possible to classify all the single field models in a plane of coordinates $(r\,,\,n_s)$ [40] and compare them to the experimental constraints. The tensor-to-scalar ratio is also a direct measure of the energy scale of inflation; by expressing $\mathcal{P}_\gamma$ through the inflaton potential V, (2.159) results in

$$V^{\frac{1}{4}} \sim \left(\frac{r}{0.01}\right)^{\frac{1}{4}} 10^{16}\,\mathrm{Gev}\,. \tag{2.162}$$

The detection of a signal of cosmological gravitational waves could be as well an indepent measurement linking to the energy scale of the inflation, as it is apparent from (2.156).

[2] In [6] is reported the value $\ln\left(10^{10}A_s\right) = 3.089 \pm 0.036$ (TT+lowP) at the pivot scale $k_0 = 0.05\,\mathrm{Mp}^{-1}$.

2.6.3 The *Planck* constraints

Important constraints on inflation have been provided by the *Planck* mission with an unprecedent level of accuracy [4]. The results infered from the *Planck* data, in reference to the parameters n_s and r discussed above, can be summarized as follows:

1. the CMB power spectra are accurately descibed by a spectrum of scalar flutuations with a spectral index, at the pivot scale $k_0 = 0.05\,\mathrm{Mp}^{-1}$, given by

$$n_s \,=\, 0.968 \,\pm\, 0.006 \quad (68\%\, CL)\,; \tag{2.163}$$

2. no evidence for tensor modes are implied in the data; as for the value of the tensor-to-scalar ratio, it has been constrained with the upper limit

$$r \,<\, 0.11 \quad (95\%\, CL)\,. \tag{2.164}$$

The results of *Planck* suggest the departure from scale invariance for the spectrum of scalar perturbations, *i.e* $n_s \neq 1$, and they are therefore consistent with the prediction (2.151) of single field inflationary models. The constraint on r can be translated into an upper bound on the energy scale of inflation [4],

$$V_* = (1.88 \times 10^{16}\,\mathrm{GeV}\,)^4 \,\frac{r}{0.10}\,, \tag{2.165}$$

where the star denotes that the quantity is evaluated at the time when the pivot scale k_0 exits the horizon. As for the model classification in the parameter space $(n_s\,,\, r)$, the analysis in [4] rules out quadratic inflationary potentials, $V(\varphi) \propto \varphi^2$, predicting $r \approx 0.16$.

Chapter 3

Primordial Non-Gaussianity: beyond the linear order

The single-field slow-roll model of inflation reviewed in Chapter 1 predicts a nearly scale invariant power spectrum of primordial scalar perturbations ([17] and section 2.151). Such a prediction has been verified to a high degree of accuracy by the data of the *Planck* mission ([6] and section 2.163). Despite its success, a vast class of inflationary models have been proposed on theoretical grounds that can lead to similar outcomes, making it difficult to select a proper candidate for the description of the early Universe.

Whithin the abundance of possible inflationary scenarios, a selective investigation on primordial fluctuations should consider the statistical higher order contribution, namely the *non-Gaussianities* in the statistics of primordial fluctuations. The non-Gaussian signatures of a model is mathematically translated into non-vanishing ($n \geq 3$)-point correlation functions that we are able to extract from it. Higher order correlation functions are in fact sensitive probes of the interactions experienced during inflation by the fields. The inclusion of such interactions id due to the coupling between different modes in time. The simple single-field slow-roll model naturally predicts (a low level of) non-Gaussian signals [2, 48].

It should be emphasized that such a model is characterized by a set of minimal conditions:

1. only one scalar field, the inflaton φ, generates both the inflationary evolution and the primordial curvature perturbations;
2. the inflaton has a canonical kinetic term, *i.e.* $\partial_\mu \varphi \partial^\mu \varphi / 2$;
3. the inflationary stage is characterized by the slowly rolling of φ down its potential $V(\varphi)$, which, consequently, must be sufficiently flat;
4. the initial condition of the evolution of φ is assumed to be the Bunch-Davies vacuum.

As we will see in section 3.3, all the above conditions lead inevitably to a slow-roll suppressed non-Gaussian signal, hence undetectable through current measurements [3]. A robust detection of primordial non-Gaussianity will therefore rule out the canonical single-field slow-roll model, indicating that a different dynamical process has occured in the inflationary stage. Viceversa, the constraints on non-Gaussianity will limit the vast array of scenarios formulated on theoretical grounds. It can be argued, therefore, that probing inflationary non-Gaussianities is in principle equivalent to reconstructing the Lagrangian of the quantum fields in the early Universe.

3.1 Primordial bispectrum: amplitude and shape

The leading signal of non-Gaussianity is a non-vanishing three-points correlation function of the gravitational potential Φ. More specifically, the analisys of *Planck* are focused on the *bispectrum*

B_Φ, which is defined in Fourier space as follows:

$$\langle \Phi_{\mathbf{k_1}} \Phi_{\mathbf{k_2}} \Phi_{\mathbf{k_3}} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}) B_\Phi(k_1, k_2, k_3) \,. \tag{3.1}$$

In this case Φ equals to the Bardeen potential (2.121) in the matter dominated era, which is related to the curvature perturbation ζ on super-horizon scales through $\Phi = \frac{3}{5}\zeta$ [41]. The delta function in (3.1) translates the momentum conservation in mathematical language, hence forcing the three wave-vectors $\mathbf{k_i}$ to form a closed triangle.

It should be also emphasized that the primordial bispectrum is evaluated at late times, namely, at the end of the inflationary stage. One of the aims of the *Planck* mission is de facto to establish observational constaints on such a signal through the analysis of the CMB data [3, 5]. We will present the *Planck* updated results in section 3.4. Nonetheless, we should say that even in case of a definitive non-detection of a primordial bispectrum, the observational searching for a non-Gaussian signature of inflationary origin can still avail itself of the four-point correlation function of primordial fluctuations (the *trispectra* in Fourier space) [3, 5]. In this thesis, the observational aspects of primordial non-Gaussianities will not be examined, limiting the discussion only on theoretical subjects.

The isotropy assumption of the fluctuations statistics dictates the form (3.1) of the primordial bispectrum. This is manifested by the dependence of B_Φ only on the magnitude of the wave-vectors. Therefore (3.1) is not the most universal form of the primordial bispectrum, but it allows the parametrization of B_Φ as a product of a dimensionless number f_{NL} and a function of the three magnitudes of the momenta [22, 45, 42, 3, 5]. By adopting the same formalism used in [3, 5], the bispectrum can thus be written as

$$B_\Phi(k_1, k_2 k_3) = f_{NL} F(k_1, k_2, k_3) \,. \tag{3.2}$$

The parameter f_{NL} is the *amplitude* of the bispectrum, while $F(k_1, k_2, k_3)$ encodes the functional dependence on the triangular configurations of the three momenta. For reasons regarding the weakness of the CMB signal [45, 28, 22], the observational test on the inflationary predictions for both the amplitude f_{NL} and the function F must be strongly model-dependent. This means that in order to compare theory and data, a set of theoretically motivated ansatzes for the bispectrum form must be provided, named *templates*. Each of these templates, or a proper superposition of them, approximates a class of models that violates one of the features (1)-(4) of the single-field slow-roll scenario. All of these models predict sizeable bispectrum amplitudes f_{NL} ([5] and references therein).

Given the above considerations on the methodology, the results of the *Planck* mission are essentially a set of constraints on the amplitude f_{NL} associated with a specific template. The *Planck* mission succeded especially in putting constraints on the *Local* form, the *Equilateral* form and the *Orthogonal* form, which will be discussed in detail in the following paragraphs.

3.1.1 Local form

The Local form, as reported in [5], consists in the following ansatz:

$$\begin{aligned} B_\Phi^{local}(k_1, k_2, k_3) &= 2 f_{NL}^{local} \Big[P_\Phi(k_1) P_\Phi(k_2) + P_\Phi(k_1) P_\Phi(k_3) + P_\Phi(k_2) P_\Phi(k_3) \Big] \\ &= 2A^2 f_{NL}^{local} \left[\frac{1}{k_1^{4-n_s} k_2^{4-n_s}} + (2\, perm.) \right] \,. \end{aligned} \tag{3.3}$$

Here $P_\Phi(k) = A/k^{4-n_s}$ represents the power spectrum of Bardeen's gravitational potential, A being a normalization factor and n_s the scalar spectral index. The local form peaks in the squeezed triangle, for which one the momenta tends to zero, *e.g.* $k_3 \approx 0$, $k_1 \approx k_2$. By considering the above limit, the result is

$$B_{\Phi}^{local}(k_1, k_1, k_3 \to 0) = 4 f_{NL}^{local} P_{\Phi}(k_1) P_{\Phi}(k_3) \, . \tag{3.4}$$

A bispectrum that peaks in the squeezed configuration is exprected in particular in multi-fields inflationary models or in models in which non-linearities develop on super-horizon scales. The physical interpretation of such a dominant configuration is that the long mode k_3, when leaving the Hubble radius, influences the smaller scales $k_{1,2}$.

3.1.2 Equilateral form

The Equilateral bispectrum shape is given by

$$\begin{aligned} B_{\Phi}^{eq}(k_1, k_2, k_3) =& 6A^2 f_{NL}^{eq} \Bigg\{ - \frac{1}{k_1^{4-n_s} k_2^{4-n_s}} - \frac{1}{k_2^{4-n_s} k_3^{4-n_s}} - \frac{1}{k_1^{4-n_s} k_3^{4-n_s}} - \frac{1}{(k_1 k_2 k_3)^{\frac{2}{3}(4-n_s)}} \\ & + \left[\frac{1}{k_1^{(4-n_s)/3} k_1^{2(4-n_s)/3} k_3^{4-n_s}} + (5\, perm.) \right] \Bigg\} \, . \end{aligned} \tag{3.5}$$

The name of such a template is due to the fact that it peaks for equilateral triangles, namely $k_1 \approx k_2 \approx k_3$. Such a template is used for single-field models in which the kinetic term is non-standard and in scenarios characterized by more general higher-derivative interactions. A detailed list of models, which the Equiateral form is used for, is reported in [42, 5, 3, 14] and in references therein.

At this point of the discussion, it is important to define a method that allows to asses the degree of independency of two bispectra. This will enable us to quantify how well a given template approximates the bispectrum under consideration. For this purpose, an inner product between bispectra it has been introduced [14, 45, 28]. Following the convention of [45], it is first necessary to introduce a dimensionless function that encodes the dependence of the bispectrum on triangles configurations, the so-called *shape function.* The latter is defined as

$$S(k_1, k_2, k_3) \equiv \frac{1}{N} (k_1 k_2 k_3)^2 B_{\Phi}(k_1, k_2, k_3) \, , \tag{3.6}$$

where N is a normalization factor, which is usually chosen such that $S(k, k, k) = 1$. The Local form, assuming $n_s = 1$ for simplicity, results therefore in the shape function given by [45]

$$S^{loc}(k_1, k_2, k_3) = \frac{1}{3} \left(\frac{k_1^2}{k_2 k_3} + \frac{k_2^2}{k_1 k_3} + \frac{k_3^2}{k_1 k_2} \right) , \tag{3.7}$$

while, for the Equilateral form, results into

$$S^{eq}(k_1, k_2, k_3) = \frac{(k_1 + k_2 - k_3)(k_2 + k_3 - k_1)(k_1 + k_3 - k_2)}{k_1 k_2 k_3} \, . \tag{3.8}$$

The degree of independency, or *orthogonality*, of two shapes is then determined by the following inner product:

$$F(S_1, S_2) \equiv \int_{\mathcal{V}_k} dk_1 dk_2 dk_3 \, S_1(k_1, k_2, k_3) S_2(k_1, k_2, k_3) \omega k_1, k_2, k_3 \, . \tag{3.9}$$

In the above expression, $\mathcal{V}_k$ is the region in the space (k_1, k_2, k_3) delimited by $k_1 + k_2 + k_3 \leq 2k_{max}$, while the weight function is $\omega(k_1, k_2, k_3) = (k_1 + k_2 + k_3)^{-1}$. Both the value of k_{mak} and the form of ω are based on observational grounds [45, 29]. Given the inner product (3.9), the so called *3D Cosine* is then naturally defined,

$$C(S_1, S_2) = \frac{F(S_1, S_2)}{\sqrt{F(S_1, S_1)} \sqrt{F(S_2, S_2)}} \, , \tag{3.10}$$

which will be a number in $[-1\, ,\, 1]$, quantifying the level of overlapping between two shapes. As reported in [45], S^{loc} an S^{eq} have $C(S^{loc}, S^{eq}) \approx 0.46$, indicating they are rather orthogonal and hence can be measured nearly independently.

3.1.3 Orthogonal form

The Orthogonal form template is constructed in such a way that its shape is nearly orthogonal to both the Local and the Equilateral ones:

$$\begin{aligned} B_{\Phi}^{ortho}(k_1, k_2, k_3) =& 6A^2 f_{NL}^{ortho} \\ & \times \left\{ -\frac{3}{k_1^{4-n_s} k_2^{4-n_s}} - \frac{3}{k_2^{4-n_s} k_3^{4-n_s}} - \frac{3}{k_1^{4-n_s} k_3^{4-n_s}} - \frac{8}{(k_1 k_2 k_3)^{2(4-n_s)/3}} \right. \\ & \left. + \left[\frac{3}{k_1^{(4-n_s)/3} k_2^{2(4-n_s)/3} k_3^{4-n_s}} + (5\, perm.) \right] \right\} . \end{aligned} \tag{3.11}$$

As the Equilateral form, this form emerges in models characterized by more general derivative interactions. A list of such models is reported in [42, 5, 3].

So far, the discussion was restricted only to the primordial scalar bispectrum. More generally, it is possible to define bispectra for other types of perturbations, such as for the vector (V^{λ}) and tensor perturbations(γ^s):

$$\langle \gamma_{\mathbf{k_1}}^{s_1} \gamma_{\mathbf{k_2}}^{s_2} \gamma_{\mathbf{k_3}}^{s_3} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}) B_{\gamma\gamma\gamma}^{s_1\, s_2\, s_3}(k_1, k_2, k_3) \,, \tag{3.12}$$

$$\langle V_{\mathbf{k_1}}^{\lambda_1} V_{\mathbf{k_2}}^{\lambda_2} V_{\mathbf{k_3}}^{\lambda_3} \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}) B_{VVV}^{\lambda_1\, \lambda_2\, \lambda_3}(k_1, k_2, k_3) \,. \tag{3.13}$$

Mixed-type of bispectra can be also considered, encoding the interactions between different perturbations:

$$B_{\zeta\zeta\gamma} \,,\; B_{\zeta\gamma\gamma} \,,\; B_{\zeta\gamma V} \,,\; B_{\zeta\zeta V} \,,\; B_{\zeta VV} \,,\; B_{\gamma\gamma V} \,,\; B_{\gamma VV} \,. \tag{3.14}$$

3.2 The computation of primordial bispectra

The computation of the primordial bispectrum involves a meticulous approach. A possible way to address the computation on non-Gaussianities consists in including second order perturbations of the fields in the Einstein equation. This method was used in [2] within the model of single-field slow-roll inflation, and also in [49] and [17], the latter providing also applications to models with additional scalar fields. The methodology can be schematize as follows. Firstly, it is necessary to expand both φ and $g_{\mu\nu}$ up to second order around their background values:

$$\varphi \approx \bar{\varphi} + \delta^{(1)}\varphi + \frac{1}{2}\delta^{(2)}\varphi \,, \tag{3.15}$$

$$g_{\mu\nu} \approx \bar{g}_{\mu\nu} + \delta^{(1)} g_{\mu\nu} + \frac{1}{2}\delta^{(2)} g_{\mu\nu} \,. \tag{3.16}$$

Once the above expansions has been inserted, the Einstein equations are then solved order by order:

$$M_P^2 \bar{G}_{\mu\nu} = \bar{T}_{\mu\nu} \,, \quad M_P^2 \delta^{(1)} G_{\mu\nu} = \delta^{(1)} T_{\mu\nu} \,, \quad M_P^2 \delta^{(2)} G_{\mu\nu} = \delta^{(2)} T_{\mu\nu} \,. \tag{3.17}$$

Quadratic terms of first order become therefore sources for the differential equations of second order perturbations. Moreover, the equations generally couple diffent types of perturbations, *e.g*, scalar perturbations of first order are sources for second order tensor perturbations [50]. Within this approach, the significant issue of the gauge freedom must be addressed. De facto, it is necessary to avail of a rigorous definition of gauge transformations of second order [26], so that the gauge fixing and the gauge-invariant variables can be unambiguously defined. Such a formalism, however, has not been developted whithin the scope of this thesis. Nonetheless, the use of the ADM parametrization, combined with the *in-in formalism*, provides an alternative methodology to approach the computation of primordial bispectra. In the next section we will present this methodology.

3.2.1 The ADM parametrization of the action

In order to extract the bispectrum of a cosmological perturbation from a specific model of inflation, might the perturbation be a scalar, a vector or a tensor, it is necessary to know the interactions between the fields involved. In general, the interactions emerge both from the gravity sector (Einstein-Hilbert Lagrangian) and from the matter sector, the latter being denoted by $\sqrt{-g}\mathcal{L}_m$ and containing the fields involved in the model.

The methodology adopted in this thesis essentially consists in two main steps, the first one being the expansion of the Lagrangian to the desired order in perturbations. In order to achieve such a result, it is convenient to use the ADM formalism, previously introduced in section 2.2.1. The full action will then become [29, 28]

$$S = \int dt d^3x N\sqrt{h}\Big\{\frac{M_p^2}{2}\Big[R^{(3)} + N^{-2}\left(E_{ij}E^{ij} - E^2\right)\Big] + \mathcal{L}_m\Big\}. \tag{3.18}$$

In this context, the 3D Ricci scalar $R^{(3)}$, as well as E_{ij} and E, given by

$$E_{ij} = \frac{1}{2}\Big(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\Big)\,, \tag{3.19}$$

$$E = E_{ij}h^{ij}\,, \tag{3.20}$$

are all quantites constructed out of the 3D metric h_{ij} . The auxiliary variables N and N^j are Lagrangian multipliers, so that they do not represent dynamical fields. Their equations of motion reproduce the Hamiltonian and momentum constraints:

$$\frac{M_p^2}{2}\Big[R^{(3)} - N^{-2}\left(E_{ij}E^{ij} - E^2\right)\Big] + \mathcal{L}_m + N\frac{\partial\mathcal{L}_m}{\partial N} = 0\,, \tag{3.21}$$

$$\frac{M_p^2}{2}\nabla_i\Big[N^{-1}\left(E^i_j - E\delta^i_j\right)\Big] + N\frac{\partial\mathcal{L}_m}{\partial N^j} = 0\,. \tag{3.22}$$

As explained in [48, 29, 56] and proved in [28], the third order action in the dynamical perturbations is obtained through the following steps:

1. the full action (3.18) has to be expanded up to the third order in all the perturbations involved;
2. the constraint equations (3.21) and (3.22) have to be solved to first order in the dynamical perturbations;
3. once those solutions have been obtained, they have to be substituted into the action.

The above procedure is generalized to higher orders as well [28]: to derive the expression of the action to order n in dynamical perturbations, it is sufficient to know the solutions of (3.21) and (3.22) to the order $(n-2)$.

At this point, the computation of cosmological correlators proceeds by using the so-called *in-in formalism*, which is curently a standard technique for the calculation of primordial non-Gaussianities. Being rooted in Quantum Field Theory (QFT), the in-in formalism is essentially based on the algebra of quantum operators.

3.2.2 The in-in master formula

The in-in formalism was developted first by J. Schwinger [38] and then by L. V. Keldysh [39] in the context of condensed matter systems. It was then applied for the first time to quantum cosmology by R. Jordan [37], E. Calzetta and B. L. Hu [27]. In 2002 J. Maldacena used the in-in formalism to compute all the three-point correlators expected in the single-field slow-roll model of inflation, in a paper that is now considered a classic on primordial non-Gaussianity [48]. In 2005, the whole formalism was posed on firmer grounds by S. Weinberg [58]. Additional insights into the subject are reported in [8], [7], [29], [28] and [56]. We will now present an overview of the in-in formalism, following the notation of [58] and [8].

As previously mentioned, the in-in formalism provides a method to calculate the n-point functions of cosmological perturbations. In order to have a concise notation, the cosmological perturbations will be collectively denoted with $\delta\phi_{\mathbf{k}} = \{\zeta_{\mathbf{k}},\, \gamma^s_{\mathbf{k}},\, V^{\lambda}_{T,\mathbf{k}}\}$. In the following expressions, the general quantum operator $Q(\tau)$ is understood as a product of cosmological perturbations, namely $Q(\tau) = \delta\phi_{\mathbf{k_1}}\delta\phi_{\mathbf{k_2}}\cdots\delta\phi_{\mathbf{k_n}}$. Its expectation value on the vacuum at fixed time τ corresponds to the following n-point correlation function:

$$\langle Q(\tau)\rangle = \langle\Omega|Q(\tau)|\Omega\rangle\,. \tag{3.23}$$

In the above expression, $|\Omega\rangle$ is the vacuum state of the full theory at some initial time τ_0 in the far past ($\tau_0 < \tau$), which differs from the vacuum of the free theory $|0\rangle$. In order to compute a non-Gaussian signal of primordial origin, the correlator (3.23) has to be evaluated at the end of the inflationary stage ($\tau = \tau_e$), while the initial time is taken in the asymptotic past ($\tau_0 = -\infty$). According to [58], it is therefore possible to compute the vacuum expectation value (3.23) through the following formula:

$$\langle\Omega|Q(\tau_e)|\Omega\rangle = \langle 0|\Big[\bar{T}e^{i\int_{-\infty(1+i\epsilon)}^{\tau_e} d\tau H^I_{int}(\tau)}\Big]Q^I(\tau_e)\Big[Te^{-i\int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau H^I_{int}(\tau)}\Big]|0\rangle. \tag{3.24}$$

The symbols T and $\bar{T}$ indicate the time and anti-time ordering [52]. H_{int} is the part of the Hamiltonian operator containing the interactions between the perturbation fields. The superscript I denotes that the quantum operators are in the interaction picture [52]. The presence of a small imaginary component in the extremes of the integrals should be highlighted, which form a contour of integration that does not close. From now on, we will refer to (3.24) as the *in-in master formula*.

Hereafter, the result (3.24) will be explained through a concise inspection of its derivation [58]. For conciseness, many technical steps will be omitted.

3.2.3 Time evolution in the Heisenberg picture

In the Heisenberg picture, it is known that the operator $Q(\tau)$ has its evolution in time determined by the Hamiltonian

$$H[\phi_a(\tau),\pi_a(\tau)] = \int d^3x\mathcal{H}[\phi_a(\tau,\mathbf{x}),\pi_a(\tau,\mathbf{x})], \tag{3.25}$$

where $\mathcal{H}$ is the Hamiltonian density. The subscript a labels a single element of the set of fields $\{\phi\}$, whose fluctuations, as previously mentioned, will collectively denote scalar, vector and tensor perturbations. The symbol π_a indicates its conjugated momentum. As usual, the following algebra holds for the equal-time commutators,

$$\Big[\phi_a(\tau,\mathbf{x}),\pi_b(\tau,\mathbf{y})\Big] = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\quad \Big[\phi_a(\tau,\mathbf{x}),\phi_b(\tau,\mathbf{y})\Big] = 0\,,\quad \Big[\pi_a(\tau,\mathbf{x}),\pi_b(\tau,\mathbf{y})\Big] = 0\,. \tag{3.26}$$

The Heisenberg equations of motion are

$$\phi'_a = i\Big[H\,,\,\phi_a\Big] \quad \text{and} \quad \pi'_a = i\Big[H\,,\,\pi_a\Big]\,. \tag{3.27}$$

In order to perform a perturbative expansion oh H, ϕ_a and π_a must be split into a background part in addition to the fluctuation, assuming that the background is a c-number:

$$\phi_a(\tau,\mathbf{x}) = \bar{\phi}_a + \delta\phi_a(\tau,\mathbf{x})\,,\qquad \pi_a(\tau,\mathbf{x}) = \bar{\pi}_a + \delta\pi_a(\tau,\mathbf{x})\,. \tag{3.28}$$

Given that all the background values are c-numbers, the perturbations have the same algebra of (3.26):

$$\Big[\delta\phi_a(\tau,\mathbf{x}),\delta\pi_b(\tau,\mathbf{y})\Big] = i\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,, \tag{3.29}$$

and all the other commutators are identically null. The expansion of the full Hamiltonian H about the background then results in the following decomposition:

$$H = H[\bar{\phi}, \bar{\pi}] + \delta H[\tau; \delta\phi, \delta\pi] = H_B + H_0 + H_{int} \,. \tag{3.30}$$

Here, $H_B \equiv H[\bar{\phi}, \bar{\pi}]$ is the part of the Hamiltonian constructed out of the classical background fields. The term $\delta H[\tau; \delta\phi, \delta\pi]$ represents the Hamiltonian for the perturbations. As explained in [58], δH starts at the second order in fluctuations. It is therefore further split in two distinct parts: the free perturbation Hamiltonian H_0, containing quadratic terms in $\delta\phi_a$ and $\delta\pi_a$, and the interaction Hamiltonian H_{int}. The latter collects all the terms of higher orders and encodes all the types of interaction between the perturbations. Finally, it should be noted that the time-dependence of δH has been inherited from the background values $\bar{\phi}$ and $\bar{\pi}$ [58]. This dependence is the sign that we are dealing with quantized fields on a time-evolving background.

The perturbed Hamiltonian δH determines the equations of motion of $\delta\phi_a$ and $\delta\pi_a$:

$$\delta\phi_a' = i\Big[\delta H \,,\, \delta\phi_a\Big] \,, \qquad \delta\pi_a' = i\Big[\delta H \,,\, \delta\pi_a\Big] \,. \tag{3.31}$$

The solutions for such equations are therefore established by

$$\delta\phi_a'(\tau, \mathbf{x}) = U^{-1}(\tau, \tau_0)\delta\phi_a(\tau_0, \mathbf{x})U(\tau, \tau_0) \,, \quad \delta\pi_a'(\tau, \mathbf{x}) = U^{-1}(\tau, \tau_0)\delta\pi_a(\tau_0, \mathbf{x})U(\tau, \tau_0) \,, \tag{3.32}$$

where the unitary operator U satisfies

$$\frac{d}{d\tau}U(\tau, \tau_0) = -i\delta H[\tau; \delta\phi, \delta\pi]U(\tau, \tau_0) \,, \tag{3.33}$$

with initial condition $U(\tau_0, \tau_0) = \mathbb{I}$.

The non-quadratic part of δH yields non-linear equation of motions for the perturbations, which normally cannot be solved. In order to bypass this problem, the perturbative Quantum Field Theory provides a scheme to deal with perturbation theory, which relies on the interaction picture.

3.2.4 Time evolution in the interaction picture

As explained in [52], the free Hamiltonian H_0 determines the leading time-evolution of the field operators. Namely,

$$\delta\phi_a^{I\,\prime} = i\Big[H_0[\delta\phi_a^I(\tau), \delta\pi_a^I(\tau); \tau] \,,\, \delta\phi_a^I(\tau)\Big] \,, \tag{3.34}$$

and simirarly for $\delta\pi_a^I$. When considering the states, $|\chi_0\rangle$ indicates a generic state at the initial time τ_0 which evolves in the interaction picture according to

$$|\chi^I(\tau)\rangle = U_0^{-1}(\tau, \tau_0)U(\tau, \tau_0)|\chi_0\rangle \equiv F(\tau, \tau_0)|\chi_0\rangle \,. \tag{3.35}$$

We are interested in deriving an expession for the unitary operator $F(\tau, \tau_0)$. For this purpose, it is first necessary to specify the initial condition on the fields in the interaction picture. We thus define the fields $\delta\phi_a^I$ and $\delta\pi_a^I$ such that they coincide with the fields of the full theory at the initial time τ_0:

$$\delta\phi_a^I(\tau_0, \mathbf{x}) = \delta\phi_a(\tau_0, \mathbf{x}) \quad , \quad \delta\pi_a^I(\tau_0, \mathbf{x}) = \delta\pi_a(\tau_0, \mathbf{x}) \,. \tag{3.36}$$

In many inflationary applications, these last two conditions are equivalent to imposition of the Bunch-Davies vacuum: at very early times ($\tau_0 = -\infty$), when the modes are deep inside the horizon, the fields do not perceive the curved space and their behaviour is the one of an harmonic oscillator. Because H_0 is quadratic, the interaction picture operators are free fields satisfying linear wave equations. They are therefore in the form

$$\delta\phi^I(\tau, \mathbf{x}) = \frac{1}{(2\pi)^3}\int d^3p \left[a_I(\mathbf{p})u(\tau, p)e^{i\mathbf{p}\cdot\mathbf{x}} + a_I^\dagger(\mathbf{p})u^*(\tau, p)e^{-i\mathbf{p}\cdot\mathbf{x}}\right] \,, \tag{3.37}$$

where $u(\tau, k)$ is the solution of the equation of motion obtained from varying the second order action. The Bunch-Davies condition is $a_I(\mathbf{p})|0\rangle = 0$.

Turning our attention to the evolution equation (3.34), we are able to choose the time at which $H_0[\delta\phi_a^I(\tau), \delta\pi_a^I(\tau); \tau]$ is evaluated. This is a consequence of the fact that H_0 commutes with itself. It is thus possible to carry out the following replacement,

$$H_0[\delta\phi_a^I(\tau), \delta\pi_a^I(\tau); \tau] \quad \rightarrow \quad H_0[\delta\phi_a(\tau_0), \delta\pi_a(\tau_0); \tau]\,, \tag{3.38}$$

where the relation (3.36) has been used. The time-dependence of H_0 in (3.38) is due to the background quantities.

At this point, the solution to (3.34) is given by the unitary operator U_0:

$$\delta\phi_a^{I\,\prime} \equiv i\Big[H_0\,, \delta\phi_a^I\Big] \quad \rightarrow \quad \delta\phi_a^I(\tau) = U_0^{-1}(\tau, \tau_0)\delta\phi_a(\tau_0)U_0(\tau, \tau_0)\,, \tag{3.39}$$

with U_0 satisfying the equation

$$\frac{d}{d\tau}U_0(\tau, \tau_0) = -iH_0[\delta\phi_a(\tau_0), \delta\pi_a(\tau_0); \tau]U_0(\tau, \tau_0)\,, \tag{3.40}$$

with inital condition $U_0(\tau_0, \tau_0) = \mathbb{I}$. As for the unitary operator $F(\tau, \tau_0)$, it is possible to prove [52] that it satisfies

$$\frac{d}{d\tau}F(\tau, \tau_0) = -iH_{int}[\delta\phi_a^I(\tau), \delta\pi_a^I(\tau); \tau]F(\tau, \tau_0)\,, \tag{3.41}$$

with $F(\tau_0, \tau_0) = 1$. The above equation has the following solution,

$$\begin{aligned} &F(\tau, \tau_0) = T\exp\Big[-i\int_{\tau_0}^{\tau} d\tau H_{int}^I(\tau)\Big]\,, \\ &F(\tau_0, \tau_0) = \mathbb{I}\,. \end{aligned} \tag{3.42}$$

Turning our attention to the expectation value (3.23), we first evolve back in time the Heisenberg operator $Q(\tau)$ through (3.32) and then we express Q in the interaction picture with (3.37):

$$\begin{aligned} \langle\Omega|Q(\tau)|\Omega\rangle =& \langle\Omega|U^{-1}(\tau, \tau_0)Q(\tau_0)U(\tau, \tau_0)|\Omega\rangle \\ =& \langle\Omega|U^{-1}(\tau, \tau_0)U_0(\tau, \tau_0)Q^I(\tau)U_0^{-1}(\tau, \tau_0)U(\tau, \tau_0)|\Omega\rangle \\ =& \langle\Omega|F^{-1}(\tau, \tau_0)Q^I(\tau)F(\tau, \tau_0)|\Omega\rangle \\ =& \langle\Omega|\left[Te^{-i\int_{\tau_0}^{\tau} d\tau' H_{int}^I(\tau')}\right]^{-1} Q^I(\tau)\left[Te^{-i\int_{\tau_0}^{\tau} d\tau' H_{int}^I(\tau')}\right]|\Omega\rangle\,. \end{aligned} \tag{3.43}$$

Here, we have used also (3.35) and (3.42). The presence of the small imaginary terms $i\epsilon$ in the contour of integration of (3.24), however, must be justified. It will be demonstrated that such a device is equivalent to turning off the interaction H_{int} in the asymptotic past.

3.2.5 Projection onto the interacting vacuum: the $i\epsilon$ prescription

The following argument is given in [56], and in [52] in the context of the scattering theory. The action of the unitary operator $U(\tau, \tau_0)$ defined in (3.33) on the vacuum state of the free theory $|0\rangle$ ca be expressed as follows:

$$U(\tau, \tau_0)|0\rangle = e^{-iH(\tau-\tau_0)}|0\rangle. \tag{3.44}$$

The vacuum of the free theory $|0\rangle$ is then expanded using a complete set of energy eigenstates of the interacting theory; for simplicity, we consider the discrete set $\big\{|n\rangle\big\}_{n=0} = \big\{|\Omega\rangle\,, \big\{|m\rangle\big\}_{m=1}\big\}$. The action of $U_0(\tau, \tau_0)$ results thus in

$$e^{-iH(\tau-\tau_0)}|0\rangle = \sum_n e^{-E_n(\tau-\tau_0)}|n\rangle\langle n|0\rangle = e^{-iE_\Omega(\tau-\tau_0)}|\Omega\rangle\langle\Omega|0\rangle + \sum_m e^{-iE_m(\tau-\tau_0)}|m\rangle\langle m|0\rangle\,, \tag{3.45}$$

where $E_\Omega = \langle\Omega|H|\Omega\rangle$ is the energy of the ground state of the interacting theory. The strategy adopted to turn off the interaction in the far past consists in rotating the time coordinate τ by a small angle ϵ in the complex plane:

$$\tau \to \tilde{\tau} = \tau(1 - i\epsilon). \tag{3.46}$$

Considering the limit $\tau_0 \to -\infty$, the result of such a rotation is that the $i\epsilon$ factor damps all the complex exponentials. Because E_Ω is the lowest value among all the energy eigenvalues, the damping effect is slower for the ground state. The above equation can therefore be solved as

$$\lim_{\substack{\tau_0\to-\infty\\ \epsilon\to 0}} e^{-iH(\tilde{\tau}-\tilde{\tau}_0)}|\Omega\rangle = \lim_{\substack{\tau_0\to-\infty\\ \epsilon\to 0}} \frac{e^{-iH(\tilde{\tau}-\tilde{\tau}_0)}|0\rangle}{\langle\Omega|0\rangle}\,. \tag{3.47}$$

By multiplying both the sides of the above equation by U_0^{-1}, we get

$$\lim_{\substack{\tau_0\to-\infty\\ \epsilon\to 0}} F(\tilde{\tau},\tilde{\tau}_0)|\Omega\rangle = \lim_{\substack{\tau_0\to-\infty\\ \epsilon\to 0}} \frac{F(\tilde{\tau},\tilde{\tau}_0)|0\rangle}{\langle\Omega|0\rangle}\,. \tag{3.48}$$

From now on the double limit $\tau_0 \to -\infty$, $\epsilon \to 0$ is implied in all future in-in formulas. The equations (3.48) and (3.43) lead to the following expresion for the expectation value of $Q(\tau)$:

$$\langle\Omega|Q(\tau)|\Omega\rangle = \frac{\langle 0|\left[Te^{-i\int_{\tilde{\tau}_0}^{\tilde{\tau}} d\tau' H^I_{int}(\tau')}\right]^{-1} Q^I(\tau)\left[Te^{-i\int_{\tilde{\tau}_0}^{\tilde{\tau}} d\tau' H^I_{int}(\tau')}\right]|0\rangle}{|\langle\Omega|0\rangle|^2}\,. \tag{3.49}$$

If $Q(\tau) = \mathbb{I}$, we can then write $|\langle|\Omega|0\rangle|^2$ as

$$|\langle\Omega|0\rangle|^2 = \frac{\langle 0|\left[Te^{-i\int_{\tilde{\tau}_0}^{\tilde{\tau}} d\tau' H^I_{int}(\tau')}\right]^{-1} Q^I(\tau)\left[Te^{-i\int_{\tilde{\tau}_0}^{\tilde{\tau}} d\tau' H^I_{int}(\tau')}\right]|0\rangle}{\langle\Omega|\Omega\rangle} = 1\,, \tag{3.50}$$

where the last equality holds if we require $|\Omega\rangle$ and $|0\rangle$ to be normalized states.

The steps discussed so far have justified the master formula (3.24). A final subtlety concerning the contour of integration should be considered: in writing explicitely the inverse operator F^{-1}, such a procedure entails not only the substitution of the T symbols with the anti-time ordering symbol $\bar{T}$, but also the sign flipping of all the imaginary units. The integration contour therefore goes from $-\infty(1 - i\epsilon)$ to the real value τ_e where the correlation function is evaluated, and then goes back to $-\infty(1 + i\epsilon)$. This contour does not close.

3.2.6 Perturbative expansion and Wick contractions

In principle, the master formula (3.24) allows for the computation of cosmological correlators to every order in H_{int}. According to the scattering theory [52], the order of the expansion of the T-products in H_{int} corresponds to the number of interaction verteces involved in the computation. As explained in [58], it is also possible to establish a set of diagrammatic rules through which one is able to organize the expansion. In this thesis, however, we restrict the computation of cosmological correlators to the first order in H_{int}. For this reason, the issues concerning higher orders will not be examined. Moreover, since we are interested in cosmological bispectra, only three-point functions will be considered in the hereafter.

In order to infer the expression of $\langle Q(\tau_e)\rangle$ to first order in H_{int}, we proceed by expanding the T ordered exponential; for conciseness, we adopt the symbols $-\infty_\mp = -\infty(1 \mp i\epsilon)$:

$$\begin{aligned} &T\exp\Big(-i\int_{-\infty_-}^{\tau_e} d\tau H^I_{int}(\tau)\Big) = \\ &\mathbb{I} - i\int_{-\infty_-}^{\tau_e} d\tau\, H^I_{int}(\tau) + \frac{i^2}{2}\int_{-\infty_-}^{\tau_e} d\tau' \int_{-\infty_-}^{\tau'} d\tau''\, H^I_{int}(\tau')H^I_{int}(\tau'') + \cdots\,. \end{aligned} \tag{3.51}$$

The anti-time ordered exponential is obtained by taking the hermitian conjugated of the above series. By inserting in (3.24) the expansions in H_{int}, we get

$$\begin{aligned}\langle\Omega|Q(\tau_e)|\Omega\rangle &= \langle 0|\Big(\mathbb{I}+i\int_{-\infty_+}^{\tau_e} d\tau\, H^I_{int}(\tau)+\cdots\Big)Q^I(\tau_e)\Big(\mathbb{I}-i\int_{-\infty_-}^{\tau_e} d\tau\, H^I_{int}(\tau)+\cdots\Big)|0\rangle \\ &= \langle 0|Q^I(\tau_e)|0\rangle + i\langle 0|\int_{-\infty_+}^{\tau_e} d\tau H^I_{int}(\tau)Q^I(\tau_e) - Q^I(\tau_e)\int_{-\infty_-}^{\tau_e} d\tau H^I_{int}(\tau)|0\rangle + \cdots\,.\end{aligned} \tag{3.52}$$

The first term of the above series is identically null. The reason is that Q^I is the product of three operators in the interaction picture, *i.e* in the form (3.37), and an odd number of annihilation/creation operators results in zero between two $|0\rangle$. As for the second term, it is possible to rearrange it in a compact form using the hermiticity of Q^I and H^I_{int}:

$$\langle 0|HQ|0\rangle = \langle 0|H^\dagger Q^\dagger|0\rangle = \langle 0|(QH)^\dagger|0\rangle = \Big(\langle 0|QH|0\rangle\Big)^*\,, \tag{3.53}$$

where the star denotes the complex conjugation. The expression of the in-in master formula at a single vertex is finally given as

$$\langle\Omega|Q(\tau_e)|\Omega\rangle = 2Im\Big[\int_{-\infty_-}^{\tau_e} d\tau\,\langle 0|Q^I(\tau_e)H^I_{int}(\tau)|0\rangle\Big]\,. \tag{3.54}$$

It is possible to apply the formula (3.54) to every inflationary theory described in terms of Lagrangian formalism, once the Hamiltonian has been properly explicitated. It is known that the Legendre tranform relates the two quantities with $H=\int d^3x(\pi\dot\phi-\mathcal{L})$, with the conjugate momentum given by $\pi=\frac{\partial\mathcal{L}}{\partial\dot\phi}$. As long as the interaction terms in $\mathcal{L}$ do not contain any time-derivative coupling, the statement $H_{int}=-\int d^3x\mathcal{L}_{int}$ is correct. However, this cannot be true in all sccenarios. We will see that the case of Solid Inflation does not present any time-derivative coupling in the third-order interactions we are intersted in.

The next step to compute the three-point function is the expansion of both H^I_{int} and Q^I in terms of interaction picture fields. To evaluate the expectation value in the vacuum of the free theory, one can shift the creation/annihilation operator, following the commutation relations. We now present an efficient scheme, developed in [7], through which the computation of cosmological correlators makes contact with the formalism of the scattering theory. In order to illustrate such a scheme, it is first necessary to introduce the notation used in [7]. We rewrite the equation (3.37) in the following form,

$$\begin{aligned}\delta\phi^I(\tau,\mathbf{x}) &= \frac{1}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\Big[a_I(\mathbf{p})u(\tau,p)+a_I^\dagger(-\mathbf{p})u^*(\tau,p)\Big] \\ &= \frac{1}{(2\pi)^3}\int d^3p\, e^{i\mathbf{p}\cdot\mathbf{x}}\Big[\delta\phi^+_{I\,\mathbf{p}}(\tau)+\delta\phi^-_{I\,-\mathbf{p}}(\tau)\Big]\,.\end{aligned} \tag{3.55}$$

Since a_I and $a_I^\dagger$ obey the usual commutation relations, for $\delta\phi^\pm_I$ the algebra is given by

$$\Big[\delta\phi^+_{I\,\mathbf{p}}(\tau)\,,\,\delta\phi^-_{I\,\mathbf{p}'}(\tau')\Big] = (2\pi)^3\,u(\tau,p)u^*(\tau',p')\delta^{(3)}(\mathbf{p}-\mathbf{p}')\,, \tag{3.56}$$

and all the other commutators are null. The Bunch-Davies condition is now simply $\delta\phi^+_{I\,\mathbf{p}}|0\rangle=0$.

In the case under consideration, the formula (3.54) yields to the computation of the expectation value of products of $\delta\phi_I$ operators in the ground state of the free theory. In this context, as per the scattering theory, it is useful to define the normal ordering operation [52], which consists in moving all the creation operators to the left of all the annihilation operators. For conciseness, the subscript I will be omitted from now on; the normal ordering operation results therefore in

$$N[\delta\phi_1\cdots\delta\phi_n]\equiv\delta\phi_1^-\delta\phi_2^-\cdots\delta\phi_n^+\,. \tag{3.57}$$

It is then possible to express a product of interaction picture operators in the form of normal ordered products. As explained in [7], we can in fact define consistently the following contraction rule:

$$\delta\overbrace{\phi_1\delta\phi}_2 \equiv \left[\delta\phi_1^+ \,,\, \delta\phi_2^-\right] . \tag{3.58}$$

In the remaining of this thesis, the above contraction will be named "Wick contraction". As per the usual Wick theorem in QFT [52], one can prove by induction that a generic product of operators is rearranged as follows,

$$\delta\phi_1\delta\phi_2\cdots\delta\phi_n = N[\delta\phi_1\delta\phi_2\cdots\delta\phi_n] + \sum_{1\ \text{contr.}} N[\delta\phi_1\cdots\delta\phi_n] + \cdots + \sum_{n\ \text{contr.}} N[\delta\phi_1\cdots\delta\phi_n] \,, \tag{3.59}$$

where the subscript "j contr." indicates that the relative summation has to be understood as the sum of all the possible terms involving j contractions in the product. Once evaluated in $|0\rangle$, the surviving terms are the last ones in the above expression, which present all the operators contracted:

$$\langle 0|\delta\phi_1\delta\phi_2\cdots\delta\phi_n|0\rangle = \sum_{n\ \text{contr.}} N[\delta\phi_1\cdots\delta\phi_n] \,. \tag{3.60}$$

It should be noted that the sum of all possible contractions contains both "connected" and "disconnected" pieces. The "connected" pieces are generated when each operator of Q^I is contracted with an operator of H^I_{int}; other types of contractions generate "disconnected" pieces. As for the reasons explained in [58], the "disconnected" pieces must be ignored.

3.3 Non-Gaussianities in single-field slow-roll inflation

As stated in the introduction of this chapter, a non-vanishing bispectra involving the curvature perturbation ζ can be computed within the single-field slow-roll model [2, 48]. These correlations originates mainly by the coupling with gravity, being the latter intrinsically non-linear, while the constribution coming from the self-interacting term $V'''(\varphi)(\delta\varphi)^3 \equiv \lambda/3!\delta\varphi^3$ results sub-dominant, as a consequence of the flatness of the potential.

We will now illustrate the computation, ascribed to Maldacena [48], of the bispectrum $B_\zeta(k_1, k_2, k_3)$ inferred from the single-field slow-roll model of inflation. Such a result has been obtained by the application of the in-in formalism. A detailed presentation of the subject would be extremely lengthy, so that only the main steps of the computation will be shown. The purpose of the present section is dual. Firstly, it will provide a concise report of the method that will be extensively applied in Chapter 5 to compute the bispectra of Solid Inflation. Secondly, the resulting bispectrum will confirm through exact formulas the suppressed non-Gaussian signal predicted by the single-field slow-roll model.

1. The theory of single-scalar slow-roll inflation is encoded in the full action, consisting of the Einstein-Hilbert term along with the Lagrangian of the inflaton minimally coupled to gravity:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\Big[\frac{M_p^2}{2}R - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\Big] \\ &= \int dtd^3xN\sqrt{h}\Big\{\frac{M_p^2}{2}\Big[{}^{(3)}R + \frac{1}{N^2}\left(E^{ij}E_{ij} - E^2\right)\Big] - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)\Big\} . \end{aligned} \tag{3.61}$$

In the second line the metric has been parametrized in the ADM formalism. The meaning of the symbols adopted has been previously explained in sections 2.2.1 and 4.4. In the following, ϵ and η correspond to

$$\epsilon = -\frac{\dot{H}}{H^2} = \epsilon_V \,, \quad \eta = \frac{\dot{\epsilon}}{\epsilon H} = 4\epsilon_V - 2\eta_V \,. \tag{3.62}$$

2. The gauge choice is the Uniform Density gauge (2.136), where the value of the scalar field fluctuation is set to $\delta\varphi = 0$. According to the conventions (2.38)-(2.39) of section 2.2.1, the constraint equations for $N = 1 + \phi$ and $N_i = a\partial_i B$ are then solved to first order in fluctuations. In Fourier space, they result in

$$\phi_{\mathbf{k}} = \frac{\dot{\zeta}_{\mathbf{k}}}{H}\,, \quad B_{\mathbf{k}} = -\frac{\zeta_{\mathbf{k}}}{H} - \frac{a^2\epsilon\dot{\zeta}_{\mathbf{k}}}{k^2}\,. \tag{3.63}$$

3. The substitution of the above expressions in the expanded Lagrangian results in the following second order action in ζ:

$$S_2 = M_p^2 \int dt \frac{d^3k}{(2\pi)^3}\, \epsilon \left(a^3\dot{\zeta}_{\mathbf{k}}^2 - ak^2\zeta_{\mathbf{k}}^2\right)\,. \tag{3.64}$$

As explained in section 3.2.4, the solution of the equation of motion (EOM) that originates from the minimization of S_2 must be identified with the field in the interaction picture. We report below the quantization of $\zeta^I_{\mathbf{k}}$ in conjunction with the solution of the equation of motion to the lowest order in slow-roll [28]:

$$\zeta^I_{\mathbf{k}} = u(\tau,k)a_I(\mathbf{k}) + u^*(\tau,k)a_I^\dagger(-\mathbf{k}) = \zeta^+_{I\,\mathbf{k}}(\tau) + \zeta^-_{I\,\mathbf{k}}(\tau)\,, \tag{3.65}$$

$$u(\tau,k) = \frac{iH}{M_P\sqrt{4\epsilon k^3}}(1 + ik\tau)e^{-ik\tau}\,, \tag{3.66}$$

where

$$\zeta^+_{I\,\mathbf{k}}|0\rangle = 0\,. \tag{3.67}$$

On a side note, the $u(\tau,k)$ generates a scale invariant power spectrum, as such a mode function is the solution of the EOM to the leading order in the slow-roll parameters. As for the third order action in ζ, it results in

$$\begin{aligned} S_3 = \int dt d^3x \Big[& a^3\epsilon^2\zeta\dot{\zeta}^2 + a\epsilon^2\zeta(\partial\zeta)^2 - 2a\epsilon\dot{\zeta}\partial_i\zeta\partial_i\chi + \partial_t\Big(-\frac{\epsilon\eta}{2}a^3\zeta^2\dot{\zeta}\Big) \\ & + \frac{a^3\epsilon}{2}\dot{\eta}\zeta^2\dot{\zeta} + \frac{\epsilon}{2a}\partial_i\partial_i\chi\partial^2\chi + \frac{\epsilon}{4a}\partial^2\zeta(\partial_i\chi)^2\Big], \end{aligned} \tag{3.68}$$

where χ is defined by

$$\partial^2\chi \equiv a^2\epsilon\dot{\zeta}\,. \tag{3.69}$$

Given the above definition of χ, one can readly realize that the second line in S_3 is suppressed by the slow-roll parameters in respect to the first line. Therefore it will be not considered in the computation of the bispectrum.

4. Once the interaction picture Hamiltonian has been obtained, it is possible to use (3.54) to compute the three-point function in Fourier space:

$$\langle\zeta_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\rangle = 2Im\Big[\int_{-\infty_-}^{\tau_e} d\tau\, a(\tau)\langle 0\zeta^I_{\mathbf{k_1}}(\tau_e)\zeta^I_{\mathbf{k_2}}(\tau_e)\zeta^I_{\mathbf{k_3}}(\tau_e)H_3^I(\tau)|0\rangle\Big]\,. \tag{3.70}$$

By following the methodology exposed in section 3.2.6, the leading three-point correlator in slow-roll will results in

$$\begin{aligned} \langle\zeta_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\rangle = {} & (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\left(\frac{H}{M_p}\right)^4 \frac{1}{4\epsilon^2}\frac{1}{(k_1k_2k_3)^3} \\ & \Big[\frac{\eta-\epsilon}{8}\sum_i k_i^3 + \frac{\epsilon}{8}\Big(\sum_{i\neq j} k_ik_j^2 + \frac{8}{K}\sum_{i<j} k_i^2k_j^2\Big)\Big]\,, \end{aligned} \tag{3.71}$$

where $K = k_1 + k_2 + k_3$.

The shape of the above primordial bispectrum can be accurately represented as a superposition of the local and equilateral shape [45], namely

$$S^{s.f.} \simeq (6\epsilon_V - 2\eta_V) S^{loc}(k_1, k_2, k_3) + \frac{5}{3}\epsilon_V S^{eq}(k_1, k_2, k_3)\,. \tag{3.72}$$

It should be also noticed that the combination appearing in (3.72) equates to the relation (2.150) between the slow-roll parameters and the scalar spectral index n_s:

$$6\epsilon_V - 2\eta_V = 1 - n_s\,. \tag{3.73}$$

Moreover, as reported in [45], the above shape overlaps with the local shape with a high value of the 3D Cosine (99.7%). Since S^{loc} peaks in the squeezed limit, the dominant signal of $S^{s.f.}$ will be given in such a configuration. The single-field slow-roll model, therefore, predicts [5]

$$B_\Phi^{s.f}(k_1 \to, k_2, k_3 = k_2) = \frac{5}{3}(1 - n_s) P_\Phi(k_1) P_\Phi(k_2)\,. \tag{3.74}$$

It is thus possible to conclude that the amplitude f_{NL} is highly suppressed:

$$f_{NL} = \mathcal{O}(1 - n_s) = \mathcal{O}(\epsilon_V, \eta_V)\,. \tag{3.75}$$

3.4 *Planck* constraints on primordial non-Gaussianities

The outcomes from the updated analysis of the *Plank* data, reported in [5], represent a great step forward in the observation of non-Gaussian signals in the CMB. In respect to the previous release on the same subject [3], by combining the map of both the CMB temperature and the polarization anisotropies, a substantial improvement has been achieved in putting constraints on primordial non-Gaussianities. With respect to the Local, Equiateral and Orthgonal forms, the new constraints on the amplitudes are

$$f_{NL}^{local} = 0.8 \pm 5.0 \quad (68\%\, CL)\,, \tag{3.76}$$

$$f_{NL}^{equil} = -4 \pm 43 \quad (68\%\, CL)\,, \tag{3.77}$$

$$f_{NL}^{ortho} = -26 \pm 21 \quad (68\%\, CL)\,. \tag{3.78}$$

The standard single-field slow-roll inflation has not been ruled out: its predicted deviation from Gaussianity is (almost) of the local type and highly suppressed ($f_{NL}^{s.g.} = \mathcal{O}(\epsilon, \eta)$). The observational value of f_{NL}^{local} is therefore in agreement with such predictions.

Chapter 4

Solid inflation

In this chapter we will discuss the dynamics of the cosmological perturbations in the so-called *Solid Inflation* model. Such a model was first proposed in [35] under the name of *Elastic Inflation.* The proposal of using in cosmology an effective field theory of a solid was also discussed in [30] and then revisited in detail in [32], where the computation of the primordial scalar bispectrum was also taken into account. The model was named *Solid Inflation* in the last reference and our review is based on it. Within this scenario, which is very different with respect to the standard models of inflation, the energy density of the Universe is dominated by a set of three scalar fields $\{\Phi^I(y, \mathbf{x})\}$. Cosmic inflation can be triggered when the above fields assume the background values

$$\bar{\Phi}^A(\mathbf{x}, t) = x^A \,. \tag{4.1}$$

Even if $\bar{\Phi}^A$ individially breaks the isotropy and homogeneity of the space, it is still possible to impose to their action proper symmetry requirements in order to make the above fields configuration compatible with an FRW background. If the above mentioned action is invariant under shift transformations and rotations of the internal fields space, space translation and rotations can be reabsorbed in the configuration (4.1). These conditions yield to the effective field theory of a homogeneous and isotropic solid [30], whose low energy effective field theory will be introduced in the next section.

4.1 The effective field theory of solids on Minkowski background

The study of a continuous medium relies on the possibility of a coarse-grained description of its microphysics. The coarse-grained description consists conceptually in a division of the volume of this medium in cells. Such cells must be big enough to contain a great number of particles, but at the same time sufficiently small to be identified as points of a continuous set of coordinates. When observing the single cell with this methodology, the details of the microphysical dynamics are lost. Nonetheless, the medium is equipped with a set of fields that define its internal coordinates within a coarse-grained description.

With regards to our case of study, the three scalars $\Phi^A(t, \mathbf{x}) \in \mathbb{R}^3$ $(A = 1, 2, 3)$ label a generic point of the solid, standing in position $\mathbf{x}$ at time t. In other words, the trajectory of the solid element is obtained by the map

$$\Phi^A \to x^i = x^i(t, \Phi^A) \,. \tag{4.2}$$

It is also supposed that each Φ^A is a smooth function, so that the inverse map $x^i \to \Phi^A(t, \mathbf{x})$ exists. Given the existence of the map (4.2) and its inverse, it is then possible to interpret the fields Φ^A as the comoving system of the medium, to which we can come back by the following spatial coordinate transformation:

$$x^i \to \tilde{x}^i = \Phi^i(t, \mathbf{x}) \,. \tag{4.3}$$

The four-velocity $u^\mu = dx^\mu/ds$ of the medium is thus naturally defined by the condition that, along the medium's four-velocity, the variations of the comoving coordinates is null; namely,

$$u^\mu \partial_\mu \Phi^A(t, \mathbf{x}) = 0 \,, \qquad u_\mu u^\mu = -1 \,. \tag{4.4}$$

Given the above relations, the configuration of the medium at rest is therefore

$$\Phi^A(t, \mathbf{x}) = x^A \,. \tag{4.5}$$

The first symmetry requirement is the homogeneity of the solid. This is equivalent to asking the invariance of its action under a shift of the internal coordinates:

$$\Phi^A \longrightarrow \Phi^A + c^A \,, \qquad \partial_\mu c^A = 0 \,. \tag{4.6}$$

This a symmetry condition under shifts forces the three scalars Φ^A to appear at least with one derivative in the action. If we restrict the discussion only to the lowest order in derivatives, the fundamental block, of which the action is constructed out, must therefore be the following matrix,

$$B^{AB} \equiv \eta^{\mu\nu} \partial_\mu \Phi^A \partial_\nu \Phi^B \,. \tag{4.7}$$

Additional symmetries may be required as well. For instance, if the medium has a crystalline structrure, the invariance under the action of certain discrete groups of rotation must be imposed. In the case under consideration, we want to recover the full isotropy, which is translated into the invariance under the tranformations

$$\Phi^A \longrightarrow R^B_A \Phi^A \,, \qquad R \in SO(3)_F \,, \tag{4.8}$$

where $SO(3)_F$ denotes that the rotation is understood in the internal space of the fields. In order to maintain the invariance under (4.6), it is necessary to construct $SO(3)_F$ invariants out of the matrix (4.7). For a 3×3 matrix, there are only three independent $SO(3)$ invariant quantities, which can be the traces of B^{AB}, its square and its cube [32]:

$$\{ [B^1] \,, [B^2] \,, [B^3] \}, \qquad [B] \equiv Tr(B) \,. \tag{4.9}$$

Alternatively, the following variables satisfy the same requirements as well:

$$X = [B] \,, \qquad Y = \frac{[B^2]}{[B]^2} \,, \qquad Z = \frac{[B^3]}{[B]^3} \,. \tag{4.10}$$

The low-energy effective action for such an homogeneous, isotropic solid is thus provided by

$$S_F = \int d^4x \, F(X, Y, Z) \,, \tag{4.11}$$

where F is a general function that encodes the specific physical properties of the medium. As a side note, it should be noticed that the determinant of the matrix B^{AB} could serve as appropriate variable for F as well as X, Y and Z. However, the determinant does not represent an additional independent variable, since it can be expressed as

$$det(B) = \frac{[B]^3 - 3[B][B^2] + 2[B^3]}{6} \,. \tag{4.12}$$

We will now consider the configuration of the solid at rest, which is characterized by the background value

$$\bar{\Phi}^A = x^A \,. \tag{4.13}$$

The fields Φ^A are therefore split into background values and fluctuations, namely

$$\Phi^A(t, \mathbf{x}) = x^A + \pi^A(t, \mathbf{x})\,. \tag{4.14}$$

Since we can interpret the background configuration of the scalars (4.1) as the comoving coordinates of the solid, we will stop differentiating between capital and latin letters from now on. In order to get the second order action of the fluctuations, one has to expand (4.11) to the second order in π^i around $\bar{\Phi}$. For such purpose, one has to consider the matrix (4.7) evaluated in the configuration (4.14), given by

$$B^{ij} = \delta^{ij} + \partial^i \pi^j + \partial^i \pi^j + \partial_\mu \pi^i \partial^\mu \pi^j\,. \tag{4.15}$$

The background values of X, Y and Z are therefore

$$\bar{X} = 3\,, \quad \bar{Y} = \frac{1}{3}\,, \quad \bar{Z} = \frac{1}{9}\,. \tag{4.16}$$

Furthermore, we introduce a convenient notation, which consists in denoting with subscripts the partial derivatives of F with respect to its variables, namely

$$\frac{\partial F}{\partial X} \equiv F_X\,, \quad \frac{\partial F}{\partial Y} \equiv F_Y\,, \quad \frac{\partial F}{\partial Z} \equiv F_Z\,. \tag{4.17}$$

In the following steps, the above derivatives will be understood as evaluated in the background configuration, while the value of the Lagrangian on the background will be denoted by $\bar{F}$. Given the above details, the free action of π^A will therefore result in [32]

$$\begin{aligned} S_2 = \int d^4x \bigg\{ -\frac{1}{3} F_X \bar{X} \dot{\vec{\pi}}^2 + \left[\frac{1}{3} F_X \bar{X} + \frac{6}{27}\Big(F_Y + F_Z\Big) \right] \left(\partial_i \pi^j\right)\left(\partial_i \pi^j\right) \\ + \left[\frac{2}{9} F_{XX} \bar{X}^2 + \frac{2}{27}\Big(F_Y + F_Z\Big) \right] (\partial_i \pi^i)^2 \bigg\}\,. \end{aligned} \tag{4.18}$$

It is then possible to decompose $\vec{\pi}$ into its longitudinal and transverse components

$$\pi^i = \pi^i_L + \pi^i_T\,, \qquad \varepsilon_{ijk}\partial_j \pi^k_L = 0\,, \qquad \partial_i \pi^i_T = 0\,, \tag{4.19}$$

in terms of which the free action (4.18) is written as

$$S_2 = \int d^4x \Big(-\frac{1}{3} F_X \bar{X} \Big) \left[\dot{\vec{\pi}}^2 - c_T^2 \left(\partial_i \pi^j\right)^2 - \left(c_L^2 - c_T^2\right)\left(\partial_i \pi^i\right)^2 \right]. \tag{4.20}$$

The newly introduced parameters represent the propagation speeds of the longitudinal and transverse components of $\vec{\pi}$ [32]:

$$c_L^2 = 1 + \frac{2}{3}\frac{F_{XX}\bar{X}^2}{\bar{F}\bar{X}} + \frac{8}{9}\frac{F_Y + F_Z}{F_X \bar{X}}\,, \qquad c_T^2 = 1 + \frac{2}{3}\frac{F_Y + F_Z}{F_X \bar{X}}\,. \tag{4.21}$$

We will come back to c_L and c_T when we will discuss their role in an inflationary background.

4.1.1 The energy-momentum tensor of the solid

We will now consider a generic space-time equipped with metric tensor $g_{\mu\nu}$ and we will analyze the form of the energy-momentum tensor of the solid. The inclusion of gravity is obtained by the "minimal coupling" prescription [59], which simply consists in the following replacements in (4.11):

$$\eta_{\mu\nu} \to g_{\mu\nu}(x)\,, \quad d^4x \to d^4x\sqrt{-g}\,. \tag{4.22}$$

At this point we are able to derive the energy-momentum tensor from the solid action in the standard way:

$$
\begin{aligned}
T_{\mu\nu} &= -\frac{2}{\sqrt{-g(x)}}\frac{\delta S_F}{\delta g^{\mu\nu}(x)} \\
&= -\frac{2}{\sqrt{-g(x)}}\int d^4y\left[\frac{-\sqrt{-g(y)}}{2}g_{\mu\nu}(y)\delta^{(4)}(x-y)F(y)+\sqrt{-g(y)}\frac{\partial F}{\partial B^{AB}}\frac{\delta B^{AB}}{\delta g^{\mu\nu}(x)}\right] \\
&= g_{\mu\nu}F - 2\left(\frac{\partial F}{\partial B^{AB}}\right)\partial_\mu\Phi^A\partial_\nu\Phi^B\,. \qquad (4.23)
\end{aligned}
$$

The choice of variables (4.10) then leads to

$$
\frac{\partial F}{\partial B^{AB}}(X,Y,Z) = F_X\frac{\partial X}{\partial B^{AB}} + F_Y\frac{\partial Y}{\partial B^{AB}} + F_Z\frac{\partial Z}{\partial B^{AB}}\,, \qquad (4.24)
$$

where

$$
\begin{aligned}
\frac{\partial X}{\partial B^{AB}} &= \frac{\partial B^K_K}{\partial B^{AB}} = \delta^K_A\delta^L_B\delta_{LK} = \delta_{AB}\,, & (4.25)\\
\frac{\partial Y}{\partial B^{AB}} &= \frac{\partial}{\partial B^{AB}}\left(\frac{B^K_L B^L_K}{X^2}\right) = \frac{2B_{AB}}{X^2} - 2\frac{Y}{X}\delta_{AB}\,, & (4.26)\\
\frac{\partial Z}{\partial B^{AB}} &= \frac{1}{\partial B^{AB}}\left(\frac{B^K_L B^L_J B^J_K}{X^3}\right) = 3\frac{B_{AK}B^K_B}{X^3} - 3\frac{Z}{X}\delta_{AB}\,. & (4.27)
\end{aligned}
$$

The form of the energy-momentum tensor of the solid therefore results in

$$
T_{\mu\nu} = g_{\mu\nu}F - 2\partial_\mu\Phi^A\partial_\nu\Phi^B M_{AB}\,, \qquad (4.28)
$$

where

$$
M_{AB} = \left[\left(F_X - 2F_Y\frac{Y}{X} - 3F_Z\frac{Z}{X}\right)\delta_{AB} + 2F_Y\frac{B_{AB}}{X^2} + 3F_Z\frac{B_{AC}B^C_B}{X^3}\right]\,. \qquad (4.29)
$$

From the above expression of $T_{\mu\nu}$, it is possible to identify the energy density, the pressure and the anisotropic stress. The four velocity u^μ, defined in (4.4), automatically solves the equation that characterizes the *energy frame* of the medium (see equation (2.75)):

$$
(g^{\mu\nu} + u^\mu u^\nu)u^\gamma T_{\gamma\nu} = 0\,. \qquad (4.30)
$$

The energy density, the pressure and the anisotropic stress of $T_{\mu\nu}$ can then be identified by applying the definitions (2.70). The energy-momentum tensor has therefore the imperfect fluid form reported in (2.76):

$$
\begin{aligned}
T_{\mu\nu} &= (\rho + p)\,u_\mu u_\nu + p g_{\mu\nu} + \pi_{\mu\nu}\,, & (4.31)\\
\rho &= -F\,, & (4.32)\\
p &= F - \frac{2}{3}M_{AB}\partial_\mu\Phi^A\partial^\mu\Phi^B\,, & (4.33)\\
\pi_{\mu\nu} &= \frac{2}{3}M_{AB}\partial_\rho\Phi^A\partial^\rho\Phi^B\,(u_\mu u_\nu + g_{\mu\nu}) - 2M_{AB}\partial_\mu\Phi^A\partial_\nu\Phi^B\,. & (4.34)
\end{aligned}
$$

4.2 The solid on an FRW background

As in every perturbative approach, the fields Φ^A have to be split in a background part and a perturbation part, where the background configuration $\bar{\Phi}^A$ must be compatible with the symmetries of the FRW metric. We know that spatial translations and rotations are isometries for such metric tensor and must be valid for $T_{\mu\nu}$ as well, at least at zeroth order. To achieve this condition in the model under consideration, we exploit the homogeneity and the isotropy imposed in the space of the internal coordinates of the medium.

The energy-momentum tensor (4.28) is in fact constructed to be form-invariant under shifts and rotations of the internal space:

$$T_{AB}(\{R\Phi\}) = R_A^C R_B^D T_{CD}(\{\Phi\}) \,, \tag{4.35}$$
$$T_{AB}(\{\Phi + c\}) = T_{AB}(\{\Phi\}) \,. \tag{4.36}$$

Therefore, if we choose as background configuration the solid at rest

$$\bar{\Phi}^A = x^A \,, \tag{4.37}$$

then the energy-momentum tensor sees (4.6) and (4.8) as space-time tranformations, under which it is form-invariant. This is exactly the condition that makes the solid's $T_{\mu\nu}$ compatible with a FRW background. As done in the previous section, we will stop differentiating between capital and latin letters from now on.

The fundamental block (4.7), when evaluated on the background, is simply given by

$$\bar{B}^{ij} = g^{\mu\nu}\partial_\mu x^i \partial_\nu x^j = g^{ij} = \frac{1}{a^2}\delta^{ij} \,. \tag{4.38}$$

Consequently, we have

$$\bar{X} = \frac{3}{a^2} \,, \qquad \bar{Y} = \frac{1}{3} \,, \qquad \bar{Z} = \frac{1}{9} \,. \tag{4.39}$$

The dependence of the X alone on the scale factor is justified by the fact that the three variables are constructed in a way that only one of them is sensitive to a scale tranformation. As expected, the energy-momentum tensor on the background results in a diagonal form:

$$\bar{T}_{\mu\nu} = \bar{g}_{\mu\nu}\bar{F} - 2F_X \delta_{ij} \,, \tag{4.40}$$
$$\bar{\rho} = -\bar{F} \,, \qquad \bar{p} = \bar{F} - 2\frac{F_X}{a^2} \,. \tag{4.41}$$

Since the energy density is positive, we get $\bar{F} < 0$. We are now able to give the Friedmann equations in terms of the Hubble parameter:

$$H^2 = \frac{\bar{\rho}}{3M_P^2} = \frac{-\bar{F}}{3M_P^2} \,, \qquad \dot{H} = -\frac{\bar{\rho} + \bar{p}}{2M_P^2} = \frac{F_X}{a^2 M_P^2} \,. \tag{4.42}$$

We will now discuss how the slow-roll conditions constraint the values of F_X and F_{XX}. The slow-roll parameters that will be considered are the ones defined in (1.45), namely

$$\epsilon = -\frac{\dot{H}}{H^2} \,, \quad \eta = \frac{\dot{\epsilon}}{\epsilon H} \,. \tag{4.43}$$

Even if nothing is "rolling" in this model, inflation is nonetheless characterized by the smallness of ϵ and η. By definition, we have

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{3}{a^2}\frac{F_X}{\bar{F}} \ll 1 \,, \tag{4.44}$$

which implies

$$\left(\frac{\partial \ln(F)}{\partial \ln(X)}\right)_{\bar{X},\bar{Y},\bar{Z}} \ll 1 \,. \tag{4.45}$$

In the above expression the background value of X in (4.39) has been used, while the subscript indicates that the quantity is evaluated on the background. We thus see that if we want our solid to drive near exponential inflation, we need a very weak dependence of F on the variable X. This is not surprising, as in this case we have a condition that resembles a cosmological constant dominating the energy density of the Universe. In our specific case, being $F = -\rho$, we are stating that the solid's energy should not change much if the dilation is over e^{60}, where $N = 60$ represents a sensible minimum number of e-foldings.

In order to justify the smallness of the slow-roll parameter, we can assume an approximate symmetry of the solid action under scale transformations of the internal coordinates, namely

$$\Phi^A \longrightarrow \lambda\,\Phi^A\,, \qquad \lambda \in \mathbb{R}\,. \tag{4.46}$$

Under the action of the above transformation the matrix B^{AB} scales by an overall factor λ^2

$$B^{AB} \longrightarrow B'^{AB} = g^{\mu\nu}\partial_\mu(\lambda\Phi^A)\partial_\nu(\lambda\Phi^B) = \lambda^2 B^{AB}\,. \tag{4.47}$$

Therefore one immeadiately sees that X alone scales with λ^2, while Y and Z remain unaffected by the transformation. An exact scale invariance like (4.46) would imply that F depends only on Y and Z,

$$F(X,Y,Z) \longrightarrow F(\lambda^2 X,Y,Z) = F(X,Y,Z)\,,$$

which implies

$$F = F(Y,Z)\,. \tag{4.48}$$

In addition, we have to consider the smallness of the second slow-roll parameter η. By definition, we have

$$\eta \equiv \frac{1}{H}\frac{\dot\epsilon}{\epsilon} \ll 1\,. \tag{4.49}$$

We then recast the above equation in terms of the solid's background values. It should be noticed that

$$\ln(\bar X) = \ln\left(\frac{3}{a^2}\right)\,, \tag{4.50}$$

which implies

$$\frac{d\ln(\bar X)}{dt} = -2\frac{da}{dt} = -2H\,. \tag{4.51}$$

Using equation (4.45), the parameter η results in

$$\eta = \frac{1}{H\epsilon}\left[-2H\frac{\partial}{\partial\ln(X)}\left(\frac{\partial\ln F}{\partial\ln(X)}\right)\right] = -\frac{2}{\epsilon}\left[X\frac{\partial}{\partial X}\left(\frac{XF_X}{F}\right)\right]\,, \tag{4.52}$$

therefore

$$\frac{F_{XX}X^2}{F} = -\epsilon - \frac{\epsilon\eta}{2} + 2\epsilon^2\,. \tag{4.53}$$

By taking the ratio between (4.53) and (4.43) we get the following relation:

$$\frac{F_{XX}\bar X^2}{F_X\bar X} = -1 + 2\epsilon - \frac{\eta}{2} \approx -1 + \mathcal{O}(\epsilon,\eta)\,. \tag{4.54}$$

Recalling the sound speeds of the longitudinal and transverse mode, respectively c_L^2 and c_T^2 in (4.21), we see that they are related in a quite simple way:

$$c_T^2 = \frac{3}{4}(1 + c_L^2 - \frac{2}{3}\epsilon + \frac{1}{3}\eta) \approx \frac{3}{4}(1 + c_L^2)\,, \tag{4.55}$$

where the second approximate equality is exact at zeroth order in the slow-roll parameters.

We now want to impose two natural conditions on the propagation speeds: they have to be sub-luminal and they have to be non-negative. These conditions automatically force the combination $(F_Y + F_Z)$ to fit in a small window of possible values. For both speeds to be sub-luminal, we need at least $c_T^2 < 1$,

$$c_T^2 \approx \frac{3}{4}(1 + c_L^2)\,, \tag{4.56}$$

so that

$$c_L^2 < \frac{1}{3}\,. \tag{4.57}$$

The last statement leads then to

$$c_L^2 = \frac{1}{3} + \frac{8}{9}\frac{F_Y + F_Z}{F_X \bar{X}} + \frac{1}{3}(4\epsilon - \eta) < \frac{1}{3}\,, \tag{4.58}$$

therefore

$$(F_Y + F_Z) > \frac{3}{8}|\bar{F}|(4\epsilon^2 - \epsilon\eta)\,, \tag{4.59}$$

since $F_X \bar{X} = -|\bar{F}|\epsilon$. As for the non-negativity of the propagation speeds, it fixes an upper bound for $(F_Y + F_Z)$:

$$\begin{cases} \frac{1}{3} - \frac{8}{9}\frac{F_Y+F_Z}{|\bar{F}|\epsilon} + \frac{1}{3}(4\epsilon - \eta) > 0 \\ 1 - \frac{2}{3}\frac{F_Y+F_Z}{|\bar{F}|\epsilon} > 0 \end{cases}\,, \tag{4.60}$$

which implies

$$\begin{cases} F_Y + F_Z < \frac{3}{8}\epsilon|\bar{F}| + \frac{3}{8}|\bar{F}|(4\epsilon^2 - \eta\epsilon) \\ F_Y + F_Z < \frac{3}{2}|\bar{F}|\epsilon \end{cases}\,. \tag{4.61}$$

Disregarding the corrections $\mathcal{O}(\epsilon^2, \eta\epsilon)$, the window of possible values for the combination of $(F_Y + F_Z)$ is given by

$$0 < F_Y + F_Z < \frac{3}{8}\epsilon|\bar{F}|\,. \tag{4.62}$$

According to the above condition, we see that in the full space of the parameters F_Y and F_Z there are two extreme cases:

- $|F_{Y,Z}| \sim |\bar{F}|$, which necessary implies that $F_Y = -F_Z + \mathcal{O}(\epsilon)$;
- $|F_Y| \sim |F_Z| \sim \epsilon|\bar{F}|$, which means that all the derivatives of F are slow-roll suppressed, like F_X is. The energy density and pressure of the solid are therefore dominated by a cosmological constant term and it does not depend greatly on the dynamics of the fields π^i.

We will discuss in Chapter 5 the importance of the two above cases for the computation of the bispectra.

Finally, it is very important to remark that the propagation speeds c_L^2 and c_T^2 are time-dependent functions. In order to characterize their behaviour on cosmic time scales, we introduce two new slow-roll parameters:

$$s_L \equiv \frac{\dot{c}_L}{c_L H}\,, \tag{4.63}$$

$$u \equiv \frac{\dot{c}_T}{c_T H}\,. \tag{4.64}$$

The smallness of the two above quantities is justified by the fact that both c_L and c_T depends on time only via the dependence on X of the Lagrangian F [32].

4.3 First order cosmological perturbation in Solid Inflation

We will now study the cosmological perturbation of the model of Solid inflation. In Chapter 2 we introduced the perturbation theory at linear order, which describes how the metric perturbation couples with the perturbation of the energy-momentum tensor. We will now apply this method in order to derive some relevant quantities of the model. More specifically, we will obtain

- the firts order perturbation of the energy-momentum tensor;

- the solutions of the Hamiltonian and momentum constraints in the spatially flat gauge, respectively given in the equations (2.106) and (2.107);
- the solution of the vector constraint equations (2.112);
- finally, equations (2.109) and (2.113) will provide differential equations for the gauge-invariant scalar perturbations ζ, $\mathcal{R}$, in addition to an auxiliary equation for S^i.

The choice of the spatially flat gauge is justified by the fact that, as explained in section 2.5.2, the Hamiltonian and momentum constraints acquire a simple form and therefore they are easy solvable.

We thus add the fluctuations of Φ^i to the background value (4.37):

$$\Phi^i = x^i + \pi^i . \tag{4.65}$$

In the ADM parametrization of the metric, the matrix (4.7) is expressed as

$$\begin{aligned} B^{ab} &= g^{00}\partial_0\Phi^a\partial_0\Phi^b + g^{0i}\left(\partial_i\Phi^a\partial_0\Phi^b + \partial_0\Phi^a\partial_i\Phi^b\right) + g^{ij}\partial_i\Phi^a\partial_j\Phi^b \\ &= -\frac{1}{N^2}\left(\dot{\Phi}^a - N^k\partial_k\Phi^a\right)\left(\dot{\Phi}^b - N^k\partial_k\Phi^b\right) + h^{km}\partial_k\Phi^a\partial_m\Phi^b . \end{aligned} \tag{4.66}$$

In the spatially flat gauge, the metric is given by

$$ds^2 = -(1+2\delta N)dt^2 + N_i dt dx^i + a(t)^2\left(e^{\gamma}\right)_{ij} dx^i dx^j , \tag{4.67}$$
$$\partial_i\gamma_{ij} = 0 , \qquad \gamma_{ii} = 0 , \tag{4.68}$$

and keeping only the first order perturbation of (4.66), we then get

$$\begin{aligned} B^{ij} = &-(1-2\delta N)\left(\dot{\pi}^i - N^i - N^k\partial_k\pi^i\right)\left(\dot{\pi}^j - N^j - N^k\partial_k\pi^j\right) \\ &+ \frac{1}{a^2}\left(\delta^{km} - \gamma^{km}\right)\left(\delta^i_k + \partial_k\pi^i\right)\left(\delta^j_m + \partial_m\pi^j\right) \\ &\approx \frac{1}{a^2}\left(\delta^{ij} - \gamma^{ij} + \partial_i\pi^j + \partial_j\pi^i\right) . \end{aligned} \tag{4.69}$$

The splitting of N^i into longitudinal and transverse parts is made according to (2.48), namely

$$N^i = \frac{\partial_i}{\sqrt{-\nabla^2}} N_L + N^i_T , \qquad \partial_i N^i_T = 0 , \tag{4.70}$$

and π^i will be split in the same manner,

$$\pi^i = \frac{\partial_i\pi_L}{\sqrt{-\nabla^2}} + \pi^i_T , \qquad \partial_i\pi^i_T = 0 . \tag{4.71}$$

In order to maintain the same notations of the analysis made in Chapter 2, we will use the variable ($\hat{\phi}$, B, S^i) defined in the spatially flat gauge, respectively in (2.130), (2.131) and (2.133). The relations between the above variables and the ADM variables are thus given by

$$\delta N = \hat{\phi} , \quad N_L = a(t)B , \quad N^i_T = \frac{S^i}{a(t)} . \tag{4.72}$$

4.3.1 Energy-momentum tensor of the Solid: first order perturbation

This subsection is devoted to the identification of the quantities that enter in the energy-momentum tensor perturbation $\delta T_{\mu\nu}$, such as the peculiar matter velocity v^i (2.82), the longitudinal and transverse components of the momentum density δq and δq^i (2.92), the perturations $\delta\rho$, δp and the anisotropic stress π_{ij} (2.84). We will see that the latter contains the scalar, the vector and the tensor components, *i.e.* $(\Pi^S, \Pi^V_i, \Pi^T_{ij})$. A thorough inspection reveals two remarkable features of the model of Solid Inflation:

1. non-decaying vector perturbations;
2. a mass term for gravitons.

Let us begin by considering the first order perturbation of (4.28), namely

$$\delta T^\mu_\nu = \delta^\mu_\nu \delta F - 2\delta g^{\mu\alpha}\delta^k_\alpha \delta^l_\nu \bar{M}_{kl} - 2\bar{g}^{\mu\alpha}\delta^k_\alpha \delta^l_\nu \delta M_{kl} - 2\bar{g}^{\mu\alpha}\bar{M}_{kl}\left(\partial_\alpha \pi^k \delta^l_\nu + \delta^k_\alpha \partial_\nu \pi^l\right), \tag{4.73}$$

where

$$\delta F = F_X \delta^{(1)} X + F_Y \delta^{(1)} Y + F_Z \delta^{(1)} Z\,. \tag{4.74}$$

We list below the linear order perturbations of X, Y and Z, which are computed in Appendix A.2.3:

$$\delta^{(1)} X = \frac{2}{a^2}\frac{\partial_i \partial^i \pi}{\sqrt{-\nabla^2}}\,, \tag{4.75}$$

$$\delta^{(1)} Y = \delta\left(\frac{B^{ij}B^{ij}}{X^2}\right) = 0\,, \tag{4.76}$$

$$\delta^{(1)} Z = \delta\left(\frac{B^{ij}B^{jk}B^{ki}}{X^3}\right) = 0\,. \tag{4.77}$$

The superscript $\delta^{(1)}$ will be omitted for simplicity, being the order of every perturbed variables taken for granted in the next computations.

The next step consists in evaluating the perturbed value of the matrix M^{AB} that appears in (4.29) around the background. We will use the background values (4.39) and the above equation (4.75), obtaining

$$\begin{aligned}\delta M_{ij} = &\left[\delta F_X - \frac{2a^2}{9}\delta F_Y - \frac{3a^2}{27}\delta F_Z - 2F_Y\delta\left(\frac{Y}{X}\right) - 3F_Z\delta\left(\frac{Z}{X}\right)\right]\delta_{ij} \\ &+2\delta F_Y\left(\frac{\bar{B}_{ij}}{\bar{X}^2}\right) + +2F_Y\delta\left(\frac{B_{ij}}{X^2}\right) + 3\delta F_Z\left(\frac{\bar{B}_{ik}\bar{B}_{kj}}{\bar{X}^3}\right) + 3F_Z\delta\left(\frac{B_{ik}B_{kj}}{X^3}\right).\end{aligned}$$

By using $\delta F_X = F_{XX}\delta X$, the previous expression becomes

$$\delta M_{ij} = \left[F_{XX} - \frac{2}{3\bar{X}^2}\left(F_Y + F_Z\right)\right]\delta X \delta_{ij} + \frac{2}{\bar{X}^2}\left(F_Y + F_Z\right)\delta B_{ij}\,. \tag{4.78}$$

The peculiar velocity of the matter fields v^i is obtained by taking the perturbed equation (4.4), which yields

$$\delta u^\mu \delta^i_\mu = -\bar{u}^\mu \partial_\mu \pi^i \tag{4.79}$$

$$v^i = -a\dot{\pi}^i\,. \tag{4.80}$$

We are thus able to provide δq and δq_i:

$$\delta q = a(\bar{\rho} + \bar{p})(v + B) = a\left(-\frac{2F_X}{a^2}\right)\frac{1}{\sqrt{-\nabla^2}}(B - a\dot{\pi}_L)\,, \tag{4.81}$$

$$\delta q_i = a(\bar{\rho} + \bar{p})(v^i_T - S^i) = -a\left(-\frac{2F_X}{a^2}\right)(S_i + a\dot{\pi}_{T,i})\,. \tag{4.82}$$

We can now compute explicitely the first order perturbation (4.73) of the energy-momentum tensor of the solid by gathering all the results obtained above. Its components are listed below:

$$\delta T^0_0 = F_X \delta X\,, \tag{4.83}$$

$$\delta T^0_i = a\left(-\frac{2F_X}{a^2}\right)(B_i - a\dot{\pi}_i)\,, \tag{4.84}$$

$$\delta T^i_j = \delta^i_j F_X \delta X - 2\delta g^i_j F_X - \frac{2}{a^2}F_X\left(\partial_i \pi^j + \partial_j \pi^i\right) - \frac{2}{a^2}\delta M^i_j\,. \tag{4.85}$$

From equation (4.83) we can read the linear perturbation of the energy density in the flat gauge, *i.e.*

$$\begin{aligned}\delta\rho_{flat} &= -F_X\delta X = -\frac{2}{a^2}F_X\frac{\nabla^2}{\sqrt{-\nabla^2}}\pi_L \\ &= -\frac{2}{3}\epsilon\bar{F}\frac{\nabla^2}{\sqrt{-\nabla^2}}\pi_L\,. \end{aligned} \tag{4.86}$$

It is important to stress the extreme insensitivity of the energy of the solid to volume expansions as represented by the equation (4.45). This feature is inherited at perturbative level, where the local fluctuations in the energy are produced by small compressions and dilations in the cells of the solid. This argument is also valid for local isotropic pressure perturbations.

By taking the trace of the spatial components (4.85) of the perturbed energy-momentum tensor, it is possible to infer the linear perturbation of the pressure:

$$\delta p_{flat} = \frac{1}{3}\delta T^i_i = \frac{1}{3}\left(F_X - 2XF_{XX}\right)\delta X = \frac{2}{9}\epsilon\bar{F}(-1+4\epsilon-\eta)\frac{\nabla^2\pi_L}{\sqrt{-\nabla^2}}\,, \tag{4.87}$$

where in the second line we have exploited the relations (4.45) and (4.54). At first order in slow-roll, it is simply

$$\delta p_{flat} \approx -\frac{2}{9}\epsilon\bar{F}\frac{\nabla^2\pi_L}{\sqrt{-\nabla^2}}\,. \tag{4.88}$$

By performing the subtraction $\delta T^i_j - \delta p\delta^i_j$, we will obtain the anisotropic stress, which, according to the decomposition in (2.84), is split into the scalar, vector and tensor parts reported below:

$$\left(\partial_i\partial_j - \delta_{ij}\frac{1}{3}\nabla^2\right)\Pi^S = -\left(\frac{\partial_i\partial_j}{\sqrt{-\nabla^2}} - \delta_{ij}\frac{1}{3}\frac{\nabla^2}{\sqrt{-\nabla^2}}\right)\left(\frac{4F_X}{a^2} + \frac{8}{9}(F_Y+F_Z)\right)\pi_L\,, \tag{4.89}$$

$$\partial_{(i}\Pi^V_{j)} = -\left(2\frac{F_X}{a^2} + \frac{4}{9}(F_Y+F_Z)\right)\left(\partial_i\pi_{T,\,j} + \partial_j\pi_{T,\,i}\right)\,, \tag{4.90}$$

$$\Pi^T_{ij} = \left(2\frac{F_X}{a^2} + \frac{4}{9}(F_Y+F_Z)\right)\gamma_{ij}\,. \tag{4.91}$$

In Fourier space we then get

$$\tilde{\Pi}^S = -\frac{4F_X\bar{X}}{3k}\left(1+\frac{2}{3}\frac{F_Y+F_Z}{F_X\bar{X}}\right)\pi_L(t,\mathbf{k}) = -\frac{4}{3k}\epsilon\bar{F}c_T^2\pi_L(t,\mathbf{k})\,, \tag{4.92}$$

$$\tilde{\Pi}^{V,\,i} = -\frac{2}{3}\left[F_X\bar{X} + \frac{2}{3}(F_Y+F_Z)\right]\pi^i_T(t,\mathbf{k}) = -\frac{2}{3}\epsilon\bar{F}c_T^2\pi^i_T(t,\mathbf{k})\,, \tag{4.93}$$

$$\tilde{\Pi}^T_{ij} = \frac{2}{3}\left[F_X\bar{X} + \frac{2}{3}(F_Y+F_Z)\right]\gamma_{ij}(t,\mathbf{k}) = \frac{2}{3}\epsilon\bar{F}c_T^2\gamma_{ij}(t,\mathbf{k})\,. \tag{4.94}$$

We can therefore summarize the consequences of the last two terms as:

- a non-vanishing $\Pi^{V\,,i}$ implies that vector perturbations do not decay during inflation, as it is illustrated in the equation (2.116);
- the term Π^T_{ij}, once been inserted in the equation (2.120), represents a mass term for gravitational waves.

4.3.2 Solutions to the constraint equations

We now have all the elements we need to solve the Hamiltonian and momentum constraints in the spatially flat gauge, as well as the vector constraint equation.

Recalling the system (2.134), it can be now be written as

$$\begin{cases} 3H^2\hat{\phi} + H\frac{\nabla^2}{a^2}\left(\frac{B}{\sqrt{-\nabla^2}}\right) = \frac{\dot{H}}{\sqrt{-\nabla^2}}\nabla^2\pi_L \\ H\hat{\phi} = \frac{a\dot{H}}{\sqrt{-\nabla^2}}(B - a\dot{\pi}_L) \end{cases}, \tag{4.95}$$

which in Fourier space has the following solutions,

$$\tilde{\phi}(t,\mathbf{k}) = -\frac{a^2\dot{H}}{kH}\frac{\dot{\pi}_L - \frac{\dot{H}}{H}\pi_L}{1-\frac{3a^2\dot{H}}{k^2}}, \qquad \tilde{B}(t,\mathbf{k}) = a\frac{\frac{-3a^2\dot{H}}{k^2}\dot{\pi}_L + \frac{\dot{H}}{H}\pi_L}{1-\frac{3a^2\dot{H}}{k^2}}. \tag{4.96}$$

As for vector perturbations, the constraint (2.112) in Fourier space gives

$$\frac{k^2\tilde{S}_i}{a} = 4a\dot{H}(\tilde{S}_i + a\dot{\pi}_{i,T}), \tag{4.97}$$

which implies

$$-\tilde{S}_i(t,\mathbf{k}) = \frac{a\dot{\pi}_{i,T}}{1-\frac{k^2}{4a^2\dot{H}}}. \tag{4.98}$$

By combining the above vector constraint with the conservation equation (2.113), they result in

$$\frac{\partial}{\partial t}\left(\frac{\tilde{S}_i}{a}\right) + 3H\left(\frac{\tilde{S}_i}{a}\right) = -16\pi G\tilde{\Pi}^V_i. \tag{4.99}$$

This latter will provide a useful auxiliary equation when we will search for the solution of the equation of motion of π^i_T.

We can now collect the above results and recast them in terms of the ADM variables $(\delta N, N_L, N^i_T)$ in Fourier space:

$$\delta N(t,\mathbf{k}) = -\frac{a^2\dot{H}}{kH}\frac{\dot{\pi}_L - \frac{\dot{H}}{H}\pi_L}{1-\frac{3a^2\dot{H}}{k^2}}, \tag{4.100}$$

$$N_L(t,\mathbf{k}) = \frac{\frac{-3a^2\dot{H}}{k^2}\dot{\pi}_L + \frac{\dot{H}}{H}\pi_L}{1-\frac{3a^2\dot{H}}{k^2}}, \tag{4.101}$$

$$N^i_T(t,\mathbf{k}) = \frac{\dot{\pi}^i_T}{1-\frac{k^2}{4a^2\dot{H}}}, \tag{4.102}$$

while the auxiliary equation (4.99) results in

$$\dot{N}^i_T(t,\mathbf{k}) + 3HN^i_T(t,\mathbf{k}) + 4\epsilon c_T^2(t)H^2\pi^i_T(t,\mathbf{k}) = 0, \tag{4.103}$$

where we have used the expression of the vector part of the anisotropic stress Π^V and the definition of the slow-roll parameter.

4.3.3 Equations for ζ and $\mathcal{R}$

We now have all the elements to give the expression of the gauge invariant scalar variables $\mathcal{R}$ and ζ in terms of matter fields. The expression of ζ in the Spatially Flat gauge is obtained by combining the equations (2.138) and (4.86):

$$\zeta = \frac{1}{3}\frac{\delta\rho_{flat}}{\bar{\rho}+\bar{p}} = \frac{1}{3}\frac{\nabla^2}{\sqrt{-\nabla^2}}\pi_L, \tag{4.104}$$

while the Fourier counterpart results in

$$\zeta_k = -\frac{k}{3}\tilde{\pi}_L. \tag{4.105}$$

As for the curvature perturbation in comoving gauge $\mathcal{R}$, recalling the definition (2.140) and the expression of δp in (4.87), we get

$$\mathcal{R} = -H\frac{\delta q_{flat}}{\bar{\rho}+\bar{p}} = \frac{aH}{\sqrt{-\nabla^2}}(a\dot{\pi}_L - B)\,. \tag{4.106}$$

The above quantity can be expressed now in terms of the matter field π_L through the solution of $\tilde{B}$ in (4.96):

$$\mathcal{R}_k = -\frac{k}{3H\epsilon}\frac{\dot{\pi}_L + \epsilon H\pi_L}{1+\frac{1}{3\epsilon}(\frac{k}{aH})^2}\,. \tag{4.107}$$

From the above equation one can easily see the relation between $\mathcal{R}_k$ and ζ_k in Solid inflation:

$$\mathcal{R}_k = \frac{1}{\epsilon H}\frac{(-\frac{k}{3}\dot{\pi}_L) + \epsilon H(\frac{-k}{3}\pi_L)}{1+\frac{1}{3\epsilon}(\frac{k}{aH})^2} = \frac{3}{H}\frac{\dot{\zeta}_k + \epsilon H\zeta_k}{\epsilon + (\frac{k}{aH})^2}\,. \tag{4.108}$$

It is possible to formulate an auxiliary equation for $\mathcal{R}_k$ that will be used in solving the equation of motion of ζ. Let us recall the Einstein equation (2.109) in spatially flat gauge. Its right hand side in Fourier space is given by

$$4\pi G\left(\delta p_k - \frac{2}{3}k^2\tilde{\Pi}^S\right) = \dot{H}c_L^2 k\pi_L(t,\mathbf{k})\,, \tag{4.109}$$

while its left hand side is obtained by the solution for $\tilde{\phi}$ of the system (4.96). The equation (2.109) is therefore expressed in terms of the field $\pi_L(t,\mathbf{k})$ as follows

$$\frac{\partial}{\partial t}\left[\frac{k}{H}\frac{\dot{\pi}_L+\epsilon H\pi_L}{1+\frac{k^2}{3\epsilon a^2H^2}}\right] + (3H-2\epsilon H)\left[\frac{k}{H}\frac{\dot{\pi}_L+\epsilon H\pi_L}{1+\frac{k^2}{3\epsilon a^2H^2}}\right] = -c_L^2\epsilon Hk\pi_L\,. \tag{4.110}$$

The above formula is quite cumbersome but it becomes simpler if expressed in terms of $\mathcal{R}_k$ and ζ_k:

$$\frac{\dot{\mathcal{R}}_k}{H} + (3-2\epsilon+\eta)\mathcal{R}_k = -3c_L^2\zeta_k\,, \tag{4.111}$$

where we have used the definition of the second slow-roll parameter $\eta \equiv \dot{\epsilon}/\epsilon H$.

This last equation, together with (4.108), forms a system of two first order differential equations of two variables. From the two equations we can provide a second order differential equation for $\mathcal{R}_k$. With some efforts, we get

$$\ddot{\mathcal{R}}_k + (3+\eta-2s_L)H\dot{\mathcal{R}}_k + \frac{1}{a^2}\left[c_L^2k^2 + a^2H^2(3\epsilon - 6s_L + 3c_L^2\epsilon)\right]\mathcal{R}_k \tag{4.112}$$

$$+\left[a^2H^2(\epsilon\eta - 2\epsilon^2 + 2s_L\eta - 4s_L\epsilon) + a^2H\frac{d}{dt}(\eta-2\epsilon)\right]\mathcal{R}_k = 0\,, \tag{4.113}$$

where we have used the slow-roll parameter defined in (4.63). In conformal time the equation results in

$$\begin{aligned}\mathcal{R}_k'' + (2+\eta-2s_L)aH\mathcal{R}_k' + \left[c_L^2k^2 + a^2H^2(3\epsilon-6s_L+3c_L^2\epsilon)\right]\mathcal{R}_k\\ +aH\left[aH(\epsilon\eta - 2\epsilon^2 + 2s_L\eta - 4s_L\epsilon) + (\eta' - 2\epsilon')\right]\mathcal{R}_k = 0\,.\end{aligned} \tag{4.114}$$

4.4 Scalar, vector and tensor perturbations: mode functions and power spectra

This section is devoted to the equations of motion, and the corresponding solutions, of the variables ζ, γ_{ij} and π_T^i. As for vector perturbations, we will report below the detailed derivation of the corresponding mode function $\pi_{T,,cl}(\tau,k)$, while the derivation of the mode functions ζ_{cl} and

γ_{cl} is reported in Appendix C.

As explained in section 3.2.1, in order to obtain the free action of scalar, vector and tensor perturbations, it is necessary to expand the full action of the system up to second order both in the matter fields and in the ADM variables. This is the approach followed in [32], which we will now summarize. The full action consists of the solid and the Einstein-Hilbert action, namely

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R + F(X,Y,Z)\right] \\ &= \int d^4x N\sqrt{h}\left\{\frac{M_P^2}{2}\left[R^{(3)} - \frac{1}{N^2}(E^{ij}E_{ij} - E^2)\right] + F(X,Y,Z)\right\}, \end{aligned} \tag{4.115}$$

where in the second line the ADM variables have been used. In the spatially flat gauge, the notation means

$$h_{ij} = a^2[e^\gamma]_{ij}, \quad h^{ij} = \frac{1}{a^2}[e^{-\gamma}]^{ij}, \quad \partial_i\gamma_{ij} = \gamma_{ii} = 0, \tag{4.116}$$

$$h = det(a^2e^\gamma) = \exp\left[Tr[\ln(a^2\mathbb{I}_3) + Tr[\gamma]]\right] = a^6, \tag{4.117}$$

$$E_{ij} = \frac{1}{2}\left[\partial_t[a^2e^\gamma]_{ij} - \nabla_iN_j - \nabla_jN_i\right], \tag{4.118}$$

$$E = E^i_i = \frac{1}{a^2}[e^{-\gamma}]^{ij}E_{ij}. \tag{4.119}$$

Here, ∇_i is the covariant derivative on spatial hypersurface (see section (3.2.1)). The Hamiltonian and momentum constraint equations, given respectively by (3.21) and (3.22), have then to be solved to linear order and their solution have to be inserted in the expanded action. At that point the non-dynamical degrees of freedom will be eliminated.

Although we did not follow the approach explained above, we were able to provide the solutions to the Hamiltonian and momentum constraints in section 4.3.2 and recast them in terms of the ADM variables:

$$\delta N(t,\mathbf{k}) = -\frac{a^2\dot{H}}{kH}\frac{\dot{\pi}_L - \frac{\dot{H}}{H}\pi_L}{1 - \frac{3a^2\dot{H}}{k^2}}, \tag{4.120}$$

$$N_L(t,\mathbf{k}) = \frac{\frac{-3a^2\dot{H}}{k^2}\dot{\pi}_L + \frac{\dot{H}}{H}\pi_L}{1 - \frac{3a^2\dot{H}}{k^2}}, \tag{4.121}$$

$$N^i_T(t,\mathbf{k}) = \frac{\dot{\pi}^i_T}{1 - \frac{k^2}{4a^2\dot{H}}}. \tag{4.122}$$

The above expressions coincide with the ones reported in [32]. Once they have been plugged into the expansion of (4.115), the final result provides the quadratic actions of γ_{ij}, π_T and π_L [32]:

$$S^{(2)} = S^{(2)}_\gamma + S^{(2)}_T + S^{(2)}_L$$

$$S^{(2)}_\gamma = \frac{1}{4}M_P^2\int dtd^3xa^3\left[\frac{1}{2}\dot{\gamma}_{ij}^2 - \frac{1}{2a^2}(\partial_k\gamma_{ij})^2 + 2\dot{H}c_T^2\gamma_{ij}^2\right], \tag{4.123}$$

$$S^{(2)}_T = M_P^2\int dt\int\frac{d^3k}{(2\pi)^3}a^3\left[\frac{\frac{k^2}{4}}{1 + \frac{k^2}{4\epsilon a^2H^2}}|\dot{\pi}^i_T|^2 + c_T^2\dot{H}k^2|\pi^i_T|^2\right], \tag{4.124}$$

$$S^{(2)}_L = M_P^2\int dt\int\frac{d^3k}{(2\pi)^3}a^3\left[\frac{\frac{k^2}{3}}{1 + \frac{k^2}{3\epsilon a^2H^2}}|\dot{\pi}_L + \epsilon H\pi_L|^2 + c_T^L\dot{H}k^2|\pi_L|^2\right]. \tag{4.125}$$

The next step is the quantization of the fields. Since we want to follow the method exposed in the Chapter 1, we have to redefine the fields in such a way that, in conformal time, they yield to the action of a harmonic oscillator with time-dependent frequency.

4.4.1 Time dependence of background quantities

In order to derive, at least approximately, the mode functions of ζ, γ_{ij} and π_T^i, it is first necessary to know the explicit time dependence of the background quantities that will appear in the equation of motion. More precisely, the dependence on the conformal time τ of a, H, c_L, c_T and of the slow-roll parameter ϵ will be provided up to the first order in slow-roll.

As explained in [32], it is convenient to consider a reference conformal time τ_c, defined as the one when the longest mode of observational relevance today (k_{min}) exits the horizon, *i.e.* τ_c is defined such that

$$|c_{L,c}k_{min}\tau_c| \simeq |c_{L,c}\tau_c a_0 H_0| = 1\,, \tag{4.126}$$

where the subscript "c" denotes that the the quantity is evaluated at the time τ_c, while a_0 and H_0 indicate the scale factor and the Hubble parameter at the present time. In [32] have been derived the following time dependence:

$$a(\tau) = a_c\left(\frac{\tau}{\tau_c}\right)^{-1-\epsilon_c} + \mathcal{O}(\epsilon^2)\,, \tag{4.127}$$

$$H(\tau) = -\frac{1+\epsilon_c}{a_c\tau_c}\left(\frac{\tau}{\tau_c}\right)^{\epsilon_c} + \mathcal{O}(\epsilon^2)\,, \tag{4.128}$$

$$\epsilon(\tau) = \epsilon_c\left(\frac{\tau}{\tau_c}\right)^{-\eta_c} + \mathcal{O}(\epsilon^3)\,. \tag{4.129}$$

The time dependence of c_L and c_T is determined by the slow-roll parameters s_L and u, respectively defined in (4.63) and in (4.64):

$$c_L(\tau) = c_{L,c}\left(\frac{\tau}{\tau_c}\right)^{-s_c} + \mathcal{O}(\epsilon^2)\,, \tag{4.130}$$

$$c_T(\tau) = c_{T,c}\left(\frac{\tau}{\tau_c}\right)^{-u_c} + \mathcal{O}(\epsilon^2)\,. \tag{4.131}$$

All the above expressions will be frequently used in solving the equations of motion in Appendix C, neglecting higher order corrections in slow-roll.

4.4.2 Scalar perturbation ζ: mode function and power spectrum

The scalar quantity will quantize is the curvature perturbation ζ. The analysis presented so far allows us to compute only $\langle\zeta\zeta\rangle$ and the power spectrum linked to it, as well as for vector and tensor perturbations. For the computation of the three point functions to be achieved, we need to go further in the Lagrangian expansion up to the third order in perturbations. This will be done in the following chapter, where the some bispectra predicted by the model of Solid inflation will be computed through the in-in formalism.

We then promote $\zeta(\tau,\mathbf{k})$ to a quantum operator by decomposing it in terms of creation/annihilation operators,

$$\zeta(\tau,\mathbf{k}) = \zeta_{cl}(\tau,k)b(\mathbf{k}) + \zeta_{cl}^*(\tau,k)b^\dagger(-\mathbf{k})\,, \tag{4.132}$$

$$\left[b(\mathbf{k}), b^\dagger(\mathbf{k}')\right] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\,. \tag{4.133}$$

The equation of motion for the mode function ζ_{cl} is derived by varying the quadratic action (4.125) with respect to π_L, recalling that $\zeta(\tau,\mathbf{k}) = -\frac{k}{3}\pi_L(\tau,\mathbf{k})$. In conformal time, the action of ζ becomes

$$S_\zeta^{(2)} = \int d\tau\frac{d^3k}{(2\pi)^3}\left[\frac{3a^2M_P^2}{1+\frac{k^2}{3\epsilon a^2H^2}}|\zeta' + \epsilon aH\zeta|^2 - 9c_L^2a^4H^2M_P^2\epsilon|\zeta|^2\right]\,. \tag{4.134}$$

We now summarize the steps necessary in order to derive the mode function ζ_{cl}.

- The action (4.134) must take the form of the action of an armonic oscillator with time dependent frequency (see (1.55)). This is achieved by the field redefinition

$$v \equiv z\zeta\,, \qquad z \equiv -\sqrt{2}\frac{\sqrt{3}aM_P}{\sqrt{1+\frac{1}{3\epsilon}\left(\frac{k}{aH}\right)^2}} = -\frac{3\sqrt{2\epsilon}a^2HM_P}{k}\left(\frac{k}{\sqrt{k^2+3\epsilon a^2H^2}}\right)\,. \tag{4.135}$$

- The equation of motion of v is then solved in deep sub-horizon regime $-k\tau \to +\infty$ and the normalization condition (1.62) must be imposed in conjunction with the Bunch-Davies vacuum condition (1.63). This way we know the asymptotic behaviour of ζ_{cl} in the deep sub-horizon regime.

- We then solve the differential equation of $\mathcal{R}$ (4.114) neglecting terms of second order in slow roll (*i.e.* the second line of the equation (4.114)), whose solution is given in terms of Hankel functions. Replacing the above solution in equation (4.111) provides finally the mode function ζ_{cl}.

The derivation of the mode function ζ_{cl} is very lenghty, therefore we will report here just the final result. However, all the details of the computation are reported in Appendix C. The mode function ζ_{cl} is thus given by

$$\begin{aligned}\zeta_{cl}(\tau,\mathbf{k}) = &-i\sqrt{\frac{\pi}{2}}e^{i\frac{\pi}{2}(\nu_s+\frac{1}{2})}\frac{c_{L,c}H_c}{3M_P\sqrt{4\epsilon_c}\sqrt{c_{L,c}^5k^3}}\left(\frac{\tau}{\tau_c}\right)^{\epsilon_c+\frac{\eta_c}{2}+\frac{5s_c}{2}} \\ &\times(1-2\epsilon_c+s_c)\left[Q^{\frac{5}{2}}H^{(1)}_{\nu_s+1}(Q)+c_{L,c}^2\epsilon_cQ^{\frac{3}{2}}H^{(1)}_{\nu_s}(Q)\right]\,,\end{aligned} \tag{4.136}$$

where

$$Q \equiv -c_L(\tau)k\tau(1+s_c)\,, \tag{4.137}$$

$$\nu_s \equiv \frac{1}{2}\left(3+5s_c-2c_{L,c}^2\epsilon_c+\eta_c\right)\,. \tag{4.138}$$

The mode function ζ_{cl} has been explicitely given for the first time in [20]. It should be noticed that (4.136) contains many corrections of second and higher order in the slow-roll parameters which must be neglected. Nonetheless it is useful when evaluating the super horizon limit because of the asymptotic behaviour of the Hankel function $H^{(1)}$ (see (B.3)).

When the super horizon regime is considered, the most sizeable contribution to (4.136) comes from the term $Q^{\frac{5}{2}}H^{(1)}_{\nu_s+1}(Q)$. On super horizon scales the above term results in

$$\begin{aligned}Q^{\frac{5}{2}}H^{(1)}_{\nu_s+1}(Q) \underset{-k\tau\to 0}{\longrightarrow} &\sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}\,2^{\nu_s+1-\frac{3}{2}}\frac{\Gamma(\nu_s+1)}{\Gamma(\frac{3}{2})}Q^{-(\nu_s+1-\frac{5}{2})} \\ = &-i3\sqrt{\frac{2}{\pi}}\left(\frac{\tau}{\tau_c}\right)^{c_{L,c}^2\epsilon_c-\frac{\eta_c}{2}-\frac{5s_c}{2}}(-c_{L,c}k\tau_c)^{c_{L,c}^2\epsilon_c-\frac{\eta_c}{2}-\frac{5s_c}{2}}+\mathcal{O}(\epsilon)\,.\end{aligned} \tag{4.139}$$

Therefore, the mode function ζ_{cl} on super horizon scales is given by

$$\zeta_{cl} \underset{-k\tau\to 0^+}{=} \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c_{T,c}^2\epsilon_c}(-c_{L,c}k\tau_c)^{c_{L,c}^2\epsilon_c-\frac{5}{2}s_c-\frac{1}{2}\eta_c}\left(\frac{H_c}{\sqrt{4\epsilon_c}M_P\sqrt{c_{L,c}^5k^3}}+\mathcal{O}(\epsilon^{1/2})\right)\,. \tag{4.140}$$

It should be stressed that the mode function ζ_{cl} of the model of Solid inflation is not constant on super horizon scales, but still evolves in time with a mild dependence, suppressed by the slow-roll parameter ϵ_c. This fact is in constrast with the single-field slow-roll model and it can be considered an unusual feature of this model. Another difference between the two models is that ζ and $\mathcal{R}$ do not coincide on super horizon scales. The last statement can be seen by a comparison with the asymptotic behaviour of the mode function of $\mathcal{R}$ [32]:

$$\mathcal{R}_{cl} \underset{-k\tau\to 0^+}{=} \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c_{T,c}^2\epsilon_c - 2s_c} (-c_{L,c}k\tau_c)^{c_{L,c}^2\epsilon_c - \frac{5}{2}s_c - \frac{1}{2}\eta_c} \left(-\frac{H_c}{\sqrt{4\epsilon_c}M_P\sqrt{c_{L,c}k^3}} + \mathcal{O}(\epsilon^{1/2})\right) . \tag{4.141}$$

At the leading order on super horizon scales they are related as follows

$$\mathcal{R}_{cl} \underset{-k\tau\to 0^+}{=} -c_L^2(\tau)\zeta_{cl} . \tag{4.142}$$

It is then possible to provide the expression of the two point function of ζ on super horizon scale:

$$\begin{aligned}\langle\zeta(\tau,\mathbf{k}_1)\zeta(\tau,\mathbf{k}_2)\rangle &= (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_1)|\zeta_{cl}(\tau,\mathbf{k}_1)|^2 \\ &\underset{-k\tau\to 0^+}{\longrightarrow} (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_1)\frac{H_c^2}{4\epsilon_c c_{L,c}^5 M_P^2}\frac{1}{k_1^3}\frac{\left(\frac{\tau}{\tau_c}\right)^{\frac{8}{3}c_{L,c}^2\epsilon_c}}{(-c_{L,c}k_1\tau_c)^{5s_c-2c_{L,c}^2+\eta_c}} ,\end{aligned} \tag{4.143}$$

which implies that the power spectrum of ζ is given by

$$P_\zeta(\tau,k) = \frac{H_c^2}{4\epsilon_c c_{L,c}^5 M_P^2}\frac{1}{k_1^3}\left(\frac{\tau}{\tau_c}\right)^{\frac{8}{3}c_{L,c}^2\epsilon_c}(-c_{L,c}k_1\tau_c)^{-(5s_c-2c_{L,c}^2+\eta_c)} . \tag{4.144}$$

The above expression is quite cumbersome but give us the possibility to read directly the spectral index. Recalling the definition (1.93), we have

$$n_\zeta = 2c_{L,c}^2\epsilon_c - 5s_c - \eta_c . \tag{4.145}$$

4.4.3 Vector perturbation π_T: mode function and power spectrum

In this section, the derivation of the mode function of the vector perturbation π_T^i will be computed in detail. The procedure is conceptually identical to the case of the scalar perturbation ζ: once the canonically normalized field[1] have been identified, the Bunch-Davies vacuum and the normalization condition yield to the proper sub-horizon behaviour of π_T. As for the scalar case, instead of trying to solve directly the equation of motion of π_T, it is easier to compute the mode function of the related variable N_T, and only then express π_T through N_T using equation. Repeated indixes are understood to be summed.

Let us begin by writing the vector modes as sum of the polarized components

$$\pi_T^i(\tau,\mathbf{k}) = \sum_{\lambda=\pm}\epsilon_\lambda^i(\mathbf{k})\pi_{T,\lambda}(\tau,\mathbf{k}) . \tag{4.146}$$

where the polarization vectors satisfy the transverse condition and form an orthonormal complete set, reported respectively as follows,

$$k_i\epsilon_\lambda^i(\mathbf{k}) = 0 , \qquad \epsilon_\lambda^i(\mathbf{k})(\epsilon_{\lambda'}^i(\mathbf{k}))^* = \delta_{\lambda\lambda'} . \tag{4.147}$$

The quantum field π_T^i is decomposed further in the usual manner:

$$\pi_{T,\lambda}(\tau,\mathbf{k}) = \pi_{T,\lambda}^{cl}(\tau,\mathbf{k})d_\lambda(\mathbf{k}) + \pi_{T,\lambda}^{cl}(\tau,\mathbf{k})^* d_\lambda^\dagger(-\mathbf{k}) , \tag{4.148}$$

$$\left[d_\lambda(\mathbf{k}) \,,\, d_{\lambda'}^\dagger(\mathbf{k}')\right] = (2\pi)^3\delta_{\lambda\lambda'}\delta^{(3)}(\mathbf{k}-\mathbf{k}') . \tag{4.149}$$

[1] The field whose quadratic action in conformal time is the one of an harmonic oscillator [52].

At this point, the equations (4.102) and (4.103) form a system of differential equation of first order relating the two vectors N^i_T and π^i_T. After that N^i_T has been expressed in terms of creation and annihilation operators in the same manner as π^i_T, we can recast the system in terms of their mode functions:

$$\dot{N}^{cl}_T + 3HN^{cl} + 4\epsilon H^2 c_T^2 \pi^{cl}_T = 0\,, \tag{4.150}$$

$$N^{cl}_T = \frac{\dot{\pi}^{cl}_T}{1+\frac{k^2}{4\epsilon a^2 H^2}}\,. \tag{4.151}$$

Eliminating π^{cl}_T, we reach a second order differential equation for N^{cl}_T, which is, in conformal time,

$$N''_{T,cl} + aH(2+2\epsilon-2u-\eta)N'_{T,cl} + \left[c^2(\tau)_T k^2 + a^2H^2(3\epsilon - 6u - 3\eta + 4\epsilon c_T^2(\tau))\right] N^{cl}_T = 0\,. \tag{4.152}$$

Here, u is the slow-roll parameter defined in (4.64). Exploiting the time-dependence of the background quantities and keeping only the terms up to the first order in slow-roll, the above equation becomes

$$N''_{T,cl} - \frac{1}{\tau}(2+4\epsilon_c-2u_c-\eta_c)N'_{T,cl} + \left[c^2(\tau)_T k^2 + \frac{1}{\tau}(3\epsilon_c - 6u_c - 3\eta_c + 4\epsilon_c c^2_{T,c})\right] N^{cl}_T = 0\,. \tag{4.153}$$

Being this last expression an equation of the same type of (C.16), although with a different combination of slow-roll parameters, we already know its solution:

$$N^{cl}_T(\tau,\mathbf{k}) = (-\tau)^{-\beta}\left[\mathcal{E}H^{(1)}_{\nu_v}(-k\tau c_T(\tau)(1+u_c)) + \mathcal{F}H^{(2)}_{\nu_v}(-k\tau c_T(\tau)(1+u_c))\right]\,, \tag{4.154}$$

$$\beta \equiv -\frac{3}{2} - 2\epsilon_c + \frac{1}{2}\eta_c + u_c\,, \qquad \nu_v \equiv \frac{1}{2}(3+5u_c+\eta_c-2c^2_{T,c}\epsilon_c)\,. \tag{4.155}$$

In order to fix the values of $\mathcal{E}$ and $\mathcal{F}$, we need to know the initial condition in the deep sub-horizon regime. This task is achieved through the imposition of the Bunch-Davies and the normalization conditions on the canonically normalized field, here indicated as $\hat{\pi}$. One recognizes that to turn the action (4.124) into the one of an harmonic oscillator in flat space, it is necessary to perform the following field redefinition:

$$\hat{\pi} = \sqrt{2\epsilon}a^2HM_P\sqrt{\frac{k^2}{k^2+4\epsilon a^2H^2}}\pi^{cl}_T \underset{k>>aH}{\longrightarrow} \sqrt{2\epsilon}a^2HM_P\pi^{cl}_T\,. \tag{4.156}$$

Since in the deep sub-horizon regime the field $\hat{\pi}$ obeys

$$\hat{\pi}''(\tau,\mathbf{k}) + c_T^2k^2\hat{\pi}(\tau,\mathbf{k}) = 0\,, \tag{4.157}$$

which is the same equation as (C.8), we know that the normalized Bunch-Davies solution is given by $\hat{\pi} \to \frac{1}{\sqrt{2c_Tk}}e^{-ic_Tk\tau(1+u_c)}$ up to a constant phase. Consequently, we have the initial condition for π^{cl}_T:

$$\pi^{cl}_T \underset{k>>aH}{\longrightarrow} \frac{1}{\sqrt{2\epsilon}a^2HM_P}\frac{1}{\sqrt{2c_Tk}}e^{-ic_Tk\tau(1+u_c)}\,. \tag{4.158}$$

Exploiting the equation (4.150), we are now able to compute the mode function π^{cl}_T up to first order in slow-roll. For the same reasons we discussed in the scalar case, we can safely set $\mathcal{F}=0$; we are therefore left with the following computation:

$$\begin{aligned}
-4\epsilon aH^2c_T^2\pi^{cl}_T &= N'_{T,cl} + 3aHN_{T,cl}\\
&= \mathcal{E}(-\tau)^{-\beta}\left[(-\tau)^{-1}(3+3\epsilon_c+\beta)H^{(1)}_{\nu_v}(Q) + \frac{d}{dQ}H^{(1)}_{\nu_v}\frac{dQ}{d\tau}\right]\\
&= \mathcal{E}(-\tau)^{-\beta-1}\left[c_Tk(-\tau)H^{(1)}_{\nu_v+1} + \underbrace{(\beta-\nu_v(1-u_c)+3\epsilon_c+3)}_{\frac{4}{3}c^2_{T,c}\epsilon_c}H^{(1)}_{\nu_v}(Q)\right]\,,
\end{aligned} \tag{4.159}$$

where we have made use of the property (B.5) and the relation $(1 + c_{L,c}^2) \approx 4c_{T,c}^2/3$. Combining (4.158) and (B.2), we get the coefficient $\mathcal{E}$ at the desired order in slow-roll:

$$
\begin{aligned}
\mathcal{E} &= \frac{-4\epsilon a H c_T^2 (-\tau)^{\beta+1} \pi_T^{cl}}{c_T k(-\tau) H^{(1)}_{\nu_v+1}(Q) + \frac{4}{3} c_{T,c}^2 \epsilon_c H^{(1)}_{\nu_v}(Q)} \\
&\underset{-k\tau\to+\infty}{\longrightarrow} (-\tau)^{\beta+1} \frac{-\sqrt{4\epsilon} H c_T^2}{a M_P \sqrt{c_T k}} \frac{e^{-ick\tau(1+u_c)}}{-c_T k\tau H^{(1)}_{\nu_v+1}(+\infty)} (1 + \mathcal{O}(\epsilon)) \\
&= -i\sqrt{\frac{\pi}{2}} \sqrt{4\epsilon_c} \frac{c_{T,c} H_c}{M_P k} \sqrt{1+u_c}\, e^{i\frac{\pi}{2}(\nu_v+1)} (-\tau_c)^{-2\epsilon_c + \frac{1}{2}\eta_c + u_c} \underbrace{\left(\frac{-1}{a_c \tau_c}\right)}_{\approx H_c(1-\epsilon_c)} (1 + \mathcal{O}(\epsilon)) \\
&= -i\sqrt{\frac{\pi}{2}} \sqrt{4\epsilon_c} \frac{c_{T,c} H_c^2}{M_P k} \left(1 + \frac{u_c}{2} - \epsilon_c\right) e^{i\frac{\pi}{2}(\nu_v+1)} (-\tau_c)^{-2\epsilon_c + \frac{1}{2}\eta_c + u_c} + \mathcal{O}(\epsilon^{5/2})\,.
\end{aligned} \tag{4.160}
$$

After some manipulations, we have the solution π_T^{cl}, expressed below in a compact way:

$$
\begin{aligned}
\pi_T^{cl} = i e^{i\frac{\pi}{2}(\nu_v+1)} \sqrt{\frac{\pi}{2}} \frac{H_c}{\sqrt{4\epsilon_c} M_P c_{T,c}^{\frac{5}{2}} k^{\frac{5}{2}}} \left(\frac{\tau}{\tau_c}\right)^{\epsilon_c + \frac{1}{2}\eta_c + \frac{5}{2}u_c} & (1 - u_c - 2\epsilon_c) \\
\times \left[Q^{\frac{5}{2}} H^{(1)}_{\nu_v+1}(Q) + \frac{4}{3} c_{T,c}^2 \epsilon_c\, Q^{\frac{3}{2}} H^{(1)}_{\nu_s}(Q) \right] &,
\end{aligned} \tag{4.161}
$$

where, in this case, $Q = -c_T(\tau) k\tau(1 + u_c)$.

The asymptotic behaviour of π_T^{cl} when the mode exits the horizon is then obtained keeping the lowest order in the slow-roll parameters, but leaving expressed the dependence on time:

$$
\begin{aligned}
&\pi_T^{cl}(\tau, k) \underset{-k\tau\to 0^+}{\longrightarrow} i e^{i\frac{\pi}{2}(\nu_v+1)} \sqrt{\frac{\pi}{2}} \frac{H_c}{\sqrt{4\epsilon_c} M_P (c_{T,c} k)^{\frac{5}{2}}} \Bigg[\sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}}\, 2^{\nu_v+1-\frac{3}{2}} \\
&\frac{\Gamma(\nu_v+1)}{\Gamma(\frac{3}{2})} \left(\frac{1}{-kc_T\tau(1+u_c)}\right)^{\nu_v+1-\frac{5}{2}} \left(\frac{\tau}{\tau_c}\right)^{\epsilon_c + \frac{\eta_c}{2} + \frac{5u_c}{2}} + \mathcal{O}(\epsilon) \Bigg] \\
&= \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c_{T,c}^2 \epsilon_c} (-c_{T,c} k\tau_c)^{c_{L,c}^2 \epsilon_c - \frac{\eta_c}{2} - \frac{5u_c}{2}} \left(\frac{-3H_c}{\sqrt{4\epsilon_c} M_P c_{T,c}^{\frac{5}{2}} k^{\frac{5}{2}}} + \mathcal{O}(\epsilon^{1/2}) \right),
\end{aligned}
$$

where we have made use of the asymptotic form (B.3) and the relation $1 + c_{L,c}^2 = \frac{4}{3}c_{T,c}^2$. In the above expression we stressed the time-dependence of the mode function on super-horizon scales, as well as the dependence on k.

As for the two point function, it therefore results in

$$
\begin{aligned}
&\langle \pi_{T,\lambda}(\tau, \mathbf{k}) \pi_{T,\lambda'}(\tau, \mathbf{k}') \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \delta_{\lambda\lambda'} \pi_T^{cl}(\tau, k) \pi_T^{cl}(\tau, k) \\
&\underset{-k\tau\to 0^+}{\longrightarrow} (2\pi)^3 \delta^{(3)}(\mathbf{k} + \mathbf{k}') \delta_{\lambda\lambda'} \frac{9H_c^2}{4\epsilon_c M_P^2 c_{T,c}^5 k^5} \left(\frac{\tau}{\tau_c}\right)^{\frac{8}{3}c_{T,c}^2 \epsilon_c} (-c_{T,c} k\tau_c)^{2c_{L,c}^2 \epsilon_c - \eta_c - 5u_c} + \mathcal{O}(\epsilon)\,.
\end{aligned} \tag{4.162}
$$

The power spectrum of vector perturbation is finally given by

$$
P_T(\tau, k) = \frac{9H_c^2}{4\epsilon_c M_P^2 c_{T,c}^5 k^5} \left(\frac{\tau}{\tau_c}\right)^{\frac{8}{3}c_{T,c}^2 \epsilon_c} (-c_{T,c} k\tau_c)^{2c_{L,c}^2 \epsilon_c - \eta_c - 5u_c} \tag{4.163}
$$

4.4.4 Tensor perturbation γ: mode function and power spectrum

The full calculation of the tensor mode function can be found by the reader in Appendix C.2. In this paragraph we will highlight just the essential steps for our discussion.

The tensor modes are decomposed into their polarized components, which are of course traceless and transverse:

$$\gamma_{ij}(\tau, \mathbf{k}) = \sum_{s=+,\times} \epsilon^s_{ij}(\mathbf{k}) \gamma^s(\tau, \mathbf{k}) , \tag{4.164}$$

$$\epsilon^s_{ij}(\epsilon^{s'}_{ij})^* = 2\delta^{ss'} , \qquad k_i \epsilon_{ij}(\mathbf{k}) = 0 , \qquad \epsilon_{ii}(\mathbf{k}) = 0 . \tag{4.165}$$

Also in this case, repeated indexes are to be considered summed.

Promoting γ^s to quantum operator, we further decompose it as

$$\gamma^s(\tau, k) = \gamma^s_{cl}(\tau, k)\, a^s(\mathbf{k}) + \gamma^{s\,*}_{cl}(\tau, k) a^{s\,\dagger}(-\mathbf{k}) , \tag{4.166}$$

where the following algebra holds for a^s and $a^{s\,\dagger}$

$$[a^s(\mathbf{k}) \, , \, a^{s'\,\dagger}(\mathbf{k}')] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \delta^{ss'} . \tag{4.167}$$

Varying the action (4.123), we get the equation of motion of the mode function $\gamma^s(\tau, k)$. Using the time-dependence of the background quantities, it results in

$$\frac{d^2}{d\tau^2}\gamma_{cl} - \frac{2}{\tau}(1 + \epsilon_c)\frac{d}{d\tau}\gamma_{cl} + \left(k^2 + \frac{4\epsilon_c c^2_{T,c}}{\tau^2} \right) \gamma_{cl} = 0 . \tag{4.168}$$

In the same way as the previous cases of scalar and vector perturbations, we need to the solve the equation of motion for the canonically normalized field, that in this case is given by $\hat{\gamma}^s = \frac{a(\tau) M_P}{\sqrt{2}} \gamma^s_{cl}$. Once the solution is obtained, we have to impose the usual conditions (*i.e.* Bunch-Davies vacuum and normalization). The final result for γ^s_{cl}, which is valid up to first order in slow roll, is given by

$$\gamma^s_{cl}(\tau, \mathbf{k}) = \sqrt{\frac{\pi}{2}} \frac{H_c}{M_P} \left(\frac{\tau}{\tau_c} \right)^{\epsilon_c} (1 - \epsilon_c) e^{i\frac{\pi}{2}(\nu_T + \frac{1}{2})} (-\tau)^{-\frac{3}{2}} H^{(1)}_{\nu_T}(-k\tau) , \tag{4.169}$$

where

$$\nu_T \equiv \frac{3}{2} + \epsilon_c - \frac{4}{3} c^2_{T,c} \epsilon_c . \tag{4.170}$$

The mode function γ^s_{cl} on large scale is then obtained by the usual asymptotic behaviour of the Hankel function $H^{(1)}$; we thus get

$$\gamma^s_{cl} \underset{-k\tau \to 0^+}{=} \left(\frac{\tau}{\tau_c} \right)^{\frac{4}{3} c^2_{T,c} \epsilon_c} (-k\tau_c)^{c^2_{L,c}\epsilon_c} \left(i \frac{H_c}{M_P \sqrt{k^3}} + \mathcal{O}(\epsilon) \right) . \tag{4.171}$$

It should be highlighted that, as in the case of the scalar perturbations ζ and $\mathcal{R}$, the tensor perturbation still evolves on super horizon scales, with a time dependence suppressed by the slow-roll parameter ϵ.

The two-point function for tensor perturbations on super-horizon scales is then

$$\begin{aligned} \langle \gamma^{s_1}(\tau, \mathbf{k_1}) \gamma^{s_2}(\tau, \mathbf{k_2}) \rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \delta^{s_1 s_2} |\gamma^{s_1}_{cl}(\tau, \mathbf{k}_1)|^2 \\ \underset{-k\tau \to 0^+}{\longrightarrow} & (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_1) \delta^{s_1 s_2} \frac{H_c^2}{M_p^2} \frac{1}{k_1^3} \frac{\left(\frac{\tau}{\tau_c} \right)^{\frac{8}{3} c^2_{T,c} \epsilon_c}}{(-k_1 \tau_c)^{-2 c^2_{L,c} \epsilon_c}} . \end{aligned} \tag{4.172}$$

We can read off the index of the power spectrum to first order in slow-roll from the above expression:

$$n_T - 1 = 2 c^2_{L,c} \epsilon_c . \tag{4.173}$$

Now that we have at our disposal also the power spectrum of tensor perturbations, it is possible to compute the tensor-to-scalar ratio r. At the lowest order in slow-roll, it results in

$$r \approx 8 \epsilon_c c^5_L . \tag{4.174}$$

It should be highlighted that the above value of the tensor-to-scalar ratio violates the consistency relation (2.161) predicted by the single-field slow-roll model.

Chapter 5

Bispectra of Solid inflation

In Chapter 3 we illustrated the methodology through which it is possible to extrapolate the primordial bispectra from the Lagrangian of a generic model. The above methodology consists of two main steps. It is firstly necessary to derive the correct third-order interactions from the action of the theory. In section 3.2.1, we discussed how the ADM formalism provides a scheme that allows to have the explicit cubic interactions between the dynamical perturbation fields. Once such interaction terms have been derived, the second step consists in applying the in-in formalism in order to obtain the vacuum expectation value of the product of three perturbation fields, *i.e* the three point correlation function.

In the present chapter the above methodology will be applied to the model of Solid Inflation in order to obtain some of the bispectra that are predicted by this model. More specifically, we are interested in computing the following three-point functions to *leading order* in the slow-roll parameters:

$$\langle\zeta\pi_T\gamma\rangle\,,\,\langle\zeta\zeta\gamma\rangle\,,\langle\pi_T\pi_T\gamma\rangle\,,\,\langle\zeta\gamma\gamma\rangle\,,\,\langle\pi_T\gamma\gamma\rangle\,,\,\langle\gamma\gamma\gamma\rangle\,. \tag{5.1}$$

where γ, ζ and π_T are respectively the tensor, scalar and vector perturbation. With the exception of $\langle\zeta\gamma\pi_T\rangle$, which has yet to appear in literature, all the correlators (5.1) have been calculated in the squezeed configuration in [31]. In the same paper, the authors did not apply the in-in formalism, but rather they applied a semi-classical method in which the long-wave mode is treated as a classical background. Such a method led to a simple and systematic derivation of almost all the Solid Inflation bispectra, although limited to the squezeed configurations. These limited outcomes encouraged us to work anew on the derivation of most of the three-point functions presented in [31], this time through the application of the in-in formalism. These original results are more comprehensive and allow to take into account the full variety of triangle configurations. The correctness of the newly calculated bispectra is based on the comparison with the outcomes of [31]: when the same long-wave mode is considered, the bispectra reported in the above paper coincide with those computed in this thesis.

All the forthcoming analysis will be performed within the spatially flat gauge, previously discussed in section 2.5.2 of this thesis.

5.1 The bispectrum $B_{\zeta\zeta\zeta}$ of Solid Inflation

A comprehensive analysis of the bispectrum of the curvature perturbation ζ has been perfomed both in [32] and in [9]. As it will be shown, the bispectrum of ζ, $B_{\zeta\zeta\zeta}$, is characterized by the dependence on the triangular configurations of the three wave-vectors $\mathbf{k}_i$ and not only on their corresponding magnitudes. As reported in [32], the 3D Cosine highlights a very small overlap of the bispectrum shape of Solid Inflation with the *Local*, the *Equilateral* and the *Orthogonal* templates.

Before proceeding in giving the expression of the bispectrum of ζ as reported in [32], a relevant aspect of Solid Inflation should be highlighted, which is at the basis of the different approach of [32] and [9] to the computation of $B_{\zeta\zeta\zeta}$. The authors of [32] derived $B_{\zeta\zeta\zeta}$ by assuming a fine tuned relation between F_Y and F_Z, namely

$$F_Y = -F_Z + \mathcal{O}(\epsilon)\,. \tag{5.2}$$

As explained in [32], the above condition is rather arbitrary, since it is not justified by any symmetry of the theory. In order to better understand the origin of the above fine tuned relation, it should be recalled that in Solid Inflation the superluminal pathology is avoided by the bounds (4.62) on the sum $F_Y + F_Z$:

$$0 < F_Y + F_Z < \frac{3}{8}\epsilon|\bar{F}|\,. \tag{5.3}$$

The authors of [32, 31] satisfied the above condition by considering F_Y and F_Z as $\mathcal{O}(1)$ parameters in slow-roll. More precisely, they assumed

$$\left|\frac{F_Y}{F}\right| = \left|\frac{F_Y}{F}\right| \sim \mathcal{O}(1)\,, \tag{5.4}$$

so that (5.3) translates into (5.2). In order to calculate $\langle\zeta\zeta\zeta\rangle$ to the leading order in the slow-roll parameters, the last relation greatly simplifies the analysis of [32], leading to a reduction of the number of interaction terms. The three-point function calculated in [32] has therefore the following expression at leading order in the slow-roll parameters:

$$\langle\zeta_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B_{\zeta\zeta\zeta}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3})\,, \tag{5.5}$$

$$B_{\zeta\zeta\zeta}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) = \frac{3}{32}\frac{F_Y}{F}\left(\frac{H}{M_P}\right)^4\frac{1}{\epsilon^3 c_L^{12}}\frac{1}{(k_1k_2k_3)^3}U(k_1,k_2,k_3)Q(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3})\,, \tag{5.6}$$

where U and Q are given by

$$\begin{aligned}
U(k_1,k_2,k_3) =& \frac{2}{k_1k_2k_3K_t^3}\Big[3\left(k_1^6+k_2^6+k_3^6\right)+20(k_1k_2k_3)^2+18\left(k_1^4k_2k_3+k_1k_2^4k_3+k_1k_2k_3^4\right)\\
&+12\left(k_1^3k_2^3+k_1^3k_3^3+k_2^3k_3^3\right)+9\left(k_1^5k_2+(5\,perm.)\right)+12\left(k_1^4k_2^2+(5\,perm.)\right)\\
&+18\left(k_1^3k_2^2k_3+(5\,perm.)\right)\Big]\,, \qquad (5.7)\\
Q(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) =& \frac{7}{81}k_1k_2k_3-\frac{5}{27}\left(k_1\frac{(\mathbf{k_2}\cdot\mathbf{k_3})^2}{k_2k_3}+k_2\frac{(\mathbf{k_1}\cdot\mathbf{k_3})^2}{k_1k_3}+k_3\frac{(\mathbf{k_1}\cdot\mathbf{k_2})^2}{k_1k_2}\right)\\
&+\frac{2}{3}\frac{(\mathbf{k_1}\cdot\mathbf{k_2})(\mathbf{k_2}\cdot\mathbf{k_3})(\mathbf{k_1}\cdot\mathbf{k_3})}{k_1k_2k_3}\,. \qquad (5.8)
\end{aligned}$$

The resulting amplitude f_{NL} of the bispectrum (5.6) is highly enhanced [32], being approximately

$$f_{NL} \sim \frac{F_Y}{F}\frac{1}{\epsilon c_L^2}\,. \tag{5.9}$$

However, as pointed out in [9], the above value appears an excessively high number to be consistent with the contraints established through the *Planck* data [5]. In order to mitigate such a tension, the author of [9] considered F_Y and F_Z as ranging in the whole allowed region of the parameters space, which can be expressed as

$$\left\{(F_Y\,,\,F_Z)\quad\middle|\quad 0 < F_Y + F_Z < \frac{3}{8}\epsilon|\bar{F}|\right\}. \tag{5.10}$$

Whithin this generic setting, the upper bound (5.3) is therefore potentially satisfied by

$$\left|\frac{F_Y}{F}\right| \sim \left|\frac{F_Y}{F}\right| \sim \mathcal{O}(\epsilon)\,. \tag{5.11}$$

The price for considering F_Y and F_Z as free parameters is that the complexity of the calculation of $B_{\zeta\zeta\zeta}$ increases considerably: one should to take into account the corrections $\mathcal{O}(\epsilon)$ coming from both the mode functions and the Lagrangian. Nonetheless, the author of [9] succeeded in the computation of $B_{\zeta\zeta\zeta}$ whitin this generic setting. The expression of his bispectrum $B_{\zeta\zeta\zeta}$ will not be reported here, since it has a very complex form. However, the advantage of the result of [9] consists in the possibility of modulating the Solid parameters F_Y and F_Z. In particular, the setting (5.11) leads to an accetable value of f_{NL} [9].

5.1.1 Specifics on method and notations

The first necessary step to address for the computation of the three-point functions (5.1) is the expansion of the Solid Lagrangian $F(X,Y,Z)$ around the background value $(\bar{X},\bar{Y},\bar{Z}) = (\frac{3}{a^2},\frac{1}{3},\frac{1}{9})$ up to third order in the dynamical fields π^i and γ_{ij}. The above expansion is performed in section A.2 of the Appendix. Aside the global factor $\sqrt{-g}$, the part of $F(X,Y,Z)$ that includes all the cubic interactions between the perturbation fields can be formally written as

$$F_3 \;=\; F_X\delta X_3+F_Y\delta Y_3+F_Z\delta Z_3+F_{XX}\delta X_1\delta X_2+\big(F_{XY}+F_{XZ}\big)\delta X_1\delta Y_2+\frac{1}{6}F_{XXX}\left(\delta X_1\right)^3\,, \tag{5.12}$$

where the subscripts in δX, δY and δZ denote only the number of perturbation fields contained in the corresponding terms. However, the above notation does not make a distinction in the specific fields interaction encoded in such terms. In order to fully express (5.12) many lenghty computations are necessary, not reported here, however summarized in the Appendix A.2. The final results of the above mentioned computations are the contributions that δX, δY and δZ individually provide to (5.12). Such contributions are listed in section A.2.3 and labeled on the basis of their order and content in fields. The contributions of (5.12) that involve a specific interaction between γ_{ij} and π^i will be labeled by a proper subscript as follows:

$$\mathcal{L}^F_{\pi\pi\gamma}\,,\quad \mathcal{L}^F_{\pi\gamma\gamma}\quad \mathcal{L}^F_{\gamma\gamma\gamma}\,. \tag{5.13}$$

As for the quantum operators, for the sake of simplicity the interaction picture fields will not be labeled by the index I. The conventions that will be adopted in the following paragraphs on the quantum field decomposition and their algebra is summarized below. As for the repeated indexes, they always need to be considered summed, whether they are lower or upper ones.

- Scalar curvature perturbation in the interaction picture:

$$\zeta(\mathbf{x},\tau) = \int \frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}}\zeta_{\mathbf{p}}(\tau)\,, \tag{5.14}$$

$$\zeta_{\mathbf{p}}(\tau) = \zeta_{cl}(p,\tau)b(\mathbf{p}) + \zeta^*_{cl}(p,\tau)b^\dagger(-\mathbf{p}) = \zeta^+_{\mathbf{p}}(\tau)+\zeta^-_{\mathbf{p}}(\tau)\,. \tag{5.15}$$

- Tensor perturbation in the interaction picture:

$$\gamma_{ij}(\mathbf{x},\tau) = \sum_{s=+,\times}\int \frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}}\epsilon^s_{ij}(\mathbf{p})\gamma^s_{\mathbf{p}}(\tau)\,, \tag{5.16}$$

$$\gamma^s_{\mathbf{p}}(\tau) = \gamma_{cl}(p,\tau)a^s(\mathbf{p}) + \gamma^*_{cl}(p,\tau)a^{s\,\dagger}(-\mathbf{p}) = \gamma^{s\,+}_{\mathbf{p}}(\tau)+\gamma^{s\,-}_{\mathbf{p}}(\tau)\,. \tag{5.17}$$

  The two polarization tensors $\epsilon^{+,\times}_{ij}$ satisfy the transverse conditions $p_i\epsilon^s_{ij}(\mathbf{p}) = 0$ and the traceless condition $\epsilon^s_{ii} = 0$

- Vector perturbation in the interaction picture:

$$\pi^i_T(\mathbf{x},\tau) = \sum_{\lambda=\pm} \int \frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}} \epsilon^i_\lambda(\mathbf{p}) \pi^\lambda_{T,\mathbf{p}}(\tau)\,, \tag{5.18}$$

$$\pi^\lambda_{T,\mathbf{p}}(\tau) = \pi_{T,cl}(p,\tau) d^\lambda(\mathbf{p}) + \pi^*_{T,cl}(p,\tau) d^{\lambda\,\dagger}(-\mathbf{p}) = \pi^{\lambda\,+}_{\mathbf{p}}(\tau) + \pi^{\lambda\,-}_{\mathbf{p}}(\tau)\,. \tag{5.19}$$

The polarization vectors $\epsilon^i_{+,-}$ satisfy the transverse condition $p_i \epsilon^i_\lambda(\mathbf{p}) = 0$.

The associated operator algebra is determined by the following commutators,

$$\left[\zeta^+_{\mathbf{p}}(\tau)\,,\, \zeta^-_{\mathbf{p}'}(\tau')\right] = (2\pi)^3 \zeta_{cl}(p,\tau)\zeta^*_{cl}(p,\tau')\delta^{(3)}(\mathbf{p}+\mathbf{p}')\,, \tag{5.20}$$

$$\left[\gamma^{s+}_{\mathbf{p}}(\tau)\,,\, \gamma^{s'-}_{\mathbf{p}'}(\tau')\right] = (2\pi)^3 \gamma_{cl}(p,\tau)\gamma^*_{cl}(p,\tau')\delta^{ss'}\delta^{(3)}(\mathbf{p}+\mathbf{p}')\,, \tag{5.21}$$

$$\left[\pi^{\lambda+}_{\mathbf{p}}(\tau)\,,\, \pi^{\lambda'-}_{\mathbf{p}'}(\tau')\right] = (2\pi)^3 \pi_{T,cl}(p,\tau)\pi^*_{T,cl}(p,\tau')\delta^{\lambda\lambda'}\delta^{(3)}(\mathbf{p}+\mathbf{p}')\,, \tag{5.22}$$

while all the other commutators are identically null. The above relations will have a prominent role in the computation of all the forthcoming three-point functions.

Since we are interested in the computation of the three-point functions (5.1) to leading order in the slow-roll parameters, $a(\tau)$ will be the scale factor of the de-Sitter space,

$$a(\tau) = -\frac{1}{H\tau}\,, \tag{5.23}$$

while the quantities H, ϵ, c_L, c_T will be considered as constants. The perturbation mode functions will be therefore approximated to leading order in the slow-roll parameters:

$$\begin{aligned}\zeta_{cl}(\tau,k) &= i\sqrt{\frac{\pi}{2}}\frac{c_L H}{3M_P\sqrt{4\epsilon c_L^5 k^3}}(-c_L k\tau)^{\frac{5}{2}} H^{(1)}_{\frac{5}{2}}(-c_L k\tau)\\ &= \frac{H}{2\sqrt{\epsilon c_L^5 k^3} M_P}\left(1 + ic_L k\tau - \frac{1}{3}c_L^2 k^2\tau^2\right) e^{-ic_L k\tau}\,,\end{aligned} \tag{5.24}$$

$$\begin{aligned}\pi_{T,cl}(\tau,k) &= i\sqrt{\frac{\pi}{2}}\frac{c_T H}{M_P\sqrt{4\epsilon c_T^5 k^5}}(-c_T k\tau)^{\frac{5}{2}} H^{(1)}_{\frac{5}{2}}(-c_T k\tau)\\ &= \frac{-3H}{2\sqrt{\epsilon c_T^5 k^5} M_P}\left(1 + ic_T k\tau - \frac{1}{3}c_T^2 k^2\tau^2\right) e^{-ic_T k\tau}\,,\end{aligned} \tag{5.25}$$

$$\begin{aligned}\gamma_{cl}(\tau,k) &= \sqrt{\frac{\pi}{2}}\frac{H}{M_P\sqrt{k^3}}(-k\tau)^{\frac{3}{2}} H^{(1)}_{\frac{3}{2}}(-k\tau)\\ &= i\frac{H}{\sqrt{k^3}M_P}\left(1 + ik\tau\right) e^{-ik\tau}\,.\end{aligned} \tag{5.26}$$

The explicit form of the Hankel functions of fractional order is derived from the general formula (B.6) reported in the Appendix B. When the bispectra (5.1) will be evaluated in the squeezed configuration, it is convenient to express the result through the power spectra at the end of inflation. To leading order in slow-roll, they result in

$$P_\zeta(k) = \frac{H^2}{4\epsilon M_p^2 c_L^5}\frac{1}{k^3}\,,\quad P_T(k) = \frac{9H^2}{4\epsilon c_T^5 M_p^2}\frac{1}{k^5}\,,\quad P_\gamma(k) = \frac{H^2}{M_p^2}\frac{1}{k^3}\,. \tag{5.27}$$

A further comment should be given about the following integral, which will appear in all the three-point functions in (5.1):

$$\int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau \frac{e^{iK_t\tau}}{\tau}\,. \tag{5.28}$$

Here, τ_e denotes the conformal time coordinate of the end of inflation ($\tau_e < 0$), while K_t is the sum of the three wave-numbers magnitudes. The expression (5.28) is simply related to the *Exponential Integral* $E_1(z)$ [1], which is defined, for complex z, as

$$E_1(z) \equiv \int_z^\infty dx\, \frac{e^{-x}}{x}\,, \qquad \big(|Arg(z)| < \pi\big)\,. \tag{5.29}$$

In the above definition it is assumed that the path of integration excludes the origin and does not cross the negative real axis. Through a $\pi/2$ rotation of the time coordinate in the complex plane, *i.e.* $\tau \to i\tau$, in conjunction with the rescaling $x = K_t\tau$, the integral (5.28) becomes

$$\int_{-i\infty(1-i\epsilon)}^{iK_t\tau_e} dx \frac{e^{-x}}{x} = -E_1(iK_t\tau_e)\,. \tag{5.30}$$

An important property of $E_1(z)$ is that it admits the following series expansion [1],

$$E_1(z) = -\gamma - \ln(z) - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{nn!}\,, \tag{5.31}$$

where γ is the Euler-Mascheroni constant ($\gamma \approx 0.577$). Since only the real part of $-E_1(iK_t\tau_e)$ evaluated at $K_t\tau_e \approx 0$ will be considered, the series expansion (5.31) provides the following relation,

$$Re\Big[-E_1(iK_t\tau_e)\Big] \approx \gamma + \ln(-K_t\tau_e) = \gamma + N_{K_t}\,, \tag{5.32}$$

where N_{K_t} represents the number of e-foldings from the time the scale corresponding to K_t has left the horizon during inflation and the end of inflation.

The remaining part of this chapter is structured with each section beginning with one of the cubic interaction Lagrangian in (5.13). In addition, each section will contain the detailed computation of the bispectra that will be extrapolated from the above mentioned cubic term. The final results will be the analytic expressions of the bispectra listed in (5.1). In line with the approach of [9], we prefer to consider F_Y and F_Z as independent parameters in the region (5.10). We will illustrate, as well, how to consistently reproduce the results of [31].

5.2 Interactions involving one γ and two π

The cubic part of the Solid Lagrangian that involves the coupling between one tensor γ_{ij} and two perturbation fields π^i is given by

$$\begin{aligned}\mathcal{L}^F_{\gamma\pi\pi} = \sqrt{-g}\Big[&F_X\delta X_{\gamma\pi\pi} + F_Y\delta Y_{\gamma\pi\pi} + F_Z\delta Z_{\gamma\pi\pi}\\ &+ F_{XX}\delta X_\pi\delta X_{\gamma\pi} + \big(F_{XY}+F_{XZ}\big)\delta X_\pi\delta Y_{\gamma\pi}\Big]\,.\end{aligned} \tag{5.33}$$

The explicit form of the above Lagrangian can be constructed out of the expressions of

$$\delta X_\pi\,, \quad \delta X_{\gamma\pi}\,, \quad \delta X_{\gamma\pi\pi}\,, \quad \delta Y_{\gamma\pi}\,, \quad \delta Y_{\gamma\pi\pi}\,, \quad \delta Z_{\gamma\pi\pi}\,, \tag{5.34}$$

which are reported in section A.2.3. By plugging those expressions in (5.33), we obtain

$$\begin{aligned}\mathcal{L}^F_{\gamma\pi\pi} =a^3\Bigg\{ &-\frac{F_X}{a^2}\gamma_{ij}\partial_i\pi^k\partial_j\pi^k - \frac{4F_{XX}}{a^4}\gamma_{ij}\partial_i\pi^j\partial_k\pi^k\\ &+F_Y\left[\frac{8}{9}\gamma_{ij}\partial_i\pi^j\partial_k\pi^k - \frac{4}{9}\gamma_{ij}\partial_i\pi^k\partial_j\pi^k - \frac{4}{9}\gamma_{ij}\partial_i\pi^k\partial_k\pi^j - \frac{2}{9}\gamma_{ij}\partial_k\pi^i\partial_k\pi^j\right]\\ &+F_Z\left[\frac{32}{27}\gamma_{ij}\partial_i\pi^j\partial_k\pi^k - \frac{5}{9}\gamma_{ij}\partial_i\pi^k\partial_j\pi^k - \frac{2}{3}\gamma_{ij}\partial_i\pi^k\partial_k\pi^j - \frac{1}{3}\gamma_{ij}\partial_k\pi^i\partial_k\pi^j\right]\\ &-\frac{8}{9a^2}(F_{XY}+F_{XZ})\gamma_{ij}\partial_i\pi^j\partial_k\pi^k + \frac{2a^2}{9}(F_Y+F_Z)\gamma_{ij}(\dot\pi^i - N^i)(\dot\pi^j - N^j)\Bigg\}\,.\end{aligned} \tag{5.35}$$

where the global factor a^3 is due to $\sqrt{-g}$, being g the determinant of the backgroung metric. The last line is $\mathcal{O}(\epsilon^2)$, since

$$\left(\dot{\pi}^i - N^i\right) \sim \mathcal{O}(\epsilon)\,, \qquad (F_Y + F_Z) \lesssim \epsilon|F|\,, \qquad (F_{XY} + F_{XZ}) \sim \mathcal{O}(\epsilon^2)\,. \tag{5.36}$$

The corresponding terms will thus be neglected in the forthcoming calculations. Through the relations

$$-F = 3M_P^2 H^2\,, \qquad \bar{X} = \frac{3}{a^2}\,, \qquad \frac{F_X \bar{X}}{F} = \epsilon\,, \qquad \frac{F_{XX}\bar{X}}{F_X} = -1 + \mathcal{O}(\epsilon)\,, \tag{5.37}$$

we obtain the contributions to the interaction Solid Lagrangian $\mathcal{L}^F_{\gamma\pi\pi}$ up to first order in ϵ, namely,

$$\begin{aligned}\mathcal{L}^F_{\gamma\pi\pi} =& a^3 M_P^2 H^2 \gamma_{ij}\Big[\left(\epsilon + \frac{4}{3}\frac{F_Y}{F} + \frac{5}{3}\frac{F_Z}{F}\right)\partial_i\pi^k\partial_j\pi^k - \left(\frac{4}{3}\epsilon + \frac{8}{3}\frac{F_Y}{F} + \frac{32}{9}\frac{F_Z}{F}\right)\partial_i\pi^j\partial_k\pi^k \\ &+ \left(\frac{2}{3}\frac{F_Y}{F} + \frac{F_Z}{F}\right)\partial_k\pi^i\partial_k\pi^j + \left(\frac{4}{3}\frac{F_Y}{F} + 2\frac{F_Z}{F}\right)\partial_i\pi^k\partial_k\pi^j\Big].\end{aligned} \tag{5.38}$$

The above formula is also reported in [9]. In the latter, the author used (5.38) in order to compute the bispectrum involving one tensor γ and two scalars ζ through the in-in formalism. However, in [9] the last steps were omitted and the squeezed limit was directly obtained. The analytic expression of $\langle\gamma\zeta\zeta\rangle$ for generic triangular configurations were not reported in the above paper. Nonetheless, the form of $B_{\gamma\zeta\zeta}$ for $k_\gamma \approx 0$ in [9] provides an additional cross-check on the validity of the result that we will derive in section 5.2.2.

By splitting the field π^i into its longitudinal and transverse components,

$$\pi^i = \pi^i_L + \pi^i_T = \frac{\partial_i \pi_L}{\sqrt{-\nabla^2}} + \pi^i_T\,, \tag{5.39}$$

the cubic terms in π^i and γ_{ij} of (5.38) will generate all the mixed-type interactions between the primordial perturbations γ, π_T and ζ, the latter being related to the longitudinal part of π^i by

$$\zeta(\mathbf{k}) = -\frac{k}{3}\pi_L(\mathbf{k})\,. \tag{5.40}$$

The above splitting will produce the interaction Lagrangians $\{\mathcal{L}^F_{\gamma\zeta\pi_T}, \mathcal{L}^F_{\gamma\zeta\zeta}, \mathcal{L}^F_{\gamma\pi_T\pi_T}\}$, from which we will extrapolate the corresponding bispectra.

5.2.1 The $\gamma\zeta\pi_T$ bispectrum

Interactions of the scalar-tensor-vector type are generated in the non-linear dynamics. Within Solid Inflation, vector perturbations do not decay since they are sourced by the anistropic stress (see (4.93)). The leading contribution of the Solid Lagrangian $F(X, Y, Z)$ to such an interaction in the spatially flat gauge are encoded in the term $\mathcal{L}^F_{\gamma\zeta\pi_T}$, which will be derived below. We are ultimately interested in the correlator $\langle\gamma\zeta\pi_T\rangle$ and its corresponding bispectrum, which does not appear in literature. Because of this reason, we choose to analyze thoroughly the calculation of the bispectrum under consideration.

In order to derive $\mathcal{L}^F_{\gamma\zeta\pi_T}$, it is necessary to implement in (5.38) the splitting of the field π^i into its longitudinal and transverse components and then keep just the terms $\gamma\pi_L\pi_T$. The following properties will be used in the calculations:

$$\gamma_{ij} = \gamma_{ji}\,, \quad \partial_k\pi^k_T = 0\,, \quad \partial_i\pi^j_L = \partial_j\pi^i_L\,. \tag{5.41}$$

Given the above considerations, the different types of tensor contractions that appear in (5.38) contribute to $\mathcal{L}^F_{\gamma\zeta\pi_T}$ with the following terms:

$$\gamma_{ij}\partial_i\pi^k\partial_j\pi^k \supseteq \gamma_{ij}\left(\partial_i\pi^k_L\partial_j\pi^k_T + \partial_i\pi^k_T\partial_j\pi^k_L\right) = 2\gamma_{ij}\partial_i\pi^k_L\partial_j\pi^k_T\,, \tag{5.42}$$

$$\gamma_{ij}\partial_i\pi^j\partial_k\pi^k \supseteq \gamma_{ij}\partial_i\pi^j_T\partial_k\pi^k_L\,, \tag{5.43}$$

$$\gamma_{ij}\partial_i\pi^k\partial_k\pi^j \supseteq \gamma_{ij}\left(\partial_i\pi^k_L\partial_k\pi^j_T + \partial_i\pi^k_T\partial_k\pi^j_L\right) = \gamma_{ij}\partial_k\pi^i_L\partial_k\pi^j_T + \gamma_{ij}\partial_i\pi^k_L\partial_j\pi^k_T\,, \tag{5.44}$$

$$\gamma_{ij}\partial_k\pi^i\partial_k\pi^j \supseteq \gamma_{ij}\left(\partial_k\pi^i_L\partial_k\pi^j_T + \partial_k\pi^i_L\partial_k\pi^j_T\right) = 2\gamma_{ij}\partial_k\pi^i_L\partial_k\pi^j_T\,. \tag{5.45}$$

The resulting interaction Lagrangian $\mathcal{L}^F_{\gamma\zeta\pi_T}$ is reported below, where the relation (5.40) between ζ and π_L is understood:

$$\begin{aligned}\mathcal{L}^F_{\gamma\zeta\pi_T} =& a^3 M_P^2 H^2 \gamma_{ij}\Big[\left(2\epsilon + 4\frac{F_Y}{F} + \frac{16}{3}\frac{F_Z}{F}\right)\partial_i\pi_L^k\partial_j\pi_T^k \\ & - \left(\frac{4}{3}\epsilon + \frac{8}{3}\frac{F_Y}{F} + \frac{32}{9}\frac{F_Z}{F}\right)\partial_k\pi_L^k\partial_i\pi_T^j + \left(\frac{8}{3}\frac{F_Y}{F} + 4\frac{F_Z}{F}\right)\partial_k\pi_L^i\partial_k\pi_T^j\Big]\,. \end{aligned} \tag{5.46}$$

In order to implement the in-in formalism, the subsequent step consists in promoting the fields that partecipate to the interaction to quantum operators in the interaction picture. The quantization of the perturbation fields is given in (5.14), (5.16) and (5.18). The Hamiltonian $H_{\gamma\zeta\pi_T}$ in the interaction picture will then be constructed out of these fields. As explained in section 3.2.6, it can be argued that

$$H_{\gamma\zeta\pi_T}(\tau) = -L_{\gamma\zeta\pi_T}(\tau) = -\int d^3x\,\mathcal{L}^F_{\gamma\zeta\pi_T}(\tau)\,, \tag{5.47}$$

since no time-derivative of the perturbation fields appear in $\mathcal{L}^F_{\gamma\zeta\pi_T}$. This is true for all the cases discussed in the present chapter.

At this point of the discussion, it is useful to express $H_{\gamma\zeta\pi_T}$ through Fourier space variables beacause such a form will be systematically applied in all the forthcoming calculations. As an example, let us consider the term $\gamma_{ij}\partial_i\pi_T^j\partial_k\pi_L^k$ in (5.46). In Fourier space, we get,

$$\begin{aligned}&\int d^3x\Big(\gamma_{ij}\partial_i\pi_T^j\partial_k\pi_L^k\Big)(\mathbf{x},\tau) = \\ &\int d^3x\left(\frac{1}{2\pi}\right)^9\int d^3p_1d^3p_2d^3p_3\,e^{i\mathbf{x}\cdot(\mathbf{p_1}+\mathbf{p_2}+\mathbf{p_3})} \\ &\sum_s \epsilon^s_{ij}(\mathbf{p_1})\gamma^s_{\mathbf{p_1}}(\tau)\left[(ip_{3i})\sum_\lambda\epsilon^\lambda_j(\mathbf{p_3})\pi^\lambda_{T,\mathbf{p_3}}(\tau)\right]\left[\frac{(ip_{2k})(ip_{2k})}{p_2}\pi_{L,\mathbf{p_2}}(\tau)\right] = \\ &\frac{-i}{(2\pi)^6}\int\prod_{n=1}^3 d^3p_n\,\delta^{(3)}\left(\sum_{a=1}^3\mathbf{p}_a\right) \\ &\sum_{s,\lambda}\Big[\epsilon^s_{ij}(\mathbf{p_1})\epsilon^\lambda_j(\mathbf{p_3})p_{3i}\Big(\widehat{p_{2k}}\widehat{p_{2k}}\Big)\gamma^s_{\mathbf{p_1}}(\tau)\pi^\lambda_{T,\mathbf{p_3}}(\tau)\Big(p_2\pi_{L,\mathbf{p_2}}(\tau)\Big)\Big]\,,\end{aligned} \tag{5.48}$$

where $\widehat{p_i} = \frac{p_i}{p}$. The three wave-numbers are now manifestly subjected to the constraint

$$p_{1i} + p_{2i} + p_{3i} = 0\,. \tag{5.49}$$

By exploiting the transversality condition

$$\epsilon^s_{ij}(\mathbf{p_1})p_{3i} = -\epsilon^s_{ij}(\mathbf{p_1})\,(p_{2i}+p_{1i}) = -\epsilon^s_{ij}(\mathbf{p_1})p_{2i}, \tag{5.50}$$

in conjunction with the replacement of π_L in favour of ζ through (5.40), the formula (5.48) acquires a simple form:

$$\begin{aligned}&\int d^3x\Big(\gamma_{ij}\partial_i\pi_T^j\partial_k\pi_L^k\Big)(\mathbf{x},\tau) = \\ &\frac{-3i}{(2\pi)^6}\sum_{s,\lambda}\int\prod_{n=1}^3 d^3p_n\,\delta^{(3)}\left(\sum_{a=1}^3\mathbf{p}_a\right)\epsilon^s_{ij}(\mathbf{p_1})\epsilon^\lambda_j(\mathbf{p_3})p_{2i}\;\gamma^s_{\mathbf{p_1}}(\tau)\pi^\lambda_{T,\mathbf{p_3}}(\tau)\zeta_{\mathbf{p_2}}(\tau)\,.\end{aligned} \tag{5.51}$$

Once the same manipulations have been performed on every term in (5.46), the sum of all these terms produces the expression of $H_{\gamma\zeta\pi_T}$ in Fourier space variables:

$$\begin{aligned}
-H_{\gamma\zeta\pi_T}(\tau) = {}& a^3 H^2 M_p^2 \frac{3i}{(2\pi)^6} \int \prod_{a=1}^{3} d^3p_a\, \delta^{(3)}\left(\sum_{b=1}^{3} \mathbf{p}_b\right) \sum_{s,\lambda} \epsilon^s_{ij}(\mathbf{p_1}) \Bigg[\\
&+ \left(\frac{8}{3}\frac{F_Y}{F} + 4\frac{F_Z}{F}\right)\left(\widehat{p_{2k}} p_{3k}\right)\epsilon^\lambda_i(\mathbf{p_3})\widehat{p_{2j}} \\
&- \left(2\epsilon + 4\frac{F_Y}{F} + \frac{16}{3}\frac{F_Z}{F}\right)\left(\epsilon^\lambda_k(\mathbf{p_3})\widehat{p_{2k}}\right)\widehat{p_{2i}} p_{2j} \\
&+ \left(\frac{4}{3}\epsilon + \frac{8}{3}\frac{F_Y}{F} + \frac{32}{9}\frac{F_Z}{F}\right)\epsilon^\lambda_i(\mathbf{p_3}) p_{2j}\Bigg] \gamma^s_{\mathbf{p_1}}(\tau)\zeta_{\mathbf{p_2}}(\tau)\pi^\lambda_{T,\mathbf{p_3}}(\tau)\,.
\end{aligned} \tag{5.52}$$

All the constituents necessary to compute the three-point function $\langle\gamma\zeta\pi_T\rangle$ evaluated at the end the inflationary stage (τ_e) are now available. By following the methodology exposed in section 3.2.6 and the simplified notation that has been explained in 5.1.1, the three-point correlator under consideration is therefore given by

$$\langle\gamma^s(\mathbf{k_1},\tau_e)\zeta(\mathbf{k_2},\tau_e)\pi^\lambda_T(\mathbf{k_3},\tau_e)\rangle = -i\int d\tau\, a(\tau)\langle\Big[\gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\pi^\lambda_{T,\mathbf{k_3}}(\tau_e)\,,\, H_{\gamma\zeta\pi_T}(\tau)\Big]\rangle. \tag{5.53}$$

We focus for the moment on the part of (5.53) that involves quantum operators, which is

$$\langle\Big[\gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\pi^\lambda_{T,\mathbf{k_3}}(\tau_e)\,,\, \gamma^{s'}_{\mathbf{p_1}}(\tau)\zeta_{\mathbf{p_2}}(\tau)\pi^{\lambda'}_{T,\mathbf{p_3}}(\tau)\Big]\rangle. \tag{5.54}$$

According to the methodology exposed in 3.2.6, the the Wick contractions which have to be considered are

$$\gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\pi^\lambda_{T,\mathbf{k_3}}(\tau_e)\gamma^{s'}_{\mathbf{p_1}}(\tau)\zeta_{\mathbf{p_2}}(\tau)\pi^{\lambda'}_{T,\mathbf{p_3}}(\tau)\,. \tag{5.55}$$

Only the above contractions survive inside the vacuum expectation value, leading to

$$\begin{aligned}
&\langle\gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\pi^\lambda_{T,\mathbf{k_3}}(\tau_e)\gamma^{s'}_{\mathbf{p_1}}(\tau)\zeta_{\mathbf{p_2}}(\tau)\pi^{\lambda'}_{T,\mathbf{p_3}}(\tau)\rangle = \\
&\Big[\gamma^{s,+}_{\mathbf{k_1}}(\tau_e)\,,\,\gamma^{s',-}_{\mathbf{p_1}}(\tau)\Big]\Big[\zeta^+_{\mathbf{k_2}}(\tau_e)\,,\,\zeta^-_{\mathbf{p_2}}(\tau)\Big]\Big[\pi^{\lambda,+}_{\mathbf{k_3}}(\tau_e)\,,\,\pi^{\lambda',-}_{\mathbf{p_3}}(\tau)\Big] = \\
&(2\pi)^9\gamma_{cl}(k_1,\tau_e)\gamma^*_{cl}(p_1,\tau)\zeta_{cl}(k_2,\tau_e)\zeta^*_{cl}(p_2,\tau)\pi_{T,cl}(k_3,\tau_e)\pi_{T,cl}(p_3,\tau) \\
&\qquad\qquad \times\delta^{ss'}\delta^{\lambda\lambda'}\delta^{(3)}(\mathbf{k_1}+\mathbf{p_1})\delta^{(3)}(\mathbf{k_2}+\mathbf{p_2})\delta^{(3)}(\mathbf{k_3}+\mathbf{p_3})\,.
\end{aligned} \tag{5.56}$$

By using the explicit form of the mode functions (5.24), (5.25) and (5.26), equation (5.53) becomes

$$\begin{aligned}
&\langle\gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\pi^\lambda_{\mathbf{k_3}}(\tau_e)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\frac{27}{16\epsilon^2 c_L^5 c_T^5}\left(\frac{H}{M}\right)^4\frac{1}{k_1^3k_2^3k_3^5} \\
&\epsilon^s_{ij}(\mathbf{k_1})\Bigg[\left(\frac{8}{3}\frac{F_Y}{F}+4\frac{F_Z}{F}\right)\left(\widehat{k_{2k}}k_{3k}\right)\epsilon^\lambda_i(\mathbf{k_3})\widehat{k_{2j}} - \left(2\epsilon+4\frac{F_Y}{F}+\frac{16}{3}\frac{F_Z}{F}\right)\left(\epsilon^\lambda_k(\mathbf{k_3})\widehat{k_{2k}}\right)\widehat{k_{2i}}k_{2j} \\
&+\left(\frac{4}{3}\epsilon+\frac{8}{3}\frac{F_Y}{F}+\frac{32}{9}\frac{F_Z}{F}\right)\epsilon^\lambda_i(\mathbf{k_3})k_{2j}\Bigg]\left\{2i\Im\Big[I_1(\bar{k}_1,\bar{k}_2,\bar{k}_3)\Big]\right\}.
\end{aligned} \tag{5.57}$$

Here, the symbol $I_1(\bar{k}_1,\bar{k}_2,\bar{k}_3)$ denotes the following integral in conformal time,

$$I_1(\bar{k}_1,\bar{k}_2,\bar{k}_3) = \int_{-\infty(1-i\epsilon)}^{\tau_e}\frac{d\tau}{\tau^4}\big(1-i\bar{k}_1\tau\big)\big(1-i\bar{k}_2\tau-\frac{1}{3}\bar{k}_2^2\tau^2\big)\big(1-i\bar{k}_3\tau-\frac{1}{3}\bar{k}_3^2\tau^2\big)e^{i\bar{K}_t\tau}, \tag{5.58}$$

while the meaning of the above notation is given by

$$\bar{k}_1\equiv k_1\,,\qquad \bar{k}_2\equiv c_Lk_2\,,\qquad \bar{k}_3\equiv c_Tk_3\,,\qquad \bar{K}_t\equiv\bar{k}_1+\bar{k}_2+\bar{k}_3\,. \tag{5.59}$$

The computation of the integral I_1 can be performed with the method of integration by parts. Two aspects of the calculation are worth to be emphasized. The first is about the contour of integration, which damps the exponential factor at early times. The second aspect is that, for $\bar{K}_t\tau_e = \Lambda \approx 0$, many primitives are of the form $\bar{K}_t\frac{e^{i\Lambda}}{\Lambda} = \frac{\bar{K}_t}{\Lambda} + i\bar{K}_t + \mathcal{O}(\Lambda)$. By discarding all the terms $\mathcal{O}(\Lambda)$, the computation of I_1 leads to

$$I_1(\bar{k}_1, \bar{k}_2, \bar{k}_3) = -\left[\frac{\bar{K}_t^3}{3\Lambda^3} + \frac{\bar{K}_t}{6\Lambda}\left(2\bar{k}_1^2 + \sum_i \bar{k}_i^2\right)\right] + i\frac{\bar{k}_1^3}{3}\int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau \frac{e^{i\bar{K}_t\tau}}{\tau}$$
$$-\frac{i}{9}\left\{\bar{K}_t\left(3\bar{k}_1^2 + \sum_i \bar{k}_i^2 - \sum_{i<j}\bar{k}_i\bar{k}_j\right) + \frac{\bar{k}_2\bar{k}_3}{\bar{K}_t^2}\left[\bar{K}_t\left(3\bar{k}_1\bar{k}_2 + 3\bar{k}_1\bar{k}_3 + \bar{k}_2\bar{k}_3\right) + \bar{k}_1\bar{k}_2\bar{k}_3\right]\right\}. \quad (5.60)$$

It should be noted the appearance of the integral (5.28) in the right hand side of the above equation. As it has been discussed in 5.1.1, the real part of such an integral is approximated by $\gamma + N_{\bar{K}_t}$, being γ the Euler-Mascheroni constant and $N_{\bar{K}_t}$ the number of e-foldings from the time the scale corresponding to $\bar{K}_t$ has left the horizon and the time of the end of inflation.

By plugging the imaginary part of (5.60) into (5.57), we finally get the expression of the three-point function and the corresponding bispectrum:

$$\langle\gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\pi^\lambda_{\mathbf{k_3}}(\tau_e)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B^{s,\lambda}_{\gamma\zeta\pi_T}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}), \quad (5.61)$$

where

$$B^{s,\lambda}_{\gamma\zeta\pi_T}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) = \frac{3i}{8\epsilon^2 c_L^5 c_T^5}\left(\frac{H}{M}\right)^4 \frac{1}{k_1^3k_2^3k_3^5}$$
$$\epsilon^s_{ij}(\mathbf{k_1})\left[\left(\frac{8}{3}\frac{F_Y}{F} + 4\frac{F_Z}{F}\right)\left(\widehat{k_{2k}}k_{3k}\right)\epsilon^\lambda_i(\mathbf{k_3})\widehat{k_{2j}} - \left(2\epsilon + 4\frac{F_Y}{F} + \frac{16}{3}\frac{F_Z}{F}\right)\left(\epsilon^\lambda_k(\mathbf{k_3})\widehat{k_{2k}}\right)\widehat{k_{2i}}k_{2j}\right.$$
$$\left.+\left(\frac{4}{3}\epsilon + \frac{8}{3}\frac{F_Y}{F} + \frac{32}{9}\frac{F_Z}{F}\right)\epsilon^\lambda_i(\mathbf{k_3})k_{2j}\right]\left\{3\bar{k}_1^3\left(\gamma + N_{\bar{K}_t}\right) - \bar{K}_t\left(3\bar{k}_1^2 + \sum_i\bar{k}_i^2 - \sum_{i<j}\bar{k}_i\bar{k}_j\right)\right.$$
$$\left.-\frac{\bar{k}_2\bar{k}_3}{\bar{K}_t^2}\left[\bar{K}_t\left(3\bar{k}_1\bar{k}_2 + 3\bar{k}_1\bar{k}_3 + \bar{k}_2\bar{k}_3\right) + \bar{k}_1\bar{k}_2\bar{k}_3\right]\right\}. \quad (5.62)$$

5.2.2 The $\gamma\zeta\zeta$ bispectrum

The correlator $\langle\gamma\zeta\zeta\rangle$ will be extrapolated from the cubic Lagrangian that involves two π_L and one γ, which in our convention is denoted by $\mathcal{L}^F_{\gamma\zeta\zeta}$. The latter is included in (5.38) and it is constructed out of the longitudinal component of π^i. One can readily obtain the expression of $\mathcal{L}^F_{\gamma\zeta\zeta}$ through the identification $\pi^i = \pi^i_L$ in (5.38), in conjunction with the property $\partial_k\pi^i_L = \partial_i\pi^k_L$. The final result is given by

$$\mathcal{L}^F_{\gamma\zeta\zeta} = a^3M_p^2H^2\gamma_{ij}\left[\left(\epsilon + \frac{10}{3}\frac{F_Y}{F} + \frac{14}{3}\frac{F_Z}{F}\right)\partial_k\pi^i_L\partial_k\pi^j_L - \left(\frac{4}{3}\epsilon + \frac{8}{3}\frac{F_Y}{F} + \frac{32}{9}\frac{F_Z}{F}\right)\partial_i\pi^j_L\partial_k\pi^k_L\right]. \quad (5.63)$$

The Hamiltonian operator $H_{\gamma\zeta\zeta}$ is then expressed by variables and fields in Fourier space as follows:

$$
\begin{aligned}
-H_{\gamma\zeta\zeta}(\tau) = \int d^3x\, \mathcal{L}^F_{\gamma\zeta\zeta} = \frac{9a^3M_P^2H^2}{(2\pi)^6} \int \prod_{a=1}^{3} d^3p_a\, \delta^{(3)}\left(\sum_{b=1}^{3} \mathbf{p}_b\right) \\
\sum_s \epsilon^s_{ij}(\mathbf{p_1}) \Bigg[\left(\epsilon + \frac{10}{3}\frac{F_Y}{F} + \frac{14}{3}\frac{F_Z}{F}\right) \widehat{p_{2i}}\widehat{p_{3j}} \left(\widehat{p_{2k}}\widehat{p_{3k}}\right) \\
- \left(\frac{4}{3}\epsilon + \frac{8}{3}\frac{F_Y}{F} + \frac{32}{9}\frac{F_Z}{F}\right) \widehat{p_{2i}}\widehat{p_{2j}} \Bigg] \gamma^s_{\mathbf{p_1}}(\tau)\zeta_{\mathbf{p_2}}(\tau)\zeta_{\mathbf{p_3}}(\tau)\,. \qquad (5.64)
\end{aligned}
$$

The above form has been obtained by using $\epsilon^s_{ij}(\mathbf{k})k_{i,j} = 0$, $\left(p_{1i} + p_{2i} + p_{3,i}\right) = 0$ and $\widehat{p_k}\widehat{p_k} = 1$. The correlator $\langle\gamma\zeta\zeta\rangle$ is then derived through

$$
\langle \gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\rangle = -i \int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau \langle \Big[\gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\,,\, H_{\gamma\zeta\zeta}(\tau)\Big]\rangle\,. \qquad (5.65)
$$

Only the Wick contractions reported below will survive in the expectation value of the vacuum state of the free theory:

$$
\begin{aligned}
&\langle \gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\gamma^{s'}_{\mathbf{p_1}}(\tau)\zeta_{\mathbf{p_2}}(\tau)\zeta_{\mathbf{p_3}}(\tau)\rangle = \\
&\langle 0|\gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\gamma^{s'}_{\mathbf{p_1}}(\tau)\zeta_{\mathbf{p_2}}(\tau)\zeta_{\mathbf{p_3}}(\tau) + \gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\gamma^{s'}_{\mathbf{p_1}}(\tau)\zeta_{\mathbf{p_2}}(\tau)\zeta_{\mathbf{p_3}}(\tau)|0\rangle = \\
&\langle 0|\Big[\gamma^{s\,+}_{\mathbf{k_1}}(\tau_e)\,,\,\gamma^{s'\,-}_{\mathbf{p_1}}(\tau)\Big]\left(\Big[\zeta^+_{\mathbf{k_2}}(\tau_e)\,,\,\zeta^-_{\mathbf{p_2}}(\tau)\Big]\Big[\zeta^+_{\mathbf{k_3}}(\tau_e)\,,\,\zeta^-_{\mathbf{p_3}}(\tau)\Big] + (p_2 \leftrightarrow p_3)\right)|0\rangle = \\
&(2\pi)^9\delta^{ss'}\delta^{(3)}(\mathbf{k_1}+\mathbf{p_1})\delta^{(3)}(\mathbf{k_2}+\mathbf{p_2})\delta^{(3)}(\mathbf{k_3}+\mathbf{p_3}) \\
&\quad \times \gamma^s_{cl}(k_1,\tau_e)(\gamma^{s'}_{cl}(p_1,\tau))^*\zeta_{cl}(k_2,\tau_e)\zeta^*_{cl}(p_2,\tau)\zeta_{cl}(k_3,\tau_e)\zeta^*_{cl}(p_3,\tau) + (p_2 \leftrightarrow p_3)\,. \qquad (5.66)
\end{aligned}
$$

By plugging the above expression in (5.65), we get the following intermediate result:

$$
\begin{aligned}
&\langle \gamma^s_{\mathbf{k_1}}(\tau_e)\zeta_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\rangle = 18M_P^2(2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\epsilon^s(\mathbf{k_1})_{ij}\Bigg[\\
&\left(\epsilon + \frac{10}{3}\frac{F_Y}{F} + \frac{14}{3}\frac{F_Z}{F}\right)\widehat{k}_{2i}\widehat{k}_{3j}\left(\widehat{k}_{2k}\widehat{k}_{3k}\right) - \left(\frac{2}{3}\epsilon + \frac{4}{3}\frac{F_Y}{F} + \frac{16}{9}\frac{F_Z}{F}\right)\left(\widehat{k}_{2i}\widehat{k}_{2j} + \widehat{k}_{3i}\widehat{k}_{3j}\right)\Bigg] \\
&\times\; i\gamma^s_{cl}(k_1,\tau_e)\zeta_{cl}(k_2,\tau_e)\zeta_{cl}(k_3,\tau_e)\int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau a^4H^2\gamma^{s*}_{cl}(k_1,\tau)\zeta^*_{cl}(k_2,\tau)\zeta^*_{cl}(k_3,\tau) + (c.c.)\,. \qquad (5.67)
\end{aligned}
$$

The last line of (5.67) will produce, in addition to a pre-factor, an integral of the same form of $I_1(\bar{k}_1,\bar{k}_2,\bar{k}_3)$ in (5.58). More specifically, we get

$$
\begin{aligned}
&iM_P^2H^2\gamma^s_{cl}(\tau_e)\zeta_{cl}(\tau_e)\zeta_{cl}(\tau_e)\int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau a^4\gamma^{s*}_{cl}(\tau)\zeta^*_{cl}(\tau)\zeta^*_{cl}(\tau) + (c.c) = \\
&\left(\frac{H}{2M_P\sqrt{\epsilon}}\right)^4 \frac{1}{c_L^{10}(k_1k_2k_3)^3}\Big\{-2\Im\Big[I_1(\bar{k}_1,\bar{k}_2,\bar{k}_3)\Big]\Big\}\,, \qquad (5.68)
\end{aligned}
$$

where

$$
\bar{k}_1 \equiv k_1\,, \qquad \bar{k}_2 \equiv c_Lk_2\,, \qquad \bar{k}_3 \equiv c_Lk_3\,, \qquad \bar{K}_t \equiv \bar{k}_1+\bar{k}_2+\bar{k}_3\,. \qquad (5.69)
$$

On the basis of the result (5.60), it is therefore possible to provide the three-point function $\langle\gamma\zeta\zeta\rangle$ and the corresponding bispectrum:

$$
\langle \gamma^s(\tau_e,\mathbf{k_1})\zeta(\tau_e,\mathbf{k_2})\zeta(\tau_e,\mathbf{k_3})\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B_{\gamma\zeta\zeta}(k_1,k_2,k_3)\,, \qquad (5.70)
$$

where

$$B_{\gamma\zeta\zeta}(k_1,k_2,k_3) = \frac{1}{4}\left(\frac{H}{M_P}\right)^4 \frac{1}{\epsilon^2 c_L^{10}}\left(\frac{1}{k_1k_2k_3}\right)^3$$
$$\epsilon^s_{ij}(\mathbf{k_1})\left[\left(\epsilon + \frac{10}{3}\frac{F_Y}{F} + \frac{14}{3}\frac{F_Z}{F}\right)\widehat{k_{2i}}\widehat{k_{3j}}\left(\widehat{k_{2k}}\widehat{k_{3k}}\right) - \left(\frac{2}{3}\epsilon + \frac{4}{3}\frac{F_Y}{F} + \frac{16}{9}\frac{F_Z}{F}\right)\left(\widehat{k_{2i}}\widehat{k_{2j}} + \widehat{k_{3i}}\widehat{k_{3j}}\right)\right]$$
$$\Bigg\{\bar{K}_t\left(3\bar{k}_1^2 + \sum_i \bar{k}_i^2 - \sum_{i<j}\bar{k}_i\bar{k}_j\right) + \frac{\bar{k}_2\bar{k}_3}{\bar{K}_t^2}\left[\bar{K}_t\Big(3\bar{k}_1\bar{k}_2 + 3\bar{k}_1\bar{k}_3 + \bar{k}_2\bar{k}_3\Big) + \bar{k}_1\bar{k}_2\bar{k}_3\right]$$
$$- 3\bar{k}_1^3\Big(\gamma + N_{\bar{K}_t}\Big)\Bigg\}\,. \tag{5.71}$$

We will now test the correctness of the above bispectrum form through a comparison with the results reported in literature. Since both in [9] and [31] the bispectrum $B_{\gamma\zeta\zeta}$ is given for long tensor modes, we have to take in (5.71) the limit $k_1 \approx 0$, $k_2 \approx k_3$:

$$\begin{aligned} B_{\gamma\zeta\zeta}(k_1\to 0,k_2,k_2) &= \frac{5}{8}\left(\frac{H}{M_P}\right)^4 \frac{1}{\epsilon^2 c_L^{10}k_1^3k_2^3}\epsilon^s_{ij}(\mathbf{k_1})\widehat{k_{2i}}\widehat{k_{2j}}\left(-\frac{1}{3}\epsilon + \frac{2}{3}\frac{F_Y}{F} + \frac{10}{9}\frac{F_Z}{F}\right) \\ &= \frac{5}{2}P_\zeta(k_2)P_\gamma(k_1)\frac{1}{\epsilon c_L^2}\left(-\frac{\epsilon}{3} + \frac{2}{3}\frac{F_Y}{F} + \frac{10}{9}\frac{F_Z}{F}\right)\epsilon^s_{ij}(\mathbf{k_1})\widehat{k_{2i}}\widehat{k_{2j}}\,. \end{aligned} \tag{5.72}$$

The above squeezed bispectrum is in agreement with the result of [9]. In order to consistently recover the result of [31], one should neglect the contribution $\epsilon/3$ in (5.72) and cease to consider F_Y and F_Z as independent parameters, by setting $F_Y = -F_Z + \mathcal{O}(\epsilon)$.

5.2.3 The $\gamma\pi_T\pi_T$ bispectrum

The cubic Lagrangian interaction $\mathcal{L}^F_{\gamma\pi_T\pi_T}$ is obtained by identifying $\pi^i = \pi^i_T$ in (5.38) and exploiting $\partial_k\pi^k_T = 0$:

$$\begin{aligned} \mathcal{L}^F_{\gamma\pi_T\pi_T} =& a^3H^2M_P^2\gamma_{ij}\Bigg[\left(\epsilon + \frac{4}{3}\frac{F_Y}{F} + \frac{5}{3}\frac{F_Z}{F}\right)\partial_i\pi^k_T\partial_i\pi^k_T \\ &+ \left(\frac{2}{3}\frac{F_Y}{F} + \frac{F_Z}{F}\right)\partial_k\pi^i_T\partial_k\pi^j_T + \left(\frac{4}{3}\frac{F_Y}{F} + 2\frac{F_Z}{F}\right)\partial_i\pi^k_T\partial_k\pi^j_T\Bigg]\,. \end{aligned} \tag{5.73}$$

The Hamiltonian operator $H_{\gamma\pi_T\pi_T}$ expressed through variables and fields in Fourier space therefore results in

$$\begin{aligned} -H_{\gamma\pi_T\pi_T} &= \int d^3x\,\mathcal{L}^F_{\gamma\pi_T\pi_T} = -\frac{a^3M_P^2H^2}{(2\pi)^6}\int\prod_{a=1}^3 d^3p_a\,\delta^{(3)}\left(\sum_{b=1}^3\mathbf{p}_b\right) \\ &\sum_s\sum_{\lambda_1\lambda_2}\epsilon^s_{ij}(\mathbf{p_1})\Bigg[\left(\epsilon + \frac{4}{3}\frac{F_Y}{F} + \frac{5}{3}\frac{F_Z}{F}\right)\left(\epsilon^{\lambda_1}_k(\mathbf{p_2})\epsilon^{\lambda_2}_k(\mathbf{p_3})\right)p_{2i}p_{3j} \\ &+ \left(\frac{2}{3}\frac{F_Y}{F} + \frac{F_Z}{F}\right)\epsilon^{\lambda_1}_i(\mathbf{p_2})\epsilon^{\lambda_2}_j(\mathbf{p_3})\Big(p_{2k}p_{3k}\Big) \\ &+ \left(\frac{4}{3}\frac{F_Y}{F} + 2\frac{F_Z}{F}\right)\left(\epsilon^{\lambda_1}_k(\mathbf{p_2})p_{3k}\right)\epsilon^{\lambda_2}_j(\mathbf{p_3})p_{2i}\Bigg]\gamma^s_{\mathbf{p_1}}(\tau)\pi^{\lambda_1}_{\mathbf{p_2}}(\tau)\pi^{\lambda_2}_{\mathbf{p_3}}(\tau)\,. \end{aligned} \tag{5.74}$$

With regard to the expectation value $\langle\gamma\pi_T\pi_T\rangle$ provided by the in-in formalism, we now focus on the Wick contractions surviving in $\langle 0|\gamma_{\mathbf{k_1}}\pi_{\mathbf{k_2}}\pi_{\mathbf{k_3}}H_{\gamma\pi_T\pi_T}|0\rangle$. We will report directly the final result, since the calculation is analogous to the one reported in the previous section:

$$\begin{aligned}
&\langle 0|\gamma^s_{\mathbf{k_1}}\pi^{\lambda_1}_{\mathbf{k_2}}\pi^{\lambda_2}_{\mathbf{k_3}}H_{\gamma\pi_T\pi_T}|0\rangle \supset \langle 0|\gamma^s_{\mathbf{k_1}}\pi^{\lambda_1}_{\mathbf{k_2}}\pi^{\lambda_2}_{\mathbf{k_3}}\gamma^{s'}_{\mathbf{p_1}}\pi^{\lambda'_1}_{\mathbf{p_2}}\pi^{\lambda'_2}_{\mathbf{p_3}}|0\rangle \\
&= (2\pi)^9\delta^{ss'}\delta^{(3)}(\mathbf{k_1}+\mathbf{p_1})\Big[\delta^{\lambda_1\lambda'_1}\delta^{\lambda_2\lambda'_2}\delta^{(3)}(\mathbf{k_2}+\mathbf{p_2})\delta^{(3)}(\mathbf{k_3}+\mathbf{p_3}) \\
&+ (p_2,\lambda'_1)\leftrightarrow(p_3,\lambda'_2)\Big]\gamma_{cl}(k_1,\tau_e)\pi_{T,cl}(k_2,\tau_e)\pi_{T,cl}(k_3,\tau_e)\gamma^*_{cl}(p_1,\tau)\pi^*_{T,cl}(p_2,\tau)\pi^*_{T,cl}(p_3,\tau)\,. \quad (5.75)
\end{aligned}$$

In the expression of the three-point function $\langle\gamma\pi_T\pi_T\rangle$ the above contractions will produce

$$\begin{aligned}
&\langle\gamma^s_{\mathbf{k_1}}(\tau_e)\pi^{\lambda_1}_{\mathbf{k_2}}(\tau_e)\pi^{\lambda_2}_{\mathbf{k_3}}(\tau_e)\rangle = -2(2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}) \\
&\epsilon^s_{ij}(\mathbf{k_1})\Bigg\{\left(\epsilon+\frac{4}{3}\frac{F_Y}{F}+\frac{5}{3}\frac{F_Z}{F}\right)(\epsilon^{\lambda_1}\cdot\epsilon^{\lambda_2})k_{2i}k_{3j} \\
&+\left(\frac{2}{3}\frac{F_Y}{F}+\frac{F_Z}{F}\right)\left[(k_2\cdot k_3)\epsilon^{\lambda_1}_i\epsilon^{\lambda_2}_j+(k_3\cdot\epsilon^{\lambda_1})k_{2i}\epsilon^{\lambda_2}_j+(k_2\cdot\epsilon^{\lambda_2})k_{3i}\epsilon^{\lambda_1}_j\right]\Bigg\} \\
&\times iM_P^2H^2\gamma_{cl}(\tau_e)\pi_{T,cl}(\tau_e)\pi_{T,cl}(\tau_e)\int_{-\infty(1-i\epsilon)}^{\tau_e}d\tau a^4\gamma_{cl}(\tau)\pi_{T,cl}(\tau)\pi_{T,cl}(\tau)+(c.c.) \quad (5.76)
\end{aligned}$$

Given that the mode function $\pi_{cl}(k,\tau)$ has the same time-dependence of $\zeta_{cl}(k,\tau)$ (see (5.24) and (5.25)), the last line of (5.76) will produce the integral $I_1(\bar{k}_1,\bar{k}_2,\bar{k}_3)$, with $\bar{k}_i$ properly defined for the case under consideration:

$$\bar{k}_1 = k_1\,,\quad \bar{k}_2 = c_T k_2\,,\quad \bar{k}_3 = c_T k_3\,. \quad (5.77)$$

The three-point function $\langle\gamma\pi_T\pi_T\rangle$ and the corresponding bispectrum are therefore given by

$$\langle\gamma^s_{\mathbf{k_1}}(\tau_e)\pi^{\lambda_1}_{t,\mathbf{k_2}}(\tau_e)\pi^{\lambda_2}_{T,\mathbf{k_3}}(\tau_e)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B^{s\lambda_1\lambda_2}_{\gamma\pi_T\pi_T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\,, \quad (5.78)$$

where

$$\begin{aligned}
&B^{s\lambda_1\lambda_2}_{\gamma\pi_T\pi_T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3) = -\frac{9}{4}\left(\frac{H}{M_P}\right)^4\frac{1}{\epsilon^2c_T^{10}}\left(\frac{1}{k_1^3k_2^5k_3^5}\right) \\
&\epsilon^s_{ij}(\mathbf{k_1})\Bigg\{\left(\epsilon+\frac{4}{3}\frac{F_Y}{F}+\frac{5}{3}\frac{F_Z}{F}\right)\Big(\epsilon^{\lambda_1}(\mathbf{k_2})\cdot\epsilon^{\lambda_2}(\mathbf{k_3})\Big)k_{2i}k_{3j} \\
&+\left(\frac{2}{3}\frac{F_Y}{F}+\frac{F_Z}{F}\right)\left[\Big(\mathbf{k}_2\cdot\mathbf{k}_3\Big)\epsilon^{\lambda_1}_i(\mathbf{k_2})\epsilon^{\lambda_2}_j(\mathbf{k_3})+\Big(\mathbf{k}_3\cdot\epsilon^{\lambda_1}\Big)k_{2i}\epsilon^{\lambda_2}_j+\Big(\mathbf{k}_2\cdot\epsilon^{\lambda_2}\Big)k_{3i}\epsilon^{\lambda_1}_j\right]\Bigg\} \\
&\Bigg\{\bar{K}_t\left(3\bar{k}_1^2+\sum_i\bar{k}_i^2-\sum_{i<j}\bar{k}_i\bar{k}_j\right)+\frac{\bar{k}_2\bar{k}_3}{\bar{K}_t^2}\left[\bar{K}_t\Big(3\bar{k}_1\bar{k}_2+3\bar{k}_1\bar{k}_3+\bar{k}_2\bar{k}_3\Big)+\bar{k}_1\bar{k}_2\bar{k}_3\right] \\
&+3\bar{k}_1^3\Big(\gamma+N_{\bar{K}_t}\Big)\Bigg\}\,. \quad (5.79)
\end{aligned}$$

In order to compare the above result with the one in [31], it is firstly necessary to consider the squeezed limit configuration with the tensor mode taken as the long mode, *i.e.* $\mathbf{q}=\mathbf{k}_1\approx\mathbf{0}$, $\mathbf{k}_2\approx-\mathbf{k}_3$:

$$\begin{aligned}
&B^{s\lambda_1\lambda_2}_{\gamma\pi_T\pi_T}(\mathbf{q},\mathbf{k},-\mathbf{k}) = -\frac{45}{8}\left(\frac{H}{M_P}\right)^4\frac{1}{\epsilon^2c_T^7}\frac{1}{q^3k^5} \\
&\epsilon^s_{ij}(\mathbf{q})\left[-\left(\epsilon+\frac{4}{3}\frac{F_Y}{F}+\frac{5}{3}\frac{F_Z}{F}\right)\Big(\epsilon^{\lambda_1}(\mathbf{k})\cdot\epsilon^{\lambda_1}(-\mathbf{k})\Big)\widehat{k}_i\widehat{k}_j-\left(\frac{2}{3}\frac{F_Y}{F}+\frac{F_Z}{F}\right)\epsilon^{\lambda_1}_i(\mathbf{k})\epsilon^{\lambda_2}_j(-\mathbf{k})\right] \\
&= \frac{5}{2}P_\gamma(q)P_T(k)\frac{1}{\epsilon c_T^2}\epsilon^s_{ij}(\mathbf{q})\left[\left(\epsilon+\frac{4}{3}\frac{F_Y}{F}+\frac{5}{3}\frac{F_Z}{F}\right)(\epsilon^{\lambda_1}\cdot\epsilon^{\lambda_1})\widehat{k}_i\widehat{k}_j+\left(\frac{2}{3}\frac{F_Y}{F}+\frac{F_Z}{F}\right)\epsilon^{\lambda_1}_i\epsilon^{\lambda_2}_j\right]. \quad (5.80)
\end{aligned}$$

In the above analysis F_Y and F_Z are assumed to be independent parameters. In order to consistently reproduce the result in [31], the contribution ϵ in (5.80) must be neglected and it must be further imposed the relation $F_Y = -F_Z + \mathcal{O}(\epsilon)$. Once these steps have been performed, (5.80) coincides with the squeezed bispectrum in [31].

5.3 Interactions involving two γ and one π

The cubic part of the Solid Lagrangian that involves the coupling between two tensors perturbations γ_{ij} and one π^i is given by

$$\mathcal{L}^F_{\gamma\gamma\pi} = \sqrt{-g}\Big[F_X\delta X_{\gamma\gamma\pi} + F_Y\delta Y_{\gamma\gamma\pi} + F_Z\delta Z_{\gamma\gamma\pi} + F_{XX}\delta X_\pi\delta X_{\gamma\gamma} + \big(F_{XY}+F_{XZ}\big)\delta X_\pi\delta Y_{\gamma\gamma}\Big]\,. \tag{5.81}$$

Because $(F_{XY}+F_{XZ})$ is $\mathcal{O}(\epsilon^2)$ the corresponding contribution hereafter will be neglected.

In order to compute explicitely (5.81) one should calculate the expressions of $\{\,\delta X_\pi,\, \delta X_{\gamma\gamma},\, \delta X_{\gamma\gamma\pi},\, \delta Y_{\gamma\gamma\pi},\, \delta Z_{\gamma\gamma\pi}\}$. These last terms are reported in section A.2.3. Once they have been inserted in (5.81), the cubic Lagrangian $\mathcal{L}^F_{\gamma\gamma\pi}$ results in

$$\begin{aligned}\mathcal{L}^F_{\gamma\gamma\pi} = a^3\Bigg[&\frac{F_X}{a^2}\gamma_{ij}\gamma_{jk}\partial_k\pi^i + \frac{2}{3}F_Y\gamma_{ij}\gamma_{jk}\partial_k\pi^i - \frac{2}{9}F_Y\gamma_{ij}\gamma_{ji}\partial_k\pi^k + \frac{8}{9}F_Z\gamma_{ij}\gamma_{jk}\partial_k\pi^i\\ &-\frac{8}{27}F_Z\gamma_{ij}\gamma_{ji}\partial_k\pi^k + \frac{F_{XX}}{a^4}\gamma_{ij}\gamma_{ji}\partial_k\pi^k\Bigg]\,.\end{aligned} \tag{5.82}$$

By using the definition of the background quantities

$$F = -3H^2M_P^2\,,\quad \bar{X} = \frac{3}{a^2}\,,\quad \frac{F_X\bar{X}}{F} = \epsilon\,,\quad \frac{F_{XX}\bar{X}^2}{F} = -\epsilon + \mathcal{O}(\epsilon^2)\,, \tag{5.83}$$

the cubic Lagrangian (5.81) up to first order in the slow-roll parameter will be thus given by

$$\mathcal{L}^F_{\gamma\gamma\pi} = a^3M_P^2H^2A\Big(\gamma_{ij}\gamma_{ij}\partial_k\pi^k - 3\gamma_{ik}\gamma_{jk}\partial_i\pi^j\Big)\,, \tag{5.84}$$

where

$$A = \frac{\epsilon}{3} + \frac{2}{3}\frac{F_Y}{F} + \frac{8}{9}\frac{F_Z}{F}\,. \tag{5.85}$$

The above expression equals the same interaction Lagrangian in [31], provided that the condition $F_Y = -F_Z + \mathcal{O}(\epsilon)$ is imposed and the contributions $\mathcal{O}(\epsilon)$ are discarded.

5.3.1 The $\gamma\gamma\zeta$ bispectrum

The Lagrangian $\mathcal{L}^F_{\gamma\gamma\zeta}$ is inferred from (5.84) by simply identifying π^i with its longitudinal component π^i_L, namely

$$\mathcal{L}^F_{\gamma\gamma\zeta} = a^3M_P^2H^2A\Big(\gamma_{ij}\gamma_{ij}\partial_k\pi^k_L - 3\gamma_{ik}\gamma_{jk}\partial_i\pi^j_L\Big)\,, \tag{5.86}$$

which readily leads to the Hamiltonian operator $H_{\gamma\gamma\zeta}$ expressed through fields and variables in Fourier space:

$$\begin{aligned}-H_{\gamma\gamma\zeta}(\tau) &= \int d^3x\mathcal{L}^F_{\gamma\gamma\zeta} = \frac{3a^3M_P^2H^2A}{(2\pi)^6}\int\prod_{a=1}^3 d^3p_a\,\delta^{(3)}\Bigg(\sum_{b=1}^3\mathbf{p}_b\Bigg)\\ &\sum_{s_1s_2}\Big[\epsilon^{s_1}_{ij}(\mathbf{p_1})\epsilon^{s_2}_{ij}(\mathbf{p_2}) - 3\epsilon^{s_1}_{ik}(\mathbf{p_1})\epsilon^{s_2}_{kj}(\mathbf{p_2})\widehat{p_{3j}}\widehat{p_{3i}}\Big]\,\gamma^{s_1}_{\mathbf{p_1}}(\tau)\gamma^{s_2}_{\mathbf{p_2}}(\tau)\zeta_{\mathbf{p_3}}(\tau)\,.\end{aligned} \tag{5.87}$$

Since the contraction rules have been extensively discussed in the previous sections, we can go directly to the outcome of the in-in formula applied to the correlator $\langle\gamma\gamma\zeta\rangle$:

$$
\begin{aligned}
&\langle\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\rangle = -i\int d\tau\, a\,\langle 0|\left[\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\,,\,H_{\gamma\gamma\zeta}(\tau)\right]|0\rangle\\
&= (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})3A\left(\frac{H}{M_P}\right)^4\frac{1}{4\epsilon c_L^5}\frac{1}{(k_1k_2)^3k_3^5}\\
&\left[\epsilon^{s_1}_{ij}(\mathbf{k_1})\epsilon^{s_2}_{ij}(\mathbf{k_2}) - 3\epsilon^{s_1}_{ik}(\mathbf{k_1})\epsilon^{s_2}_{kj}(\mathbf{k_2})\widehat{k_{3j}}\widehat{k_{3i}} + (k_1,s_1)\leftrightarrow(k_2,s_2)\right] iI_2(\bar{k}_1,\bar{k}_2,\bar{k}_3) + (c.c.).
\end{aligned}
\tag{5.88}
$$

The permutations are produced by the Wick contractions between the γ operators and they lead to a global factor 2, because of the symmetry of the above expression in $\mathbf{k_1}\leftrightarrow\mathbf{k_2}$. As for $I_2(\bar{k}_1,\bar{k}_2,\bar{k}_3)$, it denotes the following integral:

$$
I_2(\bar{k}_1,\bar{k}_2,\bar{k}_3) = \int_{-\infty(1-i\epsilon)}^{\tau_e} d\tau\frac{1}{\tau^4}\Big(1-i\bar{k}_1\tau\Big)\Big(1-i\bar{k}_2\tau\Big)\Big(1-i\bar{k}_3\tau-\frac{1}{3}\bar{k}_3^2\tau^2\Big)e^{i\bar{K}_t\tau}\,,
\tag{5.89}
$$

where

$$
\bar{k}_1 = k_1\,,\quad \bar{k}_2 = k_2\,,\quad \bar{k}_3 = c_Lk_3\,,\quad \bar{K}_t = \bar{k}_1+\bar{k}_2+\bar{k}_3\,.
\tag{5.90}
$$

By defining $\Lambda = \bar{K}_t\tau_e \approx 0$, the computation of the integral I_2 can be performed by following the same method that has been used for I_1 and reported in section 5.2.1. The result of such a computation is the expression below:

$$
\begin{aligned}
I_2(\bar{k}_1,\bar{k}_2,\bar{k}_3) = &-\left[\frac{1}{3\Lambda^3}+\frac{\bar{K}_t^2}{6\Lambda}+\frac{1}{3\Lambda}\left(\bar{k}_1^2+\bar{k}_2^2-\sum_{i<j}\bar{k}_i\bar{k}_j\right)\right]\\
&+\frac{i}{3}\left[\bar{K}_t\left(\bar{k}_1^2+\bar{k}_2^2-\sum_{i<j}\bar{k}_i\bar{k}_j\right)+\bar{k}_3^2(\bar{k}_1+\bar{k}_2)+3\bar{k}_1\bar{k}_2\bar{k}_3\right]\int_{-\infty(1-i\epsilon)}^{\tau_e}d\tau\frac{e^{i\bar{K}_t\tau}}{\tau}\\
&-i\left[\frac{\bar{K}_t^3}{9}+\frac{\bar{K}_t}{3}\left(\bar{k}_1^2+\bar{k}_2^2-\sum_{i<j}\bar{k}_i\bar{k}_j\right)+\frac{\bar{k}_1\bar{k}_2\bar{k}_3^2}{3\bar{K}_t}\right]\,.
\end{aligned}
\tag{5.91}
$$

The appearance of the integral (5.28) should be noticed, whose real part approximately corresponds to the number of e-foldings $N_{\bar{K}_t}$. In the previous bispectra, where only one γ was involved, the term $N_{\bar{K}_t}$ was suppressed in the squeezed configurations with long tensor modes. As it will be shown in a moment, the bispectra that include two or more tensor perturbations γ maintain this sizeable contribution when the mode of the third perturbation involved is taken as the long mode.

Given the above considerations, the final expression of (5.88) results in

$$
\langle\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\zeta_{\mathbf{k_3}}(\tau_e)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B^{s_1s_2}_{\gamma\gamma\zeta}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3})\,,
\tag{5.92}
$$

where

$$
\begin{aligned}
B^{s_1s_2}_{\gamma\gamma\zeta}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) = &\left(\frac{H}{M_P}\right)^4\frac{1}{\epsilon c_L^5}\frac{1}{(k_1k_2k_3)^3}\left(\epsilon+2\frac{F_Y}{F}+\frac{8}{3}\frac{F_Z}{F}\right)\\
&\times\left[\epsilon^{s_1}_{ij}(\mathbf{k_1})\epsilon^{s_2}_{ij}(\mathbf{k_2}) - 3\epsilon^{s_1}_{ik}(\mathbf{k_1})\epsilon^{s_2}_{kj}(\mathbf{k_2})\widehat{k_{3j}}\widehat{k_{3i}}\right]\\
&\times\Bigg\{-\frac{1}{3}\left[\bar{K}_t\left(\bar{k}_1^2+\bar{k}_2^2-\sum_{i<j}\bar{k}_i\bar{k}_j\right)+\bar{k}_3^2(\bar{k}_1+\bar{k}_2)+3\bar{k}_1\bar{k}_2\bar{k}_3\right]\Big(\gamma+N_{\bar{K}_t}\Big)\\
&+\left[\frac{\bar{K}_t^3}{9}+\frac{\bar{K}_t}{3}\left(\bar{k}_1^2+\bar{k}_2^2-\sum_{i<j}\bar{k}_i\bar{k}_j\right)+\frac{\bar{k}_1\bar{k}_2\bar{k}_3^2}{3\bar{K}_t}\right]\Bigg\}\,.
\end{aligned}
\tag{5.93}
$$

We will now compare (5.93) with the same bispectra reported in [31] by following the method that we have adopted so far. In the squeezed configuration $\mathbf{k_3} \equiv \mathbf{q} \approx \mathbf{0}$, $-\mathbf{k_2} \approx \mathbf{k_1} \equiv \mathbf{k}$, where the scalar perturbation ζ represents the long mode, the term $N_{\bar{K}_t}$ provides the dominant contribution:

$$\begin{aligned} B^{s_1 s_2}_{\gamma\gamma\zeta}(\mathbf{k}, -\mathbf{k}, \mathbf{q}) &= \left(\frac{H}{M_P}\right)^4 \frac{3A}{\epsilon c_L^5} \frac{1}{(k^2 q)^3} \left[\epsilon^{s_1}_{ij}(\mathbf{k})\epsilon^{s_2}_{ij}(-\mathbf{k}) - 3\epsilon^{s_1}_{ik}(\mathbf{k})\epsilon^{s_2}_{kj}(-\mathbf{k})\widehat{q_{3j}}\widehat{q_{3i}}\right] \\ &\times \left[-\frac{2}{3}k^3\Big(N_{2k} + \gamma - \frac{7}{3}\Big)\right] \\ &\approx -\frac{8}{3}\, 3A\, P_\gamma(k) P_\zeta(q) \left[\epsilon^{s_1}_{ij}(\mathbf{k})\epsilon^{s_2}_{ij}(-\mathbf{k}) - 3\epsilon^{s_1}_{ik}(\mathbf{k})\epsilon^{s_2}_{kj}(-\mathbf{k})\widehat{q_{3j}}\widehat{q_{3i}}\right] \ln(-k\tau_e)\,, \end{aligned} \quad (5.94)$$

which agrees with the result in [31] for $A = -\frac{2}{9}\frac{F_Y}{F}$.

5.3.2 The $\gamma\gamma\pi_T$ bispectrum

The cubic Lagrangian $\mathcal{L}^F_{\gamma\gamma\pi_T}$ is simpler with respect to $\mathcal{L}^F_{\gamma\gamma\zeta}$, since the first term in (5.81) disappears:

$$\mathcal{L}^F_{\gamma\gamma\pi_T} = -3a^3 M_P^2 H^2 A\, \gamma_{ik}\gamma_{jk}\partial_i \pi^j_T\,, \quad (5.95)$$

where A is the coefficient defined in (5.85). The Hamiltonian operator $H_{\gamma\gamma\pi_T}$ is then given by

$$\begin{aligned} -H_{\gamma\gamma\pi_T}(\tau) = \int d^3x \mathcal{L}^F_{\gamma\gamma\pi_T} &= -i3A\frac{a^3 M_P^2 H^2}{(2\pi)^6} \int \prod_{a=1}^{3} d^3p_a\, \delta^{(3)}\Bigg(\sum_{b=1}^{3}\mathbf{p}_b\Bigg) \\ &\sum_{s_1 s_2}\sum_{\lambda}\left[\epsilon^{s_1}_{ik}(\mathbf{p_1})\epsilon^{s_2}_{kj}(\mathbf{p_2})\epsilon^{\lambda}_{i}(\mathbf{p_3})\, p_{3j}\right]\gamma^{s_1}_{\mathbf{p_1}}(\tau)\gamma^{s_2}_{\mathbf{p_2}}(\tau)\pi^{\lambda}_{T,\,\mathbf{p_3}}(\tau)\,. \end{aligned} \quad (5.96)$$

The three-point function $\langle\gamma\gamma\pi_T\rangle$ is derived from the in-in formula in the same manner as the previous case. The mode function of $\pi_{T,cl}$ and ζ_{cl}, aside from the pre-factor, have an identical time-dependence (see (5.24) and (5.25)). As for the Wick contractions, they are identical to the previous case, with the exception of the appearance of $\delta^{\lambda\lambda'}$ that encodes the polarization information of the vector. The in-in formula therefore yields

$$\begin{aligned} &\langle\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\pi^{\lambda}_{T,\,\mathbf{k_3}}(\tau_e)\rangle = -i\int d\tau\, a\langle 0|\left[\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\pi^{\lambda}_{T,\,\mathbf{k_3}}(\tau_e)\,,\, H_{\gamma\gamma\pi_T}(\tau)\right]|0\rangle \\ &(2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\frac{9}{4}\left(\frac{H}{M_P}\right)^4\frac{3A}{\epsilon c_T^5}\frac{1}{(k_1k_2)^3k_3^5} \\ &\times 2i\left[k_{3i}\epsilon^{\lambda}_{j}(\mathbf{k_3})\left(\epsilon^{s_1}_{ik}(\mathbf{k_1})\epsilon^{s_2}_{kj}(\mathbf{k_2}) + (k_1 s_1)\leftrightarrow(k_2 s_2)\right)\right]\Im\Big[I_2(k_1,k_2,c_Tk_3)\Big]\,. \end{aligned} \quad (5.97)$$

In order to achieve the final result, one should insert the imaginary part of the integral I_2 defined in (5.89) in the above expression. By introducing the notation

$$\bar{k}_1 = k_1\,,\quad \bar{k}_2 = k_2\,,\quad \bar{k}_3 = k_3\,,\quad \bar{K}_t = \bar{k}_1 + \bar{k}_2 + \bar{k}_3\,, \quad (5.98)$$

the three-point function is therefore,

$$\langle\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\pi^{\lambda}_{T,\,\mathbf{k_3}}(\tau_e)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B^{s_1s_2\lambda}_{\gamma\gamma T}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3})\,, \quad (5.99)$$

where the corresponding bispectrum is given by

$$
\begin{aligned}
B^{s_1 s_2 \lambda}_{\gamma\gamma T}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) = i\frac{9}{2}\left(\frac{H}{M_P}\right)^4 \frac{1}{\epsilon c_T^5}\frac{1}{(k_1k_2)^3 k_3^5}\left(\epsilon + 2\frac{F_Y}{F} + \frac{8}{3}\frac{F_Z}{F}\right) \\
\times \left[k_{3i}\epsilon^{\lambda}_{j}(\mathbf{k_3})\Big(\epsilon^{s_1}_{ik}(\mathbf{k_1})\epsilon^{s_2}_{kj}(\mathbf{k_2}) + (k_1 s_1)\leftrightarrow(k_2 s_2)\Big)\right] \\
\times \left\{\frac{1}{3}\left[\bar{K}_t\left(\bar{k}_1^2+\bar{k}_2^2-\sum_{i<j}\bar{k}_i\bar{k}_j\right)+\bar{k}_3^2(\bar{k}_1+\bar{k}_2)+3\bar{k}_1\bar{k}_2\bar{k}_3\right]\Big(\gamma+N_{\bar{K}_t}\Big)\right. \\
\left. -\left[\frac{\bar{K}_t^3}{9}+\frac{\bar{K}_t}{3}\left(\bar{k}_1^2+\bar{k}_2^2-\sum_{i<j}\bar{k}_i\bar{k}_j\right)+\frac{\bar{k}_1\bar{k}_2\bar{k}_3^2}{3\bar{K}_t}\right]\right\}. \qquad (5.100)
\end{aligned}
$$

In the squeezed limit $\mathbf{k_3}\equiv\mathbf{q}\approx\mathbf{0}$, $-\mathbf{k_2}\approx\mathbf{k_1}\equiv\mathbf{k}$, the above bispectrum shows the same dominant logarithmic enhancement of the previous case, namely

$$
B^{s_1 s_2 \lambda}_{\gamma\gamma T}(\mathbf{k},-\mathbf{k},\mathbf{q}) \approx 4i\,A\,P_\gamma(k)P_T(q)\epsilon^{\lambda}_{i}(\mathbf{q})q_j\left[\epsilon^{s_1}_{ik}(\mathbf{k})\epsilon^{s_2}_{kj}(-\mathbf{k})+\epsilon^{s_2}_{ij}(-\mathbf{k})\epsilon^{s_1}_{kj}(\mathbf{k})\right]\ln(-k\tau_e)\,, \qquad (5.101)
$$

which agrees with [31] for $A = -\frac{2}{9}\frac{F_Y}{F}$.

5.4 Interaction involving three γ

The last non-linear interaction that will be discussed is encoded in the cubic part of the Solid Lagrangian involving three tensor perturbations. In line with the approximation that we have chosen, only the terms up to first order in the slow-roll parameters will be considered. We are therefore interested in the following third-order Lagrangian:

$$
\mathcal{L}^F_{\gamma\gamma\gamma} = \sqrt{-g}\Big[F_X\delta X_{\gamma\gamma\gamma}+F_Y\delta Y_{\gamma\gamma\gamma}+F_Z\delta Z_{\gamma\gamma\gamma}\Big]\,. \qquad (5.102)
$$

We are able to compute explicitely (5.102) through the expressions of $\delta X_{\gamma\gamma\gamma}$, $\delta Y_{\gamma\gamma\gamma}$ and $\delta Z_{\gamma\gamma\gamma}$ reported in section A.2.3. The resulting cubic Lagrangian is therefore given by

$$
\mathcal{L}^F_{\gamma\gamma\gamma} = \frac{a^3M_P^2H^2}{3}\left(\frac{\epsilon}{2}+\frac{F_Y}{F}+\frac{4}{3}\frac{F_Z}{F}\right)Tr(\gamma^3)\,, \qquad (5.103)
$$

where the background relations (5.83) have been used.

5.4.1 The $\gamma\gamma\gamma$ bispectrum

For the sake of simplicity, we introduce in (5.103) the factor

$$
B = \frac{1}{3}\left(\frac{\epsilon}{2}+\frac{F_Y}{F}+\frac{4}{3}\frac{F_Z}{F}\right)\,. \qquad (5.104)
$$

As per the previous cases, the Hamiltonian operator $H_{\gamma\gamma\gamma}$ is derived from $\mathcal{L}^F_{\gamma\gamma\gamma}$:

$$
\begin{aligned}
-H_{\gamma\gamma\gamma}(\tau) = \int d^3x\mathcal{L}^F_{\gamma\gamma\gamma} = \frac{a^3M^2H^2B}{(2\pi)^6}\int\prod_{a=1}^{3}d^3p_a\,\delta^{(3)}\left(\sum_{b=1}^{3}\mathbf{p}_b\right) \\
\sum_{s_1s_2s_3}\epsilon^{s_1}_{ik}(\mathbf{p_1})\epsilon^{s_2}_{kj}(\mathbf{p_2})\epsilon^{s_3}_{ji}(\mathbf{p_3})\gamma^{s_1}_{\mathbf{p_1}}(\tau)\gamma^{s_2}_{\mathbf{p_2}}(\tau)\gamma^{s_3}_{\mathbf{p_3}}(\tau)\,. \qquad (5.105)
\end{aligned}
$$

The expectation value $\langle\gamma\gamma\gamma\rangle$ is provided by the usual in-in formula,

$$\langle\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\gamma^{s_3}_{\mathbf{k_3}}(\tau_e)\rangle = -i\int d\tau\, a\langle 0|\left[\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\gamma^{s_3}_{\mathbf{k_3}}(\tau_e)\,,\, H_{\gamma\gamma\pi_T}(\tau)\right]|0\rangle\,. \tag{5.106}$$

The terms surviving in $\langle 0|\gamma^{s_1}_{\mathbf{k_1}}\gamma^{s_2}_{\mathbf{k_2}}\gamma^{s_3}_{\mathbf{k_3}}H_{\gamma\gamma\gamma}|0\rangle$ are six in total, deriving from all the possible Wick contractions. The results of these contractions are reported below, where the permutation of p_i is understood to happen in conjuction with s'_i:

$$\begin{aligned}
&\langle 0|\gamma^{s_1}_{\mathbf{k_1}}\gamma^{s_2}_{\mathbf{k_2}}\gamma^{s_3}_{\mathbf{k_3}}H_{\gamma\gamma\gamma}(\tau)|0\rangle \supset \langle 0|\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\gamma^{s_3}_{\mathbf{k_3}}(\tau_e)\gamma^{s'_1}_{\mathbf{p_1}}(\tau)\gamma^{s'_2}_{\mathbf{p_2}}(\tau)\gamma^{s'_3}_{\mathbf{p_3}}(\tau)|0\rangle \\
&= (2\pi)^9\Bigg[\delta^{s_1s'_1}\delta^{s_2s'_2}\delta^{s_3s'_3}\delta^{(3)}(\mathbf{k_1}+\mathbf{p_1})\delta^{(3)}(\mathbf{k_1}+\mathbf{p_1})\delta^{(3)}(\mathbf{k_1}+\mathbf{p_1}) \\
&+(p_1\leftrightarrow p_2)+(p_1\leftrightarrow p_3)+(p_2\leftrightarrow p_3)+\begin{pmatrix} p_1\to p_2\\ p_2\to p_3\\ p_3\to p_1\end{pmatrix}+\begin{pmatrix} p_1\to p_3\\ p_2\to p_1\\ p_3\to p_2\end{pmatrix}\Bigg] \\
&\times\gamma_{cl}(\tau_e,k_1)\gamma_{cl}(\tau_e,k_2)\gamma_{cl}(\tau_e,k_3)\gamma^*_{cl}(\tau,p_1)\gamma^*_{cl}(\tau,p_2)\gamma^*_{cl}(\tau,p_3)\,.
\end{aligned}\tag{5.107}$$

Once the above expression has been inserted in the in-in integral (5.106), despite the abundance of permutations, the cyclic property of the trace makes possible to sum all the above contributions. We therefore get

$$\int\prod_{a=1}^{3}d^3p_a\sum_{s_1s_2s_3}\epsilon^{s_1}_{ij}\epsilon^{s_2}_{jk}\epsilon^{s_2}_{ki}\langle\gamma^{s_1}\gamma^{s_2}\gamma^{s_3}\gamma^{s'_1}\gamma^{s'_2}\gamma^{s'_3}\rangle = 6Tr\left[\epsilon^{s_1}\epsilon^{s_2}\epsilon^{s_3}\right]\gamma^{s_1}_{cl}\gamma^{s_2}_{cl}\gamma^{s_3}_{cl}\left(\gamma^{s_1}_{cl}\gamma^{s_2}_{cl}\gamma^{s_3}_{cl}\right)^*\,. \tag{5.108}$$

Once the mode function (5.26) has been inserted in (5.106), the three-point function results in

$$\begin{aligned}
&\langle\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\gamma^{s_3}_{\mathbf{k_3}}(\tau_e)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})\left(\frac{H}{M_P}\right)^4\frac{6B}{(k_1k_2k_3)^3} \\
&\times Tr[\epsilon^{s_1}(\mathbf{k_1})\epsilon^{s_2}(\mathbf{k_2})\epsilon^{s_3}(\mathbf{k_3})]\left[-2\Im\Big(I_3(k_1,k_2,k_3)\Big)\right],
\end{aligned}\tag{5.109}$$

where I_3 denotes the following integral:

$$I_3(k_1,k_2,k_3) = \int_{-\infty(1-i\epsilon)}^{\tau_e}d\tau\frac{1}{\tau^4}(1-ik_1\tau)(1-ik_2\tau)(1-ik_3\tau)e^{i(k_1+k_2+k_3)\tau}\,. \tag{5.110}$$

The analytic form of I_3 can be obtained in the same manner of I_1 and I_2, respectively defined in (5.58) and (5.89). We report below only its imaginary part, where $K_t = k_1+k_2+k_3$ is undestood:

$$\begin{aligned}
\Im\Big[I_3(k_1,k_2,k_3)\Big] = &-\frac{1}{9}K_t\left(4\sum_{i=1}^{3}k_i^2+\sum_{i<j}k_ik_j\right) \\
&+\left[\frac{K_t}{3}\left(\sum_{i=1}^{3}k_i^2-\sum_{i<j}k_ik_j\right)+k_1k_2k_3\right]\Big(\gamma+\ln(-K_t\tau_e)\Big)\,.
\end{aligned}\tag{5.111}$$

The three-point correlator in (5.106) is thus, finally,

$$\langle\gamma^{s_1}_{\mathbf{k_1}}(\tau_e)\gamma^{s_2}_{\mathbf{k_2}}(\tau_e)\gamma^{s_3}_{\mathbf{k_3}}(\tau_e)\rangle = (2\pi)^3\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B^{s_1s_2s_3}_{\gamma\gamma\gamma,\,F}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3})\,, \tag{5.112}$$

where the corresponding bispectrum is given by

$$\begin{aligned}
&B^{s_1s_2s_3}_{\gamma\gamma\gamma}(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) = \left(\frac{H}{M_P}\right)^4\left(\epsilon+2\frac{F_Y}{F}+\frac{8}{3}\frac{F_Z}{F}\right)\left(\frac{1}{k_1k_2k_3}\right)^3 Tr[\epsilon^{s_1}(\mathbf{k_1})\epsilon^{s_2}(\mathbf{k_2})\epsilon^{s_3}(\mathbf{k_3})] \\
&\times\left\{\frac{2}{9}K_t\left(4\sum_{i=1}^{3}k_i^2+\sum_{i<j}k_ik_j\right)-2\left[\frac{K_t}{3}\left(\sum_{i=1}^{3}k_i^2-\sum_{i<j}k_ik_j\right)+k_1k_2k_3\right]\Big(\gamma+N_{k_t}\Big)\right\}.
\end{aligned}\tag{5.113}$$

The squeezed configuration can be analyzed by considering $\mathbf{q} \equiv \mathbf{k_3} \to 0$ and $\mathbf{k} \equiv \mathbf{k_1} \approx -\mathbf{k_2}$:

$$\begin{aligned} B^{s_1 s_2 s_3}_{\gamma\gamma\gamma}(\mathbf{k}, -\mathbf{k}, \mathbf{q}) &= \left(\frac{H}{M_P}\right)^4 \left(\epsilon + 2\frac{F_Y}{F} + \frac{8}{3}\frac{F_Z}{F}\right) \left(\frac{1}{k^2 q}\right)^3 Tr[\epsilon^{s_1}(\mathbf{k})\epsilon^{s_2}(-\mathbf{k})\epsilon^{s_3}(\mathbf{q})] \\ &\quad \times \frac{4}{3}k^3 \Big(\frac{10}{3} - \gamma - \ln(2k\tau_e)\Big) \\ &\approx -\frac{4}{3}\left(\epsilon + 2\frac{F_Y}{F} + \frac{8}{3}\frac{F_Z}{F}\right) P_\gamma(k) P_\gamma(q) Tr[\epsilon^{s_1}(\mathbf{k})\epsilon^{s_2}(-\mathbf{k})\epsilon^{s_3}(\mathbf{q})] \ln(k\tau_e)\,. \end{aligned} \tag{5.114}$$

The above result equals the one reported in [31], once that the condition $F_Y = -F_Z + \mathcal{O}(\epsilon)$ has been imposed and the $\mathcal{O}(\epsilon)$ terms has been neglected.

Conclusions

In this thesis we first studied the theory of perturbation during the inflationary stage of the Universe, by analyzing first the theory of perturbation at linear order and then the non-linear regime and its predictions. In particular, we focused on the primordial bispectra as a distinctive signature of primordial non-Gaussianity and the necessary methodology to address their computation by starting from the Lagrangian of a specific model. We ultimately applied the above methodology to the model of Solid Inflation in order to derive six of the bispectra predicted within this scenario.

The main aims of this thesis were to examine the conceptual construction of Solid Inflation and explore some of its predictions on cosmological perturbations. Solid Inflation is a theory involving a set of three scalar fields $\{\Phi^I\}$. These scalar fields represent the internal degrees of freedom of the medium, whose background values individually break the isotropy and homogeneity of the space-time, namely

$$\langle \Phi^I \rangle = x^I \,. \tag{5.115}$$

Nonetheless, we learned that this scenario can be compatible with a Friedmann-Robertson-Walker background, provided that the Lagrangian of the solid is symmetric under shift transformations and rotations of the fields $\{\Phi^I\}$. Moreover, the possibility of sustaining an inflationary expansion requires an additional approximated symmetry under the scale transformations $\Phi^I \to \lambda \Phi^I$.

We reviewed the results concerning both the background dynamics and the free action of perturbations, which were already discussed in [32]. In line with the study of the linear perturbation theory that we made in Chapter 2, we derived the expression of the first order perturbation of the energy-momentum tensor. In particular, we provided the individual explicit expressions of the scalar, vector and tensor components of the anisotropic stress π_{ij}, namely Π_S, Π_V^i and Π_T^{ij}. We remarked that vector perturbations are sustained during inflation, which is an unusual feature whithin the inflationary models. As for tensor perturbations, Π_T^{ij} corresponds to a mass term for the gravitons. From the free action of the perturbations we were able to extrapolate the respective mode functions and power spectra. The expression of the mode functions of both the scalar curvature perturbation ζ and the tensor perturbations γ were derived. The latter results were already reported in [32] and [20]. We contributed partially to the derivation of the mode function of the vector perturbations π_T to first order in slow-roll, whose analytic form has never appeared in literature. However, the method for this derivation is suggested in [32].

The main achievement of this thesis was the computation of the following three-point functions to the leading order in slow-roll through the in-in formalism:

$$\langle \zeta\zeta\gamma \rangle \,, \, \langle \zeta\pi_T\gamma \rangle \,, \, \langle \pi_T\pi_T\gamma \rangle \,, \, \langle \zeta\gamma\gamma \rangle \,, \, \langle \pi_T\gamma\gamma \rangle \,, \, \langle \gamma\gamma\gamma \rangle \,. \tag{5.116}$$

The explicit expression of the above correlators has never appeared in literature for generic triangular configurations. All the above three-point functions, with the exception of $\langle \zeta\pi_T\gamma \rangle$, are reported in [31] and [9] in the squeezed configuration only. We compared our results with the ones of the analyzed literature by considering the squeezed configuration cited in [31] and [9]. We obtained a positive confirmation for all the cases.

We saw that all the above three-point functions include a contribution proportional to N_{K_t}, where $K_t = k_1 + k_2 + k_3$. The above term is the number of e-foldings from the time when the scale corresponding to K_t leaves the horizon during inflation to the end of inflation. Typically $N_{K_t} \approx 60$ is the value for observable cosmological large scales and thus it represents an enhancement of the non-Gaussian signals. In particular, the bispectra in the squeezed configuration with two short tensor modes and a long mode, whether it be a scalar, a vector or a tensor one, maintain such an enhancement. A possible explanation of this fact could be linked to the violation of the *tensor consistency relations* in Solid Inflation [23].

The study of the non-Gaussianities predicted by Solid Inflation is linked to a vast class of developments, of which we can provide only a partial and non-exaustive list. Extensions to Solid Inflation has been considered, for example, in [18, 15], where a fourth scalar field is introduced to account also for time diffeomorphism breaking, besides the breaking of space diffeomorphism, as it happens for Solid Inflation. Another possible direction of application of Solid Inflation is in the context of anisotropic background space-time, as done in [10, 20, 21]. Whithin Solid Inflation, a natural extension of the work of this thesis could be the inclusion of the contributions from the Einstein-Hilbert Lagrangian to the three-point functions, or the computation of the scalar trispectrum.

Appendices

Appendix A

Lagrangian expansion up to second and third order

In this appendix we will expand the Lagrangian of Solid Inflation up to third order. The contributions provided by the Einstein-Hilbert Lagrangian will be also considered.

We start from the action of the theory in the ADM formalism, as done in [48] and [28]:

$$S = \int d^4x N\sqrt{h}\Big\{\frac{M_P^2}{2}\Big[R^{(3)} + \frac{1}{N^2}\left(E_{ij}E^{ij} - E^2\right)\Big] + F(X,Y,Z)\Big\} \tag{A.1}$$

where

$$h_{ij} = a^2(t)[e^{\gamma}]_{ij}\,, \tag{A.2}$$

$$h = \det(h_{ij}) = \exp\left(\frac{1}{2}Tr\big[\ln\left(a^2\mathbb{I}_3\right)\big]\right)\exp\left(\frac{1}{2}Tr\big[\gamma\big]\right) = a^3\,, \tag{A.3}$$

$$\partial_i\gamma_{ij} = \gamma_{ii} = 0\,, \tag{A.4}$$

$$E_{ij} = \frac{1}{2}\left(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i\right)\,, \tag{A.5}$$

$$E = E^i_i = h^{ij}E_{ij}\,, \tag{A.6}$$

while the 3D Ricci scalar is constructed out of the 3D metric h_{ij}:

$$^{(3)}\Gamma^k_{ij} = \frac{1}{2}h^{kl}\left(\partial_i h_{jl} + \partial_j h_{il} - \partial_l h_{ij}\right)\,, \tag{A.7}$$

$$R^{(3)}_{ij} = \partial^{(3)}_k\Gamma^k_{ij} - \partial^{(3)}_j\Gamma^k_{ik} +^{(3)}\Gamma^k_{ij}\,{}^{(3)}\Gamma^l_{kl} -^{(3)}\Gamma^l_{ik}\,{}^{(3)}\Gamma^k_{lj}\,, \tag{A.8}$$

$$R^{(3)} = h^{ij}R^{(3)}_{ij}\,. \tag{A.9}$$

The Hamiltonian and the momentum constraint equations are, in order,

$$\frac{M_P^2}{2}\Big[R^{(3)} - \frac{1}{N^2}\left(E_{ij}E^{ij} - E^2\right)\Big] + F(X,Y,Z) + N\frac{\partial F(X,Y,Z)}{\partial N} = 0\,, \tag{A.10}$$

$$\frac{M_P^2}{2}\nabla_i\Big[\frac{1}{N}\left(E^i_j - \delta^i_j E\right)\Big] + N\frac{\partial F(X,Y,Z)}{\partial N^j} = 0\,. \tag{A.11}$$

A.1 Expansion of the Einstein-Hilbert Lagrangian up to third order

A.1.1 Expansion of $R^{(3)}$

First, we compute the 3D Christoffel symbol and then we will proceed to the 3D Ricci tensor and the 3D Ricci scalar. The basic elements of the calculation are h_{ij} and h^{ij} expanded up to second order in γ_{ij}:

$$h_{ij} = a^2(t)[e^\gamma]_{ij} = a^2(t)\Big(\mathbb{I}_3 + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma_{kj}\Big) + \mathcal{O}(\gamma^3)\,, \tag{A.12}$$

$$h^{ij} = \frac{1}{a^2(t)}[e^{-\gamma}]^{ij} = \frac{1}{a^2(t)}\Big(\mathbb{I}_3 - \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma_{kj}\Big) + \mathcal{O}(\gamma^3)\,. \tag{A.13}$$

By inserting the above expansions in (A.7), we get the 3D Christoffel symbol up to second order:

$$\begin{aligned} {}^{(3)}\Gamma^k_{ij} =& \frac{1}{2}\Big\{ \left(\partial_i\gamma_{kj} + \partial_j\gamma_{ki} - \partial_k\gamma_{ij}\right) \\ & - \gamma_{ka}\left(\partial_i\gamma_{aj} + \partial_i\gamma_{aj} - \partial_i\gamma_{aj}\right) + \frac{1}{2}\Big[\partial_i\left(\gamma_{kb}\gamma_{bj}\right) + \partial_j\left(\gamma_{kb}\gamma_{bi}\right) - \partial_k\left(\gamma_{ib}\gamma_{bj}\right)\Big]\Big\}\,. \end{aligned} \tag{A.14}$$

We list below the terms in (A.8) which contribute to the 3D Ricci tensor, in conjunction with the result of the contraction with h^{ij}:

$$\begin{aligned} \partial_k^{(3)}\Gamma^k_{ij} =& \frac{1}{2}\Big\{ -\partial_k\partial_k\gamma_{ij} - \gamma_{ka}\left(\partial_k\partial_i\gamma_{aj} + \partial_k\partial_j\gamma_{ai} - \partial_k\partial_a\gamma_{ij}\right) \\ & + \frac{1}{2}\Big[\partial_k\partial_i\left(\gamma_{ka}\gamma_{ak}\right) + \partial_k\partial_j\left(\gamma_{ka}\gamma_{ai}\right) - \partial_k\partial_k\left(\gamma_{ia}\gamma_{aj}\right)\Big]\Big\}\,, \end{aligned} \tag{A.15}$$

$$h^{ij}\partial_k^{(3)}\Gamma^k_{ij} = \frac{1}{2a^2(t)}\Big(\partial_i\gamma_{ka}\partial_k\gamma_{ai} - \partial_k\gamma_{ij}\partial_k\gamma_{ij}\Big) \tag{A.16}$$

$$\begin{aligned} \partial_j^{(3)}\Gamma^k_{ik} =& \frac{1}{2}\partial_j\Big\{ -\gamma_{ka}\Big[\partial_i\gamma_{ak} + \partial_k\gamma_{ai} - \partial_a\gamma_{ik}\Big] \\ & + \frac{1}{2}\Big[\partial_i\left(\gamma_{ka}\gamma_{ak}\right) + \partial_k\left(\gamma_{ka}\gamma_{ai}\right) - \partial_k\left(\gamma_{ia}\gamma_{ak}\right)\Big]\Big\}\,, \end{aligned} \tag{A.17}$$

$$h^{ij}\partial_j^{(3)}\Gamma^k_{ik} = 0\,, \tag{A.18}$$

$${}^{(3)}\Gamma^k_{ij}\,{}^{(3)}\Gamma^n_{kn} = 0\,, \tag{A.19}$$

$$-{}^{(3)}\Gamma^n_{ik}\,{}^{(3)}\Gamma^k_{nj} = -\frac{1}{4}\Big[\partial_i\gamma_{kn}\partial_j\gamma_{kn} + 2\partial_k\gamma_{in}\partial_n\gamma_{kj} - 2\partial_n\gamma_{ik}\partial_n\gamma_{kj}\Big]\,, \tag{A.20}$$

$$-h^{ij}\,{}^{(3)}\Gamma^n_{ik}\,{}^{(3)}\Gamma^k_{nj} = -\frac{1}{4a^2(t)}\Big(2\partial_i\gamma_{jk}\partial_k\gamma_{ij} - \partial_k\gamma_{ij}\partial_k\gamma_{ij}\Big)\,. \tag{A.21}$$

By summing all the contributions to the 3D Ricci scalar, we finally get $R^{(3)}$ up to second order:

$$R^{(3)} = -\frac{1}{4a^2(t)}\partial_k\gamma_{ij}\partial_k\gamma_{ij}\,. \tag{A.22}$$

As for the expression of $R^{(3)}$ to third order in γ, we report the result given in [33]:

$$R^{(3)} \underset{(III)}{=} \frac{1}{2}\Big(\gamma_{ik}\gamma_{jl} - \frac{1}{2}\gamma_{ij}\gamma_{kl}\Big)\partial_k\partial_l\gamma_{ij}\,. \tag{A.23}$$

A.1.2 Expansion of $E^{ij}E_{ij} - E^2$

$$E_{ij} = \frac{1}{2}\Big\{2a\dot{a}[e^\gamma]_{ij} + a^2\frac{\partial}{\partial t}[e^\gamma] - \Big[\Big(\partial_i N_j -^{(3)}\Gamma^k_{ij}N_k\Big) + (i\leftrightarrow j)\Big]\Big\}\,. \tag{A.24}$$

In order to compute E^{ij} and E, it is useful to keep h_{ij} in the the exponential form (A.12), so that the Baker-Campbell-Hausdorff formula (BCH) [54], in conjunction with the identity $[e^\gamma]_{ik}[e^{-\gamma}]_{kj} = \delta^i_j$, will facilitate the calculation and the identification of the order of every term. The BCH formula gives

$$\left[\left(\frac{\partial}{\partial t}e^\gamma\right)e^{-\gamma}\right]^i_j = \dot{\gamma}^i_j + \frac{1}{2}\Big[\gamma\,,\,\dot{\gamma}\Big]^i_j + \frac{1}{3!}\Big[\gamma\,,\,\Big[\gamma\,,\,\dot{\gamma}\Big]\Big]^i_j + \mathcal{O}(\gamma^4)\,, \tag{A.25}$$

$$\left[e^{-\gamma}\left(\frac{\partial}{\partial t}e^\gamma\right)\right]^i_j = \dot{\gamma}^i_j + \frac{1}{2}\Big[\dot{\gamma}\,,\,\gamma\Big]^i_j + \frac{1}{3!}\Big[\Big[\dot{\gamma}\,,\,\gamma\Big]\,,\,\gamma\Big]^i_j + \mathcal{O}(\gamma^4)\,. \tag{A.26}$$

It should be noticed that in spatially flat gauge, being valid (A.25),(A.26) and $Tr(\gamma) = 0$, no term of the type $\gamma\gamma\gamma$ is contained in $E^{ij}E_{ij} - E^2$. The possible forms in which such a term can potentially appear, are in fact

$$Tr(\gamma)Tr(\gamma)Tr(\dot{\gamma})\,, \quad Tr(\gamma)Tr\Big(\Big[\dot{\gamma},\gamma\Big]\Big)\,, \quad Tr\Big(\gamma\Big[\dot{\gamma},\gamma\Big]\Big)\,, \quad Tr\Big(\Big[\Big[\gamma,\dot{\gamma}\Big],\gamma\Big]\Big)\,. \tag{A.27}$$

The first and the second term in the above are apparently null. One can therefore prove that the others are null as well by using the cyclic property of the trace. We can thus safely state that no third order interaction of tensors comes from $E^{ij}E_{ij} - E^2$.

By using the formulas (A.25) and (A.26), after a lenghty but rather straightforward calculaition, we can give the expression of $E^{ij}E_{ij} - E^2$ up to third order in perturbations:

$$\begin{aligned} E^{ij}E_{ij} - E^2 = &- 6H^2 + 4H\partial_k N^k \\ &+ \frac{1}{2}\Big[(\partial_a N^b)^2 + (\partial_a N^b)(\partial_b N^a) - 2(\partial_a N^a)^2\Big] + \frac{1}{4}\dot{\gamma}_{ab}\dot{\gamma}_{ab} - 4H\gamma_{ab}\partial_a N^b \\ &- \frac{1}{2}\gamma_{ij}\left(\partial_i N^b + \partial_b N^i\right)\left(\partial_j N^b + \partial_b N^j\right) + 2\gamma_{ij}\partial_i N^j \partial_k N^k \\ &- \left(\partial_a\gamma_{bk} + \partial_b\gamma_{ak} - \partial_k\gamma_{ab}\right)\partial_a N^b N^k \\ &+ 2H\gamma_{ac}\gamma_{cb}\partial_a N^b + 5H\gamma_{ab}\partial_b\gamma_{ac}N^c - 3H\gamma_{ab}\partial_c\gamma_{ab}N^c \\ &- \dot{\gamma}_{ab}\partial_a\gamma_{bc}N^c - \frac{1}{2}\dot{\gamma}_{ab}\partial_a\gamma_{bc}N^c\,. \end{aligned} \tag{A.28}$$

A.2 Expansion of the Solid Lagrangian up to third order

The expansion of $F(X,Y,Z)$ can be formally written as follows:

$$\begin{aligned} F(X,Y,Z) =& \bar{F} + F_X\Delta X + F_Y\Delta Y + F_Z\Delta Z + \frac{1}{2}F_{XX}(\Delta X)^2 + \frac{1}{2}F_{YY}(\Delta Y)^2 + \frac{1}{2}F_{ZZ}(\Delta Z)^2 \\ &+ \Big(F_{XY}\Delta X\Delta Y + F_{XZ}\Delta X\Delta Z\Big) + \frac{1}{3!}F_{XXX}(\Delta X)^3 + \cdots\,; \end{aligned} \tag{A.29}$$

The first step for the computation of (A.29) is to split ΔX, ΔY and ΔZ according to their order in perturbations, namely

$$\Delta X = \delta X_1 + \delta X_2 + \delta X_3 + \cdots\,, \tag{A.30}$$

where δX_n encodes terms at n-order in fluctuations. We now state that some simplifications will occur, so that the number of the terms that one has to compute will be reduced. By a closer inspection of the variables X, Y and Z, defined as

$$X = [B]\,, \qquad Y = \frac{[B^2]}{X^2}\,, \qquad Z = \frac{[B^3]}{X^3}\,, \tag{A.31}$$

it is in fact possible to prove that

$$\delta Y_1 = \delta Z_1 = 0\,, \tag{A.32}$$

$$\delta Y_2 = \delta Z_2\,. \tag{A.33}$$

We did not explore the possible existence of higher order relations similar to (A.32) and (A.33). However, a systematic analysis in this direction can be achieved by following the suggestions in [46, 24]. Given the above considerations, the Lagrangian $F(X,Y,Z)$, when expanded up to third order, consists of the following contributions, listed below according to the order in fluctuations:

$$F_1 = F_X\delta X_1\,, \tag{A.34}$$

$$F_2 = F_X\delta X_2 + \big(F_Y + F_Z\big)\delta Y_2 + \frac{1}{2}F_{XX}\left(\delta X_1\right)^2\,, \tag{A.35}$$

$$F_3 = F_X\delta X_3 + F_Y\delta Y_3 + F_Z\delta Z_3 + F_{XX}\delta X_1\delta X_2 + \big(F_{XY} + F_{XZ}\big)\delta X_1\delta Y_2 + \frac{1}{6}F_{XXX}\left(\delta X_1\right)^3\,. \tag{A.36}$$

The next necessary step is to express the form of $\{\delta X_{1,2,3}\,,\,\delta Y_{2,3}\,,\,\delta Z_3\}$ in function of the fields of the theory. We thus proceed in listing the intermediate passages of such a computation, starting from the matrix B^{IJ} and then moving to the traces $[B]$, $[B^2]$, $[B^3]$, which are the basic elements one needs to derive δX, δY and δZ to the desired order. Each term will be labeled with a proper subscript, which denotes the number and the type of the fields contained in it.

A.2.1 The expansion of the matrix B

When expressed in the ADM variables, the matrix $B^{IJ} = g^{\mu\nu}\partial_\mu\phi^I\partial_\nu\phi^J$ is given by

$$B^{IJ} = -\frac{1}{N^2}\left(\dot\phi^I - N^k\partial_k\phi^I\right)\left(\dot\phi^J - N^k\partial_k\phi^J\right) + h^{km}\partial_k\phi^I\partial_m\phi^J\,. \tag{A.37}$$

We then expand the fields about an FRW background as follows:

$$\phi^I = x^I + \pi^I\,, \qquad h_{ij} = a^2[e^\gamma]_{ij}\,, \qquad N = 1+\delta N \tag{A.38}$$

where, in the spatially flat gauge, γ is traceless and transverse. Since we are interested in third order interactions, we need N and N^i to first order in fluctuations; $\delta N \equiv \delta N_1$ and $N^i \equiv N^i_1$ are therefore understood. The expansion of the matrix B that captures all the cubic interactions is thus given by

$$\begin{aligned} B^{ij} =& -(1-2\delta N)\left(\dot\pi^i - N^i - N^k\partial_k\pi^i\right)\left(\dot\pi^j - N^j - N^k\partial_k\pi^j\right)\\ &+\frac{1}{a^2}\left(\delta_{km} - \gamma_{km} + \frac{1}{2}\gamma_{ka}\gamma_{am} - \frac{1}{6}\gamma_{ka}\gamma_{ab}\gamma_{bm}\right)\left(\delta^i_k\delta^j_m + \delta^i_k\partial_m\pi^j + \delta^j_m\partial_k\pi^i + \partial_k\pi^i\partial_m\pi^j\right)\,, \end{aligned} \tag{A.39}$$

where the second line is the outcome of the expansion of $e^{-\gamma}$.

We list below all the terms resulting from the expansion of B^{ij}. We will denote their order in perturbations and their field content by a proper subscript. The zeroth and first order terms are

$$B^{ij}_0 = \frac{1}{a^2}\delta^{ij}\,, \tag{A.40}$$

$$B^{ij}_\pi = \frac{1}{a^2}\partial_i\pi^j + (i\leftrightarrow j)\,, \tag{A.41}$$

$$B^{ij}_\gamma = -\frac{1}{a^2}\gamma_{ij}\,. \tag{A.42}$$

The second order terms we are interest in are given by

$$B^{ij}_{\pi\pi} = \frac{1}{a^2}\partial_k\pi^i\partial_k\pi^j - (\dot\pi^i - N^i)(\dot\pi^j - N^j)\,, \tag{A.43}$$

$$B^{ij}_{\gamma\pi} = -\frac{1}{a^2}\gamma_{ik}\partial_k\pi^j + (i\leftrightarrow j)\,, \tag{A.44}$$

$$B^{ij}_{\gamma\gamma} = \frac{1}{2a^2}\gamma_{ik}\gamma_{kj}\,. \tag{A.45}$$

As for the third order, we have

$$B^{ij}_{\gamma\gamma\gamma} = -\frac{1}{6a^2}\gamma_{ia}\gamma_{ab}\gamma_{bj}\,, \tag{A.46}$$

$$B^{ij}_{\gamma\gamma\pi} = \frac{1}{2a^2}\gamma_{ia}\gamma_{am}\partial_m\pi^j + (i\leftrightarrow j)\,, \tag{A.47}$$

$$B^{ij}_{\gamma\pi\pi} = -\frac{1}{a^2}\gamma_{km}\partial_k\pi^i\partial_m\pi^j\,, \tag{A.48}$$

$$B^{ij}_{\pi\pi\pi} = 2\delta N\left(\dot\pi^i - N^i\right)\left(\dot\pi^j - N^j\right) + \left[N^k\partial_k\pi^j\left(\dot\pi^i - N^i\right) + (i\leftrightarrow j)\right]\,. \tag{A.49}$$

A.2.2 The traces of the B^2 and B^3

The traces of B^2 and B^3 are the intermediate step to achieve δX, δY and δZ. We will list below their contributions, according to the order in perturbations and the field content. Such traces, at zeroth and first order, result in

$$[B^2]_0 = \frac{3}{a^4}\,, \tag{A.50}$$

$$[B^2]_\gamma = 2B_0^{ij}B_\gamma^{ij} = -\frac{2}{a^4}\gamma_{ii} = 0\,, \tag{A.51}$$

$$[B^2]_\pi = 2B_0^{ij}B_\pi^{ij} = \frac{4}{a^4}\partial_i\pi^i\,, \tag{A.52}$$

$$[B^3]_0 = \frac{3}{a^6}\,, \tag{A.53}$$

$$[B^3]_\gamma = 3B_0^{ik}B_0^{kj}B_\gamma^{ij} = -\frac{3}{a^6}\gamma_{ii} = 0\,, \tag{A.54}$$

$$[B^3]_\pi = 3B_0^{ik}B_0^{kj}B_\pi^{ij} = \frac{6}{a^6}\partial_i\pi^i\,. \tag{A.55}$$

As for the second order in perturbations, the relative elements in $[B^2]$ are given by

$$[B^2]_{\gamma\pi} = 2B_0^{ij}B_{\gamma\pi}^{ij} + 2B_\gamma^{ij}B_\pi^{ij} = -\frac{8}{a^4}\gamma_{ij}\partial_i\pi^j\,, \tag{A.56}$$

$$[B^2]_{\gamma\gamma} = 2B_0^{ij}B_{\gamma\gamma}^{ij} + B_\gamma^{ij}B_\gamma^{ij} = \frac{2}{a^4}\gamma_{ij}\gamma_{ij}\,, \tag{A.57}$$

$$[B^2]_{\pi\pi} = 2B_0^{ij}B_{\pi\pi}^{ij} + B_\pi^{ij}B_\pi^{ij} = \frac{4}{a^4}(\partial_i\pi^j)^2 + \frac{2}{a^4}\partial_i\pi^k\partial_k\pi^i - \frac{2}{a^2}(\dot\pi^i - N^i)^2\,. \tag{A.58}$$

while, with regards to $[B^3]$,

$$[B^3]_{\gamma\gamma} = 3B_0^{ij}B_0^{ij}B_{\gamma\gamma}^{ij} + 3B_0^{ij}B_\gamma^{ij}B_\gamma^{ij} = \frac{9}{2a^6}\gamma_{ij}\gamma_{ij}, \tag{A.59}$$

$$[B^3]_{\pi\pi} = 3B_0^{ij}B_0^{ij}B_{\pi\pi}^{ij} + 3B_0^{ij}B_\pi^{ij}B_\pi^{ij} = \frac{9}{a^6}(\partial_i\pi^j)^2 + \frac{6}{a^6}\partial_i\pi^j\partial_j\pi^i - \frac{3}{a^4}\left(\dot\pi^i - N^i\right)^2, \tag{A.60}$$

$$[B^3]_{\gamma\pi} = 3B_0^{ij}B_0^{ij}B_{\gamma\pi}^{ij} + 6B_0^{ij}B_\pi^{ij}B_\gamma^{ij} = -\frac{18}{a^6}\gamma_{ij}\partial_i\pi^j. \tag{A.61}$$

Finally, the contributions of $[B^2]$ that encodes the interactions are listed below:

$$[B^2]_{\gamma\gamma\gamma} = 2B_0^{ij}B_{\gamma\gamma\gamma}^{ij} + 2B_\gamma^{ij}B_{\gamma\gamma}^{ij} = -\frac{4}{3a^4}\gamma_{ia}\gamma_{ab}\gamma_{bi}\,, \tag{A.62}$$

$$[B^2]_{\gamma\gamma\pi} = 2B_0^{ij}B_{\gamma\gamma\pi}^{ij} + 2B_\gamma^{ij}B_{\gamma\pi}^{ij} + 2B_\pi^{ij}B_{\gamma\gamma}^{ij} = \frac{8}{a^4}\gamma_{ia}\gamma_{am}\partial_m\pi^i\,, \tag{A.63}$$

$$\begin{aligned}[B^2]_{\gamma\pi\pi} &= 2B_0^{ij}B_{\gamma\pi\pi}^{ij} + 2B_\gamma^{ij}B_{\pi\pi}^{ij} + 2B_\pi^{ij}B_{\gamma\pi}^{ij} = \frac{2}{a^2}\gamma_{ij}\left(\dot\pi^i - N^i\right)\left(\dot\pi^j - N^j\right) \\ &\quad - \frac{2}{a^4}\gamma_{ij}\left(3\partial_i\pi^k\partial_j\pi^k + 2\partial_i\pi^k\partial_k\pi^j + \partial_k\pi^i\partial_k\pi^j\right)\,,\end{aligned} \tag{A.64}$$

$$\begin{aligned}[B^2]_{\pi\pi\pi} &= 2B_0^{ij}B_{\pi\pi\pi}^{ij} + 2B_\pi^{ij}B_{\pi\pi}^{ij} = \frac{4}{a^4}\partial_i\pi^j\partial_k\pi^i\partial_k\pi^j - \frac{4}{a^2}\partial_i\pi^j\left(\dot\pi^i - N^i\right)\left(\dot\pi^j - N^j\right) \\ &\quad + \frac{4}{a^2}\delta N\left(\dot\pi^i - N^i\right)^2 + \frac{4}{a^2}N^k\partial_k\pi^i\left(\dot\pi^i - N^i\right)\,.\end{aligned} \tag{A.65}$$

As for the terms in $[B^3]$, they are

$$[B^3]_{\gamma\gamma\gamma} = 3B_0^{ik}B_0^{kj}B_{\gamma\gamma\gamma}^{ji} + 6B_0^{ik}B_\gamma^{kj}B_{\gamma\gamma}^{ji} + B_\gamma^{ik}B_\gamma^{kj}B_\gamma^{ji} = -\frac{9}{2a^6}\gamma_{ik}\gamma_{kj}\gamma_{ji}\,, \tag{A.66}$$

$$\begin{aligned}[B^3]_{\gamma\gamma\pi} &= 3B_0^{ik}B_0^{kj}B_{\gamma\gamma\pi}^{ji} + 3B_\gamma^{ik}B_\gamma^{kj}B_\pi^{ji} + 6B_0^{ik}B_\gamma^{kj}B_{\gamma\pi}^{ji} + 6B_0^{ik}B_\pi^{kj}B_{\gamma\gamma}^{ji} \\ &= \frac{27}{a^6}\gamma_{ik}\gamma_{kj}\partial_j\pi^i\,,\end{aligned} \tag{A.67}$$

$$[B^3]_{\gamma\pi\pi} = 3B_0^{ik}B_0^{kj}B_{\gamma\pi\pi}^{ji} + 3B_\gamma^{ik}B_\pi^{kj}B_\pi^{ji} + 6B_0^{ik}B_\gamma^{kj}B_{\pi\pi}^{ji} + 6B_0^{ik}B_\pi^{kj}B_{\gamma\pi}^{ji} \tag{A.68}$$

$$\begin{aligned}&= -\frac{9}{a^6}\gamma_{ij}\left(2\partial_i\pi^k\partial_j\pi^k + 2\partial_i\pi^k\partial_k\pi^j + \partial_k\pi^i\partial_k\pi^j\right) \\ &\quad + \frac{6}{a^4}\gamma_{ij}\left(\dot\pi^i - N^i\right)\left(\dot\pi^j - N^j\right)\,,\end{aligned} \tag{A.69}$$

$$\begin{aligned}[B^3]_{\pi\pi\pi} &= 3B_0^{ik}B_0^{kj}B_{\pi\pi\pi}^{ji} + 6B_0^{ik}B_\pi^{kj}B_{\pi\pi}^{ji} + B_\pi^{ik}B_\pi^{kj}B_\pi^{ji} \\ &= \frac{2}{a^6}\partial_i\pi^j\left[\partial_i\pi^k\partial_j\pi^k + 7\partial_k\pi^i\partial_k\pi^j + \partial_i\pi^k\partial_k\pi^j + \partial_j\pi^k\partial_k\pi^i\right] \\ &\quad - \frac{6}{a^4}\left(\dot\pi^i - N^i\right)\left[2\partial_i\pi^j\left(\dot\pi^j - N^j\right) - \delta N\left(\dot\pi^i - N^i\right) - N^k\partial_k\pi^i\right]\,.\end{aligned} \tag{A.70}$$

A.2.3 δX, δY and δZ up to third order

Given the above results, it is now possible to obtain δX, δY and δZ up to third order in perturbations. Since the expanded Lagrangian (A.29) is constructed out of such terms, we can formulate the second order Lagrangian of γ, π_L and π_T. Moreover, by using the results reported below, we are able to derive the third order Lagrangian relative to all the types of interaction in Solid Inflation.

By taking the traces of (A.41)-(A.49), we will obtain the values of δX at different orders. The first-order ones, δX_1, are therefore:

$$\delta X_\pi = [B]_\pi = \frac{2}{a^2}\partial_i\pi^i\,, \tag{A.71}$$

$$\delta X_\gamma = [B]_\gamma = -\frac{1}{a^2}\gamma_{ii} = 0\,; \tag{A.72}$$

as for δX_2, they are:

$$\delta X_{\pi\pi} = [B]_{\pi\pi} = \frac{1}{a^2}(\partial_i\pi^j)^2 - \left(\dot\pi^i - N^i\right)^2\,, \tag{A.73}$$

$$\delta X_{\gamma\gamma} = [B]_{\gamma\gamma} = \frac{1}{2a^2}\gamma_{ij}\gamma_{ij}, \tag{A.74}$$

$$\delta X_{\gamma\pi} = [B]_{\gamma\pi} = -\frac{2}{a^2}\gamma_{ij}\partial_i\pi^j. \tag{A.75}$$

It should be pointed out that the mixing term $\delta X_{\gamma\pi}$ does not contribute to the second-order action, in agreement with the fact that no coupling between fluctuations of different kind appears at linear order in the Einstein equations. It is in fact equal to $\partial_i(\gamma_{ij}\pi^j)$, which is a boundary term. The interaction contibutions δX_3 are then given by

$$\delta X_{\gamma\gamma\gamma} = [B]_{\gamma\gamma\gamma} = -\frac{1}{6a^2}\gamma_{ik}\gamma_{kj}\gamma_{ki}\,, \tag{A.76}$$

$$\delta X_{\gamma\gamma\pi} = [B]_{\gamma\gamma\pi} = \frac{1}{a^2}\gamma_{ik}\gamma_{kj}\partial_j\pi^i\,, \tag{A.77}$$

$$\delta X_{\gamma\pi\pi} = [B]_{\gamma\pi\pi} = -\frac{1}{a^2}\gamma_{ij}\partial_i\pi^k\partial_j\pi^k\,, \tag{A.78}$$

$$\delta X_{\pi\pi\pi} = [B]_{\pi\pi\pi} = 2\left(\dot\pi^i - N^i\right)\left[\delta N\left(\dot\pi^i - N^i\right) + N^k\partial_k\pi^i\right]\,. \tag{A.79}$$

We now consider the perturbations δY and δZ up to third order. In the following, $\bar X$ denotes the background value of X. As previously mentioned, the first order fluctuations δY_1 and δZ_1 are identically null; they are in fact given by

$$\delta Y_1 = \frac{1}{\bar{X}^2}\left[[B^2]_1 - 2\frac{[B^2]_0}{\bar{X}}\delta X_1\right], \quad \delta Z_1 = \frac{1}{\bar{X}^3}\left[[B^3]_1 - 3\frac{[B^3]_0}{\bar{X}}\delta X_1\right], \tag{A.80}$$

where

$$[B^2]_1 = 2B_0^{ij}B_1^{ij} = \frac{2}{a^2}\delta^{ij}B_1^{ij} = 2\frac{[B^2]_0}{\bar{X}}\delta X_1, \quad [B^3]_1 = 3B_0^{ik}B_0^{kj}B_1^{ji} = \frac{3}{a^4}\delta^{ik}\delta^{kj}B_1^{ji} = 3\frac{[B^3]_0}{\bar{X}}\delta X_1\,.$$

The second-order fluctuations of Y and Z are calculated by using the following expressions:

$$\delta Y_2 = \frac{1}{\bar{X}^2}\left[[B^2]_2 - \frac{2}{\bar{X}}\left([B^2]_0\delta X_2 + [B^2]_1\delta X_1\right) + 3\frac{[B^2]_0}{\bar{X}^2}\delta X_1^2\right], \tag{A.81}$$

$$\delta Z_2 = \frac{1}{\bar{X}^3}\left[[B^3]_2 - \frac{3}{\bar{X}}\left([B^3]_0\delta X_2 + [B^3]_1\delta X_1\right) + 6\frac{[B^3]_0}{\bar{X}^3}\delta X_1^2\right]. \tag{A.82}$$

A close inspection of the above formulas reveals that $\delta Y_2 = \delta Z_2$. We thus list below only the values of δY_2:

$$\delta Y_{\pi\pi} = \frac{2}{9}(\partial_i\pi^j)^2 + \frac{2}{27}(\partial_i\pi^i)^2\,, \tag{A.83}$$

$$\delta Y_{\gamma\gamma} = \frac{1}{9}\gamma_{ij}\gamma_{ij}\,, \tag{A.84}$$

$$\delta Y_{\gamma\pi} = -\frac{4}{9}\gamma_{ij}\partial_i\pi^j\,. \tag{A.85}$$

We can discard $\delta Y_{\gamma\pi}$ in the second-order action, being the same argument still valid as previously applied to $\delta X_{\gamma\pi}$. In order to compute the generic interaction term δY_3, we will use the following formula:

$$\delta Y_3 = \frac{1}{\bar{X}^2}\Big[[B^2]_3 - \frac{2}{\bar{X}}\left([B^2]_2\delta X_1 + [B^2]_1\delta X_2 + [B^2]_0\delta X_3\right) + \tag{A.86}$$

$$\frac{3}{\bar{X}^2}\left([B^2]_1\delta X_1^2 + 2[B^2]_0\delta X_1\delta X_2\right) - \frac{4}{\bar{X}^3}[B^2]_0\delta X_1^3\Big], \tag{A.87}$$

$$\tag{A.88}$$

$$\delta Z_3 = \frac{1}{\bar{X}^3}\Big[[B^3]_3 - \frac{3}{\bar{X}}\left([B^3]_2\delta X_1 + [B^3]_1\delta X_2 + [B^3]_0\delta X_3\right) + \tag{A.89}$$

$$\frac{6}{\bar{X}^2}\left([B^3]_1\delta X_1^2 + 2[B^3]_0\delta X_1\delta X_2\right) - \frac{10}{\bar{X}^3}[B^3]_0\delta X_1^3\Big]. \tag{A.90}$$

The results of the above formulas are listed below:

$$\delta Y_{\gamma\gamma\gamma} = -\frac{1}{9}\gamma_{ik}\gamma_{kj}\gamma_{ki}\,, \tag{A.91}$$

$$\delta Z_{\gamma\gamma\gamma} = -\frac{4}{27}\gamma_{ik}\gamma_{kj}\gamma_{ki}\,, \tag{A.92}$$

$$\delta Y_{\gamma\gamma\pi} = \frac{2}{3}\gamma_{ik}\gamma_{kj}\partial_j\pi^i - \frac{2}{9}\gamma_{ij}\gamma_{ji}\partial_k\pi^k\,, \tag{A.93}$$

$$\delta Z_{\gamma\gamma\pi} = \frac{8}{9}\gamma_{ik}\gamma_{kj}\partial_j\pi^i - \frac{8}{27}\gamma_{ij}\gamma_{ji}\partial_k\pi^k\,, \tag{A.94}$$

$$\begin{aligned}\delta Y_{\gamma\pi\pi} = \frac{2}{9}\gamma_{ij}\Big[4\partial_i\pi^j\partial_k\pi^k - 2\partial_i\pi^k\partial_k\pi^j - \partial_k\pi^i\partial_k\pi^j - 2\partial_i\pi^k\partial_j\pi^k\Big]& \\ + \frac{2a^2}{9}\gamma_{ij}\left(\dot\pi^i - N^i\right)\left(\dot\pi^j - N^j\right)\,,&\end{aligned} \tag{A.95}$$

$$\begin{aligned}\delta Z_{\gamma\pi\pi} = \frac{1}{3}\gamma_{ij}\Big[\frac{32}{9}\partial_i\pi^j\partial_k\pi^k - \partial_k\pi^i\partial_k\pi^j - 2\partial_i\pi^k\partial_k\pi^j - \frac{5}{3}\partial_i\pi^k\partial_j\pi^k\Big]& \\ + \frac{2a^2}{9}\gamma_{ij}\left(\dot\pi^i - N^i\right)\left(\dot\pi^j - N^j\right)\,,&\end{aligned} \tag{A.96}$$

$$\begin{aligned}\delta Y_{\pi\pi\pi} = \frac{16}{81}(\partial_k\pi^k)^3 - \frac{4}{9}\partial_k\pi^k(\partial_i\pi^j)^2 - \frac{8}{27}(\partial_m\pi^m)\partial_i\pi^k\partial_k\pi^i + \frac{4}{9}\partial_i\pi^j\partial_i\pi^k\partial_k\pi^j \\ - \frac{4a^2}{9}\left(\dot\pi^i - N^i\right)\partial_i\pi^j\left(\dot\pi^j - N^j\right) + \frac{4a^2}{27}\partial_k\pi^k\left(\dot\pi^i - N^i\right)^2\,,\end{aligned} \tag{A.97}$$

$$\begin{aligned}\delta Z_{\pi\pi\pi} = \frac{64}{243}(\partial_k\pi^k)^3 - \frac{16}{27}(\partial_k\pi^k)(\partial_i\pi^j)^2 - \frac{4}{9}(\partial_k\pi^k)(\partial_i\pi^j\partial_j\pi^i) + \frac{2}{27}(\partial_k\pi^i)(\partial_i\pi^j\partial_j\pi^k) \\ + \frac{2}{3}(\partial_i\pi^j)(\partial_i\pi^k\partial_k\pi^j) + \frac{4a^2}{27}\partial_k\pi^k(\dot\pi^i - N^i)^2 - \frac{4a^2}{9}(\dot\pi^i - N^i)\partial_i\pi^j(\dot\pi^j - N^j)\,.\end{aligned} \tag{A.98}$$

Appendix B

Properties of Hankel functions

In dealing with cosmological perturbations, in a large variety of models, their evolution in time is encoded in the form of Bessel equations. As seen in the first chapter, the Bessel equation is the one that the canonically normalized fields obey to in the case of a quasi-de Sitter space-time. Its ubiquitous presence in cosmological models, therefore, should not come as a surprise. We will focus only on the case in which the independent variable is real, as well as the parameter ν:

$$z^2 \frac{d^2}{dz^2} Z(z) + z \frac{d}{dz} Z(z) + (z^2 - \nu^2) Z(z) = 0 \,, \qquad \nu, z \in \mathbb{R} \,. \tag{B.1}$$

When the equation of motion is not directly given in the Bessel's form, to find a solution it is convenient to turn it into (B.1), by redefining field and variables, or taking into account an appropriate degree of approximation (*e.g* discarding high powers of the slow-roll parameters).

The solution of the above equation is often given as combination of *Bessel functions of the first kind* J_ν and *Bessel functions of the second kind* Y_ν []. The subscript ν is said to be the *order* of the function. The solution can be recast in terms of $H_\nu^{(1)}$ and $H_\nu^{(2)}$, the so called *Hankel functions*, respectively of first and second kind, defined as

$$H^{(1)}(z) \equiv J_\nu(z) + iY_\nu(z) \,,$$
$$H^{(2)}(z) \equiv J_\nu(z) - iY_\nu(z) \,,$$

When z is real, J_ν an Y_ν are real, and $H_\nu^{(1)}(z) = [H_\nu^{(2)}(z)]^*$.

We are particularly interested in the asymptotic behaviour of Hankel functions. The two following forms hold true when the argument acquires large values:

$$H_\nu^{(1)}(z) \underset{z\to+\infty}{\longrightarrow} \sqrt{\frac{2}{\pi z}} e^{iz - i\frac{\pi}{2}(\nu+\frac{1}{2})} \,, \qquad H_\nu^{(2)}(z) \underset{z\to+\infty}{\longrightarrow} \sqrt{\frac{2}{\pi z}} e^{-iz + i\frac{\pi}{2}(\nu+\frac{1}{2})} \,. \tag{B.2}$$

Given that for cosmological perturbations the argument is generally in the form $z = -ck\tau$, where c denotes the velocity of the perturbation, the two limits above represent the behaviour of Hankel functions in the deep sub-horizon regime. The presence of the complex exponential signals the oscillatory behavioiur of the mode in the sub-orizon regime. In this context, it is useful to recall that the equation of motion is indeed the one of an harmonic oscillator in that limit.

As for the super-horizon behaviour of the perturbation's mode function, we can derive an explicit expression for this behaviour. Thanks to the asymptotic form of Hankel functions when the argument tends to zero, $H_\nu^{(1),(2)}$ will assume the following forms:

$$H_\nu^{(1)}(z) \underset{z\to 0^+}{\longrightarrow} -i\frac{(\nu-1)!}{\pi}\left(\frac{2}{x}\right)^\nu = \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\nu-\frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} z^{-\nu} \,. \tag{B.3}$$

The latter form is useful when $\nu \simeq \frac{3}{2}$. A slightly different situation is when ν is close to $\frac{5}{2}$. To deal with that case, we recall a property of the Euler Gamma function:

$$\Gamma\left(\frac{1}{2}+n\right) = \frac{(2n)!}{4^n n!}\sqrt{\pi} \rightarrow \Gamma\left(\frac{5}{2}\right) = \sqrt{\pi}\frac{3}{4}\,. \tag{B.4}$$

Another important property of Hankel functions worth to be mentioned is their derivative with respect to the argument:

$$\frac{d}{dz}H_\nu^{(1,2)}(z) = \frac{\nu}{z}H_\nu^{(1,2)}(z) - H_{\nu+1}^{(1,2)}(z)\,. \tag{B.5}$$

It should be noted that derivative adds an Hankel function of different order but does not change the function type. In particular, it is interesting for our aim to note that the asymptotyc behaviours are not affected.

We should mention one last property of the Hankel functions, which we greatly exploit in the computation of bispectra: the particularly simple form that they acquire when the order in $\nu = n - \frac{1}{2}$. Namely,

$$H^{(1)}_{n-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}(i)^{-n}\, e^{iz} \sum_{k=0}^{n-1}(-1)^k \frac{(n+k-1)!}{k!(n-k-1)!}\left(\frac{1}{2iz}\right)^k\,. \tag{B.6}$$

For convenience, we only report the cases of interest:

$$H^{(1)}_{\frac{3}{2}}(z) = i\sqrt{\frac{2}{\pi z}}\frac{e^{iz}}{z}(1-iz) = i\sqrt{\frac{2}{\pi}}\left(\frac{1}{z}\right)^{\frac{3}{2}}(1-iz)e^{iz} \tag{B.7}$$

$$H^{(1)}_{\frac{5}{2}}(z) = -3i\sqrt{\frac{2}{\pi z}}\frac{e^{iz}}{z^2}\left(1-iz-\frac{z^2}{3}\right) = -3i\sqrt{\frac{2}{\pi}}\left(\frac{1}{z}\right)^{\frac{5}{2}}\left(1-iz-\frac{z^2}{3}\right)e^{iz} \tag{B.8}$$

Since the scalar, vector and tensor mode functions in lowest order in slow-roll consist of terms like $z^{\frac{3}{2}}H^{(1)}_{\frac{3}{2}}(z)$ or $z^{\frac{5}{2}}H^{(1)}_{\frac{5}{2}}(z)$, the above forms entail that the integrals involved in the computation of the three-point functions are exactly solvable in this approximation.

Appendix C

Solid Inflation: mode functions of scalar and tensor perturbations

This appendix is devoted to the derivation of the mode function for the scalar, vector and tensor perturbations. Once we will have obtained them, we will be able to compute the relative power spectrum. Since the scalar one is the most cumbersome of the three cases, while the other two are strictly similar to it, we are going to present a detailed and pedagogical examination only for the case of the scalar perturbation ζ.

C.1 Derivation of the mode function of ζ

We promote $\zeta(\tau, \mathbf{k})$ to a quantum operator by decomposing it in terms of creation/annihilation operators:

$$\zeta(\tau, \mathbf{k}) = \zeta_{cl}(\tau, k) b(\mathbf{k}) + \zeta^*_{cl}(\tau, k) b^\dagger(-\mathbf{k}) \,, \tag{C.1}$$

$$\left[b(\mathbf{k}), b^\dagger(\mathbf{k}')\right] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') \,. \tag{C.2}$$

The equation of motion for the mode function ζ_{cl} follows from varying the quadratic action of π_L (4.125), recalling that $\zeta(\tau, \mathbf{k}) = -\frac{k}{3}\pi_L(\tau, \mathbf{k})$. In conformal time, we get

$$S^{(2)}_\zeta = \int d\tau \frac{d^3k}{(2\pi)^3} \left[\frac{3a^2 M_P^2}{1 + \frac{k^2}{3\epsilon a^2 H^2}} |\zeta' + \epsilon a H \zeta|^2 - 9 c_L^2 a^4 H^2 M_P^2 \epsilon |\zeta|^2 \right] \,. \tag{C.3}$$

The initial condition for ζ is then obtained by computing the canonically normalized variables, that we dub as v, the action of which is the one of an harmonic oscillator with time-dependent frequency. We then set

$$v \equiv z\zeta \,, \qquad z \equiv -\sqrt{2} \frac{\sqrt{3} a M_P}{\sqrt{1 + \frac{1}{3\epsilon}\left(\frac{k}{aH}\right)^2}} = -\frac{3\sqrt{2\epsilon} a^2 H M_P}{k} \left(\frac{k}{\sqrt{k^2 + 3\epsilon a^2 H^2}} \right) \,. \tag{C.4}$$

For future convenience, we should note the relation between v and ζ_{cl} in the deep sub-horizon regime:

$$v = z\zeta_{cl} \underset{k>>aH}{\longrightarrow} -3 \frac{\sqrt{2\epsilon} a^2 H M_P}{k} \zeta_{cl} \,. \tag{C.5}$$

Turning our attention to (C.3), the action of v is obtained through the field redefinition (C.4). After some manipulations, the action becomes

$$S^{(2)}_v = \frac{1}{2} \int d\tau \frac{d^3k}{(2\pi)^3} \left[|v'|^2 - \omega^2 |v|^2 \right] \,. \tag{C.6}$$

The frequency has a quite cumbersome expression, given by

$$\begin{aligned}\omega^2(\tau,k) &= c_L^2 k^2 + 3\epsilon c_L^2 a^2 H^2 - \frac{z''}{z}\\ &= c_L^2 k^2 - a^2H^2\Bigg\{2 - 3\epsilon c_L^2 - \epsilon - \frac{1}{2}\left(\frac{k^2}{k^2+3\epsilon a^2 H^2}\right)\\ &\left[2+4\epsilon-3\eta+\frac{2}{H}\dot{\epsilon}-\frac{\dot{\eta}}{H}+(\epsilon-\eta)(2\epsilon-\eta)\right]+\frac{3}{4}\left(\frac{k^2}{k^2+3\epsilon a^2H^2}\right)^2(2\epsilon+2-\eta)^2\Bigg\} \end{aligned} \tag{C.7}$$

As it was not possible to solve exactly the equation of motion for the canonically normalized variable v, we will use an expedient to simplify the attainment of an approximate solution. We have in fact at our disposal the differential equations (4.108) and (4.114). The latter is the equation of motion of the gauge invariant variable $\mathcal{R}$, and, at first order in the slow-roll parameters, we are able to solve it in terms of Hankel functions. In order to fix the values of the two integration constants that will appear in the solution, we need to know how $\mathcal{R}$ behaves in the deep sub-horizon regime. We are able to do this by studying the behaviour of the canonically normaized field v in that regime. When inserting the solution in (4.108), we finally get the expression for ζ.

We then consider the limit $k \gg aH$ of the equation of motion of v:

$$v''(\tau,\mathbf{k}) + c_L^2(\tau)k^2 v(\tau,\mathbf{k}) = 0\,. \tag{C.8}$$

Although the latter encodes the dynamics of an harmonic oscillator, as it was expected for the canonically normalized field, it also includes a time-dependent speed which we have to take into account to solve (C.8) properly. In respect to the cases encountered in the first chapter, the above equation involves some additional subtleties that are worth to be mentioned.

In order to find a solution for (C.8), we exploit the slowly changing in time of the speed c_L^2 as illustrated in (4.63). The approximation consists then in keeping just the first-order contributions in slow-roll. To include the dependence on time of c_L, we thus define the new variable $\tilde{y} = -c_L(\tau)k\tau$. The equation (C.8) then becomes

$$(1-s_c)^2\tilde{y}^2\frac{d^2}{d\tilde{y}^2}v - s_c(1-s_c)\tilde{y}\frac{d}{d\tilde{y}}v + \tilde{y}^2 v = 0 \tag{C.9}$$

or, equivalently, using the variable $y = (1+s_c)\tilde{y}$ and discarding the terms $\mathcal{O}(s_c^2)$,

$$y^2\frac{d^2}{dy^2}v - s_c y\frac{d}{dy}v + y^2 v = 0\,. \tag{C.10}$$

Finally, the rescaled field $\tilde{v} = y^{-\frac{s_c+1}{2}}v$ obeys the Bessel equation (B.1) with $\nu^2 = \frac{(s_c+1)^2}{4}$:

$$y^2\frac{d^2}{dy^2}\tilde{v} + y\frac{d}{dy}\tilde{v} + \left[y^2 - \left(\frac{s_c+1}{2}\right)^2\right]\tilde{v} = 0\,. \tag{C.11}$$

The general solution of the above equation is given by $\tilde{v} = c_+H_\nu^{(1)}(y) + c_-H_\nu^{(2)}(y)$. For the field v we thus have

$$v(\tau,y) = y^{\frac{s_c+1}{2}}\left[c_+H_\nu^{(1)}(y) + c_-H_\nu^{(2)}(y)\right]\,. \tag{C.12}$$

The two integration constants are then fixed by the imposition of the Bunch-Davies vacuum and the normalization condition on $v(\tau,\mathbf{k})$. Recalling (B.2), the B.D. vacuum is achieved demanding that the asymptotic form matches the negative frequency solution in flat space. The limit in this case is $y \to +\infty$:

$$\begin{aligned} v &\underset{y\to+\infty}{\approx} \sqrt{\frac{2}{\pi}}y^{\frac{s_c}{2}}\left[c_+e^{iy-i\frac{\pi}{2}(\nu+\frac{1}{2})} + c_-e^{-iy+i\frac{\pi}{2}(\nu+\frac{1}{2})}\right]\\ &\Longrightarrow c_- = 0\,. \end{aligned} \tag{C.13}$$

The normalization condition then fixes the value of c_+ up to a phase:

$$v^* \frac{d}{d\tau} v - (c.c) = -i \underset{\tau\to-\infty}{\longrightarrow} -i(c_L k)\frac{4}{\pi}|c_+|^2 y^{s_c} = -i$$
$$\Longrightarrow |c_+| = \frac{\pi}{2}\frac{y^{-\frac{s_c}{2}}}{\sqrt{2c_L k}}\,.$$

The resulting behaviour of the canonically normalized fields is then given by

$$v(\tau,\mathbf{k}) \underset{k>>aH}{\longrightarrow} \frac{e^{-ic_L(\tau)k\tau(1+s_c)}}{\sqrt{2c_L(\tau)k}} \tag{C.14}$$

and hence, thanks to (C.4), we get

$$\zeta_{cl}(\tau,k) \underset{k>>aH}{\longrightarrow} -\sqrt{\frac{k}{4\epsilon c_L(\tau)}}\frac{e^{-ic_L(\tau)k\tau(1+s_c)}}{3a^2 H M_P} \tag{C.15}$$

Now that we have established the sub-horizon behaviour of ζ_{cl}, we can determin it up to first order in the slow-roll parameters. As already mentioned, we exploit the relation between ζ and $\mathcal{R}$.

If we dub $\mathcal{R}_{cl}$ the mode function of the quantum operator $\mathcal{R}$, we can recast (4.114) in terms of $\mathcal{R}_{cl}(\tau.\mathbf{k})$. With the explicit time-dependence of the background quantities, the classical equation of motion becomes

$$\mathcal{R}_{cl}'' - \frac{1}{\tau}(2+\eta_c-2s_c+2\epsilon_c)\mathcal{R}_{cl}' + \left[c_L^2k^2 + \frac{1}{\tau^2}(3\epsilon_c - 6s_c + 3c_{L,c}^2\epsilon_c)\right]\mathcal{R}_{cl} = 0 \tag{C.16}$$

where we have discarded all the terms of higher order than the first in the slow-roll parameters. We temporarily dub as δ the coefficient in front of the first derivative, that is $\delta \equiv 2+\eta_c-2s_c+2\epsilon_c$; one can see that the rescaled field $\chi \equiv (-\tau)^{-\frac{\delta+1}{2}}\mathcal{R}_{cl}$ obeys the equation:

$$\tau^2\chi'' + \tau\chi + \left[c_L^2k^2\tau^2 + \underbrace{(3\epsilon_c - 6s_c + 3c_{L,c}^2\epsilon_c) - \left(\frac{\delta+1}{2}\right)^2}_{\approx -\frac{9}{4}-\frac{3}{2}\eta_c - 3s_c + 3c_{L,c}^2\epsilon_c}\right]\chi = 0$$

As done for (C.8), we set $y = -c_L k\tau$. The resulting equation is thus

$$y^2\frac{d^2}{dy^2}\chi + y\frac{d}{dy}\chi + \left[\frac{y^2}{(1-s_c)^2} + \left(\frac{3c_{L,c}^2\epsilon_c - \frac{9}{4} - 3s_c - \frac{3}{2}\eta_c}{(1-s_c)^2}\right)\right]\chi = 0\,. \tag{C.17}$$

Setting $x = y(1+s_c)$ and discarding higher slow-roll terms, we obtain an equation of Bessel's kind:

$$x^2\frac{d^2}{dx^2}\chi + x\frac{d}{dx}\chi + \left[x^2 + \underbrace{-\frac{1}{4}(9+6\eta_c - 12c_{L,c}^2\epsilon_c - 30s_c)}_{\approx\frac{1}{4}(3+\eta_c-2c_{L,c}^2\epsilon_c-5s_c)^2}\right]\chi = 0 \tag{C.18}$$

If we then set

$$\alpha \equiv \frac{\delta+1}{2} = -\frac{1}{2}(3+2\epsilon_c+\eta_c-2s_c)\,, \qquad \nu_s \equiv \frac{1}{2}(3+5s_c-2c_{L,c}^2\epsilon_c+\eta_c) \tag{C.19}$$

one finds that the general solution of the equation of motion for $\mathcal{R}_{cl}$ is given by

$$\mathcal{R}_{cl}(\tau,\mathbf{k}) = (-\tau)^{-\alpha}\left[CH_{\nu_s}^{(1)}\left(-c_L(\tau)k\tau(1+s_c)\right) + DH_{\nu_s}^{(2)}\left(-c_L(\tau)k\tau(1+s_c)\right)\right] \tag{C.20}$$

We then insert the explicit solution (C.20) and its time derivative into (4.111), and we finally obtain the mode function ζ_{cl} at the desired order in slow-roll. We can safely set the time-independent coefficient $D = 0$. Thanks to the properties (B.2) and (B.5), one can easily see that neither $H^{(2)}$ nor its time derivative match the asymptotic exponential form with negative frequency (C.15) dictated by the Bunch-Davies vacuum.

We report the direct computation of ζ we are left with, where, for simplicity, we set $Q \equiv -k\tau c_L(\tau)(1+s_c)$ as the argument of the Hankel functions:

$$\begin{aligned}
&-3aHc_L^2(\tau)\zeta_{cl} = \mathcal{R}'_{cl} + aH(3-2\epsilon+\eta)\mathcal{R}_{cl}\\
&-3c_L^2 aH\zeta_{cl} = C\left[\alpha(-\tau)^{-\alpha-1}H^{(1)}_{\nu_s}(Q) + (-\tau)^{-\alpha}\frac{d}{d\tau}H^{(1)}_{\nu_s}(Q) + (-\tau)^{-\alpha-1}(3+\epsilon_c+\eta_c)H^{(1)}_{\nu_s}(Q)\right].
\end{aligned}$$

Recalling the relation (B.5), we get

$$\frac{d}{d\tau}\left[H^{(1)}_{\nu_s}(Q)\right] = \left[\frac{\nu_s}{Q}H^{(1)}_{\nu_s}(Q) - H^{(1)}_{\nu_s+1}(Q)\right]\frac{dQ}{d\tau} = c_L k\left(\frac{\nu_s}{c_L k\tau(1+s_c)}H^{(1)}_{\nu_s}(Q) + H^{(1)}_{\nu_s+1}(Q)\right),$$

therefore

$$-3c_L^2 aH\zeta_{cl} = C(-\tau)^{-\alpha-1}\left\{\underbrace{[\alpha-\nu_s(1-s_c)+3+\epsilon_c\eta_c]}_{\approx c^2_{L,c}\epsilon_c}H^{(1)}_{\nu_s}(Q) + c_L k(-\tau)H^{(1)}_{\nu_s+1}(Q)\right\}.$$

At this point we can fix the constant C by confronting the above expression in the sub-horizon regime with the one given in (C.15). We have thus

$$\begin{aligned}
C =& \frac{-3c_L^2 aH(-\tau)^{\alpha+1}\zeta_{cl}}{c_L k(-\tau)H^{(1)}_{\nu_s+1}(Q) - c^2_{L,c}\epsilon_c H^{(1)}_{\nu_s}(Q)}\\
&\underset{-k\tau\to+\infty}{\longrightarrow}(-\tau)^{\alpha+1}\frac{-3c_L^2 aH}{c_L k(-\tau)H^{(1)}_{\nu_s+1}(+\infty)}\left[-\sqrt{\frac{k}{4\epsilon c_L^2}}\frac{e^{-ic_L k\tau(1+s_c)}}{3a^2HM_P}\right]\Big(1+\mathcal{O}(\epsilon)\Big).
\end{aligned}$$

Recalling the asymptotic form (B.2), namely

$$H^{(1)}_{\nu_s+1}\left(-c_L k\tau(1+s_c)\right)\underset{-k\tau\to+\infty}{\longrightarrow}\sqrt{\frac{2}{\pi c_L k(-\tau)(1+s_c)}}e^{-ic_L k\tau(1+s_c)}e^{-i\frac{\pi}{2}(\nu_s+1+\frac{1}{2})}, \tag{C.21}$$

we get

$$\begin{aligned}
C &= \sqrt{\frac{\pi}{2}}ie^{i\frac{\pi}{2}(\nu_s+\frac{1}{2})}(-\tau)^{\alpha+\frac{1}{2}}\frac{\sqrt{1+s_c}c_L(\tau)}{a(\tau)\sqrt{4\epsilon(\tau)}M_P}\Big(1+\mathcal{O}(\epsilon)\Big)\\
&= \sqrt{\frac{\pi}{2}}ie^{i\frac{\pi}{2}(\nu_v+\frac{1}{2})}\frac{c_{L,c}H_c}{\sqrt{4\epsilon_c}M_P}(1+\frac{1}{2}s_c-\epsilon_c)(-\tau_c)^{s_c-\epsilon_c-\frac{1}{2}\eta_c} + \mathcal{O}(\epsilon^{3/2}),
\end{aligned}$$

where in the last step we have expressed the time-dependence of the background quantities.

We can write the mode function ζ_{cl} in a compact form:

$$\begin{aligned}
\zeta_{cl}(\tau,\mathbf{k}) = -i\sqrt{\frac{\pi}{2}}e^{i\frac{\pi}{2}(\nu_s+\frac{1}{2})}\frac{c_{L,c}H_c}{3M_P\sqrt{4\epsilon_c}\sqrt{c^5_{L,c}k^3}}\left(\frac{\tau}{\tau_c}\right)^{\epsilon_c+\frac{\eta_c}{2}+\frac{5s_c}{2}}\left(\frac{1-\epsilon_c}{(1+\epsilon_c)(1+s_c)}\right)&\\
\times\left[Q^{\frac{5}{2}}H^{(1)}_{\nu_s+1}(Q) + c^2_{L,c}\epsilon_c Q^{\frac{3}{2}}H^{(1)}_{\nu_s}(Q)\right]&
\end{aligned} \tag{C.22}$$

where, we remind, $Q \equiv -c_L(\tau)k\tau(1+s_c)$ is understood.

We can now evaluate ζ_{cl} on super-horizon scales, that corresponds to the form (B.3) of the Hankel function $H^{(1)}$. Considering (C.22), one realizes that the most sizeable contribution in that limit comes from the term $Q^{\frac{5}{2}}H^{(1)}_{\nu_s+1}(Q)$. On super-horizon scales the above term results in

$$\begin{aligned}
Q^{\frac{5}{2}}H^{(1)}_{\nu_s+1}(Q) &\to \sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}\,2^{\nu_s+1-\frac{3}{2}}\frac{\Gamma(\nu_s+1)}{\Gamma(\frac{3}{2})}Q^{-(\nu_s+1-\frac{5}{2})} \\
&= -i\sqrt{\frac{2}{\pi}}\,2^{1+\mathcal{O}(\epsilon)}\,\frac{\Gamma(\frac{5}{2}+\mathcal{O}(\epsilon))}{\Gamma(\frac{3}{2})}\left(\frac{\tau}{\tau_c}\right)^{c^2_{L,c}\epsilon_c-\frac{\eta_c}{2}-\frac{5s_c}{2}}(-c_Lk\tau_c(1+s_c))^{c^2_{L,c}\epsilon_c-\frac{\eta_c}{2}-\frac{5s_c}{2}} \\
&= -i3\sqrt{\frac{2}{\pi}}\left(\frac{\tau}{\tau_c}\right)^{c^2_{L,c}\epsilon_c-\frac{\eta_c}{2}-\frac{5s_c}{2}}(-c_{L,c}k\tau_c)^{c^2_{L,c}\epsilon_c-\frac{\eta_c}{2}-\frac{5s_c}{2}}+\mathcal{O}(\epsilon)\,.
\end{aligned}$$

It can be argued that, on super horizon scales, ζ_{cl} is not frozen, but evolves in time with a power law of exponent $(1+c^2_{L,c})\epsilon_c \approx \frac{4}{3}c^2_{T,c}\epsilon_c$:

$$\zeta_{cl}(\tau,k) \underset{-k\tau\to 0^+}{=} \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c^2_{T,c}\epsilon_c}(-c_{L,c}k\tau_c)^{c^2_{L,c}\epsilon_c-\frac{5}{2}s_c-\frac{1}{2}\eta_c}\left(\frac{H_c}{\sqrt{4\epsilon_c}M_P\sqrt{c^5_{L,c}k^3}}+\mathcal{O}(\epsilon^{1/2})\right)\,. \tag{C.23}$$

In a very similar fashion, the asymptotic form of $\mathcal{R}_{cl}$ can also be calculated:

$$\mathcal{R}_{cl}(\tau,k) \underset{-k\tau\to 0^+}{=} \left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c^2_{T,c}\epsilon_c-2s_c}(-c_{L,c}k\tau_c)^{c^2_{L,c}\epsilon_c-\frac{5}{2}s_c-\frac{1}{2}\eta_c}\left(-\frac{H_c}{\sqrt{4\epsilon_c}M_P\sqrt{c_{L,c}k^3}}+\mathcal{O}(\epsilon^{1/2})\right)\,. \tag{C.24}$$

It is also important to note that on large scales $\mathcal{R}_{cl}$ and ζ_{cl} do not coincide, but rather

$$\mathcal{R}_{cl}(\tau,k) \underset{-k\tau\to 0^+}{=} -c^2_L(\tau)\zeta_{cl}\,, \tag{C.25}$$

as seen directly from the two asymptotic forms above.

At this stage we can state the formula of the two point function for ζ on large scales, in conjunction with the power spectrum:

$$\begin{aligned}
\langle\zeta(\tau,\mathbf{k}_1)\zeta(\tau,\mathbf{k}_2)\rangle &= (2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_1)|\zeta_{cl}(\tau,\mathbf{k}_1)|^2 \\
\underset{-k\tau\to 0^+}{\longrightarrow} &(2\pi)^3\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_1)\frac{H_c^2}{4\epsilon_c c^5_{L,c}M_P^2}\frac{1}{k_1^3}\frac{\left(\frac{\tau}{\tau_c}\right)^{\frac{8}{3}c^2_{L,c}\epsilon_c}}{(-c_{L,c}k_1\tau_c)^{5s_c-2c^2_{L,c}+\eta_c}}
\end{aligned} \tag{C.26}$$

$$P_\zeta(\tau,k) = \frac{H_c^2}{4\epsilon_c c^5_{L,c}M_P^2}\frac{1}{k_1^3}\left(\frac{\tau}{\tau_c}\right)^{\frac{8}{3}c^2_{L,c}\epsilon_c}(-c_{L,c}k_1\tau_c)^{-(5s_c-2c^2_{L,c}+\eta_c)}\,. \tag{C.27}$$

The above expression is quite cumbersome, but gives us the possibility to read directly the spectral index. Recalling the definition (1.93), we have indeed

$$n_\zeta - 1 \equiv \frac{d\ln\mathcal{P}_\zeta}{d\ln k} = 2c^2_{L,c}\epsilon_c - 5s_c - \eta_c\,. \tag{C.28}$$

C.2 Derivation of the mode function of γ

The following calculation may be considered simpler than the case above. We summarize the decomposition of the tensor modes into its polarized components, that respect the traceless, transverse conditions on γ_{ij}:

$$\gamma_{ij}(\tau,\mathbf{k}) = \sum_{s=+,\times}\epsilon^s_{ij}(\mathbf{k})\gamma^s(\tau,\mathbf{k})\,, \tag{C.29}$$

$$\epsilon^s_{ij}(\epsilon^{s'}_{ij})^* = 2\delta^{ss'}\,,\qquad k_i\epsilon_{ij}(\mathbf{k}) = 0\,,\qquad \epsilon_{ii}(\mathbf{k}) = 0\,. \tag{C.30}$$

By promoting γ^s to quantum operator, we further decompose as

$$\gamma^s(\tau,\mathbf{k}) = \gamma^s_{cl}(\tau,\mathbf{k})\, a^s(\mathbf{k}) + \gamma^{s\,*}_{cl}(\tau,\mathbf{k}) a^{s\,\dagger}(-\mathbf{k})\,, \tag{C.31}$$

where the following algebra holds for a^s and $a^{s\,\dagger}$

$$[a^s(\mathbf{k})\,,\, a^{s'\,\dagger}(\mathbf{k}')] = (2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{k}')\delta^{ss'}\,. \tag{C.32}$$

Varying the action (4.123), we get the equation of motion of the mode function $\gamma^s(\tau,\mathbf{k})$. With the time-dependence of the background quantities, it results in the following expression,

$$\frac{d^2}{d\tau^2}\gamma_{cl} - \frac{2}{\tau}(1+\epsilon_c)\frac{d}{d\tau}\gamma_{cl} + \left(k^2 + \frac{4\epsilon_c c^2_{T,c}}{\tau^2}\right)\gamma_{cl} = 0\,. \tag{C.33}$$

The above equation presents a peculiar exception with respect to the case of the single scalar field, which is a mass term. It should be noted that this term is slow-roll suppressed and proportional to the transverse speed of sound c_T^2. Once the solid-fluid transition happens at the end of the inflationary epoch, c_T dissappears as well as the mass of gravitons.

When analyzing (4.123), the canonically normalized field is

$$\hat{\gamma}^s(\tau,\mathbf{k}) = \frac{a(\tau)M_P}{\sqrt{2}}\gamma^s(\tau,\mathbf{k})\,, \tag{C.34}$$

and the corresponding equation of motion is the one of a massive scalar field as studied in the first chapter:

$$\hat{\gamma}'' + \left(k^2 - \frac{a''}{a} + 4\epsilon c_T^2 a^2 H^2\right)\hat{\gamma} = 0\,. \tag{C.35}$$

Once again, the solution of the equation of motion (C.33) is given in terms of Hankel functions:

$$\gamma_{cl}(\tau,\mathbf{k}) = (-\tau)^{\frac{3}{2}+\epsilon_c}\left[\mathcal{A}H^{(1)}_{\nu_t}(-k\tau) + \mathcal{B}H^{(2)}_{\nu_t}(-k\tau)\right]\,, \qquad \nu_t = \frac{3}{2} + \epsilon_c - \frac{4}{3}c^2_{T,c}\epsilon_c\,. \tag{C.36}$$

As per the analysis conducted in the first chapter on the massive scalar fieds in a quasi- de Sitter Universe, we can easily find the values of the coefficients $\mathcal{A}$ and $\mathcal{B}$. We already know that the Bunch-Davies vacuum and the normalization condition yield

$$\hat{\gamma}(\tau,\mathbf{k}) = \frac{\sqrt{\pi}}{2}e^{i(\nu_t+\frac{1}{2})\frac{\pi}{2}}(-\tau)^{\frac{1}{2}}H^{(1)}_{\nu_t}(-k\tau)\,. \tag{C.37}$$

Consequently, we argue that

$$\mathcal{A} = \frac{1}{M_P}\sqrt{\frac{\pi}{2}}e^{i(\nu_t+\frac{1}{2})\frac{\pi}{2}}\frac{(-\tau)^{-1-\epsilon_c}}{a(\tau)} = \frac{H_c(1-\epsilon_c)}{M_P}\sqrt{\frac{\pi}{2}}e^{i(\nu_t+\frac{1}{2})\frac{\pi}{2}}(-\tau_c)^{-\epsilon_c} + \mathcal{O}(\epsilon^2)\,, \tag{C.38}$$

$$\mathcal{B} = 0\,. \tag{C.39}$$

Finally, the resulting tensor mode function is

$$\gamma_{cl}(\tau,k) = \sqrt{\frac{\pi}{2}}\frac{H_c}{M_P\sqrt{k^3}}e^{i\frac{\pi}{2}(\nu_t+\frac{1}{2})}\left(\frac{\tau}{\tau_c}\right)^{\epsilon_c}(-k\tau)^{\frac{3}{2}}H^{(1)}_{\nu_t}(-k\tau)\,. \tag{C.40}$$

Given the asymptotic form of Hankel functions (B.3) for small arguments, we can compute the expression of the mode function on super-horizon scales:

$$\begin{aligned}\gamma^s_{cl}(\tau,\mathbf{k}) \underset{-k\tau\to 0}{\sim} & \left(\sqrt{\frac{\pi}{2}}\frac{H_c(1-\epsilon_c)}{M_P}e^{i\frac{\pi}{2}(\nu_t+\frac{1}{2})}\sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}2^{\nu_t-\frac{3}{2}}\frac{\Gamma(\nu_t)}{\Gamma(\frac{3}{2})}\right)\left(\frac{\tau}{\tau_c}\right)^{\epsilon_c}(-\tau)^{\frac{3}{2}}(-k\tau)^{-\nu_t} \\ = & \left(\frac{iH_c}{\sqrt{k^3}M_P} + \mathcal{O}(\epsilon)\right)\left(\frac{\tau}{\tau_c}\right)^{\frac{4}{3}c^2_{T,c}\epsilon_c}(-k\tau_c)^{c^2_{L,c}\epsilon_c}\,,\end{aligned} \tag{C.41}$$

where in the last exponent the relation $c_{T,c}^2 \approx \frac{3}{4}(1 + c_{L,c}^2)$ has been used.

The two point function evaluated on super-horizon scales is then given by:

$$\begin{aligned}\langle \gamma^{s_1}(\tau, \mathbf{k_1}) \gamma^{s_2}(\tau, \mathbf{k_2}) \rangle &= (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \delta^{s_1 s_2} |\gamma_{cl}^{s_1}(\tau, \mathbf{k}_1)|^2 \\ \underset{-k\tau \to 0^+}{\longrightarrow} & (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \delta^{s_1 s_2} \frac{H_c^2}{M_p^2} \frac{1}{k_1^3} \frac{\left(\frac{\tau}{\tau_c}\right)^{\frac{8}{3} c_{T,c}^2 \epsilon_c}}{(-k_1 \tau_c)^{-2 c_{L,c}^2 \epsilon_c}} \,. \end{aligned} \tag{C.42}$$

The power spectrum of tensor perturbations results therefore in

$$P_\gamma(k) = \frac{H_c^2}{M_p^2} \frac{1}{k^3} \left(\frac{\tau}{\tau_c} \right)^{\frac{8}{3} c_{T,c}^2 \epsilon_c} (-k \tau_c)^{2 c_{L,c}^2 \epsilon_c} \,. \tag{C.43}$$

The spectral index at first order in slow-roll is then given by

$$n_t - 1 = 2 c_{L,c}^2 \epsilon_c \,. \tag{C.44}$$

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