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**Arbitrage theory in discrete time markets
with bid-ask spread**

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*Ai miei nonni, Ivo e Italo,
alle nonne, Anna e Pia,
e alla zia Bruna*

Abstract

This thesis studies the work of Przemysław Rola on the condition of no-arbitrage in a finite discrete time market with a money account (risk-free) and bid-ask spreads. In the first chapter, we introduce the mathematical model and we state the notions of Equivalent Bid-Ask Martingale Measure (EBAMM) and consistent price system (CPS). In the second chapter, we prove some lemmas and the fundamental theorem of asset pricing using the existence of EBAMM or superCPS and subCPS as an equivalent condition for no-arbitrage. In the last chapter, as an application of our findings, we introduce the Cox-Ross-Rubinstein model with bid-ask spreads.

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Introduction

In this thesis, we will present and analyze the model of a finite discrete time market with a bid-ask spread in order to find an equivalent condition to the absence of arbitrage. In this financial market, there are two prices for each asset: the bid and ask price and the numerical difference between these is called the bid-ask spread.

The ask price is the minimum amount a market maker is willing to sell a unit of asset: from an investor's point of view, it is the cost they would have to pay to purchase shares of the asset. On the other hand, the bid price is the highest amount a market maker is willing to buy a unit of an asset: from an investor's point of view, it is the potential gain from selling a unit of that asset. Obviously, the first one is greater than the second one: the bid-ask spread is an implicit cost of transaction that the investor accepts when he deals with assets.

The bid-ask spread can vary depending on many factors: liquidity is one of the most important. Liquidity is a measure of how easily an asset can be converted into cash without impacting its value. Currency is considered the most liquid asset, with a very small bid-ask spread, measured in fractions of pennies (one-hundredth of a per cent). In contrast, less liquid assets such as small-cap stocks may have bid-ask spreads that are a significant percentage of the asset's lowest ask price, such as 1-2%.

The difference between the bid and ask prices can also be significantly influenced by the amount of buy and sell orders, or "bids" and "asks", placed by market participants. If there are fewer buyers placing orders to purchase a security, this can lead to a decrease in bid prices, causing the spread to widen. Similarly, if there are fewer sellers placing orders to sell, this can also cause the spread to widen. For example, stocks like Google, Apple, and Microsoft, which are heavily traded, tend to have a narrow bid-ask spread. On the other hand, assets that are not well-known or popular may have a wider bid-ask spread because of lower trading volume and less investor's interest. De facto, the bid-ask spread also indicates the market maker's level of risk associated with making a trade, so it is also influenced by the potential for price volatility.

The other main topic of this thesis is the notion of no-arbitrage. In mathematical finance, arbitrage is a strategy, so the choice of purchase or sale of assets, that allows the investor to make a profit without any risk: it generates a strictly positive gain almost surely.

In absence of bid-ask spread, an equivalent condition for the absence of arbitrage for friction-less markets is given by the first Fundamental Theorem of Asset Pricing, also known as the Dalang–Morton–Willinger theorem: this states that a financial market is arbitrage-free if and only if there exists at least one equivalent martingale measure \mathbb{Q} (or

risk-neutral probability measure) that is equivalent to the original probability measure \mathbb{P} .

Finding equivalent conditions for markets with friction is more challenging: today there are a lot of relevant papers on this and we resume some of them. In article [10], Kabanov gave equivalent conditions for strict no-arbitrage for markets with proportional transaction cost and efficient friction.

In paper [18], Schachermayer states that the model satisfies the conditions of robust no-arbitrage (which is a type of no-arbitrage notion, robust with respect to small changes of bid-ask spreads) if and only if it admits a strictly consistent price system.

In this thesis, we examine a market with a limited number of assets and a money market account over a finite and discrete period. We work with the traditional concept of no-arbitrage and introduce the idea of an equivalent bid-ask martingale measure (EBAMM).

We then introduce the idea of a consistent price system (CPS) and two variations of it: the supermartingale consistent price system (supCPS) and the submartingale consistent price system (subCPS). We demonstrate that the existence of an EBAMM is equivalent to the existence of CPS.

In the second chapter, we prove two lemmas and the theorem where we verify that the existence of supCPS and subCPS implies the absence of arbitrage (this result is still valid if there exists a CPS).

In the second section of the second chapter, after two technical lemmas, we state and prove the main theorem where we give necessary and sufficient conditions for the absence of arbitrage.

In the last chapter, we give some applications where we make use of our results. We study the Cox-Ross-Rubinstein model with bid-ask spreads: investigating the conditions for the absence of arbitrage, we examine the evolution of the model and the existence of EBAMM.

Chapter 1

The mathematical model

In this chapter, we introduce the model for the financial market with bid-ask spread. At the end of this thesis in Appendix A, there are definitions and basic notions that might help the reader.

Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$ be a finite space equipped with a probability measure (see Definition 12 in Appendix A) \mathbb{P} such that $\mathbb{P}(\{\omega\}) = p_n > 0$ for $n = 1, \dots, N$. In addition to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we also fix $T \in \mathbb{N}$, $T \neq 0$, that is our final time horizon, and a filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ on Ω , an increasing sequence of σ -algebras (see Definition 11 in Appendix A).

In the financial market, we assume the existence of a money market account or a non-risky asset \mathcal{B} , which is a strictly positive adapted process $\mathcal{B} = (B_t)_{t=0}^T$. The dynamic of the title is settled by the equation:

$$\begin{cases} B_0 = 1 \\ B_t = B_{t-1}(1 + r) \end{cases}$$

where r is the risk-free rate. We can assume $B_t = 1 \quad \forall t \in \{0, 1, \dots, T\}$ without losing generality thanks to the discounting process described in [5, Section 2.1]. All transactions in our model will be calculated in units of this process.

We now introduce the stochastic process (see Definition 14 in Appendix A) that model risky assets.

Definition 1. The bid price process \underline{S} is a d -dimensional and \mathbb{F} -adapted process defined as

$$\underline{S} = (\underline{S}_t)_{t=0}^T = (\underline{S}_t^1, \dots, \underline{S}_t^d)_{t=0}^T.$$

The ask price process \overline{S} is a d -dimensional and \mathbb{F} -adapted process defined as

$$\overline{S} = (\overline{S}_t)_{t=0}^T = (\overline{S}_t^1, \dots, \overline{S}_t^d)_{t=0}^T.$$

The pair $(\underline{S}, \overline{S})$ is called the bid-ask price process.

\overline{S}_t^i models the price of the risky asset at time \tilde{t} of the i -th asset, on the other hand, \underline{S}_t^i models the income from the sale of the same asset: at any time, the investor can

buy or sell an unlimited number of the i -th price process. We suppose $\underline{S}_t^i \leq \bar{S}_t^i$ for any $i \in \{1, \dots, d\}$ and for any $t \in \{0, 1, \dots, T\}$: therefore when the investor sells a unit of titles, he earns less than what he would spend on buying a new one.

Remark 1. In some papers, the hypothesis could have slight differences. It's supposed there exists a stock process $S = (S_t)_{t=0}^T$ which models the “value” of the unit of asset. The bid and price process are defined as

$$\underline{S}_t^i := (1 + \lambda_t^i) S_t^i \quad \text{and} \quad \bar{S}_t^i := (1 - \mu_t^i) S_t^i,$$

where $\lambda_t^i \in [0, +\infty)$ and $\mu_t^i \in [0, 1)$ are real numbers.

A *trading strategy* is a stochastic process $H = (H_t)_{t=0}^T = (H_t^1, \dots, H_t^d)_{t=0}^T$ which is predictable with respect to \mathbb{F} (see Definition 14 in Appendix A). H_t^i stands for the units of the asset S^i that the investor keeps in his portfolio during the period from $t - 1$ to t . Usually, the strategy that refers to the money account is denoted by β_t . Therefore we define the *value of strategy* or the value of portfolio as

$$V_t^{(H, \beta)} := (H_t)^+ \cdot \underline{S}_t - (H_t)^- \cdot \bar{S}_t + \beta_t. \quad (1.1)$$

We calculate it using the immediate liquidation: at time t , the investor sells all units of assets he holds and pays short-selling positions.

We are interested in self-financing strategies, which are the particular type of strategies that satisfies

$$\beta_t = \beta_{t-1} - \bar{S}_{t-1} \cdot (H_t - H_{t-1})^+ + \underline{S}_{t-1} \cdot (H_t - H_{t-1})^-. \quad (1.2)$$

This can be interpreted as follows: at time $t - 1$, we hold H_{t-1} units of risky assets and β_{t-1} units of the non-risky asset and we build the strategy for the period $[t - 1, t]$ choosing H_t units of risky assets and β_t units of the non-risky asset so as not to change the total value of the portfolio. In this way, all asset transactions must be done by borrowing from or charging into the money account. With the same financial interpretation, the condition in Equation (1.2) is equivalent to

$$V_{t-1}^{(H, \beta)} = H_t^+ \cdot \underline{S}_{t-1} + H_t^- \cdot \bar{S}_{t-1} + \beta_t + \mathcal{L}_t, \quad (1.3)$$

where

$$\mathcal{L}_t := \bar{S}_{t-1} \cdot (H_t - H_{t-1})^+ - \underline{S}_{t-1} \cdot (H_t - H_{t-1})^- + \bar{S}_{t-1} \cdot (H_t^- - H_{t-1}^-) - \underline{S}_{t-1} \cdot (H_t^+ - H_{t-1}^+).$$

Remark 2. The Equation (1.3) may appear to have no financial meaning. Actually, based on the definitions given in Remark 1, it turns into

$$V_{t-1}^{(H, \beta)} = H_t \cdot S_{t-1} + \beta_t + \lambda_{t-1} S_{t-1} \cdot (H_t - H_{t-1})^+ + \mu_{t-1} S_{t-1} \cdot (H_t - H_{t-1})^+,$$

with the convention $\lambda_{t-1} S_{t-1} = (\lambda_{t-1}^1 S_{t-1}^1, \dots, \lambda_{t-1}^d S_{t-1}^d)$ and the same agreement is valid for $\mu_{t-1} S_{t-1}$. The final terms (which correspond to \mathcal{L}_t) can be interpreted as the alteration

due to the reorganisation dictated by the strategy H that will be balanced with the position in the risk-free asset. We can rewrite also the Equation (1.2) as

$$\beta_t = \beta_{t-1} - (1 + \lambda_{t-1})S_{t-1} \cdot (H_t - H_{t-1})^+ + (1 - \mu_{t-1})S_{t-1} \cdot (H_t - H_{t-1})^+,$$

where $(1 + \lambda_{t-1})S_{t-1} = ((1 + \lambda_{t-1}^1)S_{t-1}^1, \dots, (1 + \lambda_{t-1}^d)S_{t-1}^d)$ and the same convention applies to $(1 + \mu_{t-1})S_{t-1}$. It is important to note, however, that with these definitions the value of the portfolio becomes as follows

$$V_{t-1}^{(H,\beta)} := H_{t-1} \cdot S_{t-1} + \beta_{t-1}.$$

This implies that there are no transaction costs at the moment of leaving the financial market. In our model, we don't have this assumption so we calculate $V_{t-1}^{(H,\beta)}$ through the immediate liquidation.

Let \mathcal{P}_T be the set of all possible self-financing strategies. With the convention $H \geq 0$ if $H^i \geq 0 \forall i$ and, in the same way, $H \leq 0$ if $H^i \leq 0 \forall i$, we define the subsets

$$\mathcal{P}_T^+ := \{H \in \mathcal{P}_T \mid H \geq 0\} \quad \text{and} \quad \mathcal{P}_T^- := \{H \in \mathcal{P}_T \mid H \leq 0\}. \quad (1.4)$$

Definition 2. The stochastic process $x = (x_t)_{t=0}^T$ is defined as

$$x_t = x_t(H) := - \sum_{j=1}^t (\Delta H_j)^+ \cdot \bar{S}_j - 1 + \sum_{j=1}^t (\Delta H_j)^- \cdot \underline{S}_j - 1 + (H_t)^+ \cdot \underline{S}_t - (H_t)^- \cdot \bar{S}_t, \quad (1.5)$$

where \cdot is the inner product in \mathbb{R}^d and $\Delta H_j^i = H_j^i - H_{j-1}^i$ for any $i = 1, \dots, d$ and $j = 1, \dots, t$ with the convention $\Delta H_1^i = H_1^i$.

The random variable x_t models the gain or loss incurred up to time t in the market with bid-ask spreads following the strategy H and starting from 0 units in the bank account and 0 units in stock accounts.

In order to make clearer what each term stands for, we analyze $x_2(H)$ for just one risky asset ($d=1$):

$$x_2(H) = -H_1^+ \cdot \bar{S}_0 + H_1^- \cdot \underline{S}_0 - (H_2 - H_1)^+ \cdot \bar{S}_1 + (H_2 - H_1)^- \cdot \underline{S}_1 + H_2^+ \cdot \underline{S}_2 - H_2^- \cdot \bar{S}_2.$$

We examine each element:

- $-H_1^+ \cdot \bar{S}_0$: it is the price to buy $\#H_1^+$ units of the risky asset at ask price \bar{S}_0 at time $t = 0$;
- $+H_1^- \cdot \underline{S}_0$: it is the income from selling $\#H_1^-$ units of the risky asset at bid price \underline{S}_0 at time $t = 0$;
- $-(H_2 - H_1)^+ \cdot \bar{S}_1$: at time $t = 1$, the investor takes the decision to have $\#H_2$ units of the risky asset in the time interval $[1, 2]$. So if $(H_2 - H_1)$ is positive, he buys at time $t = 1$ $\#(H_2 - H_1)$ units of asset for the ask price \bar{S}_1 ;

- $+(H_2 - H_1)^- \cdot \underline{S}_1$: following the same reason as before, the investor sells $\#(H_2 - H_1)^-$ units of asset at bid price \underline{S}_1 ;
- $+H_2^+ \cdot \underline{S}_2 - H_2^- \cdot \overline{S}_2$: this is the result of the immediate liquidation. At the time $t = 2$, the investor sells all units of risky asset that he owns (so $\#H_2^+$) for the bid price \underline{S}_2 and he buys $\#H_2^-$ units of the asset at ask price \overline{S}_2 to pay off his debts.

In general, for any $t \in [0, T]$, the first sum in Equation (1.5) is the aggregate cost from buying assets up to time t while the second sum is the aggregate earnings from selling assets up to time t . In the end, in order to get the value of our portfolio, we liquidate all positions in risky assets and we pay short-selling positions: in this way we get the value in units of the money account.

Remark 3. If the trading strategy H is self-financing, the position for the money account is uniquely determined by the risky-asset strategy. So we can express β_t in terms of H_t^i .

We use the notation $L^0(\mathbb{R}^d, \mathcal{F}_t)$ for the set of \mathcal{F}_t -measurable random vector taking values in \mathbb{R}^d . When the σ -algebra is not written, it stands for $L^0(\mathbb{R}^d, \mathcal{F}_T)$, on the other hand when the dimension is not written, it stands for $L^0(\mathbb{R}, \mathcal{F}_t)$. In addition, let $L_+^0(\mathbb{R}^d, \mathcal{F}_t)$ be the subspace of $L^0(\mathbb{R}^d, \mathcal{F}_t)$ including the random vectors $H \geq 0$ with the convention fixed before. Moreover, we deal with the standard space L^1 and L^∞ (see Definition 15 in Appendix A) in the same way.

Definition 3. The set of final gains is

$$\mathcal{R}_T := \{x_T(H) \mid H \in \mathcal{P}_T\}.$$

The set of final sub-gains is

$$\mathcal{A}_T := \mathcal{R}_T - L_+^0.$$

The closure of \mathcal{A}_T in probability is as $\overline{\mathcal{A}}_T$.

We now introduce some random variables and sets that we will use later in proofs. They are similar to the ones we introduced so far but they refer to different time ranges.

Definition 4. For $1 \leq t \leq t+k \leq T$ and $H \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$, let be

$$x_{t-1, t+k}(H) := -(H)^+ \cdot \overline{S}_{t-1} + (H)^- \cdot \underline{S}_{t-1} + (H)^+ \cdot \underline{S}_{t+k} - (H)^- \cdot \overline{S}_{t+k}.$$

This process is similar to x_t : it models the gains or losses incurred when we buy(sell) $\#H$ units of the assets at time $t-1$ and then we sell(buy) them at time $t+k$.

Definition 5. For any $0 \leq j \leq t \leq T$, let be

$$\mathcal{R}_{j,t}^+ := \left\{ H \cdot (\underline{S}_t - \overline{S}_j) \mid H \in L_+^0(\mathbb{R}^d, \mathcal{F}_j) \right\},$$

$$\mathcal{R}_{j,t}^- := \left\{ H \cdot (\overline{S}_t - \underline{S}_j) \mid -H \in L_+^0(\mathbb{R}^d, \mathcal{F}_j) \right\}.$$

The first one is the set of claims achieved when, at time j , we buy $\#H$ units of assets at ask price \bar{S}_j and then we sell them for \underline{S}_t at time t . On the other hand, $\mathcal{R}_{j,t}^-$ is the set of claims made by a short-selling strategy: first we sell $\#H$ units of assets at bid price \underline{S}_j and then we re-buy at time t for \bar{S}_t .

Remark 4. In the Definition 5 we do not use the time index for the strategy since during the time interval $j \leq k \leq t$ we do not trade. If this creates confusion for the reader, according to the other notations in the thesis, they can replace H with H_{j+1} .

Furthermore, we define, for any $1 \leq t \leq t+k \leq T$, the following sets

$$\mathbb{F}_{t-1,t+k} := \mathcal{R}_{t-1,t+k}^+ + \mathcal{R}_{t-1,t+k}^- \quad \text{and} \quad F_{t-1,t+k} := \mathbb{F}_{t-1,t+k} - L_+^0(\mathcal{F}_{t+k}) \quad (1.6)$$

and for any $t = 1, \dots, T$ we define the counterpart of sets \mathcal{R}_t and \mathcal{A}_t

$$\mathbb{F}_t := \sum_{j < t} \mathcal{R}_{j,t}^+ + \sum_{j < t} \mathcal{R}_{j,t}^- \quad \text{and} \quad F_t := \mathbb{F}_t - L_+^0(\mathcal{F}_t) \quad (1.7)$$

In the end, we also introduce $\Lambda_T := \sum_{t=1}^T \mathbb{F}_t - L_+^0$.

Chapter 2

Consistent price system and equivalent bid-ask martingale measure

In this Chapter, we start exploring the main topic of this thesis: arbitrage in a market with a bid-ask spread. In a frictionless market, the first fundamental theorem of asset pricing states that a discrete market is arbitrage-free if, and only if, there exists at least one risk-neutral probability measure, also called equivalent martingale measure, that is equivalent to the original probability measure \mathbb{P} . Finding similar conditions in a market with friction is much more challenging: following the work of Przemysław in the article [15], we define a consistent price system and equivalent bid-ask martingale measure and we investigate the relations with the absence of arbitrage.

First, we give the definition of arbitrage using the random variable x_t given in Definition 2.

Definition 6. An *arbitrage* is a self-financing strategy $H \in \mathcal{P}_t$ verifying the following conditions:

- $H_0 = 0$ and $\beta_0 = 0$;
- $x_t(H) \geq 0$;
- $\mathbb{P}(x_t(H) > 0) > 0$.

An arbitrage is a trading strategy such that, starting from an investment zero, the resulting contingent gain $x_T(H)$ is non-negative and not identically equal to zero: the investor can gain without taking risks. We say that a market model is arbitrage-free if \mathcal{P}_T does not include arbitrage strategies.

We can characterize it using the set introduced in Definition 3 with the following condition:

Definition 7. There is *no arbitrage* in the market with bid-ask spreads if

$$\mathcal{A}_T \cap L_+^0 = \{0\}. \tag{NA}$$

Obviously, the condition NA is equivalent to $\mathcal{F}_T \cap L_+^0 = \{0\}$.

Lemma 1. *Assume that $\mathcal{A}_T \cap L_+^0 = \{0\}$. Then $\mathcal{A}_t \cap L_+^0(\mathcal{F}_t) = \{0\}$ for any $t \leq T$.*

Proof. If \tilde{H} is an arbitrage strategy with time horizon \tilde{t} (we liquidate all positions in risky assets in \tilde{t}), then there is also an arbitrage strategy H in a model with a larger horizon T : we can describe it as:

$$H_t = \begin{cases} \tilde{H}_t & \text{if } t \leq \tilde{t} \\ 0 & \text{if } t > \tilde{t}. \end{cases}$$

We keep following the strategy \tilde{H} until \tilde{t} , then we liquidate all risky assets and keep all the money in the money account, which is risk-free, until T . This strategy satisfies the three conditions in Definition 6. \square

Before introducing the Equivalent Bid-Ask Martingale Measure, we give the definition of an equivalent martingale measure in a market without friction. The description of some of the elements we will introduce is given in Definitions 16, 17, 18, 19 in Appendix A.

Definition 8. An *equivalent martingale measure (EMM)* is a probability measure \mathbb{Q} on (Ω, \mathcal{F}) such that:

- \mathbb{Q} is equivalent to \mathbb{P} :

$$\forall A \in \mathcal{F}, \quad \mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0;$$

- for any $n = 1, \dots, N$ we have

$$S_{n-1} = E^{\mathbb{Q}}[S_n | \mathcal{F}_{n-1}]. \quad (\text{EMM})$$

Namely, S is a \mathbb{Q} -martingale.

Equivalent martingale measures are often called risk-neutral measures. They are used to set an objective and fair cost for an asset or a financial instrument like derivatives. We use this measure to establish the price in order to remove the risk component. If real probabilities were used, the cost of each asset would need to be changed depending on the individual investor's risk appetite.

Now we are ready to introduce a measure that plays a similar role in markets with bid-ask spread. We use the same notation as in Przemysław's article [15].

Definition 9. An *equivalent bid-ask martingale measure (EBAMM)* for the bid-ask process $(\underline{S}, \overline{S})$ is a probability measure \mathbb{Q} such that is equivalent to \mathbb{P} , $\underline{S}_t, \overline{S}_t \in L^1(\mathbb{Q})$ and

$$\underline{S}_{t-1}^i \leq E^{\mathbb{Q}}(\overline{S}_t^i | \mathcal{F}_{t-1}) \quad \text{and} \quad E^{\mathbb{Q}}(\underline{S}_t^i | \mathcal{F}_{t-1}) \leq \overline{S}_{t-1}^i. \quad (\text{EBAMM})$$

The interpretation of this measure is that if we buy units of risky assets at time $t - 1$ at price \overline{S}_{t-1}^i , we do not expect on "average" (following the probability measure \mathbb{Q}) to sell them at a better price. The same thing happens when we short sale: if we sell earning \underline{S}_{t-1}^i at $t - 1$, we expect at t to re-buy at a higher price \overline{S}_t^i .

The similarity between these two definitions 8 and 9 is clear. We can see *EBAMM* as a generalization of *EMM*: if we assume $\underline{S} = \bar{S}$ (reducing our model to a market model without transaction costs), the two inequalities in *EBAMM* correspond with the equivalence in *EMM*.

We now introduce the notion of Consistent Price System.

Definition 10. A consistent price system (CPS) in the market with bid-ask spread is the pair (\tilde{S}, \tilde{P}) , where \tilde{P} is a probability measure equivalent to \mathbb{P} and $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ is a d -dimensional process adapted to the filtration \mathbb{F} such that it is a \tilde{P} -martingale and

$$\underline{S}_t^i \leq \tilde{S}_t^i \leq \bar{S}_t^i \quad \mathbb{P}\text{-a.e. for all } i = 1, \dots, d \text{ and } t = 0, \dots, T. \quad (\text{CPS})$$

If the process \tilde{S} is a \tilde{P} -supermartingale [respectively \tilde{P} -submartingale], then the pair (\tilde{S}, \tilde{P}) is a supermartingale consistent price system (*supCPS*) [submartingale consistent price system (*subCPS*)].

Lemma 2. If there exists a CPS, then there exists an EBAMM.

Proof. Let (\tilde{S}, \mathbb{Q}) be a consistent price system. Using the condition *CPS* and the monotonicity of expected value, for any $t = 1, \dots, T$ and $i = 1, \dots, d$ we have

$$\underline{S}_t^i \leq \tilde{S}_t^i \implies E^{\mathbb{Q}}\left(\underline{S}_t^i | \mathcal{F}_{t-1}\right) \leq E^{\mathbb{Q}}\left(\tilde{S}_t^i | \mathcal{F}_{t-1}\right).$$

Then \tilde{S} is a \mathbb{Q} -martingale so

$$E^{\mathbb{Q}}\left(\tilde{S}_t^i | \mathcal{F}_{t-1}\right) = \tilde{S}_{t-1}^i \leq \bar{S}_{t-1}^i$$

where we re-use the fact \tilde{S} is a *CPS*. Putting together the first and last inequality above we have

$$E^{\mathbb{Q}}\left(\underline{S}_t^i | \mathcal{F}_{t-1}\right) \leq \bar{S}_{t-1}^i$$

which is the second condition in Definition 9. For the other we retrace the same reasoning changing the direction of inequalities and reversing the bid and ask process

$$\underline{S}_{t-1}^i \leq \tilde{S}_{t-1}^i = E^{\mathbb{Q}}\left(\tilde{S}_t^i | \mathcal{F}_{t-1}\right) \leq E^{\mathbb{Q}}\left(\bar{S}_t^i | \mathcal{F}_{t-1}\right).$$

□

2.1 First Condition for No Arbitrage

Before we give the first result about the relation between the absence of arbitrage and the existence of a consistent price system, we state some auxiliary lemmas that will be useful in the proof of Theorem 1.

First, recalling the sets defined in Equation (1.4), let

$$\tilde{R}_T := \left\{ (\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T \mid \hat{H} \in \mathcal{P}_T^+, \check{H} \in \mathcal{P}_T^- \right\}$$

$$\tilde{R}_T^b := \left\{ (\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T \mid \hat{H} \in \mathcal{P}_T^+, \check{H} \in \mathcal{P}_T^- \text{ and } \hat{H}, \check{H} \text{ are bounded by } b \right\}$$

where $(H \cdot S)_t := \sum_{j=1}^t H_j \cdot \Delta S_j$ and “ \cdot ” is the inner product in \mathbb{R}^d and $\Delta S_j = S_j - S_{j-1}$.

Lemma 3. *The following are equivalent:*

(a) $\tilde{R}_T \cap L_+^0 = \{0\}$

(b) $\left\{ \hat{\eta} \cdot \Delta \hat{S}_t + \check{\eta} \cdot \Delta \check{S}_t \mid \hat{\eta}, -\check{\eta} \in L_+^0(\mathbb{R}^d, \mathcal{F}_{t-1}) \right\} \cap L_+(\mathcal{F}_t) = \{0\}$ for any $t = 1, \dots, T$.

Remark 5. Notice that condition (b) is the absence of arbitrage in a one-step model.

Proof Lemma 3. We prove each implication:

(a) \Rightarrow (b) This implication is trivial. If there are no arbitrage strategies until time T , then there are no arbitrage strategies for any $t \leq T$ by Lemma 1 so they cannot exist between two instants.

(b) \Rightarrow (a) Using the same reasoning as in book [11, Section 2.1.1], we assume $\tilde{R}_T \cap L_+^0 \neq \{0\}$ and prove the existence of arbitrage in a one-step model. Let the smallest $1 < t \leq T$ be fixed such that $\tilde{R}_t \cap L_+^0 \neq \{0\}$. Therefore we can find two strategies $\hat{H} \in \mathcal{P}_t^+$ and $\check{H} \in \mathcal{P}_t^-$ such that

$$\left(\hat{H} \cdot \hat{S} \right)_t + \left(\check{H} \cdot \check{S} \right)_t \geq 0 \quad \mathbb{P}\text{-a.e.} \quad \text{and} \quad \mathbb{P} \left(\left(\hat{H} \cdot \hat{S} \right)_t + \left(\check{H} \cdot \check{S} \right)_t > 0 \right) > 0.$$

In particular, rewriting the first condition,

$$\begin{aligned} \left(\hat{H} \cdot \hat{S} \right)_t + \left(\check{H} \cdot \check{S} \right)_t &= \sum_{j=1}^t \hat{H}_j \Delta \hat{S}_j + \sum_{j=1}^t \check{H}_j \Delta \check{S}_j \\ &= \sum_{j=1}^{t-1} \hat{H}_j \Delta \hat{S}_j + \hat{H}_t \Delta \hat{S}_t + \sum_{j=1}^{t-1} \check{H}_j \Delta \check{S}_j + \check{H}_t \Delta \check{S}_t \\ &= \left(\hat{H} \cdot \hat{S} \right)_{t-1} + \left(\check{H} \cdot \check{S} \right)_{t-1} + \hat{H}_t \Delta \hat{S}_t + \check{H}_t \Delta \check{S}_t \geq 0. \end{aligned}$$

Due to the choice of t , we have $\left(\hat{H} \cdot \hat{S} \right)_{t-1} + \left(\check{H} \cdot \check{S} \right)_{t-1} \leq 0$ \mathbb{P} -a.e.

This implies $\hat{H}_t \Delta \hat{S}_t + \check{H}_t \Delta \check{S}_t \geq 0$ \mathbb{P} -a.e., so for time t we get

$$\left\{ \hat{\eta} \cdot \Delta \hat{S}_t + \check{\eta} \cdot \Delta \check{S}_t \mid \hat{\eta}, -\check{\eta} \in L_+^0 \right\} \cap L_+^0 = \left\{ \hat{H}_t \Delta \hat{S}_t + \check{H}_t \Delta \check{S}_t \right\} \neq \{0\}.$$

□

Lemma 4. *The following are equivalent:*

$$(a) \tilde{R}_T \cap L_+^0 = \{0\}$$

$$(b) \tilde{R}_T^b \cap L_+^0 = \{0\}$$

Proof. We prove each implication:

(a) \Rightarrow (b) This implication is trivial: the set of bounded strategies is smaller than the set of all admissible strategies.

(b) \Rightarrow (a) By Lemma 3, we know that $\tilde{R}_T \cap L_+^0 = \{0\}$ is equivalent to

$$\left\{ \hat{\eta} \cdot \Delta \hat{S}_t + \check{\eta} \cdot \Delta \check{S}_t \mid \hat{\eta}, -\check{\eta} \in L_+^0(\mathbb{R}^d, \mathcal{F}_{t-1}) \right\} \cap L_+^0(\mathcal{F}_t) = \{0\} \quad \text{for any } t = 1, \dots, T.$$

Notice that if there is arbitrage in any one-step model, then $\tilde{R}_T^b \cap L_+^0 \neq \{0\}$. Fix any t and suppose that there exist two random variables $\hat{H} \in L_+^0(\mathbb{R}^d, \mathcal{F}_{t-1})$ and $(-\check{H}) \in L_+^0(\mathbb{R}^d, \mathcal{F}_{t-1})$ such that

$$\hat{H}_t \Delta \hat{S}_t + \check{H}_t \Delta \check{S}_t \geq 0 \quad \mathbb{P}\text{-a.e.} \quad \text{and} \quad \mathbb{P}(\hat{H}_t \Delta \hat{S}_t + \check{H}_t \Delta \check{S}_t > 0) > 0.$$

Define $H_t := (\hat{H}_t, \check{H}_t) \in L^0(\mathbb{R}^{2d}, \mathcal{F}_{t-1})$ and the normalized random vector

$$\bar{H}_t = \begin{cases} H_t / \|H_t\| & \text{if } H_t \neq 0 \\ 0 & \text{if } H_t = 0. \end{cases}$$

Now split \bar{H}_t into $\hat{H}_t^b := (\bar{H}_t^1, \dots, \bar{H}_t^d)$ and $\check{H}_t^b := (\bar{H}_t^{d+1}, \dots, \bar{H}_t^{2d})$. It is obvious that $\hat{H}_t^b \in L_+^0(\mathbb{R}^d, \mathcal{F}_{t-1})$ and $(-\check{H}_t^b) \in L_+^0(\mathbb{R}^d, \mathcal{F}_{t-1})$. These two processes satisfy

$$\hat{H}_t^b \Delta \hat{S}_t + \check{H}_t^b \Delta \check{S}_t \geq 0 \quad \mathbb{P}\text{-a.e.} \quad \text{and} \quad \mathbb{P}(\hat{H}_t^b \Delta \hat{S}_t + \check{H}_t^b \Delta \check{S}_t > 0) > 0$$

which means they are arbitrage strategies in the class of bounded ones, so $\tilde{R}_T^b \cap L_+^0 \neq \{0\}$.

□

Remark 6. Following Przemysław's work [15, Section 2], we state and prove Lemma 3 and 4 in order to have a clearer idea from an analytical point of view. Let be noted the usefulness of Lemma 3 because it emphasizes how the condition of no-arbitrage in the final moment implies that we cannot realize an arbitrage strategy for all previous moments. Nevertheless, we could skip these lemmas: we do not consider unlimited strategies in \mathcal{P}_T since they are not realistic. Therefore, the equivalence between the two conditions in Lemma 4 is trivial. From now on, \tilde{R}_T and \tilde{R}_T^b will be the same set.

We now finally prove the following theorem.

Theorem 1. *Suppose there exists a supCPS (\tilde{S}, \mathbb{Q}) and a subCPS (\tilde{S}, \mathbb{Q}) . Then $\tilde{R}_T \cap L_+^0 = \{0\}$ and there is no arbitrage, i.e., $\mathcal{A}_T \cap L_+^0 = \{0\}$.*

Proof. We start proving $\tilde{R}_T \cap L_+^0 = \{0\}$. Fix any random variable

$$X = (\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T \in \tilde{R}_T \cap L_+^0.$$

Hence $(\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T \geq 0$ because $X \in L_+^0$. We want to show $E^{\mathbb{Q}}[(\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T] \leq 0$. Using the law of total expectation, we can rewrite it as

$$E^{\mathbb{Q}}[(\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T] = E^{\mathbb{Q}} \left[E^{\mathbb{Q}} \left[\sum_{t=0}^T \hat{H}_t \cdot \Delta \hat{S}_t + \sum_{t=0}^T \check{H}_t \cdot \Delta \check{S}_t \middle| \mathcal{F}_0 \right] \right]. \quad (2.1)$$

For the positivity, the sign of conditional expectation is the same as the expected value. For the linearity, we can extract each sum and, analyzing the single addendum, by the tower property, we obtain:

$$E^{\mathbb{Q}}[\hat{H}_t \cdot \Delta \hat{S}_t | \mathcal{F}_0] = E^{\mathbb{Q}} \left[E^{\mathbb{Q}}[\hat{H}_t \cdot \Delta \hat{S}_t | \mathcal{F}_{t-1}] | \mathcal{F}_0 \right]$$

for any $t = 1, \dots, T$ because $\mathcal{F}_0 \subseteq \mathcal{F}_t$. The inner conditional expectation is now

$$\begin{aligned} E^{\mathbb{Q}}[\hat{H}_t \cdot \Delta \hat{S}_t | \mathcal{F}_{t-1}] &= \hat{H}_t \cdot E^{\mathbb{Q}}[\Delta \hat{S}_t | \mathcal{F}_{t-1}] \\ &= \hat{H}_t \cdot E^{\mathbb{Q}}[\hat{S}_t - \hat{S}_{t-1} | \mathcal{F}_{t-1}] \\ &= \hat{H}_t \cdot (E^{\mathbb{Q}}[\hat{S}_t | \mathcal{F}_{t-1}] - E^{\mathbb{Q}}[\hat{S}_{t-1} | \mathcal{F}_{t-1}]) \\ &= \hat{H}_t \cdot (E^{\mathbb{Q}}[\hat{S}_t | \mathcal{F}_{t-1}] - \hat{S}_{t-1}) \end{aligned}$$

where we moved the process \hat{H}_t out of the conditional expectation because it is predictable and $E^{\mathbb{Q}}[\hat{S}_{t-1} | \mathcal{F}_{t-1}] = \hat{S}_{t-1}$ because \hat{S} is an adapted process. Now we use the assumption that \hat{S}_t is a \mathbb{Q} -supermartingale, i.e.,

$$E^{\mathbb{Q}}[\hat{S}_t | \mathcal{F}_{t-1}] \leq \hat{S}_{t-1}$$

and, since $\hat{H}_t \in L_+^0$, we conclude that

$$E^{\mathbb{Q}}[\hat{H}_t \cdot \Delta \hat{S}_t | \mathcal{F}_{t-1}] \leq 0.$$

Analogously, for the other addends of the other sum, we get

$$E^{\mathbb{Q}}[\check{H}_t \cdot \Delta \check{S}_t | \mathcal{F}_{t-1}] = \check{H}_t (E^{\mathbb{Q}}[\check{S}_t | \mathcal{F}_{t-1}] - \check{S}_{t-1})$$

and, since $(-\check{H}_t) \in L_+^0$ and \check{S}_t is a \mathbb{Q} -submartingale, we get

$$E^{\mathbb{Q}}[\check{H}_t \cdot \Delta \check{S}_t | \mathcal{F}_{t-1}] \leq 0.$$

Summing up, for any t , we have $E^{\mathbb{Q}}[\hat{H}_t \cdot \Delta \hat{S}_t + \check{H}_t \cdot \Delta \check{S}_t | \mathcal{F}_{t-1}] \leq 0$ which implies

$$\sum_{t=0}^T E^{\mathbb{Q}} [\hat{H}_t \cdot \Delta \hat{S}_t + \check{H}_t \cdot \Delta \check{S}_t | \mathcal{F}_{t-1}] \leq 0$$

and, for Equation (2.1) and the positivity of the expected value, it is possible to conclude that

$$E^{\mathbb{Q}} \left[E^{\mathbb{Q}} \left[\sum_{t=0}^T E^{\mathbb{Q}} \left[\hat{H}_t \cdot \Delta \hat{S}_t + \check{H}_t \cdot \Delta \check{S}_t \middle| \mathcal{F}_{t-1} \right] \middle| \mathcal{F}_0 \right] \right] = E^{\mathbb{Q}}[(\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T] \leq 0.$$

Therefore, if a random variable X is positive and $E^{\mathbb{Q}}[X] \leq 0$, this means $X = 0$ \mathbb{Q} -a.e. and, from the equivalence of measures, $X = 0$ \mathbb{P} -a.e., proving the first assumption of Theorem.

We want now to prove the absence of arbitrage. Let $\xi \in \mathcal{A}_T \cap L_+^0$. The following inequalities are satisfied

$$0 \leq \xi \leq - \sum_{t=1}^T (\Delta H_t)^+ \bar{S}_{t-1} + \sum_{t=1}^T (\Delta H_t)^- \underline{S}_{t-1} + (H_T)^+ \underline{S}_T - (H_T)^- \bar{S}_T. \quad (2.2)$$

For any strategy $H \in \mathcal{P}_T$, there exists two strategies $\hat{H} \in \mathcal{P}_T^+$ and $\check{H} \in \mathcal{P}_T^+$ such that $\Delta H_t^i = \Delta \hat{H}_t^i + \Delta \check{H}_t^i$. We can define them by splitting the strategy H into two taking only long and short position

$$\hat{H}_t^i := (H_t^i)^+ \quad \text{and} \quad \check{H}_t^i := (H_t^i)^- \quad \text{for any } t = 1, \dots, T \quad i = 1, \dots, d.$$

Notice that it is not possible to have $\Delta \hat{H}_t^i > 0$ and $\Delta \check{H}_t^i < 0$ at the same time t : indeed, we would get simultaneously $(H_t^i)^+ > (H_{t-1}^i)^+ \geq 0$ and $(H_t^i)^- > (H_{t-1}^i)^- \geq 0$ which is a contradiction. In the same way, there cannot exist H such that $\Delta \hat{H}_t^i < 0$ and $\Delta \check{H}_t^i > 0$.

We can define \hat{H} and \check{H} also in the following way

$$\begin{aligned} \text{if } H_t^i \geq 0 \ \& \ H_{t-1}^i \geq 0 \quad \text{then} \quad \Delta \hat{H}_t^i := \Delta H_t^i, \quad \Delta \check{H}_t^i := 0 \\ \text{if } H_t^i < 0 \ \& \ H_{t-1}^i < 0 \quad \text{then} \quad \Delta \hat{H}_t^i := 0, \quad \Delta \check{H}_t^i := \Delta H_t^i \\ \text{if } H_t^i \geq 0 \ \& \ H_{t-1}^i < 0 \quad \text{then} \quad \Delta \hat{H}_t^i := H_t^i, \quad \Delta \check{H}_t^i := -H_{t-1}^i \\ \text{if } H_t^i < 0 \ \& \ H_{t-1}^i \geq 0 \quad \text{then} \quad \Delta \hat{H}_t^i := -H_{t-1}^i, \quad \Delta \check{H}_t^i := H_t^i. \end{aligned}$$

With these definitions, it is evident that

$$\begin{aligned} \Delta H_t^i > 0 \quad \implies \quad \Delta \hat{H}_t^i \geq 0 \quad \text{and} \quad \Delta \check{H}_t^i \geq 0 \\ \Delta H_t^i < 0 \quad \implies \quad \Delta \hat{H}_t^i \leq 0 \quad \text{and} \quad \Delta \check{H}_t^i \leq 0 \end{aligned}$$

for any $t = 1, \dots, T$ and $i = 1, \dots, d$. Therefore, it is always true that

$$(\Delta H_t^i)^+ = (\Delta \hat{H}_t^i)^+ + (\Delta \check{H}_t^i)^+ \quad \text{and} \quad (\Delta H_t^i)^- = (\Delta \hat{H}_t^i)^- + (\Delta \check{H}_t^i)^-. \quad (2.3)$$

By replacing $(\Delta H)^+$ and $(\Delta H)^-$ in Equation (2.2), we obtain

$$\begin{aligned} \xi &\leq - \sum_{t=1}^T (\Delta H_t)^+ \bar{S}_{t-1} + \sum_{t=1}^T (\Delta H_t)^- \underline{S}_{t-1} + (H_T)^+ \underline{S}_T - (H_T)^- \bar{S}_T \\ &= - \sum_{t=1}^T (\Delta \hat{H}_t)^+ \bar{S}_{t-1} + \sum_{t=1}^T (\Delta \hat{H}_t)^- \underline{S}_{t-1} + (\hat{H}_T)^+ \underline{S}_T - (\hat{H}_T)^- \bar{S}_T \\ &\quad - \sum_{t=1}^T (\Delta \check{H}_t)^+ \bar{S}_{t-1} + \sum_{t=1}^T (\Delta \check{H}_t)^- \underline{S}_{t-1} + (\check{H}_T)^+ \underline{S}_T - (\check{H}_T)^- \bar{S}_T. \end{aligned}$$

Now we consider inequalities CPS for \hat{S} and \check{S}

$$\underline{S}_t^i \leq \hat{S}_t^i \leq \bar{S}_t^i \quad \text{and} \quad \underline{S}_t^i \leq \check{S}_t^i \leq \bar{S}_t^i \quad \mathbb{P}\text{-a.e.}$$

for any $t = 1, \dots, T$ and, using again Equation (2.3), $i = 1, \dots, d$ we get

$$\begin{aligned} \xi &\leq - \sum_{t=1}^T (\Delta \hat{H}_t)^+ \hat{S}_{t-1} + \sum_{t=1}^T (\Delta \hat{H}_t)^- \check{S}_{t-1} + (\hat{H}_T)^+ \check{S}_T - (\hat{H}_T)^- \hat{S}_T \\ &\quad - \sum_{t=1}^T (\Delta \check{H}_t)^+ \hat{S}_{t-1} + \sum_{t=1}^T (\Delta \check{H}_t)^- \check{S}_{t-1} + (\check{H}_T)^+ \check{S}_T - (\check{H}_T)^- \check{S}_T \\ &\leq - \sum_{t=1}^T \Delta \hat{H}_t \hat{S}_{t-1} + \hat{H}_T \hat{S}_T - \sum_{t=1}^T \Delta \check{H}_t \check{S}_{t-1} + \check{H}_T \check{S}_T. \end{aligned}$$

If we expand the two sums, we can simplify some addends (we show it only for $\Delta \hat{H} \hat{S}$, for $\Delta \check{H} \check{S}$ it is the same):

$$\begin{aligned} - \sum_{t=1}^T \Delta \hat{H}_t \hat{S}_{t-1} + \hat{H}_T \hat{S}_T &= -\hat{H}_1 \hat{S}_0 - (\hat{H}_2 - \hat{H}_1) \hat{S}_1 - (\hat{H}_3 - \hat{H}_2) \hat{S}_2 - \dots - (\hat{H}_T - \hat{H}_{T-1}) \hat{S}_{T-1} + \hat{H}_T \hat{S}_T \\ &= -\hat{H}_1 \hat{S}_0 + \hat{H}_1 \hat{S}_1 - \hat{H}_2 \hat{S}_1 + \hat{H}_2 \hat{S}_2 - \dots - \hat{H}_T \hat{S}_{T-1} + \hat{H}_T \hat{S}_T \\ &= \hat{H}_1 (\Delta \hat{S}_1) + \hat{H}_2 (\Delta \hat{S}_2) + \dots + \hat{H}_T (\Delta \hat{S}_T) \\ &= (\hat{H} \cdot \hat{S})_T. \end{aligned}$$

In conclusion, we arrive at the final inequality

$$0 \leq \xi \leq (\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T$$

where, on the right, we have a random variable in the set $\tilde{R}_T \cap L_+^0$ (we take the intersection because it is surely positive). From the previous part of the proof, if $(\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T \geq 0$, then $(\hat{H} \cdot \hat{S})_T + (\check{H} \cdot \check{S})_T = 0$ \mathbb{P} -a.e and hence $\xi = 0$ \mathbb{P} -a.e. From the arbitrariness of ξ , we get $\mathcal{A}_T \cap L_+^0 = \{0\}$, the condition of no arbitrage. \square

Remark 7. If there exists a CPS, instead of supCPS and subCPS, then Theorem 1 is still valid: in the proof, since S is a \mathbb{Q} -martingale, $E^{\mathbb{Q}}[S_t | \mathcal{F}_{t-1}] = S_{t-1}$ and

$$E^{\mathbb{Q}} \left[\hat{H}_t \cdot \Delta S_t + \check{H}_t \cdot \Delta S_t | \mathcal{F}_{t-1} \right] = 0.$$

2.2 The fundamental theorem of asset pricing

Before stating the fundamental theorem, we prove some technical lemmas we will use later using the sets and the process introduced in Equation (1.6) and (1.7) and Definition 4 and 5.

Lemma 5. *Let Π belong to \mathbb{F}_t . Then there exist two d -dimensional processes $\tilde{\vartheta} = (\tilde{\vartheta}_j)_{j=1}^t$ and $\vartheta = (\vartheta_j)_{j=1}^t$, predictable and non negative, such that for any $j = 1, \dots, t$ and $i = 1, \dots, d$ we cannot have at the same time $\vartheta_j^i > 0$ and $\tilde{\vartheta}_j^i > 0$ and Π is bounded by*

$$\Pi \leq \Xi := - \sum_{j=1}^t \vartheta_j \bar{S}_{j-1} + \sum_{j=1}^t \tilde{\vartheta}_j \underline{S}_{j-1} + \sum_{j=1}^t \vartheta_j \underline{S}_t - \sum_{j=1}^t \tilde{\vartheta}_j \bar{S}_t \quad \mathbb{P}\text{-a.e.}$$

Proof. Let be Π as follows

$$\begin{aligned} \Pi &= \theta_1(\underline{S}_t - \bar{S}_0) + \theta_2(\underline{S}_t - \bar{S}_1) + \dots + \theta_t(\underline{S}_t - \bar{S}_{t-1}) + \\ &\quad + \tilde{\theta}_1(\bar{S}_t - \underline{S}_0) + \tilde{\theta}_2(\bar{S}_t - \underline{S}_1) + \dots + \tilde{\theta}_t(\bar{S}_t - \underline{S}_{t-1}). \end{aligned}$$

Reorganising the terms we get

$$\Pi = - \sum_{j=1}^t \theta_j \bar{S}_{j-1} + \sum_{j=1}^t \tilde{\theta}_j \underline{S}_{j-1} + \sum_{j=1}^t \theta_j \underline{S}_t - \sum_{j=1}^t \tilde{\theta}_j \bar{S}_t.$$

From the Definition 5 of $\mathcal{R}_{j,t}^+$ and $\mathcal{R}_{j,t}^-$, $\Theta = (\theta_j)_{j=1}^t$ and $\Theta = (\theta_j)_{j=1}^t$ are d -dimensional, predictable and non negative processes. Let be $\nu_j^i := \min\{\theta_j^i, \tilde{\theta}_j^i\} \geq 0$. We define $\vartheta, \tilde{\vartheta}$ as follows

$$\vartheta_j^i := \theta_j^i - \nu_j^i \quad \tilde{\vartheta}_j^i := \tilde{\theta}_j^i - \nu_j^i \quad \text{for } i = 1, \dots, d, j = 1, \dots, t.$$

Clearly, ϑ and $\tilde{\vartheta}$ are d -dimensional, predictable and non-negative processes and they respect the condition that we cannot have $\vartheta_j^i > 0$ and $\tilde{\vartheta}_j^i > 0$ at the same time j .

Let Ξ be

$$\Xi = - \sum_{j=1}^t \vartheta_j \bar{S}_{j-1} + \sum_{j=1}^t \tilde{\vartheta}_j \underline{S}_{j-1} + \sum_{j=1}^t \vartheta_j \underline{S}_t - \sum_{j=1}^t \tilde{\vartheta}_j \bar{S}_t.$$

Rewriting Π using ϑ and $\tilde{\vartheta}$, we obtain

$$\begin{aligned} \Pi &= - \sum_{j=1}^t (\vartheta_j + \nu_j) \bar{S}_{j-1} + \sum_{j=1}^t (\tilde{\vartheta}_j + \nu_j) \underline{S}_{j-1} + \sum_{j=1}^t (\vartheta_j + \nu_j) \underline{S}_t - \sum_{j=1}^t (\tilde{\vartheta}_j + \nu_j) \bar{S}_t \\ &= \Xi - \sum_{j=1}^t \nu_j (\bar{S}_{j-1} - \underline{S}_{j-1}) - \sum_{j=1}^t \nu_j (\bar{S}_t - \underline{S}_t). \end{aligned}$$

The selling price is greater than the buying price from the hypotheses about processes \bar{S} and \underline{S} , so the last two sums are positive and we get $\Pi \leq \Xi$ \mathbb{P} -a.e. \square

Lemma 6. *For any $t = 1, \dots, T$ we have $F_t \subset \mathcal{A}_t$.*

Proof. Since $F_t = \mathbb{F}_t - L_+^0(\mathcal{F}_t)$, it suffices to show $\mathbb{F}_t \subset \mathcal{A}_t$. Let $\Pi \in \mathbb{F}_t$. By Lemma 5, there exist two d -dimensional processes $\tilde{\vartheta} = (\tilde{\vartheta}_j)_{j=1}^t$ and $\vartheta = (\vartheta_j)_{j=1}^t$ predictable and non negative such that

$$\Pi \leq \Xi := -\sum_{j=1}^t \vartheta_j \bar{S}_{j-1} + \sum_{j=1}^t \tilde{\vartheta}_j \underline{S}_{j-1} + \sum_{j=1}^t \vartheta_j \underline{S}_t - \sum_{j=1}^t \tilde{\vartheta}_j \bar{S}_t \quad \mathbb{P}\text{-a.e.}$$

We define the strategy $H = (H_j)_{j=1}^t \in \mathcal{P}_t$ as follows

$$\Delta H_j := (\Delta H_j)^+ - (\Delta H_j)^- \quad \text{where} \quad (\Delta H_j)^+ := \vartheta_j \quad \text{and} \quad (\Delta H_j)^- := \tilde{\vartheta}_j$$

with the convention $\Delta H_j = H_j - H_{j-1}$ and $\Delta H_1 = H_1$. For the construction of ϑ and $\tilde{\vartheta}$ in the previous proof, H is a well-defined strategy.

Now we consider the random variable r defined as follows

$$\begin{aligned} r &= \sum_{j=1}^t (\Delta H_j)^- \bar{S}_t - \sum_{j=1}^t (\Delta H_j)^+ \underline{S}_t + H_t^+ \underline{S}_t - H_t^- \bar{S}_t \\ &= \underline{S}_t \left(H_t^+ - \sum_{j=1}^t (\Delta H_j)^+ \right) - \bar{S}_t \left(H_t^- - \sum_{j=1}^t (\Delta H_j)^- \right). \end{aligned}$$

Then, using the definitions of positive and negative parts, we have

$$\sum_{j=1}^t (\Delta H_j)^+ - \sum_{j=1}^t (\Delta H_j)^- = \sum_{j=1}^t \left((\Delta H_j)^+ - (\Delta H_j)^- \right) = \sum_{j=1}^t \Delta H_j = H_t = H_t^+ - H_t^-.$$

Reorganising the left-hand side and right-hand side, we obtain

$$H_t^+ - \sum_{j=1}^t (\Delta H_j)^+ = H_t^- - \sum_{j=1}^t (\Delta H_j)^-$$

and observing that

$$H_t^+ = \left(\sum_{j=1}^t \Delta H_j \right)^+ \leq \sum_{j=1}^t (\Delta H_j)^+$$

we have

$$r = \left(H_t^+ - \sum_{j=1}^t (\Delta H_j)^+ \right) (\underline{S}_t - \bar{S}_t) \geq 0$$

since the two factors are both negative and this implies $r \in L_+^0(\mathcal{F}_t)$. Hence we can consider

$$\Pi + r \leq \Xi + r = -\sum_{j=1}^t (\Delta H_j)^+ \bar{S}_{j-1} + \sum_{j=1}^t (\Delta H_j)^- \underline{S}_{j-1} + (H_t)^+ \underline{S}_t - (H_t)^- \bar{S}_t = x_t(H).$$

Therefore $\Pi \leq x_t(H) - r$ with $x_t(H) \in \mathcal{R}_t$. This implies that there exists $\tilde{r} \in L_+^0(\mathcal{F}_t)$, defined as $\tilde{r} = r + \Xi - \Pi$, such that $\Pi = x_t(H) - \tilde{r}$: here we get $\Pi \in \mathcal{A}_t$. \square

Lemma 7. For any $1 \leq t \leq t+k \leq T$ and $x \in F_{t-1,t+k}$, there exist $H_t \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$ and $r \in L^0_+(\mathcal{F}_{t+k})$ such that $x = x_{t-1,t+k}(H_t) - r$.

Proof. Fix any t, k such that $1 \leq t \leq t+k \leq T$ and consider any $x \in F_{t-1,t+k}$: since the definitions of $F_{t-1,t+k}$ and $\mathbb{F}_{t-1,t+k}$ in Equation (1.6), we can write the random variable x as

$$\begin{aligned} x &= \Pi - l \\ &= -\theta \cdot \bar{S}_{t-1} + \tilde{\theta} \cdot \underline{S}_{t-1} + \theta \cdot \underline{S}_{t+k} - \tilde{\theta} \cdot \bar{S}_{t+k} - l \end{aligned}$$

where $\theta, \tilde{\theta} \in L^0_+(\mathbb{R}^d, \mathcal{F}_{t-1})$ and $l \in L^0_+(\mathbb{R}, \mathcal{F}_{t+k})$. Reasoning in the same way as in the proof of Lemma 5, we can find two strategies $\vartheta, \tilde{\vartheta} \in L^0_+(\mathbb{R}^d, \mathcal{F}_{t-1})$ such that for any $j = 1, \dots, t$ and $i = 1, \dots, d$ we cannot have $\vartheta_j^i > 0$ and $\tilde{\vartheta}_j^i > 0$ at the same time: let be $\nu^i := \min\{\theta^i, \tilde{\theta}^i\} \geq 0$. We define $\vartheta, \tilde{\vartheta}$ as follows

$$\vartheta_j^i := \theta_j^i - \nu_j^i \quad \tilde{\vartheta}_j^i := \tilde{\theta}_j^i - \nu_j^i \quad \text{for } i = 1, \dots, d, j = 1, \dots, t.$$

In this way, $\vartheta, \tilde{\vartheta}$ are d -dimensional, non-negative and \mathcal{F}_{t-1} measurable random vectors. Furthermore, pointing out that $\bar{S}_k - \underline{S}_k \geq 0$ is the bid-ask spread, we have

$$\begin{aligned} \Pi &= -(\vartheta + \nu) \cdot \bar{S}_{t-1} + (\tilde{\vartheta} + \nu) \cdot \underline{S}_{t-1} + (\vartheta + \nu) \cdot \underline{S}_{t+k} - (\tilde{\vartheta} + \nu) \cdot \bar{S}_{t+k} \\ &= -\vartheta \cdot \bar{S}_{t-1} + \tilde{\vartheta} \cdot \underline{S}_{t-1} + \vartheta \cdot \underline{S}_{t+k} - \tilde{\vartheta} \cdot \bar{S}_{t+k} - \nu \cdot (\bar{S}_{t-1} - \underline{S}_{t-1}) - \nu \cdot (\bar{S}_{t+k} - \underline{S}_{t+k}) \\ &= \Xi - \nu \cdot (\bar{S}_{t-1} - \underline{S}_{t-1}) - \nu \cdot (\bar{S}_{t+k} - \underline{S}_{t+k}) \\ &\leq \Xi \quad \mathbb{P}\text{-a.e.} \end{aligned}$$

Notice that $\Xi = -\vartheta \cdot \bar{S}_{t-1} + \tilde{\vartheta} \cdot \underline{S}_{t-1} + \vartheta \cdot \underline{S}_{t+k} - \tilde{\vartheta} \cdot \bar{S}_{t+k} \in \mathcal{R}_{t+k}$.

As in the Lemma 6, we define $H_t := \vartheta - \tilde{\vartheta}$. It is obvious that $H_t \in L^0_+(\mathbb{R}^d, \mathcal{F}_{t-1})$ and $(H_t^i)^+ = \vartheta^i, (H_t^i)^- = \tilde{\vartheta}^i$. Defining $\tilde{l} := \Xi - \Pi \in L^0_+(\mathcal{F}_{t+k})$, we have $x = \Pi - l = \Xi - l - \tilde{l}$. Naming $r := l + \tilde{l} \in L^0_+(\mathcal{F}_{t+k})$, we get

$$\begin{aligned} x &= -\vartheta \cdot \bar{S}_{t-1} + \tilde{\vartheta} \cdot \underline{S}_{t-1} + \vartheta \cdot \underline{S}_{t+k} - \tilde{\vartheta} \cdot \bar{S}_{t+k} - l - \tilde{l} \\ &= -(H_t)^+ \cdot \bar{S}_{t-1} + (H_t)^- \cdot \underline{S}_{t-1} + (H_t)^+ \cdot \underline{S}_{t+k} - (H_t)^- \cdot \bar{S}_{t+k} - r \\ &= x_{t-1,t+k}(H_t) - r. \end{aligned}$$

□

Remark 8. It is worth noting that, for any $\Pi \in F_T$, there exists a strategy $H \in \mathcal{P}_T$ such that $\Pi = x_T(H) - r$ where r is a random variable $r \in L^0_+$.

Lemma 8. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random vectors such that $X_n \in \mathbb{R}^d$ and for almost all $\omega \in \Omega$ we have $\liminf_n \|X_n(\omega)\| < \infty$. Then there exists a sequence of random vectors Y_n taking values in \mathbb{R}^d that verify the following conditions:

- (1) Y_n converges pointwise to Y almost surely where Y is a d -dimensional random vector;
- (2) $Y_n(\omega)$ is a convergent subsequence of $X_n(\omega)$ for almost all $\omega \in \Omega$.

Proof. See the paper [12, Lemma 2] or the article [10, Lemma 1]. \square

Lemma 9 (Kreps-Yan). *Let $K \supseteq -L_+^1$ be a closed convex cone in L^1 such that $K \cap L_+^1 = \{0\}$. Then there exists a probability $\tilde{P} \sim P$ with $\frac{d\tilde{P}}{dP} \in L^\infty$ such that $\mathbb{E}^{\tilde{P}}[\xi] \leq 0$ for all $\xi \in K$.*

Proof. See the paper [12, Lemma 3] or the book [11, Theorem 2.1.4]. \square

We are now ready to state and prove the main theorem of this thesis. We would like to find a relationship between the EBAMM, the non-arbitrage condition (NA) and the CPS.

In markets without transaction costs, the non-arbitrage condition is related to EMM by the First Fundamental Theorem of Asset Pricing, also known as Dalang–Morton–Willinger theorem.

Theorem 2 (FTAP). *In a financial finite discrete time market $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ without frictions, the following conditions are equivalent:*

(a) $\mathcal{A}_T \cap L_+^0 = \{0\}$ (NA);

(b) $\mathcal{M}^e \neq \emptyset$;

where \mathcal{M}^e is the set of equivalent martingale measures.

Proof. See in book [5, Theorem 2.2.7]. \square

The fundamental theorem of asset pricing holds a crucial place in the study of pricing and hedging of derivative securities. The most common interpretation of this result is that the martingale measure is a mathematical model for a *perfectly fair* game: for any strategy $H \in \mathcal{P}_T$, we will always have $E^{\mathbb{Q}}[x_T(H)] = 0$. An interpretation of this result is that the investor expects neither to gain nor to lose in expectation under \mathbb{Q} . In contrast, a market allowing for arbitrage is a model for an *unfair* game: choosing a suitable strategy $\tilde{H} \in \mathcal{P}_T$, the investor is sure not to lose, but also he has a strictly positive probability to earn money.

Theorem 3 (Fundamental theorem). *The following conditions are equivalent:*

(a) $\mathcal{A}_T \cap L_+^0 = \{0\}$ (NA);

(b) $F_t \cap L_+^0(\mathcal{F}_t) = \{0\}$ for any $t \in \{1, \dots, T\}$;

(c) $F_{t-1,t+k} \cap L_+^0(\mathcal{F}_{t+k}) = \{0\}$ for any $1 \leq t \leq t+k \leq T$;

(d) $F_{t-1,t+k} \cap L_+^0(\mathcal{F}_{t+k}) = \{0\}$ and $F_{t-1,t+k} = \bar{F}_{t-1,t+k}$ for any $1 \leq t \leq t+k \leq T$;

(e) $\bar{F}_{t-1,t+k} \cap L_+^0(\mathcal{F}_{t+k}) = \{0\}$ for any $1 \leq t \leq t+k \leq T$;

(f) there exists an EBAMM \mathbb{Q} for the bid-ask process (\underline{S}, \bar{S}) such that $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$;

(g) there exists a supCPS (\hat{S}, \mathbb{Q}) and subCPS (\check{S}, \mathbb{Q}) such that $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$.

In our model, EBAMM plays the same role as EMM: under this probability measure, the conditional expectation of the bid price \underline{S}_t^i given \mathcal{F}_{t-1} is less than the ask price \overline{S}_{t-1}^i : so we do not expect to sell a unit of an asset at a higher price than we paid for it. In the same way, the conditional expectation of the ask price \overline{S}_t^i given \mathcal{F}_{t-1} is greater than the bid price \underline{S}_{t-1}^i .

In Theorem 3, we show that the condition (NA) is equivalent to the existence of the EBAMM. However, we will not obtain a link between these two and the existence of a CPS: instead, we will find as an equivalent condition the existence under the same probability measure of a supCPS and subCPS. Splitting a strategy H into short and long positions, supCPS and subCPS play the role of CPS in the two different cases.

The problem of whether there is a correlation between the presence of an EBAMM and a CPS in our model remains open. In literature, we have noticeable result: in article [7], Grigoriev prove this equivalence for any T in case $d = 1$; further on, in Corollary 2.2, we prove it for d assets but for a time horizon $T = 1$.

In general, it is not clear when, in our model with d assets and the final time horizon T , the existence of a supCPS (\hat{S}, \mathbb{Q}) and a subCPS (\check{S}, \mathbb{Q}) under the same probability measure implies the existence of a CPS (\tilde{S}, \mathbb{Q}) .

Proof Theorem 3. In this proof, we will retrace Przemysław's work in the article [15]. We prove each implication.

- (a) \Rightarrow (b) By Lemma 1, $\mathcal{A}_T \cap L_+^0 = \{0\}$ implies $\mathcal{A}_t \cap L_+^0(\mathcal{F}_t) = \{0\}$ for any $t \in \{1, \dots, T\}$. By the inclusion from Lemma 6, we can conclude $F_t \cap L_+^0(\mathcal{F}_t) = 0$ for any $t \in \{1, \dots, T\}$.
- (b) \Rightarrow (c) This implication is trivial from the way the sets F_t and $F_{t-1, t+k}$ are defined.
- (c) \Rightarrow (d) In this implication, we use similar reasoning as in article [12, Proof of Theorem 1] and article [17, Theorem 2.33].

The first part, $F_{t-1, t+k} \cap L_+^0(\mathcal{F}_{t+k}) = \{0\}$, is trivial. In the second part, we show that the set $F_{t-1, t+k}$ is closed in the topology generated by convergence in probability \mathbb{P} . Taking a sequence $(\xi^n)_{n \in \mathbb{N}}$ such that $\xi^n \in F_{t-1, t+k}$ and, for $n \rightarrow \infty$, $\xi^n \rightarrow \zeta$ in probability. To prove our thesis, we need only to show that $\zeta \in F_{t-1, t+k}$.

This sequence includes a subsequence which is convergent to ζ a.s. Restricting to this subsequence, then $\xi^n \rightarrow \zeta$ \mathbb{P} -a.s. By Lemma 7, for any n there are $H_t^n \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$ and $r_n \in L_+^0(\mathcal{F}_{t+k})$ such that

$$\begin{aligned} \xi^n &= x_{t-1, t+k}(H_t^n) - r_n \in F_{t-1, t+k} \\ &= -(H_t^n)^+ \cdot \overline{S}_{t-1} + (H_t^n)^- \cdot \underline{S}_{t-1} + (H_t^n)^+ \cdot \underline{S}_{t+k} - (H_t^n)^- \cdot \overline{S}_{t+k} - r_n. \end{aligned}$$

First, let us work on the set $\Omega_1 := \{\omega \in \Omega \mid \liminf_n \|H_t^n(\omega)\| < \infty\}$. Using Lemma 8, we find an increasing sequence of integer-valued \mathcal{F}_{t-1} -measurable random variables τ_n such that $H_t^{\tau_n}$ is convergent a.s. on Ω_1 and for almost all $\omega \in \Omega_1$ the sequence $H_t^{\tau_n(\omega)}(\omega)$ is a convergent subsequence of $H_t^n(\omega)$. Obviously, we

keep $H_t^{\tau_n} \in L^0(\mathbb{R}^d, \mathcal{F}_{t-1})$ and $r_{\tau_n} \in L^0_+(\mathcal{F}_{t+k})$. Now let $\tilde{H}_t := \lim_{n \rightarrow \infty} H_t^{\tau_n}$. From the convergence of $H_t^{\tau_n}$, we can state that $(H_t^{\tau_n})^+$ and $(H_t^{\tau_n})^-$ are convergent too. Therefore we have $(H_t^{\tau_n})^+ \rightarrow (\tilde{H}_t)^+$ and $(H_t^{\tau_n})^- \rightarrow (\tilde{H}_t)^-$. Furthermore, r_{τ_n} is convergent too a.s. on Ω_1 and we introduce $\tilde{r} := \lim_{n \rightarrow \infty} r_{\tau_n}$. Hence

$$\begin{aligned} \zeta &= \lim_{n \rightarrow \infty} [-(H_t^n)^+ \cdot \bar{S}_{t-1} + (H_t^n)^- \cdot \underline{S}_{t-1} + (H_t^n)^+ \cdot \underline{S}_{t+k} - (H_t^n)^- \cdot \bar{S}_{t+k} - r_n] \\ &= \lim_{n \rightarrow \infty} [-(H_t^{\tau_n})^+ \cdot \bar{S}_{t-1} + (H_t^{\tau_n})^- \cdot \underline{S}_{t-1} + (H_t^{\tau_n})^+ \cdot \underline{S}_{t+k} - (H_t^{\tau_n})^- \cdot \bar{S}_{t+k} - r_{\tau_n}] \\ &= -(\tilde{H}_t)^+ \cdot \bar{S}_{t-1} + (\tilde{H}_t)^- \cdot \underline{S}_{t-1} + (\tilde{H}_t)^+ \cdot \underline{S}_{t+k} - (\tilde{H}_t)^- \cdot \bar{S}_{t+k} - \tilde{r}. \end{aligned}$$

From the last expression, it can be clearly seen that $\zeta \in F_{t-1, t+k}$.

Now, we have to prove that the set $F_{t-1, t+k}$ is closed also on the complementary set $\Omega_2 := \{\omega \in \Omega \mid \liminf_n \|H_t^n(\omega)\| = \infty\}$. For detailed proof, check the article [15, Proof of Theorem 2]

(d) \Rightarrow (e) This implication is trivial. We insert it in order to make the reasoning more clear.

(e) \Rightarrow (f) In this implication we use some reasoning from article [14] and the construction of a measure by induction as in [17, Corollary 2.35].

Without losing generality, we assume $\underline{S}_t^i, \bar{S}_t^i$ are integrable for any $i = 1, \dots, d$ and $t = 0, \dots, T$: for any random variable $\eta \in L^1(\mathbb{P})$ there exists a probability measure $\mathbb{P}' \sim \mathbb{P}$ such that $\frac{d\mathbb{P}'}{d\mathbb{P}} \in L^\infty$ and $\eta \in L^1(\mathbb{P}')$.

We will use induction on the length of the time interval. Let be it fixed at length 1. Choose any $t \in \{1, \dots, T\}$ and define $\Psi_{t-1, t} := \bar{F}_{t-1, t} \cap L^1(\mathcal{F}_t)$, which is a closed convex cone in $L^1(\mathcal{F}_t)$. From hypothesis $\Psi_{t-1, t} \cap L^1_+(\mathcal{F}_t) = \{0\}$, so, by Lemma 9, we can find a probability measure $\mathbb{Q}^t \sim \mathbb{P}$ on (Ω, \mathcal{F}_t) such that

$$\frac{d\mathbb{Q}^t}{d\mathbb{P}} \in L^\infty(\mathcal{F}_t) \quad \text{and} \quad E^{\mathbb{Q}^t}[\xi] \leq 0 \quad \text{for any } \xi \in \Psi_{t-1, t}.$$

In particular, since $\xi \in \bar{F}_{t-1, t}$, using Equation (1.7) and Definition 5, we can explicit ξ as

$$\xi_{t-1, t}^i = H_t^i(\underline{S}_t^i - \bar{S}_{t-1}^i) \quad \text{or} \quad \xi_{t-1, t}^i = (-H_t^i)(\bar{S}_t^i - \underline{S}_{t-1}^i)$$

based on whether $\xi \in \mathcal{R}_{t-1, t}^+$ or $\xi \in \mathcal{R}_{t-1, t}^-$. Then

$$E^{\mathbb{Q}^t}[H_t^i(\underline{S}_t^i - \bar{S}_{t-1}^i)] \leq 0 \quad \text{and} \quad E^{\mathbb{Q}^t}[H_t^i(\bar{S}_t^i - \underline{S}_{t-1}^i)] \geq 0.$$

Dividing the expected value, for any $i = 1, \dots, d$, we obtain the inequalities

$$E^{\mathbb{Q}^t}[H_t^i \underline{S}_t^i] \leq E^{\mathbb{Q}^t}[H_t^i \bar{S}_{t-1}^i] \quad \text{and} \quad E^{\mathbb{Q}^t}[H_t^i \bar{S}_t^i] \leq E^{\mathbb{Q}^t}[H_t^i \underline{S}_{t-1}^i].$$

Notice that $H^i \in L^0_+(\mathcal{F}_t)$. By the fact H_t^i is predictable and \bar{S} and \underline{S} are adapted processes, we finally get

$$E^{\mathbb{Q}^t}[\underline{S}_t^i | \mathcal{F}_{t-1}] \leq E^{\mathbb{Q}^t}[\bar{S}_{t-1}^i | \mathcal{F}_{t-1}] = \bar{S}_{t-1}^i \quad \text{and} \quad E^{\mathbb{Q}^t}[\bar{S}_t^i | \mathcal{F}_{t-1}] \leq E^{\mathbb{Q}^t}[\underline{S}_{t-1}^i | \mathcal{F}_{t-1}] = \underline{S}_{t-1}^i.$$

Obviously all $\underline{S}_t, \bar{S}_t \in L^1(\mathbb{Q}^t)$ since $\underline{S}_t, \bar{S}_t \in L^1(\mathbb{P})$, $\underline{S}_t, \bar{S}_t \in \mathcal{F}_t$ and $\mathbb{Q}^t \sim \mathbb{P}$. These are the conditions in Equation (EBAMM) that allow us to state \mathbb{Q}^t is an Equivalent Bid-Ask Martingale Measure for the bid-ask process $\left((\underline{S}_j)_{j=t-1}^t, (\bar{S}_j)_{j=t-1}^t \right)$ and $\frac{d\mathbb{Q}^t}{d\mathbb{P}} \in L^\infty$.

Now let us move on to the inductive step: assume the claim is true for a long time interval k and prove it is still true for a long time interval $k + 1$.

Fix any t, k such that $1 \leq t \leq t + k \leq T$. By the induction hypothesis, \mathbb{Q}^{t+k} is an EBAMM in the market with the bid-ask process $\left((\underline{S}_j)_{j=t}^{t+k}, (\bar{S}_j)_{j=t}^{t+k} \right)$ and $\frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} \in L^\infty$. Notice that the condition (d) does not change under an equivalent probability measure. Hence we repeat the same process as in the previous part to the probability space $(\Omega, \mathcal{F}_t, \mathbb{Q}_{|\mathcal{F}_t}^{t+k})$ where $\mathbb{Q}_{|\mathcal{F}_t}^{t+k}$ stands for the measure \mathbb{Q}^{t+k} restricted to \mathcal{F}_t .

Then, by Lemma 9, there exists a probability measure $\mathbb{Q}^t \sim \mathbb{Q}_{|\mathcal{F}_t}^{t+k}$ on (Ω, \mathcal{F}_t) such that, as we showed previously, for any $i = 1, \dots, d$, we have

$$E^{\mathbb{Q}^t}(\underline{S}_t^i | \mathcal{F}_{t-1}) \leq \bar{S}_{t-1}^i \quad \text{and} \quad E^{\mathbb{Q}^t}(\bar{S}_t^i | \mathcal{F}_{t-1}) \geq \underline{S}_{t-1}^i \quad \text{and} \quad \frac{d\mathbb{Q}^t}{d\mathbb{Q}_{|\mathcal{F}_t}^{t+k}} \in L^\infty. \quad (2.4)$$

Now we define the probability measure \mathbb{Q} on $(\Omega, \mathcal{F}_{t+k})$ so that it satisfies

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{d\mathbb{Q}^t}{d\mathbb{Q}_{|\mathcal{F}_t}^{t+k}} \frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}}. \quad (2.5)$$

Therefore for any $j \in \{t + 1, \dots, t + k\}$ and $i \in \{1, \dots, d\}$, using Theorem 4 in Appendix A, we get

$$\begin{aligned} E^{\mathbb{Q}}[\underline{S}_j^i | \mathcal{F}_{j-1}] &= \frac{E^{\mathbb{P}} \left[\frac{d\mathbb{Q}^t}{d\mathbb{Q}_{|\mathcal{F}_t}^{t+k}} \frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} \underline{S}_j^i | \mathcal{F}_{j-1} \right]}{E^{\mathbb{P}} \left[\frac{d\mathbb{Q}^t}{d\mathbb{Q}_{|\mathcal{F}_t}^{t+k}} \frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} | \mathcal{F}_{j-1} \right]} = \frac{E^{\mathbb{P}} \left[\frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} \underline{S}_j^i | \mathcal{F}_{j-1} \right]}{E^{\mathbb{P}} \left[\frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} | \mathcal{F}_{j-1} \right]} \\ &= E^{\mathbb{Q}^{t+k}}[\underline{S}_j^i | \mathcal{F}_{j-1}] \leq \bar{S}_{j-1}^i. \end{aligned}$$

The last inequalities hold since \mathbb{Q}^{t+k} is an EBAMM for $\left((\underline{S}_j)_{j=t}^{t+k}, (\bar{S}_j)_{j=t}^{t+k} \right)$.

In the same way, we get also

$$\begin{aligned} E^{\mathbb{Q}}[\bar{S}_j^i | \mathcal{F}_{j-1}] &= \frac{E^{\mathbb{P}} \left[\frac{d\mathbb{Q}^t}{d\mathbb{Q}_{|\mathcal{F}_t}^{t+k}} \frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} \bar{S}_j^i | \mathcal{F}_{j-1} \right]}{E^{\mathbb{P}} \left[\frac{d\mathbb{Q}^t}{d\mathbb{Q}_{|\mathcal{F}_t}^{t+k}} \frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} | \mathcal{F}_{j-1} \right]} = \frac{E^{\mathbb{P}} \left[\frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} \bar{S}_j^i | \mathcal{F}_{j-1} \right]}{E^{\mathbb{P}} \left[\frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} | \mathcal{F}_{j-1} \right]} \\ &= E^{\mathbb{Q}^{t+k}}[\bar{S}_j^i | \mathcal{F}_{j-1}] \geq \underline{S}_{j-1}^i. \end{aligned}$$

Obviously all $\underline{S}_t, \bar{S}_t \in L^1(\mathbb{Q})$ since $\underline{S}_t, \bar{S}_t \in L^1(\mathbb{P})$, $\underline{S}_t, \bar{S}_t \in \mathcal{F}_t$ and $\mathbb{Q} \sim \mathbb{P}$.

Now consider the measure $\mathbb{Q}|_{\mathcal{F}_t}^{t+k}$ restricted to \mathcal{F}_{t-1} : it is the same measure as \mathbb{Q}^{t+k} restricted to \mathcal{F}_{t-1} since $\mathcal{F}_{t-1} \subset \mathcal{F}_t$. For this reason, using the definition of \mathbb{Q} given in Equation (2.5), we have

$$E^{\mathbb{Q}}[\bar{S}_t^i | \mathcal{F}_{t-1}] = E^{\mathbb{Q}^t}[\bar{S}_t^i | \mathcal{F}_{t-1}] \geq \underline{S}_{t-1}^i \quad \text{and} \quad E^{\mathbb{Q}}[S_t^i | \mathcal{F}_{t-1}] = E^{\mathbb{Q}^t}[S_t^i | \mathcal{F}_{t-1}] \leq \bar{S}_{t-1}^i,$$

where in the last step we used the inequalities in Equation (2.4).

In the end, \mathbb{Q} is an EBAMM in the market with the bid-ask process $((\underline{S}_j)_{j=t-1}^{t+k}, (\bar{S}_j)_{j=t-1}^{t+k})$ and $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$. By the induction, we conclude that there exists an EBAMM for the bid-ask process $((\underline{S}_t)_{t=0}^T, (\bar{S}_t)_{t=0}^T)$ such that $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty$.

(f) \Rightarrow (g) We prove this implication using the induction on the length of the time interval. Starting with a long time interval 1, fix any $t \in \{1, \dots, T\}$ and define $\hat{S} = (\hat{S}_j)_{j=t-1}^t$ and $\check{S} = (\check{S}_j)_{j=t-1}^t$ in the following way:

$$\begin{aligned} \hat{S} &:= \underline{S}_t, & \hat{S}_{t-1} &:= \max \left\{ \underline{S}_{t-1}, E^{\mathbb{Q}^t}[\hat{S}_t | \mathcal{F}_{t-1}] \right\}, \\ \check{S} &:= \bar{S}_t, & \check{S}_{t-1} &:= \min \left\{ \bar{S}_{t-1}, E^{\mathbb{Q}^t}[\check{S}_t | \mathcal{F}_{t-1}] \right\}, \end{aligned} \quad (2.6)$$

where \mathbb{Q}^t is an EBAMM in the market with the bid-ask process $((\underline{S}_j)_{j=t-1}^t, (\bar{S}_j)_{j=t-1}^t)$ and $\frac{d\mathbb{Q}^t}{d\mathbb{P}} \in L^\infty$. In this way, (\hat{S}, \mathbb{Q}^t) is a supCPS while $(\check{S}, \mathbb{Q}^t)$ is a subCPS. Now we proceed with the inductive step: we suppose the claim is true in a model with time interval of length $k \geq 1$ and prove it for a time interval of length $k+1$. By the induction hypothesis, there exists a supCPS $((\hat{S}_j)_{j=t}^{t+k}, \mathbb{Q}^{t+k})$ and a subCPS $((\check{S}_j)_{j=t}^{t+k}, \mathbb{Q}^{t+k})$ in the market with the bid-ask process $((\underline{S}_j)_{j=t}^{t+k}, (\bar{S}_j)_{j=t}^{t+k})$ such that $\frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}} \in L^\infty$. Consider the probability space $(\Omega, \mathcal{F}_t, \mathbb{Q}|_{\mathcal{F}_t}^{t+k})$ where $\mathbb{Q}|_{\mathcal{F}_t}^{t+k}$ stands for the measure \mathbb{Q}^{t+k} restricted to \mathcal{F}_t . By condition (f) there exists an EBAMM $\mathbb{Q}^t \sim \mathbb{Q}|_{\mathcal{F}_t}^{t+k}$ such that $\frac{d\mathbb{Q}^t}{d\mathbb{Q}|_{\mathcal{F}_t}^{t+k}} \in L^\infty$. Defining the processes $\hat{S} = (\hat{S}_j)_{j=t-1}^t$ and $\check{S} = (\check{S}_j)_{j=t-1}^t$ in the same way as in Equation 2.6, we already know that (\hat{S}, \mathbb{Q}^t) and $(\check{S}, \mathbb{Q}^t)$ are supCPS and subCPS.

For any $i \in \{1, \dots, d\}$, let be the stopping time

$$\tau_i := \min\{j \geq t-1 \mid \check{S}_j^i = \hat{S}_j^i\}.$$

Then by optimal stopping theory (see the book [1, Chapter 21 and in particular Proposition 21.15]), the process

$$\check{S}^\tau := (\check{S}_{j \wedge \tau})_{j=t-1}^t = \left(\check{S}_{j \wedge \tau_1}^1, \dots, \check{S}_{j \wedge \tau_d}^d \right)_{j=t-1}^t.$$

is a \mathbb{Q}^t -martingale. As previous, we define the probability measure \mathbb{Q} on $(\Omega, \mathcal{F}_{t+k})$ that it satisfies

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{d\mathbb{Q}^t}{\mathbb{Q}|_{\mathcal{F}_t}^{t+k}} \frac{d\mathbb{Q}^{t+k}}{d\mathbb{P}}. \quad (2.7)$$

Moreover, we define another process $\hat{S}' = (\hat{S}'_j)_{j=t-1}^{t+k}$ in the following way:

$$\hat{S}'_j = \begin{cases} \hat{S}_j & \text{for any } j > t \\ \check{S}_{j \wedge \tau} & \text{for } j \in \{t-1, t\}. \end{cases}$$

Notice that $\hat{S}_{t-1}^i = \check{S}_{t-1 \wedge \tau_i}^i = \check{S}_{t-1}^i$ and $\hat{S}_t^i = \check{S}_{t \wedge \tau_i}^i = \check{S}_{\tau_i}^i$, so the inequalities CPS are satisfied for $j = t-1, t$ (for $j > t$ they are satisfied since \hat{S}_j is a supCPS). Moreover, by definition of τ_i , definition of measure \mathbb{Q} in Equation 2.7 and definition of the process \hat{S} in Equation 2.6, we also obtain for any $i = 1, \dots, d$

$$E^{\mathbb{Q}}[\hat{S}_{t+1}^i | \mathcal{F}_t] = E^{\mathbb{Q}}[\check{S}_{t+1}^i | \mathcal{F}_t] = E^{\mathbb{Q}^{t+k}}[\hat{S}_{t+1}^i | \mathcal{F}_t] \leq \hat{S}_t^i \leq \bar{S}_t^i = \check{S}_t^i = \check{S}_{\tau_i}^i = \hat{S}_t^i.$$

$\hat{S}' = (\hat{S}'_j)_{j=t-1}^{t+k}$ is the desired \mathbb{Q} -supermartingale and (\hat{S}', \mathbb{Q}) is the supCPS.

In an analogous way, we can construct a \mathbb{Q} -submartingale. Define the stopping time

$$\sigma_i := \min\{j \geq t-1 \mid \hat{S}_j^i = \hat{S}_t^i\}.$$

Then

$$\hat{S}^\sigma := (\hat{S}_{j \wedge \sigma})_{j=t-1}^t = (\hat{S}_{j \wedge \sigma_1}^1, \dots, \hat{S}_{j \wedge \sigma_d}^d)_{j=t-1}^t.$$

is a \mathbb{Q}^t -martingale. Defining the measure \mathbb{Q} as in Equation 2.7 and the process $\check{S}' = (\check{S}'_j)_{j=t-1}^{t+k}$ in the following way

$$\check{S}'_j = \begin{cases} \check{S}_j & \text{for any } j > t \\ \hat{S}_{j \wedge \sigma} & \text{for } j \in \{t-1, t\}. \end{cases}$$

we obtain the \mathbb{Q} -submartingale and (\check{S}', \mathbb{Q}) is the subCPS.

(g) \Rightarrow (a) This implication is equivalent to Theorem 1. □

Corollary. *If $T = 1$, then*

$$(NA) \iff (EBAMM) \iff (CPS).$$

Proof. The first iff is given by the Theorem 3. For the other, as in the proof (f) \Rightarrow (g), let \mathbb{Q}^1 be an EBAMM and define the Snell envelope $\hat{S} = (\hat{S}_t)_{t=0}^1$ of the bid process $\underline{S} = (\underline{S}_t)_{t=0}^1$ as follows

$$\hat{S}_1 := \underline{S}_1 \quad \text{and} \quad \hat{S}_0 := \max\{\underline{S}_0, E^{\mathbb{Q}^1}[\hat{S}_1 | \mathcal{F}_0]\}.$$

It's clear that (\hat{S}, \mathbb{Q}^1) is a supCPS. Furthermore, considering the optimal stopping time

$$\tau_i := \min\{t \geq 0 \mid \underline{S}_t^i = \hat{S}_t^i\}$$

we construct the process $\tilde{S} := (\tilde{S}_t)_{t=0}^1$ where $\tilde{S}^i := \hat{S}_{t \wedge \tau_i}^i$: this is a \mathbb{Q}^1 -martingale (see the book [1], Proposition 21.15, page 335). In this way, for any $i = 1, \dots, d$

$$\tilde{S}_0^i = \hat{S}_0^i \quad \text{and} \quad \tilde{S}_1^i = \hat{S}_{\tau_i}^i = \underline{S}_1^i.$$

For $t \in \{0, 1\}$, the CPS inequalities are satisfied so $(\tilde{S}, \mathbb{Q}^1)$ is a CPS. □

Remark 9. Condition (g) of Theorem 3 states there are a supCPS and a subCPS. We define the Snell envelope \tilde{S} of \hat{S} for any $t = 1, \dots, T$

$$\tilde{S}_T := \hat{S}_T \quad \text{and} \quad \tilde{S}_{t-1} := \max \left\{ \hat{S}_{t-1}, E^{\mathbb{Q}}[\tilde{S}_t | \mathcal{F}_{t-1}] \right\}.$$

By optimal stopping theory,

$$\tau_i := \min \left\{ t \geq 0 \mid \tilde{S}_t^i = \hat{S}_t^i \right\}$$

is an optimal stopping time and

$$\tilde{S}^\tau := (\tilde{S}_{t \wedge \tau})_{t=0}^T = (\tilde{S}_{t \wedge \tau_1}^1, \dots, \tilde{S}_{t \wedge \tau_d}^d)_{t=0}^T$$

is a \mathbb{Q} -martingale (see the book [1], Proposition 21.15, page 335). On the other hand, we do not know if $(\tilde{S}^\tau, \mathbb{Q})$ is a CPS: indeed, we can only state that $\underline{S}_{t \wedge \tau_i}^i \leq \tilde{S}_{t \wedge \tau_i}^i \leq \overline{S}_{t \wedge \tau_i}^i$ but this does not imply $\underline{S}_t^i \leq \tilde{S}_{t \wedge \tau_i}^i \leq \overline{S}_t^i$.

Chapter 3

The Cox-Ross-Rubinstein model

The Cox-Ross-Rubinstein model, also known as the binomial option pricing model, is a famous discrete-time model used to determine the value of assets in finance. It models the evolution of the price of an asset over a discrete period of time, considering the possibility of upward and downward movements, as well as the associated probabilities of each movement. It assumes that the stock price can increase or decrease by a fixed amount during a time interval and that future prices are unrelated to past prices. One of the key strengths of the CRR model is its simplicity. The model uses a tree structure to represent the different possible outcomes of the price process over time: the branches of the tree represent the possible upward or downward movements. This simplicity makes the CRR model flexible: it can be modified to incorporate various assumptions about the underlying asset, the market environment, and the behaviour of financial participants. However, there are also some limitations to the CRR model. For example, it assumes that the stock price can only move in two directions, up or down, over a certain time period. This is a simplification of the real world, where stock prices change with continuity and can be influenced by a multitude of factors, such as economic indicators, geopolitical events, and company-specific news. Additionally, the model assumes that the probability of upward and downward movements is constant over time, which may not always be the case in real-world situations.

Now we introduce this model for a market with bid-ask spread. For the sake of simplicity, in this chapter we assume $d = 1$. In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, define a sequence of i.i.d. bivariate random variables $(\zeta_t)_{t=1}^T = (\underline{\zeta}_t, \bar{\zeta}_t)_{t=1}^T$ such that

$$\mathbb{P}(\zeta_t = (\underline{u}, \bar{u})) := p > 0 \quad \text{and} \quad \mathbb{P}(\zeta_t = (\underline{d}, \bar{d})) := 1 - p > 0,$$

where $\underline{u}, \bar{u}, \underline{d}, \bar{d}$ are real numbers. These values represent the increase (u) or decrease (d) of the bid-ask process: in a natural way, we fix $\underline{d} < \underline{u}$ and $\bar{d} < \bar{u}$. Consider the filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t = \sigma(\zeta_1, \dots, \zeta_t)$ and $\mathcal{F}_T = \mathcal{F}$. As in Chapter 1, we assume the money market account

$$B_t \equiv 1 \quad \text{for any } t = 1, \dots, T.$$

The dynamics of the bid-ask process (\underline{S}, \bar{S}) is determined by the equations below:

$$\underline{S}_t := (1 + \underline{\zeta}_t)\bar{S}_{t-1} \quad \text{and} \quad \bar{S}_t := (1 + \bar{\zeta}_t)\underline{S}_{t-1}.$$

Moreover, assume that \underline{S}, \bar{S} are strictly positive, $\underline{S}_t > 0$ and $\bar{S}_t > 0$ \mathbb{P} -a.e. We clarify the values of the bid-ask process, introducing these notions in the case of an “up” or “down” market:

$$\begin{aligned}\underline{S}_t^u &:= \bar{S}_{t-1}(1 + \underline{u}), & \bar{S}_t^u &:= \underline{S}_{t-1}(1 + \bar{u}), \\ \underline{S}_t^d &:= \bar{S}_{t-1}(1 + \underline{d}), & \bar{S}_t^d &:= \underline{S}_{t-1}(1 + \bar{d}).\end{aligned}$$

This is illustrated in Figure 3.1.

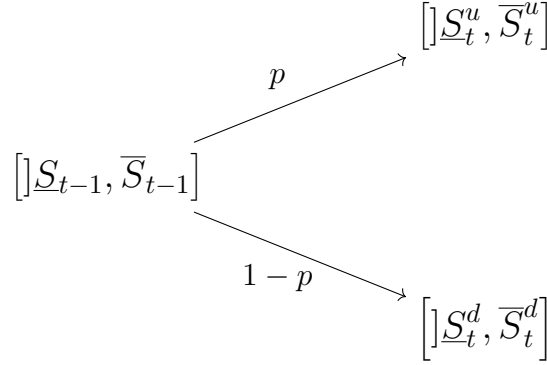


Figure 3.1: The CRR model during $[t - 1, t]$

Now we are going to study a one-step model with one risky asset: the general case with d assets and period of time $[0, T]$ is a concatenation of this one-step model.

Let us fix the initial value of the bid-ask process, i.e. \underline{S}_{t-1} and \bar{S}_{t-1} . The first condition to impose in our model is $\underline{S}_t \leq \bar{S}_t$ which is equivalent to

$$\bar{S}_{t-1}(1 + \underline{u}) \leq \underline{S}_{t-1}(1 + \bar{u}) \quad \text{and} \quad \bar{S}_{t-1}(1 + \underline{d}) \leq \underline{S}_{t-1}(1 + \bar{d}) \quad (3.1)$$

$$\begin{aligned}\Downarrow & & \Downarrow \\ \frac{\bar{S}_{t-1}}{\underline{S}_{t-1}} &\leq \frac{1 + \bar{u}}{1 + \underline{u}} & \text{and} & & \frac{\bar{S}_{t-1}}{\underline{S}_{t-1}} &\leq \frac{1 + \bar{d}}{1 + \underline{d}}.\end{aligned} \quad (3.2)$$

We try to estimate an EBAMM. Let \mathbb{Q}^* be an EBAMM. For any $t = 1, \dots, T$, let

$$q^* := \mathbb{Q}^*((\underline{\zeta}_t, \bar{\zeta}_t) = (\underline{u}, \bar{u}) | \mathcal{F}_{t-1}) \quad \text{and} \quad 1 - q^* := \mathbb{Q}^*((\underline{\zeta}_t, \bar{\zeta}_t) = (\underline{d}, \bar{d}) | \mathcal{F}_{t-1}). \quad (3.3)$$

By Definition 9, \mathbb{Q}^* has to satisfy conditions in Equation EBAMM, so

$$E^{\mathbb{Q}^*}[\underline{S}_t | \mathcal{F}_{t-1}] = \bar{S}_{t-1}(1 + \underline{u})q^* + \bar{S}_{t-1}(1 + \underline{d})(1 - q^*) \leq \bar{S}_{t-1},$$

$$E^{\mathbb{Q}^*}[\bar{S}_t | \mathcal{F}_{t-1}] = \underline{S}_{t-1}(1 + \bar{u})q^* + \underline{S}_{t-1}(1 + \bar{d})(1 - q^*) \geq \underline{S}_{t-1}.$$

From these inequalities, we obtain

$$\frac{-\bar{d}}{\bar{u} - \bar{d}} \leq q^* \leq \frac{-\underline{d}}{\underline{u} - \underline{d}}. \quad (3.4)$$

In order to make possible the chain of inequalities in Equation 3.4, we have to require that

$$\frac{-\bar{d}}{\bar{u} - \bar{d}} \leq \frac{-\underline{d}}{\underline{u} - \underline{d}}, \quad 0 < \frac{-\underline{d}}{\underline{u} - \underline{d}}, \quad \frac{-\bar{d}}{\bar{u} - \bar{d}} < 1,$$

or equivalent by

$$\underline{d}\bar{u} \leq \bar{d}\underline{u}, \quad \underline{d} < 0, \quad \bar{u} > 0. \quad (3.5)$$

These inequalities in Equation 3.5 ensure the existence of at least one $q^* \in (0, 1)$: these are necessary and sufficient conditions for the existence of an EBAMM. From Theorem 3 and Corollary 2.2, these conditions are equivalent to the absence of arbitrage and the existence of a CPS.

We show our results in some examples.

Example 1. Consider a CRR model with $T = 1$ represented in Figure 3.2 where $\underline{S}_0 = 1$ and $\bar{S}_0 = 2$. Let

$$\bar{u} = 3, \quad \bar{d} = -\frac{1}{3}, \quad \underline{u} = \frac{1}{4}, \quad \underline{d} = -\frac{4}{5}.$$

Notice that conditions in Equation 3.2 are true, i.e.

$$\frac{\bar{S}_0}{\underline{S}_0} = \frac{2}{1} \leq \frac{16}{5} = \frac{1 + \bar{u}}{1 + \underline{u}} \quad \text{and} \quad \frac{\bar{S}_0}{\underline{S}_0} = \frac{2}{1} \leq \frac{10}{3} = \frac{1 + \bar{d}}{1 + \underline{d}}$$

Equations 3.5 are satisfied:

$$\underline{d} = -\frac{4}{5} < 0, \quad \bar{u} = 3 > 0, \quad \underline{d}\bar{u} = -\frac{12}{5} \leq -\frac{1}{12} = \bar{d}\underline{u},$$

so there exist an EBAMM \mathbb{Q}^* such that $\frac{-\bar{d}}{\bar{u} - \bar{d}} = \frac{1}{10} \leq q^* \leq \frac{16}{21} = \frac{-\underline{d}}{\underline{u} - \underline{d}}$ and there is no arbitrage.

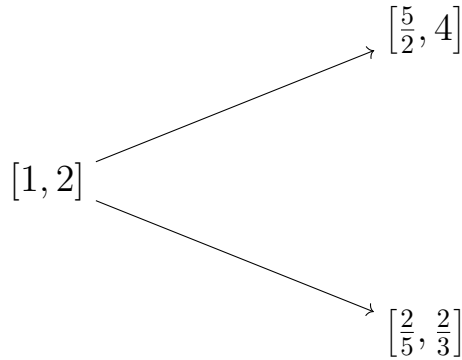


Figure 3.2: The CRR model of Example 1

Example 2. Consider a CRR model with $T = 1$ represented in Figure 3.3 where $\underline{S}_0 = 1$ and $\bar{S}_0 = 2$. Let

$$\bar{u} = 10, \quad \bar{d} = 3, \quad \underline{u} = 4, \quad \underline{d} = \frac{1}{2}.$$

Notice that conditions in Equation 3.2 are satisfied so $\underline{S}_1 \leq \bar{S}_1$. On the other hand, conditions for the existence of EBAMM in Equation 3.5 are not true since $\underline{d} > 0$.

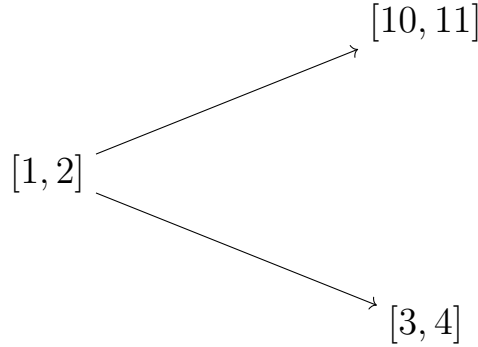


Figure 3.3: The CRR model of Example 2

It is clear that the strategy H , where the investor buys a unit of the asset at time $t = 0$ and at time $t = 1$ he liquidates all, is an arbitrage: it generates a strictly positive gain whether the market goes up or down.

Now we are in a position to consider a multi-period model with horizon T . We are interested in studying recombining trees: this property reduces the number of the nodes from 2^T to $T+1$ speeding up the calculation of the option price and avoiding the numerical issues. A tree is recombining if the paths $(up, down)$ and $(down, up)$ merge together, so the price of a unit of asset experiencing a market movement that goes *up* and then *down* is the same of a market movement that first goes *down* and then *up*. This is illustrated in Figure 3.4 with $T = 2$.

Consider a two-step model (from $t-1$ to $t+1$). In order to have the recombining property, we need to impose in the middle path

$$\underline{S}_{t-1}(1 + \bar{u})(1 + \underline{d}) = \underline{S}_{t-1}(1 + \bar{d})(1 + \underline{u}) \quad \text{and} \quad \bar{S}_{t-1}(1 + \underline{u})(1 + \bar{d}) = \bar{S}_{t-1}(1 + \underline{d})(1 + \bar{u}),$$

which are equivalent to

$$\frac{1 + \bar{u}}{1 + \underline{u}} = \frac{1 + \bar{d}}{1 + \underline{d}}.$$

In this way, Equation 3.2 becomes

$$1 + \Delta_{t-1} \leq \frac{1 + \bar{u}}{1 + \underline{u}} = \frac{1 + \bar{d}}{1 + \underline{d}}$$

and we keep $\underline{S}_t < \bar{S}_t \quad \forall t$. For the estimation of an EBAMM we make the same steps: defining q^* in the same way as in Definition 3.3, we get at time t

$$E^{\mathbb{Q}^*}[\underline{S}_t | \mathcal{F}_{t-1}] = \bar{S}_{t-1}(1 + \underline{u})q^* + \bar{S}_{t-1}(1 + \underline{d})(1 - q^*) \leq \bar{S}_{t-1},$$

$$E^{\mathbb{Q}^*}[\bar{S}_t | \mathcal{F}_{t-1}] = \underline{S}_{t-1}(1 + \bar{u})q^* + \underline{S}_{t-1}(1 + \bar{d})(1 - q^*) \geq \underline{S}_{t-1};$$

and at time $(t + 1)$

$$E^{\mathbb{Q}^*} [\underline{S}_{t+1} | \mathcal{F}_t] = \bar{S}_t(1 + \underline{u})q^* + \bar{S}_t(1 + \underline{d})(1 - q^*) \leq \bar{S}_t,$$

$$E^{\mathbb{Q}^*} [\bar{S}_{t+1} | \mathcal{F}_t] = \underline{S}_t(1 + \bar{u})q^* + \underline{S}_t(1 + \bar{d})(1 - q^*) \geq \underline{S}_t.$$

From these, it is clear that conditions in Equation 3.5 are still necessary and sufficient for the existence of an EBAMM and the absence of arbitrage by Theorem 3.

Corollary. Consider a CRR model with bid-ask spread from $t = 0$ to $t = T$. Define $\Delta = \bar{S}_0 - \underline{S}_0$. Then the conditions in Equation 3.5, namely

$$\underline{d}\bar{u} \leq \bar{d}\underline{u}, \quad \underline{d} < 0, \quad \bar{u} > 0$$

are equivalent to the existence of an EBAMM \mathbb{Q}^* and the absence of arbitrage, where

$$\mathbb{Q}^*(\zeta_t = (\underline{u}, \bar{u}) | \mathcal{F}_{t-1}) = q^* \in \left[\frac{-\bar{d}}{\bar{u} - \bar{d}}, \frac{-\underline{d}}{\underline{u} - \underline{d}} \right].$$

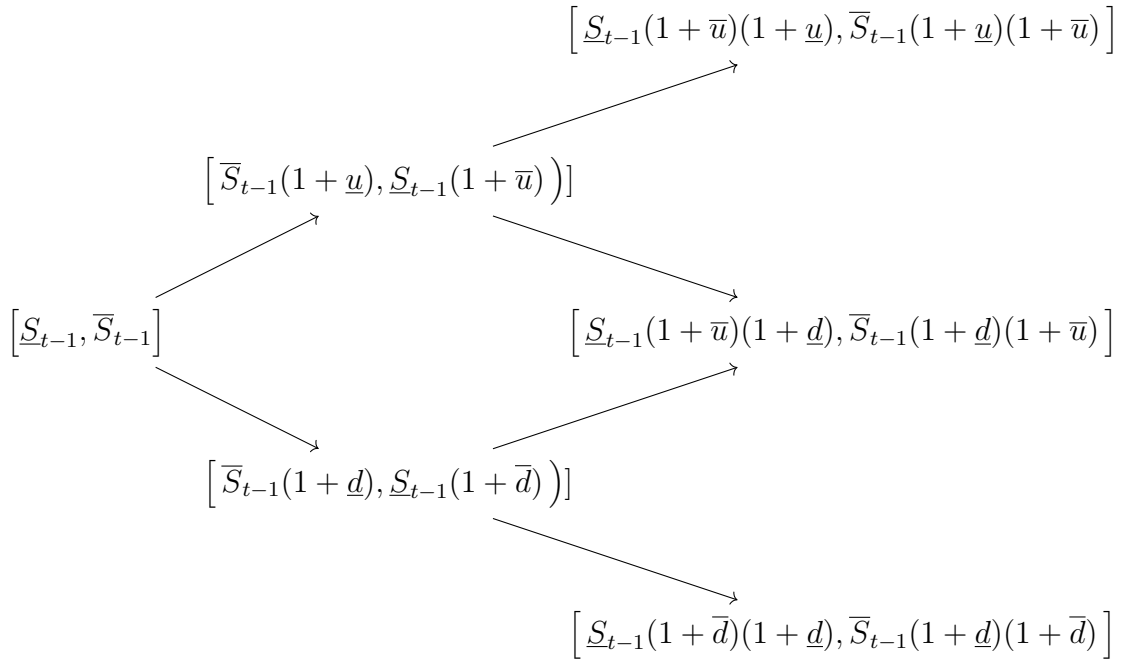


Figure 3.4: Two-steps CRR model

Appendix A

Mathematical prerequisites

In this appendix, we will provide a list of definitions to enhance the reader's understanding.

Definition 11 (σ -algebra). Let be Ω a set. Then a family of subsets $\mathcal{F} \subset P(\Omega)$ is called a σ -algebra if it satisfies the following properties:

- $\Omega \in \mathcal{F}$
- \mathcal{F} is closed under complementation: if $A \in \mathcal{F}$, then $X \setminus A \in \mathcal{F}$
- \mathcal{F} is closed under countable unions: if $A_1, A_2, A_3 \dots \in \mathcal{F}$, then $A = A_1 \cup A_2 \cup A_3 \cup \dots \in \mathcal{F}$

Definition 12 (probability space). An ordered triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space if

- Ω is a sample space (a non-empty set);
- \mathcal{F} is a σ -algebra;
- $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ is a probability measure such that it is countably additive and $\mathbb{P}(\Omega) = 1$.

Definition 13 (filtration). Given a set of index $\mathcal{T} = \{0, 1, 2, \dots, T\}$, an increasing family of σ -algebras, i.e. $\mathcal{F}_0 \subseteq \mathcal{F}_2 \subseteq \dots \subseteq \mathcal{F}_T$ is called filtration and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ a filtrated probability space. Usually, we assume $\mathcal{F}_T = \mathcal{F}$. $\mathcal{F}_0 = \{\emptyset, \Omega\}$ is called the trivial σ -algebra.

Definition 14 (stochastic process). Given a set of index $\mathcal{T} = \{0, 1, 2, \dots, T\}$ and two measurable spaces (Ω, \mathcal{F}) and (E, \mathcal{E}) , a stochastic process with values in (E, \mathcal{E}) is a family of random variable

$$X = (X_t)_{t \in \mathcal{T}} = \{X_t | t \in \mathcal{T}\}$$

Moreover, we define:

- a stochastic process X *adapted* with respect to (\mathcal{F}_t) if, for any $t \in \mathcal{T}$, X_t is \mathcal{F}_t -measurable;
- a stochastic process X *predictable* with respect to (\mathcal{F}_t) if, for any $t \in \mathcal{T}$, X_t is \mathcal{F}_{t-1} -measurable.

Definition 15. Let $(\Omega, \mathcal{F}, \mathbb{P})$ a finite probability space where $\Omega = \{\omega_1, \dots, \omega_N\}$. We define the norm for random variables

$$\|X\|_p := \left(\sum_{n=1}^N |X(\omega_n)|^p \mathbb{P}[\omega_n] \right)^{\frac{1}{p}} = E^{\mathbb{P}} [|X|^p]^{\frac{1}{p}}$$

and for $p = \infty$

$$\|X\|_{\infty} := \max_{n \in \{1, \dots, N\}} \{X(\omega_n) \mid \mathbb{P}[\omega_n] > 0\}.$$

For every p , we define the vector space L^p as

$$L^p(\Omega, \mathcal{F}, \mathbb{P}) := \{X : \Omega \rightarrow \mathbb{R} \text{ is } \mathcal{F}\text{-measurable and } \|X\|_p < \infty\}.$$

Definition 16 (conditional expectation). Let $X \in L(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subset \mathcal{F}$ be a sub- σ -algebra. We call $E[X|\mathcal{G}]$ the *conditional expectation* of X given \mathcal{G} as the unique random variable in $L(\Omega, \mathcal{G}, \mathbb{P})$ such that for all $Z \in L(\Omega, \mathcal{G}, \mathbb{P})$

$$E[XZ] = E[E[X|\mathcal{G}]Z].$$

Definition 17 (martingale). An adapted process $(X)_{t \in \mathcal{T}}$ is called *martingale* if for any $t \in \mathcal{T}$

$$E[X_t | \mathcal{F}_{t-1}] = X_{t-1}$$

Definition 18 (supermartingale). An adapted process $(X)_{t \in \mathcal{T}}$ is called *supermartingale* if for any $t \in \mathcal{T}$

$$E[X_t | \mathcal{F}_{t-1}] \leq X_{t-1}$$

Definition 19 (submartingale). An adapted process $(X)_{t \in \mathcal{T}}$ is called *submartingale* if for any $t \in \mathcal{T}$

$$E[X_t | \mathcal{F}_{t-1}] \geq X_{t-1}$$

Theorem 4. Let \mathbb{P} be a probability measure on (Ω, \mathcal{F}) and \mathbb{Q} another probability measure on (Ω, \mathcal{F}) defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := Z$$

where Z is the Radon-Nikodym derivative, a random variable \mathcal{F} -measurable such that $Z \geq 0$ a.s. and $E^{\mathbb{P}}[Z] = 1$.

Let $\mathcal{G} \subseteq \mathcal{F}$ be any sub- σ -algebra. For any \mathcal{F} -measurable random variable X we have

$$E^{\mathbb{P}}[Z | \mathcal{G}] E^{\mathbb{Q}}[X | \mathcal{G}] = E^{\mathbb{P}}[ZX | \mathcal{G}]$$

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