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Stochastic fluid dynamics equations with multiplicative noise

Relatore:

Prof. David Barbato

Candidato:

Lucio Galeati

Matricola 1148003

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Introduction

Despite having been formulated more than two centuries ago, equations of fluid dynamics still present outstanding open problems in mathematics. We refer in particular to the equations of incompressible Newtonian fluids, which are described in the viscous case by the Navier-Stokes equations

$$\begin{cases} \partial_t v - \nu \Delta v + (v \cdot \nabla)v + \nabla p = f \\ \nabla \cdot v = 0 \end{cases} \quad (1)$$

and in the inviscid case by the Euler equations

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla p = f \\ \nabla \cdot v = 0 \end{cases} \quad (2)$$

Here v is the velocity field of the fluid, p is the scalar pressure field, f is the external force field and $\nu > 0$ is the kinematic viscosity, which is a fixed parameter and corresponds to the inverse of the Reynolds number. The incompressibility condition is represented by the condition $\nabla \cdot v = 0$. The unknowns are v and p , while f is given, as well as an initial condition $v(0, x) = v_0(x)$ which, together with system (1) or (2), defines a Cauchy problem. If the equation is considered on a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, then suitable boundary conditions must be added, usually slip or no-slip conditions; otherwise if the problem is considered on the whole space $\mathbb{R}^d$, a sufficient decay at infinity must be imposed, while on $\mathbb{T} = \mathbb{R}^d/\mathbb{Z}^d$ periodic boundary conditions are assumed. In the 3-D case, $d = 3$, well-posedness of the above systems is not known: in the case of Navier-Stokes equations, local existence and uniqueness of solutions is known for any sufficiently regular initial data, while global existence of weak solutions is known for any $v_0 \in L^2$; however we don't know whether smooth solutions are defined globally or a blow-up might occur and the uniqueness of weak solutions is an open problem. For the 3-D Euler equations the situation is even worse, since only local existence and uniqueness of strong solutions for $v_0 \in H^s$, $s \geq 5/2$, is known, while there are no results regarding the existence of weak solutions and there are examples of their nonuniqueness, first obtained by Scheffer and later by Shnirelman and De Lellis, see [1], [2]. In particular for the Euler equations there is a huge gap with respect to what is known in the 2-D case, where the invariance of the enstrophy allows to obtain global existence of strong solutions for sufficiently regular initial data. For more details we refer to standard textbooks and reviews, such as [3] and [4].

In his pioneering work [5], Leray showed the existence, for any $v_0 \in L^2$, of global weak solutions of (1) satisfying an additional condition on their energy, which are usually called Leray weak

solutions. His proof was based on considering solutions of the following auxiliary systems, called Leray- α model of Navier-Stokes equations:

$$\begin{cases} \partial_t v^\alpha - \nu \Delta v^\alpha + (u^\alpha \cdot \nabla) v^\alpha + \nabla p = f \\ u^\alpha = G_\alpha * v^\alpha \\ \nabla \cdot v^\alpha = 0 \end{cases} \quad (3)$$

where $\alpha > 0$ is a fixed parameter and G_α is a smoothing kernel such that $G_\alpha \rightarrow I$ in some sense as $\alpha \rightarrow 0^+$. Leray was able to show global existence of strong solutions of (3) for any smooth initial data and then obtained the Leray weak solutions as weak limits in H^1 of (a subsequence of) $\{v^\alpha\}_{\alpha>0}$; the condition obtained on the energy is a consequence of the properties of $\{v^\alpha\}_{\alpha>0}$ and the weak limit. However, he wasn't able to show uniqueness of such solutions. In [6], a special smoothing kernel was considered, namely the Green function associated to the Helmholtz operator $I - \alpha \Delta$, so that (3) becomes

$$\begin{cases} \partial_t v^\alpha - \nu \Delta v^\alpha + (u^\alpha \cdot \nabla) v^\alpha + \nabla p = f \\ v^\alpha = (I - \alpha \Delta) u^\alpha \\ \nabla \cdot v^\alpha = 0 \end{cases} \quad (4)$$

This choice of G_α works as a kind of filter with width α and reflects a sub-grid length scale in the model; it can be regarded as a numerical regularization of (1). Indeed, it is more reliable and robust in numerical simulations and agrees with experimental data, especially at high Reynolds number. It would seem reasonable to try to follow Leray's approach also in the case of Euler equations and consider the following family of auxiliary systems, which we will call Leray- α model of Euler equations:

$$\begin{cases} \partial_t v^\alpha + (u^\alpha \cdot \nabla) v^\alpha + \nabla p = f \\ v^\alpha = (I - \alpha \Delta) u^\alpha \\ \nabla \cdot v^\alpha = 0 \end{cases} \quad (5)$$

However, differently from (4), well-posedness of (5) is not known. Local existence of strong solutions can be shown similarly to the Euler equations and global existence of weak solutions is known (see the appendix in [7] for a proof), but their uniqueness is an open problem. Moreover, we lack a regularity result needed in order to pass to the limit as $\alpha \rightarrow 0$ and obtain a weak solution of (2); in fact, if we only assume weak convergence in L^2 , then an additional term, called Reynolds stress tensor, is expected to appear in the equation, see [4].

An approach considered in recent years in order to solve systems (1) and (2) is to stochastically perturb the equations by a suitable noise. The introduction of noise has phenomenological motivations: disturbances are always present, although very small with respect to the macroscopic quantities; they usually don't appear in the models only because these are idealizations of real life phenomena in which negligible terms are omitted. It is well known that the addition of noise to a deterministic ODE can have regularizing effects in terms of well-posedness, as it was shown by Veretennikov in 1981, see [8]; such phenomenon is usually referred as regularization by noise. More recently, the same feature has been shown also for several PDEs, in which

the addition of noise may improve regularity of the solution, restore well-posedness or prevent blow-up; reviews of the results on the topic are given in [9] and [10]. A possible program of this research direction (see Section 3 of [11] for a more detailed version) is the following: first find a suitable noise whose introduction restores well-posedness (uniqueness in law is sufficient) of the problem; then understand what happens when we let the noise tend to 0. Namely, let $\{P^\varepsilon\}_{\varepsilon>0}$ denote the laws of the solutions of the perturbed problem (with respect to the same given initial conditions and external forces) where the noise is taken with a multiplicative coefficient ε . The non trivial zero-noise result would be to show that the whole family $\{P^\varepsilon\}_{\varepsilon>0}$ converges in law to a measure P . We expect the support of P to be made of (possibly weak) solutions of the original unperturbed problem; the question then becomes to understand if such solutions have a special physical meaning: maybe the mathematical problem has a defect of non-uniqueness but the physical problem behaves uniquely and the physical behaviour is the one stable with respect to infinitesimal perturbations. Unfortunately, such program is far from being developed in a general theory; only partial results, related to specific models, have been obtained.

One thing has become clear however: there isn't a unique way to stochastically perturb a model and, depending on the structure of the equation, some types of noise can perform differently. For systems of PDEs it is natural to introduce, in analogy to Veretennikov's results for ODEs, additive non degenerate noise. However, starting with [12], it has been observed that also a bilinear, multiplicative Stratonovich (thus strongly degenerate) noise can have a regularizing effect. The latter has been more successful in problems arising in fluid dynamics, especially in the inviscid case, as it introduces a parabolic term in the equation at the level of mean values of the solutions (to understand better what we mean, we refer to the general discussion of Section 3.1 of this work or to Section 5.5.2 of [10]). Another reason to prefer a Stratonovich multiplicative noise is that it may preserve useful conservation laws of the original system, which are broken by the introduction of additive noise.

At present, 3-D Euler equations seem too difficult to study, even in the presence of noise; for this reason in this work we focus instead on a stochastic perturbation of (5) of the following form:

$$\begin{cases} dv^\alpha + [(u^\alpha \cdot \nabla)v^\alpha + \nabla p] dt + \sqrt{\varepsilon} [(I - \alpha\Delta)^{-1} \circ dW \cdot \nabla]v^\alpha = f dt \\ v^\alpha = (I - \alpha\Delta)u^\alpha \\ \nabla \cdot v^\alpha = 0 \end{cases} \quad (6)$$

where W is a generalized Wiener process on L^2 and $\alpha, \varepsilon > 0$ are fixed parameters; a more detailed description of W , as well as an explanation of the meaning of system (6), are given in Sections 3.1 and 3.3. For simplicity we have restricted our attention to the (physically less meaningful, but mathematically easier to treat) case of functions defined on $\mathbb{T} = \mathbb{R}^3/\mathbb{Z}^3$ with periodic boundary conditions. Most of the time we only consider $\varepsilon = 1$, but all the results stated are easily extended to any $\varepsilon > 0$. Moreover, everything holds also in the case $d = 2$ and could be generalized to higher dimension, up to some modifications such as a different choice of the regularization G_α .

The thesis is structured as follows: the first two chapters are mainly based on [13] and [14]; we provide there, in the most concise way possible, all the tools related to probability theory and stochastic integration in Hilbert spaces which will be needed later; we underline however

that the approach we followed, treating weak solutions of (6), is in some aspects more similar to the variational formulation of SPDEs, treated for example in [15], even if we never explicitly use Gelfand triples. In Chapter 1 concepts such as Bochner integral and Gaussian measures on Hilbert spaces are introduced and several results, like existence of conditional expectation and properties of stochastic processes, are extended in this setting. Chapter 2 deals with the construction of Wiener processes and stochastic integrals in Hilbert spaces and the proof of classical results like Ito formula and Girsanov transform in this context. The last two chapters are devoted to the study of the stochastic Leray- α model of Euler equations (6). Chapter 3 is mainly based on [7]; global existence and uniqueness in law of Leray weak solutions (both in the probabilistic and variational sense) is shown for any initial data $v_0 \in L^2(\mathbb{T}; \mathbb{R}^3)$ and external force field $f \in L^2((0, T) \times \mathbb{T}; \mathbb{R}^3)$ satisfying some suitable conditions; moreover, continuity in law with respect to v_0 and f is obtained. The strategy we followed, similar to the one employed in other model of fluid dynamics like [16], is the following: we write model (6) in the correspondent Fourier space, which gives an infinite system of coupled SDEs, and we find the correspondent Ito formulation. Then we consider a linearised version of the system, for which we are able to show existence and pathwise uniqueness of solutions; uniqueness is achieved by studying the associated covariance matrices, which satisfy a closed system of ODEs for which a comparison principle holds. Existence and uniqueness in law for the nonlinear system are then obtained by a suitable Girsanov transform; continuous dependence in law on v_0 and f is shown by exploiting the strong dependence on the same data of the linear system and the explicit densities provided by the Girsanov transform. Chapter 4 explores interesting features of the model, such as pathwise regularity of the solutions and anomalous dissipation of energy. The results contained were obtained in the final phase of the thesis project and for this reason might not always be conclusive; future research on the model could lead to more detailed and satisfactory answers.

I want to express my gratitude to my thesis supervisor Prof. David Barbato, for guiding me through this project and always being available to meet in the office for clarifications and stimulating discussions. I'm deeply indebted to my family for having supported, both emotionally and economically, my studies and decisions during my entire life; this thesis wouldn't have seen the light without them. Finally, I want to thank Laura for her infinite patience and willingness to lend an ear every time I needed it.

Chapter 1

Foundations

1.1 Random variables and Bochner integral

In the following we will always, if not specified otherwise, consider the following setting:

$(\Omega, \mathcal{F}, \mathbb{P})$ denotes a standard probability space and $(E, \mathcal{B}(E))$ is a measurable Banach space, where $\mathcal{B}(E)$ is the Borel σ -algebra induced by the norm $\|\cdot\|$. E^* denotes the topological dual of E , i.e. the collection of all linear continuous functions from E to $\mathbb{R}$.

If X is a random variable from $(\Omega, \mathcal{F}, \mathbb{P})$ into $(E, \mathcal{B})$, we will denote by $\mathcal{L}(X)$ the law of X ,

$$\mathcal{L}(X)(A) := \mathbb{P}(\omega \in \Omega : X(\omega) \in A) \quad \forall A \in \mathcal{B}$$

If X is a real valued random variable, then its integral $\int_{\Omega} X(\omega) d\mathbb{P}(\omega)$ will be denoted by $\mathbb{E}[X]$.

Proposition 1.1. Let E be a separable Banach space. Then $\mathcal{B}(E) = \sigma(\{\varphi \in E^*\})$, or equivalently $\mathcal{B}(E)$ is the smallest σ -algebra generated by sets of the form

$$\{x \in E : \varphi(x) \leq \alpha\} \quad \varphi \in E^*, \alpha \in \mathbb{R} \quad (1.1)$$

Moreover, there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}} \subset E^*$ such that

$$\|x\| = \sup_{n \in \mathbb{N}} |\varphi_n(x)| \quad \text{for any } x \in E \quad (1.2)$$

Proof. Let us first show (1.2). Let $\{x_n\}_{n \in \mathbb{N}}$ be a dense set in E ; by the Hahn-Banach theorem, for each n there exists $\varphi_n \in E^*$ such that $\varphi_n(x_n) = \|x_n\|$ and $\|\varphi_n\| = 1$. We claim that $\{\varphi_n\}_n$ has the desired property. Fix $x \in E$, then $|\varphi_n(x)| \leq \|x\|$ for each n and so $\sup_n |\varphi_n(x)| \leq \|x\|$; conversely, for any $\varepsilon > 0$ there exists x_n such that $\|x - x_n\| < \varepsilon$ and so $|\varphi_n(x) - \varphi_n(x_n)| \leq \varepsilon$, which implies $|\varphi_n(x)| > \|x\| - 2\varepsilon$. By the arbitrariness of ε we can conclude.

It follows from (1.2) that

$$\overline{B}(a, r) = \{x \in E : \|x - a\| \leq r\} = \bigcap_{n \in \mathbb{N}} \{x \in E : |\varphi_n(x) - \varphi_n(a)| \leq r\}$$

Therefore the smallest σ -algebra containing all sets of form (1.1) contains closed balls of E , and thus contains $\mathcal{B}(E)$. Conversely, any $\varphi : (E, \mathcal{B}(E)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ in E^* is continuous, thus measurable, so $\mathcal{B}(E)$ contains all sets of the form (1.1). $\square$

Remark 1.2. It follows from the proposition that if E is a separable Banach space, then a map $X : \Omega \rightarrow E$ is an E -valued random variable if and only if $\varphi(X) : \Omega \rightarrow \mathbb{R}$ is a real random variable for every $\varphi \in E^*$.

Corollary 1.3. Let E be a separable Banach space. The following hold:

- (i) If $X, Y : \Omega \rightarrow E$ are random variables, $\alpha, \beta \in \mathbb{R}$, then $\alpha X + \beta Y$ is a random variable.
- (ii) If $X_n : \Omega \rightarrow E$ is a sequence of random variables s.t. $X_n \rightarrow X$ $\mathbb{P}$ -a.s., then X is a r.v.
- (iii) If X is a E -valued random variable, then $\|X\|$ is a real random variable.

Proof. (i) and (ii) follow immediately from Remark 1.2 and the fact that the statement holds for real random variables; (iii) follows from the fact that $\|\cdot\| : E \rightarrow \mathbb{R}$ is continuous and so its composition with a measurable map is still measurable. $\square$

Remark 1.4. Part (ii) of the corollary can be weakened by only assuming that $X_n \rightarrow X$ $\mathbb{P}$ -a.s. Moreover, we can use the corollary to strengthen the previous remark: in order to show that X is measurable, it suffices that $\varphi(X)$ is measurable for every φ belonging to a linearly dense subset of E^* .

Lemma 1.5. Let E be a separable Banach space, X and Y two E -valued random variables. Then $X = Y$ $\mathbb{P}$ -a.s. if and only if, for all $\varphi \in E^*$, $\varphi(X) = \varphi(Y)$ $\mathbb{P}$ -a.s.

Proof. It suffices to show that if X is an E -valued random variable such that, for all $\varphi \in E^*$, $\varphi(X) = 0$ $\mathbb{P}$ -a.s., then $X = 0$ $\mathbb{P}$ -a.s. Let $\{\varphi_n\} \subset E^*$ satisfying (1.2), then $\varphi_n(X) = 0$ $\mathbb{P}$ -a.s. and so $\sup_n |\varphi_n(X)| = \|X\| = 0$ $\mathbb{P}$ -a.s., which implies the conclusion. $\square$

The next result shows that E -valued random variables can be suitably approximated by simple functions, i.e. functions with a finite image. We state the result in full generality since it doesn't hold only for separable Banach spaces.

Lemma 1.6. Let (E, ρ) be a separable metric space and let X be an E -valued random variable. Then there exists a sequence $\{X_n\}_{n \in \mathbb{N}}$ of simple E -valued random variables such that, for any $\omega \in \Omega$, the sequence $\rho(X(\omega), X_n(\omega))$ is monotonically decreasing to 0.

Proof. Let $\{x_k\}_{k \in \mathbb{N}}$ be a countable dense subset of E . For every $n \in \mathbb{N}$, $\omega \in \Omega$ define

$$\rho_n(\omega) = \min\{\rho(X(\omega), x_k), k = 1, \dots, n\}$$

$$k_n(\omega) = \min\{k \leq n : \rho_n(\omega) = \rho(X(\omega), x_k)\}$$

$$X_n(\omega) = x_{k_n(\omega)}$$

X_n are simple random variables since $X_n(\omega) \in \{x_1, x_2, \dots, x_n\}$. Moreover, by the density of $\{x_k\}_{k \in \mathbb{N}}$, the sequence $\{\rho_n(\omega)\}_{n \in \mathbb{N}}$ is monotonically decreasing to 0 for arbitrary ω .

Since $\rho_n(\omega) = \rho(X(\omega), X_n(\omega))$, the conclusion follows. $\square$

The previous lemma allows us to define the integral for an E -valued random variable on $(\Omega, \mathcal{F}, \mathcal{P})$, where E is a separable Banach space. The idea is to first define it for simple random variables, then extend it to a larger class by approximation and show that the extension does not depend

on the chosen approximating sequence.

For a simple random variable X ,

$$X = \sum_{i=1}^N x_i \mathbb{1}_{A_i}, \quad A_i \in \mathcal{F}, x_i \in E, N \in \mathbb{N}$$

we set

$$\int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{\Omega} X d\mathbb{P} := \sum_{i=1}^N x_i \mathbb{P}(A_i)$$

Here $\mathbb{1}_A$ denotes the indicator function of A . The definition does not depend on the choice of the particular representation of X ; moreover the usual properties of additivity and linearity of the integral hold and

$$\left\| \int_{\Omega} X(\omega) d\mathbb{P}(\omega) \right\| = \left\| \sum_{i=1}^N x_i \mathbb{P}(A_i) \right\| \leq \sum_{i=1}^N \|x_i\| \mathbb{P}(A_i) = \int_{\Omega} \|X(\omega)\| d\mathbb{P}(\omega) \quad (1.3)$$

Definition 1.7. A random variable X is said to be **Bochner integrable** if

$$\int_{\Omega} \|X(\omega)\| d\mathbb{P}(\omega) < \infty \quad (1.4)$$

Note that $\|X\|$ is a real random variable, so the above integral is defined as usual.

Let X be Bochner integrable; by the previous lemma there exists a sequence $\{X_n\}$ of simple random variables such that $\|X(\omega) - X_n(\omega)\| \downarrow 0$ for all ω in Ω . It follows that

$$\left\| \int_{\Omega} X_n d\mathbb{P} - \int_{\Omega} X_m d\mathbb{P} \right\| \leq \int_{\Omega} \|X_n - X_m\| d\mathbb{P}(\omega) \leq \int_{\Omega} \|X_n - X\| d\mathbb{P} + \int_{\Omega} \|X - X_m\| d\mathbb{P} \rightarrow 0$$

as $m, n \rightarrow \infty$, by monotone convergence theorem. Therefore $\int_{\Omega} X_n d\mathbb{P}$ is a Cauchy sequence and it must admit limit in E ; we define the integral of X as

$$\int_{\Omega} X(\omega) d\mathbb{P}(\omega) := \lim_{n \rightarrow \infty} \int_{\Omega} X_n(\omega) d\mathbb{P}(\omega)$$

The limit does not depend on the sequence X_n . In fact, if $X_n, \tilde{X}_n$ are sequences of simple random variables such that $\int_{\Omega} \|X - X_n\| d\mathbb{P}, \int_{\Omega} \|X - \tilde{X}_n\| d\mathbb{P} \rightarrow 0$, then

$$\left\| \int_{\Omega} X_n d\mathbb{P} - \int_{\Omega} \tilde{X}_n d\mathbb{P} \right\| \leq \int_{\Omega} \|X_n - \tilde{X}_n\| d\mathbb{P}(\omega) \leq \int_{\Omega} \|X_n - X\| d\mathbb{P} + \int_{\Omega} \|X - \tilde{X}_n\| d\mathbb{P} \rightarrow 0$$

and so the sequences $\int_{\Omega} X_n d\mathbb{P}, \int_{\Omega} \tilde{X}_n d\mathbb{P}$ have the same limit.

Definition 1.8. Given a Bochner integrable random variable X , the quantity $\int_{\Omega} X d\mathbb{P}$ is called **Bochner integral** of X and is also denoted by $\mathbb{E}[X]$.

Bochner integral enjoys many properties of the Lebesgue integral. In particular it follows by approximation that linearity and estimate (1.3) hold for all random variables satisfying (1.4). We use the same notation for both integrals. Here are some other useful properties of the integral; we omit the proofs as they can be immediately adapted from the standard ones.

Proposition 1.9. Let X be a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a measurable space $(G, \mathcal{G})$ and μ be its law. If ψ is a measurable mapping from $(G, \mathcal{G})$ into $(E, \mathcal{B}(E))$ integrable with respect to μ , then

$$\mathbb{E}[\psi(X)] = \int_G \psi(x) d\mu(x)$$

Theorem 1.10 (Fubini-Tonelli). Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ be two probability spaces and let $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ be their product space. If E is a separable Banach space and $X : \Omega_1 \times \Omega_2 \rightarrow E$ is a Bochner integrable random variable:

$$\int_{\Omega_1 \times \Omega_2} \|X(\omega_1, \omega_2)\| d(\mathbb{P}_1 \otimes \mathbb{P}_2)(\omega_1, \omega_2) < \infty$$

then, for $\mathbb{P}_1$ -a.e. ω_1 , the map $\omega_2 \mapsto X(\omega_1, \omega_2)$ is measurable and Bochner integrable and

$$\int_{\Omega_1 \times \Omega_2} X(\omega_1, \omega_2) d(\mathbb{P}_1 \otimes \mathbb{P}_2)(\omega_1, \omega_2) = \int_{\Omega_1} \int_{\Omega_2} X(\omega_1, \omega_2) d\mathbb{P}_2(\omega_2) d\mathbb{P}_1(\omega_1)$$

Recall that a linear operator $A : D(A) \rightarrow F$, where $D(A)$ is a linear subspace of E , is **closed** if for any sequence $\{x_n\}_{n \in \mathbb{N}} \subset D(A)$ such that $x_n \rightarrow x$ in E and $Ax_n \rightarrow y$ in F , it holds that $x \in D(A)$ and $Ax = y$. Equivalently, A is closed if its graph

$$\text{graph}(A) = \{(x, y) \in E \times F : x \in D(A), y = Ax\}$$

is closed in the product space $E \times F$. Moreover, $D(A)$ endowed with the **graph norm** of A , $\|x\|_A := \|x\|_E + \|Ax\|_F$, is a Banach space.

Theorem 1.11 (Hille). Let E and F be separable Banach spaces and $A : D(A) \rightarrow F$ be a closed operator such that its domain $D(A)$ is a Borel subset of E . If $X : \Omega \rightarrow E$ is a random variable such that $X(\omega) \in D(A)$ for $\mathbb{P}$ -almost every ω , then AX is an F -valued random variable, and X is a $D(A)$ -valued random variable, where $D(A)$ is endowed with the graph norm of A . If moreover X is Bochner integrable and

$$\int_{\Omega} \|AX(\omega)\|_F d\mathbb{P}(\omega) < \infty \tag{1.5}$$

then

$$A \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{\Omega} AX(\omega) d\mathbb{P}(\omega) \tag{1.6}$$

Proof. We only prove the last statement. By Lemma 1.6, there exists a sequence $\{X_n\}_{n \in \mathbb{N}}$ of simple $D(A)$ -valued random variables s.t. $\|X - X_n\|_{D(A)} = \|X - X_n\|_E + \|AX - AX_n\|_F \downarrow 0$. Consequently

$$\int_{\Omega} \|X(\omega) - X_n(\omega)\|_{D(A)} d\mathbb{P}(\omega) = \int_{\Omega} \|X(\omega) - X_n(\omega)\|_E d\mathbb{P}(\omega) + \int_{\Omega} \|AX(\omega) - AX_n(\omega)\|_F d\mathbb{P}(\omega) \rightarrow 0$$

as well. Then

$$\int_{\Omega} X_n(\omega) d\mathbb{P}(\omega) \rightarrow \int_{\Omega} X(\omega) d\mathbb{P}(\omega), \quad \int_{\Omega} AX_n(\omega) d\mathbb{P}(\omega) \rightarrow \int_{\Omega} AX(\omega) d\mathbb{P}(\omega)$$

But

$$\int_{\Omega} AX_n(\omega) d\mathbb{P}(\omega) = A \int_{\Omega} X_n(\omega) d\mathbb{P}(\omega)$$

from the very definition of the integral, and therefore, by the closedness of A ,

$$\int_{\Omega} AX(\omega) d\mathbb{P}(\omega) = A \int_{\Omega} X(\omega) d\mathbb{P}(\omega)$$

□

Remark 1.12. It follows immediately that if X is Bochner integrable and $\varphi \in L(E, F)$, then

$$\varphi\left(\int_{\Omega} X d\mathbb{P}\right) = \int_{\Omega} \varphi(X) d\mathbb{P}$$

In particular, $\mathbb{E}[X]$ is characterized as the unique element $x \in E$ such that

$$\varphi(x) = \mathbb{E}[\varphi(X)] \quad \forall \varphi \in E^*$$

Similarly to the real case, we can define L^p spaces on E , usually called **Bochner spaces**.

We denote by $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ the set of all equivalence classes of E -valued integrable random variables (with respect to the equivalence relation $X \sim Y \Leftrightarrow X = Y$ $\mathbb{P}$ -almost surely). It can be checked that, similarly to the real case, $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ equipped with the norm $\|X\|_1 = \mathbb{E}[\|X\|]$, is a Banach space. In the same way one can define $L^p(\Omega, \mathcal{F}, \mathbb{P}; E)$ for any $p > 1$, with norm

$$\|X\|_p = \begin{cases} \mathbb{E}[\|X\|^p]^{\frac{1}{p}} & \text{if } p \in (1, +\infty) \\ \text{ess. sup}_{\omega \in \Omega} \|X(\omega)\| & \text{if } p = \infty \end{cases}$$

If $(\Omega, \mathcal{F}) = ((a, b), \mathcal{B}(a, b))$ and $\mathbb{P}$ is the Lebesgue measure on (a, b) , we write $L^p(a, b; E)$.

The following result generalizes the one regarding existence and uniqueness of conditional expectation for real random variables.

Theorem 1.13. Let E be a separable Banach space. Let X be a Bochner integrable E -valued random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{G}$ be a sub- σ -algebra of $\mathcal{F}$. There exists a unique, up to a zero measure set, integrable E -valued random variable Z , measurable with respect to $\mathcal{G}$ such that

$$\int_A X d\mathbb{P} = \int_A Z d\mathbb{P} \quad \forall A \in \mathcal{G} \tag{1.7}$$

The random variable Z will be denoted as $\mathbb{E}[X|\mathcal{G}]$ and called the **conditional expectation** of X given $\mathcal{G}$.

Proof. We first show uniqueness. Assume that there exist two random variables Z and $\tilde{Z}$ with the required properties. Then for any $\varphi \in E^*$, using Remark 1.12 and property (1.7), we get

$$\int_A \varphi(Z) d\mathbb{P} = \varphi\left(\int_A Z d\mathbb{P}\right) = \varphi\left(\int_A X d\mathbb{P}\right) = \int_A \varphi(X) d\mathbb{P} \quad \forall A \in \mathcal{G}$$

An analogue equation holds for $\tilde{Z}$; so $\varphi(Z) = \mathbb{E}[\varphi(X)|\mathcal{G}] = \varphi(\tilde{Z})$ $\mathbb{P}$ -a.s., because the result is true for real random variables. But then we conclude that $Z = \tilde{Z}$ $\mathbb{P}$ -a.s. by Lemma 1.5.

We now prove existence. If X is a simple r.v., say $X = \sum_{j=1}^n x_j \mathbb{1}_{A_j}$, then one defines

$$Z = \sum_{j=1}^n x_j \mathbb{P}(A_j|\mathcal{G})$$

where $\mathbb{P}(A_j|\mathcal{G})$ represents the classical notion of the conditional expectation of A_j given $\mathcal{G}$. Z fulfills (1.7) and moreover

$$\mathbb{E}[\|Z\|] \leq \sum_{j=1}^n \mathbb{E}[\|x_j\| \mathbb{P}(A_j|\mathcal{G})] = \sum_{j=1}^n \|x_j\| \mathbb{P}(A_j) = \mathbb{E}[\|X\|]$$

For general X , let $\{X_n\}$ be the sequence defined in Lemma 1.6 and $Z_n = \mathbb{E}[X_n|\mathcal{G}]$. Then

$$\mathbb{E}[\|Z_n - Z_m\|] \leq \mathbb{E}[\|X_n - X_m\|] \rightarrow 0 \quad \text{as } n, m \rightarrow \infty$$

Therefore $\{Z_n\}$ is a Cauchy sequence in L^1 and there exists a $\mathcal{G}$ -measurable random variable Z to which it converges. Moreover, for arbitrary $A \in \mathcal{G}$,

$$\int_A Z_n d\mathbb{P} = \int_A X_n d\mathbb{P} \quad \forall n \in \mathbb{N}$$

and taking the limit as $n \rightarrow \infty$ we get (1.7). □

Remark 1.14. It follows from the proof that $Z = \mathbb{E}[X|\mathcal{G}]$ is uniquely characterized by

$$\varphi(Z) = \mathbb{E}[\varphi(X)|\mathcal{G}] \quad \forall \varphi \in E^* \tag{1.8}$$

In particular it suffices to check such relation for a linearly dense subset of E^* . Moreover, as a consequence of the proof, we have the basic inequality

$$\|\mathbb{E}[X|\mathcal{G}]\| \leq \mathbb{E}[\|X\||\mathcal{G}] \tag{1.9}$$

The following result regarding conditional expectation will be useful later on.

Proposition 1.15. Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be two measurable spaces and $\psi : E_1 \times E_2 \rightarrow \mathbb{R}$ a bounded measurable function. Let ξ_1, ξ_2 be two random variables in $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ respectively, and let $\mathcal{G} \subset \mathcal{F}$ be a fixed σ -algebra. Assume that ξ_1 is $\mathcal{G}$ -measurable, then there is a bounded $\mathcal{E}_1 \otimes \mathcal{G}$ -measurable function $\hat{\psi}(x_1, \omega)$, $x_1 \in E_1$, $\omega \in \Omega$ such that

$$\mathbb{E}[\psi(\xi_1, \xi_2)|\mathcal{G}](\omega) = \hat{\psi}(\xi_1(\omega), \omega), \quad \omega \in \Omega \tag{1.10}$$

Moreover if ξ_2 is independent of $\mathcal{G}$, then

$$\hat{\psi}(x_1, \omega) = \hat{\psi}(x_1) = \mathbb{E}[\psi(x_1, \xi_2)], \quad x_1 \in E_1 \tag{1.11}$$

Proof. Assume first that $\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2)$, where $\psi_i : E_i \rightarrow \mathbb{R}$ are bounded measurable functions. Then

$$\mathbb{E}[\psi(\xi_1, \xi_2)|\mathcal{G}] = \mathbb{E}[\psi_1(\xi_1)\psi_2(\xi_2)|\mathcal{G}] = \psi_1(\xi_1)\mathbb{E}[\psi_2(x_2)|\mathcal{G}]$$

and it is enough to set

$$\widehat{\psi}(x_1, \omega) = \psi_1(x_1)\mathbb{E}[\psi_2(x_2)|\mathcal{G}](\omega)$$

Denote by $\mathcal{G}_1$ the family of all sets $\Gamma \in \mathcal{E}_1 \otimes \mathcal{E}_2$ such that the representation (1.10) holds for $\psi = \mathbb{1}_\Gamma$, and by $\mathcal{K}$ the family of all sets $\Gamma = \Gamma_1 \times \Gamma_2$ where $\Gamma_1 \in \mathcal{E}_1$, $\Gamma_2 \in \mathcal{E}_2$. Then $\mathcal{K}$ is a π -system, $\mathcal{G}_1$ is a d-system and $\mathcal{K} \subset \mathcal{G}_1$, therefore by Dynkin's lemma (1.10) holds for all $\Gamma \in \mathcal{E}_1 \otimes \mathcal{E}_2$. Consequently the result is true for all measurable simple functions ψ and by approximation it can be extended to all bounded measurable ψ . The proof of the representation (1.11) when ξ_2 is independent of $\mathcal{G}$ is similar. $\square$

Remark 1.16. If X and Y are square integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a separable Hilbert space H , then $\langle X, Y \rangle$ is an integrable real random variable, since

$$\mathbb{E}[|\langle X, Y \rangle|] \leq \mathbb{E}[\|X\| \|Y\|] \leq \mathbb{E}[\|X\|^2]^{1/2} \mathbb{E}[\|Y\|^2]^{1/2} < \infty$$

If Y is $\mathcal{G}$ -measurable, $\mathcal{G} \subset \mathcal{F}$, then

$$\mathbb{E}[\langle X, Y \rangle | \mathcal{G}] = \langle \mathbb{E}[X | \mathcal{G}], Y \rangle \quad (1.12)$$

In fact, given an orthonormal basis $\{e_n\}_n$ of H , it holds

$$\begin{aligned} \mathbb{E}[\langle X, Y \rangle | \mathcal{G}] &= \mathbb{E}\left[\sum_n \langle X, e_n \rangle \langle Y, e_n \rangle \middle| \mathcal{G}\right] = \sum_n \mathbb{E}[\langle X, e_n \rangle | \mathcal{G}] \langle Y, e_n \rangle \\ &= \sum_n \langle \mathbb{E}[X | \mathcal{G}], e_n \rangle \langle Y, e_n \rangle = \langle \mathbb{E}[X | \mathcal{G}], Y \rangle \end{aligned}$$

where the exchange between the series and conditional expectation is legit since the series is convergent in $L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and we used Remark 1.14.

In the case of random variables taking values in a separable Hilbert space H , it's also possible to define the covariance operator of X .

Definition 1.17. If $X, Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$, we define the **correlation operator** of (X, Y) and the **covariance operator** of X respectively as

$$\text{Cor}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X]) \otimes (Y - \mathbb{E}[Y])]$$

$$\text{Cov}(X) = \text{Cor}(X, X) = \mathbb{E}[(X - \mathbb{E}[X]) \otimes (X - \mathbb{E}[X])]$$

The definition is well posed since for any $h_1, h_2 \in H$ we have

$$\begin{aligned} |\text{Cor}(X, Y)(h_1, h_2)| &= |\mathbb{E}[\langle X - \mathbb{E}[X], h_1 \rangle \langle Y - \mathbb{E}[Y], h_2 \rangle]| \\ &\leq \|h_1\| \|h_2\| \mathbb{E}[\|X - \mathbb{E}[X]\| \|Y - \mathbb{E}[Y]\|] \\ &\leq \|h_1\| \|h_2\| \mathbb{E}[\|X - \mathbb{E}[X]\|^2]^{1/2} \mathbb{E}[\|Y - \mathbb{E}[Y]\|^2]^{1/2} \end{aligned}$$

Therefore $\text{Cor}(X, Y)$ is a continuous bilinear form and by Riesz theorem can be identified with a linear functional from H to itself; whenever it's convenient we will use one identification or the other without specifying. In particular $\text{Cov}(X)$ is symmetric and nonnegative:

$$\text{Cov}(X)(h_1, h_2) = \text{Cov}(X)(h_2, h_1) \quad \text{Cov}(X)(h_1, h_1) \geq 0 \quad \forall h_1, h_2 \in H$$

Moreover $\text{Cov}(X)$ is a nuclear operator (see the appendix for the definition): given an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ of H , we have

$$\sum_{k=1}^{\infty} \text{Cov}(X)(e_k, e_k) = \sum_{k=1}^{\infty} \mathbb{E}[\langle X - \mathbb{E}[X], e_k \rangle^2] = \mathbb{E}[\|X - \mathbb{E}[X]\|^2] < \infty$$

Remark 1.18. If we identify $\text{Cor}(X, Y)$ with the linear operator from H to itself given by

$$\text{Cor}(X, Y)(h) = \mathbb{E}[(X - \mathbb{E}[X])\langle Y - \mathbb{E}[Y], h \rangle]$$

so that, with some abuse of notation, we can write

$$\text{Cor}(X, Y)(h_1, h_2) = \langle h_1, \text{Cor}(X, Y)h_2 \rangle$$

then for any $A \in L(H, U), B \in L(H, U)$, where U is another separable Hilbert space, denoting by B^* the adjoint operator of B , it holds

$$\text{Cor}(AX, BY) = A \text{Cor}(X, Y)B^* \quad (1.13)$$

since for any $u_1, u_2 \in U$ we have

$$\begin{aligned} \langle u_1, \text{Cor}(AX, BY)u_2 \rangle &= \mathbb{E}[\langle AX - A\mathbb{E}[X], u_1 \rangle \langle BY - B\mathbb{E}[Y], u_2 \rangle] \\ &= \mathbb{E}[\langle X - \mathbb{E}[X], A^*u_1 \rangle \langle Y - \mathbb{E}[Y], B^*u_2 \rangle] \\ &= \langle A^*u_1, \text{Cor}(X, Y)B^*u_2 \rangle = \langle u_1, A \text{Cor}(X, Y)B^*u_2 \rangle \end{aligned}$$

1.2 Gaussian measures on Hilbert spaces

In this section we restrict ourselves to the case of a separable Hilbert space H . Before introducing the concept of Gaussian measure, we need some general concepts on probability distributions.

Definition 1.19. Let μ be a probability measure on $(H, \mathcal{B}(H))$. The **characteristic function** of μ is defined as

$$\hat{\mu}(h) = \varphi_{\mu}(h) = \int_E e^{i\langle h, x \rangle} d\mu(x), \quad h \in H \quad (1.14)$$

Definition 1.20. A **cylindrical set** of H is a set of the form

$$\{x \in H : (\langle h_1, x \rangle, \dots, \langle h_n, x \rangle) \in \Gamma\}$$

for some $h_1, \dots, h_n \in H$ and $\Gamma \in \mathcal{B}(\mathbb{R}^n)$.

Cylindrical sets form a π -system and by Proposition 1.1 they generate $\mathcal{B}(H)$, so if two measures coincide on cylindrical sets then by Dynkin's lemma they are equal.

Proposition 1.21. Assume that M is a linear subspace of H which generates $\mathcal{B}(H)$.

- (i) If μ and ν are probability measures such that $\varphi_\mu(h) = \varphi_\nu(h)$ for all $h \in M$, then $\mu = \nu$.
- (ii) If X and Y are two H -valued random variables such that $\mathcal{L}(\langle h, X \rangle) = \mathcal{L}(\langle h, Y \rangle)$ for all $h \in M$, then $\mathcal{L}(X) = \mathcal{L}(Y)$.

Proof. It suffices to show (i). Fix $h_1, \dots, h_n$ in M , $\lambda_1 \dots \lambda_n$ in $\mathbb{R}$. By the hypothesis we have

$$\int_H e^{i\langle \lambda_1 h_1 + \dots + \lambda_n h_n, x \rangle} d\mu(x) = \int_H e^{i\langle \lambda_1 h_1 + \dots + \lambda_n h_n, x \rangle} d\nu(x)$$

This implies that the $\mathbb{R}^n$ -valued mapping $x \mapsto (\langle h_1, x \rangle, \dots, \langle h_n, x \rangle)$ maps measures μ and ν onto measures $\hat{\mu}$ and $\hat{\nu}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with identical characteristic functions. So $\hat{\mu} = \hat{\nu}$ by the analogue finite dimensional result. But then μ and ν are equal on the π -system of all cylindrical sets generated by M , so $\mu = \nu$ on $\mathcal{B}(H)$. $\square$

Recall that a probability measure μ on $\mathbb{R}$ is Gaussian with mean $m \in \mathbb{R}$ and variance $q \geq 0$ if and only if its characteristic function is given by

$$\varphi_\mu(\lambda) = e^{i\lambda m - \frac{\lambda^2 q}{2}}$$

Similarly, a probability measure μ on $\mathbb{R}^n$ is Gaussian with mean $m \in \mathbb{R}^n$ and variance Q , $Q \in \mathbb{R}^{n \times n}$ being symmetric and nonnegative, if and only if its characteristic function is given by

$$\varphi_\mu(\lambda) = e^{i\langle \lambda, m \rangle - \frac{1}{2}\langle Q\lambda, \lambda \rangle}$$

Moreover, if μ is a Gaussian measure on $\mathbb{R}^n$, then for any $\lambda \in \mathbb{R}^n$ the pushforward measure of the function $x \mapsto \langle \lambda, x \rangle$ is still Gaussian. This motivates the following definition.

Definition 1.22. A probability measure μ on $(H, \mathcal{B}(H))$ is a **Gaussian measure** if for every $h \in H$ the map $\langle h, \cdot \rangle$, considered as a real random variable on $(H, \mathcal{B}(H), \mu)$, is Gaussian distributed. If in addition every $\langle h, \cdot \rangle$ has symmetric (zero mean) Gaussian distribution, then μ is called a **symmetric Gaussian measure**. An H -valued random variable X is a **Gaussian random variable** if $\mathcal{L}(X)$ is Gaussian; equivalently, X is Gaussian distributed if and only if $\langle h, X \rangle$ is a real Gaussian random variable for all $h \in H$.

Remark 1.23. It follows immediately from the definition that if X and Y are H -valued independent Gaussian random variables, then $X + Y$ is Gaussian as well. Moreover, if U is another separable Hilbert space and $T \in L(H, U)$, then $T(X)$ is Gaussian distributed.

If μ is Gaussian, the following functionals are well defined:

$$h \mapsto \int_H \langle h, x \rangle d\mu(x) \quad (h_1, h_2) \mapsto \int_H \langle h_1, x \rangle \langle h_2, x \rangle d\mu(x)$$

In order to show that they are continuous we need the following general lemma.

Lemma 1.24. Let ν be a probability measure on $(H, \mathcal{B}(H))$. Assume that for some $k \in \mathbb{N}$

$$\int_H |\langle z, x \rangle|^k d\nu(x) < \infty \quad \forall z \in H$$

Then there exists a constant $C > 0$ such that

$$\left| \int_H \langle h_1, x \rangle \cdots \langle h_k, x \rangle d\nu(x) \right| \leq C |h_1| \cdots |h_k| \quad \forall h_1, \dots, h_k \in H$$

Proof. Define for any $n \in \mathbb{N}$ the set

$$U_n := \left\{ z \in H : \int_H |\langle z, x \rangle|^k d\nu(x) \leq n \right\}$$

By hypothesis $H = \bigcup_n U_n$ and the U_n are closed sets. Therefore by the Baire category theorem there exists $n_0 \in \mathbb{N}$ such that U_{n_0} has nonempty interior; in particular, there exist $z_0 \in U_{n_0}$ and $r_0 > 0$ such that $\overline{B(z_0, r_0)} \subset U_{n_0}$. Hence

$$\int_H |\langle z_0 + y, x \rangle|^k d\nu(x) \leq n_0 \quad \forall y \in \overline{B(0, r_0)}$$

By the standard inequality $|a + b|^k \leq 2^{k-1}(|a|^k + |b|^k)$, for a, b real numbers, for $y \in \overline{B(0, r_0)}$

$$\int_H |\langle y, x \rangle|^k d\nu(x) \leq 2^{k-1} \int_H (|\langle z_0 + y, x \rangle|^k + |\langle z_0, x \rangle|^k) d\nu(x) \leq 2^k n_0$$

So for all $z \neq 0$ we can apply the previous inequality to $y = r_0 z / |z|$ to obtain

$$\int_H |\langle z, x \rangle|^k d\nu(x) \leq 2^k n_0 r_0^{-k} |z|^k$$

By the elementary inequality

$$|\alpha_1 \cdots \alpha_k| \leq |\alpha_1|^k + \dots + |\alpha_k|^k \quad \forall (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$$

it follows that the map

$$(h_1, \dots, h_k) \mapsto \int_H \langle h_1, x \rangle \cdots \langle h_k, x \rangle d\nu(x)$$

is continuous; since it's k -linear, the conclusion follows. $\square$

It follows from the lemma and Riesz theorem that if μ is Gaussian, then there exist $m \in H$ and $Q \in L(H, H)$ such that

$$\int_H \langle h, x \rangle d\mu(x) = \langle h, m \rangle \quad \int_H \langle h_1, x - m \rangle \langle h_2, x - m \rangle d\mu(x) = \langle Qh_1, h_2 \rangle$$

The vector m is the **mean** and the operator Q is the **covariance** of μ . It's clear that Q is symmetric and nonnegative; by the definition of Gaussian measure, we have

$$\varphi_\mu(h) = e^{i\langle h, m \rangle - \frac{1}{2}\langle Qh, h \rangle} \quad \forall h \in H$$

Therefore by Proposition 1.21 the measure μ is uniquely determined by m and Q ; it's denoted by $\mathcal{N}(m, Q)$. μ is a symmetric Gaussian measure if and only if $m = 0$; without loss of generality from now on we will always work with symmetric Gaussian measures.

Proposition 1.25. Let $\mu \sim \mathcal{N}(0, Q)$. Then Q has finite trace.

Proof. Consider the characteristic function of μ , $\varphi(h) = \int_H e^{i\langle h, x \rangle} d\mu(x) = e^{-\langle Qh, h \rangle/2}$. For arbitrary $c > 0$ we have

$$\begin{aligned} 1 - \varphi(h) &= \int_H (1 - \cos\langle h, x \rangle) d\mu(x) \\ &\leq \frac{1}{2} \int_{|x| \leq c} \langle h, x \rangle^2 d\mu(x) + 2\mu(\{x \in H : |x| > c\}) \\ &\leq \frac{1}{2} \langle Q_c h, h \rangle + 2\mu(\{x \in H : |x| > c\}) \end{aligned}$$

where Q_c is the continuous linear operator defined by

$$Q_c h = \int_{|x| \leq c} x \langle h, x \rangle d\mu(x)$$

Q_c is symmetric, nonnegative and nuclear since for any $\{e_n\}$ orthonormal basis of H we have

$$\sum_n \langle Q_c e_n, e_n \rangle = \sum_n \int_{|x| \leq c} \langle e_n, x \rangle^2 d\mu(x) = \int_{|x| \leq c} |x|^2 d\mu(x) \leq c^2$$

To show that Q has finite trace, it suffices to show that there exist $\beta > 0$ and $c > 0$ such that

$$\forall h \in H, \langle Q_c h, h \rangle \leq 1 \Rightarrow \langle Qh, h \rangle \leq \beta \quad (1.15)$$

In fact, by applying it to $\tilde{h} = h \langle Q_c h, h \rangle^{-1/2}$ we obtain $\langle Qh, h \rangle \leq \beta \langle Q_c h, h \rangle$ and so

$$\text{Tr}(Q) \leq \beta \text{Tr}(Q_c) < \infty$$

Recall that we have

$$1 - e^{-\frac{1}{2}\langle Qh, h \rangle} \leq \frac{1}{2} \langle Q_c h, h \rangle + 2\mu(\{x \in H : |x| > c\})$$

Therefore, if $\langle Q_c h, h \rangle \leq 1$, we obtain

$$e^{\frac{1}{2}\langle Qh, h \rangle} \leq \left(\frac{1}{2} - 2\mu(\{x \in H : |x| > c\}) \right)^{-1}$$

In particular (1.15) holds once we take c sufficiently big such that $\mu(\{x \in H : |x| > c\}) < \frac{1}{4}$ and the proof is concluded. $\square$

It follows that if X is an H -valued random variable, $X \sim \mathcal{N}(0, Q)$, then $\text{Tr}(Q) = \mathbb{E}[|X|^2]$ and the covariance operator Q coincides with the one defined at the end of the previous section. Since Q is a symmetric, nonnegative nuclear operator (therefore compact), by the spectral theorem there exists an orthonormal basis $\{e_k\}_{k \in \mathbb{N}}$ of eigenvectors of Q , with corresponding nonnegative eigenvalues $\{\lambda_k\}_k$. Moreover

$$\text{Tr}(Q) = \sum_{k=1}^{\infty} \lambda_k < \infty$$

For any $x \in H$, set $x_k = \langle x, e_k \rangle$; then $\{x_k\}_k$ is a sequence of independent real random variables, $x_k \sim \mathcal{N}(0, \lambda_k)$, since they are jointly Gaussian and uncorrelated by the orthogonality of $\{e_k\}$. We can use this information to obtain better estimates on μ , as the next proposition shows.

Proposition 1.26. Let $\mu = \mathcal{N}(0, Q)$. If $s < (2\text{Tr}Q)^{-1}$, then

$$\int_H e^{s|x|^2} d\mu(x) \leq \frac{1}{\sqrt{1 - 2s\text{Tr}Q}}$$

Proof. Fix $s < (2\text{Tr}Q)^{-1}$ and set

$$I_n(s) = \int_H e^{s \sum_{i=1}^n x_i^2} d\mu(x)$$

Since $x_1, \dots, x_n$ are independent and $x_i \sim \mathcal{N}(0, \lambda_i)$ we have

$$\begin{aligned} I_n(s) &= \prod_{i=1}^n \int_H e^{sx_i^2} d\mu(x) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\lambda_i}} \int_{\mathbb{R}} e^{-\frac{t^2}{2}(-2s + \frac{1}{\lambda_i})} dt \\ &= \prod_{i=1}^n \frac{1}{\sqrt{1 - 2s\lambda_i}} = \frac{1}{\sqrt{\prod_{i=1}^n (1 - 2s\lambda_i)}} \end{aligned}$$

Since $2s\lambda_i < 1$ for all $i = 1, \dots, n$, we have

$$\prod_{i=1}^n (1 - 2s\lambda_i) \geq 1 - 2s \sum_{i=1}^n \lambda_i$$

The inequality is certainly true for $s \leq 0$; if $0 < 2s\lambda_i < 1$, it follows from an induction argument. Thus

$$I_n(s) \leq \frac{1}{\sqrt{1 - 2s \sum_{i=1}^n \lambda_i}}$$

By letting $n \rightarrow \infty$ we obtain the conclusion. $\square$

Remark 1.27. It follows from the proposition that μ admits moments of any order, namely

$$\int_H |x|^k d\mu(x) < \infty \quad \forall k \in \mathbb{N}$$

Up to now we have studied properties that Gaussian measures must satisfy, assuming they exist, but we haven't addressed the problem of their existence. The following result shows how to construct Gaussian measures starting from their covariance operator.

Proposition 1.28. Let Q be a positive, symmetric, trace class operator in H and let $m \in H$. Then there exists a Gaussian measure in H with mean m and covariance Q .

Proof. Let $\{\xi_n\}$ be a sequence of independent $\mathcal{N}(0, 1)$ real variables. Set

$$\xi = m + \sum_{n=1}^{\infty} \sqrt{\lambda_n} \xi_n e_n \tag{1.16}$$

The series is convergent in L^2 since

$$\mathbb{E} \left[\left(\sum_{n=1}^{\infty} \sqrt{\lambda_n} \xi_n \right)^2 \right] = \sum_{n=1}^{\infty} \lambda_n = \text{Tr}(Q) < \infty$$

Fix $h \in H$, then

$$\begin{aligned}\mathbb{E}\left[e^{i\langle h, \xi \rangle}\right] &= e^{i\langle h, m \rangle} \lim_{n \rightarrow \infty} \mathbb{E}\left[\exp\left(i \sum_{j=1}^n \sqrt{\lambda_j} \xi_j \langle h, e_j \rangle\right)\right] \\ &= \exp\left(i\langle h, m \rangle - \frac{1}{2} \sum_{j=1}^{\infty} \lambda_j \langle h, e_j \rangle^2\right) = e^{i\langle h, m \rangle - \frac{1}{2} \langle Qh, h \rangle}\end{aligned}$$

Thus ξ is Gaussian distributed, $\mathbb{E}[\xi] = m$ and

$$\langle \text{Cov}(\xi)x, y \rangle = \mathbb{E}\left[\sum_{j=1}^{\infty} \sqrt{\lambda_j} \xi_j \langle e_j, x \rangle \sum_{k=1}^{\infty} \sqrt{\lambda_k} \xi_k \langle e_k, y \rangle\right] = \sum_{j=1}^{\infty} \lambda_j \langle e_j, x \rangle \langle e_j, y \rangle = \langle Qx, y \rangle$$

for all $x, y \in H$. Thus $\mathcal{L}(\xi) = \mathcal{N}(m, Q)$. $\square$

Proposition 1.29. Let $\mu = \mathcal{N}(0, Q)$, then for every $m \in \mathbb{N}$ there exists a constant C_m s.t.

$$\int_H |x|^{2m} d\mu(x) \leq C_m \text{Tr}(Q)^m$$

Proof. Using the same notation from the previous proposition,

$$\begin{aligned}\int_H |x|^{2m} d\mu(x) &= \mathbb{E}\left[\left|\sum_{j=1}^{\infty} \sqrt{\lambda_j} \xi_j e_j\right|^{2m}\right] = \mathbb{E}\left[\left(\sum_{j=1}^{\infty} \lambda_j \xi_j^2\right)^m\right] \\ &= \sum_{j_1, \dots, j_m}^{\infty} \lambda_{j_1} \dots \lambda_{j_m} \mathbb{E}[\xi_{j_1}^2 \dots \xi_{j_m}^2] \leq C \left(\sum_{j=1}^{\infty} \lambda_j\right)^m\end{aligned}$$

where $C := \sup_{j_1, \dots, j_m} \mathbb{E}[\xi_{j_1}^2 \dots \xi_{j_m}^2]$ is finite by Cauchy-Schwartz inequality. $\square$

1.3 Stochastic processes

In this section we are going to recall some standard results in the theory of stochastic processes, as well as give some extensions in the Banach-valued case. As before, we consider a separable Banach space E , its Borel σ -algebra $\mathcal{B}(E)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 1.30. Let I be an interval of $\mathbb{R}$. An E -valued **stochastic process** is a family $X = \{X(t)\}_{t \in I}$ of E -valued random variables $X(t)$ defined on Ω . We also say that $X(t)$ is a stochastic process on I . We set $X(t, \omega) = X(t)(\omega)$ for all $t \in I$ and $\omega \in \Omega$. Functions $X(\cdot, \omega)$ are called the **trajectories** of $X(t)$. X is **continuous** if its trajectories $X(\cdot, \omega)$ are continuous for $\mathbb{P}$ -a.e. ω ; X is **α -Holder continuous** if its trajectories $X(\cdot, \omega)$ are α -Holder continuous for $\mathbb{P}$ -a.e. ω .

Definition 1.31. A stochastic process Y is a **modification** of X if

$$\mathbb{P}(X(t) \neq Y(t)) = 0 \quad \forall t \in I$$

An important result on the existence of regular modifications is the following theorem.

Theorem 1.32 (Kolmogorov Continuity Criterion). Let $X(t)$, $t \in [0, T]$, be a stochastic process with values in a complete metric space (E, ρ) such that for some constants $C > 0$, $\varepsilon > 0$, $\delta > 1$ and all $t, s \in [0, T]$,

$$\mathbb{E}[\rho(X(t), X(s))^\delta] \leq C|t - s|^{1+\varepsilon}$$

Then there exists a modification of X which is α -Holder continuous for any $\alpha < \varepsilon/\delta$. In particular, X has a continuous modification.

Proof. Without loss of generality we can assume $T = 1$. Note that the X is stochastically uniformly continuous, in the sense that for any $\beta > 0$

$$\mathbb{P}(\rho(X(t), X(s)) \geq \beta) \leq \beta^{-\delta} \mathbb{E}[\rho(X(t), X(s))^\delta] \leq C\beta^{-\delta}|t - s|^{1+\varepsilon} \quad (1.17)$$

Let $0 < \gamma < \varepsilon/\delta$. By (1.17), for $k = 1, \dots, 2^n$, $n \in \mathbb{N}$,

$$\mathbb{P}(\rho(X(k2^{-n}), X((k-1)2^{-n})) \geq 2^{-n\gamma}) \leq C2^{-n(1+\varepsilon-\gamma\delta)}$$

and therefore

$$\begin{aligned} & \mathbb{P}\left(\max_{1 \leq k \leq 2^n} \rho(X(k2^{-n}), X((k-1)2^{-n})) \geq 2^{-n\gamma}\right) \\ & \leq \sum_{k=1}^{2^n} \mathbb{P}(\rho(X(k2^{-n}), X((k-1)2^{-n})) \geq 2^{-n\gamma}) \leq C2^{-n(\varepsilon-\gamma\delta)} \end{aligned}$$

Since $\sum_n 2^{-n(\varepsilon-\gamma\delta)} < \infty$, by Borel-Cantelli lemma there exists a set $\tilde{\Omega} \in \mathcal{F}$, $\mathbb{P}(\tilde{\Omega}) = 1$ and a random variable $\tilde{N}(\omega)$, $\omega \in \Omega$, taking values in $\mathbb{N}$, such that for $\omega \in \tilde{\Omega}$ and $n \geq \tilde{N}(\omega)$,

$$\max_{1 \leq k \leq 2^n} \rho(X(k2^{-n}), X((k-1)2^{-n})) < 2^{-n\gamma} \quad (1.18)$$

Let

$$D_n = \{k2^{-n} : 0 \leq k \leq 2^n - 1\}, \quad D = \bigcup_{n=1}^{\infty} D_n$$

Any $x = k2^{-n}$ has a unique representation of the form $x = \sum_{j=1}^m \varepsilon_j 2^{-j}$, with $\varepsilon_j = 0$ or 1 .

Now let $m > n \geq \tilde{N}(\omega)$ and $t = k2^{-m}$, $s = l2^{-m}$, $0 \leq l \leq k < 2^m$ be such that $t - s < 2^{-n}$. Then

$$t - s = (k - l)2^{-m} = \sum_{j=n+1}^m \varepsilon_j 2^{-j} \quad \text{with } \varepsilon_j = 0 \text{ or } 1$$

Consequently, by (1.18),

$$\begin{aligned} \rho(X(s), X(t)) & \leq \rho(X(s), X(s + \varepsilon_1 2^{-n-1})) + \dots + \rho(X(s + \sum_{j=n+1}^{m-1} \varepsilon_j 2^{-j}), X(t)) \\ & \leq 2^{-\gamma(n+1)} + \dots + 2^{-\gamma m} \leq 2^{-\gamma(n+1)} \frac{1}{1 - 2^{-\gamma}} \end{aligned}$$

Selecting n such that $2^{-n-1} \leq t - s < 2^{-n}$ one gets

$$\rho(X(t), X(s)) \leq \frac{(t - s)^\gamma}{1 - 2^{-\gamma}} \quad (1.19)$$

Therefore $X(t, \omega)$, $t \in D$, is a uniformly continuous function on D and has unique extension to a continuous function $\tilde{X}(t, \omega)$, $t \in [0, 1]$. Set $\tilde{X}(t, \omega) = 0$ for $t \in [0, 1]$ and $\omega \notin \tilde{\Omega}$. Stochastic continuity implies that the process $\tilde{X}$ is a modification of X and inequality (1.19) implies that trajectories of $\tilde{X}$ are γ -Holder continuous. $\square$

Let us assume that $I = [0, T]$ or $[0, +\infty)$ and that we are given a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$. We will always assume to be working with a **normal filtration**, i.e. complete and right-continuous. $\{\mathcal{F}_t\}$ is **complete** if $\mathcal{F}_0$ contains all $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$; it's **right-continuous** if $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in I$.

Definition 1.33. An E -valued process X is **adapted** to the filtration if $X(t)$ is $\mathcal{F}_t$ -measurable for every $t \in I$. X is **progressively measurable** if for every $t \in I$ the mapping $[0, t] \times \Omega \rightarrow E$, $(s, \omega) \mapsto X(s, \omega)$, is $(\mathcal{B}([0, t]) \otimes \mathcal{F}_t)$ -measurable.

Definition 1.34. An E -valued process X is a **martingale** if it is integrable, adapted to the filtration $\{\mathcal{F}_t\}_{t \in I}$ and for any $s, t \in I$, $s \leq t$, $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$.

Remark 1.35. It follows from the properties of conditional expectation that X is a martingale if and only if it is an integrable process and $\varphi(X)$ is a real martingale for all $\varphi \in E^*$.

Proposition 1.36. The following hold.

- i) If $M(t)$ is a martingale, then $\|M(t)\|$ is a submartingale.
- ii) If g is an increasing convex function from $[0, +\infty)$ to itself and $\mathbb{E}[g(\|M(t)\|)] < \infty$ for $t \in [0, T]$, then $g(\|M(t)\|)$ is a submartingale.

Proof. i) By the basic inequality of conditional expectation, see Remark 1.14, we have

$$\|M(s)\| = \|\mathbb{E}[M(t)|\mathcal{F}_s]\| \leq \mathbb{E}[\|M(t)\||\mathcal{F}_s] \quad \forall s \leq t$$

ii) Using the fact that g is increasing and Jensen inequality, together with the fact that $\|M(t)\|$ is a submartingale, we have

$$g(\|M(s)\|) \leq g(\mathbb{E}[\|M(t)\||\mathcal{F}_s]) \leq \mathbb{E}[g(\|M(t)\|)]$$

$\square$

As an immediate consequence of the proposition and the maximal inequalities for real valued submartingales, we have the following result.

Theorem 1.37 (Doob's maximal inequalities). The following hold.

- i) If $M(t)$, $t \in I$, is an H -valued martingale, I a countable set and $p \geq 1$, then for any $\lambda > 0$

$$\mathbb{P}\left(\sup_{t \in I} \|M(t)\| \geq \lambda\right) \leq \lambda^{-p} \sup_{t \in I} \mathbb{E}[\|M(t)\|^p]$$

- ii) If in addition $p > 1$, then

$$\mathbb{E}\left[\sup_{t \in I} \|M(t)\|^p\right] \leq \left(\frac{p}{p-1}\right)^p \sup_{t \in I} \mathbb{E}[\|M(t)\|^p]$$

- iii) The above estimates remain true if the set I is uncountable and the martingale M is continuous.

We now consider a separable Hilbert space H . Let us fix $T > 0$ and denote by $\mathcal{M}_T^2(H)$ the space of all H -valued, continuous, square integrable martingales M such that $M(0) = 0$. We are going to show that $\mathcal{M}_T^2(H)$ is an Hilbert space.

Proposition 1.38. The space $\mathcal{M}_T^2(H)$, equipped with the inner product

$$\langle M, N \rangle_{\mathcal{M}_T^2(H)} = \mathbb{E}[\langle M(T), N(T) \rangle_H] \quad (1.20)$$

is a Hilbert space.

Proof. It is immediate to check that $\langle M, N \rangle_{\mathcal{M}_T^2(H)}$ defines an inner product. In particular, if $\langle M, M \rangle_{\mathcal{M}_T^2(H)} = 0$ then by point (i) of Theorem 1.37 we have $\sup_{[0, T]} \|M(t)\| = 0$ $\mathbb{P}$ -a.s. Now assume that $\{M_n\}$ is a Cauchy sequence, i.e.

$$\mathbb{E}[\|M_n(T) - M_m(T)\|^2] \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

Then by part (ii) of Theorem 1.37, applied to the martingales $M_n - M_m$, we must also have

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|M_n(t) - M_m(t)\|^2 \right] \rightarrow 0 \text{ as } m, n \rightarrow \infty$$

So we can find a subsequence $\{M_{n_k}\}$ such that

$$\mathbb{P} \left(\sup_{t \in [0, T]} \|M_{n_{k+1}}(t) - M_{n_k}(t)\| \geq 2^{-k} \right) \leq 2^{-k}$$

Then Borel-Cantelli lemma implies that $\{M_{n_k}\}$ converges uniformly $\mathbb{P}$ -a.s. to a process $M(t)$, $t \in [0, T]$. So M is a continuous process and $M(0) = 0$. It's clear that, for any t , the sequence $\{M_{n_k}(t)\}$ converges to $M(t)$ in the mean square. If $0 \leq s \leq t \leq T$ and $k \in \mathbb{N}$ then

$$\mathbb{E}[M_{n_k}(t) | \mathcal{F}_s] = M_{n_k}(s)$$

and letting $k \rightarrow \infty$ we obtain that M is a martingale as well. So $M \in \mathcal{M}_T^2(H)$ and $M_n \rightarrow M$ in $\mathcal{M}_T^2(H)$. $\square$

Remark 1.39. Similarly to the case of real, square integrable martingales, using Remark 1.16 it's immediate to check that for any $s \leq t$

$$\mathbb{E}[\|M(t) - M(s)\|^2 | \mathcal{F}_s] = \mathbb{E}[\|M(t)\|^2 | \mathcal{F}_s] - \|M(s)\|^2$$

and in particular

$$\mathbb{E}[\|M(t) - M(s)\|^2] = \mathbb{E}[\|M(t)\|^2 - \|M(s)\|^2]$$

Recall that if $M \in \mathcal{M}_T^2(\mathbb{R})$, then there exists a unique increasing predictable process $[M](t)$, starting from 0, called the **quadratic variation** of M , such that the process

$$M^2(t) - [M](t), \quad t \in [0, T]$$

is a continuous martingale. We are now going to define the quadratic variation process for $M \in \mathcal{M}_T^2(H)$. Denote by $L_1(H)$ the space of all nuclear operators on H , equipped with the nuclear norm. Then $L_1 = L_1(H)$ is a separable Banach space and for each $a, b \in H$ the mapping $T \mapsto \langle Ta, b \rangle$ is a continuous functional on L_1 . An L_1 -valued process $V(\cdot)$ is said to be **increasing** if the operators $V(t)$, $t \in [0, T]$, are nonnegative and $V(s) \leq V(t)$ if $0 \leq s \leq t \leq T$.

Definition 1.40. Let $M \in \mathcal{M}_T^2(H)$. An L_1 -valued continuous, adapted and increasing process V such that $V(0) = 0$ is said to be the **quadratic variation process** of M if for any $a, b \in H$ the process

$$\langle M(t), a \rangle \langle M(t), b \rangle - \langle V(t)a, b \rangle, \quad t \in [0, T]$$

is an $\mathcal{F}_t$ -martingale, or equivalently if and only if the L_1 -valued process

$$M(t) \otimes M(t) - V(t), \quad t \in [0, T]$$

is an $\mathcal{F}_t$ -martingale.

We now show existence and uniqueness of the process $V(\cdot)$, which will be denoted by $[M](t)$.

Proposition 1.41. An arbitrary $M \in \mathcal{M}_T^2(H)$ has exactly one quadratic variation process. Moreover, for any $t \in [0, T]$ and for any sequence of subdivisions $\Delta_n[0, t]$ of the interval $[0, t]$ such that $|\Delta_n[0, t]| \rightarrow 0$, where $|\Delta_n[0, t]| = \sup_{k=1, \dots, N_n} |t_k - t_{k-1}|$, the following convergence in probability takes place in L_1 :

$$\sum_{k=1}^{N_n} (M(t_k) - M(t_{k-1})) \otimes (M(t_k) - M(t_{k-1})) \rightarrow [M](t) \quad (1.21)$$

Sketch of proof. The proof is basically an adaptation of the standard one for real valued martingales, which can be found for example in [17], Theorem 5.14 and Exercise 5.15, p. 147. Therefore we omit the complete proof and we only show how the process $[M]$ can be represented. Let $\{e_i\}$ be an orthonormal basis of H . Then the processes $M_i(t) = \langle M(t), e_i \rangle$, are continuous real square integrable martingales. Note that

$$\mathbb{E} \left[\sum_{i=1}^{\infty} [M_i](t) \right] = \mathbb{E} \left[\sum_{i=1}^{\infty} \langle M(t), e_i \rangle^2 \right] = \mathbb{E}[|M(t)|^2] < \infty$$

Consequently the sum is convergent $\mathbb{P}$ -a.s. and the formula

$$[M](t) = \sum_{i,j=1}^{\infty} [M_i, M_j](t) e_i \otimes e_j, \quad t \in [0, T] \quad (1.22)$$

defines an $L_1(H)$ -valued adapted process. It is easy to see that, for any $a, b \in H$,

$$\langle M(t), a \rangle \langle M(t), b \rangle - \langle [M](t)a, b \rangle$$

is a continuous martingale. Moreover $[M](t)$ is $\mathbb{P}$ -a.s. a nonnegative operator. One can also show that the constructed process is $L_1(H)$ -continuous and that convergence (1.21) takes place. $\square$

Remark 1.42. It follows from the proof that

$$\|M\|^2 - \text{Tr}([M]) = \sum_{i=1}^{\infty} (M_i^2 - [M_i])$$

is a martingale as well, i.e. $[\|M\|^2] = \text{Tr}([M])$.

In a similar way, one can define the **cross quadratic variation** for $M \in \mathcal{M}_T^2(U)$, $N \in \mathcal{M}_T^2(H)$ where U and H are two separable Hilbert spaces. Namely we define

$$[M, N](t) := \sum_{i,j=1}^{\infty} [M_i, N_j](t) e_i \otimes f_j$$

where $\{e_i\}$ and $\{f_j\}$ are orthonormal bases in U and H respectively. Taking a sequence of subdivisions $\Delta_n[0, t]$ as above, we still have the following convergence in probability in $L_1(H, U)$:

$$\sum_{k=1}^{N_n} (M(t_k) - M(t_{k-1})) \otimes (N(t_k) - N(t_{k-1})) \rightarrow [M, N](t)$$

If M and N take values on the same Hilbert space H , then taking the trace we also obtain that $\langle M, N \rangle - \text{Tr}([M, N])$ is a martingale, $[\langle M, N \rangle] = \text{Tr}([M, N])$.

Remark 1.43. For any fixed t , we can identify the bilinear operator $[M, N](t)$ with the element of $L(H, U)$ given by

$$[M, N](t)(h) = [M, \langle N, h \rangle](t)$$

Similarly to Remark 1.18, it can be shown that for any $A \in L(U, \tilde{U})$ and $B \in L(H, \tilde{H})$ it holds

$$[AM, BN](t) = A[M, N](t)B^*$$

The following results are direct generalizations of the analogue statements for real martingales.

Proposition 1.44. The following hold.

- i) If $M \in \mathcal{M}_T^2(H)$, and $[M] \equiv 0$ on $[0, T]$, then $\mathbb{P}$ -a.s. $M \equiv 0$ on $[0, T]$.
- ii) If $M^1 \in \mathcal{M}_T^2(H_1)$, $M^2 \in \mathcal{M}_T^2(H_2)$ are independent martingales, then $[M^1, M^2] \equiv 0$.

Theorem 1.45 (Davis inequality). If $M \in \mathcal{M}_T^2(H)$, then

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|M(t)\| \right] \leq 3 \mathbb{E} \left[(\text{Tr}[M](T))^{1/2} \right]$$

Definition 1.46. A function $f : [0, T] \rightarrow H$ is said of **bounded variation** if

$$\sup \left\{ \sum_{i=1}^n \|f(t_i) - f(t_{i-1})\| \mid n \in \mathbb{N}, 0 = t_0 < t_1 < \dots < t_n = T \right\} < \infty$$

An H -valued process X is said a **bounded variation process** if its trajectories have bounded variation $\mathbb{P}$ -a.s.

Definition 1.47. An H -valued process X is a **continuous semimartingale** if it is continuous, adapted and it can be written as $X(t) = M(t) + A(t)$, where $M \in \mathcal{M}_T^2(H)$ and A is a continuous adapted process of bounded variation.

Remark 1.48. It follows immediately from point i) of Proposition 1.44 that the decomposition $X = M + A$ is unique.

We can extend the definition of the quadratic variation process to continuous semimartingales $X = M + A$ by setting $[X] = [M]$; in particular, the quadratic variation is still uniquely determined as the limit in probability of sequences of the form (1.21). Analogue definitions and statements hold for the cross quadratic variation of two continuous semimartingales.

Definition 1.49. An H -valued stochastic process X is said to be **Gaussian** if for any $n \in \mathbb{N}$ and for any positive numbers $t_1, \dots, t_n$, the H^n -valued random variable $(X(t_1), \dots, X(t_n))$ is Gaussian.

Proposition 1.50. Let X be a Gaussian process on H . Assume that $\mathbb{E}(X(t)) = 0$, $t \geq 0$, and that there exist $M > 0$ and $\gamma \in (0, 1]$ such that

$$\mathbb{E}[|X(t) - X(s)|^2] \leq M|t - s|^\gamma \quad \forall t, s \geq 0 \quad (1.23)$$

Then X has an α -Holder continuous modification for any $\alpha \in (0, \gamma/2)$.

Proof. From (1.23) and Proposition 1.29 it follows that

$$\mathbb{E}[|X(t) - X(s)|^{2m}] \leq C_m \mathbb{E}[|X(t) - X(s)|^2]^m \leq \tilde{C}_m |t - s|^{m\gamma}, \quad \forall t, s \geq 0$$

so, by the Kolmogorov test, X has an α_m -Holder continuous modification with $\alpha_m = \frac{m\gamma-1}{2m}$. Taking $m \rightarrow \infty$ the conclusion follows. $\square$

Let X be a Gaussian process in a Hilbert space H . Let

$$m(t) = \mathbb{E}[X(t)], \quad Q(t) = \mathbb{E}[(X(t) - m(t)) \otimes (X(t) - m(t))] \quad t \geq 0$$

and

$$B(t, s) = \mathbb{E}[(X(t) - m(t)) \otimes (X(s) - m(s))], \quad t, s \geq 0$$

Definition 1.51. The process X is said to be **stationary** if

$$\mathbb{E} \left[e^{i \sum_{k=1}^n \langle X(t_k+r), h_k \rangle} \right] = \mathbb{E} \left[e^{i \sum_{k=1}^n \langle X(t_k), h_k \rangle} \right]$$

for all $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0, +\infty)$, $h_1, \dots, h_n \in H$, and $r \in [0, \infty)$.

Proposition 1.52. A Gaussian process X is stationary if and only if

- i) $m(t+r) = m(t)$, for all $t, r \geq 0$.
- ii) $B(t+r, s+r) = B(t, s)$ for all $t, s, r \geq 0$.

Proof. Let $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0, \infty)$, $\xi_1, \dots, \xi_n \in H$ and $r > 0$. Then $(\langle X(t_1), \xi_1 \rangle, \dots, \langle X(t_n), \xi_n \rangle)$ is a Gaussian $\mathbb{R}^n$ -valued random variable and

$$\mathbb{E} \left[\exp \left(i \sum_{k=1}^n \langle X(t_k), \xi_k \rangle \right) \right] = \exp \left(i \sum_{k=1}^n \langle m(t_k), \xi_k \rangle - \frac{1}{2} \sum_{k,l=1}^n \langle B(t_k, t_l) \xi_k, \xi_l \rangle \right)$$

The same equation holds for $(\langle X(t_1 + r), \xi_1 \rangle, \dots, \langle X(t_n + r), \xi_n \rangle)$ and therefore the conclusion follows. $\square$

Remark 1.53. It's easy to see from the proof that the distribution of a Gaussian process is uniquely determined by the functions m and B .

Chapter 2

Stochastic Analysis in Hilbert spaces

In this section we introduce Hilbert-valued Wiener processes and we construct the stochastic integral with respect to them. We also discuss properties of the stochastic integral and we prove results like Ito formula and Girsanov theorem.

2.1 Wiener processes in Hilbert spaces

Let U be a separable Hilbert space, $Q \in L(U)$ a symmetric, positive trace class operator. Then there exists an orthonormal complete basis $\{e_k\}$ of eigenvectors of Q relative to the eigenvalues $\{\lambda_k\}$, $\lambda_k \geq 0$, $\text{Tr } Q = \sum_k \lambda_k < \infty$.

Definition 2.1. A U -valued stochastic process $W(t)$, $t \geq 0$, is called a **Q -Wiener process** if

- (i) $W(0) = 0$;
- (ii) W has continuous trajectories;
- (iii) W has independent increments;
- (iv) $\mathcal{L}(W(t) - W(s)) = \mathcal{N}(0, (t - s)Q)$, $0 \leq s \leq t$.

Remark 2.2. It follows immediately from the definition that, if W is a Q -Wiener process, then for any $u \in U$ the process $\langle W(t), u \rangle$ is a real Brownian motion with

$$\mathbb{E}[\langle W(t), u \rangle^2] = \langle tQu, u \rangle = t|Q^{1/2}u|^2$$

In particular, if u is such that $|Q^{1/2}u| = 1$, then $\langle W(t), u \rangle$ is a standard Brownian motion.

Definition 2.3. An U -valued process W on a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ is a Q -Wiener process with respect to $\{\mathcal{F}_t\}$ if W is a Q -Wiener process, it's adapted to the filtration and for any $s \leq t$, $W(t) - W(s)$ is independent of $\mathcal{F}_s$.

Proposition 2.4. Let W be a Q -Wiener process, then it's a Gaussian process on U with

$$\mathbb{E}(W(t)) = 0, \quad \text{Cov}(W(t)) = tQ$$

Moreover, for any $t \geq 0$, W has the expansion

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k \tag{2.1}$$

where

$$\beta(t) = \frac{1}{\sqrt{\lambda_k}} \langle W(t), e_k \rangle, \quad k \in \mathbb{N} \quad (2.2)$$

are real valued independent Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ and the series in (2.2) is convergent in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$.

Proof. Let $n \in \mathbb{N}$, $0 < t_1 < \dots < t_n$ and $u_1, \dots, u_n \in U$. Let us consider Z defined by

$$Z = \sum_{k=1}^n \langle W(t_k), u_k \rangle = \sum_{k=1}^n \langle W(t_1), u_k \rangle + \sum_{k=2}^n \langle W(t_2) - W(t_1), u_k \rangle + \dots + \langle W(t_n) - W(t_{n-1}), u_n \rangle$$

Since W has independent increments, Z is Gaussian for any choice of $u_1, \dots, u_n$ and so W is a Gaussian process. Let $0 \leq s \leq t$, then by (2.1) it follows that

$$\begin{aligned} \mathbb{E}[\beta_i(t)\beta_j(s)] &= \frac{1}{\sqrt{\lambda_i\lambda_j}} \mathbb{E}[\langle W(t), e_i \rangle \langle W(s), e_j \rangle] \\ &= \frac{1}{\sqrt{\lambda_i\lambda_j}} \mathbb{E}[\langle W(s), e_i \rangle \langle W(s), e_j \rangle] \\ &= \frac{1}{\sqrt{\lambda_i\lambda_j}} s \langle Q e_i, e_j \rangle = s \delta_{ij} \end{aligned}$$

and so independence of β_i , $i \in \mathbb{N}$ follows. Observe that, for $n \leq m$,

$$\mathbb{E} \left[\left| \sum_{k=n}^m \sqrt{\lambda_k} \beta_k(t) e_k \right|^2 \right] = \sum_{k=n}^m \lambda_k$$

where $\sum_{k=1}^{\infty} \lambda_k < \infty$. Therefore the series in (2.1) is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P}; U)$ which converges pointwise to $W(t)$, and the conclusion follows. $\square$

Proposition 2.5. For any trace class, symmetric, nonnegative operator Q on a separable Hilbert space U there exists a Q -Wiener process $W(t)$, $t \geq 0$.

Proof. The existence of a process W satisfying conditions (i), (iii) and (iv) of Definition 2.1 is a consequence of the Kolmogorov extension theorem; alternatively, it can be shown constructively by taking a family $\{\beta_k\}_k$ of real standard independent Brownian motions and defining W as in (2.1). Proposition 1.50 then guarantees the existence of a modification which is α -Holder continuous for any $\alpha \in (0, \frac{1}{2})$. $\square$

Theorem 2.6. Let W be a Q -Wiener process. Then the series (2.1) is $\mathbb{P}$ -a.s. uniformly convergent on $[0, T]$ for arbitrary $T > 0$.

Proof. Consider the random variables ξ_k defined by

$$\xi_k(t) = \sqrt{\lambda_k} \beta_k(t) e_k, \quad t \in [0, T]$$

and the sequence $S_n(t) = \sum_{k=1}^n \xi_k(t)$. By Doob's maximal inequality, for any $\varepsilon > 0$

$$\mathbb{P} \left(\sup_{t \leq T} |W(t) - S_n(t)| > \varepsilon \right) = \mathbb{P} \left(\sup_{t \leq T} \left| \sum_{k \geq n} \xi_k(t) \right| > \varepsilon \right) \leq \frac{1}{\varepsilon^2} \mathbb{E} \left[\sum_{k \geq n} |\xi_k(T)|^2 \right] = \frac{T}{\varepsilon^2} \sum_{k \geq n} \lambda_k$$

So, the sequence of $C([0, T]; U)$ -valued random variables S_n converges in probability (w.r.t. to the norm of $C([0, T]; U)$) to the Wiener process W . In order to show that it converges uniformly $\mathbb{P}$ -a.s. to W , it suffices to show that the sequence S_n is $\mathbb{P}$ -a.s. a Cauchy sequence. In particular, it's enough to show that, for any $\varepsilon > 0$,

$$\mathbb{P}\left(\sup_{k \geq n} \|S_k - S_n\|_{C([0, T]; U)} > \varepsilon\right) \rightarrow 0 \text{ as } n \rightarrow \infty$$

Observe that the sequence $\{S_n\}$ is a $C([0, T]; U)$ -valued martingale w.r.t. the index n . Therefore by applying again Doob's maximal inequality we have

$$\begin{aligned} \mathbb{P}\left(\sup_{n \leq k \leq l} \|S_k - S_n\|_{C([0, T]; U)} > \varepsilon\right) &\leq \frac{1}{\varepsilon^2} \mathbb{E}\left[\|S_l - S_n\|_{C([0, T]; U)}^2\right] \\ &= \frac{1}{\varepsilon^2} \mathbb{E}\left[\sup_{[0, T]} |S_l(t) - S_n(t)|^2\right] \\ &\leq \frac{4}{\varepsilon^2} \mathbb{E}\left[|S_l(T) - S_n(T)|^2\right] = \frac{4}{\varepsilon^2} \sum_{k=n}^l \lambda_k \end{aligned}$$

Taking $l \rightarrow \infty$ we obtain that for any $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{k \geq n} \|S_k - S_n\|_{C([0, T]; U)} > \varepsilon\right) \leq \frac{4}{\varepsilon^2} \sum_{k=n}^{\infty} \lambda_k$$

and the conclusion follows. $\square$

Note that the quadratic variation of a Q -Wiener process in U is given by $[W](t) = tQ$, $t \geq 0$. We have in fact the following generalization of Levy's one dimensional result.

Theorem 2.7 (Levy's theorem). A martingale $M \in \mathcal{M}_T^2(U)$ is a Q -Wiener process on $[0, T]$ with respect to the filtration $\{\mathcal{F}_t\}$ if and only if $[M](t) = tQ$.

Proof. We only need to show that, if $[M](t) = tQ$, $t \geq 0$, then M is a Q -Wiener process with respect to $\{\mathcal{F}_t\}$. Note that, for any $k \in \mathbb{N}$, the process $M_k(t) = \langle M(t), e_k \rangle$ belongs to $M_T^2(\mathbb{R})$ and $[M_k](t) = \langle [M](t)e_k, e_k \rangle = \lambda_k t$. So by Levy's theorem, M_k is a real λ_k -Wiener process with respect to the filtration $\{\mathcal{F}_t\}$. By the same argument, the finite dimensional processes $(M_1(t), \dots, M_n(t))$ are Wiener processes with respect to $\{\mathcal{F}_t\}$ with diagonal quadratic variation $\text{Diag}(\lambda_1, \dots, \lambda_n)$. Consequently M can be written as

$$M(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k$$

where $\beta_k(t) = \frac{1}{\sqrt{\lambda_k}} M_k(t)$ are normalized independent Wiener processes with respect to $\{\mathcal{F}_t\}$ and the conclusion follows. $\square$

Generalized Wiener processes

In this section we consider a fixed filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$; without specifying each time, all Wiener processes considered are with respect to the filtration $\{\mathcal{F}_t\}$.

Let $W(t)$, $t \geq 0$, be a Q -Wiener process on a Hilbert space U . Then for each $a \in U$ we can define a real valued Wiener process $W_a(t)$, $t \geq 0$, by the formula

$$W_a(t) = \langle a, W(t) \rangle$$

The transformation $a \mapsto W_a$ is linear from U to the space $\mathcal{M}_T^2(H)$, for any $T \in (0, +\infty)$; moreover it is continuous, as for any sequence $\{a_n\} \subset U$

$$a_n \rightarrow a \text{ in } U \Rightarrow \mathbb{E}[|W_{a_n}(T) - W_a(T)|^2] \rightarrow 0 \quad (2.3)$$

This suggests the following extension of the definition of Wiener process.

Definition 2.8. Any linear transformation $a \mapsto W_a$ whose values are real valued Wiener processes on $[0, +\infty)$ satisfying property (2.3) is called a **generalized Wiener process**.

It follows from the definition that there exists a bilinear form $K(a, b)$, $a, b \in U$, such that

$$\mathbb{E}[W_a(t)W_b(s)] = t \wedge s \mathbb{E}[W_a(1)W_b(1)] = t \wedge s K(a, b), \quad t, s \geq 0, \quad a, b \in U \quad (2.4)$$

Condition (2.3) implies that K is a continuous bilinear form in U and therefore there exists $Q \in L(U, U)$ such that

$$\mathbb{E}[W_a(t)W_b(s)] = t \wedge s \langle Qa, b \rangle, \quad t, s \geq 0, \quad a, b \in U \quad (2.5)$$

Observe that Q is symmetric and nonnegative.

Definition 2.9. The operator Q is the **covariance** of the generalized Wiener process $a \mapsto W_a$. If the covariance Q is the identity operator I , then the generalized Wiener process is called a **cylindrical Wiener process** in U . The space $U_0 := Q^{1/2}(U)$, with the induced inner product $\langle a, b \rangle_0 := \langle Q^{-1/2}a, Q^{-1/2}b \rangle$, is called the **reproducing kernel** of the generalized Wiener process $a \mapsto W_a$.

Lemma 2.10. For any symmetric and nonnegative operator $Q \in L(U, U)$ there exists a generalized Wiener process with covariance Q .

Proof. Since Q is symmetric and nonnegative, it admits square root $Q^{1/2} \in L(U; U)$ (see [18], Theorem 9.4-2, p.476). Let $\{e_k\}$ be an orthonormal basis of U , $\{\beta_k\}$ a sequence of independent real valued standard Wiener processes. Define

$$W_a(t) = \sum_{k=1}^{\infty} \langle Q^{1/2}e_k, a \rangle \beta_k(t), \quad t \geq 0, \quad a \in U \quad (2.6)$$

We claim that W_a has the desired properties. Since

$$\sum_{k=1}^{\infty} \langle Q^{1/2}e_k, a \rangle^2 = \sum_{k=1}^{\infty} \langle e_k, Q^{1/2}a \rangle^2 = |Q^{1/2}a|^2 < \infty$$

the series defining W_a is convergent in $\mathcal{M}_T^2(\mathbb{R})$ and so W_a is a Wiener process for every $a \in U$. The same computation also shows that $a \mapsto W_a$ satisfies (2.3). Finally

$$\mathbb{E}[W_a(t)W_b(s)] = (t \wedge s) \sum_{k=1}^{\infty} \langle Q^{1/2}e_k, a \rangle \langle Q^{1/2}e_k, b \rangle = t \wedge s \langle Q^{1/2}a, Q^{1/2}b \rangle = t \wedge s \langle Qa, b \rangle$$

and the result follows. $\square$

Formula (2.6) seems to imply that we have $W_a = \langle W, a \rangle$, with W given by

$$W(t) = \sum_{k=1}^{\infty} Q^{1/2}e_k \beta_k(t)$$

which is fairly similar to (2.1). However Q is not a trace class operator and so the series above is not convergent in $L^2(\Omega, \mathbb{P}; U)$. The next proposition shows that we can still give meaning to the above expression by suitably enlarging the space U .

Proposition 2.11. Let U_1 be a Hilbert space such that $U_0 = Q^{1/2}(U)$ is embedded into U_1 with a Hilbert-Schmidt embedding J . Then the formula

$$W(t) = \sum_{k=1}^{\infty} Q^{1/2}e_k \beta_k(t), \quad t \geq 0 \tag{2.7}$$

defines a U_1 -valued Wiener process, by identifying $Q^{1/2}e_k \in U$ with $J(Q^{1/2}e_k) \in U_1$. Moreover, if Q_1 is the covariance of W , then the spaces $Q_1^{1/2}(U_1)$ and U_0 are identical, in the sense that $J : U_0 \rightarrow Q_1^{1/2}(U_1)$ gives an isometry between the two reproducing kernels.

Proof. The elements $g_k = Q^{1/2}e_k$ form an orthonormal basis of U_0 ; since J is Hilbert-Schmidt

$$\sum_{k=1}^{\infty} |Jg_k|_{U_1}^2 < \infty$$

Consequently the series in (2.7) defines a Wiener process in U_1 . For $a, b \in U_1$ we have

$$\begin{aligned} \langle Q_1 a, b \rangle_{U_1} &= \mathbb{E}[\langle W(1), a \rangle_{U_1} \langle W(1), b \rangle_{U_1}] = \sum_{k=1}^{\infty} \langle Jg_k, a \rangle_{U_1} \langle Jg_k, b \rangle_{U_1} \\ &= \sum_{k=1}^{\infty} \langle g_k, J^* a \rangle_{U_0} \langle g_k, J^* b \rangle_{U_0} = \langle J^* a, J^* b \rangle_{U_0} = \langle J J^* a, b \rangle_{U_1} \end{aligned}$$

Consequently, $Q_1 = J J^*$. In particular

$$|Q_1^{1/2} a|_{U_1}^2 = \langle Q_1 a, a \rangle_{U_1} = \langle J^* a, J^* a \rangle_{U_0} = |J^* a|_{U_0}^2$$

Thus by Proposition B.4, applied to the operators $Q_1^{1/2} : U_1 \rightarrow U_1$ and $J : U_0 \rightarrow U_1$, we have $Q_1^{1/2}(U_1) = J(U_0)$ and $|Q_1^{-1/2} u|_{U_1} = |u|_{U_0}$, as required. $\square$

Thus, with some abuse of language we can say that any generalized Wiener process on U is a classical Wiener process in some larger Hilbert space U_1 . There is no canonical way to define the extension U_1 , but reproducing kernels related to all these extensions are the same, which in a sense means that such extension is unique.

To complete the picture we will now show that any generalized Wiener process is of the form (2.7). Without loss of generality, we can assume Q to be nondegenerate; therefore $Q^{1/2}(U)$ is dense in U : if $x \in U$ is such that $\langle x, Q^{1/2}u \rangle = 0$ for every u , then $\langle Q^{1/2}x, u \rangle = 0$ and so $x = 0$. Now let $\{f_j\}$ be an orthonormal basis of U and set

$$V := \text{Span}(\{Q^{1/2}f_j\})$$

Then V is a dense subspace of U ; by applying the orthogonalization procedure to $\{Q^{1/2}f_j\}$, we can obtain an orthonormal basis $\{e_k\}$ of U . Note that $\text{Span}(\{e_k\}) = V \subset Q^{1/2}(U)$ and

$$U = \overline{\text{Span}(\{Q^{1/2}f_j\})} = \overline{\text{Span}(\{e_k\})}$$

Proposition 2.12. For every $a \in U$ we have

$$W_a(t) = \sum_{k=1}^{\infty} \langle Q^{1/2}e_k, a \rangle W_{Q^{-1/2}e_k}(t), \quad t \geq 0 \quad (2.8)$$

Proof. Observe that for any i, j we have

$$\mathbb{E}[W_{Q^{-1/2}e_i}(s)W_{Q^{-1/2}e_j}(t)] = t \wedge s \langle QQ^{-1/2}e_i, Q^{-1/2}e_j \rangle = t \wedge s \delta_{ij}$$

therefore $W_{Q^{-1/2}e_i}$, $i \in \mathbb{N}$, are independent standard brownian motions and so by formula (2.6) the right-hand side of (2.8), denote it by $\widetilde{W}_a$, is a generalised Wiener process with covariance Q . Now fix $a = Q^{-1/2}e_k$, then

$$\widetilde{W}_a(t) = \sum_{j=1}^{\infty} \langle Q^{1/2}e_j, Q^{-1/2}e_k \rangle W_{Q^{-1/2}e_j}(t) = \sum_{j=1}^{\infty} \langle e_j, e_k \rangle W_{Q^{-1/2}e_j}(t) = W_{Q^{-1/2}e_k}(t) = W_a(t)$$

Since the set $\{Q^{-1/2}e_k\}$ is linearly dense in U and the map $a \mapsto W_a$ is linear and continuous, the identity can be extended to all $a \in U$. $\square$

2.2 The stochastic integral

In this section we are going to define stochastic integration, first with respect to Q -Wiener processes and then also with respect to generalised ones. The procedure is basically the same as in the real case: we first define the stochastic integral for elementary processes and then we show that this provides an isometry and can be extended to more general predictable processes; by a localization procedure we then show how the definition can be further extended.

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be a filtered probability space, $\{\mathcal{F}_t\}$ a normal filtration; consider a Q -Wiener process W with respect to $\{\mathcal{F}_t\}$, taking values in U , where Q is a nuclear operator. Then we know that there exists an orthonormal basis $\{e_k\}$ of U such that W can be represented by

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k$$

We assume for simplicity that Q is non degenerate, i.e. $\lambda_k > 0$ for all $k \in \mathbb{N}$.

Definition 2.13. Let us fix $T < \infty$. An $L = L(U, H)$ -valued process $\Phi(t)$, $t \in [0, T]$, taking only a finite number of values is said to be **elementary** if there exists a sequence $0 = t_0 < t_1 < \dots < t_n = T$ and a sequence $\Phi_0, \dots, \Phi_{n-1}$ of L -valued simple random variables such that Φ_k are $\mathcal{F}_{t_k}$ -measurable and

$$\Phi(t) = \Phi_0 \mathbb{1}_{\{0\}} + \sum_{k=0}^{n-1} \Phi_k \mathbb{1}_{(t_k, t_{k+1}]}(t)$$

For elementary processes Φ we define the **stochastic integral** by the formula

$$\int_0^t \Phi(s) dW(s) = \sum_{k=0}^{n-1} \Phi_k (W_{t_{k+1} \wedge t} - W_{t_k \wedge t})$$

and denote it by $\Phi \cdot W(t)$, $t \in [0, T]$.

Recall that the space $U_0 = Q^{1/2}(U)$, with scalar product $\langle a, b \rangle_0 = \langle Q^{-1/2}a, Q^{-1/2}b \rangle$, is a Hilbert space with orthonormal complete basis given by $g_k = \sqrt{\lambda_k} e_k$. Then the space $L_2^0 = L_2(U_0, H)$ of all Hilbert-Schmidt operators from U_0 into H is a separable Hilbert space with norm

$$\|\Psi\|_{L_2^0}^2 = \sum_{i,j=1}^{\infty} \langle \Psi g_i, f_j \rangle^2 = \sum_{i,j=1}^{\infty} \lambda_i \langle \Psi e_i, f_j \rangle^2 = \|\Psi Q^{1/2}\|_{L_2}^2 = \text{Tr}((\Psi Q^{1/2})(\Psi Q^{1/2})^*)$$

where $\{f_j\}$ is any orthonormal basis of H . Observe that $L \subset L_2^0$ with continuous embedding: in fact, if $\Psi \in L$, then

$$\|\Psi\|_{L_2^0}^2 = \sum_{i,j=1}^{\infty} \lambda_i |\Psi e_i|^2 \leq \|\Psi\|_L^2 \sum_{i=1}^{\infty} \lambda_i < \infty$$

In particular, if $\Psi \in L$ then we can write

$$\|\Psi\|_{L_2^0}^2 = \text{Tr}(\Psi Q \Psi^*)$$

However not all operators from L_2^0 can be regarded as restrictions of operators from L . The space L_2^0 contains genuinely unbounded operators on U .

Let $\Phi(t)$, $t \in [0, T]$, be a measurable L_2^0 -valued process; we define the norms

$$\|\Psi\|_t = \mathbb{E} \left[\int_0^t \|\Psi(s)\|_{L_2^0}^2 ds \right]^{1/2} = \mathbb{E} \left[\int_0^t \text{Tr}((\Psi(s)Q^{1/2})(\Psi(s)Q^{1/2})^*) ds \right]^{1/2}$$

Proposition 2.14. Let Ψ be an elementary process such that $\|\Psi\|_T < \infty$, then the process $\Psi \cdot W$ is a continuous, square integrable H -valued martingale on $[0, T]$ and

$$\mathbb{E}[|\Psi \cdot W(t)|^2] = \|\Psi\|_t^2 \quad \forall t \in [0, T] \quad (2.9)$$

Proof. We will check for instance that (2.9) holds for $t = t_m \leq T$. Define $\xi_k = W(t_{k+1}) - W(t_k)$, $k = 1, \dots, m-1$. Then

$$\mathbb{E}[|\Psi \cdot W(t)|^2] = \mathbb{E} \left[\left| \sum_{i=0}^{m-1} \Phi(t_i) \xi_i \right|^2 \right] = \sum_{i=0}^{m-1} \mathbb{E}[|\Phi(t_i) \xi_i|^2] + 2 \sum_{i < j} \mathbb{E}[\langle \Phi(t_i) \xi_i, \Phi(t_j) \xi_j \rangle]$$

We will show first that

$$\mathbb{E}[|\Phi(t_i)\xi_i|^2] = (t_{i+1} - t_i)\mathbb{E}[\|\Phi(t_i)\|_{L_0^2}^2], \quad i = 0, \dots, m-1 \quad (2.10)$$

To this purpose note that the random variable $\Phi^*(t_i)f_k$ is $\mathcal{F}_{t_i}$ -measurable, and ξ_i independent of $\mathcal{F}_{t_i}$. Consequently (see Proposition 1.15)

$$\begin{aligned} \mathbb{E}[|\Phi(t_i)\xi_i|^2] &= \sum_{k=1}^{\infty} \mathbb{E}[\langle \Phi(t_i)\xi_i, f_k \rangle^2] = \sum_{k=1}^{\infty} \mathbb{E}[\langle \xi_i, \Phi^*(t_i)f_k \rangle^2] \\ &= (t_{i+1} - t_i) \sum_{k=1}^{\infty} \mathbb{E}[\langle Q\Phi^*(t_i)f_k, \Phi^*(t_i)f_k \rangle] \\ &= (t_{i+1} - t_i) \sum_{k=1}^{\infty} \mathbb{E}[|Q^{1/2}\Phi^*(t_i)f_k|^2] = (t_{i+1} - t_i)\mathbb{E}[\|\Phi(t_i)\|_{L_0^2}^2] \end{aligned}$$

This shows (2.10). Similarly one has

$$\mathbb{E}[\langle \Phi(t_i)\xi_i, \Phi(t_j)\xi_j \rangle] = 0 \quad \text{if } i \neq j$$

and the conclusion follows. $\square$

Remark 2.15. Formula (2.9) implies that the stochastic integral is an isometry from the space of all elementary processes equipped with the norm $\|\cdot\|_T$ into the space $\mathcal{M}_T^2(H)$ of H -valued martingales.

To extend the definition of the stochastic integral to more general processes, we need to understand what is the closure of the space of elementary processes with respect to the norm $\|\cdot\|_T$. To do so, it is convenient to introduce the notion of predictable processes.

Definition 2.16. Let $\mathcal{P}_T$ denote the σ -algebra generated by sets of the form

$$(s, t] \times F, \quad 0 \leq s < t \leq T, \quad F \in \mathcal{F}_s \quad \text{and} \quad \{0\} \times F, \quad F \in \mathcal{F}_0 \quad (2.11)$$

$\mathcal{P}_T$ is called the **predictable σ -algebra** and its elements the predictable sets. A measurable mapping from $([0, T] \times \Omega, \mathcal{P}_T)$ into a Banach space $(E, \mathcal{B}(E))$ is called a **predictable process**.

Clearly predictable processes are always $\mathcal{F}_t$ -adapted; elementary processes are always predictable. We are now going to show that predictable processes form quite a large class.

Proposition 2.17. An adapted process Φ with values in $L(U, H)$ such that for arbitrary $u \in U$ and $h \in H$ the process $\langle \Phi(t)u, h \rangle$, $t \geq 0$, has left continuous (respectively right continuous) trajectories is predictable.

Proof. Let us first show that $\mathcal{B}(L(U, H))$ is generated by the family of continuous linear maps $T \mapsto \langle Tu, h \rangle$. In fact, if $\{u_n\}$ and $\{h_m\}$ are countable sets respectively dense in $\{u \in U : |u| = 1\}$ and $\{h \in H : |h| = 1\}$, then

$$\|T\|_{L(U, H)} = \sup_{|u|=1} |Tu| = \sup_{n, m} \langle Tu_n, h_m \rangle$$

and the rest of the proof is analogue to the one of Proposition 1.1.

Therefore, in order to show that Φ is predictable, it suffices to show that the real process $X(t) = \langle \Phi(t)u, h \rangle$ is predictable for all $u \in U, h \in H$. By hypothesis X is adapted and left continuous; consider the processes

$$X_m(t, \omega) := \sum_{k=1}^{\infty} X_{m,k}(t, \omega), \quad t \in [0, T], \omega \in \Omega$$

where, for any $k \in \mathbb{N}$,

$$X_{m,k}(t, \omega) = X\left(\frac{k-1}{2^m}, \omega\right), \quad \text{for } t \in \left(\frac{k-1}{2^m}, \frac{k}{2^m}\right], \omega \in \Omega$$

The processes X_m are predictable and by left continuity of X they converge pointwise to it, therefore X is predictable. The proof for the right continuous case is analogue. $\square$

Consider now the space $\Omega_T = [0, T] \times \Omega$ endowed with the σ -algebra $\mathcal{P}_T$ and the product measure $\mathbb{P}_T = \mathcal{L} \otimes \mathbb{P}$, where $\mathcal{L}$ denotes the Lebesgue measure on $[0, T]$. Then it's easy to see that the norm $\|\cdot\|_T$ defined before corresponds to

$$\|\Psi\|_T^2 = \int_{\Omega_T} \|\Psi(t, \omega)\|_{L_2^0}^2 d\mathbb{P}_T(t, \omega)$$

and that elementary processes are a subspace of $L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; L_2^0)$, where L_2^0 is a separable Hilbert space on which we can define Bochner integral. We are now going to show that elementary processes are dense in $L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; L_2^0)$.

Proposition 2.18. The following hold.

- (i) If a mapping Φ from Ω_T into L is L -predictable then Φ is also L_2^0 -predictable. In particular, elementary processes are L_2^0 -predictable.
- (ii) If Φ is an L_2^0 -predictable process such that $\|\Phi\|_T < \infty$, then there exists a sequence $\{\Phi_n\}$ of elementary processes such that $\|\Phi - \Phi_n\|_T \rightarrow 0$ as $n \rightarrow \infty$.

Proof. i) We have seen before that $L \subset L_2^0$ with continuous embedding. Since the continuous image of a predictable process is still predictable, we can conclude.

ii) Since L is densely embedded into L_2^0 (for example because $g_i \otimes f_j \in L$ for every i, j), by Lemma 1.6 there exists a sequence $\{\Phi_n\}$ of L -valued predictable processes on $[0, T]$ taking only a finite numbers of values such that

$$\|\Phi(t, \omega) - \Phi_n(t, \omega)\|_{L_2^0} \downarrow 0$$

for all $(t, \omega) \in \Omega_T$. Consequently $\|\Phi - \Phi_n\|_T \downarrow 0$. It is therefore sufficient to prove that any $A \in \mathcal{P}_T$ can be arbitrarily approximated by predictable events, i.e. for any $\varepsilon > 0$ there exists a finite union Γ of disjoint sets of the form (2.11) such that

$$\mathbb{P}_T(A \triangle \Gamma) < \varepsilon$$

But predictable events form a π -system that generates $\mathbb{P}_T$, so the conclusion follows by standard corollaries of Dynkin's lemma. $\square$

We are now able to extend the definition of stochastic integral to all the elements of $L^2(\Omega_T; L_2^0)$, i.e. to all L_2^0 -predictable processes Φ such that $\|\Phi\|_T < \infty$. They form a Hilbert space which will be denoted by $\mathcal{N}_W^2(0, T; L_2^0)$, more simply $\mathcal{N}_W^2(0, T)$ or $\mathcal{N}_W^2$; by the previous proposition, elementary processes form a dense set in $\mathcal{N}_W^2(0, T)$. By Proposition 2.14 the stochastic integral $\Phi \cdot W$ is an isometric transformation from that dense set into $\mathcal{M}_T^2(H)$, therefore it can be extended by density to an isometry from $\mathcal{N}_W^2(0, T)$ to $\mathcal{M}_T^2(H)$. In particular (2.9) holds and $\Phi \cdot W$ is a continuous square integrable martingale for all $\Phi \in \mathcal{N}_W^2$.

As a final step, we extend the definition of stochastic integral to L_2^0 -predictable processes satisfying an even weaker integrability condition. The extension can be accomplished by means of a **localization procedure**.

Definition 2.19. An L_2^0 -predictable process Φ is **stochastically integrable** on $[0, T]$ if

$$\mathbb{P}\left(\int_0^T \|\Phi(s)\|_{L_2^0} ds < \infty\right) = 1 \quad (2.12)$$

Such processes form a linear space denoted by $\mathcal{N}_W(0, T; L_2^0)$, more simply $\mathcal{N}_W(0, T)$ or $\mathcal{N}_W$.

We need the following lemma.

Lemma 2.20. Assume that $\Phi \in \mathcal{N}_W^2(0, T; L_2^0)$ and that τ is an $\mathcal{F}_t$ -stopping time such that $\mathbb{P}(\tau \leq T) = 1$. Then

$$\int_0^T \mathbb{1}_{[0, \tau]}(s) \Phi(s) dW(s) = \Phi \cdot W(t \wedge \tau), \quad t \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (2.13)$$

Proof. First assume that Φ is elementary and that τ is a simple stopping time, then it's easy to see that (2.13) holds. If Φ is elementary and τ is a general stopping time, then there exists a sequence of simple stopping times $\{\tau_n\}$ such that $\tau_n \downarrow \tau$ pointwise. Therefore, by continuity of $\Phi \cdot W$, $\Phi \cdot W(\tau_n \wedge t) \rightarrow \Phi \cdot W(\tau \wedge t)$ $\mathbb{P}$ -a.s. On the other hand

$$\|\mathbb{1}_{[0, \tau]} - \mathbb{1}_{[0, \tau_n]}\|_T = \mathbb{E} \int_0^T \mathbb{1}_{[\tau, \tau_n]}(t) \|\Phi(t)\|_{L_2^0}^2 dt \downarrow 0$$

and using the fact that the stochastic integral is an isometry we get that $\mathbb{1}_{[0, \tau_n]} \Phi \cdot W \rightarrow \mathbb{1}_{[0, \tau]} \Phi \cdot W$ in $\mathcal{M}_T^2$, therefore we can conclude that $\mathbb{1}_{[0, \tau]} \Phi \cdot W(t) = \Phi \cdot W(\tau \wedge t)$.

If Φ is a general predictable process, let $\{\Phi_n\}$ be a sequence of elementary processes such that $\|\Phi - \Phi_n\|_T \rightarrow 0$, then $\|\mathbb{1}_{[0, \tau]} \Phi - \mathbb{1}_{[0, \tau]} \Phi_n\|_T \rightarrow 0$ and $\mathbb{1}_{[0, \tau]} \Phi_n \cdot W \rightarrow \mathbb{1}_{[0, \tau]} \Phi \cdot W$, which allows to conclude. $\square$

Assume that condition (2.12) holds and define

$$\tau_n = \inf \left\{ t \in [0, T] : \int_0^t \|\Phi\|_{L_2^0}^2(s) ds \geq n \right\}$$

with the convention that the infimum of an empty set is T . Then $\{\tau_n\}$ is a sequence such that

$$\mathbb{E} \left[\int_0^T \|\mathbb{1}_{[0, \tau_n]} \Phi\|_{L_2^0}^2(s) ds \right] < \infty \quad (2.14)$$

Consequently stochastic integrals $\mathbb{1}_{[0, \tau_n]} \Phi \cdot W$ are well defined for all $n \in \mathbb{N}$.

Moreover, if $n < m$, then $\mathbb{P}$ -a.s.

$$\mathbb{1}_{[0, \tau_n]} \Phi \cdot W(t) = (\mathbb{1}_{[0, \tau_n]}(\mathbb{1}_{[0, \tau_m]} \Phi)) \cdot W(t) = \mathbb{1}_{[0, \tau_m]} \Phi \cdot W(\tau_n \wedge t)$$

It follows from (2.12) that $\tau_n \uparrow T$ $\mathbb{P}$ -a.s. Therefore for any $t \in [0, T]$ we can define

$$\Phi \cdot W(t) = \mathbb{1}_{[0, \tau_n]} \Phi \cdot W(t) \quad (2.15)$$

for n sufficiently large such that $t \leq \tau_n$. Note that if also $t \leq \tau_m$ and $n \leq m$, then

$$\mathbb{1}_{[0, \tau_n]} \Phi \cdot W(t) = \mathbb{1}_{[0, \tau_m]} \Phi \cdot W(\tau_n \wedge t) = \mathbb{1}_{[0, \tau_m]} \Phi \cdot W(t)$$

and therefore definition (2.15) is consistent. By an analogous reasoning, if $\{\tilde{\tau}_n\} \uparrow T$ is another sequence satisfying (2.14), then definition (2.15) leads to an indistinguishable stochastic process. Note that for any $n \in \mathbb{N}$, $\omega \in \Omega$ and $t \in [0, T]$,

$$\Phi \cdot W(\tau_n \wedge t) = \mathbb{1}_{[0, \tau_n]} \Phi \cdot W(\tau_n \wedge t) = M_n(\tau_n \wedge t)$$

where M_n is a square integrable continuous H -valued martingale. This property will be referred as the **local martingale property** of stochastic integral.

Remark 2.21. It follows from the above construction that Lemma 2.20 holds for all $\Phi \in \mathcal{N}_W$.

Stochastic integral for generalized Wiener processes

The construction of the stochastic integral required to assume Q of finite trace; only in that case the Q -Wiener process takes values in U . We can however extend the definition of the integral to the case of generalized Wiener processes with a covariance operator Q not necessarily of trace class.

As before we denote by $U_0 = Q^{1/2}(U)$ (with the induced norm $|u|_0 = |Q^{-1/2}u|$) the reproducing kernel of W and $L_2^0 = L_2(U_0, H)$. It is not true anymore that $L(U, H) \subset L_2(U_0, H)$, but $\{f_i \otimes e_j\}$ is still a subset of $L(U, H)$ that is densely linear in $L_2(U_0, H)$. We can exploit such density in order to show that, given a U -valued random variable Z and an operator $R \in L_2^0$, RZ is still well defined as a random variable, as the following proposition shows.

Proposition 2.22. Assume that Z is a U -valued random variable with mean 0 and covariance Q , $R \in L_2^0$. If R_n is a sequence in $L(U, H) \cap L_2^0$ such that $R_n \rightarrow R$ in L_2^0 , then there exists a random variable RZ such that

$$\lim_{n \rightarrow \infty} \mathbb{E}[|RZ - R_n Z|_{L_2^0}^2] = 0$$

Moreover, RZ is independent of the approximating sequence $\{R_n\}$.

Proof. Observe that for any $S \in L(U, H) \cap L_2^0$ the following identity holds:

$$\mathbb{E}[|SZ|^2] = |S|_{L_2^0}^2 = |SQ^{1/2}|_{L_2(U, H)}^2 = \text{Tr}(SQS^*)$$

In fact,

$$\mathbb{E}[|SZ|^2] = \sum_{j=1}^{\infty} \mathbb{E}[\langle SZ, f_j \rangle^2] = \sum_{j=1}^{\infty} \mathbb{E}[\langle Z, S^* f_j \rangle^2] = \sum_{j=1}^{\infty} \langle QS^* f_j, S^* f_j \rangle$$

Then, defining $Z_n = R_n Z$, we have

$$\mathbb{E}[|Z_n - Z_m|^2] = |R_n - R_m|_{L_2^0}^2 \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

Therefore $\{Z_n\}$ is a Cauchy sequence in $L^2(\Omega; H)$. If $\{\tilde{R}_n\}$ is another sequence with all the required properties, then

$$\mathbb{E}[|Z_n - \tilde{Z}_n|^2] = |R_n - \tilde{R}_n|_{L_2^0}^2 \rightarrow 0 \text{ as } n \rightarrow \infty$$

and the result follows. $\square$

Now if W_a , $a \in U$, is a generalized Wiener process with covariance Q , then by Proposition 2.11 there exist a sequence $\{\beta_j\}$ of independent Wiener processes and an orthonormal basis $\{e_j\}$ of U such that

$$W_a(t) = \sum_{j=1}^{\infty} \langle a, Q^{1/2} e_j \rangle \beta_j(t), \quad a \in U, t \geq 0$$

Moreover the formula

$$W(t) = \sum_{j=1}^{\infty} Q^{1/2} e_j \beta_j(t), \quad t \geq 0$$

defines a Wiener process on any Hilbert space U_1 such that $U_0 \subset U_1$ with Hilbert-Schmidt embedding; by Proposition 2.11, U_0 and $Q_1(U_1)$ coincide. Therefore by applying Proposition 2.22 to the U_1 -valued random variables $W(t)$, $t \geq 0$, $\Phi W(t)$ are well defined for any $\Phi \in L_0^2$ and given by the formula

$$\Phi W(t) = \sum_{j=1}^{\infty} \Phi Q^{1/2} e_j \beta_j(t), \quad t \geq 0 \tag{2.16}$$

Thus the construction of the stochastic integral

$$\int_0^t \Phi(s) dW(s), \quad t \geq 0$$

can be done as in the case when $\text{Tr } Q < \infty$. It is enough to take into account that random variables of the form

$$\Phi_{t_j}(W_{t_{j+1}} - W_{t_j})$$

are defined in a unique way provided that $\Phi_{t_j} \in L_2^0$. The basic formula

$$\mathbb{E} \left[\left| \int_0^t \Phi(s) dW(s) \right|^2 \right] = \mathbb{E} \left[\int_0^t \|\Phi(s)\|_{L_2^0}^2 ds \right], \quad t \geq 0$$

remains the same.

Equivalently, one can repeat the definition of the stochastic integral for the U_1 -valued Wiener process W determined by W_a , $a \in U$. Again, the space of integrands and previous formula remain the same.

Approximation of stochastic integrals

We now describe a way of approximating stochastic integrals which could also serve as a different way of defining the stochastic integral with respect to a Q -Wiener process ($\text{Tr } Q \leq \infty$). Let

$$W_N(t) = \sum_{j=1}^N \sqrt{\lambda_j} \beta_j(t) e_j$$

where $\{\lambda_j, e_j\}$ is an eigensequence defined by Q , and let $\Phi \in \mathcal{N}_W(0, T)$. Notice that W_N and $W^N = W - W_N$ are Wiener processes with covariance operators respectively $Q_N = \sum_{j=1}^N \lambda_j e_j \otimes e_j$ and $Q^N(t) = \sum_{j=N+1}^\infty \lambda_j e_j \otimes e_j$. Observe that $\Phi \cdot W = \Phi \cdot W_N + \Phi \cdot W^N$: the identity is immediate in the case Φ is an elementary process and so by density it must hold for every $\Phi \in \mathcal{N}_W$. Thus

$$\mathbb{E}[|\Phi \cdot W(T) - \Phi \cdot W_N(T)|^2] = \mathbb{E}[|\Phi \cdot W^N(T)|^2] = \mathbb{E}\left[\int_0^T \|\Phi(s)(Q^N)^{1/2}\|_{L_2}^2 ds\right]$$

If $\|\Phi\|_T < \infty$, then

$$\mathbb{E}\left[\int_0^T \|\Phi(s)(Q^N)^{1/2}\|_{L_2}^2 ds\right] \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

Then by the martingale property of the stochastic integral

$$\mathbb{E}\left[\sup_{0 \leq t \leq T} |\Phi \cdot W(t) - \Phi \cdot W_N(t)|^2\right] \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

and consequently one can consider a subsequence $\{\Phi \cdot W_{N_k}\}$ converging $\mathbb{P}$ -a.s. and uniformly in $[0, T]$. Thus the stochastic integral with respect to an infinite dimensional Wiener process, also cylindrical, can be obtained, in the above sense, as a limit of stochastic integrals with respect to finite dimensional Wiener processes. The sequence $\Phi \cdot W_N$ contains a subsequence convergent $\mathbb{P}$ -a.s. uniformly with respect to $t \in [0, T]$. The limit is independent of the subsequence chosen and perhaps gives a more intuitive definition of the stochastic integral of a predictable process Φ such that $\mathbb{E}\left[\int_0^T \|\Phi(s)\|_{L_2^0}^2 ds\right] < \infty$. The case of general $\Phi \in \mathcal{N}_W(0, T; L_2^0)$ can be obtained by localization.

2.3 Properties of the stochastic integral

The following theorem summarizes some results from the previous sections also giving further information.

Theorem 2.23. For every $\Phi \in \mathcal{N}_W^2(0, T)$, the stochastic integral $\Phi \cdot W$ is a continuous square integrable martingale with quadratic variation given by

$$[\Phi \cdot W](t) = \int_0^t Q_\Phi(s) ds = \int_0^t (\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^* ds \quad (2.17)$$

If $\Phi \in \mathcal{N}_W(0, T)$, then $\Phi \cdot W$ is a local martingale.

Proof. We only need to prove formula (2.17); it suffices to show that, for any $h \in H$, the process

$$\langle \Phi \cdot W(t), h \rangle_H^2 - \int_0^t |Q^{1/2} \Phi^*(s) h|_U^2 ds$$

is a martingale. Let us first assume that Φ is a simple process, $\Phi = \sum_{k=0}^{n-1} \Phi_k \mathbb{1}_{(t_k, t_{k+1}]}$. Then we consider the process

$$X(t) = \left(\sum_{k=0}^{n-1} \langle W_{t_{k+1} \wedge t} - W_{t_k \wedge t}, \Phi_k^* h \rangle_U \right)^2 - \sum_{k=0}^{n-1} (t_{k+1} \wedge t - t_k \wedge t) |Q^{1/2} \Phi_k^* h|_U^2$$

In order to show that X is a martingale, it's enough to show that $\mathbb{E}[X(t) | \mathcal{F}_s] = X(s)$ holds for any $s, t \in [t_m, t_{m+1}]$, $s \leq t$, $m \leq n$, since we can then extend the result to any s, t using the tower property of conditional expectation. But on $[t_m, t_{m+1}]$ the process Φ is constant, so without loss of generality we can assume $\Phi(t) \equiv \Phi$. That is, we want to show that

$$\langle W(t), \Phi^* h \rangle_U^2 - t |Q^{1/2} \Phi^* h|_U^2$$

is a martingale. But this follows immediately from the definition of tQ as the quadratic variation of $W(t)$. So this concludes the proof when Φ is an elementary process. Now let $\Phi \in \mathcal{N}_W^2(0, T)$; then we can take a sequence Φ_n of elementary processes such that $\Phi_n \rightarrow \Phi$ pointwise and in $\mathcal{N}_W^2(0, T)$. Thus $\Phi_n \cdot W \rightarrow \Phi \cdot W$ in $\mathcal{M}_T^2(H)$ and so, for any $h \in H$, $t \in [0, T]$

$$\langle \Phi_n \cdot W(t), h \rangle^2 \rightarrow \langle \Phi \cdot W(t), h \rangle^2 \quad \text{in } L^1(\Omega; \mathbb{R})$$

Moreover then map $\Psi \mapsto |Q^{1/2} \Psi^* h|_U^2$, $\Psi \in L_2^0$ is continuous, so $|Q^{1/2} \Phi_n^* h|_U^2 \rightarrow |Q^{1/2} \Phi^* h|_U^2$ pointwise and

$$|Q^{1/2} \Phi_n^* h|_U^2 \leq |h|_H^2 |\Phi_n|_{L_2^0}^2 \rightarrow |h|_H^2 |\Phi|_{L_2^0}^2$$

and so we obtain that, for any $t \in [0, T]$,

$$\int_0^t |Q^{1/2} \Phi_n^*(s) h|_U^2 ds \rightarrow \int_0^t |Q^{1/2} \Phi^*(s) h|_U^2 ds \quad \text{in } L^1(\Omega, \mathbb{R})$$

Since the result holds for Φ_n , for any $0 \leq s \leq t \leq T$

$$\mathbb{E} \left[\langle \Phi_n \cdot W(t), h \rangle_H^2 - \int_0^t |Q^{1/2} \Phi_n^*(r) h|_U^2 dr \middle| \mathcal{F}_s \right] = \langle \Phi_n \cdot W(s), h \rangle_H^2 - \int_0^s |Q^{1/2} \Phi_n^*(r) h|_U^2 dr$$

Conditional expectation preserves the L^1 -convergence, therefore taking the limit as $n \rightarrow \infty$ the equality holds also for Φ , which implies the conclusion. $\square$

In a similar way, the following result can be shown.

Proposition 2.24. Let $\Phi_1, \Phi_2 \in \mathcal{N}_W^2(0, T)$. Then the martingales $\Phi_1 \cdot W$, $\Phi_2 \cdot W$ have cross quadratic variation

$$[\Phi_1 \cdot W, \Phi_2 \cdot W](t) = \int_0^t (\Phi_2(r) Q^{1/2})(\Phi_1(r) Q^{1/2})^* dr \quad (2.18)$$

We omit the proof as it's just a variation on the one of Theorem 2.23. We only highlight the following fact: since $\Phi_1, \Phi_2 \in \mathcal{N}_W^2(0, T)$, $\Phi_2(r)Q^{1/2}$ and $(\Phi_1(r)Q^{1/2})^*$ belong to $L_2(U, H)$ $\mathbb{P}$ -a.s. and so

$$\begin{aligned} \|(\Phi_2(r)Q^{1/2})(\Phi_1(r)Q^{1/2})^*\|_{L_1(U, U)} &\leq \|\Phi_2(r)Q^{1/2}\|_{L_2(U, H)} \|(\Phi_1(r)Q^{1/2})^*\|_{L_2(H, U)} \\ &= \|\Phi_1(r)\|_{L_2^0} \|\Phi_2(r)\|_{L_2^0} \end{aligned}$$

In particular by Cauchy's inequality we obtain

$$\int_0^t \|(\Phi_2(r)Q^{1/2})(\Phi_1(r)Q^{1/2})^*\|_{L_1(U)} dr \leq \left(\int_0^t \|\Phi_1(r)\|_{L_2^0}^2 dr \right) \left(\int_0^t \|\Phi_2(r)\|_{L_2^0}^2 dr \right) \leq \|\Phi_1\|_T^2 \|\Phi_2\|_T^2$$

Therefore the integral in (2.18) is well defined as a Bochner integral.

The two following corollaries follow immediately from the previous results. We state them separately as they are often useful in calculation.

Corollary 2.25. Assume that $\Phi_1, \Phi_2 \in \mathcal{N}_W^2(0, T)$. Then

$$\mathbb{E}[\Phi_i \cdot W(t)] = 0 \quad \text{and} \quad \mathbb{E}[|\Phi_i \cdot W(t)|^2] < \infty \quad \forall t \in [0, T], \quad i = 1, 2$$

Moreover the correlation operators $V(t, s) = \text{Cor}(\Phi_1 \cdot W(t), \Phi_2 \cdot W(s))$ are given by

$$V(t, s) = \mathbb{E} \int_0^{t \wedge s} (\Phi_2(r)Q^{1/2})(\Phi_1(r)Q^{1/2})^* dr$$

Corollary 2.26. Assume that $\Phi_1, \Phi_2 \in \mathcal{N}_W^2(0, T)$. Then

$$\mathbb{E}[\langle \Phi_1 \cdot W(t), \Phi_2 \cdot W(s) \rangle] = \mathbb{E} \int_0^{t \wedge s} \text{Tr}[(\Phi_2(r)Q^{1/2})(\Phi_1(r)Q^{1/2})^*] dr \quad (2.19)$$

If Φ_1 and Φ_2 are $L(U, H)$ -valued processes, then formula (2.19) can be written as

$$\mathbb{E}[\langle \Phi_1 \cdot W(t), \Phi_2 \cdot W(s) \rangle] = \mathbb{E} \int_0^{t \wedge s} \text{Tr}[\Phi_2(r)Q\Phi_1(r)^*] dr$$

Several results valid for Bochner integral have their counterparts for stochastic integrals. In particular, an analogue of Theorem 1.11 holds. Let $A : D(A) \subset H \rightarrow E$ be a closed operator, with the domain $D(A)$ a Borel subset of H and E another separable Hilbert space, and let $\Phi(t)$ be an $L_2^0(H) = L_2(U_0, H)$ -predictable process.

Proposition 2.27. If $\Phi(t)(U) \subset D(A)$ $\mathbb{P}$ -a.s. for all $t \in [0, T]$ and

$$\begin{aligned} \mathbb{P} \left(\int_0^T \|\Phi(s)\|_{L_2^0(H)}^2 ds < \infty \right) &= 1 \\ \mathbb{P} \left(\int_0^T \|A\Phi(s)\|_{L_2^0(E)}^2 ds < \infty \right) &= 1 \end{aligned}$$

then $\int_0^T \Phi(s)dW(s) \in D(A)$ $\mathbb{P}$ -a.s. and

$$A \int_0^T \Phi(s)dW(s) = \int_0^T A\Phi(s)dW(s) \quad \mathbb{P}\text{-a.s.} \quad (2.20)$$

Proof. Let us observe the following facts: $D(A)$ is still a separable Hilbert space with inner product $\langle h_1, h_2 \rangle_{D(A)} = \langle h_1, h_2 \rangle_H + \langle Ah_1, Ah_2 \rangle_E$ and norm $|h|_{D(A)}^2 = |h|_H^2 + |Ah|_E^2$; a linear operator S belongs to $L_2^0(D(A))$ if and only if $S \in L_2^0(H)$ and $AS \in L_2^0(E)$, moreover

$$\|S\|_{L_2^0(D(A))}^2 = \|S\|_{L_2^0(H)}^2 + \|AS\|_{L_2^0(E)}^2$$

In facts, for any orthonormal basis g_k of U_0 we have

$$\|S\|_{L_2^0(D(A))}^2 = \sum_{k=1}^{\infty} |Sg_k|_{D(A)}^2 = \sum_{k=1}^{\infty} |Sg_k|_H^2 + \sum_{k=1}^{\infty} |ASg_k|_E^2 = \|S\|_{L_2^0(H)}^2 + \|AS\|_{L_2^0(E)}^2$$

In particular, it follows from the hypothesis that

$$\mathbb{P}\left(\int_0^T \|\Phi(s)\|_{L_2^0(D(A))}^2 ds < \infty\right) = 1$$

Then by Theorem 1.11 Φ is an $L_2^0(D(A))$ -valued process and $\Phi \in \mathcal{N}_W(0, T; L_2^0(D(A)))$. The rest of the proof is similar to the one of Theorem 1.11: by linearity of A equality (2.20) holds whenever Φ is an elementary process; if $\Phi \in \mathcal{N}_W^2(0, T; L_2^0(D(A)))$, then we can consider a sequence of elementary processes $\Phi_n \rightarrow \Phi$ pointwise and in $\mathcal{N}_W^2(0, T; L_2^0(D(A)))$ and so by closedness of A the equality must hold. Finally, if $\Phi \in \mathcal{N}_W(0, T; L_2^0(D(A)))$, a localization procedure can be applied. $\square$

We now introduce a useful estimate, which also allows us to show that convergence in probability is preserved under stochastic integration.

Proposition 2.28. Let $\Phi \in \mathcal{N}_W(0, T)$. Then for any $a, b \in (0, +\infty)$

$$\mathbb{P}\left(\sup_{[0, T]} |\Phi \cdot W(t)| > a\right) \leq \frac{b}{a^2} + \mathbb{P}\left(\int_0^T \|\Phi(t)\|_{L_2^0}^2 dt > b\right)$$

Proof. Define

$$\tau_b = \inf \left\{ t \in [0, T] : \int_0^t \|\Phi(s)\|_{L_1^0}^2 ds > b \right\}$$

Then $\mathbb{P}(\sup_{[0, T]} |\Phi \cdot W(t)| > a) = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \mathbb{P}\left(\sup_{[0, T]} |\Phi \cdot W(t)| > a \text{ and } \int_0^T \|\Phi(s)\|_{L_1^0}^2 ds > b\right) \\ I_2 &= \mathbb{P}\left(\sup_{[0, T]} |\Phi \cdot W(t)| > a \text{ and } \int_0^T \|\Phi(s)\|_{L_1^0}^2 ds \leq b\right) \end{aligned}$$

But

$$I_2 \leq \mathbb{P}\left(\sup_{[0, T]} \left| \int_0^t \mathbb{1}_{[0, \tau_b]}(s) \Phi(s) dW(s) \right| > a\right)$$

and so by Theorem 1.37 and the definition of τ_b

$$I_2 \leq \frac{1}{a^2} \mathbb{E} \int_0^T \|\mathbb{1}_{[0, \tau_b]}(s) \Phi(s)\|_{L_2^0}^2 ds \leq \frac{b}{a^2}$$

Since $I_1 \leq \mathbb{P}(\int_0^T \|\Phi(s)\|_{L_1^0}^2 ds > b)$, the result follows. $\square$

Corollary 2.29. If Φ_n, Φ are elements of $\mathcal{N}_W(0, T)$ such that $\Phi_n \rightarrow \Phi$ in probability in $L^2(0, T; L_2^0)$, namely for any $\varepsilon > 0$

$$\mathbb{P}\left(\int_0^T \|\Phi_n(t) - \Phi(t)\|_{L_2^0}^2 dt > \varepsilon\right) \rightarrow 0 \text{ as } n \rightarrow \infty$$

then $\Phi_n \cdot W \rightarrow \Phi \cdot W$ in probability in $C(0, T; H)$.

Proof. Fix $\varepsilon > 0$; by linearity of the integral and Proposition 2.28, for any $\delta > 0$ we have

$$\mathbb{P}\left(\sup_{[0, T]} |\Phi_n \cdot W(t) - \Phi \cdot W(t)| > \varepsilon\right) \leq \frac{\delta}{\varepsilon^2} + \mathbb{P}\left(\int_0^T \|\Phi_n(t) - \Phi(t)\|_{L_2^0}^2 dt > \delta\right)$$

and taking the limit as $n \rightarrow \infty$

$$\limsup_{n \rightarrow \infty} \mathbb{P}\left(\sup_{[0, T]} |\Phi_n \cdot W(t) - \Phi \cdot W(t)| > \varepsilon\right) \leq \frac{\delta}{\varepsilon^2}$$

By arbitrariness of δ we can conclude. $\square$

We conclude this section with the following basic estimates, which allow to deduce regularity of the stochastic integral $\Phi \cdot W$ from the integrability of Φ , by means of Kolmogorov continuity criterion. The proof is omitted and the interested reader is referred to [13], section 4.6.

Theorem 2.30. For every $p > 0$ there exists $c_p > 0$ such that for every $t \geq 0$

$$\mathbb{E}\left[\sup_{s \in [0, t]} |\Phi \cdot W(s)|^p\right] \leq c_p \mathbb{E}\left[\left(\int_0^t \|\Phi(s)\|_{L_2^0}^2 ds\right)^{p/2}\right] \quad (2.21)$$

Theorem 2.31. For every $p \geq 2$ there exists $c'_p > 0$ such that for every $t \geq 0$

$$\mathbb{E}\left[\sup_{s \in [0, t]} |\Phi \cdot W(s)|^p\right] \leq c'_p \left(\int_0^t \mathbb{E}\left[\|\Phi(s)\|_{L_2^0}^p\right]^{2/p} ds\right)^{p/2} \quad (2.22)$$

Further extensions, Stratonovich integral and examples

In this section for simplicity we only consider the case of Q -Wiener processes, where Q is a trace class operator. Given an H -valued square integrable martingale M of the form $M = \Phi \cdot W$, where $\Phi \in \mathcal{N}_W^2(0, T; L_2^0)$, we can define the stochastic integral with respect to M by setting

$$\int_0^t \Psi(s) dM(s) := \int_0^t \Psi(s) \Phi(s) dW(s) \quad (2.23)$$

for any predictable process Ψ such that

$$\mathbb{P}\left(\int_0^T \text{Tr}((\Psi(s)\Phi(s)Q^{1/2})(\Psi(s)\Phi(s)Q^{1/2})^*) ds < \infty\right) = 1$$

Therefore $\Psi \cdot M$ is a local martingale with quadratic covariation given by

$$[\Psi \cdot M](t) = \int_0^t \Psi(s) Q \Phi(s) \Psi^*(s) ds \quad (2.24)$$

where

$$Q_\Phi(s) := (\Phi(s)Q^{1/2})(\Phi(s)Q^{1/2})^*$$

and

$$[M](t) = \int_0^t Q_\Phi(s)ds$$

All the results of the previous section can be adapted to this extension of the stochastic integral. The procedure of Section 2.2 can be adapted in order to define stochastic integration with respect to general martingales $M \in \mathcal{M}_T^2(H)$, see [19]. We omit this extension as it's outside the scope of this work. Another possible extension is the one with respect to continuous semimartingales and in particular the so called **Ito processes**, i.e. processes of the form

$$X(t) = X(0) + \int_0^t \Phi(s)dW(s) + \int_0^t \varphi(s)ds$$

where $X(0)$ is an H -valued, $\mathcal{F}_0$ -measurable random variable, $\Phi \in \mathcal{N}_W^2(0, T; H)$ and φ is an H -valued, predictable process such that

$$\mathbb{P}\left(\int_0^T |\varphi(s)|ds < \infty\right) = 1$$

For X as above we can define $\Psi \cdot X = \int \Psi dX$ by

$$\int_0^t \Psi(s)dX(s) := \int_0^t \Psi(s)\Phi(s)dW(s) + \int_0^t \Psi(s)\varphi(s)ds$$

for any predictable process Ψ such that

$$\mathbb{P}\left(\int_0^T \text{Tr}(\Psi(s)Q_\Phi(s)\Psi(s)^*)ds + \int_0^T |\Psi(s)\varphi(s)|ds < \infty\right) = 1$$

Another concept of integral which is often useful is the Stratonovich one.

Definition 2.32. Let Φ be a continuous, square integrable semimartingale taking values in L_2^0 and consider a sequence Δ_n of subdivisions of $[0, T]$ whose mesh $|\Delta_n| \rightarrow 0$ as n goes to infinity. We define the **Stratonovich integral** $\int \Phi \circ dW$ as the limit in probability of the processes

$$\sum_{i=1}^{N_n} \frac{1}{2} (\Phi(t \wedge t_{i+1}) + \Phi(t \wedge t_i)) (W(t \wedge t_{i+1}) - W(t \wedge t_i))$$

Similarly to the real case, using the results from Section 1.3 it's immediate to see that the limit of the above processes exists and doesn't depend on the chosen sequence of subdivisions, since

$$\begin{aligned} & \sum_{i=1}^{N_n} \frac{1}{2} (\Phi(t \wedge t_{i+1}) + \Phi(t \wedge t_i)) (W(t \wedge t_{i+1}) - W(t \wedge t_i)) \\ &= \sum_{i=1}^{N_n} \Phi(t \wedge t_i) (W(t \wedge t_{i+1}) - W(t \wedge t_i)) + \frac{1}{2} \sum_{i=1}^{N_n} (\Phi(t \wedge t_{i+1}) - \Phi(t \wedge t_i)) (W(t \wedge t_{i+1}) - W(t \wedge t_i)) \end{aligned}$$

which converge in probability respectively to $\int_0^t \Phi(s) dW(s)$ and $[\Phi, W](t)$. Therefore we obtain the formula

$$\int_0^t \Phi(s) \circ dW(s) = \int_0^t \Phi(s) dW(s) + \frac{1}{2}[\Phi, W](t) \quad (2.25)$$

We conclude this section by giving some particular examples of stochastic integrals.

Remark 2.33. Let E be a Banach space and let $B : E \times U \rightarrow H$ be a continuous bilinear form, namely there exists $C > 0$ such that

$$|B(e, u)|_H \leq C \|e\|_E \|u\|_U \quad \forall e \in E, u \in U$$

then the map from E to $L(U, H)$, $e \mapsto B(e, \cdot)$, is linear and continuous. Therefore if $X(t)$ is an E -valued predictable process, then $B(X(t), \cdot)$ is an $L(U, H)$ -valued predictable process and therefore also an L_2^0 -valued one; moreover $\|B(X(t), \cdot)\|_{L_2^0} \leq \tilde{C} \|X(t)\|_E$. In particular, if

$$\mathbb{P}\left(\int_0^T \|X(t)\|_E^2 dt < \infty\right) = 1$$

then $B(X(t), \cdot) \in \mathcal{N}_W(0, T)$ and we can define the process

$$\int_0^t B(X(s), dW(s)), \quad t \in [0, T]$$

A similar definition can be given if B is such that the map from E to L_2^0 , $e \mapsto B(e, \cdot)$, is continuous.

Example 2.34. If we take $B = \langle \cdot, \cdot \rangle_U : U \times U \rightarrow \mathbb{R}$ and X a U -valued process as above, we can define

$$\int_0^t \langle X(s), dW(s) \rangle_U \quad (2.26)$$

Moreover we have

$$\|\langle X(t), \cdot \rangle_U\|_{L_2^0}^2 = \sum_{k=1}^{\infty} \langle X(t), g_k \rangle^2 = |Q^{1/2} X(t)|_U^2$$

Therefore (2.26) can be defined for any U -valued, predictable process X such that

$$\mathbb{P}\left(\int_0^T |Q^{1/2} X(t)|_U^2 dt < \infty\right) = 1$$

and if $Q^{1/2} X \in L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; U)$, then it's a real square integrable martingale with

$$\left[\int_0^\cdot \langle X(s), dW(s) \rangle_U \right](t) = \int_0^t |Q^{1/2} X(s)|_U^2 ds$$

Stochastic integrals of the form (2.26) can also provide a different way to define stochastic integration. In fact it's easy to check, first for elementary processes and then by approximation, that the following identity holds: for any $\Phi \in \mathcal{N}_W(0, T)$ and for any $h \in H$,

$$\left\langle \int_0^t \Phi(s) dW(s), h \right\rangle = \int_0^t \langle \Phi(s)^* h, dW(s) \rangle$$

The identity above uniquely characterizes the process $\Phi \cdot W$; observe that, since $\Phi \in \mathcal{N}_W(0, T)$, Φ^*h is an H -valued predictable process such that $\int_0^T |\Phi^*(s)h|_U^2 < \infty$ $\mathbb{P}$ -a.s.

If instead we consider the linear map $B : U \times U_0 \rightarrow \mathbb{R}$, $B(u_1, u_2) = \langle u_1, Q^{-1/2}u_2 \rangle_U = \langle Q^{1/2}u_1, u_2 \rangle_0$, then similar calculations give that the integral

$$\int_0^t \langle X(s), Q^{-1/2}dW(s) \rangle_U = \int_0^t \langle Q^{1/2}X(s), dW(s) \rangle_0$$

is well defined for any U -valued predictable process X such that

$$\mathbb{P}\left(\int_0^T |X(s)|_U^2 ds\right) < \infty$$

Observe that the map $X \mapsto \int_0^\cdot \langle X, Q^{-1/2}dW \rangle_U$ defines an isometry between $L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; U)$ and $\mathcal{M}_T^2(\mathbb{R})$.

Example 2.35. If we take $B = \langle \cdot, \cdot \rangle_0 : U_0 \times U_0 \rightarrow \mathbb{R}$ and X a U_0 -valued predictable process, then $\|\langle X, \cdot \rangle\|_{L_2^0} = |X|_0^2$ and so under suitable hypothesis we can define

$$\int_0^t \langle X(s), dW(s) \rangle_0$$

This time the map $X \mapsto \int \langle X, dW \rangle_0$ is an isometry from $L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; U_0)$ to $\mathcal{M}_T^2(\mathbb{R})$.

Example 2.36. If we consider $B = \otimes : (h, u) \mapsto h \otimes u$, seen as a map from $H \times U$ to $L_2(U, H)$, then

$$\|h \otimes \cdot\|_{L_2^0(L_2(U, H))}^2 = \sum_{k=1}^{\infty} \|h \otimes g_k\|_{L_2(U, H)}^2 = \sum_{k=1}^{\infty} |h|^2 |g_k|^2 = |h|^2 \text{Tr}(Q)$$

If X is an H -valued predictable process, with suitable integrability conditions, we can then define

$$\int_0^t X(s) \otimes dW(s)$$

It can be checked, as usual first for elementary processes and then by approximation, that for any $u \in U$ and $h \in H$ it holds

$$\left\langle h, \left(\int_0^t X(s) \otimes dW(s) \right) u \right\rangle_H = \int_0^t \langle X(s), h \rangle_H d\langle W(s), u \rangle_U$$

where the right hand side is the stochastic integral of the real predictable process $\langle X, h \rangle_H$ with respect to the real continuous martingales $\langle W, u \rangle_U$.

The above examples can be extended, under suitable assumption on the process X , also in the case of stochastic integration w.r.t semimartingales or in the Stratonovich sense. In particular, for any continuous semimartingales X and Y satisfying suitable assumptions in order for everything to be well defined, the following **product rule** holds:

$$X(t) \otimes Y(t) = X(0) \otimes Y(0) + \int_0^t X(s) \otimes dY(s) + \int_0^t dX(s) \otimes Y(s) + [X, Y](t) \quad (2.27)$$

which can be also expressed as

$$d(X \otimes Y) = X \otimes dY + (dX) \otimes Y + d[X, Y]$$

In fact, for any subdivision $0 = t_0 < t_1 < \dots < t_n = t$ of $[0, t]$, we have

$$\begin{aligned} X(t) \otimes Y(t) &= X(0) \otimes Y(0) + \sum_{i=0}^{n-1} [X(t_{i+1}) \otimes Y(t_{i+1}) - X(t_i) \otimes Y(t_i)] \\ &= X(0) \otimes Y(0) + \sum_{i=0}^{n-1} X(t_i) \otimes (Y(t_{i+1}) - Y(t_i)) + \sum_{i=0}^{n-1} (X(t_{i+1}) - X(t_i)) \otimes Y(t_i) \\ &\quad + \sum_{i=0}^{n-1} (X(t_{i+1}) - X(t_i)) \otimes (Y(t_{i+1}) - Y(t_i)) \end{aligned}$$

and taking the limit as the mesh of the subdivision goes to 0 we obtain (2.27). Similarly it can be shown that

$$X(t) \otimes Y(t) = X(0) \otimes Y(0) + \int_0^t X(s) \otimes \circ dY(s) + \int_0^t \circ dX(s) \otimes Y(s) \quad (2.28)$$

Moreover, if X and Y take values on the same space U , taking the trace in (2.27) we find

$$\langle X(t), Y(t) \rangle_U = \langle X(0), Y(0) \rangle_U + \int_0^t \langle X(s), dY(s) \rangle_U + \int_0^t \langle Y(s), dX(s) \rangle_U + \text{Tr}([X, Y](t)) \quad (2.29)$$

2.4 Ito formula

Let Φ be an L_2^0 -valued stochastically integrable process on $[0, T]$, φ an H -valued integrable process on $[0, T]$ and $X(0)$ an $\mathcal{F}_0$ -measurable H -valued random variable. Then the process

$$X(t) = X(0) + \int_0^t \varphi(s) ds + \int_0^t \Phi(s) dW(s), \quad t \in [0, T]$$

is well defined. Let $F : [0, T] \times H \rightarrow \mathbb{R}$ be a function such that F and its partial derivatives F_t , F_x , F_{xx} are uniformly continuous on bounded sets of $[0, T] \times H$.

Theorem 2.37 (Ito formula). Under the above conditions, $\mathbb{P}$ -a.s. for all $t \in [0, T]$

$$\begin{aligned} F(t, X(t)) &= F(0, X(0)) + \int_0^t \langle F_x(s, X(s)), \Phi(s) dW(s) \rangle \\ &\quad + \int_0^t \left\{ F_t(s, X(s)) + \langle F_x(s, X(s)), \varphi(s) \rangle + \frac{1}{2} \text{Tr} [F_{xx}(s, X(s)) (\Phi(s) Q^{1/2}) (\Phi(s) Q^{1/2})^*] \right\} ds \end{aligned} \quad (2.30)$$

Proof. By localization, we can reduce ourselves to the case in which the process $X(t)$ and the integrals $\int_0^T |\varphi(s)| ds$, $\int_0^T \|\Phi(s)\|_{L_2^0}^2 ds$ are bounded. In fact, for any $C > 0$ consider the stopping time

$$\tau_C = \inf \left\{ t \in [0, T] : \max \left(|X(t)|, \int_0^t |\varphi(s)| ds, \int_0^t \|\Phi(s)\|_{L_2^0}^2 ds \right) \geq C \right\}$$

with the convention $\inf \emptyset = T$. If we define

$$X_C(t) = X(t \wedge \tau_C), \quad \varphi_C(t) = \mathbb{1}_{[0, \tau_C]}(t)\varphi(t), \quad \Phi_C(t) = \mathbb{1}_{[0, \tau_C]}(t)\Phi(t), \quad t \in [0, T]$$

then by Lemma 2.20

$$X_C(t) = X_C(0) + \int_0^t \varphi_C(s)ds + \int_0^t \Phi_C(s)dW(s), \quad t \in [0, T]$$

Therefore if formula (2.30) is true for X_C, φ_C, Φ_C for arbitrary $C > 0$, then again by Lemma 2.20 it holds in the general case. So we can assume

$$\mathbb{E} \int_0^T |\varphi(s)|ds < \infty, \quad \mathbb{E} \int_0^T \|\Phi(s)\|^2 ds < \infty$$

By Lemma 1.6 and Proposition 2.18 we can then restrict ourselves to the case in which φ and Φ are elementary processes; then φ and Φ are locally constant and so, up to "gluing" together the intervals, we can assume that $\varphi(s) \equiv \varphi_0$ and $\Phi(s) \equiv \Phi_0$ are constant processes. So we only need to prove formula (2.30) for X of the form

$$X(t) = X(0) + t\varphi_0 + \Phi_0 W(t) \tag{2.31}$$

Let the points $0 = t_0 < t_1 < \dots < t_k = t$ define a partition of $[0, t] \subset [0, T]$. Then

$$F(t, X(t)) - F(0, X(0)) = \sum_{j=0}^{k-1} [F(t_{j+1}, X(t_{j+1})) - F(t_j, X(t_{j+1}))] + \sum_{j=0}^{k-1} [F(t_j, X(t_{j+1})) - F(t_j, X(t_j))]$$

Applying Taylor's formula one gets (random) $\theta_{00}, \theta_{01}, \dots, \theta_{0(k-1)}, \theta_{10}, \dots, \theta_{1(k-1)} \in [0, 1]$ such that

$$\begin{aligned} F(t, X(t)) - F(0, X(0)) &= \sum_{j=0}^{k-1} F_t(t_{j+1}, X(t_{j+1}))\Delta t_j \\ &+ \sum_{j=0}^{k-1} \langle F_x(t_j, X(t_j)), \Delta X_j \rangle \\ &+ \frac{1}{2} \sum_{j=0}^{k-1} \langle F_{xx}(t_j, X(t_j)) \cdot \Delta X_j, \Delta X_j \rangle \\ &+ \sum_{j=0}^{k-1} [F_t(\tilde{t}_j, X(t_{j+1})) - F_t(t_{j+1}, X(t_{j+1}))]\Delta t_j \\ &+ \frac{1}{2} \sum_{j=0}^{k-1} \langle [F_{xx}(t_j, \tilde{X}(t_j)) - F_{xx}(t_j, X(t_j))] \cdot \Delta X_j, \Delta X_j \rangle \\ &= I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned}$$

where

$$\begin{aligned} \Delta t_j &= t_{j+1} - t_j, \quad \Delta X_j = X(t_{j+1}) - X(t_j) \\ \tilde{t}_j &= t_j + \theta_{0j}(t_{j+1} - t_j), \quad \tilde{X}_j = X_j + \theta_{1j}(X(t_{j+1}) - X(t_j)) \end{aligned}$$

Taking into account (2.31), when the mesh of the partition goes to 0 we have

$$\begin{aligned} I_1 &\rightarrow \int_0^t F_t(s, X(s))ds \quad \mathbb{P}\text{-a.s.} \\ I_2 &\rightarrow \int_0^t \langle F_x(s, X(s)), \varphi_0 \rangle ds + \int_0^t \langle F_x(s, X(s)), \Phi_0 dW(s) \rangle \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

To find the limit of I_3 observe that

$$\begin{aligned} I_3 &= \frac{1}{2} \sum_{j=0}^{k-1} \langle \Phi_0^* F_{xx}(t_j, X(t_j)) \Phi_0 \Delta W_j, \Delta W_j \rangle + \frac{1}{2} \sum_{j=0}^{k-1} \langle F_{xx}(t_j, X(t_j)) \varphi_0, \varphi_0 \rangle (\Delta t_j)^2 \\ &\quad + \sum_{j=0}^{k-1} \langle F_{xx}(t_j, X(t_j)) \Phi_0 \Delta W_j, \varphi_0 \rangle \Delta t_j = I_{31} + I_{32} + I_{33} \end{aligned}$$

We will first show that for a suitable subsequence

$$I_{31} \rightarrow \frac{1}{2} \int_0^t \text{Tr}[\Phi_0^* F_{xx}(t, X(t)) \Phi_0 Q] ds \quad (2.32)$$

Denote $Y_j = \Phi_0^* F_{xx}(t_j, X(t_j)) \Phi_0$, then

$$\begin{aligned} J &:= \mathbb{E} \left[\left(\sum_{j=0}^{k-1} \langle \Phi_0^* F_{xx}(t_j, X(t_j)) \Phi_0 \Delta W_j, \Delta W_j \rangle - \sum_{j=0}^{k-1} \text{Tr}[\Phi_0^* F_{xx}(t_j, X(t_j)) \Phi_0 Q \Delta t_j] \right)^2 \right] \\ &= \sum_{j=0}^{k-1} \mathbb{E}[\langle Y_j \Delta W_j, \Delta W_j \rangle^2 - (\text{Tr}[Y_j Q])^2 (\Delta t_j)^2] \end{aligned}$$

This follows from Proposition 1.15 and the general fact that, if $Z_0, \dots, Z_{k-1}$ are real square integrable random variables and $\mathcal{G}_0, \dots, \mathcal{G}_k$ is an increasing sequence of σ -algebras such that Z_j is $\mathcal{G}_{j+1}$ -measurable for every j , then

$$\mathbb{E} \left[\left(\sum_{j=0}^{k-1} Z_j - \sum_{j=0}^{k-1} \mathbb{E}[Z_j | \mathcal{G}_j] \right)^2 \right] = \sum_{j=0}^{k-1} \left(\mathbb{E}[Z_j^2] - \mathbb{E}[\mathbb{E}[Z_j | \mathcal{G}_j]^2] \right)$$

Let M be a constant such that $|Y_j| \leq M$ for all j , then

$$J \leq M^2 \left(\sum_{j=0}^{k-1} \mathbb{E}[|W(t_{j+1}) - W(t_j)|^4] + (\text{Tr } Q)^2 (t_{j+1} - t_j)^2 \right) = M^2 C \sum_{j=0}^{k-1} (t_{j+1} - t_j)^2$$

and we see that $J \rightarrow 0$. Consequently taking a subsequence we obtain (2.32). From the paths continuity of W and boundedness of $F_{xx}(s, X(s))$, it follows that $I_{32} \rightarrow 0$ and $I_{33} \rightarrow 0$. It remains to show that there exist subsequences of I_4 and I_5 $\mathbb{P}$ -a.s. converging to 0. The pointwise convergence to 0 of I_4 is a consequence of the uniform continuity of F_t . By the uniform continuity of F_{xx} and the fact that the sequence $\sum_{j=0}^{k-1} |X(t_{j+1}) - X(t_j)|^2$ contains a $\mathbb{P}$ -a.s. bounded subsequence it follows that there is a subsequence of I_5 that tends to 0. The proof is complete. $\square$

Remark 2.38. Recalling the definition of stochastic integral w.r.t. X , Ito formula can also be expressed as follows:

$$F(t, X(t)) = F(0, X(0)) + \int_0^t \langle F_x(s, X(s)), dX \rangle + \int_0^t \left\{ F_t(s, X(s)) + \frac{1}{2} \text{Tr}[F_{xx}(s, X(s))Q_\Phi(s)] \right\} ds$$

Moreover, similarly to the finite dimensional case, it can be shown that if F is sufficiently regular, then Stratonovich integral satisfies the chain rule, namely

$$F(t, X(t)) = F(0, X(0)) + \int_0^t \langle F_x(s, X(s)), \circ dX \rangle + \int_0^t \langle F_t(s, X(s)) ds \quad (2.33)$$

2.5 Girsanov transform

In this section we are going to prove an infinite dimensional version of Girsanov's theorem; Girsanov's transformations are particularly useful, as they allow to construct weak solutions of stochastic differential equations starting from solutions of other, usually simpler, SDEs. In order to prove the theorem we need two preliminary lemmas. As before, W is an U -valued Wiener process with covariance Q and $U_0 = Q^{1/2}(U)$ denotes its reproducing kernel; $|\cdot|_0 = |\cdot|_{U_0} = |Q^{-1/2} \cdot|_U$.

Lemma 2.39. Let X_n, X be complex integrable random variables such that $X_n \rightarrow X$ in probability and $\mathbb{E}[|X_n|] \rightarrow \mathbb{E}[|X|]$. Then $\mathbb{E}[|X - X_n|] \rightarrow 0$.

Proof. We can extract a subsequence such that $X_{n_k} \rightarrow X$ $\mathbb{P}$ -a.s. Consider the sequence

$$Y_k = |X_{n_k}| - |X - X_{n_k}|$$

Then $|Y_k| \leq |X|$ for all $k \in \mathbb{N}$ and $Y_k \rightarrow |X|$ $\mathbb{P}$ -a.s., therefore $\mathbb{E}[Y_k] \rightarrow \mathbb{E}[|X|]$ by dominated convergence. But $\mathbb{E}[|X_{n_k}|] \rightarrow \mathbb{E}[|X|]$ as well and so $\mathbb{E}[|X - X_{n_k}|] \rightarrow 0$. Since the reasoning applies for any subsequence of X_n , we conclude that $\mathbb{E}[|X - X_n|] \rightarrow 0$ as well. $\square$

Lemma 2.40. Let $\psi(t)$, $t \in [0, T]$ be a U_0 -valued $\mathcal{F}_t$ -predictable process such that

$$\mathbb{P}\left(\int_0^T |\psi(s)|_0^2 ds < \infty\right) = 1$$

Then there exists a real standard Wiener process $\beta(t)$, $t \in [0, T]$ with respect to $\mathcal{F}_t$ such that, $\mathbb{P}$ -a.s.,

$$\int_0^t |\psi(s)|_0 d\beta(s) = \int_0^t \langle \psi(s), dW(s) \rangle_0 \quad \forall t \in [0, T] \quad (2.34)$$

Proof. Fix $a \in U_0$ such that $|a|_0 = 1$ and define a process $\tilde{\psi}$ as follows:

$$\tilde{\psi}(s) = \begin{cases} \frac{\psi(s)}{|\psi(s)|_0} & \text{if } \psi(s) \neq 0 \\ a & \text{if } \psi(s) = 0 \end{cases}$$

The process $\tilde{\psi}$ is also predictable and $|\tilde{\psi}(t)|_0 = 1$ for all $t \in [0, T]$. Consequently the process

$$\beta(t) = \int_0^t \langle \tilde{\psi}(s), dW(s) \rangle_0 \quad (2.35)$$

is a square integrable martingale with quadratic variation $[\beta](t) = t$. By Levy's theorem, $\beta(t)$ is a standard real Wiener process with respect to $\mathcal{F}_t$. It follows from (2.35) that

$$\int_0^t |\psi|_0 d\beta(s) = \int_0^t |\psi|_0 \langle \tilde{\psi}(s), dW(s) \rangle_0 = \int_0^t \langle \psi(s), dW(s) \rangle_0 \quad t \in [0, T]$$

as required. $\square$

Theorem 2.41 (Girsanov Theorem). Let ψ be a U_0 -valued $\mathcal{F}_t$ -predictable process such that

$$\mathbb{E} \left[e^{\int_0^T \langle \psi(s), dW(s) \rangle_0 - \frac{1}{2} \int_0^T |\psi(s)|_0^2 ds} \right] = 1 \quad (2.36)$$

Then the process

$$\widehat{W}(t) = W(t) - \int_0^t \psi(s) ds, \quad t \in [0, T] \quad (2.37)$$

is a Q -Wiener process w.r.t. $\mathcal{F}_t$ on the probability space $(\Omega, \mathcal{F}, \widehat{\mathbb{P}})$, where

$$d\widehat{\mathbb{P}}(\omega) = e^{\int_0^T \langle \psi(s), dW(s) \rangle_0 - \frac{1}{2} \int_0^T |\psi(s)|_0^2 ds} d\mathbb{P}(\omega) \quad (2.38)$$

Proof. Assume first that $|\psi(t)|_0 \leq C$ for all $t \in [0, T]$ for some constant C . Let $g : [0, T] \rightarrow U_0$ be a bounded Borel measurable function. Then

$$\int_0^T \langle g(t), d\widehat{W}(t) \rangle_0 = \int_0^T \langle g(t), dW(t) \rangle_0 + \int_0^T \langle g(t), \psi(t) \rangle_0 dt$$

We now show that

$$\widehat{\mathbb{E}} \left[e^{\int_0^T \langle g(t), d\widehat{W}(t) \rangle_0} \right] = e^{\frac{1}{2} \int_0^T |g(t)|_0^2 dt} \quad (2.39)$$

In fact, by the definition of $\widehat{\mathbb{P}}$,

$$\begin{aligned} \widehat{\mathbb{E}} \left[e^{\int_0^T \langle g(t), d\widehat{W}(t) \rangle_0} \right] &= \mathbb{E} \left[e^{\int_0^T \langle \psi(t), dW(t) \rangle_0 - \frac{1}{2} \int_0^T |\psi(t)|_0^2 dt + \int_0^T \langle g(t), dW(t) \rangle_0 - \int_0^T \langle g(t), \psi(t) \rangle_0 dt} \right] \\ &= e^{\frac{1}{2} \int_0^T |g(t)|_0^2 dt} \mathbb{E} \left[e^{\int_0^T \langle g(t) + \psi(t), dW(t) \rangle_0 - \frac{1}{2} \int_0^T |g(t) + \psi(t)|_0^2 dt} \right] \end{aligned} \quad (2.40)$$

The expectation in the last line is 1: by Lemma 2.40, applied to the bounded process $g(t) + \psi(t)$, it can be written as

$$\mathbb{E} \left[e^{\int_0^T \gamma(t) d\beta(t) - \frac{1}{2} \int_0^T \gamma(t)^2 dt} \right]$$

where $\gamma(t) = |g(t) + \psi(t)|_0$ is a bounded real process, and so the conclusion follows. It follows from (2.39) that, for any $\lambda \in \mathbb{R}$,

$$\widehat{\mathbb{E}} \left[e^{\lambda \int_0^T \langle g(t), d\widehat{W}(t) \rangle_0} \right] = e^{\frac{\lambda^2}{2} \int_0^T |g(t)|_0^2 dt} \quad (2.41)$$

For any $z \in \mathbb{C}$, define the function

$$h(z) = \widehat{\mathbb{E}} \left[e^{z \int_0^T \langle g(t), d\widehat{W}(t) \rangle_0} \right]$$

Since $h(z)$ is finite for all real z , it is well defined for any $z \in \mathbb{C}$; it's also continuously differentiable with respect to the complex variable z and so analytic on $\mathbb{C}$. Therefore

$$h(z) = e^{\frac{z^2}{2} \int_0^T |g(t)|_0^2 dt} \quad \forall z \in \mathbb{C}$$

In particular, for any $\lambda \in \mathbb{R}$,

$$\widehat{\mathbb{E}}\left[e^{i\lambda \int_0^T \langle g(t), d\widehat{W}(t) \rangle_0}\right] = e^{-\frac{\lambda^2}{2} \int_0^T |g(t)|_0^2 dt}$$

Therefore the random variables $\int_0^T \langle g(t), d\widehat{W}(t) \rangle_0$ are Gaussian with covariance $\int_0^T |g(t)|_0^2 dt$. Moreover, by applying calculations similar to (2.40) and using the fact that the processes

$$\begin{aligned} M(\tilde{t}) &= e^{\int_t^{\tilde{t}} |g(s)+\psi(s)|_0 d\beta(s) - \frac{1}{2} \int_t^{\tilde{t}} |g(s)+\psi(s)|_0^2 ds} \quad \tilde{t} \geq t \\ N(t) &= e^{\int_0^t \langle \psi(s), dW(s) \rangle_0 - \frac{1}{2} \int_0^t |\psi(s)|_0^2 ds} \end{aligned}$$

are $\mathbb{P}$ -martingales, we obtain that for any $\Gamma \in \mathcal{F}_t$:

$$\widehat{\mathbb{E}}\left[e^{\int_t^T \langle g(s), d\widehat{W}(s) \rangle_0} \mathbb{1}_\Gamma\right] = e^{\frac{1}{2} \int_t^T |g(s)|_0^2 ds} \mathbb{E}[M(T)N(t) \mathbb{1}_\Gamma] = e^{\frac{1}{2} \int_t^T |g(s)|_0^2 ds} \widehat{\mathbb{P}}(\Gamma)$$

where the last equality holds since

$$\mathbb{E}[M(T)N(t) \mathbb{1}_\Gamma] = \mathbb{E}[\mathbb{E}[M(T)|\mathcal{F}_t]N(t) \mathbb{1}_\Gamma] = \mathbb{E}[N(t) \mathbb{1}_\Gamma] = \mathbb{E}[\mathbb{E}[N(T)|\mathcal{F}_t] \mathbb{1}_\Gamma] = \mathbb{E}[N(T) \mathbb{1}_\Gamma]$$

By the same procedure used previously we obtain that, for any $\lambda \in \mathbb{R}$ and $\Gamma \in \mathcal{F}_t$,

$$\widehat{\mathbb{E}}\left[e^{i\lambda \int_t^T \langle g(s), d\widehat{W}(s) \rangle_0} \mathbb{1}_\Gamma\right] = e^{-\frac{\lambda^2}{2} \int_t^T |g(s)|_0^2 ds} \widehat{\mathbb{P}}(\Gamma) = \widehat{\mathbb{E}}\left[e^{i\lambda \int_t^T \langle g(s), d\widehat{W}(s) \rangle_0}\right] \widehat{\mathbb{E}}[\mathbb{1}_\Gamma] \quad (2.42)$$

Therefore the random variables $\int_t^T \langle g(s), d\widehat{W}(s) \rangle_0$ are independent of $\mathcal{F}_t$ and so we can conclude. For a general process ψ not necessarily bounded, consider a sequence of bounded processes ψ_n such that

$$\lim_{n \rightarrow \infty} \int_0^T |\psi(t) - \psi_n(t)|_0^2 dt = 0 \quad \mathbb{P}\text{-a.s.}$$

and define processes

$$\widehat{W}_n(t) = W(t) - \int_0^t \psi_n(s) ds, \quad t \in [0, T], \quad n \in \mathbb{N}$$

Then by (2.42), for any $\Gamma \in \mathcal{F}_t$

$$\begin{aligned} &\mathbb{E}\left[e^{\int_0^T \langle \psi_n(s), dW(s) \rangle_0 - \frac{1}{2} \int_0^T |\psi_n(s)|_0^2 ds} e^{i\lambda \int_t^T \langle g(s), dW(s) \rangle_0 - i\lambda \int_t^T \langle g(s), \psi_n(s) \rangle_0 ds}\right] \\ &= e^{-\frac{\lambda^2}{2} \int_t^T |g(s)|_0^2 ds} \mathbb{E}\left[e^{\int_0^T \langle \psi_n(s), dW(s) \rangle_0 - \frac{1}{2} \int_0^T |\psi_n(s)|_0^2 ds} \mathbb{1}_\Gamma\right] \end{aligned} \quad (2.43)$$

Recall that by Corollary 2.29, $\int_0^T \langle \psi_n(s), dW(s) \rangle_0 \rightarrow \int_0^T \langle \psi(s), dW(s) \rangle_0$ in probability. Then using the assumption (2.36) and Lemma 2.39 we can take the limit in both sides of (2.43) and obtain that (2.42) holds for general ψ . $\square$

Lemma 2.40 also allows us to obtain conditions under which (2.36) must hold, similar to the finite dimensional case.

Proposition 2.42 (Novikov's criterion). One of the following is sufficient for (2.36) to hold:

$$\text{i) } \mathbb{E}\left[e^{\frac{1}{2} \int_0^T |\psi(t)|_0^2 dt}\right] < \infty$$

ii) There exists $\delta > 0$ such that $\sup_{[0,T]} \mathbb{E}[e^{\delta|\psi(t)|_0^2}] < \infty$

Proof. By Lemma 2.40, there exists a real Wiener process β such that (2.34) holds. The process $X(t) = \int_0^t |\psi(s)|_0 d\beta(s)$ is a local martingale with $[X](t) = \int_0^t |\psi(s)|_0^2 ds$ and so equation (2.36) can be written as

$$\mathbb{E}\left[e^{X(T) - \frac{1}{2}[X](T)}\right] = 1$$

But then i) and ii) follow from the analogue real results applied to the process $X(t)$ (see for instance [20], Proposition 1.15 p. 332 and Exercise 1.40 p. 338). $\square$

Chapter 3

The stochastic Leray- α model

3.1 A general scheme

In this section we outline the general strategy we will follow; the approach here is more heuristic, since we are more interested in showing the applicability of the method, which can be adapted also to other models, rather than giving rigorous results.

Several models from inviscid fluid dynamics, such as Euler equations, the Leray- α model associated and the dyadic model, in absence of external forces can be expressed as

$$\dot{u} = B(u, u), \quad u \in H \quad (3.1)$$

where H is a separable Hilbert space and $B : H \times D(B) \rightarrow H$ is a bilinear map such that

$$\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle \quad \forall u \in H, v, w \in D(B) \quad (3.2)$$

Actually in the next sections we will consider also the presence of external forces, but here for simplicity we restrict to the homogeneous case; observe that if we added also a linear operator in the r.h.s. of (3.1), such formulation could also include Navier-Stokes equations and their variants. It follows immediately from (3.2) that if u is a solution such that $u \in D(B)$ for all times, then energy is preserved:

$$\frac{d}{dt}|u|^2 = 2\langle \dot{u}, u \rangle = 2\langle B(u, u), u \rangle = 0 \quad (3.3)$$

However in general we do not expect solutions to be regular enough in order for $u \in D(B)$ to hold. We need therefore a concept of weak solution, which can be formulated as follows:

$$\frac{d}{dt}\langle u, v \rangle = -\langle B(u, v), u \rangle \quad \forall v \in D(B) \quad (3.4)$$

Observe that if u is a weak solution, then computation (3.3) does not hold and so we cannot deduce the invariance of $|u|^2$. We summarize this feature by saying that the energy is *formally* preserved. We can consider a stochastic version of (3.1) obtained adding a Stratonovich multiplicative noise, namely

$$du = B(u \, dt + \circ dW, u) = B(u, u)dt + B(\circ dW, u) \quad (3.5)$$

where the equation has to be understood in integral sense, i.e.

$$u(t) = u(0) + \int_0^t B(u(s), u(s))ds + \int_0^t B(\circ dW(s), u(s))$$

As before, in order for the integrals to be well defined, we would need some regularity on u and should impose a condition like

$$\mathbb{P}\left(\int_0^T \|u(s)\|_{D(B)}^2 ds < \infty\right) = 1$$

Instead we can define weak solutions of (3.5) by requiring that

$$\langle u(t), v \rangle = \langle u(0), v \rangle - \int_0^t \langle B(u(s), v), u(s) \rangle ds - \int_0^t \langle B(\circ dW(s), v), u(s) \rangle ds \quad \forall v \in D(B)$$

Due to (3.2) the energy is still formally invariant:

$$d|u|^2 = \langle \circ du, u \rangle = \langle B(u, u), u \rangle dt + \langle B(\circ dW, u), u \rangle = 0$$

The choice of perturbing the equation in such a way is made precisely in order to preserve the features of the original equation, namely the energy invariance. Other choices, like additive noise, while having valid physical and mathematical reasons to be introduced, don't have this property. Assuming we can apply Girsanov theorem, by defining $d\widehat{W} = dW + u dt$, we can reduce to study the linear equation

$$du = B(\circ d\widehat{W}, u) \tag{3.6}$$

From now on we will use W instead of $\widehat{W}$ for simplicity. Assume that $\{e_k\} \subset D(B)$ is an orthonormal basis of H such that W can be written as

$$W(t) = \sum_k \sigma_k e_k \beta_k(t)$$

where $\{\beta_k\}$ are independent standard real brownian motions and $\sigma_k \geq 0$ for all k (W can be a trace class Wiener process or a generalised one). Defined the linear operators $B_k = B(e_k, \cdot)$, equation (3.6) becomes

$$du = \sum_k \sigma_k B_k u \circ d\beta_k$$

We can now use the properties of Stratonovich integral in order to obtain the equivalent Ito formulation:

$$\begin{aligned} du &= \sum_k \sigma_k B_k u \circ d\beta_k \\ &= \sum_k \sigma_k B_k u d\beta_k + \frac{1}{2} \sum_k \sigma_k d[B_k u, \beta_k] \\ &= \sum_k \sigma_k B_k u d\beta_k + \frac{1}{2} \sum_k \sigma_k B_k d[u, \beta_k] \\ &= \sum_k \sigma_k B_k u d\beta_k + \frac{1}{2} \sum_k \sigma_k^2 B_k^2 u dt \end{aligned}$$

If we define (on the domain $D(A)$) the linear functional

$$Av = \frac{1}{2} \sum_k \sigma_k^2 B_k^2 v \quad (3.7)$$

then (3.6) in Ito form becomes

$$du = Au dt + B(dW, u) \quad (3.8)$$

Observe that A does not depend on time; moreover, it's symmetric and negative definite, since the map $v \mapsto B_k^2 v$ has these properties for any k : by property (3.2), $B_k^* = -B_k$, thus

$$\langle B_k^2 v, v \rangle = -\langle B_k v, B_k v \rangle = -|B_k v|^2 \quad \forall v \in D(A) \cap D(B)$$

$$\langle B_k^2 v, w \rangle = -\langle B_k v, B_k w \rangle = \langle v, B_k^2 w \rangle \quad \forall v, w \in D(A) \cap D(B)$$

With similar calculations, it can be shown that when u is a weak solution, (3.8) still holds, if we understand it as testing against sufficiently regular functions v .

If A is a closed operator and u is sufficiently regular, then A and expectation commute and so by (3.8) $\mathbb{E}[u]$ must satisfy

$$\frac{d}{dt} \mathbb{E}[u] = A \mathbb{E}[u]$$

Denoting by e^{tA} the semigroup generated by A (which is reasonable to exist since A is symmetric and negative definite), for any deterministic initial data $u(0)$ we have

$$\mathbb{E}[u(t)] = e^{tA} u(0) \quad (3.9)$$

If we define the process $v(t) = e^{-tA} u(t)$, then it satisfies:

$$dv = -e^{-tA} Au dt + e^{-tA} du = e^{-tA} B(dW, u) = e^{-tA} B(dW, e^{tA} v) \quad (3.10)$$

therefore v is a martingale and satisfies a closed equation; observe however that, as e^{tA} would be the semigroup associated to a diffusion, e^{-tA} would be associated to an antidiffusion, so that v is only defined in a very weak sense, i.e. as a distribution.

We can also obtain a closed equation for the operator $\mathbb{E}[u \otimes u]$: applying formula

$$d(u \otimes u) = (du) \otimes u + u \otimes (du) + d[u, u]$$

and computing

$$\begin{aligned} d[u, u] &= d \left[\int_0^\cdot B(dW, u), \int_0^\cdot B(dW, u) \right] = \sum_{k,j} \sigma_k \sigma_j d \left[\int_0^\cdot B_k u d\beta_k, \int_0^\cdot B_j u d\beta_j \right] \\ &= \sum_k \sigma_k^2 (B_k u) \otimes (B_k u) dt = - \sum_k \sigma_k^2 B_k (u \otimes u) B_k dt \end{aligned}$$

and then using the fact that $(Au) \otimes u = A(u \otimes u)$ and $u \otimes (Au) = (u \otimes u)A^* = (u \otimes u)A$, we find

$$d(u \otimes u) = B(dW, u) \otimes u + A(u \otimes u)dt + u \otimes B(dW, u) + (u \otimes u)Adt - \sum_k \sigma_k^2 B_k (u \otimes u) B_k dt$$

Taking expectation we obtain

$$\frac{d}{dt}\mathbb{E}[u \otimes u] = A \mathbb{E}[u \otimes u] + \mathbb{E}[u \otimes u]A - \sum_k \sigma_k^2 B_k \mathbb{E}[u \otimes u] B_k \quad (3.11)$$

Moreover, by the definition of A , equation (3.11) can also be written as

$$\frac{d}{dt}\mathbb{E}[u \otimes u] = \frac{1}{2} \sum_k \sigma_k^2 \left(B_k^2 \mathbb{E}[u \otimes u] + \mathbb{E}[u \otimes u] B_k^2 - 2B_k \mathbb{E}[u \otimes u] B_k \right)$$

Such formulation may seem unnecessarily complicated, but it's very similar to the one we will find later making explicit computations. In the case of weak solutions u , equation (3.11) must be interpreted as the fact that, setting $\Sigma = \mathbb{E}[u \otimes u]$ as the linear operator

$$\langle v, \Sigma w \rangle = \mathbb{E}[\langle u, v \rangle \langle u, w \rangle]$$

then for any $v, w \in D(A) \cap D(B)$ it holds

$$\frac{d}{dt}\langle v, \Sigma w \rangle = \langle Av, \Sigma w \rangle + \langle \Sigma v, Aw \rangle + \sum_k \sigma_k^2 \langle B_k v, \Sigma B_k w \rangle$$

Let Q denote the covariance operator of W , so that $Q^{1/2}e_k = \sigma_k e_k$; if we define the operators $B(Q^{1/2}, v) : w \mapsto B(Q^{1/2}w, v)$, then the above equation can also be written as

$$\frac{d}{dt}\langle v, \Sigma w \rangle = \langle Av, \Sigma w \rangle + \langle \Sigma v, Aw \rangle + \text{Tr}((B(Q^{1/2}, v))^* \Sigma B(Q^{1/2}, w))$$

We will not make use of this formulation in the calculations, but it's useful as it does not depend on the chosen orthonormal basis $\{e_k\}$. Equation (3.11) can also be written in a corresponding *mild form* (which perhaps gives a better understanding in the case u is a weak solution) as follows. By bringing the terms containing A on the l.h.s. and multiplying by e^{-tA} , (3.11) can be written as

$$\frac{d}{dt} \left(e^{-tA} \mathbb{E}[u \otimes u] e^{-tA} \right) = - \sum_k \sigma_k^2 e^{-tA} B_k \mathbb{E}[u \otimes u] B_k e^{-tA}$$

so that integrating from 0 to t and multiplying again by e^{tA} we obtain

$$\mathbb{E}[u(t) \otimes u(t)] = e^{tA} \mathbb{E}[u(0) \otimes u(0)] e^{tA} - \sum_k \sigma_k^2 \int_0^t e^{(t-s)A} B_k \mathbb{E}[u(s) \otimes u(s)] B_k e^{(t-s)A} ds \quad (3.12)$$

The strategy we will follow for the Leray α -model is the following: we first study the linear system (3.6). Energy formal invariance allows to construct *energy controlled weak solutions* (the rigorous definition will be given later) for such equation. Then by formula (3.11) we are able to show pathwise uniqueness of the solutions, and so by the Yamada-Watanabe theorem we also obtain uniqueness in law. In particular we will see that for energy controlled solutions Girsanov transform can be applied successfully and we can construct weak (in the probabilistic sense) solutions of the nonlinear system (3.5); moreover, as the laws of the solutions of (3.5) are in correspondence with those of (3.6), we obtain uniqueness in law for the nonlinear system. Continuous dependence on initial data and external forces is also studied.

3.2 The deterministic equation and functional setting

The Leray- α model of Euler equations is the following system of differential equations:

$$\begin{cases} \partial_t v + (u \cdot \nabla)v + \nabla p = f \\ v = (I - \alpha \Delta)u \\ \nabla \cdot v = 0 \end{cases} \quad (3.13)$$

Here v is the velocity field of an inviscid incompressible fluid (incompressibility given by the condition $\nabla \cdot v = 0$), p is the scalar pressure field and f is the vector field of the external forces acting on the system; α is a fixed positive constant and u is a spatial regularization of v . We might use the superscript v^α , u^α to stress the dependence of the system on the parameter α , but when it's not necessary it will be dropped. In the case $\alpha = 0$, system (3.13) reduces to Euler equations for incompressible fluids:

$$\begin{cases} \partial_t v + (v \cdot \nabla)v + \nabla p = f \\ \nabla \cdot v = 0 \end{cases} \quad (3.14)$$

The external force field f is given, while v , u and p are unknown (even if, as we will see, p does not play a significant role, so that the problem can be simplified to only determining v); suitable initial conditions at time $t = 0$ on v , u and p are given as well, so that we obtain a Cauchy problem associated to (3.13).

We will consider the case in which all functions are defined on a periodic box $\mathbb{T} = [0, 2\pi]^3$, with periodic boundary conditions. That is, u , v and f are maps from $[0, T] \times \mathbb{T}$ to $\mathbb{R}^3$, p is a map from $[0, T] \times \mathbb{T}$ to $\mathbb{R}$ and, for any fixed $t \in [0, T]$, $u(t, \cdot)$, $v(t, \cdot)$, $f(t, \cdot)$ and $p(t, \cdot)$ are periodic on $\mathbb{T}$. We will assume f to satisfy the condition $f \in L^2(0, T; L^2(\mathbb{T}; \mathbb{R}^3))$, namely

$$\int_{[0, T] \times \mathbb{T}} |f(t, x)|^2 dx dt < \infty$$

The condition $v = (I - \alpha \Delta)u$ can be written as $u = (I - \alpha \Delta)^{-1}v = G_\alpha * v$, where G_α is the Green function associated to the Helmholtz operator $I - \alpha \Delta$; we will denote $(I - \alpha \Delta)^{-1}$ by K^α . It is a well known fact from analysis that $K^\alpha : L^2(\mathbb{T}) \rightarrow H^2(\mathbb{T})$ is a linear continuous operator; also observe that, by the Sobolev embeddings, $H^2(\mathbb{T}) \hookrightarrow C(\mathbb{T})$ with compact embedding. A more precise description of K^α will be given later in Fourier space. Observe that, if $\nabla \cdot v = 0$, then $\nabla \cdot u = \nabla \cdot (K^\alpha v) = 0$ as well. Then we can use the following equality, which holds for any smooth functions f, g such that $\nabla \cdot f = 0$:

$$\nabla \cdot (f \otimes g) = (f \cdot \nabla)g$$

where the divergence of a matrix A is defined by

$$(\nabla \cdot A)_i = \sum_j \partial_j A_{ji}$$

to write system (3.13) in the following divergence form:

$$\begin{cases} \partial_t v + \nabla \cdot ((K^\alpha v) \otimes v + p I) = f \\ \nabla \cdot v = 0 \end{cases} \quad (3.15)$$

where I denotes the identity matrix. If v is a classical solution of (3.15), then using the fact that divergence terms give no contribution when integrated on $\mathbb{T}$, by the periodic boundary assumption, it holds

$$\frac{d}{dt} \int_{\mathbb{T}} v(t, x) dx = \int_{\mathbb{T}} \partial_t v(t, x) dx = \int_{\mathbb{T}} f(t, x) dx$$

Therefore

$$m(t) := \int_{\mathbb{T}} v(t, x) dx = \int_{\mathbb{T}} v(0, x) dx + \int_0^t \int_{\mathbb{T}} f(s, x) dx ds$$

If we define

$$\tilde{v}(t, x) = v(t, x + m(t)) - m(t), \quad \tilde{p}(t, x) = p(t, x + m(t)), \quad \tilde{f}(t, x) = f(t, x + m(t)) - \int_{\mathbb{T}} f(t, x) dx$$

then they satisfy

$$\begin{aligned} [\tilde{v}_t + (K^\alpha \tilde{v} \cdot \nabla) \tilde{v} + \nabla \tilde{p}](t, x) &= [v_t + m \cdot \nabla v - m' + (K^\alpha v \cdot \nabla) v - m \cdot \nabla v + \nabla p](t, x + m(t)) \\ &= f(t, x + m(t)) - m'(t) = \tilde{f}(t, x) \end{aligned}$$

Therefore $(\tilde{v}, \tilde{p})$ is still a solution for $\tilde{f}$ given. Viceversa, if we can solve the system associated to $\tilde{v}, \tilde{p}, \tilde{f}$, then we are able to obtain v and p as well. Hence without loss of generality we can assume

$$\int_{\mathbb{T}} v(t, x) dx = 0 = \int_{\mathbb{T}} f(t, x) dx \quad \forall t \geq 0$$

We now briefly discuss properties of the operator $(f, g) \mapsto (f \cdot \nabla)g$. Let $C_p^\infty(\mathbb{T}; \mathbb{R}^3)$ denote the set of all smooth maps from $\mathbb{T}$ to $\mathbb{R}^3$ which are periodic on $\mathbb{T}$ together with all their derivatives. For any $f, g, \varphi \in C_p^\infty(\mathbb{T}; \mathbb{R}^3)$ such that $\nabla \cdot f = 0$, it holds

$$\begin{aligned} \langle (f \cdot \nabla)g, \varphi \rangle_{L^2} &= \sum_{i=1}^3 \int_{\mathbb{T}} f_i(x) \langle \partial_i g(x), \varphi(x) \rangle dx = \sum_{i,j=1}^3 \int_{\mathbb{T}} f_i(x) \partial_i g_j(x) \varphi_j(x) dx \\ &= - \sum_{i,j=1}^3 \int_{\mathbb{T}} g_j(x) \partial_i [f_i(x) \varphi_j(x)] dx = - \sum_{i,j=1}^3 \int_{\mathbb{T}} f_i(x) g_j(x) \partial_i \varphi_j(x) dx \\ &= - \langle (f \cdot \nabla)\varphi, g \rangle_{L^2} \end{aligned}$$

The above computation can be formulated as the antisymmetry of the bilinear operator $(g, h) \mapsto \langle (f \cdot \nabla)g, h \rangle_{L^2}$. Observe that, when we compose with the operator K^α and so we consider the operator $(f, g) \mapsto ((K^\alpha f) \cdot \nabla)g$, in order for the last term to be a well defined element of $L^2(\mathbb{T}; \mathbb{R}^3)$ it suffices to require $f \in L^2(\mathbb{T}; \mathbb{R}^3)$ and $g \in H^1(\mathbb{T}; \mathbb{R}^3)$. Indeed by the properties of the operator K^α , $K^\alpha f$ is an element of $L^\infty(\mathbb{T}; \mathbb{R}^3)$ and $\|K^\alpha f\|_\infty \leq C|f|_{L^2}$, while $\nabla g \in L^2(\mathbb{T}; \mathbb{R}^{3 \times 3})$, therefore

$$|((K^\alpha f) \cdot \nabla)g|_{L^2} \leq \tilde{C}|f|_{L^2}|g|_{H^1}$$

In particular, we find that for any $f \in L^2(\mathbb{T}; \mathbb{R}^3)$ such that $\nabla \cdot f = 0$ (see later for the general definition) and for any $g, h \in H^1(\mathbb{T}; \mathbb{R}^3)$ it holds

$$\langle ((K^\alpha f) \cdot \nabla)g, h \rangle_{L^2} = - \langle ((K^\alpha f) \cdot \nabla)h, g \rangle_{L^2}$$

This also allows to define $((K^\alpha f) \cdot \nabla)g$ for any $f, g \in L^2(\mathbb{T}; \mathbb{R}^3)$ as an element of the dual space of $H^1(\mathbb{T}; \mathbb{R}^3)$ by setting

$$\left(((K^\alpha f) \cdot \nabla)g \right)(h) = -\langle ((K^\alpha f) \cdot \nabla)h, g \rangle_{L^2} \quad \forall h \in H^1(\mathbb{T}; \mathbb{R}^3)$$

The above relations already suggest a possible definition of weak solutions for system (3.13); before giving the precise definition we need to introduce suitable function spaces.

Definition 3.1. A function $f \in L^2(\mathbb{T}; \mathbb{R}^3)$ is said to have **divergence 0 in the sense of distributions**, or to be divergence-free, and we write $\nabla \cdot f = 0$, if

$$\int_{\mathbb{T}} \langle f(x), \nabla \varphi(x) \rangle dx = 0 \quad \forall \varphi \in C_p^\infty(\mathbb{T}) \quad (3.16)$$

Remark 3.2. It's easy to see integrating by parts that, if f is a smooth function with divergence 0, then it also has divergence 0 in the sense of distributions. It follows from the definition that the elements of $L^2(\mathbb{T}; \mathbb{R}^3)$ satisfying (3.16) form a closed subspace; moreover, identity (3.16) can be extended to all $\varphi \in H^1(\mathbb{T}; \mathbb{R}^3)$, so that it can be interpreted as an orthogonality relation w.r.t. to the subspace

$$G := \left\{ \nabla \varphi : \varphi \in H^1(\mathbb{T}; \mathbb{R}^3) \text{ such that } \int_{\mathbb{T}} \varphi(x) dx = 0 \right\}$$

which is again a closed subspace of $L^2(\mathbb{T}; \mathbb{R}^3)$ by Poincaré inequality.

Definition 3.3. We define H as the closed subspace of $L^2(\mathbb{T}; \mathbb{R}^3)$ given by

$$H := \left\{ f \in L^2(\mathbb{T}; \mathbb{R}^3) : \nabla \cdot f = 0, \int_{\mathbb{T}} f(x) dx = 0 \right\}$$

H is an Hilbert space, endowed with the scalar product of $L^2(\mathbb{T}; \mathbb{R}^3)$; we denote by P the projector from $L^2(\mathbb{T}; \mathbb{R}^3)$ to H , called the **Leray-Helmholtz projector**. We define $V := H \cap H^1(\mathbb{T}; \mathbb{R}^3)$, which is an Hilbert space endowed with the scalar product $\langle \varphi, \psi \rangle_V = \langle \nabla \varphi, \nabla \psi \rangle_{L^2}$. If we denote by $L_0^2(\mathbb{T}; \mathbb{R}^3)$ the subspace of $L^2(\mathbb{T}; \mathbb{R}^3)$ given by the functions f such that $\int_{\mathbb{T}} f dx = 0$, then by the previous remark we have the orthogonal decomposition

$$L_0^2(\mathbb{T}; \mathbb{R}^3) = H \oplus G$$

Let v be a classical solution of (3.13); then since $\nabla \cdot v = 0$ for all t , $\nabla \cdot \partial_t v = 0$ as well. Applying the projector P to both sides (we relabel Pf by f for simplicity) we obtain the following system:

$$\begin{cases} \partial_t v + P((K^\alpha v) \cdot \nabla)v = f \\ \nabla \cdot v = 0 \end{cases} \quad (3.17)$$

Observe that the pressure p has now disappeared and we have a closed system for v . Moreover, p is completely determined once v is known: taking the divergence on both sides of (3.13)₁, we obtain

$$\Delta p = \nabla \cdot f - \nabla \cdot [((K^\alpha v) \cdot \nabla)v]$$

which uniquely determines p once we impose the condition $\int_{\mathbb{T}} p(t, x) dx = 0$ for all $t \geq 0$. For this reason from now on we will focus primarily on solving system (3.17) and will only consider

initial conditions for v , as these imply corresponding initial conditions on u and p . The second equation of (3.17) can now be formulated as the fact that, for any t , $v(t, \cdot) \in H$; as for the first one, if we define

$$B(v, w) := -P(((K^\alpha v) \cdot \nabla)w)$$

then it can be written as

$$\partial_t v = f + B(v, v)$$

Remark 3.4. In the particular case $f \equiv 0$ (no external forces acting on the system) the equation becomes

$$\partial_t v = B(v, v)$$

which has the structure (3.1) by the previous remarks. Here we can take $D(B) = H^1(\mathbb{T}; \mathbb{R}^3)$.

We are now ready to give a definition of weak solution for system (3.13).

Definition 3.5. A map $v : [0, T] \rightarrow H$ is a **weak solution** of the Cauchy problem

$$\begin{cases} \partial_t v = B(v, v) + f \\ v(0) = v_0 \end{cases} \quad (3.18)$$

where $v_0 \in H$ and $f \in L^2(0, T; H)$, if $v(t) \rightharpoonup v_0$ weakly in H as $t \rightarrow 0$ and for any $\varphi \in V$ it holds

$$\langle v(t), \varphi \rangle_H - \langle v_0, \varphi \rangle_H = \int_0^t \left[-\langle B(v(s), \varphi), v(s) \rangle_H + \langle f(s), \varphi \rangle_H \right] ds$$

Equivalently, v as above is a weak solution if and only if, for any $\varphi \in C_p^\infty(\mathbb{T}; \mathbb{R}^3)$ such that $\nabla \cdot \varphi = 0$, it holds

$$\begin{aligned} \int_{\mathbb{T}} \langle v(t, x), \varphi(x) \rangle dx - \int_{\mathbb{T}} \langle v_0(x), \varphi(x) \rangle dx = \\ - \int_0^t \int_{\mathbb{T}} \langle [((K^\alpha v)(s, x) \cdot \nabla) \varphi(x)], v(s, x) \rangle dx ds + \int_0^t \int_{\mathbb{T}} \langle f(s, x), \varphi(x) \rangle dx ds \end{aligned} \quad (3.19)$$

We say that v is a **strong solution** if v is a weak solution, $v(t) \in V$ for all $t \in (0, T]$ and $v(t) \rightarrow v(0)$ in H as $t \rightarrow 0$.

If v is a strong solution of (3.18), then

$$\frac{d}{dt} |v|_H^2 = 2\langle \partial_t v, v \rangle_H = 2\langle B(v, v), v \rangle_H + 2\langle f, v \rangle_H = 2\langle f, v \rangle_H$$

which implies that the function

$$E(t) = \frac{1}{2} |v(t)|_H^2 - \int_0^t \langle v(s), f(s) \rangle_H ds \quad (3.20)$$

is constant. In particular we have the **energy identity**

$$\frac{1}{2} |v(t)|_H^2 - \int_0^t \langle v(s), f(s) \rangle_H ds = \frac{1}{2} |v_0|_H^2 \quad \forall t \in [0, T] \quad (3.21)$$

However, if v is a weak solution, then in general $B(v, v)$ is not an element of H and therefore the computation above does not hold; we need to weaken it to an inequality.

Definition 3.6. A function v is a **Leray weak solution** of (3.18) if it is a weak solution and satisfies the following **energy inequality**:

$$\frac{1}{2}|v(t)|_H^2 - \int_0^t \langle v(s), f(s) \rangle_H ds \leq \frac{1}{2}|v_0|_H^2 \quad \forall t \in [0, T] \quad (3.22)$$

If the inequality (3.22) is strict, we say that **anomalous dissipation** of energy has occurred. This is due to the fact that no friction terms appear in the equation and therefore there is no precise physical explanation of why the energy balance is not satisfied. However, if the solution is sufficiently regular, as for example if v is a strong solution, then the energy identity must hold; the problem of understanding the precise regularity conditions needed for a weak solution in order for the energy identity to hold is similar to Onsager's conjecture for Euler equations.

Remark 3.7. From the energy inequality and the assumption $f \in L^2(0, T; H)$ we can obtain a uniform bound on $|v(t)|_H$: by the basic inequality $2\langle a, b \rangle \leq |a|^2 + |b|^2$ it follows that

$$\frac{1}{2}|v(t)|^2 \leq \frac{1}{2}|v_0|^2 + \int_0^t \langle v(s), f(s) \rangle \leq \frac{1}{2} \left[|v_0|^2 + \int_0^t |f(s)|^2 ds + \int_0^t |v(s)|^2 ds \right]$$

and so by Gronwall's lemma

$$|v(t)|^2 \leq |v_0|^2 + e^t \int_0^t |f(s)|^2 ds$$

In particular, if $T < \infty$, then taking $C = e^T$ we obtain the following uniform bound:

$$\sup_{t \in [0, T]} |v(t)|_H^2 \leq C \left(|v_0|_H^2 + \int_0^T |f(s)|_H^2 ds \right) \quad (3.23)$$

which holds for any initial value v_0 and external force field $f \in L^2(0, T; H)$. In a similar fashion, using the inequality

$$\frac{1}{2}|v(t)|^2 \leq \frac{1}{2}|v_0|^2 + \int_0^t |v(s)| |f(s)| ds$$

and a Gronwall type inequality (see [21], Theorem 5, p. 4), if $f \in L^1(0, +\infty; H) \cap L_{loc}^2(0, +\infty; H)$ and the energy controlled solution v is defined for all positive times, then

$$\sup_{t \geq 0} |v(t)|_H \leq |v_0|_H + \int_0^\infty |f(s)|_H ds \quad (3.24)$$

In the remaining part of this section we are going to "translate" all the concepts introduced in Fourier spaces, which is the formulation we will adopt when making explicit calculations. For more details on the setting we refer to [22].

Recall that any function $g \in L^2(\mathbb{T}; \mathbb{R}^3)$ can be decomposed as its Fourier series, that is

$$g(x) = \sum_{k \in \mathbb{Z}^3} g_k e^{i\langle k, x \rangle}, \quad \text{where } g_k = \frac{1}{(2\pi)^3} \int_{\mathbb{T}} g(x) e^{-i\langle k, x \rangle} dx \text{ for all } k \in \mathbb{Z}^3$$

Here for convenience we consider $L^2(\mathbb{T}; \mathbb{C}^3)$ endowed with the rescaled scalar product and norm

$$\langle f, g \rangle_{L^2} = \frac{1}{(2\pi)^3} \int_{\mathbb{T}} f(x) \bar{g}(x) dx, \quad |f|_{L^2}^2 = \frac{1}{(2\pi)^3} \int_{\mathbb{T}} |f(x)|^2 dx$$

So that, defined $e_k = e^{i\langle k, \cdot \rangle}$ for any $k \in \mathbb{Z}^3$, $\{e_k\}_{k \in \mathbb{Z}^3}$ is an orthonormal basis of $L^2(\mathbb{T}; \mathbb{C})$; in particular it holds

$$|g|_{L^2}^2 = \sum_{k \in \mathbb{Z}^3} |g_k|^2$$

A function g belongs to $L^2(\mathbb{T}; \mathbb{R}^3)$ if and only if $g_{-k} = \bar{g}_k$ for all $k \in \mathbb{Z}^3$.

Observe that $\{e_k\}$ gives an orthonormal basis of eigenvalues of $(I - \alpha\Delta)$:

$$(I - \alpha\Delta)e_k = (1 + \alpha|k|^2)e_k \quad \forall k \in \mathbb{Z}^3$$

This allows to define, for any $\alpha > 0$, the operator $K^\alpha = (I - \alpha\Delta)^{-1}$ as

$$g = \sum_k g_k e_k \mapsto K^\alpha g := \sum_k \frac{1}{1 + \alpha|k|^2} g_k e_k$$

K^α is a symmetric Hilbert-Schmidt operator, since

$$\sum_{k \in \mathbb{Z}^3} |K^\alpha e_k|_{L^2}^2 = \sum_{k \in \mathbb{Z}^3} \frac{1}{(1 + \alpha|k|^2)^2} < \infty$$

Moreover, for any $g \in L^2(\mathbb{T}; \mathbb{C}^3)$, $K^\alpha g \in H^2(\mathbb{T}; \mathbb{C}^3)$, since

$$|K^\alpha g|_{H^2}^2 = \sum_k (1 + |k| + |k|^2) |(K^\alpha g)_k|^2 = \sum_k \frac{1 + |k| + |k|^2}{1 + \alpha|k|^2} |g_k|^2 \leq C \sum_k |g_k|^2 = C |g|_{L^2}^2$$

If $g \in H^1(\mathbb{T}; \mathbb{C}^3)$, then its divergence has Fourier series

$$\nabla \cdot g = \sum_k \nabla \cdot (g_k e_k) = i \sum_k \langle g_k, k \rangle e_k$$

By the uniqueness of the Fourier expansion it follows that

$$\nabla \cdot g = 0 \Leftrightarrow \langle g_k, k \rangle = 0 \quad \forall k \in \mathbb{Z}^3$$

It's immediate to check that the above condition can be generalized to $g \in L^2(\mathbb{T}; \mathbb{C}^3)$ and gives a characterization of g being a divergence-free vector field. Therefore the space H corresponds in Fourier series to

$$H = \left\{ g = (g_k)_{k \in \mathbb{Z}^3} : \sum |g_k|^2 < \infty, \ g_{-k} = \bar{g}_k, \ g_0 = 0, \ \langle g_k, k \rangle = 0 \right\}$$

while V is given by

$$V = \left\{ g = (g_k)_{k \in \mathbb{Z}^3} : \sum |k|^2 |g_k|^2 < \infty, \ g_{-k} = \bar{g}_k, \ g_0 = 0, \ \langle g_k, k \rangle = 0 \right\}$$

For any $k \in \mathbb{Z}^3$, let P_k denote the orthogonal projection on $k^\perp$ (with the convention $P_0 \equiv 0$), then any $g \in L_0^2(\mathbb{T}; \mathbb{C}^3)$ can be decomposed as

$$g = \sum_k g_k e_k = \sum_k P_k(g_k) e_k + \sum_k \lambda_k k e_k$$

where the first term is a divergence-free vector field and the λ_k are complex coefficients such that

$$\sum_k |\lambda_k|^2 |k|^2 \leq \sum_k |g_k|^2 < \infty$$

therefore the above is exactly the decomposition of g as a sum of an element of H and one of G and the Leray-Helmholtz projector P in the Fourier space is given by

$$g = \sum_k g_k e_k \mapsto Pg = \sum_k P_k(g_k) e_k$$

We now want to find the equivalent weak formulation of system (3.18) in terms of Fourier coefficients. Since any $\varphi \in C_p^\infty(\mathbb{T}; \mathbb{R}^3)$ can be arbitrarily approximated by trigonometric polynomials, it suffices to check that (3.19) holds when $\varphi = e_k$; but testing (3.19) against $\{e_k\}_k$ is equivalent to comparing the Fourier coefficients on both sides of the equation. In this way we find an infinite system of coupled ODEs for the Fourier coefficients of v . If

$$v(t, x) = \sum_{k \in \mathbb{Z}^3} v_k(t) e_k(x) \quad \text{with } \langle v_k(t), k \rangle = 0 \quad \forall t, k \text{ and } v_0 \equiv 0$$

then, defined $\sigma_k = (1 + \alpha|k|^2)^{-1}$, we have

$$\partial_t v = \sum_k \dot{v}_k e_k, \quad K^\alpha v = \sum_k \sigma_k v_k e_k, \quad \nabla v = i \sum_k (v_k \otimes k) e_k$$

and

$$\begin{aligned} B(v, v) &= (K^\alpha v \cdot \nabla) v = \sum_{i=1}^3 (K^\alpha v)_i \partial_i v = i \sum_k \langle K^\alpha v, k \rangle v_k e_k = i \sum_{k, h} \sigma_h \langle v_h, k \rangle e_h v_k e_k \\ &= i \sum_{k, h} \sigma_h \langle v_h, k \rangle v_k e_{k+h} = i \sum_k \left(\sum_h \sigma_h \langle v_h, k \rangle v_{k-h} \right) e_k \end{aligned}$$

where in the last passage we have used the substitution $\tilde{k} = k + h$ and $\langle v_h, k - h \rangle = \langle v_h, k \rangle$, since v is divergence-free. Then applying the projector P we obtain

$$P((K^\alpha v \cdot \nabla) v) = i \sum_k P_k \left(\sum_h \sigma_h \langle v_h, k \rangle v_{k-h} \right) e_k = i \sum_k \left(\sum_h \sigma_h \langle v_h, k \rangle P_k(v_{k-h}) \right) e_k$$

Since f is also of the form

$$f(t, x) = \sum_{k \in \mathbb{Z}^3} f_k(t) e_k(x) \quad \text{with } \langle f_k(t), k \rangle = 0 \quad \forall t, k \text{ and } f_0 \equiv 0$$

then substituting everything in the equation (3.18) we obtain

$$\sum_k \dot{v}_k e_k + i \sum_k \left(\sum_h \sigma_h \langle v_h, k \rangle v_{k-h} \right) e_k = \sum_k f_k e_k$$

Comparing the Fourier coefficients we obtain the following infinite dimensional system:

$$\begin{cases} \dot{v}_k = f_k - i \sum_{h \in \mathbb{Z}^3} \sigma_h \langle v_h, k \rangle P_k(v_{k-h}) & \forall k \neq 0 \\ v_0 \equiv 0, \quad \langle v_k, k \rangle = 0 \\ v_{-k} = \bar{v}_k \end{cases} \quad (3.25)$$

Observe that the series in the first equation is convergent, since for any fixed k

$$\sum_h \sigma_h |\langle v_h, k \rangle P_k(v_{k-h})| \leq |k| \sum_h |v_h| |v_{k-h}| \leq |k| \left(\sum_h |v_h|^2 \right)^{1/2} \left(\sum_h |v_{k-h}|^2 \right)^{1/2} = |k| \sum_h |v_h|^2$$

Let us show how the formal energy identity (3.21) translates in Fourier components: since

$$\frac{1}{2} |v(t)|_{L^2}^2 = \frac{1}{2} \sum_k |v_k(t)|^2 \quad \langle f(t), v(t) \rangle_{L^2} = \sum_k \langle f_k(t), \bar{v}_k(t) \rangle$$

we can define the function

$$E(t) = \frac{1}{2} \sum_k \langle v_k(t), \bar{v}_k(t) \rangle - \int_0^t \sum_k \langle f_k(s), \bar{v}_k(s) \rangle ds \quad (3.26)$$

and so deriving (assuming everything is regular enough) we obtain

$$\begin{aligned} \dot{E} &= \sum_k \Re(\langle \dot{v}_k, \bar{v}_k \rangle) - \sum_k \langle f_k, \bar{v}_k \rangle \\ &= \sum_k \Re(\langle f_k, \bar{v}_k \rangle) - \sum_k \langle f_k, \bar{v}_k \rangle + \sum_{k,h} \Re\left(-i\sigma_h \langle v_h, k \rangle \langle P_k(v_{k-h}), \bar{v}_k \rangle\right) \\ &= -\sum_k \Im(\langle f_k, \bar{v}_k \rangle) + \sum_{k,h} \Im\left(\sigma_h \langle v_h, k \rangle \langle v_{k-h}, \bar{v}_k \rangle\right) \end{aligned}$$

where we have used the fact that $\langle P_k(v_{k-h}), \bar{v}_k \rangle = \langle v_{k-h}, P_k(\bar{v}_k) \rangle = \langle v_{k-h}, \bar{v}_k \rangle$. Observe that $\Im\langle f_k, \bar{v}_k \rangle = -\Im\langle f_{-k}, \bar{v}_{-k} \rangle$, so the first series is 0 as it contains terms that cancel each other. For the second series we also have mutual cancellation since

$$\sigma_{h'} \langle v_{h'}, k' \rangle \langle v_{k'-h'}, \bar{v}_{k'} \rangle = \overline{\sigma_h \langle v_h, k \rangle \langle v_{k-h}, \bar{v}_k \rangle}$$

for the choice $h' = -h$, $k' = k - h$. Therefore formally E is preserved. The problem with this reasoning is that it requires the series

$$\sum_{k,h} \sigma_h \langle v_h, k \rangle \langle v_{k-h}, \bar{v}_k \rangle$$

to be absolutely convergent; however the condition

$$\sum_k |v_k|^2 < \infty$$

corresponding to $v \in H$, is not sufficient to ensure it. Instead if for example $v \in V$, then the above series is convergent, since

$$\begin{aligned} \left| \sum_{k,h} \sigma_h \langle v_h, k-h \rangle \langle v_{k-h}, \bar{v}_k \rangle \right| &\leq \sum_h \sigma_h |v_h| \left(\sum_k |k-h| |v_{k-h}| |v_k| \right) \\ &\leq \sum_h \sigma_h |v_h| \left(\sum_k |k-h|^2 |v_{k-h}|^2 \right)^{1/2} \left(\sum_k |v_k|^2 \right)^{1/2} \\ &\leq \left(\sum_h \sigma_h^2 \right) \left(\sum_k |v_k|^2 \right) \left(\sum_k |k|^2 |v_k|^2 \right)^{1/2} < \infty \end{aligned}$$

In this setting the energy inequality can be formulated as

$$\frac{1}{2} \sum_k |v_k(t)|^2 - \int_0^t \sum_k \langle f_k(s), \bar{v}_k(s) \rangle ds \leq \frac{1}{2} \sum_k |v_k(0)|^2 \quad \forall t \in [0, T] \quad (3.27)$$

Similarly to Remark 3.7, the above inequality can be used to find uniform estimates on $\sum_k |v_k(t)|^2$.

3.3 The stochastic Leray- α model

We are now going to introduce the model we will study, which is a stochastic perturbation of the deterministic model of the previous section, obtained by adding a Stratonovich multiplicative noise. Throughout the section we consider a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$ and the Wiener processes considered are all with respect to $\mathcal{F}_t$. We do not specify for now whether the space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$ and the drivers are fixed a priori or if we can choose them (namely if we are considering strong or weak solutions in the probabilistic sense). Such distinction will be made clear in the next sections.

We want to study the equation

$$dv = f dt + B(v dt + \circ dW, v) = [f + B(v, v)]dt + B(\circ dW, v) \quad v \in H \quad (3.28)$$

where v is an H -valued process and W is an H -valued cylindrical Wiener process of the form

$$W(t) = \sum_k W_k(t) e_k$$

where W_k are a family of $\mathbb{C}^3$ -valued brownian motions satisfying $W_{-k} = \overline{W}_k$, $W_0 \equiv 0$ and $\langle W_k, k \rangle = 0$. A more detailed description of the construction of W will be given later. Equivalently equation (3.28) can be written as the system

$$\begin{cases} dv + [(K^\alpha v \cdot \nabla) v + \nabla p - f]dt + ((K^\alpha \circ dW) \cdot \nabla) v = 0 \\ \nabla \cdot v = 0 \end{cases}$$

and it must be interpreted as an integral equation, i.e. v is an H -valued process such that

$$v(t) = v(0) + \int_0^t [f(s) + B(v(s), v(s))]ds + \int_0^t B(\circ dW(s), v(s)) \quad \forall t \in [0, T] \quad (3.29)$$

Remark 3.8. The term $K^\alpha \circ dW$ appearing in the second formulation could be written as $\circ d\widehat{W}$, where $\widehat{W} = K^\alpha W$ is a trace class Wiener process with covariance $(K^\alpha)^2$, so that we could work with trace class Wiener processes instead of generalised ones. However we prefer to use W as above as it does not depend on the parameter α chosen and allows to consider multiple systems, depending on different choices of α , defined on the same probability space.

As before, we can define weak (in the variational sense) solutions of (3.29) by requiring the relation to hold when we test it against functions in V .

Definition 3.9. An H -valued process $\{v(t)\}_{t \in [0, T]}$ is a **weak solution** of the Cauchy problem

$$\begin{cases} dv = f + B(v) dt + \odot dW, v \\ v(0) = v_0 \end{cases} \quad (3.30)$$

where v_0 is an $\mathcal{F}_0$ -measurable H -valued random variable and $f \in L^2(0, T; H)$ is a deterministic function, if $\mathbb{P}$ -a.s. $v(t) \rightharpoonup v_0$ weakly in H and for any deterministic $\varphi \in V$ it holds

$$d\langle v(t), \varphi \rangle_H = [\langle f(t), \varphi \rangle_H - \langle B(v(t), \varphi), v(t) \rangle_H] dt - \langle B(\odot dW(t), \varphi), v(t) \rangle_H$$

With calculations analogue to the deterministic case, it can be checked that E is still formally preserved, thanks to the properties of Stratonovich integral. Therefore it makes sense to formulate the energy inequality and the notion of Leray solutions.

Definition 3.10. A process v is a **Leray weak solution** of the Cauchy problem (3.30) if it's a weak solution and satisfies the **energy inequality**

$$\sup_{t \in [0, T]} \left[\frac{1}{2} |v(t)|_H^2 - \int_0^t \langle f(s), v(s) \rangle_H ds \right] \leq \frac{1}{2} |v(0)|_H^2 \quad \mathbb{P}\text{-a.s.}$$

If the energy inequality is satisfied, then the uniform bound (3.23) still holds. Passing to the formulation in Fourier coefficients, with similar calculations to before we obtain the following infinite system of SDEs:

$$\begin{cases} dY_k(t) = f_k(t) dt - i \sum_h \sigma_h P_k(Y_{k-h}(t)) \langle Y_h(t) dt + \odot dW_h(t), k \rangle \quad \forall k \neq 0 \\ Y_0 \equiv 0, \quad \langle Y_k(t), k \rangle = 0 \\ Y_{-k}(t) = \overline{Y}_k(t) \end{cases} \quad (3.31)$$

Observe that $\langle \odot dW_h(t), k \rangle = \odot d\langle W_h(t), k \rangle$, so that it can be considered as an integration with respect to a $\mathbb{C}$ -valued process. For more details on the conventions we adopted when extending stochastic integration in the case of complex process, we refer to the appendix.

Let us now show the construction of the generalised Wiener process W . Set

$$J = \{k \in \mathbb{Z}^3 : k_1 > 0 \text{ or } (k_1 = 0, k_2 > 0) \text{ or } (k_1 = k_2 = 0, k_3 > 0)\}$$

so that J and $-J$ form a partition of $\mathbb{Z}^3 \setminus \{0\}$. Let $\{\widetilde{W}_k\}_{k \in J}$ be a collection of independent $\mathbb{C}^3$ -valued standard Brownian motions (a $\mathbb{C}^3$ -valued standard Brownian motion has the form $B = B^{(1)} + iB^{(2)}$, where $B^{(1)}$ and $B^{(2)}$ are independent $\mathbb{R}^3$ -valued standard Brownian motions). Set

$$W_k = P_k(\widetilde{W}_k) \quad \forall k \in J, \quad W_k = \overline{W}_{-k} \quad \forall k \in -J, \quad W_0 \equiv 0$$

Then

$$W(t) = \sum_{k \in \mathbb{Z}^3} W_k(t) e_k$$

has the desired properties. It can also be written as

$$W(t) = 2 \sum_{k \in J} (\cos kx \Re W_k(t) + \sin kx \Im W_k(t))$$

Observe that by construction

$$\langle W_h, k \rangle = \langle P_h(\widetilde{W}_h), k \rangle = \langle \widetilde{W}_h, P_h(k) \rangle$$

with the last process being a $\mathbb{C}$ -valued Brownian motion with variance $|P_h(k)|^2$. In particular it can be written as

$$\langle W_h, k \rangle(t) = |P_h(k)|B_{h,k}(t)$$

where $\{B_{h,k}\}$ is a family of standard $\mathbb{C}$ -valued Brownian motions. By construction $B_{h,k}$ and $B_{h',k'}$ are independent whenever $h' \neq \pm h$, while $B_{-h,k} = \overline{B}_{h,k}$. The equations of system (3.31) can be written as

$$dY_k(t) = f_k(t)dt - i \sum_h \sigma_h \langle Y_h(t), k \rangle P_k(Y_{k-h}(t))dt - i \sum_h \sigma_h |P_h(k)| P_k(Y_{k-h}(t)) \circ dB_{h,k}(t) \quad (3.32)$$

This formulation is very useful in order to compute the corresponding system of SDEs in Ito form.

Theorem 3.11. Let $\{Y_k\}_{k \in \mathbb{Z}^3 \setminus \{0\}}$ be a sequence of continuous and adapted processes defined on $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$ such that $\int_0^T \sum_k |Y(t)|^2 dt < \infty$ $\mathbb{P}$ -a.s. Then they solve (3.32) if and only if they solve the following system:

$$\begin{aligned} dY_k(t) = & f_k(t)dt - i \sum_h \sigma_h \langle Y_h(t), k \rangle P_k(Y_{k-h}(t))dt - i \sum_h \sigma_h |P_h(k)| P_k(Y_{k-h}(t))dB_{h,k}(t) \\ & - \sum_h \sigma_h^2 |P_h(k)|^2 P_k(P_{k-h}(Y_k(t)))dt \quad \forall k \neq 0 \end{aligned} \quad (3.33)$$

Proof. We are going to show that, if $\{Y_k\}$ solve system (3.32), then they solve system (3.33). The other implication is analogue and only requires to go through the same calculations backwards. Recall that, by the properties of Stratonovich integral, we have

$$P_k(Y_{k-h}(t)) \circ dB_{h,k}(t) = P_k(Y_{k-h}(t) \circ dB_{h,k}(t)) = P_k(Y_{k-h}(t)dB_{h,k}(t) + \frac{1}{2}d[Y_{k-h}, B_{h,k}](t))$$

Therefore it suffices to show that, for any h , we have

$$d[Y_{k-h}, B_{h,k}](t) = -2i\sigma_h |P_h(k)| P_{k-h}(Y_k(t))dt$$

Using the fact that Y_{k-h} satisfies (3.32), we obtain

$$\begin{aligned} d[Y_{k-h}, B_{h,k}] &= -i \sum_l \sigma_l |P_l(k-h)| \left[\int_0^\cdot P_{k-h}(Y_{k-h-l}) \circ dB_{l,k-h}, B_{h,k} \right] \\ &= -i \sum_l \sigma_l |P_l(k-h)| P_{k-h} \left[\int_0^\cdot Y_{k-h-l} dB_{l,k-h}, B_{h,k} \right] \\ &= -i\sigma_h |P_h(k-h)| P_{k-h}(Y_{k-2h})d[B_{h,k-h}, B_{h,k}] \\ &\quad - i\sigma_{-h} |P_{-h}(k-h)| P_{k-h}(Y_k)d[B_{-h,k-h}, B_{h,k}] \\ &= -i\sigma_h |P_h(k)| P_{k-h} \left(Y_{k-2h}d[B_{h,k-h}, B_{h,k}] + Y_kd[B_{-h,k-h}, B_{h,k}] \right) \end{aligned}$$

where we used the fact that $B_{h,k}$ and $B_{h',k'}$ are independent whenever $h' \neq \pm h$. Observe that by construction $B_{h,k} = B_{h,k-h}$ for any h ; since $B_{h,k}$ is a standard $\mathbb{C}$ -valued Brownian motion, and $B_{-h,k} = \overline{B}_{h,k}$, we have

$$[B_{h,k-h}, B_{h,k}] = [B_{h,k}, B_{h,k}] = 0, \quad [B_{-h,k-h}, B_{h,k}] = [\overline{B}_{h,k}, B_{h,k}] = 2t$$

and so we obtain

$$d[Y_{k-h}, B_{h,k}](t) = -2i\sigma_h|P_h(k)|P_{k-h}(Y_k(t))dt$$

□

3.4 The linear system

We are now going to study the following linearized version of system (3.33), together with an initial condition $Y(0) = y$. Here y is the representation in Fourier components of a $\mathcal{F}_0$ -measurable, square integrable H -valued random variable. Equivalently, it can be thought as a collection $\{y_k\}_{k \in \mathbb{Z}^3}$ of $\mathbb{C}^3$ -valued, $\mathcal{F}_0$ -measurable random variables such that $y_0 \equiv 0$, $y_{-k} = \overline{y}_k$, $\langle y_k, k \rangle = 0$ for every $k \neq 0$ and

$$\mathbb{E}[|y|_H^2] = \sum_k \mathbb{E}[|y_k|^2] < \infty$$

So we are interested in studying the system

$$\begin{cases} dY_k = f_k dt - i \sum_h \sigma_h |P_h(k)| P_k(Y_{k-h}) dB_{h,k} - \sum_h \sigma_h^2 |P_h(k)|^2 P_k(P_{k-h}(Y_k)) dt \\ Y_0 \equiv 0, \quad \langle Y_k, k \rangle = 0 \\ Y_{-k} = \overline{Y}_k \\ Y_k(0) = y_k \end{cases} \quad (3.34)$$

In the next sections we will see how Girsanov transform allows to pass from the linear system to the nonlinear one. Notice that, if $\int_0^T \mathbb{E}[|Y(t)|_H^2] dt < \infty$, then the r.h.s. of equation (3.34)₁ is well defined. Indeed, for the Ito integrals, using the fact that $[B_{h,k}, \overline{B}_{l,k}](t) = 2t\delta_{h,l}$, we have

$$\begin{aligned} & \mathbb{E} \left[\left| \sum_h \sigma_h |P_h(k)| \int_0^t P_k(Y_{k-h}) dB_{h,k} \right|^2 \right] \\ &= \mathbb{E} \left[\sum_{h,l} \sigma_h \sigma_l |P_h(k)| |P_l(k)| \int_0^t \langle P_k(Y_{k-h}), P_k(\overline{Y}_{k-l}) \rangle d[B_{h,k}, \overline{B}_{l,k}] \right] \\ &= 2 \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t \mathbb{E}[|P_k(Y_{k-h})|^2] dt \\ &\leq 2|k|^2 \sum_h \int_0^T \mathbb{E}[|Y_{k-h}|^2] dt < \infty \end{aligned}$$

and for the deterministic integrals

$$\begin{aligned}
& \left| \int_0^t \sum_h \sigma_h^2 |P_h(k)|^2 P_k(P_{k-h}(Y_k)) ds \right| \\
& \leq \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t |P_k(P_{k-h}(Y_k))| ds \\
& \leq |k|^2 \left(\sum_h \sigma_h^2 \right) \int_0^T |Y_k| ds < \infty
\end{aligned}$$

In this section we shall deal with solutions of the stochastic system (3.34) that are strong in the probabilistic sense, i.e. with a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and the Brownian motions $\{\widetilde{W}_k\}_{k \in J}$ given a priori (and $B_{h,k}$ constructed from $\{\widetilde{W}_k\}$ as before).

Definition 3.12. Given y initial data as above, an **energy controlled solution** for the Cauchy problem associated to system (3.34) is an adapted H -valued stochastic process, or equivalently a family of continuous and adapted $\mathbb{C}^3$ -valued stochastic processes $\{Y_k\}_{k \in \mathbb{Z}^3}$, such that for all $t \geq 0$

$$\begin{cases} Y_k(t) = y_k + \int_0^t f_k(s) ds - i \sum_h \sigma_h |P_h(k)| \int_0^t P_k(Y_{k-h}(s)) dB_{h,k}(s) \\ \quad - \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t P_k(P_{k-h}(Y_k(s))) ds \\ Y_0 \equiv 0, \langle Y_k, k \rangle = 0 \\ Y_{-k} = \overline{Y}_k \end{cases} \quad (3.35)$$

and such that

$$\sup_{t \in [0, T]} \left\{ \sum_k |Y_k(t)|^2 - \int_0^t \sum_k \langle f_k, \overline{Y}_k \rangle ds \right\} \leq \sum_k |y_k|^2 \quad \mathbb{P}\text{-a.s.}$$

Theorem 3.13. For any $\mathcal{F}_0$ -measurable initial data such that $\mathbb{E}[|y|_H^2] < \infty$ and deterministic $f \in L^2(0, T; H)$, there exists an energy controlled solution of (3.35).

Proof. We construct the solution by means of a Faedo-Galerkin scheme. That is, we construct a sequence of solutions of finite dimensional systems associated to (3.35) and we show that they admit a convergent subsequence, whose limit is an energy controlled solution.

For any $N > 0$, consider the sets $\Gamma_N^k = \{h \in \mathbb{Z}^3 : 0 < |h| < N, 0 < |h - k| < N\}$ and the system

$$\begin{cases} dY_k = f_k dt - i \sum_{h \in \Gamma_N^k} \sigma_h |P_h(k)| P_k(Y_{k-h}) dB_{h,k} - \sum_{h \in \Gamma_N^k} \sigma_h^2 |P_h(k)|^2 P_k(P_{k-h}(Y_k)) dt \\ Y_0 \equiv 0, \langle Y_k, k \rangle = 0 \\ Y_{-k} = \overline{Y}_k \\ Y_k(0) = y_k \end{cases} \quad (3.36)$$

for each $k \in \mathbb{Z}^3$ such that $0 < |k| < N$. We denote by y^N the initial data with all components with $|k| > N$ set to 0. By linearity, the finite dimensional system of SDEs, together with the

$\mathcal{F}_0$ -measurable initial data y^N , has a unique global strong solution $Y^N = \{Y_k^N, 0 < |k| < N\}$; every component is a continuous and adapted process and the energy identity holds:

$$\sup_{t \in [0, T]} \left\{ |Y^N(t)|_H^2 - \int_0^t \sum_{|k| < N} \langle f_k(s), \bar{Y}_k^N(s) \rangle ds \right\} = |y^N|_H^2 \leq |y|_H^2 \quad \mathbb{P}\text{-a.s.} \quad (3.37)$$

Therefore by estimate (3.23) we obtain that for some $C > 0$, independent of N , $\mathbb{P}$ -a.s.

$$\sup_{[0, T]} |Y^N(t)|_H^2 \leq C \left(|y^N|_H^2 + \int_0^T |f(s)|_H^2 ds \right) \leq C \left(|y|_H^2 + \int_0^T |f(s)|_H^2 ds \right) \quad (3.38)$$

Since y is square integrable, it holds

$$\mathbb{E} \left[\int_0^T |Y^N(t)|_H^2 dt \right] \leq CT \left(\mathbb{E}[|y|_H^2] + \int_0^T |f(s)|_H^2 ds \right) \quad (3.39)$$

This implies that the sequence $\{Y^N\}$ is uniformly bounded in $L^2(\Omega_T, \mathcal{P}_T, \mathbb{P}_T; H)$; this space was introduced in Section 2.2 and from now on will be denoted by $L^2(\Omega_T; H)$. Since $L^2(\Omega_T; H)$ is reflexive (it's an Hilbert space), there exists a subsequence (not relabelled) Y^N and an element Y such that $Y^N \rightharpoonup Y$ weakly in $L^2(\Omega_T; H)$. In particular, for all $k \in \mathbb{Z}^3$,

$$Y_k^N \rightharpoonup Y_k \text{ in } L^2(\Omega_T; \mathbb{C}^3) \quad (3.40)$$

We now consider the convergence of the integrals on the r.h.s. of (3.36)₁. Observe that, for any fixed k , the operator from $L^2(\Omega_T; H)$ to $L^2(\Omega_T; \mathbb{C}^3)$, defined by

$$Z \mapsto \sum_h \sigma_h |P_h(k)| \int_0^\cdot P_k(Z_{k-h}(s)) dB_{h,k}(s)$$

is linear and strongly continuous, since

$$\begin{aligned} & \int_0^T \mathbb{E} \left| \sum_h \sigma_h |P_h(k)| \int_0^t P_k(Z_{k-h}(s)) dB_{h,k}(s) \right|^2 dt \\ &= 2 \int_0^T \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t \mathbb{E}[|Z_{k-h}(s)|^2] ds dt \\ &\leq 2T |k|^2 \sum_h \int_0^T \mathbb{E}[|Z_{k-h}(s)|^2] ds = 2T |k|^2 \|Z\|_{L^2(\Omega_T; H)}^2 \end{aligned}$$

Therefore it's also a weakly continuous operator, which implies that, for any k ,

$$\sum_{h \in \Gamma_N^k} \int_0^\cdot \sigma_h |P_h(k)| P_k(Y_{k-h}^N) dB_{h,k} \rightharpoonup \sum_{h \in \mathbb{Z}^3} \int_0^\cdot \sigma_h |P_h(k)| P_k(Y_{k-h}) dB_{h,k} \quad \text{in } L^2(\Omega_T; \mathbb{C}^3)$$

For the deterministic integrals, by similar reasonings we have

$$\sum_{h \in \Gamma_N^k} \sigma_h^2 |P_h(k)|^2 \int_0^\cdot P_k(P_{k-h}(Y_k^N)) dt \rightharpoonup \sum_{h \in \mathbb{Z}^3} \sigma_h^2 |P_h(k)|^2 \int_0^\cdot P_k(P_{k-h}(Y_k)) dt$$

Since $y^N \rightarrow y$ pointwise in H and $\|y^N\|_H \leq \|y\|_H$, by dominated convergence $y^N \rightarrow y$ in $L^2(\Omega; H)$. Therefore taking the limit on both sides of equation (3.36)₁ we find that Y satisfies (3.35). Moreover, for any k the process

$$\sum_h \sigma_h |P_h(k)| \int_0^\cdot P_k(Y_{k-h}(s)) dB_{h,k}(s)$$

is a continuous martingale, since it's the limit in $L^2(\Omega_T; \mathbb{C}^3)$ of the finite series, which are continuous martingales by the properties of stochastic integral. Therefore Y is such that Y_k are continuous and adapted processes. This shows that Y is a solution; it remains to show that it is energy controlled. Consider the set

$$A = \left\{ Z \in L^2(\Omega_T; H) : |Z(t)|_H^2 - \int_0^t \langle f(s), Z(s) \rangle_H ds \leq |y|_H^2 \text{ for } \mathbb{P}_T\text{-a.e. } (\omega, t) \right\}$$

It's easy to check that A is a convex set. It's also strongly closed: if a sequence $Z_n \rightarrow Z$ in $L^2(\Omega_T; H)$, $Z_n \in A$, then $\int_0^\cdot \langle f(s), Z_n(s) \rangle_H ds \rightarrow \int_0^\cdot \langle f(s), Z(s) \rangle_H ds$ in $L^2(\Omega_T; \mathbb{R})$ as well and so we can extract a subsequence such that they both converge pointwise for $\mathbb{P}_T\text{-a.e. } (\omega, t)$; since $Z_n \in A$, this implies that $Z \in A$ as well. Since A is convex and strongly closed, it's also weakly closed; by equation (3.37), $Y^N \in A$ for all N and therefore $Y \in A$ as well. In particular, for $\mathbb{P}\text{-a.e. } \omega$, the trajectories $Y_k(\cdot, \omega)$ are continuous and there exists a set $\Gamma(\omega) \subset [0, T]$ of full measure (thus dense) such that

$$|Y(t, \omega)|_H^2 - \int_0^t \langle f(s), Y(s, \omega) \rangle_H ds \leq |y|_H^2 \quad \forall t \in \Gamma(\omega) \quad (3.41)$$

Now let $t \in [0, T] \setminus \Gamma(\omega)$, then there exists a sequence $t_n \rightarrow t$, $t_n \in \Gamma(\omega)$. By Fatou's lemma, since the trajectories $Y_k(\cdot, \omega)$ are continuous,

$$|Y(t, \omega)|_H^2 = \sum_k |Y_k(t, \omega)|^2 \leq \liminf_{n \rightarrow \infty} \sum_k |Y_k(t_n, \omega)|^2 = \liminf_{n \rightarrow \infty} |Y(t_n, \omega)|_H^2$$

and since the process $\int_0^\cdot \langle f(s), Y(s) \rangle_H ds$ has continuous trajectories, similarly

$$\int_0^t \langle f(s), Y(s, \omega) \rangle_H ds = \lim_{n \rightarrow \infty} \int_0^{t_n} \langle f(s), Y(s, \omega) \rangle_H ds$$

Therefore by taking the limit as $n \rightarrow \infty$, by density of $\Gamma(\omega)$ we obtain that inequality (3.41) extends to all $t \in [0, T]$. Since the reasoning holds for $\mathbb{P}\text{-a.e. } \omega$ we obtain the conclusion. $\square$

Remark 3.14. It follows from the fact that Y is an energy controlled solution that it satisfies the energy bound

$$\sup_{t \in [0, T]} |Y(t)|_H^2 \leq C \left(|y|_H^2 + \int_0^T |f(s)|_H^2 ds \right) \quad \mathbb{P}\text{-a.s.}$$

In particular, if the initial data $y \in L^\infty(\Omega; H)$, then $Y \in L^\infty(\Omega_T; H)$. Moreover, in this case the proof of Theorem 3.13 can be carried out taking a sequence Y^N that converges weakly- $\star$ to Y in $L^\infty(\Omega_T; H)$.

Remark 3.15. Since the process Y is such that, for $\mathbb{P}$ -a.e. ω , its components Y_k are continuous and the trajectory $Y(\cdot, \omega)$ is uniformly bounded in H by Remark 3.14, we can also deduce that the process Y has $\mathbb{P}$ -a.s. weakly continuous trajectories in H .

Before concluding this section, we make a few observations on the structure of system (3.34); for simplicity we restrict to the case $f \equiv 0$. Define the matrices

$$M_k := \sum_h \sigma_h^2 |P_h(k)|^2 P_k P_{k-h} P_k$$

where for any k fixed the series is absolutely convergent since $\sum_h \sigma_h^2 < \infty$; define the operator $M : D(M) \rightarrow H$ by

$$g = \sum_k g_k e_k \mapsto Mg = \sum_k M_k g_k e_k$$

For any k , M_k is a symmetric, nonnegative matrix, therefore M is a symmetric and nonnegative operator as well. Observe that the last term in $(3.34)_1$ corresponds to the action of $-M$ on Y . Indeed, $-M$ corresponds to the operator A defined by formula (3.7) in the first section. What is actually interesting and unexpected is the fact that operators K^α and M commute on the domain of definition, since they share the same eigenspaces. It follows from (3.34) that in the case $f \equiv 0$ the functions $\mathbb{E}[Y_k]$ satisfy

$$\frac{d}{dt} \mathbb{E}[Y_k(t)] = -M_k \mathbb{E}[Y_k(t)]$$

which implies

$$\mathbb{E}[Y_k(t)] = e^{-tM_k} \mathbb{E}[y_k]$$

which corresponds to equation (3.9). In particular, since M_k is strictly positive on $k^\perp$, this implies that $\mathbb{E}[Y_k]$ is converging to 0 exponentially fast.

3.5 The covariance matrices

In this section we are going to show strong uniqueness (in the probabilistic sense) of the energy controlled solutions of the linear model, whose existence was just shown. For simplicity we will restrict ourselves to the homogeneous case $f \equiv 0$, since given two solutions Y and Z of the inhomogeneous system, their difference $Y - Z$ is a solution of the homogeneous one.

In order to show uniqueness of solutions, we will study the evolution in time of the **covariance matrices** $\{A_k\}_{k \in \mathbb{Z}^3}$ defined by

$$A_k(t) = \mathbb{E}[\Re(Y_k(t) \otimes \bar{Y}_k(t))] = \mathbb{E}[\Re Y_k(t) \otimes \Re Y_k(t)] + \mathbb{E}[\Im Y_k(t) \otimes \Im Y_k(t)]$$

Observe that, for any $t \geq 0$, $A_k(t)$ is symmetric and semipositive definite; since $\langle Y_k(t), k \rangle = 0$ $\mathbb{P}$ -a.s., $A_k(t)k = 0$, which can also be expressed as $P_k A_k(t) = A_k(t) P_k = A_k(t)$. Condition $Y_{-k} = \bar{Y}_k$ implies $A_k = A_{-k}$; moreover, by definition of A_k , $\text{Tr}(A_k(t)) = \mathbb{E}[|Y_k(t)|^2]$. Therefore the energy inequality yields

$$\sum_k \text{Tr}(A_k(t)) \leq \mathbb{E}[|y|_H^2] \quad \forall t \geq 0 \tag{3.42}$$

We are now going to show the non trivial fact that the matrices $\{A_k\}$ fullfill a closed system of linear differential equations.

Proposition 3.16. For each $k \neq 0$, A_k satisfies

$$\frac{dA_k}{dt}(t) = \sum_h \sigma_h^2 |P_h(k)|^2 \left(2P_k A_{k-h}(t) P_k - P_k P_{k-h} A_k(t) - A_k(t) P_{k-h} P_k \right) \quad (3.43)$$

Remark 3.17. Observe that the r.h.s. of (3.43) is well defined whenever $\{A_k\}$ is a uniform bounded sequence, since $\|P_h\| = 1$ and $\sum_h \sigma_h^2 < \infty$. We know that $\{A_k\}$ enjoys better properties given by the inequality (3.42), but the equation would still be well defined without this assumption.

Proof. We will actually show that the system (3.43) is satisfied by $A_k = \mathbb{E}[Y_k \otimes \bar{Y}_k]$; then the conclusion follows by taking the real parts on both sides of the system. Recall that the tensor product $\otimes$ of Ito processes satisfies the property

$$Y_k(t) \otimes \bar{Y}_k(t) = y_k \otimes \bar{y}_k + \int_0^t Y_k(s) \otimes d\bar{Y}_k(s) + \int_0^t dY_k(s) \otimes \bar{Y}_k(s) + [Y_k, \bar{Y}_k](t)$$

Using the fact that Y is a solution of (3.35) with $f \equiv 0$, the identity becomes

$$\begin{aligned} Y_k(t) \otimes \bar{Y}_k(t) &= y_k \otimes \bar{y}_k - \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t Y_k(s) \otimes P_k(P_{k-h}(\bar{Y}_k(s))) ds \\ &\quad - \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t P_k(P_{k-h}(Y_k(s))) \otimes \bar{Y}_k(s) ds + M(t) + [Y_k, \bar{Y}_k](t) \end{aligned}$$

where $M(t)$ is a martingale starting from 0, given by a convergent series of stochastic integrals. Using the property $(Au) \otimes (Bv) = A(u \otimes v)B^T$ of the tensor product and taking expectation, we find

$$\begin{aligned} \mathbb{E}[Y_k(t) \otimes \bar{Y}_k(t)] &= \mathbb{E}[y_k \otimes \bar{y}_k] - \sum_h \sigma_h^2 |P_h(k)|^2 \left(\int_0^t \mathbb{E}[Y_k(s) \otimes \bar{Y}_k(s)] ds \right) P_{k-h} P_k \\ &\quad - \sum_h \sigma_h^2 |P_h(k)|^2 P_k P_{k-h} \left(\int_0^t \mathbb{E}[Y_k(s) \otimes \bar{Y}_k(s)] ds \right) + \mathbb{E}[[Y_k, \bar{Y}_k](t)] \end{aligned}$$

It remains to compute $\mathbb{E}[[Y_k, \bar{Y}_k](t)]$. Y is a solution of (3.35), therefore

$$\begin{aligned} [Y_k, \bar{Y}_k](t) &= \sum_{h,l} \sigma_h \sigma_l |P_h(k)| |P_l(k)| \int_0^t P_k(Y_{k-h}(s)) \otimes P_k(\bar{Y}_{k-l}(s)) d[B_{h,k}, \bar{B}_{l,k}](s) \\ &= 2 \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t P_k(Y_{k-h}(s) \otimes \bar{Y}_{k-h}(s)) P_k ds \end{aligned}$$

where we have used again $[B_{h,k}, \bar{B}_{l,k}](t) = 2t\delta_{h,l}$ and the property $(Au) \otimes (Bv) = A(u \otimes v)B^T$. Taking expectation and substituting in the previous equation we find that $A_k = \mathbb{E}[Y_k \otimes \bar{Y}_k]$ satisfy

$$A_k(t) = \mathbb{E}[y_k \otimes \bar{y}_k] + \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t (2P_k A_{k-h}(s) P_k - A_k(s) P_{k-h} P_k - P_k P_{k-h} A_k(s)) ds$$

which gives the conclusion. $\square$

Remark 3.18. It can be shown with similar calculations that, defined

$$A_{k,l}(t) := \mathbb{E}[\Re(Y_k(t) \otimes \bar{Y}_l(t))]$$

so that $A_k = A_{k,k}$, the matrices $A_{k,l}$ satisfy the following system of ODEs:

$$\frac{dA_{k,l}}{dt} = - \sum_h \sigma_h^2 |P_h(k)|^2 P_k P_{k-h} A_{k,l} - \sum_h \sigma_h^2 |P_h(l)|^2 A_{k,l} P_{l-h} P_l + 2 \sum_h \sigma_h^2 \langle P_h(k), l \rangle P_k A_{k-h,l-h} P_l$$

Observe the similarity of this equation and (3.43) to formula (3.11) which we found in the first section; however an unexpected simplification has occurred, since the "diagonal" terms A_k of the covariance operator $\mathbb{E}[Y \otimes \bar{Y}]$ satisfy a closed system in which the non diagonal terms $A_{k,l}$ with $k \neq l$ do not appear.

In order to show uniqueness of solution for system (3.35), we are going to show first uniqueness of solutions for the equations (3.43). This is performed by showing a suitable comparison theorem; to this aim we need to introduce some notions. Here ρ denotes the spectral radius and $A \geq B$ means that the matrix $A - B$ is semipositive definite.

Definition 3.19. A family $\{A_k\}_{k \in \mathbb{Z}^3 \setminus \{0\}} \subset C^1([0, T], \mathbb{R}^{3 \times 3})$ is **admissible** if $A_k = A_k^T$, $A_k k = 0$, $A_{-k} = A_k$ for all k and

$$\int_0^T \sum_k \rho(A_k(t)) dt < \infty$$

An admissible family is a **supersolution** of system (3.43) if, for all k ,

$$\frac{dA_k}{dt}(t) \geq \sum_h \sigma_h^2 |P_h(k)|^2 \left(2P_k A_{k-h}(t) P_k - P_k P_{k-h} A_k(t) - A_k(t) P_{k-h} P_k \right)$$

Analogously, an admissible family is a **subsolution** of system (3.43) if, for all k ,

$$\frac{dA_k}{dt}(t) \leq \sum_h \sigma_h^2 |P_h(k)|^2 \left(2P_k A_{k-h}(t) P_k - P_k P_{k-h} A_k(t) - A_k(t) P_{k-h} P_k \right)$$

Remark 3.20. It's clear by linearity of (3.43) that A_k is a supersolution if and only if $-A_k$ is a subsolution, and it's a solution if and only if it's both a supersolution and a subsolution. In the admissibility condition we've used ρ , but we could have used any operator norm, since they are all equivalent; however we can't use the trace in general, since we are not requiring A_k to be semipositive definite.

Theorem 3.21 (Comparison Theorem). Let $\{M_k\}$ and $\{N_k\}$ be respectively a supersolution and a subsolution of (3.43) such that $M_k(0) \geq N_k(0)$ for all k . Then $M_k(t) \geq N_k(t)$ for all k and for all $t \in [0, T]$.

Proof. Define $A_k := M_k - N_k$, then by linearity A_k is still a supersolution of (3.43) and $A_k(0) \geq 0$ for all k ; the conclusion is equivalent to showing that $A_k(t) \geq 0$ for all k and $t \in [0, T]$. The proof is by contradiction: we will assume that there exist k and $t \in [0, T]$ such that $A_k(t)$ has at

least one negative eigenvalue; we will show that we can then construct inductively a sequence of indexes $\{k_n\}$ and eigenvectors v_n of A_{k_n} , with $|v_n| = 1$, such that $(x^- = \max\{-x, 0\})$

$$\int_0^T \sum_n \langle v_n, A_{k_n}(s)v_n \rangle^- ds = +\infty$$

which contradicts the admissibility condition, since

$$\int_0^T \sum_n \langle v_n, A_{k_n}(s)v_n \rangle^- ds \leq \int_0^T \sum_{k \in \mathbb{Z}^3} \rho(A_k(s)) ds < \infty$$

For any v , k and t we introduce the following notations:

$$\begin{aligned} \varphi_k(v, s) &:= -\langle v, A_k(s)v \rangle^- \\ \psi_k(v) &:= \int_0^T \varphi_k(v, s) ds \\ \Gamma_k(v) &:= \{s \in [0, T] : \varphi_k(v, s) < 0\} \end{aligned}$$

Observe that $\varphi_k(v, \cdot)$ is strictly negative on $\Gamma_k(v)$ and 0 outside; $\psi_k(v) < 0$ if and only if there exists some $t \in [0, T]$ such that $\langle v, A_k(t)v \rangle < 0$, which implies that $A_k(t)$ has a negative eigenvalue. By hypothesis $\varphi_k(v, 0) = 0$ for all k and v ; moreover $\varphi_k(v, T) \leq 0$, so we can deduce that

$$\int_{\Gamma_k(v)} \frac{\partial \varphi_k}{\partial t}(v, s) ds \leq 0 \quad (3.44)$$

Now suppose we have an index $k = k_n$ such that φ_k admits negative values and let $v_n = v$ be a unitary vector that minimizes φ_k (among u such that $|u| = 1$). Since $A_k k = 0$, we can assume $v \in k^\perp$. By applying v to both sides of (3.43) and integrating over $\Gamma_k(v)$ we obtain:

$$\begin{aligned} 0 &\geq \int_{\Gamma_k(v)} \frac{\partial \varphi_k}{\partial t}(v, s) ds = \int_{\Gamma_k(v)} \left\langle v, \frac{dA_k}{dt}(s)v \right\rangle ds \\ &\geq \sum_h \sigma_h^2 |P_h(k)|^2 \int_{\Gamma_k(v)} \left(2\langle v, P_k A_{k-h}(s) P_k v \rangle - \langle v, P_k P_{k-h} A_k(s) v \rangle - \langle v, A_k(s) P_{k-h} P_k v \rangle \right) ds \\ &= 2 \sum_h \sigma_h^2 |P_h(k)|^2 \int_{\Gamma_k(v)} \left(\langle v, A_{k-h}(s) v \rangle - \langle P_{k-h} v, A_k(s) v \rangle \right) ds \end{aligned}$$

If we define the matrices

$$B_h := \int_{\Gamma_k(v)} A_h(s) ds$$

then by linearity of the integral we obtain

$$\sum_h \sigma_h^2 |P_h(k)|^2 \left(\langle v, B_{k-h} v \rangle - \langle P_{k-h} v, B_k v \rangle \right) \leq 0 \quad (3.45)$$

Observe that, by the properties of A_h , B_h are such that $B_h h = 0$, $B_h = B_h^T$ and

$$\sum_h \rho(B_h) \leq \sum_h \int_{\Gamma_h(v)} \rho(A_h(s)) ds \leq \sum_h \int_0^T \rho(A_h(s)) ds < \infty$$

In particular this implies that B_{k-h} is infinitesimal as h goes to infinity and so in the series in equation (3.45) some strictly negative terms must appear. Moreover, we chose v minimizing $\psi_k(u)$ and it holds $\psi_k(v) = \langle v, B_k v \rangle \leq \psi(u) \leq \langle u, B_k u \rangle$ for all unitary u , therefore v must be an eigenvector of B_k , $B_k v = \lambda v$ where $\lambda = \lambda_n < 0$. Therefore there must exist some index h such that

$$\langle v, B_{k-h} v \rangle - \langle P_{k-h} v, B_k v \rangle = \langle P_{k-h} v, B_{k-h} P_{k-h} v \rangle - \lambda |P_{k-h} v|^2 < 0$$

where we used the fact that $P_{k-h} B_{k-h} P_{k-h} = B_{k-h}$. Defined $k_{n+1} = \tilde{k} := k-h$ and $\tilde{v} := \frac{P_{k-h} v}{|P_{k-h} v|}$, it holds

$$\langle \tilde{v}, B_{\tilde{k}} \tilde{v} \rangle < \lambda$$

therefore $\psi(\tilde{v}) < \lambda$ and we can start again the procedure; this way we construct a sequence of indexes k_n and unitary vectors v_n such that the sequence

$$\int_0^T \langle v_n, A_{k_n}(s) v_n \rangle^- ds$$

is strictly decreasing, which gives the desired conclusion. $\square$

Corollary 3.22. The following hold:

- i) (Positivity) If $\{A_k\}$ is a supersolution s.t. $A_k(0) \geq 0$ for all k , then $A_k(t) \geq 0$ for all k, t .
- ii) (Uniqueness) For any admissible initial data $\{a_k\}$, i.e. such that $a_k = a_k^T$, $a_k k = 0$, $a_{-k} = a_k$ and $\sum_k \rho(a_k) < \infty$, there exists at most one solution of (3.43) such that $A_k(0) = a_k$ for all k .

Proof. Part i) is straightforward; for part ii), let A_k and B_k be two solutions starting at a_k , then by applying the Comparison Theorem, first with A_k supersolution and B_k subsolution, then conversely, we obtain $A_k(t) \geq B_k(t)$ and $B_k(t) \geq A_k(t)$ for all k, t , which implies $A \equiv B$. $\square$

We are now able to state a uniqueness result for system (3.35).

Theorem 3.23. There exists at most one energy controlled solution of system (3.35), that is given a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, a family of Brownian motions $\{\widetilde{W}_k\}_{k \in J}$, an initial data y and a deterministic function f as in Theorem 3.13, then any two energy controlled strong solutions $Y^{(1)}$ and $Y^{(2)}$ to system (3.35) defined on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and with same initial data y are such that, for any $t \geq 0$,

$$Y^{(1)}(t) = Y^{(2)}(t) \quad \mathbb{P}\text{-a.s.}$$

In particular, if $Y^{(1)}$ and $Y^{(2)}$ are weakly continuous processes, then they are indistinguishable.

Proof. Define the process $Y = Y^{(1)} - Y^{(2)}$; by linearity Y solves system (3.35) with $f \equiv 0$ and initial data 0. Moreover $Y^{(1)}$ and $Y^{(2)}$ both satisfy the energy inequality and therefore bound (3.39), which implies that

$$\int_0^T \mathbb{E}[|Y(t)|_H^2] dt < \infty$$

Therefore the covariance matrices A_k associated to Y are an admissible solution to system (3.43) with initial data 0. The uniqueness result for these solutions implies that $A_k \equiv 0$ for all k , thus

$$\mathbb{E}[|Y^{(1)}(t) - Y^{(2)}(t)|_H^2] = \mathbb{E}[|Y(t)|_H^2] = \sum_k \text{Tr}(A_k(t)) = 0$$

and the conclusion follows. $\square$

Remark 3.24. For simplicity in the statement of Theorem (3.23) we have only considered energy controlled solutions, but the proof actually shows that uniqueness holds under the weaker assumption that Y is a strong solution of (3.35) satisfying the condition $Y \in L^2(\Omega_T; H)$, namely

$$\int_0^T \mathbb{E}[|Y(t)|_H^2] dt < \infty \quad (3.46)$$

This also implies a (simple) regularity result: if Y is a weakly continuous solution such that $Y \in L^2(\Omega_T; H)$, then it must necessarily satisfy the energy inequality.

We can now put together Theorems 3.13 and 3.23 and state the following comprehensive result.

Theorem 3.25. In the setting of Theorem 3.23, for any initial data y and deterministic function f satisfying the usual assumptions, there exists a strongly unique energy controlled solution Y to system (3.35). Moreover, such solution Y has $\mathbb{P}$ -a.s. weakly continuous trajectories. The solutions continuously depend on the initial data y and the function f , namely if Y and Z are solution of (3.35) respectively w.r.t. functions f and g and initial data y and z , then

$$\sup_{t \in [0, T]} |Y(t) - Z(t)|_H^2 \leq C \left(|y - z|_H^2 + \int_0^T |f(t) - g(t)|_H^2 dt \right) \quad \mathbb{P}\text{-a.s.} \quad (3.47)$$

In particular, in the case of the homogeneous system associated to (3.35), i.e. with $f \equiv 0$, there is linear continuous dependence on initial data.

Proof. Existence, uniqueness and weak continuity of trajectories have already been shown. If Y and Z are energy controlled solutions of (3.35), then by linearity $W = Y - Z$ is a solution of (3.35) with initial data $y - z$ and inhomogeneous term $f - g$. But W satisfies an energy bound of the form (3.46) since Y and Z do so, hence by Remark 3.24 it is the solution and so the energy inequality must hold, which yields (3.47). In the homogeneous case there must be linear dependence on the initial data, since if Y and Z are solutions starting respectively at y and z , then by linearity $\alpha Y + Z$ is a solution of the homogeneous system with initial data $\alpha y + z$ and by the previous reasoning it is the unique solution for such initial data. $\square$

We can reformulate the previous result by introducing a suitable Banach space; this will also be useful when discussing convergence in distribution of solutions in Section 3.6. Given a separable Banach space E , consider the space of all weakly continuous functions:

$$C([0, T]; E_w) = \left\{ f : [0, T] \rightarrow E \text{ weakly continuous} \right\}$$

endowed with the norm

$$\|f\|_\infty = \sup_{t \in [0, T]} \|f(t)\|_E$$

Observe that, if $f \in C([0, T]; E_w)$, then by the Banach-Steinhaus theorem necessarily $\|f\|_\infty < \infty$. The following holds.

Lemma 3.26. $(C([0, T]; E_w), \|\cdot\|_\infty)$ is a Banach space.

Proof. The proof is quite standard, but we give it for the sake of completeness. It's clear that $\|\cdot\|_\infty$ is a norm, so we only need to show completeness. Let $\{f_n\}$ be a Cauchy sequence, then for any $t \in [0, T]$ $\{f_n(t)\}$ is a Cauchy sequence in E and so it admits limit. So we can define pointwise the function

$$f(t) = \lim_{n \rightarrow \infty} f_n(t)$$

Since $\{f_n\}$ is a Cauchy sequence, for any $\varepsilon > 0$ there exists n such that for any $m \geq n$, $\|f_m(t) - f_n(t)\|_E \leq \varepsilon$ for all $t \in [0, T]$. Taking the limit as $m \rightarrow \infty$ we obtain

$$\|f(t)\|_E \leq \|f(t) - f_n(t)\|_E + \|f_n(t)\|_E = \lim_{m \rightarrow \infty} \|f_m(t) - f_n(t)\|_E + \|f_n(t)\|_E \leq \varepsilon + \|f_n(t)\|_E$$

which shows that $\sup_{t \in [0, T]} \|f(t)\|_E < \infty$. In a similar way it can be shown that

$$\sup_{t \in [0, T]} \|f(t) - f_n(t)\|_E \rightarrow 0 \text{ as } n \rightarrow \infty$$

In order to show weak continuity of f , it suffices to show that, for any $\varphi \in E^*$, $t \mapsto \varphi(f(t))$ is weakly continuous. By hypothesis, $\varphi \circ f_n$ are continuous functions, and

$$\sup_{t \in [0, T]} |\varphi(f(t)) - \varphi(f_n(t))| \leq \|\varphi\|_{E^*} \sup_{t \in [0, T]} \|f(t) - f_n(t)\|_E \rightarrow 0$$

and so by the analogue real result we can deduce that $\varphi \circ f$ is continuous as it's the uniform limit of continuous functions; since this holds for any $\varphi \in E^*$, we can conclude. $\square$

Since we have shown that, for any initial data $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $f \in L^2(0, T; H)$, the solutions Y are pathwise weakly continuous, we can regard them as random variables in $C([0, T]; H_w)$. Theorem 3.25 and in particular equation (3.47) tell us that the map

$$L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H) \times L^2(0, T; H) \rightarrow L^2(C([0, T]; H_w), \quad (y, f) \mapsto (Y(t))_{t \in [0, T]}$$

that associates to (y, f) the unique energy controlled solution of system (3.35), is linear and continuous.

In the remaining part of this section we are going to strengthen Theorem 3.21. While this is not necessary in order to obtain the uniqueness result, it still gives information on the dynamics.

Let us first rewrite the system for the covariance matrices, which is given by

$$\frac{dA_k}{dt}(t) = \sum_h \sigma_h^2 |P_h(k)|^2 \left(2P_k A_{k-h}(t) P_k - P_k P_{k-h} A_k(t) - A_k(t) P_{k-h} P_k \right) \quad (3.48)$$

Recall that we defined the symmetric, nonnegative matrices

$$M_k = \sum_h \sigma_h^2 |P_h(k)|^2 P_k P_{k-h} P_k$$

M_k are such that $P_k M_k = M_k P_k = M_k$; using the fact that $A_k P_k = P_k A_k = A_k$, we can write equation (3.48) as

$$\frac{dA_k}{dt}(t) = 2 \sum_h \sigma_h^2 |P_h(k)|^2 P_k A_{k-h}(t) P_k - M_k A_k(t) - A_k(t) M_k$$

By taking all the terms containing A_k on the r.h.s. and multiplying on both sides by e^{tM_k} we obtain

$$e^{tM_k} \left(M_k A_k(t) + \frac{dA_k}{dt}(t) + A_k(t) M_k \right) e^{tM_k} = 2 \sum_h \sigma_h^2 |P_h(k)|^2 e^{tM_k} P_k A_{k-h}(t) P_k e^{tM_k}$$

Using the fact that P_k and e^{tM_k} commute since P_k and M_k do so and properties of differentiation, the above equality becomes

$$\frac{d}{dt} \left(e^{tM_k} A_k(t) e^{tM_k} \right) = 2 \sum_h \sigma_h^2 |P_h(k)|^2 P_k e^{tM_k} A_{k-h}(t) e^{tM_k} P_k \quad (3.49)$$

and then integrating and multiplying on both sides by e^{-tM_k} it follows that

$$A_k(t) = e^{-tM_k} A_k(0) e^{-tM_k} + 2 \sum_h \sigma_h^2 |P_h(k)|^2 P_k \left(\int_0^t e^{(s-t)M_k} A_{k-h}(s) e^{(s-t)M_k} ds \right) P_k \quad (3.50)$$

This integral formulation of equation (3.48) is useful as there is no dependence on A_k in the series on the r.h.s.

Remark 3.27. For simplicity we have done all calculation in the case of A_k solution of (3.48), but equation (3.50) holds (respectively with $\geq$ and $\leq$) also for supersolutions and subsolutions. This is due to the fact that all operations applied in the calculations are order preserving; in particular, since M_k is symmetric, e^{tM_k} is symmetric as well, and multiplying on both sides by a symmetric matrix leaves the order invariant (more generally, if $A \geq B$, then for any C it holds $CAC^T \geq CBC^T$).

Theorem 3.28. Let A_h and B_h be respectively a supersolution and a subsolution of (3.43) such that $A_h(0) \geq B_h(0)$ for all h . If there exist $t > 0$ and k such that $A_k(t) = B_k(t)$, then

$$A_h(s) = B_h(s) \quad \forall h, \forall s \in [0, t]$$

Proof. As in the proof of Theorem 3.21, it suffices to show that if A_h is a supersolution such that $A_h(0) \geq 0$ for all h and there exist $t > 0$ and k such that $A_k(t) = 0$, then $A_h(s) = 0$ for all h and for all $s \in [0, t]$. We already know, by Theorem 3.21, that $A_h(s) \geq 0$ for all h and s ; then equation (3.50) applied to $A_k(t)$ gives

$$0 = e^{-tM_k} A_k(0) e^{-tM_k} + 2 \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t e^{(s-t)M_k} P_k A_{k-h}(s) P_k e^{(s-t)M_k} ds$$

Observe that the first term and the integrands on the r.h.s. are both semipositive definite, so in order for the r.h.s. to be 0 they must be 0; since e^{cM_k} is invertible for any c , this implies that $A_k(0) = 0$ and $P_k A_{k-h}(s) P_k = 0$ for any h such that $|P_h(k)| \neq 0$, i.e. for any $h \notin \text{span}(k)$. This implies, by applying again equation (3.50) to $A_k(s)$ with $s < t$, that $A_k(s) = 0$ as well.

We claim that, for any h such that $P_k A_{k-h}(s) P_k = 0$ and $k - h \notin k^\perp$, that is for any h such that

$$h \notin \Gamma_k := \text{span}(k) \cup \{k + k^\perp\}$$

we must have $A_{k-h}(s) = 0$. Indeed, we know that $A_{k-h}(s)(k-h) = 0$ and $k-h \notin k^\perp$; $P_k A_{k-h}(s) P_k = 0$ implies that $A_{k-h}(k^\perp) \subset \text{span}(k)$. If $A_{k-h}(k^\perp) = \{0\}$, then $\ker A_{k-h}$ has dimension 3 and we can conclude. If $A_{k-h}(k^\perp) \neq \{0\}$, then $\ker A_{k-h}$ has dimension 2 and $\text{Im} A_{k-h} = A_{k-h}(k^\perp) = \text{span}(k)$. Since A_{k-h} is symmetric, it is diagonalizable and therefore its image must correspond to the eigenspace relative to the only non zero eigenvalue. But then k is an eigenvector for A_{k-h} ; since the orthogonal of an eigenspace must be invariant under A_{k-h} , this implies that $A_{k-h}(k^\perp) \subset k^\perp$. But $A_{k-h}(k^\perp) = \text{span}(k)$, absurd.

So we have shown that, for any $h \notin \Gamma_k$, $A_{k-h}(s) = 0$ for all $s \in [0, t]$.

By iterating the same reasoning, this time starting from $A_{k-h}(t) = 0$, for h as above, we find the conclusion. $\square$

We immediately obtain the following corollary, which shows that equation (3.43) enjoys the property of **infinite speed of signal**, that is any component A_k is affected at any time t by the initial value of any other component $A_h(0)$, no matter how big the distance $|k-h|$.

Corollary 3.29. Let A_h be a solution of (3.43) with non negative initial value, i.e. $A_h(0) \geq 0$ for all h , such that $A_k \neq 0$ for at least one k . Then $A_h(t) \neq 0$ for all h and for all $t \in [0, T]$.

Remark 3.30. This implies immediately that a similar property also holds for any Y_k solution to system (3.35) with $f \equiv 0$, since in this case $A_k = \mathbb{E}[\Re(Y_k \otimes \bar{Y}_k)]$ is a solution of (3.43). That is, if the initial data y is not identically 0, then at any positive time t there cannot be components $Y_k(t)$ such that $Y_k(t) = 0$ $\mathbb{P}$ -a.s.

3.6 Girsanov transform and the nonlinear system

We now consider the nonlinear system in Ito form

$$\left\{ \begin{array}{l} dY_k = f_k dt - i \sum_h \sigma_h \langle Y_h, k \rangle P_k(Y_{k-h}) dt - i \sum_h \sigma_h |P_h(k)| P_k(Y_{k-h}) dB_{h,k} \\ \quad - \sum_h \sigma_h^2 |P_h(k)|^2 P_k(P_{k-h}(Y_k)) dt \\ Y_0 \equiv 0, \quad \langle Y_k, k \rangle = 0 \\ Y_{-k} = \bar{Y}_k \\ Y_k(0) = y_k \end{array} \right. \quad (3.51)$$

We shall deal with solutions on any fixed finite time interval $[0, T]$; we will need to impose on the initial data y a condition of the form

$$\mathbb{E}_Q[e^{\delta |y|^2}] < \infty \text{ for some } \delta > 0$$

as this is the easiest condition needed on y in order for Novikov's test (in its second version) to be successful, as we will see; observe in particular that deterministic initial data $y \in H$, as well as uniformly bounded initial data y , trivially satisfy such condition.

Definition 3.31. Given y as above and $f \in L^2(0, T; H)$, a **weak solution** of system (3.51) is a filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \in [0, T]}, Q)$, an H -valued cylindrical Wiener process W' and an

H -valued stochastic process $Y = (Y_k)_{k \in \mathbb{Z}^3}$ on $(\Omega, \{\mathcal{F}_t\}_{t \in [0, T]}, Q)$, with continuous and adapted components Y_k such that for all $t \in [0, T]$

$$\begin{cases} Y_k(t) = y_k + \int_0^t f_k(s) ds - i \sum_h \sigma_h \int_0^t \langle Y_h(s), k \rangle P_k(Y_{k-h}(s)) ds \\ \quad - i \sum_h \sigma_h |P_h(k)| \int_0^t P_k(Y_{k-h}(s)) dB'_{h,k}(s) - \sum_h \sigma_h^2 |P_h(k)|^2 \int_0^t P_k(P_{k-h}(Y_k(s))) ds \\ Y_0 \equiv 0, \quad \langle Y_k(t), k \rangle = 0 \\ Y_{-k}(t) = \overline{Y}_k(t) \end{cases}$$

where the Brownian motions $B'_{h,k}$ are constructed from W' as before. We denote this solution by $((\Omega, \{\mathcal{F}_t\}_{t \in [0, T]}, Q), Y, W')$. Moreover, it is called an **energy controlled weak solution** if it satisfies the energy inequality

$$\sup_{t \in [0, T]} \left\{ \sum_k |Y_k(t)|^2 - \int_0^t \sum_k \langle f_k, \overline{Y}_k \rangle ds \right\} \leq \sum_k |y_k|^2 \quad Q\text{-a.s.}$$

We now apply the Girsanov transform, which was introduced in Chapter 2. Let $Y = \{Y_k\}_k$ be the strong energy controlled solution of (3.35) with respect to Wiener process W . Define the process

$$\varphi(t) = \int_0^t \langle Y(s), dW(s) \rangle_H = \sum_h \int_0^t \langle Y_h(s), d\overline{W}_h(s) \rangle$$

Observe that, since $Y_{-h} = \overline{Y}_h$ and $W_{-h} = \overline{W}_h$, φ is a real process; by the properties of stochastic integral it's a martingale. Using as before the fact that W_h and W_k are independent whenever $h \neq \pm k$, $[W_k, W_k] = 0$ and $[W_k, \overline{W}_k](t) = 2tP_k$, we can compute

$$[\varphi, \varphi](t) = 2 \int_0^t \sum_k |Y_k(s)|^2 ds = 2 \int_0^t |Y(s)|_H^2 ds$$

We have the following.

Proposition 3.32. Let Y be the strong solution of system (3.35) with respect to a given filtered probability space $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ and Wiener process W . Then

$$W'(t) = W(t) - \int_0^t Y(s) ds$$

defines an H -valued cylindrical Wiener process on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, Q)$, where the measure Q is equivalent to $\mathbb{P}$ and

$$\frac{dQ}{d\mathbb{P}} = \exp \left(\varphi(T) - \frac{1}{2} [\varphi, \varphi](T) \right) = \exp \left(\int_0^T \langle Y(s), dW(s) \rangle_H ds - \int_0^T |Y(s)|_H^2 ds \right) \quad (3.52)$$

Proof. In order to apply Girsanov transform, it suffices to show that Novikov's criterion is satisfied. Since Y is an energy controlled solution, we know that

$$\sup_{t \in [0, T]} |Y(t)|_H^2 \leq |y|_H^2 + C \int_0^T |f(s)|_H^2 ds \quad \mathbb{P}\text{-a.s.}$$

where C can be taken as $C = e^T$. Therefore

$$\sup_{t \in [0, T]} \mathbb{E}_{\mathbb{P}}[e^{\delta |Y(t)|_H^2}] \leq \exp\left(\delta C \int_0^T |f(s)|^2 ds\right) \mathbb{E}_{\mathbb{P}}[e^{\delta |y|_H^2}] < \infty$$

Therefore Novikov's criterion (in its second version) can be applied and the conclusion follows. $\square$

Remark 3.33. Observe that, since $e^{\varphi(t) - 1/2[\varphi, \varphi](t)}$ is a martingale starting at 1 and y is $\mathcal{F}_0$ -measurable, then

$$\mathbb{E}_Q[e^{\delta |y|^2}] = \mathbb{E}_{\mathbb{P}}[e^{\delta |y|^2} e^{\varphi(T) - 1/2[\varphi, \varphi](T)}] = \mathbb{E}_{\mathbb{P}}[e^{\delta |y|^2}]$$

so that the condition imposed on the initial data y is left invariant by the Girsanov transform.

Theorem 3.34. For any initial data y and function f as above, system (3.51) has an energy controlled weak solution. Moreover, this solution is unique in law.

Proof. As far as existence is concerned, system (3.35) has a unique strong energy controlled solution defined on $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$. By applying the Girsanov transform as in the previous proposition, we obtain a weak solution $((\Omega, \{\mathcal{F}_t\}, Q), Y, W')$ of (3.51). Moreover, since $\mathbb{P}$ and Q are equivalent, the energy inequality must also hold Q -a.s. and so it's a weak energy controlled solution. Regarding uniqueness, if there were two different energy controlled weak solutions of (3.51), then again by Girsanov transform each of them would give rise to a weak solution of system (3.35) by Girsanov transform. On the other side, by the Yamada-Watanabe theorem (see the appendix), pathwise uniqueness for system (3.35) also implies uniqueness in law. By the equivalence of Q and $\mathbb{P}$ given by the density (3.52) we deduce that uniqueness in law must also hold for (3.51). $\square$

We can actually strengthen the result, by observing that the law of the solution must depend continuously on y and f .

Theorem 3.35. Let $((\Omega, \{\mathcal{F}_t\}, Q^n), Y^n, \tilde{W}^n)$, be a sequence of weak energy controlled solutions associated to initial values y_n and functions $f_n \in L^2(0, T; H)$. Assume there exist $\delta > 0$ and $C > 0$ such that

$$\mathbb{E}_{Q^n}[e^{\delta |y_n|^2}] \leq C \quad \forall n$$

and

$$y_n \rightarrow y \text{ in distribution,} \quad f_n \rightarrow f \text{ in } L^2(0, T; H)$$

Then $Y^n \rightarrow Y$ in distribution in $C([0, T]; H_w)$, where $((\Omega, \{\mathcal{F}_t\}, Q), Y, \tilde{W})$ is a weak solution with initial data y and function f . Moreover, for any $t \in [0, T]$, $Y^n(t) \rightarrow Y(t)$ in distribution.

Proof. Since $y_n \rightarrow y$ in distribution and H is a Polish space, by Skorokhod's Theorem we can consider a sequence $\{z_n\}$, z of H -valued random variables defined on the same probability space $(\Omega', \mathcal{G}, \mathbb{P})$ such that y_n is distributed as z_n , y is distributed as z and $z_n \rightarrow z$ $\mathbb{P}$ -a.s. It holds

$$\mathbb{E}_{\mathbb{P}}[e^{\delta |z|^2}] \leq \liminf_{n \rightarrow \infty} \mathbb{E}_{\mathbb{P}}[e^{\delta |z_n|^2}] = \liminf_{n \rightarrow \infty} \mathbb{E}_{Q^n}[e^{\delta |y_n|^2}] \leq C$$

Now let W be a cylindrical Wiener process on H , on the probability space $(\Omega', \mathcal{G}, \mathbb{P})$, independent of the sigma algebra $\mathcal{G}_0$ generated by the variables y_n , and consider $\{\mathcal{H}_t\}$ the augmented filtration generated by $\mathcal{G}_0$ and W . Since $\mathbb{E}_{\mathbb{P}}[|z_n|^2] < \infty$ for all n , we can consider the energy controlled

solutions Z^n of the linearized system with initial data z_n , functions f_n and driver W ; similarly for Z with initial data z and function f . By the continuous dependence on the data and the hypothesis we have that

$$\sup_{t \in [0, T]} |Z^n(t) - Z(t)|_H^2 \leq C \left(|z_n - z|_H^2 + \int_0^T |f_n(s) - f(s)|_H^2 ds \right) \rightarrow 0 \quad \mathbb{P}\text{-a.s.} \quad (3.53)$$

that is, $\mathbb{P}$ -a.s. $Z^n \rightarrow Z$ in $C([0, T]; H_w)$. We can now construct weak energy controlled solutions of the nonlinear system by applying Girsanov transform to Z^n and Z ; the associated distributions are given by

$$\frac{d\tilde{Q}^n}{d\mathbb{P}} = \exp \left(\int_0^T \langle Z^n(t), dW(t) \rangle_H - \int_0^T |Z^n(t)|_H^2 dt \right)$$

and similarly for Z . By the uniqueness in law, the solutions constructed this way must have the same distribution as $((\Omega, \{\mathcal{F}_t\}, Q^n), Y^n, \tilde{W}^n)$ and so it suffices to show convergence in distribution for the first sequence. It follows from (3.53) and the fact that the stochastic integral preserves convergence in probability that

$$\frac{d\tilde{Q}^n}{d\mathbb{P}} \rightarrow \frac{d\tilde{Q}}{d\mathbb{P}} \quad \text{in probability}$$

But then $d\tilde{Q}^n/d\mathbb{P}$ is a sequence of positive random variables such that

$$\mathbb{E}_{\mathbb{P}} \left[\frac{d\tilde{Q}^n}{d\mathbb{P}} \right] = 1, \quad \mathbb{E}_{\mathbb{P}} \left[\frac{d\tilde{Q}}{d\mathbb{P}} \right] = 1, \quad \frac{dQ'_n}{d\mathbb{P}} \rightarrow \frac{dQ'}{d\mathbb{P}} \quad \text{in probability w.r.t. } \mathbb{P}$$

and therefore $d\tilde{Q}^n/d\mathbb{P} \rightarrow d\tilde{Q}/d\mathbb{P}$ in $L^1(\Omega', \mathbb{P})$. Now let g be any continuous and bounded map from $C([0, T]; H_w)$ to $\mathbb{R}$, then

$$\mathbb{E}_{\tilde{Q}^n}[g(Z^n)] = \mathbb{E}_{\mathbb{P}} \left[g(Z^n) \frac{d\tilde{Q}^n}{d\mathbb{P}} \right] \rightarrow \mathbb{E}_{\mathbb{P}} \left[g(Z) \frac{d\tilde{Q}}{d\mathbb{P}} \right] = \mathbb{E}_{\tilde{Q}}[g(Z)]$$

where we have used the facts that g is bounded, $Z_n \rightarrow Z$ $\mathbb{P}$ -a.s. and the densities converge in L^1 . Therefore we can conclude that $Y^n \rightarrow Y$ in distribution as well.

The last part of the statement follows from the fact that, for any $t \in [0, T]$, the evaluation map

$$\delta_t : C([0, T]; H_w) \rightarrow H \quad f \mapsto f(t)$$

is continuous, and convergence in distribution is preserved by continuous functions. $\square$

Remark 3.36. Observe in particular that the hypothesis of the theorem hold when we consider deterministic initial data $y_n \in H$ such that $y_n \rightarrow y$ in H .

We conclude this section with the following comprehensive remark; here we are not only referring to results those of this section but also Sections 3.4 and 3.5. In the Fourier space formulation of the Leray- α model, the only consequence of the choice of the operator K^α was the appearance of the constants $\sigma_h = (1 + \alpha|h|^2)^{-1}$ which have the key property that

$$\sum_{h \in \mathbb{Z}^3} \sigma_h^2 < \infty$$

which is equivalent to K^α being a Hilbert-Schmidt operator. This was a fundamental feature in order for the stochastic integrals to be well defined and the series to be convergent. It's easy to see however that all the proofs can be easily generalized to the choice of any other Hilbert-Schmidt operator K whose eigenvectors are $\{e_k\}_k$. For example we could have considered

$$K^\alpha = (I - \alpha\Delta)^{-\beta}, \quad K^\alpha = (I - \alpha\Delta^\beta)^{-1} \quad (3.54)$$

for any $\beta > 3/4$. More generally, given any nonnegative Hilbert-Schmidt operator K as above, one can define a family $\{K^\alpha\}_{\alpha>0}$ of Hilbert-Schmidt operators such that $K^\alpha \rightarrow I$ as $\alpha \rightarrow 0$ by setting

$$K^\alpha = (I + \alpha K^{-1})^{-1}$$

In fact, if $Ke_k = \lambda_k e_k$, $\sum_k \lambda_k^2 < \infty$, then

$$K^\alpha e_k = \frac{\lambda_k}{\alpha + \lambda_k}, \quad \sum_k \left(\frac{\lambda_k}{\alpha + \lambda_k} \right)^2 \leq \frac{1}{\alpha^2} \sum_k \lambda_k^2 < \infty \quad \forall \alpha > 0$$

All the results also easily generalize in other dimensions, once the operators K^α are Hilbert-Schmidt, which might depend on the dimension of the space. For example, in $d = 2$, if we chose K^α as in (3.54), then we would need $\beta > 1/2$; more generally in $\mathbb{R}^d$ the condition becomes $\beta > d/4$. The only result in which it was explicitly used the fact that we are considering dimension $d = 3$ or less, and which would require a new proof in higher dimension, was Theorem 3.28 and the subsequent Corollary 3.29.

Chapter 4

Further results

4.1 Regularity of solutions

In this section we are interested in studying pathwise regularity of the solutions; we consider the linear system (3.34), but the same results must also hold for the nonlinear one, by the equivalence of the measures $\mathbb{P}$ and Q defined by the Girsanov transform. We already know that the solutions can be regarded as random variables in $C([0, T]; H_w)$. In order to find better regularity results, we first need a suitable reformulation of the problem; the only reason this formulation had not been adopted before, it's because we were not aware of it when the previous chapters had been developed. Recall that we have defined the spaces

$$\begin{aligned} H &= \left\{ u \in L^2(\mathbb{T}; \mathbb{R}^3) : \int_{\mathbb{T}} u \, dx = 0, \nabla \cdot u = 0 \right\} \\ V &= \left\{ u \in H^1(\mathbb{T}; \mathbb{R}^3) : \int_{\mathbb{T}} u \, dx = 0, \nabla \cdot u = 0 \right\} \end{aligned}$$

which can be identified in Fourier space as

$$\begin{aligned} H &= \left\{ u = \sum_k u_k e_k : u_{-k} = \bar{u}_k, u_0 = 0, u_k \cdot k = 0, \sum_k |u_k|^2 < \infty \right\} \\ V &= \left\{ u = \sum_k u_k e_k : u_{-k} = \bar{u}_k, u_0 = 0, u_k \cdot k = 0, \sum_k |k|^2 |u_k|^2 < \infty \right\} \end{aligned}$$

More generally, we can define the family of Hilbert spaces

$$\mathbb{H}^\alpha = \left\{ u = \sum_k u_k e_k : u_{-k} = \bar{u}_k, u_0 = 0, u_k \cdot k = 0, \sum_k |k|^{2\alpha} |u_k|^2 < \infty \right\}$$

with inner product

$$\langle u, v \rangle_{\mathbb{H}^\alpha} = \sum_k |k|^{2\alpha} \langle u_k, \bar{v}_k \rangle$$

where α is a real parameter. We have $\mathbb{H}^0 = H$, $\mathbb{H}^1 = V$ and for any $\alpha < \beta$, $\mathbb{H}^\beta \hookrightarrow \mathbb{H}^\alpha$ with continuous and compact embedding, see [22] for more details. In particular, for any $\alpha > 0$, $\mathbb{H}^\alpha \hookrightarrow H$ implies that $H^* \hookrightarrow (\mathbb{H}^\alpha)^*$ with continuous and compact embedding as well. If we

identify H with its dual by Riesz theorem, then $(\mathbb{H}^\alpha)^*$ is identified with $\mathbb{H}^{-\alpha}$. In particular

$$V^* = \left\{ u = \sum_k u_k e_k : u_{-k} = \bar{u}_k, u_0 = 0, u_k \cdot k = 0, \sum_k \frac{|u_k|^2}{|k|^2} < \infty \right\}$$

Recall that the linearized system can be written in abstract form as

$$dY = f dt + B(\circ dW, Y) = [f + AY]dt + B(dW, Y)$$

where the operator B is defined by

$$B(u, v) = -P[((K^\alpha u) \cdot \nabla)v]$$

In general $B(u, v)$ is not well defined as an element of H if $v \notin V$. However we can regard it as an element of V^* by setting

$$B(u, v)(\varphi) = \langle ((K^\alpha u) \cdot \nabla)\varphi, v \rangle_H \quad \forall \varphi \in V$$

In particular, for any fixed $v \in H$, the map $u \mapsto B(u, v)$, from H to V^* , is linear and continuous:

$$\sup_{|\varphi|_V=1} |B(u, v)(\varphi)| = \sup_{|\varphi|_V=1} |\langle ((K^\alpha u) \cdot \nabla)\varphi, v \rangle_H| \leq \sup_{|\varphi|_V=1} |\nabla \varphi|_{L^2} |v|_H |K^\alpha u|_{L^\infty} \leq C|v|_H |u|_H$$

where we used the fact that $K^\alpha : H \rightarrow \mathbb{H}^2$ is linear and continuous and $\mathbb{H}^2 \hookrightarrow C(\mathbb{T}; \mathbb{R}^3)$ by the Sobolev embeddings. The above estimate also shows that the map $v \mapsto B(\cdot, v)$ is linear and continuous from H to $L(H, V^*)$. With a slight abuse of notation, from now on we will denote by B also the map $v \mapsto B(v) = B(\cdot, v)$. We can actually strengthen the previous estimate by showing that B is a continuous map from H to $L_2(H, V^*)$. Recall that

$$u = \sum_k u_k e_k, \quad v = \sum_k v_k e_k \Rightarrow B(u, v) = -i \sum_k \left(\sum_h \sigma_h \langle u_h, k \rangle P_k(v_{k-h}) \right) e_k$$

Therefore for any fixed $u \in \mathbb{C}^3$ and for any $l \in \mathbb{Z}^3 \setminus \{0\}$,

$$|B(u e_l, v)|_{V^*}^2 = \sum_k \sigma_l^2 \frac{\langle u, k \rangle^2}{|k|^2} |P_k(v_{k-l})|^2$$

If for any $l \in \mathbb{Z}^3$ we consider two orthonormal vectors $u_{l,1}, u_{l,2}$ in $\mathbb{C}^3$ such that $\text{span}\{u_{l,1}, u_{l,2}\} = l^\perp$, then summing over l we find

$$\begin{aligned} \|B(v)\|_{L_2(H, V^*)}^2 &= \sum_l \sum_{i=1}^2 |B(u_{l,i} e_l, v)|_{V^*}^2 = \sum_{l,k} \sigma_l^2 \frac{|P_l(k)|^2}{|k|^2} |P_k(v_{k-l})|^2 \\ &\leq \sum_{l,k} \sigma_l^2 |v_{k-l}|^2 = \left(\sum_l \sigma_l^2 \right) |v|_H^2 = C|v|_H^2 \end{aligned}$$

This fact now allows to give a proper meaning to the stochastic integral in the equation

$$dY = [f + AY] dt + B(dW, Y) = [f + AY] dt + B(Y) dW \quad (4.1)$$

Indeed, if Y is a predictable H -valued process, then $B(Y)$ is a predictable $L_2(H, V^*)$ -valued process and so a sufficient condition for

$$\int_0^\cdot B(Y(s)) dW(s) \quad (4.2)$$

to be a continuous martingale in V^* is

$$\int_0^T \mathbb{E}[\|B(Y(s))\|_{L_2(H, V^*)}^2] ds \leq C \int_0^T \mathbb{E}[|Y(s)|_H^2] ds < \infty$$

and we know that the above condition is satisfied by the energy inequality, which holds for weak solutions of (4.1). Recall that the operator A is given by

$$v = \sum_k v_k e_k \mapsto Av = - \sum_k M_k v_k e_k \quad (4.3)$$

where

$$M_k = \sum_h \sigma_h^2 |P_h(k)|^2 P_k P_{k-h} P_k$$

A is a linear and continuous operator from V to V^* , and more generally from $\mathbb{H}^\alpha$ to $\mathbb{H}^{\alpha-2}$, since there exists a constant C such that $\|M_k\| \leq C|k|^2$ and therefore

$$|Av|_{\mathbb{H}^{\alpha-2}}^2 = \sum_k |k|^{2\alpha-4} |M_k v_k|^2 \leq C^2 \sum_k |k|^{2\alpha} |v_k|^2 = C^2 |v|_{\mathbb{H}^\alpha}^2$$

We can regard A and B as operators $A : D(A) \rightarrow V^*$ and $B : D(B) \rightarrow L_2(H, V^*)$, where $D(A) = V$ and $D(B) = H$ are considered as subspaces of V^* . In this way, the results we have proved for system (3.34) can be actually formulated as the existence and uniqueness of weak solutions of system

$$dY = [f + AY] dt + B(Y) dW$$

with respect to the notion of weak solution considered in [13], namely a V^* valued process satisfying the above equation in a suitable sense and such that $Y \in D(B)$ $\mathbb{P}_T$ -a.s. and

$$\mathbb{P}\left(\int_0^T [|Y(s)|_{V^*} + \|B(Y(s))\|_{L_2}^2] ds < \infty\right) = 1$$

Moreover, the solutions of (4.1) are strong, in the sense defined in [13], if and only if in addition $Y \in D(A) = V$ $\mathbb{P}_T$ -a.s. and

$$\mathbb{P}\left(\int_0^T |AY(s)|_{V^*} ds < \infty\right) = 1$$

which would be satisfied if

$$\mathbb{P}\left(\int_0^T |Y(s)|_V ds < \infty\right) = 1$$

which is consistent with our intuitive notion of what a strong solution should be, as well as the corresponding definition in the deterministic setting. In [13], a third notion of solution of problem (4.1) is considered, the so called **mild solution**. In order to define it, we first need to study the semigroup generated by the operator A . We need the following lemma.

Lemma 4.1. There exists constants $0 < c < C$ such that, for all $k \in \mathbb{Z}^3 \setminus \{0\}$,

$$c|k|^2 P_k \leq M_k \leq C|k|^2 P_k \quad (4.4)$$

Proof. We have

$$\sum_h \sigma_h^2 |P_h(k)|^2 P_{k-h} \leq |k|^2 \left(\sum_h \sigma_h^2 \right) I$$

which implies, by multiplying both sides by P_k , that the inequality on the right of (4.4) holds for

$$C = \sum_h \sigma_h^2 < \infty$$

Regarding the other inequality, observe that for any fixed h , the matrix $P_k P_{k-h} P_k$ has eigenvalues 0, 1 and

$$\cos^2 \theta_{k,k-h} = \left\langle \frac{k}{|k|}, \frac{k-h}{|k-h|} \right\rangle^2$$

with the eigenspaces associated to 0 and 1 being generated respectively by k and $k \wedge (k-h)$, whenever k and $k-h$ are independent. In particular, since M_k is defined by a series of semipositive definite matrices, and for each fixed k is always possible to find h such that $0 < \cos^2 \theta_{k,k-h} < 1$, we can conclude that for fixed k there exists c such that (4.4) holds; the same argument applies for a finite collection of k . Therefore it suffices to show that there exists c such that (4.4) holds for all k sufficiently big, i.e. for all k such that $|k| > N$ for some N . Let us fix k and consider a vector x such that $|x| = 3$ and $\langle x, k \rangle = 0$. Let $h(k)$ denote (possibly one of) the vector with integer entries closest to x . Then necessarily $|x - h(k)| \leq 1$ and

$$\begin{aligned} M_k &\geq \sigma_{h(k)}^2 |P_{h(k)}(k)|^2 P_k P_{k-h(k)} P_k \\ &\geq \frac{1}{(1 + \alpha |h(k)|^2)^2} |k|^2 \left(1 - \left\langle \frac{k}{|k|}, \frac{h(k)}{|h(k)|} \right\rangle^2 \right) \left\langle \frac{k}{|k|}, \frac{k-h(k)}{|k-h(k)|} \right\rangle^2 P_k \\ &\geq \frac{3}{4(1 + 16\alpha)^2} |k|^2 \left\langle \frac{k}{|k|}, \frac{k-h(k)}{|k-h(k)|} \right\rangle^2 P_k \end{aligned}$$

where we used the fact that $2 \leq |h(k)| \leq 4$ and $\langle k, h(k) \rangle = \langle k, h(k) - x \rangle \leq |k|$. Taking the limit as k goes to infinity we obtain

$$\liminf_{k \rightarrow \infty} \left(\frac{1}{|k|^2} M_k - \frac{3}{4(1 + 16\alpha)^2} P_k \right) \geq 0$$

This implies that, taking $\tilde{c} < 3/4(1 + 16\alpha)^2$, we can find N sufficiently big such that (4.4) holds with $\tilde{c}$ for all $|k| > N$. By the above reasoning we can then conclude. $\square$

Corollary 4.2. $A : D(A) \subset \mathbb{H}^\alpha \rightarrow \mathbb{H}^\alpha$ is a closed operator, with $D(A) = \mathbb{H}^{\alpha+2}$; in particular there exist constants $0 < c < C < \infty$ such that

$$c|v|_{\mathbb{H}^{\alpha+2}} \leq |v|_{D(A)} \leq C|v|_{\mathbb{H}^{\alpha+2}} \quad \forall v \in \mathbb{H}^{\alpha+2}$$

We now recall a few concepts of semigroup theory, see for instance [23] or appendix A of [13] and the references therein.

Definition 4.3. Given a Banach space E , a family $\{S(t), t \geq 0\}$ in $L(E)$ is a **semigroup** if

$$S(t+s) = S(t)S(s) \quad \forall t, s \geq 0, \quad S(0) = I$$

It's a **strongly continuous semigroup** if in addition, for any $x \in E$, the map $t \mapsto S(t)x$ is continuous. A strongly continuous semigroup is **contractive** if

$$\|S(t)\| \leq 1 \quad \forall t \geq 0$$

Given a semigroup S , the associated **infinitesimal generator** $A : D(A) \subset E \rightarrow E$ is given by

$$Ax := \lim_{t \rightarrow 0^+} \frac{S(t)x - x}{t} \quad \text{for all } x \text{ such that the limit exists}$$

We also say that A **generates** the semigroup S , since the family $\{S(t), t \geq 0\}$ is uniquely determined by A . Moreover, for any $x \in D(A)$, the unique solution of the Cauchy problem

$$\begin{cases} \dot{u}(t) = Au(t), & t \geq 0 \\ u(0) = x \end{cases}$$

is given by $u(t) = S(t)x$.

Definition 4.4. Given an operator $A : D(A) \subset E \rightarrow E$, we denote by $\rho(A)$ its resolvent set and by $R(\lambda, A) = (\lambda I - A)^{-1}$ the associated resolvent operator. We say that A is **sectorial** if there exist $\omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$ and $M > 0$ such that

$$\text{i) } \rho(A) \supset S_{\theta, \omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}$$

$$\text{ii) } \|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|} \quad \forall \lambda \in S_{\theta, \omega}$$

Given a strongly continuous semigroup S , we say that it's an **analytical semigroup** if its infinitesimal generator A is sectorial.

Definition 4.5. Let $A : D(A) \subset X \rightarrow X$ be an operator on a (complex) Hilbert space X , with $D(A)$ dense in X . Then we define the **adjoint** of A , $A^* : D(A^*) \subset X \rightarrow X$, by setting

$$D(A^*) = \{y \in X : |\langle Ax, y \rangle| \leq C_y |x| \quad \forall x \in D(A) \text{ for some constant } C_y\}$$

and A^*y is defined as the unique element z such that $\langle Ax, y \rangle = \langle x, z \rangle$ for all $x \in D(A)$. An operator A is said to be **symmetric** if

$$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in D(A)$$

A is **self-adjoint** if it is symmetric and $D(A) = D(A^*)$. A self-adjoint operator A is **nonnegative** (resp. nonpositive) if $\langle Ax, x \rangle \geq 0$ for all $x \in D(A)$ (resp. ≤ 0).

Definition 4.6. $A : D(A) \subset X \rightarrow X$ is a **variational operator** on an Hilbert space X if

- i) There exists a Hilbert space V densely embedded in X , a continuous bilinear form $a : V \times V \rightarrow \mathbb{R}$ and some constants $\alpha > 0$, $\lambda_0 \geq 0$ such that

$$-a(v, v) \geq \alpha |v|_V^2 - \lambda_0 |v|_X^2 \quad \forall v \in V \quad (4.5)$$

- ii) $D(A) = \{u \in V : a(u, \cdot) \text{ is continuous in the topology of } X\}$
- iii) $a(u, v) = \langle Au, v \rangle \quad \forall u \in D(A), v \in V$

We also recall the following fundamental result, whose proof is omitted.

Proposition 4.7. Let A be a variational operator in H such that (4.5) holds. Then A generates an analytic semigroup S such that $\|S(t)\| \leq e^{\lambda_0 t}$, $t \geq 0$. Moreover, if A is symmetric, then A is self-adjoint.

We are now ready to give a comprehensive statement on the operator A we are interested in.

Proposition 4.8. Consider the operator $A : D(A) \subset V^* \rightarrow V^*$ defined by (4.3). Then A is a self-adjoint, nonpositive, variational operator which generates an analytical, contractive semigroup given by

$$v = \sum_k v_k e_k \mapsto S(t)v = \sum_k e^{-tM_k} v_k \quad (4.6)$$

Proof. The expression (4.6) for the semigroup follows immediately from the fact that the equation

$$\dot{v}(t) = Av(t)$$

implies that each component v_k must satisfy

$$\dot{v}_k(t) = -M_k v_k(t)$$

which gives $v_k(t) = e^{-tM_k} v_k(0)$ and so expression (4.6). The fact that A is symmetric follows immediately from $M_k = M_k^T$ for all k and nonpositivity is given by

$$\langle Av, v \rangle = - \sum_k \langle M_k v_k, \bar{v}_k \rangle = - \sum_k |M_k^{1/2} v_k|^2 \leq 0$$

By Proposition 4.7, in order to conclude it suffices to show that A is a variational operator with $\lambda_0 = 0$. Recall that H is continuously and densely embedded in V^* ; consider the bilinear form $a : H \times H \rightarrow \mathbb{R}$ given by

$$a(u, v) = - \sum_k \frac{1}{|k|^2} \langle M_k u_k, \bar{v}_k \rangle$$

It's easy to check that a is bilinear and continuous; by Lemma 4.1

$$-a(u, u) = \sum_k \frac{1}{|k|^2} \langle M_k u_k, \bar{u}_k \rangle \geq c \sum_k |u_k|^2 = c \|u\|_H^2$$

It's also immediate to check that points ii) and iii) of Definition 4.5 are satisfied for $D(A) = V$. $\square$

We can now go back to problem (4.1). Since A generates a strongly continuous semigroup S , any weak solution of equation (4.1), with initial data y and external forces f given, must also be a **mild solution**, namely it must satisfy the fixed point equation

$$Y(t) = S(t)y + \int_0^t S(t-s)f(s) ds + \int_0^t S(t-s)B(Y(s)) dW(s) \quad (4.7)$$

where the last term on the r.h.s. is a **stochastic convolution**. We won't discuss here in detail properties of stochastic convolutions, for which we refer to [13]. Let us only stress that it's not a standard stochastic integral, since the integrand depends on t as well, and in fact the process

$$Z(t) := \int_0^t S(t-s)B(Y(s)) dW(s)$$

is not a martingale. However, for fixed t , $Z(t) = \tilde{Z}(t)$, where the process $\tilde{Z}$ is given by

$$\tilde{Z}(s) = \int_0^s S(t-r)B(Y(r)) dW(r)$$

and the last term is now a well defined stochastic integral. Indeed, since $S(t-r)$ are deterministic operators, there is no problem with the predictability of the integrand; by the inequality

$$\|S(t-s)B(Y(s))\|_{L_2(H,V^*)} \leq \|S(t-s)\|_{L(V^*)} \|B(Y(s))\|_{L_2(H,V^*)} \leq \|B(Y(s))\|_{L_2(H,V^*)}$$

it's easy to see that the conditions required on the integrand are satisfied. This has the immediate consequence that $\mathbb{E}[Z(t)] = 0$ and that several properties of stochastic integrals, like Ito isometry and some basic estimates, still hold for stochastic convolutions. It follows from equation (4.7) that the solution Y also satisfies

$$Y(t) - Y(s) = (S(t-s) - I)Y(s) + \int_s^t S(t-r)f(r) dr + \int_s^t S(t-r)B(Y(r)) dW(r) \quad (4.8)$$

We can now use this identity, together with Kolmogorov continuity criterion, to obtain regularity results for the solutions. Observe that, for any $v \in H$, the map $t \mapsto S(t)v$ is $1/2$ -Holder continuous in the V^* -topology. In fact, for any $s \leq t$ and for any $v \in H$,

$$\begin{aligned} |S(t)v - S(s)v|_{V^*}^2 &\leq |S(t-s)v - v|_{V^*}^2 \\ &= \sum_k \frac{1}{|k|^2} |(e^{-(t-s)M_k} - I)v_k|^2 \\ &\leq 2 \sum_k \frac{1}{|k|^2} \|e^{-(t-s)M_k} - I\| |v_k|^2 \\ &\leq 2|t-s| \sum_k \frac{\|M_k\|}{|k|^2} |v_k|^2 \\ &\leq 2C|t-s| \sum_k |v_k|^2 \end{aligned}$$

where we used the fact that, for any nonnegative matrix M , it holds

$$\|e^{-tM} - I\| = \left\| \int_0^t (-M)e^{-sM} ds \right\| \leq \|M\| t$$

This implies that we can estimate the first term on the r.h.s. of (4.8) with

$$|(S(t-s) - I)Y(s)|_{V^*} \leq |t-s|^{1/2} |Y(s)|_H$$

Regarding the second term, by hypothesis $f \in L^2(0, T; H)$, therefore

$$\left| \int_s^t S(t-r)f(r) dr \right|_H \leq |t-s|^{1/2} \left(\int_s^t |f(r)|_H^2 dr \right)^{1/2} \leq |t-s|^{1/2} \|f\|_{L^2}$$

and clearly a similar estimate also holds for the V^* -norm. For the last term, we can use Theorem (2.30) to obtain

$$\begin{aligned} \mathbb{E} \left[\left| \int_s^t S(t-r)B(Y(r)) dW(r) \right|_{V^*}^p \right] &\leq C \mathbb{E} \left[\left(\int_s^t \|S(t-r)B(Y(r))\|_{L_2}^2 dr \right)^{p/2} \right] \\ &\leq \tilde{C} \mathbb{E} \left[\left(\int_s^t |Y(r)|_H^2 dr \right)^{p/2} \right] \end{aligned}$$

In particular it follows from the energy inequality that, if $y \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, $p > 2$ then we have a uniform bound on $|Y(r)|_H^2$ and so for a suitable constant we obtain

$$\mathbb{E} \left[\left| \int_s^t S(t-r)B(Y(r)) dW(r) \right|_{V^*}^p \right] \leq C|t-s|^{p/2}$$

Putting together the previous estimates and the inequality $|a+b+c|^p \leq 3^{p-1}(|a|^p + |b|^p + |c|^p)$, as well as Kolmogorov continuity criterion, we obtain the following result.

Proposition 4.9. Let Y be the energy controlled solution of system (3.34) (respectively (3.51)) with given data y and f . Then for any $p \geq 2$, if $y \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, there exists a constant $C = C(p, y, f)$ such that

$$\mathbb{E}[|Y(t) - Y(s)|_{V^*}^p] \leq C|t-s|^{p/2}$$

In particular, Y is α -Holder continuous in V^* for any $\alpha < \frac{1}{2} - \frac{1}{p}$.

Remark 4.10. If y admits moments of any order, in particular if it's deterministic, $y \in L^\infty(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ or there exists $\delta > 0$ such that $\mathbb{E}[e^{\delta|y|_H^2}] < \infty$, then we can conclude that $Y \in C^{\frac{1}{2}-}([0, T]; V^*)$. This is actually a very intuitive result, since for the deterministic Leray- α of Euler equations the existence of weak solutions Lipschitz in V^* is known, see [7]. By the properties of stochastic integral, it's reasonable to expect the solutions of the stochastic system to have "half" the regularity of the deterministic ones. Observe however that in the previous calculations we used quite rough estimates: most of the time we only used contractivity of S , which however has a very strong regularising effect. Therefore it is not to exclude that better regularity results can be found, at least for sufficiently regular data y and f . For example, we are still not able to show that the solutions are pathwise continuous in H .

Remark 4.11. The formulation (4.1) of problem (3.34) introduced in this section allows to have a clear picture of the reason why introducing the operator K^α and a multiplicative noise $\circ dW$ works so well in this setting. It is only thanks to the presence of the operator K^α that we can define the map $B : H \rightarrow L_2(H, V^*)$, which wouldn't have this regularity otherwise; passing from the Stratonovich formulation to the Ito one gives rise to the operator A , which as we've seen is negative definite and generates an analytic semigroup. Indeed, A plays the role of a Laplacian Δ : their domain of definition is the same, their graph norms are equivalent and they have a common orthonormal basis of eigenvectors. This also implies that the interpolation spaces that can be defined starting from the associated analytical semigroups (see [23] for details) coincide.

4.2 Continuity on average and transition probabilities

In the previous section we have studied pathwise regularity of solutions, but we haven't been able to determine whether the solutions are pathwise (strongly) continuous in H . Here we provide a weaker result, namely that the solutions are continuous in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ with respect to time. In this section, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the norm and scalar product in H . As usual we consider a fixed filtered probability space $L^2(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P}; H)$ and an $\mathcal{F}_t$ -Wiener process W ; for any $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $f \in L^2(0, T; H)$, we denote by $\{Y(t; y, f), t \in [0, T]\}$ the unique solution of the linear system (3.34) with initial data y , external forces f and driver W . However, when there is no ambiguity we will only write $Y(t)$. Recall that for such solution, the energy inequality holds:

$$\sup_{t \in [0, T]} \left\{ |Y(t; y, f)|^2 - \int_0^t \langle f(s), Y(s; y, f) \rangle ds \right\} \leq |y|^2 \quad \mathbb{P}\text{-a.s.}$$

Observe that, for any fixed t , if we define the process $Z(s) := Y(t + s; y, f)$, then Z is again a solution of system (3.34), this time with respect to the filtration $\{\mathcal{G}_s, s \in [0, T - t]\}$ given by $\mathcal{G}_s = \mathcal{F}_{t+s}$, driver $\bar{W}(s) = W(s + t) - W(t)$, initial data $Y(t; y, f) \in \mathcal{G}_0$ and external forces $\tau(t)f(\cdot) = f(t + \cdot) \in L^2(0, T - t; H)$. In particular by uniqueness, Z must satisfy the energy inequality as well, which implies that for any fixed $t \in [0, T]$, $\mathbb{P}$ -a.s.

$$\sup_{s \in [t, T]} \left\{ |Y(s; y, f)|^2 - \int_0^s \langle f(r), Y(r; y, f) \rangle dr \right\} \leq |Y(t; y, f)|^2 - \int_0^t \langle f(r), Y(r; y, f) \rangle dr \quad (4.9)$$

This allows to prove the following lemma.

Lemma 4.12. For any initial data $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and for any $f \in L^2(0, T; H)$, the **energy function**

$$E(t) = \mathbb{E} \left[|Y(t)|^2 - \int_0^t \langle f(r), Y(r) \rangle dr \right]$$

is a decreasing, right continuous function.

Proof. The fact that E is decreasing is an immediate consequence of (4.9). In order to show that a decreasing function is right continuous, it suffices to show that it's lower semicontinuous. The function

$$t \mapsto \mathbb{E} \left[\int_0^t \langle f(r), Y(r) \rangle dr \right] = \int_0^t \langle f(r), \mathbb{E}[Y(r)] \rangle dr$$

is continuous, since by Cauchy inequality and the energy bound

$$\langle f(\cdot), \mathbb{E}[Y(\cdot)] \rangle \in L^1(0, T; \mathbb{R})$$

Therefore it's enough to show lower semicontinuity of $t \mapsto \mathbb{E}[|Y(t; y, f)|^2]$. Since $Y(\cdot; y, f)$ is pathwise weakly continuous, by properties of weak convergence $|Y(\cdot; y, f)|$ is pathwise lower semicontinuous. Then by Fatou's lemma

$$\mathbb{E}[|Y(t_0)|^2] \leq \mathbb{E} \left[\liminf_{t \rightarrow t_0} |Y(t)|^2 \right] \leq \liminf_{t \rightarrow t_0} \mathbb{E}[|Y(t)|^2]$$

which gives the conclusion. $\square$

It follows by properties of decreasing functions that E can have at most countable points of discontinuity. In particular, since the second term in the definition of E is continuous, it follows that the map

$$t \mapsto \mathbb{E}[|Y(t)|^2]$$

is right continuous and admits at most countable points of discontinuity. This is a key fact in the proof of the following result.

Theorem 4.13. Let $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $f \in L^2(0, T; H)$ and consider the associated solution $Y(t)$. Then the map $t \mapsto Y(t)$ is continuous from $[0, T]$ to $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$.

Proof. We divide the proof in several steps.

Step 1. The map $t \mapsto Y(t)$ is weakly continuous from $[0, T]$ to $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$. Namely, if $t \rightarrow t_0$, then $Y(t) \rightharpoonup Y(t_0)$. This is equivalent to showing that, for any $X \in L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$, it holds

$$\lim_{t \rightarrow t_0} \mathbb{E}[\langle X, Y(t) \rangle] = \mathbb{E}[\langle X, Y(t_0) \rangle]$$

Since Y is pathwise lower semicontinuous, $\langle X, Y(t) \rangle \rightarrow \langle X, Y(t_0) \rangle$ $\mathbb{P}$ -a.s.; by the energy bound

$$|\langle X, Y(t) \rangle| \leq C|X|(1 + |y|) \in L^1(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$$

and so by dominated convergence we conclude.

Step 2. Recall that for general Hilbert spaces, $x_n \rightarrow x$ if and only if $x_n \rightharpoonup x$ and $|x_n| \rightarrow |x|$. Therefore in order to show that $Y(t) \rightarrow Y(t_0)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$, by the previous step it suffices to show that $\mathbb{E}[|Y(t)|^2] \rightarrow \mathbb{E}[|Y(t_0)|^2]$. Namely we only need to show that the function

$$t \mapsto \mathbb{E}[|Y(t)|^2] \tag{4.10}$$

is continuous on $[0, T]$. Since we already know that such function is right continuous and admits at most countable points of discontinuity, we deduce that the same must hold for $t \mapsto Y(t)$.

Step 3. We claim that, if $t \mapsto Y(t)$ is continuous at t_0 , then it must be continuous on $[t_0, +\infty)$. Let us first show how the claim implies the conclusion: since the map can have at most a countable number of discontinuities, for any $\varepsilon > 0$ we can find a continuity point in $(0, \varepsilon)$ and therefore by the claim, the map must be continuous on $[\varepsilon, \infty)$. As the reasoning holds for any $\varepsilon > 0$, we find continuity on $(0, +\infty)$, and continuity at 0 is given by right continuity of the map.

Step 4. It remains to prove the claim. We first give an heuristic idea of the proof and then formalize it properly. Recall that, for any fixed t , the map $(y, f) \mapsto Y(t; y, f)$ is continuous (in the usual spaces). Now assume that we have continuity at t_0 , namely $Y(t; y, f) \rightarrow Y(t_0; y, f)$ as $t \rightarrow t_0$. Then we can apply again the above map, for some $s > 0$, to find that $Y(s; Y(t; y, f), f) \rightarrow Y(s; Y(t_0; y, f), f)$ as well. But by uniqueness of the solutions we know that

$$Y(s; Y(t; y, f), f) = Y(t + s; y, f)$$

which implies that $Y(t + s; y, f) \rightarrow Y(t_0 + s; y, f)$, which implies continuity at $t_0 + s$. As the reasoning holds for any $s > 0$, we can conclude.

The main reason why the above argument is incorrect is that the map $(y, f) \mapsto Y(s; y, f)$ is defined only for $y \in \mathcal{F}_0$ and so it cannot be applied again to $Y(t; y, f)$ as predictability of the process would fail. In order to make it rigorous, we use a coupling argument. Fix $\varepsilon > 0$ and consider the filtration $\mathcal{G}_s = \mathcal{F}_{s+t_0+\varepsilon}$ and the $\mathcal{G}_s$ -Wiener process $\widetilde{W}_s = W(s+t_0+\varepsilon) - W(t_0+\varepsilon)$. Then, for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$, $Y(t; y, f)$ is $\mathcal{G}_0$ -measurable and is an admissible initial data for system (3.34) with respect to the filtered probability space $(\Omega, \{\mathcal{G}_s\}, \mathcal{F}, \mathbb{P})$ and the driver $\widetilde{W}$. We can therefore consider the unique solution of system (3.34) with initial data $Y(t; y, f)$ and external forces $\tau_t(f)$, which we denote by $\widetilde{Y}(\cdot; Y(t; y, f), \tau_t(f))$. By hypothesis $Y(t; y, f) \rightarrow Y(t_0; y, f)$ in $L^2(\Omega, \mathcal{G}_0, \mathbb{P}; H)$ and by the properties of translations $\tau_t(f) \rightarrow \tau_{t_0}(f)$ as $t \rightarrow t_0$, therefore for any $s > 0$ it holds

$$\widetilde{Y}(s; Y(t; y, f), \tau_t(f)) \rightarrow \widetilde{Y}(s; Y(t_0; y, f), \tau_{t_0}(f))$$

and in particular

$$\mathbb{E}[|\widetilde{Y}(s; Y(t; y, f), \tau_t(f))|^2] \rightarrow \mathbb{E}[|\widetilde{Y}(s; Y(t_0; y, f), \tau_{t_0}(f))|^2]$$

But by uniqueness in law $\widetilde{Y}(s; Y(t; y, f), \tau_t(f))$ must be distributed as $Y(t+s; y, f)$ and similarly $\widetilde{Y}(s; Y(t_0; y, f), \tau_{t_0}(f))$ as $Y(t_0+s; y, f)$, therefore

$$\mathbb{E}[|Y(t+s; y, f)|^2] \rightarrow \mathbb{E}[|Y(t_0+s; y, f)|^2]$$

which implies by Step 2 that $Y(t+s; y, f) \rightarrow Y(t_0+s; y, f)$ as $t \rightarrow t_0$, namely continuity at $t_0 + s$. As the reasoning holds for any $s > 0$, we find the conclusion. $\square$

Remark 4.14. It follows from the previous proof that the same result holds for the solutions of the nonlinear system (3.51), for any initial data y such that $\mathbb{E}[e^{\delta|y|^2}] < \infty$ and $f \in L^2(0, T; H)$. Indeed, since the solutions are still pathwise weakly continuous and the energy bound holds, it suffices to show that the map $t \mapsto \mathbb{E}[|Y(t)|^2]$ is continuous. Since this is a condition on the law of the solution, which is unique, we only need to show it for a suitable weak solution of our choice. In particular, if we construct the solution starting from the one of the linear system and applying Girsanov transform, since the measures $\mathbb{P}$ and Q are equivalent, convergence in probability is preserved. It follows then from $Y(t) \rightarrow Y(t_0)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ that $Y(t) \rightarrow Y(t_0)$ in probability with respect to Q ; then the energy bound allows to apply dominated convergence to find that $Y(t) \rightarrow Y(t_0)$ in $L^2(\Omega, \mathcal{F}, Q; H)$ as well.

A simple application of the previous result regards the transition semigroup associated to the dynamics; for simplicity we only consider it in the linear case. Let us recall that, for any $s < t$, the solutions Y satisfy the identity

$$Y(t) = S(t-s)Y(s) + \int_s^t S(t-r)f(r)dr + \int_s^t S(t-r)B(Y(r))dW(r)$$

Then taking expectation, by the properties of stochastic integral we obtain

$$\mathbb{E}[Y(t)|\mathcal{F}_s] = S(t-s)Y(s) + \int_s^t S(t-r)f(r)dr = \mathbb{E}[Y(t)|Y(s)]$$

which shows that, for any initial data y and external forces f , the solutions $\{Y(t), t \in [0, T]\}$ are **markovian**. We can also consider the maps

$$P_t : C_b(H) \rightarrow C_b(H), \quad (P_t g)(x) = \mathbb{E}[g(Y(t; x))] \quad (4.11)$$

where $C_b(H)$ denotes the space of continuous and bounded functions from H to $\mathbb{R}$, endowed with the supremum norm, and $Y(t; x)$ is the solution starting at $x \in H$ with external forces $f \equiv 0$. The family $\{P_t\}_{t \geq 0}$ is called the **transition semigroup** associated to system (3.34). Let us actually show that it is a semigroup; first of all, it's clear that $P_0 = I$. We must now show that, for any $t > 0$ and $g \in C_b(H)$, $P_t g \in C_b(H)$ as well. Indeed, if $x_n \rightarrow x$, then for any $t \geq 0$ we know that $Y(t; x_n) \rightarrow Y(t; x)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$, thus by dominated convergence ($\|g\|_\infty < \infty$)

$$(P_t g)(x_n) = \mathbb{E}[g(Y(t; x_n))] \rightarrow \mathbb{E}[g(Y(t; x))] = (P_t g)(x)$$

which shows that $P_t g \in C_b(H)$. Linearity of P_t is immediate and

$$\|P_t g\|_\infty \leq \|g\|_\infty$$

which shows that $P_t \in L(C_b(H))$ for all $t \geq 0$. It remains to show the semigroup property, namely $P_t \circ P_s = P_{t+s}$. Let us make a quick remark: when defining the semigroup as in (4.11), in order to construct the solutions $Y(t; x)$ we are choosing a stochastic basis $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$ and an $\mathcal{F}_t$ -Wiener process; however the definition is independent of such choice, since uniqueness in law holds and the definition of P_t only relies on the law of the solution $Y(t; x)$. In particular, the semigroup property is a consequence of uniqueness in law: if we take consider a solution $Y(t; x)$ and then we construct another solution, starting at $Y(t; x)$, with respect to a Wiener process independent of $\mathcal{F}_t$, then the obtained process $\tilde{Y}(s; Y(t; x))$ must be distributed as $Y(t + s; x)$.

It follows from Theorem 4.13 that, for any fixed $g \in C_b(H)$ and $x \in H$, the map

$$t \mapsto (P_t g)(x)$$

is continuous. Indeed, for any fixed t_0 , we know that $Y(t; x) \rightarrow Y(t_0; x)$ in $L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ as $t \rightarrow t_0$, therefore by dominated convergence

$$(P_t g)(x) = \mathbb{E}[g(Y(t; x))] \rightarrow \mathbb{E}[g(Y(t_0; x))] = (P_{t_0} g)(x) \quad \text{as } t \rightarrow t_0$$

Together with the fact that $P_t g$ is continuous, this can be summarized as the fact that the map

$$(t, x) \mapsto (P_t g)(x)$$

is continuous for any fixed $g \in C_b(H)$. It can also be shown that the semigroup $\{P_t\}$ also acts on $B_b(H)$, where $B_b(H)$ denotes the set of all bounded Borel maps from H to $\mathbb{R}$, endowed with the $\|\cdot\|_\infty$ norm. The proof is standard and is taken from [14].

Lemma 4.15. $P_t \varphi \in B_b(H)$ for all $\varphi \in B_b(H)$ and $t \geq 0$.

Proof. We start with $\varphi = \mathbb{1}_C$, where C is a closed subset of H . For all $n \geq 1$ define

$$C^n = \{x \in H : d(x, C) \geq 1/n\}, \quad \varphi_n(x) = \frac{d(x, C^n)}{d(x, C) + d(x, C^n)}$$

Then φ^n is a sequence in $C_b(H)$ such that, for fixed x , $\varphi_n(x) \downarrow \varphi(x)$ as $n \rightarrow \infty$. Consequently by monotone convergence it holds

$$(P_t \varphi^n)(x) \downarrow (P_t \varphi)(x) \text{ as } n \rightarrow \infty \forall x \in H, t \geq 0$$

But $\varphi^n \in C_b(H)$, therefore $\varphi^n \in B_b(H)$ and we can conclude that $\varphi \in B_b(H)$ as it's the pointwise limit of Borel maps. The rest of the proof is based on approximation techniques: we can extend the result first to $\varphi \in \mathbb{1}_\Gamma$, for any Borel set Γ , then by linearity to any simple function and finally to any $\varphi \in B_b(H)$. $\square$

It's clear that, for any $t \geq 0$, P_t is a continuous and linear map also from $B_b(H)$ to itself. The fact that P_t acts from $C_b(H)$ to itself is usually referred as **Feller property**. Further properties of the transition semigroup generated by system (3.34) (and system (3.51)) are yet to be studied and could provide fundamental information on the long time behaviour of the system, such as irreducibility and ergodicity. It is not known whether $\{P_t\}$ satisfies the **strong Feller property**, namely $P_t \varphi \in C_b(H)$ for all $t > 0$ and $\varphi \in B_b(H)$.

4.3 Anomalous dissipation of energy

In this section we consider the linear system (3.34) in the homogeneous case $f \equiv 0$ and initial data $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$. As before, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the scalar product and norm in H and $\{Y(t; y), t \geq 0\}$ denotes the solution starting at y . We know that

$$\sup_{t \geq 0} |Y(t; y)| \leq |y| \quad \mathbb{P}\text{-a.s.}$$

This is a consequence of the energy inequality, which was initially introduced as a weakened version of the energy invariance, which only holds *formally*, namely if the solutions are regular enough. It's therefore natural to investigate whether the energy is actually preserved by the dynamics or if there exist some initial data y and time $t > 0$ such that

$$\mathbb{P}(|Y(t; y)| < |y|) > 0$$

In this case we say that **anomalous dissipation** of energy has occurred. This problem was initially studied for the deterministic Euler equations, where the existence of weak solutions displaying such phenomenon was found by Shnirelman and later De Lellis, see [24] and [25]. From a physical point of view, anomalous dissipation is actually expected to happen, since the solutions of Euler equations are believed to be **turbulent**, i.e. obtained as a vanishing viscosity limit of the solutions of the corresponding Navier-Stokes equations. Numerical simulations hint that the rate of dissipation of energy of the solutions of Navier-Stokes does not tend to 0 as the viscosity does, which suggests that turbulent solutions of Euler equations should have a positive rate of dissipation as well. Since uniqueness of weak solutions of Euler equations does not hold, anomalous dissipation could then play the role of a selection principle of physically meaningful solutions. It also gives important information on the dynamics, since a dissipative solution cannot be too regular, in particular it can't be a strong solution for all times; if anomalous dissipation were to be shown for all initial data, this would imply that regular solutions blow-up in finite

time. Evidence suggesting that our system should display anomalous dissipation is also given by [26], where it was shown for a stochastic dyadic model with Stratonovich multiplicative noise which enjoys properties very similar to our model. An heuristic explanation of dissipation of energy can be given when looking at the dynamics in Fourier space: the energy is believed to pass from lower to higher modes, faster and faster, in a **cascade mechanism** which leads to some amount of it "escaping at infinity" in finite time; this can be also seen as some kind of weak convergence which is not also strong and thus creates an energy gap. Anomalous dissipation of energy could even be the cause behind well-posedness of our system: in models arising from fluid dynamics, uniqueness may be broken by spontaneous generation of energy, such as the appearance of vortexes, which could be caused by energy "entering from infinity". The role of the noise could be preventing this type of phenomena to occur forcing the energy to "go in the other direction", namely decrease in time.

Even if we haven't been able to find a full answer for our system, we collect here the partial results obtained so far.

Definition 4.16. We say that, for a given initial data y , the solution $Y(\cdot, y)$ is **energy preserving** if $\mathbb{E}[|Y(t; y)|^2] = \mathbb{E}[|y|^2]$ for all $t \geq 0$. Otherwise, the solution shows **anomalous dissipation** of energy.

Remark 4.17. It's easy to see that such definition is equivalent to the previous one: since $|Y(t; y)| \leq |y|$ with probability 1,

$$\mathbb{E}[|Y(t; y)|^2] = \mathbb{E}[|y|^2] \Leftrightarrow |Y(t; y)| = |y| \quad \mathbb{P}\text{-a.s.}$$

However, Definition 4.16 is preferable for several reasons: it's related to the deterministic function $t \mapsto \mathbb{E}[|Y(t; y)|^2]$, which is continuous and decreasing, and can be expressed in function of a deterministic system, namely the covariance matrices of Section 3.5. Observe that the above definition only depends on the law of $Y(\cdot, y)$ and not the particular stochastic basis or Wiener process considered, so that it's a well defined property of the initial data y (actually, of its law). For this reason, even if we will restrict to $\mathcal{F}_0$ -measurable initial data, the definition could be extended to any square integrable, H -valued random variable, as it's always possible to construct a cylindrical Wiener process W independent of it and then consider the solutions of system (3.34) starting at y with driver W .

Remark 4.18. Even if for simplicity we consider the linear system (3.34), an initial data $y \in H$ gives rise to an energy preserving solution of (3.34) if and only if it does for the nonlinear system (3.51). Indeed, even in the nonlinear case the definition only relies on the law of the solution, so that we can restrict to consider solutions constructed via Girsanov transform; since the measures $\mathbb{P}$ and Q are equivalent,

$$\mathbb{P}(|Y(t; y)| = |y|) = 1 \Leftrightarrow Q(|Y(t; y)| = |y|) = 1$$

which also implies that

$$\mathbb{E}_{\mathbb{P}}[|Y(t; y)|^2] = \mathbb{E}_{\mathbb{P}}[|y|^2] \Leftrightarrow \mathbb{E}_Q[|Y(t; y)|^2] = \mathbb{E}_Q[|y|^2]$$

The same holds for any initial data y such that $\mathbb{E}_{\mathbb{P}}[e^{\delta|y|^2}] < \infty$ for some $\delta > 0$.

Since the function $t \mapsto \mathbb{E}[|Y(t; y)|^2]$ is decreasing, for any $y \in L^2(\Omega, \mathcal{F}, \mathbb{P}; H)$ the following function is well defined:

$$F : L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H) \rightarrow \mathbb{R}, \quad F(y) = \lim_{t \rightarrow \infty} \mathbb{E}[|Y(t; y)|^2]$$

Moreover, since the solutions depend continuously on the initial data, the same holds for the limit and F is a continuous map. It's also locally Lipschitz: if y and z are such that $\|y\|_{L^2}, \|z\|_{L^2} \leq N$, then

$$\begin{aligned} |\mathbb{E}[|Y(t; y)|^2 - |Y(t; z)|^2]| &\leq \mathbb{E}[|Y(t; y) - Y(t; z)| |Y(t; z) + Y(t; y)|] \\ &\leq \mathbb{E}[|Y(t; y - z)|^2]^{\frac{1}{2}} \mathbb{E}[|Y(t; y + z)|^2]^{\frac{1}{2}} \\ &\leq \mathbb{E}[|y - z|^2]^{\frac{1}{2}} \mathbb{E}[|y + z|^2]^{\frac{1}{2}} \\ &\leq 2N \|y - z\|_{L^2} \end{aligned}$$

and the inequality still holds when passing to the limit as $t \rightarrow \infty$. By the energy inequality $F(y) \leq \|y\|_{L^2}^2$ and by definition, $Y(t; y)$ is energy preserving if and only if $F(y) = \|y\|_{L^2}^2$. We can then investigate the properties of some level sets of F .

Proposition 4.19. Consider the subsets of $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ given by

$$\begin{aligned} \Gamma &:= \{y : Y(\cdot; y) \text{ is energy preserving}\} \\ \Theta &:= \{y : F(y) = 0\} \end{aligned}$$

then Γ and Θ are closed subspaces of $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$.

Proof. It's clear that Γ and Θ are closed by continuity of F , we only need to show that they are subspaces. In the case of Θ , by the basic inequality $|a + b|^2 \leq 2|a|^2 + 2|b|^2$ and linearity of the solutions it follows that

$$F(\alpha y + \beta z) \leq 2(\alpha^2 F(y) + \beta^2 F(z))$$

which shows that Θ is a subspace. Consider now Γ . Recall that in the proof of Theorem 3.13, when we constructed the solution $Y(t; y)$ on any finite interval $[0, T]$, we employed a Galerkin scheme, constructing a sequence Y^N of solutions of finite dimensional problems such that (in the case $f \equiv 0$)

$$\int_0^T \mathbb{E}[|Y^N(t)|^2] dt = T \mathbb{E}[|y^N|^2]$$

where y^N denotes the projection of y on a finite dimensional space, and $Y^N \rightharpoonup Y$ in $L^2(\Omega_T; H)$. Since $L^2(\Omega_T; H)$ is an Hilbert space and $Y^N \rightharpoonup Y$, $Y^N \rightarrow Y$ if and only if the relative norms converge. But by hypothesis $Y(\cdot; y)$ is an energy preserving solution, so that

$$\int_0^T \mathbb{E}[|Y(t; y)|^2] dt = T \mathbb{E}[|y|^2]$$

and $y^N \rightarrow y$ in $L^2(\Omega_T; H)$, thus $\int_0^T \mathbb{E}[|Y^N(t; y)|^2] dt \rightarrow T \mathbb{E}[|y|^2]$ and we can deduce that Y^N converge to Y in $L^2(\Omega_T; H)$. This gives a characterization of energy preserving solutions: $Y(\cdot; y)$ is energy preserving if and only if it's the strong limit in $L^2(\Omega_T; H)$ of the sequence $Y^N(\cdot; y)$,

for any $T > 0$. Since the map $y \mapsto Y^N(\cdot; y)$ is linear for any N , this implies that if $Y^N(\cdot; y) \rightarrow Y(\cdot; y)$ and $Y^N(\cdot; z) \rightarrow Y(\cdot; z)$, then

$$Y^N(\cdot; \alpha y + \beta z) = \alpha Y^N(\cdot; y) + \beta Y^N(\cdot; z) \rightarrow \alpha Y(\cdot; y) + \beta Y(\cdot; z) = Y(\cdot; \alpha y + \beta z)$$

which implies that Γ is a linear subspace. $\square$

Remark 4.20. If we extended the definition of F also to non $\mathcal{F}_0$ -measurable initial data, then F would be invariant under the dynamics: given a solution $Y(\cdot; y)$, for any $t \geq 0$ the process $Y(t + \cdot; y)$ is still a solution, with initial data $Y(t; y)$ and driver $\widehat{W}(s) = W(t + s) - W(t)$, so that

$$F(Y(t; y)) = \lim_{s \rightarrow \infty} \mathbb{E}[|Y(t + s; y)|^2] = F(y)$$

It follows that, if we extended the definition of Γ and Θ as well, being level sets of F , they would be invariant under the dynamics as well. The problem with this reasoning is that it's unclear, if we considered initial data which are not measurable with respect to the same $\mathcal{F}_0$, how to treat their sum as a random variable; i.e. if we extended the definition of Γ and Θ , it would be unclear how to treat them as subspaces. Anyway, even restricting to $\mathcal{F}_0$ -measurable initial data, the above result is quite surprising, especially in the case of Γ : we are able to deduce, from properties of the laws of two initial data y_1 and y_2 , that the same property must also hold for their sum $y_1 + y_2$, regardless of their joint distribution.

Remark 4.21. In a similar fashion, we can define the following subsets of H :

$$\begin{aligned} \widehat{\Gamma} &:= \{y \in H : Y(\cdot; y) \text{ is energy preserving}\} \\ \widehat{\Theta} &:= \{y \in H : F(y) = 0\} \end{aligned}$$

and by the same proof, $\widehat{\Gamma}$ and $\widehat{\Theta}$ are closed subspaces of H . It's easy to see that, if $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ takes values in $\widehat{\Gamma}$, then $y \in \Gamma$: it can be checked that it holds for y simple and then it can be extended to any y as above by approximation. Similarly, if y takes values in $\widehat{\Theta}$, then $y \in \Theta$. However, we don't have a proof of the converse implications.

We can now use the characterization of the energy preserving solutions obtained in the previous proof to show that they enjoy a form of "stability" with respect to all other solutions.

Proposition 4.22. Let $y \in \Gamma$. Then for any $z \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and for any $T > 0$ it holds

$$\mathbb{E}[\langle y, z \rangle] = \frac{1}{T} \int_0^T \mathbb{E}[\langle Y(t; y), Y(t; z) \rangle] dt \quad (4.12)$$

Proof. We observed initially that the linear system (3.34), if $f \equiv 0$, formally preserves energy; this was the starting point for the introduction of the concept of energy controlled solutions. A closer inspection reveals that the system formally preserves scalar products, that is if $Y(t; y)$ and $Y(t; z)$ are solutions, then formally $\langle Y(t; y), Y(t; z) \rangle = \langle y, z \rangle$. In particular, when we consider the solutions $Y^N(t; y)$ of the Galerkin scheme, the previous equality actually holds and so

$$\int_0^T \mathbb{E}[\langle Y^N(t; y), Y^N(t; z) \rangle] dt = T \mathbb{E}[\langle y^N, z^N \rangle]$$

Since $y \in \Gamma$, we know that $Y^N(\cdot; y) \rightarrow Y(\cdot; y)$ in $L^2(\Omega_T; H)$. Moreover, $Y^N(\cdot; z) \rightharpoonup Y(\cdot; z)$ and so by properties of weak convergence it follows that

$$\langle Y^N(\cdot; y), Y^N(\cdot; z) \rangle_{L^2(\Omega_T; H)} \rightarrow \langle Y(\cdot; y), Y(\cdot; z) \rangle_{L^2(\Omega_T; H)}$$

that is

$$\lim_{N \rightarrow \infty} T \mathbb{E}[\langle y^N, z^N \rangle] = \lim_{N \rightarrow \infty} \int_0^T \mathbb{E}[\langle Y^N(t; y), Y^N(t; z) \rangle] dt = \int_0^T \mathbb{E}[\langle Y(t; y), Y(t; z) \rangle] dt$$

Using the fact that $y^N \rightarrow y$ and $z^N \rightarrow z$ in $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, the conclusion follows. $\square$

Corollary 4.23. For any $y \in \Gamma$ and $z \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$,

$$\mathbb{E}[\langle Y(t; y), Y(t; z) \rangle] = \mathbb{E}[\langle y, z \rangle] \quad \forall t \geq 0$$

In particular, Γ and Θ are orthogonal subspaces of $L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$; similarly, $\widehat{\Gamma}$ and $\widehat{\Theta}$ are orthogonal subspaces of H .

Proof. Fix $t > 0$; then by considering as usual $\widetilde{W}(s) = W(t+s) - W(t)$, $\widetilde{\mathcal{F}}_s = \mathcal{F}_{t+s}$, $Y(t+s; y)$ and $Y(t+s; z)$ are solutions with driver $\widetilde{W}$ and initial data $Y(t; y), Y(t; z) \in \widetilde{\mathcal{F}}_0$. Moreover, since $y \in \Gamma$, $Y(t; y) \in \widehat{\Gamma}$, which is defined as Γ but with $\widetilde{\mathcal{F}}_0$ -measurable initial data. We can therefore apply (4.12) to $Y(t; y)$ and $Y(t; z)$ to obtain

$$\mathbb{E}[\langle Y(t; y), Y(t; z) \rangle] = \frac{1}{T} \int_0^T \mathbb{E}[\langle Y(t+s; y), Y(t+s; z) \rangle] ds = \frac{1}{T} \int_t^{T+t} \mathbb{E}[\langle Y(s; y), Y(s; z) \rangle] ds$$

and (4.12) also holds for $\mathbb{E}[\langle y, z \rangle]$. But then, observing that for any fixed $t > 0$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_t^{T+t} \mathbb{E}[\langle Y(s; y), Y(s; z) \rangle] ds - \frac{1}{T} \int_0^T \mathbb{E}[\langle Y(s; y), Y(s; z) \rangle] ds = 0$$

the conclusion follows. In particular, if $z \in \Theta$ (respectively $\widehat{\Theta}$) then $\mathbb{E}[|Y(t; z)|^2] \rightarrow 0$, which implies that

$$\mathbb{E}[\langle y, z \rangle] = \lim_{t \rightarrow \infty} \mathbb{E}[\langle Y(t; y), Y(t; z) \rangle] = 0$$

$\square$

Up to now, we have treated the problem of anomalous dissipation in a quite abstract way. This is useful as the results obtained in this way are likely to be true also for similar models, but has the drawback of not fully exploiting the structure of this particular one. Instead, from now on we will approach the problem by studying the corresponding covariance matrices, which were introduced in Section 3.5. Recall that they are defined as $A_k = \Re(\mathbb{E}[Y_k(t; y) \otimes \overline{Y}_k(t; y)])$, satisfy the properties $A_k = A_{-k}$, $A_k = A_k^T$ and $A_k P_k = P_k A_k = A_k$ and solve the system

$$\frac{dA_k}{dt} = 2 \sum_h \sigma_h^2 |P_h(k)|^2 P_k A_{k-h} P_k - M_k A_k - A_k M_k \quad (4.13)$$

where

$$M_k = \sum_h \sigma_h^2 |P_h(k)|^2 P_k P_{k-h} P_k$$

The covariance matrices are a fundamental tool in the study of anomalous dissipation because the energy function is entirely determined by them, since

$$\mathbb{E}[|Y(t; y)|^2] = \sum_k \text{Tr}(A_k(t)) \quad \forall t \geq 0$$

In Section 3.5 we have already proved several results for system (4.13), in particular two comparison principles. We now need the following additional one.

Lemma 4.24. Let $\{A_h(t)\}_h$ be a supersolution of (4.13) with non zero, nonnegative initial data $\{A_h(0)\}_h$. If there exist $t > 0$ and k such that $A_k(t)$ has rank at most one, then

$$A_h(s) = 0 \quad \text{for all } h \in \mathbb{Z}^3 \setminus 0 \text{ and } s \in [0, t]$$

Proof. We know by the comparison principle that $A_h(s) \geq 0$ for all h and s . In order to obtain the conclusion, by Corollary 3.29, it suffices to show that $A_k(t) = 0$. $A_k(t)$ is a symmetric matrix and $A_k(t)k = 0$, therefore it has rank at most one if and only if there exists a vector v , $|v| = 1$, such that $\langle v, k \rangle = 0$ and $A_k(t)v = 0$. Recall that, being $\{A_k\}_k$ a supersolution of (4.13), it holds

$$A_k(t) \geq e^{-tM_k} A_k(0) e^{-tM_k} + \sum_h \sigma_h^2 |P_h(k)|^2 P_k \left(\int_0^t e^{-(t-s)M_k} A_{k-h}(s) e^{-(t-s)M_k} ds \right) P_k \quad (4.14)$$

Also recall that, for any symmetric semipositive definite matrix B ,

$$Bu = 0 \Leftrightarrow \langle u, Bu \rangle = 0$$

Applying (4.14) to v we obtain

$$0 \geq \langle e^{-tM_k} v, A_k(0) e^{-tM_k} v \rangle + \sum_h \sigma_h^2 |P_h(k)|^2 \left(\int_0^t \langle e^{-(t-s)M_k} v, A_{k-h}(s) e^{-(t-s)M_k} v \rangle ds \right)$$

Using the fact that all the terms are nonnegative and the property mentioned above, as well as the continuity of $t \mapsto A_h(t)$, we obtain:

$$A_{k-h}(s) e^{-(t-s)M_k} v = 0 \quad \forall h \text{ such that } |P_h(k)|^2 \neq 0 \text{ and } \forall s \in [0, t]$$

In particular, $A_{k-h}(t)v = 0$ for all h not belonging to $\text{span}(k)$. If $v \notin \text{span}(k-h)$, this implies that $A_{k-h}(t)$ has rank at most one as well. Observe that if $v \in \text{span}(k-h)$, i.e. $v = \alpha(k-h)$ for some $\alpha \neq 0$, then $0 = \langle v, k \rangle = \alpha(|k|^2 - \langle h, k \rangle)$, which implies that h belongs to $k + k^\perp$. Therefore we deduce that $A_{k-h}(t)$ has rank at most one for all h such that

$$h \notin \Gamma_k = \text{span}(k) \cup (k + k^\perp)$$

Actually, the above reasoning holds even in the case we do not assume $v \perp k$, with the only difference that we can conclude that $P_k(v) \in \ker(A_{k-h})(t)$ for all $h \notin \Gamma_k$. In particular we can apply again the reasoning above, this time starting from $v \in \ker(A_{k-h}(t))$, to find that $P_{k-h}(v) \in \ker(A_k(t))$, since

$$h \notin \Gamma_k \Rightarrow -h \notin \Gamma_{k-h}$$

If we are able to choose h such that $k, v, P_{k-h}(v)$ are independent, then since they all belong to the kernel of $A_k(t)$ we can conclude that $A_k(t) = 0$. In order to find such h , we give a characterization of all h such that $P_{k-h}(v) \in \text{span}(k, v)$. Recall that, since k and v are orthogonal and $|v| = 1$, a vector u belongs to $\text{span}(k, v)$ if and only if

$$u = \langle u, v \rangle v + \langle u, k \rangle \frac{k}{|k|^2} \quad (4.15)$$

and this is also equivalent to

$$|u|^2 = \langle u, v \rangle^2 + \left\langle u, \frac{k}{|k|} \right\rangle^2 \quad (4.16)$$

Also recall that

$$P_{k-h}v = v - \left\langle v, \frac{k-h}{|k-h|} \right\rangle \frac{k-h}{|k-h|}, \quad |P_{k-h}v|^2 = 1 - \left\langle v, \frac{k-h}{|k-h|} \right\rangle^2$$

Therefore $P_{k-h}v$ belongs to $\text{span}(k, v)$ if and only if

$$\begin{aligned} |P_{k-h}(v)|^2 &= \langle P_{k-h}(v), v \rangle^2 + \left\langle P_{k-h}(v), \frac{k}{|k|} \right\rangle^2 \\ &= |P_{k-h}(v)|^4 + \left\langle v, \frac{k-h}{|k-h|} \right\rangle^2 \left\langle \frac{k-h}{|k-h|}, \frac{k}{|k|} \right\rangle^2 \end{aligned}$$

Explicit calculations then lead to

$$\left\langle v, \frac{k-h}{|k-h|} \right\rangle^2 = \left\langle v, \frac{k-h}{|k-h|} \right\rangle^2 \left(\left\langle v, \frac{k-h}{|k-h|} \right\rangle^2 + \left\langle \frac{k-h}{|k-h|}, \frac{k}{|k|} \right\rangle^2 \right)$$

The equation is clearly satisfied if $\langle v, k-h \rangle = -\langle v, h \rangle = 0$, i.e. if $h \perp v$. Otherwise we can simplify and obtain

$$1 = \left\langle v, \frac{k-h}{|k-h|} \right\rangle^2 + \left\langle \frac{k-h}{|k-h|}, \frac{k}{|k|} \right\rangle^2$$

But this is equation (4.16) applied to $u = \frac{k-h}{|k-h|}$, which implies that $k-h \in \text{span}(k, v)$ and therefore also $h \in \text{span}(k, v)$. Therefore we can conclude that

$$P_{k-h}(v) \in \text{span}(k, v) \Leftrightarrow h \in (v^\perp \cup \text{span}(k, v))$$

In particular if we take $h \in \mathbb{Z}^3$ such that

$$h \notin \Gamma_k \cup v^\perp \cup \text{span}(k, v)$$

which is always possible, since the above set is the finite union of affine subspaces of dimension at most 2, then $k, v, P_{k-h}v$ are three independent elements of $\ker(A_k(t))$ and we can conclude. $\square$

Remark 4.25. It follows immediately that, if $y \neq 0$, then the corresponding solution $Y(t; y)$ is such that the associated covariance matrices $\{A_k(t)\}$ must have rank 2 for all k and $t > 0$.

We are now ready to prove the following result.

Theorem 4.26. The following hold:

- i) If $\Theta \neq \{0\}$, then $\widehat{\Theta} = H$.
- ii) If $\Gamma \neq \{0\}$, then $\widehat{\Gamma} = H$.

Proof. i) Suppose there exists a random initial data $y \in \Theta \setminus \{0\}$. In order to conclude, it suffices to show that $\widehat{\Theta}$ contains a basis of H . The matrices A_k associated to $Y(t; y)$ must satisfy system (4.13); by the previous result, we know that $rk(A_k(t)) = 2$ for all k and $t > 0$. Since $y \in \Theta$, this implies that

$$\lim_{t \rightarrow \infty} \sum_k \text{Tr}(A_k(t)) = 0 \quad (4.17)$$

We can shift the solution A_k by setting $\widetilde{A}_k(t) = A_k(t + s)$ for some $s > 0$ in order to find a solution of (4.13) satisfying condition (4.17) and such that its initial data $\widetilde{A}_k(0) = A_k(s)$ is made entirely of matrices of rank 2. From now on we will denote $\widetilde{A}_k$ by A_k for simplicity. We now construct a deterministic initial data $y \in H$ such that

$$A_k(0) = \Re(y_k \otimes \bar{y}_k) = \Re(y_k) \otimes \Re(y_k) + \Im(y_k) \otimes \Im(y_k) \quad \forall k \in \mathbb{Z} \setminus \{0\} \quad (4.18)$$

Since $A_k(0)$ is a nonnegative, symmetric matrix of rank 2, there exists two orthonormal eigenvectors $v_k^{(1)}, v_k^{(2)}$ with correspondent eigenvalues $\lambda_k^{(1)}, \lambda_k^{(2)} > 0$ such that

$$A_k(0) = \lambda_k^{(1)} v_k^{(1)} \otimes v_k^{(1)} + \lambda_k^{(2)} v_k^{(2)} \otimes v_k^{(2)}$$

Observe that $A_k = A_{-k}$, so that we can take $v_k^{(i)} = v_{-k}^{(i)}$, and $v_k^{(1)}, v_k^{(2)}$ form a basis of $k^\perp$. Take $J \subset \mathbb{Z}^3$ such that J and $-J$ form a partition of $\mathbb{Z}^3 \setminus \{0\}$ and define $y \in H$ by setting

$$y_k = \sqrt{\lambda_k^{(1)}} v_k^{(1)} + i \sqrt{\lambda_k^{(2)}} v_k^{(2)} \text{ for } k \in J, \quad y_k = \bar{y}_{-k} \text{ for } k \in -J \quad (4.19)$$

Then y is a well defined element of H , since

$$\sum_k |y_k|^2 = \sum_k (\lambda_k^{(1)} + \lambda_k^{(2)}) = \sum_k \text{Tr}(A_k(0)) < \infty$$

and it's easy to see that (4.18) holds. We now claim that, for any fixed k , the elements

$$z_k = \sqrt{\lambda_k^{(1)}} (v_k^{(1)} e_k + v_k^{(1)} e_{-k}), \quad \tilde{z}_k = \sqrt{\lambda_k^{(2)}} (i v_k^{(2)} e_k - i v_k^{(2)} e_{-k})$$

belong to $\widehat{\Theta}$. Indeed, it's immediate to check that the matrices $\{B_h(t)\}$ and $\{C_h(t)\}$ associated to z_k and $\tilde{z}_k$ are such that

$$B_h(0) \leq A_h(0), \quad C_h(0) \leq A_h(0) \quad \forall h$$

which implies by the comparison principle that the inequality holds for all $t \geq 0$ and therefore

$$\sum_h \text{Tr}(B_h(t)) \leq \sum_h \text{Tr}(A_h(t)) \rightarrow 0 \text{ as } t \rightarrow \infty$$

and similarly for C_h . This implies that $z_k, \tilde{z}_k \in \widehat{\Theta}$. The above reasoning also holds for

$$w_k = \sqrt{\lambda_k^{(2)}} (v_k^{(2)} e_k + v_k^{(2)} e_{-k}), \quad \tilde{w}_k = \sqrt{\lambda_k^{(1)}} (i v_k^{(1)} e_k - i v_k^{(1)} e_{-k}) \in \widehat{\Theta}$$

and this implies the conclusion, since $\{z_k, \tilde{z}_k, w, \tilde{w}_k\}_{k \in \mathbb{Z}^3 \setminus \{0\}}$ form a basis of H .

ii) Reasoning as in part i), starting from an element of Γ , we can construct $y \in H$ of the form (4.19) such that $y \in \widehat{\Gamma}$. As before, in order to conclude it suffices to show that $\widehat{\Gamma}$ contains a basis of H , since we know that it is a closed subspace of H . Now observe that, for fixed k , if we consider the elements of H given by $z_k = y_k e_k + y_{-k} e_{-k}$, $w_k = y - y_k$ then the matrices $\{B_h\}, \{C_h\}$ associated satisfy

$$A_h(0) = B_h(0) + C_h(0) \quad \forall h \in \mathbb{Z}^3 \setminus \{0\}$$

Linearity of system (4.13) and uniqueness of solutions imply that the above identity must also hold for all $t \geq 0$; taking the traces and summing over h we then obtain that

$$\mathbb{E}[|Y(t; y)|^2] = \mathbb{E}[|Y(t; z_k)|^2] + \mathbb{E}[|Y(t; w_k)|^2] \quad \forall t \geq 0$$

In particular, since no dissipation occurs on the l.h.s., neither can occur on the r.h.s., which implies that $z_k \in \widehat{\Gamma}$ for all k . Now observe that we can decompose z_k as $z_k = z_k^{(1)} + z_k^{(2)}$, where

$$z_k^{(1)} = \Re(y_k)(e_k + e_{-k}), \quad z_k^{(2)} = i\Im(y_k)(e_k - e_{-k})$$

and it still holds that

$$\Re(z_k \otimes \bar{z}_k) = \Re(z_k^{(1)} \otimes \bar{z}_k^{(1)}) + \Re(z_k^{(2)} \otimes \bar{z}_k^{(2)})$$

which implies, reasoning as before, that necessarily $z_k^{(1)}, z_k^{(2)} \in \widehat{\Gamma}$. Analogously it can be shown that

$$w_k^{(1)} = \Im(y_k)(e_k + e_{-k}), \quad w_k^{(2)} = i\Re(y_k)(e_k - e_{-k}) \in \widehat{\Theta}$$

which gives the conclusion as $\{z_k^{(1)}, z_k^{(2)}, w_k^{(1)}, w_k^{(2)}\}_{k \in \mathbb{Z}^3 \setminus \{0\}}$ form a basis of H . $\square$

It's clear that, if $\widehat{\Gamma} = H$ (resp. $\widehat{\Theta}$), then $\Gamma = L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ (resp. Θ). Moreover, reasoning as before, it's easy to extend the result to the nonlinear system (3.51). We would be tempted to conclude that either the energy is truly invariant under the dynamics or 0 is the unique fixed point, which asymptotically attracts all the solutions. Unfortunately, we are not able to prove that at least one between Γ and Θ is different from $\{0\}$; a priori the system could exhibit a third behaviour, in which the energy is never preserved but never fully dissipated and solutions only "dissipate partially". Intuitively this clashes with markovianity of the system - if there were a positive probability that some fixed amount of energy is dissipated between 0 and T , then by iterating it on the intervals $[nT, (n+1)T]$ it should follow that the energy goes to 0 as time goes to infinity; however, we haven't been able to make this argument rigorous. Still, we can try to collect information on the system under this third scenario, in the hope that it will lead to a contradiction in the future. Let us define the bilinear map

$$T : H \times H \rightarrow \mathbb{R}, \quad T(x, y) := \lim_{t \rightarrow \infty} \mathbb{E}[\langle Y(t; x), Y(t; y) \rangle]$$

where the limit is well defined since by linearity

$$\mathbb{E}[\langle Y(t; x), Y(t; y) \rangle] = \frac{1}{4} \left(\mathbb{E}[|Y(t; x+y)|^2] - \mathbb{E}[|Y(t; x-y)|^2] \right)$$

The map T is continuous since

$$|\mathbb{E}[\langle Y(t; x), Y(t; y) \rangle]| \leq \mathbb{E}[|Y(t; x)| |Y(t; y)|] \leq |x| |y| \quad \forall t \geq 0$$

By Riesz theorem, we can then regard T as a linear map from H to itself, by setting (with a slight abuse of notation) $\langle Tx, y \rangle = T(x, y)$ for all x and y . It's immediate to check that T is self-adjoint, positive definite and $\|T\| \leq 1$. The hypothesis $\hat{\Gamma} = \hat{\Theta} = \{0\}$ implies that

$$0 < \langle Tx, x \rangle < 1 \quad \forall x \in H \setminus \{0\} \quad (4.20)$$

We can also show that $\|T\| = 1$. To do so, we employ the following fact, which can be found in the proof of Theorem 4.26: for any $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, there exists $\tilde{y} \in H$ such that $\mathbb{E}[|Y(t; y)|^2]$ and $\mathbb{E}[|Y(t; \tilde{y})|^2]$ coincide for all $t \geq 0$. Now let y be any element of H ; by continuity of the map $t \mapsto \mathbb{E}[|Y(t; y)|^2]$, for any $\varepsilon > 0$ we can find t_ε such that

$$\mathbb{E}[|Y(t_\varepsilon; y)|^2] = (1 + \varepsilon) \lim_{t \rightarrow \infty} \mathbb{E}[|Y(t; y)|^2]$$

Since $Y(t_\varepsilon; y)$ is the initial data of the solution $Y(t_\varepsilon + \cdot; y)$ w.r.t. to the usual shifted Wiener process, there exists $\tilde{y}_\varepsilon \in H$ such that

$$\mathbb{E}[|Y(s; \tilde{y}_\varepsilon)|^2] = \mathbb{E}[|Y(t_\varepsilon + s; y)|^2] \quad \forall s \geq 0$$

which implies that

$$\frac{\langle T\tilde{y}_\varepsilon, \tilde{y}_\varepsilon \rangle}{|\tilde{y}_\varepsilon|^2} = \lim_{s \rightarrow \infty} \frac{\mathbb{E}[|Y(t_\varepsilon + s; y)|^2]}{\mathbb{E}[|Y(t_\varepsilon; y)|^2]} = \frac{1}{1 + \varepsilon}$$

and by the arbitrariness of ε we deduce that $\|T\| = 1$. Now let us define the following subspaces of H :

$$V_k = \left\{ ve_k + \bar{v}e_{-k} : v \in \mathbb{C}^3, \langle v, k \rangle = 0 \right\}, \quad k \in \mathbb{Z}^3 \setminus \{0\}$$

For each k , V_k is the subspace associated to the Fourier k -th mode; it's a collection of 4-dimensional real subspaces which gives an orthogonal decomposition of H . Again looking at the proof of Theorem 4.26, we can deduce that

$$\langle Tx, y \rangle = 0 \quad \text{for all } x \in V_k \text{ and } y \in V_h, \quad k \neq \pm h$$

which implies that each V_k is invariant under T . $T : V_k \rightarrow V_k$ is a symmetric, positive definite map, which is therefore diagonalisable. As this holds for any V_k , it follows that there exists an orthogonal basis of H made of eigenvectors $\{e_n\}$ of T , with associated eigenvalues $\{\lambda_n\} \subset (0, 1)$. Moreover, 1 must be an accumulation point of $\{\lambda_n\}$, since $\|T\| = 1$. Also observe that, since V_k are invariant under T , this operator commutes Δ ; we do not know however if it commutes with other operators playing a key role in the dynamics, as for example A . It follows from the definition of T that the functional $x \mapsto \langle Tx, x \rangle$ is invariant on average under the dynamics:

$$\mathbb{E}[\langle TY(t; x), Y(t; x) \rangle] = \lim_{s \rightarrow \infty} \mathbb{E}[|Y(t + s; x)|^2] = \langle Tx, x \rangle \quad \forall t \geq 0$$

Moreover, $\langle T\cdot, \cdot \rangle$ can be regarded as an inner product on H and so

$$\|x\|_T := \sqrt{\langle Tx, x \rangle} = |T^{1/2}x|$$

is a well defined norm on H . However pay attention that we do not know if H is an Hilbert space, namely if it is complete, w.r.t. $\|\cdot\|_T$. By the open mapping theorem, $(H, \|\cdot\|_T)$ is an Hilbert space if and only if there exists $c > 0$ such that $\|x\|_T \geq c|x|$, namely if $T^{1/2}$ is open or equivalently if $\inf_n \lambda_n > 0$. Observe that $T^{1/2}$ (resp. T) cannot be compact, since 1 is an accumulation point of $\{\lambda_n\}$. In any case, $\|\cdot\|_T^2$ can be interpreted as some kind of energy. This means that, under the hypothesis that $|\cdot|^2$ is not preserved and 0 is not attractive, we are still able to find a suitable energy which is preserved (on average) along the dynamics.

4.4 A vanishing noise limit

In this section, we are going to study what happens to the dynamics of the linear system (3.35) when we consider a suitable small noise limit. Instead of (3.35), we are now interested in considering the linear equation in H given by:

$$dY = f dt + \sqrt{\varepsilon} B^\alpha(Y) \circ dW$$

where the index B^α stresses the dependence on the fixed parameter $\alpha > 0$, which determines K^α . The above equation in Fourier components becomes the infinite system

$$dY_k = f_k dt - i\sqrt{\varepsilon} \sum_h \sigma_h |P_h(k)| P_k(Y_{k-h}) \circ dB_{h,k}$$

and can be written in Ito form as

$$dY_k = f_k dt - i\sqrt{\varepsilon} \sum_h \sigma_h |P_h(k)| P_k(Y_{k-h}) dB_{h,k} - \varepsilon \sum_h \sigma_h^2 |P_h(k)|^2 P_k(P_{k-h}(Y_k)) dt \quad (4.21)$$

The components Y_k also satisfy the usual conditions $Y_0 \equiv 0$, $\langle Y_k, k \rangle = 0$, $Y_{-k} = \bar{Y}_k$. Here ε is a positive real parameter and the dependence on α is given by the constants $\sigma_h = (1 + \alpha|h|^2)^{-1}$; system (4.21) would not be well defined for $\alpha = 0$, since the series

$$\sum_h |P_h(k)|^2 P_k(P_{k-h}(Y_k))$$

is not absolutely convergent. As in Chapter 3, it can be shown that for any initial data $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and function $f \in L^2(0, T; H)$ there exists a unique energy controlled solution of system (4.21); as usual, it follows from the energy inequality that

$$\sup_{t \in [0, T]} |Y(t)|_H^2 \leq C \left(|y|_H^2 + \int_0^T |f(s)|^2 ds \right) \quad \mathbb{P}\text{-a.s.} \quad (4.22)$$

where the constant C does not depend on the parameters ε and α chosen. We denote by $Y^{\alpha, \varepsilon}$ the unique weakly continuous, energy controlled solution of system (4.21) associated to y and f . As we are not able to define $Y^{0, \varepsilon}$, we investigate instead the following problem: is there a suitable definition of $\varepsilon = \varepsilon(\alpha)$ such that the sequence of solutions $Y^\alpha = Y^{\alpha, \varepsilon(\alpha)}$ admits a non trivial limit in some sense as $\alpha \rightarrow 0^+$?

Observe that the first term on the r.h.s. of (4.21) doesn't seem to pose problems, as it is uniformly

bounded, even without the coefficient $\sqrt{\varepsilon}$, thanks to (4.22). Instead for the second term, which become ill-defined for $\alpha = 0$, a natural way to make it at least uniformly bounded is setting

$$\varepsilon(\alpha) = \left(\sum_h \sigma_h(\alpha)^2 \right)^{-1} = \left(\sum_h \frac{1}{(1 + \alpha|h|^2)^2} \right)^{-1} \quad (4.23)$$

as in this way

$$\int_0^T \left| \int_0^t \varepsilon(\alpha) \sum_h \sigma_h^2 |P_h(k)|^2 P_k(P_{k-h}(Y_k(s))) ds \right| dt \leq T|k|^2 \int_0^T |Y_k(s)| ds \leq \tilde{C}|k|^2$$

for a suitable constant $\tilde{C}$ independent of α . Observe that by definition $\varepsilon(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0^+$, so that in the limit we expect the noise to disappear. However, this does not imply that the sequence $\{Y^\alpha\}$ will converge as $\alpha \rightarrow 0^+$ to the solution of (4.21) with $\alpha = \varepsilon = 0$, as we will see at the end of the section. Let us define

$$M_k^\alpha := \varepsilon(\alpha) \sum_h \sigma_h^2 |P_h(k)|^2 P_k P_{k-h} P_k$$

so that as usual system (4.21), with the choice $\varepsilon = \varepsilon(\alpha)$, can be written as

$$dY^\alpha = f_k dt - i\sqrt{\varepsilon(\alpha)} \sum_h \sigma_h |P_h(k)| P_k(Y_{k-h}^\alpha) dB_{h,k} - M_k^\alpha(Y_k) dt$$

We want to understand if the matrices M_k^α admit limit as $\alpha \rightarrow 0^+$, namely if

$$M_k := \lim_{\alpha \rightarrow 0} M_k^\alpha = \lim_{\alpha \rightarrow 0} \left(\sum_h \sigma_h^2 \right)^{-1} \sum_h \sigma_h^2 |P_h(k)|^2 P_k P_{k-h} P_k \quad (4.24)$$

are well defined for all k ; we would also like to compute them explicitly. To this end, we need the following lemma.

Lemma 4.27. The following hold:

- i) Let $\{x_n\}_n$ be a real sequence such that $x_n \rightarrow x \in \mathbb{R}$; consider, for any $\alpha > 0$, a non negative sequence $\{p_n(\alpha)\}_n$ such that: for fixed α , $\sum_n p_n(\alpha) = 1$; for fixed n , $p_n(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0^+$. Then

$$\lim_{\alpha \rightarrow 0^+} \sum_n p_n(\alpha) x_n = x$$

- ii) Let $\{x_n\}_{n \geq 1}$ be as in i) and consider a non negative collection $\{p_n(\alpha)\}_{n \geq 1}$ such that: for fixed α , $\sum_n p_n(\alpha) < \infty$; for fixed n , $p_n(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0^+$. Then

$$\lim_{\alpha \rightarrow 0^+} \left(\sum_n p_n(\alpha) \right)^{-1} \left(\sum_n p_n(\alpha) x_n \right) = x$$

- iii) Let $\{p_n(\alpha)\}$ be as in ii) and such that, for any fixed $\alpha > 0$, the sequence $\{p_n(\alpha)\}_n$ is decreasing; let $\{x_n\}_n$ be a bounded sequence such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N x_n = x$$

Then

$$\lim_{\alpha \rightarrow 0^+} \left(\sum_n p_n(\alpha) \right)^{-1} \left(\sum_n p_n(\alpha) x_n \right) = x$$

Proof. i) For any $\varepsilon > 0$, let $n(\varepsilon)$ be such that, for all $n \geq n(\varepsilon)$, $x_n \leq x + \varepsilon$. Then

$$\sum_n p_n(\alpha) x_n \leq \sum_{k=1}^{n(\varepsilon)} p_k(\alpha) x_k + (x + \varepsilon) \sum_{n > n(\varepsilon)} p_n(\alpha) \leq \sum_{k=1}^{n(\varepsilon)} p_k(\alpha) x_k + x + \varepsilon$$

Taking the limit as $\alpha \rightarrow 0^+$ we obtain

$$\limsup_{\alpha \rightarrow 0^+} \sum_n p_n(\alpha) x_n \leq x + \varepsilon$$

and with the same line of reasoning

$$\liminf_{\alpha \rightarrow 0^+} \sum_{n \geq 1} p_n(\alpha) x_n \geq x - \varepsilon$$

and so by the arbitrariness of $\varepsilon > 0$ we conclude. For part ii), consider

$$\tilde{p}_n(\alpha) := \left(\sum_k p_k(\alpha) \right)^{-1} p_n(\alpha)$$

It follows from the hypothesis on $p_n(\alpha)$ that $\tilde{p}_n(\alpha)$ satisfies assumptions of part i) and the conclusion follows. For part iii), consider $y_n := \frac{1}{n} \sum_{k=1}^n x_k$, so that $y_n \rightarrow x$; define $\tilde{p}_n(\alpha)$ as in ii) and

$$\hat{p}_n(\alpha) = n(\tilde{p}_n(\alpha) - \tilde{p}_{n+1}(\alpha))$$

For fixed $\alpha > 0$, $\{\hat{p}_n(\alpha)\}_{n \geq 1}$ is non negative, since $p_n(\alpha)$ is decreasing; for fixed n , $\hat{p}_n(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0^+$ since $\tilde{p}_n(\alpha)$ does. Moreover for fixed α it holds

$$\sum_{n=1}^{\infty} \hat{p}_n(\alpha) = \sum_{n=1}^{\infty} n(\tilde{p}_n(\alpha) - \tilde{p}_{n+1}(\alpha)) = \sum_{n=1}^{\infty} \sum_{k=1}^n (\tilde{p}_n(\alpha) - \tilde{p}_{n+1}(\alpha)) = \sum_k \tilde{p}_k(\alpha) = 1$$

Therefore $\{\hat{p}_n(\alpha)\}$ satisfies the assumption of part i). Observe that

$$\sum_{n=1}^{\infty} y_n \hat{p}_n(\alpha) = \sum_{n=1}^{\infty} n y_n (\tilde{p}_n(\alpha) - \tilde{p}_{n+1}(\alpha)) = \sum_{n=1}^{\infty} \sum_{k=1}^n x_k (\tilde{p}_n(\alpha) - \tilde{p}_{n+1}(\alpha)) = \sum_{k=1}^{\infty} x_k \tilde{p}_k(\alpha)$$

But then it follows from part i) that

$$\lim_{\alpha \rightarrow 0^+} \sum_{k=1}^{\infty} x_k \tilde{p}_k(\alpha) = \lim_{\alpha \rightarrow 0^+} \sum_{n=1}^{\infty} y_n \hat{p}_n(\alpha) = x$$

□

Now consider the sequence $\{\sigma_h(\alpha)\}$, with the vectors $h \in \mathbb{Z}^3 \setminus \{0\}$ ordered in an increasing way w.r.t. the norm (i.e. such that $n \geq m$ implies $|h_n| \geq |h_m|$). Then $\{\sigma_h(\alpha)\}$ with this ordering satisfies assumption of part iii) of Lemma 4.27; defined $\Gamma_n = \{h \in \mathbb{Z}^3 : 0 < |h| \leq n\}$, if we are able to show that the limits

$$N_k = \lim_{n \rightarrow \infty} \frac{1}{|\Gamma_n|} \sum_{h \in \Gamma_n} |P_h(k)|^2 P_{k-h}$$

exist, for all k , then by Lemma 4.27 we must have

$$M_k = P_k N_k P_k$$

(The lemma was stated for real sequences for simplicity, but it can be adapted to matrices by looking at each component). For any $u \in \mathbb{R}^3$,

$$P_{k-h}(u) = u - \frac{\langle u, k-h \rangle^2}{|k-h|^2} = u - \frac{\langle u, h \rangle^2}{|h|^2} + o(1) = P_h(u) + o(1) \quad \text{as } h \rightarrow \infty$$

Therefore the limit N_k exists if and only if the following limit exists, and in that case they must be equal:

$$N_k = \lim_{n \rightarrow \infty} \frac{1}{|\Gamma_n|} \sum_{h \in \Gamma_n} |P_h(k)|^2 P_h$$

Since P_h is a symmetric matrix, the limit N_k , if it exists, must be symmetric as well; therefore it is characterized uniquely by the values $\langle a, N_k a \rangle$, for a varying in $\mathbb{R}^3$, and it must hold

$$\langle a, N_k a \rangle = \lim_{n \rightarrow \infty} \frac{1}{|\Gamma_n|} \sum_{h \in \Gamma_n} |P_h(k)|^2 |P_h(a)|^2 \quad (4.25)$$

Observe that, by definition of projection, $P_h = P_{h/|h|}$, where $h/|h| \in S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$.

We now exploit the following general fact.

Theorem 4.28 (Equidistribution on the sphere). The sequence $\{h/|h|\}_{h \in \mathbb{Z}^3 \setminus \{0\}}$, ordered in an increasing way w.r.t. $|h|$, is equidistributed on the sphere. Namely, for any continuous $f : S^2 \rightarrow \mathbb{R}^3$ it holds

$$\lim_{n \rightarrow \infty} \frac{1}{|\Gamma_n|} \sum_{h \in \Gamma_n} f\left(\frac{h}{|h|}\right) = \frac{1}{4\pi} \int_{S^2} f(x) d\sigma(x)$$

Applying Theorem 4.28 we obtain

$$\langle a, N_k a \rangle = \frac{1}{4\pi} \int_{S^2} |P_x(k)|^2 |P_x(a)|^2 d\sigma(x) \quad (4.26)$$

Since

$$|P_x(k)|^2 = |k|^2 - \langle k, x \rangle^2, \quad |P_x(a)|^2 = |a|^2 - \langle a, x \rangle^2$$

equation (4.26) becomes

$$\langle a, N_k a \rangle = |k|^2 |a|^2 - \frac{|k|^2}{4\pi} \int_{S^2} \langle a, x \rangle^2 d\sigma(x) - \frac{|a|^2}{4\pi} \int_{S^2} \langle k, x \rangle^2 d\sigma(x) + \frac{1}{4\pi} \int_{S^2} \langle k, x \rangle^2 \langle a, x \rangle^2 d\sigma(x) \quad (4.27)$$

We can now compute separately the terms appearing in (4.27). Observe that by symmetry

$$\frac{1}{4\pi} \int_{S^2} x_i x_j d\sigma = \frac{1}{3} \delta_{ij}$$

So that

$$\frac{1}{4\pi} \int_{S^2} \langle a, x \rangle^2 d\sigma(x) = \frac{1}{4\pi} \sum_{i,j} a_i a_j \int_{S^2} x_i x_j d\sigma(x) = \frac{1}{3} \sum_i a_i^2 = \frac{|a|^2}{3}$$

and similarly

$$\frac{1}{4\pi} \int_{S^2} \langle k, x \rangle^2 d\sigma(x) = \frac{|k|^2}{3}$$

It only remains to compute the last term in (4.27). Again by symmetry it holds

$$\int_{S^2} \langle k, x \rangle^2 x_i x_j d\sigma = 0 = \int_{S^2} x_i x_j x_l^2 d\sigma \quad \forall i \neq j$$

and so

$$\begin{aligned} \frac{1}{4\pi} \int_{S^2} \langle x, k \rangle^2 \langle x, a \rangle^2 d\sigma &= \frac{1}{4\pi} \sum_{i,j} a_i a_j \int_{S^2} \langle x, k \rangle^2 x_i x_j d\sigma = \frac{1}{4\pi} \sum_j a_j^2 \int_{S^2} \langle x, k \rangle^2 x_j^2 d\sigma \\ &= \frac{1}{4\pi} \sum_{i,j,l} k_i k_l a_j^2 \int_{S^2} x_i x_l x_j^2 d\sigma = \frac{1}{4\pi} \sum_{i,j} k_i^2 a_j^2 \int_{S^2} x_i^2 x_j^2 d\sigma \end{aligned}$$

By symmetry it holds:

$$\begin{aligned} \frac{1}{4\pi} \int_{S^2} x_1^4 d\sigma &= \frac{1}{4\pi} \int_{S^2} x_2^4 d\sigma = \frac{1}{4\pi} \int_{S^2} x_3^4 d\sigma \\ \frac{1}{4\pi} \int_{S^2} x_1^2 x_2^2 d\sigma &= \frac{1}{4\pi} \int_{S^2} x_2^2 x_3^2 d\sigma = \frac{1}{4\pi} \int_{S^2} x_1^2 x_3^2 d\sigma \\ 1 &= \frac{1}{4\pi} \int_{S^2} (x_1^2 + x_2^2 + x_3^2)^2 d\sigma = \frac{3}{4\pi} \int_{S^2} x_1^4 d\sigma + \frac{6}{4\pi} \int_{S^2} x_1^2 x_2^2 d\sigma \end{aligned}$$

Finally by explicit calculations we find

$$\frac{1}{4\pi} \int_{S^2} x_1^4 d\sigma = \frac{1}{5}, \quad \frac{1}{4\pi} \int_{S^2} x_1^2 x_2^2 d\sigma = \frac{1}{15}$$

Therefore

$$\langle a, N_k a \rangle = \frac{1}{3} |k|^2 |a|^2 + \sum_i \left(\sum_j k_j^2 \frac{1}{4\pi} \int_{S^2} x_i^2 x_j^2 d\sigma \right) a_i^2$$

that is, N_k is diagonal with diagonal entries given by

$$(d_1, d_2, d_3) = \left(\frac{1}{3} |k|^2 + \frac{1}{5} k_1^2 + \frac{1}{15} k_2^2 + \frac{1}{15} k_3^2, \frac{1}{3} |k|^2 + \frac{1}{15} k_1^2 + \frac{1}{5} k_2^2 + \frac{1}{15} k_3^2, \frac{1}{3} |k|^2 + \frac{1}{15} k_1^2 + \frac{1}{15} k_2^2 + \frac{1}{5} k_3^2 \right)$$

which can be written as

$$N_k = \frac{2}{5} |k|^2 I + \frac{2}{15} \begin{pmatrix} k_1^2 & 0 & 0 \\ 0 & k_2^2 & 0 \\ 0 & 0 & k_3^2 \end{pmatrix} \quad (4.28)$$

Therefore we have finally found that

$$\lim_{\alpha \rightarrow 0^+} \left(\sum_h \sigma_h^2 \right)^{-1} \sum_h \sigma_h^2 |P_h(k)|^2 P_{k-h} = N_k = \frac{2}{5} |k|^2 I + \frac{2}{15} \begin{pmatrix} k_1^2 & 0 & 0 \\ 0 & k_2^2 & 0 \\ 0 & 0 & k_3^2 \end{pmatrix} \quad (4.29)$$

and a similar expression holds for $M_k = P_k N_k P_k$. Observe that, since in finite dimensional spaces all norms are equivalent, we have obtained that, for any $k \in \mathbb{Z}^3 \setminus \{0\}$,

$$\lim_{\alpha \rightarrow 0^+} \|M_k^\alpha - M_k\| = 0$$

where by $\|\cdot\|$ we are denoting the operator norm for 3×3 matrices. The operator $M : D(M) \rightarrow H$ defined in Fourier components by

$$g = \sum_k g_k e_k \mapsto Mg := \sum_k M_k g_k e_k$$

corresponds to

$$M = -\frac{2}{5}\Delta - \frac{2}{15}P(\partial_1^2, \partial_2^2, \partial_3^2)$$

where P is the Leray-Helmholtz projector and the last expression must be interpreted as

$$(\partial_1^2, \partial_2^2, \partial_3^2)(g_1, g_2, g_3) = (\partial_1^2 g_1, \partial_2^2 g_2, \partial_3^2 g_3)$$

We are now ready to prove the main result of this section.

Theorem 4.29. Let $y \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ and $f \in L^2(0, T; H)$ and denote by Y^α the solution of system (4.21) with initial data y , external forces f , parameter α and $\varepsilon(\alpha)$ given by (4.23). Then the sequence $\{Y^\alpha\}_{\alpha>0}$ converges weakly in $L^2(\Omega_T; H)$ as $\alpha \rightarrow 0^+$ to the process Y given by

$$Y_k = e^{-tM_k} y_k + \int_0^t e^{(s-t)M_k} f_k ds \quad (4.30)$$

Proof. The sequence $\{Y^\alpha\}_{\alpha>0}$ is uniformly bounded in $L^2(\Omega_T; H)$ thanks to the energy bound (4.22). Therefore we can extract a subsequence (not relabelled for simplicity) such that $Y^\alpha \rightharpoonup Y$ weakly for some $Y \in L^2(\Omega_T; H)$. Recall that Y^α satisfies

$$Y_k^\alpha(t) = y_k + \int_0^t f_k(s) ds - i\sqrt{\varepsilon(\alpha)} \sum_h \sigma_h |P_h(k)| \int_0^t P_k(Y_{k-h}^\alpha(s)) dB_{h,k}(s) - \int_0^t M_k^\alpha(Y_k^\alpha(s)) ds$$

We now want to study the weak convergence of the terms on the r.h.s. Recall that $\varepsilon(\alpha) \rightarrow 0$ as $\alpha \rightarrow 0^+$, while

$$\begin{aligned} & \int_0^T \mathbb{E} \left[\left| \sum_h \sigma_h |P_h(k)| \int_0^t P_k(Y_{k-h}^\alpha(s)) dB_{h,k}(s) \right|^2 \right] dt \\ &= 2 \int_0^T \int_0^t \sum_h \sigma_h^2 |P_h(k)|^2 \mathbb{E}[|P_k(Y_{k-h}^\alpha(s))|^2] ds dt \\ &\leq 2T|k|^2 \int_0^T \sum_h \mathbb{E}[|Y_{k-h}^\alpha(s)|^2] ds = 2T|k|^2 \|Y^\alpha\|_{L^2(\Omega_T; H)}^2 \leq C \end{aligned}$$

which implies that

$$i\sqrt{\varepsilon(\alpha)} \sum_h \sigma_h |P_h(k)| \int_0^t P_k(Y_{k-h}^\alpha(s)) dB_{h,k}(s) \rightarrow 0 \text{ strongly in } L^2(\Omega_T; H) \text{ as } \alpha \rightarrow 0^+$$

Now let us write the last term as

$$\int_0^t M_k^\alpha(Y_k^\alpha(s)) ds = \int_0^t M_k(Y_k^\alpha(s)) ds + \int_0^t (M_k^\alpha - M_k)(Y_k^\alpha(s)) ds$$

Since $Y^\alpha \rightharpoonup Y$, $Y_k^\alpha \rightharpoonup Y_k$ in $L^2(\Omega_T; \mathbb{C}^3)$. The operator from $L^2(\Omega_T; \mathbb{C}^3)$ to itself defined by

$$Z_k \mapsto \int_0^\cdot M_k(Z_k(s)) ds$$

is linear and continuous, since

$$\int_0^T \mathbb{E} \left[\left| \int_0^t M_k(Z_k(s)) ds \right|^2 \right] dt \leq T^2 \|M_k\|^2 \int_0^T \mathbb{E}[|Z_k(s)|^2] ds$$

thus it's also weakly continuous. Instead for the other term it holds

$$\int_0^T \mathbb{E} \left[\left| \int_0^t (M_k^\alpha - M_k)(Y_k^\alpha(s)) ds \right|^2 \right] \leq T^2 \|M_k^\alpha - M_k\|^2 \int_0^T \mathbb{E}[|Y_k^\alpha(s)|^2] ds \leq C \|M_k^\alpha - M_k\|^2 \rightarrow 0$$

which implies that

$$\int_0^\cdot M_k^\alpha(Y_k^\alpha(s)) ds \rightharpoonup \int_0^\cdot M_k(Y_k(s)) ds \text{ weakly in } L^2(\Omega_T; \mathbb{C}^3)$$

Therefore the limit Y must satisfy the system of equation

$$Y_k(t) = y + \int_0^t f_k(s) ds - \int_0^t M_k(Y_k(s)) ds \quad (4.31)$$

whose unique solution is given by (4.30). Since the reasoning holds for any subsequence of $\{Y^\alpha\}_{\alpha>0}$, we can conclude that the entire family converges weakly in $L^2(\Omega_T; H)$ to Y . $\square$

Remark 4.30. In the case of a deterministic initial data $y \in H$, the limit Y is a deterministic function which solves the (uncoupled) system of equations

$$\dot{Y}_k = f_k - M_k(Y_k)$$

which correspond in the Fourier space to the partial differential equation

$$\partial_t v - \frac{2}{5} \Delta v - \frac{2}{15} (\partial_1^2, \partial_2^2, \partial_3^2) v + \nabla p = f$$

Observe that $M_k \geq \frac{2}{5} |k|^2 I$ on $k^\perp$, therefore the solution v of the above equation is smooth at all stricly positive times.

We can give an alternative version of the previous result, which is the following.

Theorem 4.31. Let $y, f, \{Y^\alpha\}$ and Y as above. Then there exists a sequence $\alpha_n \rightarrow 0^+$ s.t.

$$Y^{\alpha_n} \rightharpoonup Y \text{ weakly in } L^2(0, T; H) \text{ } \mathbb{P}\text{-a.s.}$$

Proof. It follows from the energy bound (4.22) and the computations made in the proof of Theorem 4.29 that we can consider a sequence $\alpha_n \rightarrow 0^+$ and a set $\Gamma \subset \Omega$ such that $\mathbb{P}(\Gamma) = 1$ and for all $\omega \in \Gamma$ it holds

$$\sqrt{\varepsilon(\alpha_n)} \sum_h \sigma_h |P_h(k)| \int_0^\cdot P_k(Y_{k-h}^{\alpha_n}(\omega, s)) dB_{h,k}(\omega, s) \rightarrow 0 \text{ in } L^2(0, T; H) \text{ as } n \rightarrow \infty$$

$$\sup_n \sup_{t \in [0, T]} |Y^{\alpha_n}(\omega, t)|_H^2 \leq C$$

Now let us fix $\omega \in \Gamma$. The second equation implies that the sequence $\{Y^{\alpha_n}(\omega, \cdot)\}$ is bounded in $L^2(0, T; H)$. Therefore we can extract a subsequence (not relabelled for simplicity) such that it admits weak limit, say $Z \in L^2(0, T; H)$. Recall that, as Y^{α_n} are solutions of (4.21), it holds

$$\begin{aligned} Y_k^{\alpha_n}(\omega, \cdot) = & y(\omega) + \int_0^\cdot f_k(s) ds - i\sqrt{\varepsilon(\alpha_n)} \sum_h \sigma_h |P_h(k)| \int_0^\cdot P_k(Y_{k-h}^{\alpha_n}(\omega, s)) dB_{h,k}(\omega, s) \\ & - \int_0^\cdot M_k^{\alpha_n}(Y_k^{\alpha_n}(\omega, s)) ds \end{aligned}$$

The first two terms on the r.h.s. do not depend on n , so there is no problem with them when taking the limit. As for the third term, it goes to 0 as $\omega \in \Gamma$; it remains to study the last term. As in the previous proof, we write

$$\int_0^\cdot M_k^{\alpha_n}(Y_k^{\alpha_n}(\omega, s)) ds = \int_0^\cdot M_k(Y_k^{\alpha_n}(\omega, s)) ds + \int_0^\cdot (M_k^{\alpha_n} - M_k)(Y_k^{\alpha_n}(\omega, s)) ds \quad (4.32)$$

Regarding the first term on the r.h.s., since $Y_k^{\alpha_n}(\omega, \cdot) \rightharpoonup Z_k$ weakly in $L^2(0, T; \mathbb{C}^3)$ and the map from $L^2(0, T; \mathbb{C}^3)$ to itself given by

$$X \mapsto \int_0^\cdot M_k(X(s)) ds$$

is linear and strongly continuous, hence weakly continuous, it follows that

$$\int_0^\cdot M_k(Y_k^{\alpha_n}(\omega, s)) ds \rightharpoonup \int_0^\cdot M_k(Z(s)) ds \text{ weakly in } L^2(0, T; \mathbb{C}^3)$$

As for the second term, since $\omega \in \Gamma$,

$$\left\| \int_0^\cdot (M_k^{\alpha_n} - M_k)(Y_k^{\alpha_n}(\omega, s)) ds \right\|_{L^2(0, T; H)}^2 \leq CT^2 \|M_k^{\alpha_n} - M_k\|^2 \rightarrow 0 \text{ as } n \rightarrow \infty$$

Therefore, passing to the limit as $n \rightarrow \infty$, Z must satisfy

$$Z_k(t) = y(\omega) + \int_0^t f_k(s) ds - \int_0^t M_k(Z_k(s)) ds \quad t \in [0, T], \quad k \in \mathbb{Z}^3$$

and so $Z = Y(\omega, \cdot)$. Since the reasoning holds for any subsequence of $\{Y^{\alpha_n}(\omega, \cdot)\}$, it follows that $Y^{\alpha_n}(\omega, \cdot) \rightharpoonup Y(\omega, \cdot)$ weakly in $L^2(0, T; H)$. As this holds for any $\omega \in \Gamma$, we obtain the conclusion. $\square$

Remark 4.32. It can be shown with similar techniques that, for fixed y and f , the map from $(0, +\infty)$ to $L^2(\Omega_T; H)$, that associates to α the solution Y^α , is weakly continuous. It is then possible to show, similarly to the last part of the proof of Theorem 3.13, that the following enhanced version of the energy inequality holds:

$$\sup_{\alpha > 0} \sup_{t \in [0, T]} \left[\frac{1}{2} |Y^\alpha(t)|_H^2 - \int_0^t \langle f(s), Y^\alpha(s) \rangle ds \right] \leq \frac{1}{2} |y|_H^2 \quad \mathbb{P}\text{-a.s.}$$

In particular, if $y \in L^\infty(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ (for example if y is deterministic), then there exists a constant K such that

$$\sup_{\alpha > 0} \|Y^\alpha\|_{L^2(0, T; H)} \leq K \quad \mathbb{P}\text{-a.s.} \quad (4.33)$$

Recall that $B(0, K) := \{g \in L^2(0, T; H) \mid \|g\|_{L^2(0, T; H)} \leq K\}$, endowed with the weak topology τ_w of $L^2(0, T; H)$, is a metrizable compact space; it will be denoted by $\tilde{H}$. It follows from (4.33) that we can consider Y^α as $\tilde{H}$ -valued random variables. Then the last result can be rephrased as the fact that there exists a sequence $\alpha_n \rightarrow 0^+$ such that $Y^{\alpha_n} \rightarrow Y$ in $\tilde{H}$ $\mathbb{P}$ -a.s. Moreover, it can be shown similarly that any sequence $\alpha_n \rightarrow 0^+$ admits a subsequence with the above property. We can therefore conclude that the family $\{Y^\alpha\}_{\alpha>0}$ converges in probability in $\tilde{H}$ to Y as $\alpha \rightarrow 0^+$.

Remark 4.33. In all the computation we have taken $\sigma_h = (1 + \alpha|h|^2)^{-1}$ as these are the coefficients related to the operator K^α . However all the computations above hold for any sequence $\{\sigma_h(\alpha)\}_{h \in \mathbb{Z}^3}$ such that the hypothesis of part iii) of Lemma 4.27 hold, namely such that

- $\sum_h \sigma_h^2(\alpha) < \infty$ for all $\alpha > 0$
- For α fixed, the sequence $\{\sigma_h(\alpha)\}_h$ is decreasing with respect to its norm
- For α fixed, $\sigma_h(\alpha) \rightarrow 0$ as $h \rightarrow \infty$
- For h fixed, $\sigma_h(\alpha) \rightarrow 1$ as $\alpha \rightarrow 0^+$

The only difference in the calculations is that $\varepsilon(\alpha)$ must be defined accordingly to (4.23), as a function of the chosen $\{\sigma_h(\alpha)\}$; instead the limit matrices M_k and the weak convergence to Y given by (4.30) remain unchanged. This suggest that the limit Y has some kind of universality. An example of a different choice of $\sigma_h(\alpha)$ we could have used is

$$\sigma_h = \frac{1}{1 + \alpha|h|^{2\beta}}, \quad \beta > \frac{3}{4}$$

which correspond to the operator $K = (I - \alpha\Delta^\beta)^{-1}$, Δ^β being the fractional Laplacian.

Remark 4.34. The system considered in this section is fairly simple and the techniques employed do not allow a generalization of the same results to the nonlinear case; still, it provides an easy example of a family of systems which are formally energy preserving whose associated solutions converge to the solution of a PDE in which a clearly dissipative term, namely the operator M , appears. As the convergence is weak, we cannot deduce anything about the real behaviour of the energy of the solutions of systems (4.21) for α small; yet it gives another reason to believe that anomalous dissipation might occur.

Appendix A

Nuclear and Hilbert-Schmidt operators

Definition A.1. Let E, G be Banach spaces; an element $T \in L(E, G)$ is said to be a **nuclear** or **trace class operator** if there exist two sequences $\{a_j\} \subset G, \{\varphi_j\} \subset E^*$ such that

$$\sum_{j=1}^{\infty} \|a_j\| \|\varphi_j\| < \infty$$

and T has representation

$$Tx = \sum_{j=1}^{\infty} a_j \varphi_j(x) \quad \forall x \in E$$

Remark A.2. The series $\sum_j a_j \varphi_j$ is absolutely convergent in the operator norm, so the sequence $T_n = \sum_{j=1}^n a_j \varphi_j$ converges to T in $L(E, G)$. In particular T is the limit of a sequence of finite rank operators and so it's a compact operator.

The spaces of all nuclear operators from E into G , endowed with the norm

$$\|T\|_1 = \inf \left\{ \sum_{j=1}^{\infty} \|a_j\| \|\varphi_j\| < \infty : Tx = \sum_{j=1}^{\infty} a_j \varphi_j(x) \right\}$$

is a Banach space and will be denoted as $L_1(E, G)$; $L_1(E, E)$ is denoted by $L_1(E)$.

Remark A.3. Let K be another Banach space; if $T \in L_1(E, G)$ and $S \in L(G, K)$, then $ST \in L_1(E, K)$ and $\|ST\|_1 \leq \|S\| \|T\|_1$. In fact, if $Tx = \sum_j a_j \varphi_j(x)$ with $\sum_j \|a_j\| \|\varphi_j\| < \infty$, then $STx = \sum_j Sa_j \varphi_j(x)$ with

$$\sum_j \|Sa_j\| \|\varphi_j\| \leq \sum_j \|S\| \|a_j\| \|\varphi_j\| \leq \|S\| \sum_j \|a_j\| \|\varphi_j\| < \infty$$

and taking the infimum we obtain $\|ST\|_1 \leq \|S\| \|T\|_1$.

Let now H be a separable Hilbert space; by the Riesz representation theorem, $T \in L_1(H)$ if and only if there exist two sequences $\{a_j\}, \{b_j\} \subset H$ such that $Tx = \sum_j a_j \langle b_j, x \rangle$ and $\sum_j \|a_j\| \|b_j\| < \infty$. Let $\{e_k\}$ be a complete orthonormal system in H ; for $T \in L_1(H)$, we define the **trace** of T as

$$\text{Tr } T = \sum_{k=1}^{\infty} \langle Te_k, e_k \rangle$$

Proposition A.4. If $T \in L_1(H)$, then $\text{Tr } T$ is a well defined number, independent of the choice of the orthonormal basis $\{e_k\}$.

Proof. Let $\{a_j\}, \{b_j\} \subset H$ be such that $Tx = \sum_j a_j \langle b_j, x \rangle$. Then

$$\langle Te_k, e_k \rangle = \sum_{j=1}^{\infty} \langle a_j, e_k \rangle \langle b_j, e_k \rangle$$

Moreover

$$\begin{aligned} \sum_{k=1}^{\infty} |\langle Te_k, e_k \rangle| &= \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} \langle a_j, e_k \rangle \langle b_j, e_k \rangle \right| \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\langle a_j, e_k \rangle \langle b_j, e_k \rangle| \\ &\leq \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |\langle a_j, e_k \rangle|^2 \right)^{1/2} \left(\sum_{k=1}^{\infty} |\langle b_j, e_k \rangle|^2 \right)^{1/2} = \sum_{j=1}^{\infty} \|a_j\| \|b_j\| < \infty \end{aligned}$$

And so $\text{Tr } T$ is well defined. Finally

$$\sum_{k=1}^{\infty} \langle Te_k, e_k \rangle = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \langle a_j, e_k \rangle \langle b_j, e_k \rangle = \sum_{j=1}^{\infty} \langle a_j, b_j \rangle$$

therefore $\text{Tr } T$ does not depend on the choice of $\{e_k\}$. $\square$

Remark A.5. It follows that for all $T \in L_1(H)$, $|\text{Tr } T| \leq \|T\|_1$: for any $\{a_j\}, \{b_j\}$ such that $Tx = \sum_j a_j \langle b_j, x \rangle$ and $\sum_j \|a_j\| \|b_j\| < \infty$ we have

$$|\text{Tr } T| \leq \sum_{j=1}^{\infty} |\langle a_j, b_j \rangle| \leq \sum_{j=1}^{\infty} \|a_j\| \|b_j\|$$

and taking the infimum we get the conclusion.

Proposition A.6. If $T \in L_1(H)$ and $S \in L(H)$, then $TS, ST \in L_1(H)$ and $\text{Tr } TS = \text{Tr } ST$.

Proof. It suffices to show the equality. If $Tx = \sum_j a_j \langle b_j, x \rangle$, then $STx = \sum_{j=1}^{\infty} Sa_j \langle b_j, x \rangle$, $TSx = \sum_{j=1}^{\infty} a_j \langle b_j, Sx \rangle = \sum_{j=1}^{\infty} a_j \langle S^* b_j, x \rangle$ and

$$\text{Tr } TS = \sum_{j=1}^{\infty} \langle a_j, S^* b_j \rangle = \sum_{j=1}^{\infty} \langle Sa_j, b_j \rangle = \text{Tr } ST$$

$\square$

Proposition A.7. A symmetric nonnegative operator $T \in L(H)$ is of trace class if and only if for an orthonormal basis $\{e_k\}$ on H

$$\sum_{j=1}^{\infty} \langle Te_j, e_j \rangle < \infty$$

Moreover in this case $\text{Tr } T = \|T\|_1$.

Proof. We will first show that T is compact. Let $T^{1/2}$ denote the nonnegative square root of T (any symmetric nonnegative operator on an Hilbert space admits a square root, see for example [18], Theorem 9.4-2, p.476). Then

$$T^{1/2}x = \sum_{j=1}^{\infty} \langle T^{1/2}x, e_j \rangle e_j$$

and

$$\begin{aligned} \left| T^{1/2}x - \sum_{j=1}^N \langle T^{1/2}x, e_j \rangle e_j \right|^2 &= \sum_{k=N+1}^{\infty} |\langle T^{1/2}x, e_k \rangle|^2 \\ &\leq |x|^2 \sum_{k=N+1}^{\infty} |T^{1/2}e_k|^2 = |x|^2 \sum_{k=N+1}^{\infty} \langle Te_k, e_k \rangle \end{aligned}$$

So $T^{1/2}$ is the limit in the operator norm of a sequence of finite rank operators. Therefore $T^{1/2}$ is compact and $T = T^{1/2}T^{1/2}$ is a compact, symmetric operator. Then by the spectral theorem there exists an orthonormal basis $\{f_k\}$ of eigenvectors of T with eigenvalues $\lambda_k \geq 0$,

$$Tx = \sum_{k=1}^{\infty} \lambda_k \langle x, f_k \rangle f_k \quad (\text{A.1})$$

Since

$$\langle Te_j, e_j \rangle = \sum_{k=1}^{\infty} \lambda_k \langle e_j, f_k \rangle^2$$

we have

$$\text{Tr } T = \sum_{j=1}^{\infty} \langle Te_j, e_j \rangle = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \lambda_k \langle e_j, f_k \rangle^2 = \sum_{k=1}^{\infty} \lambda_k < \infty$$

From this and the expansion (A.1) we conclude that T is nuclear and $\text{Tr } T \geq |T|_1$. But then by Remark A.5 we get the conclusion. $\square$

Definition A.8. Let U and H be separable Hilbert spaces with orthonormal bases $\{e_k\}$, $\{f_j\}$, respectively. $T \in L(U, H)$ is said to be **Hilbert-Schmidt** if

$$\|T\|_2 := \sum_{k=1}^{\infty} |Te_k|^2 < \infty$$

Remark A.9. Since

$$\sum_{k=1}^{\infty} |Te_k|^2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \langle Te_k, f_j \rangle^2 = \sum_{j=1}^{\infty} |T^*f_j|^2$$

the definitions of Hilbert-Schmidt operator and $\|T\|_2$ are independent of the choice of the basis. Moreover $\|T\|_2 = \|T^*\|_2$.

The set $L_2(E, F)$ of all Hilbert-Schmidt operators from E into F , equipped with the norm

$$\|T\|_2 = \left(\sum_{k=1}^{\infty} |Te_k|^2 \right)^{1/2}$$

is a separable Hilbert space, with the scalar product

$$\langle S, T \rangle_2 = \sum_{k=1}^{\infty} \langle S e_k, T e_k \rangle$$

The double sequence of operators $\{f_j \otimes e_k\}_{j,k \in \mathbb{N}}$ is a complete orthonormal basis in $L_2(E, F)$.

Proposition A.10. Let E, F, G be separable Hilbert spaces. If $T \in L_2(E, F)$ and $S \in L_2(F, G)$ then $ST \in L_1(E, G)$ and

$$\|ST\|_1 \leq \|S\|_2 \|T\|_2$$

Proof. Note that for any x in E we have

$$STx = \sum_{j=1}^{\infty} \langle Tx, f_j \rangle S f_j$$

where $\{S f_j\}_j \subset G$ and $x \mapsto \langle Tx, f_j \rangle = \langle x, T^* f_j \rangle$ is an element of E^* with operator norm $|T^* f_j|$. Thus by the definition of nuclear norm

$$\|ST\|_1 \leq \sum_{j=1}^{\infty} |T^* f_j| |S f_j| \leq \left(\sum_{j=1}^{\infty} |T^* f_j|^2 \right)^{1/2} \left(\sum_{j=1}^{\infty} |S f_j|^2 \right)^{1/2}$$

□

Appendix B

Comparison of images of linear operators

Assume we are given three separable Hilbert spaces E_1 , E_2 and E and two linear bounded operators $A_1 : E_1 \rightarrow E$, $A_2 : E_2 \rightarrow E$. We will denote by $A_1^* : E \rightarrow E_1$ and $A_2^* : E \rightarrow E_2$ their adjoint operators. For any $x \in A_1(E_1)$, $x \neq 0$, consider the set

$$A_1^{-1}(x) = \{x_1 \in E_1 : A_1 x_1 = x\}$$

Observe that the set is convex, closed and does not contain 0. Therefore there exists a unique point in $A_1^{-1}(x)$ that minimizes the distance from the origin.

Definition B.1. The **pseudo-inverse** of A_1 , denoted by A_1^{-1} , is defined by

$$A_1^{-1}x = \left\{ y \in A_1^{-1}(x) : |y|_{E_1} \leq |z|_{E_1} \ \forall z : A_1 z = x \right\}$$

Remark B.2. Observe that $A_1^{-1}(E) = \{A_1^{-1}x : x \in E\}$ is the orthogonal subspace to $\text{Ker}(A_1)$. Moreover, A_1^{-1} is a closed operator: let $\{x_n\} \subset A_1(E)$ be a sequence such that $x_n \rightarrow x$ and $A_1^{-1}x_n \rightarrow y$. Then, since A_1 is continuous and by definition $A_1(A_1^{-1}x_n) = x_n$, we must have $x = A_1 y$; moreover $A_1^{-1}x_n \perp \text{Ker}(A_1)$ for each n , so $y \perp \text{Ker}(A_1)$ as well and $y = A_1^{-1}x$.

We need the following lemma.

Lemma B.3. Let E , F , G be Banach spaces, $B : D(B) \rightarrow G$ with $D(B) \subset F$ be a closed operator and $A \in L(E, F)$ such that $A(E) \subset D(B)$. Then $B \circ A : E \rightarrow G$ is continuous.

Proof. By the closed graph theorem, since $B \circ A$ is defined on the whole space, it suffices to show that it is a closed operator. Let x_n be a sequence in E such that $x_n \rightarrow x$ in E and $B A x_n \rightarrow z$ in G . By continuity of A , $A x_n \rightarrow A x$ in F ; but B is closed, therefore we must have $z = B A x$. $\square$

Proposition B.4. The following hold.

- i) $A_1(E_1) \subset A_2(E_2)$ if and only if there exists $c > 0$ such that $|A_1^*| \leq c|A_2^*|$ for all $h \in E$.
- ii) If $|A_1^* h| = |A_2^* h|$ for all h , then $A_1(E_1) = A_2(E_2)$ and $|A_1^{-1} h| = |A_2^{-1} h|$ for all $h \in A_1(E_1)$.

Proof. i) We first show that $A_1(E_1) \subset A_2(E_2)$ if and only if there exists $c > 0$ such that

$$\{A_1u : |u| \leq 1\} \subset \{A_2v : |v| \leq c\} \quad (\text{B.1})$$

The "only if" implication is trivial, let us show the "if" implication. By hypothesis we have $A_1(E_1) \subset D(A_2^{-1})$ with A_2^{-1} a closed operator, therefore by the previous lemma we have that $A_2^{-1}A_1$ is a continuous operator; then there exists a constant $c > 0$ such that

$$|A_2^{-1}A_1u| \leq c \quad \forall u \in \overline{B(0,1)}$$

and the conclusion follows. Secondly we prove that (B.1) holds if and only if $|A_1^*h| \leq c|A_2^*h|$ for all $h \in E$. In fact if (B.1) holds, then

$$|A_1^*h| = \sup_{|u| \leq 1} |\langle h, A_1u \rangle| \leq \sup_{|v| \leq c} |\langle h, A_2v \rangle| = c|A_2^*h|$$

Conversely assume $|A_1^*h| \leq c|A_2^*h|$ for every $h \in E$ and that by contradiction there exists $u_0 \in E_1$ such that $|u_0| \leq 1$ and $A_1u_0 \notin \{A_2v : |v| \leq c\}$. Observe that by reflexivity of E , the set $\{A_2v : |v| \leq c\}$ is closed: if y_n is a contained sequence such that $y_n \rightarrow y$, then let x_n be a corresponding sequence in E_2 such that $|x_n| \leq c$ and $A_2x_n = y_n$. By reflexivity we can extract a subsequence $x_{n_k} \rightharpoonup x$, $|x| \leq c$ and so $A_2x_{n_k} \rightharpoonup A_2x = y$.

So $A_1u_0 \notin \{A_2v : |v| \leq c\}$, which is convex and closed; by the Hahn-Banach theorem there exists $h \in E$ such that

$$\langle h, A_1u_0 \rangle > 1 \quad \text{and} \quad \langle h, A_2v \rangle \leq 1 \quad \forall v : |v| \leq c$$

Thus $|A_1^*h| > 1$ and $|A_2^*h| \leq c$, contradiction.

ii) The first statement follows immediately from part (i); let us prove the second one. We can assume $\text{Ker} A_1 = \text{Ker} A_2 = 0$ (otherwise we can take the restriction on the orthogonal complement of the respective kernels). We have to show that if $e \in E$ is such that $e = A_1h_1 = A_2h_2$, then $|h_1| = |h_2|$. Assume by contradiction that $|h_1| > |h_2|$, then

$$\frac{e}{|h_2|} = A_2\left(\frac{h_2}{|h_2|}\right) \in \{A_2v : |v| \leq 1\} = \{A_1u : |u| \leq 1\}$$

But $\frac{e}{|h_2|} = A_1\left(\frac{h_1}{|h_2|}\right)$ and $\left|\frac{h_1}{|h_2|}\right| > 1$, contradiction. □

Appendix C

Complex valued martingales

Here we define the notions we have adopted, throughout Chapter 3, regarding complex valued martingales and complex valued stochastic integration. These may not be the usual ones, since some of the extensions from the real notions are done by bilinearity instead of sesquilinearity.

If X and Y are $\mathbb{C}$ -valued random variable, then their covariance is given by

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(\overline{Y} - \mathbb{E}[\overline{Y}])]$$

The map $(X, Y) \mapsto Cov(X, Y)$ is sesquilinear and

$$Var(X) = Cov(X, X) = \mathbb{E}[|X - \mathbb{E}[X]|^2]$$

If X is a $\mathbb{C}^n$ -valued random variable, then its covariance matrix $Cov(X)$ is given by

$$Cov(X)_{ij} = Cov(X_i, X_j) \quad \forall i, j = 1, \dots, n$$

In particular, for any $v, w \in \mathbb{C}^n$ it holds

$$v^T Cov(X) \overline{w} = \sum_{i,j} v_i Cov(X_i, X_j) \overline{w}_j = Cov(\langle X, v \rangle, \langle X, w \rangle)$$

where $\langle a, b \rangle = \sum_i a_i b_i$ denotes the $\mathbb{R}^n$ product extended by bilinearity to $\mathbb{C}^n$ (and it's not their hermitian product, that would correspond to $\langle a, \overline{b} \rangle$). Observe that, by the above relation, $Cov(X)$ is an hermitian, semipositive matrix. If we consider $\otimes$ to be the real tensor product extended by bilinearity to $\mathbb{C}^n$, that is

$$a \otimes b = \Re a \otimes \Re b - \Im a \otimes \Im b + i \Re a \otimes \Im b + i \Im a \otimes \Re b$$

then the covariance matrix can be written shortly as

$$Cov(X) = \mathbb{E}[(X - \mathbb{E}[X]) \otimes (\overline{X} - \mathbb{E}[\overline{X}])]$$

The elements on the diagonal of $Cov(X)$ are real, nonnegative numbers and

$$\Re[Cov(X)] = Cov(\Re X) + Cov(\Im X)$$

A $\mathbb{C}$ -valued process is a martingale if and only if, once we identify it with an $\mathbb{R}^2$ -valued process by the isometry $x + iy \mapsto (x, y)$, we obtain an $\mathbb{R}^2$ -valued martingale. Therefore M is a $\mathbb{C}$ -valued martingale if and only if $\Re M$ and $\Im M$ are real martingales. Similarly, M is a $\mathbb{C}^n$ -valued martingale if and only if $\Re M$ and $\Im M$ are $\mathbb{R}^n$ -valued martingales.

Let M and N be continuous, square integrable $\mathbb{C}$ -valued martingales, then their real and imaginary parts are still continuous and square integrable and so it makes sense to define their cross quadratic variation as

$$[M, N](t) = [\Re M, \Re N](t) - [\Im M, \Im N](t) + i[\Re M, \Im N](t) + i[\Im M, \Re N](t)$$

Moreover, since

$$M(t)N(t) = \Re M(t)\Re N(t) - \Im M(t)\Im N(t) + i\Re M(t)\Im N(t) + i\Im M(t)\Re N(t)$$

it follows immediately from the analogue real results that $[M, N]$ is the unique predictable process of bounded variation, starting at 0, such that

$$MN - [M, N]$$

is a $\mathbb{C}$ -valued martingale. If M and N are $\mathbb{C}^n$ -valued martingales, then their cross quadratic variation is defined analogously. Observe that, since the extension is bilinear, it's still true that, given A and $B \in \mathbb{C}^{n \times n}$,

$$[AM, BN](t) = A[M, N](t)B^T$$

Observe that it's not true anymore that, if $[M, M](t) = 0$, then M is almost surely constant on $[0, t]$. In fact

$$[M, M] = 0 \Leftrightarrow \begin{cases} [\Re M, \Re M] = [\Im M, \Im M] \\ [\Re M, \Im M] = 0 \end{cases}$$

which implies that $\Re M$ and $\Im M$ must be independent, but not necessarily constant. For example, if B_1 and B_2 are $\mathbb{R}^n$ -standard independent Brownian motions, $B := B_1 + iB_2$, then $[B, B] = 0$. However, if M is such that $[M, \overline{M}] = 0$, then M must be almost surely constant. Moreover $|M|^2 - [M, \overline{M}]$ is a real martingale.

We now define stochastic integration with respect to $\mathbb{C}$ -valued martingales. M is a square integrable, continuous martingale if and only if the real processes $\Re M$ and $\Im M$ have the same properties; similarly, a $\mathbb{C}$ -valued process X is predictable if and only if $\Re X$ and $\Im X$ are, and it satisfies suitable integrability conditions if and only if $\Re X$ and $\Im X$ do so. Therefore it makes sense, under the usual assumptions, to define

$$\int_0^t X dM := \int_0^t \Re X d\Re M - \int_0^t \Im X d\Im M + i \int_0^t \Im X d\Re M + i \int_0^t \Re X d\Im M$$

It follows from the definition that the stochastic integral is still a $\mathbb{C}$ -valued martingale and it can still be seen as a sort of Stieltjes integral, i.e. given a sequence of partitions of $[0, t]$ with mesh Δt tending to 0, then

$$\int_0^t X dM = \lim_{n \rightarrow \infty} \sum_{k=0}^{N_n-1} X(t_k^{(n)})(M(t_{k+1}^{(n)}) - M(t_k^{(n)})) \quad \text{in probability}$$

It can be checked, comparing the respective real and imaginary parts, that the Kunita-Watanabe identity still holds, that is for any M, N continuous, square integrable complex martingales, and for any X predictable process (satisfying suitable integrability conditions),

$$\left[\int_0^\cdot X dM, N \right](t) = \int_0^t X(s) d[M, N](s)$$

It follows from the identity that, for any X and M as above,

$$\mathbb{E} \left[\left| \int_0^t X dM \right|^2 \right] = \int_0^t |X(s)|^2 d[M, \overline{M}](s)$$

In particular if B is a standard $\mathbb{C}$ -valued Brownian motion, then $[B, B] = 0$, $[B, \overline{B}](t) = 2t$ and

$$\mathbb{E} \left[\left| \int_0^t X dB \right|^2 \right] = 2 \int_0^t |X(s)|^2 ds$$

Similarly, if B is a $\mathbb{C}^n$ -valued Brownian motion, then $[B, B] = 0$, $[B, \overline{B}](t) = 2tI$.

The definition of the Stratonovich integral $\int_0^t X \circ dM$, as a bilinear extension, and its formulation as a limit of sums are analogue.

Appendix D

The Yamada-Watanabe Theorem

The aim of this section is to present a suitable version of the Yamada-Watanabe theorem. The main difficulty, not present in the finite dimensional case, is given by the existence of multiple notions of solution of a stochastic PDE; for example in [13] a distinction is made between *strong*, *weak* and *mild* solutions. Instead in [15] the concept of *variational* solutions is considered. This implies that different versions of the theorem have to be employed, depending on the notion of solution adopted. Here we restrict to the variational framework and we follow the exposition given in [27]; for a proof in the framework of mild solutions, see [28]. We also stress the fact that, in order to state a fairly general result, we will adopt a slightly different framework from the one considered in Chapter 3; however the latter can be easily adapted to fit in the following one. For this reason, the Hilbert spaces V , H , E and U considered in the following are general spaces and not the ones defined in Chapter 3.

We start with the following preliminary lemma.

Lemma D.1. Let V and H be two separable Hilbert spaces such that $V \subset H$ continuously and densely. Then $V \in \mathcal{B}(H)$ and $\mathcal{B}(V) = \mathcal{B}(H) \cap V$.

Proof. In order to show that $V \in \mathcal{B}(H)$, it suffices to show that

$$B_N := \{v \in V : |v|_V \leq N\}$$

are closed sets of H ; if that's the case, then $V = \bigcup_{N=1}^{\infty} B_N$ is a Borel set of H . Now fix N and consider a sequence v_n in B_N such that $v_n \rightarrow v$ in H . Since v_n is a bounded sequence in V , we can consider a subsequence weakly converging to $\tilde{v} \in V$. Since V is continuously embedded in H , this implies that $v_n \rightharpoonup \tilde{v}$ weakly in H as well, thus $\tilde{v} = v$. Since the reasoning holds for any subsequence, this implies that $v_n \rightharpoonup v$ weakly in V and so, by the properties of weak convergence, $v \in B_N$. This shows that B_N is a closed subset of H for every N . For the second statement, using the fact that V is densely embedded in H , analogously to the proof of Lemma 1.1 we can find a sequence $\{v_n\}_n$ in V such that $\mathcal{B}(V) = \sigma\{\langle v_n, \cdot \rangle_V, n \in \mathbb{N}\}$ and $\mathcal{B}(H) = \sigma\{\langle v_n, \cdot \rangle_H, n \in \mathbb{N}\}$, which gives the conclusion. $\square$

Let us now consider three separable Hilbert spaces V , H , E such that

$$V \subset H \subset E$$

continuously and densely. By the previous lemma, $V \in \mathcal{B}(H)$, $H \in \mathcal{B}(E)$, $\mathcal{B}(V) = \mathcal{B}(H) \cap V$ and $\mathcal{B}(H) = \mathcal{B}(E) \cap H$. Setting $|v|_V = +\infty$ for all $v \in H \setminus V$, we can regard $|\cdot|_V$ and a convex, lower semicontinuous measurable function on H . Hence we can define the following path space:

$$\mathbb{B} := \left\{ w \in C(\mathbb{R}_+; H) : \int_0^T |w|_V dt < \infty \text{ for all } T \in [0, +\infty) \right\}$$

equipped with the metric

$$\rho(w_1, w_2) := \sum_{k=1}^{\infty} 2^{-k} \left[\left(\int_0^k |w_1(t) - w_2(t)|_V dt + \sup_{t \in [0, k]} |w_1(t) - w_2(t)|_H \right) \wedge 1 \right]$$

which makes it a complete metric space. We denote by $\mathcal{B}_t(\mathbb{B})$ the σ -algebra generated by the maps $\pi_s : \mathbb{B} \rightarrow H$, $s \leq t$, where $\pi_s(w) = w(s)$.

Let U be another separable Hilbert space and let $L_2(U, H) = L_2$ denote as usual the space of Hilbert-Schmidt operators from U to H , with norm $\|\cdot\|_{L_2}$. Let

$$b : \mathbb{R}_+ \times \mathbb{B} \rightarrow E, \quad \sigma : \mathbb{R}_+ \times \mathbb{B} \rightarrow L_2$$

be measurable maps ($\mathbb{R}_+ \times \mathbb{B}$ endowed with the σ -algebra $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{B})$) such that, for all $t \geq 0$,

$$b(t, \cdot) : (\mathbb{B}, \mathcal{B}_t(\mathbb{B})) \rightarrow (E, \mathcal{B}(E)), \quad \sigma(t, \cdot) : (\mathbb{B}, \mathcal{B}_t(\mathbb{B})) \rightarrow (L_2, \mathcal{B}(L_2))$$

are measurable maps. Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be a stochastic basis, namely a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a normal filtration $\{\mathcal{F}_t\}$. Let W be a cylindrical $\mathcal{F}_t$ -Wiener process on U . If we fix an orthonormal basis $\{e_k\}_k$ of U , then W can be informally written as

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k$$

where β_k , $k \in \mathbb{N}$ are independent $\mathcal{F}_t$ -Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$.

We consider the following stochastic evolution equation:

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad t \in [0, +\infty) \quad (\text{D.1})$$

Definition D.2. A pair (X, W) , where X is an $\mathcal{F}_t$ -adapted process with paths in $\mathbb{B}$ and W is a cylindrical Wiener process on a stochastic basis $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ is a **weak solution** of (D.1) if

i) For any $t \geq 0$,

$$\mathbb{P} \left(\int_0^t |b(s, X(s))|_E ds + \int_0^t \|\sigma(s, X(s))\|_{L_2}^2 ds < \infty \right) = 1$$

ii) As a stochastic equation on E we have $\mathbb{P}$ -a.s.

$$X(t) = X(0) + \int_0^t b(s, X(s)) ds + \int_0^t \sigma(s, X(s)) dW(s) \quad \text{for all } t \in [0, +\infty)$$

Remark D.3. By the measurability assumptions on b and σ , if X is as in the previous definition, then both processes $b(\cdot, X)$ and $\sigma(\cdot, X)$ are $\mathcal{F}_t$ -adapted.

Definition D.4. We say that **weak uniqueness** holds for (D.1) if, whenever (X, W) and (X', W') are two weak solutions with stochastic bases $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ and $(\Omega', \{\mathcal{F}'_t\}, \mathcal{F}', \mathbb{P}')$, such that

$$\mathcal{L}(X(0)) = \mathcal{L}(X'(0))$$

as measures on $(H, \mathcal{B}(H))$, then

$$\mathcal{L}(X) = \mathcal{L}(X')$$

as measures on $(\mathbb{B}, \mathcal{B}(\mathbb{B}))$.

Definition D.5. We say that **pathwise uniqueness** holds for (D.1) if, whenever (X, W) and (X', W') are two weak solutions on the same stochastic basis $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ and with the same cylindrical Wiener process W , such that $X(0) = X'(0)$ $\mathbb{P}$ -a.s., then $\mathbb{P}$ -a.s.

$$X(t) = X'(t) \quad \text{for all } t \in [0, +\infty)$$

Recall that, if J is an Hilbert-Schmidt embedding from U to another Hilbert space $\overline{U}$, then W can be regarded as a Q -Wiener process on $\overline{U}$, $Q = JJ^*$, by setting

$$\overline{W}(t) = \sum_{k=1}^{\infty} \beta_k(t) J e_k$$

and in this case

$$\int_0^t \sigma(s, X(s)) dW(s) = \int_0^t \sigma(s, X(s)) \circ J^{-1} d\overline{W}(s)$$

so that the definition of the stochastic integral is independent of the choice of $\overline{U}$ and J . Below we shall fix $\overline{U}$ and J . We define

$$\mathbb{W}_0 = \{w \in C(\mathbb{R}_+; \overline{U}) : w(0) = 0\}$$

endowed with the metric

$$\rho(w_1, w_2) := \sum_{k=1}^{\infty} 2^{-k} \left(\sup_{t \in [0, k]} |w_1(t) - w_2(t)|_{\overline{U}} \wedge 1 \right)$$

and the σ -algebra $\mathcal{B}(\mathbb{W}_0)$. For any $t \geq 0$, as before $\mathcal{B}_t(\mathbb{W}_0)$ denotes the σ -algebra generated by $\pi_s : \mathbb{W}_0 \rightarrow \overline{U}$, $s \leq t$, where $\pi_s(w) = w(s)$.

In order to define strong solutions, we need to introduce the following class of maps. Let $\mathcal{E}$ denote the set of all maps $F : H \times \mathbb{W}_0 \rightarrow \mathbb{B}$ such that for every probability measure μ on $(H, \mathcal{B}(H))$ there exists a measurable map $F_\mu : H \times \mathbb{W}_0 \rightarrow \mathbb{B}$ such that, for μ -a.e. $x \in H$,

$$F(x, w) = F_\mu(x, w) \quad \text{for } \mathbb{P}^Q\text{-a.e. } w \in \mathbb{W}_0$$

Here $H \times \mathbb{W}_0$ is endowed with the completion of $\mathcal{B}(H) \otimes \mathcal{B}(\mathbb{W}_0)$ with respect to $\mu \otimes \mathbb{P}^Q$, where $\mathbb{P}^Q$ denotes the distribution of the Q -Wiener process $\overline{W}$ on $(\mathbb{W}_0, \mathcal{B}(\mathbb{W}_0))$. F_μ is uniquely determined $\mu \otimes \mathbb{P}^Q$ -a.s.

Definition D.6. A weak solution (X, W) to (D.1) on $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ is a **strong solution** if there exists $F \in \mathcal{E}$ such that for any $x \in H$, $w \mapsto F(x, w)$ is measurable from $\overline{\mathcal{B}_t(\mathbb{W}_0)}^{\mathbb{P}^Q}$ to $\mathcal{B}_t(\mathbb{B})$ for every $t \geq 0$ and

$$X = F_{\mathcal{L}(X(0))}(X(0), \overline{W}) \quad \mathbb{P}\text{-a.s.}$$

Definition D.7. We say that **strong uniqueness** holds for (D.1) if there exists $F \in \mathcal{E}$ satisfying the adaptiveness condition in Definition D.6 and such that:

- i) For every cylindrical Wiener process W on a stochastic basis $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ and any $\mathcal{F}_0$ -measurable H -valued random variable ξ the continuous process

$$X = F_{\mathcal{L}(\xi)}(\xi, \overline{W})$$

is such that (X, W) is a weak solution to (D.1) and $X(0) = \xi$ $\mathbb{P}$ -a.s.

- ii) For any weak solution (X, W) to (D.1) we have

$$X = F_{\mathcal{L}(X(0))}(X(0), \overline{W}) \quad \mathbb{P}\text{-a.s.}$$

Remark D.8. Since $X(0)$ is $\mathbb{P}$ -independent of $\overline{W}$,

$$\mathcal{L}((X(0), \overline{W})) = \mathcal{L}(X(0)) \otimes \mathbb{P}^Q$$

therefore strong uniqueness implies weak uniqueness.

In this setting, the Yamada-Watanabe Theorem can be formulated as follows.

Theorem D.9. Let b and σ as above. Then equation (D.1) has a unique strong solution if and only if the following hold:

- i) For every probability measure μ on H there exists a weak solution (X, W) of (D.1) such that $\mathcal{L}(X(0)) = \mu$.
- ii) Pathwise uniqueness holds for (D.1).

We omit the proof, for which we refer to [27].

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