Università degli Studi di Padova

Dipartimento di Fisica e Astronomia *Galileo Galilei*
Corso di Laurea Triennale in Fisica

Cosmological Perturbation Theory beyond shell-crossing: Schrödinger equation approach

Laureando:
Pasquale Tiziano Ursino
1028542

Relatore:
Prof. Sabino Matarrese

Anno Accademico 2016 - 2017

Abstract

In this dissertation, in order to study the growth of the density fluctuations of the Cold Dark Matter, the standard perturbation techniques, such as Eulerian perturbation theory and Zel'dovich approximation, have been reviewed. In the second part of our work, we introduce a novel approach to the study of large–scale structure formation in which the Cold Dark Matter is modelled by a complex scalar field whose dynamics are ruled by coupled Schrodinger and Poisson equations. In the last part, we show that results predicted by "tree-level" perturbation theory for the cold dark matter are perfectly recovered.

Contents

Introduction

The local Universe presents a rich hierarchical pattern of galaxy clustering covering a broad range of length scales, which result into rich clusters, super-clusters and filaments. The large-scale structure of the universe is the result of a process known as *gravitational instability*, according to which small density fluctuations in the early universe stands at the origin of the formation of galaxies.
The discovery of some fluctuations in the cosmic microwave background (CMB) temperature are the experimental evidence that assure the validity of the theory of the gravitational instability.

In order to construct an appropriate scenario for the origin of cosmic protostructure, cosmologists built their theories around the idea that the universe is dominated by non-baryonic dark matter, which are weakly interacting and collisionless particles. Therefore, there are two possible scenarios: The Hot Dark Matter (HDM) scenario, characterised by the assumption that these particles are relativistic and the Cold Dark Matter (CDM) scenario characterised by the assumption that we have non-relativistic particles.

The Cold Dark Matter is the scenario in which the most common analytical methods to understand structure formation will be discussed later on in this work. These methods, which are based on the idea that cold dark matter can be treated as a self-gravitating pressureless fluid, can be divided into two wide classes:

- The Eulerian Perturbation Theory (EPT), which considers macroscopic fluid quantities as the density field, and is based on applying the perturbation theory to the equations of motion of the pressureless fluid. This approach ranges from first order - *linearized fluid approach* - to higher order approaches;
- The Lagrangian Perturbation Theory (LPT), which considers linear perturbations in the trajectories of individual elements of the fluid, the so-called *Zeldovich approximation.*

Despite the fact that the growth of the density fluctuations are well understood when density variations are significantly smaller than the density average (linear regime), it becomes analitically intractable when the non-linear regime occurs showing internal weaknesses. Indeed, they do not assure a density field that is positive everywhere and this could lead to absurdities. Secondly, they might possibly totally break down like the *Zeldovich approximation* does when particle trajectories cross. When this phenomenon, known as *shell-crossing*, happens the density field generates a singularity called *caustic*. In this work we are introducing a novel approach where CDM is modelled by a complex scalar field whose dynamics are ruled by coupled Schrödinger and Poisson equations. This approach overcomes the above mentioned problems that affect standard perturbation methods.

The layout of our work is as follows. In Chapter 1, the cosmological background and its fundamental concepts will be introduced, as they are quite useful for the following dissertation. In Chapter 2, we present the standard cosmological tecniques and its own limits. Finally, in Chapter 3, we propose a new approach based on the correspondence limit of the Schrödinger equation.

Chapter 1

The Cosmological Background

On large scales, i.e. on a distance of hundreds of Mpc, the universe is homogeneous and isotropic for a comoving[1] observer at a fixed cosmic time. This idea is of such importance in cosmology that it has been elevated to the status of principle, known as *Cosmological Principle* [1].

1.1 The Robertson–Walker metric

The most general spacetime that describes a universe in which the *Cosmological Principle* is valid is the Robertson–Walker metric:

$$\mathrm{d}s^2 = (cdt)^2 - a(t)^2\left[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta + sin^2\theta d\varphi^2)\right] \tag{1.1}$$

Where t is the cosmological proper time, a(t) is the cosmic *scale factor*, κ is the spatial curvature, an adimensional constant, that takes only the values ± 1 if we have positive or negative curvature, respectively and 0 in the case of flat space sections. In this context, the equations of motion are determined by the Einstein's field equations which assume the following form:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{a^2} + \frac{\Lambda c^2}{3}, \tag{1.2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) + \frac{\Lambda c^2}{3}, \tag{1.3}$$

$$\dot{\rho} = -3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right). \tag{1.4}$$

Where ρ is the density matter and Λ is the cosmological constant.

[1] we will see in the next pages the meaning of this term

1.2 Einstein-De Sitter model: dust dominated universe

Our work is taking place in an Einsten-De Sitter universe, that is flat ($\kappa = 0$) and characterised by the fact that the cosmological constant Λ is equal to zero. In order to solve Friedmann equations we need to consider a perfect fluid with an equation of state $P = P(\rho)$ (barotropic fluid) of the form:

$$P = w\rho c^2$$

where ω is an adimensional constant that lies in the range, known as Zel'dovich interval:

$$0 \leq w \leq 1$$

If we insert this relations in the third Friedmann equation, we obtain:

$$\dot{\rho} = -3H(1+\omega)\rho \quad \Rightarrow \quad \frac{\dot{\rho}}{\rho} = -3\frac{a}{\dot{a}}(1+\omega) \quad \Rightarrow \quad \rho \propto a^{-3(1+\omega)}$$

In this work we are treating, as mentioned above, a pressureless fluid that means $\omega = 0$. This is the case of a *dust* or *matter-dominated universe*, in which the relation between the density matter and the scale factor is the following:

$$\rho a^3 = const.$$

Chapter 2

Cosmological Structure Formation

In this chapter we run through the standard basics of gravitational instability in an expanding universe. First, we treat the first order Eulerian perturbation theory which well describes the gravitational amplifications when they are still small [2]. Then we penetrate the weakly non-linear regime with the *Zel'dovich approximation* and, afterwards, with *tree-level* PT giving the relative results.

2.1 Fluid treatment

The Cold Dark Matter can be described as a pressureless fluid. In this section we investigate the behaviour of density fluctuations of CDM in the linear regime ($\delta << 1$)[1]. First of all we have to introduce a set of coordinates that have property to be inertial for an observer who moves with the Hubble expansion:

$$\vec{r} = a(t)\vec{x}$$

These coordinates set us up in the *comoving reference frame.* One has:

$$\vec{w} \equiv \dot{\vec{r}} = \frac{\dot{a}}{a}\vec{r} + a\frac{dx}{dt} \equiv H\vec{r} + \vec{v} \tag{2.1}$$

$$\nabla_{\vec{r}} = \frac{1}{a}\nabla_{\vec{x}} \tag{2.2}$$

[1] We consider that perturbations are adiabatic, i.e. entropy is constant.

with $\vec{v}$ the *peculiar velocity* $\vec{v} \equiv a\frac{dx}{dt}$. The convective derivative of a generic function $f(\vec{r}, t)$ is:

$$\frac{Df(\vec{r},t)}{Dt} = \frac{\partial f}{\partial t}\bigg|_{\vec{r}} + (\vec{w} \cdot \nabla_{\vec{r}})f = \frac{\partial f}{\partial t}\bigg|_{\vec{r}} + H(\vec{r} \cdot \nabla_{\vec{r}})f + (\vec{v} \cdot \nabla_{\vec{r}})f$$

Now we take $f(\vec{x}, t)$ we have:

$$\frac{Df(\vec{x},t)}{Dt} = \frac{\partial f}{\partial t}\bigg|_{\vec{x}} + \frac{1}{a}(\vec{v} \cdot \nabla_{\vec{x}})f$$

But $\frac{Df(\vec{r},t)}{Dt} = \frac{Df(\vec{x},t)}{Dt}$, so:

$$\frac{\partial f}{\partial t}\bigg|_{\vec{x}} = \frac{\partial f}{\partial t}\bigg|_{\vec{r}} + H(\vec{r} \cdot \nabla_{\vec{r}})f \tag{2.3}$$

Once introduced comoving coordinates, we can introduce the equations of motion of the Cold Dark Matter:

- The *Euler equation*

$$\frac{\partial \vec{w}}{\partial t}\bigg|_{\vec{r}} + (\vec{w} \cdot \nabla_{\vec{r}})\vec{w} + \frac{1}{\rho}\nabla_{\vec{r}}p + \nabla_{\vec{r}}\Phi = 0; \tag{2.4}$$

- The *continuity equation*

$$\frac{\partial \rho}{\partial t}\bigg|_{\vec{r}} + \nabla_{\vec{r}}(\rho\vec{w}) = 0; \tag{2.5}$$

- The *Poisson equation*

$$\nabla^2_{\vec{r}}\Phi - 4\pi G\rho = 0. \tag{2.6}$$

2.2 Linear Eulerian Perturbation Theory

In order to get the perturbative solutions at first order of the Eulerian PT, we have to place the fluctuation of the Cold Dark Matter in the Friedmann-Robertson-Walker background:

$$\rho \equiv \rho_b + \delta\rho \equiv \rho_b(1 + \delta), \quad \Phi \equiv \Phi_b + \phi \tag{2.7}$$

where ρ_b is the mean background density, $\delta \equiv \frac{\rho - \rho_b}{\rho_b}$ is the *density contrast*[2] and ϕ is the peculiar Newtonian gravitational potential, i.e. the fluctuations in potential with rispect to the background. At this point, we eliminate all background terms and rephrase in

[2]It's necessary remind, as written above, that the perturbation δ can assume positive or negative values.

the new comoving coordinates our set of three equations. We begin with Euler equation that, using (2.1), it becomes:

$$\left.\frac{\partial \vec{v}}{\partial t}\right|_{\vec{r}} + \left.\frac{\partial (H\vec{r})}{\partial t}\right|_{\vec{r}} + H^2(\vec{r}\cdot\nabla_{\vec{r}})\vec{r} + H(\vec{v}\cdot\nabla_{\vec{r}})\vec{r} + (H\vec{r}\cdot\nabla_{\vec{r}})\vec{v} + (\vec{v}\cdot\nabla_{\vec{r}})\vec{v} = -\frac{1}{\rho}\nabla_{\vec{r}}p - \nabla_{\vec{r}}\Phi_b - \nabla_{\vec{r}}\phi$$

we have to equate to zero all background terms, i.e. ignoring those terms without peculiar velocity $\vec{v}$:

$$\left.\frac{\partial (H\vec{r})}{\partial t}\right|_{\vec{r}} + H^2(\vec{r}\cdot\nabla_{\vec{r}})\vec{r} + \nabla\Phi_b = 0$$

it is possible to obtain:

$$(\dot{H} + H^2)\vec{r} = -\frac{4\pi G}{3}\rho_b\vec{r}$$

which is satisfied by Friedmann equations (1.2) and (1.4), leading to:

$$\dot{H} = -4\pi G\rho.$$

Hence we have:

$$\left.\frac{\partial \vec{v}}{\partial t}\right|_{\vec{r}} + H(\vec{v}\cdot\nabla_{\vec{r}})\vec{r} + (H\vec{r}\cdot\nabla_{\vec{r}})\vec{v} + (\vec{v}\cdot\nabla_{\vec{r}})\vec{v} = -\frac{1}{\rho}\nabla_{\vec{r}}p - \nabla_{\vec{r}}\phi.$$

Then using $\nabla_{\vec{r}}\vec{r} = \mathbb{1}$ and (2.3) we obtain:

$$\frac{\partial \vec{v}}{\partial t} + H\vec{v} + \frac{1}{a}(\vec{v}\cdot\nabla_x)\vec{v} = -\frac{1}{a\rho}\nabla_x p - \frac{1}{a}\nabla_x\phi. \tag{2.8}$$

Regarding continuity equation, using (2.1) and simplifying, it results in:

$$\left.\frac{\partial \rho}{\partial t}\right|_{\vec{x}} + H\rho\nabla_{\vec{r}}\cdot\vec{r} + \rho\nabla_{\vec{r}}\cdot\vec{v} + \nabla_{\vec{r}}\rho\cdot\vec{v} = 0$$

therefore using (2.2) and trivial vectorial identity, we have the continuity equation in comoving coordinates:

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\nabla_x(\rho\vec{v}) = 0, \tag{2.9}$$

about Poisson's equation, let's keep in mind that $\nabla^2_{\vec{r}} = \frac{1}{a^2}\nabla^2_{\vec{x}}$, so

$$\nabla^2_x\phi - 4\pi G a^2\delta\rho = 0. \tag{2.10}$$

Once we have our equations in comoving coordinates, we can proceed to linearize them going to Fourier space. Indeed, we have to think of a perturbation as a superposition of plane waves, which evolve linearly, while the fluctuation grows indipendently of each other. So the most natural way to represent this superposition of plane waves is the Fourier representation:

$$\delta(\vec{x},t) = \frac{1}{(2\pi)^3}\int e^{(i\vec{k}\cdot\vec{x})}\delta_{\vec{k}}(t)d^3k,$$

$$\vec{v}(\vec{x},t) = \frac{1}{(2\pi)^3}\int e^{(i\vec{k}\cdot\vec{x})}\vec{v}_{\vec{k}}(t)d^3k,$$

$$\vec{\phi}(\vec{x},t) = \frac{1}{(2\pi)^3}\int e^{(i\vec{k}\cdot\vec{x})}\vec{\phi}_{\vec{k}}(t)d^3k.$$

Before Fourier expanding our equations we have to adjust them. About Euler equation we neglect the non-linear term $\frac{1}{a}(\vec{v}\cdot\nabla_x)\vec{v}$ obtaining:

$$\frac{\partial\vec{v}}{\partial t} + H\vec{v} = -\frac{1}{a\rho}\nabla_{\vec{x}}p - \frac{1}{a}\nabla_{\vec{x}}\phi. \tag{2.11}$$

For the continuity equation we substitute the first term of (2.7):

$$\frac{\partial\rho_b(1+\delta)}{\partial t} + 3H\rho_b(1+\delta) + \frac{1}{a}\nabla_x\big(\rho_b(1+\delta)\vec{v}\big) =$$

$$= \frac{\partial\rho_b\delta}{\partial t} + (1+\delta)\big(\frac{\partial\rho_b}{\partial t} + 3H\rho_b\big) + \frac{1}{a}\nabla_x(\rho_b(1+\delta))\cdot\vec{v} + \frac{1}{a}(\rho_b(1+\delta))\nabla_x\cdot\vec{v} = 0$$

Keeping in mind that the Friedmann equation (1.4) for $P = 0$ is $\dot{\rho} = -3H\rho$ and eliminating all background terms, we get:

$$\frac{\partial\delta}{\partial t} + \frac{1}{a}\nabla_{\vec{x}}\cdot\vec{v} = 0 \tag{2.12}$$

and finally for Poisson equation using $\rho = \rho_b\delta$:

$$\nabla^2_{\vec{x}}\phi = 4\pi G a^2\rho_b\delta. \tag{2.13}$$

Only now, we can Fourier expand (2.11), (2.12) and (2.13). Then the Euler equation becomes:

$$\frac{1}{(2\pi)^3}\int e^{(i\vec{k}\cdot\vec{x})}\Big[\dot{\vec{v}}_{\vec{k}}(t) + H\vec{v}_{\vec{k}}(t)\Big]d^3k = -\frac{1}{(2\pi)^3}\int e^{(i\vec{k}\cdot\vec{x})}i\vec{k}\Big[\frac{1}{a\rho_b}\frac{\partial P}{\partial\rho}\rho_b\delta_{\vec{k}}(t) + \frac{1}{a}\vec{\phi}_{\vec{k}}(t)\Big]d^3k$$

where $\frac{\partial P}{\partial\rho} = c_s^2$ that is the speed of sound,

$$\frac{1}{(2\pi)^3}\int e^{(i\vec{k}\cdot\vec{x})}\Big[\dot{\vec{v}}_{\vec{k}}(t) + H\vec{v}_{\vec{k}}(t)\Big]d^3k = -\frac{1}{(2\pi)^3}\int e^{(i\vec{k}\cdot\vec{x})}i\vec{k}\Big[\frac{c_s^2}{a}\delta_{\vec{k}}(t) + \frac{1}{a}\vec{\phi}_{\vec{k}}(t)\Big]d^3k,$$

$$\dot{\vec{v}}_{\vec{k}} + H\vec{v}_{\vec{k}} = -\frac{ik}{a}c_s^2\left(\delta_{\vec{k}} + \vec{\phi}_{\vec{k}}\right). \tag{2.14}$$

Instead about continuity equation we have:

$$\frac{1}{(2\pi)^3}\int e^{(i\vec{k}\cdot\vec{x})}\dot{\delta}_{\vec{k}}d^3k = -\frac{1}{(2\pi)^3}\int \nabla_{\vec{x}}\left[e^{(i\vec{k}\cdot\vec{x})}\right]\vec{v}_{\vec{k}}d^3k$$

$$\dot{\delta}_{\vec{k}} + \frac{ik\cdot\vec{v}_{\vec{k}}}{a} = 0 \tag{2.15}$$

Finally, Poisson equation is as follow:

$$\frac{1}{(2\pi)^3}\int \nabla^2_{\vec{x}}e^{(i\vec{k}\cdot\vec{x})}\phi_{\vec{k}}d^3k = \frac{1}{(2\pi)^3}\int e^{(i\vec{k}\cdot\vec{x})}4\pi Ga^2\rho_b\delta_{\vec{k}}d^3k,$$

$$k^2\phi_{\vec{k}} = -4\pi Ga^2\rho_b\delta_{\vec{k}}. \tag{2.16}$$

Once we obtained the linearized equations of motion we need to use the *Kelvin circulation theorem*, that says:
"The vorticity is conserved along fluid lines in the absence of dissipative processes"

$$\dot{\vec{v}}_\perp + H\vec{v}_\perp = 0 \quad \Rightarrow \vec{v}_\perp \propto \frac{1}{a}$$

where $\vec{v}_\perp$ is the perpendicular component of peculiar velocity. As a consequence of the theorem, we are allowed to consider only the component of the equations relative to the peculiar velocity parallel to $\vec{k}$. Differentiating (2.15) we get:

$$\ddot{\delta}_{\vec{k}} + \frac{ik}{a}\dot{\vec{v}}_{\vec{k}} - \frac{ik}{a}H\vec{v}_{\vec{k}} = 0 \tag{2.17}$$

Replacing (2.14) in(2.17) we obtain:

$$\ddot{\delta}_{\vec{k}} - \frac{ik}{a}H\vec{v}_{\vec{k}} + \left(\frac{kc_s}{a}\right)^2\delta_{\vec{k}} + \left(\frac{k}{a}\right)^2\phi_{\vec{k}} - \frac{ik}{a}H\vec{v}_{\vec{k}} = 0$$

Now using (2.16) and (2.15):

$$\ddot{\delta}_{\vec{k}} - \frac{ik}{a}H\vec{v}_{\vec{k}} + \left(\frac{kc_s}{a}\right)^2\delta_{\vec{k}} + \left(\frac{k}{a}\right)^2\phi_{\vec{k}} - \frac{ik}{a}H\vec{v}_{\vec{k}} = 0,$$

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\vec{\delta}}_{\vec{k}} + \left[\left(\frac{c_sk}{a}\right)^2 - 4\pi G\rho_b\right]\delta_{\vec{k}} = 0.$$

We define a *comoving Jeans wavenumber* as:

$$k_j \equiv a \frac{\left(4\pi G\rho_b\right)^{1/2}}{c_s}. \tag{2.18}$$

Hence, for $k \ll k_j$ we have the following approximate equation:

$$\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{\vec{k}} - 4\pi G\rho_b\delta_{\vec{k}} \approx 0 \tag{2.19}$$

In an Einstein-De Sitter universe we have two solutions for $\delta_{\vec{k}}$ and $\vec{v}_{\vec{k}}$:

$$a \propto t^{2/3}, \quad H = \frac{2}{3t}, \quad \rho_b = \frac{1}{6\pi G t^2}$$

$$\ddot{\delta}_{\vec{k}} + \frac{4}{3t}\dot{\delta}_{\vec{k}} - \frac{2}{3t^2}\delta_{\vec{k}} \approx 0$$

We look for a solution of the form $\delta \propto t^{\alpha}$ of our equation that becomes:

$$3\alpha^2 + \alpha - 2 = 0$$

This equation leads to two different solutions:

$$\alpha = 2/3 \Longrightarrow \delta_{\vec{k}} \propto t^{2/3} \quad \textit{"growing mode"}$$

$$\alpha = -1 \Longrightarrow \delta_{\vec{k}} \propto t^{-1} \quad \textit{"decaying mode"}$$

This latter result expects that fluctuations decrease by time. We will focus on the *growing-mode* solution, instead, in which the density fluctuations, $\delta \ll 1$, in the early universe grews by time.

2.3 Weakly non-linear regime

The linearized equations of motion give a great description of gravitational instability until density fluctuations are small ($\delta \ll 1$). When density contrast grews ($\delta < 1$) the linear perturbation theory breaks down marking the beginning of weakly non-linear (or quasilinear) regime.

2.3.1 Zel'dovich approximation

The Zel'dovich approximation, as written above, is a Lagrangian approach in which individual particles' trajectories are considered.

First of all we introduce new variables:

$$\eta \equiv \frac{\rho}{\rho_b} = 1 + \delta, \quad \vec{u} \equiv \frac{\vec{v}}{a\dot{a}}, \quad \varphi \equiv \frac{3t_*^2}{2a_*^3}\phi, \quad a(t) = a_*(t/t_*)^{2/3}. \tag{2.20}$$

Replacing these variables in the equations (2.9), (2.8) and (2.10) we have:

$$\frac{\partial \vec{u}}{\partial a} + \vec{u} \cdot \nabla \vec{u} + \frac{3}{2a}\vec{u} = -\frac{3}{2a}\nabla\varphi, \tag{2.21}$$

$$\frac{\partial \eta}{\partial a} + \vec{u} \cdot \nabla \eta + \eta \nabla \cdot \vec{u} = 0, \tag{2.22}$$

$$\nabla^2 \varphi = \frac{\delta}{a}. \tag{2.23}$$

Recalling the results of the previous chapter, in the growing mode we have:

$$\delta_{\vec{k}} \propto t^{2/3}, \quad \vec{v} \propto t^{1/3}, \quad \phi \propto const.$$

This implies that $\vec{u} = const$ and thus

$$\frac{D\vec{u}}{Da} \equiv \frac{\partial \vec{u}}{\partial a} + \vec{u} \cdot \nabla \vec{u} = 0 \tag{2.24}$$

$$\Rightarrow \vec{u} = -\nabla\varphi \tag{2.25}$$

The Zel'dovich's *Ansätze* is to assume that (2.24) and (2.25) are valid even beyond linear theory. Indeed, going to Fourier space:

$$\phi_{\vec{k}} \propto \frac{\delta_{\vec{k}}}{k^2}, \quad \vec{u}_{\vec{k}} \propto k\phi_{\vec{k}}.$$

Thus, these two quantities keep on a linear level on smaller scales and for longer times than the density fluctuation field $\delta_{\vec{k}}$. So we have a new set of equations:

$$\frac{D\vec{u}}{Da} = 0, \tag{2.26}$$

$$\frac{D\eta}{Da} + \eta\big(\nabla \cdot \vec{u}\big) = 0. \tag{2.27}$$

We note that Poisson equation is used only for initial conditions. The first equation rules the behaviour of a set of collisionless particles which move under their "inertia". Its solution is:

$$\vec{u}(\vec{x}, a) = \vec{u}_0(\vec{q}) \tag{2.28}$$

with $\vec{u}_0(\vec{q})$ is the initial velocity in lagrangian position $\vec{q}$ of the infinitesimal fluid element which is in the eulerian position $\vec{x}$ at time $a(t)$. Integrating equation (2.28) we obtain the particle's trajectory:

$$\vec{x}(\vec{q}, a) = \vec{q} + (a - a_0)\vec{u}_0(\vec{q}).$$

Following the Zel'dovich's *Ansätze* we know that (2.25) is valid then, setting initial scale factor $a_0 = 0$, we have:

$$\vec{x}(\vec{q}, a) = \vec{q} - a\nabla_q \varphi_0(\vec{q}) \quad \Rightarrow \vec{u}(\vec{x}(\vec{q}, a), a) = \vec{u}_0(\vec{q}) = \frac{\vec{x} - \vec{q}}{a}$$

the particles go on straight lines. About the continuity equation (2.27) we can obtain a solution by integration:

$$\eta(\vec{x}, a) = \eta_0(\vec{q}) e^{-\int_{a_o}^{a} da' \nabla \cdot \vec{u}[\vec{x}(\vec{q}, a'), a']}$$

Nevertheless, it's more suitable to consider mass conservation of individual infinitesimal fluid element whereby

$$\eta(\vec{x}, a) d^3x = \eta_0(\vec{q}) d^3q$$

But this relation could be equally obtained considering the fact that when we found the particle's trajectory we have defined nothing else than a unique map between Eulerian and Lagrangian coordinates, so:

$$\eta(\vec{x}(\vec{q}, a), a) = (1 + \delta_0(\vec{q})) \left| \frac{\partial \vec{x}}{\partial \vec{q}} \right|^{-1}$$

with $\left| J(\vec{r}, t) \right| = \left| \frac{\partial \vec{x}}{\partial \vec{q}} \right|$ is the Jacobian determinant of the mapping. For $a_0 \to 0$, so at early times we have, recalling (2.23), $\delta_0 \to 0$, then

$$\eta(\vec{x}(\vec{q}, a), a) = \left| \frac{\partial \vec{x}}{\partial \vec{q}} \right|^{-1},$$

the matrix of the change of cordinates $\vec{q} \to \vec{x}$ has components[3]

$$\frac{\partial x^i}{\partial q^j} = \delta^i_j - a \frac{\partial^2 \varphi_0(\vec{q})}{\partial q_i \partial q^j} \quad (i, j = 1, 2, 3)$$

where $\frac{\partial^2 \varphi_0(\vec{q})}{\partial q_i \partial q^j}$ is the *deformation tensor* $D_{0,ij}(\vec{q})$.
Now, we can locally diagonalize this tensor going to principal axes Q_1, Q_2, Q_3 with eigen-

[3] according to the *Einsten summation notation.*

values $\lambda_1(\vec{q}), \lambda_2(\vec{q}), \lambda_3(\vec{q})$. Write now

$$\eta(\vec{x}(\vec{q},a),a) = \frac{1}{(1-\lambda_1(\vec{q})a)(1-\lambda_2(\vec{q})a)(1-\lambda_3(\vec{q})a)} \tag{2.29}$$

Now taking the highest positive eigenvalue λ_i of the deformation tensor $D_{0,ij}(\vec{q})$, at the time $a_{sc} = \frac{1}{\lambda_i(\vec{q})}$ the equation (2.29) indicates that a singularity, called *shell-crossing* occurs because of the density field which becomes locally infinite. Indeed what happens is that trajectories of particles cross and then two points with different Lagrangian coordinates get to the same Eulerian coordinate. Hence the Jacobian $\left|\frac{\partial \vec{x}}{\partial \vec{q}}\right|$ is ill defined and diverges. This singularity occurs via one-dimensional collapse[4] - creating the so-called *pancake.* If there is more than one positive eigenvalue, then collapse will occur along the axis corresponding to the most positive one.
The validity of Zel'dovich categorically breaks down after shell-crossing: particles entering a pancake from either side merely sail through it and pass out the opposite side, the pancake appears only instantaneously and rapidly smeared out. In reality, the matter that enters in the *caustic* -the region in which shell-crossing occurs- would remain there because of the strong gravity. However the Zel'dovich approximation is only kinematic so it doesn't take into account of gravitational interaction.

2.3.2 "Tree-level" Perturbation Theory

In the previous section we described the Eulerian perturbation theory in the linear regime, now we penetrate the non-linear regime through the use of a perturbative expansion [7]. First of all, we consider two different formulations of continuity and Eulerian equation:

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot \big[(1+\delta)\vec{v}\big] = 0, \tag{2.30}$$

$$\frac{\partial \vec{v}}{\partial t} + \mathcal{H}\vec{v} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla\phi. \tag{2.31}$$

With $d\tau = dt/a$ is the *conformal time* and $\mathcal{H} = \frac{d\ln a}{d\tau} = Ha$ is the *conformal expansion rate.* Now going to Fourier space (2.31) and (2.30) read

$$\frac{\partial \tilde{\delta}(\vec{k},\tau)}{\partial \tau} + \tilde{\theta}(\vec{k}.\tau) = -\int d^3k_1 d^3k_2 \delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)\alpha(\vec{k}_1,\vec{k}_2)\tilde{\theta}(\vec{k}_1,\tau)\tilde{\delta}(\vec{k}_2,\tau) \tag{2.32}$$

$$\frac{\partial \tilde{\theta}(\vec{k},\tau)}{\partial \tau} + \mathcal{H}\tilde{\theta}(\vec{k},\tau) + \frac{3}{2}\mathcal{H}^2\tilde{\delta}(\vec{k},\tau) = -\int d^3k_1 d^3k_2 \delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)\beta(\vec{k}_1,\vec{k}_2)\tilde{\theta}(\vec{k}_2,\tau) \tag{2.33}$$

[4]The Zel'dovich approximation becomes exact if φ_0 depends only on one coordinate q_i.

where δ_D is the three-dimensional Dirac delta distribution and $\alpha(\vec{k}_1, \vec{k}_2)$, $\beta(\vec{k}_1, \vec{k}_2)$ are the *mode coupling* functions:

$$\alpha(\vec{k}_1, \vec{k}_2) \equiv \frac{(\vec{k}_1 + \vec{k}_2) \cdot \vec{k}_1}{k_1^2}, \quad \beta(\vec{k}_1, \vec{k}_2) \equiv \frac{(\vec{k}_1 + \vec{k}_2)^2 (\vec{k}_1 \cdot \vec{k}_2)}{2k_1^2 k_2^2}. \tag{2.34}$$

These functions contain all non-linear terms of Eulerian and continuity equation encoding, de facto, the non-linearity of the evolution. These equations are very hard to solve because they are coupled integrodifferential equations. In order to find a solution, we make a perturbative expansion which lets formally solve equations (2.32) and (2.33):

$$\tilde{\delta}(\vec{k}, \tau) = \sum_{n=1}^{\infty} a^n(\tau) \delta^{(n)}(\vec{k}, \tau), \quad \tilde{\theta}(\vec{k}, \tau) = \sum_{n=1}^{\infty} -\mathcal{H}(\tau) \theta^{(n)}(\vec{k}, \tau). \tag{2.35}$$

At small $a(\tau)$, the series are dominated by their first terms, and since we obtain from the continuity equation that $\delta_1(\vec{k}) = -\theta_1(\vec{k})$, $\delta_1(\vec{k})$ totally defines the linear fluctuations. The equations (2.32) and (2.33) determine $\delta_n(\vec{k})$ and $\theta_n(\vec{k})$ in term of linear fluctuations:

$$\delta_n(\vec{k}) = \int d^3q_1 ... \int d^3q_n \delta_D(\vec{k} - \vec{q}_{1...n}) F_n(\vec{q}_1, ..., \vec{q}_n) \delta_1(\vec{q}_1) ... \delta_1(\vec{q}_n), \tag{2.36}$$

$$\theta_n(\vec{k}) = \int d^3q_1 ... \int d^3q_n \delta_D(\vec{k} - \vec{q}_{1...n}) G_n(\vec{q}_1, ..., \vec{q}_n) \delta_1(\vec{q}_1) ... \delta_1(\vec{q}_n). \tag{2.37}$$

Where $F_n(\vec{q}_1, ..., \vec{q}_n)$ and $G_n(\vec{q}_1, ..., \vec{q}_n)$ are homogeneous functions, known as *kernels*, that are constructed from the fundamental mode coupling functions $\alpha(\vec{k}_1, \vec{k}_2)$ and $\beta(\vec{k}_1, \vec{k}_2)$. Now, given the perturbative expansion, if we take the first non trivial term of our perturbative expansions (tree-level), we must consider its corresponding kernels as well:

$$F_2(\vec{q}_1, \vec{q}_2) = \frac{5}{7} + \frac{1}{2} \frac{\vec{q}_1 \cdot \vec{q}_2}{q_1 q_2} \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{2}{7} \frac{(\vec{q}_1 \cdot \vec{q}_2)^2}{q_1^2 q_2^2}, \tag{2.38}$$

$$G_2(\vec{q}_1, \vec{q}_2) = \frac{3}{7} + \frac{1}{2} \frac{\vec{q}_1 \cdot \vec{q}_2}{q_1 q_2} \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) + \frac{4}{7} \frac{(\vec{q}_1 \cdot \vec{q}_2)^2}{q_1^2 q_2^2}. \tag{2.39}$$

Hence, higher order of "tree-level" PT involve more complex kernels. These functions are used to calculate quantities such as power spectrum, bispectrum, skewness, cumulants et cetera. The necessity for a statistical approach gets stronger and stronger and it is due to the fact that is impossible to have direct observational access to primordial fluctuations. Then, we briefly introduce some statistical tools that are indispensable to describe the results of "tree-level" perturbation theory:

- The *two-point correlation function*, which is the joint ensemble average of the density of two different positions

$$\xi(r) = \langle \delta(\vec{x})\delta(\vec{x}+\vec{r})\rangle \tag{2.40}$$

this definition can be extended to Fourier space leading to the density *power spectrum* $P(k)$:

$$\xi(r) = \int d^3k P(k) e^{i\vec{k}\cdot\vec{r}} \tag{2.41}$$

or equivalently,

$$\langle \delta(\vec{k_1})\delta(\vec{k_2})\rangle = \delta_D(\vec{k}_1+\vec{k}_2)P(k) \tag{2.42}$$

where $\delta_D(\vec{k}_1+\vec{k}_2+\vec{k}_3)$ is a term that comes out when we go to Fourier space. It is possible to define higher-order correlation functions, for instance, *three-point correlation function*, which is defined as the connected part of the joint ensemble average density at three different location and its counterpart in Fourier space that is the *bispectrum* $B(\vec{k}_1,\vec{k}_2)$:

$$\langle \delta(\vec{k_1})\delta(\vec{k_2})\delta(\vec{k_3})\rangle_c = \delta_D(\vec{k}_1+\vec{k}_2+\vec{k}_3)B(\vec{k}_1,\vec{k}_2); \tag{2.43}$$

- The *moments* and *cumulants* and their respective generating functions. The *moments* provide a specific quantitative measure of the shape of the probability density. The first moment is the mean, the second one is the variance, the third central moment is the skewness, and the fourth one is the kurtosis. A function from which all moments can be generated is the *moment generating functions* defined by

$$\mathcal{M}(t) \equiv \sum_{p=0}^{\infty} \frac{\langle \delta^p \rangle}{p!} t^p; \tag{2.44}$$

The *cumulants* of a probability distribution are a set of quantities that represent an alternative to the moments. The function from which all cumulants can be obtained is the *cumulant generating functions*

$$\mathcal{C}(t) \equiv \sum_{p=2}^{\infty} \frac{\langle \delta^p \rangle_c}{p!} t^p = \log[\mathcal{M}(t)]; \tag{2.45}$$

- The *vertices* ν_n and μ_n, that are the spherical average of the PT kernels F_n and G_n, defined as follow:

$$\nu_n \equiv n! \int \frac{d\Omega_1}{4\pi} ... \frac{d\Omega_n}{4\pi} F_n(\vec{k}_1, ..., \vec{k}_n), \tag{2.46}$$

$$\mu_n \equiv n! \int \frac{d\Omega_1}{4\pi} ... \frac{d\Omega_n}{4\pi} G_n(\vec{k}_1, ..., \vec{k}_n). \tag{2.47}$$

Using relation (2.46) and using kernel F_n we can calculate the first four vertices:

$$\nu_1 = 1, \quad \nu_2 = \frac{34}{21}, \quad \nu_3 = \frac{682}{189}, \quad \nu_4 = \frac{446,440}{43,659}. \tag{2.48}$$

As written above, the moments are the suitable mean through which one can give a fully description of gravitational fluctuation in the weakly non-linear regime. The first relevant moments that arise from mode coupling function are the third moment, the skewness S_3, and the fourth moment, the kurtosis S_4. The skewness, in particular, measures the propensity of gravitational clustering to generate an asymmetry between overdense and underdense region. They are defined as follows:

$$S_3 \equiv \frac{\langle \delta^3 \rangle}{\langle \delta^2 \rangle^2}, \quad S_4 \equiv \frac{\langle \delta^4 \rangle_c}{\langle \delta^2 \rangle^3}. \tag{2.49}$$

However the direct calculation of S_p parameters beyond kurtosis becomes extremely difficult because the complexity of kernels F_n and G_n. Nevertheless, it's possible to use a relationship that exists between the vertices ν_p and S_p parameters to calculate the latter. Using (2.48), we give the explicit relation until S_5:

$$S_3 = 3\nu_2 = \frac{34}{7}, \tag{2.50}$$

$$S_4 = 4\nu_3 + 12\nu_2^2 = \frac{60,712}{1323}, \tag{2.51}$$

$$S_5 = 5\nu_4 + 60\nu_3\nu_2 + 60\nu_2^3 = \frac{200,575,880}{305,613}. \tag{2.52}$$

These results, that were obtained taking until third-order of "tree-level" PT, allow us to compare the effectiveness of the new approach to gravitational instability based on the use of Schrödinger equation.

Chapter 3

SPT: Schrödinger perturbation theory

In the linear regime, the Eulerian PT through the linearization of fluid dynamics equations of motion and in the weakly non-linear regime, the Lagrangian approaches, such as the Zel'dovich approximation, establish a consistent standard framework within which the evolution of gravitational fluctuations can be described and understood. When fluctuations become strongly non-linear the perturbation theories break down and we has to resort to numerical simulation to study their evolution. In this chapter we are introducing a new approach to the study of the growth of density fluctuations of CDM suggested by Widrow and Kaiser [5]. This approach overcomes the limits of the standard perturbation theory such as the problem to have a density field that is not positive everywhere or the fact that methods, such as the Zel'dovich approximation, break down at shell-crossing.

3.1 The Widrow-Kaiser approach

The new approach to the study of collisionless matter is based on the coupled Schrödinger and Poisson equations:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + mV\psi \quad \nabla^2 V = 4\pi G\psi\psi^* \tag{3.1}$$

where the squared modulus of the wavefunction ψ corresponds to the density matter. The validity of this approach is due to the equivalence between the classical mechanics of point particles and wave mechanics in the geometric limit. In the next section we will show, step by step, the procedure to get Schrödinger's equation from fluid dynamics equations.

3.1.1 From fluid dynamics to Schrödinger's equation

We assume a pressureless fluid in a static Universe with an irrotational velocity field, i.e. $\vec{w} = \nabla\phi$. Hence the equations (2.5) and (2.4), become:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\nabla\phi) = 0 \tag{3.2}$$

$$\frac{\partial\nabla\phi}{\partial t} + (\nabla\phi\cdot\nabla)\nabla\phi = -\nabla V \tag{3.3}$$

with $V \equiv \Phi$.
Then, we focus on the equation above to show how $\frac{1}{2}\partial_i(\partial_j\phi\partial^j\phi) = (\nabla\phi\cdot\nabla)\nabla\phi$:

$$\frac{1}{2}\partial_i(\partial_j\phi\partial^j\phi) = \frac{1}{2}(\partial_i\partial_j\phi\partial^j\phi + \partial_j\phi\partial_i\partial^j\phi) = \partial_j\phi\partial_i\partial^j\phi = w_j\partial_i w^j = (\nabla\phi\cdot\nabla)\nabla\phi.$$

Once we obtain this relation we can substitute in (3.3):

$$\partial_i\frac{\partial\phi}{\partial t} + \frac{1}{2}\partial_i(\partial_j\phi\partial^j\phi) = -\partial_i V.$$

Now using the *divergence theorem* we find:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = -V \tag{3.4}$$

This equation, known as *Bernoulli equation*, togheter with (3.2) provide our starting set of equations. First of all, we make the so-called *Madelung transformation*[1] [8] - in honour of who first noticed that the Schrödinger equation can be put into a fluid dynamical form:

$$\begin{cases} \psi(r,t) = R(x,t)e^{\frac{i\phi}{\nu}} \\ \rho = \psi^*\psi = R^2 \end{cases} \tag{3.5}$$

with ν a new parameter of dimension $\frac{L^2}{T}$. Now we calculate $\nabla\psi$ and $\nabla^2\psi$ that are necessary for our calculations:

$$\nabla\psi = e^{\frac{i\phi}{\nu}}\left[\nabla R + \frac{iR}{\nu}\nabla\phi\right] \tag{3.6}$$

and

$$\nabla^2\psi = e^{\frac{i\phi}{\nu}}\left[\nabla^2 R + \frac{i}{\nu}(2\nabla R\cdot\nabla\phi + R\nabla^2\phi) - \frac{R}{\nu^2}(\nabla\phi)^2\right]. \tag{3.7}$$

[1] It's interesting observe that ψ includes in itself both position and velocity information.

Therefore we start substituting the density field of (3.5) in (3.2):

$$\frac{\partial R^2}{\partial t} + \nabla \cdot (R^2 \nabla \phi) = 2R\dot{R} + 2R\nabla R \cdot \nabla \phi + R^2 \nabla^2 \phi$$

$$\implies 2\nabla R \cdot \nabla \phi + R\nabla^2 \phi = -2\dot{R} \tag{3.8}$$

We substitute the relation (3.8) in (3.7) to get $(\nabla \phi)^2$

$$\nabla^2 \psi = e^{\frac{i\phi}{\nu}} \left[\nabla^2 R - \frac{2i}{\nu}\dot{R} - \frac{R}{\nu^2}(\nabla \phi)^2 \right]$$

$$\implies (\nabla \phi)^2 = \nu^2 \frac{\nabla^2 R}{R} - 2i\nu \frac{\dot{R}}{R} - \frac{\nu^2}{R} \nabla^2 \psi e^{-\frac{i\phi}{\nu}} \tag{3.9}$$

Let's consider the time derivative $\frac{\partial \psi}{\partial t}$

$$\frac{\partial \psi}{\partial t} = e^{\frac{i\phi}{\nu}} \left(\dot{R} + \frac{iR}{\nu}\dot{\phi} \right).$$

We can invert and find $\frac{\partial \phi}{\partial t}$

$$\frac{\partial \phi}{\partial t} = -\frac{i\nu}{R} e^{-\frac{i\phi}{\nu}} \dot{\psi} + i\nu \frac{\dot{R}}{R}. \tag{3.10}$$

Now we can finally substitute (3.10) and (3.9) in the Bernoulli equation obtaining:

$$-\frac{i\nu}{R} e^{-\frac{i\phi}{\nu}} \dot{\psi} + i\nu \frac{\dot{R}}{R} + \frac{1}{2} \left(\nu^2 \frac{\nabla^2 R}{R} - 2i\nu \frac{\dot{R}}{R} - \frac{\nu^2}{R} \nabla^2 \psi e^{-\frac{i\phi}{\nu}} \right) = -V. \tag{3.11}$$

Then,

$$\cancel{i\nu \frac{\dot{R}}{R}} - \frac{i\nu}{R} e^{-\frac{i\phi}{\nu}} \dot{\psi} + \frac{\nu^2}{2} \frac{\nabla^2 R}{R} \cancel{- i\nu \frac{\dot{R}}{R}} - \frac{\nu^2}{2R} \nabla^2 \psi e^{-\frac{i\phi}{\nu}} = -V$$

with some rearrangements we arrive to the Schrödinger equation:

$$i\nu \frac{\partial \psi}{\partial t} = -\frac{\nu^2}{2} \nabla^2 \psi + \left(V + \frac{\nu^2}{2} \frac{\nabla^2 R}{R} \right) \psi. \tag{3.12}$$

We immediatly notice that there is an additive term in the right side of the equation: the *quantum pressure* term. This name is due to the fact that it resembles a pressure gradient. Indeed, if we introduce our wave function in the usual Schrodinger's equation and we work backwards, we arrive at Bernoulli equation with an extra term that is exactly the same [4]. It's evident how this novel approach totally overcomes the issues of the standard perturbation theory. Indeed the way we defined the wave function ψ causes the density ρ to assume only positive values. Secondly, Widrow and Kaiser do not consider trajectories of single particles but a complex scalar field, hence this approach doesn't break down at shell-crossing. Moreover in the wave function no singularities occur at any time.

3.2 Perturbation Theory with the Schrödinger equation

In this section, we work in an expanding universe. Our aim is to obtain the statistical tools that allow us to compare the Schrödinger perturbation theory with the tree-level Eulerian perturbation theory results. We start from equations (2.21), (2.22) and (2.23), assuming an irrotational velocity field $\vec{v} = \nabla\Phi$, then the Euler and continuiy equation assume the following form:

$$\frac{\partial \eta}{\partial a} + \nabla \cdot (\eta \nabla \Phi) = 0, \tag{3.13}$$

$$\frac{\partial \nabla \Phi}{\partial a} + (\nabla\Phi \cdot \nabla)\nabla\Phi + \frac{3}{2a}\nabla\Phi = -\frac{3}{2a}\nabla\varphi. \tag{3.14}$$

As demonstrated in the previous section, $\frac{1}{2}\partial_i(\nabla\Phi)^2 = (\nabla\Phi \cdot \nabla)\nabla\Phi$ then we substitute it in (3.14) and using the divergence theorem we arrive at:

$$\frac{\partial \Phi}{\partial a} + \frac{1}{2}(\nabla\Phi)^2 + \frac{3}{2a}\Phi = -\frac{3}{2a}\varphi. \tag{3.15}$$

Now we can proceed to look for Schrödinger equation with the following set of equations:

$$\begin{cases} \frac{\partial \Phi}{\partial a} + \frac{1}{2}(\nabla\Phi)^2 + \frac{3}{2a}\Phi = -\frac{3}{2a}\varphi \\ \frac{\partial \eta}{\partial a} + \nabla \cdot (\eta \nabla \Phi) = 0 \end{cases} \tag{3.16}$$

For this calculations we use a complex scalar field, that represents the CDM, of the following form:

$$\psi(r,t) = e^{A(r,t) + \frac{i}{\hbar}B(r,t)} \tag{3.17}$$

where $A(r,t)$ and $B(r,t)$ are two scalar fields. the density of the c, with this definition of the wave function, is:

$$\delta = \eta - 1 = e^{2A} - 1. \tag{3.18}$$

Once we introduced our wave function we can proceed as done in the previous section calculating $\nabla\psi$ and $\nabla^2\psi$:

$$\nabla\psi = \psi\left[\nabla A + \frac{i}{\hbar}\nabla B\right] \tag{3.19}$$

and

$$\nabla^2\psi = \psi\left[\nabla^2 A + |\nabla A|^2 + \frac{i}{\hbar}(2\nabla A \cdot \nabla B + \nabla^2 B) - \frac{1}{\hbar^2}|\nabla B|^2\right]. \tag{3.20}$$

We fix $\Phi(r,t) = B(r,t)$ and use the continuity equation (3.13) in the following way:

$$\frac{\partial \eta}{\partial a} + \nabla\cdot(\eta\nabla B) = 2e^{2A}\frac{\partial A}{\partial a} + \Big(2\nabla A\cdot\nabla B + \nabla^2 B\Big)e^{2A} = 0$$

we find that:

$$\implies \quad 2\nabla A\cdot\nabla B + \nabla^2 B = -2\frac{\partial A}{\partial a}$$

We insert this relation in the equation (3.20). Once we did it we can express $|\nabla B|^2$ in function of $\nabla^2\psi$ obtaining:

$$|\nabla B|^2 = \hbar^2\Big(\nabla^2 A - \frac{2i}{\hbar}\frac{\partial A}{\partial a} + |\nabla A|^2 - \frac{\nabla^2\psi}{\psi}\Big). \tag{3.21}$$

In order to find $\frac{\partial B}{\partial a}$ we derive $\frac{\partial \psi}{\partial a}$ and as done previously we arrive at:

$$\frac{\partial B}{\partial a} = -i\hbar\Big(\frac{1}{\psi}\frac{\partial\psi}{\partial a} - \frac{\partial A}{\partial a}\Big). \tag{3.22}$$

At the end our wave function (3.17) can be inverted to find:

$$B = \frac{\hbar}{2i}\ln\Big(\frac{\psi}{\psi^*}\Big). \tag{3.23}$$

Finally, keeping in mind that $\Phi = B(r,t)$, we substitute (3.22), (3.21) and (3.23) in equation (3.15) obtaining:

$$-i\hbar\Big(\frac{1}{\psi}\frac{\partial\psi}{\partial a} - \frac{\partial A}{\partial a}\Big) + \frac{\hbar^2}{2}\Big(\nabla^2 A - \frac{2i}{\hbar}\frac{\partial A}{\partial a} + |\nabla A|^2 - \frac{\nabla^2\psi}{\psi}\Big) + \hbar\frac{3}{4ai}\ln\Big(\frac{\psi}{\psi^*}\Big) = -\frac{3}{2a}\varphi$$

multiplying and eliminating the opposite terms:

$$-i\hbar\frac{1}{\psi}\frac{\partial\psi}{\partial a} + \cancel{i\hbar\frac{\partial A}{\partial a}} + \frac{\hbar^2}{2}(\nabla^2 A + |\nabla A|^2) - \cancel{i\hbar\frac{\partial A}{\partial a}} - \frac{\hbar^2}{2}\frac{\nabla^2\psi}{\psi} + \hbar\frac{3}{4ai}\ln\Big(\frac{\psi}{\psi^*}\Big) = -\frac{3}{2a}\varphi$$

with some calculations and ignoring the quantum pressure term $\frac{\hbar^2}{2}\big(\nabla^2 A + |\nabla A|^2\big)$, we obtain:

$$i\hbar\frac{\partial\psi}{\partial a} = -\frac{\hbar^2}{2}\nabla^2\psi + V\psi \tag{3.24}$$

where we introduced a *general potential* V defined as follow:

$$V = \frac{3}{2a}(B+\varphi) = \frac{3}{2a}\Big(\frac{\hbar}{2i}\ln\big(\frac{\psi}{\psi^*}\big) + \varphi\Big). \tag{3.25}$$

Once we have shown how to arrive at Schrodinger equation in an expanding universe we proceed to work with it. We substitute the wave function $\psi = e^{A(r,t)+\frac{i}{\hbar}B(r,t)}$, that we previously defined, in the Schrodinger equation:

$$i\hbar\left[\frac{\partial A}{\partial a}+\frac{i}{\hbar}\frac{\partial B}{\partial a}\right]\psi = -\frac{\hbar^2}{2}\left[\nabla^2 A+|\nabla A|^2+\frac{i}{\hbar}(2\nabla A\cdot\nabla B+\nabla^2 B)-\frac{1}{\hbar^2}|\nabla B|^2\right]\psi + V\psi = 0,$$

$$i\hbar\frac{\partial A}{\partial a}-\frac{\partial B}{\partial a} = -\frac{\hbar^2}{2}\big(\nabla^2 A+|\nabla A|^2\big)-\frac{i\hbar}{2}(2\nabla A\cdot\nabla B+\nabla^2 B)+\frac{1}{2}|\nabla B|^2+V.$$

After some simple rearrangements we obtained an equation that can be split in two coupled equations respectively for imaginary and real part.
Then for the imaginary part we have:

$$\frac{\partial A}{\partial a} = -\frac{1}{2}\big(\nabla^2 B+2\nabla A\cdot\nabla B\big). \tag{3.26}$$

Instead for the real one:

$$\frac{\partial B}{\partial a} = \frac{\hbar^2}{2}\big(\nabla^2 A+|\nabla A|^2\big)-\frac{1}{2}|\nabla B|^2-V. \tag{3.27}$$

Before going to Fourier space in order to operate in the most natural space for the perturbation, as we explained in section 2.2, we want focus on the Poisson equation (2.23). Indeed, keeping in mind our change of coordinates $V = \frac{3}{2a}\Bigg(\frac{\hbar}{2i}\ln\left(\frac{\psi}{\psi^*}\right)+\varphi\Bigg)$ we can invert it and find the expression of φ:

$$\varphi = \frac{2a}{3}\left(V+\frac{3i\hbar}{4a}\ln\left(\frac{\psi}{\psi^*}\right)\right),$$

we substitute in the Poisson equation

$$\nabla^2\varphi \equiv \nabla^2\left[\frac{2a}{3}\left(V+\frac{3i\hbar}{4a}\ln\left(\frac{\psi}{\psi^*}\right)\right)\right] = \frac{(e^{2A}-1)}{a} \tag{3.28}$$

that in the correspondence limit, i.e. $\hbar \to 0$, become:

$$\nabla^2\varphi \equiv \frac{2a}{3}\nabla^2 V = \frac{(e^{2A}-1)}{a}.$$

We work in the correspondence limit for the others two equations as well:

$$\frac{\partial A}{\partial a} = -\frac{1}{2}\big(\nabla^2 B+2\nabla A\cdot\nabla B\big),$$

$$\frac{\partial B}{\partial a} = -\frac{1}{2}|\nabla B|^2 - V.$$

Now going to Fourier space we obtain the following expressions:

$$\frac{\partial A_k}{\partial a} = -\frac{1}{2}\left(k^2 B(\vec{k}) + 2\int d^3k_1 d^3k_2 \delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)\vec{k}_1\cdot\vec{k}_2 A(\vec{k}_1)B(\vec{k}_2)\right) \tag{3.29}$$

$$\begin{aligned}\frac{\partial B_k}{\partial a} =& -\frac{1}{2}\int d^3k_1 d^3k_2 \delta_D(\vec{k}-\vec{k}_1-\vec{k}_2)\vec{k}_1\cdot\vec{k}_2 B(\vec{k}_1)B(\vec{k}_2) - \frac{3H^2a^2}{2k^2} \\ &\times \sum_{N\geq 1}\frac{2^N}{N!}\int d^3k_1...d^3k_N \delta_D(\vec{k}-\vec{k_1}-...-\vec{k_N})A(\vec{k}_1)...A(\vec{k}_N)\end{aligned} \tag{3.30}$$

where δ_D is the three-dimensional Dirac delta. We show how we obtained the last term of the second equation[2] recalling:

$$\varphi \equiv \frac{3t_*^2}{2a_*^3}\phi, \quad a_*^3 = a^3\frac{t_*^2}{t^2}, \quad t = \frac{2}{3H}.$$

So, we can transform Poisson equation (3.28) in this manner:

$$\nabla^2\varphi = \frac{3t_*^2}{2a_*^3}\nabla^2\phi = \frac{(e^{2A}-1)}{a}$$

Now Fourier transforming and using the definition of the exponential:

$$\frac{3t_*^2}{2a_*^3}k^2\phi_k = \frac{1}{a}\sum_{N\geq 1}\frac{2^N}{N!}\int d^3k_1...d^3k_N \delta_D(\vec{k}-\vec{k_1}-...-\vec{k_N})A(\vec{k}_1)...A(\vec{k}_N) \tag{3.31}$$

$$\Longrightarrow \phi_k = \frac{3H^2a^2}{2k^2}\sum_{N\geq 1}\frac{2^N}{N!}\int d^3k_1...d^3k_N \delta_D(\vec{k}-\vec{k_1}-...-\vec{k_N})A(\vec{k}_1)...A(\vec{k}_N) \tag{3.32}$$

In order to render the equations (3.29) and (3.30) homogeneous in a and H we procede to make a perturbative expansion of our scalar field using the *Ansätze* given by Szapudi and Kaiser [3]:

$$\begin{cases} A_k = \sum A_k^{(N)} a^N \\ B_k = -H\sum B_k^{(N)} a^{N+2} \end{cases} \tag{3.33}$$

where the usual kernels that has the following definition appear:

$$A_k^{(N)} = \int d^3k_1 ... \int d^3k_n \delta_D(\vec{k}-\vec{k}_{1...n})F^{(N)}(\vec{k}_1,...,\vec{k}_n)A_{k_1}^{(1)}...A_{k_N}^{(1)}, \tag{3.34}$$

[2]here we use ϕ (peculiar gravitational potential) instead of φ (gravitational potential) that, after applied the correspondence limit, is related to V (general potential) through: $\nabla^2\varphi \equiv \frac{2a}{3}\nabla^2 V$.

$$B_k^{(N)} = \frac{2}{k^2}\int d^3k_1 ... \int d^3k_n \delta_D(\vec{k} - \vec{k}_{1...n}) G^{(N)}(\vec{k}_1, ..., \vec{k}_n) A_{k_1}^{(1)} ... A_{k_N}^{(1)}, \tag{3.35}$$

substituting to the equations (3.29) and (3.30) we obtain two recursive relations for the two respective kernels $F^{(N)}$ and $G^{(N)}$. Here we give explicitly the N=2 case just for F:

$$F_2(\vec{k}_1, \vec{k}_2) = \frac{3}{7} + \frac{10}{7}\alpha(k_1, k_2) + \frac{2}{7}\beta(k_1, k_2). \tag{3.36}$$

The kernels are constructed from *mode coupling functions* that are similar to the Eulerian case:

$$\alpha(q_1, q_2) = \frac{(q_1 q_2)}{k_2^2}, \quad \beta(q_1, q_2) = k^2 \frac{(q_1 q_2)}{(q_1^2 q_2^2)}. \tag{3.37}$$

3.2.1 Connection with "tree-level" PT

Once we have obtained the expressions for the kernels we can introduce the respective vertices, that we defined in the subsection 2.3.2:

$$\nu_1 = 1, \quad \nu_2 = \frac{26}{21}, \quad \nu_3 = \frac{568}{189}, \quad \nu_4 = \frac{473,744}{43,659}. \tag{3.38}$$

From these quantities we can proceed to calculate first the tree-level cumulants of the scalar field A that are:

$$S_3^A = 3\nu_2 = \frac{26}{7}, \tag{3.39}$$

$$S_4^A = 4\nu_3 + 12\nu_2^2 = \frac{40,240}{1323}, \tag{3.40}$$

$$S_5^A = 5\nu_4 + 60\nu_3\nu_2 + 60\nu_2^3 = \frac{119,609,680}{305,613}. \tag{3.41}$$

Then, we can find the cumulants of the density field δ, that is connected to A through $\delta = e^{2A} - 1$, using the formula of Fry and Gatzañaga [6]. Indeed, they assume that the galaxy density can be written as a function of the mass density of dark matter and they express this function as a Taylor series. For instance, the first coefficient of the expansion is the linear bias factor b ($\delta_g = b\delta_\rho$). They showed that under a certain limit the cumulants of δ_ρ are connected with the cumulants of δ_g and they obtained the following recursive equations that we can use for our purpose due to the fact that we have an analogous relation between our two fields. Then, the cumulants of our density field are:

$$S_3 = b^{-1}(S_3^A + 3c_2) = \frac{34}{7}, \tag{3.42}$$

$$S_4 = b^{-2}(S_4^A + 12c_2 S_3^A + 4c_3 + 12c_2^2) = \frac{60,712}{1323}, \tag{3.43}$$

$$S_5 = b^{-3}[S_5^A + 20c_2 S_4^A + 15c_2 S_3^A + (30c_3 + 120c_2)S_3^A + 5c_4 + 60c_3c_2] = \frac{200,575,880}{305,613}; \quad (3.44)$$

where in our case the coefficients are: $b = 2$ and $c_N = b_N/b = 2^{N-1}$. These are exactly the same results, that we have presented in subsection 2.3.2, for the "tree-level" perturbation theory.

Conclusions

In this work, we have presented a novel approach to the study of the evolution of Cold Dark Matter under the influence of gravity based on the correspondence limit of the Schrödinger equation. First, we investigated the growth of gravitational amplifications in the linear regime, i.e. when the density variations are very small than the unity, using the linear eulerian perturbation theory.
Then we penetrated the weakly non-linear regime with the Zel'dovich approximation, which follows the particle's trajectory giving a formidable comprehension of the behaviour of Cold Dark Matter fluid until the shell-crossing occurs. Then it totally breaks down falling into a singularity.
Therefore we introduced the "tree-level" perturbation theory with some statical tools that allow us to verify the validity and the power of the approach suggested by Widrow and Kaiser.
In the second part of our dissertation we focused on the new formalism that we derive from the equations of the fluid dynamical in a static universe. This approach exceeds the issues that affects the perturbation theories mentioned above. In the end, we deduced the Schrödinger equation in an expanding universe and following the *Ansätze* by Widrow and Kaiser we showed how they have recovered the "tree-level" perturbation theory results. Hence, this technique can be a useful tool to pursue the growth of the density fluctuations in the mildly non-linear regime.

Bibliography

[1] P. Coles and F. Lucchin, *Cosmology* (Wiley, 2002).

[2] S. Matarrese, *Notes on gravitational instability*, (2002–2005).

[3] I. Szapudi and N. Kaiser, Astrophys. J. **583**, L1-L4 (2003).

[4] P. Coles and K. Spencer, Mon. Not. R. Astron. Soc. **342**, 176-184 (2003).

[5] L. M. Widrow and N. Kaiser, Astrophys. J. **416**, L71-L74 (1993).

[6] J. N. Fry and E. Gaztañaga, Astrophys. J. **413**, 447-452 (1993).

[7] F. Bernardeau, S. Colombi, E. Gaztañaga, and R. Scoccimarro, Phys. Rep. **367**, 1-248 (2002).

[8] E. A. Spiegel, Physica D **1**, 236-240 (1980).