UNIVERSITÀ DEGLI STUDI DI PADOVA

DIPARTIMENTO DI MATEMATICA TULLIO LEVI-CIVITA

Bachelor’s degree in mathematics

The Adjoint Functor Theorem and some applications

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Introduction

We're no strangers to the need for a weaker definition of equivalence between mathematical objects. For example, when we define a retraction by deformation between two topological spaces, we do not require the two to be isomorphic (homeomorphic), we just need them to be "equivalent enough", that being, to have the same fundamental group; that is done through a pair of morphisms that are not exactly the inverse of one another. We love this kind of definitions because they're usually pretty easy to find and give a simpler description of otherwise hard to check "something-preserving-property".

Our Goal for this script is, in fact, to introduce the Adjoint functor theorem and some of its applications in mathematics: this theorem characterizes adjunctions (sort-of-equivalences between categories) in terms of continuity.

In applying this theorem in its "if and only if" form there is no privileged hypothesis and thesis, we use continuity to prove the presence of adjunctions and we use the presence of adjunctions to prove continuity.

As you probably already know, the rules aren't simple and require us to abandon our usual perspective using groups, rings, fields and so on, giving up the strong base that is set theory and the understanding of objects as "sets with properties" as we usually do. I'll explain in as much detail as I can but a full understanding of the topic is too big of a commitment for a mere bachelor's thesis and too costly in terms of time, space and effort.

My approach to the drafting of this thesis is meant to be Top-down-inside-bottom-up, meaning the succession of chapters and sections follows a bottom-up scheme, going from the basics to the more advanced topics while the explaining inside the sections tries to be top-down, following the flow of discovery and disassembling what we see to better explain how definitions and hypotheses came to be as they are.

What I'm thinking of is a brief recap of the base stuff, meaning Categories, Functors, natural transformations and how to interpret usual set-theoretical concepts in the language of categories, both with categorization (e.g. groups as 1-object groupoids) and through contextualization (e.g. groups as objects in the category of groups).

We'll then spend a chapter on Yoneda's lemma, one of the most important results of the whole subject. So important in fact that it's used to define a lot of the language that will follow, for example we will define limits as initial objects in an element category, a concept that arises from the lemma itself.

We'll learn about limits, what they mean and how they are used to build new objects with useful properties, like products and equalizers.

Finishing with a chapter about adjunctions and the adjoint functor theorem, a theorem giving us a bridge between the continuity of functors and adjunctions.

Up to this point I've been following E. Riehl's approach to the subject (in fact my main source is [Rie16]). Not every step of the ladder is strictly necessary to reach the top but I beliveve they build the necessary understanding to handle the concepts more fluently.

In the last chapter I will explore some first-level applications of the adjoint functor theorem, two being similar theorem regarding -in order- the relationship between continuous and adjoint functors between Grothendieck Topoi and the relationship between exact and cocontinuous functors between module categories. The last application characterizes the algebraicness of compact hausdorff spaces.

Side Note: While writing I was worried you readers wouldn't get behind this style, since my lexic tends to be less formal than what's normally used, but I ultimately decided to leave it like this: I enjoy both reading and writing as informal as possible (doesn't hurt that it feels less like copying from my main source or any other). I don't know if I should have focused on putting down information in a plainer way, but if we ask Guy de maupassant, "[...]Black words on a white page are the soul laid bare". I'll interpret his "are" as "should be".

Side Note 2: I'm writing in english for a couple of reasons: first being that category theory's translations are not always unified and most of the documentation I'm using is in english, second I just think some of the translation are bad. Some are funny, though.

I wanna tell you how bad these get, I can't help the way I'm feeling about them. If I gotta make you hear one: the actual translation for "flabby sheaf" is "fascio flaccido", translated back "flaccid fascist".

We are not going to talk about nor I fully understand flabby sheaves but come on, you get the point.

Abstract

The first four chapters are dedicated to building some basic category theory language and notation:

In chapter 1 we define categories, functors and natural transformations, how they interact and (as a little treat) how they assemble into a 2-category.

In chapter 2 we explore the Yoneda lemma, the representability of functors and what derives from them: category of elements, yoneda embeddings, presheaves and universal elements.

In chapter 3 we take a look at limits and colimits and how they define objects we already know from a categorial standpoint. We also look at how functors interact with limits and define Continuous functors (functors that preserve limits)

In chapter 4 we talk about Adjoint functors, adjunctions and the two flavors of adjoint functor theorems giving us a "a functor is right adjoint if and only if it's continuous" under some conditions: the first (Freyd's) being more general but with a difficult-to-check condition, the second one (Special) more restrictive but very straight-forward.

In the fifth and last chapter we explore three applications to said Adjoint functor theorem(s):

In Topos Theory we apply the adjoint functor theorem to a statement parallel to the SAFT but regarding Grothendieck topoi (a kind of topoi that arise from algebraic geometry), showing that any Grothendieck Topoi actually satisfies the conditions of the SAFT.

In Module Theory we apply the dual SAFT to prove a theorem about exactness and continuity in functors between R-Modules, again proving that any R-Module category satisfies the conditions of the SAFT

Lastly, in Topological Algebra we have our first non-strictly-categorial application in which we show that the category of compact Hausdorff algebras **KHaus** $-A$ is algebraic (actually monadic) by applying the AFT to find the monad that acts like the Free-Forgetful pair over the category **KHaus** $-A$

CHAPTER 1

Categories, functors and natural transformations

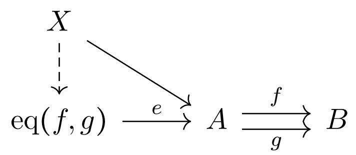
As stated in the introduction, the concepts we will use are the ones of category theory. Category theory has been described as the view of Mathematics through context instead of content: when we want to define a new object in category theory we usually describe the way the object interacts with others.

For an easy example when we define an equalizer using sets we write:

Given $f, g : A \rightrightarrows B$,

$$A \supseteq \mathrm{eq}(f, g) := \{a \in A : f(a) = g(a)\}$$

whereas in category theory an equalizer is



intuitively, the biggest object that satisfies $fe = ge$ where e is the inclusion $eq(f, g) \hookrightarrow a$ (We'll get to a more precise definition in a couple of chapters)

The latter might not be as immediate as the former but you might notice that not once an element of the set was called, meaning that potentially we can describe objects that aren't sets and the definition would work just as well.

1.1 Categories

The main objects of category theory are, unsurprisingly, categories. Let's define them:

Definition 1.1.1 (Category).
*A **category** C consists of a collection of objects $x, y, z, \dots$ denoted $\mathrm{ob}(C)$ and a collection of morphisms (also called maps or arrows) $f, g, h, \dots$ denoted $\mathrm{mor}(C)$ so that:*

1. *Each morphism has specified domain and codomain objects:*
   $\forall f \in \mathrm{mor}(C)\ \exists\, x, y \in \mathrm{ob}(C) \mid f : x \to y$,
2. *Each object has an identity morphism:*
   $\forall x \in \mathrm{ob}(C)\ \exists\, 1_x : x \to x \in \mathrm{mor}(C)$,
3. *It's closed under composition of morphisms:*
   $\forall f : x \to y, g : y \to z\ \exists\, gf : x \to z$,
4. *Identities are actual identities:*
   $\forall f : x \to y\ 1_y f = f 1_x = f$,
5. *Composition of morphisms is associative:*
   $\forall f : x \to y, g : y \to z, h : z \to w \in \mathrm{mor}(C)\ h(gf) = (hg)f =: hgf.$

In this definition objects are given more importance than they actually have: in reality we could define a category without even talking about objects at all and using identity-morphisms instead.

It is more practical, though, to have objects in hand but the main focus of every definition, theorem, and so on will be how the arrows interact with each other.

We'll distinguish between concrete and abstract categories:

Definition 1.1.2 (Concrete and Abstract categories).
*We call a category **concrete** if its objects have an underlying set and some structure and its morphisms are maps preserving that structure (I'll wait to give a more rigorous definition until we talk about functors).*

*An **abstract** category is a category that isn't concrete.*

Some examples will be useful:

Example 1.1.3 (Concrete categories).

- ***Set*** *where objects are sets and morphisms are functions,*
- ***Grp*** *where objects are Groups and morphisms are Group homomorphisms,*
- ***Top*** *where objects are topological spaces and morphisms are continuous maps.*

Example 1.1.4 (Abstract categories).

- ***Htpy*** *where objects are again topological spaces but morphisms are the homotopy classes of continuous maps,*

- *Every group G defines a category* ***BG*** *with one object and an isomorphism for each item in the group,*
- ***Mat***$_R$ *(where R is a unital ring) where objects are natural numbers and morphisms* $n \to m$ *are R-valued* $m \times n$ *matrices.*

We also need to distinguish between sizes of categories: the forementioned **BG** for any finite group G is quite a lot smaller than **Top** for example.

Definition 1.1.5 (Small categories)**.**
We'll say that a category is ***small*** *if it only has a set's worth of morphisms.*

Note. *This means it also has only a set's worth of objects:*

$\mathrm{ob}(C) \simeq \{$*identity-morphisms*$\} \subseteq \mathrm{mor}(C)$.

Another definition that will be quite useful is local smallness:

Definition 1.1.6 (Locally small categories)**.**
A category is ***locally small*** *if for each pair of objects* $x, y \in \mathrm{ob}(C)$ *there's only a set's worth of morphisms* $x \to y$.

We'll call that set $\mathrm{Hom}(x, y)$ *or, more often* $C(x, y)$.

It's quite trivial to show that every small category is locally small but not every locally small category is small: $C(x, y) \subseteq \mathrm{mor}(C)\ \forall x, y$ shows that Small $\implies$ Locally small.

A counterexample we pick **Set**, a locally small category (the set of functions $x \to y$ is a set) that isn't small (Russell's "paradox" tells us that ob(**Set**) is too big to be a set).

Since we'll work a lot with morphisms let's define some types:

Definition 1.1.7 (Mono,Epi and Isomorphisms)**.**

- *A* ***Monomorphism*** *in C is a morphism* $f : x \to y$*s.t.*
  $\forall h, k : z \rightrightarrows x\, fh = fk \implies h = k$ *alternative notations are mono (for short) or in adjectival form "monic";*
- *An* ***Epimorphism*** *in C is a morphism* $f : x \to y$ *s.t.*
  $\forall h, k : y \rightrightarrows z\, hf = kf \implies h = k$ *alternative notations are epi (for short) or in adjectival form "epic";*
- *A* ***Split Monomorphism*** *in C is a morphism* $f : x \to y$ *s.t.* $\exists\, g : y \to x \mid gf = 1_x$.
- *A* ***Split Epimorphism*** *in C is a morphism* $f : x \to y$ *s.t.* $\exists\, g : y \to x \mid fg = 1_y$;
- *An* ***Isomorphism*** *in C is a morphism* $f : x \to y$ *s.t.* $\exists\, g : y \to x \mid fg = 1_x\, ,\, gf = 1_y$.

We can show that:

Theorem 1.1.8.

- *Every Split mono is monic;*

- *Every Split epi is epic;*
- *Every morphism that is a both split mono and epic is an isomorphism;*
- *Every morphism that is both a split epi and monic is an isomorphism.*

We'll posticipate the proof to this fact to the moment we introduce the opposite category, effectively cutting in half the work we need to do.

Let's give a couple of definitions:

Definition 1.1.9 (Subcategory).
*Given a category C a **subcategory** D ≤ C is defined by choosing a subcollection of objects and morphisms in C so that they form a category.*

For example the maximal groupoid of a category is always a subcategory:

Definition 1.1.10 (Groupoids, Maximal Groupoid).
*A **groupoid** is a category where every morphism is an isomorphism.*

*The **Maximal Groupoid** of a category C is the subcategory of C where the only morphisms are C's isomorphisms.*

Note. *We need to prove that the maximal groupoid is actually a subcategory: let's call M the maximal groupoid of C.*

Then

- *Each morphism has specified domain and codomain:*

  it does since ob(M)=ob(C) and mor(M)⊆mor(C);
- *Each object has an identity morphism:*

  1_x *is an isomorphism* $\forall x$ *since* $1_x 1_x = 1_x$, *meaning it is its own left and right inverse;*
- *It's closed under composition:*

  if f *and* g *are composable isomorphism with inverses* f^{-1} *and* g^{-1} *then* $(gf)(f^{-1}g^{-1}) = g(ff^{-1})g^{-1} = g1_y g^{-1} = gg^{-1} = 1_z$ *meaning* gf *is too an isomorphism;*
- *Identities work as identities:*

  this is given from being inside C;
- *Composition is associative:*

  again as an heredity from C.

1.2 Duality

The concept of duality is nothing new to us: dual vector spaces, dual graphs, dual projective spaces and so on: mathematical objects like to be found in pairs.

Categories are no different: For each category C exists a dual category C^{op} with opposite properties.

Definition 1.2.1 (Dual Category)**.**
*Let C be a category. The **opposite** (or **dual**) category C^{op} of C is made of*

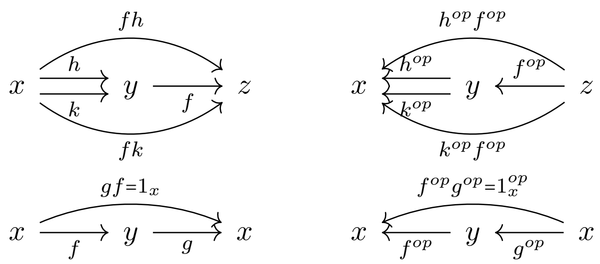
- *ob(C^{op})=ob(C),*
- *A morphism $f^{op} : y \to x$ for each $f : x \to y$.*

We can give it a category structure by using 1^{op}s as identities and by defining composition as $f^{op}g^{op} = (gf)^{op}$.

Intuitively, C^{op} is C but with arrows pointing the opposite direction.

Let's put our duality principle in practice by proving **Theorem 1.1.8**

*Proof of **Theorem 1.1.8**.*
Let's first see that we can use duality, i.e. that every (split) mono in C is (split) epi in C^{op} (and vice versa):



The four represent a monomorphism, an epimorphism, a split monomorphism and a split epimorphism. I believe the "arrow-flipping duality" between the left and right pairs is pretty clear.

We just need to prove that every split mono is mono and that mono + split epi yields an isomorphism, the other two points will follow from duality:

split mono $\implies$ mono:

f split mono $\implies \exists g : gf = 1_X$ so $fh = fk \implies gfh = gfk \implies 1_X h = 1_X k \implies h = k$.

mono + split epi $\implies$ iso:

To prove this we find useful to prove

- If $f : x \to y$ and $g : y \to z$ are mono then $gf : x \to z$ too is monic,
- If $f : x \to y$ and $g : y \to z$ are morphism so that $gf : x \to z$ is monic then f is monic;

and dually

- If $f : x \to y$ and $g : y \to z$ are epi then $gf : x \to z$ too is epic,

- If $f : x \to y$ and $g : y \to z$ are morphism so that $gf : x \to z$ is epic then g is epic;

as usual we'll prove just the first one and argue that the second one also holds by duality

Proof.

- $gfh = gfk \implies g(fh) = g(fk) \implies fh = fk \implies h = k$,
- $gfh = gfk \implies h = k$:

  If f wasn't monic then $\exists\, h, k : fh = fk$ but $h \neq k$. that means $gfh = gfk$ but $h \neq k$, contraddicting the first statement.

$\square$

Now we know that $fg = 1_y$, we need to prove that $gf = 1_x$ too f is mono so g is unique,

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ & {\scriptstyle 1_X} \searrow & \downarrow g \\ & & x \end{array}$$

1_x is an isomorphism, in particular an epimorphism, meaning that (using the fact we just proved) g is epic.

$(gf)g = g(fg) = g1_y = g = 1_xg \implies gf = 1_x$. $\square$

We'll prove the last lemma before moving on:

Lemma 1.2.2.
TFAE:

1. $f : x \to y$ *is an isomorphism in C.*
2. $\forall\, c \in C$

$$\begin{aligned} f_* : C(c, x) &\to C(c, y) \\ g &\mapsto fg \end{aligned}$$

   is a bijection.
3. $\forall\, c \in C$

$$\begin{aligned} f^* : C(y, c) &\to C(x, c) \\ g &\mapsto gf \end{aligned}$$

   is a bijection.

Proof. $1 \iff 2$:

$1 \implies 2$: f bijection $\implies \exists\, g : y \to x$ s.t. $gf = 1_x,\ fg = 1_y$. That means that we can define a map $g_* : C(c, y) \to C(c, x)$.

$g_*f_* : C(c,x) \to C(c,x)$ sends $h \mapsto gfh = h \implies g_*f_* = 1_{C(c,x)}$ and $f_*g_* : C(c,y) \to C(c,y)$ sends $h \mapsto fgh = h \implies f_*g_* = 1_{C(c,y)}$.

$2 \implies 1$: f_* bijection $\implies \exists\, g \in C(y,x)$ s.t. $f_* : C(y,x) \to C(y,y)$ maps $g \mapsto 1_y$. Now by associativity, the elements $gf$and 1_x have the same image under $f_* : C(x,x) \to C(x,y)$ meaning $gf = 1_x$.

$2 \iff 3$: $f_*^{op} : C^{op}(c,y) \to C^{op}(c,x)$is an isomorphism$\forall c \in C^{op} \iff f^* : C(y,c) \to C(x,c)$ is an isomorphism $\forall c \in C$.

The last steps follow trivially from duality. □

1.3 Functors

What we had in the last lemma was a strong relation between a (locally small) category and its Hom-sets. There are other cases of this strong relations between categories: the duality $C \simeq C^{op}$, the embedding **Ab**↪**Grp** and many more.

How do we encode this morphisms between categories?

Definition 1.3.1 (Functors)**.**
Given two categories C, D, a ***functor*** $F : C \to D$ *is a map with the following properties:*

1. *Moves objects:*
   $\exists$ *an object* $Fc \in D \ \forall c \in C$;
2. *Moves morphisms:*
   $\exists$ *a morphism* $Ff : Fc \to Fc' \in D \ \forall\, f : c \to c' \in C$;
3. *Mantains composition:*
   $Fg \cdot Ff = F(g \cdot f) \forall\, f, g \in C$;
4. *Mantains identities:*
   $F(1_c) = 1_{Fc} \forall\, c \in C$.

We call this type of Functors "Covariant" to distinguish them from contravariant ones:

Definition 1.3.2 (Contravariant Functors)**.**
A ***contravariant functor*** $F : C \to D$ *is a (covariant) functor* $F : C^{op} \to D$

We'll think of a covariant functor as "maintaining morphisms in the right direction" and a contravariant functor as "flipping morphisms". Showing it in a simple diagram:

(a) Covariant functor (b) Contravariant functor

Functors are morphisms between categories, and they behave as such:

Lemma 1.3.3 (Functors preserve isomorphisms).
Let A, B be categories and $F : A \to B$ a functor, then if $f \in \mathrm{mor}(A)$ is an isomorphism, $Ff \in \mathrm{mor}(B)$ is too an isomorphism.

Proof.
If f is an isomorphism with inverse g we have that

$F(g)F(f) = F(gf) = F(1_x) = 1_{Fx} \implies Fg$ is a left inverse of Ff exchanging the role of f and g shows that it also is a right inverse. □

We've seen the functor $C(c, \_)$ before (even though we didn't properly call it a functor).

Lemma 1.3.4 (Pair of functors represented by an object).
Given a locally small category C we can define a pair of functors

$C(c, \_) : C \to$ ***Set*** *(covariant) and*
$C(\_, c) : C^{op} \to$ ***Set*** *(contravariant) acting like we've seen in* ***Lemma 1.2.2.***

Proof.
We need to prove that this is actually a functor:

1. Moves objects:

   C is locally small $\implies C(c, x) \in$ **Set**$\forall x \in C$;

2. Moves Morphisms:

   $C(c, f) = f^*$ is a morphism between sets $\forall f \in C$;

3. Mantains composition:

   $f^* g^* = (fg)*$ (already proven);

4. Mantains identity:

   $1_c^* = 1$;

and dually for the other functor. □

A nicer way to encode the same information is through a **bifunctor**, i.e. a functor of two variables. To define that, though, we first need to define the product category:

Definition 1.3.5 (Product category).
Let C and D be categories, the ***product category*** *$C \times D$ is a category whose objects are pairs $(c, d) : c \in C$, $d \in D$ and whose morphisms are pairs $(f, g) : f \in C$, $g \in D$ simple like that.*

Using this new definition we can encode the information in $C(c,_)$ and $C(,C)$ into a **two-sided represented functor**

$$\begin{array}{rl} C(\_,\_) : C \times C^{op} \to & \textbf{Set} \\ (x,y) \mapsto & C(x,y) \quad g \\ {}_{(f,h)} \downarrow \mapsto & \downarrow_{(f^*,h_*)} \quad \downarrow \\ (z,w) \mapsto & C(z,w) \; hgf \end{array}$$

From two functors $F : A \to C$ and $G : B \to C$ we can create what's called a Comma category:

Definition 1.3.6 (Comma category)**.**
The Comma category $F \downarrow G$ is the category whose objects are triplets (x, y, f) assembled as

$$Fx \xrightarrow{f} Gy$$

and whose morphisms are pairs (h,k) so that

$$\begin{array}{ccc} Fx & \xrightarrow{f} & Gy \\ {\scriptstyle Fh}\downarrow & & \downarrow{\scriptstyle Gk} \\ Fx' & \xrightarrow{f'} & Gy' \end{array}$$

commutes.

$F \downarrow G$ is indeed a category, since

- (By definition) every morphism has domain and codomain;
- Composition is respected (because F and G are functors);
- Has identity-morphisms (the couples $(1_x, 1_y)$);
- Associativity is respected (because F and G are functors).

We defined categories as objects and morphisms; we now have categories and morphisms between categories, it's natural to ask ourselves: "does the category of all categories exist?"

Definition 1.3.7 (Cat and CAT)**.**
We'd like to define the category of categories but we are limited by the logical constraint that no object should contain itself (Russell, again).

***Cat** is the category whose objects are small categories and whose morphisms are functors,*

***CAT** is the category whose objects are locally small categories and whose morphisms are functors.*

Note that **Cat** is locally small but not small, and **CAT** is not locally small, meaning the constraints we put are well placed.

Definition 1.3.8 (Some classes of functors).
Given two locally small categories C, D, a functor $F : C \to D$ is:

- ***Faithful*** *if $C(x, y) \to D(Fx, Fy)$ is injective $\forall x, y \in ob(C)$,*
- ***Full*** *if $C(x, y) \to D(Fx, Fy)$ is surjective $\forall x, y \in ob(C)$,*
- ***Essentially surjective on objects*** *if $\forall\, d \in D \exists\, c \in C : d \simeq Fc$,*
- ***Fully faithful*** *if it's full and faithful,*
- *An* ***embedding*** *if it's faithful and injective on objects,*
- *A* ***full embedding*** *if it's full and an embedding.*

We'll see a couple of important embeddings in the next chapter.

1.4 Naturality and equivalence

We can talk about transformation between functors too:

Definition 1.4.1 (Natural transformations).
Given a pair of functors $F, G : C \rightrightarrows D$ a ***natural transformation*** *$\alpha : F \Rightarrow G$ consists of an arrow $\alpha_c : Fc \to Gc$ for each object $c \in C$ so that*

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow{\alpha_{c'}} & Gc' \end{array}$$

commutes.

A ***natural isomorphism*** *is a natural transformation where every α_c is an isomorphism.*

We'll also draw natural transformations with naturality diagrams

$$C \underset{G}{\overset{F}{\underset{\alpha \Downarrow}{\rightrightarrows}}} D$$

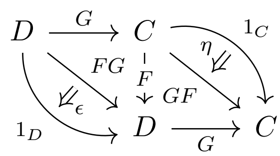
when we need to emphasize that its "morphism between functors"' nature.

Natural transformations are useful for various reasons, one of them is defining equivalences: in the entirety of mathematics we use isomorphisms as "pretty much identities", like if two groups are isomorphic we treat them as the same group.

We use naturality to give a similar, rigorous understanding of isomorphism between categories.

Definition 1.4.2 (Equivalences between categories)**.**
*A natural **equivalence** between two categories C and D is a pair of functors $F : C \rightleftarrows D : G$ equipped with two natural isomorphisms $\epsilon : FG \Rightarrow 1_D$ and $\eta : 1_C \Rightarrow GF$.*

*If two categories possess this pair of functors and naturals they are called **equivalent** and we write $C \simeq D$.*

As a diagram 

Of course

Lemma 1.4.3.
Natural equivalence is an equivalence relation.

Proof.

- $C \simeq C$: trivially using the identity for both functors and the identity for both the naturals,
- $C \simeq D \implies D \simeq C$: trivially by exchanging the role of F and G,
- $C \simeq D, D \simeq E \implies C \simeq E$:

  let's call F, G, η, ϵ the equivalence $C \simeq D$ and F', G', η', ϵ' the equivalence $D \simeq E$. The two functors that we are looking for our equivalence are $F'F$ and GG'

  $C \underset{G}{\overset{F}{\rightleftarrows}} D \underset{G'}{\overset{F'}{\rightleftarrows}} E$ now $F'FGG'x \simeq_\eta F'G'x \simeq_{\eta'} 1_E x \ \forall x$

  and $GG'F'Fx \simeq_{\epsilon'} GFx \simeq_\epsilon 1_C x \ \forall x$.

□

We have an easier way to characterize a cat equivalence:

Theorem 1.4.4 (Characterization of an equivalence functor)**.**
Any functor defining an equivalence is faithful, full and essentially surjective on objects.

Moreover, (assuming the axiom of choice) any functor with said properties defines an equivalence between categories.

Proof.
Suppose F, G, η, ϵ are the two couples of functors and naturals that realize the equivalence as per the definition:

$\forall\, d \in D$ the component $\eta_d : FGd \simeq d$ demonstrates that F is essentially surjective. consider $f, g : c \rightrightarrows c'$ in C. $Ff = Fg \implies$

$$\begin{array}{ccc} c & \xrightarrow[\simeq]{\epsilon_c} & GFc \\ {\scriptstyle f\, or\, g}\big\downarrow & & \big\downarrow{\scriptstyle GFf = GFg} \\ c' & \xrightarrow[\simeq]{\epsilon_{c'}} & GFc' \end{array}$$

commutes, expressing that the naturality of ϵ commutes.

using that ϵ is a natural isomorphism $\exists!\, c \to c'$ with this property, whence $f = g$.

This means F is faithful, and by simmetry so is G. Similarly

$$\begin{array}{ccc} c & \xrightarrow[\simeq]{\epsilon_c} & GFc \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle Gk\, or\, GFh} \\ c' & \xrightarrow[\simeq]{\epsilon_{c'}} & GFc' \end{array}$$

by the naturality of ϵ we have that $GFh = Gk \implies Fh = k$ (G is faithful) $\implies F$ faithful and essentially surjective.

For the converse suppose $F : C \to D$ is full, faithful and essentially surjective on objects. Using essential surjectivity and the axiom of choice $\forall\, d \in D$ we can choose an object $Gd \in C$ and an isomorphism $\eta_d : FGd \simeq d$. $\forall\, l : d \to d'$. We have an unique morphism

$$\begin{array}{ccc} FGd & \xrightarrow[\simeq]{\eta_d} & d \\ \big\downarrow & & \big\downarrow{\scriptstyle l} \\ FGd' & \xrightarrow{\eta_{d'}} & d' \end{array}$$

since F is fully faithful $\exists!\, Gd \to Gd'$ with this image under F, which we define to be Gl. This definition is arranged so that the chosen isomorphisms assemble into the components of a natural transformation $\eta : FG \Rightarrow 1_D$. It remains to prove that the assignment of arrows $l \mapsto Gl$ is functorial and to define the natural isomorphism $\epsilon : 1_C \Rightarrow GF$.

The functoriality of G is another natural consequence of the faithfulness of F. The morphisms $FG1_d$ and $F1_{Gd}$ both make

$$\begin{array}{ccc} FGd & \xrightarrow[\simeq]{\eta_d} & d \\ {\scriptstyle FG1_d\, or\, F1_{Gd}}\big\downarrow & & \big\downarrow{\scriptstyle 1_d} \\ FGd & \xrightarrow[\simeq]{\eta_d} & d \end{array}$$

commute. This implies $G1_d = 1_{Gd}$. Similarly given $l' : d' \to d''$, both $F(Gl' \cdot Gl)$ and $FG(l'l)$ make

$$\begin{array}{ccc} FGd & \xrightarrow[\simeq]{\eta_d} & d \\ {\scriptstyle F(Gl' \cdot Gl)\, or\, FG(l'l)}\big\downarrow & & \big\downarrow{\scriptstyle l'l} \\ FGd'' & \xrightarrow[\simeq]{\eta''_d} & d'' \end{array}$$

commute, whence $Gl' \cdot Gl = G(l'l)$.

Finally by full faithfulness of F we may define the isomorphisms $\epsilon_c : c \to GFc$ by specifying $F\epsilon_c : Fc \to FGFc$. Define $F\epsilon_c$ to be ϵ_{Fc}^{-1}. $\forall\, f : c \to c'$ the outer rectangle

$$\begin{array}{ccccc} Fc & \xrightarrow{F\eta_c} & FGFc & \xrightarrow{\eta_{Fc}} & Fc \\ \big\downarrow{\scriptstyle Ff} & & \big\downarrow{\scriptstyle FGFf} & & \big\downarrow{\scriptstyle Ff} \\ Fc' & \xrightarrow{F\eta_{c'}} & FGFc' & \xrightarrow{\eta_{Fc'}} & Fc' \end{array}$$

commutes, both composites being Ff.

The right hand square commutes by naturality of η. since $\eta_{Fc'}$ is an isomorphism, this implies that the left hand square commutes; the faithfulness of F tells us that $\epsilon_{c'} \cdot f = GFf \cdot \epsilon_c$, i.e. ϵ is a natural transformation. □

1.5 Diagrams

We've drawn quite a lot of shapes with arrows and objects, but what do they mean?

These so called "commutative diagrams" are a very common and useful way to express properties, induce reasoning and even prove theorems: It's about time we introduce them formally.

Definition 1.5.1 (Diagram).
*A **diagram** in a category C is a functor $D : J \to C$ where J, the so-called **indexing category** is a small category.*

Diagrams do not exist in a void: what if we found an useful diagram in a category and want to have it in another category?

Lemma 1.5.2.
Functors preserve commutative diagrams.

Proof.
Given $F : A \to B$ a functor and $D : J \to A$ a diagram in A, then $FD : J \to B$ is a diagram in B. □

Lemma 1.5.3.
If $F : A \to B$ is faithful, then any diagram in A whose image commutes in B also commutes in A.

Proof.
A diagram in A is (WLOG) a directed graph with composable morphisms in A so that $Ff_n \cdots Ff_1 = Fg_m \cdots Fg_1$ in B,

by faithfulness and functoriality of F, $f_n \cdots f_1 = g_m \cdots g_1$ in A. □

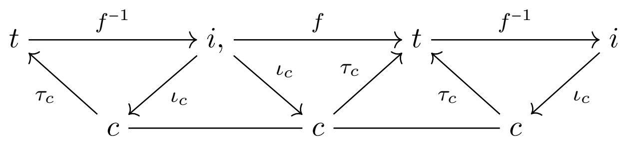
1.5.1 Initial, Terminal objects, Examples

Definition 1.5.4 (Initial, terminal and Zero objects)**.**
Given a category C

- *An object $i \in C$ is **initial** if $\forall\, c \in C \exists!\ i \to c$,*
- *An object $t \in C$ is **terminal** if $\forall\, c \in C \exists!\ c \to t$,*
- *An initial and terminal object is called a **zero object**.*

Lemma 1.5.5.
Any map $t \to i$ is an isomorphism.

Proof.
$\forall c \in C \exists! \tau_c : t \to c$ and $\exists!\ \iota_c : i \to c$

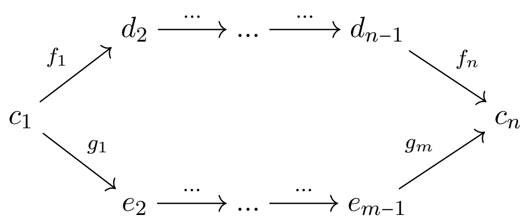


$f f^{-1} \tau_c = \tau_c$ and $\iota_c f^{-1} f = \iota_c$. Since t is terminal

$t \underset{\exists! \tau_c}{\overset{f}{\rightleftarrows}} c \implies \tau_c f = 1_t \implies t$ split epi $\implies$ epi dually ι_c split mono $\implies$ mono.

$\implies f f^{-1} = 1_t$ and $f^{-1} f = 1_i$. □

Lemma 1.5.6.
Let ${}_1, \dots, f_n$ and $g_1, \dots, g_m$ be composable sequences of morphisms so that the domain of f_1 equals the domain of g_1 and the codomain of f_n equals the codomain ofg_m. If this common codomain is a terminal object, or if this common domain is an initial object, then $f_n \cdots f_1 = g_m \cdots g_1$.



Proof.
Follows trivially from the uniqueness of the morphisms in the definition of initial and terminal objects. □

Diagrams let us define general objects that basically behave the same way regardless of the category they're in, and the fact that functors preserve commutative diagrams serves as a powerful way to comunicate between categories.

Example 1.5.7.

- *A **Monoid** is an object $M \in$ **Set** with a pair of morphisms $\mu : M \times M \to M$; $\eta : 1 \to M$ so that*

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{1_M \times \mu} & M \times M \\ {\scriptstyle \mu \times 1_M}\downarrow & & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow{\mu} & M \end{array} \qquad \begin{array}{ccccc} M & \xrightarrow{\eta \times 1_M} & M \times M & \xleftarrow{1_M \times \eta} & M \\ & {\scriptstyle 1_M}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_M} & \\ & & M & & \end{array}$$

commute.

- *A **Topological Monoid** is an object $M \in$ **Top** with a pair of morphisms $\mu : M \times M \to M$; $\eta : 1 \to M$ so that*

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{1_M \times \mu} & M \times M \\ {\scriptstyle \mu \times 1_M}\downarrow & & \downarrow{\scriptstyle \mu} \\ M \times M & \xrightarrow{\mu} & M \end{array} \qquad \begin{array}{ccccc} M & \xrightarrow{\eta \times 1_M} & M \times M & \xleftarrow{1_M \times \eta} & M \\ & {\scriptstyle 1_M}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_M} & \\ & & M & & \end{array}$$

commute.

- *A **Coalgebra** for an endofunctor $T : C \to C$ is an object $c \in C$ equipped with a map $\gamma : c \to Tc$. a morphism of coalgebras $f : (c, \gamma) \to (c', \gamma')$ is a map $f : c \to c'$ so that the square*

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma'} \\ Tc & \xrightarrow{Tf} & Tc' \end{array}$$

commutes.

1.6 Extra: the 2-category of categories

For completeness we'll now venture a bit into "higher-cat" territory. it's not strictly necessary for the subject of this thesis but we may use its terminology to describe operations between natural transformations.

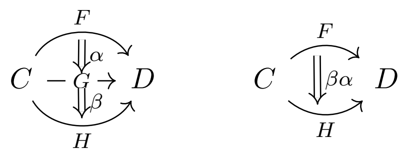
Even though adjunctions *do form a 2-category* we will not talk about it.

We usually have a category when we have objects and morphisms; a 2-category has objects, morphisms, and 2-morphisms. i.e. objects, morphisms between objects and morphisms between morphisms. It's easy to see how this can be generalized talking about "n-categories", "∞-categories" and (maybe someday) "(n,r)-categories" (which put some constrictions on their k-morphisms. There are some fun examples of them, like the (∞,1)-category of "all mathematical concepts").

We can't talk about 2-categories without "vertical and horizontal composition". Let's introduce them:

Lemma 1.6.1 (Vertical composition)**.**
Suppose $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ are natural transformations between parallel functors $F, G, H : C \to D$. Then there is a natural transformation $\beta\alpha : F \Rightarrow H$ defined as $(\beta\alpha)_c := \beta_c \cdot \alpha_c$.

We'll call this ***vertical composition*** *and draw its diagram as follows*



Proof.
Naturality of α and β implies that for any $f : c \to c'$ in the domain category each square (and thus each rectangle) commutes

$$\begin{array}{ccccc} Fc & \xrightarrow{\alpha_c} & Gc & \xrightarrow{\beta_c} & Hc \\ \big\downarrow{\scriptstyle Ff} & & \big\downarrow{\scriptstyle Gf} & & \big\downarrow{\scriptstyle Hf} \\ Fc' & \xrightarrow{\alpha_{c'}} & Gc' & \xrightarrow{\beta_{c'}} & Hc' \end{array}$$

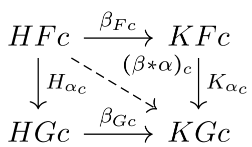
□

Corollary 1.6.2.
For any pair of categories C and D the functors $C \to D$ and natural transformation define a category D^C.

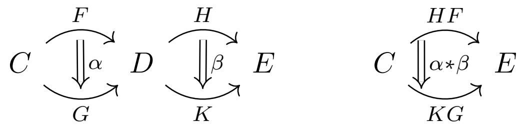
Proof.
Vertical composition is associative and unital 'cause of associativity and unitality of composition in D. □

It's pretty natural to guess that vertical composition is not the only composition we have.

Lemma 1.6.3 (Horizontal composition)**.**
Given a pair of natural transformations $\alpha : F \Rightarrow G, \beta : H \Rightarrow K$ between functors $F, G : C \to D$ and $H, K : D \to E$ there is a natural transformation $\beta * \alpha : HF \to KG$ defined as



*We'll call this **horizontal composition** and draw the diagram as follows*



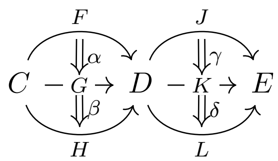
Proof.

$$\begin{array}{ccccc} HFc & \xrightarrow{H\alpha_c} & HGc & \xrightarrow{\beta_{Gc}} & KGc \\ \downarrow{\scriptstyle HFf} & & \downarrow{\scriptstyle HGf} & & \downarrow{\scriptstyle KGf} \\ HFc' & \xrightarrow{H\alpha_{c'}} & HGc' & \xrightarrow{\beta_{Gc'}} & KGc' \end{array}$$

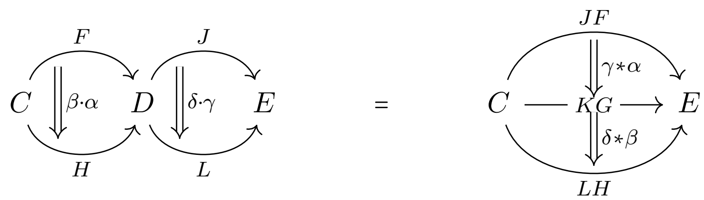
The left hand square commutes by naturality of α, the right hand square by naturality of β; H preserves commutative diagrams meaning that $KGf \cdot (\beta * \alpha)_c = (\beta * \alpha)_{c'} \cdot HFf \ \forall f : c \to c'$ in C. □

Let's see how those compositions interact with one another.

Lemma 1.6.4 (Middle four interchange).
Given functors and natural transformation as in the diagram



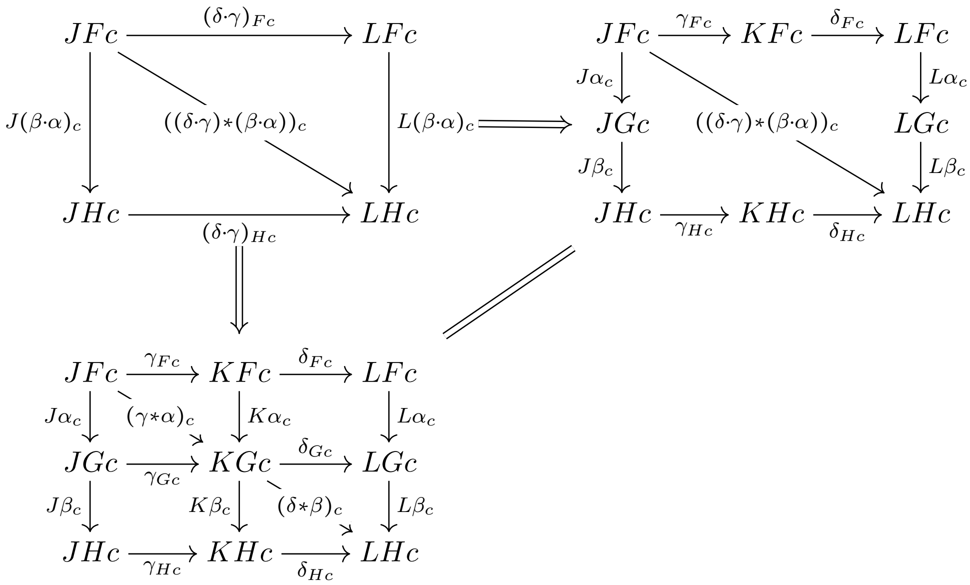
then



i.e. $(\delta \cdot \gamma) * (\beta \cdot \alpha) = (\delta * \beta) \cdot (\gamma * \alpha)$.

Proof.

Let's develop the diagrams



all those diagrams are equivalent, meaning the diagonal arrow is the same arrow.

$((\delta\cdot\gamma)*(\beta\cdot\alpha))_c = ((\delta*\beta)\cdot(\gamma*\alpha))_c \ \forall c$.

The two are the same transformation. □

All that's left is to define a 2-category:

Definition 1.6.5 (2-category)**.**
*A **2-category** is comprised of*

1. *A collection of objects,*
2. *A collection of 1-morphisms (between objects),*
3. *A collection of 2-morphisms (between 1-morphisms), so that*
    a) *The objects and 1-morphisms form a category,*
    b) *Morphisms and 2-morphisms form a category under vertical composition,*
    c) *Morphisms and 2-morphisms form a category under horizontal composition,*
    d) *The law of middle four interchange holds.*

CHAPTER 2

Universal properties, representability, the Yoneda lemma

The Yoneda lemma is probably the most important theorem in this whole subject. Functors are hard to use by their own, things get pretty big pretty fast. There are, however some special functors, that behave like the Represented functors we already encountered.

The Yoneda lemma encodes natural transformation between functors in a quite simple set.

We'll see its potential in the following chapters, especially the fourth.

2.1 Representable functors

We already encountered representable functors in the previous chapter, $C(c, \_)$ sending an object into its hom-set with c and $C(\_, c)$, its contravariant sibling.

Those functors are pretty important for our study of categories, so much that we define Representable functors as functors that act like them:

Definition 2.1.1 (Representable functors and Representations)**.**

1. *A functor $F : C \to D$ (where C and D are locally small categories) is called* ***representable*** *if there is an object $c \in C$ and a natural isomorphism between F and $C(c, \_)$ (or $C(\_, c)$, depending on F's variance).*

   In that case we say that F is ***represented*** *by c.*

2. *A **representation** for a functor F is a choice of object $c \in C$ together with a natural isomorphism $C(c, \_) \simeq F$ (or $C(\_, c) \simeq F$, again depending on F's variance).*

We can characterize objects using representations:

Definition 2.1.2 (Representble characterization of initial and terminal objects)**.**

- *An object $c \in C$ is **initial** if $C(c, \_) : C \to$ **Set** is naturally isomorphic to the costant functor $* : C \to$ **Set** which sends every object to the singleton set.*
- *An object $c \in C$ is **terminal** if $C(\_, c) : C^{op} \to$ **Set** is naturally isomorphic to the costant functor $* : C^{op} \to$ **Set** which sends every object to the singleton set.*

2.2 The Yoneda lemma

Theorem 2.2.1 (Yoneda lemma)**.**
*For any functor $F : C \to$ **Set**, whose domain C is locally small and any object $c \in C$ there is a bijection*

$$Hom(C(c, \_), F) \simeq Fc$$

that associates a natural transformation $\alpha : C(c, \_) \Rightarrow F$ to the element $\alpha_c(1_c) \in Fc$.

Moreover this correspondence is natural in both F and c.

Note. *In this version F is supposed covariant but the contravariant version uses $C(\_, c)$ the exact same way.*

Note also that C isn't necessarily small, implying $Hom(C(c, \_), F)$ needn't be a set. The smallness of this Hom-set is one of the strong points of this lemma.

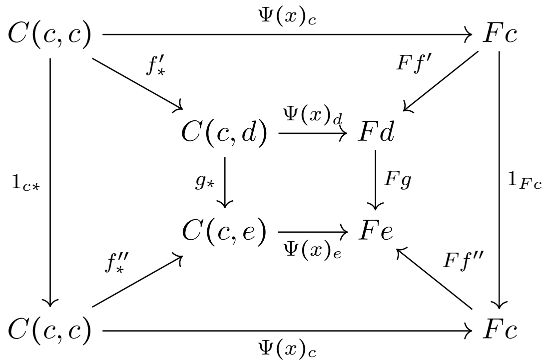
Proof of the bijection.
There is clearly a function $\Phi : Hom(C(c, \_), F) \to Fc$ that maps a natural transformation to the image of 1_c, i.e. $\Phi(\alpha) = \alpha_c(1_c)$.

To prove it is a bijection we need to find an inverse: let's define $\Psi : Fc \to Hom(C(c, \_), F)$ that constructs a natural transformation $\Psi(x) : C(c,_) \Rightarrow F$ for every $x \in Fc$. To achieve this we define $\Psi(x)_d : C(c, d) \to Fd$ so that

$$\begin{array}{ccc} C(c,c) & \xrightarrow{\Psi(x)_c} & Fc \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle Ff} \\ C(c,d) & \xrightarrow{\Psi(x)_d} & Fd \end{array}$$

commutes.

We have that



also commutes, proving the naturality of $\Psi(x)$. □

Proof of naturality.
First we prove that the isomorphism is natural in the choice of F, i.e. given $\beta : F \Rightarrow G$ the element of Gc representing the composite natural transformation $\beta\alpha : C(c,\_) \Rightarrow F \Rightarrow G$ is the image under $\beta_c : Fc \to Gc$ of the element Fc representing $C(c,\_) \Rightarrow F$, implying that the diagram

$$\begin{array}{ccc} Hom(C(c,\_),F) & \xrightarrow[\simeq]{\Phi_F} & Fc \\ \downarrow{\scriptstyle \beta_*} & & \downarrow{\scriptstyle \beta_c} \\ Hom(C(c,\_),G) & \xrightarrow[\simeq]{\Phi_G} & Gc \end{array}$$

commutes in **Set**. By definition $\Phi_G(\beta\alpha) = (\beta\alpha)_c(1_c) = \beta_c(\alpha_c(1_c)) = \beta_c(\Phi_F(\alpha))$.

Now we need to prove naturality in the choice of c, i.e. given a morphism $f : c \to d$ in C, the element of Fd representing the composite natural $\alpha f^* : C(d,\_) \Rightarrow C(c,\_) \Rightarrow F$ is the image under $Ff : Fc \to Fd$ of the element Fc representing α, meaning that the diagram

$$\begin{array}{ccc} Hom(C(c,\_),F) & \xrightarrow[\simeq]{\Phi_c} & Fc \\ \downarrow{\scriptstyle (f^*)^*} & & \downarrow{\scriptstyle Ff} \\ Hom(C(d,\_),F) & \xrightarrow[\simeq]{\Phi_d} & Fd \end{array}$$

commutes.

Here the image of $\alpha \in Hom(C(c,\_),F)$ along the top-right composite is $Ff(\alpha_c(1_c))$ and the image along the left bottom composite is $(\alpha f^*)_d(1_d)$. We know that $\alpha f^* =$

$$\begin{array}{ccccc} C(d,d) & \xrightarrow{f^*} & C(c,d) & \xrightarrow{\alpha_d} & Fd \\ & & & & \\ 1_d & \longmapsto & f & \longmapsto & \alpha_d(f) \end{array}$$

we know that $\alpha_d(f) = Ff(\alpha_c(1_c))$ thus proving the second square commutes. □

This theorem lets us have two fundamental embeddings for categories:

Corollary 2.2.2 (Yoneda embeddings).
The functors

$$C \xhookrightarrow{y} \textbf{\textit{Set}}^{C^{op}} \qquad\qquad C^{op} \xhookrightarrow{y} \textbf{\textit{Set}}^{C}$$

$$\begin{array}{ccc} c & \longmapsto & C(\_,c) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_*} \\ d & \longmapsto & C(\_,d) \end{array} \qquad\qquad \begin{array}{ccc} c & \longmapsto & C(c,\_) \\ {\scriptstyle f}\downarrow & & \uparrow \\ d & \longmapsto & C(d,\_) \end{array}$$

are fully faithful embeddings.

Proof.
We need to prove that

$$C(c,d) \to Hom(C(\_,c),C(\_,d)) \text{ and } C(c,d) \to Hom(C(d,\_),C(c,\_)$$

are bijections. We already know from the first chapter that they are injections, seen that distinct morphisms $c \rightrightarrows d$ induce distinct natural transformations.

Yoneda's lemma implies that every natural transformation arises that way, i.e. every $\alpha : C(d,\_) \Rightarrow C(c,\_)$ correspond to elements of $C(c,d)$, which are the morphisms $C \to d$. The natural transformation $f^* : C(d,\_) \Rightarrow C(c,\_)$ (pre-composition by f) sends 1_d to f, thus $\alpha = f^*$ is unique, proving its surjectivity.

The analogue using pre-composition holds for the $C(\_,x)$. □

It's not strictly necessary but I feel like it's cute to show an important application of this lemma:

Example 2.2.3 (Cailey's theorem).
Every group is isomorphic to a subgroup of a permutation group.

Proof.
Regarding a group G as a one-object groupoid BG we can look at the image of the covariant Yoneda embedding $BG \hookrightarrow$ **Set**$^{BG^{op}}$: Consider a $G - set\ X : BG \to$ **Set**. a natural transformation $G \Rightarrow X$ is exactly a $G - equivariant$ map $\phi : G \to X$, implying that the aforementioned image is a right G-set G. The corollary tells us that the only G-equivariant endomorphisms of the right G-set G are those maps defined by left multiplication with a fixed element of G. In particular any G-equivariant endomorphism of G must be an isomorphism too.

In this way the Yoneda embedding defines an isomorphism between G and the automorphism group of the right G-set G, an object in **Set**$^{BG^{op}}$. Composing with the faithful forgetful functor $\mathbf{Set}^{BG^{op}} \to Set$ we obtain an embedding $G \hookrightarrow Sym(G)$, the first isomorphism theorem tells us that means $G \simeq$ subgroup of $Sym(G)$ which is a permutation group. □

2.3 Universal properties and universal elements

We can use representability (actually, representable functors) to describe some interesting elements and properties in a category.

Proposition 2.3.1 (Representable isomorphisms).
Consider a pair of objects x and y in a locally small category C, then

$C(x, \_) \simeq C(y, \_)$ *or* $C(\_, x) \simeq C(\_, y) \implies x \simeq y,$

in particular, if x and y represent the same functor then $x \simeq y$.

Proof.
The Yoneda embeddings $C \hookrightarrow \mathbf{Set}^{C^{op}}$ and $C^{op} \hookrightarrow \mathbf{Set}^{C}$ are both fully faithful, meaning they create isomorphisms. an isomorphism between represented functors is induced by a unique isomorphism between their representing objects, meaning they must be isomorphic. □

From this we can extract a corollary:

Corollary 2.3.2.
The full subcategory of C spanned by its terminal objects is either empty or is a contractible groupoid.

Where

Definition 2.3.3 (Contractible Groupoids).
*A **Contractible Groupoid** is a category equivalent to the terminal category $\mathbb{1}$, i.e. a category with exactly one morphism in each hom-set.*

Proof.
We know that $C(\_, t) \Rightarrow C(\_, t') \simeq C(t, t') \simeq$ singleton set.

The last proposition tells us that the only morphism in there must be an isomorphism, meaning that t, t' are terminal if and only iff they represent the functor $* : C^{op} \to \mathbf{Set}$ (constant at the singleton set). □

The Yoneda Lemma also gives us access to what we call universal properties:

Definition 2.3.4 (Universal Properties).
*A **Universal Property** of an object $c \in C$ is expressed by a functor F together with an element $x \in Fc$ which we'll call a **Universal Element** that defines a natural isomorphism $C(c, \_) \simeq F$ or $C(\_, c) \simeq F$ depending on F's variance as usual.*

Example 2.3.5 (Tensor Product).
*Fix 2 $\Bbbk$–vector spaces $V, W \in$ **Vect**$_{\Bbbk}$ and consider the functor*

$$\mathrm{Bilin}(V, W; \_) : \textbf{\textit{Vect}}_{\Bbbk} \to \textbf{\textit{Set}}$$

*sending a vector space to its bilinear maps $V \times W \to U$ $U \mapsto \mathrm{Bilin}(V, W; U)$. We know there is one space realizing the **Vect**$_{\Bbbk}(c, \_) \Rightarrow \mathrm{Bilin}(V, W; \_)$ as an isomorphism, we'll call it $V \otimes_{\Bbbk} W$, i.e.*

$$\exists! V \otimes_{\Bbbk} W \ s.t. \textbf{\textit{Vect}}_{\Bbbk}(V \otimes_{\Bbbk} W, U) \simeq \mathrm{Bilin}(V, W; U).$$

This isomorphism is determined by a universal element of $\mathrm{Bilin}(V, W; V \otimes_{\Bbbk W}$ i.e. the bilinear map $\otimes : V \times W \to V \otimes_{\Bbbk} W$.

2.4 The category of elements

The last tool we will define in this section is going to be the category of elements: We spent some lines describing how an object $c \in C$ describes elements of $Fc \in$ **Set** via a co(contra)variant functor $F : C \to$ **Set**.

We can describe F using a category that we will define this way:

Definition 2.4.1 (Category of elements).
*The **Category of elements** of a co(contra)variant functor $F : C \to Set$ is defined as follows:*

$\int^{c:C} F$ or $\int F$ is the category whose objects are pairs $(c, x) : c \in C, x \in Fc$ and whose morphisms $(c, x) \to (c', x')$ are the morphisms $f : c \to c' \in C$ so that $Ff(x) = x'$ ($Ff(x') = x$ for the contravariant case) every element category has an obvious forgetful functor $\Pi : \int F \to C$ that sends $(c, x) \mapsto c$ and $(f : (c, x) \to (c', x')) \mapsto (f : c \to c')$

We can use this category to get a condition for representability:

Proposition 2.4.2 (Universal elements are universal elements).
*A co(contra)variant functor $F : C \to$ **Set** (or $C^{op} \to$ **Set**) is representable if and only if its category of elements has an initial (terminal) object.*

Proof.
$\Longrightarrow$ is quite trivial: $\int F \simeq \int C(c, \_) \simeq c \downarrow 1_C$ which has initial object $1_c \in c \downarrow 1_C$

$\Longleftarrow$ uses the Yoneda lemma:

Consider a functor $F : C \to$ **Set** and suppose $(c, x) \in \int F$ is initial We will show that the natural transformation $\Psi(x) : C(c, \_) \Rightarrow F$ defined by the Yoneda lemma is a natural isomorphism: For any $y \in Fd$ initiality of $x \in Fc$ states that $\exists! f : (c, x) \to (d, y) \implies \exists! f : c \to d \in C$ s.t. $Ff(x) = y$.

That means $\Psi(x)_d : C(c, d) \to Fd$ is an isomorphism.

Reversing the argument, a natural isomorphism $\alpha : C(c, \_) \simeq F$ defines an object $\alpha_c(1_c) \in Fc$. For each $\alpha_d : C(c, d) \simeq Fd \implies \forall y \in Fd\ \exists! f : c \to d : Ff(\alpha_c(1_c)) = y$. thus

$\forall (d, y) \in \int F \exists! f : (c, \alpha_c(1_c)) \to (d, y) \in \int F$ meaning $(c, \alpha_c(1_c))$ is initial in $\int F$. □

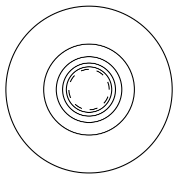
CHAPTER 3

Limits and Colimits

Aside from being necessary to the understanding of the theory itself, the main matter of this thesis - the adjoint functor theorem - has a lot to do with limits. We encountered Limits in analysis, then in topology, then in algebra but what are they?

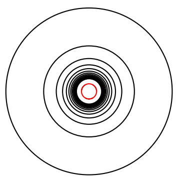
Let $F : \mathbb{N} \to Set$ (regarding $\mathbb{N}$ as a preorder) be a diagram shaped as a chain of inclusions

$$\cdots \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow X_0$$



The limit of this sequence, intuitively is the object at the start of this sequence, i.e. the biggest object X_∞ so that the diagram

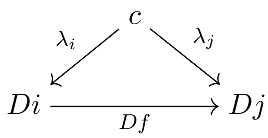
$$\begin{array}{ccccccc} & & {\color{red}X_\infty} & & & & \\ & \swarrow & \downarrow & \searrow & & \searrow & \\ \cdots & \hookrightarrow & X_2 & \hookrightarrow & X_1 & \hookrightarrow & X_0 \end{array}$$



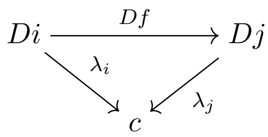
commutes. In other words if there is another object satisfying the commutative diagram it must exist an arrow pointing toward the limit.

3.1 Limits and colimits as universal cones

Definition 3.1.1 (Cones).
*A **cone over** a diagram $D : J \to C$ with **summit** or **apex** $c \in C$ is a natural transformation $\lambda : c \Rightarrow F$ where c is the constant functor at c. The components $\lambda_i : c \to Fi$ are called **legs** of the cone.*



*Dually we call a **cone under** or **cocone over** $D : J \to C$ with **nadir** $c \in C$ a natural transformation $\lambda : D \Rightarrow c$; the **legs** of the diagram are the arrows $\lambda_i : Fi \to c$.*



Note. *the use of the prefix "co" in "cocone" is justified as any cocone over $D : J \to C$ is exactly a cone over $D : J^{op} \to C^{op}$. I will use the term "cocone" more often than "cone under" since I greatly appreciate the elegance of duality and -frankly- I like that it sounds a little silly.*

This eases us to define limits and colimits:

Definition 3.1.2 (Limits and Colimits).
*A **Limit** of a diagram $F : J \to C$ is the terminal object in the category of cones over F $\int \mathrm{Cone}(\_, F)$.*

*Dually, the **Colimit** of a diagram $F : J \to C$ is the initial object in the category of Cocones over F $\int \mathrm{Cone}(F, \_)$.*

We call these Limit and Colimit cones and their apices and nadirs Limit and Colimit objects.

Definition 3.1.3 (Limits and Colimits II).
*The limit of a diagram gives us a representation for the functor $\mathrm{Cone}(\_, F)$: We can use this as an alternate definition: the **Limit** of a diagram $F : J \to C$ is a representation for $\mathrm{Cone}(\_, F)$; by the Yoneda Lemma, this consists of an object $LimF$ together with a natural transformation $\lambda : \mathrm{Lim}\, F \Rightarrow F$ so that $C(\_, \mathrm{Lim}\, F) \simeq \mathrm{Cone}(\_, F)$.*

*Dually the **Colimit** is a representation for $\mathrm{Cone}(F, \_)$, consisting of an object $\mathrm{CoLim}\, F$ together with a natural $\lambda : F \Rightarrow \mathrm{CoLim}\, F$ so that $C(\mathrm{CoLim}\, F, \_) \simeq \mathrm{Cone}(F, \_)$.*

Proposition 2.4.2 grants us that the two definitions are in fact equivalent.

Even though we introduced them (at the beginning of the chapter) using the "classical approach", Limits and Colimits aren't limited in use by it.

Example 3.1.4 (Common limits).

- *A **terminal object** is regarded as a trivial case of a limit, where the indexing category is empty,*
- *A **product** is the limit of a diagram indexed by a discrete category with only identity-morphisms,*
- *An **equalizer** is the limit of a diagram indexed by a parallel pair of morphisms* $\bullet \rightrightarrows \bullet$*,*
- *A **pullback** is the limit of a diagram indexed by* $\bullet \to \bullet \leftarrow \bullet$*.*

Example 3.1.5 (Common colimits).

- *An **initial object** is regarded as a trivial case of a colimit, where the indexing category is empty,*
- *A **coproduct** is the colimit of a diagram indexed by a discrete category with only identity-morphisms,*
- *A **coequalizer** is the limit of a diagram indexed by a parallel pair of morphisms* $\bullet \rightrightarrows \bullet$*,*
- *A **pushout** is the colimit of a diagram indexed by* $\bullet \leftarrow \bullet \to \bullet$*.*

We must note that the existence of limits is not always granted:

Example 3.1.6 (Limits do not always exist). *[Lei14]*
In the discrete category with two objects (and only identity morphisms) $\mathbb{2}$ *there is no product, i.e. there's not a limit for the diagram* $\bullet \quad \bullet$*.*

No need for an actual proof, there are no cones.

3.2 Limits in Set

It's useful to see -mainly as a mind model to fall back on- what limits are in a set category, to be precise the category **Set** (no pun intended).

Definition 3.2.1 (Complete and Cocomplete categories).
Said that a diagram $F : J \to C$ *is **Small** if and only if* J *itself is small, we call a category C **Complete** if every C-valued small diagram has a limit and **Cocomplete** if every C-valued small diagram has a colimit.*

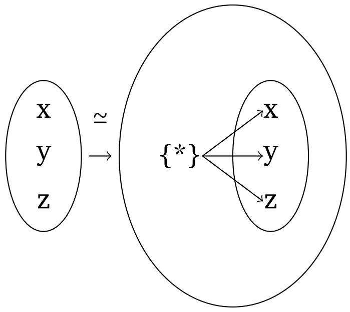
Needless to say we want to prove that **Set** is complete.

Let's consider a small diagram $F : J \to$ **Set**. A limit is a representation **Set**$(x, \operatorname{Lim} F) \simeq \operatorname{Cone}(x, F)$ of the functor sending a set x to the set of cones over F having x as its' apex.

Specializing to a singleton set 1, representing the identity functor in **Set**:

$$\operatorname{Lim} F \simeq \textbf{Set}(1, \operatorname{Lim} F) \simeq \operatorname{Cone}(1, F)$$

Since obviously the embeddings of a singleton to a set's element will result in something isomorphic to the set itself.

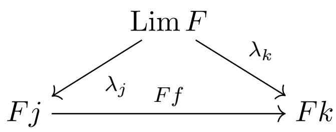


Defining $\operatorname{Lim} F := \operatorname{Cone}(1, F)$ with $\lambda_j : \operatorname{Lim} F \to Fj$ we just need to prove that it is indeed a limit.

Note. *We use the strong hypothesis that if J is small then* $\textbf{Set}^J$ *is locally small, meaning that - in particular -* $\operatorname{Cone}(1, F)$ *is a set.*

Theorem 3.2.2 (Completeness of **Set**).
The category of sets is complete.

Proof.
First we need to prove that



commutes.

For any element $\mu : 1 \Rightarrow F$ of the set $\operatorname{Lim} F$

$$Ff(\lambda_j(\mu)) = Ff(\mu_j) = \mu_k = \lambda_k(\mu)$$

where the middle equality holds because $\mu : 1 \Rightarrow F$ defines a cone. This is proof that the λ_j define a cone over F.

To prove now that the cone is an universal cone consider $\zeta : x \Rightarrow F$ with a generic summit. We must show that this cone factors uniquely through λ along a function $r : x \to \operatorname{Lim} F$

Since we look at $\xi \in x$ as $\xi : 1 \to x$ there is a cone $\zeta\xi : 1 \Rightarrow F$ defined by restricting the cone ζ along ξ. Defining $r(\xi) = \zeta\xi$, the legs of the limit cone λ abide

$$\lambda_j(r(\xi)) = \lambda_j(\zeta\xi) = (\zeta\xi)_j = \zeta_j\xi$$

meaning that not only ζ factors along r through λ but also that the factorization is unique, since defining $r(\xi) = \zeta\xi$ is necessary to the proof's completion. □

This means that we can describe elements in the categorical constructions to find how they equal our usual constructions:

Example 3.2.3 (Binary product).
*Given $a, b \in$ **Set** we define $a \times b := \{(x, y) : x \in a \ , y \in b\}$;*

$a \times b$ is also the limit of the diagram $a \quad b$; as we saw just a few lines

$$\operatorname{Lim} F \simeq \operatorname{Cone}(1, F)$$

The legs of this cone are maps $a \leftarrow 1 \rightarrow b$ meaning the elements of $\operatorname{Cone}(1, F)$ must be pairs (x, y) so that $x \in a$ and $y \in b$, giving us the usual definition.

Limits in set help us connect our usual intuition of objects and their categorical counterpart.

Let's analyze that further:

3.2.1 Products, Terminal Objects, Pullbacks and Equalizers in Set

We already described binary products in **Set**, general products are constructed the same way.

(The following examples are contained in [Lan71])

Set has the singleton set as its terminal object. we can easily see that given $\varnothing$ the empty diagram $\operatorname{Lim} \varnothing \simeq \operatorname{Cone}(1, \varnothing) \simeq 1$ as the only morphism present is the identity.

Set has pullbacks shapen as $\{(x, y) : x \in a, y \in b, fx = gy\}$

the legs in

$$\begin{array}{ccc} 1 & \longrightarrow & a \\ \downarrow & \lrcorner & \downarrow f \\ b & \xrightarrow{g} & c \end{array}$$

are included in the product diagram, meaning each object of P is a subset of $X \times Y$. we also need that its image under f and g must be the same, giving us the aforementioned defintion.

Set has equalizers shapen as $\{x \in a : fx = gx\}$

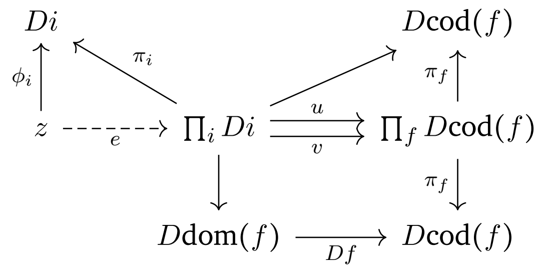
the legs in $1 \longrightarrow a \overset{f}{\underset{g}{\rightrightarrows}} b$ are elements $x \in a$ equipped with the condition that $fx = gx$.

An useful theorem that we will use in the future is:

Theorem 3.2.4 (Completeness via Products and equalizers).
A category A is Complete if it has all small products and binary equalizers.

(dually a category is cocomplete if it has all small coproducts and binary coequalizers).

Sketch of proof.
Given a diagram $D : J \to A$, the proof is contained in the commutativity of the following diagram



□

3.3 Preservation, Reflection and Creation of limits

Now we are interested in seeing how functors and limits interact.

A functor that preserves limits will be called continuous, for obvious reasons.

Definition 3.3.1 (Preservation, Reflection and Creation of limits).
Given a functor $F : C \to D$ we say

- *F **preserves** limits if for any diagram $A : J \to C$ $F\,\mathrm{Lim}(A) = \mathrm{Lim}(FA)$,*
- *F **reflects** limits if for any diagram $A : J \to C$ $Fc = \mathrm{Lim}(FA) \implies c = \mathrm{Lim}(A)$,*
- *F **creates** limits if for any diagram $A : J \to C$ $d = \mathrm{Lim}(FA) \implies \exists c : Fc = d$ and that c is the limit of A.*

Of course creates $\implies$ reflects, since the existence of the lift c implies $Fc = d$

It is true, even if less immediate, that a functor that creates limits also preserves them.

Formally

Proposition 3.3.2.
If $F : C \to D$ creates limits for a class of diagrams in C and D has limits of those diagrams then C admits those limits and F preserves them.

Proof.
For any diagram $A : J \to C$ there is a limit cone $\mu : d \Rightarrow FA$. Being that F creates limits, there is a limit cone $\lambda : c \Rightarrow A$ so that $F\lambda \simeq \mu$ meaning C admits those limits.

To see that F preserves them we use the uniqueness under isomorphisms of the limit, meaning that if there were another limit $\lambda' : c' \Rightarrow A$ then $\lambda \simeq \lambda' \implies F\lambda' \simeq \mu$ meaning $F\lambda'$ is itself a limit cone. $\square$

Of course equivalences between categories preserve, reflect and create limits of any kind.

CHAPTER 4

Adjoint functors and the Adjoint functor theorem

We've come to the main matter:

We've seen in Galois theory, topology, algebraic geometry and so on that sometimes objects have strong relationships that let us jump back and forth from different branches:

Looking back to Galois Theory, we know that given a galois extension Ω/K over a field K if we draw the diagrams for the intermediate extensions and the subgroups of Gal(Ω/K) we find out that they have the same shape but the inclusions are "flipped"

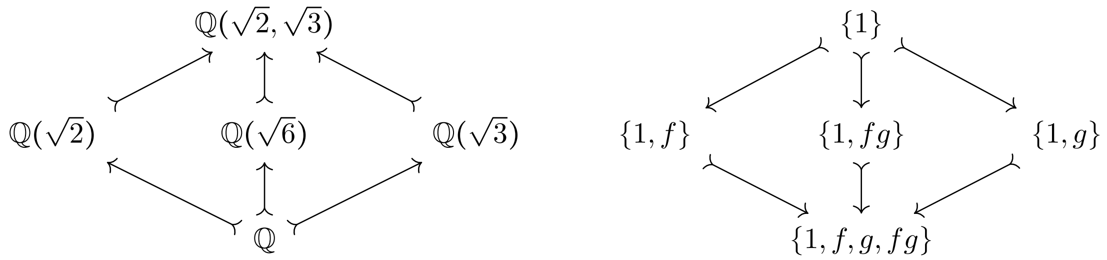


Figure 4.1: Example: Extensions of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

meaning that Gal$(\Omega/_)$ and Fix(_) (the maps sending respectively a field to the galois group over it and an automorphism group to its fixed points) are - in this context - kind of inverse transformations between a group preorder and a field preorder.

4.1 Adjunctions and adjoint functors

Let's give some definitions:

Definition 4.1.1 (Adjoint Functors, Adjunctions).
Let A, B be categories and let $F : A \to B$, $G : B \to A$ be functors.

*We say that F is **left adjoint** to G, or G is **right adjoint** to F and we write $G \vdash F$ if*

$$B(Fx, y) \simeq A(x, Gy)$$

where $\simeq$ is natural in both x and y.

We can express this naturality with a diagram

$$A^{op} \times B \underset{A(\_,G\_)}{\overset{B(F\_,\_)}{\rightrightarrows}} \textbf{\textit{Set}} \qquad (\Downarrow \simeq)$$

*We call the pair of functors F, G as described above together with a choice of $\simeq$ an **adjunction**.*

Definition 4.1.2 (Transposed morphisms).
Under the natural isomorphism $B(F\_, \_) \simeq A(\_, G\_)$ above we have

$$Fx \xrightarrow{f^\sharp} y \;\overset{\simeq}{\longleftrightarrow}\; x \xrightarrow{f^\flat} Gy$$

we say $f^\sharp$ and $f^\flat$ are transposes of each other.

Note that if F, G is an equivalence between categories then it's an adjunction, but the converse isn't true. We will prove this equivalence-like nature of adjunction in the next section. before though, let's give a characterization of the adjunction between functors. The first one is that

$$Gk \cdot f^\flat = (k \cdot f^\sharp)^\flat$$

meaning that composing with "flat" morphisms after G is the same as "flattening" the composition of "sharp" morphisms before G, i.e.

$$\begin{array}{ccc} B(Fx, y) & \xrightarrow{\simeq} & A(x, Gy) \\ {\scriptstyle k_*}\downarrow & & \downarrow{\scriptstyle Gk_*} \\ B(Fx, y') & \xrightarrow{\simeq} & A(x, Gy') \end{array}$$

commutes.

Dually if we look at pre-composition we have that

$$f^\flat \cdot h = (f^\sharp \cdot Fh)^\flat$$

meaning that composing with "flat" morphisms before F is the same as "flattening" the composition of "sharp" morphisms after F, i.e.

$$\begin{array}{ccc} B(Fx,y) & \xrightarrow{\simeq} & A(x,Gy) \\ {\scriptstyle Fh^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ B(Fx',y) & \xrightarrow{\simeq} & A(x',Gy) \end{array}$$

commutes.

This leaves us with the last condition

$$kf^\sharp = g^\sharp Fh \iff Gkf^\flat = g^\flat h$$

easily proven by the series of implications:

$$\begin{aligned} Gkf^\flat = g^\flat h &\iff g^\flat h = (kf^\sharp)^\flat \\ &\iff (g^\sharp Fh)^\flat = (kf^\sharp)^\flat \\ &\iff g^\sharp Fh = kf^\sharp \end{aligned}$$

□

Making use of the first condition, then the second, and last that $\simeq$ is in fact an isomorphism.

4.2 Unit and Counit

As previously mentioned Adjunctions are a different form of equivalences between categories: Hom-sets before and after the functors are in fact isomorphic: where in an equivalence $F : A \leftrightarrows B : G,\ FG \simeq 1_B,\ GF \simeq 1_A$ we have

$$A(x,y) \simeq B(Fx,Fy) \simeq A(GFx,GFy) \simeq B(FGFx,FGFy)$$

in an adjunction $F : A \leftrightarrows B : G$ we have

$$G : B(Fx,y) \mapsto A(GFx,Gy) \simeq A(x,Gy) \ \ and \ \ F : A(x,Gy) \mapsto B(Fx,FGy) \simeq B(Fx,y)$$

It is not a proper inversion but works in a similar fashion. This is the same as saying that the object $Fx \in D$ represents $A(x,G\_) : D \to$ **Set**. By the Yoneda Lemma this means that the natural isomorphism $B(Fx,\_) \simeq A(x,G\_)$ is represented by an element of $A(x,GFx)$.

This means $A(x,GFx) \simeq B(Fx,Fx)$.

We denote $(1_{Fx})^\sharp := \eta_x$, letting us assemble the η_x's into the components of a natural transformation $1_x \Rightarrow GF$.

Lemma 4.2.1.
Given an adjunction $F \dashv G$ there is a natural transformation $\eta : 1_x \Rightarrow GF$, whose componets $\eta_x : x \to GFx$ are the transposes of the identity morphisms 1_{Fx}.

Proof.
We need to prove that η is a natural isomorphism, i.e. that the square

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & GFx \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\ x' & \xrightarrow{\eta_{x'}} & GFx' \end{array}$$

commutes. That is almost trivial using the last lemma of the previous chapter:

$$kf^\sharp = g^\sharp Fh \iff Gkf^\flat = g^\flat h$$

meaning

$$\begin{aligned} \eta_{x'} \cdot f = GFf \cdot \eta_x &\iff (\eta_{x'})^\sharp \cdot Ff = Ff \cdot (\eta_x)^\sharp \\ &\iff 1_{Fx'} \cdot Ff = Ff \cdot 1_{Fx} \end{aligned}$$

□

Dually $Gy \in A$ represents $B(F\_, y)$, meaning that there is a natural transformation $\epsilon : FG \Rightarrow 1_B$ whose components ϵ_y are the transposes of 1_{Gy}.

Definition 4.2.2 (Unit and Counit)**.**
Given an adjunction $F \dashv G$ we call the two natural isomorphims described above $\eta : 1_A \Rightarrow GF$ sending $x \to GFx$ as the transpose of the identity $1_F x$ and $\epsilon : FG \Rightarrow 1_B$ sending $FGx \to x$ as the transpose of the identity $1_G x$ respectively the ***unit*** *and* ***counit*** *of the adjunction.*

With these notions we acquired we can give Adjunctions an alternative definition:

Definition 4.2.3 (Adjunctions II)**.**
An ***adjunction*** *consists of a pair of functors $F : A \leftrightarrows B : G$ and two natural transformations $\eta : 1_A \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_B$ so that the two triangles*

$$\begin{array}{ccc} F & \xRightarrow{F\eta} & FGF \\ & {\scriptstyle 1_F}\searrow & \Downarrow{\scriptstyle \epsilon F} \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xRightarrow{\eta G} & GFG \\ & {\scriptstyle 1_G}\searrow & \Downarrow{\scriptstyle G\epsilon} \\ & & G \end{array}$$

commute in B^A and A^B respectively.

Since adjunction is a "sort-of equivalence" we expect adjoints to be unique:

Proposition 4.2.4 (Uniqueness of the left adjoint)**.**
If F and F' are left adjoint to G then $\exists! \theta : F \simeq F'$ natural isomorphism commuting with units and counits of the adjunctions:

$$\begin{array}{ccc} 1_A & \xRightarrow{\eta} & GF \\ & {\scriptstyle \eta'}\searrow & \Downarrow{\scriptstyle G\theta} \\ & & GF' \end{array} \qquad \begin{array}{ccc} FG & \xRightarrow{\epsilon} & 1_B \\ {\scriptstyle \theta G}\Downarrow & \nearrow{\scriptstyle \epsilon'} & \\ F'G & & \end{array}$$

Proof.
To define a natural transformation $\theta : F \Rightarrow F'$ we just need to define a transposed natural transformation $\eta' : 1_A \to GF'$

$$\begin{array}{ccc} Fx & \xrightarrow{\theta_x} & F'x \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle F'f} \\ Fx' & \xrightarrow[\theta_{x'}]{} & F'x' \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} x & \xrightarrow{\eta'_x} & GF'x \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GF'f} \\ x' & \xrightarrow[\eta'_{x'}]{} & GF'x' \end{array}$$

We can define a couple of natural transformations $\theta : F \Rightarrow F'$ and $\theta : F' \Rightarrow F$ as

$$\theta : F \xRightarrow{F\eta'} FGF' \xRightarrow{\epsilon F'} F'$$

$$\theta' : F' \xRightarrow{F'\eta} F'GF \xRightarrow{\epsilon' F} F$$

and do the legwork to prove that they're the inverse of one another, i.e. prove that $\eta : 1 \Rightarrow GF$ equals the composite

$$1 \xRightarrow{\eta} GF \xRightarrow{GF\eta'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

by naturality of η this composite equals

$$1 \xRightarrow{\eta'} GF' \xRightarrow{\eta GF'} GFGF' \xRightarrow{G\epsilon F'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

Since $G\epsilon \cdot \eta G = 1_G$ we can reduce to

$$1 \xRightarrow{\eta'} GF' \xRightarrow{GF'\eta} GF'GF \xRightarrow{G\epsilon' F} GF$$

by naturality of η' this equals

$$1 \xRightarrow{\eta} GF \xRightarrow{\eta' GF} GF'GF \xRightarrow{G\epsilon' F} GF$$

We can reduce it once more using the triangle identity $G\epsilon' \cdot \eta' G = 1_G$

$$1 \xRightarrow{\eta} GF$$

Lastly we need to prove that the two triangles above commute: Using the definition of θ

$$1 \xRightarrow{\eta} GF \xRightarrow{GF\eta'} GFGF' \xRightarrow{G\epsilon F'} GF'$$

$$1 \xRightarrow{\eta'} GF' \xRightarrow{\eta GF'} GFGF' \xRightarrow{G\epsilon F'} GF'$$

$$1 \xRightarrow{\eta'} GF'$$

proves the first triangle, and duality deals with the second;

This means that the transpose of θ across $F \dashv G$ is η', thereby proving uniqueness. □

We now can talk about the relationship between equivalences and adjunctions:

In an equivalence you have a couple of natural isomorphisms $\eta : 1_A \simeq GF$ and $\epsilon : FG \simeq 1_D$ meaning that the composite

$$\gamma : G \xRightarrow{\eta G} GFG \xRightarrow{G\epsilon} G$$

is too an isomorphism. It isn't - however - an identity.

Proposition 4.2.5 (Equivalences and adjunctions).
Any equivalence $F : A \leftrightarrows B : G$, $\eta : 1_A \simeq GF$, $\epsilon : FG \simeq 1_B$ can be improved to be an adjunction by changing either the unit or the counit.

Proof.
Keeping the notation used above, let $\epsilon' \coloneqq \epsilon \cdot F\gamma^{-1}$ by naturality of η the diagram

$$\begin{array}{ccccccc} G & \xRightarrow{\eta G} & GFG & \xRightarrow{GF\gamma} & GFG & \xRightarrow{G\epsilon} & G \\ & {\scriptstyle \gamma^{-1}} \searrow & & \nearrow {\scriptstyle \eta G} & & \nearrow {\scriptstyle \gamma} & \\ & & G & & & & \end{array}$$

commutes, meaning $G\epsilon' \cdot \eta G = 1_G$.

By naturality of η and ϵ' and by the first triangle identity we have that

$$\begin{array}{ccccc} F & \xRightarrow{F\eta} & FGF & \xRightarrow{\epsilon'_F} & F \\ {\scriptstyle F\eta}\Downarrow & & \Downarrow{\scriptstyle FGF\eta} & & \Downarrow{\scriptstyle F\eta} \\ FGF & \xRightarrow{F\eta_{GF}} & FGFGF & \xRightarrow{\epsilon'_{FGF}} & FGF \\ & \searrow{=} & \Downarrow{\scriptstyle FG\epsilon'_F} & & \Downarrow{\scriptstyle \epsilon'_F} \\ & & FGF & \xRightarrow{\epsilon'_F} & F \end{array}$$

commutes, namely $\epsilon'_F \cdot F\eta = 1_F$. □

For completeness we also define morphisms of adjunctions:

Definition 4.2.6 (Morphisms of adjunctions).
*A **Morphisms of adjunctions** $(F \dashv G) \to (F' \dashv G')$ is comprised of a pair of functors H, K so that*

$$\begin{array}{ccc} A & \xrightarrow{H} & A' \\ {\scriptstyle F}\downarrow \dashv \uparrow{\scriptstyle G} & & {\scriptstyle F'}\downarrow \dashv \uparrow{\scriptstyle G'} \\ B & \xrightarrow[K]{} & B' \end{array}$$

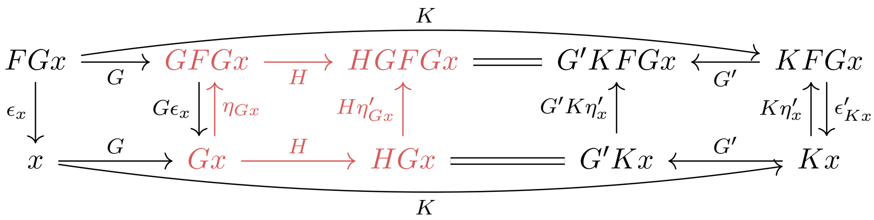
commutes both along the left adjoints and the right adjoints ($KF = F'H$ and $HG = G'K$) and (equivalently)

1. $H\eta = \eta' H$ *(where* η *and* η' *are the units)*
2. $K\epsilon = \epsilon' K$ *(where* ϵ *and* ϵ' *are the counits)*

Of course we need to prove that those three conditions are in fact equivalent:

Proof.

- $1 \implies 2$:



- $2 \implies 1$: is the dual diagram of the one above.

□

4.3 Adjunctions, limits and colimits

We have limits and we have functors, we feel like we're in the beginning of a basic topology course. Paying respect to terminology:

Definition 4.3.1 (Continuous and Cocontinuous functors)**.**
A functor $F : A \to B$ *is called* ***continuous*** *if for every (small) diagram* $D : J \to C$

$$F(LimD) \simeq LimFD$$

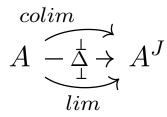
and ***cocontinuous*** *if for every (small) diagram* $D : J \to C$

$$F(ColimD) \simeq ColimFD$$

(WLOG using the characterization in ***Theorem 3.2.4*** *a functor is continuous* $\iff$ *preserves all products and binary equalizers, cocontinuous* $\iff$ *preserves all coproducts and binary coequalizers).*

Limits and adjunctions are very much intertwined, since you can define limits and colimits as the left and right adjoints of the functor that sees an object as the apex - or nadir - of a set cone - or cocone- i.e.:

Proposition 4.3.2 (Limits and adjunctions).
a category A admits limits and colimits of J-shaped diagrams if and only if the constant diagram functor $\Delta : A \to A^J$ admits respectively right and left adjoints.



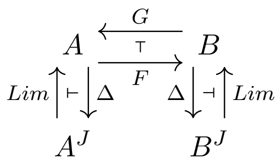
Note that we can take this as an alternate definition for limits and colimits.

Proof.
From the definition of limit we know that $Cone(x, D) \simeq A(x, LimD)$ is natural, rewriting the category of J-indexed cones over D as $A^J(\Delta x, D)$ (natural transformations between the constant x and the diagram) we have that $A^J(\Delta x, D) \simeq A(x, LimD)$ is natural, prooving that $Lim : A^J \to A$ is right adjoint to Δ.

$Lim : A^J \to A$ forms a functor if and only if every J-shaped diagram D has limit.

Duality gives us the same thing for the colimit. □

Looking at this square



we can guess the two theorems:

Theorem 4.3.3 (RAPL). *Right adjoints preserve Limits.*

and

Theorem 4.3.4 (LAPC). *Left adjoints preserve colimits.*

Proof.
We just need to prove (RAPL) and let duality take care of (LAPC).

Given an adjunction $F \dashv G$ then given a diagram $D : J \to B$ then $G(LimD)$ is a limit for $G(D)$

we know that

$$A^J(\Delta x, GD) \simeq_1 B^J(F\Delta x, D) \simeq_2 B^J(\Delta Fx, D) \simeq_3 B(Fc, LimD) \simeq_4 A(x, GLimD)$$

and where every $\simeq$ is natural because:

1. Lifting an adjunction results in an adjunction: since $F \dashv G$ we have that post-composition with F and G in A^J or B^J trivially respect the natural triangle of **Definition 4.2.4**,

2. $F\Delta x = \Delta F x$,
3. Definition of limit,
4. Definition of adjunction.

□

Now, is the converse true? does any continuous/cocontinuous functor give birth to an adjunction?

4.4 The Adjoint functor theorem

Freyd's Adjoint functor theorem, main matter of this thesis states that

Given a locally small, complete category A, if a continuous functor $G : A \to X$ satisfies the **solution set condition**

$\forall x \in X\ \exists \Phi_x = \{f_i : x \to Ga_i\} \in$ **Set** so that any $f : x \to Ga$ factors through some $f_i \in \Phi_x$ along an arrow $(a_i \to a) \in A$

$$\begin{array}{ccccc} x & \xrightarrow{f} & Ga & \quad & a \\ & \searrow^{f_i} & \uparrow{\scriptstyle Gg_i} & & \uparrow{\scriptstyle g_i} \\ & & Ga_i & & a_i \end{array}$$

then it admits a left adjoint.

To be fair, the theorem is an $\iff$ but we're only interested in the $\implies$ side since the important part of the $\impliedby$ side has been covered in the previous section.

To prove it we lean onto a lemma characterizing adjunction through comma categories:

Lemma 4.4.1.
A functor $G : A \to B$ has a left adjoint if and only if $\forall\ y \in B$ the comma category $y \downarrow G$ has an initial object.

Reminding that the comma category $y \downarrow G$ is a category whose objects are pairs (a, f)

$$y \xrightarrow{f} Ga$$

and whose morphisms are maps $h : a \to a'$ so that

$$\begin{array}{ccc} y & \xrightarrow{f} & Ga \\ & \searrow^{f'} & \downarrow{\scriptstyle Gh} \\ & & Ga' \end{array}$$

commutes.

Proof.
We know that $y \downarrow G \simeq \int B(y, G\_)$. If a left adjoint $F \vdash G$ then the component of the unit at y defines an initial object $\eta_y : y \to GFy$ in this category:

recalling that $\int B(y, G\_)$ has as objects pairs $(a \in A, f \in B(y, Ga))$ and morphisms $g : a \to a'$ so that $B(y, Ggf) = f'$,

meaning that the object in the category of elements is actually $(Fy, \eta_y : y \to GFy)$ it is initial $\iff \forall(a, f : y \to Ga) \in \int B(y, G\_) \exists! g : Fy \to a$ so that $f = Gg \cdot \eta_y$.

$$\begin{array}{ccccc} (Fy, & & y & \xrightarrow{\eta_y} & GFy) \\ {\scriptstyle g}\downarrow & & \| & & \downarrow{\scriptstyle Gg} \\ (a, & & y & \xrightarrow[f]{} & Ga) \end{array}$$

This is true because of the adjunction: f is by definition the transpose of g.

Conversely, suppose $y \downarrow G$ admits an initial object for each y, we can basically try to trace back the previous proof: we want to prove that the initial objects $\eta_y : y \to GFy$ assemble into the unity of an adjunction $F \vdash G$.

We need to prove that F is indeed a functor.

That is true if we define $Ff : Fy \to Fy'$ to be the unique morphism closing the square

$$\begin{array}{ccc} y & \xrightarrow{\eta_y} & GFy \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GFf} \\ y' & \xrightarrow{\eta_{y'}} & GFy' \end{array}$$

The initiality of η_y means that map exists and is unique, meaning $F : B \to A$ is indeed a functor and $\eta : 1_B \Rightarrow GF$ is a natural transformation, meaning we can define a natural transformation

$$\Phi : A(F\_, \_) \Rightarrow B(\_, G\_)$$

with components $\Phi_{y,a} : A(Fy, a) \to B(y, Ga)$. defining

$$\Phi_{y,a} := y \xrightarrow{\eta_y} GFy \xrightarrow{Gg} Ga$$

The initiality of the η_ys gives us injectivity and surjectivity of the $\Phi_{y,a}$s, meaning it is a natural isomorphism, meaning $F \vdash G$ is indeed an adjunction. □

This theorem suggests why we need the solution set condition, since we want to take advantage of the initial object in the comma category.

The solution set is a stronger condition than having an initial arrow in the condition of the theorem though, if we build the comma $x \downarrow G$ we notice that Φ_x is the set of "initial object without uniqueness".

Let's give them a name:

Definition 4.4.2 (Weakly initial objects, Jointly weakly initial sets)**.**

- *An object $c \in C$ is **weakly initial** if $\forall\ x \in C \exists c \to x$,*
- *A set $\Phi : \{c_i \in C : i \in I\}$ is **jointly weakly initial** if $\forall\ x \in C \exists c_i \in \Phi : \exists c_i \to x$.*

And in the right conditions this is stronger than having an initial object:

Lemma 4.4.3.
If a category C is complete, locally small and has a jointly weak initial set Φ, then C has an initial object.

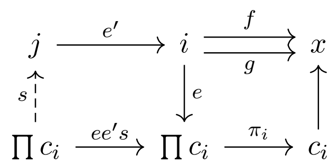
Proof. [Lan71]
Since C is complete we can build the product object $\prod c_i$ where $\Phi = \{c_i\}$.

$\forall x \in C\ \exists\ c_i \to x$ meaning for each $x \in C$ we can compose a morphism $\lambda_x : \prod c_i \to c_i \to x$.

Let's look at the hom-set $C(\prod c_i, \prod c_i)$ and construct the equalizer of all the endomorphisms $\prod c_i \to \prod c_i$ $e : i \to \prod c_i$

$\forall x \in C\ \exists i \to \prod c_i \to x$ means i is weakly initial.

to prove that i is initial suppose there were two distinct $f, g : i \to x$ then we can pick the equalizer $e' = eq(f, g)$, the diagram



commutes.

We want to prove that $ee's$ is the identity. that is true because e was defined as the equalizer of all the endomorphisms of $\prod c_i$, so

$$ee'se = 1_{\prod c_i} e = e 1_i$$

Now e is an equalizer, in particular it's monic, meaning $e'se = 1_i$ meaning e' has right inverse, making it an isomorphism.

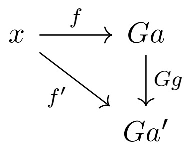
this concludes the proof, since this way $f = g$, i.e. i is initial in C. □

And like that the proof of Freyd's adjoint functor theorem is basically done: we just need to do some stirring.

I'll repeat here the statement:

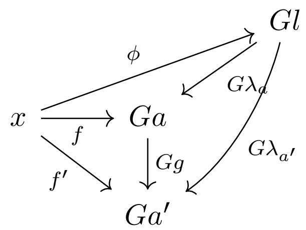
Theorem 4.4.4 (Freyd's Adjoint Functor Theorem)**.**
Given a locally small, complete category A, any continuous functor $G : A \to X$ that satisfies the solution set condition admits a left adjoint

Proof.
The solution set condition is equivalent to having a jointly weakly initial set in the category $x \downarrow G$. C is locally small and complete meaning $x \downarrow G$ is locally small and complete: $x \downarrow G((a, f), (a', f'))$ is the collection of morphisms $g : a \to a'$ so that



commutes, meaning it's contained in $A(a, a')$.

Since A is complete and G is continuous let's pick a cone over a diagram $D' : J \to x \downarrow G$



This is clearly equal to $x \Rightarrow G(\mathrm{Cone}(l, D))$ where $D : J \to A$ is a diagram in A, meaning that $(\mathrm{Lim}\, D, \phi : x \to G\,\mathrm{Lim}\, D) = \mathrm{Lim}\, D'$, meaning $x \downarrow G$ is complete too.

We have a jointly weakly initial set in the locally small complete category $x \downarrow G$, thus we have an initial object in $x \downarrow G$, meaning G admits left adjoint. □

4.5 The Special Adjoint Functor Theorem

Our new objective is to get rid of the solution set condition: for that we need to define some objects and properties:

Definition 4.5.1 (Subobjects)**.**
Given a category C, a ***subobject*** *of $x \in ob(C)$ is an isomorphism class of monomorphisms $s \hookrightarrow x$ (two morphims $a \to x$ $b \to x$ are said to be isomorphic if exists an isomorphism $a \to b$ that makes the triangle commute).*

Definition 4.5.2 (Well-powered categories)**.**
A category is ***well-powered*** *if every object has a small poset of subobjects, i.e. if the category $sub_C(x)$ of its subobjects is a small poset (antisymmetric preorder, i.e. there is no pair of morphisms $a \leftrightarrows b$ for any pair of objects a, b).*

Definition 4.5.3 (Generating families)**.**
A ***generating family*** *for a category C is a family of objects $S := \{s_i\}_{i \in I}$ so that every object $x \in C$ can be written as the quotient of a S-indexed coproduct, i.e.*

for each object $x \in C$ exists a map

$$\epsilon_x : \coprod s_i \to x$$

and that map is an epimorphism.

The dual concept is

Definition 4.5.4 (Cogenerating families)**.**
*A **cogenerating family** for a category C is a family of objects $S := \{s_i\}_{i \in I}$ so that every object $x \in C$ can be included in a S-indexed product, i.e.*

for each object $x \in C$ exists a map

$$\eta_x : x \to \prod s_i$$

and that map is a monomorphism.

Theorem 4.5.5 (Special Adjoint Functor Theorem)**.** *[Bor94]*
Let A be a complete, well-powered category with a cogenerating family, $F : A \to B$ a functor, then

F is continuous $\implies$ F admits left adjoint.

Proof.
We just need to prove the solution set condition for every object $b \in B$. To do this we pick a cogenerating family in A $(u_i)_{i \in I}$, then look at the product

$$P_b := \prod_{i \in I} u_i^{|B(b, Fu_i)|}$$

(where $u^{|B(b,Fu)|}$ means as many copies of u as elements of the set $B(b, Fu)$) and define

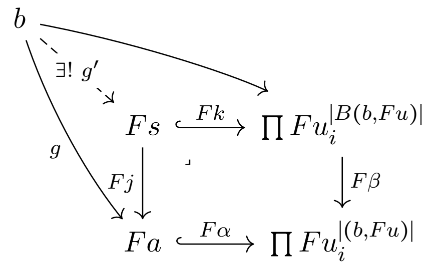
$$S_b := \{\text{subobjects of} P_b\}$$

We want to prove that S_b is a solution set for b: let $a \in A,\ g : b \to Fa$, we need to find an $\mathrm{mor}(S_b) \ni j : s \to a,\ g' : b \to Fs$ so that $Fj \cdot g' = g$. The pull-back

$$\begin{array}{ccc} s & \overset{k}{\hookrightarrow} & \prod u_i^{|B(b,Fu)|} \\ {\scriptstyle j}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ a & \overset{\alpha}{\hookrightarrow} & \prod u_i^{|(b,Fu)|} \end{array}$$

of α, β gives us an element $s \in \mathrm{ob}(S_b)$ with a morphism $j : s \to a$

meaning that applying F (which is continuous) to the diagram means that Fs is still a pullback, i.e. $\forall b \in \mathrm{ob}(B)$ including our chosen b



Any $g : s \to a$ factors uniquely through a morphism $Fj : Fs \to Fa$. □

CHAPTER 5

Applications of the Adjoint functor theorem

5.1 In Geometry and Topos Theory

5.1.1 Topoi

We're not new to non-set structures (we talked about abstract categories before) but we like **Set** since it has nice properties, like computing limits pointwise, completeness, subsets, etc.

In the category of Sets a stronger version of Freyd's theorem holds:

Theorem 5.1.1 (AFT in **Set**).
Given a functor $F : \mathbf{Set} \to \mathbf{Set}$, F is continuous $\iff$ it has left adjoint.

That is because, of course, **Set** has all the properties descripted in the SAFT. Can we generalize the category of sets to some kind of categories in which the same is true?

Definition 5.1.2 (Topoi). *[Wra90]*
*A **Topos** (pl. **Topoi**) is a category that*

- *Has finite limits,*
- *Is cartesian closed,*
- *Has a subobject classifier.*

Where having finite limits means that it contains the limit of any diagram indexed by a finite category (category with a finite number of morphisms).

We are left to define what's a cartesian closed category and what's a subobject classifier:

Definition 5.1.3 (Cartesian closed categories)**.** *[Pet89]*
*A category C is **Cartesian closed** if for each $x \in ob(C)$ the endofunctor $x \times \_ : C \to C$ has right adjoint.*

This definition is not as intuitive as we usually like them to be, let's develop it:

The functor $\_ \times x$ has a right adjoint, meaning that there is a functor that we'll call $\_^x$ so that $\forall y, z \in ob(\textbf{Set})$ $\textbf{Set}(x \times y, z) \simeq \textbf{Set}(y, z^x)$.

That's what we usually call "currying":

$$C(x \times y, z) \xrightarrow{\simeq} C(y, z^x)$$

$$(x, y) \mapsto f(x, y) \qquad\qquad (y) \mapsto f(x)(y)$$

Meaning that the right adjoint we're look for is the exponential object; If we're dealing with **Set** or a concrete category the exponential object is exactly the set z^x of morphisms from z to x or its preimage under the forgetful functor if it exists. in fact the functor $\textbf{Set}(x, \_)$ is left adjoint to $x \times \_$.

And

Definition 5.1.4 (Subobject classifier)**.** *[Sau93]*
*In a category C with finite limits, a **Subobject classifier** is a monomorphism $\top : * \to \omega$ (where $*$ is the terminal object of C) so that for each monomorphism $u \to x \in C$ there is an unique morphism $\chi_u : x \to \omega$ so that*

$$\begin{array}{ccc} u & \longrightarrow & * \\ \big\downarrow & \lrcorner & \big\downarrow \top \\ x & \xrightarrow[\chi_u]{} & \omega \end{array}$$

is a pullback diagram.

Again, this is convoluted. Let's deconstruct it:

In **Set** we have subsets: $y' \subset y \iff (x \in y' \implies x \in y)$

I.e. there is a function $\chi_{y'} : y \to \{\top, \bot\}$

$$\chi_{y'}(x) = \begin{cases} \top \text{ if } \ x \in y' \\ \bot \text{ if } x \in y \setminus y' \end{cases}$$

This uses Boolean, binary logic, but it doesn't matter: we don't even need the false element.

We can modify the function as $\chi_{y'} : y \to \omega$

$$\chi_{y'}(x) \equiv \top \ \forall x \in y'$$

meaning that looking at the inclusion $y' \hookrightarrow y$ then picking out any element in y' $(y' \to *)$ results in it being "true" (inside y), thus the commutativity of the square.

We would like **Set**'s AFT to hold for any pair of topos but sadly it doesn't.

This leads us to talk about Grothendieck topoi, which intuitively are "not-too-big-not-too-small" topoi.

5.1.2 Sheaves and Sites

(Most of this section is derived from [Car])

Definition 5.1.5 (Presheaves on a topological space).
*A **Presheaf** $\mathcal{F}$ on a topological space (X, τ) consists of*

- *A set $\mathcal{F}(U) \forall U \in \tau$,*
- *A morphism $r_{u,v} : \mathcal{F}(U) \to \mathcal{F}(V)$ for each $U \subseteq V$ so that $r_{U,U} = 1_{\mathcal{F}(U)}$ and $\forall U \subseteq V \subseteq W$ $r_{u,w} = r_{v,w} \cdot ru, v$.*

Morphisms of presheaves $\mathcal{F} \to \mathcal{G}$ are collections of maps $\mathcal{F}(U) \to \mathcal{G}(U)$ compatible with the restrictions $r_{\_,\_}$.

We can see the functorial nature of these items. A stronger notion is the Sheaf:

Definition 5.1.6 (Sheaves on a topological space).
*A **Sheaf** $\mathcal{F}$ on a topological space (X, τ) is a presheaf on (X, τ) so that $\forall U \in \tau$ if $\{V_i\}_{i \in I}$ is an open covering of U then*

- *Taken a pair of elements $s, t \in \mathcal{F}(U)$, $s_{|V_i} = t_{|V_i}$ $\forall i \implies s = t$,*
- *Taken an element s_i for each V_i, if $s_{i|(V_i \cap V_j)} = s_{j|(V_i \cap V_j)}$ then $\exists s \in \mathcal{F}(U)$ so that $s_{|V_i} = s_i$.*

Morphisms of sheaves are morphisms of the underlying presheaves.

Categorizing these notions, for any topological space (X, τ) exists a category $\mathcal{O}(X)$, the poset category corrresponding at the lattice of open sets of the topological space X with the inclusion, meaning that sheaves and presheaves are functors $\mathcal{O}(X)^{op} \to$ **Set**.

A morphism of presheaves is nonother than a natural transformation between presheaves (morphism that mantains structure), meaning that $\mathbf{Set}^{\mathcal{O}(X)^{op}}$ (presheaves on a topological space) is a category.

Trying to move from topological spaces regarded as categories to categories in general. We have that inclusions generalize seemlessly to arrows, while coverings are a little less immediate.

Definition 5.1.7 (Presieves and Sieves).
*Given a category C a **Presieve** P_c is a collection of arrows with codomain c, a **Sieve** S_c is a presieve in which $\forall f : a \to c \in S \forall g : b \to a \in \mathrm{mor}(C),\ fg \in S$.*

This means we can define the categorical equivalent to a topology.

Definition 5.1.8 (Grothendieck Topologies).
*A **Grothendieck Topology** on a small category C is a map $J : c \mapsto \{collections\ of\ sieves\ on\ c\}$ so that*

- *If S_c is a sieve in $J(c)$ then $\forall\ g : d \to c$ the pullback $g * S_c$ is a sieve in $J(d)$,*
- *If S_c is a sieve on c so that $S_d = \bigcup_d \{g : d \to c | g * S_c covers d\}$ contains a sieve of $J(c)$ then $S_c \in J(c)$,*
- *The maximal sieve (collection of all arrows with codomain c, corresponding to the functor $C(\_, c)$) is in $J(c)$;*

Meaning that a presheaf becomes just a functor $C^{op} \to$ **Set**.

Definition 5.1.9 (Presheaves on a category).
*A **Presheaf** on a category C is a functor $P : C^{op} \to$ **Set**.*

The last generalization is the one regarding the two points in the definition of sheaves on a topological space:

Definition 5.1.10 (Matching families and Amalgamations).
*Fixed a Presheaf $P : C^{op} \to$ **Set** and a sieve S_c we call a **Matching family** for S_c a map $\alpha : (f : d \to c) \mapsto x_f \in P(d)$ so that $\forall g : e \to d\ Pg\ x_f = x_{fg}$.*

*Given a matching family, an **Amalgamation** is a single element $x \in Pc\ :\ Pf\ x = x_f \forall f \in S_c$.*

Definition 5.1.11 (Sites, Sheaves on a site).
*We call a **Site** the pair (C, J) where C is a category and J a Grothendieck Topology.*

*Given a site (C, J), a presheaf on C is called a **J-sheaf** if every matching family for any sieve in J has an unique amalgamation.*

*Finally we call **Sh**(C, J) of sheaves on the site (C, J) the full subcategory of **Set**$^{C^{op}}$ formed by the J-sheaves.*

5.1.3 Adjoint Functor Theorem for Grothendieck Topoi

All this is to finally define the Grothendieck Topos:

Definition 5.1.12 (Grothendieck Topoi).
*A **Grothendieck Topos** is a category equivalent to the category of the sheaves on a site.*

It's quite interesting to show that this is in fact a topos. First though we show that The presheaf category is a topos:

Theorem 5.1.13 (The Presheaf Category is a Topos). *[Ell]*
*Given a small category C, the functor category **Set**$^{C^{op}}$ (also called presheaf category on C) is a topos.*

Proof.
We need to prove that $\mathbf{Set}^{C^{op}}$ has a subobject classifier, finite limits and is cartesian closed. To prove that we have a **subobject classifier** we use the Sieves: fixed an object c Since M_c is a sieve we can define a map (a natural transformation) $\top : 1 \to \Omega$ where Ω is the collection of sieves over c.

This actually works as a subobject classifier: Suppose $\mathcal{G} \leq \mathcal{F}$ are presheaves. For each morphism $f : c \to d \ \in \mathrm{mor}(C)$ we have a function $\mathcal{F}d \to \mathcal{F}d$ (the restriction $x_{|f}$) in **Set** which may or may not take an element $\mathcal{F}d$ into $\mathcal{G}c \leq \mathcal{F}c$. Given $x \in \mathcal{F}d$ we write

$$\phi_d(x) = \{g|\text{cod}(g) = d,\ x_{|g} \in \mathcal{G}(\text{dom}(g))\}$$

then $\phi_d(x)$ is a sieve on d and $\phi : \mathcal{F} \to \Omega$ is a natural transformation of presheaves. Moreover $\phi_d(x)$ is the maximal sieve $M_d \iff x \in \mathcal{G}d$ meaning the subfunctor $\mathcal{G}$ is the pullback

$$\begin{array}{ccc} \mathcal{G} & \longrightarrow & 1 \\ \big\downarrow & \lrcorner & \big\downarrow \top \\ \mathcal{F} & \underset{\phi}{\longrightarrow} & \Omega \end{array}$$

meaning that ϕ, if unique, is a subobject classifier in $\mathbf{Set}^{C^{op}}$.

ϕ is unique, since for any other candidate map ψ, the pullback condition implies $x_{|f} \in \mathcal{G}c \iff \psi_c(x_{|f}) = \top_c(1) = M_c$ by the naturality of ψ this is equivalent to $\psi_d(x)_{|f} = M_c$ meaning $f \in \psi_d(x)$ i.e. $\psi_d = \phi_d\ \forall d$.

Now we need **exponential objects**: the obvious choice would be $\mathcal{G}^{\mathcal{F}}c = \hom(\mathcal{F}c, \mathcal{G}c)$ but sadly it's not a functor of C. looking at the definition we need that

$$\hom(\mathcal{H} \times \mathcal{F}, \mathcal{G}) \simeq \hom(\mathcal{H}, \mathcal{G}^{\mathcal{F}})$$

Using the yoneda embedding, we can define

$$\mathcal{G}^{\mathcal{F}}c = \hom(C(c, \_) \times F, G)$$

which clearly acts as an exponential object. □

The fact that the category of presheaves over a small category is a topos means that its reflective subcategory of sheaves over a small site is a topos too.

Theorem 5.1.14 (AFT For Grothendieck Topoi)**.**
Let A, B be Grothendieck Topoi, $F : A \to B$ a functor. if F is continuous then F has left adjoint.

Sketch of proof.
We need to show that $\mathbf{Sh}(C, J)$ is complete, well powered and has a cogenerating family:

- **Completeness** is a given: we can compute limits pointwise like we did for **Set** in chapter 3, using $1_{\mathbf{Sh}(C,J)}$ = the functor sending every object to the singleton set.

- To show that **Sh**(C, J) is **Well-Powered** we can show that the subobjects (characterized before in **Set**$^{C^{op}}$) form a complete Heyting Algebra, there's a complete proof in section $III.8$ of [Sau94].
- As a **Cogenerating family** we use the inclusions in the product of power objects Ω^c where c ranges over representables functors for (C, J) and Ω is the Truth value object described in the latest sections.

□

5.2 In Module Theory

5.2.1 Abelian categories

Another interesting application comes from the algebra of modules.

What we've seen so far are "plain" categories, categories where the arrows didn't have any structure.

We know more than one example where that's not the whole picture: given U, V vector spaces $\hom(U, V)$ is itself a vector space.

Definition 5.2.1 (Enrichment).
*Given a category C, a C-**enriched** category D is a category for which $\forall x, y \in \mathrm{ob}(D)$ $D(x, y) \in \mathrm{ob}(C)$.*

For example, we saw a few lines upward that **Vect** is a **Vect**-enriched category.

The enrichment on **Ab**, the category of abelian groups and abelian group homomorphisms allows us to define preadditive and additive categories:

Definition 5.2.2 (Preadditive and Additive categories).
*A **Preadditive** category C is an **Ab**-enriched category in which the composition of morphisms distributes on the sum, i.e.*

- $C(a, b) \in$***Ab*** $\forall\ a, b \in \mathrm{ob}(C)$,
- $\forall f, g : a \to b\ h, k : c \to a$
  $f(h + k) = fh + fk\ ,\ (f + g)h = fh + gh$.

*An **Additive** category is a preadditive category with finite coproducts and a zero-object.*

In this context finite products and coproducts coincide, we call those "biproducts" or direct sums, using the symbol $\oplus$.

We want one more thing, we want kernels, cokernels, images and coimages as we are used to with abelian groups. let's define them:

Definition 5.2.3 (Kernel, Cokernel, Image, Coimage).
Given a map $f : a \to b$ in an additive category C we call

- ***Kernel*** $\ker(f)$ *the pullback*

$$\begin{array}{ccc} ker(f) & \longrightarrow & a \\ \downarrow & \lrcorner & \downarrow f \\ 0 & \longrightarrow & b \end{array}$$

- ***Cokernel*** $\operatorname{coker}(f)$ *dually, the pushout*

$$\begin{array}{ccc} a & \longrightarrow & 0 \\ \downarrow{\scriptstyle f} & \ulcorner & \downarrow \\ b & \longrightarrow & \mathit{coker}(f) \end{array}$$

- ***Image*** $\operatorname{im}(f)$ *the equalizer*

$$b \rightrightarrows b \sqcup_a b \qquad \text{where} \qquad \begin{array}{ccc} a & \xrightarrow{f} & b \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ b & \longrightarrow & b \sqcup_a b \end{array}$$

- ***Coimage*** $\operatorname{coim}(f)$ *dually, the coequalizer*

$$a \times_b a \rightrightarrows a \qquad \text{where} \qquad \begin{array}{ccc} a \times_b a & \longrightarrow & a \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ a & \xrightarrow[f]{} & b \end{array}$$

It's easier though to think of the image as the quotient $b/\ker(f)$ and the coimage as $a/\operatorname{coker}(f)$, even though we haven't really defined what a quotient space means in this context.

Now we can define preabelian and abelian categories:

Definition 5.2.4 (Preabelian and Abelian categories).
*A **Preabelian** category C is an additive category in which $\forall\ f \in \operatorname{mor}(C)\ \exists\ \ker(f), \operatorname{coker}(f)$ and the diagram*

$$\begin{array}{ccccccc} \mathit{ker}(f) & \longrightarrow & a & \xrightarrow{f} & b & \longrightarrow & \mathit{coker}(f) \\ & & \downarrow & & \uparrow & & \\ & & \mathit{coim}(f) & \xrightarrow{\bar{f}} & \mathit{im}(f) & & \end{array}$$

commutes.

*An **Abelian** category is a preabelian category in which the $\bar{f}$ defined in the diagram is an isomorphism.*

As we usually do, we define objects and morphisms, so what's a morphism between abelian categories?

Definition 5.2.5 (Additive functors).
*Given two preadditive categories C, D, a functor $F : C \to D$ is **Additive** if $C(x, y) \to D(Fx, Fy)$ is a morphism between abelian groups.*

and

Definition 5.2.6 (Exact Functors).
A functor $F : A \to B$ between abelian categories is

left-exact *if it preserves direct sums and kernels and*

right-exact *if it preserves direct sums and cokernels.*

5.2.2 Modules over a Ring

We know how vector spaces work: we can build a n-dimensional space from a field by setting elements of a field in different dimensions.

For example, a 2-D vector space on $\mathbb{R}$ is $\{\alpha(1,0) + \beta(0,1) : \alpha, \beta \in \mathbb{R}\}$.

If we loosen the condition of having a field, what we obtain doesn't have the same properties: to say one we don't have commutativity as a given anymore.

This sort of vector spaces on a Ring R are called R–Modules, and they -obviously- have their own category $\mathrm{Mod}-R$.

Lemma 5.2.7 (Abelianity of $\mathrm{Mod}-R$).
$\mathrm{Mod}-R$ *is an abelian category.*

Proof.
This proof is omitted but can be found in [Dat]. □

We want to show some important properties of $\mathrm{Mod}-R$:

Lemma 5.2.8 (Cocompleteness of $\mathrm{Mod}-R$).
$\mathrm{Mod}-R$ *is cocomplete.*

Sketch of proof.
We need to show that $\mathrm{Mod}-R$ has all coproducts and all binary coequalizers.

Coproducts in Mod_R are direct sums as we're used to in vector spaces: given two modules $m = \{(m_1, ..., m_k, ...)\}, n = \{(n_1, ..., n_l, ...)\} \in \mathrm{Mod}-R$, we define

$$m \oplus n = (m_1, n_1, ...m_k, n_k, ..., m_l, n_l, ...)$$

and we know that this kind of reasoning works for direct sums of any size.

Coequalizers in preadditive categories are defined as cokernels: $\mathrm{coeq}(f,g) = \mathrm{coker}(g - f)$ which we can quickly make sense of by dualizing $\mathrm{eq}(f,g) = \ker(g - f)$, trivial.

Since in an abelian category we have all cokernels, we have all coequalizers. □

A similar argument can prove completeness but we don't care right now.

Lemma 5.2.9 (Co-well-poweredness of $\mathrm{Mod}-R$).
$\mathrm{Mod}-R$ is co-well-powered, i.e. $\mathrm{Mod}-R^{op}$ is well-powered.

Sketch of proof.
This is true if and only if any object of $\mathrm{Mod}-R$ has a small poset of sup-objects. we know that $\mathrm{Mod}-R$ forms a poset with inclusions, the category $m \oplus \mathrm{Mod}-R$ of the direct sums with an object m is again a poset of objects bigger than m. □

Lemma 5.2.10.
$\mathrm{Mod}-R$ has a generating family.

Sketch of proof.
Any large enough direct sum of R with itself acts as a generator (any map onto any object is a projection $\implies$ an epimorphism), meaning $\{R\}_{i \in I}$ is a generating family. □

5.2.3 AFT in module categories

The Special Adjoint functor theorem can be used to prove a theorem about adjunctions between module categories.

A more complete version of the theorem with an alternative proof (that doesn't use the SAFT) can be found in [Ste75] as proposition 10.1.

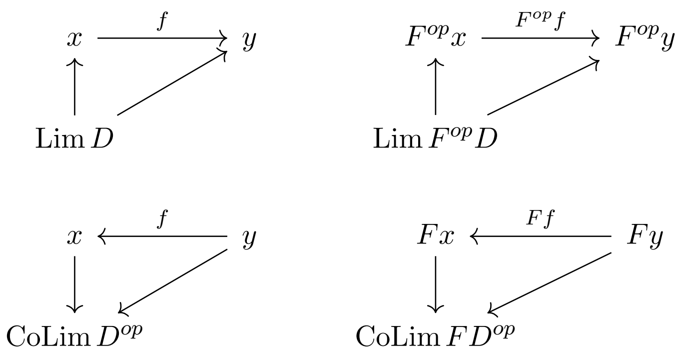
Theorem 5.2.11.
A functor $S : \mathrm{Mod}-A \to \mathrm{Mod}-B$ has a right adjoint $\iff$ S is right-exact

Proof.
We want to use the dual SAFT: given a cocomplete, co-well-powered category A with a generating family:

$F : A \to B$ has right adjoint $\iff$ it is cocontinuous.

The dual SAFT holds since if we define $F^{op} : A^{op} \to B^{op}$ where A is cocomplete, co-well-powered and with a generating family, A^{op} will be complete, well-powered and with a cogenerating family, meaning it satisfies the conditions for the SAFT, giving that F^{op}

continuous $\iff$ it has left adjoint. $\forall$ diagram $D : J \to A$ (and $D^{op} : J \to A^{op}$)



clearly F^{op} cocontinuous $\iff$ F cocontinuous, and a right adjoint $G^{op} \vdash F^{op}$

$$B^{op}(F^{op}x, y) \simeq A^{op}(x, G^{op}y)$$

yields

$$B(y, Fx) \simeq A(Gy, x)$$

meaning G is left adjoint to F.

We've already proven that $\mathrm{Mod}-R$ is cocomplete, co-well-powered with a generating family for any ring R, we just need to see that a right-exact functor is cocontinuous.

We know that a functor is cocontinuous if it preserves direct sums and binary coequalizers, since we defined direct sums as coproducts and coequalizers as cokernels in the previous sections we know it is true. $\square$

5.3 In Topological Algebra

In section 5.1 we did some topology to algebraic structures (categories in this instance). What about the converse? Can we "do algebra" on topological structures?

In this section we will discuss Algebraic categories and their topological counterparts.

Intuitively an algebraic category is a category of "sets with some algebraic structure": in this sense the categories of groups, rings and vector spaces are all algebraic categories. How do we formalize this notion in a categorical sense?

5.3.1 Monads and Beck's theorem

First we need to talk about monads and specifically monads that arise from adjunctions.

Definition 5.3.1 (Monads).
*Given a category A, a **Monad** on A is an endofunctor $T : A \to A$ equipped with two natural transformations $\eta : 1_A \Rightarrow T, \mu : T^2 \Rightarrow T$ so that both*

$$\begin{array}{ccc} T^3 & \overset{T\mu}{\Longrightarrow} & T^2 \\ {\scriptstyle \mu T}\Downarrow & & \Downarrow{\scriptstyle \mu} \\ T^2 & \underset{\mu}{\Longrightarrow} & T \end{array} \qquad \begin{array}{ccc} T & \overset{\eta T}{\Longrightarrow} & T^2 \\ {\scriptstyle T\eta}\Downarrow & \diagdown\!\!\diagdown & \Downarrow{\scriptstyle \mu} \\ T^2 & \underset{\mu}{\Longrightarrow} & T \end{array}$$

(where $T^n x = T(T(...Tx))$(n times)) commute.

The name is not random at all: this is in fact a monoid in A^A.

One could think that such a behaviour could arise from composing adjoint functors $F \dashv G$, using $T = GF, \eta = \eta$ (the unit of the adjunction) and $\mu = T\epsilon$ (where ϵ is the counit) but unfortunately not every adjunction can be promoted to a monad. In fact we call a functor monadic if its adjoint composes into a monad.

Definition 5.3.2 (Monadic functors).
*A functor $F : A \to B$ is **Monadic** if it has left adjoint G and $(GF, \eta, G\epsilon F)$ (where η, ϵ are unit and counit of $G \vdash F$) is a monad.*

Theorem 5.3.3 (Beck's monadicity theorem).
A functor $F : A \to B$ is monadic if

- *It has left adjoint,*
- *It creates coequalizers of F-split pairs, where an F-split pair is a parallel pair $f, g : x \rightrightarrows y \in \mathrm{mor}(A)$ so that the pair $Ff, Fg : Fx \rightrightarrows Fy$ has a split coequalizer (coequalizer which is also a split epimorphism) in B.*

Proof.
An accurate (if very long) proof is contained in [Lan71] in section VII.7.1 □

5.3.2 Algebraic categories

The categories we would like to call algebraic (groups, rings, vector spaces, modules, algebras, ...) have a very strong monad, formed from the adjunction "Free$\dashv$Forgetful" but not every algebraic category is monadic. We don't give an accurate definition for what an algebraic category is since we will never use it and we just need to know that every monadic category is algebraic.

For example **Grp** is algebraic: We know the Free group on a set

$$F_s := \{\text{words with elements of s as letters, product by concatenation and cancellation}\} :$$

the monad $T : x \to \text{Free}(x)$ sends a group to the free group on its elements. η is the map including a group x to the free group as strings of length 1 and the μ is the natural concatenation of strings.

For completeness I will give the definition of Algebraic category but I will not elaborate on it, we just need the fact that every monadic category is algebraic.

Definition 5.3.4 (Algebraic categories).
*A concrete category A, $U : A \to$ **Set** is **Algebraic** if*

- *It has all binary coequalizers,*
- *It has free objects (objects that act like the adjoints of U),*
- *U preserves and reflects extremal epimorphisms (an epimorphism e is extremal if $e = mn$ where m is a monomorphism $\implies$ m is an isomorphism).*

5.3.3 Compact Hausdorff spaces

The theorem we're going to prove in the following section makes use of topological spaces and particularly compact topological spaces.

Knowing **Top** is the category of topological spaces and continuous maps, We can define the category of compact hausdorff spaces:

Definition 5.3.5 (KHaus).
*We call **KHaus** the subcategory of **Top** where the objects are Compact Hausdorff spaces .*

We can also apply this restriction to other categories:

Definition 5.3.6 (Top $-A$, KHaus $-A$).
*Let A be an algebraic category, we call **Top** $-A$ the category of topological algebras on A so that the algebra operations are continuous, and **KHaus** $-A$ its full subcategory made of Compact Hausdorff spaces.*

Lemma 5.3.7.
*The forgetful functor **KHaus** $\to$ **Set** has a left adjoint β : **Set** $\to$ **KHaus** .*

Proof.
The left adjoint β is precisely the functor sending a set into the Stone-Cech compactification of the corresponding discrete space. This fact is stated and proven as a corollary of theorem 2.1 of [Joh82]. □

Theorem 5.3.8. *[Joh82]*
*For any algebraic category A the category **KHaus** $-A$ is algebraic.*

Proof.
First we use the AFT to verify the existence of a left adjoint to the forgetful U : **KHaus** $-A \to$ **Set**. It is clear that this functor creates arbitrary limits since the forgetful functors **KHaus** $\to$ **Set** and $A \to$ **Set** both do so;

The ssc is verified by inducing a map between algebras:

let F : **Set** $\to A$ denote the free functor for A and consider $f : x \to a$ sending a set to a compact hausdorff algebra a. a is an algebra, meaning that f extends uniquely to an homomorphism of algebras $f' : Fx \to a$ whose image is the subalgebra a'.

the closure a'' of a' is too a subalgebra of a.

If we further extend f' to $f'' : \beta(Fx) \to a$ by the adjunction, the image of f'' is precisely $a'' \in \mathrm{ob}($**KHaus** $-A)$, showing that any map $f : x \to a$ factors through one for which the induced map $\beta(Fx) \to a$ is surjective. There is only a set of non-isomorphic maps $x \to a$ with this property, since there's only a set of surjective images of $\beta(Fx)$.

We're now using Beck's theorem, meaning we have left to prove that U creates coequalizers for pairs of maps which become contractible in **Set**, but given a pair of maps $a \rightrightarrows b$ like that, its coequalizer $b \to c$ inherits a unique algebra structure and a unique hausdorff topology since the forgetful functors of A and **KHaus** are both monadic.

The two structures are compatible since $b^n \to c^n$ is a quotient map is a quotient map, meaning that the algebra operations $c^n \to c^n$ are continuous. □

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