Università degli Studi di Padova

DIPARTIMENTO DI FISICA E ASTRONOMIA "G. Galilei"
Corso di laurea triennale in Fisica

Tesi di Laurea

THE GENERATING FUNCTIONAL FOR SCALAR FIELD THEORIES

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ANNO ACCADEMICO 2015 - 2016

1 Introduction

In 1931 Dirac published a paper concerning key similarities between the classical Hamilton-Jacobi theory and the transition amplitudes in quantum mechanics. In particular, he derived the following relation

$$\langle q, t|Q, T\rangle \sim e^{\frac{i}{\hbar}\int_T^t dtL}\,. \tag{1.1}$$

In 1948, Feynman developed Dirac's suggestion and succeeded in deriving a new formulation of quantum mechanics, quite different from the standard one. This formulation does not require the use of operators and of the Schrödinger equation to express the quantum mechanical amplitudes. The physical idea is that the probability amplitude to find a particle at the space-time point (Q, T), knwowing that it was at (q, t), is given by the sum of all the possible paths between the two space-time points, each one contributing with an appropriate weight.
In this work it is firstly presented (Sec. 2) the Dirac original idea and then how Feynman developed his path-integral formulation of quantum mechanics. Moreover, it is explained how this approach can be generalized to quantum field theory. In particular we focus on the case of a scalar field (Sec. 3), introducing the generating functional, a basic tool to compute Green's functions without the use of Feynman diagrams. Finally, in Sec. 4 and 5, we illustrate two alternative representations of the generating functional, developed in the Ref.[2]. The first one is expressed as

$$W[J] = T[\phi_c] = \frac{N}{N_0}\exp(-U_0[\phi_c])\exp\left(\frac{1}{2}\frac{\delta}{\delta\phi_c}\Delta\frac{\delta}{\delta\phi_c}\right)\exp\left(-\int V(\phi_c)\right), \tag{1.2}$$

where ϕ_c is defined as $\phi_c(x) = \int d^DyJ(y)\Delta(y-x)$. This dual representation is used to express Schwinger-Dyson equation, obtaining

$$\left[\frac{\delta}{\delta\phi_c} + e^{U_0[\phi_c]}\int\frac{\delta V}{\delta\phi}\left(\Delta\frac{\delta}{\delta\phi_c}\right)e^{-U_0[\phi_c]}\right]e^{\frac{1}{2}\frac{\delta}{\delta\phi_c}\Delta\frac{\delta}{\delta\phi_c}}e^{-\int V(\phi_c)} = 0\,. \tag{1.3}$$

It is also possible to note the presence of a deep connection between the above dual representation and the Hermite polynomials. Then, we express $T[\phi_c]$ in terms of "covariant" derivatives acting on 1

$$T[\phi_c] = \frac{N}{N_0}\exp\left(-U_0[\phi_c]\right)\exp\left(-\int V(\mathcal{D}^-_{\phi_c})\right)\cdot 1\,, \tag{1.4}$$

where $\mathcal{D}^{\pm}_{\phi}(x) = \mp\Delta\frac{\delta}{\delta\phi}(x) + \phi(x)$. These "covariant" derivatives simplify the form of the equations. For example the Schwinger-Dyson equation becomes

$$\left(\frac{\delta}{\delta\phi_c(x)} + \int\frac{\delta V}{\delta\phi}\left(\mathcal{D}^-_{\phi_c}\right)\right)\exp\left(-\int V(\mathcal{D}^-_{\phi_c})\right)\cdot 1 = 0\,. \tag{1.5}$$

We also see how they make "more comfortable" some explicit calculations.

2 The path-integral

The Dirac formulation. The key initial idea, that led to the concept of path-integral, is due to Dirac who was looking for an alternative formulation of quantum mechanics provided by the Lagrangian. He believed that the Lagrangian formulation of classical dynamics is more fundamental than the Hamiltonian one for the following reasons. First of all the Lagrangian method allows to find the equations of motion, thanks to the stationary property of a certain action function. Secondly, the Lagrangian method could be easily expressed relativistically since the action is Lorentz invariant. For doing this, Dirac worked on the analogy between the classical theory of Hamilton-Jacobi and the transition amplitude in quantum mechanics. To show this analogy, let us consider a one-dimensional classical system of only one particle. Let q be the coordinate and p the momentum. $\mathcal{H}$ is the Hamiltonian of the system. The Hamilton variational principle

$$\delta \int_{t_0}^{t} dt \left(p\frac{dq}{dt} - \mathcal{H}(q,p,t) \right) = 0 \,, \tag{2.1}$$

allows to describe the time evolution of $q(t)$, $p(t)$ by the Hamilton equations, expressed through the Poisson brackets as follows

$$\frac{dq}{dt} = \{q, \mathcal{H}\} \,, \qquad\qquad \frac{dp}{dt} = \{p, \mathcal{H}\} \,. \tag{2.2}$$

A canonical transformation is a transformation of q and p into new variables Q and P leaving Hamilton's equations form invariant. It is well known that a function $G(t,q,Q)$ called generating function exists, such that

$$p = \frac{\partial G}{\partial q} \,, \qquad P = -\frac{\partial G}{\partial Q} \,, \qquad \tilde{\mathcal{H}} = \mathcal{H} + \frac{\partial G}{\partial t} \,, \tag{2.3}$$

where $\tilde{\mathcal{H}}$ is the Hamiltonian of the new system.
Denote $S(t,q,Q)$ the special canonical transformation such as $\dot{Q} = \dot{P} = 0$. By (2.3), it follows that

$$\mathcal{H}\left(q, p = \frac{\partial S}{\partial q}, t \right) = -\frac{\partial S}{\partial t} \,. \tag{2.4}$$

Since

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q}\frac{dq}{dt} = -\mathcal{H} + p\frac{dq}{dt} \,, \tag{2.5}$$

we have

$$S = \int_{t_0}^{t} dt' \mathcal{L} \,. \tag{2.6}$$

Note that once S has been evaluated on the solution of the equation of motions, it can be interpreted as a functional of $q(t)$ and $q(t_0) = Q$. In this way, the action is the generating function of the canonical transformation, transforming the system variables from t_0 to t.
Describing the same one-dimensional system in quantum mechanics it is possible to introduce two independent representations for the system, $|q\rangle$ and $|Q\rangle$, and look for a $\langle q|Q\rangle$ connecting the two representations. If F is any function of the

dynamical variables, it will have a "mixed" representative $\langle q|\hat{F}|Q\rangle$ and thanks to the completeness relation

$$\int dq\, |q\rangle \langle q| = 1 \tag{2.7}$$

we get

$$\langle q|\hat{F}|Q\rangle = \int \langle q|\hat{F}|q'\rangle \langle q'|Q\rangle\, dq' = \int \langle q|Q'\rangle \langle Q'|\hat{F}|Q\rangle\, dQ' , \tag{2.8}$$

From these relations we obtain

$$\langle q|\hat{q}|Q\rangle = q \langle q|Q\rangle \ , \qquad \langle q|\hat{p}|Q\rangle = -i\hbar \frac{\partial}{\partial q} \langle q|Q\rangle \ , \tag{2.9}$$

$$\langle q|\hat{Q}|Q\rangle = Q \langle q|Q\rangle \ , \qquad \langle q|\hat{P}|Q\rangle = i\hbar \frac{\partial}{\partial Q} \langle q|Q\rangle \ . \tag{2.10}$$

However, since $\hat{Q}$ and $\hat{q}$ do not necessarily commute, if $F = F[q,Q]$ the "mixed" representative $\langle q|F[\hat{q},\hat{Q}]|Q\rangle$, may be not well defined. The generic function $F = F[q,Q]$ is called well-ordered if it can be expressed as $F[q,Q] = \sum_k f_k^1(q) f_k^2(Q)$. Then if F is well-ordered, so the above "mixed" representative is well defined. Therefore, setting $\langle q|Q\rangle = e^{\frac{i}{\hbar}U(q,Q)}$ into equations above we obtain

$$\langle q|\hat{p}|Q\rangle = \frac{\partial U(q,Q)}{\partial q} \langle q|Q\rangle \ , \qquad \langle q|\hat{P}|Q\rangle = -\frac{\partial U(q,Q)}{\partial Q} \langle q|Q\rangle \ . \tag{2.11}$$

Finally supposing $\frac{\partial U}{\partial q}$ and $\frac{\partial U}{\partial Q}$ are well-order we find

$$\hat{p} = \frac{\partial \hat{U}}{\partial q} , \qquad \hat{P} = -\frac{\partial \hat{U}}{\partial Q} . \tag{2.12}$$

So U is the analogue of the classical function S and in this way Dirac concluded that

$$\langle q,t|Q,T\rangle \sim e^{\frac{\text{ß}}{\hbar}\int_T^t dtL} . \tag{2.13}$$

The Feynman path-integral. As Dirac emphasized, the "$\sim$" above is just a loose connection. As matter of fact, a "$=$" would not be correct in the previous relation as long as $T-t$ is a finite time interval. Feynman started from (2.13) and he assumed it as an equality (up to a constant) only for an infinitesimal time interval:

$$\langle q'_t|q_{t+\delta t}\rangle = C \exp\left(-\frac{i}{\hbar}\delta t L(q'_t, q_{t+\delta t})\right) . \tag{2.14}$$

Now split the time interval $T-t$ into N infinitesimal time intervals $t_a = t + a\epsilon$, $N\epsilon = T - t$, using the completeness relation (2.7), we find

$$\langle q'_t|q_T\rangle = \int dq_1 dq_2 \cdots dq_{N-1} \langle q'_t|q_1\rangle \langle q_1|q_2\rangle \cdots \langle q_{N-1}|q_T\rangle \ , \tag{2.15}$$

which is an exact quantum mechanical relation. Replacing Eq.(2.14) into Eq.(2.15) we can conclude that

$$\langle q'_t|q_t\rangle = \lim_{N\to\infty} A^N \int \left(\prod_{i=1}^{N-1} dq_i\right) e^{\frac{i}{\hbar}\int_T^t dtL(q,\dot{q})} . \tag{2.16}$$

Such an expression is not fully corrected, yet. It can be proved that to "exactly" formulate the transition amplitude the action must be expressed through the Hamiltonian formalism, so we can finally define the transition amplitude as

$$\langle q_t' | q_T \rangle = \int \mathcal{D}q \mathcal{D}p \exp\left\{ i \int_T^t dT \left[p\frac{dq}{dt} - H(p,q) \right] \right\} . \tag{2.17}$$

3 The path-integral for a scalar field

Let us apply the functional method of path-integral to the theory of a real scalar field $\phi(x)$. The Lagrangian density of the theory is the following

$$\mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}_0(\phi, \partial_\mu \phi) - V(\phi) = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 - V(\phi) , \tag{3.1}$$

where $V(\phi)$ is the potential. The first step is to build the Hamiltonian density $\mathcal{H}$, performing a Legendre transformation:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \phi\right)} = \partial_0 \phi \equiv \dot{\phi} , \tag{3.2}$$

$$\mathcal{H}(\pi, \phi, \vec{\nabla}\phi) = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left(\pi^2 + \left(\vec{\nabla}\phi\right)^2 + m^2\phi^2 \right) + V(\phi) , \tag{3.3}$$

where $\pi(x)$ is the canonical momentum. Eq.(2.17), extended to field theory, defines the transition amplitude from $\phi_a(0, \mathbf{x})$ to $\phi_b(T, \mathbf{x}) = T$ as

$$\langle \phi_b(\mathbf{x}) | \, e^{-iHT} \, | \phi_a(\mathbf{x}) \rangle = N \int \mathcal{D}\phi \mathcal{D}\pi \exp\left[i \int_0^T d^4x \left(\pi\dot{\phi} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right) \right] , \tag{3.4}$$

where N is the normalization constant and $\phi(x)$, over which we integrate, has the following boundary conditions

$$\phi(x) = \begin{cases} \phi(\mathbf{x}) = \phi_a(\mathbf{x}) & \text{if } x^0 = 0 , \\ \phi(\mathbf{x}\ ) = \phi_b(\mathbf{x}) & \text{if } x^0 = T . \end{cases} \tag{3.5}$$

The integration over π is trivial, just completing the square on the exponent and integrating, we obtain

$$\langle \phi_b(\mathbf{x}) | \, e^{-iHT} \, | \phi_a(\mathbf{x}) \rangle = N' \int \mathcal{D}\phi \exp\left[i \int_0^T d^4x \mathcal{L} \right] , \tag{3.6}$$

where N' is a normalization constant. Hereafter we will give up the Hamiltonian formalism, and take Eq.(3.6) to define the Hamiltonian dynamics.
Although the relation (3.6) is a very elegant one, physicists are mostly concerned with computing quantities that can be measured, like cross sections and decay rates. These quantities can be related to the S-matrix, which can be computed from the connected correlation functions trough LSZ (Lehmann, Symanzik, Zimmermann) reduction formula. Then, we need a formula to compute the correlation functions. Let us consider the following object

$$\int \mathcal{D}\phi(x)\phi(x_1)\phi(x_2) \exp\left[i \int_{-T}^T d^4x \mathcal{L}(\phi) \right] , \tag{3.7}$$

where $\phi(x)$ is constrained by the boundary condition

$$\phi(x) = \begin{cases} \phi(\mathbf{x}) = \phi_a(\mathbf{x}) & \text{if } x^0 = -T\,, \\ \phi(\mathbf{x}) = \phi_b(\mathbf{x}) & \text{if } x^0 = T\,. \end{cases} \tag{3.8}$$

With some manipulation it can be proved that (3.7) is equal to

$$\langle \phi_b | \, e^{-iHT} T \left\{ \phi_H(x_1) \phi_H(x_2) \right\} e^{-iHT} \, |\phi_\alpha\rangle \ , \tag{3.9}$$

where[1] T is the time ordering operator and the operators ϕ_H are expressed through the Heisemberg picture. Eq.(3.7) can be extended to the case of correlation functions from $t = -\infty$ to $t = \infty$; this quantity is very important in quantum field theory. Therefore, we have to take the limit for $T \to \infty(1 - i\epsilon)$ which selects the lowest energy level, indicated with $|\Omega\rangle$. Eq.(3.7) becomes

$$\langle \Omega | \, T \phi_H(x_1) \phi_H(x_2) \, | \Omega \rangle = \lim_{T \to \infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi \phi(x_1) \phi(x_2) \exp\left[i \int_{-T}^{T} d^4x \mathcal{L} \right]}{\int \mathcal{D}\phi \exp\left[i \int_{-T}^{T} d^4x \mathcal{L} \right]} \ , \tag{3.10}$$

that is the desired formula. Higher correlation functions can be obtained, just inserting additional factors ϕ_H and ϕ, respectively on the left-hand and on the right-hand sides of the previous equation. Another feature of Eq.(3.10) is that it is manifestly Lorentz invariant ad it preserves also all the symmetries the Lagrangian $\mathcal{L}$ may have.

3.1 The generating functional

Generalizing the above equations we can introduce the central object of this work: the *generating functional* $W[J]$ of the Green functions, defined as follows

$$\begin{aligned} W[J] \equiv \frac{\langle \Omega | \Omega \rangle_J}{\langle \Omega | \Omega \rangle} &= N \int \mathcal{D}\phi \mathcal{D}\pi e^{i \langle \pi \dot{\phi} - \mathcal{H} + J\phi \rangle} \\ &= N' \int \mathcal{D}\phi \exp\left[i \left\langle \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 - V(\phi) + J\phi \right\rangle \right] \ , \end{aligned} \tag{3.11}$$

where $\langle f(x_1) \cdots f(x_N) \rangle \equiv \int d^D x_1 \ldots d^D x_N f(x_1, \ldots, x_N)$ and N, N' are normalization constants.

This path-integral is not well-defined because of the oscillatory integrand; we can remedy to this problem introducing a damping term or working in the Euclidean space. In this work it will be often used the latter method. Therefore, setting the new variables $x_0 = -i\bar{x}_0, d^4x = -id^4\bar{x}, \partial_\mu \phi \partial^\mu \phi = -\bar{\partial}_\mu \phi \bar{\partial}_\mu \phi$; then the generating functional in Euclidean space is the following:

$$W_E[J] = N_E \int \mathcal{D}\phi \exp\left[- \left\langle \frac{1}{2} \bar{\partial}_\mu \phi \bar{\partial}_\mu \phi + \frac{1}{2} m^2 \phi^2 + V(\phi) - J\phi \right\rangle \right] \ . \tag{3.12}$$

This object is very important since it allows to compute the Green functions, defined as the coefficients of the functional expansion

$$W[J] = \sum_{N=0}^{\infty} \frac{i^N}{N!} \left\langle J_1 \cdots J_N G^{(N)}(1, \ldots, N) \right\rangle \ , \tag{3.13}$$

[1] See appendix A

$$G^{(N)}(1,\ldots,N)=\frac{1}{i^N}\frac{\delta}{\delta J_1}\cdots\frac{\delta}{\delta J_N}W[J]\bigg|_{J=0}\,. \tag{3.14}$$

Green's functions $G^{(N)}$ in Minkowsky space are identified with correlation functions. However, we can still use W_E to construct Green's functions $G_E^{(N)}$ in Euclidean space, but in this case we have to relate them to $G^{(N)}$ through analytic continuation (Wick rotation), which presuppose no singularities are encountered in the process of contour rotation.

The Feynman propagator. Let us evaluate the generating functional for a free theory ($V=0$), working in the Minkowsky space and putting the damping term $e^{-\frac{1}{2}\epsilon\phi}$ for the convergence problem; at the end of the calculation we have to take the limit $\epsilon\to 0$ ($\epsilon>0$). Therefore the new generating functional is

$$W_{0,\epsilon}\equiv N\int\mathcal{D}\phi\exp\left[i\left\langle\frac{1}{2}\partial_\mu\phi\partial_\mu\phi-\frac{1}{2}(m^2-i\epsilon)\phi^2-J\phi\right\rangle\right]\,. \tag{3.15}$$

The standard method to compute this integral is to work in the momentum space, given the Fourier transform and anti-transform

$$\tilde{F}(p)=\int_{-\infty}^{+\infty}\frac{d^4x}{(2\pi)^2}e^{-ip\cdot x}F(x)\,,\qquad F(x)=\int_{-\infty}^{+\infty}\frac{d^4p}{(2\pi)^2}e^{ip\cdot x}\tilde{F}(p)\,. \tag{3.16}$$

Then introducing the new field ($\tilde{\phi}$ is the Fourier transorm of $\phi(x)$)

$$\tilde{\phi}'(p)=\tilde{\phi}(p)+\left[p^2-m^2+i\epsilon\right]^{-1}\tilde{J}(p)\,, \tag{3.17}$$

the generating functional becomes

$$W_0[J]=\exp\left[-\frac{i}{2}\int d^4p\frac{|\tilde{J}(p)|^2}{p^2-m^2+i\epsilon}\right]\int\mathcal{D}\phi' e^{i\left\langle\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi'-\frac{1}{2}(m^2-i\epsilon)\phi'^2\right\rangle}\,, \tag{3.18}$$

where $\mathcal{D}\phi'$ differs from $\mathcal{D}\phi$ only for an omitted multiplicative constant. Then the following relation is evident

$$W_0[J]=W_0[0]\exp\left[-\frac{i}{2}\int d^4p\frac{\tilde{J}(p)\tilde{J}(-p)}{p^2-m^2+i\epsilon}\right]\,, \tag{3.19}$$

using the Fourier anti-transform it follows that

$$W_0[J]=W_0[0]e^{-\frac{i}{2}\langle J_1\Delta_{F12}J_2\rangle}\,, \tag{3.20}$$

where $\Delta_{F12}\equiv\Delta_F(x_1-x_2)$ is the Feynman propagator

$$\Delta_F(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}\,. \tag{3.21}$$

It could be more convenient to set

$$W_0[J]=e^{iZ_0[J]}\,, \tag{3.22}$$

where

$$Z_0[J]=\left\langle\frac{i}{2}J(x)\Delta_F(x-y)J(y)\right\rangle\,. \tag{3.23}$$

3.2 Generating functional of connected Green functions

As in the free case seen above, we set

$$W[J] = e^{iZ[J]} \,. \tag{3.24}$$

The term $Z[J]$ plays a key role in quantum field theory since it is the generating functional of the connected Green functions. Now, we prove it with reference to [4]. Let $G_c^{(N)}$ denote the N-point connected Green functions. The general $G^{(N)}$ contains σ_K copies of $G^{(K)}$ ($K \leq N$). Then $G^{(N)}$ may be expanded in the form

$$G^{(N)} = \sum_{\{\sigma_1,\sigma_2,\ldots,\sigma_N\}} \sum_P P\left[G_c^{(1)}\cdots G_c^{(1)}\right]\cdots\left[G_c^{(N)}\cdots G_c^{(N)}\right] , \tag{3.25}$$

where the occupation number σ_i are constrained by $1\sigma_1 + \ldots + N\sigma_N = N$. P denotes all possible distinct permutations of the N variables. Then

$$\begin{aligned} W[J] = e^{iZ[J]} &= \sum_{N=0}^{\infty} \frac{i^N}{N!}\int d^Dx_1\cdots d^Dx_N G^N(x_1,\cdots,x_N)J(x_1)\cdots J(x_N) \\ &= \sum_{N=0}^{\infty} i^N \sum_{\{\sigma_1,\ldots,\sigma_N\}} \prod_{j=1}^{N} \frac{\left[\int d^Dx_1\cdots d^Dx_j G_c^{(j)}\cdots G_c^{(j)} J(x_1)\cdots J(x_j)\right]^{\sigma_j}}{\sigma_j!(j!)^{\sigma_j}} \,. \end{aligned} \tag{3.26}$$

Noting that $\sum_{N=0}^{\infty}\sum_{\{\sigma_1,\ldots,\sigma_N\}} = \sum_{\sigma_k}$, where the summation on the right hand side has no restriction, we obtain

$$\begin{aligned} W[J] &= \prod_{j=1}^{\infty}\sum_{\sigma_j=0}^{\infty}\frac{1}{\sigma_j!}\left[\frac{i}{j!}\int d^Dx_1\cdots d^Dx_j G_c^{(j)}(x_1,\ldots,x_j)J(x_1)\cdots J(x_j)\right]^{\sigma_j} \\ &= \exp\sum_{N=1}^{\infty}\frac{i^N}{N!}\int d^Dx_1\cdots d^Dx_N G_c^{(N)}(x_1,\ldots,x_N)J(x_1)\cdots J(x_N)\,, \\ &= \exp\left(iZ[J]\right) \,. \end{aligned} \tag{3.27}$$

4 Alternative representation for the generating functional

In this section we will introduce a different representation for the generating functional, working in D dimensional Euclidean space. Hereafter the subscript E will be omitted. First of all it is necessary to introduce some notations. For every even function or distribution G and for any functions or operators f_1 and f_2, we set:

$$f_1 G f_2 = \langle f_1(x)G(x-y)f_2(y)\rangle \,, \quad \frac{\delta}{\delta f_1}G\frac{\delta}{\delta f_2} = \left\langle \frac{\delta}{\delta f_1(x)}G(x-y)\frac{\delta}{\delta f_2(y)}\right\rangle \,. \tag{4.1}$$

The starting form of $W[J]$ is the following

$$W[J] \equiv e^{-Z[J]} = N \int \mathcal{D}\phi \exp\left[-\int d^D x \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi) - J\phi\right)\right], \tag{4.2}$$

where $Z[J]$ is the generating functional for connected Green functions and N is the normalization constant.

$$N = \left(\int \mathcal{D}\phi \exp(-S[\phi])\right)^{-1}, \tag{4.3}$$

where

$$S[\phi] = \int d^D x \left(\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + V(\phi)\right). \tag{4.4}$$

Schwinger representation. Suppose $V(\phi)$ can be expanded as

$$V(\phi) = \sum_{n=0}^{\infty} c^n \phi^n . \tag{4.5}$$

Using

$$\frac{\delta}{\delta J(x)} e^{\langle J\phi\rangle} = \phi(x) e^{\langle J\phi\rangle}, \tag{4.6}$$

we have

$$\begin{aligned} W[J] &= N \int \mathcal{D}\phi e^{\langle -V(\phi)\rangle} e^{-\left\langle \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 - J\phi\right\rangle} \\ &= N e^{-\langle V(\frac{\delta}{\delta J})\rangle} \int \mathcal{D}\phi e^{-\left\langle \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 - J\phi\right\rangle} \\ &= \frac{N}{N_0} \exp\left(-\int V\left(\frac{\delta}{\delta J}\right)\right) W_0[J], \end{aligned} \tag{4.7}$$

where

$$N_0 = \left(\int \mathcal{D}\phi \exp(-S_0[\phi])\right)^{-1}. \tag{4.8}$$

Expression (4.7) takes the name of Schwinger representation for the generating functional.

4.1 Dual representation for $W[J]$

As we have seen the connection between the path-integral formalism and the operator one is the following

$$W[J] = \frac{\langle \Omega|\Omega\rangle_J}{\langle \Omega|\Omega\rangle}, \tag{4.9}$$

Note that we have

$$W[J] = N \langle 0| T \exp\left[\int (-V(\hat{\phi}) + J\hat{\phi})\right] |0\rangle , \tag{4.10}$$

where $|0\rangle$ is the free vacuum. Let us introduce the field $\phi_c(x)$, defined as

$$\phi_c(x) = \int d^D y J(y) \Delta(y-x) \,, \tag{4.11}$$

which satisfies the following equation

$$\left(-\partial^2 + m^2\right) \phi_c(x) = J(x) \,. \tag{4.12}$$

Replacing ϕ by $\phi + \phi_c$ into Eq.(4.10), it follows that (up to a constant)

$$\begin{aligned} W[J] &= \langle 0| T \exp\left[\int (-V(\hat{\phi}+\phi_c)) + J(\hat{\phi}+\phi_c)\right] |0\rangle \\ &= e^{-Z_0[J]} \langle 0| T \exp\left[\int (-V(\hat{\phi}+\phi_c))\right] |0\rangle \ . \end{aligned} \tag{4.13}$$

Note that, thanks to the Wick theorem (A.11),

$$\begin{aligned} \langle 0| T F[\hat{\phi}+f] |0\rangle &= \langle 0| \exp\left(\frac{1}{2}\frac{\delta}{\delta f}\Delta \frac{\delta}{\delta f}\right) : F[\hat{\phi}+f] : |0\rangle \\ &= \exp\left(\frac{1}{2}\frac{\delta}{\delta f}\Delta \frac{\delta}{\delta f}\right) \langle 0| F[f] |0\rangle \\ &= \exp\left(\frac{1}{2}\frac{\delta}{\delta f}\Delta \frac{\delta}{\delta f}\right) F[f] \,. \end{aligned} \tag{4.14}$$

Finally, applying Eq.(4.14) to the right hand side of Eq.(4.13), we get

$$W[J] = \exp(-Z_0[J]) \exp\left(\frac{1}{2}\frac{\delta}{\delta J}\Delta^{-1}\frac{\delta}{\delta J}\right) \exp\left[-\int d^D x V\left(\int d^D z J(z) \Delta(z-x)\right)\right] \,, \tag{4.15}$$

that, can be expressed through ϕ_c as

$$W[J] = T[\phi_c] = \frac{N}{N_0} \exp(-U_0[\phi_c]) \exp\left(\frac{1}{2}\frac{\delta}{\delta\phi_c}\Delta \frac{\delta}{\delta\phi_c}\right) \exp\left(-\int V(\phi_c)\right) \,, \tag{4.16}$$

where

$$U_0[\phi_c] = -\frac{1}{2}\phi_c \Delta^{-1} \phi_c \,, \tag{4.17}$$

and $\Delta^{-1}(x) = \int d^D p (p^2 + m^2) e^{ipx}$.

4.2 Schwinger-Dyson equation in the dual representation

Here we shortly present the Schwinger-Dyson equation, then we will express this equation through the dual representation we have just introduced.

Schwinger-Dyson equation. This equation is the quantum equation of motion for Green's functions. In classical mechanics the equation of motion could be derived by imposing that the action has to be stationary under an infinitesimal variation

$$\phi(x) \to \phi(x) + \epsilon(x) \,. \tag{4.18}$$

The appropriate generalization to quantum field theory is to consider this variation as an infinitesimal change of variables

$$\phi(x) \to \phi(x) + \epsilon F[\phi, x] \,, \tag{4.19}$$

which does not change the measure ($\mathcal{D}\phi = \mathcal{D}\phi'$) and the value of the path-integral. $F[\phi, x]$ is an arbitrary functional of ϕ (we suppose it admits an expansion in powers of ϕ). The generating functional, expanded to the first order in ϵ, becomes:

$$W_\epsilon[J] = \int \mathcal{D}\left[1 + \epsilon\left\langle\frac{\delta F}{\delta\phi}\right\rangle\right]\left\{1 - \epsilon\int d^Dx\left[\frac{\delta\langle\mathcal{L}\rangle}{\delta\phi} - J\phi\right]F\right\}\exp\left(-\langle\mathcal{L} - J\phi\rangle\right) . \tag{4.20}$$

Collecting the terms proportional to ϵ, imposing that the path-integral does not change and using Eq.(4.6), we could find

$$\int d^DxF\left(\frac{\delta}{\delta J}, x\right)\left[\frac{\delta\langle\mathcal{L}\rangle}{\delta\phi}\left(\frac{\delta}{\delta J}\right) - J(x)\right]W[J] = 0 \,. \tag{4.21}$$

If $F = F(x)$, then (4.19) is just a translation of ϕ. The above equation reduces to

$$\left[\int\frac{\delta\langle\mathcal{L}\rangle}{\delta\phi}\left(\frac{\delta}{\delta J}\right) - J\right]W[J] = 0 \,, \tag{4.22}$$

that could be expressed as

$$\left[\Delta^{-1}\frac{\delta}{\delta J}(x) + \int\frac{\delta V}{\delta\phi(x)}\left(\frac{\delta}{\delta J}\right) - J(x)\right]W[J] = 0 \,, \tag{4.23}$$

where

$$\Delta^{-1}\frac{\delta}{\delta J}(x) \equiv \int d^Dy\Delta^{-1}(y - x)\frac{\delta}{\delta J}(y) \,. \tag{4.24}$$

With the dual representation, the above equation becomes

$$\begin{aligned}\left[\frac{\delta}{\delta\phi_c} + \int\frac{\delta V}{\delta\phi}\left(\Delta\frac{\delta}{\delta\phi_c}\right)\right]e^{-U_0[\phi_c]}e^{\frac{1}{2}\frac{\delta}{\delta\phi_c}\Delta\frac{\delta}{\delta\phi_c}}e^{-\int V(\phi_c)} &= \\ \left[\frac{\delta}{\delta\phi_c} + e^{U_0[\phi_c]}\int\frac{\delta V}{\delta\phi}\left(\Delta\frac{\delta}{\delta\phi_c}\right)e^{-U_0[\phi_c]}\right]e^{\frac{1}{2}\frac{\delta}{\delta\phi_c}\Delta\frac{\delta}{\delta\phi_c}}e^{-\int V(\phi_c)} &= 0 \,.\end{aligned} \tag{4.25}$$

As we will see there is a deep connection with the Hermite polynomials.

Relation with the Hermite polynomials. The standard representation of the "probabilistic" Hermite polynomials is given by

$$He_n(x) = (-1)^n e^{\frac{x^2}{2}} D^n e^{-\frac{x^2}{2}} \,. \tag{4.26}$$

Thanks to Eq.(B.7) the right hand side of (4.26) becomes

$$(-1)^n e^{\frac{x^2}{2}} D^n e^{-\frac{x^2}{2}} = e^{-\frac{D^2}{2}} x^n \,. \tag{4.27}$$

Replacing x with ix into (4.27) we get

$$e^{-\frac{x^2}{2}} D^n e^{\frac{x^2}{2}} = e^{\frac{D^2}{2}} x^n \,. \tag{4.28}$$

Then, supposing $f(x)$ can be expanded in power of x, we find

$$e^{-\frac{x^2}{2}} f(D) e^{\frac{x^2}{2}} = e^{\frac{D^2}{2}} f(D) \,. \tag{4.29}$$

This provides the following expansion

$$e^{\frac{D^2}{2}} f(x) = \sum_{n=0}^{\infty} (-i)^n c_n He_n(ix) \,. \tag{4.30}$$

This equation can be used in quantum field theory. As matter of fact, (4.16) involves $\exp\left(\frac{1}{2}\delta_{\phi_c}\Delta\delta_{\phi_c}\right)$ acting on ϕ_c. So in the perturbative expansion there appear terms like

$$\exp\left(\frac{1}{2}\delta_{\phi_c}\Delta\delta_{\phi_c}\right)\phi_c^n \,. \tag{4.31}$$

Note that

$$\delta_{\phi_c}\Delta\delta_{\phi_c}\phi_c^n(x) = n(n-1)\Delta(0)\phi_c^{n-2}(x) \,, \tag{4.32}$$

is the functional version of

$$\Delta(0)\partial^2_{\phi_c}\phi_c^n = n(n-1)\Delta(0)\phi_c^{n-2} \,, \tag{4.33}$$

thanks to (4.30) we obtain

$$\exp\left(\frac{1}{2}\delta_{\phi_c}\Delta\delta_{\phi_c}\right)\phi_c^n(x) = (-i)^n \Delta^{\frac{n}{2}}(0) He_n\left(\frac{i\phi_c(x)}{\Delta^{\frac{1}{2}}(0)}\right) \,. \tag{4.34}$$

Eq.(4.34) suggests a connection of the Schwinger-Dyson equation with the Hermite polynomials. We start from (4.27), through which we can find

$$\begin{aligned} e^{U_0[\phi_c]}\frac{\delta^n}{\delta\phi_c^n(x)}e^{-U_0[\phi_c]} &= \left[\sum_{k=0}^{n}\binom{n}{k}\frac{\delta^{n-k}}{\delta\phi_c^{n-k}(x)}e^{-U_0[\phi_c]}\right]\frac{\delta^k}{\delta\phi_c^k(x)} \\ &= \left[\frac{1}{2}\exp\left(\delta_{\phi_c}\Delta\delta_{\phi_c}\right)\sum_{k=0}^{n}\binom{n}{k}\left(\Delta^{-1}\phi_c\right)^{n-k}(x)\right]\frac{\delta^k}{\delta\phi_c^k(x)} \,. \end{aligned} \tag{4.35}$$

Using (4.34) we can express the Schwinger-Dyson equation for $V = \frac{\lambda}{n!}\phi^n$ as follows

$$\left[\frac{\delta}{\delta\phi_c(x)} + \sum_{k=0}^{n-1}\frac{\lambda(-i)^k\Delta^{\frac{k}{2}}(0)}{(n-k-1)!k!}He_k\left(\frac{i\phi_c(0)}{\Delta^{\frac{1}{2}}(0)}\right)\left(\Delta\frac{\delta}{\delta\phi_c}\right)^{n-k-1}(x)\right]e^{\frac{1}{2}\frac{\delta}{\delta\phi_c}\Delta\frac{\delta}{\delta\phi_c}}e^{-\int V(\phi_c)} = 0 \,. \tag{4.36}$$

It interesting to consider a normal ordered potential

$$:V(\phi):= \frac{\lambda}{n!}:\phi^n:= \frac{\lambda}{n!}\exp\left(-\frac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)\phi^n \,,$$

in this case (4.25) becomes

$$\left[\frac{\delta}{\delta\phi_c(x)} + \frac{\lambda}{(n-1)!}\sum_{k=0}^{n-1}\binom{n-1}{k}\phi_c^k(x)\left(\Delta\frac{\delta}{\delta\phi_c}\right)^{n-k-1}(x)\right]e^{\frac{1}{2}\frac{\delta}{\delta\phi_c}\Delta\frac{\delta}{\delta\phi_c}}e^{-\int :V(\phi_c):} = 0 \,, \tag{4.37}$$

that compared with (4.36) shows how the terms $e^{\pm U_0[\phi_c]}$ compensate the contribution coming from the normal ordering regularization of the potential.

4.3 $T[\phi_c]$ and normal ordered potentials

Let us consider only the case of a normal ordered potential, indicated with $: V :$. It is useful to set

$$\mathcal{D}_j = \frac{1}{2}\frac{\delta}{\delta\phi_{c_j}}\Delta\frac{\delta}{\delta\phi_{c_j}}, \qquad \mathcal{D}_{jk} = \frac{\delta}{\delta\phi_{c_j}}\Delta\frac{\delta}{\delta\phi_{c_k}}, \qquad \mathcal{D} = \frac{1}{2}\frac{\delta}{\delta\phi_c}\Delta\frac{\delta}{\delta\phi_c}\,. \tag{4.38}$$

The expression (4.16) becomes

$$T[\phi_c] = \frac{N}{N_0}\exp(-U_0[\phi_c])\exp(\mathcal{D})\exp\left(-\int : V(\phi_c) :\right)\,. \tag{4.39}$$

Considering a generic functional $F[\phi]$, we want to express $e^{\mathcal{D}}F[\phi]$ as

$$\exp(\mathcal{D})\exp\left(F[\phi]\right) = \exp\left(\sum_{N=1}^{\infty}\frac{Q_N}{N!}\right)\,, \tag{4.40}$$

where $\{Q_N\}$ is a set of connected functionals, defined as

$$\begin{aligned} Q_N[\phi] &= e^{\mathcal{D}}F^N[\phi]\big|_{\text{conn}} \\ &= \prod_{i>j=1}^{N} e^{\mathcal{D}_{ij}}\prod_{i=1}^{N} e^{\mathcal{D}_i}F[\phi_i]\big|_{\text{conn}\,\phi_i\to\phi}\,. \end{aligned} \tag{4.41}$$

The subscript "connect" means that at least one linkage operator ($e^{D_{ij}}$) must be retained between each pairs of[2] $F[\phi_i]$. We need a similar decomposition; then we set

$$T[\phi_c] = \frac{N}{N_0}\exp\left(-U_0[\phi_c] + \sum_{k=1}^{\infty}\frac{Q_k[\phi_c]}{k!}\right)\,, \tag{4.42}$$

where now

$$Q_N[\phi_c] = e^{\mathcal{D}}\left(\int : V(\phi_c) :\right)^N\Bigg|_{\text{conn}}\,. \tag{4.43}$$

Note that, like $U[\phi_c]$, the Q_N generate connected functions. Rescaling the potential by a constant μ we find

$$\exp(\mathcal{D})\exp\left(-\mu\int : V(\phi_c) :\right) = \exp\left(\sum_{K=1}^{\infty}\frac{\mu^k}{k!}Q_k[\phi_c]\right)\,, \tag{4.44}$$

and so

$$Q_k[\phi_c] = \partial_\mu^k \ln\left[\exp(\mathcal{D})\exp\left(-\mu\int : V(\phi_c) :\right)\right]\Bigg|_{\mu=0}\,. \tag{4.45}$$

Now, the relation (B.1), expressed below through the appropriate variables,

$$e^{\mathcal{D}}F[\phi_c]G[\phi_c] = e^{\mathcal{D}_{12}}\left(e^{\mathcal{D}_1}F[\phi_{c_1}]e^{\mathcal{D}_2}G[\phi_{c2}]\right)\big|_{\phi_{c_1}=\phi_{c_2}=\phi_c} \tag{4.46}$$

allows to make some considerable simplification. To show this, let us calculate Q_1 and Q_2

$$Q_1 = -e^{\mathcal{D}}\int : V := -\int V \tag{4.47}$$

[2]This method is clearly exposed in [3].

and

$$Q_2 = \left[e^{\mathcal{D}_{12}} - 1\right] \left[\left(e^{\mathcal{D}_1} \int : V[\phi_{c_1}] : \right) \left(e^{\mathcal{D}_2} \int : V[\phi_{c_2}] : \right) \right] \Bigg|_{\phi_{c_1}=\phi_{c_2}=\phi_c}$$
$$= \left[e^{\mathcal{D}_{12}} - 1\right] \left[\int V(\phi_{c_1}) \int V(\phi_{c_2}) \right] \Bigg|_{\phi_{c_1}=\phi_{c_2}=\phi_c} . \qquad (4.48)$$

Then

$$Q_n[\phi_c] = (-1)^n \prod_{j>k}^{n} e^{\mathcal{D}_{jk}} \prod_{i=1}^{n} \int V(\phi_{c_i}) \Bigg|_{c,\, \phi_{c_1}=\phi_{c_2}=\ldots=\phi_c} , \qquad (4.49)$$

where the subscript c indicates that non connected terms must be discharged. So the generating functional of the connected Green function can be expressed as follows

$$U[\phi_c] = \ln \frac{N}{N_0} + U_0[\phi_c] + \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p!} \prod_{j>k}^{n} e^{\mathcal{D}_{jk}} \prod_{i=1}^{n} \int V(\phi_{c_i}) \Bigg|_{c,\, \phi_{c_1}=\phi_{c_2}=\ldots=\phi_c} . \qquad (4.50)$$

5 Generating functional and “covariant” derivatives

In this section we will express the generating functional through “covariant” derivatives. The key expression is the following operator identity

$$\exp\left(-\frac{1}{2} IMI\right) F[\delta_I] \exp\left(\frac{1}{2} IMI\right) = F\left[\mathcal{D}_{MI}\right] , \qquad (5.1)$$

where F is a functional, I and M are functions (or distributions), $\mathcal{D}_{MI}(x)$ denotes the “covariant derivative”

$$\mathcal{D}_{MI} = \frac{\delta}{\delta I(x)} + MI(x) . \qquad (5.2)$$

In our case we define

$$\mathcal{D}^{\pm}_{\phi}(x) = \mp \Delta \frac{\delta}{\delta \phi}(x) + \phi(x) . \qquad (5.3)$$

It can be easily proved that these operators satisfy the following commutation relations

$$\left[\mathcal{D}^{-}_{\phi}(x), \mathcal{D}^{+}_{\phi}(y)\right] = 2\Delta(x-y) , \qquad \left[\mathcal{D}^{\pm}_{\phi}(x), \mathcal{D}^{\pm}_{\phi}(y)\right] = 0 . \qquad (5.4)$$

Another fundamental relation comes from the use of (5.2) into the operatorial version of (B.7)

$$\exp\left(\frac{1}{2} \delta_I M^{-1} \delta_I\right) F[MI] = F[\mathcal{D}_{MI}] \cdot 1 . \qquad (5.5)$$

Thanks to the above relations, (4.16) becomes

$$T[\phi_c] = \frac{N}{N_0} \exp\left(-U_0[\phi_c]\right) \exp\left(-\int V(\mathcal{D}^-_{\phi_c})\right) \cdot 1\,. \tag{5.6}$$

This new representation still simplifies the form of the Schwinger-Dyson equation (4.25), reducing it to

$$\left(\frac{\delta}{\delta\phi_c(x)} + \int \frac{\delta V}{\delta\phi}\left(\mathcal{D}^-_{\phi_c}\right)\right) \exp\left(-\int V(\mathcal{D}^-_{\phi_c})\right) \cdot 1 = 0\,. \tag{5.7}$$

Now we will try to calculate the Green function through covariant derivatives. To do this, note the expression below

$$\frac{\delta}{\delta J(x)} \exp\left(-U_0[\phi_c]\right) = \exp\left(-U_0[\phi_c]\right) \mathcal{D}^-_{\phi_c}(x)\,, \tag{5.8}$$

so the N-point Green function is

$$\begin{aligned} \frac{\delta^n W[J]}{\delta J(x_1)\ldots\delta J(x_N)} &= \exp(-U_0[\phi_c])\mathcal{D}^-_{\phi_c}(x_1)\ldots\mathcal{D}^-_{\phi_c}(x_N) \exp\left(-\int V(\mathcal{D}^-_{\phi_c})\right) \cdot 1 \\ &= \exp(-U_0[\phi_c]) \left(-\int V(\mathcal{D}^-_{\phi_c})\right) \mathcal{D}^-_{\phi_c}(x_1)\ldots\mathcal{D}^-_{\phi_c}(x_N) \cdot 1\,. \end{aligned} \tag{5.9}$$

The above representation makes easier the explicit calculation as we will see later.
Another feature of this representation concerns the case of a normal ordered potential. According to the Wick theorem (A.11) and to (5.1), we can write

$$: F[\phi] := F[\mathcal{D}^+_{\phi_c}] \cdot 1\,. \tag{5.10}$$

For example let us try to calculate $:\phi^4:$. We use the notation $\Delta(x_1 - x_2) = \Delta_{12}$ and $\phi(x_i) = \phi_i$.

$$\begin{aligned} :\phi^2(x): &= \prod_{k=1}^{2} \mathcal{D}_\phi(x_k)\cdot 1\Big|_{x_1=x_2=x} = \prod_{k=1}^{2} \phi(x_k) + \Delta_{12}\Big|_{x_1=x_2=x} = \phi^2(x) - \Delta(0)\,, \\ :\phi^3(x): &= \prod_{k=1}^{3} \mathcal{D}_\phi(x_k)\cdot 1\Big|_{x_k=x} = \prod_{k=1}^{3} \phi(x_k) - \Delta_{12}\phi_3 - \Delta_{13}\phi_2 - \Delta_{23}\phi_1\Big|_{x_i=x} \\ &= \phi^3(x) - 3\Delta(0)\phi(x)\,, \\ :\phi^4(x): &= \prod_{k=1}^{4} \mathcal{D}_\phi(x_k)\cdot 1\Big|_{x_k=x} = \prod_{k=1}^{4} \phi_k - \Delta_{12}\phi_3\phi_4 - \Delta_{13}\phi_2\phi_4 - \Delta_{14}\phi_2\phi_3 - \\ &- \Delta_{23}\phi_1\phi_4 - \Delta_{24}\phi_1\phi_3 - \Delta_{34}\phi_1\phi_2 + \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{23}\Delta_{14}\big|_{x_k=x} \\ &= \phi^4(x) - 6\Delta(0)\phi^2(x) + 3\Delta^2(0)\,. \end{aligned} \tag{5.11}$$

Since $\mathcal{D}^+_\phi$ and $\mathcal{D}^-_\phi$ differ only by the sign of $\Delta(x-y)$, we can easily get the expression of $\prod_k^n \mathcal{D}^-_\phi$ from (5.11). For example let us calculate $T[\phi_c]$ for $V(\phi) =$

$\frac{\lambda}{4!}$, to the first order in λ. Using Eq.(5.6) and thanks to (5.11)

$$\begin{aligned} T[\phi_c] &= \frac{N}{N_0}\exp\left(-U_0[\phi_c]\right)\left[1-\frac{\lambda}{4!}\int d^D x \mathcal{D}_{\phi_\rfloor}^{-\,4}(x)+...\right]\cdot 1\,, \\ &= \frac{N}{N_0}\exp\left(-U_0[\phi_c]\right)\left[1-\frac{\lambda}{4!}\left(\int d^D x\left(\phi_c^4(x)+6\phi_c^2(x)\Delta(0)+3\Delta^2(0)\right)\right)+\ldots\right]. \end{aligned} \tag{5.12}$$

The generating functional of connected Green functions. Let us employ this representation to $Z[J]$. With reference to section 4.3 we will consider a generic potential $V[\phi]$ which can be expanded in powers of ϕ (V is not normal ordered as in the previous case). Then the generating functional is

$$\begin{aligned} T[\phi_c] &= \frac{N}{N_0}\exp\left(-U_0[\phi_c]\right)\exp\left(\frac{1}{2}\frac{\delta}{\delta\phi_c}\Delta\frac{\delta}{\delta\phi_c}\right)\exp\left(-\int V(\phi_c)\right) \\ &= \frac{N}{N_0}\exp\left(-U_0[\phi_c]+\sum_{k=1}^{\infty}\frac{Q_k[\phi_c]}{k!}\right) = \exp\left(-U[\phi_c]\right)\,, \end{aligned} \tag{5.13}$$

where Q_N are connected functionals. Rescaling the potential with μ, we could obtain

$$Q_k[\phi_c] = \partial_\mu^k \ln\left[\exp(\mathcal{D})\exp\left(-\mu\int V(\phi_c)\right)\right]\Bigg|_{\mu=0}\,. \tag{5.14}$$

Introducing the covariant derivatives and thanks to the following relations

$$\exp\left(\pm\frac{1}{2}\frac{\delta}{\delta\phi}\Delta\frac{\delta}{\delta\phi}\right)\exp\left(-\int V(\phi_c)\right) = \exp\left(-\int V(\mathcal{D}_{\phi_c}^{\mp})\right)\cdot 1\,, \tag{5.15}$$

$$\exp(\mathcal{D})F[\phi]G[\phi] = F[\mathcal{D}_\phi^-]G[\mathcal{D}_\phi^-]\cdot 1\,, \tag{5.16}$$

we are able to make some relevant simplifications. Let us try to compute Q_1 and Q_2

$$Q_1 = -e^{\mathcal{D}}\int V = -\int V\left(\mathcal{D}_{\phi_c}^-\right)\cdot 1\,, \tag{5.17}$$

$$Q_2 = e^{\mathcal{D}}\int V\int V = \int V\left(\mathcal{D}_{\phi_c}^-\right)\int V\left(\mathcal{D}_{\phi_c}^-\right)\cdot 1\,. \tag{5.18}$$

So generalizing, we can express the generating functional as follows:

$$U[\phi_c] = \ln\frac{N}{N_0}+U_0[\phi_c]+\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p!}\left(\int V\left(\mathcal{D}_{\phi_c}^-\right)\right)^p\cdot 1\Bigg|_c\,, \tag{5.19}$$

where the subscript c means that terms non connected by at least one propagator must be discharged. Eq.(5.19) can be expanded to the case of normal ordered potential $:V(\phi):$ as

$$\begin{aligned} U[\phi_c] &= \ln\frac{N}{N_0}+U_0[\phi_c]+\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p!}\left(\int V\left(\mathcal{D}_{\phi_c}^+\right)\cdot 1\Bigg|_{\phi_c=\mathcal{D}_{\phi_c}^-}\right)^p\cdot 1\Bigg|_c \\ &= \ln\frac{N}{N_0}+U_0[\phi_c]+\sum_{p=1}^{\infty}\frac{(-1)^{p+1}}{p!}\left(\int V\left(\mathcal{D}_{\phi_c}^+\right)\cdot 1\Bigg|_{\phi_c=\mathcal{D}_{\phi_c}^-}\right)^{p-1}\cdot\int V(\phi_c)\Bigg|_c\,. \end{aligned} \tag{5.20}$$

Now, we calculate $G_c^{(2)}$ and $G_c^{(4)}$ for $\frac{\lambda}{4!}\phi^4$ theory to the second order in λ, to show how this alternative representation works. Eq.(5.19) gives expression to the generating functional. Using the following notations: $\Delta(x_i - x_j) \equiv \Delta_{ij}$, $\phi_c(x_i) \equiv \phi_i$, $\phi_c(y) \equiv \phi_y$, the generating functional, expanded to the first order in λ is

$$U[\phi_c] = \ln \frac{N}{N_0} + U_0[\phi_c] + \frac{\lambda}{4!} \int \left(\mathcal{D}^-_{\phi_c}\right)^4 \cdot 1 \bigg|_c . \tag{5.21}$$

But we have already calculated $\left(\mathcal{D}^-_{\phi_c}\right)^4 \cdot 1$, and so we have

$$\left(\mathcal{D}^-_{\phi_c}\right)^4 \cdot 1 = \phi_x^4 + 6\phi_x^2\Delta_{xx} + 3\Delta_{xx}^2 . \tag{5.22}$$

Now, we proceed to calculate the generating functional to the second order in λ

$$\begin{aligned} &\mathcal{D}^-_{\phi_y} \cdot \left(\phi_x^4 + 6\phi_x^2\Delta_{xx} + 3\Delta_{xx}^2\right) \\ &= \phi_x^4\phi_y + 4\phi_x^3\Delta_{xy} + 6\phi_x^2\phi_y\Delta_{xx} + 12\phi_x\Delta_{xx}\Delta_{xy} + 3\phi_y\Delta_{xx}^2 , \end{aligned} \tag{5.23}$$

$$\begin{aligned} &\left(\mathcal{D}^-_{\phi_y}\right)^2 \cdot \left(\phi_x^4 + 6\phi_x^2\Delta_{xx} + 3\Delta_{xx}^2\right) = \\ &\phi_x^4\phi_y^2 + 8\phi_x^3\phi_y\Delta_{xy} + \phi_x^4\Delta_{yy} + 12\phi_x^2\Delta_{xx}^2 + 6\phi_x^2\phi_y^2\Delta_{xx} + 12\phi_x\phi_y\Delta_{xx}\Delta_{xy} + \\ &6\phi_x^2\Delta_{xx}\Delta_{yy} + 12\phi_x\phi_y\Delta_{xx}\Delta_{xy} + 12\Delta_{xx}\Delta_{xy}^2 + 3\phi_y^2\Delta_{xx}^2 + 3\Delta_{xx}^2\Delta_{xy} , \end{aligned} \tag{5.24}$$

$$\begin{aligned} &\left(\mathcal{D}^-_{\phi_y}\right)^3 \cdot \left(\phi_x^4 + 6\phi_x^2\Delta_{xx} + 3\Delta_{xx}^2\right) = \\ &\phi_x^4\phi_y^3 + 12\phi_x^3\phi_y^2\Delta_{xy} + 3\phi_x^4\phi_y\Delta_y y + 36\phi_x^2\phi_y\Delta_{xy}^2 + 12\phi_x^3\Delta_{xy}\Delta_{yy} + \\ &24\phi_x\Delta_{xy}^3 + 18\phi_x^2\phi_y\Delta_{xx}\Delta_{yy} + 36\phi_x\Delta_{xx}\Delta_{xy}\Delta_{yy} + 6\phi_x^2\phi_y^3\Delta_{xx} + \\ &36\phi_x\phi_y^2\Delta_{xx}\Delta_{xy} + 36\phi_y\Delta_{xx}\Delta_{xy}^2 + 3\phi_y^3\Delta_{xx}^2 + 6\phi_y\Delta_{xx}^2\Delta_{yy} + 3\phi_y\Delta_{xx}^2\Delta_{xy} . \end{aligned} \tag{5.25}$$

Our final task is to compute $G_c^{(2)}$ and $G_c^{(4)}$. Let us start with the 2-point Green's function and to find it we use the following relation:

$$G_c^{(2)}(x_1, x_2) = -\left.\frac{\delta^2 Z[J]}{\delta J_1 \delta J_2}\right|_{J=0} = -\int d^D y_1 d^D y_2 \Delta_{x_1 y_1}\Delta_{x_2 y_2} \left.\frac{\delta^2 U[\phi_c]}{\delta\phi_c(y_1)\delta\phi_c(y_2)}\right|_{\phi_c=0} . \tag{5.26}$$

It is not necessary to explicitly calculate $\left(\mathcal{D}^-_{\phi_y}\right)^4 \cdot \left(\phi_x^4 + 6\phi_x^2\Delta_{xx} + 3\Delta_{xx}^2\right)$ because many of its terms do not contribute to the 2-point Green's function. We just consider the terms of the generating functional which meet our purpose. Then, we have to discharge the disconnected terms and the terms that do not contain ϕ_y^2 or ϕ_x^2 or $\phi_x\phi_y$. We get

$$\begin{aligned} &U[\phi_c]|_2 = \ln \frac{N}{N_0} + U_0[\phi_c] + \frac{\lambda}{4!}\left(\int d^D x \phi_x^4 + 6\phi_x^2\Delta_{xx} + 3\Delta_{xx}^2\right) \\ &-\frac{\lambda^2}{2(4!)^2}\int d^D x d^D y \left(96\phi_x\phi_y\Delta_{xy}^3 + 144\phi_x\phi_y\Delta_{xx}\Delta_{yy}\Delta xy + 72\phi_x^2\Delta_{yy}\Delta_{xy}^2 + 72\phi_y^2\Delta_{xx}\Delta_{xy}^2\right) \\ &+o(\lambda^2) , \end{aligned} \tag{5.27}$$

where the subscript 2 means that this expression can just lead to $G_c^{(2)}$. Therefore, by (5.26) applied to (5.27), we obtain:

$$G_c^{(2)}(x_1,x_2) = \Delta_{x_1x_2} - \frac{\lambda}{2}\int d^Dx\Delta_{x_1x}\Delta_{xx}\Delta_{x_2x} + \frac{\lambda^2}{6}\int d^Dxd^Dy\Delta_{x_1x}\Delta_{xy}^3\Delta_{x_2y}+$$
$$\frac{\lambda^2}{4}\int d^Dxd^Dy\Delta_{x_1x}\Delta_{xy}^2\Delta_{yy}\Delta_{x_2x} + \frac{\lambda^2}{4}\int d^Dxd^Dy\Delta_{x_1x}\Delta_{xx}\Delta_{xy}\Delta_{yy}\Delta_{x_2y} + o(\lambda^2)\,. \tag{5.28}$$

Finally we work on $G_c^{(4)}$, that can be expressed as follows:

$$G_c^{(4)} = -\left.\frac{\delta^4 Z[J]}{\delta J_1\ldots\delta J_4}\right|_{J=0} = -\int d^Dy_1\cdots d^Dy_4\Delta_{x_1y_1}\cdots\Delta_{x_4y_4}\left.\frac{\delta^4 U[\phi_c]}{\delta\phi_c(y_1)\ldots\delta\phi_c(y_4)}\right|_{\phi_c=0}\,. \tag{5.29}$$

Following a similar procedure, we obtain the generating functional to calculate 4-point Green's function (to the second order in λ) and it is expressed as

$$U[\phi_c]|_4 = \ln\frac{N}{N_0} + U_0[\phi_c] - \frac{\lambda}{4!}\left(\int d^Dx\phi_x^4 + 6\phi_x^2\Delta_{xx} + 3\Delta_{xx}^2\right)$$
$$+\frac{\lambda^2}{2(4!)}\int d^Dxd^Dy\left(72\phi_x^2\phi_y^2\Delta_{xy}^2 + 48\phi_x\phi_y^3\Delta_{xx}\Delta_{xy} + 48\phi_x^3\phi_y\Delta_{yy}\Delta_{xy}\right) + o(\lambda^2)\,. \tag{5.30}$$

We obtain

$$G_c^{(4)}(x_1,x_2,x_3,x_4) = -\lambda\int d^Dx\Delta_{x_1x}\Delta_{x_2x}\Delta_{x_3x}\Delta_{x_4x}+$$
$$\frac{\lambda^2}{6}\int d^Dxd^Dy\left(\Delta_{xy}^2\left[\Delta_{x_1x}\Delta_{x_2x}\Delta_{x_3y}\Delta_{x_4y} + \Delta_{x_1x}\Delta_{x_3x}\Delta_{x_2y}\Delta_{x_4y} + \Delta_{x_1x}\Delta_{x_4x}\Delta_{x_2y}\Delta_{x_3y}\right]\right) +$$
$$+\frac{\lambda^2}{2}\int d^Dxd^Dy\left(\Delta_{yy}\Delta_{xy}\left[\Delta_{x_1x} + \Delta_{x_2x} + \Delta_{x_3x} + \Delta_{x_4y} + \text{cyclic permutations}\right]\right) + o(\lambda^2)\,, \tag{5.31}$$

and it is the result we expected to find (for example see [7]).

A Wick theorem

Let us consider the 2-field correlation function expressed through the operatorial formalism [3]

$$\langle 0| T\left\{\phi(x_1)\phi(x_2)\right\}|0\rangle \ . \tag{A.1}$$

We would like to rewrite it in a form that it is easy to evaluate and that can also be expanded to the case of more than two fields. First of all, we can write the field as following

$$\phi(x) = \phi^+(x) + \phi^-(x)\,, \tag{A.2}$$

where

$$\phi^+ = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a_{\mathbf{p}}e^{-ip\cdot x}; \qquad \phi^- = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a^\dagger_{\mathbf{p}}e^{+ip\cdot x}\,. \tag{A.3}$$

This decomposition is very useful, because thanks to $a_{\mathbf{p}}e^{-ip\cdot x}$ and $a^\dagger_{\mathbf{p}}$ follows

$$\phi^+(x)\,|0\rangle\,, \qquad\qquad \langle 0|\,\phi^-(x) = 0\,. \tag{A.4}$$

A term like $a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}a_{\mathbf{k}}a_{\mathbf{l}}$ is said to be *normal ordered* and has a vanish vacuum expectation value. Let us define the normal ordering symbol $N()$ whose action is to make into normal order the operators it contains. We introduce, now, one more quantity, the *contraction* of two field, defined as follows:

$$\overbrace{\phi(x)\phi}(y) = \begin{cases} [\phi^+(x), \phi^-(y)] & \text{for}\, x^0 > y^0\,, \\ [\phi^+(y), \phi^-(x)] & \text{for}\, y^0 > x^0\,. \end{cases} \tag{A.5}$$

This quantity is exactly the Feynman propagator

$$\overbrace{\phi(x)\phi}(y) = \Delta(x-y)\,, \tag{A.6}$$

Now, supposing $x_0 > y_0$, the time-ordered product is

$$\begin{aligned} T\phi(x)\phi(y) &= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \\ &= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) + \left[\phi^+(x), \phi^-(y)\right] \,. \end{aligned} \tag{A.7}$$

The relation between the time-ordering and the normal-ordering is the following

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y)\} + \overbrace{\phi(x)\phi}(y) = N\{\phi(x)\phi(y)\} + \langle 0|\phi(x)\phi(y)|0\rangle \ . \tag{A.8}$$

The generalization to many arbitrary field takes the name of *Wick's theorem* and it is the following (for example, see [6])

$$T\{\phi(x_1)\phi(x_2)\ldots\phi(x_n)\} = N\{\phi(x_1)\phi(x_2)\ldots\phi(x_n) + \text{all possible contractions}\}\,. \tag{A.9}$$

This rule admits a functional form that could make easier the explicit calculations. For example (A.8) can be written as

$$T\left(\phi(x_1)\phi(x_2)\right) = \left[1 \pm \frac{1}{2}\int d^Dy_1 d^Dy_2 \frac{\delta}{\delta\phi(y_1)}\Delta(y_1-y_2)\frac{\delta}{\delta\phi y_2}\right] N(\phi(x_1)\phi(x_2))\,. \tag{A.10}$$

[3]The fields are operator expressed through the Heisemberg picture

The equation above could be generalized to the case of many arbitrary fields (an excellent reference is [8]); the result is the following

$$TF[\hat{\phi}] = \exp\left(\frac{1}{2}\frac{\delta}{\delta\hat{\phi}}\Delta\frac{\delta}{\delta\hat{\phi}}\right) : F[\hat{\phi}] : . \tag{A.11}$$

B Some relations

In this appendix we detail the proofs of some relations used in this work.

B.1

Let $\phi_i(x_i)$, $\Delta(x_i, x_j), J_i(x_i)$ be functions or distributions, let F and G be functionals. The relation we want to prove is the following

$$e^{\mathcal{D}} F[\phi]G[\phi] = e^{\mathcal{D}_{12}}[(e^{\mathcal{D}_1}F[\phi_1])(e^{\mathcal{D}_2}G[\phi_2])]\big|_{\phi_1=\phi_2=\phi} \ , \tag{B.1}$$

where

$$\mathcal{D}_j = -\frac{i}{2}\int \frac{\delta}{\delta\phi_j}\Delta\frac{\delta}{\delta\phi_j} \ , \qquad\qquad \mathcal{D}_{ij} = -i\int \frac{\delta}{\delta\phi_i}\Delta\frac{\delta}{\delta\phi_j} \ . \tag{B.2}$$

Suppose $F[\phi], G[\phi]$ can be expanded in powers of ϕ, then the starting point is the relation below

$$F[\phi] = F\left[-i\frac{\delta}{\delta J}\right] \cdot \exp\left(i\int J\phi\right)\bigg|_{J=0} \ . \tag{B.3}$$

Generalizing this equations to the product of two functionals and considering that $e^{\mathcal{D}}$ commutes with $F\left[-i\frac{\delta}{\delta J}\right]$, we get

$$e^{\mathcal{D}} F[\phi]G[\phi] = F\left[\frac{1}{i}\frac{\delta}{\delta J_1}\right] G\left[\frac{1}{i}\frac{\delta}{\delta J_2}\right] e^{\left(-\frac{i}{2}\int \frac{\delta}{\delta\phi}A\frac{\delta}{\delta\phi}\right)} e^{[i\int \phi(J_1+J_2)]}\Bigg|_{J_1=J_2=0} \ . \tag{B.4}$$

Now, noting that

$$\exp\left(-\frac{i}{2}\int \frac{\delta}{\delta\phi}A\frac{\delta}{\delta\phi}\right)\exp\left(i\int J\phi\right) = \exp\left(i\int JAJ + i\int J\phi\right) \ , \tag{B.5}$$

we obtain

$$e^{\mathcal{D}} F[\phi]G[\phi] =$$
$$F\cdot G\cdot\exp\left(\frac{i}{2}\int J_1\Delta J_1 + \frac{i}{2}\int J_2\Delta J_2 + i\int J_1\Delta J_2\right)\cdot\exp\left(i\int \phi(J_1+J_2)\right)\bigg|_{J_1=J_2=0} \ , \tag{B.6}$$

that can be rearranged to Eq.(B.1) .

B.2

Let I and L be functions, let F be a functional and let M be an even function or distribution. Then, we have

$$\exp\left(-\frac{1}{2}IMI\right)F[\delta I]\exp\left(\frac{1}{2}IMI\right) = \left(\frac{1}{2}\delta_I M^{-1}\delta_I\right)F[MI]\,. \tag{B.7}$$

We are will not prove it, but we will prove its more general operatorial version. We define

$$\overline{F}[MI] \equiv \exp\left(\delta_I M^{-1}\delta_I\right)F[MI]\,. \tag{B.8}$$

The operatorial version of Eq.(B.7) is

$$\exp\left(-\frac{1}{2}IMI\right)F[\delta_I]\exp\left(\frac{1}{2}IMI\right) = \exp\left(-IMI\right)\overline{F}[\delta_I]\exp\left(LMI\right)\Big|_{L=I}\,. \tag{B.9}$$

It immediately follows that if we let the right hand side of the above equation acting on 1, we obtain the Eq.(B.7). Indeed

$$\exp\left(-IMI\right)\overline{F}[\delta_I]\exp\left(LMI\right)\Big|_{L=I}\cdot 1 = \overline{F}[\delta_I]\Big|_{\delta_I\equiv MI} = \overline{F}[MI]\,. \tag{B.10}$$

Then, let us demonstrate Eq.(B.9). To start with we introduce the Laplace transform

$$\mathcal{L}\{f\}(s) \equiv \int_{-\infty}^{\infty} dt e^{-st} f(t)\,, \tag{B.11}$$

through which we can express $\overline{F}[MI]$ as follows

$$\overline{F}[MI] = e^{\frac{1}{2}\delta_I M^{-1}\delta_I}\int DJ e^{IMJ}\hat{F}[MJ] = \int DJ e^{\frac{1}{2}JMJ+IMJ}\hat{F}[MJ]\,. \tag{B.12}$$

then we have

$$\overline{F}[\delta_I] = \int DJ e^{\frac{1}{2}JMJ+J\delta_I}\hat{F}[MJ]\,. \tag{B.13}$$

Now, we will have the Eq.(B.9) acting on a generic functional $G[MI]$

$$\begin{aligned}
e^{-\frac{1}{2}IMI}F[\delta_I]e^{\frac{1}{2}IMI}G[MI] &= e^{-\frac{1}{2}IMI}\int DJ e^{J\delta_I}\hat{F}[MJ]e^{\frac{1}{2}IMI}G[MI] \\
&= e^{-\frac{1}{2}IMI}\int DJ\hat{F}[MJ]e^{\frac{1}{2}(J+I)M(J+I)}G[M(I+J)] \\
&= e^{-IMI}\int DJ e^{\frac{1}{2}JMJ}\hat{F}[MJ]e^{LM(I+J)}\Big|_{L=I}G[M(I+J)] \\
&= e^{-IMI}\int DJ e^{\frac{1}{2}JMJ+J\delta_I}\hat{F}[MJ]e^{LMI}\Big|_{L=I}G[MI] \\
&= e^{-IMI}\overline{F}[\delta_I]e^{LMI}\Big|_{L=I}G[MI]\,.
\end{aligned} \tag{B.14}$$

and so we have proved Eq.(B.9).

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