# UNIVERSITÀ DEGLI STUDI DI PADOVA

**Dipartimento di Fisica e Astronomia "Galileo Galilei"**

**Master Degree in Physics**

**Final Dissertation**

# Topological defects in conformal field theory

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**Academic Year 2022/2023**

# Contents

## Abstract

This thesis objective is to characterize the topological defects of a particular two dimensional conformal quantum field theory. Topological defects can be interpreted as generalizations of global symmetries and, between other things, give rise to selection rules in amplitudes.

In this thesis, starting with an in-depth introduction to two dimensional conformal field theories, we will discuss the general properties of topological defects in such theories. Finally, we will explicitly describe the topological defects in a particular conformal field theory that is relevant for applications to string theory, as it describes a string moving in a space with a non-trivial topology and geometry, namely a K3 surface.

# Chapter 1

# Introduction

This thesis sets out with the main objective of talking about defects in conformal field theories. Let us briefly introduce these two subjects and motivate why their study can be considered important.

## 1.1   CFTs and defects

Conformal field theories: they are field theories which are invariant under conformal transformations. These transformations are characterized by the fact that they act on the metric multiplicating it by a positive function of the position; intuitively they are all the transformations which keep the angles fixed but not the distances. This kind of theories pop up in a lot of different areas of physics: when talking about the renormalization group in quantum field theories, when considering critical phenomena in statistical mechanics and in the quantization of strings. In this thesis we will mainly talk about field theories defined in two dimensions which apply in particular to the last case, therefore let us explain how the two study fields intersect ( [5], [6]).

The main difference between a point particle and a string is that in the first case its motion in spacetime forms a world-line while in the second it forms on a world sheet. The position of the string will then be parameterized by two quantities: $\tau$ and $\sigma$, obtaining the following fields $X^\mu(\tau, \sigma)$. If the string is closed $\sigma$ is taken periodic giving rise to a theory defined on the cylinder.

Now the main idea, in the quantization process, is to consider this theory as a quantum field theory defined in two dimensions. This checks out with what we said before, but we still have to explain where the conformal symmetry comes from.



Fig.1.1: Closed string path

Such symmetry can be demonstrated to be the byproduct of two more fundamental local symmetries: the reparametrization invariance and the Weyl invariance. Between these two gauge symmetries the first one is obvious and everyone meets it when studying the point particle. The second one comes from an extra redundancy in the Polyakov formulation of the string which we will not treat in this thesis.

Actually the connection is a bit more complicated than it looks and this becomes evident in the Polyakov path integral.

With the cylinder we described a single string moving in spacetime, but there will be more than that: the string can divide in two more strings, then reconnect and in general produce all those kind of phenomena which looks very similar to the perturbative expansions in Feynman diagrams for point particles. This fact can be formalized and one actually finds a perturbative expansion in the geometries on which the conformal field theory is defined.
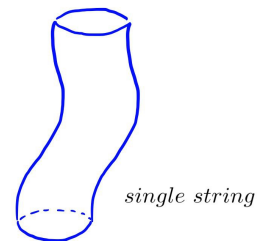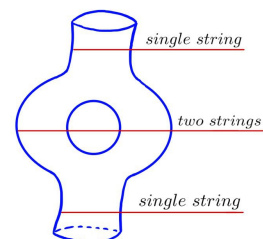


Fig.1.2: Closed string path

Therefore one can associate conformal field theories to pieces of the perturbative expansion of string theory.

To end this part let us note that conformal invariance is special in the two dimensional case compared to the higher dimensional ones. In particular it is much more restrictive having an infinite number of charges. This fact makes working with these theories much easier, so much that a lot of the recent mathematical effort in constructing actual interacting quantum field theories, with vertex operator algebras, has been developed in them.

After that let us bring up the defects ( [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26] ). First of all one can describe them as operators not localized on a point, but on a manifold of an higher dimension.
From this large set of operators we are interested in those with two particular characteristics: they are topological and they are defined on hypersurfaces. For a defect to be topological means that when inserted in a correlator, its deformations do not change the correlation function, unless the defect crosses the support of other operators. The reason why we restrict ourselves to these properties is because it is always possible from a symmetry to define a defect with these characteristics, but not all the defects of this type correspond to symmetries. This kind of defects, in analogy with symmetries, will impose ulterior constraints on the correlators.

Since we are mainly talking about theories in two dimensions the premier choice will be one dimensional operators. These line defects will represent a generalization of symmetries, which one can use when fixing the value of some correlators or establishing relations between them.

## 1.2 The theory of interest

After a necessary introduction on all these concepts, which are by themselves interesting, we will also show how to apply them in practice for a certain theory, which is of particular interest to string theory.

In order to arrive to the main point we have to start with a problem which arises in string theory. One of the main properties of string theory is that there are special conditions on the dimension of the space in which the string is defined. In particular, in the quantization procedure, the Lorentz group becomes anomalous and such anomaly can be cancelled out only if the space we are in has a special dimension. For the superstring, which is one of the most important type of strings (since for example it does not predict tachyons), this dimension is $D = 10$.
Clearly experimental evidence so far tells us that only four dimensions are detectable, therefore for the theory to be consistent the other six dimensions must be "hidden". How to carry out such a procedure can be difficult to explain, but the main idea is to have the six additional dimensions compactified. Briefly, when considering the spacetime manifold, it is possible to assign the first four dimensions to the usual flat geometry as we know it, while constraining the other six on a compact manifold, for example like a six dimensional torus.
It turns out that the best results are obtained by the following two choices for the compact manifold: orbifolds and Calaby-Yau manifolds. The first one is a procedure in which one identifies the coordinates under a certain transformation. One of the most common is the following $\mathbb{Z}_2 : X \equiv -X$. The second possibility corresponds to implementing the theory in particular types of geometries called Calaby-Yau manifolds.
These two approaches are not completely separated, as in our case. In fact when considering the space of moduli of a certain type of Calaby-Yau manifolds, which is given by the space of parameters or metrics that one can choose for a manifold of a given topology (for example the radius in the torus), under certain limits one can arrive to orbifold theories.

The theory we are interested in is a non linear sigma model on K3 [7]. First of all a non linear sigma model means a theory whose fields are valued in a manifold with non-trivial metric, which is to be expected if the space-time we are in is not trivial. A K3 surface, instead, is a particular 4 dimensional (2 complex dimensions) Calaby-Yau manifold.

The plan is to divide the 6 remaining dimension into a K3 surface and a two dimensional manifold which could be the torus $T^2$ or the orbifold of the torus $T^2/\mathbb{Z}_2$. The last part of this geometry is very well understood therefore we are left with the K3 surface.

At this point one could ask why are we considering exactly this manifold. This is because one can demonstrate that the only manifolds that preserve at least half of the supersymmetries of the superstring are either a K3 surface or $T^4$. Since $T^4$ is not that interesting we take the other possibility.

With this introduction we understand why one may want to study this theory, but we did not explain the reason behind the focus on their defects. Recent works have analyzed the symmetries of all the moduli space of the K3 surfaces even though most of these theories are actually very obscure. Since defects are generalizations of symmetries the hope is that in the future we can do the same with them, but to start off we need to find the defects of at least one particular theory.

The simplest choice is $T^4/\mathbb{Z}_2$ which is an orbifold in the moduli space of K3 theories. That case is described by a conformal supersymmetric theory with number of supercharges $N = 4$ (actually this is true for any K3 surface), but can also be put in many different forms which, by the way, were also used to study its symmetries [8]. Chosen a particular geometry for this theory one can show that it can be written in the following form:

$$\mathfrak{su}(2)_1^6 \times \overline{\mathfrak{su}(2)}_1^6, \tag{1.2.1}$$

which is a tensor product of affine algebras. What exactly is an affine algebra will be explained in the next section in which we will present conformal field theories. Such explanation will be complete in the sense that everything needed to understand our analysis will be there, but not much more after that. After all this topic is very extensive and to treat it in full it would require much more time and many pages. The same is to be said about the presentation of defects: the main points are presented in a way that guides the reader to the calculation.

After these two sections we will continue with some examples of defect analysis, showing how it can be carried out, and specializing at last to the case we have here motivated.

Before starting the new chapter a last note would be that even though we motivated this effort using string theory, it is by no means necessary in the calculation. Our analysis will be completely general and any use of the conformal field theory above mentioned can refer to this study when concerned with defects.

# Chapter 2

# Conformal field theory

## 2.1 CFTs in d-dimensions

We start by talking about these theories in a general way, defining the symmetry group in any dimension and looking for the direct general results of this. We will focus in particular on the case in which $d > 3$, that is the least particular example from which we can start and, on a later section, consider the $d = 2$ case: the one that we are interested in.

This chapter follows the references: [1], [2], [3], [4].

### Global conformal invariance

Conformal field theories are theories which are conformal invariant. It is clear that to study these theories we have to start from defining what are conformal transformations.

The *conformal group* is formed by globally defined and invertible coordinate transformations with the following property:

$$g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x) \qquad \Lambda(x) > 0, \tag{2.1.1}$$

where $g_{\mu\nu}$ is the metric tensor of our d-dimensional theory.

First of all it is clear that Poincaré transformations are the subgroup with $\Lambda(x) = 1$. While Poincaré transformations are isometries of the space which keep the distances and the angles fixed, one can instead conceive the conformal group as the one that retain only angles still.

Then let us analyze the infinitesimal coordinate transformation $x' = x + \epsilon(x)$:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}(x) = (\delta^\alpha_\mu - \partial_\mu\epsilon^\alpha)(\delta^\beta_\nu - \partial_\nu\epsilon^\beta)g_{\alpha\beta}(x) = g_{\mu\nu}(x) - (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu). \tag{2.1.2}$$

The requirement for this transformation to be conformal is to have

$$(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) = f(x)g_{\mu\nu}. \tag{2.1.3}$$

Now we assume for simplicity that $g_{\mu\nu} = \eta_{\mu\nu} = diag(1, 1, 1, ...)$ is the Euclidean metric, which will also be our main case of interest (if we took instead a Minkowskian metric there would not be much to change in the discussion).

So we manipulate this equation to find more convenient forms. First of all taking the trace

$$f(x) = \frac{2}{d}\partial_\rho\epsilon^\rho. \tag{2.1.4}$$

Adding instead a derivative and permuting the indices one arrives at

$$2\partial_\mu\partial_\nu\epsilon_\rho = \eta_{\mu\rho}\partial_\nu f + \eta_{\nu\rho}\partial_\mu f - \eta_{\mu\nu}\partial_\rho f. \tag{2.1.5}$$

Multiplying by $\eta^{\mu\nu}$

$$2\partial^2\epsilon_\mu = (2 - d)\partial_\mu f. \tag{2.1.6}$$

Applying $\partial_\nu$ to this last equation and using (2.1.3) and again (2.1.6)

$$(2 - d)\partial_\mu\partial_\nu f = \eta_{\mu\nu}\partial^2 f. \tag{2.1.7}$$

Finally multiplying again by $\eta^{\mu\nu}$ we get

$$(d - 1)\partial^2 f = 0. \tag{2.1.8}$$

Now that we have everything we need we can make some considerations. First of all when $d = 1$ these equations do not put any constraint: any smooth transformation is conformal. This is coherent with the fact that in one dimensions we do not have any notion of angles. Another special case is $d = 2$ and we will treat it later since it is going to be the one we will use in this thesis.

Then we start with $d \geq 3$. Equations (2.1.7) and (2.1.8) tell us that $\partial_\mu\partial_\nu f = 0$, so at most $f(x) = A + B_\mu x^\mu$.
Substituting this equation in (2.1.5) we get $\partial_\mu\partial_\nu\epsilon = const$ which implies

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \qquad\qquad c_{\mu\nu\rho} = c_{\mu\rho\nu}. \tag{2.1.9}$$

The coefficients in this equation must satisfy some constraints.
Since all the expressions derived before are "derivative" we do not put any on $a_\mu$. Instead from equation (2.1.3) and (2.1.4) we have the following restriction:

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b_\lambda^\lambda \, \eta_{\mu\nu}, \tag{2.1.10}$$

which implies that $b_{\mu\nu}$ must be written in the following way:

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu} \qquad\qquad m_{\mu\nu} = -m_{\nu\mu}. \tag{2.1.11}$$

Finally using (2.1.5) and (2.1.6) we find:

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu \qquad\qquad b_\mu = \frac{1}{d}c^\sigma_{\ \sigma\mu}. \tag{2.1.12}$$

These are the full infinitesimal transformations, when exponentiated they correspond to:

$$
\begin{array}{lll}
\text{translation} & x'^\mu = x^\mu + a^\mu, & \\
\text{dilation} & x'^\mu = \alpha\, x^\mu, & \\
\text{rotation} & x'^\mu = M^\mu_\nu\, x^\nu, & \quad(2.1.13) \\
\text{SCT} & x'^\mu = \dfrac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}. &
\end{array}
$$

It is well known that translations and rotations are generated by $a_\mu$ and $m_{\mu\nu}$, the fact that the dilations come from $\alpha$ is also easily demonstrated. As for the special conformal transformations (SCT), to convince ourselves that that is the right form, we can calculate the infinitesimal transformation and confront it with the one we found. Moreover transforming the metric with this transformation one finds a scale factor $\Lambda(x) = (1 - 2b \cdot x + b^2 x^2)$.
Special conformal transformations also have an interesting property: they are not globally defined if our space is $\mathbb{R}^{d,0}$ (or $\mathbb{R}^{d-1,1}$). In fact there always exists an $x$ for which $1 - 2b \cdot x + b^2 x^2 = 0$. The solution to this is provided by the implementation of the conformal compactification of our space, which basically means that we add the point $\{\infty\}$ to our space.

After this discussion we can continue by studying the algebra of the transformations still in the $d \geq 3$ case.

### Conformal algebra

We want to study the representations of the conformal group on fields $\Phi(x)$ and to do that we use the algebra. To discuss the algebra one has to find the representations of the generators $G_a$ which enter the following formula $\Phi'(x) - \Phi(x) \approx -iw_a G_a \Phi(x)$ (we are expanding up to first order in the infinitesimal parameter $w_a$).

We start then by considering the simplest possible field transformation: $\Phi'(x') = \Phi(x)$. We find for this the following table of generators:

$$
\begin{array}{lll}
\text{translation} & P_\mu = -i\partial_\mu, & \\
\text{dilation} & D = -ix^\mu \partial_\mu, & \\
\text{rotation} & L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), & \\
\text{SCT} & K_\mu = -i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu). &
\end{array}
\tag{2.1.14}
$$

From this one can demonstrate that the algebra can be written in a compact form if we define the following antisymmetric tensor:

$$
\begin{aligned}
J_{\mu\nu} &= L_{\mu\nu} & J_{-1\mu} &= \frac{1}{2}(P_\mu - K_\mu), \\
J_{-10} &= D & J_{0\mu} &= \frac{1}{2}(P_\mu + K_\mu).
\end{aligned}
\tag{2.1.15}
$$

Then, if the spacetime is euclidean, we can use $\eta = diag(-1, 1, 1, ...)$ in d+2 dimensions to write

$$
[J_{ab}, J_{cd}] = i(\eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}).
\tag{2.1.16}
$$

This form also proves that the conformal group in d dimensions is isomorphic to $SO(d+1, 1)$.

Now we proceed to study any possible infinitesimal conformal transformation of the fields:

$$
\Phi'(x') \approx (1 - iw_a T_a)\Phi(x).
\tag{2.1.17}
$$

First of all we now consider only the Poincarè transformations and take those that leave $x = 0$ invariant: the Lorentz group. Using the definition of the representations of the generators and equation (2.1.17) one finds that:

$$
\Phi'(0) - \Phi(0) \approx -iw_a G_a \Phi(0) \approx -iw_a T_a \Phi(0).
\tag{2.1.18}
$$

In this way we can define the spin operators as the following operator:

$$
L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0).
\tag{2.1.19}
$$

Then such operator can be translated to a non zero value of $x$ using this transformation:

$$
e^{ix^\rho P_\rho} L_{\mu\nu} e^{-ix^\rho P_\rho} = S_{\mu\nu} - x_\mu P_\nu + x_\mu P_\nu.
\tag{2.1.20}
$$

The right side of this equation was calculated using the previous commutation rules and the Hausdorff formula.
If now we ask for the field to transform under translation like this:

$$
\Phi'(x + a) = \Phi(x),
\tag{2.1.21}
$$

which is a reasonable condition, the representation of the generator $P_\mu$ will be the same as the one we found previously. In total, one finds:

$$
\begin{aligned}
P_\mu \Phi(x) &= -i\partial_\mu \Phi(x), \\
L_{\mu\nu}\Phi(x) &= i(x_\mu \partial_\nu - x_\nu \partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x).
\end{aligned}
\tag{2.1.22}
$$

This exact same procedure can be carried out for the full conformal group.

Considering the subgroup which leaves the origin invariant we get the following matrices: $S_{\mu\nu}$ as before for the rotations, $\tilde{\Delta}$ for the dilations and $k_\mu$ for the special conformal transformations.
This matrices must form a representation of the conformal algebra so that, with the Hausdorff formula, we can calculate the following general transformations:

$$
\begin{aligned}
D\Phi(x) &= (-ix^\nu \partial_\nu + \tilde{\Delta})\Phi(x), \\
k_\mu \Phi(x) &= \{k_\mu + 2x_\mu \tilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu \partial_\nu + ix^2 \partial_\mu\}\Phi(x).
\end{aligned}
\tag{2.1.23}
$$

In QFT the usual demand is that $\Phi(x)$ belongs to an irreducible representation of the Lorentz group. This has consequences for our matrices $\tilde{\Delta}$ and $k_\mu$. We are looking in particular at the following commutation relations:

$$
\begin{aligned}
[\tilde{\Delta}, S_{\mu\nu}] &= 0, \\
[\tilde{\Delta}, k_\mu] &= -ik_\mu.
\end{aligned}
\tag{2.1.24}
$$

The first equation tells us that $\tilde{\Delta}$ commutes with $S_{\mu\nu}$; if now we know that $S_{\mu\nu}$ is irreducible, by Schur's lemma, one must have that any matrix commuting with it must be a multiple of the identity. We deduce from this that $\tilde{\Delta}$ is basically a number, we define $\tilde{\Delta} = -i\Delta\mathbb{I}$ and call $-i\Delta$ the dimension of the field.
The next equation instead, now that we know that $\tilde{\Delta}$ is a multiple of the identity, has only one solution $k_\mu = 0$.

In principle from the generators calculated above we could derive the changes of $\Phi$ under any finite transformation, but we will present now only the case of a spin-less field. Under the conformal transformation $x \rightarrow x'$:

$$
\Phi(x) \rightarrow \Phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \Phi(x),
\tag{2.1.25}
$$

where $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of the conformal transformation of the coordinates.

A field transforming as the above is called *quasi-primary*. This type of fields will be very important in our study, in particular when we will define primary fields.

## Correlation functions

A direct effect of conformal invariance is that the correlation functions of the theory are constrained. In particular we will consider correlators of quasi-primary fields.

Before that we seize the opportunity to explain a detail on the terminology that we are going to use. When we will talk about fields we will not mean that it figures independently into the function integral measure, but also composite quantities like the energy-momentum tensor are called as such, given that they are local quantities.
In the theories that we will study in two dimensions we are going to consider only the correlators and their symmetry properties; so "How many continuous, independent degrees of freedom there are" becomes often not so clear. Useful instead in the analysis will be to consider "How many and which primary fields (or quasi-primary) are there".

Let us now return to our previous topic and write the definition of a two point correlation function of quasi-primary fields:

$$
\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \frac{1}{Z} \int [d\mathbf{\Phi}] \; \Phi_1(x_1)\Phi_2(x_2) \; e^{-S[\mathbf{\Phi}]}.
\tag{2.1.26}
$$

The invariance of the action and the functional integration measure leads to the following transformation of the correlator:

$$
\langle \Phi_1(x_1)\Phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{\Delta_1/d}_{x=x_1} \left|\frac{\partial x'}{\partial x}\right|^{\Delta_2/d}_{x=x_2} \langle \Phi_1(x_1')\Phi_2(x_2')\rangle,
\tag{2.1.27}
$$

which, if we specialize to a scale transformation $x \to \lambda x$, becomes

$$\langle \Phi_1(x_1)\Phi_2(x_2) \rangle = \lambda^{\Delta_1+\Delta_2} \langle \Phi_1(x_1')\Phi_2(x_2') \rangle. \tag{2.1.28}$$

It is now already known that rotation and translation invariance have as a consequence

$$\langle \Phi_1(x_1)\Phi_2(x_2) \rangle = f(|x_1 - x_2|). \tag{2.1.29}$$

Requiring equation (2.1.28) (which is to ask for $f$ to be homogeneous of a certain degree) fixes further the expression to

$$\langle \Phi_1(x_1)\Phi_2(x_2) \rangle = \frac{C_{1,2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{2.1.30}$$

where $C_{1,2}$ is a constant coefficient.

At last we consider the SCT:

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{\gamma^d} \qquad\qquad \gamma = (1 - 2b \cdot x + b^2 x^2). \tag{2.1.31}$$

They lead to the equality

$$\frac{C_{1,2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{1,2}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{2.1.32}$$

which is satisfied only if $\Delta_1 = \Delta_2$.

This can be "parameterized" by $C_{1,2} = 0$ if $\Delta_1 \neq \Delta_2$.

The same exact procedure can be carried out for the three point correlation function obtaining:

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3) \rangle = \frac{C_{1,2,3}}{x_{1,2}^{\Delta_1+\Delta_2-\Delta_3} x_{2,3}^{\Delta_2+\Delta_3-\Delta_1} x_{1,3}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{2.1.33}$$

where $x_{i,j} = |x_i - x_j|$.

Unfortunately this stops here, from four (or more) point functions it is in fact possible to construct *anharmonic ratios*, which are conformal invariant. Any function of these quantities will be possible in the correlator.

For example the four point function will have the following expression:

$$\langle \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4) \rangle = f(a,b) \prod_{i<j}^{4} x_{i,j}^{\Delta/3-\Delta_i-\Delta_j} \qquad \Delta = \sum_{i=1}^{4} \Delta_i,$$

$$a = \frac{x_{1,2}x_{3,4}}{x_{1,3}x_{2,4}} \qquad b = \frac{x_{1,2}x_{3,4}}{x_{2,3}x_{1,4}}, \tag{2.1.34}$$

where $f(a,b)$ is a generic function of $a$ and $b$ which are anharmonic ratios.

## 2.2 CFTs in 2-dimensions

This particular case of conformal theory requires special attention. As we will see in two dimensions there are an infinite number of coordinate transformations that are locally conformal. Among this set, of course, there are the ones we studied in the previous section, which are all the ones globally defined.

The theory should nevertheless be sensitive to the infinite number of local symmetries and therefore appear more constrained.

## Conformal group and algebra

Recollecting now equations (2.1.3) and (2.1.4) we get either one of the two following conditions on conformal transformations in two dimensions:

$$\begin{aligned} \partial_0 \epsilon_0 = \partial_1 \epsilon_1 \qquad & \partial_0 \epsilon_1 = -\partial_1 \epsilon_0, \\ \partial_0 \epsilon_0 = -\partial_1 \epsilon_1 \qquad & \partial_0 \epsilon_1 = \partial_1 \epsilon_0. \end{aligned} \qquad (2.2.1)$$

From these we clearly recognize the Cauchy-Riemann equations appearing in complex analysis. A good idea would then be to use complex coordinates:

$$\begin{aligned} z = x_0 + i x_1 \qquad & \epsilon = \epsilon_0 + i \epsilon_1 \qquad & \partial_z = \frac{1}{2}(\partial_0 - i\partial_1), \\ \overline{z} = x_0 - i x_1 \qquad & \overline{\epsilon} = \epsilon_0 - i \epsilon_1 \qquad & \partial_{\overline{z}} = \frac{1}{2}(\partial_0 + i\partial_1). \end{aligned} \qquad (2.2.2)$$

Note that now in terms of these variables:

$$\begin{aligned} g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad & g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \\ \epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix} \qquad & \epsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}. \end{aligned} \qquad (2.2.3)$$

Before carrying on with the analysis of the algebra, let us notice that for certain manipulations it can be useful to extend our variable space making $z$ and $\overline{z}$ independent. Then one can always find the physical space in the two dimensional plane $z^* = \overline{z}$.

Returning to equation (2.2.1) we will have that any holomorphic and antiholomorphic function on the complex plane defines an infinitesimal conformal transformation. Actually since we are talking about infinitesimal transformations it is enough for the functions to be meromorphic.

Now we know that any meromorphic function can be Laurent expanded as

$$\epsilon(z) = \sum_{-\infty}^{\infty} c_n z^{n+1}. \qquad (2.2.4)$$

Then calculating the representations of the generators for the simple field conformal transformation $\Phi'(z', \overline{z}') = \Phi(z, \overline{z})$ one finds the following:

$$\begin{aligned} \Phi'(z', \overline{z}') &= \Phi(z, \overline{z}) \\ &= \Phi(z', \overline{z}') - \epsilon(z')\partial'\Phi(z', \overline{z}') - \overline{\epsilon}(\overline{z}')\overline{\partial}'\Phi(z', \overline{z}') \\ \delta\Phi &= -\epsilon(z)\partial\Phi(z, \overline{z}) - \overline{\epsilon}(\overline{z})\overline{\partial}\Phi(z, \overline{z}) \\ &= \sum_n c_n l_n \Phi(z, \overline{z}) + \overline{c}_n \overline{l}_n \Phi(z, \overline{z}), \end{aligned} \qquad (2.2.5)$$

where we introduced:

$$l_n = -z^{n+1}\partial \qquad\qquad \overline{l}_n = -\overline{z}^{n+1}\overline{\partial}. \qquad (2.2.6)$$

These two infinite sets of generators form two commuting *Witt algebras*:

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n}, \\ [\overline{l}_m, \overline{l}_n] &= (m-n)\overline{l}_{m+n}, \\ [l_m, \overline{l}_n] &= 0. \end{aligned} \qquad (2.2.7)$$

It is interesting now to note which of these generators produce globally defined transformations. Let us focus only on one of the copies of the algebra, the other part will follow this discussion identically.

Take the $l_n$s, they are not all everywhere well defined: first of all in $z = 0$ they are non-singular only for $n \geq -1$.

The other ambiguous point would be $z = \infty$ which is part of the Riemann sphere (remember conformal compactification). To study it we need to perform a change of variables (or change of chart thinking from a differential manifold point of view): $z = -\frac{1}{w}$.

$$l_n = -\left(-\frac{1}{w}\right)^{n-1} \partial_w \tag{2.2.8}$$

which in $w = 0$ ($z = \infty$) is non singular only for $n \leq 1$.

In conclusion one finds that the conformal transformations on the Riemann sphere are generated by $\{l_{-1}, l_0, l_1\} \cup \{\bar{l}_{-1}, \bar{l}_0, \bar{l}_1\}$.

One can investigate what they correlate to and find that:

- $l_{-1}$ corresponds quite clearly to translations;

- $l_0$ must instead be combined with $\bar{l}_0$ to form: $(l_0 + \bar{l}_0)$ dilations and $i(l_0 - \bar{l}_0)$ rotations;

- $l_1$ can be demonstrated to correspond to SCT.

Moreover, one can analyze these transformations looking at finite ones. Exponentiation of an holomorphic function gets another holomorphic function. Complex analysis then tells us that invertible holomorphic functions globally defined on the Riemann sphere are only of the following form:

$$f(z) = \frac{az + b}{cz + d} \qquad\qquad ad - bc \neq 0. \tag{2.2.9}$$

Note now that we can completely characterize this group with the set of matrices:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad det(A) = 1 \qquad\qquad A \equiv -A, \tag{2.2.10}$$

equipped with its standard matrix multiplication.

So the conformal group will be isomorphic to the *Möbius group* $SL(2, \mathbb{C})/\mathbb{Z}_2$.

## Central extension

Let us go back to the Witt algebra. It turns out that this algebra admits a so-called *central extension*. This is a very important topic, in fact these extensions of the algebra are closely related to projective representations which are needed in Quantum Mechanics.

Moreover, if one "blindly" tries to quantize some of the most common classical conformally invariant theories, they will not arrive to a representation of the Witt algebra but to one of its extensions. At last if one tries to quantize the same theories on a curved manifold, instead of on a flat one, one will find that the conformal symmetry becomes anomalous and such anomaly will be connected to the central extension.

So first of all let us briefly explain what is a central extension.

The main idea is that we want to add a constant number to the usual commutation relations of a certain algebra $\mathfrak{g}$. To do so what we will do is to add one new operator, $C$, which has a constant eigenvalue on any irreducible representation of our original algebra (the general theory of central extensions admits the possibility to add more than only one of these operators). By Schur's lemma we have that this property is satisfied by $C$ if it commutes with all the elements of $\mathfrak{g}$.

Note that this is exactly why it is called central extension: the new element $C$ will be part of the center of the new algebra, which we will call $\hat{\mathfrak{g}}$.

Now we implement this in Witt's algebra, the new commutation relations will be:

$$\begin{aligned}
[L_m, L_n] &= (m - n)L_{m+n} + p(m, n)C, \\
[L_m, C] &= 0, \\
[C, C] &= 0.
\end{aligned} \tag{2.2.11}$$

This algebra has to satisfy some properties, the value of $p(m, n)$ is not arbitrary. The trivial solution is $p(m, n) = 0$ for any given $m, n$. This is the least interesting case, since the new algebra will be the direct sum of Witt's algebra and the one dimensional algebra.

We look then for an alternative solution:

- first of all the commutator antisymmetry imposes $p(n, m) = -p(m, n)$;

- then we can always fix $p(1, -1) = p(n, 0) = 0$ redefining $\hat{L}_n = L_n + \frac{Cp(n,0)}{n}$ for $n \neq 0$ and $\hat{L}_0 = L_0 + \frac{Cp(1,-1)}{2}$;

- Using then the following Jacobi identity:

$$0 = [[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n]$$
$$0 = (m - n)cp(m + n, 0) + ncp(n, m) - mcp(m, n) \qquad (2.2.12)$$
$$0 = (m + n)p(n, m),$$

  where c is the eigenvalue for the operator C, one finds that for $n \neq -m$ $p(n, m) = 0$;

- At last we use this Jacobi equality:

$$0 = [[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n]$$
$$0 = (-2n + 1)cp(1, -1) + (n + 1)cp(n - 1, -n + 1) + (n - 2)cp(-n, n), \qquad (2.2.13)$$

  from which we get a recursive relation that yields the result:

$$p(n, -n) = \frac{n + 1}{n - 2}p(n - 1, -n + 1) = \frac{1}{12}(n + 1)(n - 1)n, \qquad (2.2.14)$$

  having normalized $p(2, -2) = \frac{1}{2}$ while keeping c free of constraints.

In total then we obtain a very important algebra:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \qquad (2.2.15)$$

which is called *Virasoro algebra* and c is called *central charge*.

## Primary fields

First of all we define a quasi-primary field in two dimensions; it is a field that under any global conformal transformation $z \to w(z)$, $\overline{z} \to \overline{w}(\overline{z})$ transforms like

$$\Phi'(w, \overline{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\overline{w}}{d\overline{z}}\right)^{-\overline{h}} \Phi(z, \overline{z}). \qquad (2.2.16)$$

This generalizes to a *primary field* if $\Phi$ transforms like this under any conformal transformation, both local and global. By definition it is immediate to observe that any primary field is quasi-primary, but not necessarily the reverse.

Fields which are neither primary nor quasi-primary exist and are called *secondary*.

Notice that in this expression we have introduced two new variables: $h$ and $\overline{h}$. They are called *conformal weights* and their introduction comes with the fact that, differently than in the $d > 2$ case, these fields can have spin different from zero; in particular we have

$$h = \frac{1}{2}(\Delta + s) \qquad\qquad \overline{h} = \frac{1}{2}(\Delta - s). \qquad (2.2.17)$$

In fact in a complex variable environment the transformation $w(z) = \lambda z$ is both a dilation and a rotation. Decomposing $\lambda = |\lambda|e^{i\theta}$ and imposing that we stay on the physical plane $w^* = \overline{w} \to \lambda^* = \overline{\lambda}$ we get the following expression:

$$\Phi'(w, \overline{w}) = \lambda^{-\Delta/2 - s/2}\overline{\lambda}^{-\Delta/2 + s/2}\Phi(z, \overline{z}) =$$
$$= \left(|\lambda|e^{i\theta}\right)^{-\Delta/2 - s/2} \left(|\lambda|e^{-i\theta}\right)^{-\Delta/2 + s/2} \Phi(z, \overline{z}) = |\lambda|^{-\Delta}e^{-i\theta s}\Phi(z, \overline{z}). \qquad (2.2.18)$$

The field is both dilated and rotated in the complex plane.

Since primary fields will be very important let us write down how they change under an infinitesimal conformal transformation:

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(z,\bar{z}) = -(h\partial\epsilon + \epsilon\partial + \overline{h\partial\epsilon} + \bar{\epsilon}\bar{\partial})\Phi(z,\bar{z}). \tag{2.2.19}$$

At last another important field property in these theories is the following. A field is called *chiral* if it only depends on $z$ ($\Phi(z)$) and *antichiral* if it only depends on $\bar{z}$ ($\Phi(\bar{z})$).

### Energy-momentum tensor and Ward identities

Since we have these symmetries let us look at the conserved currents from the Noether theorem. It is well known that for translations and rotations the conserved currents are:

$$\begin{array}{ll} Traslations & j_T^{\mu\nu} = T^{\mu\nu}, \\ Rotations & j_R^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu, \end{array} \tag{2.2.20}$$

where $T^{\mu\nu}$ is the symmetrized energy-momentum tensor.
For the dilations one can show that the Noether current can be written as

$$j_D^\mu = T_\nu^\mu x^\nu. \tag{2.2.21}$$

From these we want to find the Ward identities. Their general form is

$$\partial_\mu\langle j_a^\mu(x)\Phi(x_1)...\Phi(x_n)\rangle = -i\sum_{i=1}^n \delta(x-x_i)\langle\Phi(x_1)...G_a\Phi(x_i)...\Phi(x_n)\rangle, \tag{2.2.22}$$

with $G_a$ being the representation of the generator of the symmetry and $j_a^\mu$ the conserved current. With some calculations one arrives at:

$$\begin{array}{ll} Traslations & \partial_\mu\langle T_\nu^\mu(x)X\rangle = -\sum_{i=1}^n \delta(x-x_i)\partial_{\nu i}\langle X\rangle, \\[2em] Rotations & \epsilon_{\mu\nu}\langle T^{\mu\nu}(x)X\rangle = -i\sum_{i=1}^n s_i\delta(x-x_i)\langle X\rangle, \\[2em] Dilations & \langle T_\mu^\mu(x)X\rangle = -\sum_{i=1}^n \delta(x-x_i)\Delta_i\langle X\rangle, \end{array} \tag{2.2.23}$$

where $X$ stands for a product of n primary fields.
One may think that we forgot something: the special conformal transformations. Actually it can be demonstrated that they do not bring any additional conserved charge. The only addition, from Poincaré invariance to conformal invariance, is the tracelessness of the energy-momentum tensor.

Until now we have studied everything using the usual spatial coordinates (not the complex ones). This is because our discussion actually holds for other dimensions as well (adding just some minor modifications). Now we go to the complex coordinates specializing to the two dimensional case. In order to do so we must write the Dirac delta function in complex coordinates:

$$\delta(x) = \frac{1}{\pi}\partial_{z^*}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{z^*}. \tag{2.2.24}$$

These are demonstrable equalities. To understand why intuitively this is so, look at $\frac{1}{z}$, it is a meromorphic function, therefore $\partial_{z^*}\frac{1}{z} = 0$ for $z \neq 0$. The behaviour at $z = 0$ instead can be reconduced to a delta function.

The Ward identities explicitly written in complex coordinates are then:

$$2\pi\partial_z\langle T_{\overline{z},z}X\rangle + 2\pi\partial_{\overline{z}}\langle T_{z,z}X\rangle = -\sum_{i=1}^{n}\partial_{\overline{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle,$$

$$2\pi\partial_z\langle T_{\overline{z},\overline{z}}X\rangle + 2\pi\partial_{\overline{z}}\langle T_{z,\overline{z}}X\rangle = -\sum_{i=1}^{n}\partial_z\frac{1}{\overline{z}-\overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle,$$

$$2\pi\langle T_{\overline{z},z}X\rangle = -\sum_{i=1}^{n}\partial_{\overline{z}}\frac{1}{z-w_i}h_i\langle X\rangle,$$

$$2\pi\langle T_{z,\overline{z}}X\rangle = -\sum_{i=1}^{n}\partial_z\frac{1}{\overline{z}-\overline{w}_i}\overline{h}_i\langle X\rangle,$$

$$(2.2.25)$$

where the last two Ward identities, corresponding to rotations and dilations, have been added and subtracted. In total these equations give the following equalities:

$$\partial_{\overline{z}}\left[\langle T(z,\overline{z})X\rangle - \sum_{i=1}^{n}\left(\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right)\right] = 0,$$

$$\partial_z\left[\langle \overline{T}(z,\overline{z})X\rangle - \sum_{i=1}^{n}\left(\frac{1}{\overline{z}-\overline{w}_i}\partial_{\overline{w}_i}\langle X\rangle + \frac{\overline{h}_i}{(\overline{z}-\overline{w}_i)^2}\langle X\rangle\right)\right] = 0,$$

$$(2.2.26)$$

where we have introduced the new quantities: $T = -2\pi T_{z,z}$ and $\overline{T} = -2\pi T_{\overline{z},\overline{z}}$.
From the above equalities we find first of all that $T$ is chiral field, while $\overline{T}$ is antichiral. Then we get the following expression

$$\langle T(z)X\rangle = \sum_{i=1}^{n}\left[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\right] + reg.; \qquad (2.2.27)$$

followed by a corresponding one for $\overline{T}$.
The *reg.* means a regular holomorphic function of $z$ without any poles.

## 2.3 The operator formalism

Until now we have studied the general consequences of a conformal symmetry without worrying too much about which was the theory in consideration. To continue our analysis is now useful, instead, to specify to a certain extent the object of our studies.
The theories that we want to consider are the ones on the cylinder, which mean a periodic spatial direction. These are ones of the most studied theories due to its connection to string theory. As mentioned in the introduction, if one thinks at the world-sheet, which is generated by a single string moving in spacetime, it can be parameterized by a cylinder.

### Radial quantization

First of all one can take any radius but for simplicity we take $R = 1$. Then in complex coordinates the periodicity condition is

$$w \sim w + 2\pi i. \qquad (2.3.1)$$

Now we map this cylinder to the complex plane with the following transformation

$$z = e^{w}. \qquad (2.3.2)$$

This mapping sends the $t \to -\infty$ to $z = 0$ and the $t \to \infty$ to the infinite point on the Riemann sphere. The periodic condition is integrated in the various fields when they are single valued on the plane. In the exact same way other conditions, like the antiperiodic one, are easily integrated using functions which are not single valued on the complex plane.
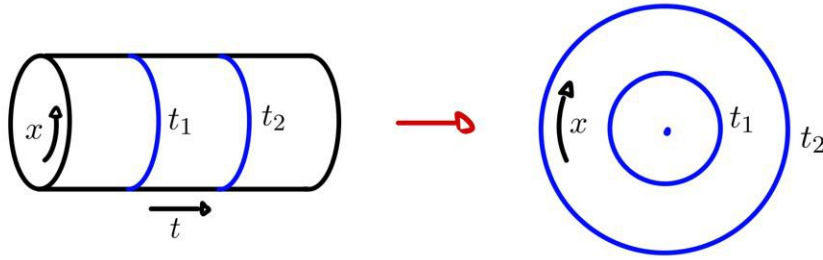
Fig.2.1: Mapping from the cylinder to the complex plane

Note that the passage from the cylinder to the plane is only necessary from a Minkowski/string point of view (we are talking about the Minkowski with imaginary time which becomes euclidean and makes for a well defined path integral). In the context of statistical mechanics, using euclidean spacetime, one could have arbitrarily chosen the time direction to be "radial", getting directly the theory on the plane.
This will be a general property of our analysis. Our results will be justified in the infinite cylinder framework, but everything we say holds also if we put all the premises as definition and construct, in this way, a general euclidean conformal field theory in two dimensions.

Now we begin to build the theory starting by some properties of the operators and in particular primary fields.
In fact, regarding the latter one, let us consider their Hermitian conjugation. It takes a form a bit different than what we are used to:

$$[\Phi(z,\overline{z})]^\dagger = \overline{z}^{-2h}z^{-2\overline{h}}\Phi\left(\frac{1}{\overline{z}},\frac{1}{z}\right).\tag{2.3.3}$$

This is not surprising, as a matter of facts considering the map from the cylinder to the complex plane as a conformal transformation we have

$$\Phi(w,\overline{w}) = z^h\overline{z}^{\overline{h}}\Phi(z,\overline{z}).\tag{2.3.4}$$

Then imposing the usual Hermitian conjugation property for euclidean time in the physical cylinder:

$$[\Phi(w,w^*)]^\dagger = \Phi(w',w'^*),\tag{2.3.5}$$

with $w' = -\tau + ix$, time inversion in complex coordinates, we get

$$z^{*h}z^{\overline{h}}[\Phi(z,z^*)]^\dagger = z^{*-h}z^{-\overline{h}}\Phi\left(\frac{1}{z^*},\frac{1}{z}\right),\tag{2.3.6}$$

since $\frac{1}{z^*} = e^{w'}$.

After that we consider the Laurent expansion of primary fields:

$$\Phi(z,\overline{z}) = \sum_{n,m\in\mathbb{Z}} z^{-m-h}\overline{z}^{-n-\overline{h}}\Phi_{m,n},$$
$$\Phi_{m,n} = \frac{1}{2\pi i}\oint dz z^{m+h-1}\frac{1}{2\pi i}\oint d\overline{z}\,\overline{z}^{n+\overline{h}-1}\Phi(z,\overline{z}).\tag{2.3.7}$$

This expressions can be reconduced to the Fourier expansion of the field on the cylinder (given its periodic behaviour) using the conformal map between it and the complex plane.
Using the Hermitian conjugation property above, we find that

$$\Phi(z,\overline{z})^\dagger = \sum_{n,m\in\mathbb{Z}} \overline{z}^{-m-h}z^{-n-\overline{h}}\Phi_{m,n}^\dagger,$$
$$\Phi_{m,n}^\dagger = \Phi_{-m,-n}.\tag{2.3.8}$$

Within radial quantization one finds out that time ordering, which is used to define correlators, becomes radial ordering:

$$R[\Phi_1(z)\Phi_2(w)] = \begin{cases} \Phi_1(z)\Phi_2(w) & |z| > |w| \\ \Phi_2(w)\Phi_1(z) & |z| < |w| \end{cases}. \tag{2.3.9}$$

Note that now we are considering chiral fields, on the physical plane. Adding an antiholomorphic part does not cause any problem.

Let us make a last consideration that is going to be useful later on. In our previous explanation, we used the transformation of primary fields from the cylinder to the complex plane, but, since the energy-momentum tensor is just a quasi-primary field, its transformation properties differ:

$$T(w) = z^2 T(z) - \frac{c}{24}. \tag{2.3.10}$$

When going from the Riemann sphere to the cylinder we gain a vacuum expectation value of $-\frac{c}{24}$ for the holomorphic part and a similar one for the antiholomorphic one.

We will introduce the Laurent expansion of the energy-momentum tensor in the next section but the treatment is not different from the one for the primary fields. The difference is that the Fourier expansion of $T$ on the cylinder and the Laurent expansion on the complex plane do not have exactly the same modes due to the appearance of the new term we introduced before. What is generally natural to do then is to make the following association:

$$L_0^{cylinder} = L_0^{plane} - \frac{c}{24}, \tag{2.3.11}$$

so that now one has a ground state energy different form zero on the cylinder.

## The operator product expansion

Before writing down the Hilbert space of our radially quantized theory let us discuss one of the most important topics in conformal field theories, which is the *operator product expansion* (OPE).
First of all, take a look at equation (2.2.27): on the left side we have a correlator of $T$ with $X$, while on the left side only a correlator of $X$. The number of fields in the correlation function diminished. In particular consider two points $z$ and $w$, then send $z \to w$. In such case one finds that

$$T(z)\Phi(w,\overline{w}) \sim \frac{h}{(z-w)^2}\Phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\Phi(w,\overline{w}), \tag{2.3.12}$$

and a similar expression for $\overline{T}$.
It is important to stress that this kind of expressions have to be considered always inside a correlator.

This fact sets the road to an interesting proposition: what if in a CFT two operators, like $A(z)$ and $B(w)$, always respected a formula like the following one:

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{[AB]_n(w)}{(z-w)^n}, \tag{2.3.13}$$

for a certain set of fields $[AB]_n(w)$ which are non singular at $w = z$.
It turns out that this is the case and this property is of fundamental importance since with it one can solve the theory. As we will see the continuous use of (2.3.13) inside the correlators completely determine their value.

For example let us study the OPE between two energy-momentum tensors and, by doing so, also determine the relation between the OPE and the commutation relations of the modes.

First of all it is quite easy to see that the charge for the conformal symmetry is

$$Q = \frac{1}{2\pi i} \oint (dz T(z)\epsilon(z) + d\bar{z}\overline{T}(\bar{z})\bar{\epsilon}(\bar{z})), \tag{2.3.14}$$

where these are equal time integrals and their orientation is appropriately chosen.

Considering only the chiral part (the other one is identical), we expand both $T(z)$ and $\epsilon(z)$ in modes:

$$\begin{aligned}
T(z) &= \sum_{n\in\mathbb{Z}} z^{-n-2} L_n & L_n &= \frac{1}{2\pi i} \oint dz z^{n+1} T(z), \\
\epsilon(z) &= \sum_{n\in\mathbb{Z}} z^{n+1} \epsilon_n & Q &= \sum_{n\in\mathbb{Z}} \epsilon_n L_n,
\end{aligned} \tag{2.3.15}$$

where the integral is again an equal time one.

In quantum field theory the charge generates the symmetry using the commutator. Then $L_n$ must respect the Virasoro algebra as previously stated.

In total we get:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{n+m,0}, \\
[L_m, \overline{L}_n] &= 0, \\
[\overline{L}_m, \overline{L}_n] &= (m-n)\overline{L}_{m+n} + \frac{\bar{c}}{12}m(m^2-1)\delta_{n+m,0}.
\end{aligned} \tag{2.3.16}$$

Now we will relate the OPE to the commutation relations.

We define $\oint_w$ as an integral along a path that encircles $w$. Then we can write

$$\begin{aligned}
\oint_w dz R(a(z)b(w)) &= \oint_{C_1} dz a(z)b(w) - \oint_{C_2} dz b(w)a(z) = [A, b(w)] \\
A &= \oint a(z)dz,
\end{aligned} \tag{2.3.17}$$

where $C_1$ and $C_2$ are fixed time contour with radii $|w| + \epsilon$ and $|w| - \epsilon$.

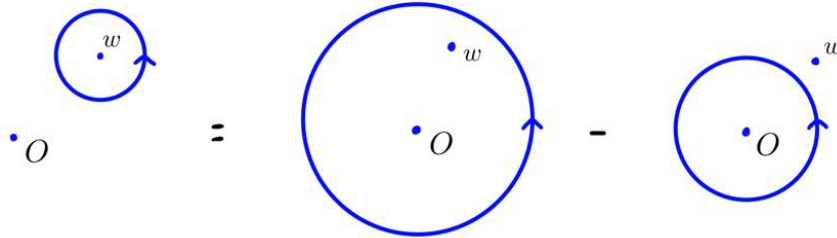The first equality correspond to the contour equality in the image below.



Fig.2.2: Contours subtraction

After that one sends $\epsilon \to 0$ obtaining something like an equal time commutator.

Similarly one finds that

$$\begin{aligned}
[A, B] &= \oint_0 dw \oint_w dz R(a(z)b(w)), \\
A &= \oint a(z)dz \qquad B = \oint b(z)dz.
\end{aligned} \tag{2.3.18}$$

This last equation can be used to demonstrate the validity of the following OPE:

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}. \tag{2.3.19}$$

We have already fixed the commutation relation between the energy-momentum tensor modes in equation (2.3.16). What is left to do is to verify that equation (2.3.19) inserted in (2.3.18) gives the right result:

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{(2\pi i)^2} \oint dz \oint dw\, z^{m+1} w^{n+1} [T(z), T(w)] \\
&= \frac{1}{(2\pi i)^2} \oint_0 dw\, w^{n+1} \oint_w dz\, z^{m+1} R(T(z)T(w)) \\
&= \frac{1}{(2\pi i)^2} \oint_0 dw\, w^{n+1} \oint_w dz\, z^{m+1} \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \\
&= \frac{1}{2\pi i} \oint_0 dw\, w^{n+1} \oint_w dz\, z^{m+1} \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \\
&= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{n+m,0}.
\end{aligned}
\tag{2.3.20}
$$

Notice that confronting (2.3.19) with (2.3.12) we find that $T(z)$ cannot be primary, but it could be quasi-primary. Since with this OPE we can also calculate how $T(z)$ transforms infinitesimally under a conformal transformation, it is possible to verify if it is quasi-primary or not.

$$
\delta_\epsilon T(w) = -[Q, T(w)] = -\frac{1}{2\pi i} \oint_w dz\, \epsilon(z) T(z) T(w) = -\frac{c}{12}\partial_w^3 \epsilon(w) - 2T(w)\partial_w \epsilon(w) - \epsilon(w)\partial_w T(w).
\tag{2.3.21}
$$

Now it is possible to demonstrate that the exponentiation of this expression is

$$
T'(w) = \left(\frac{dw}{dz}\right)^{-2} [T(z) - \frac{c}{12}\{w; z\}] \qquad\qquad \{w; z\} = \frac{d_z^3 w}{d_z w} - \frac{3}{2}\frac{d_z^2 w}{d_z w},
\tag{2.3.22}
$$

where $\{w; z\}$ is called *Schwarzian derivative*. With an explicit calculation one finds that

$$
\{\frac{az+b}{cz+d}; z\} = 0 \qquad\qquad (ad - bc = 1).
\tag{2.3.23}
$$

Therefore $T(z)$ is definitely a quasi-primary field of conformal dimension 2.

### The Hilbert space of radial quantization

First of all we need a vacuum state. In a free field theory it is easy to define it, otherwise one does the usual assumptions about the interaction "turning on" and "off".

In this framework a field at infinity can be considered free, then we define an in state as

$$
|\Phi_{in}\rangle = \lim_{z, \overline{z} \to 0} \Phi(z, \overline{z})|0\rangle,
\tag{2.3.24}
$$

since on the complex plane the origin corresponds to the point at $-\infty$ in time on the cylinder.

Using now the Laurent expansion of the field given in formula (2.3.7) we find that for this in state to be well defined one must have that

$$
\Phi_{m,n}|0\rangle = 0 \qquad\qquad (m > -h, n > -\overline{h}),
\tag{2.3.25}
$$

otherwise we encounter divergences.

If we consider in particular the in states of $T(z)$ we get the conditions:

$$
L_n|0\rangle = 0 \qquad\qquad \overline{L}_n|0\rangle = 0 \qquad\qquad n \geq -1,
\tag{2.3.26}
$$

which by the way also imply the invariance of the vacuum under global conformal transformations, since they are generated by $n = 1, 0, -1$. This property also implies that its vacuum expectation value vanishes.

From the OPE of $T(z)$ and a general primary field $\Phi(w, \overline{w})$ we get the following equation

$$[L_n, \Phi(w, \overline{w})] = h(n+1)w^n \Phi(w, \overline{w}) + w^{n+1} \partial \Phi(w, \overline{w}) \qquad n \geq -1 \tag{2.3.27}$$

and a similar one for the antiholomorphic part.

Denoting now the following state by its conformal weights

$$|h, \overline{h}\rangle \equiv \Phi(0, 0)|0\rangle, \tag{2.3.28}$$

we get the following properties:

$$\begin{aligned} L_0|h, \overline{h}\rangle = h|h, \overline{h}\rangle && \overline{L}_0|h, \overline{h}\rangle = \overline{L}|h, \overline{h}\rangle, \\ L_n|h, \overline{h}\rangle = 0 && \overline{L}_n|h, \overline{h}\rangle = 0 && n > 0. \end{aligned} \tag{2.3.29}$$

These looks a lot like highest weight representations of an algebra, with Cartan subalgebra given by $L_0$ and $C$ the central charge. This is reinforced by the following commutator too

$$[L_0, L_{-m}] = mL_{-m}. \tag{2.3.30}$$

This fact can be demonstrated, the triangular decomposition of this algebra proves the existence of highest weight representations.

To each primary field then corresponds a representation with highest weight $|h\rangle$; descendants of this states are obtained as in the following expression:

$$L_{-k_1} L_{-k_2} ... L_{-k_n}|h\rangle. \tag{2.3.31}$$

Such a state will be an eigenvector of $L_0$ with eigenvalue $h' = h + k_1 + k_2 + ... + k_n \equiv h + N$.

These representations are called *Verma modules*.

## Normal ordering

We want now to introduce a notion of normal ordering in this context. The idea is that we want to consider only the regular part in the product of two local operators inserted at the same point. Involving the OPE, consider the expansion

$$A(z)B(w) = \sum_{n=-\infty}^{N} \frac{[AB]_n(w)}{(z-w)^n}. \tag{2.3.32}$$

Then we can define and denote the normal order product in the following way

$$(AB)(w) = [AB]_0(w). \tag{2.3.33}$$

If we think at the OPE as a Laurent expansion of the first operator in $w$ we have another useful representation of the normal ordered product:

$$(AB)(w) = \frac{1}{2\pi i} \oint_w \frac{dz}{z-w} A(z)B(w). \tag{2.3.34}$$

We now use this formula to write an expression for the modes of the normal ordered product. First of all

$$\begin{aligned} (AB)(w) &= \sum_{n \in \mathbb{Z}} w^{-n-h_A-h_b}(AB)_n, \\ (AB)_n &= \oint_0 \frac{dw}{2\pi i} w^{n+h_A+h_B-1}(AB)(w). \end{aligned} \tag{2.3.35}$$

Then

$$\begin{aligned} (AB)_n &= \oint_0 \frac{dw}{2\pi i} w^{n+h_A+h_B-1} \oint_w \frac{dz}{2\pi i} \frac{A(z)B(w)}{z-w} \\ &= \oint_0 \frac{dw}{2\pi i} w^{n+h_A+h_B-1} \left( \oint_{|z|>|w|} \frac{dz}{2\pi i} \frac{A(z)B(w)}{z-w} - \oint_{|z|<|w|} \frac{dz}{2\pi i} \frac{B(w)A(z)}{z-w} \right). \end{aligned} \tag{2.3.36}$$

Divided as it is in two pieces, we start from the first one. Expanding $A$ and $B$ in their Laurent series we get

$$\oint_{|z|>|w|} \frac{dz}{2\pi i} w^{n+h_A+h_B-1} \frac{A(z)B(w)}{z-w} = \oint_{|z|>|w|} \frac{dz}{2\pi i} \sum_{p\geq 0} \sum_{r,s} z^{-r-h_A-p-1} w^{-s-h_B+p} A_r B_s, \qquad (2.3.37)$$

using $\frac{1}{z-w} = \frac{1}{z(1-w/z)}$ and the geometric series expansion (since $|z| > |w|$).
After that the integral over $z$ giver a delta function setting $r = -h_A - p$, while from the the one over $w$ the only contributing term is the one with $s = p + n + h_A$. This yields the total result of

$$\mathfrak{I}_1 = \sum_{k\leq -h_A} A_k B_{n-k}. \qquad (2.3.38)$$

For the second term $\mathfrak{I}_2$ a similar calculation produces a similar result.
We obtain the following formula

$$(AB)_n = \mathfrak{I}_1 + \mathfrak{I}_2 = \sum_{k\leq -h_A} A_k B_{n-k} + \sum_{k>-h_A} B_{n-k} A_k. \qquad (2.3.39)$$

Now consider equation (2.3.25), one can think at the modes having $m > -h$ as being annihilation operators, the others can instead be thought as creation operators.
In the expression we have just found, we can see that annihilation operators are moved to the right while the creation ones remain on the left. Therefore one can interpret normal ordering as in the usual way with the placement of creation/annihilation operators.

Some technical notes about this. The normal ordered product is both non-commutative and non-associative. Moreover our treatment was for the case of bosonic fields, but a similar one can be applied to fermionic fields.

The normal ordered product comes in handy right away.
In fact one can write the regular part of the OPE as

$$A(z)B(w) = sing + \sum_{k\geq 0} \frac{(z-w)^k}{k!} (\partial^k AB)(w). \qquad (2.3.40)$$

This can be easily verified. If we write the general form for the OPE and apply first the operator $\frac{\partial_z^k}{k!}$ and then the normal ordering, we will obtain exactly the term $[A,B]_k(w)$ in the expansion.

### Solving a theory

Finally after finding all these properties we can use them in order to almost solve our theory. By solving a theory what we mean in this context is the calculation of all the correlators in the theory. The fact that this is possible is a very important consequence of the conformal symmetry which, makes these theories a very controlled and constrained environment in which one can work.

We start by considering correlators in which we have a string of primary fields $X = \Phi_1(w_1)...\Phi_N(w_N)$ and what we will call a descendant field $\Phi^{-n}(w) = L_{-n}\Phi(w)$ (connected to descendant states by its application to the vacuum inserted in $w = 0$):

$$\langle \Phi^{-n} X \rangle = \frac{1}{2\pi i} \oint_w dz (z-w)^{1-n} \langle T(z)\Phi(w)X \rangle$$

$$= -\frac{1}{2\pi i} \oint_{\{w_i\}} dz (z-w)^{1-n} \sum_i \left[ \frac{1}{z-w_i} \partial_{w_i} \langle \Phi(w)X \rangle + \frac{h_i}{(z-w_i)^2} \langle \Phi(w)X \rangle \right] \qquad (2.3.41)$$

$$= \mathcal{L}_{-n} \langle \Phi(w)X \rangle,$$

wherein we defined the differential operator

$$\mathcal{L}_{-n} = \sum_i \left[ \frac{(n-1)h_i}{(w_i - w)^n} - \frac{1}{(w_i - w)^{n-1}} \partial_{w_i} \right]. \tag{2.3.42}$$

Without any difficulty this result is generalizes to

$$\langle \Phi^{(-k_1, \dots, -k_n)}(w) X \rangle = \mathcal{L}_{k_1} \dots \mathcal{L}_{k_n} \langle \Phi(w) X \rangle, \tag{2.3.43}$$

where $\Phi^{(-k_1, \dots, -k_n)}(w)$ is obtained applying various energy-momentum tensor modes in order. Therefore correlators of descendants fields are always reducible to those with only primary fields which will be our main object of study.

So, first of all, let us look at the simplest correlator: the two point function. We already know that, for fields that do not have the same conformal dimensions, it will be null. Let us take some fields which instead have the same dimension

$$\langle \Phi_\alpha(w, \overline{w}) \Phi_\beta(z, \overline{z}) \rangle = \frac{C_{\alpha\beta}}{(w - z)^{2h}(\overline{w} - \overline{z})^{2\overline{h}}}. \tag{2.3.44}$$

Now $C_{\alpha\beta}$ is symmetric therefore we can always combine and rescale the fields so that $C_{\alpha\beta} = \delta_{\alpha\beta}$. The orthogonality of the fields also means that their *conformal families* (sets of a primary field and all its descendants) are orthogonal by formula (2.3.43).

To solve all the other correlators we actually just need two other informations.
First of all in complete generality the OPE takes the following formula

$$\Phi_1(z, \overline{z}) \Phi_2(0, 0) = \sum_p \sum_{[k, \overline{k}]} C_{12}^{p[k, \overline{k}]} z^{h_p - h_1 - h_2 + K} \overline{z}^{\overline{h}_p - \overline{h}_1 - \overline{h}_2 + \overline{K}} \Phi_p^{[k, \overline{k}]}(0, 0), \tag{2.3.45}$$

where $K = \sum_i k_i$, the expression $[k]$ means a set of $k_i$ and the same for $\overline{k}$. The fact that the OPE of quasi-primary and primary fields only contains such fields and their descendants can be demonstrated but it is quite non trivial.
Then let us consider the other fully determined correlator of any conformal field theory: the three point function.

$$\langle \Phi_r | \Phi_1(z, \overline{z}) | \Phi_2 \rangle = \lim_{w, \overline{w} \to \infty} w^{2h_r} \overline{w}^{2\overline{h}_r} \langle \Phi_r(w, \overline{w}) \Phi_1(z, \overline{z}) \Phi_2(0, 0) \rangle = \frac{C_{r12}}{z^{h_1 + h_2 - h_r} \overline{z}^{\overline{h}_1 + \overline{h}_2 - \overline{h}_r}}. \tag{2.3.46}$$

Using the OPE one finds that

$$C_{12}^{p[0,0]} \equiv C_{12}^p = C_{p12}, \tag{2.3.47}$$

where the last one is the one we found in expression (2.3.46). Moreover since the correlators of descendants are built on the correlators of primaries, as seen in (2.3.43), one can write

$$C_{12}^{p[k, \overline{k}]} = C_{12}^p \beta_{12}^{p[k]} \overline{\beta}_{12}^{p[\overline{k}]} \tag{2.3.48}$$

with $\beta_{12}^{p[0]} = \overline{\beta}_{12}^{p[0]} = 1$.
The important property of these $\beta$ coefficients is that they are completely determined by the Virasoro algebra. We do not calculate them here or show their formula, but it is enough to know that such things can be found.

From this analysis a theory then is specified solely by the following parameters: the central charges, the conformal dimensions of the primary fields and the three point function coefficient $C_{pmn}$. Any correlator of such a theory can be calculated by successive reduction of the product of primary fields.

### Bootstrap

There is actually the hope that we can say more about the parameters that we are left with from the analysis in the previous section, this approach is called *bootstrap*.

In order to do this we are going to need the four point function

$$G(\mathbf{z}, \overline{\mathbf{z}}) = \langle \Phi_i(z_1, \overline{z}_1) \Phi_j(z_2, \overline{z}_2) \Phi_l(z_3, \overline{z}_3) \Phi_m(z_4, \overline{z}_4) \rangle. \tag{2.3.49}$$

We know that such function has a dependence on the positions of the insertions determined by conformal invariance and a "free" dependence on the crossing ratios (anharmonic ratios). We are only interested in the second one and that is the part we will consider. Let us define:

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}} \qquad\qquad \overline{x} = \frac{\overline{z}_{12}\overline{z}_{34}}{\overline{z}_{13}\overline{z}_{24}}, \tag{2.3.50}$$

where $z_{ij} = z_i - z_j$.

In order to solve the correlator we said we needed to use the OPE, so first we contract $\Phi_i$ with $\Phi_j$ and then $\Phi_l$ with $\Phi_m$. We are left with a two point function of which we know the exact formula. Explicitly the result is:

$$G(\mathbf{z}, \overline{\mathbf{z}}) = \sum_p C_{ij}^p C_{lm}^p \mathcal{F}_{ij}^{lm}(p|x) \overline{\mathcal{F}}_{ij}^{lm}(p|\overline{x}). \tag{2.3.51}$$

$\mathcal{F}_{ij}^{lm}(p|x)$ and its antiholomorphic partner are called *conformal blocks*, they have quite complicated formulas but given the central charges and the conformal dimensions they are fully determined.

Notice now that one can solve the same correlator in a different way: first contracting $\Phi_j$ with $\Phi_l$ and then $\Phi_i$ with $\Phi_m$. Effectively this means exchanging $\Phi_j$ and $\Phi_m$ which at the level of crossing ratios is achieved by $x \to 1 - x$.
The expression one finds after an explicit calculation this time is:

$$G(\mathbf{z}, \overline{\mathbf{z}}) = \sum_p C_{im}^p C_{jl}^p \mathcal{F}_{im}^{jl}(p|1-x) \overline{\mathcal{F}}_{im}^{jl}(p|1-\overline{x}). \tag{2.3.52}$$

One can depict the fact that these two expressions are the same using the figure below.
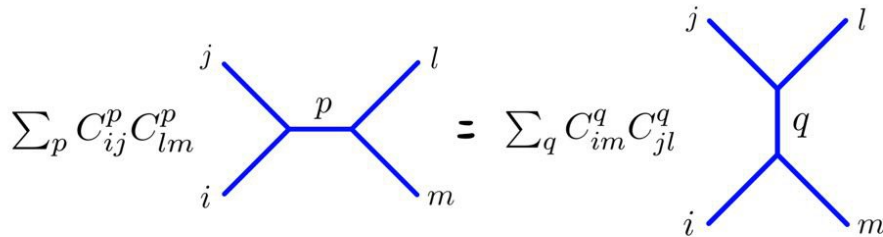


Fig.2.3: On the left we have the first expression, on the right the second

This is the reason why these are called crossing symmetry conditions.

Actually we have a last way to evaluate this function which is contracting first $\Phi_j$ with $\Phi_m$. One then finds:

$$G(\mathbf{z}, \overline{\mathbf{z}}) = \sum_p C_{il}^p C_{jm}^p \mathcal{F}_{il}^{jm}\left(p|\frac{1}{x}\right) \overline{\mathcal{F}}_{il}^{jm}\left(p|\frac{1}{\overline{x}}\right). \tag{2.3.53}$$

The hope for this kind of analysis would be to determine the coefficients $C_{ij}^p$ and the conformal dimensions of the theory thought these constraints even though there is no proof that in general this happens.

Let us end this section by saying that there are cases in which we can say a little bit more. This is the case of rational conformal field theories (RCFT) which we will study in section (2.4).

In that case there are only a finite number of conformal families and the conditions can be written as matrix equalities which are a little bit easier to solve.

### Operators and states

As the last part of our discussion about the operator formalism in conformal field theories, let us talk briefly about another one of their important aspects.

In general, in a quantum field theory there is on one hand the space of states, while on the other, a set of local operators. What happens in our case is that there is a simple useful isomorphism between these two.

To explain how this works let us look at the following consideration. First of all take the infinite cylinder, in a path integral formalism one has to specify the initial state of our theory at $t \to -\infty$. What one is able to demonstrate is that the path integral is the same as the one on the plane with a particular local operator inserted at $z = 0$. This equality is extremely evident in our previous construction of the Hilbert space.

One of the interesting corollaries one can get from this observation is the justification of the OPE.

First of all consider two operators inserted in a correlator. With a conformal transformation we can always bring one at the origin and the other one at a certain point $z$ with $|z| < 1$ (as long as one does not have any operator between them):

$$\mathcal{A}_i(z)\mathcal{A}_j(0). \tag{2.3.54}$$

It is possible then to divide the path integral in two parts, one over the interior of the unit disk and one over the exterior.
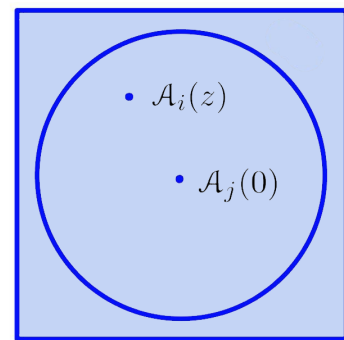


Fig.2.4: Insertion of two local operators

Computing first the one on the interior we get a path integral on the exterior of the disk with a certain kind of boundary condition/initial state at the unit disk. This is compatible with a path integral on the full plane with a specific initial condition and therefore also with a path integral with a particular operator inserted at the center by the operator-state correspondence which we previously described.
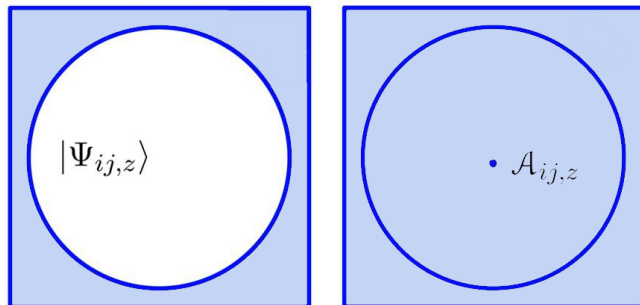


Fig.2.5 Path integral with boundary condition on the left, with operator insertion on the right

This discussion clearly justifies the OPE.

Another way to see this is the following: if one does not have any operator between two other operators a conformal transformation can bring these two operators as close as one wants, therefore it kind of makes sense to have the product of these two operators represented by a single operator.

Also, from this argument, it looks as if the same theory constructed on spaces of the same homotopy type gives the same results. For example a theory constructed on a contractible space would be the same as one on a point.

## 2.4 Models

In this section let us introduce some theorise and some methods to construct them. They will be the main objects on which we will work when looking for the defects.

**Free fermion**

First of all the action of the free Majorana fermion with metric $h_{\alpha\beta} = diag(+1, -1)$ is

$$S = \frac{1}{4\pi k} \int dx^0 dx^1 \sqrt{|h|} (-i) \overline{\Psi} \gamma^\alpha \partial_\alpha \Psi, \tag{2.4.1}$$

where $k$ is a normalization constant, $\Psi$ is a two dimensional vector, $\overline{\Psi} = \Psi^\dagger \gamma^0$ and

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.4.2}$$

With this choice of $\gamma$ matrices the Majorana condition becomes that the components of the spinor are real.

Now we perform a Wick rotation: $x_0 \to x_0$, $x_1 \to ix_1$, which entails $\partial_1 \to -i\partial_1$. What we find with that is

$$\gamma^0 \gamma^\alpha \partial_\alpha = \begin{pmatrix} \partial_0 + i\partial_1 & 0 \\ 0 & \partial_0 - i\partial_1 \end{pmatrix} = 2 \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix}. \tag{2.4.3}$$

After that we rewrite everything in complex variables and extend $\bar{z}$ to the whole plane. Setting the notation for the components of the spinor as $\Psi^T = \Psi^\dagger = (\psi(z,\bar{z}), \overline{\psi}(z,\bar{z}))$ one obtains the following form for the action:

$$S = \frac{1}{4\pi k} \int dz d\bar{z} \sqrt{|g|} 2\Psi^\dagger \begin{pmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{pmatrix} \Psi$$
$$= \frac{1}{4\pi k} \int dz d\bar{z} (\psi \overline{\partial} \psi + \overline{\psi} \partial \overline{\psi}). \tag{2.4.4}$$

Note that, as in equation (2.2.3), the components of the metric $g$ are obtained through the change of coordinate to the complex ones:

$$g_{ab} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad\qquad g^{ab} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{2.4.5}$$

The equation of motions for the theory are obtained varying the action with respect to $\psi$ and $\overline{\psi}$. First of all for $\psi$ we get:

$$0 = \delta_\psi S = \frac{1}{4\pi k} \int dz d\bar{z} [\delta\psi \overline{\partial} \psi + \psi \overline{\partial}(\delta\psi)]$$
$$= \frac{1}{2\pi k} \int dz d\bar{z} \delta\psi \overline{\partial} \psi. \tag{2.4.6}$$

The variation with respect to $\overline{\psi}$ yields a very similar result. In total then

$$\partial \overline{\psi} = \overline{\partial} \psi = 0, \tag{2.4.7}$$

which means that these fields are chiral $\psi = \psi(z)$ and antichiral $\overline{\psi} = \overline{\psi}(\bar{z})$.

Note now that the action is conformally invariant if $\psi$ is a primary field with conformal dimension $(h, \overline{h}) = (\frac{1}{2}, 0)$ and $\overline{\psi}$ with $(h, \overline{h}) = (0, \frac{1}{2})$. Moreover the theory is free therefore the engineering dimension is the same as the actual one in the quantum theory.

There are two possibilities for the behaviour under rotations by $2\pi$:

$$\psi(e^{2\pi i} z) = +\psi(z) \qquad \text{Neveu-Schwarz sector (NS)},$$
$$\psi(e^{2\pi i} z) = -\psi(z) \qquad \text{Ramond sector (R)}. \tag{2.4.8}$$

We are focusing on the chiral sector since the antichiral one has exactly the same properties. Then we Laurent expand the field:

$$\Psi(z) = \sum_r \psi_r z^{-r-\frac{1}{2}} \qquad \psi_r = \oint \frac{dz}{2\pi i} \psi(z) z^{r-\frac{1}{2}},$$
$$r \in \mathbb{Z} + \frac{1}{2} \text{ (NS)} \qquad r \in \mathbb{Z} \text{ (R)}. \tag{2.4.9}$$

Notice that the expansion is coherent with the field being single or multi valued on the plane.

We start by analyzing the Neveu-Schwarz sector.
One can verify that the propagators are:

$$\langle\psi(z)\psi(w)\rangle = \frac{k}{z-w},$$
$$\langle\overline{\psi}(\overline{z})\overline{\psi}(\overline{w})\rangle = \frac{k}{\overline{z}-\overline{w}}, \tag{2.4.10}$$
$$\langle\psi(z)\overline{\psi}(\overline{w})\rangle = 0.$$

With this we can calculate the anticommutator between the modes:

$$\begin{aligned}\{\psi_r,\psi_s\} &= \oint_0 \frac{dw}{2\pi i}w^{s-\frac{1}{2}}\oint_w \frac{dz}{2\pi i}z^{r-\frac{1}{2}}R(\psi(z)\psi(w))\\ &= k\oint_0 \frac{dw}{2\pi i}w^{r+s-1}\\ &= k\delta_{r+s,0}\end{aligned} \tag{2.4.11}$$

The first equality comes from formula (2.3.18), wherein for the fermions we have the anticommutator instead of the commutator. Then we used the propagator in (2.4.10).

The energy-momentum tensor can be derived using the following generic well known equation:

$$T_{\mu\nu} = 8\pi kg\left(-\eta_{\mu\nu}\mathcal{L} + \sum_i \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi_i)}\partial_\nu\phi_i\right). \tag{2.4.12}$$

We get:
$$T_{zz} = g\psi\partial\psi \qquad T_{z\overline{z}} = -g\overline{\psi}\partial\overline{\psi} \qquad T_{\overline{z}z} = -g\psi\overline{\partial}\psi \qquad T_{\overline{z}\overline{z}} = g\overline{\psi}\overline{\partial}\overline{\psi}. \tag{2.4.13}$$

Even though it does not look symmetric, the terms off of the diagonal vanish using the equations of motion. We know from equation (2.2.26) that the diagonal terms are one chiral and the other antichiral. We will focus on the chiral one since for the other the behaviour is identical.

$$T(z) = g(\psi\partial\psi)(z). \tag{2.4.14}$$

Let us now write the Laurent modes:

$$L_m = g\sum_{s>-\frac{3}{2}}\psi_{m-s}\psi_s\left(s+\frac{1}{2}\right) - g\sum_{s\leq-\frac{3}{2}}\psi_s\psi_{m-s}\left(s+\frac{1}{2}\right). \tag{2.4.15}$$

Using the anticommutator between the fermion modes we find

$$[L_m,\psi_r] = gk(-m-2r)\psi_{m+r}. \tag{2.4.16}$$

Then choosing $gk=\frac{1}{2}$ we get
$$[L_m,\psi_r] = (-\frac{m}{2}-r)\psi_{m+r}, \tag{2.4.17}$$

which can be easily demonstrated to be the commutation relation for a primary field of conformal dimension $h=\frac{1}{2}$, as expected.
From now on we will choose the usual normalization $k=1$.

Finally let us look at the central charge of the theory. It is possible to show that:

$$\langle 0|L_2 L_{-2}|0\rangle = \langle 0|[L_2,L_{-2}]|0\rangle = \frac{c}{2}. \tag{2.4.18}$$

Then explicitly:

$$L_{-2}|0\rangle = \frac{1}{2}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0\rangle \qquad \langle 0|L_2 = \frac{1}{2}\langle 0|\psi_{\frac{3}{2}}\psi_{\frac{1}{2}}. \tag{2.4.19}$$

Finally:

$$
\begin{aligned}
\frac{c}{2} &= \frac{1}{4}\langle 0|\psi_{\frac{3}{2}}\psi_{\frac{1}{2}}\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}|0\rangle = \frac{1}{4}\langle 0|\psi_{\frac{1}{2}}\{\psi_{\frac{3}{2}}\psi_{-\frac{3}{2}}\}\psi_{-\frac{1}{2}}|0\rangle \\
&= \frac{1}{4}\langle 0|\psi_{\frac{1}{2}}\psi_{-\frac{1}{2}}|0\rangle = \frac{1}{4}\langle 0|\{\psi_{\frac{1}{2}}\psi_{-\frac{1}{2}}\}|0\rangle = \frac{1}{4}.
\end{aligned}
\tag{2.4.20}
$$

Therefore the free fermion theory has central charge $c = \frac{1}{2}$.


Let us now consider the Ramond sector instead.
It is possible to show, using for example the mode decomposition, that this time the propagators are:

$$
\begin{aligned}
\langle \psi(z)\psi(w)\rangle &= \frac{1}{2}\frac{\sqrt{z/w} + \sqrt{w/z}}{z - w}, \\
\langle \overline{\psi}(\overline{z})\overline{\psi}(\overline{w})\rangle &= \frac{1}{2}\frac{\sqrt{\overline{z}/\overline{w}} + \sqrt{\overline{w}/\overline{z}}}{\overline{z} - \overline{w}}, \\
\langle \psi(z)\overline{\psi}(\overline{w})\rangle &= 0,
\end{aligned}
\tag{2.4.21}
$$

where we have already fixed $k = 1$.
This does not change the discussion on the anticommutator of the modes which in fact yields the same result.
The energy-momentum tensor has the same expression but we encounter some difficulties when looking at the modes. The main problem is given by $\psi_0$ which has the following property:

$$
\{\psi_0, \psi_0\} = 1 \qquad \psi_0\psi_0 = \frac{1}{2}
\tag{2.4.22}
$$

This fact makes the normal ordering prescription we used for $L_0$ inexact, since there is not a "right" choice when ordering the pair $\psi_0\psi_0$. What we can instead say is that the expression will be of the form:

$$
L_0 = \frac{1}{2}\sum_{s>-\frac{3}{2}}\psi_{-s}\psi_s\left(s + \frac{1}{2}\right) - \frac{1}{2}\sum_{s\le-\frac{3}{2}}\psi_s\psi_{-s}\left(s + \frac{1}{2}\right) + a,
\tag{2.4.23}
$$

where we added a new constant term $a$ which takes into account a possible reorganization of the terms. Such constant term can be calculated using the commutation relations between the $L_n$s, but in this case there is a faster method:

$$
\langle T(z)\rangle = \lim_{w\to z}\left[-\frac{1}{2}\partial_w\left(\frac{\sqrt{z/w} + \sqrt{w/z}}{z - w}\right) + \frac{1}{2(z - w)^2}\right] = \frac{1}{16z^2}.
\tag{2.4.24}
$$

The first term corresponds to the correlator $\langle \psi(z)\partial_w\psi(w)\rangle$, then, to take the normal ordering of this composition of fields, we subtract its singular part and then send $w \to z$, resulting in expression (2.4.24). We have just obtained is the vacuum energy, which tells us that the generator $L_0$ can be written in the following way:

$$
L_0 = \frac{1}{2}\sum_{s>0}\psi_{-s}\psi_s\left(s + \frac{1}{2}\right) - \frac{1}{2}\sum_{s<0}\psi_s\psi_{-s}\left(s + \frac{1}{2}\right) + \frac{1}{16}.
\tag{2.4.25}
$$

Notice now that the ground state of the Ramond sector has conformal dimension $\frac{1}{16}$ since

$$
L_0|0\rangle_R = \frac{1}{16}|0\rangle_R.
\tag{2.4.26}
$$

Moreover the ground state is degenerate. Since $\psi_0$ commutes with the whole Virasoro algebra, its application to the ground state gives a representation identical to the first one except for the different fermion number.

**RCFT**

When we talked about the Hilbert space we did not consider two possibilities: we can have null norm states and negative norm states.

First of all let us talk about null vectors. When considering all the states in (2.3.31) one finds that sometimes the representation of the Virasoro algebra is reducible; this happens in particular when a state exists such that $L_n|\chi\rangle = 0$ for $n > 0$. That state will be called null state and will be the highest weight of a full representation of the algebra.

The state $|\chi\rangle$ and the all its descendants can be demonstrated to be also perpendicular to the whole Verma module:

$$\langle \chi|L_{-k_1}...L_{-k_n}|h\rangle = \langle h|L_{k_n}...L_{k_1}|\chi\rangle^* = 0 \tag{2.4.27}$$

and in particular one has that $\langle \chi|\chi\rangle = 0$. For $|\chi\rangle$s descendants one just has to use the commutation relations between $L_n$.

From these representations one can obtain irreducible ones by quotienting out the null submodes, identifying states that differ by states of zero norm. For these reasons it is important to study null vectors.

To do that we start by considering a general vector; it can always be written in a certain basis as the following: $|v\rangle = \sum_a \lambda_a|a\rangle$. Then its norm is:

$$||v||^2 = \sum_{a,b} \lambda_a \langle a|b\rangle \lambda_b \equiv \sum_{a,b} \lambda_a M_{ab} \lambda_b. \tag{2.4.28}$$

This expression can be zero if and only if $\lambda_b$ is an eigenvector of $M$ with eigenvalue zero and this only happens when $det M = 0$.

In our case this process uses the so-called Kač-determinant at level N, that is the determinant of the matrix $M_N(h,c)$ composed by the following entries:

$$\langle h| \prod_i L_{k_i} \prod_j L_{-m_j} |h\rangle \qquad\qquad \sum_i k_i = \sum_j m_j = N. \tag{2.4.29}$$

V. Kač found and proved the following formula:

$$det M_N(c,h) = \alpha_N \prod_{0 < p,q \leq N} (h - h_{p,q}(c))^{P(N-pq)},$$

$$h_{p,q} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)} \qquad m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}, \tag{2.4.30}$$

where $P(n)$ is the number of partitions of n and $p, q$ are integers.

For the square root one usually chooses the branch $m \in (0, \infty)$ even though the determinant is independent from the choice thanks to the the symmetry of $h_{p,q}$: $\{p \to m - p, q \to m + 1 - q\}$.

Now let us look at negative norm states. Clearly we do not want them in any field theory, we call this property unitarity. The first consideration is that since

$$\langle h|L_1 L_{-1}|h\rangle = 2\langle h|L_0|h\rangle = 2h \tag{2.4.31}$$

we must have $h \geq 0$.

After that the considerations becomes more complicated, so we will just summarize the results:

- For $c > 1$ and $h \geq 0$ there are no null states and no negative norm states.
- For $c = 0$ one finds $det M_N = 0$ for $h = \frac{n^2}{4}$, $n \in \mathbb{Z}$.
- At last in the region $c < 1$ and $h \geq 0$ for sure all points not laying on a the curve $h_{p,q}(c)$ are not unitary. A more careful analysis shows that negative states are absent only on certain intersection points of such curves.

In the last case we considered, one can parameterize the set of unitary representations in the following way:

$$c = 1 - \frac{6}{m(m+1)} \qquad m = 3, 4, \dots$$

$$h_{p,q}(m) = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}$$

$$(2.4.32)$$

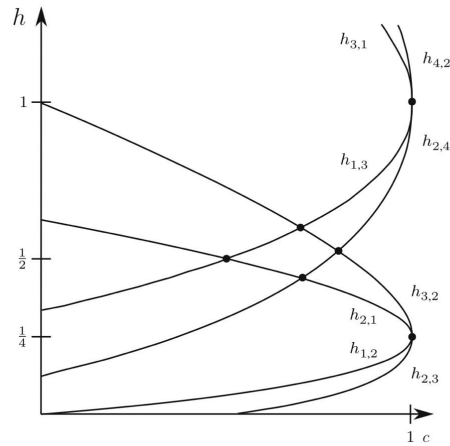with $1 \leq p \leq m - 1$ and $1 \leq q \leq m$ integers.



Fig.2.6: Curve of vanishing Kač determinant (image from [2])

After this considerations, because of these constraints, only a discrete sets of $c$ survives with a finite number of highest weight representations. These theories are called Rational CFT because they admit only rational values of the central charges. Another way to call these theories, especially in statistical mechanics is minimal models.

### Kač-Moody algebras

We have seen what concerns the Virasoro algebra in the previous section, but it is not always true that such an algebra appears alone. There are a lot of cases in which the conformal symmetry is encapsulated in a bigger symmetry group or in general we have that the conformal algebra is encapsulated in a bigger algebra.

This also usually means that the representations of the Virasoro algebra are contained in the representations of these bigger algebras too. As we will see, this is interesting because it permits to analyze, using this bigger algebras, theories that have an infinite number of representations of our original algebra, contrarily to the RCFT which have only a finite number of possible representations.

Showing where these algebras may come from will help us understand why they are very relevant, therefore let us start with that.

First of all consider a finite dimensional simple Lie algebra $\bar{\mathfrak{g}} = \{\bar{j}^a | a = 1, 2, \dots, d\}$ on $\mathbb{C}$, with it we want to construct a Loop algebra. A loop algebra is an infinite dimensional algebra composed by the maps from the circle $S_1$ to $\bar{\mathfrak{g}}$ ($P(\theta) \in \bar{\mathfrak{g}}$, $\theta \in S_1$). If these maps are taken single valued the algebra will be called untwisted ($P(0) = P(2\pi)$), if instead they are not the algebra will be called twisted ($P(2\pi) = T[P(0)]$, where $T$ is a map $\bar{\mathfrak{g}} \to \bar{\mathfrak{g}}$).

Focusing on the untwisted case, Fourier analysis tells us that a basis for this new algebra is

$$\hat{\mathfrak{g}} = \{j_n^a = \bar{j}^a \otimes z^n = \bar{j}^a \otimes e^{2\pi i n t} | a = 1, 2, \dots, d; n \in \mathbb{Z}\}. \qquad (2.4.33)$$

And knowing

$$[\bar{j}^a, \bar{j}^b] = \sum_{c=1}^{d} f^{ab}_{\ c} \bar{j}^c. \qquad (2.4.34)$$

The commutation relations for the new algebra will be:

$$[j_m^a, j_n^b] = \sum_{c=1}^{d} f^{ab}_{\ c} j_{m+n}^c. \qquad (2.4.35)$$

This is not yet an affine algebra (or Kač-Moody algebra), as we have seen for the Virasoro algebra central extensions are usually needed in quantum theories.

One can demonstrate then that the only non trivial central extension of this algebra by a central element $K$ will be

$$[j_m^a, j_n^b] = \sum_{c=1}^{d} f^{ab}{}_c j_{m+n}^c + m\delta_{m+n,0}\delta^{ab}K \tag{2.4.36}$$

supposing that the Killing form of $\overline{\mathfrak{g}}$ is $\delta^{ab}$ which in many cases is true.

A last technicality is actually needed to conclude this treatment, we just need to know that, in order to make the Killing form of the resulting algebra non-degenerate, we must add another element: the derivation D.

In total then one find that we can describe our algebra as:

$$\mathfrak{g} = \hat{\mathfrak{g}} \oplus \mathbb{C}K \oplus \mathbb{C}D, \tag{2.4.37}$$

with commutation rules:

$$
\begin{aligned}
[j_m^a, j_n^b] &= \sum_{c=1}^{d} f^{ab}{}_c j_{m+n}^c + m\delta_{m+n,0}\delta^{ab}K, \\
[D, j_m^a] &= m j_m^a, \\
[K, j_m^a] &= 0, \\
[D, K] &= 0.
\end{aligned}
\tag{2.4.38}
$$

Clearly this is an algebra of Laurent modes, we can then write them starting from a certain field:

$$j^a = \sum_{n\in\mathbb{Z}} j_n^a z^{-n-1} \qquad\qquad j_n^a = \oint \frac{dz}{2\pi i} z^n j^a(z). \tag{2.4.39}$$

Then from the commutation rules of the modes one can extract the OPE, which is:

$$j^a(z)j^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \sum_c \frac{if^{abc}}{z-w}j^c(w), \tag{2.4.40}$$

where k is the eigenvalue of $K$ in the representation.

An algebra like this is also naturally associate a certain energy-momentum tensor. This method is called Sugawara construction.

First of all our ansatz will be:

$$
\begin{aligned}
T(z) &= \gamma \sum_{a=1}^{d} (j^a j^a)(z), \\
L_m &= \gamma \sum_{a=1}^{d} \Big( \sum_{l\leq -1} j_l^a j_{m-l}^a \sum_{l>-1} j_{m-l}^a j_l^a \Big).
\end{aligned}
\tag{2.4.41}
$$

As previously mentioned using the OPE between the energy-momentum tensor and a primary field $\Phi(z)$ one gets

$$[L_m, \Phi_n] = ((h-1)m - n)\Phi_{m+n}. \tag{2.4.42}$$

It turns out that using the structure constant properties in an explicit calculation one arrives at

$$[L_m, j_n^a] = -2\gamma n(k + C_{\mathfrak{g}})j_{m+n}^a, \tag{2.4.43}$$

where the dual Coxeter number can be defined by: $\sum_{b,c} f^{bac}f^{bcd} \equiv= -2C_{\mathfrak{g}}\delta^{ad}$.

If we fix $\gamma = (2(k + C_{\mathfrak{g}}))^{-1}$ we obtain that $j^a(z)$ will be a primary field of conformal dimension one.

Therefore in total:

$$T(z) = \frac{1}{2(k + C_{\mathfrak{g}})} \sum_{a=1}^{d} (j^a j^a)(z). \tag{2.4.44}$$

This theory also has a determined central charge, which can be calculate it explicitly:

$$\frac{c}{2} = \langle 0|L_2 L_{-2}|0\rangle = \gamma^2 \sum_{a,b} \langle 0|j_1^b j_1^b j_{-1}^a j_{-1}^a|0\rangle = \frac{kd}{2(k + C_{\mathfrak{g}})}. \tag{2.4.45}$$

This means that the central charge of the conformal theory associated to the affine algebra $\mathfrak{g}_k$ (it is costume to write k as a subscript) through the Sugawara energy-momentum tensor is

| Algebra | $C_{\mathfrak{g}}$ |
|---------|--------------------|
| $\mathfrak{su}(N)$ | $N$ |
| $\mathfrak{so}(N)$ | $N - 2$ |

$$c = \frac{kd}{(k + C_{\mathfrak{g}})}. \tag{2.4.46}$$

Dual Coxeter number relevant examples

## Coset construction

We will now introduce a method to produce new theories from known ones. One can also use this procedure to decompose a theory in different, smaller parts.

Let us start with an affine algebra $\mathfrak{g}_{k_g}$ originating from the simple Lie algebra $\overline{\mathfrak{g}}$. Suppose that $\overline{\mathfrak{g}}$ contains a subalgebra $\overline{\mathfrak{h}} \subset \overline{\mathfrak{g}}$; we will call the affine algebra deriving from such sub algebra $\mathfrak{h}_{k_h}$.
From the Sugawara construction we get:

$$T_{\mathfrak{g}}(z) = \frac{1}{2(k_{\mathfrak{g}} + C_{\mathfrak{g}})} \sum_{a=1}^{dim\mathfrak{g}} (j_{\mathfrak{g}}^a j_{\mathfrak{g}}^a)(z),$$

$$T_{\mathfrak{h}}(z) = \frac{1}{2(k_{\mathfrak{h}} + C_{\mathfrak{h}})} \sum_{a=1}^{dim\mathfrak{h}} (j_{\mathfrak{h}}^a j_{\mathfrak{h}}^a)(z). \tag{2.4.47}$$

Observe now that $j_{\mathfrak{h}}^a$ is a primary field of dimension $h = 1$ both with respect to $T_{\mathfrak{g}}$ and $T_{\mathfrak{h}}$ and that $T_{\mathfrak{h}}$ is constructed only using the $j_{\mathfrak{h}}^a$s, then:

$$(T_{\mathfrak{g}} - T_{\mathfrak{h}})(z) j_{\mathfrak{h}}^a = reg.$$
$$(T_{\mathfrak{g}} - T_{\mathfrak{h}})(z) T_{\mathfrak{h}}(w) = reg. \tag{2.4.48}$$

After that, we split

$$T_{\mathfrak{g}} = T_{\mathfrak{g}/\mathfrak{h}} + T_{\mathfrak{h}} \qquad\qquad T_{\mathfrak{g}/\mathfrak{h}} = T_{\mathfrak{g}} - T_{\mathfrak{h}} \tag{2.4.49}$$

and use $T_{\mathfrak{g}/\mathfrak{h}}$ as the energy-momentum tensor of a new theory: $\mathfrak{g}_{k_g}/\mathfrak{h}_{k_h}$.

In order to understand how this new theory will be like we look at the singular part of the OPE between two energy-momentum tensors:

$$T_{\mathfrak{g}/\mathfrak{h}} T_{\mathfrak{g}/\mathfrak{h}} \sim T_{\mathfrak{g}} T_{\mathfrak{g}} - T_{\mathfrak{h}} T_{\mathfrak{h}}. \tag{2.4.50}$$

From this we immediately understand that

$$c_{\mathfrak{g}/\mathfrak{h}} = c_{\mathfrak{g}} - c_{\mathfrak{h}} = \frac{k_{\mathfrak{g}} dim\mathfrak{g}}{k_{\mathfrak{g}} + C_{\mathfrak{g}}} - \frac{k_{\mathfrak{h}} dim\mathfrak{h}}{k_{\mathfrak{h}} + C_{\mathfrak{h}}}. \tag{2.4.51}$$

This formula will be quite important since, in many cases, it identifies in a very clear way what theory we are obtaining from this process.

Finally we specify the field content of what we will call the coset theory: all the fields in $\mathfrak{g}_{k_g}$ which have a non singular OPE with the operators in $\mathfrak{h}_{k_h}$. From a modes point of view we are asking them to commute with the algebra $\mathfrak{h}_{k_h}$.

An important case of coset constructions is the diagonal cosets: $(\mathfrak{g}_{k_1} \times \mathfrak{g}_{k_2})/\mathfrak{g}_{k_1+k_2}$. As a mater of facts in the tensor product algebra one always finds the following operators $j_n^a = j_{1,n}^a + j_{2,n}^a$ where the $j_{1,n}^a$ generate $\mathfrak{g}_{k_1}$ and the $j_{1,n}^a$ $\mathfrak{g}_{k_2}$. Then one easily shows that

$$[j_m^a, j_n^b] = \sum_{c=1}^d f^{ab}{}_c j_{m+n}^c + m\delta_{m+n,0}\delta^{ab}(k_1 + k_2). \tag{2.4.52}$$

Therefore we have found the $\mathfrak{g}_{k_1+k_2}$ subalgebra justifying the diagonal coset construct.

Another important property is that the coset construction determines the decomposition of the highest weight representations of $\mathfrak{g}$ into highest weight representations of $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$. The fact that this is true is highly non trivial and interesting. These rules are called branching rules and usually, given a subalgebra $\mathfrak{h}$, they are very hard to find.
In general one can write:

$$R_\lambda^{\mathfrak{g}} = \bigoplus_{R_\lambda^{\mathfrak{h}}} R_\lambda^{\mathfrak{h}} \otimes R_\lambda^{\mathfrak{g}/\mathfrak{h}}, \tag{2.4.53}$$

where $R_\lambda^{\mathfrak{g}}$ corresponds to the representation of $\mathfrak{g}$ with highest weight $\lambda$.
We will explain better how this practically works when we will introduce the characters, which is in section (2.5).

### Affine algebra example

We now finally apply everything we learned in this section to the theory we will work with the most. It is the case of the affine algebra constructed from the $su(2)$ algebra. To set the convention let us write explicitly the commutation relations:

$$[j_m^a, j_n^b] = i\sqrt{2} \sum_{c=1}^d \epsilon^{abc} j_{m+n}^c + m\delta_{m+n,0}\delta^{ab}k, \tag{2.4.54}$$

where $\epsilon^{abc}$ is the usual totally antisymmetric tensor. We define then the raising and lowering operators and write down the resulting commutation rules:

$$\hat{j}_m^3 = \frac{1}{\sqrt{2}}j_m^3 \qquad\qquad \hat{j}_m^\pm = \frac{1}{\sqrt{2}}(j_m^1 \pm ij_m^2),$$

$$[\hat{j}_m^3, \hat{j}_n^3] = \frac{mk}{2}\delta_{m+n,0} \qquad [\hat{j}_m^3, \hat{j}_n^\pm] = \pm\hat{j}_{m+n}^\pm \qquad [\hat{j}_m^+, \hat{j}_n^-] = km\delta_{m+n,0} + 2\hat{j}_{m+n}^3. \tag{2.4.55}$$

We can now write the following conditions for a highest weight state:

$$\begin{aligned}
\hat{j}_n^3|h,q\rangle &= \hat{j}_n^\pm|h,q\rangle = 0 \qquad n > 0, \\
\hat{j}_0^3|h,q\rangle &= \frac{q}{2}|h,q\rangle, \\
\hat{j}_0^+|h,q\rangle &= 0.
\end{aligned} \tag{2.4.56}$$

The triangular decomposition is obvious from these conditions: any operator with $n > 0$ is an annihilation operator and any operator with $n < 0$ will be a creation operator, while for the zero mode we keep the usual decomposition.

We can analyze these representations further. First of all one can immediately see that the zero modes form a $\mathfrak{su}(2)$ subalgebra of $\mathfrak{su}(2)_k$. If we ask for the representation of the zero modes to be a finite dimensional spin $\frac{l}{2}$ representation we have the usual condition: $q = l \in \mathbb{Z}$.
Apart from the highest weight state, we denote the other states of this sub-representation as

$$|h, q_\alpha\rangle \equiv (\hat{j}_0^-)^\alpha|h,q\rangle, \tag{2.4.57}$$

were $q_\alpha = l - 2\alpha$.

One also finds that the zero mode one is not the only $\mathfrak{su}(2)$ subalgebra, actually we have an infinite number of them:

$$
\begin{aligned}
\tilde{j}^+_{(n)} &= \hat{j}^+_{-n}, \\
\tilde{j}^-_{(n)} &= \hat{j}^-_n, \\
\tilde{j}^3_{(n)} &= \hat{j}^3_0 - \frac{nk}{2}.
\end{aligned}
\tag{2.4.58}
$$

If we ask for all of these to be spin representations, identically to the first one, we get the condition $k \in \mathbb{Z}$.

There is actually one more constraint on these theories, consider the norm of the state $\tilde{j}^+_{(1)}|h,l\rangle$:

$$
\begin{aligned}
\langle h,l|\tilde{j}^-_{(1)}\tilde{j}^+_{(1)}|h,l\rangle &= \langle h,l|[\tilde{j}^-_{(1)},\tilde{j}^+_{(1)}]|h,l\rangle \\
&= \langle h,l| -2\tilde{j}^3_{(1)}|h,l\rangle \\
&= -2\langle h,l|\hat{j}^3_0 - \frac{k}{2}|h,l\rangle \\
&= -l + k.
\end{aligned}
\tag{2.4.59}
$$

Since this value must be non negative we have to require two things: $k \in \mathbb{Z}^+$ and $0 \leq l \leq k$.

Given the representation of the zero modes, everything else is determined, including the conformal dimensions of all the states. Knowing that for this group $C_\mathfrak{g} = 2$ and $d = 3$ we can write:

$$
L_0 = \frac{1}{2(k+2)} \sum_{a=1}^{3} \Big( \sum_{l \leq -1} j^a_{+l} j^a_{-l} \sum_{l > -1} j^a_{-l} j^a_{+l} \Big).
\tag{2.4.60}
$$

Then the highest weight conformal dimension is:

$$
\begin{aligned}
L_0|h,q\rangle &= \frac{1}{2(k+2)} \sum_{a=1}^{3} j^a_0 j^a_0 |h,q\rangle \\
&= \frac{1}{(k+2)} \sum_{a=1}^{3} \hat{j}^a_0 \hat{j}^a_0 |h,q\rangle = \frac{l(l+2)}{4(k+2)}|h,q\rangle.
\end{aligned}
\tag{2.4.61}
$$

From that one can calculate the conformal dimension of all the other states using equation (2.4.43).

This algebra is also used in one of the most important diagonal coset constructions:

$$
\frac{\mathfrak{su}(2)_k \times \mathfrak{su}(2)_1}{\mathfrak{su}(2)_{k+1}}.
\tag{2.4.62}
$$

The central charge of the resulting theory is

$$
c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)},
\tag{2.4.63}
$$

which is the same as (2.4.32) with $m = k+2$. This means that, with this coset construction, we obtain all the minimal models.

As for the branching rules one can demonstrate that they are:

$$
R^k_\lambda \otimes R^1_\mu = \sum_{\substack{0 \leq \nu \leq k+1 \\ \lambda+\mu+\nu = 0 \mod 2}} R^{k+1}_\nu \otimes R^{k+2}_{h_{\lambda+1,\nu+1}}
\tag{2.4.64}
$$

Where $\lambda, \mu, \nu$ are the $q$ values of the highest weight state of the corresponding representation of $\mathfrak{su}(2)_j$ with $j = k, 1, k+1$ (2.4.56). Instead $R^m_{h_{p,q}}$ corresponds to the Virasoro representation with conformal weight calculated by the equation (2.4.30).

## 2.5 CFT on the torus

So far we have considered theories on the Riemann sphere. In string theory such theories are used to calculate the tree level amplitude, while other terms of the perturbative expansion correspond to theories defined on higher genus Riemannian surfaces. In particular the second term (one-loop) corresponds to the torus. This is the reason why evaluating the behaviour of conformal field theories on the torus is important, it will be the first step to the construction of the theory on an arbitrary surface.

Of course in statistical mechanics, while a general Riemannian surface may not be interesting, the torus corresponds to a plane with periodic conditions in the two directions, making this section important for this other field as well.

As we will see the most important implication of this construction will be the fact that unlike before the holomorphic and antiholomorphic part will not decouple. Before we generally considered the chiral and antichiral part as independent while now there will be rules on how to combine them. Exactly how will be clarified in the next section.

### Modular invariance

The torus can be described by a complex plane on which one puts a lattice and then identifies the various zones of that lattice. This corresponds to the condition:

$$w \sim w + m\alpha_1 + n\alpha_2, \qquad (2.5.1)$$

where $(\alpha_1, \alpha_2)$ is a pair of complex numbers with different phases. The quantity describing the shape of the torus $\tau = \frac{\alpha_2}{\alpha_1} = \tau_1 + i\tau_2$ is called complex structure or modular parameter.
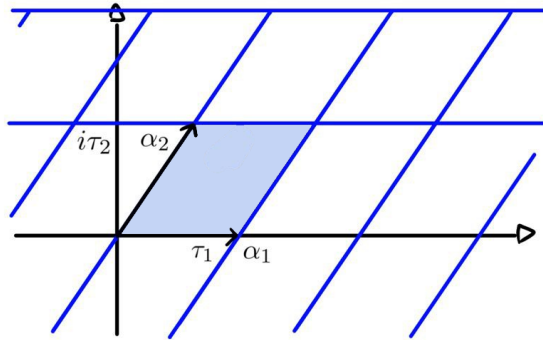


Fig.2.7: Lattice on the complex plane

There are multiple lattices describing the same torus, in particular if $(\beta_1, \beta_2)$ is equivalent to $(\alpha_1, \alpha_2)$ a matrix must exist such that:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \qquad a,b,c,d \in \mathbb{Z} \qquad (2.5.2)$$

and in a similar fashion the inverse must be true too

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \qquad (2.5.3)$$

with integer entries. This imposes the condition $ad - bc = \pm 1$ which is the same as saying that the unit cell in each basis must have the same area (up to a sign). Furthermore the lattice spanned by $(\alpha_1, \alpha_2)$ and $(-\alpha_1, -\alpha_2)$ have the same base cell, which means that they do not count as different parametrizations.

From this considerations we find that the modular group which relates different ways to describe the same lattice is $SL(2, \mathbb{Z})/\mathbb{Z}_2$.

Choosing then $(\alpha_1, \alpha_2) = (1, \tau)$ we get the following transformations for the parameter $\tau$:

$$\tau \to \frac{a\tau + b}{c\tau + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})/\mathbb{Z}_2. \qquad (2.5.4)$$

It is possible, but not easy, to demonstrate that every modular transformation is generated by the following two transformations:

- The first one is the modular T-transformation:

$$T : \tau \to \tau + 1. \tag{2.5.5}$$

- The second one is the modular S-transformation, this one will be the most important and the one that constrains the theory the most:

$$S : \tau \to -\frac{1}{\tau}. \tag{2.5.6}$$

An interesting property of this transformation is $S^2 = \mathbb{I}$.

## The partition function

Now let us look into the application where the modular invariance becomes relevant.

The fact that the partition functions is important both in statistical mechanics and as a generating functional in a quantum field theory is well known. In string theory it is relevant as well, since it is used in the calculation of the one-loop amplitude which, as we have already said, corresponds to the torus.

We define then a partition function for the torus with the following process:

1. Start from a certain state $|\psi\rangle$;
2. Evolve such state around the torus until it returns to the starting point $e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P} |\psi\rangle$;
3. Take the product between the resulting state and the one we started with $\langle\psi| e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P} |\psi\rangle$;
4. Finally sum over all the states in the Hilbert space $\sum_\psi \langle\psi| e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P} |\psi\rangle$.

Notice that in the third step we had to evolve the state both in time ($\tau_2$) and in space ($\tau_1$), this is coherent with the identification we carried out using the lattice.

The final result is:

$$Z(\tau) = Tr_{\mathcal{H}}(e^{-2\pi\tau_2 H} e^{2\pi\tau_1 P}), \tag{2.5.7}$$

where the $2\pi$ is just a convention.

In the complex plane dilations are time translations and rotations are spatial translations therefore:

$$
\begin{aligned}
H_{plane} &= L_0 + \overline{L}_0, \\
P_{plane} &= i(L_0 - \overline{L}_0).
\end{aligned}
\tag{2.5.8}
$$

This expressions can be determined using the considerations we made in the section about the conformal group and algebra in two dimensions (2.2).

From the plane to the cylinder we have seen that the only difference is, looking at equation (2.3.11), the fact that we have a ground state energy:

$$
\begin{aligned}
H_{cylinder} &= L_0 + \overline{L}_0 + \frac{c + \bar{c}}{24}, \\
P_{cylinder} &= i(L_0 - \overline{L}_0 + \frac{c - \bar{c}}{24}).
\end{aligned}
\tag{2.5.9}
$$

Since for the torus we have the same expression as for the cylinder, in total we get

$$Z(\tau) = Tr_{\mathcal{H}}(q^{L_0 - \frac{c}{24}} \overline{q}^{-\overline{L}_0 - \frac{\bar{c}}{24}}), \tag{2.5.10}$$

with $q = e^{2\pi i \tau}$.

As we have seen before, a modular transformation does not change the lattice. In the same way our theory must be invariant under such a transformation and in particular the partition function must be invariant.

This is the main way in which we obtain constraints on the possible combinations of the holomorphic and antiholomorphic representations.

Let us look at the best way to confirm this property. Since we are usually considering various highest weight representations of the chiral and antichiral algebra, which could both be Virasoro algebras or one of its extensions, the partition functions will be expressed as a sum of characters.

A character is defined as

$$\chi_i(\tau) \equiv Tr_{R_i}(q^{L_0 - \frac{c}{24}}), \tag{2.5.11}$$

where $R_i$ is the space of one of our representations.

Then generally the partition function will be written as

$$Z(\tau, \overline{\tau}) = \sum_{i,j} \chi_i M_{ij} \overline{\chi}_j \qquad M_{ij} \in \mathbb{N}, \tag{2.5.12}$$

where $M_{ij}$ is the multiplicity of the $R_i \otimes R_j$ representation in our algebra.

In a lot of theories one has that under the T transformation the characters only obtain a phase, making the partition function invariance easy to verify. For the S transformation instead, in many cases, one can represent it as in the following:

$$\chi_m\left(-\frac{1}{\tau}\right) = \sum_{m'} S_{mm'} \chi_{m'}(\tau). \tag{2.5.13}$$

Then, under that transformation,

$$Z\left(-\frac{1}{\tau}, -\frac{1}{\overline{\tau}}\right) = \sum_{i,j,m,m'} \chi_i S_{im}^T M_{mm'} S_{m'j}^* \overline{\chi}_j. \tag{2.5.14}$$

It is possible to demonstrate that $S = S^T$, then in total the condition for modular invariance will be

$$SMS^\dagger = M. \tag{2.5.15}$$

Another property, that is possible to demonstrate, is that this matrix is also unitary $SS^\dagger = \mathbb{I}$ and, while it does not necessarily square to identity, $S^2 = C$ which is the conjugation matrix and $C^2 = \mathbb{I}$.

### Example theories

Since $\mathfrak{su}(2)_k$ and RCFTs will be the main objects of our study let us talk about their partition functions, characters and S matrices.

First of all we start with $\mathfrak{su}(2)_k$. From the so called Weyl-Kač character formula it is possible to determine that the character with highest weight $0 \leq l \leq k$ has the following form:

$$\chi_l^{(k)}(\tau, z) = \frac{\Theta_{l+1,k+2}(\tau, z) - \Theta_{-l-1,k+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)} \qquad \Theta_{l,k}(\tau, z) = \sum_{n \in \mathbb{Z} + \frac{l}{2k}} q^{kn^2} e^{-2\pi i n k z} \tag{2.5.16}$$

with $z \to 0$.

At $z = 0$ the theta-functions have a well known S modular matrix in literature which is

$$S_{l,l'}^{(k+2)} = \frac{1}{\sqrt{2(k+2)}} \exp\left(-\pi i \frac{ll'}{k+2}\right). \tag{2.5.17}$$

Using some trigonometric identities it is possible to arrive to the expression for the S modular matrix of our characters:

$$S_{l,l'}^{(k)} = \sqrt{\frac{2}{(k+2)}} \sin\left(\frac{\pi}{k+2}(l+1)(l'+1)\right). \tag{2.5.18}$$

Notice that in this case the matrix has real entries, therefore we have $S^* = S$ and $S^2 = \mathbb{I}$.

While for the T-transformation invariance one just needs $h_l - \overline{h}_{l'} \in \mathbb{Z}$, it turns out that there is a complete classifications of the matrices that respect:

$$S^{(k)} M S^{(k)} = M. \qquad (2.5.19)$$

This is known as A-D-E classification, the table on the right shows all the possible cases.

| Level | Partition function | Name |
|---|---|---|
| $k = n$ | $Z = \sum_{l=0}^{n} |\chi_l|^2$ | $A_{n+1}, n \geq 1$ |
| $k = 4n$ | $Z = \sum_{l=0}^{n-1} |\chi_{2l} - \chi_{k-2l}|^2 + 2|\chi_{k/2}|^2$ | $D_{2n+2}, n \geq 1$ |
| $k = 4n-2$ | $Z = \sum_{l=0}^{k/2} |\chi_{2l}|^2 + \sum_{l=0}^{2n-2} \chi_{2l+1}\overline{\chi}_{k-2l-1}$ | $D_{2n+1}, n \geq 2$ |
| $k = 10$ | $Z = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2$ | $E_6$ |
| $k = 16$ | $Z = |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 +$ <br> $(\chi_2 + \chi_{14})\overline{\chi}_8 + \chi_8(\overline{\chi}_2 + \overline{\chi}_{14}) + |\chi_8|^2$ | $E_7$ |
| $k = 28$ | $Z = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 +$ <br> $|\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2$ | $E_8$ |

The analysis of minimal models arises naturally from the one we just did; in fact remembering the branching functions from the coset construction (2.4.64) we can write the following equality for the characters:

$$\chi^{(k)}_{(p-1)}(\tau)\chi^{(1)}_{\epsilon}(\tau) = \sum_{\substack{0 \leq (q-1) \leq k+1 \\ p - q + \epsilon = 0 \quad mod2}} \chi^{(k+1)}_{(q-1)}(\tau)\chi^{Vir(k+2)}_{(p,q)}(\tau), \qquad (2.5.20)$$

where $Vir(k+2)$ corresponds to the minimal model with $m = k + 2$ (2.4.32).

Since a core element of the characters formula is the sum over the representation, knowing how these representations decompose gives us the possibility to rewrite such sum in different ways and hence find equalities as the one we just wrote.

Applying then the modular transformation to both sides, one arrives at the following formula:

$$S^{Vir(k+2)}_{(p,q),(p',q')} = \sqrt{\frac{2}{(k+2)(k+3)}}(-1)^{(p-q)(p'-q')} \sin\left(\frac{\pi}{k+2}pp'\right) \sin\left(\frac{\pi}{k+3}qq'\right). \qquad (2.5.21)$$

As for $\mathfrak{su}(2)_k$ a classification of modular invariant partition functions exists. In particular we have that such classification boils down to the combination of the possibilities for our previous case. This is always thanks to the branching function formula.

For this case we will not go into any deeper detail since it will not be needed for our purposes.

# Chapter 3

# Topological defects

Now we dive into our main object of study, first in a general way, then specifically in conformal field theories. Here we will introduce all the elements needed for the next section in which we will attempt to find the defects of some particular theories.

This chapter follows the reference [10].

## 3.1 Defects in a general theory

When studying a theory usually we are interested in correlators of local operators like

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)O_4(x_4)...\rangle. \qquad (3.1.1)$$

It is possible to get more informations about a theory considering instead

$$\langle O_1(x_1)O_2(C_1)O_3(\Sigma_1)...\rangle, \qquad (3.1.2)$$

which are correlators of operators that are supported on higher dimensional manifolds, like lines or surfaces.



Fig.3.1: Insertions for correlation functions

For example, it is well known that, in a U(1) gauge 4-dimensional theory, it is interesting to consider the following operators $W_q(\gamma) = e^{iq \oint_\gamma A}$, which are called Wilson loops.

In our discussion we will be interested in characterizing the operators with respect to how they behave inside correlation functions.

An operator $O(M)$ supported on a $d$-dimensional closed manifold $M$ is said to be topological if any correlator containing it,

$$\langle ...O(M)...\rangle, \qquad (3.1.3)$$

is invariant under deformations of $M$ which do not cross the support of the other insertions. Another name for these operators is *topological defects*.

The main example of these objects is global symmetries. To describe how they give rise to topological operators we start with an example.

Consider a U(1) symmetry (or a general continuous symmetry), such a symmetry, by Noether theorem, has a conserved current $d * J = 0$. Quantistically this is represented by the expression $\langle d * J...\rangle = 0$ which holds outside of contact points.

Let us now define the following operator: $U_\alpha(M^{d-1}) = e^{i\alpha \int_{M^{d-1}} *J}$. This operator is topological, in fact if one considers a deformation of the support to $M'^{d-1}$ we will have that

$$U_\alpha(M'^{d-1}) - U_\alpha(M^{d-1}) = e^{i\alpha \int_{M'} *J} - e^{i\alpha \int_M *J}$$
$$= e^{i\alpha \int_M *J}(e^{i\alpha \int_{M'} *J - i\alpha \int_M *J} - 1) = e^{i\alpha \int_M *J}(e^{i\alpha \int_N d*J} - 1) = 0, \qquad (3.1.4)$$

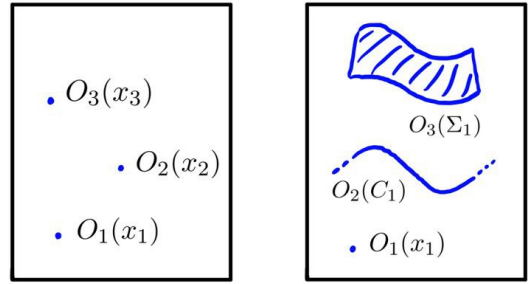where we call $N$ the $d$ dimensional manifold with the property: $\partial N = M' - M$.

Remember that quantistically the above equation is true only if in the correlator there are no operators inserted in $N$ (and therefore we do not have contact terms). This is consistent with our definition of topological operator since we asked for the supports of different operators to not cross.
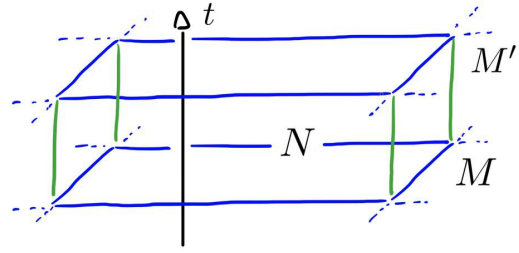
Fig.3.2: Representation of the manifold we are considering

In this particular case, if we cross a point-like insertion, we can expect the following effect:

$$\langle U_\alpha(M'^{d-1})O_q(x)...\rangle = e^{i\alpha q}\langle O_q(x)U_\alpha(M^{d-1})...\rangle, \tag{3.1.5}$$

where we are supposing that $O_q(x)$ has charge $q$ under our symmetry.

One can show the veracity of the statement by deforming $M^{d-1}$ and $M'^{d-1}$ to be space-like, then $\int_M *J = Q$, where $Q$ is the charge of the state which arises after all the operators before $U_\alpha(M^{d-1})$ have acted on the vacuum.

We can generalize this kind of "symmetry operator" to any symmetry, both continuous and discrete, in the following way:

$$\langle U_g(M)O_1(x_1)O_2(x_2)...\rangle = \langle(gO_1)(x_1)(gO_2)(x_2)...\rangle, \tag{3.1.6}$$

where $M$ is an hypersurface enclosing the other points in the correlator. Such operator will be topological by definition.

We have just discovered that each symmetry gives a topological operator, but not all topological operators are symmetries. We are interested in these structures exactly for this reason, they enlarge the constrictions on correlators from just symmetries to all topological operators.

Now, we look at some general properties of these topological defects and also find a way to characterize the ones coming from symmetries:

1. At the very least we will have an identity operator coming from the trivial identity "symmetry".

2. The orientation of the underlying manifold matters.
   For example
   $$U_\alpha(\overline{M}) = e^{i\alpha \int_{\overline{M}} *J} = e^{-i\alpha \int_M *J} = U_{-\alpha}(M), \tag{3.1.7}$$
   where $\overline{M}$ is the orientation reversed manifold.

3. We can define a trivial sum between defects in the following way:
   $$\langle(\mathcal{L}_1 + \mathcal{L}_2)O_1(x_1)O_2(x_2)...\rangle = \langle\mathcal{L}_1 O_1(x_1)O_2(x_2)...\rangle + \langle\mathcal{L}_2 O_1(x_1)O_2(x_2)...\rangle. \tag{3.1.8}$$

4. A *fusion product* between line defects also exists (we are considering the case in which there is no operator between the defects), corresponding to the limit where two topological defects get closer until they coincide.
   Just applying the definition one find that, if the defects come from group elements: $\mathcal{L}_1 = \mathcal{L}_{g_1}$ and $\mathcal{L}_2 = \mathcal{L}_{g_2}$, $\mathcal{L}_3 = \mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_{g_1 \circ g_2}$.

In general this product is associative and has a neutral element, the identity, but one does not necessarily obtain a group. While every element coming from a symmetry group has an inverse such that $\mathcal{L}\mathcal{L}^{-1} = \mathcal{L}^{-1}\mathcal{L} = \mathcal{L}_{Id}$, every element not coming from a symmetry does not have to have one. Actually one can demonstrate that this property is characterizing: an element does not have an inverse if and only if it does not come from a symmetry.
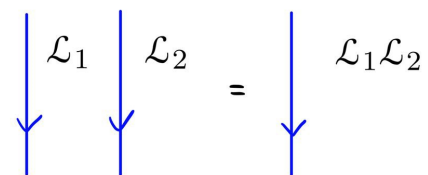
Fig.3.3: Fusion of defects

We have then found an immediate way to characterized the defects we are most interested in, which are the non invertible ones.

## 3.2  Inserting defects in a 2-dim CFT

First of all notice that in this case the symmetry operators are one dimensional. Apart from line defects the only other possibility would be to have area operators with a compact two dimensional support. We will focus on the first case since, as said before, we are very interested in the concept of symmetries' extensions.

In a two dimensional conformal field theory we have two conserved charges which arise from the Noether theorem applied to the conformal symmetry: $T(z)$ and $\overline{T}(\overline{z})$.
Since these charges generates all the coordinate transformations, we can adapt what it means for a defect to be topological. In this context a defect is called topological if it is "transparent" with respect to $T(z)$ and $\overline{T}(\overline{z})$, as in the image on the right.
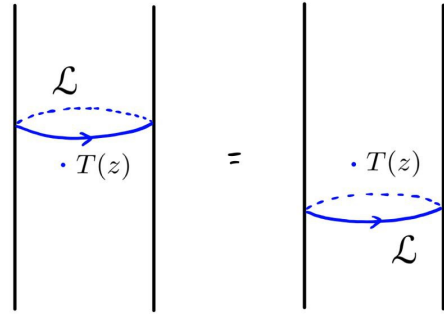


Fig.3.4: Transparency property

In figure (3.4) we also have the first way in which one can insert a defect: horizontally.
Now we go from the infinite cylinder to the plane. We can set the initial state to be $\Psi$ with an operator $O_\Psi$ inserted at the origin of the plane, obtaining the image on the right.
Then, it is possible to contract the defect on the initial state since it is topological; this is the same as sending the defect to $t \to -\infty$ on the infinite cylinder. The result of this operation is to modify the initial state with a linear operator associated to the defect: $\Psi \to \hat{\mathcal{L}}\Psi$.
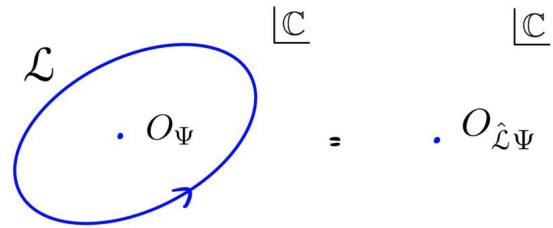


Fig.3.5: Defect contraction

An example of an associated linear operator comes from the symmetry defects, in that case it will just act as the symmetry on the state.
Then the transparency property can be written in the following way:

$$[L_n, \hat{\mathcal{L}}] = [\overline{L}_n, \hat{\mathcal{L}}] = 0, \tag{3.2.1}$$

where $L_n$ and $\overline{L}_n$ are the modes of $T(z)$ and $\overline{T}(\overline{z})$.
We have already seen that $L_n$ and $\overline{L}_n$ generate the Virasoro algebra, therefore along each irreducible representation of the algebra $\hat{\mathcal{L}}$ will be constant by Schur's lemma.

There is actually another very important way one can insert a defect. We place it vertically, even though it looks similar to the previous case the results are very different.
As usual the symmetry defect example clarifies what we mean. Looking at its definition one can rephrase it as: consider $\mathcal{L}_g$, every operator passing through it gets transformed with $g$ and $\hat{\mathcal{L}}_g|0\rangle = |0\rangle$. In this regard the vertical insertion of the defect, which we see on the right, can be interpreted as a twisted theory.
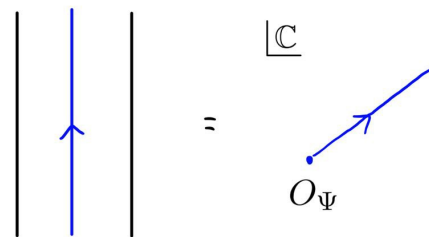


Fig.3.6: Vertical defects

We mentioned twisted theories in the section about Kač-Moody algebras (2.4), basically the idea is that they are not single valued: a rotation around the infinite cylinder (or around the origin in the complex plane) does not give the same operator. In the symmetry case it is evident that the condition is $O(ze^{2\pi i}) = (gO)(z)$.

Clearly, as it is a new theory, it will have a completely different Hilbert space. Remembering the fact that the defect is transparent with respect to the energy-momentum tensor, we can determine that it is still single valued on the plane. This permits us to say that the new space will still be a representation of the Virasoro algebras (holomorphic and antiholomorphic).

Let us now look at some properties of these spaces which we will call $\mathcal{H}_{\mathcal{L}}$:

- In general what might happen is that $\mathcal{H}_{\mathcal{L}} = \emptyset$, but it is possible to demonstrate that if the CFT is unitary, modular invariant and has an unique vacuum $\mathcal{H}_{\mathcal{L}} \neq \emptyset$.

- Since $\mathcal{H}_{\mathcal{L}}$ is a representation of the Virasoro algebras, on the torus, one will have that

$$Z_{\mathcal{L}} = \sum_{i,j} M_{ij} \chi_i \overline{\chi}_j \qquad\qquad M_{ij} \in \mathbb{N}, \qquad\qquad (3.2.2)$$

  where $\chi_i$ and $\overline{\chi}_j$ are characters of representations of the Virasoro algebra.

- We defined before the sum of defects, then

$$\mathcal{H}_{\mathcal{L}_1 + \mathcal{L}_2} = \mathcal{H}_{\mathcal{L}_1} \oplus \mathcal{H}_{\mathcal{L}_2}. \qquad\qquad (3.2.3)$$

  Since it is the direct sum of two spaces, the correlators in $\mathcal{H}_{\mathcal{L}_1 + \mathcal{L}_2}$ will be the sum of the correlators in $\mathcal{H}_{\mathcal{L}_1}$ and $\mathcal{H}_{\mathcal{L}_2}$.

## 3.3   Defects in partition functions

The following considerations are fundamental for our analysis on defects, they give the first constraints on the shape of our defects.

Consider the partition function of a certain theory with Hilbert space $\mathcal{H}$ and insert a defect in it producing the following object:

$$Z(\mathcal{L}) = Tr_{\mathcal{H}}(\mathcal{L} q^{L_0 - \frac{c}{24}} \overline{q}^{-\overline{L}_0 - \frac{\overline{c}}{24}}). \qquad\qquad (3.3.1)$$

The insertion of the defect in this way gives figure (3.7A), in which we are representing the torus as a square. This is similar to figure (2.7) in the sense that the opposite sides of the square are identified.

It is possible to demonstrate now that the modular $S$ transformation, which is $\tau \to -\frac{1}{\tau}$, applied to this first partition function gives the partition function in figure (3.7B).
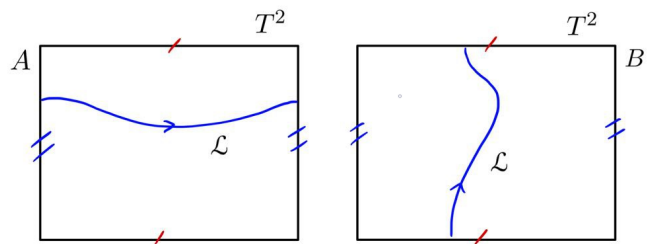


Fig.3.7: Defect on the torus

We can show that this is true using the definition of $\tau = \frac{\alpha_2}{\alpha_1}$, where $\alpha_2, \alpha_1$ come from equation (2.5.1). To invert $\tau$ would then mean to exchange $\alpha_2$ and $\alpha_1$ producing exactly the mentioned figure.

Reading now figure (3.7B) as a partition function, we get:

$$Z_{\mathcal{L}} = Tr_{\mathcal{H}_{\mathcal{L}}}(q^{L_0 - \frac{c}{24}} \overline{q}^{-\overline{L}_0 - \frac{\overline{c}}{24}}) = \sum_{i,j} M_{ij} \chi_i \overline{\chi}_j \qquad\qquad M_{ij} \in \mathbb{N}. \qquad (3.3.2)$$

This is the partition function of the theory twisted with the defect.

The correspondence we just draw is often used to put constraints on which defects can appear in our theory.

The first step in order to do that is to notice that expression (3.3.1) can be written as

$$Z(\mathcal{L}) = \sum_{i,j} \tilde{L}_{ij} \chi_i \overline{\chi}_j \qquad\qquad \tilde{L}_{ij} \in \mathbb{C}, \qquad\qquad (3.3.3)$$

where the presence of the defect is embodied by $\tilde{L}_{ij}$.

This is because, as we said before, the defect is constant along each irreducible representation of the Virasoro algebras and $\chi_i, \overline{\chi}_j$ are exactly that.

Now a clarification is in order. Schur's lemma states: any liner map that commutes with an algebra will act on the irreducible representations of such an algebra either as an isomorphism or as the zero map.

In the particular case of the map being an automorphism, one finds that the only possibility is for the map to be a multiple of the identity, which give rise to the idea of "acting as a constant". The maps between different representations (different highest weights) must be the zero maps since no isomorphism exits between those. The last possibility is to have $N_{ij} > 1$, in that case one has $N_{ij}$ identical representations which are naturally isomorphic.

Then one can always write such isomorphism as:

$$\hat{\mathcal{L}}_{i,j}^{\alpha,\alpha'} = L_{i,j}^{\alpha,\alpha'} \sum_{n,\overline{n}} (|i,j,n,\overline{n}\rangle)^{(\alpha)} (\langle i,j,n,\overline{n}|)^{(\alpha')} \qquad L_{i,j}^{\alpha,\alpha'} \in \mathbb{C}; \alpha, \alpha' \in \{1,2,...,N_{ij}\}, \qquad (3.3.4)$$

where $(|i,\overline{j},n,\overline{n}\rangle)^{(\alpha)}$ is the $(n,\overline{n})$-th member of a basis of the vector space of the irreducible $\alpha$-th representation with highest weights corresponding to $i,j$.

To explain better how this can be visualized, let us consider the example in which $N_{ij} = 2$ and write the following expression

$$\hat{\mathcal{L}} \begin{pmatrix} R_{ij}^{(1)} \\ R_{ij}^{(2)} \end{pmatrix} = \begin{pmatrix} L_{ij}^{1,1} & L_{ij}^{1,2} \\ L_{ij}^{2,1} & L_{ij}^{2,2} \end{pmatrix} \begin{pmatrix} R_{ij}^{(1)} \\ R_{ij}^{(2)} \end{pmatrix}, \qquad\qquad (3.3.5)$$

where we labelled the representations as $R_{ij}^{(\alpha)}$.

For each highest weight then the action of the defect should be thought not as a constant value, but as a constant square matrix. Of course the matrix could be one dimensional giving rise to the constant value interpretation.

Noticing that Schur's lemma by itself already restricts the space of defects with which we have to work, we now return to our previous discussion.

Looking at equation (3.3.1), it is quite easy to see that the $\tilde{L}_{ij}$s, appearing in expression (3.3.3), correspond to:

$$\tilde{L}_{ij} = Tr_\alpha(L_{ij}) = \sum_\alpha L_{ij}^{\alpha\alpha}. \qquad\qquad (3.3.6)$$

Then to complete the previously discussed procedure, we make an $S$ transformation of (3.3.3) giving

$$Z_{\mathcal{L}} = \sum_{i_1,i_2,j_1,j_2} \chi_{i_1} S_{i_1 i_2} \tilde{L}_{i_2 j_2} S_{j_2 j_1}^\dagger \overline{\chi}_{j_1} = \sum_{i,j} \chi_i M_{ij} \overline{\chi}_j, \qquad\qquad (3.3.7)$$

where the last equality refers exactly to the argument on figure (3.7).

Equation (3.3.7) then becomes

$$\sum_{i_2,j_2} S_{i_1 i_2} \tilde{L}_{i_2 j_2} S_{j_2 j_1}^\dagger = M_{i_1 j_1}. \qquad\qquad (3.3.8)$$

This last equation is our constraint, since even if $M_{i_1 j_1}$ is not known, one must have $M_{i_1 j_1} \in \mathbb{N}$. There are cases in which this fixes all the defects of the theory. The practical aspect of this procedure will be extensively explained when we will encounter the calculations.

## 3.4 Defects properties

We now look at some other properties of defects which will be used in our successive analysis.

First of all we demonstrate that $\langle \mathcal{L} \rangle \geq 0$, where

$$\langle \mathcal{L} \rangle |0\rangle = \hat{\mathcal{L}} |0\rangle. \tag{3.4.1}$$

$\langle \mathcal{L} \rangle$ is called *quantum dimension* of the defect.

Consider the partition function $Z_{\mathcal{L}}(\tau)$ and calculate it in $\tau = it$. This means that $q = e^{-2\pi t}$, therefore the trace becomes a sum of positive numbers which is, again, positive.

We use this fact in the following equation:

$$Z(\mathcal{L}) \left( -\frac{1}{it} \right) = Z_{\mathcal{L}}(it) \geq 0. \tag{3.4.2}$$

Then one can combine this last inequality with

$$Z(\mathcal{L}) \left( -\frac{1}{it} \right) = Tr_{\mathcal{H}}(\mathcal{L} q^{L_0 - \frac{c}{24}} \bar{q}^{-\bar{L}_0 - \frac{\bar{c}}{24}}) = \langle \mathcal{L} \rangle e^{\frac{2\pi}{t} \frac{c+\bar{c}}{24}} + ..., \tag{3.4.3}$$

where $\langle \mathcal{L} \rangle e^{\frac{2\pi}{t} \frac{c+\bar{c}}{24}}$ arises, in the trace, from the vacuum state and as $t \to \infty$ will be the only relevant piece. In total we get

$$\langle \mathcal{L} \rangle e^{\frac{2\pi}{t} \frac{c+\bar{c}}{24}} \geq 0 \to \langle \mathcal{L} \rangle \geq 0, \tag{3.4.4}$$

which is what we set out to show.

After that we have that many theories respect semi-simplicity. This property means that we have a certain number of simple defects $\mathcal{L}_i$ (which in most theories is finite) and that every other defect is semi-simple. To be semi-simple means that such defect can be written as

$$\mathcal{L} = \sum_i N^i \mathcal{L}_i \qquad\qquad N^i \in \mathbb{N}. \tag{3.4.5}$$

Even though the definition of a simple defect is a bit more complicated that this, we can consider simple a defect that cannot be written as a sum of other defects.

Semi-simplicity has a very interesting consequence.

First of all we can always write

$$\mathcal{L}_1 \mathcal{L}_2 = \sum_i N_{12}^i \mathcal{L}_i \qquad\qquad N_{12}^i \in \mathbb{N}, \tag{3.4.6}$$

since, as we said before, the product of two defects is always a defect.

In the particular case of the product between a defect and its orientation reversed it is possible to demonstrate that:

$$\mathcal{L}\overline{\mathcal{L}} = \mathcal{L}_{Id} + \sum_i n^i \mathcal{L}_i \qquad\qquad n^i \in \mathbb{N}. \tag{3.4.7}$$

This fact can be used to demonstrate that $\langle \mathcal{L} \rangle \geq 1$.

Consider the correlators $\langle \mathcal{L} \rangle$ and $\langle \overline{\mathcal{L}} \rangle$ which are represented in the image on the right. Since the two correlators are connected by the conformal transformation $w = \frac{1}{z}$ they are identical. This shows that

$$\langle \mathcal{L} \rangle = \langle \overline{\mathcal{L}} \rangle. \tag{3.4.8}$$
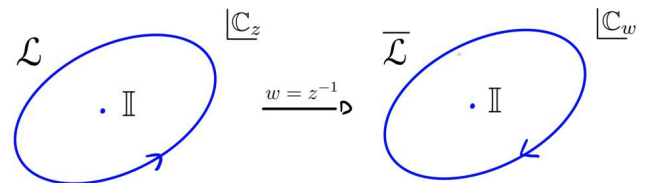


Fig.3.8: Defect correlators

Then from (3.4.7) we get the following equation:

$$\langle \mathcal{L} \rangle^2 = 1 + \sum_i n^i \langle \mathcal{L}_i \rangle, \qquad (3.4.9)$$

that combined with equation (3.4.4) gives

$$\langle \mathcal{L} \rangle \geq 1, \qquad (3.4.10)$$

which is what we wanted to find.

Specifically if the defect is not invertible:

$$\langle \mathcal{L} \rangle^2 \geq 1 + \sum_i n^i \rightarrow \langle \mathcal{L} \rangle^2 > 1 \rightarrow \langle \mathcal{L} \rangle > 1, \qquad (3.4.11)$$

while if the defect is invertible:

$$\langle 0 | \overline{\mathcal{L}} \mathcal{L} | 0 \rangle = 1 \rightarrow \langle \mathcal{L} \rangle^2 = 1 \rightarrow \langle \mathcal{L} \rangle = 1. \qquad (3.4.12)$$

This fact will become useful when trying to recognize such defects.

At last note that the same arguments in reverse gives from $\langle \mathcal{L} \rangle \geq 1 \rightarrow Z_{\mathcal{L}} \neq 0$, which in turn means $\mathcal{H}_{\mathcal{L}} \neq \emptyset$. Therefore we also demonstrate the property that we stated the section before the last.

## 3.5 Junctions

We now introduce junctions. We will not go into a very detailed discussion, but mostly state properties as facts without demonstrating them. This is because the role of this section is to provide context to some of the things we will do later, but it will not be used in the calculations.

First of all, let us consider the possibility of having an operator start a defect in a different position than the origin. The correlator in image (3.9), in radial quantization, can be described as in the following: an operator inserted at the origin creates a state in $\mathcal{H}$ at $t \rightarrow -\infty$, which is evolved until at a certain point in time, represented by the equal time circle in red, then the operator $O_\Psi$ maps it to a state in $\mathcal{H}_{\mathcal{L}}$. An important observation to make here is that while the defect itself can be moved without any consequence, moving the operator that starts the defect changes the correlator.
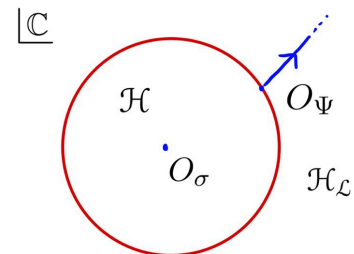


Fig.3.9: Theory gluing

The only case in which nothing changes is when the operator's conformal charges are null and therefore any coordinate transformation leaves the operator invariant.

We can add now another possibility, an operator that starts more than one defect. The treatment is exactly the same as before: one finds a new theory with a new Hilbert space $\mathcal{H}_{\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_k}$. The same method applies because fusing all the defects one obtains a theory with a single vertical defect: $\mathcal{H}_{\mathcal{L}_1 \mathcal{L}_2 \ldots \mathcal{L}_k}$. Notice that the order of the defects is important because to exchange they must cross each other. It is possible to show that a such Hilbert space and one obtained by a cyclic permutation of the defects (for example $\mathcal{H}_{\mathcal{L}_2, \ldots, \mathcal{L}_k, \mathcal{L}_1}$) are isomorphic, but with a possibly non trivial isomorphism.
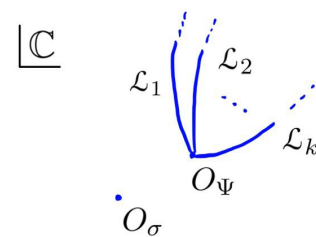


Fig.3.10: Multiple defects

The last possibility which we have not mentioned is easy to guess: we can have operators that send spaces with a certain number of defects $k_1$ to a space with another number of defects $k_2$.

Knowing all this, we can now define what a *junction* is: in general a point in which defects intersect. Now all these operators, which correspond to functions $\mathcal{H} \to \mathcal{H}_{\mathcal{L}_1,\ldots,\mathcal{L}_k}$, are in a one to one correspondence with the states in $\mathcal{H}_{\mathcal{L}_1,\ldots,\mathcal{L}_k}$. In particular, to each operator corresponds the state into which the vacuum is sent when acted upon:

$$|\Psi\rangle = O_{\mathcal{L}_1,\ldots,\mathcal{L}_k}(\Psi)|0\rangle. \tag{3.5.1}$$

Now the most used junctions are the topological ones, which send the vacuum to the subspace of conformal weights $h = \bar{h} = 0$: $V_{\mathcal{L}_1,\ldots,\mathcal{L}_k} \subset \mathcal{H}_{\mathcal{L}_1,\ldots,\mathcal{L}_{k_2}}$.

The reason why we can assume most of the junctions are topological can be explained in the following way.

First of all, let us consider a single defect: if $\mathcal{L} \neq \mathcal{L}_{Id}$ then $V_{\mathcal{L}} = \emptyset$.

This is because, if the operators starting and ending the defect are topological, then one could transform a defect on an closed line into a defect on a open line as shown in figure (3.11), without changing a correlation function. This implies that the defect encircling a local operator $O$ can be opened and then closed around the vacuum, as shown in figure (3.12).
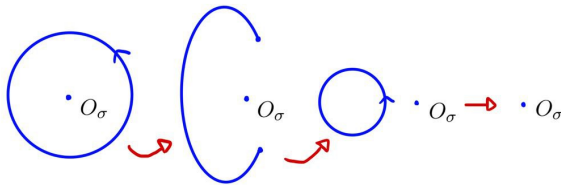


Fig.3.12: Single defect acting on an operator
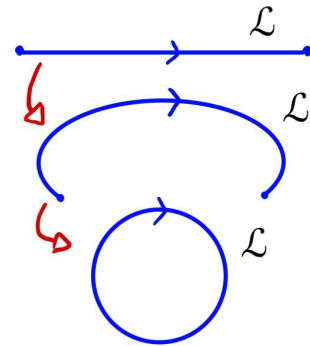


Fig.3.11: Single defect with topological operators

This is clearly a problem because it means that the action of the defect is trivial on any operator, which is in contradiction with the statement that such defect is not $\mathcal{L}_{Id}$.

We found therefore that every operator starting a non trivial defect is not topological. The fact that the focus is on topological junctions can be explained by showing that every other defect can be made into a topological defect and an operator starting a single defect line.
In figure (3.13) we are executing the following steps:

1. First we have a non topological junction;
2. In the second image we do a partial fusion of the zone circled in red. We still have not treated partial fusion, but for now take it for granted. What we are talking about will be explained later;
3. At last, we find a certain defect operator starting a single defect ($\mathcal{L}$) which connects to a topological junction ($u$) of six defects.



Fig.3.13: Partial fusion of a non topological juction

The reason why the last junction is topological comes from a partial fusion property which we will explain in the next section.

Before moving on, we add a last consideration.

Consider a junction as the one in the image on the right. Since we have for sure $O = \mathbb{I}$ the vector space of topological junctions has $dim V_{\mathcal{L},\overline{\mathcal{L}}} \geq 1$.
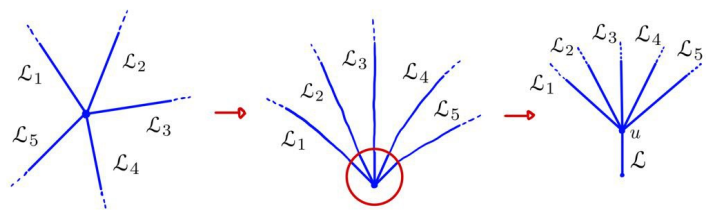
It is possible to show that if the defect is simple $dim V_{\mathcal{L},\overline{\mathcal{L}}} = 1$ otherwise it is strictly greater than one. In fact if one consider a semi-simple defect $\mathcal{L}_{ss}$, $O$ could at least be the projectors on all the spaces of the defects in witch $\mathcal{L}_{ss}$ is written: $\mathcal{L}_{ss} = \sum_i \mathcal{L}_i$.
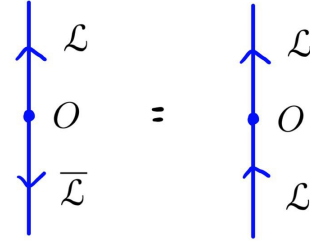


Fig.3.14: Junction of a defect $\mathcal{L}$ and $\overline{\mathcal{L}}$

Note that in the last case we have to exclude operator that transform a simple defect into another one. This is because if such an operator existed we would not be able to distinguish the two defects in any correlator function, arriving at considering them as one.

## 3.6 Partial fusion

When talking about junctions a fundamental operation one could do is the partial fusion. One of the reasons why it is important is because it permits us to build identities for junctions-defects configurations which in return gives constraints on the defects. The only problem with these constraints is that usually they are very hard to solve, making them most of the times unusable.

Let us start by introducing, how the *partial fusion* arises. Consider two defects $\mathcal{L}_1$ and $\mathcal{L}_2$ which in a certain region of space have parallel support. Then, what one can do, is to deform the supports of the defects, in the region in which they are parallel, so that they get closer and closer until they fuse. By the description we have just done, we can clearly understand why the resulting junctions are topological, in fact the portion of the support where the two defects get close to each other is arbitrary and the result must be the same since nothing in the description changes until they are so close they fuse.
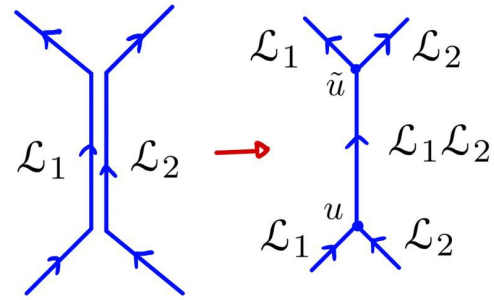


Fig.3.15: Partial fusion

Note also that the topological junctions $u$ and $\tilde{u}$ in figure (3.15) give an isomorphism between these two Hilbert spaces: $\mathcal{H}_{\mathcal{L}_1,\mathcal{L}_1} \cong \mathcal{H}_{\mathcal{L}_1\mathcal{L}_1}$.

Before continuing with this discussion let us make a clarification about these junctions. We saw that, when the junction is of the type $\mathcal{H} \to \mathcal{H}_{\mathcal{L}_1,...,\mathcal{L}_k}$, it is specified by a vector in $\mathcal{H}_{\mathcal{L}_1,...,\mathcal{L}_k}$.

In this case we have $\mathcal{H}_{\mathcal{L}_1,\mathcal{L}_2} \to \mathcal{H}_{\mathcal{L}_1\mathcal{L}_2}$, one can show that these transformations are in a one to one correspondence with the states in $\mathcal{H}_{\mathcal{L}_1,\mathcal{L}_2,\overline{\mathcal{L}_1\mathcal{L}_2}}$.

Let us suppose now that $\mathcal{L}_1$ and $\mathcal{L}_2$ are two simple defects and that $\mathcal{L}_1\mathcal{L}_2 = \sum_i N_{12}^i \mathcal{L}_i$. It is clear that locally the defect product looks a lot like the partial fusion, the only difference is that, in the first case, the portion of the support we use for the fusion is the whole defect. It is possible to show that the partial fusion can be decomposed as in figure (3.16), where $v_i$ and $\tilde{v}_i$ are vectors respectively in the spaces $V_{\overline{\mathcal{L}}_1,\overline{\mathcal{L}}_2,\mathcal{L}_i}$ and $V_{\mathcal{L}_1,\mathcal{L}_2,\overline{\mathcal{L}_i}}$ and their tensor product is completely determined by the theory.
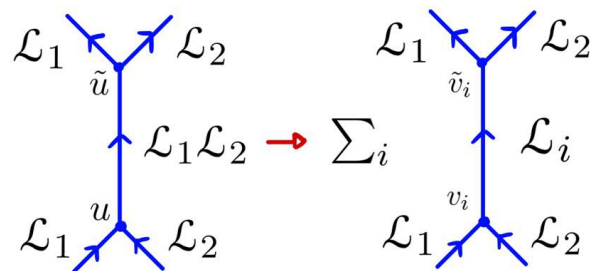


Fig.3.16: Partial fusion decomposed

To end this part about junctions, we list two other possible operation one can do with defects without explaining in detail what they entail.

The first one is the one on the left of figure (3.17): it is the contraction of a defect circling around a local operator. The difference with the usual "action of the defect on the operator" come from the fact that now the defect circling around has a junction. The result will be a new operator which is not local anymore but starts a defect.

The second one, instead, is the one on the right of figure (3.17): it is the contraction of a "bubble" between two defects. The result will be the connection of the two external defects with a junction, which, in a certain sense, will be the composition of the two junctions attaching those defects to the circle.
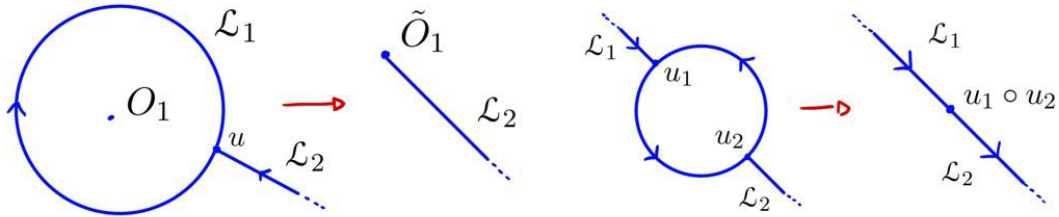


Fig.3.17: Contraction of a defect with a junction on the left, collapsing two junctions on the right

# Chapter 4

# Defect calculations

In the end we study the practical way in which one usually finds defects.

We will start with an example theory: the Ising model, which corresponds to the Virasoro minimal model of central charge $c = \frac{1}{2}$. After that we will proceed towards the original calculations of this thesis. First of all considering $\mathfrak{su}(2)_1^6 \times \overline{\mathfrak{su}(2)}_1^6$ and then going more in depth on a particular decomposition of this theory, which is useful for this kind of analysis.

## 4.1 Ising model

This RCFT is one of the minimal models considered in section 2.4, in particular it is the one with $m = 3$. In formula (2.4.32) we have already shown how one can calculate the three representations of this Virasoro algebra:

$$h = 0: \quad |0\rangle \qquad h = \frac{1}{2}: \quad |1/2\rangle \qquad h = \frac{1}{16}: \quad |1/16\rangle, \tag{4.1.1}$$

where h is the conformal charge of the representation.

We want our theory to have also an antichiral part, which is constrained by modular invariance if we want to be consistent on the torus. We make the easiest choice:

$$\mathcal{H} = [R_0 \otimes \overline{R_0}] \oplus [R_{\frac{1}{2}} \otimes \overline{R_{\frac{1}{2}}}] \oplus [R_{\frac{1}{16}} \otimes \overline{R_{\frac{1}{16}}}], \tag{4.1.2}$$

which is a diagonal theory. It is standard to call these representations as:

$$|0; \overline{0}\rangle = |0\rangle \qquad |1/2; \overline{1/2}\rangle = |\epsilon\rangle \qquad |1/16; \overline{1/16}\rangle = |\sigma\rangle. \tag{4.1.3}$$

Supposing that we do not know anything else about the theory let us try to find the defects.

First of all we know that a general defect $\mathcal{L}$ will act in the following way

$$\hat{\mathcal{L}}|0\rangle = a|0\rangle \qquad \hat{\mathcal{L}}|\epsilon\rangle = b|\epsilon\rangle \qquad \hat{\mathcal{L}}|\sigma\rangle = c|\sigma\rangle \tag{4.1.4}$$

on the primary fields, but also on all of the corresponding secondary fields, since the defect is topological.

This means that the partition function with the defect inserted becomes

$$Z = \chi_0 \overline{\chi}_0 + \chi_{1/2} \overline{\chi}_{1/2} + \chi_\sigma \overline{\chi}_\sigma \longrightarrow Z(\mathcal{L}) = a\chi_0 \overline{\chi}_0 + b\chi_{1/2} \overline{\chi}_{1/2} + c\chi_{1/16} \overline{\chi}_{1/16}. \tag{4.1.5}$$

After that, one wants to use the $S$ matrix so that we can obtain $Z_\mathcal{L}$ as showed in formula (3.3.7). In the last part of the section on CFTs on the torus, we have already met the formula for the S modular matrix (2.5.21) of this model. Working it out for the three representations explicitly, we get

$$S_{Vir_{m=3}} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}. \tag{4.1.6}$$

In the calculation one must remember that in formula (2.5.21) the indices $q$ and $p$ assume all the possible values while the actual representations are half the amount due to the symmetry: $\{p \to m-p, q \to m+1-q\}$. Basically one just has to multiply by two the result for each single representation. Then, with this, we arrive at the equation:

$$
Z_{\mathcal{L}} = \frac{1}{4} \begin{pmatrix} \chi_0 & \chi_{1/2} & \chi_{1/16} \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \overline{\chi}_0 \\ \overline{\chi}_{1/2} \\ \overline{\chi}_{1/16} \end{pmatrix}
$$

$$
Z_{\mathcal{L}} = \frac{1}{4} \begin{pmatrix} \chi_0 & \chi_{1/2} & \chi_{1/16} \end{pmatrix} \begin{pmatrix} a+b+2c & a+b-2c & \sqrt{2}(a-b) \\ a+b-2c & a+b+2c & \sqrt{2}(a-b) \\ \sqrt{2}(a-b) & \sqrt{2}(a-b) & 2(a+b) \end{pmatrix} \begin{pmatrix} \overline{\chi}_0 \\ \overline{\chi}_{1/2} \\ \overline{\chi}_{1/16} \end{pmatrix}
\tag{4.1.7}
$$

Now $\mathcal{H}_{\mathcal{L}}$ must be a sum of representations of the Virasoro algebra and therefore any combination $\chi_i \overline{\chi}_j$ has to appear in $Z_{\mathcal{L}}$ multiplied by a non negative integer, which will be the multiplicity of the corresponding representation of $Vir_{m=3} \times \overline{Vir}_{m=3}$ in $\mathcal{H}_{\mathcal{L}}$. One can set up the following identity:

$$
\frac{1}{4} \begin{pmatrix} a+b+2c & a+b-2c & \sqrt{2}(a-b) \\ a+b-2c & a+b+2c & \sqrt{2}(a-b) \\ \sqrt{2}(a-b) & \sqrt{2}(a-b) & 2(a+b) \end{pmatrix} = \begin{pmatrix} n_1 & n_2 & n_3 \\ n_2 & n_1 & n_3 \\ n_3 & n_3 & n_4 \end{pmatrix} \qquad n_1, n_2, n_3, n_4 \in \mathbf{N}. \tag{4.1.8}
$$

This equality is exactly the same as equation (3.3.8).
Then it translates into the following system of equations:

$$
\begin{cases} a+b+2c = n_1 \\ a+b-2c = n_2 \\ \frac{1}{2}(a-b) = \sqrt{2}n_3 \\ \frac{1}{2}(a+b) = n_4 \end{cases} \longrightarrow \begin{cases} a = n_1 + n_2 + \sqrt{2}n_3 \\ b = n_1 + n_2 - \sqrt{2}n_3 \\ c = n_1 - n_2 \\ n_1 + n_2 = n_4 \end{cases}, \tag{4.1.9}
$$

which, with some manipulations, can be put in the form on the right.
These conditions on $a$, $b$ and $c$ tell us all there is to know on our defects. In particular the decomposition in simple defects is clear: we have three fundamental defects: $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, while any other defect decomposes as $\mathcal{L} = n_1 \mathcal{L}_1 + n_2 \mathcal{L}_2 + n_3 \mathcal{L}_3$.
Let us give these defects a name and show how they act on our Hilbert space:

$$
\begin{aligned}
&\mathcal{L}_1 : I \\
&\hat{I}|0\rangle = |0\rangle \qquad\qquad \hat{I}|\epsilon\rangle = |\epsilon\rangle \qquad\qquad \hat{I}|\sigma\rangle = |\sigma\rangle, \\
&\mathcal{L}_2 : \eta \\
&\hat{\eta}|0\rangle = |0\rangle \qquad\qquad \hat{\eta}|\epsilon\rangle = |\epsilon\rangle \qquad\qquad \hat{\eta}|\sigma\rangle = -|\sigma\rangle, \\
&\mathcal{L}_3 : N \\
&\hat{N}|0\rangle = \sqrt{2}|0\rangle \qquad\qquad \hat{N}|\epsilon\rangle = -\sqrt{2}|\epsilon\rangle \qquad\qquad \hat{N}|\sigma\rangle = 0.
\end{aligned}
\tag{4.1.10}
$$

Looking at how they act on the vacuum we can already tell that the first two defects are invertible while the last one is not. In fact the first defect corresponds to the identity while the second one is associated to the well known $\mathbb{Z}_2$ symmetry of the theory.
The fact that $\eta$ is a representation of that group can be also shown comparing the composition rule of the group in question with the algebra of the defect. Let us write the algebra:

$$
\hat{\eta}^2 = \hat{I} \qquad\qquad \hat{N}^2 = \hat{I} + \hat{\eta} \qquad\qquad \hat{N}\hat{\eta} = \hat{\eta}\hat{N} = \hat{N}. \tag{4.1.11}
$$

The composition of two $\eta$s gives the identity and this is in conformity with the $\mathbb{Z}_2$ group as expected. Note that in this particular case the defects are orientation independent. The orientation reversed of a defect $\overline{\mathcal{L}}$ can be recognized by the following property: if the defect $\mathcal{L}$ is invertible, $\overline{\mathcal{L}}$ is the inverse defect; if instead $\mathcal{L}$ is not invertible, $\overline{\mathcal{L}}$ is such that the simple defect expansion of $\overline{\mathcal{L}}\mathcal{L}$ contains the identity.

From this, it results clear that reversing the orientation of each defect gives the defect we started with.

The most interesting defect for our purposes is $N$ since it is the only non invertible one. Let us have a brief look at how can we use this defect to impose constraints on the correlators of our theory.

First of all consider the situation in which we have $N$ besides an operator $O$ as in figure (3.18). It is now possible to "bring the defect on the other side". When doing a partial fusion, the defect we will find, is the same defects one finds in the total fusion. As argued before locally they are the same thing. What instead changes is the presence of two junctions that connect all our defects. In the end collapsing $N$ on our operator gives a certain operator $\tilde{O}$ that starts the defect it is attached to.
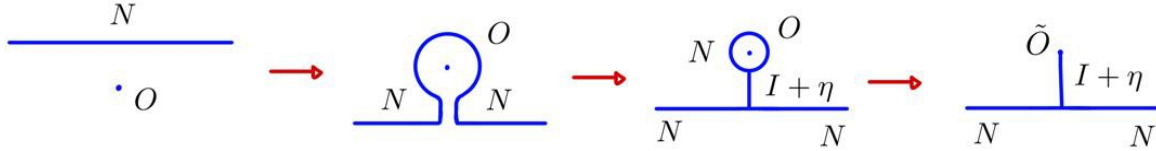


Fig.3.18: Passing the defect $N$ through an operator $O$

Knowing this one can carry out a transformation like the one in figure (3.19) on a correlator.
Consider a four points correlator, adding $N$ around the four points corresponds to multiplying the correlator by $\sqrt{2}$. In fact, to send the defect to infinity, is the same as applying it to $|0\rangle$.
After that one can bring the defect inside the four points, in the same way as we have just done in figure (3.18), and then collapse it. This last step, as before, gives a contribute of $\sqrt{2}$.
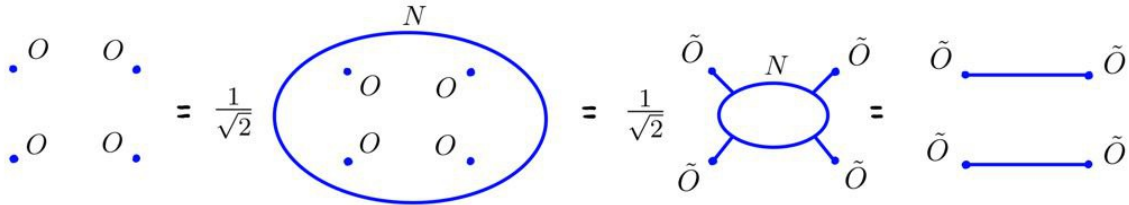


Fig.3.19: Correlators equalities

Finally we obtained what we set out to do: a non trivial equality between correlators.

Even though the Ising model is a completely solved theory in general one might conduct the analysis in an unsolved one. Therefore in principle this kind of equalities may help us in determining all the correlators of a theory or at least explain why certain correlators might be equal or even vanish.

## 4.2 Defects preserving $\mathfrak{su}(2)_1^6 \times \overline{\mathfrak{su}(2)}_1^6$

Now that we have made an example of a complete analysis on defects, let us start to look at the theory we want to do the calculation on: $\mathfrak{su}(2)_1^6 \times \overline{\mathfrak{su}(2)}_1^6$.

As the first step, we start by considering a simpler theory which is the diagonal one of the algebra $\mathfrak{su}(2)_1 \times \overline{\mathfrak{su}(2)}_1$:

$$Z = \chi_0 \overline{\chi}_0 + \chi_1 \overline{\chi}_1. \tag{4.2.1}$$

This is the partition function with the representations we have already seen in section (2.4).

Before going on with the discussion about the defects let us point out an important difference between this case and the previous one. In the algebra we are considering it is possible to show that its representations, $\lambda = 0$ and $\lambda = 1$, contain an infinite number of representations of the Virasoro algebra. This fact makes it impossible to divide the partition functions in Virasoro characters and therefore follow with the same analysis we completed for the Ising model.

What we can instead do is to restrict ourselves to the defects that are not only topological but also commute with the full algebra $\mathfrak{su}(2)_1$. In such case the defect will act as a constant on its two representations, exactly as it does when we have a finite number of representations of the Virasoro algebra.

Now one could use the same technique we used in the previous section, but for diagonal theories a theorem helps us. Such theorem states that, if a theory is diagonal and has a finite number of representations of its algebra $A \times \overline{A}$, the simple defects commuting with $A \times \overline{A}$ are the *Verlinde lines*. Such defects $\mathcal{L}_i$ are in one to one correspondence with the irreducible representations of $A$ and acts on the primary states as:

$$\hat{\mathcal{L}}_i|j,\overline{j}\rangle = \frac{S_{ij}}{S_{0j}}|j,\overline{j}\rangle, \qquad (4.2.2)$$

where $S$ is the matrix responsible for the the corresponding modular transformation. More informations on this argument can be found in article [10].

In this case using formula (2.5.18) we find two defects:

$$
\begin{aligned}
&I \\
&\hat{I}|0\rangle = |0\rangle \qquad\qquad \hat{I}|1\rangle = |1\rangle, \\
&\mathcal{L}_1 \\
&\hat{\mathcal{L}}_1|0\rangle = |0\rangle \qquad\qquad \hat{\mathcal{L}}_1|1\rangle = -|1\rangle.
\end{aligned}
\qquad (4.2.3)
$$

Notice that they are both invertible and the algebra is straightforward: $\hat{\mathcal{L}}_1^2 = I$.
$\mathcal{L}_1$ comes from a $\mathbb{Z}_2$ symmetry, in agreement with its algebra. This does not bring anything new to what we already know about the theory, in fact, as mentioned in the introduction, all its symmetries have already been studied.

Now we take into consideration the full algebra: a $\mathfrak{su}(2)_1^6 \times \overline{\mathfrak{su}(2)}_1^6$ diagonal theory.
The partition function is:

$$Z = \sum_{i_1,i_2,i_3,i_4,i_5,i_6=0}^{1} \chi_{i_1 i_2 i_3 i_4 i_5 i_6} \overline{\chi}_{i_1 i_2 i_3 i_4 i_5 i_6}. \qquad (4.2.4)$$

The various representations are marked by six indices $i_k$ and each of them can assume the values 0 or 1.
As before, to find the defects, one just needs to use the Verlinde lines:

$$\hat{\mathcal{L}}_{i_1 i_2 i_3 i_4 i_5 i_6}|j_1 j_2 j_3 j_4 j_5 j_6\rangle = \frac{S_{i_1 i_2 i_3 i_4 i_5 i_6 j_1 j_2 j_3 j_4 j_5 j_6}}{S_{000000 j_1 j_2 j_3 j_4 j_5 j_6}}|j_1 j_2 j_3 j_4 j_5 j_6\rangle. \qquad (4.2.5)$$

The S modular matrix can be easily found by taking the tensor product of the S modular matrices of each $\mathfrak{su}(2)_1$ factor. Then:

$$\hat{\mathcal{L}}_{i_1 i_2 i_3 i_4 i_5 i_6}|j_1 j_2 j_3 j_4 j_5 j_6\rangle = \prod_{k=1}^{6} \frac{\sin(\frac{\pi}{3}(i_k+1)(j_k+1))}{\sin(\frac{\pi}{3}(j_k+1))}|j_1 j_2 j_3 j_4 j_5 j_6\rangle. \qquad (4.2.6)$$

Now let us simplify the expression:

$$\frac{\sin(\frac{\pi}{3}(i_k+1)(j_k+1))}{\sin(\frac{\pi}{3}(j_k+1))} = \begin{cases} 1 & i_k = 0 \\ 1 & i_k = 1 \ j_k = 0 \\ -1 & i_k = 1 \ j_k = 1 \end{cases} = (-1)^{i_k j_k}. \qquad (4.2.7)$$

In total then one gets:

$$\hat{\mathcal{L}}_{i_1 i_2 i_3 i_4 i_5 i_6}|j_1 j_2 j_3 j_4 j_5 j_6\rangle = \prod_{k=1}^{6}(-1)^{i_k j_k}|j_1 j_2 j_3 j_4 j_5 j_6\rangle = (-1)^{\sum_{k=1}^{6} i_k j_k}|j_1 j_2 j_3 j_4 j_5 j_6\rangle. \qquad (4.2.8)$$

Since these defects act as the identity on $|000000\rangle$, they are all invertible. To understand what group they form let us look at the algebra of these defects:

$$
\begin{aligned}
\hat{\mathcal{L}}_{i_1^1 i_2^1 i_3^1 i_4^1 i_5^1 i_6^1} \hat{\mathcal{L}}_{i_1^2 i_2^2 i_3^2 i_4^2 i_5^2 i_6^2} |j_1 j_2 j_3 j_4 j_5 j_6\rangle &= (-1)^{\sum_{k=1}^6 (i_k^1 + i_k^2) j_k} |j_1 j_2 j_3 j_4 j_5 j_6\rangle \\
&= (-1)^{\sum_{k=1}^6 i_k^3 j_k} |j_1 j_2 j_3 j_4 j_5 j_6\rangle \\
&= \hat{\mathcal{L}}_{i_1^3 i_2^3 i_3^3 i_4^3 i_5^3 i_6^3} |j_1 j_2 j_3 j_4 j_5 j_6\rangle,
\end{aligned} \tag{4.2.9}
$$

with $i_k^3 = (i_k^1 + i_k^2) \mod 2$.
This is exactly the algebra of the group

$$
\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2. \tag{4.2.10}
$$

As one could have expected from the simplified analysis, we could not find anything new, no non-invertible defect. The reason behind this may be because asking for the defects to commute with the whole $\mathfrak{su}(2)_1^6 \times \overline{\mathfrak{su}(2)}_1^6$ algebra is too restrictive to get anything interesting. In the next section we will try to address this point.

## 4.3   Free fermion construction of $Vir_{m=3} \times \mathfrak{su}(2)_2$

First of all let us explain the main points of the analysis we are going to carry on from this point.
Since the previous constraints were to severe, we were not able to find any useful defect. On the other hand we are not able to describe all topological defects commuting with the Virasoro algebra, because the spectrum contains an infinite number of Virasoro representations. What we will do is to find a middle ground between these two possibilities, looking for a sub algebra of $\mathfrak{su}(2)_1^6 \times \overline{\mathfrak{su}(2)}_1^6$ that is still rational, which means to have a finite number of representations.

We will find that the main challenge of the theory is that the partition function will not be diagonal and therefore we will not be able to use the Verlinde lines. Nevertheless, there are other constraints one can use to determine the theory's defects.
Finally, to simplify the calculations, we will start by considering only $\mathfrak{su}(2)_1^2 \times \overline{\mathfrak{su}(2)}_1^2$.

Now that everything has been clarified, let us make the following consideration: $\mathfrak{su}(2)_1^2 \times \overline{\mathfrak{su}(2)}_1^2$ can be easily decomposed using the coset construction.
We have already seen that

$$
\frac{\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1}{\mathfrak{su}(2)_2} \to Vir_{m=3}. \tag{4.3.1}
$$

Using equation (2.4.64) it is then possible to calculate the branching rules which, used in the characters, give the following expressions:

$$
\begin{aligned}
\lambda_{00} &= \chi_{(1,1)0} + \chi_{(2,1)2}; \\
\lambda_{01} &= \chi_{(2,2)1}; \\
\lambda_{10} &= \chi_{(2,2)1}; \\
\lambda_{11} &= \chi_{(2,1)0} + \chi_{(1,1)2},
\end{aligned} \tag{4.3.2}
$$

where $\lambda_{ij}$ with $i,j \in \{0,1\}$ are characters of $\mathfrak{su}_1^2$, while $\chi_{(p,q),k}$ are characters of $Vir_{m=3} \times \mathfrak{su}(2)_2$.

In particular the pair $(p,q)$ determine the Virasoro minimal model representation through equation (2.4.32), while $k$ determines the representation of $\mathfrak{su}(2)_2$.
Also we are taking the convention $3q < 4p$. This selects one of the two possible ways in which such a representation can be written using the symmetry $\{p \to m-p, q \to m+1-q\}$.

To be more concrete let us make the following considerations: $Vir_{m=3}$ is the Ising model, $(p,q) = (1,1)$ corresponds to the $h = 0$ representation, $(p,q) = (2,1)$ to the $h = \frac{1}{2}$ and $(p,q) = (2,2)$ to the $h = \frac{1}{16}$.

Then, relabeling the characters of $Vir_{m=3}$ with the conformal weights instead of the two numbers $(p, q)$, we get:

$$
\begin{aligned}
\lambda_{00} &= \chi_{0,0} + \chi_{\frac{1}{2},2}; \\
\lambda_{01} &= \chi_{\frac{1}{16},1}; \\
\lambda_{10} &= \chi_{\frac{1}{16},1}; \\
\lambda_{11} &= \chi_{\frac{1}{2},0} + \chi_{0,2}.
\end{aligned}
\tag{4.3.3}
$$

Now we should write the $\mathfrak{su}(2)_1^2 \times \overline{\mathfrak{su}(2)}_1^2$ diagonal partition function and then decompose it using equation (4.3.3):

$$
\begin{aligned}
Z &= \lambda_{00}\overline{\lambda}_{00} + \lambda_{10}\overline{\lambda}_{10} + \lambda_{01}\overline{\lambda}_{01} + \lambda_{11}\overline{\lambda}_{11} \\
&= (\chi_{0,0} + \chi_{\frac{1}{2},2})(\overline{\chi}_{0,0} + \overline{\chi}_{\frac{1}{2},2}) + \chi_{\frac{1}{16},1}\overline{\chi}_{\frac{1}{16},1} + \chi_{\frac{1}{16},1}\overline{\chi}_{\frac{1}{16},1} + \\
&\quad (\chi_{\frac{1}{2},0} + \chi_{0,2})(\overline{\chi}_{\frac{1}{2},0} + \overline{\chi}_{0,2}) \\
&= \chi_{0,0}\overline{\chi}_{0,0} + \chi_{0,0}\overline{\chi}_{\frac{1}{2},2} + \chi_{\frac{1}{2},2}\overline{\chi}_{0,0} + \chi_{\frac{1}{2},2}\overline{\chi}_{\frac{1}{2},2} + 2\chi_{\frac{1}{16},1}\overline{\chi}_{\frac{1}{16},1} + \\
&\quad \chi_{\frac{1}{2},0}\overline{\chi}_{\frac{1}{2},0} + \chi_{\frac{1}{2},0}\overline{\chi}_{0,2} + \chi_{0,2}\overline{\chi}_{\frac{1}{2},0} + \chi_{0,2}\overline{\chi}_{0,2}.
\end{aligned}
\tag{4.3.4}
$$

It is better to write partition functions in matrix form. We start with

$$
Z = \begin{pmatrix} \lambda_{00} & \lambda_{01} & \lambda_{10} & \lambda_{11} \end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\overline{\lambda}_{00} \\
\overline{\lambda}_{01} \\
\overline{\lambda}_{10} \\
\overline{\lambda}_{11}
\end{pmatrix}.
\tag{4.3.5}
$$

Calling then

$$
V = \begin{pmatrix} \chi_{0,0} & \chi_{\frac{1}{2},2} & \chi_{\frac{1}{16},1} & \chi_{\frac{1}{2},0} & \chi_{0,2} & \chi_{0,1} & \chi_{\frac{1}{16},0} & \chi_{\frac{1}{2},1} & \chi_{\frac{1}{16},2} \end{pmatrix}
\tag{4.3.6}
$$

we also write:

$$
Z = V
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\overline{V}^T \equiv \sum_{i,j} V_i M_{ij}^{Th} \overline{V}_j.
\tag{4.3.7}
$$

where the last step is a definition for $M^{Th}$.

In the next section we are going to find the defects, but before that, to make the following consideration, will turn out to be useful. Both the algebras $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$, $\mathfrak{su}(2)_2 \times Vir_{m=3}$ and their representation can be described using a CFT of four free Majorana fermions. We will show this starting with the first theory then passing to the second one which is the one we are mainly interested in.

Let us consider four Majorana fermions $\psi^1$, $\psi^2$, $\psi^3$ and $\psi^4$. Each fermion will be described by a theory identical to the one in section (2.4).

It is possible to organize $\psi^1$, $\psi^2$, $\psi^3$ and $\psi^4$ in the following complex fermions:

$$
\text{with} \quad \Psi_j = \frac{1}{\sqrt{2}}(\psi_{2j-1}(z) + i\psi_{2j}(z)) \quad \Psi_j^* = \frac{1}{\sqrt{2}}(\psi_{2j-1}(z) - i\psi_{2j}(z)) \quad j = 1, 2.
\tag{4.3.8}
$$

Using the OPEs for the Majorana fermions in section (2.4), one can show that:

$$\Psi_i(z)\Psi_j^*(w) \sim \frac{\delta^{ij}}{z-w} \qquad \Psi_i^*(z)\Psi_j^*(w) \sim 0 \qquad \Psi_i(z)\Psi_j(w) \sim 0. \qquad (4.3.9)$$

Then, with an easy calculation, we find that the following currents have OPEs which correspond to the algebra $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$:

$$
\begin{aligned}
J^{3,1}(z) &= \frac{1}{2}((\Psi_1^*\Psi_1)(z) + (\Psi_2^*\Psi_2)(z)); \\
J^{+,1} &= i(\Psi_1^*\Psi_2^*)(z) \qquad J^{-,1} = i(\Psi_1\Psi_2)(z); \\
J^{3,2}(z) &= \frac{1}{2}((\Psi_1^*\Psi_1)(z) - (\Psi_2^*\Psi_2)(z)); \\
J^{+,2} &= i(\Psi_1^*\Psi_2)(z) \qquad J^{-,2} = i(\Psi_1\Psi_2^*)(z);
\end{aligned}
\qquad (4.3.10)
$$

where the generators labelled with 1 give rise to the first $\mathfrak{su}(2)_1$ and commute with all the generators labelled with 2, which instead form the second $\mathfrak{su}(2)_1$.

With these currents, one can apply the Sugawara construction in order to obtain an energy-momentum tensor. What one can show is that such tensor will be equal to the energy-momentum tensor of the theory with the four fermions.

Furthermore one can describe all representations of $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$ in terms of fermions.

First of all let us consider the (NS) sector of the fermionic theory.

In this sector the modes of the various currents are:

$$
\begin{aligned}
J_n^{3,1} &= \frac{i}{2}\Big[ \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^2_k - \sum_{k\leq-\frac{1}{2}} \psi^2_k\psi^1_{n-k} + \sum_{k>-\frac{1}{2}} \psi^3_{n-k}\psi^4_k - \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^3_{n-k}\Big], \\
J_n^{+,1} &= \frac{i}{2}\Big[ \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^3_k - \sum_{k\leq-\frac{1}{2}} \psi^3_k\psi^1_{n-k} - \sum_{k>-\frac{1}{2}} \psi^2_{n-k}\psi^4_k + \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^2_{n-k} + \\
&\quad -i\big( \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^4_k - \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^1_{n-k}\big) - i\big( \sum_{k>-\frac{1}{2}} \psi^2_{n-k}\psi^3_k - \sum_{k\leq-\frac{1}{2}} \psi^3_k\psi^2_{n-k}\big)\Big], \\
J_n^{-,1} &= \frac{i}{2}\Big[ \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^3_k - \sum_{k\leq-\frac{1}{2}} \psi^3_k\psi^1_{n-k} - \sum_{k>-\frac{1}{2}} \psi^2_{n-k}\psi^4_k + \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^2_{n-k} + \\
&\quad +i\big( \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^4_k - \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^1_{n-k}\big) + i\big( \sum_{k>-\frac{1}{2}} \psi^2_{n-k}\psi^3_k - \sum_{k\leq-\frac{1}{2}} \psi^3_k\psi^2_{n-k}\big)\Big], \\
J_n^{3,2} &= \frac{i}{2}\Big[ \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^2_k - \sum_{k\leq-\frac{1}{2}} \psi^2_k\psi^1_{n-k} - \sum_{k>-\frac{1}{2}} \psi^3_{n-k}\psi^4_k + \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^3_{n-k}\Big], \\
J_n^{+,2} &= \frac{i}{2}\Big[ \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^3_k - \sum_{k\leq-\frac{1}{2}} \psi^3_k\psi^1_{n-k} + \sum_{k>-\frac{1}{2}} \psi^2_{n-k}\psi^4_k - \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^2_{n-k} + \\
&\quad +i\big( \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^4_k - \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^1_{n-k}\big) - i\big( \sum_{k>-\frac{1}{2}} \psi^2_{n-k}\psi^3_k - \sum_{k\leq-\frac{1}{2}} \psi^3_k\psi^2_{n-k}\big)\Big], \\
J_n^{-,2} &= \frac{i}{2}\Big[ \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^3_k - \sum_{k\leq-\frac{1}{2}} \psi^3_k\psi^1_{n-k} + \sum_{k>-\frac{1}{2}} \psi^2_{n-k}\psi^4_k - \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^2_{n-k} + \\
&\quad -i\big( \sum_{k>-\frac{1}{2}} \psi^1_{n-k}\psi^4_k - \sum_{k\leq-\frac{1}{2}} \psi^4_k\psi^1_{n-k}\big) + i\big( \sum_{k>-\frac{1}{2}} \psi^2_{n-k}\psi^3_k - \sum_{k\leq-\frac{1}{2}} \psi^3_k\psi^2_{n-k}\big)\Big].
\end{aligned}
\qquad (4.3.11)
$$

Then, using these expressions, we want to find the highest weights of the representations using the conditions posed in (2.4.56).

Let $|0\rangle_{NS}$ denote the $(NS)$ vacuum of the free theory, then it is easy to verify that:

$$
\begin{aligned}
J_n^{3,i}|0\rangle_{NS} = J_n^{\pm,i}|0\rangle_{NS} = 0 \qquad & n > 0; \\
J_0^{3,i}|0\rangle_{NS} = 0; & \\
J_0^{+,i}|0\rangle_{NS} = 0 \qquad & i = 1,2.
\end{aligned}
\tag{4.3.12}
$$

This implies that $|0\rangle_{NS}$ corresponds to a primary field $\phi_{00}$ for $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$, which is the vacuum representation, corresponding to $\lambda_{00}$.

An intuitive attempt to find the other representations would be to apply the following modes to the vacuum: $\psi^1_{-\frac{1}{2}}$, $\psi^2_{-\frac{1}{2}}$, $\psi^3_{-\frac{1}{2}}$ and $\psi^4_{-\frac{1}{2}}$. Indeed, these states are not $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$ descendants of the vacuum, therefore they must belong to some other representation.

Moreover the conformal weight of the states $\psi^1_{-\frac{1}{2}}|0\rangle$, $\psi^2_{-\frac{1}{2}}|0\rangle$, $\psi^3_{-\frac{1}{2}}|0\rangle$ and $\psi^4_{-\frac{1}{2}}|0\rangle$ is $\frac{1}{2}$. This is the same weight as $\phi_{11}|0\rangle$, where we are defining in a general way $\phi_{ij}$ as the primary field corresponding to the character $\lambda_{ij}$, calculated using equation (2.4.61). Therefore, from these four states, we expect to find $\phi_{11}|0\rangle$.

First of all let us make the following consideration:

$$
\begin{aligned}
J_n^{3,i}\psi^j_{-\frac{1}{2}}|0\rangle_{NS} = [J_n^{3,i},\psi^j_{-\frac{1}{2}}]|0\rangle_{NS} \qquad & n > 0; \\
J_n^{\pm,i}\psi^j_{-\frac{1}{2}}|0\rangle_{NS} = [J_n^{\pm,i},\psi^j_{-\frac{1}{2}}]|0\rangle_{NS}; & \\
J_0^{3,i}\psi^j_{-\frac{1}{2}}|0\rangle_{NS} = [J_0^{3,i},\psi^j_{-\frac{1}{2}}]|0\rangle_{NS} \qquad & i = 1,2; \\
J_0^{+,i}\psi^j_{-\frac{1}{2}}|0\rangle_{NS} = [J_0^{+,i},\psi^j_{-\frac{1}{2}}]|0\rangle_{NS} \qquad & j = 1,2,3,4.
\end{aligned}
\tag{4.3.13}
$$

Then it turns out that the commutator between any mode $n$ of any generator with any field $\psi^i_{-\frac{1}{2}}$ will give a linear combination of fields of the type $\psi^j_{n-\frac{1}{2}}$. Such fields obey the following relation: $\psi^j_{n-\frac{1}{2}}|0\rangle = 0$ for $n > 0$. Therefore, to check equation (2.4.56), we just have to consider the $n = 0$ modes.

With a long calculation one finds:

$$
\begin{aligned}
[J_0^{3,j},\psi^1_{-\frac{1}{2}}] = -\frac{i}{2}\psi^2_{-\frac{1}{2}} \qquad & [J_0^{3,j},\psi^2_{-\frac{1}{2}}] = \frac{i}{2}\psi^1_{-\frac{1}{2}}, \\
[J_0^{3,j},\psi^3_{-\frac{1}{2}}] = (-)^j\frac{i}{2}\psi^4_{-\frac{1}{2}} \qquad & [J_0^{3,j},\psi^4_{-\frac{1}{2}}] = (-)^{j-1}\frac{i}{2}\psi^3_{-\frac{1}{2}}.
\end{aligned}
\tag{4.3.14}
$$

This can be reorganized as in equation (4.3.8):

$$
\begin{aligned}
[J_0^{3,1},\Psi^{1,*}_{-\frac{1}{2}}] = \frac{1}{2}\Psi^{1,*}_{-\frac{1}{2}} \qquad & [J_0^{3,2},\Psi^{1,*}_{-\frac{1}{2}}] = \frac{1}{2}\Psi^{1,*}_{-\frac{1}{2}}, \\
[J_0^{3,1},\Psi^{2,*}_{-\frac{1}{2}}] = \frac{1}{2}\Psi^{2,*}_{-\frac{1}{2}} \qquad & [J_0^{3,2},\Psi^{2,*}_{-\frac{1}{2}}] = -\frac{1}{2}\Psi^{2,*}_{-\frac{1}{2}}, \\
[J_0^{3,1},\Psi^2_{-\frac{1}{2}}] = -\frac{1}{2}\Psi^2_{-\frac{1}{2}} \qquad & [J_0^{3,2},\Psi^2_{-\frac{1}{2}}] = \frac{1}{2}\Psi^2_{-\frac{1}{2}}, \\
[J_0^{3,1},\Psi^1_{-\frac{1}{2}}] = -\frac{1}{2}\Psi^1_{-\frac{1}{2}} \qquad & [J_0^{3,2},\Psi^1_{-\frac{1}{2}}] = -\frac{1}{2}\Psi^1_{-\frac{1}{2}}.
\end{aligned}
\tag{4.3.15}
$$

This clearly looks like the 0 mode representation corresponding to the character $\lambda_{11}$.

At last one easily verifies that

$$
[J_0^{+,1},\Psi^{1,*}_{-\frac{1}{2}}] = 0 \qquad [J_0^{+,2},\Psi^{1,*}_{-\frac{1}{2}}] = 0,
\tag{4.3.16}
$$

confirming that $\Psi^{1,*}_{-\frac{1}{2}}|0\rangle_{NS}$ is indeed $\phi_{11}|0\rangle_{NS}$.

To find the last two representation: $\phi_{10}$ and $\phi_{01}$ of $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$ we have to use the (R) sector. As already mentioned in section (2.4) the ground state of this sector is degenerate. When we have four fermions the construction of the degenerate states is as follows.

First of all we note the following anticommutation rules:

$$
\begin{aligned}
\{\Psi_0^a, \Psi_0^{b,*}\} &= \delta^{ab}, \\
\{\Psi_0^a, \Psi_0^b\} &= \{\Psi_0^{a,*}, \Psi_0^{b,*}\} = 0.
\end{aligned}
\tag{4.3.17}
$$

In particular $(\Psi_0^a)^2 = (\Psi_0^{a,*})^2 = 0$.

Now we can form a representation for the ground states first of all defining a lowest weight state $|0\rangle_R$ such that

$$
\Psi_0^{a,*}|0\rangle_R = 0 \qquad a = 1, 2.
\tag{4.3.18}
$$

Then the other ground states will be the followings:

$$
\Psi_0^1|0\rangle_R \qquad \Psi_0^2|0\rangle_R \qquad \Psi_0^1\Psi_0^2|0\rangle_R.
\tag{4.3.19}
$$

After that we have to consider the modes of the various currents. For the most part they are the same as in the $(NS)$ sector, but $J_0^{3,1}$ and $J_0^{3,2}$ suffer from the same ordering ambiguity we had when we were determining the zero mode of the Virasoro algebra for the free fermion. The ambiguity can be fixed by requiring $J_0^{3,1}$ and $J_0^{3,2}$ to obey the expected commutation relations:

$$
[J_1^{+,1}, J_{-1}^{-,1}] = 1 + 2J_0^{3,1} \qquad [J_1^{+,2}, J_{-1}^{-,2}] = 1 + 2J_0^{3,2}.
\tag{4.3.20}
$$

Imposing such relations one is able to determine the following expressions:

$$
\begin{aligned}
J_0^{3,1} &= \frac{i}{2}\Big(\sum_{k\geq 0}\psi_{-k}^1\psi_k^2 - \sum_{k<0}\psi_k^2\psi_{-k}^1 + \sum_{k\geq 0}\psi_{-k}^3\psi_k^4 - \sum_{k<0}\psi_k^4\psi_{-k}^3\Big), \\
J_0^{3,2} &= \frac{i}{2}\Big(\sum_{k\geq 0}\psi_{-k}^1\psi_k^2 - \sum_{k<0}\psi_k^2\psi_{-k}^1 - \sum_{k\geq 0}\psi_{-k}^3\psi_k^4 + \sum_{k<0}\psi_k^4\psi_{-k}^3\Big).
\end{aligned}
\tag{4.3.21}
$$

Now since the (R) ground states of the free fermion theory have the same conformal weight as $\phi_{10}$ and $\phi_{01}$, which is $\frac{1}{4}$, let us look for the highest weights within those states.

First of all, as they are ground states, it is easy to show that any mode $J_n^{i,k}$ with $n > 0$ applied to them gives zero. What remains to check are, again, the $n = 0$ modes:

$$
\begin{aligned}
J_0^{3,1}|0\rangle_R &= \frac{1}{2}|0\rangle_R & J_0^{3,2}|0\rangle_R &= 0, \\
J_0^{3,1}\Psi_0^1\Psi_0^2|0\rangle_R &= -\frac{1}{2}\Psi_0^1\Psi_0^2|0\rangle_R & J_0^{3,2}\Psi_0^1\Psi^2|0\rangle_R &= 0, \\
J_0^{3,1}\Psi_0^2|0\rangle_R &= 0 & J_0^{3,2}\Psi_0^2|0\rangle_R &= \frac{1}{2}\Psi_0^2|0\rangle_R, \\
J_0^{3,1}\Psi_0^1|0\rangle_R &= 0 & J_0^{3,2}\Psi_0^1|0\rangle_R &= -\frac{1}{2}\Psi_0^1|0\rangle_R.
\end{aligned}
\tag{4.3.22}
$$

These look like the 0 mode representations corresponding to the characters $\chi_{10}$ and $\chi_{01}$.

As a last check, one can verify that

$$
\begin{aligned}
J_0^{+,1}|0\rangle_R &= 0 & J_0^{+,2}|0\rangle_R &= 0, \\
J_0^{+,1}\Psi_0^2|0\rangle_R &= 0 & J_0^{+,2}\Psi_0^2|0\rangle_R &= 0,
\end{aligned}
\tag{4.3.23}
$$

confirming that $|0\rangle_R = \phi_{10}|0\rangle_{NS}$ and $\Psi_0^2|0\rangle_R = \phi_{01}|0\rangle_{NS}$.

Now that we have determined the representations of $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$ we would like to do the same for $\mathfrak{su}(2)_2 \times Vir_{m=3}$. First of all we have to determine how the generators of the algebra are composed by fermions.

One can easily verify that the $\mathfrak{su}(2)_2$ subalgebra of $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$ is given by:

$$
\begin{aligned}
J^3(z) &= J^{3,1}(z) + J^{3,2}(z) = (\Psi_1^* \Psi_1)(z) = i(\psi_1 \psi_2)(z), \\
J^+(z) &= J^{+,1}(z) + J^{+,2}(z) = i(\Psi_1^* \Psi_2^*)(z) + i(\Psi_1^* \Psi_2)(z) = i[(\psi_1 \psi_3)(z) - i(\psi_2 \psi_3)(z)], \\
J^-(z) &= J^{-,1}(z) + J^{-,2}(z) = i(\Psi_1 \Psi_2)(z) + i(\Psi_1 \Psi_2^*)(z) = i[(\psi_1 \psi_3)(z) + i(\psi_2 \psi_3)(z)].
\end{aligned}
\tag{4.3.24}
$$

In these equations we can notice the complete absence of $\psi_4(z)$. Then if we write

$$
T(z) = \frac{1}{2} \psi_4 \partial \psi_4
\tag{4.3.25}
$$

we find a field that both commutes with the full $\mathfrak{su}(2)_2$ algebra and generates the Virasoro algebra $c = \frac{1}{2}$. Therefore all the generators of $\mathfrak{su}(2)_2 \times Vir_{m=3}$ have now been written.

As before we are interested in the representations and equation (4.3.3) is going to help us. The calculations may be a bit lengthy but, to verify the statements we are going to make, one just have to use the same procedures we showed for the representations of $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$.

Firstly, regarding the notation, we will call $\Phi_{i,j}$ the primary field corresponding to the character $\chi_{i,j}$. Then we start with:

$$
\begin{aligned}
\Phi_{0,2}|0\rangle_{NS} &= \Psi_{-\frac{1}{2}}^{1,*}|0\rangle_{NS}, \\
\Phi_{\frac{1}{2},0}|0\rangle_{NS} &= (\Psi_{-\frac{1}{2}}^{1} - \Psi_{-\frac{1}{2}}^{1,*})|0\rangle_{NS} = \sqrt{2}i\psi_{-\frac{1}{2}}^{4}|0\rangle_{NS}.
\end{aligned}
\tag{4.3.26}
$$

As expected they both come from the representation of $\phi_{11}$.

Next for $\Phi_{\frac{1}{16},1}^{(1)}$ and $\Phi_{\frac{1}{16},1}^{(2)}$ they are exactly respectively the representations $\phi_{10}$ and $\phi_{01}$:

$$
\begin{aligned}
\Phi_{\frac{1}{16},1}^{(1)}|0\rangle_{NS} &= |0\rangle_R, \\
\Phi_{\frac{1}{16},1}^{(2)}|0\rangle_{NS} &= \Psi_0^2|0\rangle_R.
\end{aligned}
\tag{4.3.27}
$$

The last one requires more work. As $|0\rangle_{NS}$ is obviously the representation for $\phi_{0,0}|0\rangle_{NS}$, we need to find $\phi_{\frac{1}{2},2}$.

First of all notice that every $J_n^{+,i}$ respect the commutation relation $[J_0^3, J_n^{+,i}] = J_n^{+,i}$, which means that the states $J_n^{+,i}|0\rangle_{(NS)}$ potentially could all give rise to $l = 2$ representations of $\mathfrak{su}(2)_2$.

In order to determine the $\mathfrak{su}(2)_2 \times Vir_{m=3}$ primary state, we impose $[J_m^-, a_1 J_{n_1}^{+,i} + a_2 J_{n_2}^{+,i} + ...]|0\rangle_{NS} = 0$ for $m > 0$. In total one finds

$$
\Phi_{\frac{1}{2},2}|0\rangle_{NS} = (J_{-1}^{+,1} - J_{-1}^{+,2})|0\rangle_{NS} = (\psi_{-\frac{1}{2}}^{1} - i\psi_{-\frac{1}{2}}^{2})\psi_{-\frac{1}{2}}^{4}|0\rangle_{NS}.
\tag{4.3.28}
$$

As we have determined all the highest weights we conclude this section. In the next section we will start with the analysis of defects.

## 4.4 Defects preserving $Vir_{m=3} \times \mathfrak{su}(2)_2$

Let us look back at the partition function we wrote in (4.3.7). Notice that it is not diagonal and therefore we cannot use the Verlinde lines. As per usual the first condition we can put on the defects is the one in equation (3.3.8) and, in order to do that, we need to know the S modular matrix.

The S modular matrix for $\mathfrak{su}(2)_2 \times Vir_{m=3}$ will be the tensor product of the S modular matrices of the two algebras $\mathfrak{su}(2)_2$ and $Vir_{m=3}$. Let us write it:

$$
S_{(p,q)l(p',q')l'}^{TOT} = S_{ll'}^2 S_{(p,q)(p',q')}^{Vir} = \frac{1}{\sqrt{3}}(-1)^{(p+q)(p'+q')} \sin(\frac{\pi}{4}(l+1)(l'+1)) \sin(\frac{\pi}{3}pp') \sin(\frac{\pi}{4}qq').
\tag{4.4.1}
$$

$S^2$ is the S modular matrix for $\mathfrak{su}(2)_2$ and $S^{Vir}$ is the one for $Vir_{m=3}$.

Calculating the matrix values for the various representation in the same order as the vector $V$ in equation (4.3.6), we get the following result:

$$
S^{TOT} = \frac{1}{4}
\begin{pmatrix}
1 & 1 & 2 & 1 & 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} \\
1 & 1 & 2 & 1 & 1 & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & 2 & 0 & -2 & -2 & 0 & 0 & 0 & 0 \\
1 & 1 & -2 & 1 & 1 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \\
1 & 1 & -2 & 1 & 1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} \\
\sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} & 0 & 2 & 0 & -2 \\
\sqrt{2} & -\sqrt{2} & 0 & -\sqrt{2} & \sqrt{2} & 2 & 0 & -2 & 0 \\
\sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} & 0 & -2 & 0 & 2 \\
\sqrt{2} & -\sqrt{2} & 0 & -\sqrt{2} & \sqrt{2} & -2 & 0 & 2 & 0
\end{pmatrix}
\tag{4.4.2}
$$

Now we parameterize the defects imposing that they commute with the $\mathfrak{su}(2)_2 \times Vir_{m=3} \times \overline{\mathfrak{su}(2)}_2 \times \overline{Vir}_{m=3}$ algebra.

As explained in section (3.3), the defects that commute with the algebra, when acting on a primary field with $\hat{\mathcal{L}}$, will send such field in a primary field of the same representation.

The decomposition in (4.3.7) of the partition function shows that there are eight representations of $\mathfrak{su}(2)_2 \times Vir_{m=3} \times \overline{\mathfrak{su}(2)}_2 \times \overline{Vir}_{m=3}$ and that seven of them have multiplicity 1 while $((2,2)1) \times \overline{((2,2)1)}$ has multiplicity 2. This means that $\hat{\mathcal{L}}$ will act on the non degenerate primary fields just multiplying them by a constant, while for the two $((2,2)1) \times \overline{((2,2)1)}$ primary fields it will act as a $2 \times 2$ matrix. We can therefore write:

$$
Z(\mathcal{L}) = V
\begin{pmatrix}
a & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
c & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & f & g & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & h & q & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\overline{V}^T ,
\tag{4.4.3}
$$

where $e = TrE$ with $E$ being a $2 \times 2$ matrix.

By the relation $Z_{\mathcal{L}}(\tau) = Z(\mathcal{L})(-1/\tau)$ and using using the modular transformation of the characters we find:

$$
\begin{cases}
a + b + c + d + 4e + f + g + h + q = 16n_1 \\
a + b + c + d - f - g - h - q = 8n_2 \\
a + b + c + d - 4e + f + g + h + q = 16n_3 \\
a + b + c + d + f + g + h + q = 4n_4 \\
\sqrt{2}(a - b + c - d + f - g + h - q) = 16n_5 \\
\sqrt{2}(a - b + c - d - f + g - h + q) = 16n_6 \\
\sqrt{2}(a + b - c - d + f + g - h - q) = 16n_7 \\
\sqrt{2}(a + b - c - d - f - g + h + q) = 16n_8 \\
a - b - c + d + f - g - h + q = 8n_9 \\
a - b - c + d - f + g + h - q = 8n_{10}
\end{cases}
\qquad
Z_{\mathcal{L}} = V
\begin{pmatrix}
n_1 & n_1 & n_2 & n_3 & n_3 & n_5 & n_6 & n_5 & n_6 \\
n_1 & n_1 & n_2 & n_3 & n_3 & n_5 & n_6 & n_5 & n_6 \\
n_2 & n_2 & n_4 & n_2 & n_2 & 2n_6 & 2n_5 & 2n_6 & 2n_5 \\
n_3 & n_3 & n_2 & n_1 & n_1 & n_5 & n_6 & n_5 & n_6 \\
n_3 & n_3 & n_2 & n_1 & n_1 & n_5 & n_6 & n_5 & n_6 \\
n_7 & n_7 & 2n_8 & n_7 & n_7 & n_9 & n_{10} & n_9 & n_{10} \\
n_8 & n_8 & 2n_7 & n_8 & n_8 & n_{10} & n_9 & n_{10} & n_9 \\
n_7 & n_7 & 2n_8 & n_7 & n_7 & n_9 & n_{10} & n_9 & n_{10} \\
n_8 & n_8 & 2n_7 & n_8 & n_8 & n_{10} & n_9 & n_{10} & n_9
\end{pmatrix}
\overline{V}^T ,
$$

$$\tag{4.4.4}$$

where $n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9$ and $n_{10}$ are non negative integers, related to $a, b, c, d, e, f, g, h$ and $q$ by the interpretation of $Z_{\mathcal{L}}$ as a sum of characters of representations of the algebra $\mathfrak{su}(2)_2 \times Vir_{m=3} \times \overline{\mathfrak{su}(2)}_2 \times \overline{Vir}_{m=3}$ .

With some manipulations one can arrive at the following form

$$
\begin{cases}
a = n_1 + n_2 + n_3 + n_9 + n_{10} + \sqrt{2}(n_5 + n_6) + \sqrt{2}(n_7 + n_8) \\
b = n_1 + n_2 + n_3 - n_9 - n_{10} - \sqrt{2}(n_5 + n_6) + \sqrt{2}(n_7 + n_8) \\
c = n_1 + n_2 + n_3 - n_9 - n_{10} + \sqrt{2}(n_5 + n_6) - \sqrt{2}(n_7 + n_8) \\
d = n_1 + n_2 + n_3 + n_9 + n_{10} - \sqrt{2}(n_5 + n_6) - \sqrt{2}(n_7 + n_8) \\
e = 2(n_1 - n_3) \\
f = n_1 - n_2 + n_3 + n_9 - n_{10} + \sqrt{2}(n_5 - n_6) + \sqrt{2}(n_7 - n_8) \\
g = n_1 - n_2 + n_3 - n_9 + n_{10} - \sqrt{2}(n_5 - n_6) + \sqrt{2}(n_7 - n_8) \\
h = n_1 - n_2 + n_3 - n_9 + n_{10} + \sqrt{2}(n_5 - n_6) - \sqrt{2}(n_7 - n_8) \\
q = n_1 - n_2 + n_3 + n_9 - n_{10} - \sqrt{2}(n_5 - n_6) - \sqrt{2}(n_7 - n_8) \\
n_4 = 2(n_1 + n_3)
\end{cases}
\tag{4.4.5}
$$

From the last expression it may look like we have nine simple defects, but we should not forget that these equations put constraints only on $e = TrE$, but otherwise leave the matrix $E$ unconstrained.

Representing the defects with its values

$$
\begin{pmatrix} a & b & c & d & E & f & g & h & q \end{pmatrix},
\tag{4.4.6}
$$

a "defect of type $n_i$" is one with the corresponding following form:

$$
\begin{array}{llll}
n_1: & \begin{pmatrix} 1 & 1 & 1 & 1 & E_1 & 1 & 1 & 1 & 1 \end{pmatrix} & tr E_1 = 2, \\
n_2: & \begin{pmatrix} 1 & 1 & 1 & 1 & E_2 & -1 & -1 & -1 & -1 \end{pmatrix} & tr E_2 = 0, \\
n_3: & \begin{pmatrix} 1 & 1 & 1 & 1 & E_3 & 1 & 1 & 1 & 1 \end{pmatrix} & tr E_3 = -2, \\
n_5: & \begin{pmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & E_5 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \end{pmatrix} & tr E_5 = 0, \\
n_6: & \begin{pmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & E_6 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{pmatrix} & tr E_6 = 0, \\
n_7: & \begin{pmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & E_7 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \end{pmatrix} & tr E_7 = 0, \\
n_8: & \begin{pmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & E_8 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix} & tr E_8 = 0, \\
n_9: & \begin{pmatrix} 1 & -1 & -1 & 1 & E_9 & 1 & -1 & -1 & 1 \end{pmatrix} & tr E_9 = 0, \\
n_{10}: & \begin{pmatrix} 1 & -1 & -1 & 1 & E_{10} & -1 & 1 & 1 & -1 \end{pmatrix} & tr E_{10} = 0.
\end{array}
\tag{4.4.7}
$$

As we were saying for each "type $n_i$" there might be different defect, which differ from one another by the specific form of the matrix $E$. Let us call $m_i$ the number of non equivalent defects of type $n_i$ and denote such defects as $\mathcal{L}_i^{(1)}, \mathcal{L}_i^{(2)}, ..., \mathcal{L}_i^{(m_i)}$.

In order to calculate each $m_i$ we have to use techniques more sophisticated than the constraints applied so far. For example, in article [9], one considers correlators of theories on the torus in which one inserts two parallel defects. Imposing then modularity and consistency on these correlators, one finds out ulterior constraints on the defects.
We will not report the argument in detail, but just use the formula (4.4.9).

First of all let us introduce the following notation for the partition functions:

$$
Z = \sum_{i,j} V^i Z_{ij} \overline{V}^j \qquad\qquad Z_{\mathcal{L}} = \sum_{i,j} V^i Z_{ij}^{\mathcal{L}} \overline{V}^j.
\tag{4.4.8}
$$

Then the following equality holds

$$
\sum_{i_3 j_3} N_{i_1 i_2}^{i_3} N_{j_1 j_2}^{j_3} Z_{i_3 j_3} = \sum_{\mathcal{L}} Z_{i_1 j_1}^{\mathcal{L}} Z_{i_2 j_2}^{\overline{\mathcal{L}}}
$$

$$
\text{with} \quad N_{ij}^k = \sum_m \frac{S_{im} S_{jm} S_{mk}^*}{S_{0m}}.
\tag{4.4.9}
$$

$S$ is the S modular matrix of the theory and $\sum_{\mathcal{L}}$ is the sum over all simple defects.

At the end of our analysis we will be able to write explicitly all of our defects, with that it will be easy to show that in our case $Z_{\mathcal{L}} = Z_{\overline{\mathcal{L}}}$. Knowing this, we can write the previous equation in the following form

$$\sum_{i_3 j_3} N^{i_3}_{i_1 i_2} N^{j_3}_{j_1 j_2} Z_{i_3 j_3} = \sum_{i=1, i \neq 4}^{10} m_i Z^{n_i}_{i_1 j_1} Z^{n_i}_{i_2 j_2}, \tag{4.4.10}$$

where $Z_{n_i}$ is the partition function of a simple defect of type $n_i$ .

Now the only things left to determine in equation (4.4.10) are the various values of $m_i$. Using smart choices for the four numbers $(i_1, i_2, j_1, j_2)$ we can even isolate each $m_i$ like:

$$(i_1, i_2, j_1, j_2) = (1, 1, 1, 1) \quad \rightarrow \quad 1 = m_1 \tag{4.4.11}$$

Without doing explicitly the full calculation we will just report the results. It turns out that we have two defects of the types $n_2$, $n_9$ and $n_{10}$, while all the other types only get one defect.
In total then we have twelve defects which we will call: $\mathcal{L}_1$, $\mathcal{L}_2^{(1)}$, $\mathcal{L}_2^{(2)}$, $\mathcal{L}_3$, $\mathcal{L}_5$, $\mathcal{L}_6$, $\mathcal{L}_7$, $\mathcal{L}_8$, $\mathcal{L}_9^{(1)}$, $\mathcal{L}_9^{(2)}$, $\mathcal{L}_{10}^{(1)}$ and $\mathcal{L}_{10}^{(2)}$.

Notice that the parameter $a$ is just the quantum dimension $\langle \mathcal{L} \rangle$. From this observation and by equation (4.4.7), we can already recognize which defects are non invertible: $\mathcal{L}_5$, $\mathcal{L}_6$, $\mathcal{L}_7$ and $\mathcal{L}_8$.

As the last step will use different techniques in order to determine the unknown parameters in the matrices of which, up to now, we only know the trace.

The first consideration we can make is that the only defect compatible with the identity is $\mathcal{L}_1$. Then we must have

$$\mathcal{L}_1 = \mathcal{L}_{Id} : \quad \begin{pmatrix} 1 & 1 & 1 & 1 & E_1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.4.12}$$

We also know that, among the topological defects that preserve $\mathfrak{su}(2)_2 \times Vir_{m=3}$, there are the ones that preserve the larger algebra $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$.
Because the theory is diagonal with respect to $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$ the simple defects are only the Verlinde lines $\mathcal{L}_{ij}$, where $i, j \in \{0, 1\}$. Since this is identical to the $\mathfrak{su}(2)_1^6$ analysis in section (4.2), we just show the result:

$$\begin{aligned}
\hat{\mathcal{L}}_{10}|00\rangle = |00\rangle \quad & \hat{\mathcal{L}}_{10}|01\rangle = |01\rangle \quad & \hat{\mathcal{L}}_{10}|10\rangle = -|10\rangle \quad & \hat{\mathcal{L}}_{10}|11\rangle = -|11\rangle, \\
\hat{\mathcal{L}}_{01}|00\rangle = |00\rangle \quad & \hat{\mathcal{L}}_{01}|01\rangle = -|01\rangle \quad & \hat{\mathcal{L}}_{01}|10\rangle = |10\rangle \quad & \hat{\mathcal{L}}_{01}|11\rangle = -|11\rangle, \\
\hat{\mathcal{L}}_{11}|00\rangle = |00\rangle \quad & \hat{\mathcal{L}}_{11}|01\rangle = -|01\rangle \quad & \hat{\mathcal{L}}_{11}|10\rangle = -|10\rangle \quad & \hat{\mathcal{L}}_{11}|11\rangle = |11\rangle.
\end{aligned} \tag{4.4.13}$$

while $\mathcal{L}_{00} = \mathcal{L}_{Id}$. The states $|ij\rangle$ are the highest weight states corresponding to the representations $R_{ij} \times \overline{R}_{ij}$ of the algebra.

Using now equations (4.3.3) and (4.3.4) to compare $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1$'s representations with $\mathfrak{su}(2)_2 \times Vir_{m=3}$'s, we see that the following defects must be present in the theory we are studying right now:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & A & 1 & 1 & 1 & 1 \end{pmatrix} \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & B & -1 & -1 & -1 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.4.14}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & C & -1 & -1 & -1 & -1 \end{pmatrix} \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For compatibility reasons one easily determines that, in order, these defects must be $\mathcal{L}_3$, $\mathcal{L}_2^{(1)}$ and $\mathcal{L}_2^{(2)}$.

The next constraint is the fact that the defect algebra must be closed under the fusion product.

Take a correlator with two concentric defect lines which act on the representations in the following way:

$$\begin{aligned} \mathcal{L}_a : & \quad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & E_a & a_5 & a_6 & a_7 & a_8 \end{pmatrix}, \\ \mathcal{L}_b : & \quad \begin{pmatrix} b_1 & b_2 & b_3 & b_4 & E_b & b_5 & b_6 & b_7 & b_8 \end{pmatrix}. \end{aligned} \tag{4.4.15}$$

Now, by definition, the fusion product tells us that such correlator is identical to one with a single defect line in which the defect is the product of the original ones. Then, in the first configuration, we can act on the operator encircled by the defects first with the nearest one and that with the second, while, in the second configuration, we just act with the only defect present.
The resulting equality tells us that the defect which arises from the fusion product act on the various representations as

$$\mathcal{L}_b \mathcal{L}_a : \quad \begin{pmatrix} b_1 a_1 & b_2 a_2 & b_3 a_3 & b_4 a_4 & E_b E_a & b_5 a_5 & b_6 a_6 & b_7 a_7 & b_8 a_8 \end{pmatrix}, \tag{4.4.16}$$

where we have taken $\mathcal{L}_a$ to be be the nearest defect.
Notice that in general the fusion product is not commutative and its eventual commutativity depends on the matrices $E$ of the two defects.

Knowing that the result of the fusion product must be a defect and therefore a sum of simple defects, we can get some constraints. The first one comes from the following product

$$\mathcal{L}_3 \mathcal{L}_5 : \quad \begin{pmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & -E_5 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \end{pmatrix}. \tag{4.4.17}$$

Looking at all the parameters, except for the matrix which is still undetermined, the result has to a be a simple defect and not a sum of them. In particular the defect must be none other than $\mathcal{L}_5$ itself. This determines $E_5$ as

$$-E_5 = E_5 \quad \rightarrow \quad E_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.4.18}$$

The exact same argument holds for $\mathcal{L}_6$, $\mathcal{L}_7$ and $\mathcal{L}_8$, therefore: $E_6 = E_7 = E_8 = E_5$.

We are left with just four defects to determine. Let us start with $\mathcal{L}_9^{(1)}$ and $\mathcal{L}_9^{(2)}$.
As before we calculate

$$\begin{aligned} \mathcal{L}_3 \mathcal{L}_9^{(1)} : & \quad \begin{pmatrix} 1 & -1 & -1 & 1 & -E_9^{(1)} & 1 & -1 & -1 & 1 \end{pmatrix}, \\ \mathcal{L}_3 \mathcal{L}_9^{(2)} : & \quad \begin{pmatrix} 1 & -1 & -1 & 1 & -E_9^{(2)} & 1 & -1 & -1 & 1 \end{pmatrix}. \end{aligned} \tag{4.4.19}$$

If we had $E_9^{(1)} = E_9^{(2)} = E_5$ as before $\mathcal{L}_9^{(1)}$ would then be the same defect as $\mathcal{L}_9^{(2)}$. But, from our previous discussion, we know that we need two distinct defects of type $n_9$. This means that the only possible solution for the algebra, in this case, is $\mathcal{L}_3 \mathcal{L}_9^{(1)} = \mathcal{L}_9^{(2)}$ which implies $-E_9^{(1)} = E_9^{(2)}$.

We should also consider the fact that both defects $\mathcal{L}_9^{(1)}$ and $\mathcal{L}_9^{(2)}$ are invertible.
Looking at the determined parameters, the only possibilities are either:

$$E_9^{(1)} E_9^{(1)} = \mathbb{I} \quad \text{or} \quad E_9^{(1)} E_9^{(2)} = -E_9^{(1)} E_9^{(1)} = \mathbb{I} \quad \rightarrow \quad E_9^{(1)} E_9^{(1)} = -\mathbb{I}, \tag{4.4.20}$$

thus

$$E_9^{(1)} E_9^{(1)} = (-)^{q_9} \mathbb{I} \qquad Tr E_9 = 0, \tag{4.4.21}$$

where $q_9 \in \{0, 1\}$ is an unknown parameter.

The general solution to these constraints is

$$E_9^{(1)} = \begin{pmatrix} \sqrt{(-)^{q_9} - b_9 c_9} & b_9 \\ c_9 & -\sqrt{(-)^{q_9} - b_9 c_9} \end{pmatrix}, \tag{4.4.22}$$

for unknown parameters $b_9, c_9 \in \mathbb{C}$ and $q_9 \in \{0, 1\}$.

Let us now take into consideration $\mathcal{L}_{10}^{(1)}$ and $\mathcal{L}_{10}^{(2)}$. First of all, the previous discussion hold in this case as well with the result: $-E_{10}^{(1)} = E_{10}^{(2)}$.

Now consider the following defect

$$\mathcal{L}_2^{(1)} \mathcal{L}_9^{(1)} : \qquad \begin{pmatrix} 1 & -1 & -1 & 1 & E_2^{(1)} E_9^{(1)} & -1 & 1 & 1 & -1 \end{pmatrix}. \tag{4.4.23}$$

It is clear that $\mathcal{L}_2^{(1)} \mathcal{L}_9^{(1)}$ is either $\mathcal{L}_{10}^{(1)}$ or $\mathcal{L}_{10}^{(2)}$. Since both defects are still not completely determined we can choose which one is it. We choose $\mathcal{L}_2^{(1)} \mathcal{L}_9^{(1)} = \mathcal{L}_{10}^{(1)}$ and for this algebraic rule to be consistent we must have that

$$E_2^{(1)} E_9^{(1)} = E_{10}^{(1)} \qquad \rightarrow \qquad Tr[E_2^{(1)} E_9^{(1)}] = Tr[E_{10}^{(1)}] = 0. \tag{4.4.24}$$

Imposing this constraint one finally finds the following parametrizations:

$$E_9^{(1)} = \begin{pmatrix} 0 & (-)^{q_9} c_9^{-1} \\ c_9 & 0 \end{pmatrix} \qquad E_{10}^{(1)} = \begin{pmatrix} 0 & (-)^{q_9} c_9^{-1} \\ -c_9 & 0 \end{pmatrix}. \tag{4.4.25}$$

Automatically, then, the other constraint that we previously put on $E_9^{(1)}$ is verified for $E_{10}^{(1)}$:

$$E_{10}^{(1)} E_{10}^{(1)} = \begin{pmatrix} (-)^{q_9+1} & 0 \\ 0 & (-)^{q_9+1} \end{pmatrix}. \tag{4.4.26}$$

We are left therefore with exactly two parameters, $q_9$ and $c_9$, to determine, but there are not any more constraints coming just from the fusion algebra. As a last step we will use the OPE between our representations to complete this analysis.

The OPE is a relation, which applies inside correlators, between two neighbouring fields.
The main strategy here is to consider a correlator with two operators encircled by a defect and some other fields. Such correlator will be equivalent to the one in which we used the OPE between the two encircled fields:

$$\langle ...\mathcal{L} A(z) B(w) ... \rangle = \langle ...\mathcal{L} \sum_k (z - w)^k [AB]_k(w) ... \rangle. \tag{4.4.27}$$

Collapsing then the defect on the operators, if the OPE is known, we will find a condition on the defect.

Even though the contraction of a defect on two operators $A(z)$ and $B(w)$ may be non trivial, in our case we will consider a defect which corresponds to a symmetry. In such case, the defect will act on the two operators as the symmetry it corresponds to.

For our scopes it will be enough to consider the OPE between just the two $\Phi_{\frac{1}{16},1}^{(i)}$ primary fields. Let us write the Ramond fields in the following way:

$$|0\rangle_R = (\xi_1^- \xi_2^-)(0)|0\rangle_{NS} \qquad \Psi_0^2 |0\rangle_R = (\xi_1^- \xi_2^+)(0)|0\rangle_{NS}. \tag{4.4.28}$$

It is possible to show that each highest weight $|H\rangle$ in a one to one correspondence to a primary field $\Phi(z)$ defined on the whole plane. In particular we have the relation:

$$\Phi(0)|0\rangle = |H\rangle. \tag{4.4.29}$$

In this sense we can extend the field values to all the plane obtaining the primary fields: $(\xi_1^- \xi_2^-)(z)$ and $(\xi_1^- \xi_2^+)(z)$.

From [8] we know that the following OPEs hold:

$$\xi_k^\pm(z)\xi_k^\mp(w) \sim (z-w)^{-\frac{1}{4}}\left(1 \pm \frac{1}{2}(z-w)(\Psi_k\Psi_k^*)(w)\right),$$

$$\xi^+(z)\xi_k^+(w) \sim \Psi_k(w)(z-w)^{\frac{1}{4}}c, \tag{4.4.30}$$

$$\xi^-(z)\xi_k^-(w) \sim \Psi_k^*(w)(z-w)^{\frac{1}{4}}c,$$

where $c \in \mathbb{C}$ such that $c^2 = i$. Also the $\sim$ symbol means equality up to a regular part.

So, first of all, let us calculate the following OPEs:

$$\Phi_{\frac{1}{16},1}^{(1)}(z)\Phi_{\frac{1}{16},1}^{(1)}(w) = (\xi_1^-\xi_2^-)(z)(\xi_1^-\xi_2^-)(w) = -:\xi_1^-(z)\xi_1^-(w)\xi_2^-(z)\xi_2^-(w):$$

$$\sim -:(\Psi_1^*(w)(z-w)^{\frac{1}{4}}c)(\Psi_2^*(w)(z-w)^{\frac{1}{4}}c) := -i(z-w)^{\frac{1}{2}}(\Psi_1^*\psi_3)(w) - (z-w)^{\frac{1}{2}}(\Psi_1^*\psi_4)(w),$$

$$\Phi_{\frac{1}{16},1}^{(2)}(z)\Phi_{\frac{1}{16},1}^{(2)}(w) = (\xi_1^-\xi_2^+)(z)(\xi_1^-\xi_2^+)(w) = -:\xi_1^-(z)\xi_1^-(w)\xi_2^+(z)\xi_2^+(w):$$

$$\sim -:(\Psi_1^*(w)(z-w)^{\frac{1}{4}}c)(\Psi_2(w)(z-w)^{\frac{1}{4}}c) := -i(z-w)^{\frac{1}{2}}(\Psi_1^*\psi_3)(w) + (z-w)^{\frac{1}{2}}(\Psi_1^*\psi_4)(w),$$

$$\Phi_{\frac{1}{16},1}^{(1)}(z)\Phi_{\frac{1}{16},1}^{(2)}(w) = (\xi_1^-\xi_2^-)(z)(\xi_1^-\xi_2^+)(w) = -:\xi_1^-(z)\xi_1^-(w)\xi_2^-(z)\xi_2^+(w): \tag{4.4.31}$$

$$\sim -:(\Psi_1^*(w)(z-w)c)\left(1 - \frac{1}{2}(z-w)(\Psi_2\Psi_2^*)(w)\right) := -c\Psi_1^*(w) - \frac{ci}{2}(z-w)(\Psi_1^*\psi_3\psi_4)(w),$$

$$\Phi_{\frac{1}{16},1}^{(2)}(z)\Phi_{\frac{1}{16},1}^{(1)}(w) = (\xi_1^-\xi_2^+)(z)(\xi_1^-\xi_2^-)(w) = -:\xi_1^-(z)\xi_1^-(w)\xi_2^+(z)\xi_2^-(w):$$

$$\sim -:(\Psi_1^*(w)(z-w)c)\left(1 + \frac{1}{2}(z-w)(\Psi_2\Psi_2^*)(w)\right) := -c\Psi_1^*(w) + \frac{ci}{2}(z-w)(\Psi_1^*\psi_3\psi_4)(w).$$

The notation $:...:$ means that the quantity has to be normal ordered.

Right now we have just written the first terms of these OPEs, but similarly as what we said about equation (2.3.48) the rest of the expansion can be completely determined using the algebra.

Looking always at the same section one could wonder why the expression we found is quite different from the one in (2.3.45), this is because we did not choose the same normalization as in equation (2.3.44). Note also that the important part of these OPEs is not the actual coefficients, since with a simple rescaling of the fields they would change, but, as we will see, how they relate to each other.

Let us consider the highest weights we found in the previous section, it is immediate to see which primary fields respect the relation in (4.4.29). Then one easily arrives to the following identifications:

$$(\Psi_1^*\psi_3)(w) = -\frac{i}{\sqrt{2}}(J^+\Phi_{0,0})(w),$$

$$(\Psi_1^*\psi_4)(w) = \frac{1}{\sqrt{2}}\Phi_{\frac{1}{2},2}(w), \tag{4.4.32}$$

$$\Psi_1^*(w) = \Phi_{0,2}(w),$$

$$(\Psi_1^*\psi_3\psi_4)(w) = -\frac{1}{2}(J^+\Phi_{\frac{1}{2},0})(w).$$

Therefore we can rewrite the OPEs as:

$$\Phi_{\frac{1}{16},1}^{(1)}(z)\Phi_{\frac{1}{16},1}^{(1)}(w) \sim -\frac{1}{\sqrt{2}}(z-w)^{\frac{1}{2}}(J^+\Phi_{0,0})(w) - \frac{1}{\sqrt{2}}(z-w)^{\frac{1}{2}}\Phi_{\frac{1}{2},2}(w),$$

$$\Phi_{\frac{1}{16},1}^{(2)}(z)\Phi_{\frac{1}{16},1}^{(2)}(w) \sim -\frac{1}{\sqrt{2}}(z-w)^{\frac{1}{2}}(J^+\Phi_{0,0})(w) + \frac{1}{\sqrt{2}}(z-w)^{\frac{1}{2}}\Phi_{\frac{1}{2},2}(w),$$

$$\Phi_{\frac{1}{16},1}^{(1)}(z)\Phi_{\frac{1}{16},1}^{(2)}(w) \sim -c\Phi_{0,2}(w) + \frac{ci}{4}(z-w)(J^+\Phi_{\frac{1}{2},0})(w),$$

$$\Phi_{\frac{1}{16},1}^{(2)}(z)\Phi_{\frac{1}{16},1}^{(1)}(w) \sim -c\Phi_{0,2}(w) - \frac{ci}{4}(z-w)(J^+\Phi_{\frac{1}{2},0})(w). \tag{4.4.33}$$

Up to now we have just talked about chiral fields, but for the antichiral case we obtain exactly the same results. Now let us compose holomorphic and antiholomorphic part finding:

$$\Phi^{(1)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(1)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w}) \sim$$
$$\sim \frac{1}{2}|z-w|[(J^+\overline{J}^+\Phi_{0,0;\overline{0},\overline{0}})(w,\overline{w}) + \Phi_{\frac{1}{2},2;\overline{\frac{1}{2}},\overline{2}}(w,\overline{w}) + J^+\Phi_{0,0;\overline{\frac{1}{2}},\overline{2}}(w,\overline{w}) + \overline{J}^+\Phi_{\frac{1}{2},2;\overline{0},\overline{0}}(w,\overline{w})],$$
$$\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w}) \sim$$
$$\sim \frac{1}{2}|z-w|[(J^+\overline{J}^+\Phi_{0,0;\overline{0},\overline{0}})(w,\overline{w}) + \Phi_{\frac{1}{2},2;\overline{\frac{1}{2}},\overline{2}}(w,\overline{w}) - J^+\Phi_{0,0;\overline{\frac{1}{2}},\overline{2}}(w,\overline{w}) - \overline{J}^+\Phi_{\frac{1}{2},2;\overline{0},\overline{0}}(w,\overline{w})],$$
$$(4.4.34)$$

$$\Phi^{(1)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w}) \sim i\Phi_{0,2;\overline{0},\overline{2}}(w,\overline{w}) - \frac{i}{16}|z-w|^2(J^+\overline{J}^+\Phi_{\frac{1}{2},0;\overline{\frac{1}{2}},\overline{0}})(w,\overline{w}) +$$
$$+ \frac{1}{4}(z-w)(J^+\Phi_{\frac{1}{2},0;\overline{0},\overline{2}})(w,\overline{w}) + \frac{1}{4}(\overline{z}-\overline{w})(\overline{J}^+\Phi_{0,2;\overline{\frac{1}{2}},\overline{0}})(w,\overline{w}),$$
$$\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(1)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w}) \sim i\Phi_{0,2;\overline{0},\overline{2}}(w,\overline{w}) - \frac{i}{16}|z-w|^2(J^+\overline{J}^+\Phi_{\frac{1}{2},0;\overline{\frac{1}{2}},\overline{0}})(w,\overline{w}) +$$
$$- \frac{1}{4}(z-w)(J^+\Phi_{\frac{1}{2},0;\overline{0},\overline{2}})(w,\overline{w}) - \frac{1}{4}(\overline{z}-\overline{w})(\overline{J}^+\Phi_{0,2;\overline{\frac{1}{2}},\overline{0}})(w,\overline{w}).$$
$$(4.4.35)$$

Then consider a correlator where the defect $\mathcal{L}_9^{(1)}$ is inserted on a circle enclosing the pair of fields $[\Phi^{(1)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(1)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w})]$, we obtain

$$(c_9)^2\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w}) \sim$$
$$\sim \frac{1}{2}|z-w|[(J^+\overline{J}^+\Phi_{0,0;\overline{0},\overline{0}})(w,\overline{w}) + \Phi_{\frac{1}{2},2;\overline{\frac{1}{2}},\overline{2}}(w,\overline{w}) - J^+\Phi_{0,0;\overline{\frac{1}{2}},\overline{2}}(w,\overline{w}) - \overline{J}^+\Phi_{\frac{1}{2},2;\overline{0},\overline{0}}(w,\overline{w})].$$
$$(4.4.36)$$

Comparing this to the OPE $[\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w})]$ we get the constraint $(c_9)^2 = 1$.

Since $E_9^{(2)} = -E_9^{(1)}$ we can choose $c_9 = 1$, the case with $c_9 = -1$ will correspond to $E_9^{(2)}$.

At last we consider a correlator where $\mathcal{L}_9^{(1)}$ now encircles the pair of fields $[\Phi^{(1)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w})]$, obtaining:

$$(-)^{q_9}\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(1)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w}) \sim i\Phi_{0,2;\overline{0},\overline{2}}(w,\overline{w}) - \frac{i}{16}|z-w|^2(J^+\overline{J}^+\Phi_{\frac{1}{2},0;\overline{\frac{1}{2}},\overline{0}})(w,\overline{w}) +$$
$$- \frac{1}{4}(z-w)(J^+\Phi_{\frac{1}{2},0;\overline{0},\overline{2}})(w,\overline{w}) - \frac{1}{4}(\overline{z}-\overline{w})(\overline{J}^+\Phi_{0,2;\overline{\frac{1}{2}},\overline{0}})(w,\overline{w}).$$
$$(4.4.37)$$

Again comparing this to the OPE $[\Phi^{(2)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(z,\overline{z})\Phi^{(1)}_{\frac{1}{16},1;\overline{\frac{1}{16}},\overline{1}}(w,\overline{w})]$ we finally find $q_9 = 0$.

So everything about the defects has been determined, let us write them all:

$$\mathcal{L}_{Id}: \quad \begin{pmatrix} 1 & 1 & 1 & 1 & E_1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\mathcal{L}_2^{(1)}: \quad \begin{pmatrix} 1 & 1 & 1 & 1 & E_2^{(1)} & -1 & -1 & -1 & -1 \end{pmatrix} \quad E_2^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\mathcal{L}_2^{(2)}: \quad \begin{pmatrix} 1 & 1 & 1 & 1 & E_2^{(2)} & -1 & -1 & -1 & -1 \end{pmatrix} \quad E_2^{(2)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\mathcal{L}_3: \quad \begin{pmatrix} 1 & 1 & 1 & 1 & E_3 & 1 & 1 & 1 & 1 \end{pmatrix} \quad E_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$(4.4.38)$$

$$
\mathcal{L}_5: \qquad \begin{pmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & E_5 & \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} \end{pmatrix} \qquad E_5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
\mathcal{L}_6: \qquad \begin{pmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & E_6 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{pmatrix} \qquad E_6 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
\mathcal{L}_7: \qquad \begin{pmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & E_7 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \end{pmatrix} \qquad E_7 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
\mathcal{L}_8: \qquad \begin{pmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & E_8 & -\sqrt{2} & -\sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix} \qquad E_8 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
\mathcal{L}_9^{(1)}: \qquad \begin{pmatrix} 1 & -1 & -1 & 1 & E_9^{(1)} & 1 & -1 & -1 & 1 \end{pmatrix} \qquad E_9^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

$$
\mathcal{L}_9^{(2)}: \qquad \begin{pmatrix} 1 & -1 & -1 & 1 & E_9^{(2)} & 1 & -1 & -1 & 1 \end{pmatrix} \qquad E_9^{(2)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},
$$

$$
\mathcal{L}_{10}^{(1)}: \qquad \begin{pmatrix} 1 & -1 & -1 & 1 & E_{10}^{(1)} & -1 & 1 & 1 & -1 \end{pmatrix} \qquad E_{10}^{(1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
$$

$$
\mathcal{L}_{10}^{(2)}: \qquad \begin{pmatrix} 1 & -1 & -1 & 1 & E_{10}^{(2)} & -1 & 1 & 1 & -1 \end{pmatrix} \qquad E_{10}^{(2)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

$$\text{(4.4.39)}$$

As for the fusion algebra one can easily show that the invertible elements form the dihedral group (D4), also known as the octic group, with relations:

$$
\begin{aligned}
(\mathcal{L}_{10}^{(k)})^4 &= \mathbb{I} & k &= 1, 2; \\
(\mathcal{L}_2^{(k)})^2 &= (\mathcal{L}_9^{(k)})^2 = (\mathcal{L}_3)^2 = \mathbb{I}; \\
(\mathcal{L}_{10}^{(k)})^2 &= \mathcal{L}_3.
\end{aligned}
\qquad (4.4.40)
$$

This group corresponds to the theory's symmetries.

Let us also write the most important fusion products for the non invertible defects:

$$
\begin{aligned}
(\mathcal{L}_5)^2 &= \mathcal{L}_{Id} + \mathcal{L}_3 & (\mathcal{L}_6)^2 &= \mathcal{L}_{Id} + \mathcal{L}_3 \\
(\mathcal{L}_7)^2 &= \mathcal{L}_{Id} + \mathcal{L}_3 & (\mathcal{L}_8)^2 &= \mathcal{L}_{Id} + \mathcal{L}_3 \\
\mathcal{L}_5\mathcal{L}_6 = \mathcal{L}_6\mathcal{L}_5 &= \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} & \mathcal{L}_5\mathcal{L}_7 = \mathcal{L}_7\mathcal{L}_5 &= \mathcal{L}_9^{(1)} + \mathcal{L}_9^{(2)} \\
\mathcal{L}_5\mathcal{L}_8 = \mathcal{L}_8\mathcal{L}_5 &= \mathcal{L}_{10}^{(1)} + \mathcal{L}_{10}^{(2)} & \mathcal{L}_6\mathcal{L}_7 = \mathcal{L}_7\mathcal{L}_6 &= \mathcal{L}_{10}^{(1)} + \mathcal{L}_{10}^{(2)} \\
\mathcal{L}_6\mathcal{L}_8 = \mathcal{L}_8\mathcal{L}_6 &= \mathcal{L}_9^{(1)} + \mathcal{L}_9^{(2)} & \mathcal{L}_7\mathcal{L}_8 = \mathcal{L}_8\mathcal{L}_7 &= \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)}.
\end{aligned}
\qquad (4.4.41)
$$

All the other rules of the algebra can be easily calculated using the expressions in (4.4.38).

## 4.5  Connection to the full theory

Let us now explain how the analysis in the previous section relates to the full theory with the algebra $\mathfrak{su}(2)_1^4 \times \mathfrak{su}(2)_2 \times Vir_{m=3} \times \overline{\mathfrak{su}(2)}_1^4 \times \overline{\mathfrak{su}(2)}_2 \times \overline{Vir}_{m=3}$.

First of all we introduce the concept of a theory which is the tensor product of two. With this we do not only refer to the subdivision of the theory algebra into the tensor product two subalgebras, but to the full separation of the Hilbert space, and therefore also of the operatorial content of the theory, into the tensor product of two.

The most immediate consequence of this definition is the factorization of all correlators into the product of two correlators in the corresponding two theories.

A characterizing property of this kind of phenomenon is the following: a theory with Hilbert space $\mathcal{H}$ is the tensor product of two theories with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ if and only if the theory's partition function can be factorized into the product of the partition functions of the other two theories.

$$
Z_{\mathcal{H}} = Z_{\mathcal{H}_1} Z_{\mathcal{H}_2}. \qquad (4.5.1)
$$

This fact can be easily demonstrate using $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

We are going to show that this is true in our case. In particular the two pieces will be the representations of the $\mathfrak{su}(2)_1^4 \times \overline{\mathfrak{su}(2)}_1^4$ algebra and the ones of the $\mathfrak{su}(2)_2 \times Vir_{m=3} \times \overline{\mathfrak{su}(2)}_2 \times \overline{Vir}_{m=3}$ algebra. First of all, by definition (2.5.11), the characters always factorize in the following way:

$$\chi_{R_i \otimes R_j} = \chi_{R_i} \chi_{R_j}, \tag{4.5.2}$$

which in our case means:

$$\chi_{i_1 i_2 i_3 i_4; h, l} = \chi^{(1)}_{i_1 i_2 i_3 i_4} \chi^{(2)}_{h, l}, \tag{4.5.3}$$

where the indices $i_1 i_2 i_3 i_4$ correspond to a representation of $\mathfrak{su}(2)_1^4$ as in section (4.2), while $h, l$ are respectively the conformal weight of the $Vir_{m=3}$ representation and the highest weight label of the $\mathfrak{su}(2)_2$ representation as in section (4.3). Notice also that we are going to label the theory and the characters corresponding to the $\mathfrak{su}(2)_1^4 \times \overline{\mathfrak{su}(2)}_1^4$ algebra with (1) and the other with (2).

Then the full partition function can be written as:

$$Z = \sum_{i_1 i_2 i_3 i_4 i j_1 j_2 j_3 j_4 j} \chi^{(1)}_{i_1 i_2 i_3 i_4} V^{(2)}_i N_{i_1 i_2 i_3 i_4 i j_1 j_2 j_3 j_4 j} \overline{\chi}^{(1)}_{j_1 j_2 j_3 j_4} \overline{V}^{(2)}_j, \tag{4.5.4}$$

with

$$N_{i_1 i_2 i_3 i_4 i j_1 j_2 j_3 j_4 j} = \delta_{i_1 j_1} \delta_{i_2 j_2} \delta_{i_3 j_3} \delta_{i_4 j_4} M^{Th}_{ij}, \tag{4.5.5}$$

where $V$ and $M^{Th}$ were defined in section (4.3), equations (4.3.7) and (4.3.6).

It is easy to see then that the partition function factorizes into

$$Z = Z^{(1)} Z^{(2)} = (\sum_{i_1 i_2 i_3 i_4 j_1 j_2 j_3 j_4} \chi^{(1)}_{i_1 i_2 i_3 i_4} \delta_{i_1 j_1} \delta_{i_2 j_2} \delta_{i_3 j_3} \delta_{i_4 j_4} \overline{\chi}^{(1)}_{j_1 j_2 j_3 j_4})(\sum_{ij} V^{(2)}_i M^{Th}_{ij} \overline{V}^{(2)}_j), \tag{4.5.6}$$

which is exactly what we wanted to show.

Now, in order to find the full theory defects, let us proceed with the following construction: given a defect $\mathcal{L}^{(1)}_1$ in theory (1) and $\mathcal{L}^{(2)}_2$ in theory (2) we can define the tensor product between defects $\mathcal{L}^{(1)}_1 \otimes \mathcal{L}^{(2)}_2$ as the defect such that on the various representations acts as $\hat{\mathcal{L}}^{(1)}_1 \otimes \hat{\mathcal{L}}^{(2)}_2$.

From the factorization of the partition function, and in general of the theory, we will have that this definition gives, indeed, a topological defects of the full theory. Moreover, what one can show, is that they are all the topological defects of the theory.
Finally, due to this last property, all the defects can be written as a sum of tensor products between simple defect of the two theories. This means that

$$\{\mathcal{L}^{(1)}_i \otimes \mathcal{L}^{(2)}_j | \mathcal{L}^{(1)}_i \text{ is a simple defect in theory (1) and } \mathcal{L}^{(2)}_j \text{ is a simple defect in theory (2)}\} \tag{4.5.7}$$

is the set of simple defects in the full theory.

In our case we have that the defects of the theory (2) were studied in the previous section, while in theory (1), in analogy with section (4.3), the simple defect are the Verlinde lines.

If we take a step back and consider the diagonal theory of the algebra $\mathfrak{su}(2)_1^6 \times \overline{\mathfrak{su}(2)}_1^6$, one can easily show that it can be decomposed into the tensor product of six diagonal theories with algebra $\mathfrak{su}(2)_1 \times \overline{\mathfrak{su}(2)}_1$. In this sense (4.5.6) is a direct consequence of the fact that the full theory can be written as a tensor product of a diagonal $\mathfrak{su}(2)_1^4 \times \overline{\mathfrak{su}(2)}_1^4$ theory and a diagonal $\mathfrak{su}(2)_1^2 \times \overline{\mathfrak{su}(2)}_1^2$ theory.

In the end, knowing this last information, we can take the analysis even further.

Let us consider the full theory as the tensor product of three $\mathfrak{su}(2)_1^2 \times \overline{\mathfrak{su}(2)}_1^2$ theories. After that we can use equation (4.3.3) to decompose every representation of $\mathfrak{su}(2)_1^2$ into representations of $Vir_{m=3} \times \mathfrak{su}(2)_2$.

From these considerations one determines the simple defects that commute with the algebra $(Vir_{m=3} \times \mathfrak{su}(2)_2)^3 \times (\overline{Vir}_{m=3} \times \overline{\mathfrak{su}(2)}_2)^3$ are:

$$\{\mathcal{L}_{i_1}^{(1)} \otimes \mathcal{L}_{i_2}^{(2)} \otimes \mathcal{L}_{i_3}^{(3)} | \mathcal{L}_{i_j}^{(j)} \text{ is a simple defect in the j-th theory with algebra } \overline{Vir}_{m=3} \times \overline{\mathfrak{su}(2)}_2\}, \quad (4.5.8)$$

noticing that the defect analysis for each of these theories follows identically to section (4.4).

This concludes our analysis.

# Chapter 5

# Conclusion

Let us recap what we have seen so far.

First of all, we introduced conformal field theories basically from scratch. Then we brought up topological defects and demonstrated their usefulness. Finally we showed actual calculations in example theories and in actually relevant ones.

The results of this are the followings:

- We were not able to determine any new information from the $\mathfrak{su}(2)_1^6 \times \overline{\mathfrak{su}(2)}_1^6$ diagonal theory as it was;
- We were instead able to find the decomposition of the algebra, and of the representations, $\mathfrak{su}(2)_1^2 \times \overline{\mathfrak{su}(2)}_1^2 = \mathfrak{su}(2)_2 \times Vir_{m=3} \times \overline{\mathfrak{su}(2)}_2 \times \overline{Vir}_{m=3}$ ;
- This decomposition allowed us lessen the constraints on the defects and find various non invertible ones;
- After that we reconducted the results of the smaller theory on the algebra $\mathfrak{su}(2)_2 \times Vir_{m=3} \times \overline{\mathfrak{su}(2)}_2 \times \overline{Vir}_{m=3}$ to the actual theory which has the algebra $\mathfrak{su}(2)_1^4 \times \mathfrak{su}(2)_2 \times Vir_{m=3} \times \overline{\mathfrak{su}(2)}_1^4 \times \overline{\mathfrak{su}(2)}_2 \times \overline{Vir}_{m=3}$;
- As the last point we also showed how the full algebra decomposes into $(Vir_{m=3} \times \mathfrak{su}(2)_2)^3 \times (\overline{Vir}_{m=3} \times \overline{\mathfrak{su}(2)}_2)^3$ and demonstrated how to calculate the theory's defects which commute with such algebra.

In total we could say that this gives us an overview both on the problem on hand, but also on the general possible uses for this kind of tool.

Let us be reminded what was the long term objective of this line of work.
The theory we took into consideration is one of the possible ways to write a specific non linear sigma model on a K3 surface. The symmetries of all moduli space of these theories have already been determined, even though some of this theories are very obscure. Our aim is to take this one step further and determine also the defects of all the moduli space of these theories, in this way we might be able to understand these theories better.

In this thesis we started with analyzing the defects of just one of these non liner sigma models on K3. The next steps in order to take the analysis further would be the following ones:

- The decomposition using diagonal cosets can be iterated, obtaining a $\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_5 \times Vir_{m=6} \times Vir_{m=5} \times Vir_{m=4} \times Vir_{m=3}$ algebra, where the antiholomorphic part is identical;
- With some inspiration one may also find finer decomposition with still a finite number of representations;
- Looking at this progression, it may give us an insight on what kind of topological defects, which only commute with the Virasoro algebra, we can expect in the theory.

As stated in the introduction every sigma model is described by a certain supersymmetric $N = 4$ algebra. The problem is that this kind of algebras have an infinite number of representations and therefore are hard to deal with when looking for defects.

Following this procedure we hope to find more and more defects, which will give us a glimpse of what the total defect content may look like.

In the last decomposition mentioned we left the last $\mathfrak{su}(2)_1$ untouched, this is because one can show that the $N = 4$ supersymmetric theory has a $\mathfrak{su}(2)_1$ subalgebra.

The next step, that we have just mentioned, should follow the guidelines we traced in this thesis, the only worrying point would be the increasing number of representations one has to deal with. A last advice for anyone who might want to take on this work would be to try to automatize the defect calculations in a way that makes the computer do the heavy lifting.

# Bibliography

[1] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer-Verlang, New York (1997).

[2] R. Blumenhagen and E. Plauschinn, *Introduction to Conformal Field Theory: With Applications to String Theory*, Springer, Berlin (2009).

[3] M. R. Gaberdiel, *An Introduction to Conformal Field Theory*, Rep. Prog. Phys (1999), arXiv:9910156 [hep-th].

[4] J. Fuchs and C. Schweigert, *Symmetries, Lie Algebras and Representations: A graduate course for physicist*, Cambridge University Press, Cambridge (1997).

[5] J. Polchinski, *String Theory Volume I*, Cambridge University Press, New York (1998).

[6] J. Polchinski, *String Theory Volume II*, Cambridge University Press, New York (1998).

[7] W. Nahm and K. Wendland, *A Hiker's Guide ti K3. Aspects of N=(4,4) Superconformal Field Theory with Central Charge $c = 6$*, (2000), arXiv:9912067 [hep-th].

[8] M. R. Gaberdiel, A. Taormina, R. Volpato and K. Wendland, *A K3 sigma model with $\mathbb{Z}_2^8 : \mathbb{M}_{20}$ symmetry*, JHEP (2013), arXiv:1309.4127 [hep-th].

[9] V. B. Petkova and J.-B. Zuber, *Generalised twisted partition functions*, (2000), arXiv:0011021 [hep-th].

[10] C.-M. Chang, Y.-H. Lin, S.-H. Shao, Y. Wang and X. Yin, *Topological Defect Lines and Renormalization Group Flows in Two Dimensions*, (2019), arXiv:1802.04445 [hep-th].

[11] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators 1: Partition functions*, Nucl. Phys. B646 (2002) 353–497, [hep-th/0204148].

[12] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators 2: Unoriented world sheets*, Nucl. Phys. B678 (2004) 511–637, [hep-th/0306164].

[13] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators 3: Simple currents*, Nucl. Phys. B694 (2004) 277–353, [hep-th/0403157].

[14] J. Fuchs, I. Runkel, and C. Schweigert, *TFT construction of RCFT correlators 4: Structure constants and correlation functions*, Nucl. Phys. B715 (2005) 539–638, [hep-th/0412290].

[15] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, *Kramers-Wannier duality from conformal defects*, Phys. Rev. Lett. 93 (2004) 070601, [cond-mat/0404051].

[16] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, *Duality and defects in rational conformal field theory*, Nucl. Phys. B763 (2007) 354–430, [hep-th/0607247].

[17] J. Fuchs, I. Runkel, and C. Schweigert, *The fusion algebra of bimodule categories*, Appl. Categ. Struct. 16 (2008) 123–140, [math/0701223].

[18] J. Fuchs, M. R. Gaberdiel, I. Runkel, and C. Schweigert, *Topological defects for the free boson CFT*, J. Phys. A40 (2007) 11403, [arXiv:0705.3129].

[19] N. Carqueville and I. Runkel, *Orbifold completion of defect bicategories*, Quantum Topol. 7 (2016) 203, [arXiv:1210.6363].

[20] I. Brunner, N. Carqueville, and D. Plencner, *A quick guide to defect orbifolds*, Proc. Symp. Pure Math. 88 (2014) 231–242, [arXiv:1310.0062].

[21] L. Bhardwaj and Y. Tachikawa, *On finite symmetries and their gauging in two dimensions*, [arXiv:1704.02330].

[22] I. Brunner, N. Carqueville, and D. Plencner, *Discrete torsion defects*, Commun. Math. Phys. 337 (2015), no. 1 429–453, [arXiv:1404.7497].

[23] J. Wang, X.-G. Wen, and E. Witten, *Symmetric Gapped Interfaces of SPT and SET States: Systematic Constructions*, [arXiv:1705.06728].

[24] Y. Tachikawa, On gauging finite subgroups, arXiv:1712.09542

[25] N. Carqueville, *Lecture notes on 2-dimensional defect TQFT*, 2016. [arXiv:1607.05747].

[26] J. Frohlich, J. Fuchs, I. Runkel, and C. Schweigert, *Defect lines, dualities, and generalised orbifolds*, 16th International Congress on Mathematical Physics (ICMP09): Prague, Czech Republic, August 3-8, 2009, [arXiv:0909.5013].