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Unitarity-based methods and Integration-by-parts identities for Feynman Integrals

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Abstract

In this thesis, we present a novel idea to address the evaluation of multi-loop Feynman integrals, inspired by *unitarity of S-matrix*.

In the first part of this work, we present the Feynman integral formalism. Within the *dimensional regularization scheme*, Feynman integrals are known to obey *integration-by-parts identities* (IBPs), yielding the identification of an independent integral basis, dubbed *master integrals*. We describe the currently adopted strategy for the amplitudes decomposition, known as reduction algorithm, which is based on IBPs for integrands with denominators that depend quadratically on the loop momenta (quadratic denominators).

In the second part of this thesis, we present a novel strategy to decompose Feynman integrals based on the use of partial fractions decomposition of its integrand. Within this approach any multi-loop integrand is first decomposed into a combination of integrands that contain just a minimal, irreducible number of quadratic denominators and several other denominators that carry a linear dependence of the loop momenta (linear denominators). After partial fractioning, IBPs are applied to integrals with linear denominators, in order to identify an alternative set of master integrals. Finally, the obtained relations are combined back to restore the IBPs for the original integrals containing quadratic denominators only. We examine the underlying algebraic structure of dimensionally regulated integrals with linear denominators, classifying all spurious, vanishing classes of integrals that may emerge after partial fractioning.

In the last part of the thesis, we present the implementation of the novel algorithm within a Mathematica code called PARSIVAL (PARtial fractions-baSed method for feynman Integral eVALuation), which has been interfaced to the public package REDUZE, for the IBPs decomposition. Preliminary results for the application of PARSIVAL+REDUZE framework to 1-loop integrals for $2 \to n$ ($n = 1, 2, 3$) scattering processes, and to 2-loop integrals, corresponding to planar and non-planar diagrams for $2 \to n$ ($n = 1, 2$) scattering amplitudes are given in the final chapter.

The proposed algorithm is suitable for parallelization, and the preliminary results show that its effectiveness can be improved by exploiting the symmetries of the integrand under redefinition of loop-momenta, not accounted for in the present version of the code.

The proposed strategy is very general and it can be applied to any scattering reduction.

Contents

Introduction

Since its origins, the main objective of Physics has been the understanding of the fundamental constituents of Nature. Passing from the Classical Determinism to Probabilistic principles of Quantum Mechanics, matter and forces at microscopic scale behaves differently from the macroscopic scale. *Quantum Field Theory* (QFT), which unifies Special Relativity and Quantum Mechanics, represents the ideal framework to investigate Nature at microscopic level.

Nowadays, its exploration has led to the formulation of the *Standard Model of Particle Physics* (SM), the best QFT model which describes matter and forces as interacting elementary particles. The most recent success of SM relies in the discovery of the Higgs boson in 2012[1], predicted in 1964 to explain the mass of vector bosons[2], which is a scalar particle responsible for the spontaneous symmetry breaking and for all the masses of SM particles.

Anyway, there are evidences that suggest SM as a low-energy limit of a more general theory: gravity still not be in its description; the evidence of neutrino oscillation, confirmed at Super-Kamiokande experiment[3], tells us that neutrinos have small mass; the presence in the Universe of dark matter, dark energy[4] and baryon/anti-baryon asymmetry[5, 6].

High energy particles collisions are the best way to investigate microscopic phenomena and to test our physical theories. Within the *Perturbative Quantum Field Theory* framework, collisions are quantified by interaction probabilities between n particles, which are observables of QFT encoded in the *scattering cross section*.

Scattering Amplitudes are the heart of scattering cross sections: they relate theoretical QFT predictions and collider experiments. Single contributions to Scattering Amplitude are called *Feynman diagrams*.

Feynman diagrams are pictorial representations of specific scattering processes, in which occur external and internal legs. These last kind of lines represent *virtual particles*, which do not satisfy the on-shell condition. The full contribution to an n-point amplitude is the sum of all the n-external legs Feynman diagrams which can be built from the Feynman rules .

Contributions to scattering amplitudes, at every order of its perturbative expansion, come from two kinds of diagrams:

- *tree-level Feynman diagrams*, related to the *leading order* to the total amplitude. This kind of diagrams can be splitted in two connected subdiagrams by cutting an internal line;

- *l-loop Feynman diagrams*, related to $\hbar^l$ quantum correction of the total amplitudes. Every l-loop Feynman diagrams is an integral in l internal momenta, *loop momenta*.

In general, loop diagrams correspond to integrals that may *diverge* for space-time dimensions $d = 4$, either for large values (Ultraviolet region) or for large values (Infrared region) of the integration momenta. To regulate this behaviour, a regularisation scheme is mandatory. In this work we adopt the *dimensional regularization* scheme: leaving the space-time dimension as a continuous parameter d and performing the d-dimensional integrations, divergent terms appear as poles in the Laurent expansion parameter $\epsilon = \frac{4-d}{2}$.

Testing theories means comparing experimental measurements and theoretical predictions. This comparison can be done through two different approaches: *direct searches*, employing the highest accessible energy to find selected particles in the final state; *indirect searches*, involving high precision measurements to find selected particles occuring in virtual corrections of the scattering amplitude. With the recent improvements in collider technology, cross sections can be measured at very high precisions, for which, at the theoretical level, the evaluation of corrections beyond the leading order (LO) is a necessary requirement. Currently, to compare results at the nowadays experimental precision, from indirect searches point of view, calculating next-to next-to leading order (NNLO) amplitudes is mandatory.

Every n particles process at NNLO-order implies the evaluation of scattering amplitudes of three different types:

- *virtual corrections*, represented by 2-loops and n external legs Feynman diagrams.

- *real-virtual correction*, represented by an 1-loop Feynman diagrams with $n+1$ external legs.

- *real correction*, represented by tree-level Feynman diagrams with $n+2$ external legs.

Tree-level and 1-loop Feynman amplitudes calculations received a tremendous improvements in the last twenty years[7, 8, 9, 10]: the key to achieve a efficient automation relied in *unitarity-based methods*[11]. They are a class of identities and relations which come from the unitarity of the scattering matrix: $S^+ S = \mathbb{I}$. This identity gives arise to the *optical theorem*, the *unitarity cuts*[12] and the *BCFW recurrence relation*[13, 14]. The implementation of unitarity-based methods in automation algorithms for 1-loop amplitudes calculation had a great impact on collider phenomenology, allowing the study of processes involving an high number of particles.

In this last few years it was also developed various approaches to face 2-loop virtual corrections[15, 16, 17, 18], like analytical and numerical unitarity-based methods for $2 \to 2$ amplitudes[19, 20], but an efficient evaluation and its automation are still an open problem: limitations occur even at $2 \to 2$ 2-loop non-planar amplitudes with either massive external or internal lines, and $2 \to 3$ amplitudes represents a cap for automatic calculations.

This work will focuses on an aspect of the evaluation of virtual corrections. Calculating a loop-correction of a scattering amplitude through a direct integration is prohibitive task, even with the standard techniques of Feynman parameters and alpha-parametrization[11]: it involves the evaluation of multi-variate d-dimensional integrals.

For these reasons, a different strategy to evaluate this functions is mandatory.

The modern approach to evaluate single Feynman amplitudes[21] is divided in three stages:

1. Perform the *tensor reduction*, in order to decompose a single Feynman amplitude in combination of *scalar Feynman integrals*[22];

2. decompose scalar Feynman integrals in a "basis", which elements are called *master integrals*[23, 11];

3. evaluate each master integral.

The second stage of this algorithm is called *reduction* of Feynman integrals.

Feynman integrals, within the dimensional regularization scheme, obeys to a set of non-trivial relations, deriving from the *integration-by-parts identities*[24, 23]. These identities, known as *IBPs*, come from the d-dimensional Gauss's divergence theorem and can be exploited to form a linear system of equations. Performing the Gaussian elimination on the system of IBPs, it is possible to find that exists a set of integral which are linearly independent. By definition, elements of this set are called *Master integrals*.

There are several codes which automatically generate and perform Gauss-elimination on the system of IBPs[25] and select a basis of master integrals. This method is known as Laporta algorithm[26, 21]. Some public routines which implements this algorithm are AIR[27], REDUZE[28], FIRE[29], AZURITE[25] and KIRA[30]. They also implements other tools: the evaluation of IBPs on the maximal cut[21]; the generation of a system of differential equation which master integrals satisfy[31, 32].

The optimization of these routines is an open problem:

1. IBPs generation gets harder with increasing number of loop momenta and mass scale parameters (external momenta and masses);

2. It is possible to generate several thousands of IBPs for a single Feynman integral: the Gaussian elimination on such a system can be extremely time consuming. Computationally, solving a system of n equations with n unknowns involves $\sim n^3$ operations, implying an arithmetic complexity of $O(n^3)$.

These issues occur even for $2 \rightarrow 2$ two-loop amplitudes with massive external and virtual particles[30]. To overcome these limitations it is appropriate to investigate a way to work at integrand level of Feynman integrals, in analogy with the improvement in 1-loop calculation through unitarity-based methods.

With this in mind, in this work we investigate a new algorithm aimed at an amelioration of the modern system-solving strategy.

In general, the integrand of a Feynman integral is a rational function with a certain number of denominators D_j (inverse propagators), which are quadratic polynomial in the loop momenta. An integrand of this form is decomposable in a combination of *partial fractions*: each partial fraction is a new rational function, which now contains differences between denominators $D_i - D_j$. It can be shown that *the difference between two inverse propagators is linear in the loop momenta*[33, 13]. This decomposition is inspired by unitarity-based methods for Feynman amplitude

calculations, in particular by the BCFW recurrence relation for tree-level amplitudes[13]. From the automation point of view, it could be more convenient to deal with linear propagator instead of quadratic ones.

Within the exploitation of the partial fractioning and its application to Feynman integrals (subsequently, we will refer to them as *quadratic Feynman integral*), we faced two particular aspects:

1. The action of partial fractioning on a Feynman integral generates a decomposition of it in terms of the so called *linearized Feynman integrals*. We generalized the partial fractioning formula, for generical rational function and studied the key properties of linearized Feynman integrals. These new integral functions have non-trivial relations between them, and carried unexpected properties: most of the linearized Feynman integrals are related by *crossing symmetries*, relations arising from the switch of external particles; moreover, it can be shown that linearized Feynman integrals containing only linear propagators vanishes. The understanding of this properties was the key to optimize the IBPs generation for linearized Feynman integrals and the reduction of the quadratic Feynman integral in terms of *linearized master integrals*.

2. The recollection of linearized master integrals to build back the quadratic ones brought non-trivial problems. By inverting partial fractioning relations, we expected to get the quadratic reduction. Actually, not all the linearized master integrals can be related to quadratic MIs. The occuring of these "spurious" terms represented a real issues of the recollection. We overcomed this problem using crossing symmetries: inserting these relations between spurious linearized master integrals, we showed that their combination vanishes, leaving only linearized MIs which are partial fractions of quadratic MIs.

The understanding of these aspects of partial fractioning allowed us to define our algorithm systematically. Starting from a generic Feynman integral:

- decompose the integrand in partial fractions and build linearized Feynman integrals;

- reduce each linearized Feynman integrals in terms of linearized master integrals;

- combine linear reductions to get the quadratic one.

We implemented this algorithm in a MATHEMATICA code, called PARSIVAL (PARtial fractions-based Strategy for Feynman Integral eVALuation). For this work, we interfaced it with REDUZE, which perform IBPs generation and reductions on linearized Feynman integrals.

The strenght of PARSIVAL is its full generality and its flexibility: it can be used with any future IBP generator in order to improve their efficiency. In this sense, PARSIVAL is meant to be a sort of *app*, or a booster, intended to enhance IBP generators.

This thesis is structured to cover every aspect of the reduction of scattering amplitudes, focusing on the state-of-the-art multi-loop calculation methods and

unitarity-based methods for Feynman integrals. Then, we present our new algorithm: the partial fractioning applied on Feynman integrals, the explicit reduction procedure passing through linearized Feynman integrals and its implementation in PARSIVAL.

In the Chapter 1 we expose the decomposition procedure of a Feynamn amplitude in *Feynman integrals*, both starting from the single Feynman amplitude and with an assiomatic approach. After this we show a set of relation and symmetries which the Feynman integrals satisfy, with particular attention to integration-by-parts identities[24, 23].

In the Chapter 2 we define what is a master integral[11], their properties and the reduction of Feynman integrals in the basis which they form. Here we also present methods to explicitely evaluate master integrals[11, 16, 32].

In the Chapter 3 we show a preparatory example on the automation of the reduction algorithm using the computer program REDUZE[28]. Lastly we present strenght and weakness of the odiern automation and try to investigate a method to enhance the already existing routines.

In Chapter 4 we present the unitarity-based methods involved in Feynman integrals calculation, the partial fractioning of generical rational functions, and the explicit definition of linearized Feynman integrals and their key properties. After doing this, we present our new algorithm, the reduction of linearized Feynman integrals and their recollection to get the quadratic reduction.

In Chapter 5 we describe the functioning of PARSIVAL (interfaced with REDUZE), showing how it handles the decomposition of a Feynman integrals in linearized Feynman integrals and the recollection of linearized reductions to get the quadratic reduction.

In the Chapter 6 we report 1-loop and 2-loop tests on the effectiveness of PARSIVAL with a comparison among REDUZE running times and the system PARSIVAL+REDUZE running times.

Let's recall that this method is completely general, and it can be applied on every n-points l-loop amplitude. At the odiern state of research, this code is meant to be used for ameliorate routines in the NNLO corrections calculation of QCD amplitudes. One of the most immediate application can be the virtual contribution to the $gg \rightarrow ggg$ 2-loop amplitude, for which we hope to have results as soon as possible.
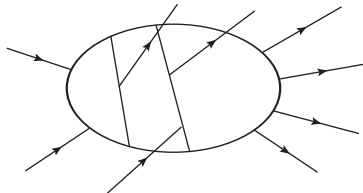
Chapter 1

Feynman integrals

Feynman integrals are the heart of the scattering amplitudes. They encode all the topological information of the Feynamn diagram related to the starting Feynman amplitudes, so that kinematics, number of loop momenta and virtual particles and their masses. In this Chapter we will give some basic definitions about diagram topologies and Feynman integrals and will show a set of relations between these objects. In particular we will present one of the core elements of this thesis: the Integration-by-parts identities.

1.1 Definitions

In this work, we're going to present a large number of the sets, whose usually are finite: for every set A of cardinality $|A|$, we will define an ordering as $i_A : \{1, \ldots, |A|\} \to A$, $i_A : j \mapsto A_j$. So, for simplicity, we will refer to a specific element by indexing it, so that an element of A is denoted as A_j.

1.1.1 Tensor decomposition

A generic Feynman diagram of n external legs and l loops is a pictorial representation of a function $\mathcal{F}^l(p_1^\mu, \ldots, p_n^\mu)$, depending on the independent momenta. The indice l stands for the number of loop integration momenta.

$$\mathcal{F}^l(p_1^\mu, \ldots, p_n^\mu) = \quad \text{(1.1)}$$

This function is decomposable in three factors: one carrying the external particles polarizations and their polarization, $E^{i,\nu}(p_1^\mu, \cdots, p_n^\mu)$; one carrying the pure tensorial and spinorial part, $B_{k,i,\nu}(p_1^\mu, \cdots, p_n^\mu)$; the last one is an integral factor, enclosing the internal lines properties, called *scalar form factor* $I_c^l(p_1^\mu, \cdots, p_n^\mu)$. This decomposition[22, 34] can be expressed as following

$$\mathcal{F}^l(p_1^\mu, \ldots, p_n^\mu) = E^{i,\nu}(p_1^\mu, \cdots, p_n^\mu) \sum_{c=1}^{C} B_{\nu,i,c}(p_1^\mu, \cdots, p_n^\mu) I_c^l(p_1^\mu, \cdots, p_n^\mu) \quad \text{(1.2)}$$

where C is the number of indipendent Lorentz tensors. This decomposition is possible for any diagram, in full generality, because of the Lorentz invariance and the gauge invariance of any physical Lagrangian.

1.1.2   Feynman integrals

Let's focus on the form factor $I_c^l(p_1^\mu, \cdots, p_n^\mu)$. Consider a Feynman diagram with $n+1$ external legs, l loops, and h internal lines: it has $l+n$ indipendent momenta. Let's start with some definitions: $K = \{k_1^\mu, \ldots, k_l^\mu\}$, $P = \{p_1^\mu, \ldots, p_n^\mu\}$, $\mathbf{M} = \{M_1, \ldots, M_{n+1}\}$ and $\mathbf{m} = \{m_1, \ldots, m_h\}$. The last quantities are the set of external and internal masses.

Once given those definitions, we can build some auxiliary sets:

$$\boldsymbol{\sigma} = \{p_i \cdot p_j \mid p_i, p_j \in P\};$$

$$\bar{\boldsymbol{\sigma}} = \{v_i \cdot v_j \mid v_i, v_j \in K \cup P\};$$

$$\mathbf{m}^2 = \{m_i^2 \mid m_i \in \mathbf{m}\};$$

$$\mathbf{M}^2 = \{M_i^2 \mid M_i \in \mathbf{M}\};$$

Lastly, we define $\boldsymbol{s} = \{s_{ij} = \frac{1}{1+3\delta_{ij}}(p_i + p_j)^2 \mid p_i, p_j \in P\}$ and the set of the *kinematic invariants* $\mathcal{I} = \mathbf{s} \sqcup \mathbf{M}^2 \cup \mathbf{m}^2$. It is clear that the on-shell implies that $s_{ii} = M_i^2$: this is the reason we choose to define $\mathcal{I}$ with a disjoint union.

The scalar form factor depends on all the kinematics invariants: it has the following general shape[1][16]

$$I_c^l(P) = I_c^l(\mathcal{I}) = A_c \int \left(\prod_{i=1}^{l} \frac{d^d k_i}{(2\pi)^{d-2}} \right) \frac{\prod_{j=1}^{N_s} S_j^{b_j}(K, P)}{\prod_{i=1}^{h} D_i(K, P, \mathbf{m})} \tag{1.3}$$

In this definition:

- A_c is a numerical coefficient;

- $D_i(K, P, \mathbf{m}) = [V_i(K, P) \cdot V_i(K, P) + m_i^2]$ is a polynomial of second order in the external and internal momenta. V_i is a sum of momenta, with at least one loop momenta, in formulas $V_i^\mu(K, P) = \sum_{j=1}^{l} w_{ij} k_j^\mu + \sum_{k=1}^{n} w'_{ik} p_k^\mu$ where: $w_{ij} \in \{0, 1\}$ and at least one of w_{ij} must be equal to 1; $w'_{ik} \in \{0, 1\}$; m_i is the mass associated to the internal line. From this moment, we will refer to $D_i(K, P, \mathbf{m})$ as *denominator*, $D = \{D_1(K, P, \mathbf{m}), \ldots, D_h(K, P, \mathbf{m})\}$, and to $V_i^\mu(K, P)$ as *momenta current*.

- $S_j(K, P)$ is a scalar product between momenta, which has at least one loop momentum. In formulas, the set of the scalar products of all possible couples of the momenta, $S(K, P) = \bar{\boldsymbol{\sigma}} \backslash \boldsymbol{\sigma}$. The set $S(K, P) = \{S_1(K, P), \cdots, S_{N_s}(K, P)\}$ is the one built by ordering $S(K, P)$, and $N_s = |S(K, P)|$ is the total number of scalar product in $S(K, P)$:

$$N_s = nl + \frac{l(l+1)}{2} \tag{1.4}$$

---

[1] We already performed the Wick rotation

- d is the dimensional regulator.

Later, we will often omit the functional dependence of $D_i(K, P)$ and $S_i(K, P)$ for a lighter notation.

The expression (1.3) can be simplified more. In general, a diagram could contain internal lines with same momenta currents and same mass. Moreover, we know that the denominators D_j are combinations of scalar product of momenta. This means that, rearranging the scalar product at the numerator, S_j, we can simplify h of them, leaving only an irreducible rational integrand.

To express this correctly, we have to write only indipendent denominators, applying a power to each of them:

$$\frac{1}{\prod_{j=1}^{h} D_j} = \frac{1}{D_1 \cdots D_h} \rightarrow \frac{1}{D_1^{a_1} \cdots D_t^{a_t}} = \frac{1}{\prod_{j=1}^{t} D_j^{a_j}} \tag{1.5}$$

where $t < h$ is the number of indipendent denominators appearing in the integrand, and $\sum_{j=1}^{t} a_j = h$.

Now we have t denominators of the form $D_i = (V_i^2 + m_i^2)$. We can express some of our scalar product S_j as combination of denominators and kinematic invariants $\mathcal{I}$:

$$S_j = \sum_{i=1}^{t} \alpha_{ji} D_i + \sum_{k=1}^{|\mathcal{I}|} \beta_{jk} s_k \tag{1.6}$$

where $s_k \in \mathcal{I}$.

Example 1.1. (Reduce scalar products) Let's suppose to have a scalar product S_i that could be written as $S_i = D_j - m_j^2$. An integrand become

$$\frac{S_1 \cdots S_i \cdots S_{N_s}}{D_1^{a_1} \cdots D_j^{a_j} \cdots D_t^{a_t}} = \frac{S_1 \cdots (D_j - m_j^2) \cdots S_{N_s}}{D_1^{a_1} \cdots D_j^{a_j} \cdots D_t^{a_t}} =$$
$$= \frac{S_1 \cdots \hat{S}_i \cdots S_{N_s}}{D_1^{a_1} \cdots D_j^{a_j - 1} \cdots D_t^{a_t}} - m_j^2 \frac{S_1 \cdots \hat{S}_i \cdots S_{N_s}}{D_1^{a_1} \cdots D_j^{a_j - 1} \cdots D_t^{a_t}} \tag{1.7}$$

where the 'hat' notation stand for 'missing term'. ∎

At this point, we are ready for some formal definitions, that will help us to define correctly our Feynman integrals. We define:

$$T = \left\{ D_i \in D \mid D_i \neq D_j, \ \forall \ j \in \{1, \ldots, h\} \right\} \tag{1.8}$$

$$\text{ISP} = \left\{ S_j \in S \mid S_j \neq \sum_{i=1}^{t} \alpha_{ji} D_i + \sum_{k=1}^{|\mathcal{I}|} \beta_{jk} s_k, \ \forall \ \alpha_{ji}, \beta_{jk} \right\} \tag{1.9}$$

We need also to define $\mathcal{P}(T)$, the power set of T, and

$$\mathcal{D} : \mathbb{N} \rightarrow \mathcal{P}(T) \setminus \emptyset \tag{1.10}$$

$$\mathcal{D} : r \mapsto \mathcal{D}(r) = \{\tau \in \mathcal{P}(T) \setminus \emptyset \text{ such that } |\tau| = r\} \tag{1.11}$$

In other words, $\mathcal{D}(r) \subset \mathcal{P}(T)$ is the set of elements of $\mathcal{P}(T)$ with exactly r denominators.
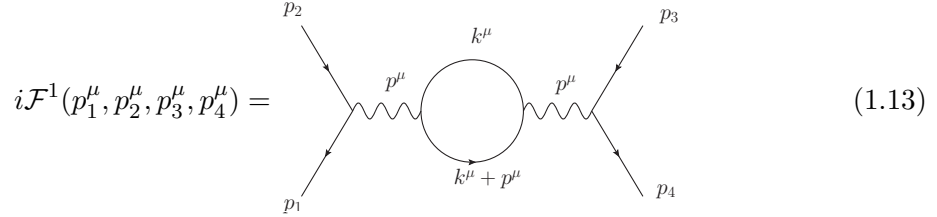
With these last definitions, the decomposition of the form factor (1.3) results[2]

$$
I_c^l(\mathcal{I}) = \sum_{r=1}^{t} \sum_{q=1}^{|\mathcal{D}(r)|} A_{c,r,q} \int \left(\prod_{i=1}^{l} \frac{d^d k_i}{(2\pi)^{d-2}} \right) \frac{\prod_{j=1}^{N_s-t} S_j^{b_j}(K,P)}{\prod_{D_i \in \mathcal{D}_q(r)} D_i^{a_i}(K,P,\mathbf{m})} \tag{1.12}
$$

The integral objects in (1.12) are called *Feynman integrals*[16, 23], T is the *topology* of the Feynman integral, $\mathcal{D}_q(r)$ is called *sector* or *subtopology*.

This is the most general definition of the form factor $I_c^l(\mathcal{I})$, constructible starting from the scattering amplitude. We could define all this quantities in another way, more userful in the context which we will face later.

Example 1.2. [QED 1-loop Vacuum polarization] Let's start from 4-point 1-loop Feynman amplitude of $e^- e^+ \to \mu^+ \mu^-$ scattering[16]

$$
i\mathcal{F}^1(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu) = \tag{1.13}
$$

which has analytic expression

$$
\begin{aligned}
\mathcal{F}^1(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu) &= \bar{v}(p_2)(e\gamma^\mu)u(p_1)\frac{\delta_{\mu\rho}}{p^2}\Pi^{\rho\sigma}(p^2)\frac{\delta_{\sigma\nu}}{p^2}\bar{u}(p_3)(e\gamma^\nu)v(p_4) \\
&= \frac{e^2}{(p^2)^2}\bar{v}(p_2)\gamma_\mu u(p_1)\Pi^{\mu\nu}(p^2)\bar{u}(p_3)\gamma_\nu v(p_4)
\end{aligned} \tag{1.14}
$$

where $p^\mu = p_1^\mu + p_2^\mu$ and

$$
\Pi^{\mu\nu}(p^2) = -\int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr}[(e\gamma^\mu)(-\slashed{k}+m)(e\gamma^\nu)(-(\slashed{k}-\slashed{p})+m)]}{[k^2+m^2][(k-p)^2+m^2]} \tag{1.15}
$$

Here we used the QED Feynman rules in Euclidean space. The explicit derivation is exposed in Appendix A.

Firstly, we have to perform the tensor reduction: $\Pi^{\mu\nu}(p^2)$ is a Lorentz-invariant tensor, so

$$
\Pi^{\mu\nu}(p^2) = \alpha\delta^{\mu\nu} + \beta p^\mu p^\nu \tag{1.16}
$$

contract this object with another $\delta_{\mu\nu}$:

$$
\delta_{\mu\nu}\Pi^{\mu\nu}(p^2) = \alpha d + \beta p^2 \tag{1.17}
$$

QED is a gauge invariant theory, and as consequence, its amplitudes obey at the Ward-Takahashi identities:

$$
p_\mu p_\nu \Pi^{\mu\nu}(p^2) = 0 \implies \alpha p^2 + \beta p^4 = 0 \implies \alpha = -\beta p^2 \tag{1.18}
$$

So

$$
\Pi^{\mu\nu}(p^2) = (p^\mu p^\nu - p^2\delta^{\mu\nu})\Pi(p^2) \tag{1.19}
$$

---

[2]For every set $\mathcal{D}(r)$, we indexed each element of it: $\mathcal{D}(r) = \{\mathcal{D}_1(r), \ldots, \mathcal{D}_Q(r)\}$. We emphasize the fact that $\mathcal{D}_q(r) \subset T$.

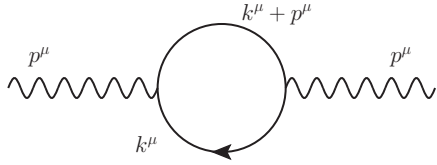where $\Pi(p^2) = \beta$. Looking at the contraction with $\delta^{\mu\nu}$:

$$\beta = \frac{\delta_{\mu\nu}\Pi^{\mu\nu}(p^2)}{p^2(1-d)} = \Pi(p^2) \tag{1.20}$$

We identify $\Pi(p^2)$ with the form factor previously defined, namely:

$$\mathcal{F}^1(p_1, p_2, p_3, p_4) = e^2 \bar{v}(p_2)\gamma_\mu u(p_1)\frac{p^\mu p^\nu - p^2 \delta^{\mu\nu}}{(p^2)^2}\bar{u}(p_3)\gamma_\nu v(p_4)\Pi(p^2) \tag{1.21}$$

In the language built before:

$$E^{ijkr}(P) = \bar{v}^i(p_2)u^j(p_1)\bar{u}^k(p_3)v^r(p_4)$$

$$B_{ijkr}(P) = e^2(\gamma_\mu)_{ij}\frac{p^\mu p^\nu - p^2 g^{\mu\nu}}{(p^2)^2}(\gamma_\nu)_{kr} \tag{1.22}$$

$$I^1(\mathcal{I}) = \Pi(p^2)$$

$$l = 1, \quad n = 1, \quad h = 2 = t$$
$$K = \{k^\mu\}, \quad P = \{p^\mu\}$$
$$\mathbf{M} = \{\sqrt{s}, \sqrt{s}\}, \quad \mathbf{m} = \{m, m\}$$

$$\boldsymbol{\sigma} = \{p^2\}, \quad \bar{\boldsymbol{\sigma}} = \{p^2, p\cdot k, k^2\}$$
$$\bar{\boldsymbol{\sigma}}\backslash\boldsymbol{\sigma} = \{p\cdot k, k^2\}$$
$$\mathbf{s} = \{p^2\} = \{s\}$$
$$\mathcal{I} = \{s, m^2\}$$

After the tensorial decomposition, we can define all of the quantities for the form factor $\Pi(p^2)$. At this point, let's work on the scalar form factor[35, 16]:

$$\begin{aligned}\Pi(s) &= -\frac{e^2}{s(1-d)}\int\frac{d^d k}{(2\pi)^d}\frac{\mathrm{Tr}[\gamma^\mu(-\slashed{k}+m)\gamma_\mu(-\slashed{k}+\slashed{p}+m)]}{(k^2+m^2)[(k-p)^2+m^2]}\\ &= -\left(\frac{\alpha}{\pi}\right)\frac{1}{s(1-d)}\int\frac{d^d k}{(2\pi)^{d-2}}\frac{\mathrm{Tr}[\gamma^\mu\slashed{k}\gamma_\mu(\slashed{k}-\slashed{p})+\gamma^\mu\gamma_\mu m^2]}{(k^2+m^2)[(k-p)^2+m^2]}\\ &= \left(\frac{\alpha}{\pi}\right)\frac{\mathrm{Tr}[\mathbb{I}_d]}{s(1-d)}\int\frac{d^d k}{(2\pi)^{d-2}}\frac{(d-2)(k^2-k\cdot p)+dm^2)}{(k^2+m^2)[(k-p)^2+m^2]}\end{aligned} \tag{1.23}$$

Here we can see that all the denominators are indipendent, so

$$T = \{k^2 + m^2, (k-p)^2 + m^2\} = \{D_1, D_2\} \tag{1.24}$$

and all elements of $\bar{\boldsymbol{\sigma}}\backslash\boldsymbol{\sigma}$ can be written as combination of denominators and kinematics invariants $\mathcal{I}$:

$$k^2 = D_1 - m^2, \quad k\cdot p = \frac{1}{2}[D_1 + s - D_2] \implies \mathrm{ISP} = \emptyset \tag{1.25}$$

Lastly, the power set of T is:

$$\mathcal{P}(T)\backslash\emptyset = \{\{D_1, D_2\}, \{D_1\}, \{D_2\}\} \tag{1.26}$$

and the function $\mathcal{D} : \mathbb{N} \to \mathcal{P}(T)$ act as

$$\mathcal{D}(1) = \{\{D_1\}, \{D_2\}\}, \quad \mathcal{D}(2) = \{\{D_1, D_2\}\} \tag{1.27}$$

So, after a few algebra, we arrive at the last decomposition of the form factor:

$$
\begin{aligned}
\Pi(s) &= \left(\frac{\alpha}{\pi}\right)\frac{\mathrm{Tr}[\mathbb{I}_d]}{s(1-d)} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{(\frac{d-2}{2})D_1 + (\frac{d-2}{2})D_2 - (\frac{d-2}{2})s + 2m^2}{D_1 D_2} \\
&= \left(\frac{\alpha}{\pi}\right)\frac{\mathrm{Tr}[\mathbb{I}_d]}{s(1-d)} \int \frac{d^d k}{(2\pi)^{d-2}} \left\{ \left(\frac{d-2}{2}\right)\frac{1}{D_2} + \left(\frac{d-2}{2}\right)\frac{1}{D_1} - \frac{(\frac{d-2}{2})s - 2m^2}{D_1 D_2} \right\}
\end{aligned}
\tag{1.28}
$$

so:

$$
\begin{aligned}
\Pi(s) = -\left(\frac{\alpha}{\pi}\right)\frac{\mathrm{Tr}[\mathbb{I}_d]}{s(1-d)} &\left\{ \left(\frac{2-d}{2}\right)\left[\int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{D_1} + \int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{D_2}\right] + \right. \\
&\left. + \left[\left(\frac{d-2}{2}\right)s - 2m^2\right]\int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{D_1 D_2} \right\}
\end{aligned}
\tag{1.29}
$$

that reflects the form (1.12). The integral terms are Feynman integrals for the QED vacuum polarization. Graphically:

$$
\begin{aligned}
\Pi(s) = -\left(\frac{\alpha}{\pi}\right)\frac{\mathrm{Tr}[\mathbb{I}_d]}{s(1-d)} &\left\{ \left(\frac{2-d}{2}\right)\left[\ \overline{\bigcirc}\ + \ \underline{\bigcirc}\ \right] + \right. \\
&\left. + \left[\left(\frac{d-2}{2}\right)s - 2m^2\right]\ {-\!\!\bigcirc\!\!-}\ \right\}
\end{aligned}
\tag{1.30}
$$

$\blacksquare$

Plugging this decomposition in (1.14), we get the full decomposition of a Feynman amplitude in scalar Feynman integrals.

1.1.3 Feynman integrals: an assiomatic approach

We can face the Feynman diagram calculation with another approach: we can neglect the fact that we're evaluating a full amplitude, by considering that the tensor decomposition isn't actually a difficult step; It can be easily automated. The hardest part of the calculation is all in the Feynman integral evaluation. So, all we need to know to evaluate a scattering amplitude is: the number of external legs, the number of loop and the number of internal propagators.

Suppose to have an l-loop Feynman diagram with $n + 1$ external legs and h internal propagators. By a rapid analysis on the momentum conservation, we can select t number of internal lines which are inter-independent. Now, we candefine two sets of momenta:

- $P = \{p_1^\mu, \ldots, p_n^\mu\}$;

- $K = \{k_1^\mu, \ldots, k_l^\mu\}$;

and two sets of masses

- $\mathbf{M} = \{M_1, \ldots, M_{n+1}\} \implies \mathbf{M}^2 = \{M_1^2, \ldots, M_{n+1}^2\}$;

- $\mathbf{m} = \{m_1, \ldots, m_t\} \implies \mathbf{m}^2 = \{m_1^2, \ldots, m_t^2\}$

With this informations we can build userful combination of momenta and masses:

- $\bar{\boldsymbol{\sigma}} = \{v_i \cdot v_j \mid v_i, v_j \in P \cup K\}$;

- $\boldsymbol{\sigma} = \{p_i \cdot p_j \mid p_i, p_j \in P\}$;

- $\bar{\Sigma} = \bar{\boldsymbol{\sigma}} \backslash \boldsymbol{\sigma}$

- $\mathbf{s} = \left\{\frac{1}{1+3\delta_{ij}}(p_i + p_j)^2 \mid p_i, p_j \in P\right\}$

- $\mathcal{I} = \mathbf{s} \sqcup \mathbf{M}^2 \cup \mathbf{m}^2$

The number of scalar products with at least one loop momenta is $|\bar{\Sigma}| = N_s$.

Recalling a previous definition, we set $V_i^\mu = \sum_{j=1}^l w_{ij} k_j^\mu + \sum_{k=1}^n w'_{ik} p_k^\mu$ such that $k_j^\mu \in K$, $p_k^\mu \in P$, $w'_{ik}, w_{ij} \in \{0, 1\}$ with at least one of $w_{ij} = 1$ $\forall i \in \{1, \ldots, t\}$. This combination of momenta is called *momentum current*.

An *external line* is a couple $E_i = (p_i^\mu, M_i)$, where $p_i^\mu \in P$ and $M_i \in \mathbf{M}$ are momenta and masses associated to each external line. We impose $E_{n+1} = (\sum_{i=1}^n p_i^\mu, M_{n+1})$. Their set is $E = \{E_1, \ldots, E_{n+1}\}$.

In the same way, an *internal line* is triplet $\Delta_j = (V_j^\mu, W_j^\mu, m_j^2)$, where V_j^μ, W_j^μ are momentum currents. The set of all internal lines[3] buildable with the momentum currents V_j^μ and W_j^μ and a mass m_j is denoted as $\mathfrak{I}$. A selection of t internal lines $\Delta_j \in \mathfrak{I}$ is $\Delta = \{\Delta_1, \ldots, \Delta_t\}$, so $\Delta \subset \mathfrak{I}$ and $|\Delta| = t$. An internal line of the form $\Delta_j = (V_j^\mu, V_j^\mu, m_j^2)$ is called *quadratic internal line*.

Taking a look at the previous definition of denominator, we said that $D_j = V_j \cdot V_j + m_j^2$. Started from the notion of internal line, we can define a *denominator* as a function

$$
\begin{aligned}
D &: \mathfrak{I} \to \mathbb{R} \\
D &: \Delta_j \mapsto D(\Delta_j) = V_j \cdot W_j + m_j^2 \equiv D_j
\end{aligned}
\tag{1.31}
$$

The similarity between Δ and the topology T defined before is given by D. In particular, $D(\Delta) = T$.

Lastly, an *irreducible scalar product* (ISP) S_j is an element of $\bar{\Sigma}$ such that

$$
S_j \neq \sum_{\Delta_i \in \Delta} \alpha_j(\Delta_i) D(\Delta_i) + \sum_{s_i \in \mathcal{I}} \beta_j(s_i) s_i
\tag{1.32}
$$

for every values of $\alpha_j(\Delta_i)$, $\beta_j(s_i)$. The set of all S_j is denoted by Σ, the sets of ISPs[4]. Clearly $\Sigma \subset \bar{\Sigma}$, and $|\Sigma| = N_s - t = N_{\text{ISP}}$.

At this point, we are ready for the first definition:

---

[3]We often will neglect this set: for many applications it matters only the knowledge of $\Delta \subset \mathfrak{I}$. In Chapter 4. we will define some objects for which is crucial the full generality of $\mathfrak{I}$.

[4]Here we presented the ISPs and the denominators in two different shape. Actually, $S \in \Sigma$ is a scalar product between momenta, for example $S = p_j \cdot k_l$. If we define two momentum currents $V^\mu = p_j^\mu$ and $W^\mu = k_l^\mu$ and an internal line $\Delta_S = (p_j^\mu, k_l^\mu, 0)$, it's easy to see that $S = D(\Delta_S)$. This feature will get great importance for the automation process, and it's based on a different definition of propagator, as we will see in Chapter 3..
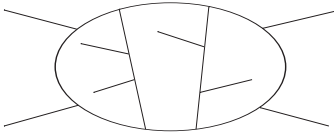
Definition 1.1 (Topology)**.** A *topology* is the couple $\mathcal{T} = (E, \Delta)$, where E is a set of external lines and $\Delta \subset \mathfrak{I}$. Elements of $\mathcal{T}$ are represented by continuous lines being part of a diagram respecting the following rules:

- $E_i \in E$ is represented by a line where one and only one extremal point is connected to other lines;

- $\Delta_j \in \Delta$ is represented by a line where both of the extremal points are connected to other lines;

- The junction between two or more lines are called *vertices*;

- Every vertices satisfies the momentum conservation: the total incoming momentum in a vertex must be equal to the outgoing one;

$$\sum_{i \text{ incoming}} k_i = \sum_{j \text{ outgoing}} k_j \tag{1.33}$$

- The graph is connected.

so that a topology can be represented as

$$\mathcal{T} = \tag{1.34}$$

The symbol $\mathbb{T}$ denotes the set of all topologies.

Every topology could generates other kind of topologies, by selecting only a subset of its internal lines:

Definition 1.2 (Subtopology)**.** Let $\mathcal{T} \in \mathbb{T}$ be a topology, with Δ the set of its internal line. The set $\mathcal{S}_{\mathcal{T}}$ of $\mathcal{T}$ is the set of the couples $\mathcal{S}_{\mathcal{T}} = \big\{(E, \bar{\Delta}_j), \ \bar{\Delta}_j \in \mathcal{P}(\Delta) \backslash \{\Delta, \emptyset\}\big\}$. $\mathcal{S}_{\mathcal{T}}$ is called *subtopology tree*. An element $(E, \bar{\Delta}) \in \mathcal{S}_{\mathcal{T}}$ is called *subtopology*, and we will denote it with $\tau_i \in \mathcal{S}_{\mathcal{T}}$. Subtopologies must obey at the same graphic rules of the parent topology.

In other words, given a Feynman graph with a topology $\mathcal{T}$, all subtopolgies $\mathcal{S}_{\mathcal{T}}$ related to $\mathcal{T}$, can be obtained graphically by *shrinking* internal lines in every possible combinations.

It's easy to show that, if we pick a $\tau_i \in \mathcal{S}_{\mathcal{T}}$, it also defines a topology. Being itself a topology, τ_i has an it own subtopology tree $\mathcal{S}_{\tau_i}$.

Proposition 1.1.1. *Let $\mathcal{T} \in \mathbb{T}$ be a topology, $\mathcal{S}_{\mathcal{T}}$ be its subtopology tree and $\tau_i \in \mathcal{S}_{\mathcal{T}}$. Then, $\mathbb{T} \ni \tau_i$ is a topology and $\mathcal{S}_{\tau_i} \subset \mathcal{S}_{\mathcal{T}} \subset \mathbb{T}$.*

Proof. If $\mathcal{T} \in \mathbb{T}$, it can be written as $\mathcal{T} = (E, \Delta)$. For definition, $\tau_i \in \mathcal{S}_{\mathcal{T}}$ can also be written as $\tau_i = (E, \Delta')$, where $\mathfrak{I} \ni \Delta' \subset \Delta$. So, by definition, a subtopology τ_i is defined by a couple of external legs E and a set of internal lines Δ'; the graph associated to τ_i is connected because it comes from a connected object by shrinking

Figure 1.1: Vertex shrinking: momentum conservation is not affected by this operation.

lines, not by "cutting" them. Lastly, we have to be sure that each vertex still have momenta conservation.

Looking at the Figure (1.1), in (1) the momentum conservation at each vertex states that

$$\sum_{j=1}^{i} p_j - k = 0, \quad k - \sum_{j=i+1}^{n} p_j = 0 \implies \sum_{j=1}^{i} p_j = \sum_{j=i+1}^{n} p_j \tag{1.35}$$

The implication is nothing but the conservation for (2). So, shrinking internal lines doesn't affects the momentum conservation. □

Further on, in the absence of ambiguity, we might call "topology" both $\mathcal{T}$ and $\tau_i \in \mathcal{S}_{\mathcal{T}}$, because of the previous proposition.

Now we can define what is an addend of (1.12):

Definition 1.3 (Feynman integral)**.** Let $\mathcal{T} = (E, \Delta) \in \mathbb{T}$ be a topology. A *Feynman integral* is and integral function of the variables P (external momenta), with integration measure $\frac{d^d k_i}{(2\pi)^{d-2}}$, with $k_i \in K$, of a rational function of both topology and irreducible scalar products $S_j \in \Sigma$:

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \int \left(\prod_{i=1}^{l} \frac{d^d k_i}{(2\pi)^{d-2}} \right) \frac{\prod_{j=1}^{N_{\mathrm{ISP}}} S_j^{b_j}}{\prod_{i=1}^{t} D(\Delta_i)^{a_i}} \tag{1.36}$$

This integral is represented by a diagram with internal lines Δ and external lines E. In this definition $a_i \geq 1$ and $b_i \geq 0$. Here, $\bar{a} \in \mathbb{N}^t$ and $\bar{b} = \mathbb{N}^{N_{\mathrm{ISP}}}$ are the vectors of powers of denominator and ISPs[16].

When a denominator $D(\Delta_i)$ has power $a_i > 1$, it will be represented by a dotted line, with $a_i - 1$ dots.

$$I_{\bar{a}}^{\bar{b}} \left(\includegraphics \right) = \includegraphics \tag{1.37}$$

If $a_i = 1$ and $b_j = 0$ for all $i \leq t$ and $j \leq N_{\mathrm{ISP}}$, $I_{\bar{1}}^{\bar{0}}(\mathcal{T})$ is called *corner integral*; those integrals are graphically represented by the same diagram of the topology $\mathcal{T}$.

$$I_{\bar{1}}^{\bar{0}} \left(\includegraphics \right) = \includegraphics \tag{1.38}$$

In (1.12), we may recognize that every addend of $I_c^l(\mathcal{I})$ contains a Feynman integrals: looking at the summatory, the term with $r = t$ is an integral built on a topology $\mathcal{T}$, and for $r < t$ is an integral built on its subtopology tree.

This means that a generic Feynman diagrams is a combination of Feynman integrals build on the topologies $\mathcal{T} \cup \mathcal{S}_{\mathcal{T}}$.

Grafically, it is represented by a Feynman diagram which respects the Definition (1.1). To understand what are the other addend of (1.12), we need another definition:

A Feynman integrals can be build with the only knowledge of $\mathcal{T}$. At this point, is clear how to construct all Feynman integrals from a given topology:

1. Build two sets $P = \{p_1^\mu, \ldots, p_n^\mu\}$, $K = \{k_1^\mu, \ldots, k_l^\mu\}$ of momenta and the others auxiliary sets;

2. Write a set E of couples (p_i^μ, M_i) for $i \in \{1, \ldots, n+1\}$;

3. Write another set Δ of triplets $(V_j^\mu, W_j^\mu, m_j^2)$ for $j \in \{1, \ldots, t\}$, such that V_j^μ and W_j^μ are momentum currents;

4. Now, $\mathcal{T} = (E, \Delta)$, making sure that the chosen momenta respect the momentum conservation;

5. Write Σ;

6. Compute $\mathcal{S}_{\mathcal{T}}$;

7. Associate $\tau_i \mapsto I_{\bar{a}}^{\bar{b}}(\tau_i)$ for all $\tau_i \in \mathcal{T} \cup \mathcal{S}_{\mathcal{T}}$ and for any choice of the powers $\bar{a}$ and $\bar{b}$.

It is possible that, for a topology $\mathcal{T}$, all the Feynman integral which can build on $\mathcal{T}$ vanish. This bring us to give the following definition:

Definition 1.4 (Trivial topologies)**.** Let $\mathcal{T} \in \mathbb{T}$ be a topology. If

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = 0 \quad \forall\, \bar{a} \in \mathbb{N}^t, \quad \forall\, \bar{b} \in \mathbb{N}^{N_{\mathrm{ISP}}} \tag{1.39}$$

then $\mathcal{T}$ is called *trivial topology.*

Moreover, setting $\mathcal{S}_{\mathcal{T}}$, we define $\mathcal{Z}_{\mathcal{T}} \subset \mathcal{S}_{\mathcal{T}}$ the set of all trivial subtopology of $\mathcal{T}$.

Feynman integrals are the building blocks for the evaluation of multi-loop Feynman diagrams. Thanks to the tensor reduction, in a evaluation of an amplitude, the tensorial and the external particle dependence is factorized, so, the only thing that remains is the information about the loop structure

Example 1.3. [QED 1-loop Vacuum polarization] Vacuum polarization is characterized by a diagram having: $n = 1$, $l = 1$, $t = 2$. So $P = \{p^\mu\}$, $K = \{k^\mu\}$, $\mathbf{M} = \{\sqrt{s}, \sqrt{s}\}$ and $\mathbf{m} = \{m_1, m_2\}$. At this poing we can build:

$$\boldsymbol{\sigma} = \{k^2, k \cdot p, p^2\}$$
$$\bar{\boldsymbol{\sigma}} = \{p^2\}$$
$$\bar{\Sigma} = \{k^2, k \cdot p\}$$

$$\mathbf{s} = \{s\} = \{p^2\}$$
$$\mathcal{I} = \{s, m_1^2, m_2^2\}$$

We have two external lines, which in vacuum polarization are two off-shell photons:

$$E = \big\{ (p^\mu, \sqrt{s}), (p^\mu, \sqrt{s}) \big\} \tag{1.40}$$

Now, we have two internal lines:

$$\Delta = \big\{ (k^\mu, k^\mu, m_1^2), (k^\mu + p^\mu, k^\mu + p^\mu, m_2^2) \big\} = \{\Delta_1, \Delta_2\} \tag{1.41}$$

Since that $|\bar{\Sigma}| - t = 0$, we have no ISPs:

$$\Sigma = \emptyset \tag{1.42}$$

And we are ready to define the topology $\mathcal{T}$ whose Feynman integrals of vacuum polarization amplitudes belong:

$$\mathcal{T} = \Big(\big\{ (p^\mu, \sqrt{s}), (p^\mu, \sqrt{s}) \big\}, \big\{ (k^\mu, k^\mu, m_1^2), (k^\mu + p^\mu, k^\mu + p^\mu, m_2^2) \big\} \Big) \tag{1.43}$$

Graphically, $\mathcal{T}$ is represented as

$$\mathcal{T} = \quad \text{—} \bigcirc \text{—} \tag{1.44}$$

Moreover, we define also $\mathcal{S}_\mathcal{T}$, the subtopology tree of $\mathcal{T}$:

$$\mathcal{S}_\mathcal{T} = \left\{ \left(\begin{array}{c} \{ (p^\mu, \sqrt{s}), (p^\mu, \sqrt{s}) \}, \\ \{ (k^\mu, k^\mu, m_1^2) \} \end{array} \right), \left(\begin{array}{c} \{ (p^\mu, \sqrt{s}), (p^\mu, \sqrt{s}) \}, \\ \{ (k^\mu + p^\mu, k^\mu + p^\mu, m_2^2) \} \end{array} \right) \right\} \tag{1.45}$$

Again, graphically:

$$\mathcal{S}_\mathcal{T} = \left\{ \underline{\bigcirc} \ , \ \overline{\bigcirc} \right\} \tag{1.46}$$

With these definitions, we can write all the Feynman integrals involving in the evaluation of 1-loop vacuum polarization:

$$I_{a_1, a_2} \big(\text{—} \bigcirc \text{—} \big) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D(\Delta_1)^{a_1} D(\Delta_2)^{a_2}}$$

$$I_{a_1} \big(\overline{\bigcirc} \big) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D(\Delta_1)^{a_1}} \tag{1.47}$$

$$I_{a_2} \big(\underline{\bigcirc} \big) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D(\Delta_2)^{a_2}}$$

$\blacksquare$

One loop structure has some simplification compared to generic multi-loop topologies. Every 1-loop diagrams with $n + 1$ external legs has exactly $n + 1$ internal indipendent legs. As a consequence, every one loop topology has no irreducible scalar product: in fact $N_s^{1-loop} = n + 1$ and $N_{\text{ISP}} = 0$.

1.2 Dimensional regularization

Feynman integrals $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ defined in the last Section have been regulated in UV region through the *dimensional regularization prescription*. It is based on the replacement of the 4-dimensional loop integration with a d-dimensional integration, where d is a continuous parameter:

$$\frac{d^4 k}{(2\pi)^2} \to \mu^{-2\epsilon} \frac{d^d k}{(2\pi)^{d-2}} \tag{1.48}$$

The parameter μ has mass dimension $[\mu] = 1$, and it has been introduced to restore the mass dimension of the new measure introduced. Here, $\epsilon = \frac{4-d}{2}$, which is our expansion parameter: expanding a Feynman integral around $\epsilon = 0$, we get an expression where the divergent behaviour of UV regions appears as *poles* of the Laurent expansion of $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$.

Example 1.4. [Tadpole integral] The tadpole topology is

$$\underline{\bigcirc} = \left(\left\{ \begin{matrix} (p, \sqrt{s}), \\ (p, \sqrt{s}) \end{matrix} \right\}, \left\{ (k^\mu, k^\mu, m^2) \right\} \right) \tag{1.49}$$

we want to evaluate explicitely the Laurent expansion around $d = 4$ of the Feynman integral

$$I_1 \left(\underline{\bigcirc} \right) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2 + m^2} \tag{1.50}$$

The analytic integration is

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2 + m^2} = \frac{1}{2(2\pi)^{d-2}} \int d\Omega_d \int_0^\infty dk^2 \frac{(k^2)^{\frac{d}{2}-1}}{k^2 + m^2} \tag{1.51}$$

The angular integration gives

$$I_1 \left(\underline{\bigcirc} \right) = \frac{1}{4(4\pi)^{\frac{d}{2}-2}} \frac{1}{\Gamma(\frac{d}{2})} \int_0^\infty dk^2 \frac{(k^2)^{\frac{d}{2}-1}}{k^2 + m^2} \tag{1.52}$$

and factorizing out the mass dependence

$$I_1 \left(\underline{\bigcirc} \right) = \frac{1}{4(4\pi)^{\frac{d}{2}-2}} \frac{(m^2)^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})} \int_0^\infty dx\, x^{\frac{d}{2}-1}(x+1)^{-1} \tag{1.53}$$

It can be shown that

$$\int dx\, x^{\alpha-1}(x+1)^{-\alpha-\gamma} = \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \tag{1.54}$$

then

$$I_1 \left(\underline{\bigcirc} \right) = \frac{1}{4(4\pi)^{\frac{d}{2}-2}} \frac{(m^2)^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})} \frac{\Gamma(\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(1)} = \frac{(m^2)^{\frac{d}{2}-1}}{4(4\pi)^{\frac{d}{2}-2}} \Gamma\left(1 - \frac{d}{2}\right) \tag{1.55}$$

Now, using $d = 4 - 2\epsilon$:

$$I_1 \left(\underline{\bigcirc} \right) = \frac{m^2}{4} \left(\frac{4\pi}{m^2}\right)^\epsilon \Gamma(-1 + \epsilon) \tag{1.56}$$

Expanding around $\epsilon = 0$ (i.e. $d = 4$), we have

$$\Gamma\left(-1 + \epsilon\right) \simeq -\frac{1}{\epsilon} - \psi(2) + o(\epsilon)$$

$$\left(\frac{4\pi}{m^2}\right)^{\epsilon} \simeq 1 + \epsilon \log\left(\frac{4\pi}{m^2}\right) + o(\epsilon^2) \tag{1.57}$$

$$I_1\left(\underline{\bigcirc}\right) = -\frac{m^2}{4}\left[\frac{1}{\epsilon} + \psi(2) + \log\left(\frac{4\pi}{m^2}\right)\right] + o(\epsilon) \tag{1.58}$$

As we can see, the divergent behaviour of the tadpole integral appears as a simple pole in the Laurent expansion. It is possible to insert a counterterm which cancel out this pole. This cancellation can be made using *renormalization prescriptions*.

Another way to explicit poles not from the Laurent expansion is through using the properties of Euler's Gamma: $\Gamma(3 + n) = (2 + n)(1 + n)\Gamma(1 + n)$:

$$I_1\left(\underline{\bigcirc}\right) = m^2\left(\frac{4\pi}{m^2}\right)^{2 - \frac{d}{2}}\frac{\Gamma\left(3 - \frac{d}{2}\right)}{(d - 4)(d - 2)} \tag{1.59}$$

Again, at $d = 4$ we find a simple pole. ∎

Through dimensional regularization scheme, we can expose properties of massless Feynman integral.

Proposition 1.2.1. *The integral*

$$I(d, a) = \int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{\left(k^2\right)^{\frac{a}{2}}} \tag{1.60}$$

vanishes in dimensional regularization if $d > a$:

$$I(d, a) \overset{dimreg}{=} 0, \qquad d > a \tag{1.61}$$

Proof. This integral is divergent in IR region ($|k| \to 0$), so that we have to introduce a mass regulator:

$$I(d, a) = \lim_{m \to 0} \int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{\left(k^2 + m^2\right)^{\frac{a}{2}}} \tag{1.62}$$

Now, this integral is well-known in dimensional regularization, which gives us

$$I(d, a) = \lim_{m \to 0} \frac{1}{4(4\pi)^{\frac{d}{2}-2}}\frac{\Gamma\left(\frac{a-d}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}\left(m^2\right)^{\frac{d-a}{2}} \propto \left(m\right)^{d-a} \tag{1.63}$$

Hence, in the limit $m \to 0$, this integral vanishes if

$$d > a \tag{1.64}$$

□

It is possible to show that the identity in the Proposition (1.2.1) can be generalized at general $d > 0$.

Proposition 1.2.2. *The integral*

$$I(d, a) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{(k^2)^{\frac{a}{2}}} \tag{1.65}$$

vanishes in dimensional regularization:

$$I(d, a) \overset{dimreg}{=} 0, \qquad \forall d, a \in \mathbb{C} \tag{1.66}$$

This identity is known as Veltman's formula

Proof. A naive proof of this identity can be given through dimensional analysis: the mass dimension of $I(d, a)$ is

$$[I(d, a)] = \left[\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{(k^2)^{\frac{a}{2}}} \right] = [k^\mu] - [|k|]^a = d - a \tag{1.67}$$

as expected. Since $I(d, a)$ does not depends on external momenta or any dimensional parameter, its mass dimension has to be zero. Hence, $I(d, a) = 0$ for $d \neq a$, due to dimensional analysis.

In $d = a$ this argument seems to not even hold. Actually, we can use the analiticity of dimensional regularization to extend the validity of $I(d, a) = 0$.

This identity can be proven rigorously for $d, a \in \mathbb{C}$[36].

A last proof of this identity is the following: suppose to evaluate the following integral

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{m^2}{k^2(k^2 + m^2)} \tag{1.68}$$

The direct integration of this function gives

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{m^2}{k^2(k^2 + m^2)} = -\frac{(m^2)^{\frac{d}{2}-1}}{4(4\pi)^{\frac{d}{2}}} \Gamma\left(1 - \frac{d}{2}\right) \tag{1.69}$$

The integral can be written in another way:

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{m^2}{k^2(k^2 + m^2)} = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2} - \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2 + m^2} \tag{1.70}$$

Integrating the second term of the combination, we get

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2 + m^2} = \frac{(m^2)^{\frac{d}{2}-1}}{4(4\pi)^{\frac{d}{2}}} \Gamma\left(1 - \frac{d}{2}\right) \tag{1.71}$$

and comparing this result with Equation (1.69), we have that

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2} = 0, \quad \forall d \in \mathbb{C} \tag{1.72}$$

Decomposing this last identity in spherical coordinates, we arrive at the identity

$$\int dk (k^2)^{\frac{d-3}{2}} = 0, \quad \forall d \in \mathbb{C} \tag{1.73}$$

and due to the generality of the power of the integrand, we arrive at

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{(k^2)^{\frac{a}{2}}} = 0 \tag{1.74}$$

$\square$

It is important to underline that $I(d, a)$ is nothing but the *massless tadpole integral*. Its topology is

$$\bigcirc = \left(\left\{ \begin{matrix} (p, \sqrt{s}), \\ (p, \sqrt{s}) \end{matrix} \right\}, \{(k^\mu, k^\mu, 0)\} \right) \tag{1.75}$$

and

$$I_{\frac{a}{2}} \left(\bigcirc \right) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{(k^2)^{\frac{a}{2}}} = I(d, a) \tag{1.76}$$

From the proposition (1.2.1) follows a corollary:

Corollary 1.2.3. *The integral*

$$I(d, 0) = \int \frac{d^d k}{(2\pi)^{d-2}} \tag{1.77}$$

is zero:

$$I(d, 0) = 0 \tag{1.78}$$

Proof. Looking at the condition (1.64), this time it states that

$$d > 0 \tag{1.79}$$

which is always satisfied. Hence

$$I(d, 0) = 0 \tag{1.80}$$

$\square$

This corollary shows that the generic 1-loop Feynman integral

$$I_0(\mathcal{T}_{1-\text{loop}}) = \int \frac{d^d k}{(2\pi)^{d-2}} = 0, \qquad \forall \mathcal{T} \in \mathbb{T} \tag{1.81}$$

Hence, general 1-loop Feynman integrals must have at least 1 propagator.

This corollary can be extended to include general l-loop topologies.

Proposition 1.2.4. *Let $\mathcal{T}$ be a topology. Each Feynman integral with no denominators vanishes:*

$$I_{\bar{0}}^{\bar{b}}(\mathcal{T}) = \int [dk] \Pi_{j=1}^{N_{ISP}} S_k^{b_k} = 0 \tag{1.82}$$

where $[dk] = \prod_{i=1}^{l} \frac{d^d k_i}{(2\pi)^{d-2}}$ and $\bar{b} = \{b_1, \ldots, b_{N_{ISP}}\}$

Proof. Irreducible scalar product appear two forms: $(p_i \cdot k_j)$ and $(k_i \cdot k_j)$. We firstly have to understand the integrals of these quantities separately, then we can combine the results to get the proof.

Consider the integral

$$I(1) = \int \frac{d^d k}{(2\pi)^{d-2}} p \cdot k \tag{1.83}$$

Due to the symmetry of the domain of integration, this integral is zero:

$$I(1) = p^\mu \int \frac{d^d k}{(2\pi)^{d-2}} k_\mu = 0 \tag{1.84}$$

Consider now an other integral:

$$I(2) = \int \frac{d^d k}{(2\pi)^{d-2}} (p \cdot k)^2 = p^\mu p^\nu \int \frac{d^d k}{(2\pi)^{d-2}} k_\mu k_\nu = p^\mu p^\nu I_{\mu\nu}(2) \tag{1.85}$$

$I_{\mu\nu}(2)$ has to be a Lorentz invariant tensor, so that

$$I_{\mu\nu}(2) = A\eta_{\mu\nu} \implies A = \frac{\eta^{\mu\nu}}{d} I_{\mu\nu}(2) \tag{1.86}$$

Hence:

$$A = \frac{1}{d} \int \frac{d^d k}{(2\pi)^{d-2}} k^2 \tag{1.87}$$

Noting that

$$A = \frac{1}{d} I(d, -2) = 0 \tag{1.88}$$

This integral vanishes, due to the Proposition (1.2.1).

Generalizing these results at l-loop topologies, we have the integral

$$I_0^{\bar{b}}(\mathcal{T}) = \int [dk] \prod_{ij} (p_i \cdot k_j)^{b_{ij}} \tag{1.89}$$

This kind of $I_0^{\bar{b}}(\mathcal{T})$ is factorizable. Notice that if the integrand do not contain dependence on one or more loop momenta, it turns to be null, due to the Corollary (1.2.3).

At this point we can focus our attention on one factor:

$$\begin{aligned} I_0^{\bar{b}}(\mathcal{T}) &= \int \frac{d^d k_j}{(2\pi)^{d-2}} \Pi_i (p_i \cdot k_j)^{b_{ij}} = \\ &= p_1^{\mu_1} \cdots p_n^{\mu_b} \int \frac{d^d k_j}{(2\pi)^{d-2}} k_{j\mu_1} \cdots k_{j\mu_b} = \\ &= p_1^{\mu_1} \cdots p_n^{\mu_b} I_{\mu_1 \cdots \mu_b}(b) \end{aligned} \tag{1.90}$$

where $b = \sum_{ij} b_{ij}$

It can be shown that a general invariant and symmetric Lorentz tensor with vanishing mass dimension is a made by sum and products of metrics $\eta^{\mu\nu}$:

$$I_{\mu_1 \cdots \mu_b}(b) = C \sum_{\sigma(i_j)} \eta_{\mu_{i_1}\mu_{i_2}} \cdots \eta_{\mu_{i_{b-1}}\mu_{i_b}} \tag{1.91}$$

with even b.

If b is odd, we have an antisymmetric function integrated over a symmetric domain. Hence, for odd b, $I_{\mu_1 \cdots \mu_b}(b) = 0$.

Now, for even b, we have to evaluate

$$C = \frac{1}{d^{\frac{b}{2}}} \eta^{\mu_1 \mu_2} \cdots \eta^{\mu_{b-1} \mu_b} I_{\mu_1 \cdots \mu_b}(b) \tag{1.92}$$

which means

$$C = \frac{1}{d^{\frac{b}{2}}} \int \frac{d^d k_j}{(2\pi)^{d-2}} (k_j^2)^{\frac{b}{2}} = \frac{1}{d^{\frac{b}{2}}} I(d, -b) \tag{1.93}$$

Again, $I(d, -b) = 0$ because of the Proposition (1.2.1). So:

$$I_{\bar{0}}^{\bar{b}}(\mathcal{T}) \tag{1.94}$$

The same arguments can be exploited to show that the following integrals vanishes:

$$
\begin{aligned}
J(1) &= \int \frac{d^d k_i}{(2\pi)^{d-2}} \frac{d^d k_i}{(2\pi)^{d-2}} (k_i \cdot k_j) = \\
&= \eta_{\mu\nu} \left(\int \frac{d^d k_i}{(2\pi)^{d-2}} k_i^\mu \right) \left(\int \frac{d^d k_i}{(2\pi)^{d-2}} k_j^\nu \right) = 0
\end{aligned}
\tag{1.95}
$$

and

$$
\begin{aligned}
J(2) &= \int \frac{d^d k_i}{(2\pi)^{d-2}} \frac{d^d k_i}{(2\pi)^{d-2}} (k_i \cdot k_j)^2 = \\
&= \eta_{\mu\nu} \eta_{\rho\sigma} \left(\int \frac{d^d k_i}{(2\pi)^{d-2}} k_i^\mu k_i^\rho \right) \left(\int \frac{d^d k_i}{(2\pi)^{d-2}} k_j^\nu k_j^\sigma \right) = \\
&= \eta_{\mu\nu} \eta_{\rho\sigma} \left(\frac{1}{d} I(d, -2) \eta^{\mu\rho} \right) \left(\frac{1}{d} I(d, -2) \eta^{\nu\sigma} \right) = 0
\end{aligned}
\tag{1.96}
$$

Generalizing this result, we get

$$
\begin{aligned}
I_{\bar{0}}^{\bar{b}}(\mathcal{T}) &= \int [dk] \prod_{ij} (k_i \cdot k_j)^{b_{ij}} = \\
&= \eta_{\mu_1 \nu_1} \cdots \eta_{\mu_b \nu_b} \prod_{j=1}^{l} \left(\int \frac{d^d k_j}{(2\pi)^{d-2}} k_j^{\mu_{\sigma(1)}} \cdots k_j^{\nu_{\sigma(b_j)}} \right) = \\
&= \eta_{\mu_1 \nu_1} \cdots \eta_{\mu_b \nu_b} \prod_{j=1}^{l} I^{\mu_{\sigma(1)} \cdots \nu_{\sigma(b_j)}}(b_j)
\end{aligned}
\tag{1.97}
$$

We already saw that this last integral is zero: $I_{\bar{0}}^{\bar{b}}(\mathcal{T})$.

From these preparatory integrals, we can see that, because of the structure of the integrand, $I_{\bar{0}}^{\bar{b}}(\mathcal{T})$ is a factorized integral.

Combining the results of (1.94) and (1.97), we notice that

$$I_{\bar{0}}^{\bar{b}}(\mathcal{T}) = \int [dk] \prod_{j=1}^{N_{\text{ISP}}} S_j^{b_j} \propto \int \frac{d^d k_j}{(2\pi)^{d-2}} (k_j^2)^{\frac{b_j}{2}} = 0 \tag{1.98}$$

$\square$

As a corollary, it follows that

$$I_0^0(\mathcal{T}_{l-\text{loop}}) = \int \prod_{i=1}^{l} \frac{d^d k_i}{(2\pi)^{d-2}} = 0 \tag{1.99}$$

Lastly, we give a general proposition which states that Feynman integral whose integrands are function of solely loop momenta vanishes

Proposition 1.2.5. *Every massless Feynman integral $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ on a topology $\mathcal{T}$*

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \int \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^{d-2}} f(K) \tag{1.100}$$

whose integrand is a function $f(K)$ depending only on loop momenta with mass dimension $[f(K)] \neq -d$ vanishes[37].

Proof. This proof follows from the previous Propositions, in fact, naming $[dk] = \prod_{j=1}^{l} \frac{d^d k_j}{(2\pi)^{d-2}}$:

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \int [dk] f(K) \propto \int \frac{d^d k_j}{(2\pi)^{d-2}} (k_j^2)^{\frac{\alpha_j}{2}} = 0 \tag{1.101}$$

due to the Proposition (1.2.1), where $\alpha_j > 0$. $\qquad\qquad\square$

Because of the dimensional regularization, it makes sense to consider only diagrams with powers of denominators $a_i \geq 0$, such that at least one of them has $a_j \geq 1$.

1.3 Relations between topologies

In order to evaluate a Feynman diagram, we have to identify the topology and all of its subtopologies, subsequently associate a Feynman diagram for each of them. In general, a Feynman diagram generates Feynman integrals with powers of denominators $a_i \geq 1$ and powers of numerators $b_j \geq 1$. We can define $a = \sum_i (a_i - 1)$ and $b = \sum_j (b_j - 1)$. Recalling that $t = |\Delta|$ and $N_{\text{ISP}} = |\Sigma|$, for a given topology, there are

$$N_{I(\mathcal{T})} = \binom{a - t - 1}{t - 1} \binom{b - N_{\text{ISP}} - 1}{N_{\text{ISP}} - 1} \tag{1.102}$$

Obviously, those integrals need to be evaluated. Actually, not all of those are independent between them. There exists some symmetries that relates a Feynman integral to another one, decreasing the number of object to be evaluated.

We can start from noticing that the function D induces an equivalence relation $\sim_D$ such that

$$\mathcal{T}_1 \sim_D \mathcal{T}_2 \iff E_1 = E_2 \wedge D(\Delta_1) = D(\Delta_2) \tag{1.103}$$

An immediate corollary states that $(-V^\mu, -W^\mu, m^2) \sim_D (V^\mu, W^\mu, m^2)$.

1.3.1   Change of variables and symmetries

Feynman integrals are integral functions, with integration variables $K = \{k_1, \ldots, k_l\}$. Clearly, integrals has shift invariance: ridefining a loop momenta shifted by an external momenta, leave the integral invariant[23]. Generalizing, there could be a set of trasformation that will leave the set $D(\Delta)$ invariant, module a different ordering. This ridefinition of all momenta generates an equivalence relation $\sim$ of all Feynman integrals related from them by a change of variables.

In a more precise way: let $\mathcal{T}_1$ be a topology $g \in \text{Diff}(\mathbb{R}^{dl})$ a change of variables and $f_g : \mathbb{T} \to \mathbb{T}$, $f_g : \mathcal{T}_1 \mapsto f_g(\mathcal{T}_1) = \mathcal{T}_2$ the transformation between topologies induced by the change of variable g such that $I(\mathcal{T}_1) = cI(f_g(\mathcal{T}_1)) = cI(\mathcal{T}_2)$, where c is a constant. Then, g induces an equivalence relation $\sim_g$, called *symmetry* between topologies (later we shall call them *sector symmetry*) through f_g, such that $\mathcal{T}_1 \sim_g \mathcal{T}_2$.

We can identify through $\mathcal{T}_1$ an equivalence class:

$$[\mathcal{T}_1] = \{f_g(\mathcal{T}_1), \ g \in \text{Diff}(\mathbb{R}^{dl}) \mid I(\mathcal{T}_1) = cI(f_g(\mathcal{T}_1))\} \tag{1.104}$$

Recalling that $\mathcal{T} = (E, \Delta)$, define the projector $\pi_\Delta : \mathbb{T} \to \mathfrak{I}$ such that $\pi_\Delta(\mathcal{T}) \mapsto \Delta$. Thanks to this definition, the equivalence relation induced by f_g is

$$\mathcal{T}_1 \sim \mathcal{T}_2 \iff \exists \, g \in \text{Diff}(\mathbb{R}^{dl}) \mid D(\pi_\Delta[f_g(\mathcal{T}_1)]) = D(\pi_\Delta[\mathcal{T}_2]) \tag{1.105}$$

Definition 1.5 (Indipendent topologies)**.** The equivalence classes defined early are called *independent topologies*. In other words $\mathcal{T}_1$ and $\mathcal{T}_2$ are independent topologies if and only if $\mathcal{T}_2 \notin [\mathcal{T}_1]$.

This equivalence is well expressed by diagrams associated to each topology. Most of the basic momenta shift are suggested by the global diagram, by reflecting or rotating it. But there are some topologies, not related by those simple graphical actions, that could be related by a variable changing.

Example 1.5. [2-loop massless bubble] Suppose to have two topologies

$$\mathcal{T}_1 = \left(\left\{ \begin{array}{c} (p^\mu, \sqrt{s}), \\ (p^\mu, \sqrt{s}) \end{array} \right\}, \left\{ \begin{array}{c} (k_1^\mu, k_1^\mu, m_1^2), \\ (k_1^\mu - p^\mu, k_1^\mu - p^\mu, m_2^2), \\ (k_2^\mu, k_2^\mu, m_3^2), \\ (k_2^\mu + k_1^\mu, k_2^\mu + k_1^\mu, m_4^2) \end{array} \right\} \right) = \text{—}\!\!\bigcirc\!\!\bigcirc\!\!\text{—} \tag{1.106}$$

$$\mathcal{T}_2 = \left(\left\{ \begin{array}{c} (p^\mu, \sqrt{s}), \\ (p^\mu, \sqrt{s}) \end{array} \right\}, \left\{ \begin{array}{c} (k_1^\mu, k_1^\mu, m_2^2), \\ (k_1^\mu - p^\mu, k_1^\mu - p^\mu, m_1^2), \\ (k_2^\mu - p^\mu, k_2^\mu - p^\mu, m_3^2), \\ (k_2^\mu - k_1^\mu, k_2^\mu - k_1^\mu, m_4^2) \end{array} \right\} \right) = \text{—}\!\!\bigcirc\!\!\bigcirc\!\!\text{—} \tag{1.107}$$

Let's write two generic Feynman integrals.

$$
\begin{aligned}
I_{\bar{a}}^{\bar{0}}(\mathcal{T}_1) &= \int \frac{d^d k_1 d^d k_2/(2\pi)^{2d-4}}{[k_1^2 + m_1^2]^{a_1}[(k_1-p)^2 + m_2^2]^{a_2}[k_2^2 + m_3^2]^{a_3}[(k_2+k_1)^2 + m_4^2]^{a_1}} \\
I_{\bar{a}}^{\bar{0}}(\mathcal{T}_2) &= \int \frac{d^d k_1 d^d k_2/(2\pi)^{2d-4}}{[k_1^2 + m_1^2]^{a_2}[(k_1-p)^2 + m_2^2]^{a_1}[(k_2-p)^2 + m_3^2]^{a_3}[(k_2-k_1)^2 + m_4^2]^{a_4}}
\end{aligned}
\tag{1.108}
$$

We can see that they are easily related by a variable change: $g(k_1, k_2) = (-k_1 + p, k_2 + p)$. This variable change induces a transformation between the topologies:

$$f_g(\mathcal{T}_2) = \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}), \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (-k_1 + p, -k_1 + p, m_2^2), \\ (-k_1, -k_1, m_1^2), \\ (k_2, k_2, m_3^2), \\ (k_1 + k_2, k_1 + k_2, m_4^2) \end{matrix} \right\} \right) \sim_D \mathcal{T}_1 \qquad (1.109)$$

This means that the two topologies are not independent: the equivalence relation is given through f_g.

$$I_{\bar{a}}^{\bar{0}}(\mathcal{T}_2) = I_{\bar{a}}^{\bar{0}}(f_g(\mathcal{T}_1)) \implies \quad \text{—}\bigcirc\kern-0.3em\bigcirc\text{—} = \text{—}\bigcirc\kern-0.3em\bigcirc\text{—} \qquad (1.110)$$

∎

The previous example shows that two graphs that related by a reflection, are easily related by some momenta shift or inversion. In general, not all Feynman integrals have a relation if their graph are the same module reflection/rotation.

Example 1.6. [5-loop massless bubble] Let's consider two massless 5-loop bubble diagrams, belonging to two topologies:

$$\mathcal{T}_1 = \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}), \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k_1 - p, k_1 - p, m^2), & (k_1, k_1, m^2), \\ (k_2 - k_1, k_2 - k_1, m^2), & (k_2, k_2, m^2), \\ (k_3 - k_2, k_3 - k_2, m^2), & (k_3, k_3, m^2), \\ (k_4 + k_1 - p, k_4 - k_1 + p, m^2), & (k_4, k_4, m^2), \\ (k_5 - k_4, k_5 - k_4, m^2), & (k_5, k_5, m^2) \end{matrix} \right\} \right) = (E, \Delta^1)$$

$$(1.111)$$

$$\mathcal{T}_2 = \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}), \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k_1 - p, k_1 - p, m^2), & (k_1, k_1, m^2), \\ (k_2 - k_1, k_2 - k_1, m^2), & (k_2, k_2, m^2), \\ (k_3 - k_2, k_3 - k_2, m^2), & (k_3, k_3, m^2), \\ (k_4 - k_1 + p, k_4 - k_1 + p, m^2), & (k_4, k_4, m^2), \\ (k_5 - k_4, k_5 - k_4, m^2), & (k_5, k_5, m^2) \end{matrix} \right\} \right) = (E, \Delta^2)$$

$$(1.112)$$

and consider the integrals

$$I_{\bar{1}}^{\bar{0}}(\mathcal{T}_1) = \quad \text{—}\bigcirc\kern-0.3em\bigcirc\text{—} = \int \left(\prod_{i=1}^{5} \frac{d^d k_i}{(2\pi)^{d-2}} \right) \frac{1}{\prod_{j=1}^{10} D(\Delta_j^1)} \qquad (1.113)$$

$$I_{\bar{1}}^{\bar{0}}(\mathcal{T}_2) = \quad \text{—}\bigcirc\kern-0.3em\bigcirc\text{—} = \int \left(\prod_{i=1}^{5} \frac{d^d k_i}{(2\pi)^{d-2}} \right) \frac{1}{\prod_{j=1}^{10} D(\Delta_j^2)} \qquad (1.114)$$
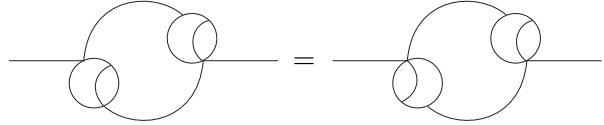
This two diagrams are not related by a simple graphical symmetry, but by performing the change of variables

$$g(k_1^\mu, k_2^\mu, k_3^\mu, k_4^\mu, k_5^\mu) = (k_1^\mu, k_2^\mu, k_3^\mu, -k_4^\mu, -k_5^\mu) \qquad (1.115)$$

we induce a function f_g, that relates our two topologies:

$$f_g(\mathcal{T}_1) = \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}) \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k_1 - p, k_1 - p, m^2), & (k_1, k_1, m^2), \\ (k_2 - k_1, k_2 - k_1, m^2), & (k_2, k_2, m^2), \\ (k_3 - k_2, k_3 - k_2, m^2), & (k_3, k_3, m^2), \\ (-k_4 + k_1 - p, -k_4 + k_1 - p, m^2), & (-k_4, -k_4, m^2), \\ (-k_5 + k_4, -k_5 + k_4, m^2), & (-k_5, -k_5, m^2) \end{matrix} \right\} \right) \tag{1.116}$$

Through f_g, we can see that $I(\mathcal{T}_2) = I(f_g(\mathcal{T}_1))$, so:

 $\tag{1.117}$

■

This happens because there's a subdiagram of the parent topology $\mathcal{T}_1$ attached to the rest of the graph by only two vertices and having no external lines. A change of variables like the one we made before, is reflected diagrammatically by detaching the subgraph and reattaching it swapping the vertices. In general, at the Feynman integrals level, it become the equality

$$I(\mathcal{T}_1) = (-1)^{(s_2 + N_{\mathrm{ISP}2})} I(\mathcal{T}_2) \tag{1.118}$$

where s and N_{ISP} are defined above.

1.3.2   Factorization of Feynman Integrals

A Feynman integral such that, ridefining the loop momenta, could become factorizable[23], we could threat it like a product of two Feynman integrals. In a more precise way, writing a Feynman integrand as $f(k_1^\mu, \ldots, k_l^\mu)$, the factorization claims that $f(k_1^\mu, \ldots, k_l^\mu) = f_1(k_1^\mu, \ldots, k_i^\mu) f_2(k_{i+1}^\mu, \ldots, k_l^\mu)$. A factorizable Feynman integral is often associated to a diagram that has at least one subdiagram attached at the rest of the graph by only one vertex.

Graphically, the factorization is represented by the splitting of the main graph in the product of two subgraphs. Instead of the vertex that joined them, they will have an external line with momenta flowing equal to the sum of all incoming momenta in that vertex.

Example 1.7. [3-loop bubble] Let's consider the topology

$$\mathcal{T} = \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}) \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k_1, k_1, m^2), & (k_1 + p, k_1 + p, m^2), \\ (k_2, k_2, m^2), & (k_1 + k_2 + p, k_1 + k_2 + p, m^2), \\ (k_3, k_3, m^2), & (k_1 - k_3, k_1 - k_3, m^2) \\ & (k_2 + k_3 + p, k_2 + k_3 + p, m^2) \end{matrix} \right\} \right) \tag{1.119}$$

and its subtopology $\tau \in \mathcal{S}_\mathcal{T}$:

$$\tau = \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}) \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k_1, k_1, m^2), & (k_1 + p, k_1 + p, m^2), \\ (k_2, k_2, m^2), & (k_3, k_3, m^2), \\ & (k_2 + k_3 + p, k_2 + k_3 + p, m^2) \end{matrix} \right\} \right) \tag{1.120}$$

and the Feynman diagram

$$I_{\bar{1}}^{\bar{0}}(\tau) = \int \left(\prod_{i=1}^{3} \frac{d^d k_i}{(2\pi)^{d-2}} \right) \frac{1}{\prod_{j=1}^{5} D(\Delta_j)} = \underbrace{\hspace{3cm}}_{} \qquad (1.121)$$

is factorizable in the obvious way

$$\begin{aligned}
I_{\bar{1}}^{\bar{0}}(\tau) = \int \frac{d^d k_1}{(2\pi)^{d-2}} &\frac{1}{[k_1^2 + m^2][(k_1 + p)^2 + m^2]} \times \\
&\times \int \frac{d^d k_2 d^d k_3}{(2\pi)^{2(d-2)}} \frac{1}{[k_2^2 + m^2][k_3^2 + m^2][(k_2 + k_3 + p)^2 + m^2]}
\end{aligned} \qquad (1.122)$$

Graphically, this is the following product.

$$\underbrace{\hspace{3cm}}_{} = \underbrace{\hspace{1.5cm}}_{} \times \underbrace{\hspace{1.5cm}}_{} \qquad (1.123)$$

∎

1.3.3 Integration-By-Parts identities

Integration-by-parts identities (IBP)[24, 23] are a set of relations between Feynman integrals that represent the most remarkable property of dimensional regularizated integrals. IBPs comes from the idea of applying the divergence theorem at a modified integrand: for simplicity, let's take a massive vacuum tadpole:

$$\bigcirc = \int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{(k^2 + m^2)} \qquad (1.124)$$

If we multiply the integrand for a vector (in this case, the only vector applyable is k^μ), we obtain an integral of a divergence by writing

$$\int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^{d-2}} \frac{\partial}{\partial k^\mu} \left[\frac{k^\mu}{(k^2 + m^2)} \right] \qquad (1.125)$$

and by using the Stokes theorem:

$$\int_{\mathbb{R}^d} \frac{d^d k}{(2\pi)^{d-2}} \frac{\partial}{\partial k^\mu} \left[\frac{k^\mu}{(k^2 + m^2)} \right] = \int_{\partial \mathbb{R}^d} \frac{d^{d-1}(\delta k)}{(2\pi)^{d-2}} \frac{k \cdot \hat{n}}{(k^2 + m^2)} \qquad (1.126)$$

Where we denoted with δk the coordinates on the 'boundary' $\partial \mathbb{R}^d$. Concluding:

$$\begin{aligned}
\int_{\partial \mathbb{R}^d} \frac{d^{d-1} \delta k}{(2\pi)^{d-2}} \frac{k \cdot \hat{n}}{(k^2 + m^2)} &= \int_{\Omega_d} \frac{d\Omega_d}{(2\pi)^{d-2}} \frac{|k|^d}{(k^2 + m^2)} \cos \theta_{k,\hat{n}} = \\
&= \lim_{|k| \to \infty} \frac{|k|^d}{(k^2 + m^2)} \Omega_d
\end{aligned} \qquad (1.127)$$

The last limit vanishes if $d < 2$. This inequality may scare a reader, because the truthness of IBPs seems to be not a general feature, but strongly depending on the

dimensional parameter. On the other side, there are usually a lot of denominators, so the inequality is always satisfied; moreover, IBP relations are proved to be true in a rigorous way.

Now, we can give a more precise definition of what is an integration-by-parts identity. IBPs are connected to the invariance of a Feynman diagram under the ridefinition of the integration momenta.

A generical loop momenta trasformation can be written as

$$(K', P') = g(K, P) = \mathbb{M}(K, P)^T = \begin{pmatrix} \mathbb{A}_{l \times l} & \mathbb{B}_{l \times n} \\ \mathbb{O}_{n \times l} & \mathbb{I}_n \end{pmatrix} \begin{pmatrix} K \\ P \end{pmatrix} \tag{1.128}$$

that is

$$K' = g(K) = \mathbb{A}K + \mathbb{B}P \tag{1.129}$$

This trasformation, clearly, has to be invertible, that is

$$\det(\mathbb{M}) = \det(\mathbb{A}) \neq 0 \tag{1.130}$$

In other words, $\mathbb{A} \in \mathrm{GL}(l)$.

Now, consider a Feynman integral[5]

$$I(P) = \int dK f(K, P) \tag{1.131}$$

Let's see how $\mathbb{M}$ acts on the integrand of $I(P)$ in an infinitesimal way:

$$\mathbb{M} \sim \mathbb{I}_{(l+n)} + \begin{pmatrix} \alpha_{l \times l} & \alpha_{l \times n} \\ \mathbb{O}_{n \times l} & \mathbb{O}_{n \times n} \end{pmatrix} = \mathbb{I}_{(l+n)} + \alpha$$
$$f(K', P') \sim f(K, P) + \frac{\partial f}{\partial v_i^\mu}(K, P) \alpha_{ij} v_j^\mu \tag{1.132}$$

where $v_i^\mu \in K \cup P$.

The Jacobian of the transformation is

$$dK' = |\det(\mathbb{I}_{l \times l} + \alpha_{l \times l})| dK = (1 + d\mathrm{Tr}(\alpha_{l \times l})) dK \tag{1.133}$$

At this point:

$$I(P') = \int \left[f(K, P) + \frac{\partial f}{\partial v_i^\mu}(K, P) \alpha_{ij} v_j^\mu \right] \left(1 + d\mathrm{Tr}(\alpha_{l \times l}) \right) dK \tag{1.134}$$

The invariance of a Feynman integral under this ridefinition is stated in the following equality:

$$\delta I = I(P') - I(P)|_{o(\alpha^2)} = 0 \tag{1.135}$$

so

$$\int \left[d\mathrm{Tr}(\alpha_{l \times l}) f(K, P) + \frac{\partial f}{\partial v_i^\mu}(K, P) \alpha_{ij} v_j^\mu \right] dK = 0 \tag{1.136}$$

---

[5]For this section we will denote the integration measure as $dK = \prod_{i=1}^{l} d^d k_i / (2\pi)^{d-2}$, and we neglect the exponents $\bar{a}$ and $\bar{b}$: a Feynman integral will be a function of P, so $I_{\bar{b}}^{\bar{a}}(\mathcal{T}) = I(P)$.

Noting that $\text{Tr}(\alpha_{l \times l}) = \alpha_{ij}\delta_{ij}$ for $i \in \{1, \ldots, l\}$:

$$\int \alpha_{ij} \Big[d\delta_{ij} + v_j^\mu \frac{\partial}{\partial v_i^\mu} \Big] f(K.P) dK = 0, \quad i \in \{1, \ldots, l\} \tag{1.137}$$

and we can write, in a more compact way, the general IBP identity, for $i \in \{1, \ldots, l\}$:

$$\int \frac{\partial}{\partial k_i^\mu} \Big[v_j^\mu f(K, P) \Big] dK = 0 \tag{1.138}$$

We can also observe that $O_{ij} = d\delta_{ij} + v_j^\mu \frac{\partial}{\partial v_i^\mu} = \frac{\partial}{\partial v_i^\mu} \cdot v_j^\mu$ is a set of operator that generates a Lie algebra:

$$[O_{ij}, O_{kl}] = \delta_{il} O_{kj} - \delta_{kj} O_{il} \tag{1.139}$$

Properties of IBPs

The IBP identities have some important properties, that makes them the most userful tool to generate relations.

1. Given a Feynman integral, IBP identities generates $l(l + n - 1)$ relations. For a given topology, we can write an infinite set of IBP: there's a set of IBP for every choice of powers $\bar{a}$ and $\bar{b}$.

2. IBP identities are relations between Feynman integrals belonging to the same topology $\mathcal{T}$ or at his subtopology tree $\mathcal{S}_{\mathcal{T}}$. Suppose to act on a_i or and b_j with an IBP[6]: it generates combination of $I_{a_i}^{b_j}(\mathcal{T})$, $I_{a_i}^{b_j-1}(\mathcal{T})$, $I_{a_i+1}^{b_j}(\mathcal{T})$, $I_{a_i-1}^{b_j-1}(\mathcal{T})$ and $I_{a_i+1}^{b_j+1}(\mathcal{T})$, according to the way in which derivatives act on the integrand. This means that IBPs enstablish relation between Feynman integrals built on a topology and its subtopoly tree, with different powers $\bar{a}$ and $\bar{b}$.

3. An IBP relates only Feynman integrals with the same loop number: this is true for dimensional regularization. If an IBP eliminates a loop momenta dipendence from the integrand, the Feynman integral become factorizable.

In order to explain exactly what those properties mean, we present some example.

Example 1.8. [IBPs for 1-loop bubble] Momenta flowing in the 1-loop bubble are $K = \{k^\mu\}$, $P = \{p^\mu\}$. The topology is

$$\mathcal{T} = \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}), \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k^\mu, k^\mu, m^2), \\ (k^\mu + p^\mu, k^\mu + p^\mu, m^2) \end{matrix} \right\} \right) = \left(E, \left\{ \begin{matrix} \Delta_1, \\ \Delta_2 \end{matrix} \right\} \right)$$

$$= \!\!\!\text{———}\bigcirc\text{———} \tag{1.140}$$

and its subtopology is:

$$\mathcal{S}_{\mathcal{T}} = \left\{ \begin{matrix} \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}), \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \{ (k^\mu, k^\mu, m^2) \} \right), \\ \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}), \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \{ (k^\mu + p^\mu, k^\mu + p^\mu, m^2) \} \right) \end{matrix} \right\} = \left\{ \begin{matrix} \tau_1, \\ \tau_2 \end{matrix} \right\}$$

$$= \left\{ \underline{\bigcirc} , \overline{\bigcirc} \right\} \tag{1.141}$$

---

[6]the action of an IBP could be relateed on its action on the powers of ISP and denominators.

Consider the Feynman integral $I_{1,1}(\mathcal{T})$:

$$I_{\bar{1}}\left(\ -\!\bigcirc\!-\ \right) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{(k^2 + m^2)[(k+p)^2 + m^2]} \tag{1.142}$$

Obviously, it's possible to make the derivative only for k^μ. Firsly, let's use as IBP momenta p^μ:

$$0 = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\partial}{\partial k^\mu} \left[\frac{p^\mu}{(k^2 + m^2)[(k+p)^2 + m^2]} \right] =$$
$$= -\int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{2k \cdot p}{(k^2+m^2)^2[(k+p)^2+m^2]} + \frac{2p \cdot (k+p)}{(k^2+m^2)[(k+p)^2+m^2]^2} \right] \tag{1.143}$$

We can replace $D(\Delta_1) = k^2 + m^2 = D_1$ and $D(\Delta_2) = (k+p)^2 + m^2 = D_2$, and we can express the numerator as:

$$-\int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{D_2 - D_1 - p^2}{D_1^2 D_2} + \frac{D_2 - D_1 + p^2}{D_1 D_2^2} \right] =$$
$$= -\int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{1}{D_1^2} - \frac{p^2}{D_1^2 D_2} - \frac{1}{D_2^2} + \frac{p^2}{D_1 D_2^2} \right] = 0 \tag{1.144}$$

It's easy to see, through a change of variable $g(k^\mu) = k^\mu + p^\mu$, that

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1^2} = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_2^2} \tag{1.145}$$

So:

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1^2 D_2} = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2^2} \implies$$
$$\implies I_{2,1}\left(\ -\!\bigcirc\!-\ \right) = I_{1,2}\left(\ -\!\bigcirc\!-\ \right) \tag{1.146}$$

Graphically, our IBP can be read like

$$\tag{1.147}$$

This is a trivial IBP, since those two Feynman integrals are related by a variables change.
In the same way, we can evaluate the last IBP:

$$0 = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\partial}{\partial k^\mu} \left[\frac{k^\mu}{(k^2 + m^2)[(k+p)^2 + m^2]} \right] =$$
$$= \int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{d}{(k^2 + m^2)[(k+p)^2 + m^2]} - \right.$$
$$\left. - \frac{2k^2}{(k^2+m^2)^2[(k+p)^2+m^2]} - \frac{2k \cdot (k+p)}{(k^2+m^2)[(k+p)^2+m^2]^2} \right] \tag{1.148}$$

Again, $k^2 + m^2 = D_1$ and $(k+p)^2 + m^2 = D_2$

$$\int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{d}{D_1 D_2} - \frac{2D_1 - 2m^2}{D_1^2 D_2} - \frac{D_1 + D_2 - 2m^2 - p^2}{D_1 D_2^2} \right] =$$
$$= \int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{d-3}{D_1 D_2} + \frac{2m^2}{D_1^2 D_2} - \frac{1}{D_2^2} + \frac{2m^2 + p^2}{D_1 D_2^2} \right] \tag{1.149}$$

Now, using the previous shift, we can replace the integrand $\frac{1}{D_2^2}$ with $\frac{1}{D_1^2}$. Another variable change is $g'(k) = -k - p$. Using this change of variable, we obtain

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1^2 D_2} = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2^2} \tag{1.150}$$

so:

$$\int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{d}{D_1 D_2} - \frac{2D_1 - 2m^2}{D_1^2 D_2} - \frac{D_1 + D_2 - 2m^2 - p^2}{D_1 D_2^2} \right] =$$
$$= \int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{d-3}{D_1 D_2} - \frac{1}{D_1^2} + \frac{4m^2 + p^2}{D_1 D_2^2} \right] = 0$$
$$\implies \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2^2} = \frac{1}{4m^2 + p^2} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1^2} -$$
$$- \frac{(d-3)}{4m^2 + p^2} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2} \tag{1.151}$$

This is a non-trivial IBP, which relates Feynman integrals belonging to the bubble topology and its relative subtopologies:

$$I_{2,1}(\mathcal{T}) = \frac{1}{4m^2 + p^2} I_2(\tau_1) - \frac{(d-3)}{4m^2 + p^2} I_{1,1}(\mathcal{T}) \tag{1.152}$$

and graphically, this means that:

$$\tag{1.153}$$

This second IBP is non-trivial: it relates a dotted integral to the combination of one integral the same topology and another one of this subtopology tree.

Therefore, because of its derivative nature, an IBP cannot relates integrals belonging to other indipendent topologies. ∎

We can also show what happens if an IBP factorize a Feynman integral.

Example 1.9. (Factorized integrand) Suppose we have an IBP that factorizes a Feynman integral $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$, which means that the action of an IBP on this integral makes the integrand independent from a loop momenta. Suppose to have in integral whose integrand doesn't depend on k_1^μ:

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \int \frac{d^d k_1}{(2\pi)^{d-2}} \int \left(\prod_{j=2}^{l} \frac{d^d k_j}{(2\pi)^{d-2}} \right) f(k_2, \dots, k_l; P) \tag{1.154}$$

The k_1^μ integral vanishes for dimensional regularization: it could be represented by a vertex not attached at no external line. It doesn't depend on external momenta, and it's massless:

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = I_{\bar{0}}^{\bar{0}}(\tau) \times I_{\bar{a}}^{\bar{b}}(\tau') \tag{1.155}$$

with $\tau, \tau' \in \mathcal{S}_{\mathcal{T}}$.

The 1-loop Feynman integral

$$I_0(\tau) = \int \frac{d^d k_1}{(2\pi)^{d-2}} \tag{1.156}$$

vanishes because of the Proposition (1.2.1) ∎

1.3.4   Lorentz-invariance identities

Feynman integrals are scalar function for the Lorentz group $O(1;3)$, depending only on the external momenta $P = (p_1^\mu, \ldots, p_n^\mu)$. This means that, applying a Lorentz transfomation on the external momenta P, it transforms with the trivial representation[7] if $\Lambda \in O(1;3)$

$$I(\Lambda^\mu{}_\nu p_1^\nu, \ldots, p_n^\mu) = \mathbb{I}_\Lambda I(p_1^\nu, \ldots, p_n^\mu) = I(p_1^\nu, \ldots, p_n^\mu) \tag{1.157}$$

where $\mathbb{I}_\Lambda$ is the trivial representation of $O(1;3)$.

Suppose we perform a infinitesimal Lorentz-transformation on the momenta p_j^μ:

$$p_j'^\mu = p_j^\mu + \omega^\mu{}_\nu p_j^\mu, \quad \omega \in \mathrm{Lie}(O(1;3)) \tag{1.158}$$

Naming $P' = (p_1^\mu, \ldots, p_j'^\mu, \ldots, p_n^\mu)$ and $P = (p_1^\mu, \ldots, p_n^\mu)$, the Feynman integral will transform like

$$I(P') = I(P) + \frac{\partial I}{\partial p_j^\mu}(P)\omega^\mu{}_\nu p_j^\nu \tag{1.159}$$

Lorentz-invariance states that $I(P') = I(P)$, and using the fact that $\omega \in \mathrm{Lie}(O(1;3))$, which means that $\omega_{\mu\nu} = -\omega_{\nu\mu}$:

$$\begin{aligned}
0 &= \frac{\partial I}{\partial p_j^\mu}(P)\omega^\mu{}_\nu p_j^\nu = \frac{1}{2}\omega^{\mu\nu}\left[\frac{\partial I}{\partial p_j^\mu}(P)p_{j\nu} - \frac{\partial I}{\partial p_j^\nu}(P)p_{j\mu}\right] \implies \\
&\implies \left[p_{j\nu}\frac{\partial}{\partial p_j^\mu} - p_{j\mu}\frac{\partial}{\partial p_j^\nu}\right]I(P) = 0
\end{aligned} \tag{1.160}$$

This identity holds for all $j \in \{1, \ldots, n\}$.

We can now sum over all indipendent momenta:

$$\sum_{j=1}^n \left[p_{j\nu}\frac{\partial}{\partial p_j^\mu} - p_{j\mu}\frac{\partial}{\partial p_j^\nu}\right]I(P) = 0 \tag{1.161}$$

In order to have a scalar quantity, we can contract this expression for an antisymmetric tensor build by external indipendent momenta:

$$(p_a^\mu p_b^\nu - p_a^\nu p_b^\mu)\sum_{j=1}^n \left[p_{j\nu}\frac{\partial}{\partial p_j^\mu} - p_{j\mu}\frac{\partial}{\partial p_j^\nu}\right]I(P) = 0 \tag{1.162}$$

or, in a more compact way:

$$(p_a^{[\mu} p_b^{\nu]})\sum_{j=1}^n \left[p_{j[\mu}\frac{\partial}{\partial p_j^{\nu]}}\right]I(P) = 0 \tag{1.163}$$

Those identities are called *Lorentz-invariance identities*[23] (LI). We can also note that the operator in front of $I(P)$ is the generator of the rotations $L_{\mu\nu}$ in operatorial representation, acting on a Lorentz scalar:

$$\Lambda \in SO(1;3)^+ \implies L_{\mu\nu}(\Lambda) = p_{j[\mu}\frac{\partial}{\partial p_j^{\nu]}} \tag{1.164}$$

---

[7]Again, in this section we will focus on the dependence on external momenta of a Feynman integrals, so $I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = I(p_1^\mu, \ldots, p_n^\mu) = I(P)$

We can proof that all the Lorentz-invariance identites could be generate from a combination of IBPs. Let's apply the definition of Lorentz-invariance identity on the integrand:

$$\int dK \left(p_a^{[\mu} p_b^{\nu]} \right) \sum_{j=1}^{n} \left[p_{j[\mu} \frac{\partial}{\partial p_j^{\nu]}} \right] f(K; P) = 0 \qquad (1.165)$$

now, we can add and subtract $k_{j[\mu} \frac{\partial}{\partial k_j^{\nu]}}$:

$$\int dK \left(p_a^{[\mu} p_b^{\nu]} \right) \sum_{j=1}^{n} \left[v_{j[\mu} \frac{\partial}{\partial v_j^{\nu]}} - k_{j[\mu} \frac{\partial}{\partial k_j^{\nu]}} \right] f(K; P) = 0 \qquad (1.166)$$

The first addend is the representation of the generator of the rotations, and acting on the scalar $f(K; P)$ gives zero. The second addend, expanding the antisymmetric part:

$$
\begin{aligned}
0 &= \int dK \left(p_a^{[\mu} p_b^{\nu]} \right) \sum_{j=1}^{n} \left[k_{j[\mu} \frac{\partial}{\partial k_j^{\nu]}} \right] f(K; P) = \\
&= \int dK \left(p_a^{\mu} p_b^{\nu} - p_a^{\nu} p_b^{\mu} \right) \sum_{j=1}^{n} \left[k_{j\mu} \frac{\partial}{\partial k_j^{\nu}} \right] f(K; P) = \qquad (1.167) \\
&= \sum_{j=1}^{n} \int dK \frac{\partial}{\partial k_j^{\nu}} \left[(p_b^{\nu} p_a \cdot k_j - p_a^{\nu} p_b \cdot k_j) f(K; P) \right]
\end{aligned}
$$

This shows that the Lorentz-invariance identities are nothing more that combinations of IBPs, with the add of a scalar product in the numerator.

So, making a fast recap, the way to evaluate a Feynman diagram is to decompose it:

1. Perform the tensorial decomposition, separating external particles and tensorial factors from the scalar form factor;

2. Indentify the Feynman integrals involved in the calculation;

3. Apply the Feynman integrals relations to decrease the number of integral to evaluate

This is the most used approach to evaluate Feynman integrals, not only for his well-defined method, but also because it's easily implementable in an iterative way. The number of IBPs, as just said, grows rapidly with the complexity of the topology, and their generation without an automatization is almost impossible.

Today, there are some routine that can produce IBPs and LIs automatically, with the only request of the knowledge of the topology. In this work, as we'll explain after, we have been used a program called *Reduze*, which is one of the standard routine used for evaluate Feynman integrals.

1.3.5 Mass derivatives identities

Recalling that a Feynman integral $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ for a topology $\mathcal{T}$ depends also on the masses $m_1^2, \ldots, m_t^2$. It is possible to factorize the mass dimensionality of $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ in the present way: for every denominator, $D(\Delta_j) = V_j \cdot W_j + m_j^2$, we can write all masses as $m_j = m x_j$, where x_j is an adimensional parameter. Redefining the loop momenta in the right way, we obtain:

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = m^{\left(ld - 2\sum_i a_i - \sum_j b_j\right)} \tilde{I}_{\bar{a}}^{\bar{b}}(\mathcal{T}) \tag{1.168}$$

where $\tilde{I}$ represent the adimensionalized Feynman integral, and all its dimensionality is carried by the first factor.

We can choose a reference mass m, factorize the mass dependence from our Feynman integral, and perform the derivative of (1.168) by m^2 :

$$-\sum_{i=1}^{t} a_i x_i I_{a_1, \ldots, a_{i+1}, \ldots, a_t}^{b_1, \ldots, b_{\text{NISP}}}(\mathcal{T}) = \left(\frac{ld}{2} - \sum_i a_i - \frac{1}{2}\sum_j b_j\right)\frac{1}{m^2} I_{\bar{a}}^{\bar{b}}(\mathcal{T}) \tag{1.169}$$

This equation relates integrals of the same belonging to the same topology but with different powers of the denominators. This relations are called *mass derivative identities*[23, 16].

Example 1.10. [1-loop massive bubble] Starting from the topology $\mathcal{T} = (\{(p^\mu, \sqrt{s}), (p^\mu, \sqrt{s})\}, \{(k^\mu, k^\mu, m_1^2), (k^\mu + p^\mu, k^\mu + p^\mu, m_2^2)\})$, let's write the generical Feynman integral:

$$I_{a_1, a_2}(\mathcal{T}) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m_1^2]^{a_1}[(k+p)^2 + m_2^2]^{a_2}} \tag{1.170}$$

Let's write all masses as $m_j = m x_j$, so

$$\begin{aligned}
I_{a_1, a_2}(\mathcal{T}) &= \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m^2 x_1^2]^{a_1}[(k+p)^2 + m^2 x_2^2]^{a_2}} \\
&= m^{-2(a_1 + a_2)} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[\frac{k^2}{m^2} + x_1^2]^{a_1}[\frac{(k+p)^2}{m^2} + x_2^2]^{a_2}}
\end{aligned} \tag{1.171}$$

and changing the variable $y^\mu = \frac{k^\mu}{m}$:

$$I_{a_1, a_2}(\mathcal{T}) = m^{d - 2(a_1 + a_2)} \int \frac{d^d y}{(2\pi)^{d-2}} \frac{1}{[y^2 + x_1^2]^{a_1}[(y + \frac{p}{m})^2 + x_2^2]^{a_2}} \tag{1.172}$$

Now, we are able to evaluate the two members of (1.169):

$$\begin{aligned}
\text{I)} \quad &- a_1 x_1 \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m_1^2]^{a_1+1}[(k+p)^2 + m_2^2]^{a_2}} - \\
&\qquad - a_2 x_2 \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m_1^2]^{a_1}[(k+p)^2 + m_2^2]^{a_2+1}}
\end{aligned} \tag{1.173}$$

$$\text{II)} \quad \left(\frac{d}{2} - a_1 - a_2\right) \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m_1^2]^{a_1}[(k+p)^2 + m_2^2]^{a_2}}$$

Graphically

$$-a_1 x_1 \quad \underset{a_1+1}{\overset{a_2}{\bigcirc}} \quad - a_2 x_2 \quad \underset{a_1}{\overset{a_2+1}{\bigcirc}} \quad =$$

$$= \left(\frac{d}{2} - a_1 - a_2\right) \quad \underset{a_1}{\overset{a_2}{\bigcirc}}$$

(1.174)

∎

Mass derivative equality are often connected with the IBPs: for most of the known cases, like LIs, mass derivatives are combinations of IBPs for the chosen topology, but this proposition has not yet been proven.

Chapter 2

Evaluating Feynman integrals

In the previous chapter, we described how to decompose a Feynman diagram in a combination of Feynman integrals belong to a specific topology and the relative subtopology. We saw that, by using IBPs, LIs and change of variables, we can reduce the number of integral involving in the combination. Obviously, this number can't reach zero: there's a set of integrals that form a sort of "basis", a set such that every constructible Feynman integral could be written as a combination of them. In this Chapter we describe what is and how to obtain the set of *master integrals* for a certain topology. Moreover we will present some methods to evaluate this objects. This last part is not strictly connected to the proposal of this work, but it is important to show that once we decomposed a scattering amplitude in master integral we have several methods to evaluate them.

2.1 Master Integrals

Proposition 2.1.1 (Master Integral). *Let $\mathcal{T} \in \mathbb{T}$ be a topology, $\mathcal{S}_\mathcal{T} \subset \mathbb{T}$ its subtopology tree and $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ a Feynman integral on $\mathcal{T}$. There exists a set of Feynman integrals $\mathcal{M}_\mathcal{T} = \left\{ J_{\bar{a}_j}^{\bar{b}_j}(\tau_j) \mid \tau_j \in \mathcal{S}_\mathcal{T}, \ j \in \{1, \ldots, N_{MI}\} \right\}$ such that*

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \sum_{j=1}^{N_{MI}} A_j J_{\bar{a}_j}^{\bar{b}_j}(\tau_j) \tag{2.1}$$

$\mathcal{M}_\mathcal{T}$ *is called* set of master integrals, $J_{\bar{a}_j}^{\bar{b}_j}(\tau_j)$ *is a* master integrals for the topology τ_j *(MI)*[1] *[23]. A_j is a function of all the dimensional parameters of the topology and of the dimension d.*

For a topology $\mathcal{T}$, we could have more than one master integral, one for each different couples $\{\bar{a}, \bar{b}\}$.

If a topology $\mathcal{T}_0$ has no master integral (i.e. there's no master integral in $\mathcal{M}_\mathcal{T}$ built using $\mathcal{T}_0$ as topology), then $\mathcal{T}_0$ is said to be reducible, *and its master integrals are built using only element belonging to its subtopology $\mathcal{S}_{\mathcal{T}_0}$.*

It has been proved that $\mathcal{M}_\mathcal{T}$ is a finite basis[38]. Moreover, the choice of $\mathcal{M}_\mathcal{T}$ is not unique, and could be difficult select a basis adapted to the problem you want to

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[1]Because of the proposition 1.1.1, we recall that a subtopology is also a topology.

deal with. For a smart choice of $\mathcal{M}_{\mathcal{T}}$, it has to not contain "complicated" integrals, where "complicated", in many context, means integrals with arbitrary high power of denominators and ISP, namely high values for $\bar{a}$ and $\bar{b}$.

A first choice of $\mathcal{M}_{\mathcal{T}}$ could be given by the *Laporta algorithm*[26], which assigns to every integral a "weight" function $W(\bar{a}, \bar{b})$. It depends on the powers of ISP and denominators; in particular, it is monotone increasing in $\bar{a}$. This means that Laporta algorithm select a basis made of integral with the lowest powers of the denominators.

Example 2.1. [1-loop vacuum massive tadpole] Let's choose the topology[23] $\mathcal{T} = (\emptyset, \{(k^{\mu}, k^{\mu}, m^2)\})$, $\mathcal{S}_{\mathcal{T}} = \emptyset$, and

$$I_a(\mathcal{T}) = \raisebox{-1em}{\includegraphics[width=2em]{circle}} = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{(k^2 + m^2)^a} \tag{2.2}$$

We can use the only IBP identity for this topology

$$0 = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\partial}{\partial k^{\mu}} \frac{k^{\mu}}{(k^2 + m^2)^a} = \int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{d}{(k^2 + m^2)^a} - \frac{2ak^2}{(k^2 + m^2)^{a+1}} \right] \tag{2.3}$$

as usual, we write $D(\Delta) = k^2 + m^2$:

$$0 = \int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{d}{D(\Delta)^a} - \frac{2aD(\Delta) - 2am^2}{D(\Delta)^{a+1}} \right] = \int \frac{d^d k}{(2\pi)^{d-2}} \left[\frac{d - 2a}{D(\Delta)^a} + \frac{2am^2}{D(\Delta)^{a+1}} \right] \implies$$
$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D(\Delta)^{a+1}} = -\frac{d - 2a}{2am^2} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D(\Delta)^a} \tag{2.4}$$

And graphically, we get

$$\raisebox{-1em}{\includegraphics[width=2em]{circle}} = \frac{a - \frac{d}{2}}{am^2} \raisebox{-1em}{\includegraphics[width=2em]{circle}} \tag{2.5}$$

We choose an arbitrary a, so this relation holds for every integral constructible with this topology. If we want to express all integrals belonging to the topology $\mathcal{T}$ in terms of an unique integral, it is suffice to iterate the result of (2.5):

$$\raisebox{-1em}{\includegraphics[width=2em]{circle}} = \frac{\Gamma(a - \frac{d}{2})}{m^{2(a-1)}\Gamma(a)\Gamma(1 - \frac{d}{2})} \raisebox{-1em}{\includegraphics[width=2em]{circle}} \tag{2.6}$$

where we used the fact that $a! = \Gamma(a + 1)$.

This means that

$$\mathcal{M}_{\bigcirc} = \left\{ I_1 \left(\bigcirc \right) \right\} \tag{2.7}$$

The 1-loop massive tadpole topology has only one master integral. ∎

Obviously, in general it's not that easy to find a good master Feynman integral. Moreover, the number of master integrals per sectors (topologies) is not immediate.

The current methods to identify some candidates for being MI is to generate IBPs for a set of powers $\bar{a}$ and $\bar{b}$, then identify the set of recurring Feynman integrals.

Once identified the set $\mathcal{M}_\mathcal{T}$, we can use it to reduce every integral built with the topology $\mathcal{M}_\mathcal{T}$. So, we passed from evaluate a single Feynman integral $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ to evaluate a set $J_{\bar{a}_j}^{\bar{b}_j}(\tau_j)$ of master integrals.

2.2 Master integrals evaluation

2.2.1 Feynman parameters

To evaluate some easier master integral, we can use the well-known *Feynman parameter trick*[39], used to express a rational function as an integral function in some parameters x_i in the following way:

$$\frac{1}{D_1^{a_1} \cdots D_t^{a_t}} = \frac{\Gamma(\sum_{i=1}^t a_i)}{\prod_{j=1}^t \Gamma(a_i)} \int_0^1 \cdots \int_0^1 \delta\Big(1 - \sum_{i=1}^t x_i\Big) \frac{\prod_{j=1}^t x_t^{a_i-1} dx_j}{[\sum_{j=1}^t (x_j D_j)]^{(\sum_{j=1}^t a_j)}} \quad (2.8)$$

This method allow us to integrate over the momenta in an easier way.

Example 2.2. [1-loop massive bubble] Suppose we have the usual bubble topology

$$\mathcal{T} = \Big(\{(p^\mu, \sqrt{s}), (p^\mu, \sqrt{s})\}, \{(k^\mu, k^\mu, m^2), (k^\mu + p^\mu, k^\mu + p^\mu, m^2)\}\Big) \quad (2.9)$$

and we want to evaluate the Feynman integral:

$$(2.10)$$

We know that for $\mathcal{T}$ we have two master integrals:

$$(2.11)$$

and for (1.138)

$$(2.12)$$

So, in order to evaluate (1.138), it is suffice to find the value of the set $\mathcal{M}_\mathcal{T}$. Let's start from the bubble: using the Feynman parameters:

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m^2][(k+p)^2 + m^2]} =$$

$$= \int \frac{d^d k}{(2\pi)^{d-2}} \int_0^1 \int_0^1 \frac{\delta(1 - x_1 - x_2)}{\{[k^2 + m^2]x_1 + [(k+p)^2 + m^2]x_2\}^2} dx_1 dx_2 \quad (2.13)$$

Inserting $\theta(1 - x_2) - \theta(-x_2)$, we can extend the domain of integration for x_2, and also this allow us to integrate the Dirac delta:

$$\int \frac{d^d k}{(2\pi)^{d-2}} \int_0^1 \int_{-\infty}^{\infty} \frac{\delta(1 - x_1 - x_2)[\theta(1 - x_2) - \theta(-x_2)]}{\{[k^2 + m^2]x_1 + [(k+p)^2 + m^2]x_2\}^2} dx_1 dx_2 =$$

$$= \int \frac{d^d k}{(2\pi)^{d-2}} \int_0^1 \frac{\theta(x_1) - \theta(x_1 - 1)}{\{[k^2 + m^2]x_1 + [(k+p)^2 + m^2](1 - x_1)\}^2} dx_1 = \qquad (2.14)$$

$$= \int \frac{d^d k}{(2\pi)^{d-2}} \int_0^1 \frac{1}{\{[k^2 + m^2]x_1 + [(k+p)^2 + m^2](1 - x_1)\}^2} dx_1$$

where $\theta(x_1) - \theta(x_1 - 1) = 1$ if $x_1 \in (0,1)$. Doing some algebra in the denominator, and renaming $x_1 \rightarrow x$:

$$\int \frac{d^d k}{(2\pi)^{d-2}} \int_0^1 \frac{1}{[k^2 + 2k \cdot p(1 - x) + p^2(1 - x) + m^2]^2} dx =$$

$$= \int \frac{d^d k}{(2\pi)^{d-2}} \int_0^1 \frac{1}{[(k + p(1 - x))^2 + p^2 x(1 - x) + m^2]^2} dx \qquad (2.15)$$

Now, we shift the integration variable $k \rightarrow k - p(1 - x)$, and call $\bar{m}^2 = p^2 x(1 - x) + m^2$:

$$\int \frac{d^d k}{(2\pi)^{d-2}} \int_0^1 \frac{1}{[(k + p(1 - x))^2 + p^2 x(1 - x) + m^2]^2} dx =$$

$$= \int \frac{d^d k}{(2\pi)^{d-2}} \int_0^1 \frac{1}{[k^2 + \bar{m}^2]^2} dx \qquad (2.16)$$

The last integral is well-known in dimensional regularization: in general

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 + \bar{m}^2]^a} = i \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(a - \frac{d}{2})}{\Gamma(a)} \bar{m}^{\frac{d}{2} - a} \qquad (2.17)$$

so

$$(2\pi)^2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 \frac{1}{[k^2 + \bar{m}^2]^2} dx = \frac{i}{4} (4\pi)^{2 - \frac{d}{2}} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 \bar{m}^{-\left(2 - \frac{d}{2}\right)} dx \qquad (2.18)$$

For $d = 4 - \epsilon$:

$$I_{1,1}(\mathcal{T}) = \frac{i}{4} (4\pi)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 [p^2 x(1 - x) + m^2]^{-\frac{\epsilon}{2}} dx \qquad (2.19)$$

The remaining integral can be evaluate only if of $p^2 = 0$ or $m^2 = 0$. It gives the exact value of our Feynman integral.

The other MI is much simpler to evaluate:

$$I_1\left(\bigcirc \right) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2 + m^2} = \frac{i}{4} (4\pi)^{\frac{\epsilon}{2}} \Gamma\left(\frac{\epsilon}{2} - 1\right) m^{\frac{\epsilon}{2} - 1} \qquad (2.20)$$

And finally, we may replace the right hand side of (1.138) with the two evaluated MI, in order to have the exact value of that integral, or more in general all the integrals for the topology $\mathcal{T}$. ∎

2.2.2 Alpha-representation

There exist another userful representation for Feynman integrals, which represents an alternative standard representation, the called α-representation[38, 40]. It is based on the fact that, in Minkowski space:

$$\frac{1}{k^2 - m^2 + i0} = -i \int_0^\infty e^{i(k^2 - m^2)\alpha} d\alpha \qquad (2.21)$$

so, we can express all the propagator in a Feynman integral by using some α parameters. We will use a more general formula:

$$\frac{1}{(k^2 - m^2 + i0)^a} = \frac{e^{i\frac{3\pi}{2}a}}{\Gamma(a)} \int_0^\infty \alpha^{a-1} e^{i(k^2 - m^2)\alpha} d\alpha \qquad (2.22)$$

The feature that makes α-representation that userful is the fact that, allows us to integrate over the loop momenta. Suppose to have a topology l-loop $\mathcal{T}$ with t internal lines, n external lines, and a Feynman integrals $I_{\bar{a}}^{\bar{0}}$:

$$I_{\bar{a}}^{\bar{0}}(\mathcal{T}) = \int \frac{d^d k_1 \cdots d^d k_l}{(2\pi)^{l(d-2)}} \frac{1}{D(\Delta_1)^{a_1} \cdots D(\Delta_t)^{a_t}} \qquad (2.23)$$

Let's now apply the substituition (2.22). Denoting $a = \sum_{i=1}^t a_i$:

$$I_{\bar{a}}^{\bar{0}}(\mathcal{T}) = i^a \frac{e^{i\pi a}}{\prod_{i=1}^t \Gamma(a_i)} \int \frac{d^d k_1 \cdots d^d k_l}{(2\pi)^{l(d-2)}} \prod_{i=1}^t \int \alpha^{a_i - 1} e^{i\alpha_i D(\Delta_i)} d\alpha_i \qquad (2.24)$$

The exponential term can be written in a more explicative way:

$$\prod_{i=1}^t \exp\left(i\alpha_i D(\Delta_i)\right) = \exp\left(i \sum_{i=1}^t \alpha_i D(\Delta_i)\right) \qquad (2.25)$$

$D(\Delta_i)$ is a quadratic polynomial in the momenta. We can write the argument of the exponential as:

$$\sum_{i=1}^t \alpha_i D(\Delta_i) = \frac{1}{2} \sum_{i,j=1}^l A_{ij}(\bar{\alpha}) k_i k_j + \sum_{j=1}^l \sum_{i=1}^n B_{ij}(\alpha) p_i k_j + \frac{1}{2} \sum_{i,j=1}^n C_{ij}(\bar{\alpha}) p_i p_j - \sum_{i=1}^t \alpha_i m_i^2 \qquad (2.26)$$

So, we can perform a ld-dimensional gaussian integral (using Einstein convention):

$$\int \frac{d^d k_1 \cdots d^d k_l}{(2\pi)^{l(d-2)}} \exp\left(\frac{i}{2} A_{ij} k_i k_j + i B_{ij} p_i k_j\right) = e^{il\frac{\pi}{2}(1-\frac{d}{2})} \frac{(2\pi)^{l(2-d/2)}}{|\det A|^{d/2}} \exp(i p_i^T B_{ik}^T A_{kr}^{-1} B_{rj} p_j) \qquad (2.27)$$

Plugging (2.27) in (2.24):

$$I_{\bar{a}}^{\bar{0}}(\mathcal{T}) = \frac{e^{i\frac{\pi}{2}(3a + l(1-\frac{d}{2}))a}(2\pi)^{l(2-d/2)}}{\prod_{i=1}^t \Gamma(a_i)} \times$$

$$\times \int_0^\infty \prod_{i=1}^t \alpha_i^{a_i - 1} \frac{\exp\left(i p_i^T [B_{ik}^T A_{kr}^{-1} B_{rj} + \frac{1}{2} C_{ij}] p_j\right)}{|\det A|^{d/2}} e^{-i\sum_{i=1}^t \alpha_i m_i^2} d\alpha_1 \cdots d\alpha_t \qquad (2.28)$$

We note that $\det A$ and the argument of the exponential are two polynomials, depending on momenta P and K. In literature, we find that those polynomial are written in this way:

$$I_{\bar{a}}^{\bar{0}}(\mathcal{T}) = \frac{e^{i\frac{\pi}{2}(3a+l(1-\frac{d}{2}))}(2\pi)^{l(2-d/2)}}{\prod_{i=1}^{t}\Gamma(a_i)} \int_0^{\infty} \prod_{i=1}^{t} \alpha_i^{a_i-1} \mathcal{U}^{-\frac{d}{2}} e^{i\mathcal{V}/\mathcal{U}-i\sum_{j=1}^{t}\alpha_j m_j^2} \qquad (2.29)$$

Our polynomials $\mathcal{U}$ and $\mathcal{V}$ can be found with a graphical method: picking a Feynman graph, we have to delete the less number of internal lines, until we get a *tree T*. It is clear that we can build more than one tree. This definition bring us the definition of the first polynomial:

$$\mathcal{U} = \sum_{\text{trees } T} \prod_{i \notin T} \alpha_i \qquad (2.30)$$

Another kind of trees can be built starting from a tree T: deleting other internal lines, we can split a tree T in two connected components. This new type of tree are called $2-tree$. The polynomial $\mathcal{V}$ is defined in terms of $2-$trees:

$$\mathcal{V} = \sum_{2-\text{trees } T} \prod_{i \notin T} \alpha_i (p^T)^2 \qquad (2.31)$$

where p^T is the sum of the external momenta flowing in a connected component (no matter which component we choose, due to momenta conservation).

So, we can give a exact definition of the α-representation:

Definition 2.1 (α-representation)**.** The α-representation is a way to express a Feynman integral $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ for a topology $\mathcal{T}$ in terms of two polynomial $\mathcal{U}(\mathcal{T})$ and $\mathcal{V}(\mathcal{T})$. In alpha representation, a Feynman integral[2] is written as

$$I_{\bar{a}}^{\bar{0}}(\mathcal{T}) = \frac{e^{i\frac{\pi}{2}(3a+l(1-\frac{d}{2}))}(2\pi)^{l(2-d/2)}}{\prod_{i=1}^{t}\Gamma(a_i)} \int_0^{\infty} \prod_{i=1}^{t} d\alpha_i \alpha_i^{a_i-1} \mathcal{U}(\mathcal{T})^{-\frac{d}{2}} e^{i\mathcal{V}(\mathcal{T})/\mathcal{U}(\mathcal{T})-i\sum_{j=1}^{t}\alpha_j m_j^2}$$
$$(2.32)$$

$\mathcal{U}(\mathcal{T})$ and $\mathcal{V}(\mathcal{T})$ have degree l and $l+1$ in the α parameters. They encode all the informations about the topology $\mathcal{T}$ and do not depend on dimensionality d or exponent of denominators.

α-representation offer a powerful method to both evaluate Feynamn integrals and find symmetries between integrals.

Proposition 2.2.1. *Let's suppose to have two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ both with t internal lines. We can build four polynomials: $\mathcal{U}_1 = \mathcal{U}(\mathcal{T}_1)$, $\mathcal{U}_2 = \mathcal{U}(\mathcal{T}_2)$, $\mathcal{V}_1 = \mathcal{V}(\mathcal{T}_1)$ and $\mathcal{V}_2 = \mathcal{V}(\mathcal{T}_2)$. If we find a permutation $\sigma \in Perm(t)$ such that $\sigma(\alpha_i) = \alpha_j$ and*

$$\mathcal{U}_1(\sigma(\alpha_i)) = \mathcal{U}_2(\alpha_j) \qquad (2.33)$$
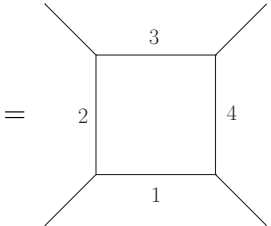$$\mathcal{V}_1(\sigma(\alpha_i)) = \mathcal{V}_2(\alpha_j) \qquad (2.34)$$

then $\mathcal{T}_1 \sim \mathcal{T}_2$.

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[2]There's a generalization of this definition for generical $\bar{b}$ coefficients, but for our purpose it is sufficient define all quatities for $\bar{b} = \bar{0}$. The only thing which a $\bar{b} \neq 0$ may affect is the form of the $\mathcal{U}$ and $\mathcal{V}$ polynomial.

Find a permutation which satisfies (2.33) and (2.34) is equivalent to find a change of variables (symmetries) which maps internal lines in theirself. This particular feature of the α-representation gives us a method to find all symmetries in a very efficient way. At computational level, it's easier to manage the permutation group instead of change of variables.

Example 2.3. (1-loop massless box topology) Suppose to have the topology:

$$
\mathcal{T} = \left(\left\{ \begin{array}{c} (p_1^\mu, 0), \\ (p_2^\mu, 0), \\ (p_3^\mu, 0), \\ (p_1^\mu + p_2^\mu + p_3^\mu, 0) \end{array} \right\}, \left\{ \begin{array}{c} (k^\mu, k^\mu, m^2), \\ (k^\mu + p_1^\mu, k^\mu + p_1^\mu, m^2), \\ (k^\mu + p_1^\mu + p_2^\mu, k^\mu + p_1^\mu + p_2^\mu, m^2), \\ (k^\mu + p_1^\mu + p_2^\mu + p_3^\mu, k^\mu + p_1^\mu + p_2^\mu + p_3^\mu, m^2) \end{array} \right\} \right)
$$
(2.35)

$$=$$

By the only knowledgement of $\mathcal{T}$, we can draw its trees:

(2.36)

To build $\mathcal{U}$, we have to sum over the trees the product of α_i which index doesn't appear in the tree:

$$
\mathcal{U}(\mathcal{T}) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4
$$
(2.37)

Now, let's build the 2-trees:

(2.38)

So, the $\mathcal{V}$ polynomial is:

$$
\begin{aligned}
\mathcal{V} = {} & \alpha_1 \alpha_2 (p_1)^2 + \alpha_1 \alpha_4 (p_1 + p_2 + p_3)^2 + \alpha_3 \alpha_4 (p_3)^2 + \\
& + \alpha_2 \alpha_3 (p_2)^2 + \alpha_1 \alpha_3 (p_1 + p_2)^2 + \alpha_2 \alpha_4 (p_2 + p_3)^2
\end{aligned}
$$
(2.39)

The external lines are on-shell: this means that $p_i^2 = 0$. So, only the last two terms of the sum contributes to $\mathcal{V}$:

$$
\mathcal{V} = \alpha_1 \alpha_3 s + \alpha_2 \alpha_4 u
$$
(2.40)

Now, we can build any Feynman integral we want in α-representation:

$$
I_{2,1,1,1}(\mathcal{T}) = e^{i\frac{\pi}{2}(16 - \frac{d}{2})} (2\pi)^{(2-d/2)} \int_0^\infty \frac{\alpha_1 e^{i\frac{\alpha_1 \alpha_3 s + \alpha_2 \alpha_4 u}{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}}}{(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)^{\frac{d}{2}}} d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4
$$
(2.41)

∎

2.2.3 Differential equations

We briefly present here the most efficient method to evaluate MIs, used for its versatility and its capacity of evaluate multi-loop MIs in a simpler way[16] than the Feynman parameter trick.

Every MI (being itself a Feynman integral) depends on the external independent momenta. This dependence both in the numerator and in the denominator.

Moreover, master integrals are scalar objects: for this reason, the result of the integration of an MI is a function the kinematic invariants $\mathcal{I}$ of the process. As usual, naming $\mathcal{T}$ our topology, $\tau_k \in \mathcal{S}_\mathcal{T}$ a subtopology of $\mathcal{T}$ and $J_{\bar{a}_k}^{\bar{b}_k}(\tau_k) \in \mathcal{M}_\mathcal{T}$, an MI depends on $\mathcal{I}$:

$$J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)(P) = J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)(\mathbf{s}) \tag{2.42}$$

where we recall that $\mathcal{I} = \mathbf{s} \sqcup \mathbf{M}^2 \cup \mathbf{m}^2$ is the set of all kinematic invariants. In particular, a MI depends on $\mathbf{s}$, kinematic invariants build by external momenta. We can apply a derivative with respect to a momenta p_j and express it as a derivative with respect a kinematic invariant:

$$\frac{\partial J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)}{\partial p_j^\mu}(\mathbf{s}) = \frac{\partial s_{ij}}{\partial p_j^\mu}\frac{\partial J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)}{\partial s_{ij}}(\mathbf{s}) = (1 + \delta_{ij})p_i^\mu\frac{\partial J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)}{\partial s_{ij}}(\mathbf{s}) \tag{2.43}$$

and by multiplying for p_i^μ:

$$p_j^\mu\frac{\partial J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)}{\partial p_j^\mu}(\mathbf{s}) = (1 + \delta_{ij})s_{ij}\frac{\partial J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)}{\partial s_{ij}}(\mathbf{s}) \tag{2.44}$$

The l.h.s. will give us a combination of Feynman integrals with different powers $\bar{a}$ and $\bar{b}$, in general not MIs, of both topology and its subtopology tree:

$$p_j^\mu\frac{\partial J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)}{\partial p_j^\mu}(\mathbf{s}) = \sum_h B_{kh}I_{\bar{a}_{kh}}^{\bar{b}_{kh}}(\tau_k) + \sum_l C_{kl}I_{\bar{a}_{kl}}^{\bar{b}_{kl}}(\bar{\tau}_{kl}), \quad \bar{\tau}_{kl} \in \mathcal{S}_{\tau_k} \tag{2.45}$$

At this point, we could use IBPs to express all the integrals in (2.45) in terms of MIs. Supposing to have H MIs for the topology τ_k and $N_{MI} - H$ MIs for the subtopology of τ_k

$$p_j^\mu\frac{\partial J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)}{\partial p_j^\mu}(\mathbf{s}) = \sum_{h=1}^{H} B'_{kh}J_{\bar{a}_{kh}}^{\bar{b}_{kh}}(\tau_k) + \sum_{l=1}^{m-H} C'_{kl}J_{\bar{a}_{kl}}^{\bar{b}_{kl}}(\bar{\tau}_{kl}) \tag{2.46}$$

and, in order to have a differential equation, we use (2.44), and renaming the coefficients:

$$\frac{\partial J_{\bar{a}_k}^{\bar{b}_k}(\tau_k)}{\partial s_{ij}}(\mathbf{s}) = \sum_{h=1}^{H} B_{kh}J_{\bar{a}_{kh}}^{\bar{b}_{kh}}(\tau_k) + \sum_{l=1}^{m-H} C_{kl}J_{\bar{a}_{kl}}^{\bar{b}_{kl}}(\bar{\tau}_{kl}) \tag{2.47}$$

Note that B_{kh} and C_{kl} are rational coefficient depending on the external parameters, so $B_{kh}(E)$ and $C_{kl}(E)$. Moreover, they are element of two matrices: $(\mathbf{B})_{kh} = B_{kh}$ and $(\mathbf{C}_1, \mathbf{C}_2)_{kl}^T = C_{kl}$.

For a certain topology, we want to be able to evaluate all the integrals in $\mathcal{M}_{\mathcal{T}}$. So, separating the MIs for the topology $\mathcal{T}$ from the one of $\mathcal{S}_{\mathcal{T}}$ as $\mathcal{M}_{\mathcal{T}} = (\mathbf{J}(\mathcal{T}), \mathbf{J}(\tau))$:

$$\frac{\partial}{\partial s_{ij}} \begin{pmatrix} \mathbf{J}(\mathcal{T}) \\ \mathbf{J}(\tau) \end{pmatrix} (\mathbf{s}) = \begin{pmatrix} \mathbf{B} & \mathbf{C}_1 \\ \mathbb{O} & \mathbf{C}_2 \end{pmatrix} \begin{pmatrix} \mathbf{J}(\mathcal{T}) \\ \mathbf{J}(\tau) \end{pmatrix} = \mathbb{A} \mathcal{M}_{\mathcal{T}} \qquad (2.48)$$

MIs satisfy a system of differential equations[16, 32, 31], where the matrix $\mathbb{A}$ is block-triangular.

Example 2.4. (1-loop massive bubble) The topology we're considering is the usual

$$\mathcal{T} = (\{(p^\mu, \sqrt{s}), (p^\mu, \sqrt{s})\}, \{(k^\mu, k^\mu, m), (k^\mu + p^\mu, k^\mu + p^\mu, m^2)\})$$
$$\mathcal{S}_{\mathcal{T}} = \left\{ \begin{array}{l} (\{(p^\mu, \sqrt{s}), (p^\mu, \sqrt{s})\}, \{(k^\mu, k^\mu, m^2)\}), \\ (\{(p^\mu, \sqrt{s}), (p^\mu, \sqrt{s})\}, \{(k^\mu + p^\mu, k^\mu + p^\mu, m^2)\}) \end{array} \right\} = \{\tau_1, \tau_2\} \qquad (2.49)$$

The only kinematic invariant constructible with p^μ is $\mathbf{s} = \{p^2\} = \{s\}$. Let's call $D_1 = D((k^\mu, k^\mu, m^2)) = k^2 + m^2$, and $D_2 = D((k^\mu + p^\mu, k^\mu + p^\mu, m^2)) = (k + p)^2 + m^2$. As already showed, $\mathcal{T}$ has two master integrals:

$$\mathcal{M}_{\mathcal{T}} = \left\{ \quad \bigcirc \quad , \quad \multimap\!\!\bigcirc\!\!\multimap \quad \right\} = \{J_1(\tau_1), J_{1,1}(\mathcal{T})\} \qquad (2.50)$$

The derivative of $J_{1,1}(\mathcal{T})$ is

$$\frac{\partial}{\partial p^\mu} J_{1,1}(\mathcal{T}) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\partial}{\partial p^\mu} \left[\frac{1}{D_1 D_2} \right] = - \int \frac{d^d k}{(2\pi)^{d-2}} \frac{2(k_\mu + p_\mu)}{D_1 D_2^2} \qquad (2.51)$$

Now, multiply it for p^μ:

$$p^\mu \frac{\partial}{\partial p^\mu} J_{1,1}(\mathcal{T}) = - \int \frac{d^d k}{(2\pi)^{d-2}} \frac{2(k \cdot p + p^2)}{D_1 D_2^2} = - \int \frac{d^d k}{(2\pi)^{d-2}} \frac{D_2 - D_1 + p^2}{D_1 D_2^2} =$$
$$= \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_2^2} - \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2} - p^2 \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2^2} \qquad (2.52)$$

We have two IBPs to replace in this equation:

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2^2} = \frac{1}{p^2 + 4m^2} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1^2} - \frac{(d-3)}{4m^2 + p^2} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2}$$
$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1^2} = - \frac{(d-2)}{2m^2} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1} \qquad (2.53)$$

So

$$p^\mu \frac{\partial}{\partial p^\mu} J_{1,1}(\mathcal{T}) = - \frac{(d-2)}{2m^2} \left[1 - \frac{p^2}{4m^2 + p^2} \right] \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1} -$$
$$- \left[1 - p^2 \frac{(d-3)}{4m^2 + p^2} \right] \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2} \qquad (2.54)$$

but

$$p^\mu \frac{\partial}{\partial p^\mu} J_{1,1}(\mathcal{T}) = 2p^2 \frac{\partial}{\partial p^2} J_{1,1}(\mathcal{T}) \qquad (2.55)$$

and we obtain the differential equation for $J_{1,1}(\mathcal{T})$;

$$\frac{\partial}{\partial p^2} J_{1,1}(\mathcal{T}) = -\frac{(d-2)}{4m^2}\left[\frac{1}{p^2} - \frac{1}{4m^2 + p^2}\right]\int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{D_1} -$$
$$-\frac{1}{2}\left[\frac{1}{p^2} - \frac{(d-3)}{4m^2 + p^2}\right]\int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{D_1 D_2} \tag{2.56}$$

Diagrammatically, this differential equation is written as

$$\frac{\partial}{\partial p^2} \quad \bigcirc \quad = -\frac{1}{2}\left[\frac{1}{p^2} - \frac{(d-3)}{4m^2 + p^2}\right] \quad \bigcirc \quad -$$
$$-\frac{(d-2)}{4m^2}\left[\frac{1}{p^2} - \frac{1}{4m^2 + p^2}\right] \quad \bigcirc \tag{2.57}$$

The differential equation for $J_1(\tau_1)$, is trivial: it is independent from p, so:

$$\frac{\partial}{\partial p^2} \quad \bigcirc \quad = 0 \tag{2.58}$$

Then, the system of differential equations for $\mathcal{M}_{\mathcal{T}}$ is

$$\frac{\partial}{\partial p^2}\begin{pmatrix} \bigcirc \\ \bigcirc \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\left[\frac{1}{p^2} - \frac{(d-3)}{4m^2+p^2}\right] & -\frac{(d-2)}{4m^2}\left[\frac{1}{p^2} - \frac{1}{4m^2+p^2}\right] \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \bigcirc \\ \bigcirc \end{pmatrix} \tag{2.59}$$

$\blacksquare$

2.2.4 Boundary conditions

To solve a system of N_{MI} differential equation, of course we need N_{MI} boundary conditions, that fixes the value of the free parameter coming from the equation.

A simple method to find those conditions is to evaluate the MI by choosing special values for the kinematics invariant.

Example 2.5. (1-loop massive bubble) Looking at the Example (2.4)[16], we know that a master integral if an analytic function of p^2. This implies that the soft limit $p^2 \in 0$ exists and has to be zero.

$$\lim_{p^2 \to 0} p^2 \frac{\partial J_{1,1}(\mathcal{T})}{\partial p^2}(p^2) = 0 \tag{2.60}$$

so, looking at the first differential equation, and taking the limit above:

$$\lim_{p^2 \to 0} p^2 \frac{\partial}{\partial p^2} \quad \bigcirc \quad = -\lim_{p^2 \to 0}\frac{1}{2}\left[1 - \frac{p^2(d-3)}{4m^2 + p^2}\right] \quad \bigcirc \quad -$$
$$-\lim_{p^2 \to 0}\frac{(d-2)}{4m^2}\left[1 - \frac{p^2}{4m^2 + p^2}\right] \quad \bigcirc \quad = 0 \tag{2.61}$$

and we obtain the boundary condition we are looking for:

$$\lim_{p^2 \to 0} J_{1,1}(\mathcal{T})(p^2) = J_{1,1}(\mathcal{T})(0) = -\frac{(d-2)}{2m^2} J_1(\tau_1) \tag{2.62}$$

$$\lim_{p^2 \to 0} \quad \text{———} \bigcirc \text{———} \quad = -\frac{(d-2)}{2m^2} \quad \bigcirc \tag{2.63}$$

■

Chapter 3

Automation

In the previous chapters we described the modern method to evaluate scattering amplitude, via evaluating a great number of Feynman integrals, which complexity derives from the number of external particles and the requested precision of the final value of the amplitude (number of loops).

Summarizing the procedure that allow us to evaluate a Feynman diagram $\mathcal{F}^l_{\lambda,i}(P)$, the actual algorithm is structured as follows:

1. Perform the tensor decomposition in order to express a Feynman diagram $\mathcal{F}^l(P)$ in a combination of tensor object multiplied by form factors $I^l_c(\mathcal{I})$;

2. Every $I^l_c(\mathcal{I})$ is decomposable in a combination of Feynman integrals $I^{\bar{b}}_{\bar{a}}(\mathcal{T})$ associated at the topology $\mathcal{T}$ of original amplitude $\mathcal{F}^l(P)$;

3. For each Feynman integral $I^{\bar{b}}_{\bar{a}}(\mathcal{T})$, generate IBPs;

4. Identify a set of MIs $\mathcal{M}_\mathcal{T}$ by looking at how many independent integrals are in the r.h.s. of IBPs;

5. Use the IBPs to express all of the $I^{\bar{b}}_{\bar{a}}(\mathcal{T})$ in terms of MIs;

6. Generate the systems of differential equations for the set $\mathcal{M}_\mathcal{T}$;

7. Solve the system;

8. Collect all the results and build back the amplitude $\mathcal{F}^l(P)$.

In this Chapter we will describe an implementation of the point 3., the generation of IBPs, through the routine whose we exploit in this work. At the end we will suggest a method to improve the efficiency of these codes, giving an hint of what the strategy might be.

3.1 Notations

In this section we're going to define a sort of vocabulary, which translates the rigorous definition gave in the previous chapters to a new one, oriented to the automatization and the code implementation.

Let's start to define a *propagator*[28, 23] P as[1]

$$P = \frac{1}{V \cdot W - m^2} \tag{3.1}$$

where V^μ and W^μ are two different momenta currents. We can see that, defining an internal line $\Delta_j = (V_j^\mu, W_j^\mu, m^2)$,

$$P_j = \frac{1}{D(\Delta_j)} \tag{3.2}$$

A set F of an ordered set of propagators, $F = \{P_1, \ldots, P_{N_s}\}$ is said *integral family*. A propagator, in the sense meant by F, could be both the inverse $D(\Delta_i)$ and an irreducible scalar product $S_i \in \Sigma$. S_i is a scalar product: $D((V^\mu, W^\mu, 0))$ is also a scalar product. This means that we can write an irreducible scalar product as a massless denominator.

The goal of having an integral family is to have a set of propagators P_i such that every scalar product $k_i \cdot v_j \in \bar{\Sigma}$ can be build as combinations of P_i and kinematic invariants.

Taking t propagators from F defines a *sector* of the integral family F. It has an obvious corrispondence with the notion of topology $\mathcal{T}$ enunced previously. Suppose to choose the propagators $P_{j_1}, \ldots, P_{j_t}$ with $j_1, \ldots, j_t \in \{1, \ldots, N_s\}$. We define the *identification number* ID as

$$\mathrm{ID} = \sum_{i=1}^{t} 2^{j_i - 1} \tag{3.3}$$

Now, denoting with $\mathcal{T}_{\mathrm{ID}}^t$ topology corrisponding to the t-sector with identification number ID, the number of t-sectors we could build with the integral family F is $\binom{N_s}{t}$. A t'-sector such that $\mathcal{T}_{\mathrm{ID}'}^{t'} \subset \mathcal{T}_{\mathrm{ID}}^t$ is said to be a *subsector* of $\mathcal{T}_{\mathrm{ID}}^t$. Recalling the first definition of topology T and the function $\mathcal{D} : \mathbb{N} \to \mathcal{P}(\mathcal{T})$, in 1-loop cases this is the same number we could get if we would evaluate $|\mathcal{D}(t)|$ on the set $\mathcal{T}$. This is nothing else that the number of sectors at fixed number of denominator t.

We can easily map these notions with the one gave in the previous chapters: a right choice of t propagators from the family F give us the equivalent of a topology $\mathcal{T}$. The set of subsectors could easily be mapped in the notion of $\mathcal{S}_\mathcal{T}$, the set of the subtopologies of $\mathcal{T}$. The other $N_s - t$ propagators are related to the set Σ of irreducible scalar products.

After choosing a sector $\mathcal{T}_{ID}^t$, a generic Feynman integral for it is

$$\mathrm{INT}(F, t, \mathrm{ID}, r, s, \bar{r}, \bar{s}) = \int \prod_{i=1}^{l} d^d k_i \, P_{j_1}^{r_1} \cdots P_{j_t}^{r_t} P_{j_{t+1}}^{-s_1} \cdots P_{r_t}^{-s_{N_s - t}} \tag{3.4}$$

Obviously, in this new vocabulary $\mathrm{INT}(F, t, ID, r, s, \bar{r}, \bar{s}) = I_{\bar{r}}^{\bar{s}}(\mathcal{T}_{\mathrm{ID}}^t)$. It will be userful to define $r = \sum_{i=1}^{t} r_i$, and $s = \sum_{i=1}^{n-t} s_i$.

This integral form is exactly the same we defined assiomatically, in the sense that the integral form (3.4) is homologous to the one gave in the previous chapter.

---

[1]The code we use in this thesis works in Minkowski space, that's why we defined P with the minus sign.

Lastly, a *corner integral* is the one with $r = t$ (so $\bar{r} = (1, \ldots, 1) = \bar{1}$) and $s = 0$:

$$\text{INT}(F, t, \text{ID}, t, 0, \bar{1}, \bar{0}) = \int \prod_{i=1}^{l} d^d k_i P_{j_1} \cdots P_{j_t} \tag{3.5}$$

3.2 *Reduze 2*

There are algorithm[30, 28] which implement the notation exposed in the previous section. The one we will use in this work is called REDUZE[28]. It's a routine that requires only the declaration of the integral family and the external kinematics: it can generates IBPs, LIs, sector symmetries, and can also find the number of MIs, reduce Feynman integrals and generate the differential equations for each MI.

3.2.1 Input files

Here, we briefly show the form of the input required by REDUZE. It will be propedeutic for the purpose of this work. It offer also a mapping between the rigorous approach presented in Chapter 1. and the REDUZE syntax.

`kinematics.yaml`

In this file, we have to specify the external kinematics: incoming and outgoing momenta (P), name of the kinematic invariants, their mass dimension and the scalar product between the external momenta ($\boldsymbol{\sigma}$).

Example 3.1. (2-loop 3 external legs diagram) Suppose we have to specify the kinematics of the 2-loop QED vertex (Figure 3.1). We have $P = \{p_1, p_2\}$, $\mathbf{s} = \{p_1^2, p_2^2, p_1 \cdot p_2\}$, $\mathbf{M} = \{m, m, \sqrt{s}\}$. So, $E = \{(p_1^\mu, m), (p_1^\mu, m), (p_1^\mu + p_2^\mu, \sqrt{s})\}$.
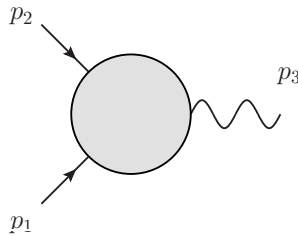
Figure 3.1: 2-loop QED vertex external kinematic

Let's define the symbols $p_1 = $ `p1`, $p_2 = $ `p2`, $p_3 = $ `p3` as the external momenta, `m` the electron mass, $s = (p_1 + p_2)^2$ the only kinematic invariant for the external kinematics (combination of elements of $\mathcal{I}$).

A scalar product, in the syntax of REDUZE, is written as $p_1 \cdot p_2 = $ `[p1,p2]`. We have also to specify the momentum conservation and the mass dimension of all kinematic invariants: s has mass dimension $[s] = 2$, so we have to write `[s,2]`. Here, instead of impose $(p_1 + p_2)^2 = s$, we report the same identity as $p_1 \cdot p_2 = \frac{1}{2}(s - 2m^2)$ (Figure 3.2). ∎

`kinematics.yaml` is not sensitive to the topology of the specific diagram chosen.

```
kinematics :
  incoming_momenta: [p1,p2,p3]
  outgoing_momenta: []
  momentum_conservation: [p3,-p1-p2]
  kinematic_invariants:
    - [s, 2]
    - [m, 1]
  scalarproduct_rules:
    - [[p1,p1], m^2]
    - [[p2,p2], m^2]
    - [[p1,p2], 1/2*(s-2*m^2)]
```

Figure 3.2: `kinematics.yaml` for a 2-loop QED vertex diagram

`integralfamilies.yaml`

In this file, we specify the internal lines of the diagram we're considering (Figure 3.4). To specify it, we have to give it a name, define loop momenta (K) and the list of propagator (Δ), as already said.

Example 3.2. [2-loop 3 external legs diagram] Let's continue to write the REDUZE files for the QED vertex (Figure 3.3). In this case, $K = \{k_1^\mu, k_2^\mu\}$ and,

$$\Delta = \begin{Bmatrix} (k_1^\mu, k_1^\mu, m^2), & (k_1^\mu + p_1^\mu, k_1^\mu + p_1^\mu, 0), \\ (k_1^\mu + p_1^\mu + p_2^\mu, k_1^\mu + p_1^\mu + p_2^\mu, m^2), & (k_2^\mu, k_2^\mu, m^2), \\ (k_2^\mu - p_1^\mu - p_2^\mu, k_2^\mu - p_1^\mu - p_2^\mu, m^2), & (k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0) \end{Bmatrix} \tag{3.6}$$

Having a 2-loop graph, we define $k_1 = $ `k1`, $k_2 = $ `k2` as the loop variables.

Figure 3.3: 2-loop QED vertex diagram

The syntax to define a propagator with momentum current $V^\mu = W^\mu$ and mass m:

$$\frac{1}{D(\Delta_j)} = \frac{1}{(V \cdot V + m^2)} \rightarrow [\text{V, "m^2"}] \tag{3.7}$$

Now, the number N_s of the scalar product with at least one loop momenta is $N_s = 2 \cdot 2 + \frac{2(2+1)}{2} = 7$, and $t = 6$: this means that we have one ISP. Then, setting $k_2 \cdot p_2$ as the ISP, we have to

write it as a "propagator". Actually, Reduze threats an ISP as a propagator, assign it a negative power, leading it at the denominator.

We already saw that Reduze implements a generalization of the standard propagator: if V^μ and W^μ are two different momenta, and m is a "mass", we have

$$\frac{1}{D(\Delta_j)} = \frac{1}{(V \cdot W + m^2)} \to \{\texttt{bilinear:} \quad \texttt{[[V,W],"m\^{}2"]}\} \qquad (3.8)$$

Then, our ISP is denoted as $p_2 \cdot k_2 = \{\texttt{bilinear:} \quad \texttt{[[k2,p2], 0]}\}$.

```
integralfamilies:
  - name: "QEDvertex"
    loop_momenta: [k1,k2]
    propagators:
      - [k1, "m^2"]
      - [k1+p1, 0]
      - [k1+p1+p2, "m^2"]
      - [k2, "m^2"]
      - [k2-p1-p2, "m^2"]
      - [k1+k2, 0]
      - {bilinear: [[k2,p2], 0]}
```

Figure 3.4: `integralfamilies.yaml` for a 2-loop QED vertex diagram.

In Figure 3.4, we wrote an ordered propagators list: the first six position are occupied by the "true" propagator, the last one by the ISP, the "fake" propagator. ∎

myintegrals

Once defined the two previous files, we have completely characterized a topology. Now, we have to tell to Reduze what Feynman integrals we want to consider. A Feynman integral for the topology $\mathcal{T}_{ID}^t$ defined in the previous section, is denoted like

$$I_{\bar{r}}^{\bar{s}}(\mathcal{T}_{ID}^t) = \texttt{INT[fam\_name,t,ID,r,s,\{r_1,\ldots,r_t,s_1,\ldots,s_{N_s-t}\}]} \qquad (3.9)$$

where `fam_name` is the name chosen in the `integralfamilies.yaml`, `t` the number of denominators, `r` the sum of powers of denominators, `s` the sum of powers of the ISPs.

`myintegrals` is a set of "target" integrals, namely a list of integrals we want to evaluate (Figure 3.16).

Example 3.3. [2-loop 3 external legs diagram] Again on the QED vertex topology. We want to evaluate some integrals with $r = 7$ and one with the ISP ($s = 1$).

As we can see, in this file we can write integrals in a more compact syntax. ∎

masters.curr.m

This file contains a list of possible master integrals for a chosen topology, selected by the user (Figure 3.6). This list could be written both before and after the effective knowledge of the number of master integrals for each topology/subtopology.

```
{
  INT["QEDvertex", {2,1,1,1,1,1,0}],
  INT["QEDvertex", {1,2,1,1,1,1,0}],
  INT["QEDvertex", {1,1,2,1,1,1,0}],
  INT["QEDvertex", {1,1,1,2,1,1,0}],
  INT["QEDvertex", {1,1,1,1,2,1,0}],
  INT["QEDvertex", {1,1,1,1,1,2,0}],
  INT["QEDvertex", {1,1,1,1,1,1,1}],
  INT["QEDvertex", {1,1,1,1,1,1,-1}]
}
```

Figure 3.5: List of Feynman integrals we want to evaluate.

Example 3.4. [2-loop 3 external legs diagram] To complete the setup to run REDUZE for the QED vertex, we have to compile the list of MIs.

```
{
INT["QEDvertex", {0,0,1,1,0,1,0}],
INT["QEDvertex", {0,1,1,1,1,1,0}],
INT["QEDvertex", {1,0,0,0,1,1,0}],
INT["QEDvertex", {1,0,1,1,1,0,0}],
INT["QEDvertex", {0,0,1,0,1,1,0}],
INT["QEDvertex", {1,0,0,1,0,1,0}],
INT["QEDvertex", {0,1,0,1,1,1,0}],
INT["QEDvertex", {1,0,1,1,1,1,0}],
INT["QEDvertex", {1,1,0,1,1,1,0}]
}
```

Figure 3.6: List of candadiates for being MIs.

This list is clearly not complete, but REDUZE is able to select the ones that actually are MIs.   ∎

3.2.2   Job system

In REDUZE 2 are implemented a set of algorithm[41] about automatic generation and calculation of quantities related to our topologies: we call them *jobs*. Jobs can be grouped in files .yaml[42] and we can run them sequentially. Here we list briefly the jobs which are userful for this work.

- setup_sector_mappings

This is a job that notices some relation between sectors, with the only knowledgement of the topology. It make use of a graphical method, like QGRAF. It has several options, all flagged true or false, which allow us to select the relations to be used (Figure 3.7).

```
        - setup_sector_mappings:
            conditional: false
            find_sector_relations: false
            find_sector_symmetries: false
            find_zero_sectors: true
            find_graphless_zero_sectors: false
            minimize_graphs_by_twists: false
            construct_minimal_graphs: false
            minimize_target_crossings: false
            allow_general_crossings: false
```

Figure 3.7: Job to set the relations between sectors of a chosen topology.

Some of those options may sounds familiar: `find_sector_relations` finds all possible change of variables that can map a sector $\mathcal{T}_{\mathrm{ID}}^t$ in another sector $\mathcal{T}_{\mathrm{ID}}^{t'}$. The subset of change of variables that map $\mathcal{T}_{\mathrm{ID}}^t$ in itself can be found with `find_sector_symmetries`. In order to find these sector symmetries, REDUZE uses an algorithm which implements the polynomials $\mathcal{U}$ and $\mathcal{V}$ defined previously, and through them it find all the symmetries between topologies.

`find_zero_sectors` identifies all sectors for which all Feynman integrals are zero. This could happen if, for example, the so called *corner integral* for a sector $\mathcal{T}_{\mathrm{ID}}^t$ is zero.

Those three options are the most important in this discussion; other options are associated to *crossing symmetries*, which inserts relations by acting on the external legs, or linked at the so-called *twists* of a graph.

- `reduce_sectors`

This is the core job of our work. It generates all the identities for a selection of integrals by varying r and s (Figure 3.8).

It select the topology $\mathcal{T}_{\mathrm{ID}}^t = $ `[QEDvertex,ID]`, with identification number ID; then it generates IBPs, LIs and sector symmetries for a selected range (in this specific case $r \in [t, 8]$ and $s \in [0, 2]$); lastly it inverts those relations to express integrals with a selected r (here $r = 9$).

The output of this job is a directory which contains a file for each not-zero sectors: every file is a list of IBPs, LIs and sector symmetries.

- `print_reduction_info_sectors`

This job, after the generation of the various identities, scans all the identities for each sector and count the number of master integrals occurs there (Figure 3.9).

The output of this job reports the number of master integrals for each sector. Giving us only a number, we have got the freedom to choose that number of independent Feynman integrals, for that specific sector.

```
- reduce_sectors:
    sector_selection:
      select_recursively:
        - [QEDvertex, 63]
    identities:
      ibp:
        - {r: [t,8], s: [0, 2]}
      lorentz:
        - {r: [t, 8], s: [0, 2]}
      sector_symmetries:
        - {r: [t, 8], s: [0, 2]}
    solutions:
      requested_solutions:
        - {r: [t, 9], s: [0, 2]}
    reduzer_options:
      num_equations_per_subjob: 1
    num_seeds_for_identities_auxjobs: 1000
    delete_temporary_files: true
```

Figure 3.8: Job to generate IBPs, LIs, symmetries for the topology `QEDvertex`.

```
- print_reduction_info_sectors:
    sector_selection:
      select_recursively:
        - [QEDvertex, 63]
    check_for_masters: [{r: [0, 8], s: [0, 2]}]
```

Figure 3.9: Job for count the number of master integrals for the topology `QEDvertex`.

- `reduce_files`

Finally, this job is the one that apply the reduction method. It picks the integrals occuring in `myintegrals` and express them in the basis choosen in `master.curr.m` (Figure 3.10).
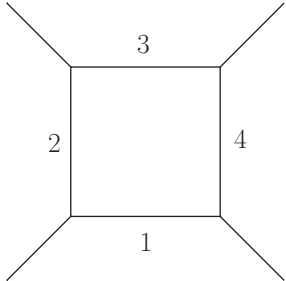
```
- reduce_files:
    equation_files: ["tmp/myintegrals.tmp", "tmp/sel.masters.curr"]
    output_file: "tmp/myintegrals.sol.tmp"
    preferred_masters_file: "tmp/all.masters.m"
```

Figure 3.10: Job to reduce the target integrals in the MIs basis.

3.2.3 1-loop box topology: a simple example

In the light-by-light scattering, we have a box topology with massless external legs. The topology $\mathcal{T}$ is:

$$
\mathcal{T} = \left(\left\{ \begin{array}{c} (p_1^\mu, 0), \\ (p_2^\mu, 0), \\ (p_3^\mu, 0), \\ (-p_1^\mu - p_2^\mu - p_3^\mu, 0) \end{array} \right\}, \left\{ \begin{array}{c} (k^\mu, k^\mu, m^2), \\ (k^\mu + p_1^\mu, k^\mu + p_1^\mu, m^2), \\ (k^\mu + p_1^\mu + p_2^\mu, k^\mu + p_1^\mu + p_2^\mu, m^2), \\ (k^\mu + p_1^\mu + p_2^\mu + p_3^\mu, k^\mu + p_1^\mu + p_2^\mu + p_3^\mu, m^2) \end{array} \right\} \right)
$$

$$
= (E, \{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}) =
$$

(3.10)

where we explicitly show the order of internal lines.

The first thing to do is to create the directory `config`, and the two configuration files: `kinematics.yaml` and `integralfamilies.yaml` (Figure 3.11).

```
kinematics :
  incoming_momenta: [p1,p2,p3,p4]
  outgoing_momenta: []
  momentum_conservation: [p4,-p1-p2-p3]
  kinematic_invariants:
    - [s, 2]
    - [m, 1]
  scalarproduct_rules:
    - [[p1,p1], 0]
    - [[p2,p2], 0]
    - [[p3,p3], 0]
    - [[p1,p2], 1/2*s]
    - [[p1,p3], 1/2*t]
    - [[p2,p3], -1/2*(s+t)]
```

```
integralfamilies:
  - name: "1loop_box"
    loop_momenta: [k1]
    propagators:
      - [k1, "m^2"]
      - [k1+p1, "m^2"]
      - [k1+p1+p2, "m^2"]
      - [k1+p1+p2+p3, "m^2"]
```

Figure 3.11: Configuration files for the 1-loop box topology.

These files define the topology $\mathcal{T}$, which has ID$= 2^0 + 2^1 + 2^2 + 2^3 = 15$.

Now, we have to specify the integrals we want to reduce. Suppose we want to reduce all integrals $I_{\vec{r}}^{\vec{s}}(\mathcal{T})$ with $r = \sum_i r_i = 6$. So, we have to generate all Feynman integrals by distribuiting the degree $r = 6$ in $t = 4$ denominators. At this point we are ready to generate IBPs (Figure 3.12).

The generation of IBPs implies the use of the `reduce_files` job. As a preliminary study, we can identify zero sectors and all symmetries among integrals without generating IBPs. In Figure (3.17) we present the standard collection of jobs dedicated to the IBP generation used in this work.
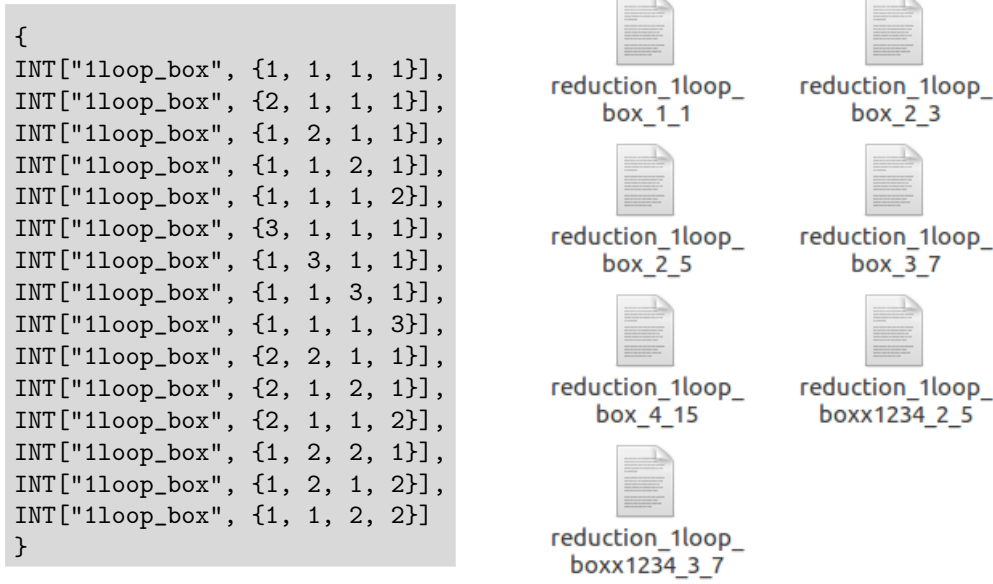
```
{
INT["1loop_box", {1, 1, 1, 1}],
INT["1loop_box", {2, 1, 1, 1}],
INT["1loop_box", {1, 2, 1, 1}],
INT["1loop_box", {1, 1, 2, 1}],
INT["1loop_box", {1, 1, 1, 2}],
INT["1loop_box", {3, 1, 1, 1}],
INT["1loop_box", {1, 3, 1, 1}],
INT["1loop_box", {1, 1, 3, 1}],
INT["1loop_box", {1, 1, 1, 3}],
INT["1loop_box", {2, 2, 1, 1}],
INT["1loop_box", {2, 1, 2, 1}],
INT["1loop_box", {2, 1, 1, 2}],
INT["1loop_box", {1, 2, 2, 1}],
INT["1loop_box", {1, 2, 1, 2}],
INT["1loop_box", {1, 1, 2, 2}]
}
```

reduction_1loop_box_1_1

reduction_1loop_box_2_3

reduction_1loop_box_2_5

reduction_1loop_box_3_7

reduction_1loop_box_4_15

reduction_1loop_boxx1234_2_5

reduction_1loop_boxx1234_3_7

Figure 3.12: `myintegrals` and `reduction` for the 1-loop box topology

After the generation, we can see that there are some new directories: one of those contains the IBP relations, `reductions`.

Files contained in `reductions` are inputs for the jobs which reduce integrals, generates differential equations and so on.

```
r=7:                          4x


Checking (1loop_box:4:15) for masters in range:
---
- {r: [0, 6], s: [0, 1]}

Missing information for at least one (r,s) in the range

Integrals appearing on r.h.s. of reductions in the range:
(1loop_box:1:1): 1
(1loop_box:2:5): 1
(1loop_boxx1234:2:5): 1
(1loop_box:3:7): 1
(1loop_boxx1234:3:7): 1
(1loop_box:4:15): 1
in total: 6
Completed Job(8) [Time: 0s, CPU: 0s]
Starting job(1): print reduction info for sectors

Completed Job(1) [Time: 0s, CPU: 0s]
Completed all jobs
Total run time: 1 s
```

Figure 3.13: Master integrals for each sectors

Once generated all the selected IBPs, we have to find the number of master integrals for each sectors. This can be done with the `check_for_masters` job. In Figure (3.13) we can see that the box topology has six master integrals, one for each

of the following subtopologies:

$$\left\{ \quad \right\} \subset \mathcal{S}_{\mathcal{T}} \qquad (3.11)$$

Let's choose the corner integral $I_{\bar{1}}(\tau_j)$ for each τ_j belonging to the master topologies (3.11):

```
{
INT["1loop_box", {1, 1, 1, 1}],
INT["1loop_box", {1, 1, 0, 1}],
INT["1loop_box", {1, 1, 1, 0}],
INT["1loop_box", {0, 1, 0, 1}],
INT["1loop_box", {0, 1, 0, 1}],
INT["1loop_box", {1, 0, 0, 0}]
}
```

Figure 3.14: Selection of master integrals for the 1-loop box topology

In the REDUZE language, we have to write the `master.curr` file (Figure 3.14). This contain all the candidate as master integrals we can choose. If this set is not independent, REDUZE will select master integrals using Laporta algorithm in order to fill the basis.

At the end, it's time to apply IBPs on our target integrals. We have to use the last job, `reduce_files`. This jobs picks the target integrals in `myintegrals` and express them in terms of master integrals by applying the IBPs. The output is printed in a file called `myintegrals.sol.mma`. In Figure 3.15 we present the reduction for $I_{1,1,1,2}(\mathcal{T})$ only.

```
INT["1loop_box",4,15,5,0,{1,1,1,2}] ->
  INT["1loop_box",4,15,4,0,{1,1,1,1}] *
      (-(-5+d)*s*(4*m^2*t-s*t-s^2)^(-1)) +
  INT["1loop_box",3,11,3,0,{1,1,0,1}] *
      (2*(-4+d)*(4*m^2*t-s*t-s^2)^(-1)) +
  INT["1loop_box",3,7,3,0,{1,1,1,0}] *
      (1/2*m^(-2)*(s*d-4*s-4*m^2*d+16*m^2)*(4*m^2*t-s*t-s^2)^(-1)) +
  INT["1loop_box",2,10,2,0,{0,1,0,1}] *
      (-4*(-3+d)*(s+4*m^2+t)^(-1)*(4*m^2*t-s*t-s^2)^(-1)) +
  INT["1loop_box",2,5,2,0,{1,0,1,0}] *
      (m^(-2)*(-3+d)*(4*m^2*t-s*t-s^2)^(-1)) +
  INT["1loop_box",1,1,1,0,{1,0,0,0}] *
      (-1/2*(s*d-2*s+d*t-2*t)*m^(-4)*(s+4*m^2+t)^(-1)*(4*m^2*t-s*t-s^2)^(-1))
```

Figure 3.15: Reduction for $I_{1,1,1,2}(\mathcal{T})$ from `myintegrals.sol.mma`

3.2.4 Usage and issues

REDUZE is a powerful tool to evaluate Feynman integrals. In Figure 3.16 is represented a flowchart of its reduction algorithm. It requires the only knowledge of the

topology of our diagram, and virtually, by building the REDUZE setup files listed below, we are able to reduce every Feynman integral belonging to any topology, having any momenta routing, and any mass value.

Actually, the generation of the identities could take a very long time. This depends on: the kinematics, namely how many external legs has a diagram; the topology, namely the number of internal legs, the momentum currents and the number of loop. For example, it could take days to generate IBPs for topologies with four external legs and two loops, strongly depending on the mass configuration. Moreover, the gaussian elimination on IBP systems is a very expensive algorithm, which can slow down the reduction, depending on the number of IBPs in the system. Lastly, the reduction running time depends also on the current computing power.
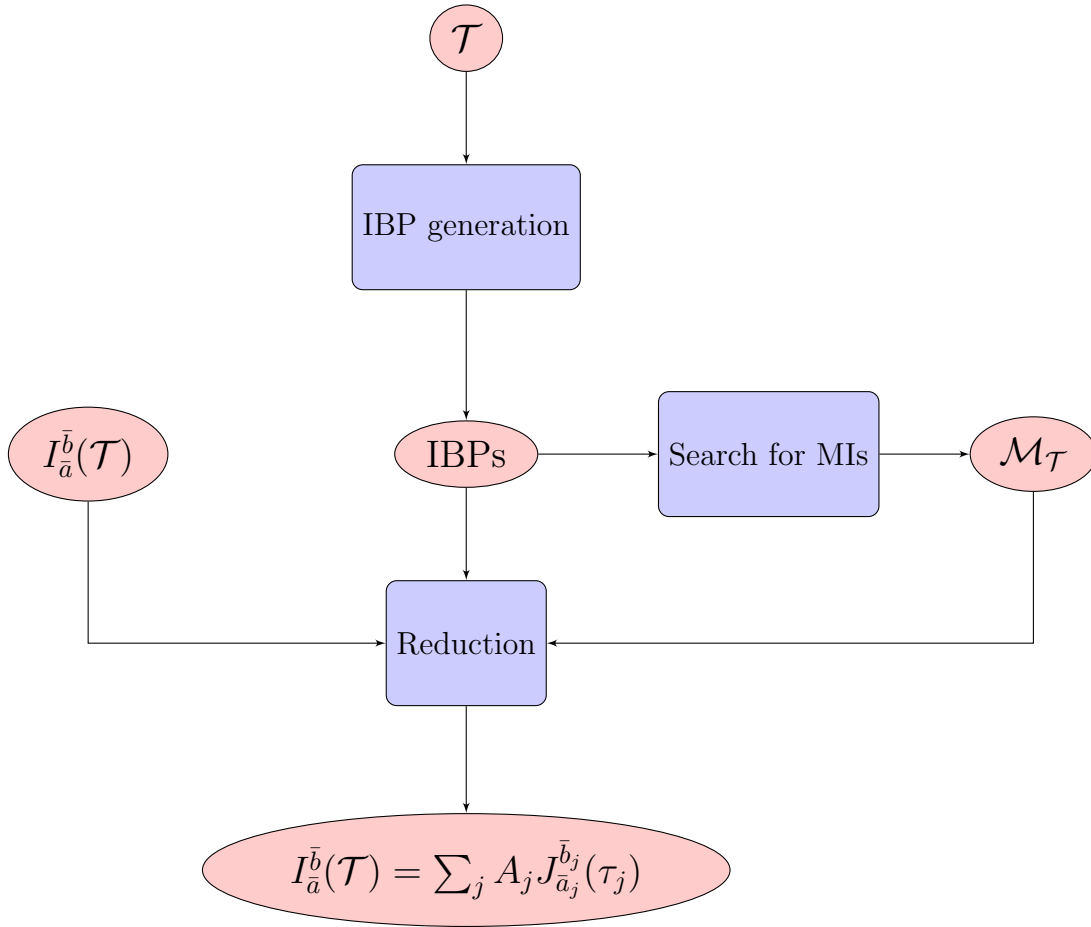

Figure 3.16: Flow chart of the basic reduction algorithm for REDUZE

The goal of this work is to present a way to improve the current reduction method, with the implementation of a new algorithm.

Since we have several other codes which can be used to evaluate Feynman integrals, the simplest idea to speed up the reduction algorithm is to modify the input through splitting it in simpler integrals. This allow us to parallelise calculations over each part of the decomposition, hoping that their running time may be less than the starting integral. Once we do this step, we have to collect the single outputs and to think how recompose them to get the right result.

Summarizing, a possible way to speed up the calculation can be madi in dfdf steps:

1. Split the input;

2. Parallelize the calculation;

3. Build back the results

The way we found is inspired by the unitarity of scattering amplitudes, in particular by the BCFW recursion relation and unitarity cuts, as we will see in the next Chapter.

```
jobs:
  - setup_sector_mappings:
      conditional: false
      find_sector_relations: true
      find_sector_symmetries: true
      find_zero_sectors: true
      find_graphless_zero_sectors: true
      minimize_graphs_by_twists: true
      construct_minimal_graphs: true
      minimize_target_crossings: true
      allow_general_crossings: true
  - setup_sector_mappings_alt:
      conditional: false
      source_sectors:
        select:
          []
        select_recursively:
          [[1loop_box, 15]]
        deselect:
          []
        deselect_recursively:
          []
        t_restriction: [-1, -1]
      find_zero_sectors: true
      find_sector_relations: true
      find_sector_symmetries: true
  - reduce_sectors:
      sector_selection:
        select_recursively: [[1loop_box, 15]]
      identities:
        ibp:
          - {r: [t, 6], s: [0, 1]}
        lorentz:
          - {r: [t, 6], s: [0, 1]}
        sector_symmetries:
          - {r: [t, 6], s: [0, 1]}
      solutions:
        requested_solutions:
          - {r: [t, 6], s: [0, 1]}
      reduzer_options:
        num_equations_per_subjob: 1
      num_seeds_for_identities_auxjobs: 1000
      delete_temporary_files: true
```

Figure 3.17: `jobs_1_reduction.yaml`

Chapter 4

Novel decomposition for Feynman integrals

In this Chapter we explain the idea behind this work. Firstly we describe the basic ingredients coming from the unitarity of scattering amplitudes. Then we present the *partial fractioning method* for the integral calculus, related to the unitarity of the scattering amplitudes, and it's application to topologies and Feynman integrals. In particular, there we present a generalized partial fractioning formula which extends its action to an arbitrary number of denominators and denominator exponents. This is one of the main results of this work.

Then, we present a novel reduction algorithm which combines partial fractioning (unitarity) and IBPs. The strenght of this method relies on get a new set of Feynman integrals built on new topologies with *linear* internal lines. Feynman integral with linear denominators has total degree (of the denominator) lower than the starting Feynman integrals.

Lastly we show some peculiar properties brought by the linearity of the denominator.

4.1 Unitary methods for scattering amplitudes

Let's recall that the *scattering matrix* S is unitary[39]: $SS^+ = \mathbb{I}$. By subtracting the non-interacting part of S, we can define $S = \mathbb{I} + iT$. Applying this splitting to the unitary relation we get:

$$-i(T - T^+) = T^+T \tag{4.1}$$

T is the part of S which is the sum of Feynman diagrams.

4.1.1 Optical theorem and Unitary cuts

A Feynman integral could be *cutted*, in the sense suggested in the *optical theorem*[39], direct conseguence of (4.1), which for a $2 \to 2$ scattering is:

$$2\mathrm{Im}\mathcal{F}(p_1p_2 \to p_1p_2) = \sum_i \int d\Pi_i \mathcal{F}^*(p_1p_2 \to \{f_i\})\mathcal{F}(p_1p_2 \to \{f_i\}) \tag{4.2}$$

where $d\Pi_i = \prod_{j=1}^{i} \frac{d^3 q_j}{(2\pi)^3} \frac{1}{2E_j} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_j q_j)$. At 1-loop, it relates the imaginary part of the total cross section at a sum of tree-level amplitudes. Diagrammatically, it can be expressed like:

$$\tag{4.3}$$

It seems that, in the r.h.s., all virtual particles are putted on-shell.

The particle could be created only when $s > s_0 = m_{th}^2$, the so-called *threshold*. It can be shown that for $s > m_{th}^2$, an amplitudes developes an imaginary part, so $\mathcal{F}(s)$ has the following properties:

- $\mathcal{F}(s - i\epsilon) = \mathcal{F}(s + i\epsilon)$, for $s < s_0$;

- $\mathrm{Re}\mathcal{F}(s + i\epsilon) = \mathrm{Re}\mathcal{F}(s - i\epsilon)$, for $s > s_0$;

- $\mathrm{Im}\mathcal{F}(s + i\epsilon) = -\mathrm{Im}\mathcal{F}(s - i\epsilon)$, for $s > s_0$;

Moreover, $\mathcal{F}(s)$ is analytic for $s < s_0$, so we can perform the analytic continuation outside from the real axis. For $s < m_{th}^2$, the amplitude begins to splits in two *branch cuts*, which bring a discontinuity through the real axis. This discontinuity could be evaluate in the following way:

$$\mathrm{Disc}\mathcal{F}(s) = \mathcal{F}(s + i\epsilon) - \mathcal{F}(s - i\epsilon) = 2i\mathrm{Im}\mathcal{F}(s + i\epsilon) \tag{4.4}$$

that is nothing else than the r.h.s. of the optical theorem. So, the physical discontinuity of an amplitude is related to the optical theorem[43].

Example 4.1. (1-loop scalar fish diagram) In the scalar $\lambda\phi^4$ theory we can write the following Feynman diagram $F(p^2)$, where $p = p_1 + p_2$, and suppose to expressing its kinematics in the center of mass ($p^\mu = (p^0, \bar{0})$).

$$\tag{4.5}$$

$$= \frac{(i\lambda)^2}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{[(\frac{p}{2} + k)^2 - m^2 + i\epsilon]} \frac{i}{[(\frac{p}{2} - k)^2 - m^2 + i\epsilon]}$$

This integral has four poles: $k_0 = \pm\frac{p^0}{2} \pm (\omega_k - i\epsilon)$, where $\omega_k^2 = k^i k_i + m^2$. Integrating on k_0, we obtain

$$\mathcal{F}(p^2) = \frac{\lambda^2}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{(2p^0)(2\omega_k)} \left(\frac{1}{\omega_k - \frac{p^0}{2} - i\epsilon} - \frac{1}{\omega_k + \frac{p^0}{2} - i\epsilon} \right) \tag{4.6}$$

The rise of an imaginary part begins from he threshold $\omega_k = \frac{p^0}{2}$, that could happens only if $p^0 \geq 2m$.

Using the identity

$$\frac{1}{x - i\epsilon} = P\left(\frac{1}{x}\right) + i\pi\delta(x) \tag{4.7}$$

the imaginary part $\text{Im}\mathcal{F}(p^2)$ is

$$\begin{aligned}
\text{Im}\mathcal{F}(p^2) &= \frac{1}{8\pi^2 p^0}\frac{\lambda^2}{2}\int_m^\infty d\omega_k \sqrt{\omega_k^2 - m^2}\pi\delta(\omega_k - \frac{p^0}{2}) \\
&= \frac{\lambda^2}{16\pi^2 p^0}\sqrt{\left(\frac{p^0}{2}\right)^2 - m^2}\ \theta(p_0 - 2m)
\end{aligned} \tag{4.8}$$

where $d^3k = 4\pi k^2 dk = 4\pi\sqrt{\omega_k^2 - m^2}\omega_k dk$. The presence of the threshold is encoded in the theta function.

We also observe that:

$$\begin{aligned}
\delta\left(\left(\frac{p}{2} - k\right)^2 - m^2\right) &= \delta\left(\left(\frac{p^0}{2} - k^0\right)^2 - \omega_k^2\right) \\
&= \frac{\delta\left(\left(\frac{p^0}{2} - k^0\right) - \omega_k\right)}{2\omega_k} + \frac{\delta\left(\left(\frac{p^0}{2} - k^0\right) + \omega_k\right)}{2\omega_k}
\end{aligned} \tag{4.9}$$

Multiplying it for $\delta((\frac{p}{2} - k)^2 - m^2) = \delta((\frac{p^0}{2} - k^0)^2 - \omega_k^2)$, the second term of (4.9) does not contribute:

$$\delta\left(\left(\frac{p}{2} - k\right)^2 - m^2\right)\delta\left(\left(\frac{p^0}{2} - k^0\right)^2 - \omega_k^2\right) = \frac{\delta\left(\left(\frac{p^0}{2} - k^0\right)^2 - \omega_k^2\right)\delta\left(\left(\frac{p^0}{2} + k^0\right) - \omega_k\right)}{2\omega_k} \tag{4.10}$$

Integrating this product in dk^0, we get:

$$\int dk^0 \delta\left(\left(\frac{p}{2} - k\right)^2 - m^2\right)\delta\left(\left(\frac{p^0}{2} - k^0\right)^2 - \omega_k^2\right) = \frac{\delta\left((p^0)^2 - 2p^0\omega_k\right)}{2\omega_k} \tag{4.11}$$

and expressing the delta with respect to ω_k:

$$\int dk^0 \delta\left(\left(\frac{p}{2} - k\right)^2 - m^2\right)\delta\left(\left(\frac{p}{2} + k\right)^2 - m^2\right) = \frac{\delta\left(\omega_k - \frac{p^0}{2}\right)}{(2p^0)(2\omega_k)} \tag{4.12}$$
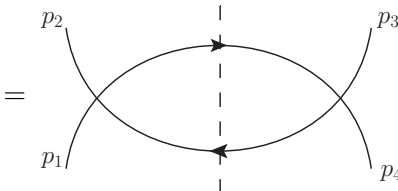
Now, we have found another way to evaluate the imaginary part of a diagram.

$$\begin{aligned}
2i\text{Im}\mathcal{F}(p^2) &= \frac{i\lambda^2}{8\pi^2}\int_m^\infty d\omega_k 4\pi\sqrt{\omega_k^2 - m^2}\omega_k\frac{\delta\left(\omega_k - \frac{p^0}{2}\right)}{(2\omega_k)(2p^0)} \\
&= \frac{i\lambda^2}{8\pi^2}\int d^3\mathbf{k}\frac{\delta\left(\omega_k - \frac{p^0}{2}\right)}{(2\omega_k)(2p^0)} \\
&= \frac{\lambda^2}{2i}\int\frac{d^4k}{(2\pi)^4}(-2\pi i)^2\delta\left(\left(\frac{p}{2} - k\right)^2 - m^2\right)\delta\left(\left(\frac{p}{2} + k\right)^2 - m^2\right) \\
&= \text{Disc}\mathcal{F}(p^2)
\end{aligned} \tag{4.13}$$

We get at an expressions close to the original integrals $i\mathcal{F}(p^2)$, but all the propagators are replaced by

$$\frac{i}{\left(\frac{p}{2}+k\right)^2-m^2} \to (-2\pi i)\delta\left(\left(\frac{p}{2}+k\right)^2-m^2\right)$$

$$\frac{i}{\left(\frac{p}{2}-k\right)^2-m^2} \to (-2\pi i)\delta\left(\left(\frac{p}{2}-k\right)^2-m^2\right)$$

(4.14)

Feynman integrals with the replacements above are denoted as

$$i\mathrm{Disc}\mathcal{F}(p^2) = \frac{(i\lambda)^2}{2}\int\frac{d^4k}{(2\pi)^4}(-2\pi i)^2\delta\left(\left(\frac{p}{2}-k\right)^2-m^2\right)\delta\left(\left(\frac{p}{2}+k\right)^2-m^2\right)$$

(4.15)

$$=$$

This example shows that we can evaluate the imaginary part of a diagram by performing a so-called *unitary cut*[44, 39]. Cutting a propagator means to impose the on-shell condition on each of them. So, we can list an algorithm, the *Cutkosky rules*, which allow us to evaluate $\mathrm{Disc}\mathcal{F}$:

1. Cutting a diagram so that two propagators can simultaneously be put on-shell;

2. For each cut propagator, replace

$$\frac{i}{V^2-m^2+i\epsilon} \to -2\pi i\delta(V^2-m^2)$$

(4.16)

3. Perform the loop integration;

4. Sum the solution of all possible cuts.

Cutting rules are strongly connected with the unitarity of the theory, and they offer a powerful method to evaluate discontinuities generated by the branch cuts running along the real axis.

With a generalization of the notion of cutting rules, we are able to prove the optical theorem at each perturbative order: this shows that cuts and optical theorem are properties of both scattering amplitude and field theories discending on their unitarity. Perform a cut is equivalent to impose the unitarity of the theory.

4.1.2 Dispersion relation

Suppose to have an analytic function $f(z)$. We know from the complex analysis that $f(z)$ satisfies the requirement to be represented as

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z')}{z'-z}dz'$$

(4.17)

where $z \in \mathbb{C}$ is a regular point surrounded by the closed path γ; moreover, γ has not to cross singularities. This is called *Cauchy's representation theorem*.

A Feynman amplitude is an analytic function of the kinematics variables. Let's take the 2-point amplitude $\mathcal{F}(p^2)$ as example, evaluate its integral representation along a closed path γ_{Λ^2}. Recalling the fact that a 2-point amplitude has a branch cut that starts from $s = 4m^2$:

$$\mathcal{F}(p^2) = \frac{1}{2\pi i} \oint_{\gamma_{\Lambda^2}} \frac{\mathcal{F}(s)}{s - p^2} ds = \frac{1}{2\pi i} \int_{4m^2}^{\Lambda^2} \frac{\mathcal{F}(s + i\epsilon) - \mathcal{F}(s - i\epsilon)}{s - p^2} ds +$$
$$+ \frac{1}{2\pi i} \oint_{|s| = \Lambda^2} \frac{\mathcal{F}(s)}{s - p^2} ds \tag{4.18}$$

The last identity is true for the theorem of residues. If the second term of the r.h.s. of the (4.18) vanishes

$$\lim_{\Lambda^2 \to \infty} \frac{1}{2\pi i} \oint_{|s| = \Lambda^2} \frac{\mathcal{F}(s)}{s - p^2} ds = 0 \tag{4.19}$$

then we get

$$\mathcal{F}(p^2) = \frac{1}{2\pi i} \int_{4m^2}^{\infty} \frac{\mathcal{F}(s + i\epsilon) - \mathcal{F}(s - i\epsilon)}{s - p^2} ds \tag{4.20}$$

The numerator is exactly the discontinuity along the branch cut:

$$\mathcal{F}(p^2) = \frac{1}{2\pi i} \int_{4m^2}^{\infty} \frac{\text{Disc}\mathcal{F}(s)}{s - p^2} ds = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im}\mathcal{F}(s)}{s - p^2} ds \tag{4.21}$$

This shows that, in order to evaluate a Feynman amplitude, it is sufficient to find the imaginary part of it. As we saw before, we can evaluate this imaginary part in an easier way instead of the full amplitude, and this feature provides a powerful tool for our calculation.

This relation goes under the name of *dispersion relation*[43]. It could be proved at all perturbative order[39], and once again, it is direct consequence of the unitarity of the theory.

Anyway, in general (4.19) is not satisfied. we can overcome to this fact by evaluating p^2 at some value $p_0^2 < 4m^2$. By doing this, after some algebra we get the so-called *once-subtracted dispersion relation*[44]

$$\mathcal{F}(p^2) = \mathcal{F}(p_0^2) + \frac{p^2 - p_0^2}{\pi} \int_{4m^2}^{\infty} \frac{\text{Im}\mathcal{F}(s)}{(s - p_0^2)(s - p^2)} ds \tag{4.22}$$

4.1.3 Generalized unitarity

Looking at the previous definition of *cut*, we remark the fact that this kind of operation is made at the level of scattering amplitudes. Cutkowski cuts divide a scattering amplitude in two diagrams: at 1-loop, they split a graph in two tree-level amplitudes.

We may think of generalizing the notion of "cut", in order to apply it on Feynman integrals, simply by replacing an arbitrary number of denominators with the same number of delta functions[45, 46]:

$$\frac{1}{V^2 - m^2 - i\epsilon} \to \delta\left(V^2 - m^2\right) \tag{4.23}$$

This time, we may cut one only denominator or cut all denominators, whatever we want. Cutting all propagators in a diagram is said to go on the *maximal cut*.

Once performed our number of cuts, we can use a standard delta identity:

$$\delta\left(V^2 - m^2\right) = \frac{1}{2\pi i}\left(\frac{1}{V^2 - m^2 - i\epsilon} - \frac{1}{V^2 - m^2 + i\epsilon}\right) \tag{4.24}$$

In Minkowski space, our propagators are written with the ϵ-prescription. If we replace a delta function with (4.24) we get propagators with both $+i\epsilon$ and $-i\epsilon$, that makes our cutted diagram obviously different from the one we started with. The benefit of this new way to cut arises when we generate IBP identities on a cutted Feynman integral.

IBP identities are differential operator which act at the integrand level: clearly, they are not sensitive to the presence of $\pm i\epsilon$, appearing before replacing (4.24) on the cut. In the following example is shown this particular feature.

Example 4.2. [Maximal cut on 1-loop bubble diagram] Let's perform a double cut on the usual 1-loop bubble diagram, $I_{1,1}(\!-\!\bigcirc\!-\!)$. Graphically it's represented by a dashed line that intersect each cutted propagator:

$$\int \frac{d^d k}{(2\pi)^{d-2}}\delta\left(k^2 - m^2\right)\delta\left((k+p)^2 - m^2\right) \tag{4.25}$$

Now, let's apply on the integrand in the r.h.s. the identity (4.24) and split all new integrals that appear

$$= \frac{1}{2\pi i}\int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{k^2 - m^2 + i\epsilon}\frac{1}{(k+p)^2 - m^2 + i\epsilon} -$$
$$- \frac{1}{2\pi i}\int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{k^2 - m^2 + i\epsilon}\frac{1}{(k+p)^2 - m^2 - i\epsilon} - \tag{4.26}$$
$$- \frac{1}{2\pi i}\int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{k^2 - m^2 - i\epsilon}\frac{1}{(k+p)^2 - m^2 + i\epsilon} +$$
$$+ \frac{1}{2\pi i}\int \frac{d^d k}{(2\pi)^{d-2}}\frac{1}{k^2 - m^2 - i\epsilon}\frac{1}{(k+p)^2 - m^2 - i\epsilon}$$

On each integral in the r.h.s. we clearly can perform IBPs. We choose this IBP:

$$\int \frac{d^d k}{(2\pi)^{d-2}}\frac{\partial}{\partial k^\mu}\left[\frac{k^\mu}{[k^2 - m^2 \pm i\epsilon][(k+p)^2 - m^2 \pm i\epsilon]}\right] = 0 \tag{4.27}$$

As already found before, adding an index to underline the presence of $+i\epsilon$ or $-i\epsilon$:

$$= \frac{(d-3)}{4m^2 - p^2} \quad\quad - \frac{(d-2)}{2m^2(4m^2 - p^2)} \tag{4.28}$$

$$\int \frac{d^d k}{(2\pi)^{d-2}} \delta(k^2 - m^2) = \frac{1}{2\pi i} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{2\pi i} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{k^2 - m^2 - i\epsilon}$$

$$(4.32)$$

we could combine those IBPs as $(4.28) - (4.29) - (4.30) + (4.31)$: this combination doesn't contain any tadpole, because of (4.32). So, we get the following expression:

$$(4.33)$$

Now, we have to understand what is the l.h.s. of this combination.

Let's take a look to the dotted bubble, where now we explicitly show the mass dependence: $I_{2,1}(m^2, m^2)$. Naming $D_1 = k^2 - m^2 + i\epsilon$ and $D_2 = (k+p)^2 - m^2 + i\epsilon$ clearly we can write:

$$I_{2,1}(m^2, m^2) = \lim_{\tilde{m} \to m} I_{2,1}(\tilde{m}^2, m^2) \tag{4.34}$$

We also have:

$$I_{2,1}(\tilde{m}^2, m^2) = \frac{\partial}{\partial \tilde{m}^2} I_{1,1}(\tilde{m}^2, m^2) \tag{4.35}$$

So, we can define a cut on a dotted propagator as

$$= \lim_{\tilde{m} \to m} \frac{\partial}{\partial \tilde{m}^2} \tag{4.36}$$

Let's apply again the identity (4.24) on the r.h.s. of this equation:

$$(4.37)$$

so

$$(4.38)$$

Plugging this last identity, the starting IBPs combination become

$$\qquad \qquad = \frac{(d-3)}{4m^2 - p^2} \qquad \qquad \qquad \qquad \qquad (4.39)$$

∎

This example has shown how powerful is cutting a Feynman integral. If we have a topology with t denominators, every Feynman integrals $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ could be cutted c times: its reduction could contain only master integrals with at least c propagators. Moreover, the coefficients of every cutted master integral are equal to the one of the uncutted reduction. Denoting a Feynman integral cutted c times as $\mathrm{Cut}_c I_{\bar{a}}^{\bar{b}}(\mathcal{T})$:

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \sum_{i=1}^{N_{MI}} A_i J_{\bar{a}_i}^{\bar{b}_i}(\tau_i) \implies \mathrm{Cut}_c I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \sum_{i=1}^{N_{MI}} A_i \mathrm{Cut}_c J_{\bar{a}_i}^{\bar{b}_i}(\tau_i) \qquad (4.40)$$

Example 4.3. [Double cut of bubble diagram] Let us start from the bubble topology

$$\!\!-\!\!\bigcirc\!\!-\!\! = \left(\left\{ \begin{matrix} (p^\mu, \sqrt{s}), \\ (p^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k^\mu, k^\mu, m^2), \\ (k^\mu + p^\mu, k^\mu + p^\mu, m^2) \end{matrix} \right\} \right) \tag{4.41}$$

and from the corner integral

$$I_{\bar{1}}\left(\!\!-\!\!\bigcirc\!\!-\!\! \right) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m_1^2][(k+p)^2 + m_2^2]} \tag{4.42}$$

The double cut of this integral is

$$\!\!-\!\!\bigcirc\!\!-\!\! = \int \frac{d^d k}{(2\pi)^{d-2}} \delta(k^2 + m_1^2)\delta((k+p)^2 + m_2^2) \tag{4.43}$$

Evaluating it for $p^\mu = (\sqrt{s}, 0, 0, 0)$:

$$\begin{aligned}
\!\!-\!\!\bigcirc\!\!-\!\! &= \int \frac{d^d k}{(2\pi)^{d-2}} \delta(k^2 + m_1^2)\delta(2k \cdot p + p^2 + m_2^2 - m_1^2) \\
&= \int \frac{dk_0 d^{d-1}\bar{k}}{(2\pi)^{d-2}} \frac{\delta(k^2 + m_1^2)}{2\sqrt{s}} \delta\left(k_0 + \frac{s + m_2^2 - m_1^2}{2\sqrt{s}} \right) \\
&= \int \frac{d^{d-1}\bar{k}}{(2\pi)^{d-2}} \frac{1}{2\sqrt{s}} \delta\left(|\bar{k}|^2 + \frac{(s + m_2^2 - m_1^2)^2}{4s} + m_1^2 \right)
\end{aligned} \tag{4.44}$$

The $d - 1$-dimensional integration gives

$$\begin{aligned}
\!\!-\!\!\bigcirc\!\!-\!\! &= \int \frac{d\Omega_{d-2}}{(2\pi)^{d-2}} \int d|\bar{k}| \frac{|\bar{k}|^{d-1}}{2\sqrt{s}} \delta\left(|\bar{k}|^2 + \frac{(s + m_2^2 - m_1^2)^2}{4s} + m_1^2 \right) \\
&= \frac{(4\pi)^{\frac{3-d}{2}}}{4\Gamma(\frac{d-1}{2})} \int d|\bar{k}|^2 \frac{(|\bar{k}|^2)^{\frac{d-3}{2}}}{\sqrt{s}} \delta\left(|\bar{k}|^2 + \frac{(s + m_2^2 - m_1^2)^2}{4s} + m_1^2 \right) \\
&= \frac{(4\pi)^{\frac{3-d}{2}}}{4\Gamma(\frac{d-1}{2})} s^{1-\frac{d}{2}} \left(-(s + m_2^2 - m_1^2)^2 - 4s m_1^2 \right)^{\frac{d-3}{2}} \\
&= \frac{(4\pi)^{\frac{3-d}{2}}}{4\Gamma(\frac{d-1}{2})} s^{1-\frac{d}{2}} \left(-(s + m_2^2)^2 - m_1^4 + 2m_1^2(m_2^2 - s) \right)^{\frac{d-3}{2}}
\end{aligned} \tag{4.45}$$

∎

Generalized unitarity and residues

Suppose to have a corner integral for a 1-loop topology $\mathcal{T}$

$$I_{\bar{1}}(\mathcal{T}) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2 \cdots D_t} \tag{4.46}$$

Applying the single cut on the denominator D_1:

$$\text{Cut}_{D_1}(I_{\bar{1}}(\mathcal{T})) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\delta(D_1)}{D_2 \cdots D_t} \tag{4.47}$$

Due to the structure of denominators, $\delta(D_1)$ acts on the integrand in the following way:

$$D_j|_{D_1=0} = D_j - D_1, \quad j \geq 2 \tag{4.48}$$

so we can write

$$\mathrm{Cut}_{D_1}(I_{\bar{1}}(\mathcal{T})) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\delta(D_1)}{(D_2 - D_1)\cdots(D_t - D_1)} \tag{4.49}$$

This result recalls the one we can get from the calculation of the *residue* of the integrand of $I_{\bar{1}}(\mathcal{T})$ at the simple pole $D_1 = 0$:

$$\mathrm{Res}_{D_1=0}\left(\frac{1}{D_1 D_2 \cdots D_t}\right) = \lim_{D_1 \to 0} \frac{D_1}{D_1 D_2 \cdots D_t} = \frac{1}{(D_2 - D_1)\cdots(D_t - D_1)}\bigg|_{D_1=0} \tag{4.50}$$

Example 4.4. [Single cut of 1-loop bubble] Let us start from the topology

$$-\!\!\bigcirc\!\!- = \left(\left\{\begin{matrix}(p^\mu, \sqrt{s}),\\(p^\mu, \sqrt{s})\end{matrix}\right\}, \left\{\begin{matrix}(k^\mu, k^\mu, m^2),\\(k^\mu + p^\mu, k^\mu + p^\mu, m^2)\end{matrix}\right\}\right) \tag{4.51}$$

and write the corner integral

$$I_{\bar{1}}\left(-\!\!\bigcirc\!\!-\right) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m^2][(k+p)^2 + m^2]} \tag{4.52}$$

The single cut of D_1 gives

$$-\!\!\bigcirc\!\!- = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\delta(k^2 + m^2)}{(k+p)^2 + m^2} = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\delta(k^2 + m^2)}{2k \cdot p + p^2} \tag{4.53}$$

With a little abuse of notation, we can denote

$$-\!\!\bigcirc\!\!- = \frac{1}{2k \cdot p + p^2}\bigg|_{D_1=0} \tag{4.54}$$

which is $\frac{1}{D_2 - D_1}$ evaluated on the cut.

∎

4.1.4 BCFW recursion relations

Let't suppose to have a tree level amplitude $i\mathcal{F}(P)$ for a scalar Feynman diagram. For each of the t propagators, we have a momentum current V_i^μ, and the amplitude is:

$$i\mathcal{F}(P) = \frac{1}{V_1^2 - m_1^2} \cdots \frac{1}{V_t^2 - m_t^2} \tag{4.55}$$

Now, suppose to have an analytic complex function $f(z)$ without poles at $|z| \to \infty$. We know for the *theorem of residues*, the integral

$$\lim_{r \to \infty} \oint_{C_r} f(z)dz = (2\pi i) \sum_{j=1}^{k} \mathrm{Res}_{z_j}(f) = 0 \tag{4.56}$$

Choosing

$$f(z) = \frac{1}{z(z - z_1) \cdots (z - z_t)} \tag{4.57}$$

and applying the theorem of residues:

$$
\begin{aligned}
\frac{(-1)^t}{z z_1 z_2 \cdots z_t} &= \frac{1}{z_1(z_2 - z_1) \cdots (z_t - z_1)} + \\
&+ \frac{1}{(z_1 - z_2) z_2 \cdots (z_t - z_2)} + \\
&+ \cdots \cdots \cdots + \\
&+ \frac{1}{(z_1 - z_t)(z_2 - z_t) \cdots (z_{t-1} - z_t) z_t}
\end{aligned}
\tag{4.58}
$$

The first observation is: if we succeed to write our amplitude in a form close to $f(z)$, we reach a powerful decomposition.

The way to transform $i\mathcal{F}(P)$ in a sort of $f(z)$ is to shift the momentum currents with a momentum η such that $\eta^2 = 0$ (light-like) with coefficient a complex number z_i:

$$V_i^2 - m_i^2 \rightarrow (V_i - z_i \eta)^2 - m_i^2 \tag{4.59}$$

Now, through the imposition of the on-shell condition, we can cut the shifted propagator:

$$(V_i - z_i \eta)^2 - m_i^2 = 0 \implies z_i = \frac{V_i^2 - m_i^2}{2 V_i \cdot \eta} \tag{4.60}$$

And for $i \neq j$:

$$(V_i - z_j \eta)^2 - m_i^2 = 2\eta \cdot V_i (z_i - z_j) \tag{4.61}$$

So, with these relations, we can replace (4.60) and (4.61) in (4.58):

$$
\begin{aligned}
\frac{(-1)^t}{(V_1^2 - m_1^2) \cdots (V_t^2 - m_t^2)} &= \frac{1}{(V_1^2 - m_1^2)[(V_2 - z_1 \eta)^2 - m_1^2] \cdots [(V_t - z_1 \eta)^2 - m_t^2]} + \\
&+ \frac{1}{[(V_1 - z_2 \eta)^2 - m_1^2](V_2^2 - m_2^2) \cdots [(V_t - z_2 \eta)^2 - m_t^2]} + \\
&+ \cdots \cdots \cdots + \\
&+ \frac{1}{[(V_1 - z_{t-1} \eta)^2 - m_1^2] \cdots (V_{t-1}^2 - m_{t-1}^2)[(V_t - z_{t-1} \eta)^2 - m_t^2]} \\
&+ \frac{1}{[(V_1 - z_t \eta)^2 - m_1^2] \cdots [(V_{t-1} - z_t \eta)^2 - m_{t-1}^2](V_t^2 - m_t^2)}
\end{aligned}
\tag{4.62}
$$

It is nothing but the partial fractioning decomposition[33]. This method is called *BFCW recursion relation*[13] (Figure 4.1).

Summarizing, at the tree-level the unitarity brought by the unitary cuts involve the partial fractioning of a rational function with t factors in the denominator.

$$\tag{4.63}$$

Figure 4.1: BFCW recurrence relation at tree-level. Every n-amplitude $\mathcal{F}$ is a sum of products of two amplitude with lower external legs. Hatted momenta are complex valued and on-shell.

4.2   Partial fractioning

Suppose we have an integral of a rational function:

$$I = \int \frac{1}{D_1(x) D_2(x)} dx \tag{4.64}$$

It is known that the integrand can be expanded as

$$\frac{1}{D_1 D_2} = \frac{A}{D_1} + \frac{B}{D_2} \tag{4.65}$$

for A and B coefficients. By imposing $A + B = 0$:

$$\frac{1}{D_1 D_2} = \frac{1}{D_1(D_2 - D_1)} + \frac{1}{(D_1 - D_2)D_2} \tag{4.66}$$

This method is often used to evaluate some complex integrals with products of several denominators.

For the rational function above, partial fractioning is performing in a simple way. Generally, for a generic number of denominators, partial fractioning decomposition says:

$$\begin{aligned}
\frac{1}{D_1 D_2 \cdots D_{t-1} D_t} &= \frac{1}{D_1(D_2 - D_1) \cdots (D_{t-1} - D_1)(D_t - D_1)} + \\
&+ \frac{1}{(D_1 - D_2)D_2 \cdots (D_{t-1} - D_2)(D_t - D_2)} + \\
&+ \cdots \cdots + \\
&+ \frac{1}{(D_1 - D_{t-1})(D_2 - D_{t-1}) \cdots D_{t-1}(D_t - D_{t-1})} + \\
&+ \frac{1}{(D_1 - D_t)(D_2 - D_t) \cdots (D_{t-1} - D_t)D_t}
\end{aligned} \tag{4.67}$$

We can make a further generalization of the previous formula. If we accept the presence of a denominator with power a, namely the first, partial fractioning

become:

$$\frac{1}{D_1^a D_2 \cdots D_t} = \sum_{i=0}^{a-1} \sum_{j_2,\ldots,j_t=0}^{j_2+\cdots+j_t=i} \frac{(-1)^i}{D_1^{a-i}(D_2 - D_1)^{1+j_2} \cdots (D_t - D_1)^{1+j_t}} +$$
$$+ \frac{1}{(D_1 - D_2)^a D_2 \cdots (D_t - D_2)} + \tag{4.68}$$
$$+ \cdots\cdots\cdots +$$
$$+ \frac{1}{(D_1 - D_t)^a (D_2 - D_t) \cdots D_t}$$

The first sum decrease the exponent a, the second one distributes i through all the denominators.

Further generalization of (4.68) needed some consideration of combinatorics. We found that, for a generic rational function with t denominators with any powers, partial fractioning become:

$$\frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_t^{a_t}} = \sum_{i=0}^{a_1-1} \sum_{j_2,\ldots,j_t=0}^{j_2+\cdots+j_t=i} \frac{\prod_{k=2}^{t}(-1)^{j_k}\binom{a_k+j_k-1}{a_k-1}}{D_1^{a_1-i}(D_2 - D_1)^{a_2+j_2} \cdots (D_t - D_1)^{a_t+j_t}} +$$

$$+ \sum_{i=0}^{a_2-1} \sum_{j_1,\hat{j}_2,\ldots,j_t=0}^{j_1+\hat{j}_2+\cdots+j_t=i} \frac{\prod_{k=1,k\neq2}^{t}(-1)^{j_k}\binom{a_k+j_k-1}{a_k-1}}{(D_1 - D_2)^{a_1+j_1} D_2^{a_2-i} \cdots (D_t - D_2)^{a_t+j_t}} + \tag{4.69}$$

$$+ \cdots\cdots\cdots +$$

$$+ \sum_{i=0}^{a_t-1} \sum_{j_1,\ldots,j_{t-1}=0}^{j_1+\cdots+j_{t-1}=i} \frac{\prod_{k=1}^{t-1}(-1)^{j_k}\binom{a_k+j_k-1}{a_k-1}}{(D_1 - D_t)^{a_1+j_1}(D_2 - D_t)^{a_2+j_2} \cdots D_t^{a_t-i}}$$

In a more compact way:

$$\frac{1}{D_1^{a_1} D_2^{a_2} \cdots D_t^{a_t}} = \sum_{p=1}^{t} \sum_{i=0}^{a_p-1} \sum_{j_1,\ldots,\hat{j}_p,\ldots,j_t=0}^{j_1+\cdots+\hat{j}_p+\cdots+j_t=i} \frac{\prod_{k=1,k\neq p}^{t}(-1)^{j_k}\binom{a_k+j_k-1}{a_k-1}}{(D_1 - D_p)^{a_1+j_1} \cdots D_p^{a_p-i} \cdots (D_p - D_t)^{a_t+j_t}}$$
$$\tag{4.70}$$

This is the most general partial fractioning formula for a rational function which has arbitrary number of denominators and denominators exponent. It represents one of the most remarkable results of this thesis, which allow us to perform the algorithm we're going to show and extends its validity to any kind of rational integrand.

We know that the integrand of a Feynman integral is a rational function of propagators and ISPs: it clearly recalls the form (4.70), and this is one of the key points of our algorithm.

4.2.1 Partial fractioning for Feynman integrals

Suppose to have a Feynman integral with two denominators, for example the corner integral of the 1-loop bubble topology:

$$I_{1,1}\left(\longrightarrow\!\!\bigcirc\!\!\longrightarrow \right) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m_1^2][(k+p)^2 + m_2^2]} \tag{4.71}$$

We can apply the partial fractioning decomposition:

$$
\frac{1}{[k^2 + m_1^2][(k+p)^2 + m_2^2]} = \frac{1}{[k^2 + m_1^2][(k+p)^2 - k^2 + (m_2^2 - m_1^2)]} +
$$
$$
+ \frac{1}{[k^2 - (k+p)^2 + (m_1^2 - m_2^2)][(k+p)^2 + m_2^2]}
$$
$$
= \frac{1}{[k^2 + m_1^2][2k \cdot p + p^2 + \Delta m^2]} +
$$
$$
+ \frac{1}{[-2k \cdot p - p^2 - \Delta m^2][(k+p)^2 + m_2^2]} \tag{4.72}
$$

where we defined $m_2^2 - m_1^2 = \Delta m^2$. In a more suggestive form:

$$
\frac{1}{[k^2 + m_1^2][(k+p)^2 + m_2^2]} = \frac{1}{[k^2 + m_1^2][p \cdot (2k+p) + \Delta m^2]} +
$$
$$
+ \frac{1}{[-p \cdot (2k+p) - \Delta m^2][(k+p)^2 + m_2^2]} \tag{4.73}
$$

At integral level:

$$
\tag{4.74}
$$

The partial fractioning of bubble diagram has some important features. Let's name $D_1 = k^2 + m_1^2$, $D_1 = (k+p)^2 + m_2^2$ so that $D_2 - D_1 = p \cdot (2k+p) + \Delta m^2$.

- $D_2 - D_1$ is *linear in the loop momentum k*. For this reason, we refer to $D_2 - D_1$ as *linear denominator*;

- the momentum current of $D_2 - D_1$ is a scalar product between two combinations of momenta;

- partial fractions has similarities with generalized unitarity: let's perform single cut to $I_{1,1}(\ -\bigcirc-\ )$ (in Euclidean space):

$$
= \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\delta(D_1)}{D_2} = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\delta(D_1)}{D_2|_{k^2+m_1^2=0}} =
$$
$$
= \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\delta(D_1)}{D_2 - D_1} = \frac{1}{D_2 - D_1} \tag{4.75}
$$

We note that the single cut generates linear propagators, similarly to perform the partial fractioning decomposition.

- Partial fractioning preserves the number of denominators: the *i*-th term of the decomposition can be found by substituing

$$
D_j \to \begin{cases} D_j - D_i, & j > i \\ D_i - D_j, & j < i \\ D_i, & j = i \end{cases} \tag{4.76}
$$

So, this decomposition preserves the structure of its parent topology. Diagrammatically, each graph of the decomposition is drawn in the same way of the original one replace some continuous line with dotted ones:

$$D_j \to \pm(D_j - D_i), \qquad \underline{j} \to \_\_\_\_\_{j}\_\_\_\_\_ \qquad (i \neq j) \qquad (4.77)$$

So, the partial fractioning method is related to unitarity of the scattering amplitudes and it decompose a Feynman integral in a sum of integrals with "linearized" denominators. Partial fractions of 1-loop topologies share linearized progapagators with single cuts, with the difference that this last one doesn't get rid of the information brought by subtopologies.

Example 4.5. [Partial fractioning and single cut at 1-loop] Let's perform the single cut on the 1-loop triangle topology:

$$
\triangleright = \left(\left\{ \begin{matrix} (p_1^\mu, m_1), \\ (p_2^\mu, m_2), \\ (p_3^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k^\mu, k^\mu, m_1^2), \\ (k^\mu + p_1^\mu, k^\mu + p_1^\mu, m_2^2), \\ (k^\mu + p_1^\mu + p_2^\mu, k^\mu + p_1^\mu + p_2^\mu, m_3^2) \end{matrix} \right\} \right)
$$
$$
= \left(E, \left\{ \begin{matrix} \Delta_1, \\ \Delta_2, \\ \Delta_3 \end{matrix} \right\} \right)
\qquad (4.78)
$$

The denominators are

$$
\begin{aligned}
D_1 &= D(\Delta_1) = k^2 + m_1^2 \\
D_2 &= D(\Delta_2) = (k + p_1)^2 + m_2^2 \\
D_3 &= D(\Delta_3) = (k + p_1 + p_2)^2 + m_3^2
\end{aligned}
\qquad (4.79)
$$

The corner integral is

$$
I_{\bar{1}}\left(\triangleright \right) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 D_2 D_3}
\qquad (4.80)
$$

Let's apply the three possible single cuts:

$$
\triangleright = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{\delta(D_1)}{D_2 D_3}
\qquad (4.81)
$$

Acting with $\delta(k^2 + m_1^2)$ on the denominators, we get

$$
\begin{aligned}
D_2|_{D_1=0} &= 2k \cdot p_1 + p_1^2 + m_2^2 - m_1^2 = D_2 - D_1 \\
D_3|_{D_1=0} &= 2k \cdot (p_1 + p_2) + (p_1 + p_2)^2 + m_3^2 - m_1^2 = D_3 - D_1
\end{aligned}
\qquad (4.82)
$$

so:

$$
\triangleright = \frac{1}{(D_2 - D_1)(D_3 - D_1)}\bigg|_{D_1=0}
\qquad (4.83)
$$

Analogous calculations for the other two single cuts bring

$$
\triangleright = \frac{1}{(D_1 - D_2)(D_3 - D_2)}\bigg|_{D_2=0}
$$
$$
\triangleright = \frac{1}{(D_1 - D_3)(D_2 - D_3)}\bigg|_{D_3=0}
\qquad (4.84)
$$

These residues are related to the partial fractioning by the following relation:

$$\frac{1}{D_1 D_2 D_3} = \frac{1}{D_1} \quad + \frac{1}{D_2} \quad + \frac{1}{D_3} \quad \tag{4.85}$$

∎

This example showed that it is possible symbolically express partial fractioning of Feynman integrals in terms of single cuts. It is important to underline that a cutted Feynman integral has a specific loop momentum, fixed by the cut; instead, partial fractioning keeps the loop momentum off-shell, preserving all informations on the starting topology.

Suppose to have a 1-loop corner integral

$$I_{\bar{1}} \left(\right) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_1 \cdots D_t} \tag{4.86}$$

Its partial fractioning can be written as:

$$= \sum_{i=1}^{t} \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{D_i} \tag{4.87}$$

4.2.2   Action of partial fractioning on a 1-loop topology

Partial fractioning induces an injective function of $\mathbb{T}$ in itself, $\mathrm{Pf}^1 : \mathbb{T} \to \mathbb{T}$ such that $\mathrm{Pf}^1(\mathcal{T}) \subset \mathbb{T}$. The index "1" stands for 1-loop.

Our induced function Pf^1 acts in the following way: suppose to have $\mathcal{T} = (E, \Delta)$. It acts on E like the identity function $\mathbb{I}$, and on Δ as the *internal line linearization* $\mathcal{L}_i^1 : \mathfrak{I} \to \mathfrak{I}$, function for which

$$\mathrm{Pf}^1 : (E, \Delta) \mapsto \{(E, \mathcal{L}_i^1(\Delta)), i \in \{1, \ldots, t\}\} \tag{4.88}$$

where the action of $\mathcal{L}_i^1$ on an internal line Δ_j is

$$\mathcal{L}_i^1 : \Delta_j = (V_j^\mu, W_j^\mu, m_j^2) \mapsto \begin{cases} (V_i^\mu + V_j^\mu, W_i^\mu - W_j^\mu, m_i^2 - m_j^2), & \text{for } i > j \\ (V_j^\mu, W_j^\mu, m_j^2), & \text{for } i = j \\ (V_j^\mu + V_i^\mu, W_j^\mu - W_i^\mu, m_j^2 - m_i^2) & \text{for } i < j \end{cases} \tag{4.89}$$

Graphically, $\mathcal{L}_i^1$ acts as

$$\mathcal{L}_i^1(\underset{j}{\underline{\quad\quad\quad}}) = \begin{cases} - - \overset{j}{-} - - & (i \neq j) \\ \underset{j}{\underline{\quad\quad\quad}} & (i = j) \end{cases} \tag{4.90}$$

For a generic 1-loop topology $\mathcal{T}$, $\mathrm{Pf}^1(\mathcal{T})$ generates t partial fractioned topologies $\mathcal{T}_i^l$ for $i \in \{1, \ldots, t\}$. In particular:

$$I_{\bar{1}}(\mathcal{T}_{1-\mathrm{loop}}) = \sum_j I_{\bar{1}}(\mathcal{T}_j^l) \tag{4.91}$$

where $\mathcal{T}_j^l$ is the *linearized topology*.

Example 4.6. (1-loop massive triangle) Consider the topology

$$
\mathcal{T} = \left(\left\{ \begin{array}{c} (p_1^\mu, M_1), \\ (p_2^\mu, M_2), \\ (p_1^\mu + p_2^\mu, \sqrt{s}) \end{array} \right\}, \left\{ \begin{array}{c} (k^\mu, k^\mu, m^2), \\ (k^\mu + p_1^\mu, k^\mu + p_1^\mu, m^2), \\ (k^\mu + p_1^\mu + p_2^\mu, k^\mu + p_1^\mu + p_2^\mu, m^2) \end{array} \right\} \right)
$$

$$(4.92)$$

$$ = \quad \triangleright $$

The partial fractioning for the corner integral:

$$(4.93)$$

$$ I_{1,1}(\mathcal{T}) = I_{1,1}(\mathcal{T}_1^l) + I_{1,1}(\mathcal{T}_2^l) + I_{1,1}(\mathcal{T}_3^l) $$

$\mathcal{T}_i^l = (E, \Delta_i^l)$ represents a new topology; the index i stands for the position of the quadratic propagator in the ordered set Δ_i^l. In particular:

$$
\mathcal{T}_1^l = \left(\left\{ \begin{array}{c} (p_1^\mu, M_1), \\ (p_2^\mu, M_2), \\ (p_1^\mu + p_2^\mu, \sqrt{s}) \end{array} \right\}, \left\{ \begin{array}{c} (k^\mu, k^\mu, m^2), \\ (2k^\mu + p_1^\mu, p_1^\mu, 0), \\ (2k^\mu + p_1^\mu + p_2^\mu, p_1^\mu + p_2^\mu, 0) \end{array} \right\} \right)
$$

$$
\mathcal{T}_2^l = \left(\left\{ \begin{array}{c} (p_1^\mu, M_1), \\ (p_2^\mu, M_2), \\ (p_1^\mu + p_2^\mu, \sqrt{s}) \end{array} \right\}, \left\{ \begin{array}{c} (2k^\mu + p_1^\mu, -p_1^\mu, 0), \\ (k^\mu + p_1^\mu, k^\mu + p_1^\mu, m^2), \\ (2k^\mu + 2p_1^\mu + p_2^\mu, p_2^\mu, 0) \end{array} \right\} \right)
$$

$$(4.94)$$

$$
\mathcal{T}_3^l = \left(\left\{ \begin{array}{c} (p_1^\mu, M_1), \\ (p_2^\mu, M_2), \\ (p_1^\mu + p_2^\mu, \sqrt{s}) \end{array} \right\}, \left\{ \begin{array}{c} (2k^\mu + p_1^\mu + p_2^\mu, -p_1^\mu - p_2^\mu, 0), \\ (2k^\mu + 2p_1^\mu + p_2^\mu, -p_2^\mu, 0), \\ (k^\mu + p_1^\mu + p_2^\mu, k^\mu + p_1^\mu + p_2^\mu, m^2) \end{array} \right\} \right)
$$

So, $\mathrm{Pf}^1(\mathcal{T}) = \{\mathcal{T}_1^l, \mathcal{T}_2^l, \mathcal{T}_3^l\}$. Clearly, Pf it's an injective function. ∎

Each linearized topology is also a topology: it could be possible to build Feynman integrals $I_{\bar{a}}^{\bar{b}}(\mathcal{T}_i^l)$ and to generate IBPs for each topology $\mathcal{T}_i^l$.

4.2.3   Classification of multi-loops topologies

Suppose to have a topology $\mathcal{T}$ built with l-loop momenta $K = \{k_1, \ldots, k_l\}$. Every internal line $\Delta_i \in \Delta$ is made of momentum currents V^μ and W^μ: each momentum current has a dependence on a subset of K.

We group denominators with the same dependence on a specific combination of momenta.

Definition 4.1 (Branch). Let $\mathcal{T} \in \mathbb{T}$ be a topology, $\Delta \subset \mathfrak{I}$ its set of internal lines, $K = \{k_1 \ldots, k_l\}$ a set of l loop momenta, $\mathcal{P}(K)$ the power set of K and $b_r \in \mathcal{P}(K)$. A *branch* $B_{b_r} \subset \Delta$ is a subset of Δ made of internal lines which depend on the specific combination of loop momenta b_r. This means that $B = \{B_{b_r} \mid b_r \in \mathcal{P}(K)\}$ is a *partition* of Δ. The number of branches for an l-loop topology is $N_b = |\mathcal{P}(K) \backslash \emptyset|$.

Branches are a natural way to split internal lines of $\mathcal{T}$, and they allow us to extend the partial fractions decomposition at l-loop Feynman integrals. From the definition of partition, we have

$$\Delta = \bigcup_{r=1}^{N_b} B_{b_r} \tag{4.95}$$

Figure 4.2: Example of branches partition on general 2-loop topologies

In Figure (4.2) we give a graphical interpretation of what exactly branches represent: Figure (A) represents a general structure of a 2-loop topology, which has internal lines $\Delta = \{\Delta_1, \ldots, \Delta_t\}$. The partition B splits Δ in three different branches: $\Delta = \{B_{k_1}, B_{k_2}, B_{k_1 k_2}\}$. A branch is a set of internal lines which depend on the same combination of loop momenta: graphically, it is a line that starts and ends with vertices attached at two or more other internal lines. In Figure (B), we marked in blue, green and red the branches in the general 2-loop topology.

Example 4.7. [2-loop vertex diagram] Clearly, 2-loops means $K = \{k_1^\mu, k_2^\mu\}$ and $\mathcal{P}(K) = \{\{k_1^\mu\}, \{k_2^\mu\}, \{k_1^\mu, k_2^\mu\}\}$. Let's write the 2-loop vertex topology:

$$
\begin{aligned}
&= \left(\left\{ \begin{array}{c} (p_1^\mu, M_1), \\ (p_2^\mu, M_2), \\ (p_1^\mu + p_2^\mu, \sqrt{s}) \end{array} \right\}, \left\{ \begin{array}{c} (k_1^\mu, k_1^\mu, m^2), \\ (k_1^\mu + p_1^\mu, k_1^\mu + p_1^\mu, 0), \\ (k_1^\mu + p_1^\mu + p_2^\mu, k_1^\mu + p_1^\mu + p_2^\mu, m^2), \\ (k_2^\mu, k_2^\mu, m^2), \\ (k_2^\mu - p_1^\mu - p_2^\mu, k_2^\mu - p_1^\mu - p_2^\mu, m^2), \\ (k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0) \end{array} \right\} \right)
\end{aligned} \tag{4.96}
$$

The branch partition for this topology gives:

$$
\begin{aligned}
B_{k_1} &= \left\{ \begin{array}{c} (k_1^\mu, k_1^\mu, m^2), \\ (k_1^\mu + p_1^\mu, k_1^\mu + p_1^\mu, 0), \\ (k_1^\mu + p_1^\mu + p_2^\mu, k_1^\mu + p_1^\mu + p_2^\mu, m^2) \end{array} \right\} \\
B_{k_2} &= \left\{ \begin{array}{c} (k_2^\mu, k_2^\mu, m^2), \\ (k_2^\mu - p_1^\mu - p_2^\mu, k_2^\mu - p_1^\mu - p_2^\mu, m^2) \end{array} \right\} \\
B_{k_1 k_2} &= \{ (k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0) \}
\end{aligned} \tag{4.97}
$$

∎

4.2.4   Action of partial fractioning on l-loop topologies

The definition of branches permits the extention of the partial fractioning at any number of loops. We want to preserve the linearity of partial fractions denominator.

Suppose to have two quadratic denominators depending on two differents loop momenta: $D_1 = k_1^2 + m^2 \;\; D_2 = k_2^2 + m^2$. The linearized denominator $D_2 - D_1$ is

$$D_2 - D_1 = k_2^2 - k_1^2 \tag{4.98}$$

and the 1-loop internal lines linearization gives

$$\mathcal{L}_2^1(k_1^\mu, k_1^\mu, m^2) = (k_2^\mu + k_1^\mu) \cdot (k_2^\mu - k_1^\mu) \tag{4.99}$$

It still be quadratic in the loop momenta k_1^μ and k_2^μ. In order to avoid this unintended dependence, it is appropriate to combine branch partition and partial fractioning: this make us sure that the 1-loop internal lines linearization function $\mathcal{L}^1$ generates linear internal lines.

Proposition 4.2.1. *The partial fractioning decomposition for a generic l-loop topology $\mathcal{T} = (E, \Delta) \in \mathbb{T}$ is performed in this way:*

- *Make the partition B of the set of internal lines Δ of $\mathcal{T}$ (namely split Δ in $N_b = 2^l - 1$ branches);*

- *For every branch $B_{b_r} \in B$, apply $\mathcal{L}_i : B \to \mathfrak{I}$ on every $\Delta_j \in B_{b_r}$ such that*

$$\mathcal{L}_i : (V_j^\mu, W_j^\mu, m_j^2) \mapsto \begin{cases} (V_i^\mu + V_j^\mu, W_i^\mu - W_j^\mu, m_i^2 - m_j^2), & for\ i > j \\ (V_j^\mu, W_j^\mu, m_j^2), & for\ i = j \\ (V_j^\mu + V_i^\mu, W_j^\mu - W_i^\mu, m_j^2 - m_i^2) & for\ i < j \end{cases} \tag{4.100}$$

graphically

$$\mathcal{L}_i^1(\ \underset{}{\overset{j}{\rule{3cm}{0.4pt}}} \ ) = \begin{cases} - - \overset{j}{-\!-\!-} - - & (i \neq j) \\ \underset{}{\overset{j}{\rule{2cm}{0.4pt}}} & (i = j) \end{cases} \tag{4.101}$$

so that for every branch we obtain a set of $|B_{b_r}|$ internal lines $\Delta_{b_r}^l$;

- *Build every set of internal lines by picking only one set $\mathcal{L}_{i_r}(B_{b_r}) \subset \mathfrak{I}$ for each branch B_{b_r}.*

- *The general partial fractioning is a function $\mathrm{Pf} : \mathbb{T} \to \mathbb{T}$ which acts on $\mathcal{T} = (E, \Delta)$ as:*

$$\mathrm{Pf} : \mathcal{T} \mapsto \mathrm{Pf}(\mathcal{T}) = \left\{ \mathcal{T}_{i_1, \ldots, i_{Nb}}^l \mid i_r \in \{1, \ldots, |B_{b_r}|\}, \forall\ 1 < r \leq N_b \right\} \tag{4.102}$$

where

$$\mathcal{T}_{i_1, \ldots, i_b}^l = \left(E, \bigcup_{r=1}^{N_b} \mathcal{L}_{i_r}(B_{b_r}) \right) \tag{4.103}$$

is called linearized topology.

Every Feynman integral built on a linearized topology $\mathcal{T}_{i_1, \ldots, i_b}^l$ is called linearized Feynman integral.

Every l-loop topology generates $N_{\mathcal{L}} = \prod_{r=1}^{N_b} |B_{b_r}|$ linearized topologies. It might seems curious to have defined the linearized topologies like that, but it's simply a conseguence of the factorization brought by every branch.

In a Feynman integral, the integrand is made by the product of denominators belonging to every branch. Partial fractions of the integrand can be found in the following way:

- Pick one quadratic denominator $D_i^{b_r}$ for each branch;

- Generate all linearized internal lines leaving $D_i^{b_r}$ quadratic;

- Make the product of all branches.

Graphically, starting from a Feynman integral, each linearized Feynman integral can be found by selecting an internal line for each branch and dashing all the other lines. In Figure (4.3) the graphical method to build linearized topologies is presented in a 2-loop example.
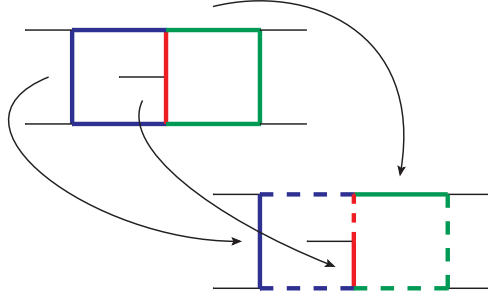

Figure 4.3: 2-loop example of the graphical rule used to extract linearized topologies from a quadratic one.

Example 4.8. [2-loop vertex diagram] Taking a look at the previous example, the topology:

$$\mathcal{T} = \quad \rhd \quad \tag{4.104}$$

has the following branches

$$
\begin{aligned}
B_{k_1} &= \left\{ \begin{array}{c} (k_1^\mu, k_1^\mu, m^2), \\ (k_1^\mu + p_1^\mu, k_1^\mu + p_1^\mu, 0), \\ (k_1^\mu + p_1^\mu + p_2^\mu, k_1^\mu + p_1^\mu + p_2^\mu, m^2) \end{array} \right\} \\
B_{k_2} &= \left\{ \begin{array}{c} (k_2^\mu, k_2^\mu, m^2), \\ (k_2^\mu - p_1^\mu - p_2^\mu, k_2^\mu - p_1^\mu - p_2^\mu, m^2) \end{array} \right\} \\
B_{k_1 k_2} &= \{(k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0)\}
\end{aligned}
\tag{4.105}
$$

Now, we can apply Pf to our topology.

$$
\mathrm{Pf}\left(\rhd \right) = \left\{ \rhd, \rhd, \rhd, \rhd, \rhd, \rhd \right\} = \left\{ \begin{array}{cc} \mathcal{T}_{111}, & \mathcal{T}_{111}, \\ \mathcal{T}_{311}, & \mathcal{T}_{121}, \\ \mathcal{T}_{221}, & \mathcal{T}_{321} \end{array} \right\}
\tag{4.106}
$$

For semplicity, we write only the linearized internal lines:

$$
\mathcal{L}(B) = \left\{
\left\{
\begin{array}{c}
(k_1^\mu, k_1^\mu, m^2), \\
(p_1^\mu, 2k_1^\mu + p_1^\mu, -m^2), \\
(p_1^\mu + p_2^\mu, 2k_1^\mu + p_1^\mu + p_2^\mu, 0), \\
(k_2^\mu, k_2^\mu, m^2), \\
(-p_1^\mu - p_2^\mu, 2k_2^\mu - p_1^\mu - p_2^\mu, 0), \\
(k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0)
\end{array}
\right\},
\left\{
\begin{array}{c}
(-p_1^\mu, 2k_1^\mu, m^2), \\
(k_1^\mu + p_1^\mu, k_1^\mu + p_1^\mu, 0), \\
(p_2^\mu, 2k_1^\mu + 2p_1^\mu + p_2^\mu, m^2), \\
(k_2^\mu, k_2^\mu, m^2), \\
(-p_1^\mu - p_2^\mu, 2k_2^\mu - p_1^\mu - p_2^\mu, 0), \\
(k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0)
\end{array}
\right\},
\right.
$$

$$
\left.
\left\{
\begin{array}{c}
(-p_1^\mu - p_2^\mu, 2k_1^\mu + p_1^\mu + p_2^\mu, 0), \\
(-p_2^\mu, 2k_1^\mu + 2p_1^\mu + p_2^\mu, -m^2), \\
(k_1^\mu + p_1^\mu + p_2^\mu, k_1^\mu + p_1^\mu + p_2^\mu, m^2), \\
(k_2^\mu, k_2^\mu, m^2), \\
(-p_1^\mu - p_2^\mu, 2k_2^\mu - p_1^\mu - p_2^\mu, 0), \\
(k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0)
\end{array}
\right\},
\left\{
\begin{array}{c}
(k_1^\mu, k_1^\mu, m^2), \\
(p_1^\mu, 2k_1^\mu + p_1^\mu, -m^2), \\
(p_1^\mu + p_2^\mu, 2k_1^\mu + p_1^\mu + p_2^\mu, 0), \\
(p_1^\mu + p_2^\mu, 2k_2^\mu - p_1^\mu - p_2^\mu, 0), \\
(k_2^\mu - p_1^\mu - p_2^\mu, k_2^\mu - p_1^\mu - p_2^\mu, 0), \\
(k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0)
\end{array}
\right\},
\right.
$$

$$
\left.
\left\{
\begin{array}{c}
(-p_1^\mu, 2k_1^\mu, m^2), \\
(k_1^\mu + p_1^\mu, k_1^\mu + p_1^\mu, 0), \\
(p_2^\mu, 2k_1^\mu + 2p_1^\mu + p_2^\mu, m^2), \\
(p_1^\mu + p_2^\mu, 2k_2^\mu - p_1^\mu - p_2^\mu, 0), \\
(k_2^\mu - p_1^\mu - p_2^\mu, k_2^\mu - p_1^\mu - p_2^\mu, 0), \\
(k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0)
\end{array}
\right\},
\left\{
\begin{array}{c}
(-p_1^\mu - p_2^\mu, 2k_1^\mu + p_1^\mu + p_2^\mu, 0), \\
(-p_2^\mu, 2k_1^\mu + 2p_1^\mu + p_2^\mu, -m^2), \\
(k_1^\mu + p_1^\mu + p_2^\mu, k_1^\mu + p_1^\mu + p_2^\mu, m^2), \\
(p_1^\mu + p_2^\mu, 2k_2^\mu - p_1^\mu - p_2^\mu, 0), \\
(k_2^\mu - p_1^\mu - p_2^\mu, k_2^\mu - p_1^\mu - p_2^\mu, 0), \\
(k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0)
\end{array}
\right\}
\right\}
$$

$$(4.107)$$

∎

4.2.5 Generalized unitarity and Partial fractioning

We stated before that partial fractioning of 1-loop topologies can be related to single cuts of a Feynman integral. We can generalize this relation at higher loop.

Suppose to have a 2-loop topology $\mathcal{T}$ and the corner integral

$$
I_{\bar{1}}^{\bar{0}}(\mathcal{T}) = \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6 D_7} \tag{4.108}
$$

where we denoted $D(\Delta_1) = D_1$.

Suppose that internal lines of $\mathcal{T}$ can be partitioned as

$$
B_{k_1} = \{\Delta_1, \Delta_2, \Delta_3\}, \quad B_{k_2} = \{\Delta_4, \Delta_5\}, \quad B_{k_1 k_2} = \{\Delta_6, \Delta_7\} \tag{4.109}
$$

so that

$$
I_{\bar{1}}^{\bar{0}}(\mathcal{T}) = \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \left(\frac{1}{D_1 D_2 D_3}\right) \left(\frac{1}{D_4 D_5}\right) \left(\frac{1}{D_6 D_7}\right) \tag{4.110}
$$

Now, we can perform a cut in each branch: performing the triple cut

$$
\begin{aligned}
I_{\bar{1}}^{\bar{0}}(\mathcal{T}) &= \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \left(\frac{\delta(D_1)}{D_2 D_3}\right) \left(\frac{\delta(D_4)}{D_5}\right) \left(\frac{\delta(D_6)}{D_7}\right) \\
&= \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \left(\frac{\delta(D_1)}{(D_2 - D_1)(D_3 - D_1)}\right) \left(\frac{\delta(D_4)}{(D_5 - D_4)}\right) \left(\frac{\delta(D_6)}{(D_7 - D_6)}\right) = \\
&= \frac{1}{(D_2 - D_1)(D_3 - D_1)}\bigg|_{D_1 = 0} \frac{1}{(D_5 - D_4)}\bigg|_{D_4 = 0} \frac{1}{(D_7 - D_6)}\bigg|_{D_6 = 0}
\end{aligned}
$$

$$(4.111)$$

which is the part of a partial fraction with only linearized denominators evaluated on the cut.

Example 4.9. [Partial fractioning and multiple cuts at 2-loop] Consider a 2-loop topology

$$
-\!\!\!\!\diagdiamond\!\!\!\!- = \left(\left\{ \begin{matrix} (p_1^\mu, M_1), \\ (p_2^\mu, M_2), \\ (p_1^\mu + p_2^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} \Delta_1, \\ \Delta_2, \\ \Delta_3, \\ \Delta_4, \\ \Delta_5, \\ \Delta_6 \end{matrix} \right\} \right) \tag{4.112}
$$

The corner integral is

$$
I_{\bar 1}^{\bar 0} \left(-\!\!\!\diagdiamond\!\!\!- \right) = \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6} \tag{4.113}
$$

where $D(\Delta_i) = D_i$.

We can perform a triple cut by putting on-shell one denominator for each branch:

$$
\begin{aligned}
-\!\!\!\diagdiamond\!\!\!- &= \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \frac{\delta(D_2)}{D_1} \frac{\delta(D_3)}{D_4} \frac{\delta(D_6)}{D_5} \\
&= \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \frac{\delta(D_2)}{(D_1 - D_2)} \frac{\delta(D_3)}{(D_4 - D_3)} \frac{\delta(D_6)}{(D_5 - D_6)} = \\
&= \frac{1}{(D_1 - D_2)} \frac{1}{(D_4 - D_3)} \frac{1}{(D_5 - D_6)} \Bigg|_{D_2 = D_3 = D_6 = 0}
\end{aligned} \tag{4.114}
$$

which is a product of linearized denominators evaluated on the cut. So, we can simbolically write a linearized Feynman integral associated at this triple cut:

$$
-\!\!\!\diagdiamond\!\!\!- = \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \frac{1}{D_2 D_3 D_6} \;\; -\!\!\!\diagdiamond\!\!\!- \tag{4.115}
$$

where we intend the cutted diagram in the r.h.s. not even evaluated, leaving the linear dependence from k_1 and k_2 of denominators. Hence, the partial fractioning of $I_{\bar 1}^{\bar 0} \left(-\!\!\!\diagdiamond\!\!\!- \right)$ can be written as

$$
-\!\!\!\diagdiamond\!\!\!- = \sum_{i,j,k}^{3-\text{cuts}} \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \frac{1}{D_i D_j D_k} \;\; -\!\!\!\diagdiamond\!\!\!-_{ijk} \tag{4.116}
$$

∎

Generalizing this feature at general 2-loop Feynman integrals, we arrive at the following identity:

$$
\diagdiamond = \sum_{i,j,k}^{3-\text{cuts}} \int \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}} \frac{1}{D_i D_j D_k} \;\; \diagdiamond_{ijk} \tag{4.117}
$$

4.3 Integration-by-parts identities and Partial fractioning method

In Chapter 2 we saw that every Feynman integral can be decompose in a combination of master integrals which can be found through the generation of a great number of IBPs.

IBP relations are generated by an automatic algorithm, already discussed in Chapter 3. The more complex is the topology under investigation, the more difficult is to generate all IBPs required for the calculation.

A Feynman integral for a topology $\mathcal{T} = (E, \Delta)$ with t internal lines is the integral of a rational function; its denominator is a complete polynomial of degree $2t$ in loop momenta:

$$\mathrm{Deg}_{k_i} \left(\prod_{j=1}^{t} D(\Delta_i) \right) = 2t \tag{4.118}$$

The idea behind this work is the following: partial fractioning splits an integral with denominator of degree $2t$ in a combination of other integrals with denominators $\mathcal{L}_i(B_{b_r}) \ni \Delta_i^l$ of degree:

$$\mathrm{Deg}_{k_i} \left(\prod_{j=1}^{t} D \left(\Delta_i^l \right) \right) = \prod_{r=1}^{N_b} [1 + |B_{b_r}|] \le 2t \tag{4.119}$$

where the equality occurs when each branch has exactly one internal line.

So, the hope is that generating IBP for Feynman integrals with linearized internal lines would be more efficient than the ones with quadratic lines.

It remains to be seen how this decomposition works with IBPs.

4.3.1 New reduction algorithm

Finally, we can present our novel reduction algorithm. Firstly, let's see how this algorithm works in a mathematical way. It is addressed in two stages:

1. Decomposition

We start from a topology $\mathcal{T}$ and a Feynman integral $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$. We can reduce $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ in Feynman integrals:

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \sum_{j=1}^{N_{\mathrm{MI}}} A_j J_{\bar{a}_j}^{\bar{b}_j}(\tau_j), \quad t_j \in \mathcal{T} \cup \mathcal{P}(\mathcal{T}) \tag{4.120}$$

The partial fractioning decomposition on the Feynman integral $I_{\bar{a}}^{\bar{b}}(\mathcal{T})$ acts as:

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \sum_i C_i I_{\bar{a}_i}^{\bar{b}_i}(\mathcal{T}_i^l) \tag{4.121}$$

where $i = \{i_1, \ldots, i_b\}$ is a multi-index such that $1 \le i_r \le |B_{b_r}|$. With this decomposition we get a set of new Feynman integrals $I_{\bar{a}_i}^{\bar{b}_i}(\mathcal{T}_i^l)$. Each of them can be

decomposed in master integrals, through the application of their IBP relation:

$$I_{\bar{a}_i}^{\bar{b}_i}(\mathcal{T}_i^l) = \sum_{k=1}^{N_{MI}^i} A_{i,k}^l J_{\bar{a}_{i,k}}^{\bar{b}_{i,k}}(\tau_{i,k}^l) \tag{4.122}$$

so that, by combining (4.120) and (4.121) we obtain a reduction in terms of master integrals with linearized denominators:

$$I_{\bar{a}}^{\bar{b}}(\mathcal{T}) = \sum_i \sum_{k=1}^{N_{MI}^i} C_i A_{i,k}^l J_{\bar{a}_{i,k}}^{\bar{b}_{i,k}}(\tau_{i,k}^l) \tag{4.123}$$

In a graphical fashion, the decomposition algorithm can be showed at 1-loop: having a Feynman integral, we can decompose it in master integrals

$$\tag{4.124}$$

The same integral is decomposable in partial fractions:

$$\tag{4.125}$$

We can reduce the linearized Feynman integrals:

$$\tag{4.126}$$

Finally, we can express a Feynman integral in term of linearized Feynman inte-

gral.

$$(4.127)$$

2. Recollection

In this stage we have to show that this decomposition doesn't affect the reduction invariate. In other words, we have sum up all the r.h.s. in (4.123) in order to obtain the reduction in (4.120):

$$\sum_{j=1}^{m} A_j J_{\bar{a}_j}^{\bar{b}_j}(\tau_j) = \sum_{i} \sum_{k=1}^{N_{MI}^i} C_i A_{i,k}^l J_{\bar{a}_{i,k}}^{\bar{b}_{i,k}}(\tau_{i,k}^l) \qquad (4.128)$$

This equality shows that master integrals in the r.h.s. of (4.128) are the ones belonging to the partial fractioning of the MIs in l.h.s.

Let's express an MIs in terms of its partial fractioning:

$$J_{\bar{a}_j}^{\bar{b}_j}(\tau_j) = \sum_{r} C'_{j,r} J_{\bar{a}_{j,r}}^{\bar{b}_{j,r}}(\tau_{j,r}^l) \qquad (4.129)$$

where this time r is a multi-index. Then we invert this expression:

$$J_{\bar{a}_{k,j}}^{\bar{b}_{k,j}}(\tau_{k,j}^l) = C'^{-1}_{k,j} \left(J_{\bar{a}_j}^{\bar{b}_j}(\tau_j) - \sum_{r} C'_{k,r} J_{\bar{a}_{k,r}}^{\bar{b}_{k,r}}(\tau_{k,r}^l) \right) \qquad (4.130)$$

putting this information in (4.128) an addend of the l.h.s.:

$$\sum_{j=1}^{N_{MI}} A_j J_{\bar{a}_j}^{\bar{b}_j}(\tau_j) = \sum_{i} \sum_{k=1}^{N_{MI}^i} C_i A_{i,k}^l C'^{-1}_{k,j} J_{\bar{a}_j}^{\bar{b}_j}(\tau_j) - \sum_{i} \sum_{k=1}^{N_{MI}^i} \sum_{r} C_i A_{i,k}^l C'^{-1}_{i,k} C'_{k,r} J_{\bar{a}_{k,r}}^{\bar{b}_{k,r}}(\tau_{k,r}^l)$$
$$(4.131)$$

Looking at this equation, the first term of the r.h.s. has the same master integrals of the l.h.s., so the second term has to be zero:

$$\begin{cases} \sum_i \sum_{k=1}^{N_{MI}^i} \sum_r C_i A_{i,k}^l C'^{-1}_{i,k} C'_{k,r} = 0 \\ \sum_i \sum_{k=1}^{N_{MI}^i} C_i A_{i,k}^l C'^{-1}_{k,j} = \sum_{j=1}^{N_{MI}} A_j \end{cases} \qquad (4.132)$$

Continuing to present the algorithm in our graphical way, with reference to Equation (4.127), we can apply the partial fractioning decomposition to master integrals such that

$$
\square = C'_{11}\,[\text{box}] + C'_{12}\,[\text{box}] + C'_{13}\,[\text{box}] + C'_{14}\,[\text{box}]
$$

$$
\triangleright = C'_{21}\,[\text{triangle}] + C'_{22}\,[\text{triangle}] + C'_{23}\,[\text{triangle}]
\tag{4.133}
$$

$$
[\text{bubble}] = C'_{31}\,[\text{bubble}] + C'_{32}\,[\text{bubble}]
$$

Expressing the linearized Feynman integrals in terms of the quadratic one minus the others:

$$
[\text{box}] = C'^{-1}_{11}\left(\square - C'_{12}\,[\text{box}] - C'_{13}\,[\text{box}] - C'_{14}\,[\text{box}]\right)
$$

$$
[\text{triangle}] = C'^{-1}_{21}\left(\triangleright - C'_{22}\,[\text{triangle}] - C'_{23}\,[\text{triangle}]\right)
\tag{4.134}
$$

$$
[\text{bubble}] = C'^{-1}_{31}\left([\text{bubble}] - C'_{32}\,[\text{bubble}]\right)
$$

Plugging the Equations (4.134) in Equations (4.127), we obtain

$$
[\text{4-point diagram}] =
$$

$$
= C_1 A^l_{1,1} C'^{-1}_{11}\,[\text{box}] + (C_1 A^l_{1,2} C'^{-1}_{21} + C_2 A^l_{2,2} C'^{-1}_{21})\,[\text{triangle}] +
$$

$$
+ (C_1 A^l_{1,3} C'^{-1}_{31} + C_2 A^l_{2,3} C'^{-1}_{31})\,[\text{bubble}] + (C_1 A^l_{1,4} + C_2 A^l_{2,4} + C_3 A^l_{3,4} + C_4 A^l_{4,4})\,[\text{tadpole}] +
$$

$$
+ (C_2 A^l_{2,1} - C_1 A^l_{1,1} C'^{-1}_{11} C'_{12})\,[\text{box}] + (C_3 A^l_{3,1} - C_1 A^l_{1,1} C'^{-1}_{11} C'_{13})\,[\text{box}] +
\tag{4.135}
$$

$$
+ (C_4 A^l_{4,1} - C_1 A^l_{1,1} C'^{-1}_{11} C'_{14})\,[\text{box}] + (C_3 A^l_{3,2} - C_1 A^l_{1,2} C'^{-1}_{21} C'_{22} - C_2 A^l_{2,2} C'^{-1}_{21} C'_{22})\,[\text{triangle}] +
$$

$$
+ (C_4 A^l_{4,2} - C_1 A^l_{1,2} C'^{-1}_{21} C'_{23} - C_2 A^l_{2,2} C'^{-1}_{21} C'_{23})\,[\text{triangle}] +
$$

$$
+ (C_3 A^l_{3,3} + C_4 A^l_{4,3} - C_1 A^l_{1,3} C'^{-1}_{31} C'_{32} - C_2 A^l_{2,3} C'^{-1}_{31} C'_{32})\,[\text{bubble}] +
$$

$$
+ (C_1 A^l_{1,5} + C_2 A^l_{2,5})\,[\text{triangle}] + (C_3 A^l_{3,5} + C_4 A^l_{4,5})\,[\text{bubble}]
$$

Hence, comparing the equation (4.124) and (4.135), we get

$$C_1 A_{1,1}^l C_{11}'^{-1} \quad \boxed{} \quad = A_1 \quad \boxed{}$$

$$(C_1 A_{1,2}^l C_{21}'^{-1} + C_2 A_{2,2}^l C_{21}'^{-1}) \quad \rhd \quad = A_2 \quad \rhd$$

$$(C_1 A_{1,3}^l C_{31}'^{-1} + C_2 A_{2,3}^l C_{31}'^{-1}) \quad -\!\bigcirc\!- \quad = A_3 \quad -\!\bigcirc\!-$$

$$(C_1 A_{1,4}^l + C_2 A_{2,4}^l + C_3 A_{3,4}^l + C_4 A_{4,4}^l) \quad \bigcirc \quad = A_4 \quad \bigcirc$$

$$(C_2 A_{2,1}^l - C_1 A_{1,1}^l C_{11}'^{-1} C_{12}') \quad = 0$$

$$(C_3 A_{3,1}^l - C_1 A_{1,1}^l C_{11}'^{-1} C_{13}') \quad = 0$$

$$(C_4 A_{4,1}^l - C_1 A_{1,1}^l C_{11}'^{-1} C_{14}') \quad = 0 \qquad (4.136)$$

$$(C_3 A_{3,2}^l - C_1 A_{1,2}^l C_{21}'^{-1} C_{22}' - C_2 A_{2,2}^l C_{21}'^{-1} C_{22}') \quad = 0$$

$$(C_4 A_{4,2}^l - C_1 A_{1,2}^l C_{21}'^{-1} C_{23}' - C_2 A_{2,2}^l C_{21}'^{-1} C_{23}') \quad = 0$$

$$(C_3 A_{3,3}^l + C_4 A_{4,3}^l - C_1 A_{1,3}^l C_{31}'^{-1} C_{32}' - C_2 A_{2,3}^l C_{31}'^{-1} C_{32}') \quad -\!\bigcirc\!- \quad = 0$$

$$(C_1 A_{1,5}^l + C_2 A_{2,5}^l) \quad = 0$$

$$(C_3 A_{3,5}^l + C_4 A_{4,5}^l) \quad -\!\bigcirc\!- \quad = 0$$

The equations (4.132) and their graphical interpretation (4.136) are the core of the recollection stage: they state that standard reduction and the reduction through partial fractioning, at mathematical level, give the same results. This is the magic of the reduction through partial fractioning method.

This represents the second remarkable result of this work, which justifies the adoption of partial fractioning decomposition as an alternative reduction approach.

4.3.2   Selection of master integrals

We have a certain freedom of choosing master integrals, so we can select our set of MIs in an affordable way. In general, for multi-loop topologies, we might have more than one MI for the same topology: let $\mathcal{T}$ be a topology and $\mathcal{M}_\mathcal{T}$ a set of master integrals for $\mathcal{T}$. Suppose to have two MIs for the topology $\mathcal{T}$:

$$\{J_{\bar{a}_1}^{\bar{b}_1}(\mathcal{T}), J_{\bar{a}_2}^{\bar{b}_2}(\mathcal{T})\} \subset \mathcal{M}_\mathcal{T} \qquad (4.137)$$

Laporta algorithm chooses master integrals which have the powers $\bar{b}_i$ and such that

$$\bar{b}_i = \bar{0} = \underbrace{\{0, \ldots, 0\}}_{N_{\mathrm{ISP}}}, \quad \bar{a}_1 = \bar{1} = \underbrace{\{1, \ldots, 1\}}_{t}, \quad \bar{a}_2 = \bar{a} \tag{4.138}$$

where $\bar{a}$ has the least degree such that $J_{\bar{a}}^{\bar{0}}(\mathcal{T})$ is a master integral different from $J_{\bar{1}}^{\bar{0}}(\mathcal{T})$.

$$\{J_{\bar{1}}^{\bar{0}}(\mathcal{T}), J_{\bar{a}}^{\bar{0}}(\mathcal{T})\} \subset \mathcal{M}_{\mathcal{T}}^{Lap} \tag{4.139}$$

Decomposing the master integral found by Laporta algorithm in linearized Feynman integrals, we have to use the Equation (4.70):

$$\begin{aligned} C_j(\bar{a}) &= \prod_{k=1, k \neq p}^{t} (-1)^{j_k} \binom{a_k + j_k - 1}{a_k - 1} \\ J_{\bar{a}}^{\bar{0}}(\mathcal{T}) &= \sum_j C_j(\bar{a}) J_{\bar{a}_j}^{\bar{0}}(\mathcal{T}_j^l) \end{aligned} \tag{4.140}$$

where we explicitly write the dependence of C_j from the power of the denominators $\bar{a}$. This choice doesn't fit properly our needs: the decomposition of $J_{\bar{a}}^{\bar{0}}(\mathcal{T})$ gives us non-trivial coefficients $C_j(\bar{a})$, as we can verify by looking at the Eq. (4.70).

Due to the binomial coefficient contained in $C_j(\bar{a})$, we note that

$$C_j(\bar{1}) = \prod_{k=1, k \neq p}^{t} (-1)^{j_k} \binom{j_k}{0} = \prod_{k=1, k \neq p}^{t} (-1)^{j_k} \frac{j_k!}{0! j_k!} = 1 \tag{4.141}$$

where $\sum_k j_k = 0$, so that $j_k = 0$. Moreover, partial fractioning decomposition is not sensitive to arbitrary powers of ISPs (it does not depends on $\bar{b}$).

This means that partial fractioning for Feynman integrals with powers of denominators equal to one, $J_{\bar{1}}^{\bar{b}}(\mathcal{T})$ is a simple sum of linearized Feynman integrals $J_{\bar{1}}^{\bar{b}}(\mathcal{T}_i^l)$, not even a general linear combination:

$$J_{\bar{1}}^{\bar{b}}(\mathcal{T}) = \sum_j J_{\bar{1}}^{\bar{b}_j}(\mathcal{T}_j^l) \tag{4.142}$$

Instead of using Laporta algorithm, from the partial fractions point of view, it is more convenient to choose master integrals with $\bar{a} = \bar{1}$ and $\bar{b} \neq \bar{0}$:

$$\{J_{\bar{1}}^{\bar{0}}(\mathcal{T}_j^l), J_{\bar{1}}^{\bar{b}_j}(\mathcal{T}_j^l)\} \subset \mathcal{M}_{\mathcal{T}}^{PF} \tag{4.143}$$

Finally, our choice of the set of master integrals $\mathcal{M}_{\mathcal{T}}^{\mathrm{PF}}$ is:

$$J_{\bar{1}}^{\bar{b}_j}(\tau_j^l) \in \mathcal{M}_{\mathcal{T}}^{\mathrm{PF}} \tag{4.144}$$

This can be easily extended to MIs belonging to the subtopologies $\mathcal{S}_{\mathcal{T}}$ because of the Proposition (1.1.1).

Example 4.10. [Selection of masters for vertex topology]
    Let us start from 2-loop vertex topology

$$
\text{\reflectbox{$\triangleright$}}\!\!\!= \left(\left\{ \begin{array}{c} (p_1^\mu, m^2), \\ (p_2^\mu, m^2), \\ (p_1^\mu + p_2^\mu, \sqrt{s}) \end{array} \right\}, \left\{ \begin{array}{c} (k_1^\mu, k_1^\mu, m^2), \\ (k_1^\mu + p_1^\mu, k_1^\mu + p_1^\mu, 0), \\ (k_1^\mu + p_1^\mu + p_2^\mu, k_1^\mu + p_1^\mu + p_2^\mu, m^2), \\ (k_2^\mu, k_2^\mu, m^2), \\ (k_2^\mu - p_1^\mu - p_2^\mu, k_2^\mu - p_1^\mu - p_2^\mu, m^2), \\ (k_1^\mu + k_2^\mu, k_1^\mu + k_2^\mu, 0) \end{array} \right\} \right) \tag{4.145}
$$

Using REDUZE we can find the number of MIs for each sector.

```
Integrals appearing on r.h.s. of reductions in the range:
(t2l:2:9): 1
(t2l:3:13): 1
(t2l:3:42): 1
(t2l:3:44): 2
(t2l:4:29): 1
(t2l:4:46): 1
(t2l:4:58): 1
(t2l:5:59): 2
in total: 10
```

Figure 4.4: Output of the REDUZE job `check_for_masters` for the vertex topology

Looking at the Figure (4.4), we note that, for the subtopology with ID=59, we have two master integrals.

$$
\text{\reflectbox{$\triangleright$}} \in \mathcal{S}_{\triangleright} \tag{4.146}
$$

We can choose the master integrals:

$$
\left\{ \quad , \quad \right\} \subset \mathcal{M}_{\triangleright} \tag{4.147}
$$

Denoting $D(\Delta_j) = D_j$ and $[dk] = \frac{d^d k_1}{(2\pi)^{d-2}} \frac{d^d k_2}{(2\pi)^{d-2}}$, we have

$$
\text{\reflectbox{$\triangleright$}} = \int [dk] \frac{1}{D_1 D_2^2 D_4 D_5 D_6} \tag{4.148}
$$

We can decompose this integral in partial fractions:

$$
\begin{aligned}
&= \int [dk]\frac{1}{D_1(D_2-D_1)^2 D_4(D_5-D_4)D_6}+ \\
&+ \int [dk]\frac{1}{(D_1-D_2)D_2^2 D_4(D_5-D_4)D_6}- \\
&- \int [dk]\frac{1}{(D_1-D_2)^2 D_2 D_4(D_5-D_4)D_6}+ \\
&+ \int [dk]\frac{1}{D_1(D_2-D_1)^2 (D_4-D_5)D_5 D_6}+ \\
&+ \int [dk]\frac{1}{(D_1-D_2)D_2^2 (D_4-D_5)D_5 D_6}- \\
&- \int [dk]\frac{1}{(D_1-D_2)^2 D_2 (D_4-D_5)D_5 D_6}+
\end{aligned}
\tag{4.149}
$$

The third and the last term of the combination has coefficient -1; in addition, we decomposed this master integral in six linearized master integrals. We can make a better choice of MIs:

$$
{}^{(p_1\cdot k_2)} = \int [dk]\frac{(p_1\cdot k_2)}{D_1 D_2 D_4 D_5 D_6}
\tag{4.150}
$$

The decomposition of this integral is:

$$
\begin{aligned}
{}^{(p_1\cdot k_2)} &= \int [dk]\frac{(p_1\cdot k_2)}{D_1(D_2-D_1)D_4(D_5-D_4)D_6}+ \\
&+ \int [dk]\frac{(p_1\cdot k_2)}{(D_1-D_2)D_2 D_4(D_5-D_4)D_6}+ \\
&+ \int [dk]\frac{(p_1\cdot k_2)}{D_1(D_2-D_1)(D_4-D_5)D_5 D_6}+ \\
&+ \int [dk]\frac{(p_1\cdot k_2)}{(D_1-D_2)D_2 (D_4-D_5)D_5 D_6}
\end{aligned}
\tag{4.151}
$$

This choice of master integral, from the partial fractioning point of view, is cheaper than the Laporta selection.

Then, a possible choice of linearized master integrals could be

$$
\left\{ \quad , \quad {}^{(p_1\cdot k_2)} \right\} \subset \mathcal{M}^{\mathrm{PF}}
\tag{4.152}
$$

from which we can build back the quadratic master integrals. ∎

Example 4.11. [Novel algorithm acting on 1-loop bubble diagram] To show how this new algorithm works, let's consider the usual 1-loop bubble topology:

$$
= \left(\left\{ \begin{aligned} &(p^\mu,\sqrt{s}), \\ &(p^\mu,\sqrt{s}) \end{aligned} \right\}, \left\{ \begin{aligned} &(k^\mu, k^\mu, m^2), \\ &(k^\mu+p^\mu, k^\mu+p^\mu, m^2) \end{aligned} \right\} \right) = \left(E, \left\{ \begin{aligned} &\Delta_1 \\ &\Delta_2 \end{aligned} \right\} \right)
\tag{4.153}
$$

and consider the Feynman integral $I_{\bar{1}}(\quad)$. By generating an IBP (with RE-DUZE, which works in the Minkowski space) for it, we can reduce the dotted bubble

$I_{2,1}($ —◯— $)$:

$$-\!\!\bigcirc\!\!- = -\frac{(d-3)}{p^2-4m^2}\;-\!\!\bigcirc\!\!-\; +\frac{(d-2)}{2m^2(p^2-4m^2)}\;\underline{\bigcirc} \tag{4.154}$$

also we have the following set of master integrals

$$\mathcal{M}_{\mathcal{T}} = \left\{\; -\!\!\bigcirc\!\!-\; ,\; \underline{\bigcirc}\; \right\} \tag{4.155}$$

We are interested in finding this same reduction by applying the partial fractioning method.

Firstly, the linearized topologies of —◯— are:

$$-\!\!\bigcirc\!\!- = \left(\left\{\begin{matrix}(p^\mu,\sqrt{s}),\\(p^\mu,\sqrt{s})\end{matrix}\right\}, \left\{\begin{matrix}(k^\mu,k^\mu,m^2),\\(p^\mu,2k^\mu+p^\mu,0)\end{matrix}\right\}\right)$$
$$-\!\!\bigcirc\!\!- = \left(\left\{\begin{matrix}(p^\mu,\sqrt{s}),\\(p^\mu,\sqrt{s})\end{matrix}\right\}, \left\{\begin{matrix}(-p^\mu,2k^\mu+p^\mu,0),\\(k^\mu+p^\mu,k^\mu+p^\mu,m^2)\end{matrix}\right\}\right) \tag{4.156}$$

The Feynman integral $I_{2,1}($ —◯— $)$ can be decomposed in a combination of integrals built on those linearized topologies. Using (4.68) and naming $D_1 = D(\Delta_1)$ and $D_2 = D(\Delta_2)$:

$$\int \frac{d^dk}{(2\pi)^{d-2}}\frac{1}{D_1^2 D_2} = \int \frac{d^dk}{(2\pi)^{d-2}}\frac{1}{D_1^2(D_2-D_1)} - \int \frac{d^dk}{(2\pi)^{d-2}}\frac{1}{4D_1(D_2-D_1)^2} +$$
$$+ \int \frac{d^dk}{(2\pi)^{d-2}}\frac{1}{(D_1-D_2)^2 D_2} \tag{4.157}$$

$$-\!\!\bigcirc\!\!- = -\!\!\bigcirc\!\!- - -\!\!\bigcirc\!\!- + -\!\!\bigcirc\!\!- \tag{4.158}$$

At this point, we have to reduce $I_{2,1}($ —◯— $)$, $I_{1,2}($ —◯— $)$ and $I_{2,1}($ —◯— $)$. As usual, firstly we generate IBPs for our two linearized topologies; then we have to make the right choice of master integrals $\mathcal{M}_{\,\text{—◯—}}$ and $\mathcal{M}_{\,\text{—◯—}}$; lastly we can reduce those three Feynman integrals. Once making this steps, we obtain

$$-\!\!\bigcirc\!\!- = -\frac{2(d-3)}{p^2-4m^2}\;-\!\!\bigcirc\!\!-\; +\frac{d-2}{2m^2(p^2-4m^2)}\;\underline{\bigcirc} \tag{4.159}$$

$$-\!\!\bigcirc\!\!- = -\frac{d-3}{p^2-4m^2}\;-\!\!\bigcirc\!\!-\; +\frac{d-2}{p^2(p^2-4m^2)}\;\underline{\bigcirc} \tag{4.160}$$

$$-\!\!\bigcirc\!\!- = -\frac{d-3}{p^2-4m^2}\;-\!\!\bigcirc\!\!-\; +\frac{d-2}{p^2(p^2-4m^2)}\;\overline{\underline{\bigcirc}} \tag{4.161}$$

It's easy to see that with a simple shift of the integration momentum we get

$$\overline{\bigcirc} = \underline{\bigcirc} \tag{4.162}$$

and by the combination (4.159)-(4.160)+(4.161) we find:

$$
= -\frac{d-3}{p^2-4m^2}\left(\qquad + \qquad \right) +   \tag{4.163}
$$

$$
+ \frac{d-2}{2m^2(p^2-4m^2)}
$$

The l.h.s. is clearly the partial fractioning of $I_{2,1}(\,\,)$. In the r.h.s. we can see that there's a tadpole $I_1(\,\,)$ with quadratic denominator, and two integrals which might be part of the partial fractioning of a bubble:

$$
= \qquad + \qquad   \tag{4.164}
$$

Let's invert this relation and plug it in the combination (4.163):

$$
= \qquad - \qquad   \tag{4.165}
$$

so

$$
= -\frac{d-3}{p^2-4m^2} \qquad + \frac{d-2}{2m^2(p^2-4m^2)} \qquad   \tag{4.166}
$$

which is the same reduction we would have got in the standard reduction method. ∎

It is necessary an observation: in the previous examples, $\qquad$ and $\qquad$ are not independent topologies: doing the change of variable $g(k) = -k - p$,

$$
\sim   \tag{4.167}
$$

which implies

$$
\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2-m^2](p^2+2k\cdot p)} = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{(-p^2-2k\cdot p)[(k+p)^2-m^2]}
$$

$$
= \tag{4.168}
$$

This means that the partial fractioning methods for $I_{\bar{1}}(\,\,)$ gives us

$$
= 2 \qquad   \tag{4.169}
$$

This is a very curious equality: it tells us that the Feynman diagram in the l.h.s. is equal to one of its partial fractioned integrals (module a coefficient).

Moreover, they satisfies the same differential equation:

$$\frac{\partial}{\partial p^2} \ \ \bigcirc \ \ = \frac{(d-4)p^2 + 4m^2}{2p^2(p^2 - 4m^2)} \ \ \bigcirc \ \ - \frac{d-2}{p^2(p^2 - 4m^2)} \ \ \bigcirc \tag{4.170}$$

and

$$\frac{\partial}{\partial p^2} \ \ \bigcirc \ \ = \frac{(d-4)p^2 + 4m^2}{2p^2(p^2 - 4m^2)} \ \ \bigcirc \ \ - \frac{d-2}{2p^2(p^2 - 4m^2)} \ \ \bigcirc \tag{4.171}$$

4.3.3   Crossing symmetries for linearized topologies

In general, we can easily see that partial fractioned topologies could be related by *crossing symmetry* as well as change of variable: in order to how crossing symmetries acts on Feynman integrals, we have to state what exactly we means with crossing symmetry.
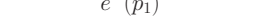
Definition 4.2 (Crossing symmetries). *Crossing symmetries* are a set of relations among scattering amplitudes. Let $\mathcal{F}(\phi_1(p_1), \ldots, \phi_i(p_i) \to \phi_{i+1}(p_{i+1}), \ldots, \phi_{n+1}(p_{n+1}))$ be a scattering amplitude for an $i \to n - i$ process, where we have denoted the particle with momenta p_j as $\phi_j(p_j)$. We can switch an incoming particle $\phi_j(p_j)$ with an outgoing anti-particle $\bar{\phi}_j(-p_j)$ with opposite momenta:

$$\mathcal{F}(\phi(p_1)\phi(p_2) \to \phi(p_3)\phi(p_4)) = \mathcal{F}(\phi(p_2) \to \bar{\phi}(-p_1)\phi(p_3)\phi(p_4)) \tag{4.172}$$
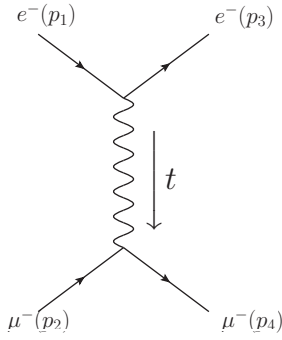
This statement has a physical meaning: anti-particles with momenta p_j may be interpreted as particles moving backward in time.

This symmetry has some important consequences: for examples, two Feynman diagrams (representing two different scattering amplitudes) which are equal module a rotation, are related by a crossing symmetry. Moreover, they share the same functional dependence from their respectively kinematics variables. We can see this better by looking at the following example.

Example 4.12. [Tree-pair production vs. tree-elastic scattering in QED] Let's focus our attention on two QED scatterings:

$$i\mathcal{F}_s(e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)) = \tag{4.173}$$

$$i\mathcal{F}_t(e^-(p_1)\mu^+(p_2) \to e^-(p_3)\mu^+(p_4)) = \qquad (4.174)$$

Using the Feynman rules of QED, we get

$$i\mathcal{F}_s(e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)) = i\frac{e^2}{s}\bar{v}(p_2)\gamma^\mu u(p_1)\bar{u}(p_3)\gamma_\mu v(p_4) \qquad (4.175)$$

$$i\mathcal{F}_t(e^-(p_1)\mu^+(p_2) \to e^-(p_3)\mu^+(p_4)) = i\frac{e^2}{t}\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2) \qquad (4.176)$$

It is clear that:

$$\begin{cases} p_2 \leftrightarrow -p_3 \\ \bar{u}(p_3) \leftrightarrow \bar{v}(p_2) \\ s \leftrightarrow t \end{cases} \implies i\mathcal{F}_s \leftrightarrow i\mathcal{F}_t \qquad (4.177)$$

If we evaluate the unpolarizated square amplitude, we can also write the functional dependence of our scattering amplitudes as a function F such that:

$$\sum_{\text{pol}} |\mathcal{F}_s|^2 = F(p_1, p_2, p_3, p_4), \quad \sum_{\text{pol}} |\mathcal{F}_t|^2 = F(p_1, -p_3, -p_2, p_4) \qquad (4.178)$$
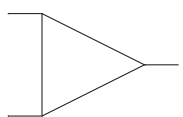
Lastly, we can show this by applying the definition of crossing symmetry on $i\mathcal{F}_s$ and $i\mathcal{F}_t$:

$$i\mathcal{F}_s(e^-(p_1)e^+(p_2) \to \mu^-(p_3)\mu^+(p_4)) = i\mathcal{F}_s(e^-(p_1)\mu^+(-p_3) \to e^-(-p_2)\mu^+(p_4)) \quad (4.179)$$
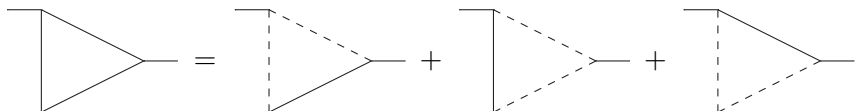
and, renaming $p_2 \leftrightarrow -p_3$ we relate the two scattering amplitudes. ∎

Crossing symmetries can be extended to Feynman integrals, and they insert some equivalence relations between Feynman integrals: in some special cases this relations could be "degenere", in the sense that the equivalence relation might become a true equality.

Example 4.13. [1-loop triangle topology] Let's take a look to the topology

$$\mathcal{T} = \left(\left\{ \begin{matrix} (p_1^\mu, M_1), \\ (p_2^\mu, M_2), \\ (-p_1^\mu - p_2^\mu, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k^\mu, k^\mu, m^2), \\ (k^\mu + p_1^\mu, k^\mu + p_1^\mu, m^2), \\ (k^\mu + p_1^\mu + p_2^\mu, k^\mu + p_1^\mu + p_2^\mu, m^2) \end{matrix} \right\} \right)$$

$$(4.180)$$

$$= \qquad$$

The partial fractioning of $I_{\bar{1}}(\mathcal{T})$ gives us

$$(4.181)$$

and the action of Pf on $\mathcal{T}$

$$
\text{Pf}(\mathcal{T}) = \left\{
\begin{array}{l}
\mathcal{T}_1^l = \left(\left\{
\begin{array}{c}
(p_1^\mu, M_1), \\
(p_2^\mu, M_2), \\
(-p_1^\mu - p_2^\mu, \sqrt{s})
\end{array}
\right\}, \left\{
\begin{array}{c}
(k^\mu, k^\mu, m^2), \\
(2k^\mu + p_1^\mu, p_1^\mu, 0), \\
(2k^\mu + p_1^\mu + p_2^\mu, p_1^\mu + p_2^\mu, 0)
\end{array}
\right\} \right), \\[18pt]
\mathcal{T}_2^l = \left(\left\{
\begin{array}{c}
(p_1^\mu, M_1), \\
(p_2^\mu, M_2), \\
(-p_1^\mu - p_2^\mu, \sqrt{s})
\end{array}
\right\}, \left\{
\begin{array}{c}
(2k^\mu + p_1^\mu, -p_1^\mu, 0), \\
(k^\mu + p_1^\mu, k^\mu + p_1^\mu, m^2), \\
(2k^\mu + 2p_1^\mu + p_2^\mu, p_2^\mu, 0)
\end{array}
\right\} \right), \\[18pt]
\mathcal{T}_3^l = \left(\left\{
\begin{array}{c}
(p_1^\mu, M_1), \\
(p_2^\mu, M_2), \\
(-p_1^\mu - p_2^\mu, \sqrt{s})
\end{array}
\right\}, \left\{
\begin{array}{c}
(2k^\mu + p_1^\mu + p_2^\mu, -p_1^\mu - p_2^\mu, 0), \\
(2k^\mu + 2p_1^\mu + p_2^\mu, -p_2^\mu, 0), \\
(k^\mu + p_1^\mu + p_2^\mu, k^\mu + p_1^\mu + p_2^\mu, m^2)
\end{array}
\right\} \right)
\end{array}
\right\} \quad (4.182)
$$

The diagrams for $\mathcal{T}_1^l$, $\mathcal{T}_2^l$ and $\mathcal{T}_3^l$ are the same, up to a rotation. This fact suggest that they may be related by a crossing symmetry. Now, let's focus our attention on $I_{\bar{1}}(\mathcal{T}_1^l)$ and $I_{\bar{1}}(\mathcal{T}_2^l)$:

$$
I_{\bar{1}}(\mathcal{T}_1^l) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[k^2 + m^2][(k+p_1)^2 + m^2][(k+p_1+p_2)^2 + m^2]} \quad (4.183)
$$

$$
I_{\bar{1}}(\mathcal{T}_2^l) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[-p_1 \cdot (2k+p_1) + m^2][(k+p_1)^2 + m^2][p_2 \cdot (2k+2p_1+p_2)^2 + m^2]} \quad (4.184)
$$
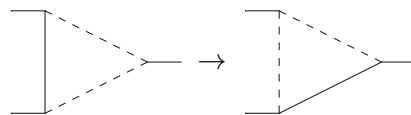
Let's perform a shift $k \to k - p_1$ on the integral $I_{\bar{1}}(\mathcal{T}_2^l)$:

$$
I_{\bar{1}}(\mathcal{T}_2^l) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[-p_1 \cdot (2k-p_1) + m^2][k^2 + m^2][p_2 \cdot (2k+p_2)^2 + m^2]} \quad (4.185)
$$

Lastly, if we switch the external lines in the following way
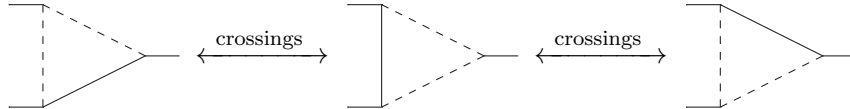
$$
\begin{cases}
(p_1^\mu, M_1) & \to (-p_1^\mu - p_2^\mu, \sqrt{s}) \\
(p_2^\mu, M_2) & \to (p_1^\mu, M_1) \\
(-p_1^\mu - p_2^\mu, \sqrt{s}) & \to (p_2^\mu, M_2)
\end{cases} \quad (4.186)
$$

we find the Feynman integral for $I_{\bar{1}}(\mathcal{T}_1^l)$

$$
I_{\bar{1}}(\mathcal{T}_2^l) \to I_{\bar{1}}(\mathcal{T}_1^l)
$$

$$(4.187)$$

Clearly, this trasformation among integrals is a conseguence of a relation at the level of topologies, such that

$$
\mathcal{T}_1^l \xleftrightarrow{\text{crossings}} \mathcal{T}_2^l \xleftrightarrow{\text{crossings}} \mathcal{T}_3^l \quad (4.188)
$$

$$(4.189)$$

For 1-loop topologies, crossing symmetries can relate all their linearized topologies: he decomposition of a 1-loop topology gives us a set of linearized topologies which have only one quadratic denominator. Because of this, there will always be crossings which relates all topologies which and allow us to express all 1-loop linearized topologies as one of them module crossings.

In bubble topologies, crossings are trivial: they have one only independent external leg; in this case, general crossings is demoted to change of integration variables.
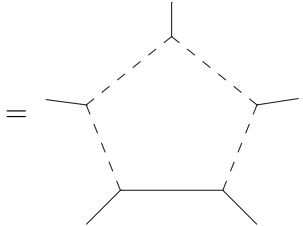
4.3.4   Zero sectors of linearized topologies

In the last section we saw that linearized topologies have a lot of non-trivial relations: some of them can simplify the IBPs generation for each topology.

Zero sectors of 1-loop linearized topologies

We know that it's not necessary to generate IBPs in order to find subsectors for which Feynman integrals are zero. At this point we are ready to use REDUZE on some test topologies:

Example 4.14. [1-loop pentagon linearized topology] Let's find the trivial subtopologies of the following topology:

$$
\mathcal{T}_1^l = \left(\left\{ \begin{matrix} (p_1^\mu, M_1) \\ (p_2^\mu, M_2) \\ (p_3^\mu, M_3) \\ (p_4^\mu, M_4) \\ (p_5^\mu, M_5) \end{matrix} \right\}, \left\{ \begin{matrix} (k^\mu, k^\mu, m_1^2) \\ (2k^\mu + p_1^\mu, p_1^\mu, m_2^2) \\ (2k^\mu + p_1 + p_2^\mu, p_1^\mu + p_2^\mu, m_3^2) \\ (2k^\mu + p_1 + p_2 + p_3^\mu, p_1^\mu + p_2^\mu + p_3^\mu, m_4^2) \\ (2k^\mu + p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu, p_1^\mu + p_2^\mu + p_3^\mu + p_4^\mu, m_5^2) \end{matrix} \right\} \right)
$$

$$(4.190)$$

This is a topology which can be found by applying the partial fractioning function Pf to the topology $\mathcal{T}$.

In Figure (4.5) are listed the IDs of the zero sectors (trivial topologies):

$t = 1$ For one denominator topologies, the only non zero sector is the one with the quadratic denominator, which is a massive quadratic tadpole. All the other are linearized tadpole, and they are trivial subtopologies.

$t = 2$ For bubble topologies, we can observe that every ID of the zero sectors is even: this means that ID $= 2^i + 2^j$ with $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$. Again, all of the zero bubbles are the ones without the quadratic denominator.

$t = 3$ Again, for triangle topologies, IDs for zero sector satisfies the following relation: ID $= 2^1 + 2^j + 2^k$ with $i, j, k \in \{1, 2, 3, 4\}$ and $i \neq j \neq k$. So, all the zero sectors are the one without the quadratic denominator.

$t = 4$ Lastly, the only zero box is the one with ID $= 30 = 2^1 + 2^2 + 2^3 + 2^4$, one more time without the quadratic denominator
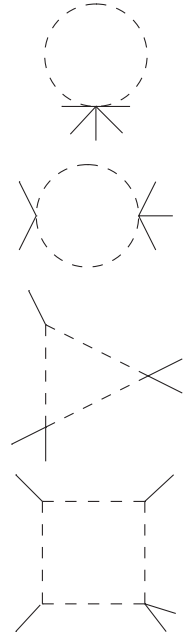
The 1-loop example above shows us that fact that zero sector of linearized 1-loop topologies doesn't have quadratic denominator. The same result can be found by

```
sectormappings:
  name: penta_1
  zero_sectors:
    t=0: [0]
    t=1: [2, 4, 8, 16]
    t=2: [6, 10, 12, 18, 20, 24]
    t=3: [14, 22, 26, 28]
    t=4: [30]
  sectors_without_graph:
    t=1: [2, 4, 8, 16]
    t=2: [3, 5, 6, 9, 10, 12, 17, 18, 20, 24]
    t=3: [7, 11, 13, 14, 19, 21, 22, 25, 26, 28]
    t=4: [15, 23, 27, 29, 30]
    t=5: [31]
  sector_relations:
    {}
  crossed_sector_relations:
    {}
  sector_symmetries:
    {}
```

Figure 4.5: Output of `setup_sector_mappings`

running the zero sectors of other 1-loop topologies, like the hexagon topology. This fact can be generalized for any kind of 1-loop linearized topology.

Proposition 4.3.1 (Zero sectors for 1-loop linearized topologies). *Let $\mathcal{T}$ a 1-loop topology, $\mathcal{T}_i^1$ a linearized topology of $\mathcal{T}$ and $\mathcal{Z}_i^1 \subset \mathcal{S}_{\mathcal{T}}$ the set of subtopologies of $\mathcal{T}$ without the quadratic internal line. Then*

$$I_{\bar{a}}(\zeta) = 0 \tag{4.191}$$

for every $\zeta \in \mathcal{Z}_i^1$ and $\bar{a} \in \mathbb{N}^t$.

Proof. Let's consider a subtopology $\zeta \in \mathcal{Z}_i$ with t internal lines. The general form of its internal line is

$$\Delta_j = (u_j^\mu, \alpha_j'' k^\mu + w_j^\mu, m_j^2) \in \mathfrak{I} \tag{4.192}$$

where u_j^μ and w_j^μ are combination of external momenta P and α_j' a numerical constant. The action of the function *denominator* D on Δ_j is:

$$D(\Delta_j) = \alpha_j'' u_j \cdot k + u_j \cdot w_j + m_j^2 = \alpha_j'(P) \cdot k + \beta_j'(P, \bar{m}^2) \tag{4.193}$$

where we renamed the coefficient in order to explicit the linearity of the denominator in k and $\bar{m}^2 = \{m_1^2, \ldots, m_t^2\}$.
    A general Feynman integral on ζ takes the form

$$I_{\bar{a}}(\zeta) = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{\prod_{j=1}^t D^{a_j}(\Delta_j)} \tag{4.194}$$

We can use the Feynman parameter representation:

$$I_{\bar{a}}(\zeta) = C \int_0^1 \left[\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{\left\{ \sum_{j=1}^t [\alpha_j'(P) \cdot k + \beta_j'(P, \bar{m}^2)] x_j \right\}^a} \right] \prod_{i=1}^t x^{a_i-1} \delta \left(1 - \sum_{j=1}^t x_j \right) dx_i$$

$$\text{(4.195)}$$

with $a = \sum_{i=1}^t a_i$. It is clear that, in order to show that $I_{\bar{a}}(\zeta) = 0$, we have to prove that

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{\left[\sum_{j=1}^t (\alpha_j'(P) \cdot k + \beta_j'(P, \bar{m}^2)) x_j \right]^a} = 0 \qquad \text{(4.196)}$$

We can rewrite the denominator in a simpler way: by naming $\bar{x} = \{x_1, \ldots, x_t\}$:

$$\sum_{j=1}^t [\alpha_j'(P) \cdot k + \beta_j'(P, \bar{m}^2)] x_j = \alpha(P, \bar{x}) \cdot k + \beta(P, \bar{m}^2, \bar{x}) \qquad \text{(4.197)}$$

and we get

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{\left[\alpha(P, \bar{x}) \cdot k + \beta(P, \bar{m}^2, \bar{x}) \right]^a} \qquad \text{(4.198)}$$

Due to the form of the denominator, we can choose an arbitrary component of the loop momenta (we will choose k_1) and perform the shift $k_1 \to k_1 - \frac{\beta(P, \bar{x})}{\alpha_1(P, \bar{m}^2, \bar{x})}$: so

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{\left[\alpha(P, \bar{x}) \cdot k + \beta(P, \bar{m}^2, \bar{x}) \right]^a} = \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{\left[\alpha(P, \bar{x}) \cdot k \right]^a} \qquad \text{(4.199)}$$

Let's perform IBPs on this integral:

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{\partial}{\partial k^\mu} \left\{ \frac{p_j^\mu}{\left[\alpha(P, \bar{x}) \cdot k \right]^a} \right\} = -a \int \frac{d^d k}{(2\pi)^{d-2}} \frac{p_j \cdot \alpha(P, \bar{x})}{\left[\alpha(P, \bar{x}) \cdot k \right]^{a+1}} = 0 \qquad \text{(4.200)}$$

and

$$\int \frac{d^d k}{(2\pi)^{d-2}} \frac{\partial}{\partial k^\mu} \left\{ \frac{k^\mu}{\left[\alpha(P, \bar{x}) \cdot k \right]^a} \right\} = \int \frac{d^d k}{(2\pi)^{d-2}} \left\{ \frac{d}{\left[\alpha(P, \bar{x}) \cdot k \right]^a} - a \frac{\alpha(P, \bar{x}) \cdot k}{\left[\alpha(P, \bar{x}) \cdot k \right]^{a+1}} \right\}$$

$$= \int \frac{d^d k}{(2\pi)^{d-2}} \left\{ \frac{d}{\left[\alpha(P, \bar{x}) \cdot k \right]^a} - \frac{a}{\left[\alpha(P, \bar{x}) \cdot k \right]^a} \right\} = 0$$

$$\text{(4.201)}$$

So that, the IBPs show that

$$\begin{cases} - \int \frac{d^d k}{(2\pi)^{d-2}} \frac{a}{[\alpha(P, \bar{x}) \cdot k]^{a+1}} = 0 \\ \int \frac{d^d k}{(2\pi)^{d-2}} \frac{d-a}{[\alpha(P, \bar{x}) \cdot k]^a} = 0 \end{cases} \implies \int \frac{d^d k}{(2\pi)^{d-2}} \frac{1}{[\alpha(P, \bar{x}) \cdot k]^a} = 0 \qquad \text{(4.202)}$$

So we proved that

$$I_{\bar{a}}(\zeta) = 0, \quad \forall \, \zeta \in \mathcal{Z}_i^1 \qquad \text{(4.203)}$$

or graphically:

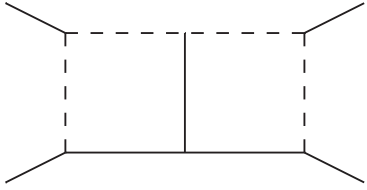$$\qquad \qquad = 0 \qquad \text{(4.204)}$$

$\square$

So, we have proved that each 1-loop linearized topology made only of linear internal line belongs to the zero sector.

Zero sectors of l-loop linearized topologies

At this point, we are ready to see what happens in l-loop case.

Example 4.15. [2-loop box topology] Let's find the trivial subtopologies of the following linearized vertex topology:

$$
\mathcal{T} = \left(\left\{ \begin{matrix} (p_1^\mu, M_1), \\ (p_2^\mu, M_2), \\ (p_3^\mu, M_3), \\ (p_1^\mu + p_2^\mu + p_3^\mu, M_4) \end{matrix} \right\} , \left\{ \begin{matrix} (k_1^\mu, k_1^\mu, m_1^2) \\ (p_1^\mu, 2k_1^\mu + p_1^\mu, m_2^2) \\ (p_1^\mu + p_2^\mu, 2k_1^\mu + p_1^\mu + p_2^\mu, m_3^2) \\ (k_2^\mu, k_2^\mu, m_4^2) \\ (p_1^\mu + p_2^\mu, 2k_2^\mu + p_1^\mu + p_2^\mu, m_5^2) \\ (p_1^\mu + p_2^\mu + p_3^\mu, 2k_2^\mu + p_1^\mu + p_2^\mu + p_3^\mu, m_6^2) \\ (k_2^\mu - k_1^\mu, k_2^\mu - k_1^\mu, m_7^2) \end{matrix} \right\} \right) \tag{4.205}
$$

such that $t = t$. As before, we have ordered the list of internal lines from 1 to t. In this list, linear propagators have position 2,3,5,6.

By running Reduze, we get the zero sectors showed in Figure (4.6).

Again, looking at the list of the zero sector we can observe the following facts:

$t = 1$ This is a trivial case: every $l - 1$-loop subtopology of a l-loop topology are zero sectors.
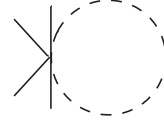
$t = 2$ The only a priori not-zero subtopologies have ID= $\{18, 34, 20, 36\}$: looking at the Figure (4.6), we can see that the topologies listed here are the zero sectors.
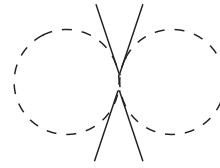
$t = 3$ Three-lines topologies containing only linear internal lines have ID= $\{22, 38, 50, 52\}$: again, we can find these ID Figure (4.6).
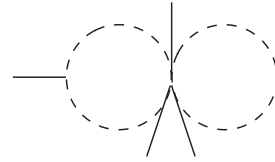
$t = 4$ The only four-linear lines subtopology has ID= 54. This last sector is a zero sector for the topology $\mathcal{T}$.   ∎

This is related to the fact that, by picking only the linearized internal lines, we get a *factorized* topology, and the factorization gives us a product of two 1-loop topologies: due to the Proposition (4.3.1), this is a trivial subtopology.

```
sectormappings:
  name: box2loop_1_1_1
  zero_sectors:
    t=0: [0]
    t=1: [1, 2, 4, 8, 16, 32, 64, 128, 256]
    t=2: [3, 5, 6, 10, 12, 17, 18, 20, 24, 33, 34,
36, 40, 48, 66, 68, 80, 96, 129, 130, 132, 136,
144, 160, 192, 257, 258, 260, 264, 272, 288, 320, 384]
    t=3: [7, 14, 19, 21, 22, 26, 28, 35, 37, 38, 42,
44, 49, 50, 52, 56, 70, 84, 112, 131, 133, 134, 138,
140, 145, 146, 148, 152, 161, 162, 164, 168, 176, 208,
224, 259, 261, 262, 266, 268, 273, 274, 276, 280, 289,
290, 292, 296, 304, 322, 324, 385, 386, 388, 392, 400, 416]
    t=4: [23, 30, 39, 46, 51, 53, 54, 58, 60, 135,
142, 147, 149, 150, 154, 156, 163, 165, 166, 170,
172, 177, 178, 180, 184, 240, 263, 270, 275, 277,
278, 282, 284, 291, 293, 294, 298, 300, 305, 306,
308, 312, 326, 387, 389, 390, 394, 396, 401, 402,
404, 408, 417, 418, 420, 424, 432]
    t=5: [55, 62, 151, 158, 167, 174, 179, 181, 182,
186, 188, 279, 286, 295, 302, 307, 309, 310, 314,
316, 391, 398, 403, 405, 406, 410, 412, 419, 421,
422, 426, 428, 433, 434, 436, 440]
    t=6: [183, 190, 311, 318, 407, 414, 423, 430,
435, 437, 438, 442, 444]
    t=7: [439, 446]
```

Figure 4.6: Output of `setup_sector_mappings`

Corollary 4.3.2 (Zero sectors for l-loop linearized topologies)**.** *Let $\mathcal{T}$ a l-loop topology and $\mathcal{T}_{j_1 \dots j_{N_b}}$ a linearized topology of $\mathcal{T}$. If the subtopology $\mathcal{Z}_{j_1 \dots j_{N_b}}$ built by selecting only the linearized internal lines of $\mathcal{T}_{j_1 \dots j_{N_b}}$ has the following properties;*

1. *$\mathcal{T}_{j_1 \dots j_{N_b}}$ is* factorizable*;*

2. *it has at least one factor represented by 1-loop diagram*

then $\mathcal{Z}_{j_1 \dots j_{N_b}}$ is a trivial subtopology.

Proof. We will give a simple graphic demonstration, by looking at the graphs, instead of proving it analytically by write a generical $\mathcal{Z}_{j_1 \dots j_{N_b}}$. So

$$\mathcal{Z}_{j_1 \dots j_{N_b}} = \quad \begin{matrix} i \searrow \diagup \diagdown \diagup i+1 \\ \left(\quad l-loop \quad \right) \\ 1 \diagup \diagdown \diagdown n \end{matrix} \qquad (4.206)$$

Without loss of generality, it is sufficient by looking at the following factorization:

$$(4.207)$$

For the Proposition (4.3.1) we know that

$$(4.208)$$

So, we have proved that $\mathcal{Z}_{j_1\ldots j_{N_b}}$ is a trivial subsector of $\mathcal{T}_{j_1\ldots j_{N_b}}$. $\square$

It is possible to generalize the Proposition (4.3.1) at every l-loop linearized topology without linearized internal lines is a zero sector.

Proposition 4.3.3 (Zero sectors for l-loop linearized topologies)**.** *Let $\mathcal{T}$ a l-loop topology, $\mathcal{T}_i^l$ a linearized topology of $\mathcal{T}$ and $\mathcal{Z}_i^l \subset \mathcal{S}_{\mathcal{T}_i^l}$ the set of subtopologies of $\mathcal{T}_i^l$ without the quadratic internal line. Then*

$$I_{\bar{a}}^{\bar{b}}(\zeta) = 0 \tag{4.209}$$

for every $\zeta \in \mathcal{Z}_i^l$, $\bar{a} \in \mathbb{N}^t$ and $\bar{b} \in \mathbb{N}^{N_{ISP}}$.

Proof. It is necessary a preliminary consideration: let $\zeta \in \mathbb{T}$ be a topology. It is possible to show that if the IBPs of $I_{\bar{1}}^{\bar{0}}(\zeta)$ states that $I_{\bar{1}}^{\bar{0}}(\zeta) = 0$ (i.e. the corner integral of ζ vanishes), then ζ is a zero sector[23]. So

$$I_{\bar{1}}^{\bar{0}}(\zeta) \overset{\text{IBP}}{=} 0 \implies I_{\bar{b}}^{\bar{a}}(\zeta) = 0 \qquad \forall \bar{a} \in \mathbb{N}^t, \quad \bar{b} \in \mathbb{N}^{N_{\text{ISP}}} \tag{4.210}$$

Due to this fact, we can focus on the corner integral of $\zeta \in \mathcal{Z}_i^l$. Suppose that ζ has t linearized internal lines

$$\Delta_j = \left(u_j^\mu, \alpha_j'' q_j^\mu + w_j^\mu, m_j^2\right) \in \mathcal{I} \tag{4.211}$$

where u_j^μ and w_j^μ are combination of external momenta P, q_j^μ a combination of loop momenta K and α_j'' a numerical constant. Acting with the function *denominator* on Δ_j, we obtain

$$D(\Delta_j) = \alpha_j'' u_j \cdot q_j + u_j \cdot w_j + m_j^2 = \alpha_j'(P) \cdot q_j + \beta_j'(P, \mathbf{m}^2) \tag{4.212}$$

where $\mathbf{m}^2 = \{m_1^2, \ldots, m_t^2\}$.

The corner integral of ζ is

$$I_{\bar{1}}^{\bar{0}}(\zeta) = \int [dk] \frac{1}{\Pi_{j=1}^t D(\Delta_j)} \tag{4.213}$$

where we denote the measure $[dk] = \Pi_{i=1}^t \frac{d^d k_i}{(2\pi)^{d-2}}$.

Using Feynman parametrization on $I_{\bar{1}}^{\bar{0}}(\zeta)$, we obtain

$$I_{\bar{1}}^{\bar{0}}(\zeta) = C \int_0^1 \left[\int [dk] \frac{1}{[\sum_{j=1}^t D(\Delta_j) x_j]^t} \right] \delta \left(1 - \sum_{j=1}^t x_j \right) dx_1 \cdots dx_t \qquad (4.214)$$

where C is a numerical constant. This means that if the integral in $[dk]$ in the r.h.s. vanishes, then $I_{\bar{1}}^{\bar{0}}(\zeta)$ is zero:

$$\int [dk] \frac{1}{[\sum_{j=1}^t D(\Delta_j) x_j]^t} = 0 \implies I_{\bar{1}}^{\bar{0}}(\zeta) = 0 \qquad (4.215)$$

Using the linearity of $D(\Delta_j)$, we can write

$$\sum_{j=1}^t [\alpha_j'(P) \cdot q_j + \beta_j'(P, \mathbf{m}^2)] x_j = \sum_{n=1}^l [\alpha_n(P, \bar{x}) \cdot k_n + \beta_n(P, \mathbf{m}^2, \bar{x})] \qquad (4.216)$$

Moreover, shifting the first component of each loop momenta k_j as

$$k_n^0 \to k_n^0 - \frac{\beta_n(P, \mathbf{m}^2, \bar{x})}{\alpha_{0n}(P, \bar{x})} \qquad (4.217)$$

we get

$$\sum_{n=1}^l [\alpha_n(P, \bar{x}) \cdot k_n + \beta_n(P, \mathbf{m}^2, \bar{x})] \to \sum_{n=1}^l \alpha_n(P, \bar{x}) \cdot k_n \qquad (4.218)$$

so that

$$\int [dk] \frac{1}{[\sum_{n=1}^t D(\Delta_n) x_n]^t} = \int [dk] \frac{1}{[\sum_{n=1}^l \alpha_n(P, \bar{x}) \cdot k_n]^t} \qquad (4.219)$$

Performing IBPs on this integral, and neglecting the dependences of α, we have

$$\int [dk] \frac{\partial}{\partial k_i^\mu} \left[\frac{p_j^\mu}{[\sum_{n=1}^l \alpha_n \cdot k_n]^t} \right] = - \int [dk] \frac{t \alpha_i \cdot p_j}{[\sum_{n=1}^l \alpha_n \cdot k_n]^{t+1}} = 0 \qquad (4.220)$$

and

$$\int [dk] \frac{\partial}{\partial k_i^\mu} \left[\frac{k_j^\mu}{[\sum_{n=1}^l \alpha_n \cdot k_n]^t} \right] = \int [dk] \left(\frac{d \delta_{ij}}{[\sum_{n=1}^l \alpha_n \cdot k_n]^t} - \frac{t \alpha_i \cdot k_j}{[\sum_{n=1}^l \alpha_n \cdot k_n]^{t+1}} \right) = 0 \qquad (4.221)$$

Summarizing our IBPs, we obtained

$$\begin{cases} \int [dk] \frac{1}{[\sum_{n=1}^l \alpha_n \cdot k_n]^{t+1}} = 0, & \\ \int [dk] \frac{\alpha_i \cdot k_j}{[\sum_{n=1}^l \alpha_n \cdot k_n]^{t+1}} = 0, & i \neq j \\ \int [dk] \frac{t \alpha_i \cdot k_i}{[\sum_{n=1}^l \alpha_n \cdot k_n]^{t+1}} = \int [dk] \frac{d}{[\sum_{n=1}^l \alpha_n \cdot k_n]^t}, & i = j \end{cases} \qquad (4.222)$$
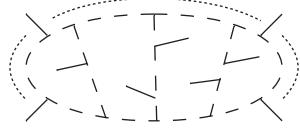
We can sum over the index i the last IBP:

$$\int [dk] \frac{t \sum_{i=1}^l \alpha_i \cdot k_i}{[\sum_{n=1}^l \alpha_n \cdot k_n]^{t+1}} = \int [dk] \frac{ld}{[\sum_{n=1}^l \alpha_n \cdot k_n]^t} \qquad (4.223)$$

and simplifying the integrand in the l.h.s.:

$$\int [dk] \frac{t - ld}{[\sum_{n=1}^{l} \alpha_n \cdot k_n]^t} = 0 \implies \int [dk] \frac{1}{[\sum_{n=1}^{l} \alpha_n \cdot k_n]^t} = 0 \tag{4.224}$$

Hence, we proved that $I_{\bar{1}}^{\bar{0}}(\zeta) = 0$ due to its IBPs, and then $\zeta \in \mathcal{Z}_j^l$ is a zero sector. Graphically:

$$ = 0 \tag{4.225}$$

$\square$

Chapter 5

Implementation: PARSIVAL

The aim of this thesis work is to present computer code which can automate the algorithm shown in the previous section.

We chose symbolic programming, due to its natural propensity to face the grammar of IBP generators; for this reason, we have written the code with MATHEMATICA 11[47]. We called this program PARSIVAL, which stands for PARtial fractions-baSed method for Integral eVALuation, and it can read the YAML format[42], the same markup language used by REDUZE and KIRA[30] input files.
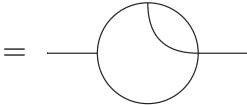
Because of the fact that, in this work, PARSIVAL interfaces with REDUZE, it needs also other packages like GINAC[48] and FERMAT[49], for some algebraic manipulations or polynomial computations.

This code will be explained step to step using as example a simple 2-loop bubble topology and a Feynman integral build on this topology, giving also some code lines.

5.1 The algorithm

In this section, we will see how PARSIVAL works through it's action on the 2-loop bubble topology

$$
\mathcal{T} = \left(\left\{ \begin{matrix} (p^{\mu}, \sqrt{s}) \\ (-p^{\mu}, \sqrt{s}) \end{matrix} \right\}, \left\{ \begin{matrix} (k_1^{\mu}, k_1^{\mu}, m^2) \\ (k_1^{\mu} - p_1^{\mu}, k_1^{\mu} - p_1^{\mu}, m^2) \\ (k_2^{\mu}, k_2^{\mu}, m^2) \\ (k_1^{\mu} + k_2^{\mu}, k_1^{\mu} + k_2^{\mu}, 0) \end{matrix} \right\} \right) \tag{5.1}
$$

$$
= \quad \text{—◯⧀—}
$$

and we assume to have created the directory `~/2-loop_bubble`, where there are stored all input files of REDUZE for $\mathcal{T}$. This will be our working directory.

PARSIVAL is composed by three `.m` files, which has to be placed in `~/2-loop_bubble` with the other input files:

1. `Parsival_Methods.m`, where there are stored all PARSIVAL functions;

2. `Parsival_Global_Variables.m`, which contains the set of imported and evaluated variables of the input file;

3. `Parsival.m` is the executable code, which calls the previous two files as header and perform the algorithm.

5.1.1   Notation and reading routines implementation

The first task to do is allowing PARSIVAL to read the YAML input files, so `kinematics.yaml` and `integralfamilies.yaml`, and store the values in a set of variables. MATHEMATICA implements the *list* variables, which are naturally ordered.

An important precisation: looking at the Pf function defined in the Subsection (4.2.2), we see that it acts like the identyty function on the external legs E. This is the reason that PARSIVAL doesn't have routines to encode the external kinematics, represented by the `kinematics.yaml` file. So, the only file we have to manipulate is the `integralfamilies.yaml`.

In REDUZE, we describe the topology $\mathcal{T}$ by writing the `integralfamilies.yaml` file, showed in Figure (5.1) (for now we neglect the `kinematics.yaml`) one.

```
integralfamilies:
  - name: "bubble_2_loop"
    loop_momenta: [k1,k2]
    propagators:
      - ["k1", m^2]
      - ["k1-p1", m^2]
      - ["k2", m^2]
      - ["k1+k2", 0]
#ISP
      - {bilinear: [[k2,p1], 0]}
#END
```

Figure 5.1: `kinematics.yaml` for a 2-loop QED vertex diagram

We implemented in PARSIVAL the formal construction defined in Section (1.1.3), with some little difference: instead of define an internal line as the triplet (V^μ, W^μ, m^2), we define two set of variables: the list of masses and the list of momenta, treated separately.

$$
\begin{array}{l}
\texttt{["k1", m\^{}2]} \\
\texttt{["k1-p1", m\^{}2]} \\
\texttt{["k2", m\^{}2]} \\
\texttt{["k1+k2", 0]}
\end{array}
\;\rightarrow\;
\begin{array}{l}
\texttt{momlist = \{k1,k1-p1,k2,k1+k2\}} \\
\texttt{masslist = \{m,m,m,0\}}
\end{array}
\tag{5.2}
$$

This splitting can be done because of the natural ordering of the arrays defined in MATHEMATICA. Moreover, to face multi-loop problems, we also have to implement the branch partition. By looking at the dependence of the momentum currents, we split the array of momenta and the one of masses in "list of lists":

$$
\begin{array}{l}
\texttt{momlist = \{k1,k1-p1,k2,k1+k2\}} \\
\texttt{masslist = \{m,m,m,0\}}
\end{array}
\;\rightarrow\;
\begin{array}{l}
\texttt{momenta = \{\{k1,k1-p1\},\{k2\},\{k1+k2\}\}} \\
\texttt{masses = \{\{m,m\},\{m\},\{0\}\}}
\end{array}
\tag{5.3}
$$

We can select the ordered branch B_{b_i} by calling `momenta[[i]]` and `masses[[i]]`.

5.1.2   Partial fractioning for topologies

At this point we can implement in a general routine the injection Pf= $(\mathbb{I}, \mathcal{L}) : \mathbb{T} \to \mathbb{T}$ (5.2).

```
Partialfractioning[nB_,qden_,lden_]:=Module[{PFB,PFBranch,i,j,k},
        Table[PFB[i,j]={},{i,1,Length[nB]},{j,1,nB[[i]]}];

  For[i=1,i<=Length[nB],i++,
    For[j=1,j<=nB[[i]],j++,
      For[k=1,k<=nB[[i]],k++,
        If[j==k,
          PFB[i,j]=Append[PFB[i,j],qden[i,j]],
          PFB[i,j]=Append[PFB[i,j],lden[i,k,j]];
        ];
      ];
    ];
  ];

  For[i=1,i<=Length[nB],i++,
    PFBranch[i]=Table[PFB[i,j],{j,1,nB[[i]]}];
  ];

  Return[PFBranch];
];

PFSubstituition[pfmat_,mom_,mass_,qden_,lden_]:=
Module[{pf,pfvec,sQ,sL,P,Q,Dm,i,j},
  P[v1_,v2_]:=v1+v2;
  Q[v1_,v2_]:=v1-v2;
  Dm[i_,j_,k_]:=mass[[i,j]]^2-mass[[i,k]]^2;
  sQ=qden[i_,j_]:>{mom[[i,j]],mass[[i,j]]^2};
  sL=lden[i_,j_,k_]:>{P[mom[[i,j]],mom[[i,k]]],
      Q[mom[[i,j]],mom[[i,k]]],Dm[i,j,k]};

  If[VectorQ[pfmat],
    pfvec=pfmat;
    For[i=1,i<=Length@pfmat,i++,
      Table[pfvec[[i]][j]=pfvec[[i]][j]/.{sQ,sL},{j,1,Length@mom}];
    ];
    Return[pfvec];,
    pf=pfmat;
    Table[pf[j]=pf[j]/.{sQ,sL},{j,1,Length@mom}];
    Return[pf];
  ];
];
```

Figure 5.2: Implementation in Parsival of Pf function

It is clear that we are interested in write a function of $\mathcal{L}_j^1$ for two reasons:

1. we're neglecting the external kinematics;

2. Pf for multi-loop topology is define by applying the Pf[1] function on each branch, and then by taking all combinations of possible internal lines.

In this way, we can define the internal lines linearization $\mathcal{L}^1$ with a function which associate `momenta` and `masses` to their partial fractioned topologies. This function runs all the internal lines in a branch and build the linear internal lines as

$$\mathcal{L}_i : \{\text{V}_j, \text{m}_j\} \rightarrow \begin{cases} \{\text{V}_i\text{-V}_j, \text{m}_i\text{-m}_j\}, & i < j \\ \{\text{V}_j, \text{m}_j\}, & i = j \\ \{\text{V}_j\text{-V}_i, \text{m}_j\text{-m}_i\}, & i > j \end{cases} \tag{5.4}$$

Supposing that the branch B_{b_i} has t_i denominators:

$$\begin{Bmatrix} \texttt{momenta[[i]]} \\ \texttt{masses[[i]]} \end{Bmatrix} \rightarrow \begin{Bmatrix} \texttt{qden[i][1]} & \cdots & \texttt{lden[i][1,}t_i\texttt{]} \\ \vdots & \ddots & \vdots \\ \texttt{lden[r][}t_i\texttt{,1]} & \cdots & \texttt{qden[i][}t_i\texttt{]} \end{Bmatrix} \tag{5.5}$$

where `qden[[r]][j]` is a quadratic denominator and `lden[[r]][j,k]` is a linear one. Looking at the example for the topology $\mathcal{T}$:

$$\begin{matrix} \texttt{Partial} \\ \texttt{Fractioning} \end{matrix} : \begin{Bmatrix} \{\texttt{k1,k1-p1}\} \\ \{\texttt{m,m}\} \end{Bmatrix} \rightarrow \begin{Bmatrix} \{\texttt{k1,m*m}\} & \{\texttt{-p1,2*k1-p1,0}\} \\ \{\texttt{p1,2*k1-p1,0}\} & \{\texttt{k1-p1,m*m}\} \end{Bmatrix} \tag{5.6}$$

Each row of the matrix in the r.h.s. represent a different linearized branch.. In this context, we distinguish the quadratic propagator from the linear ones by the length of the two lists, as well as REDUZE has two different syntax for quadratic and linear propagators.

Acting on all the branches, and then making the all possible combinations of linearized internal lines (one for each branch), we obtain two linearized topologies for $\mathcal{T}$:

$$\begin{aligned} \texttt{lintop[1,1,1]} &= \begin{Bmatrix} \{\texttt{k1,m*m}\} \\ \{\texttt{-p1,2*k1-p1,0}\} \\ \{\texttt{k2,m*m}\} \\ \{\texttt{k1+k2,0}\} \end{Bmatrix} \\ \texttt{lintop[2,1,1]} &= \begin{Bmatrix} \{\texttt{p1,2*k1-p1,0}\} \\ \{\texttt{k1,m*m}\} \\ \{\texttt{k2,m*m}\} \\ \{\texttt{k1+k2,0}\} \end{Bmatrix} \end{aligned} \tag{5.7}$$

At this point, we can recognize that `lintop[1,1,1]` a `lintop[2,1,1]` have a more familiar shape. The string of number after the topology name represents the position of the quadratic propagator in the list of branches. This notation allow us to define a new family name for each partial fractioned topology:
`<topology_name>_<ordered_quadr_props_position>`.

$$\begin{array}{c} \texttt{bubble\_2\_loop\_1\_1\_1} \\ \texttt{bubble\_2\_loop\_2\_1\_1} \end{array} \tag{5.8}$$

```
kinematics :
  incoming_momenta: [p,p2]
  outgoing_momenta: []
  momentum_conservation: [p2,-p]
  kinematic_invariants:
    - [m, 1]
    - [s, 2]
  scalarproduct_rules:
    - [[p,p], 1/2*s]
```

Figure 5.3: `kinematics.yaml` for a 2-loop QED vertex diagram

```
integralfamilies:
  - name: "bubble_2_loop"
    loop_momenta: [k1,k2]
    propagators:
      - ["k1", m^2]
      - {bilinear:[[-p1,2*k1-p1],0]}
      - ["k2", m^2]
      - ["k1+k2", 0]
#ISP
      - {bilinear: [[k2,p1], 0]}
#END
```

```
integralfamilies:
  - name: "bubble_2_loop"
    loop_momenta: [k1,k2]
    propagators:
      - {bilinear:[[p1,2*k1-p1],0]}
      - ["k1-p1", m^2]
      - ["k2", m^2]
      - ["k1+k2", 0]
#ISP
      - {bilinear: [[k2,p1], 0]}
#END
```

Figure 5.4: `integralfamilies.yaml` generated for the linearized topologies, respectively $\mathcal{T}_1^l$ and $\mathcal{T}_2^l$, of $\mathcal{T}$

5.1.3   Creating .Yaml inputs

Once built the integral family file, we have to define the set of job to run.

Let's recall that with the ID we specifies uniquely a sector (subtopology) of $\mathcal{T}$. Moreover, t_{ID} is the number of denominator for the sector ID, `r` and `s` are respectively the sum of powers of denominators and ISPs.

Firstly, the directory `config` have to contain `integralfamilies.yaml`, built with the informations found in the previous section, and the `kinematics.yaml` (Figure 5.3 and Figure 5.4). This last one, because of the fact that a topology and it's partial fractioned topologies share the same external kinematics, can be copied in each partial fractioned topology directory.

At this point,we have to perform the reduction, so we need four ingredients:

1. the jobs `setup_sector_mappings` and `reduce_sectors`, which generates the IBP set;

2. the file `master.curr`, which lists candidates to be master integrals;

3. the file `myintegrals`, which contains the list of integrals we want to reduce.

4. the job `reduce_files`, which expresses the integrals stored in `myintegrals` in the basis

Generating files for 1. and 2. is very simple: all we have to know is the ID of every linearized topology, which is equal to the one we started from, and the range of power of denominator and numerator r_{max} and s_{max}. REDUZE will distribute the degrees r and s among the denominators and ISPs and generates IBPs for each of this Feynman integral. We have to underline that the only option of `setup_sector_mappings` we use is the `find_zero_sectors`, because of the different sets of sector symmetries and sector relation that linearized topologies have with respect to the starting topology.

In PARSIVAL there are templates of jobs file, which can easily be filled with our values and printed in a file.

`myintegral` is built by noting that, by looking at the general partial fractioning formula (4.70), each term has the same degree of the one in the l.h.s.. In addiction, the sum in r.h.s. may contain different distributions of degree on the same set of denominators. Moreover, we have to recall that Equation (4.70) is true in each branch.

```
{
  ...
  INT["bubble_2_loop", {2,1,1,1,0}],
  ...
}
```

Figure 5.5: Example of Feynman integral in `myintegrals` for $\mathcal{T}$.

Because of this, every file `myintegrals` for linearized topologies have to include all the Feynman integral whose, in every branch, they have the same degree. For example, in Figure (5.5), we have an integral for the topology T which has degree 3 in the first branch and degree 1 in the second and third branches. Then, `myintegrals` files of linearized topologies have to contain all Feynman integrals with degree 3 in the first branch and degree 1 in the second and third branches (Figure 5.6).

```
{
  ...
  INT["bubble_2_loop_1_1_1", {2,1,1,1,0}],
  INT["bubble_2_loop_1_1_1", {1,2,1,1,0}],
  ...
}
```
```
{
  ...
  INT["bubble_2_loop_2_1_1", {2,1,1,1,0}],
  ...
}
```

Figure 5.6: `myintegrals` files for the linearized topologies of $\mathcal{T}$

Lastly, we have choose the canditates for being master integrals in each linearized topology input files.

We stated in the Section (4.3.2) that it is more convenient to have a set of MIs, for a specified sector with t_{ID}, which has $\mathtt{r=t_{ID}}$ (so that each denominator has exponent 1). In the case we have more than one master integral for a certain sector ID, we can pick another integral of the same sector by adding powers $\mathtt{s}$ of ISPs.

Then, we create the file $\mathtt{masters.curr}$ by generating all integrals with $\mathtt{r=t_{ID}}$ and $\mathtt{s \le s_{ID}}$ (Figure 5.7).

```
{
 ...
 INT["bubble_2_loop_1_1_1", {1,0,1,0,0}],
 INT["bubble_2_loop_1_1_1", {1,0,1,0,-1}],
 INT["bubble_2_loop_1_1_1", {1,0,1,0,-2}],
 INT["bubble_2_loop_1_1_1", {1,1,1,1,0}],
 INT["bubble_2_loop_1_1_1", {1,1,1,1,-1}],
 INT["bubble_2_loop_1_1_1", {1,1,1,1,-2}],
 ...
}
```

```
{
 ...
 INT["bubble_2_loop_2_1_1", {0,1,1,0,0}],
 INT["bubble_2_loop_2_1_1", {0,1,1,0,-1}],
 INT["bubble_2_loop_2_1_1", {0,1,1,0,-2}],
 INT["bubble_2_loop_2_1_1", {1,1,1,1,0}],
 INT["bubble_2_loop_2_1_1", {1,1,1,1,-1}],
 INT["bubble_2_loop_2_1_1", {1,1,1,1,-2}],
 ...
}
```

Figure 5.7: $\mathtt{masters.curr}$ files for the linearized topologies of $\mathcal{T}$

At this point we have all the ingredients for creating our input directories for all linearized topologies of $\mathcal{T}$.

We generate a directory within the working directory, called $\mathtt{PF\_<topology\_name>}$, and, for each linearized topology, Parsival creates a set of directories nameed as $\mathtt{<topology\_name>\_<ordered>\_<quadr>\_<props>\_<position>}$. They are our new working directories, and here we print all the configuration files and jobs found in this section.

5.1.4 Running Reduze

In order to parallelize the process, we aim to use a CPU for each generation. Mathematica can split process among CPU with the method $\mathtt{StartProcess}$. So, the right choice is to launch all our jobs with $\mathtt{StartProcess}$, and wait for the end. In Figure (5.8) are represented the launching function implemented in Parsival.

The first job to launch is clearly $\mathtt{reduce\_sectors}$, belonging to $\mathtt{jobs\_1\_reduction.yaml}$. It runs Reduze in parallel and waits until all the processes are over. In Figure 5.8,

this method is called `RunReduction` (Figure 5.8).

Once generated IBPs for each linearized topologies, we have to apply the reduction algorithm, in order to express the integrals in `myintegrals` in terms of combination of master integrals, stored in `masters.curr`. It is now sufficient launch `RunMI` method in Figure 5.8, which perform this decomposition.

At the end of this phase, we have a `.mma` file for every linearized topology directory, called `myintegrals.sol.mma`, in which occurs the reduction of the linearized Feynman integrals in `myintegrals`.

5.1.5   Recollection algorithm

Lastly we have to recollect all the "linear" reductions in order to recover the "quadratic" reductions.

We have a set of reductions for all of the linearized topologies we generated. In Figures 5.9 and 5.10 are presented the reductions of four linearized Feynman integrals.

The first step of the recollection is to relate each Feynman integral in `myintegrals` in the working directory to the integrals stored in `myintegrals` files of the linearized topologies. The way to do this is through the partial fractioning. Taking as example `INT[bubble_2_loop,2,1,1,1,0]` and by applying (4.70) we get

$$
\begin{aligned}
&\texttt{INT["bubble\_2\_loop\_1\_1\_1",3,7,3,0,\{1,1,1,0,0\}]=} \\
&\quad \texttt{INT["bubble\_2\_loop\_1\_1\_1",3,7,3,0,\{2,1,1,1,0\}]-} \\
&\quad \texttt{-INT["bubble\_2\_loop\_1\_1\_1",3,7,3,0,\{1,2,1,1,0\}]+} \\
&\quad \texttt{+INT["bubble\_2\_loop\_2\_1\_1",3,7,3,0,\{2,1,1,1,0\}]}
\end{aligned}
\tag{5.9}
$$

The translation from the grammar of Reduze to the analytic form has already been faced in Chapter (3).

Now, looking at the Equation (5.9), in order to get a right recollection, we have to combine the linearized Feynman integrals in the r.h.s..

The first thing to do is to replace the r.h.s. of (5.9) with the reductions performed in the previous section. In this way, we get an equality which expresses a quadratic Feynman integrals in terms of a combination of *linearized master integrals*. In Parsival, we wrote a function called QuadReduction (Figure 5.11) which automatically reads the linear reduction and expresses all Feynman integrals in `myintegrals` in terms of linearized master integrals.

Now, we have to build back the quadratic master integrals in order to obtain the "quadratic" reduction.

As explained in Chapter 4., we choose the linearized master integral so that the partial fractioning can be performed with the equation (4.70). The strenght of this choice is that the partial fractioning doesn't affect the powers of denominators. This allow us to associate uniquely a linearized master integral with its corrisponding quadratic integral

$$
\texttt{INT["bubble\_2\_loop\_1\_1\_1",\{1,1,1,0,0\}]->INT["bubble\_2\_loop",\{1,1,1,0,0\}]}
\tag{5.10}
$$

So, we get the set of quadratic candidates for being master integrals just by looking

```
RunReduction[name_,wd_,redpath_,idx_,flag_]:=
Module[{dir,i,proc={}},
  If[flag=="yes",
    For[i=1,i<=Length@idx,i++,
      dir=FileNameJoin[
            {wd,"PF"<>name,name<>"_"<>ToString[Row[idx[[i]],"_"]]}
          ];
      SetDirectory[dir];
      proc=Append[
              StartProcess[
                {redpath,"jobs_1_reduction.yaml"}
              ]
          ]@proc;
    ];
    SetDirectory[wd];
    While[True,
      If[AllTrue[proc,ProcessStatus[#]=="Finished"&],
        Break[]
      ]
    ];
    Return[];,
    Return[];
  ];
];
```

```
RunMI[name_,wd_,redpath_,idx_,flag_]:=
Module[{dir,i,mi={}},
  If[flag=="yes",
    For[i=1,i<=Length@idx,i++,
      dir=FileNameJoin[
            {wd,"PF"<>name,name<>"_"<>ToString[Row[idx[[i]],"_"]]}
          ];
      SetDirectory[dir];
      mi=Append[
            StartProcess[{redpath,"jobs_4_reduction_basis.yaml"}]
          ]@mi;
    ];
    SetDirectory[wd];
    While[True,
      If[AllTrue[mi,ProcessStatus[#]=="Finished"&],
        Break[]
      ]
    ];
    Return[];,
    Return[];
  ];
];
```

Figure 5.8: Parallelization routines of PARSIVAL

```
{
 ...
 INT["bubble_2_loop_1_1_1",4,15,5,0,{2,1,1,1,0}] ->
   INT["bubble_2_loop_1_1_1",3,14,3,1,{0,1,1,1,-1}] *
      (-2*(32*m^2-8*m^2*d+d*s-4*s)^(-1)*(-2+d)*(-3+d)^(-1)*m^(-4)*
      (104*m^2-32*m^2*d+d*s-4*s)*(8*m^2-s)^(-1)) +
   INT["bubble_2_loop_1_1_1",3,14,3,0,{0,1,1,1,0}] *
      (-1/4*(d^2*s^2+96*m^2*d*s-16*m^2*d^2*s+256*m^4*d^2-6*d*s^2+
      8*s^2-128*m^2*s-1536*m^4*d+2240*m^4)*(32*m^2-8*m^2*d+d*s-4*s)^(-1)
      *(-3+d)^(-1)*m^(-4)*(8*m^2-s)^(-1)) +
   INT["bubble_2_loop_1_1_1",3,7,3,0,{1,1,1,0,0}] *
      (1/4*(-3+d)^(-1)*m^(-4)*(64*m^2+8*m^2*d^2-48*m^2*d+d*s-2*s)
      *(8*m^2-s)^(-1)) +
   INT["bubble_2_loop_1_1_1",2,5,2,0,{1,0,1,0,0}] *
      (-1/2*(-3+d)^(-1)*m^(-4)*(4+d^2-4*d)*(8*m^2-s)^(-1)),

 INT["bubble_2_loop_1_1_1",4,15,5,0,{1,2,1,1,0}] ->
   INT["bubble_2_loop_1_1_1",3,14,3,1,{0,1,1,1,-1}] *
      (8*(-2+d)*m^(-2)*(16*m^2+s)*s^(-1)*(8*m^2-s)^(-2)) +
   INT["bubble_2_loop_1_1_1",3,14,3,0,{0,1,1,1,0}] *
      ((88*m^2-32*m^2*d+d*s-2*s)*m^(-2)*(8*m^2-s)^(-2)) +
   INT["bubble_2_loop_1_1_1",3,7,3,0,{1,1,1,0,0}] *
      ((-2+d)*m^(-2)*(8*m^2-s)^(-1)) +
   INT["bubble_2_loop_1_1_1",2,5,2,0,{1,0,1,0,0}] *
      (-2*(-3+d)^(-1)*m^(-2)*(4+d^2-4*d)*s^(-1)*(8*m^2-s)^(-1))
 ...
}
```

Figure 5.9: Some reduction of the linearized topology `bubble_2_loop_2_1_1`

at the list of powers.

$$
\begin{array}{c}
\texttt{INT["bubble\_2\_loop",\{0,1,1,1,-2\}]} \\
\texttt{INT["bubble\_2\_loop",\{0,1,1,1,-1\}]} \\
\texttt{INT["bubble\_2\_loop",\{0,1,1,1,0\}]} \\
\texttt{INT["bubble\_2\_loop",\{1,1,1,0,0\}]} \\
\texttt{INT["bubble\_2\_loop",\{0,1,1,0,0\}]} \\
\texttt{INT["bubble\_2\_loop",\{1,0,1,0,0\}]}
\end{array}
\tag{5.11}
$$

By looking at (5.11), we did not included two integrals:

```
INT["bubble_2_loop_2_1_1",{1,0,1,1,0}]->INT["bubble_2_loop",{1,0,1,1,0}]
INT["bubble_2_loop_2_1_1",{1,0,1,1,-1}]->INT["bubble_2_loop",{1,0,1,1,-1}]
```
(5.12)

This can be understood by looking at the powers of the first branch: {1,0}. It is clear from the Equation (4.68) that the partial fractioning acts by leaving a quadratic denominator untouched and making differences between it and the other denominators.

The linearized integrals in (5.12) has no quadratic denominator in the first branch: `bubble_2_loop_2_1_1` is the linearized topology for which, in the first branch, we have the quadratic denominator in position 2, but it has power 0.

This implies that it can't belong to any possible partial fractioning: although it is a master integral for the linearized topology `bubble_2_loop_2_1_1`, *it cannot be considered a true linearized master integral.* For the moment, we will neglect these terms and will see in what way they affects the calculation.

Now, let's apply the partial fractioning on our master integrals above and invert the partial fractioning, so that the r.h.s. of (5.9) become combination of both

```
{
...
  INT["bubble_2_loop_2_1_1",4,15,5,0,{1,2,1,1,0}] ->
    INT["bubble_2_loop_2_1_1",3,14,3,2,{0,1,1,1,-2}] *
      (-2*m^(-4)*(112*m^2*d*s-560*m^4-48*m^4*d^2+320*m^4*d+4*s^2-
      232*m^2*s-d*s^2-12*m^2*d^2*s)*(s-8*m^2)^(-2)*s^(-1)*
      (s-2*m^2)^(-1)*(-4+d)^(-1)*(-4+3*d)*(-3+d)^(-1)) +
    INT["bubble_2_loop_2_1_1",3,14,3,1,{0,1,1,1,-1}] *
      (1/2*m^(-4)*(1536*m^6*d-704*m^4*s-2048*m^6+32*s^3+1808*m^4*d*s
      +1360*m^2*d*s^2-256*m^6*d^2+5*d^2*s*s^3-420*m^2*d^2*s^2-28*d*s^3-
      960*m^4*d^2*s+36*m^2*d^3*s^2+144*m^4*d^3*s-1216*m^2*s^2)*
      (s-8*m^2)^(-2)*s^(-1)*(s-2*m^2)^(-1)*(-4+d)^(-1)*(-3+d)^(-1)) +
    INT["bubble_2_loop_2_1_1",3,14,3,0,{0,1,1,1,0}] *
      (-1/4*m^(-4)*(s-8*m^2)^(-2)*s^(-1)*(s-2*m^2)^(-1)*(5264*m^4*d*
      s^2+12288*m^8*d+960*m^6*d^2*s-224*m^2*d*s^3-5632*m^4*s^2-4352*
      m^8*d^2+6*d*s^4-96*m^6*d^3*s+512*m^8*d^3+3968*m^6*s-d^2*s^4+
      144*m^4*d^3*s^2+56*m^2*d^2*s^3-11520*m^8-8*s^4+256*m^2*s^3-
      1552*m^4*d^2*s^2-3296*m^6*d*s-4*m^2*d^3*s^3)*
      (-4+d)^(-1)*(-3+d)^(-1)) +
    INT["bubble_2_loop_2_1_1",3,13,3,1,{1,0,1,1,-1}] *
      (-2*m^(-4)*(s-8*m^2)^(-2)*(-4+d)^(-1)*(-2+d)*(4*s-d*s-104*m^2+
      32*m^2*d)*(-3+d)^(-1)) +
    INT["bubble_2_loop_2_1_1",3,13,3,0,{1,0,1,1,0}] *
      (1/4*m^(-4)*(96*m^2*d*s+2240*m^4+256*m^4*d^2-1536*m^4*d
      +d^2*s^2+8*s^2-128*m^2*s-6*d*s^2-16*m^2*d^2*s)*(s-8*m^2)^(-2)*
      (-4+d)^(-1)*(-3+d)^(-1)) +
    INT["bubble_2_loop_2_1_1",3,7,3,0,{1,1,1,0,0}] *
      (1/4*m^(-4)*(s-8*m^2)^(-1)*(2*s-8*m^2*d^2-d*s-64*m^2+48*m^2*d)*
      (-3+d)^(-1)) +
    INT["bubble_2_loop_2_1_1",2,6,2,0,{0,1,1,0,0}] *
      (m^(-2)*(4*d^3*s^2+64*m^4*d^3+280*m^2*d*s-960*m^4-480*m^4*d^2+
      1184*m^4*d-33*d^2*s^2-72*s^2-384*m^2*s-4*m^2*d^3*s+86*d*s^2-
      36*m^2*d^2*s)*(s-8*m^2)^(-2)*s^(-1)*(s-2*m^2)^(-1)*(-4+d)^(-1)*
      (-3+d)^(-1)),
...
}
```

Figure 5.10: Some reduction of the linearized topology `bubble_2_loop_1_1_1`

```
QuadReduction[integrals_,IntReduzeList_,IntFamName_,PFPows_,inds_,branches_]:=
Module[{quad,subpflist,reductionlist,sublist},
   quad=ReduzeQuadInts[integrals]; (*reads the integrals in myintegrals*)
   subpflist=ReduzePFSubsList[quad,IntReduzeList,PFPows,inds,Length@branches];
           (*perform the partial fractioning of the previous integrals*)
   reductionlist=ReadReductions[IntFamName,inds]; (*collects the linearized reductions*)
   sublist=Substitutions[subpflist,reductionlist,inds]; (*replaces integrals in
           r.h.s. of subpflist with the linearized reductions*)
   Return[sublist];
];
```

Figure 5.11: Function of PARSIVAL which expresses Feynman integrals in terms of linearized master integrals.

quadratic and linearized MIs. By doing this, we get

```
INT["bubble_2_loop_2_1_1",{0,1,1,1,-2}] -> INT["bubble_2_loop",{0,1,1,1,-2}]
INT["bubble_2_loop_2_1_1",{0,1,1,1,-1}] -> INT["bubble_2_loop",{0,1,1,1,-1}]
INT["bubble_2_loop_2_1_1",{0,1,1,1,0}] -> INT["bubble_2_loop",{0,1,1,1,0}]
INT["bubble_2_loop_1_1_1",{1,1,1,0,0}] -> INT["bubble_2_loop",{1,1,1,0,0}]-
                                          -INT["bubble_2_loop_2_1_1",{1,1,1,0,0}]
INT["bubble_2_loop_2_1_1",{0,1,1,0,0}] -> INT["bubble_2_loop",{0,1,1,0,0}]
INT["bubble_2_loop_1_1_1",{1,0,1,0,0}] -> INT["bubble_2_loop",{1,0,1,0,0}]
```

$$(5.13)$$

In the equation above, most of the integrals are already quadratics: they have one only denominator in each branch (we recall that the branch decomposition made in the first subsection still holds, so that {0,1,1,1,-2}={{0,1},{1},{1},-2}). A linearized Feynman integral with one denominator per branch is actually quadratic.

The last thing to do is replacing (5.13) to the linearized master integrals. As we saw in the previous chapter, this substitution leaves only the quadratic master integrals; the remaining linearized master integrals are summed to zero. The output of the replacements are stored in a .mma file: PF_myintegrals.sol.mma.

```
{
  ...
  INT["bubble_2_loop",{2,1,1,1,0}] ->
    INT["bubble_2_loop", {0,1,1,0,0}]*
      (-1)*(-2+d)*(32*(3-4*d+d^2)*m^4+4*(17-5*d)*d*m^2*s+(12-11*d+2*d^2)*s^2)*
      ((-4+d)*(-3+d)*(2*m^2-s)*s*(-8*m^3+m*s)^2)^(-1)-
    INT["bubble_2_loop",{0,1,1,1,-2}]*
      2*(-4+3*d)*(16*(-1+d)*m^4+4*(22-7*d)*m^2*s+(-4+d)*s^2)*((-4+d)*(-3+d)*
      m^4*(2*m^2-s)*s*(-8*m^2+s)^2)^(-1)+
    INT["bubble_2_loop",{0,1,1,1,-1}]*
      (256*(8-6*d+d^2)*m^6-16*(100-79*d+15*d^2)*m^4*s+8*(80-74*d+15*d^2)*
      m^2*s^2+(-32+28*d-5*d^2)*s^3)*(2*(-4+d)*(-3+d)*m^4*(2*m^2-s)*s*
      (-8*m^2+s)^2)^(-1)-
    INT["bubble_2_loop",{0,1,1,1,0}]*
      (-256*(15-11*d+2*d^2)*m^8+32*(44-31*d+5*d^2)*m^6*s+16*(88-67*d+12*d^2)*
      m^4*s^2-20*(8-6*d+d^2)*m^2*s^3+(8-6*d+d^2)*s^4)*(4*(-4+d)*(-3+d)*m^4*
      (2*m^2-s)*s*(-8*m^2+s)^2)^(-1)+
    INT["bubble_2_loop",{1,0,1,0,0}]*
      (-2+d)^2*(4*m^2-s)*(2*(-3+d)*m^4*(8*m^2-s)*s)^(-1)+
    INT["bubble_2_loop",{1,1,1,0,0}]*
      (-2+d)*(4*(-5+d)*m^2+s)*(4*(-3+d)*m^4*(8*m^2-s))^(-1)-
    INT["bubble_2_loop_1_1_1",{0,1,1,1,-1}]*
      (2*(-2+d)*(64*(12-7*d+d^2)*m^4+4*(-14+d+d^2)*m^2*s-(-4 + d)*s^2)*
      ((-4+d)*(-3+d)*m^4*s*(-8*m^2+s)^2)^(-1)+
    INT["bubble_2_loop_1_1_1",{0,1,1,1,0}]*
      (-2+d)*(32*(31-23*d+4*d^2)*m^4-4*(-4-3*d+d^2)*m^2*s+(-4+d)*s^2)*
      (4*(-4+d)*(-3+d)*m^4*(-8*m^2+s)^2)^(-1)-
    INT["bubble_2_loop_2_1_1",{1,0,1,1,-1}]*
      (2*(-2+d)*(64*(12-7*d+d^2)*m^4+4*(-14+d+d^2)*m^2*s-(-4+d)*s^2)*
      ((-4+d)*(-3+d)*m^4*s*(-8*m^2+s)^2)^(-1)+
    INT["bubble_2_loop_2_1_1",{1,0,1,1,0}]*
      ((-2+d)*(32*(31-23*d+4*d^2)*m^4-4*(-4-3*d+d^2)*m^2*s+(-4+d)*s^2)*
      (4*(-4+d)*(-3+d)*m^4*(-8*m^2+s)^2)^(-1)
  ...
}
```

Figure 5.12: Output of recollection algorithm of PARSIVAL

It can be shown that, for any topology, the standard reduction and the reduction through PARSIVAL bring the same result: let us write the decomposition in Figures (5.12) and (5.13) as

```
Parsival["bubble_2_loop",{2,1,1,1,0}] =
   A*INT["bubble_2_loop",{0,1,1,1,-2}] + B*INT["bubble_2_loop",{0,1,1,1,-1}]+
   C*INT["bubble_2_loop",{0,1,1,1,0}] + D*INT["bubble_2_loop",{1,1,1,0,0}] +
   E*INT["bubble_2_loop",{0,1,1,0,0}] + F*INT["bubble_2_loop",{1,0,1,0,0}]
   + Spurious
```

$$(5.14)$$

```
{
  ...
  INT["bubble_2_loop",{2,1,1,1,0}] ->
    INT["bubble_2_loop",{0,1,1,1,-2}]*
      (-2*(-4+3*d)*(s-2*m^2)^(-1)*(-3+d)^(-1)*m^(-4)*(4*s^2+28*m^2*d*s+
      16*m^4-d*s^2-16*m^4*d-88*m^2*s)*(s^2-8*m^2*s)^(-1)*(d*s-8*m^2*d-4*s+
      32*m^2)^(-1))+
    INT["bubble_2_loop",{0,1,1,1,-1}]*
      (1/2*(s-2*m^2)^(-1)*(-3+d)^(-1)*m^(-4)*(s^2-8*m^2*s)^(-1)*(d*s-8*m^2*d-
      4*s+32*m^2)^(-1)*(1600*m^4*s+592*m^2*d*s^2+32*s^3-1264*m^4*d*s-2048*m^6-
      28*d*s^3-256*m^6*d^2+5*d^2*s^3+240*m^4*d^2*s+1536*m^6*d-640*m^2*s^2-
      120*m^2*d^2*s^2))+
    INT["bubble_2_loop",{0,1,1,1,0}]*
      (1/4*(s-2*m^2)^(-1)*(-3+d)^(-1)*m^(-4)*(-4+d)^(-1)*(s^2-8*m^2*s)^(-1)*
      (120*m^2*d*s^3+d^2*s^4+160*m^6*d^2*s+1408*m^6*s-1072*m^4*d*s^2+
      1408*m^4*s^2-512*m^8*d^2-992*m^6*d*s-6*d*s^4+192*m^4*d^2*s^2+8*s^4-
      20*m^2*d^2*s^3-3840*m^8-160*m^2*s^3+2816*m^8*d)*(s-8*m^2)^(-1)) +
    INT["bubble_2_loop",{1,1,1,0,0}]*
      (-1/4*(-3+d)^(-1)*m^(-4)*(d*s-28*m^2*d+4*m^2*d^2-2*s+40*m^2)*
      (s-8*m^2)^(-1))+
    INT["bubble_2_loop",{0,1,1,0,0}]*
      (-(24*s^2+136*m^2*d*s+192*m^4-34*d*s^2-108*m^2*d^2*s+15*d^2*s^2
      -352*m^4*d-32*m^4*d^3-2*d^3*s^2+20*m^2*d^3*s+192*m^4*d^2)*
      (s-2*m^2)^(-1)*(-3+d)^(-1)*m^(-2)*s^(-1)*(d*s-8*m^2*d-4*s+
      32*m^2)^(-1)*(s-8*m^2)^(-1))+
    INT["bubble_2_loop",{1,0,1,0,0}] *
      (-1/2*(-3+d)^(-1)*m^(-4)*(s-4*m^2)*(s^2-8*m^2*s)^(-1)*(-4+4*d-d^2))
  ...
}
```

Figure 5.13: Reduction without PARSIVAL

for the recollected reduction and

```
Reduze_Only["bubble_2_loop",{2,1,1,1,0}] =
   A'*INT["bubble_2_loop",{0,1,1,1,-2}] + B'*INT["bubble_2_loop",{0,1,1,1,-1}]+
   C'*INT["bubble_2_loop",{0,1,1,1,0}] + D'*INT["bubble_2_loop",{1,1,1,0,0}]+
   E'*INT["bubble_2_loop",{0,1,1,0,0}] + F'*INT["bubble_2_loop",{1,0,1,0,0}]
```
$$(5.15)$$

With `Spurious` we denote any combination of linearized Feynman integrals which still remains after the inversion of partial fractioning decomposition.

$\dfrac{\texttt{Parsival}}{\texttt{Reduze\_Only}}$	A	B	C	D	E	F
A'	1	·	·	·	·	·
B'	·	1	·	·	·	·
C'	·	·	1	·	·	·
D'	·	·	·	1	·	·
E'	·	·	·	·	1	·
F'	·	·	·	·	·	1

for the standard one. Making the ratios between the corresponding coefficients of the master integrals, we get 1. It can be shown that the ratio between coefficient is 1 for any test topology.

In general, `Spurious`, which carries the master integrals belonging to the linearized topologies whose are not linearized master integrals, is in principle different from zero. It contains linearized master integrals, so we are not allowed to set to zero.

But we can see that `Spurious` is null. Recalling that, in order to have a right recollection, we setted symmetries flags in reduction jobs as `false`, REDUZE does not do any internal replacements that could affect our calculation or, in this case, manage the equivalent topologies occuring in this calculation.

So, let us take as example

$$\texttt{INT["bubble\_2\_loop\_1\_1\_1",} \{0,1,1,1,-1\}] =$$
$$= \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2(d-2)}} \frac{p \cdot k_2}{(-p) \cdot (2k_1 - p)(k_2^2 - m^2)(k_1 + k_2)^2} \tag{5.16}$$

Firstly, we make the inversions $k_1 \to -k_1$ and $k_2 \to -k_2$:

$$\int \frac{d^d k_1 d^d k_2}{(2\pi)^{2(d-2)}} \frac{p \cdot k_2}{(-p) \cdot (2k_1 - p)(k_2^2 - m^2)(k_1 + k_2)^2} =$$
$$\int \frac{d^d k_1 d^d k_2}{(2\pi)^{2(d-2)}} \frac{p \cdot k_2}{(-p) \cdot (2k_1 + p)(k_2^2 - m^2)(k_1 + k_2)^2} \tag{5.17}$$

then, we apply the crossing $p \to -p$:

$$\int \frac{d^d k_1 d^d k_2}{(2\pi)^{2(d-2)}} \frac{p \cdot k_2}{(-p) \cdot (2k_1 + p)(k_2^2 - m^2)(k_1 + k_2)^2} =$$
$$- \int \frac{d^d k_1 d^d k_2}{(2\pi)^{2(d-2)}} \frac{p \cdot k_2}{p \cdot (2k_1 - p)(k_2^2 - m^2)(k_1 + k_2)^2} \tag{5.18}$$

This means that

```
INT["bubble_2_loop_1_1_1",{0,1,1,1,-1}]
    = -INT["bubble_2_loop_2_1_1",{1,0,1,1,-1}]
```
(5.19)

The same calculation can be done for `INT["bubble_2_loop_1_1_1",0,1,1,1,0]` and we get

```
INT["bubble_2_loop_1_1_1",{0,1,1,1,0}]
    = -INT["bubble_2_loop_2_1_1",{1,0,1,1,0}]
```
(5.20)

These identities show us that `Spurious`= 0, and through the recollection we reconstruct the quuadratic reduction, without unwanted non-vanishing terms. In Figure (5.14) we present a flowchart of the algorithm implemented in PARSIVAL.
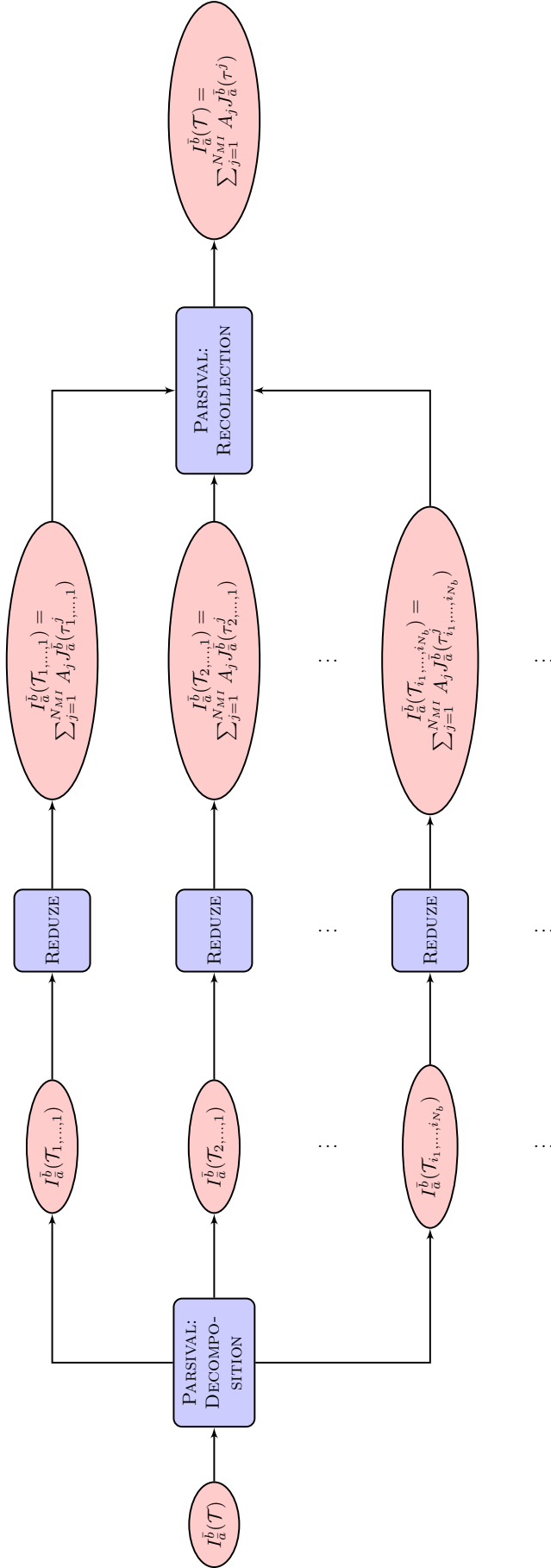
Figure 5.14: Flow chart of the working of PARSIVAL

Chapter 6

Tests and Results

In this Chapter we present tests of our new algorithm, where we show the comparison between running times of REDUZE and PARSIVAL. For each topology, we fixed the number of denominator t, the ID, the max degree of denominators r_{gen} for the IBP generation, the max degree of denominators r_{sol} for the reduction and the max degree of ISPs s. All these runs are tested on a virtual machine made available by the *Cloud computing and storage service*[50] of the section of Padova of the "Istituto Nazionale di Fisica Nucleare" (INFN).

We present the running times for a IBP reduction with REDUZE only and the same one with PARSIVAL+REDUZE, calculating the ratios:

$$\mathbf{Ratio} = \frac{t_{\text{linearized}}}{t_{\text{quadratic}}}$$

In the next pages are exposed the results of tests for:
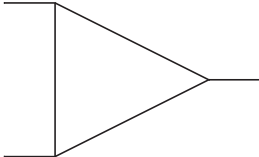
- Massive 1-loop triangle topology;

- Massive 1-loop box topology;

- Massless 1-loop pentagon topology;

- Massless 2-loop planar box;

- Massless 2-loop non-planar box;

- Massless 2-loop planar box with one external massive line;

- Massless 2-loop non-planar box with one external massive line;

6.1   1-loop tests

1-loop triangle topology
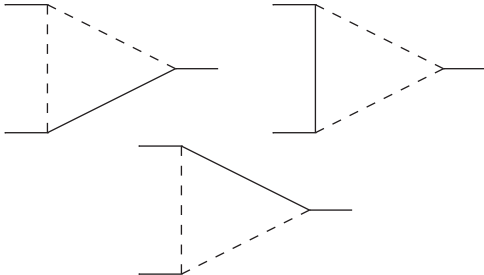
Generation Parameters				
ID	t	r_{gen}	r_{sol}	s
7	3	5	5	1

Topology

scalar_products: propagators:

[[p1,p1], M1^2]   ["k", "m1^2"]

[[p2,p2], M2^2]   ["k+p1", "m2^2"]

[[p1,p2], 1/2(s-M1^2-M2^2)]   ["k+p1+p2", "m3^2"]

Quadratic reduction

Running time
$2192s$

Linearized reductions

Running time	
$427s$	$1400s$
	$326s$

Ratios	
0.195	0.639
0.002	

1-loop box topology

Generation Parameters				
ID	t	r_{gen}	r_{sol}	s
15	4	6	6	1

Topology

```
scalar_products:
[[p1,p1], 0]              propagators:
[[p2,p2], 0]              ["k", "m^2"]
[[p3,p3], 0]              ["k+p1", "m^2"]
[[p1,p2], 1/2*s]          ["k+p1+p2", "m^2"]
[[p1,p3], 1/2*t]          ["k+p1+p2+p4", "m^2"]
[[p2,p3], -1/2(s+t)]
```
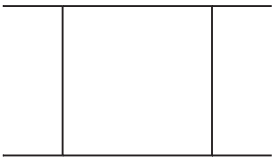

Quadratic reduction

Running time
$37s$
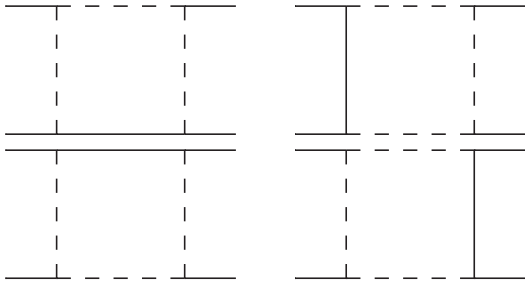
Linearized reductions

Running times	
$5s$	$8s$
$7s$	$15s$

Ratios	
0.14	0.22
0.19	0.41

1-loop massless pentagon topology

Generation Parameters				
ID	t	r_{gen}	r_{sol}	s
31	5	7	7	1

Topology

```
scalar_products:
[[p1,p1], 0]
[[p2,p2], 0]
[[p3,p3], 0]                          propagators:
[[p1,p2], 1/2*s]                      ["k", 0]
[[p1,p3], 1/2*t1]                     ["k+p1", 0]
[[p1,p4], 1/2*t2]                     ["k+p1+p2", 0]
[[p2,p3], 1/2*u1]                     ["k+p1+p2+p4", 0]
[[p2,p4], 1/2*u2]                     ["k+p1+p2+p4+p5", 0]
[[p3,p4], -1/2*(s+t1+t2+u1+u2)]
```


Quadratic reduction

Running time
$12551s$
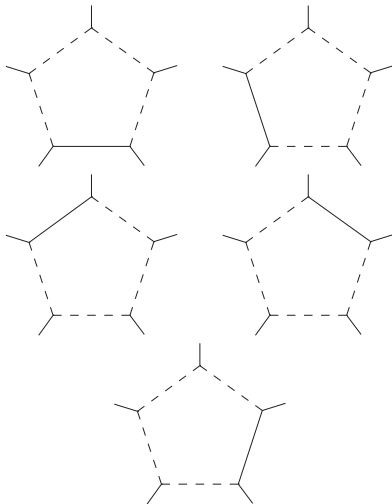
Linearized reduction

Running time	
$2023s$	$2852s$
$7289s$	$1048s$
	$4162s$

Ratios	
0.16	0.23
0.58	0.08
	0.33

6.2 2-loop cases

2-loop massless planar box topology

Generation Parameters				
ID	t	r_{gen}	r_{sol}	s
127	7	8	9	2

Topology

scalar_products:
[[p1,p1], 0]
[[p2,p2], 0]
[[p3,p3], 0]
[[p1,p2], 1/2*s]
[[p1,p3], 1/2*t]
[[p2,p3], -1/2(s+t)]

propagators:
["k1", 0]
["k1+p1", 0]
["k1+p1+p2", 0]
["k2", 0]
["k2-p1-p2", 0]
["k2-p1-p2-p3", 0]
["k1+k2", 0]

Quadratic Reduction

Running time
$20399s$

Linearized Reduction

Running Times	
$6936s$	$5448s$
$15001s$	$14080s$
$5497s$	$9842s$
$5897s$	$16150s$
	$6275s$

Ratios	
0.34	0.27
0.74	0.69
0.27	0.48
0.29	0.79
	0.31

2-loop massless non-planar box topology

Generation Parameters				
ID	t	r_{gen}	r_{sol}	s
127	7	8	9	2

Topology

scalar_products:
[[p1,p1], 0]
[[p2,p2], 0]
[[p3,p3], 0]
[[p1,p2], 1/2*s]
[[p1,p3], 1/2*t]
[[p2,p3], -1/2(s+t)]

propagators:
["k1", 0]
["k1+p1", 0]
["k1+p1+p2", 0]
["k2", 0]
["k2-p1-p2-p3", 0]
["k1+k2", 0]
["k1+k2-p3", 0]

Quadratic Reduction

Running time
$37235s$

Linearized Reduction

Running Times	
$3963s$	$2627s$
$11023s$	$6589s$
$17967s$	$7124s$
$4924s$	$12570s$
$4929s$	$18957s$
$4154s$	$7850s$

Ratios	
0.11	0.07
0.30	0.18
0.48	0.19
0.13	0.34
0.13	0.51
0.11	0.21

2-loop massless planar box topology

Generation Parameters				
ID	t	r_{gen}	r_{sol}	s
127	7	8	9	2

Topology

scalar_products:
[[p1,p1], 0]
[[p2,p2], 0]
[[p3,p3], 0]
[[p1,p2], 1/2*s]
[[p1,p3], 1/2*t]
[[p2,p3], 1/2(M^2-s-t)]

propagators:
["k1", 0]
["k1+p1", 0]
["k1+p1+p2", 0]
["k2", 0]
["k2-p1-p2", 0]
["k2-p1-p2-p3", 0]
["k1+k2", 0]

Quadratic Reduction

Running time
$243739s$

Linearized Reduction

Running Times	
$280785s$	$299702s$
$496989s$	$36347s$
$23507s$	$23997s$
$668366s$	$501484s$
$717957s$	

Ratios	
1.15	1.23
2.04	0.03
0.10	0.10
2.74	2.06
2.95	

2-loop non-planar box topology

Generation Parameters				
ID	t	r_{gen}	r_{sol}	s
127	7	8	9	2

Topology

scalar_products:

[[p1,p1], 0]

[[p2,p2], 0]

[[p3,p3], 0]

[[p1,p2], 1/2*s]

[[p1,p3], 1/2*t]

[[p2,p3], 1/2(M^2-s-t)]

propagators:

["k1", 0]

["k1+p1", 0]

["k1+p1+p2", 0]

["k2", 0]

["k2-p1-p2-p3", 0]

["k1+k2", 0]

["k1+k2-p3", 0]

Quadratic Reduction

Running time
$107277s$

Linearized Reduction

Running Times	
$50571s$	$63103s$
$83474s$	$104584s$
$103021s$	$111266s$
$91578s$	$75797s$
$84110s$	$94146s$
$56200s$	$58803s$

Ratios	
0.47	0.59
0.78	0.97
0.96	1.04
0.85	0.71
0.78	0.88
0.52	0.55

Conclusions

Scattering amplitudes are a powerful tool to investigate the microscopic behaviour of Nature. They encode all informations about particles interactions. Ambitious experimental programmes in high-energy collider physics demands theory predictions with an ever increasing level of precision. Within perturbation theory, including higher orders corrections to Scattering Amplitudes means computing *Feynman integrals* that correspond to diagrams that contain many loops, and many final-state particles.

Feynman integrals can take two different forms: tree-level diagrams, related to the leading order contributions to the full amplitude; loop diagrams, associated to quantum corrections to the scattering amplitude and to the presence of *virtual particles*. Within the *dimensional regularization scheme*, each loop occuring in the diagram represent a d-dimensional integral with respect a *loop momenta*.

Virtual contributions are decomposable in a combination of tensorial parts times scalar factors, namely the *Feynman integrals*. Hence, evaluating Feynman diagram involves the calculation of multi-variate d-dimensional integrals. A direct integration of these function gets harder with the increasing number of particles and loop momenta. In the last decade, new evaluation techniques of such quantities were developed, allowing automation through computer codes. Automation brought a great impact on the theoretical predictions and allowed them to support the LHC experimental precisions.

The current Feynman integrals calculation strategy is addressed in two stages:

1. decomposition of Feynman integrals in an "integral basis", whose elements are the so-called *master integrals* (MIs);

2. evaluation of each master integral.

The first step of this procedure is called *reduction*. Within the dimensional regularization scheme, Feynman integrals are known to obey *integration-by-parts identities* (IBPs). The solution of the linear system of equations which IBPs generate yields the identification of the basis of MIs.

For any process, and at a given loop order, the identification of the master integrals, and the corresponding amplitude decomposition constitute a heavy computational effort, requiring the manipulation of large, algebraic expressions, which in presence of many loops and/or many scales can become prohibitive. Therefore, it constitutes a bottleneck to the availability of theoretical predictions for $2 \to n$ ($n \geq 2$) scattering processes involving massive particles either in the loops or in the final states. Automating the reduction algorithm involves the developements of

codes which solve the IBPs system. For 2-loop Feynman integral, IBPs can generate thousands of equations.

Actually, the reduction algorithm is well-automated for 1-loop amplitudes, and the key to reach a good automation relied in *unitarity-based methods*.

The goal of this work was the presentation of a novel algorithm aimed at improve the current system-solving strategy inspired by unitarity-based methods for Feynman integrals evaluation.

We presented a review of unitarity-based methods, such as optical theorem, unitarity cuts and BCFW recurrence relations; we noticed that this last one turns to be the partial fractions decomposition of tree-level amplitudes. Taking this results as an inspiration, we applied partial fractioning on Feynman integrals, and developed a new reduction strategy based on such decomposition.

We had shown the action of partial fractioning on Feynman integrals. It provides a combination of new Feynman integrals with *linear* propagators in the loop momenta: we called them *linearized Feynman integrals*.

Hence, our novel partial fraction-based reduction algorithm is addressed in three stages:

1. Decompose Feynman integrals with quadratic propagators in linearized Feynman integrals;

2. Use the IBPs reduction method on each linearized Feynman integral, obtaining expressions which contains MIs with linear propagators;

3. Express the starting Feynman integral in terms of the reductions provided in 2. and use the partial fractioning relation to build back MIs with quadratic denominators.

In this work we achieved a *symbolical form of the partial fractioning decomposition*, allowing its application at general multi-loop Feynman integrals. In this way, partial fractioning does not even depend on the explicit expression of the integrand denominator, avoiding simple yet tedious algebra.

Within the recombination of the single reductions of linearized Feynman integrals, *spurious terms* arised. These terms contain master integral with linear propagators which cannot be generated by partial fractioning. The classification of lienarized Feynman integrals brought us to identify a class of vanishing and crossing-related integrals. Moreover, we proved a set of theorems within the recollection stage which state that these spurious terms have vanishing combination.

These were the two main mathematical results of this work, which rigorously justified the use of partial fractioning decomposition within the reduction algorithm for Feynman integrals evaluation.

In order to test this new strategy, we implemented it in a Mathematica code: Parsival (PARtial fractions-baSed method for feynman Integral eVALuation). It is intended to be an *app*, which can be interfaced with existing IBPs reduction routines (e.g. new tools as Kira, Fire, Azurite). We tested Parsival, interfaced with the public code Reduze, on a set of test 1-loop and 2-loop Feynman integrals.

For 1-loop topologies, we tested Feynman integrals belonging to 3-points, 4-points and 5-points amplitudes: Parsival improved the running times of the re-

duction using only Reduze in each test. These 1-loop topologies were also used as stability test for Parsival.

For 2-loop topologies, we tested planar and non-planar boxes, belonging to QCD $gg \rightarrow gg$ and $gg \rightarrow gH$ scattering amplitudes. In this cases, Parsival showed a dependence on the choice of internal momenta configuration: this suggests that an optimal choice of the routing of internal momenta flowing into the Feynman graph could ameliorate the reduction strategy of our new code.

In conclusions, we investigated our novel reduction algorithm through the development of the code Parsival, and it has to be considered as a starting point for the presentation of a public code which can be adapted to any reduction algorithm, in order to ameliorate its Feynman integral reduction strategy.

Once reached the full optimization of Parsival, it can be exploited to reduce NNLO amplitudes, which represents a bottleneck of the current automation program. As example, we mention non-planar boxes for $gg \rightarrow t\bar{t}$ scattering and $gg \rightarrow ggg$ amplitude, for which we are currently working on.

Appendix A

QED Feynman rules in Euclidean space

In this work, all definition are given in Euclidean space through Wick rotation:

$$x_0 = ix_{0E}, \qquad k_0 = ik_{0E} \tag{A.1}$$

which implies to replate the Minkowski metric with the Euclidean metric

$$\eta_{\mu\nu} \to -\delta_{\mu\nu} \tag{A.2}$$

Feynman rules in Euclidean space are slightly different from ones in Minkowski space. Starting from the photon propagator in Minkowski space (in $d = 4$)

$$
\begin{aligned}
\Delta_{\mu\nu}(x - y) &= \int \frac{d^4k}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{k^2} e^{-ik\cdot(x-y)} \\
&= \int \frac{dk_0 d^3\bar{k}}{(2\pi)^4} \frac{-i\eta_{\mu\nu}}{k_0^2 - |\bar{k}|^2} e^{-ik_0\cdot(x_0-y_0)+i\bar{k}\cdot(\bar{x}-\bar{y})}
\end{aligned}
\tag{A.3}
$$

we make the replacements (A.1) and (A.2):

$$
\begin{aligned}
\Delta_{\mu\nu}^E(x - y) &= \int \frac{idk_{0E} d^3\bar{k}}{(2\pi)^4} \frac{i\delta_{\mu\nu}}{-k_{0E}^2 - |\bar{k}|^2} e^{+ik_{0E}\cdot(x_{0E}-y_{0E})+i\bar{k}\cdot(\bar{x}-\bar{y})} \\
&= \int \frac{d^4k_E}{(2\pi)^4} \frac{\delta_{\mu\nu}}{k_E^2} e^{ik_E\cdot(x-y)_E}
\end{aligned}
\tag{A.4}
$$

For the fermion propagator

$$S_{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{i(\slashed{k} + m)}{k^2 - m^2} e^{-ik\cdot(x-y)} \tag{A.5}$$

After the Wick rotation

$$
\begin{aligned}
S_{\mu\nu}^E(x - y) &= \int \frac{id^4k_E}{(2\pi)^4} \frac{i(ik_{0E}\gamma^0 - \bar{k}_j\gamma^j + m)}{-k_E^2 - m^2} e^{ik_E\cdot(x-y)_E} \\
&= \int \frac{d^4k_E}{(2\pi)^4} \frac{-\slashed{k}_E + m}{k_E^2 + m^2} e^{ik_E\cdot(x-y)_E}
\end{aligned}
\tag{A.6}
$$

where $\not{k}_E = k_E \cdot \gamma_E = -k_{0E}\gamma_E^0 - k_i\gamma^i$. Passing to the Euclidean space, gamma matrices trasforms in this way:

$$\gamma^0 = i\gamma_E^0 \tag{A.7}$$

In Euclidean space, gamma matrices satisfies a different anticommutation relation

$$\{\gamma_E^\mu, \gamma_E^\nu\} = -2\delta^{\mu\nu} \tag{A.8}$$

so, traces of Euclidean gamma matrices are different:

$$\begin{aligned}
\mathrm{Tr}[\gamma_E^\mu\gamma_E^\nu] &= \frac{1}{2}\mathrm{Tr}[\{\gamma_E^\mu, \gamma_E^\nu\}] = -\delta^{\mu\nu}\mathrm{Tr}[\mathbb{I}_d] \\
\mathrm{Tr}[\gamma_E^\mu\gamma_{\mu E}] &= \delta_{\mu\nu}\mathrm{Tr}[\gamma_E^\mu\gamma_E^\nu] = -\delta_{\mu\nu}\delta^{\mu\nu}\mathrm{Tr}[\mathbb{I}_d] = -d\mathrm{Tr}[\mathbb{I}_d]
\end{aligned} \tag{A.9}$$

The trace of four Euclidean gamma matrices is

$$\begin{aligned}
\mathrm{Tr}[\gamma_E^\mu\gamma_E^\nu\gamma_E^\rho\gamma_E^\sigma] &= -2\delta^{\mu\nu}\mathrm{Tr}[\gamma_E^\rho\gamma_E^\sigma] - \mathrm{Tr}[\gamma_E^\nu\gamma_E^\mu\gamma_E^\rho\gamma_E^\sigma] = \\
&= 2\delta^{\mu\nu}\delta^{\rho\sigma}\mathrm{Tr}[\mathbb{I}_d] + 2\delta^{\mu\rho}\mathrm{Tr}[\gamma_E^\nu\gamma_E^\sigma] + \mathrm{Tr}[\gamma_E^\nu\gamma_E^\rho\gamma_E^\mu\gamma_E^\sigma] = \\
&= 2\delta^{\mu\nu}\delta^{\rho\sigma}\mathrm{Tr}[\mathbb{I}_d] - 2\delta^{\mu\rho}\delta^{\nu\sigma}\mathrm{Tr}[\mathbb{I}_d] - 2\delta^{\mu\sigma}\mathrm{Tr}[\gamma_E^\nu\gamma_E^\rho] - \mathrm{Tr}[\gamma_E^\nu\gamma_E^\rho\gamma_E^\sigma\gamma_E^\mu]
\end{aligned} \tag{A.10}$$

so

$$\mathrm{Tr}[\gamma_E^\mu\gamma_E^\nu\gamma_E^\rho\gamma_E^\sigma] = \mathrm{Tr}[\mathbb{I}_d](\delta^{\mu\nu}\delta^{\rho\sigma} - \delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}) \tag{A.11}$$

and

$$\mathrm{Tr}[\gamma_E^\mu\gamma_E^\nu\gamma_{\mu E}\gamma_E^\sigma] = -\mathrm{Tr}[\mathbb{I}_d](d-2)\delta^{\nu\sigma} \tag{A.12}$$

Vertices contain a gamma matrices which have to be replaced: $-ie\gamma^\mu \to e\gamma_E^\mu$. Summarizing, Feynman rules for QED in Euclidean space are:

$$\begin{aligned}
\text{(photon propagator, momentum } p\text{)} &= \frac{\delta_{\mu\nu}}{p_E^2} \\
\text{(fermion propagator, momentum } p\text{)} &= \frac{-\not{p}_E + m}{p_E^2 + m^2} \\
\text{(vertex)} &= e\gamma_E^\mu
\end{aligned} \tag{A.13}$$

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