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New approaches to positive realizations and model reduction for linear systems

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To my family,
friends,
and to myself

Abstract

Positive systems are used to model physical phenomena in which the variables must take non-negative values. This thesis first introduces the system-theoretic tools needed to pose the problem of model reduction for positive linear systems in the state-space representation, where all the matrices associated to the system have non-negative entries. Next, the analysis focuses on robust positive model reduction, a newly proposed sub-problem of the positive model reduction that seek to reduce a positive system in a more robust way with respect to the standard model reductions techniques, that is using non-negative reduction matrices. In order to tackle this problem, the theory of monotone matrices is leveraged. In fact, monotone matrices admit non-negative inverses, a key property that will be exploited to find sufficient and necessary conditions to ensure a robust positive model reduction. An algebraic approach to robust positive model reduction, that consists in enclosing the reachable space to a bigger space that admits a positive reduction, is also presented. Finally, the different approaches will be compared, giving a brief consideration on the optimality of both approaches, that in this context consist in the smaller dimension of the reduced system that a technique can achieve.

Sommario

I sistemi positivi sono di interesse vista la loro utilità nel modellare fenomeni fisici in cui le variabili in gioco devono assumere valori non-negativi. Questa tesi vengono richiamati alcuni fondamenti della teoria dei sistemi per poi porre il problema delle riduzioni del modello per sistemi positivi, i quali nella loro rappresentazione in spazio di stato presentano matrici con elementi non-negativi. Successivamente viene analizzato il problema delle riduzioni del modello robuste, un nuovo sottoproblema delle riduzioni di modello che cerca di ridurre il modello usando matrici di riduzione anch'esse non-negative. A questo proposito sono di interesse le matrici monotone, matrici che ammettono un'inversa non-negativa. Quest'ultima proprietà viene sfruttata per trovare condizioni sufficienti e necessarie riguardo l'esistenza di una riduzione di modello positiva robusta. Viene inoltre presentato un approccio algebrico che consiste nel chiudere lo spazio raggiungibile in uno spazio che lo contenga e che ammetta una riduzione di modello positiva. Vengono infine confrontati i due approcci, facendo una breve riflessione sull'ottimalità di quest'ultimi che in questo contesto corrisponde alla minor dimensione del modello ridotto che i due metodi possono ottenere.

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RPMR Robust Positive Model Reduction

1 Introduction

Positive systems are dynamical systems in which the state, the output and the input only assume non-negative values. This property makes them the ideal tool to model dynamic systems whose variables correspond to physical quantities that are indeed non-negative. Examples of these models are the Leontif model used economists for predicting productions and prices [11], the Leslie model used to study age-structured population dynamics [12], and the compartmental models commonly used in physiology [10] and epidemiology [2]. Moreover, they naturally emerge in stochastic models that describe probabilistic evolutions, such as the Markov Chains [5] and the Hidden Markov Models [16], as well as phase-type distributions [14]. However, as one could expect, this type of models are as interesting as complex. The minimal positive realization is one of the open problems to our interest, and it will be discussed further. The work of Benvenuti [4] provides an overview on the state-of-the-art of this topic, where general result and different mathematical approaches are presented.

This thesis however focuses more on the problem of positive model reduction. Being capable of reducing the dimension of a system has a great impact on the possibility of simulate and even physically implement such systems: a smaller model might need a smaller number of sensors and actuator to fully control or observe the system.

All the system theoretical notions that will be needed through the course of this work will be introduced in the first Section, starting from reachability and observability, realization theory and the standard model reduction technique used for general linear system. The positive model reduction problem will be posed, and the reasons why it still stands open briefly discussed. In the light of this, the general postive model-reduction problem will be specialized to a more constrained one, which we name *robust positive model reduction*. This work is aimed to characterize this problem, identifying when it is solvable and how to construct its solution. To this purpose, the key chapters will build a new approach to robust positive model reduction that relies on non-negative monotone matrices.

Before presenting this approach, we introduce the notion of *monotone* matrix and we present the key results of [13]. Moreover, in our case, only non-negative matrices are of interest and hence we will specialize the results of [13] to the non-negative monotone matrices. After gathering all tools needed, we will present the monotone matrices approach and the algorithm that perform, when possible, the robust positive model reduction.

Finally, recent results of Ticozzi and Grigoletto [8] regarding hidden markov models will be reviewed and extended to the general linear system setting in Section 4. This approach relies on an algebraic approach that propose a special enclosure of the reachable space to a vector algebra. Moreover, this approach will manifest similar properties to the monotone matrices approach. Consequently we will compare the two and investigate whether one or the other leads to better solutions to the robust positive model reduction problem.

In the end, we will share some ideas about possible future work and the directions this approach could lead to. The MatLab code to implement the algorithm proposed in the course of this work is included in the appendix.

2

Linear system theory - a brief recap

2.1 Reachability for linear systems

In order to make the work self contained, we start recalling the main concepts of linear system theory that will be used throughout this thesis. All the material presented here is discussed in greater detail [7] and in [3], where all the proofs of the results included in this chapter can be found. The notions of reachability and observability play a central role in standard model reduction techniques. However, we will see that the the canonical reduction is not always a viable option when dealing with positive systems. For simplicity, we will consider only discrete-time systems, but as one may know the continuous-time case is conceptually analogous.

Consider then the following state space model (Σ) for a linear system in discrete-time

$$\Sigma : \quad \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \tag{2.1}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$.

Definition 2.1.1 (Reachability matrix in k steps)**.** The reachability matrix in k steps of the systems (Σ) is defined as

$$\mathcal{R}_k = \begin{bmatrix} B \; AB \; \ldots \; A^{k-1}B \end{bmatrix}$$

Moreover, the reachable space in k is defined as its image, namely

$$X_k^R = \mathrm{Im}\mathcal{R}_k$$

In other words, the reachable space in k steps is the set of states that can be reached by using a sequence of k inputs. It is clear that X_k^R is a vector subspace, and it enjoys some important properties. Indeed, the subspaces reachable in 1, 2, ... steps satisfy the trivial chain of inclusions

$$X_1^R \subseteq X_2^R \subseteq \ldots \subseteq X_i^R \subseteq X_{i+1}^R \ldots \tag{2.2}$$

However, the reachable subspaces will eventually stop "expanding", and this important lemma proves that when they chain of subspaces (2.2) is stationary at one step, it will remain stationary also for the next steps.

Lemma 2.1.1. *If in the chain (2.2) two consecutives subspaces X_i^R and X_{i+1}^R coincide, then the following subspaces $X_{i+2}^R, X_{i+3}^R, \ldots$ coincide with X_i^R too.*

As mentioned before, there is a limit of steps for which the reachable subspace can continue to expand. This coincide to the dimension of the system.

Proposition 2.1.2. *In a system of dimension n, the chain of subspaces (2.2) is stationary (at least) starting from the n-th step, namely*

$$X_1^R \subseteq X_2^R \subseteq \ldots \subseteq X_n^R = X_{n+1}^R \ldots$$

Moreover, $X^R := X_n^R$ is called the reachable space and it is the set of states which is possible to steer the system starting from the zero state by applying suitable input sequences.

It is reasonable then to call the *reachability matrix* the reachability matrix in n steps, namely

$$\mathcal{R} = \begin{bmatrix} B \; AB \; \ldots \; A^{n-1}B \end{bmatrix} \tag{2.3}$$

We say that a system is *reachable* if $X^R = \mathbb{R}^n$. Thus a necessary and sufficient condition for the system to be reachable is $\mathrm{rank}\mathcal{R} = n$.

Remark. Recall that by construction, the reachability subspace X^R is the smallest A-invariant subspace that includes $\mathrm{Im}B$.

We will almost always deal with unreachable systems, thus it is worth to define the truncated reachability matrix.

Definition 2.1.2. Consider a system of dimension n where the reachable space has dimension $q < n$, namely

$$\text{rank}\mathcal{R} = \text{rank}\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} = q$$

We call the truncated reachability matrix any rectangular full-rank matrix composed by q columns of $\mathcal{R}$. For single-input-systems, one can trivially take

$$P = \begin{bmatrix} B & AB & \dots & A^{q-1}B \end{bmatrix}$$

Remark. Note that the image of the reachability matrix and its truncated version are equal. This follows directly from Proposition 2.2.1, that is because when the reachable subspace chain becomes stationary it will remain so.

2.2 Observability for linear systems

The concept of observability refer to the possibility of determining the state of a systems by using only the input and the output data. More precisely, an observability problem is the estimation of the state at time t when the available input and output data correspond to time instants subsequent to t. Consider again the system (Σ).

Definition 2.2.1 (Non-observable states)**.** A state $x \in \mathbb{R}^n$ is not observable in $[0, k-1]$ if for every input sequence

$$u(0), u(1), \dots, u(k-2)$$

the output sequences $y(\cdot)$ and $y'(\cdot)$ obtained using that input sequence and starting from the initial state x and $x_0 = 0$, respectively coincide at time $0, 1, \dots, k-1$, namely if

$$y(t) = \sum_{j=0}^{t-1} CA^{t-1-j}Bu(j) + CA^t x = \sum_{j=0}^{t-1} CA^{t-1-j}Bu(j) = y'(t)$$

In other words, if the output sequence obtained starting from x does not display any difference from the output sequence obtained starting from 0, then it is impossibile to determine whether the initial state of the system is x or 0. One can notice that the above condition, independently of the input sequence, is equivalent to check if $y(t) - y'(t) = CA^t x = 0$, and thus x is not observable in $[0, k-1]$ if and only if $x \in \ker[CA^t]$ for all $t = 0, \dots, k-1$.

Definition 2.2.2 (Observability matrix corresponding to an interval)**.** The observability matrix corresponding to the interval $[0, k-1]$ is defined as

$$\mathcal{O}_{[0,k-1]} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} \tag{2.4}$$

Moreover, the non-observable subspace in $[0, k-1]$ is defined as its kernel, namely

$$X^{no}_{[0,k-1]} = \ker \mathcal{O}_{[0,k-1]}$$

Again by construction, the subspaces will satisfy the chain of inclusions

$$X^{no}_{[0,0]} \supseteq X^{no}_{[0,1]} \supseteq \dots X^{no}_{[0,k-1]} \supseteq X^{no}_{[0,k]} \supseteq \dots \tag{2.5}$$

Also for these subspaces, there are interesting properties related not to the expansion, but to the shrinking of these observability spaces.

Proposition 2.2.1. *In an n dimensional system, the subspace chain (2.5) becomes stationary (at least) from the n-th steps onward*

$$X^{no}_{[0,0]} \supseteq X^{no}_{[0,1]} \supseteq \dots X^{no}_{[0,n-1]} = X^{no}_{[0,k]} \supseteq \dots$$

Moreover, $X^{no} = X^{no}_{[0,n-1]}$ is called the non-observable subspace of the system and it consists of the initial states that produce zero unforced output evolution.

We then call the *observability* matrix the observability matrix corresponding to the interval $[0, n-1]$, namely

$$\mathcal{O} = \mathcal{O}_{[0,n-1]}$$

2.3 Duality

The study of reachability and observability has been carried out along lines that exhibit some form of similarity, both in the definitions and in the corresponding characterisations. This correspondence can be formally justified by resorting to the notion of dual system.

Definition 2.3.1. Consider a system $\Sigma = (A, B, C)$. We call $\Sigma_d = (A^\top, C^\top, B^\top)$ the *dual system* of Σ.

There is a direct correspondence between the reachability and observability matrices of the dual system and the ones of the original system.

Proposition 2.3.1. *Let $\mathcal{R}_d$ and $\mathcal{O}_d$ be the reachability and observability matrices of the dual system Σ_d. Then*

$$\mathcal{R}_d = \left[C^\top \; A^\top C^\top \; \ldots (A^\top)^{n-1} C^\top\right] = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^\top = \mathcal{O}^\top$$

Analogously,

$$\mathcal{O}_d = \mathcal{R}^\top$$

This means that reachability and observability properties are reversed in the dual system. In the following, we will exploit this result when we need to reduce our system to a reachable and observable one.

2.4 Realization theory and minimal realizations

The state-space representation allow to develop systematic procedures to design a controller, a regulator, to perform optimal control and many other control aspects. However, often the system specification and quality indices are related to the input/output map, better known as the transfer function of the system. In continuous time the transfer function is described using the Laplace transform, while the $\mathcal{Z}$-transform is its discrete time counterpart. More precisely, the $\mathcal{Z}$-transform is a linear operator mapping sequences in $\mathbb{Z}_+$ (sequences of time instants) to functions of the complex variable $z \in \mathbb{C}$. The realization problem focus is on determining a compatible state-space representation for a transfer function. There may be multiple state-space that are compatible with a certain transfer function and as we will see this is strictly dependent on the dimension of the realization. As we are investigating model reduction techniques, a clear vision of this topic is crucial.

Consider the system $\Sigma = (A, B, C)$. Its transfer function can be expressed as a function of the system matrices and of the complex variable $z \in \mathbb{C}$ in the following way

$$W_\Sigma(z) = C(zI - A)^{-1}B$$

The forced output behaviour can be described in function of the transfer function and the input as

$$Y(z) = W_\Sigma(z)U(z) \tag{2.6}$$

Moreover, the transfer function can be written in a Laurent series

$$W_\Sigma(z) = M_0 + M_1 z^{-1} + M_2 z^{-2} + \dots$$

where the M_i's are the Markov coefficients (or parameters) of the system Σ.

An important theorem links the Markov's coefficients to the realization of a transfer function.

Theorem 2.4.1. *The set (A, B, C) is a realization of a transfer function $W(z)$ if and only if*

$$M_i = CA^{i-1}b \qquad i = 1, 2, \dots$$

We now recall the definition of minimal realizations and the properties of different minimal realizations.

Definition 2.4.1 (Minimal realization)**.** Given a rational matrix transfer function $W(z)$, a realization $\Sigma = (A, B, C)$ is called minimal if for any $\Sigma' = (A', B', C')$ of $W(z)$ one has

$$\mathrm{dim}\Sigma \leq \mathrm{dim}\Sigma'$$

where $\mathrm{dim}\Sigma$ is the dimension of the state-space (A, B, C).

The minimality property of a realization is strictly related to reachability and observability.

Proposition 2.4.2 (Reachable and observable realizations)**.** *A realization $\Sigma = (A, B, C)$ of a rational transfer function $W(z)$ is minimal if and only if it is reachable and observable.*

Moreover, it is possible to prove that all minimal realizations have the same dimension.

Proposition 2.4.3 (Algebraic equivalence of minimal realizations). *Let $\Sigma_1 = (A_1, B_1, C_1)$ and $\Sigma_2 = (A_2, B_2, C_2)$ be minimal realization, of dimension n, of $W(z)$. Then there exists a non-singular matrix $T \in \mathbb{R}^{n \times n}$ such that*

$$A_1 = T^{-1} A_2 T, \qquad B_1 = T^{-1} B_2, \qquad C_1 = C_2 T$$

Moreover, the Markov's coefficient of $W(z)$ are related to the minimal realizations as follows

$$M_{k+1} = C_1 A_2^k B_1 = C_2 A_2^k B_2$$

In words, the above proposition says that all minimal realization are equivalent, namely they realize the same transfer function and hence they have the same impulse response. When dealing with reduction techniques, is crucial to check whether these do not alter the input/output behaviour, otherwise the reduction would not be viable. In the following we present different model reductions techniques, but first we define the notion of model reduction in a more general sense.

Definition 2.4.2 (Model reduction). Consider a general system (Σ). A model reduction is a pair of linear maps (matrices) $\{E, J\}$, where $E \in \mathbb{R}^{q \times n}$, $J \in \mathbb{R}^{n \times q}$, $q < n$ such that $EJ = I_q$ and such that

$$\tilde{\Sigma} : \quad \tilde{A} = EAJ \qquad \tilde{B} = EB \qquad \tilde{C} = CJ$$

has the same input/output relation as (Σ), i.e.

$$W_{\tilde{\Sigma}}(z) = \tilde{C}(zI_q - \tilde{A})^{-1}\tilde{B} = W_{\Sigma}(z) = C(zI_n - A)^{-1}B$$

In other words, a model reduction consists in changing the representation of a system, going from a high dimensional basis to a lower dimensional one, preserving the input/output relation. We present the general technique used to transform a realization into a minimal one. By Proposition 2.4.4, it is clear that this consists in looking only at the reachable and observable dynamics of a realization. For this purpose, we show the standard method where we denote with † any left-inverse matrix that only needs to satisfy the condition $T^{\dagger}T = I$.

Theorem 2.4.4. *Consider (Σ) to be a non reachable and unobservable system of dimension n. Consider a (non necessarily orthogonal) projection Π_R on its reachable space, and*

$\Pi_R = JE$ a non square factorization in two full-rank matrices. A reachable realization is obtained by

$$A_r = EAJ \qquad B_r = EB \qquad C_r = CJ$$

where the reachable system $\Sigma_r = (A_r, B_r, C_r)$ has dimension $q < n$. In particular one can consider $J = P$, the truncated reachability matrix $P \in \mathbb{R}^{n\times q}$, and $E = P^\dagger$ one of its left inverses. If (Σ_r) is not observable, applying the same procedure to the dual system $\Sigma_r^D = (A_r^\top, C_r^\top, B_r^\top)$ yield a reachable and observable dual system $\Sigma_{ro}^D = (A_{ro}^\top, C_{ro}^\top, B_{ro}^\top)$. Finally, computing the dual system of the dual, $\Sigma_{ro} = (A_{ro}, B_{ro}, C_{ro})$ is a minimal realization for the original system Σ.

Proof. By duality, it is enough to prove that projecting onto the reachable space provides a reduction. Consider the projector and its factorization

$$\Pi_R = JE$$

. Recall that to to have a reduction, the reduced system has to have the same I/O behaviour of the original system, namely the same transfer function. Moreover, it is possible to prove that it is enough to prove that they have the same Markov's coefficients. Recall also that the reachable space is the smallest A-invariant subspace that contains B, thus $\Pi_R B = B$. The Markov's coefficient of the system are

$$M_{k-1} = CA^kB = CA^k\Pi_R B = C\Pi_R A^k \Pi_R B = (C\Pi_R)(\Pi_R A \Pi_R)^k(\Pi_R B)$$

and thus the systems (A, B, C) and $(\Pi_R A\Pi_R, \Pi_R B, C\Pi_R)$ are equivalent. Moreover this allow for a reduction. Indeed, exploiting the factorization, it holds that

$$M_{k-1} = C\Pi_R A^k \Pi_R B = (CJ)(EA^kJ)(EB)$$

and hence (Σ_r) is a reachable system of dimension $q < n$. □

2.5 Positive systems and problem formalization

Positive systems are dynamical systems in which the state and output variables assume positive (or at least non-negative) values for all times, for any non-negative initial state and non-negative input. This feature makes positive systems an appropriate modeling tool for dynamic phenomena whose describing variables correspond to the quantities or concentrations of any type of resource or substance. Moreover, positive systems are commonly used to model stochastic phenomena as well, since probabilities are non-negative. As one can understand, positive systems are remarkably useful in several applications in very different fields of science, ranging from biology and medicine to civil and electronic engineering. Although the problem of determining the existence of a positive realization and its computation has been solved, the characterization of minimality for positive systems is still an open problem. For these reasons, also the dimension reduction problem of positive systems is open as well. Recall that minimality is often a key issue in applications. For example, when implementing a filter, one wishes to reduce space occupation and power consumption, and hence a positive realization with minimal dimension is desirable.

For the sake of notation and terminology, we say that a matrix A is non-negative if every entry of that matrix is greater or equal than zero, and we will denote it as $A \geq 0$.

Before we dive into the positive dimension reduction problem, we define positive systems and present their main properties.

Definition 2.5.1 (Positive system)**.** Consider a discrete-time, time-invariant system of the form

$$\Sigma : \quad \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \tag{2.7}$$

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$.

- The system is said to be *externally* positive if and only if its forced output (i.e. the output corresponding to a zero initial state) is non-negative for any non-negative input function.
- The system is said to be a *internally positive system* if the state and output sequences x_k and y_k are non-negative at any time for any non-negative input sequence u_k and for any non-negative initial state $x(0) = x_0$.

It is clear that external positivity is weaker than internal positivity, and as we shall see it means that internal positive systems will enjoy stronger properties.

Proposition 2.5.1. *A linear system is externally positive if and only if its impulse response is non-negative, i.e.*

$$h(k) = CA^{k-1}B \geq 0 \qquad k \geq 1$$

Proof. Consider a discrete-time linear system with zero initial state. The output is the discrete convolution of the input and the impulse response, namely

$$y(k) = \sum_{j=0}^{k} h(j)u(k-j)$$

Therefore, if the impulse response is non-negative, the output is non-negative for every non-negative input, so that the system is externally positive. On the other hand, if the system is externally positive, $h(k)$ must be non-negative. Indeed, if this were not the case, $h(k)$ would be negative at least at one point j and hence the output would be negative for an input function that is positive at least at $k-j$, contradicting external positivity of the system. □

For internally positive system, it follows that the non-negativity of the impulse response is only a necessary condition. A necessary and sufficient condition is the following.

Proposition 2.5.2. *A discrete-time linear system $\Sigma = (A, B, C)$ is (internally) positive if and only if $A \geq 0$, $B \geq 0$, $C \geq 0$.*

Proof. ($\Leftarrow$) Letting $x(0) = 0$, positivity implies $x(1) = Bu(0) \geq 0$ for every $u(0) \geq 0$, that is $B \geq 0$. Moreover, $y(0) = Cx(0)$, and hence positivity ($y(0) \geq 0$ $\forall x(0) \geq 0$) implies $C \geq 0$. Finally, choosing $u(k) = 0$ $\forall k$, $x(1) = Ax(0)$ has to be non-negative for any $x(0)$, hence $A \geq 0$.
($\Rightarrow$) Trivial: for any $x(0) \geq 0$ and for any $u(k) \geq 0$, the non-negativity of the matrices is enough to ensure that the state and output are non-negative at any time. □

Note that as mentioned before, the non-negativity of the impulse response can be obtained with system matrices that are not non-negative, so internal positive systems are also externally positive, but the converse does not hold. The non-negativity of the system matrices have consequences on various system properties, most importantly on the reachability and observability. Indeed, the reachability and observability matrices also enjoy a similar property.

Proposition 2.5.3. *Given a positive system (Σ), both the reachability matrix and the observability matrix are non-negative matrices.*

Proof. The power of a non-negative matrix is a non-negative matrix. The product of two non-negative matrices is trivially a non-negative matrix. By construction, it follows that $\mathcal{R}$ and $\mathcal{O}$ are non-negative. □

The problem we want to front is, starting from a positive system, to find model reduction technique that preserve the positivity of the system matrices and transform the system into an equivalent one of smaller dimension.

Problem 1 (Positive model reduction)**.** Consider the positive system (Σ) of dimension n. Suppose now that it is not reachable, i.e. its reachability matrix has rank $q < n$

$$\text{rank}\mathcal{R} = \text{rank}\begin{bmatrix} B \ AB \ \dots \ A^{n-1}B \end{bmatrix} = q \tag{2.8}$$

we want to find a reduced system $(\tilde{\Sigma})$ of dimension q, satisfying the positivity constraints, such that the two systems are equivalent.

As one may expect, the general method used to obtain a minimal realization from a realization of greater dimension does not work. Indeed, although both the reachability and observability matrix of a positive system are non-negative, there may not exists respective non-negative pseudo-inverses. However, we show through an example that this is not enough to prevent the reduction to be positive.

Example 2.5.1. Consider the system

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Its reachability matrix is

$$\mathcal{R} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which has rank equal to 2. Hence, the truncated reachability matrix and its pseudo-inverse are

$$V = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad V^{\dagger} = \begin{bmatrix} -1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

where $V^{\dagger} \leq 0$. However, the reduced system

$$\tilde{A} = V^{\dagger}AV = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \tilde{B} = V^{\dagger}B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is positive.

Despite these special cases may be interesting, they are also fragile. As a matter of fact, small uncertainties on the system's model might cause the reduction to not be positive.

Example 2.5.2. Consider the same system of Example 2.5.1. We now show that introducing noise in some specific entries can make the model reduction non positive. Consider

$$A^{*} = \begin{bmatrix} 1 & 1+\epsilon & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 1 \end{bmatrix} \qquad B^{*} = \begin{bmatrix} 1 \\ 1+\epsilon \\ 0 \\ 0 \end{bmatrix}$$

Take now $\epsilon = 0.1$. The reduced system using the reduction matrices V and $V^{\dagger}$ will result in a non positive system

$$V^{\dagger}A^{*}V = \begin{bmatrix} -0.1 & 0.9 \\ 1.1 & 1.1 \end{bmatrix} \qquad V^{\dagger}B^{*} = \begin{bmatrix} 1.2 \\ -0.1 \end{bmatrix}$$

We will focus on a more "robust" type of reduction, namely the cases in which the reduction matrix and its inverse are both non-negative.

Problem 2 (Robust positive reachable model reduction (RPMR))**.** Consider the positive system (Σ) of dimension n. Suppose now that it is not reachable, i.e. its reachability matrix has rank $q < n$

$$\text{rank}\mathcal{R} = \text{rank}\begin{bmatrix} B \; AB \; \dots \; A^{n-1}B \end{bmatrix} = q \tag{2.9}$$

we want to find a non-negative reduction matrix $J \in \mathbb{R}^{q \times n}$ that admits a non-negative inverse E and such that the reduced system (Σ_+)

$$\Sigma_+ : \quad A_+ = EAJ \qquad B_+ = EB \qquad C_+ = CJ \tag{2.10}$$

is positive and equivalent to (Σ).

One can easily understand that using the duality theory properties revised in section 2.3 these two problems can be extended also to unobservable systems. Moreover, also the techniques that we derive can be trivially extended to these systems by duality. We later see how to address this problem when dealing with unobservable and unreachable systems. In the following, for the better understanding of the reader and to avoid redundancies, we only deal with observable positive systems.

3

Monotone matrices approach to Robust positive model reduction

3.1 Monotone matrices for RPMR

Monotone matrices are a family of matrices that enjoy a key property for our goal: the inverse of a monotone matrix is non-negative. This means that if we can find a monotone reduction matrix, the robust positive reduction problem is solved. Nonetheless, these matrices have a particular form that prevent us to develop a general systematic method to get a robust positive reduction. We study the cases in which the problem can be solved, trying to expand the systems that allow such reduction.

3.1.1 Monotone matrices and non-negative monotone matrices

First and foremost, we recall the definition and main properties of monotone matrices and ultimately we extend the work of [6] and [13] to the class of matrices in our interest, non-negative monotone matrices.

A $n \times q$ real matrix A is said to be of monotone kind if

$$Ax \geq 0 \implies x \geq 0$$

The characterization of rectangular monotone matrices have been derived in

[13]:

Theorem 3.1.1 ([13]). *Let A be a $n \times q$ rectangular real matrix, $q < n$. Then the following statements are equivalent:*

(a) *A is (left)-monotone;*

(b) *A has a non-negative left inverse. In other words, there exists an $q \times n$ matrix $Y \geq 0$ such that $YA = I$;*

(c) $\mathrm{cone}(A) \supseteq \mathbb{R}^q_+$

where $\mathrm{cone}(A)$ *denote the conical hull of the rows of A, i.e.*

$$\mathrm{cone}(A) = \{c \in \mathbb{R}^q | c = A^\top x,\ x \geq 0\}$$

Proof. ($a \Rightarrow b$) Suppose that A is monotone, i.e. $Ax \geq 0$ implies $x \geq 0$. Denote by y_i the i-th row of Y. Then, recalling that $YA = I$, it follows that $y_i A \geq 0$ is the the i-th canonical vector, and hence $y_i \geq 0$ for every row y_i of Y, that is $Y \geq 0$.
($b \Rightarrow a$) For the reverse implication, suppose A admits a left-inverse $Y \geq 0$. Then, if $Ax \geq 0$, $x = YAx \geq Y0 = 0$ and hence A is monotone.
($b \Leftrightarrow c$) Suppose that $Y \geq 0$ such that $YA = I$. Let y_i be the i-th row of Y. Then $y_i A$ is equal to the i-th canonical vector, and it has to hold for all i meaning that all canonical vectors can be obtained by a conical combination of the rows of A since $y_i \geq 0\ \forall i$. But this is equivalent to the statement that each canonical vector is contained in $\mathrm{cone}(A)$, which implies that (c) holds. On the other hand, if (c) holds there exist a conical combination of the rows that gives each canonical vector. Stacking in a matrix Y the coefficients of such linear combinations (with the i-th row corresponding to the coefficients that return the i-th canonical vector) we thus obtain a left inverse. □

In other words, these matrices induce the non-negativity on their left-inverse. We provide some examples of monotone and non-monotone matrices:

Example 3.1.1 (Monotone matrices).

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -1 & 3 \\ 2 & -4 \end{bmatrix}$$

A_1: it is sufficient to look at the rows 1,3,4 alone positively span $\mathbb{R}^3$.
A_2: for this case, note that $A_2^{-1} = \begin{bmatrix} 2 & 3/2 \\ 1 & 1/2 \end{bmatrix}$. Clearly since A_2 is non-singular and it is a square matrix it admits one unique inverse.

Example 3.1.2 (Non-monotone matrices).

$$A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

A_1: the rows do not positively span $\mathbb{R}^3$. Indeed $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \notin \mathrm{cone}(A_1)$
A_2: Recall that the conical combinations have only positive coefficients. $\begin{bmatrix} 1 & 0 \end{bmatrix} \notin \mathrm{cone}(A_1)$.

Remark. Note that a (left) rectangular monotone matrix do not admits a unique inverse. Clearly non-singular square monotone matrices admits a unique inverse.

We now try to characterize better this property for non-negative matrices, since we are dealing with a positive system and hence with a non-negative reachability matrix. We will see that these are simpler to understand and have strong properties that make easy to check whether a matrix is monotone even in high dimensions.

Firstly, let us give some definitions.

Definition 3.1.1. Let $x \in \mathbb{R}^n$ be a real vector of dimension n. Its support is defined as

$$\text{supp}(x) = \mathrm{span}\{e_i \in \mathbb{R}^n \mid e_i^\top x \neq 0\}$$

where e_i are the canonical vectors. We also define the support of a vector space. Let $\mathcal{X} \subseteq \mathbb{R}^n$. Its support is defined as

$$\mathrm{supp}(\mathcal{X}) = \mathrm{span}\{e_i \in \mathbb{R}^n \mid \exists x \in \mathcal{X} : e_i^\top x \neq 0\}$$

For the next result, we firstly have to investigate the notion of orthogonality when treating non-negative vectors. What follows is a simple but important characterization to our aims.

Proposition 3.1.2. *Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ be two non-negative vectors. Then they are orthogonal under the standard scalar product, i.e.*

$$\langle u, v \rangle = u^\top v = 0$$

if and only if they have disjointed support, namely

$$\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$$

We next need to introduce the concept of frame for a cone and define a generalized permutation matrix.

Definition 3.1.2. A frame is a minimal set of generators of a cone.

One may be familiar with the definition of permutation matrix, which is a matrix that has exactly one entry of 1 in each row and each column and zeroes elsewhere. A generalized permutation matrix, as the name suggest, is defined in the following way:

Definition 3.1.3. A generalized permutation matrix is a matrix with only one non-zero entry on each row and each column. Moreover, a non-negative generalized permutation matrix is a generalized permutation matrix where the non-zero entries are positive.

With these notions we are ready to present the main results that are crucial for our analysis and will clarify the constraints of the problem.

Proposition 3.1.3. *Let $A \in \mathbb{R}^{n \times n}$ be a square non-negative matrix. Then it is monotone if and only if it is a non-negative generalized permutation matrix.*

Proof. ($\Leftarrow$) Recall that a non-generalized permutation matrix has only one positive entry for each row and column, hence the rows are vectors proportional to the canonical vectors, and this is enough to prove that the conical hull of the rows is equal to $\mathbb{R}^n_+$. We thus conclude using Theorem 3.1.1.
($\Rightarrow$) Since we are dealing with a square non-negative matrix and $\mathbb{R}^n_+$ is the biggest positively generated space by a set of n non-negative vectors, the third condition of Theorem 3.1.1 is equivalent to $\mathrm{cone}(A) = \mathbb{R}^n_+$. The main observation is that conical combinations of non-negative vectors can only expand the dimension of the support. More precisely, call $u, v \in \mathbb{R}^n_+$ and call $\dim(\mathrm{supp}(u))$ the dimension of $\mathrm{supp}(u)$. Then $\dim(\mathrm{supp}(f)) \geq \max\{\dim(\mathrm{supp}(u)), \dim(\mathrm{supp}(v))\}$, where

$f = \alpha u + \beta v$, $\alpha, \beta \in \mathbb{R}_+$. This can clearly be expanded to conical combinations of n non-negative vectors. This inevitably imply that to positively generate $\mathbb{R}^n_+$ with non-negative vectors, these have to be proportional to the canonical base. But then monotonicity imply that A has to be non-singular and have columns (and rows) proportional to the canonical vectors. This is indeed a non-negative generalized permutation matrix. □

Putting together the last properties, we can give a specific characterization for non-negative rectangular monotone matrices.

Proposition 3.1.4. *Let $A \in \mathbb{R}^{n \times q}$ with $q < n$ be a full-rank non-negative matrix. Then it is monotone if and only if it contains a set of q distinct orthogonal rows. This is equivalent to say that it is monotone if and only if it contains q distinct rows that are non-negative and proportional to the canonical vectors.*

Proof. (⇒) Suppose that A is monotone, hence $\mathrm{cone}(A) \supseteq \mathbb{R}^q_+$. This is equivalent to say that q rows of A generate the cone $\mathbb{R}^q_+$ since all others $n - q$ rows are linearly dependent and hence are inside the cone generated by these q rows. Stacking these q rows to the top, we obtain a square $q \times q$ monotone matrix, and by Proposition 3.1.3 it is a non-negative generalized permutation matrix and hence its rows are orthogonal.
(⇐) Suppose now that A contains a set of q distinct orthogonal rows. A permutation of the rows of A is equivalent to left-multiply A by a permutation matrix P. This operations does not alter the row space, namely the span of the rows of A. Indeed $\mathrm{span}(A^\top) = \mathrm{span}(A^\top P^\top)$. Then, stacking the q orthogonal rows to the top, we would get a matrix of the form

$$A = \left[\begin{array}{c} G \\ \hline * \end{array}\right]$$

where $G \in \mathbb{R}^{q \times q}$ is square a non-negative generalized permutation matrix with orthogonal rows, and hence a monotone matrix by Proposition 3.1.3. Note that a square non-negative generalized permutation matrix has rows that are proportional to the canonical vectors. □

3.1.2 Robust positive reductions for positive reachable spaces

One can now imagine how this result can be applied to systems that have a truncated reachability matrix that is monotone. What follows is that for these systems the standard reduction method provides a positive reduction.

Remark. It is very important to understand that in the case of multi-input systems, the choice of the truncated reachability matrix P is **not unique**. Indeed, by definition, it is a rectangular full-rank matrix composed by any $\operatorname{rank}\mathcal{R} = q$ columns of $\mathcal{R}$. This give us the choice to choose the q linearly independent columns of $\mathcal{R}$. However, how we will see further into this thesis, it is possible to overcome this problem.

By Theorem 2.4.4, this corollary follows.

Corollary 3.1.4.1. *Consider an unreachable system $\Sigma = (A, B, C)$ of dimension n. Suppose that its reachable subspace has dimension $q < n$ and that its truncated reachability matrix P is non-negative monotone. Then (Σ) admits the following reduced positive system of dimension q*

$$\Sigma_+ : \quad A_+ = P^\dagger AP \qquad B_+ = P^\dagger B \qquad C_+ = CP$$

Proof. From Theorem 2.4.4, $\Pi_R = PP^\dagger$ is a projection onto the reachable space. Moreover, by monotonicity of P it follows that $P^\dagger \geq 0$. Trivially (Σ_+) is a positive system. □

There is still one important remark yet to be made. One may ask if a rectangular monotone matrix can admit different non-negative inverses. We show that this is the case and we explain what is the effect of choosing different left-inverses through an example.

Example 3.1.3. Consider the following system

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Its reachability matrix and its truncated version are

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Note that

$$N_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad N_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \qquad N_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

are left inverse for P. Clearly for our purpose we consider only the non-negative ones. The different projectors are

$$\Pi_R^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \Pi_R^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Pi_R^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

Notice that by construction, $\mathrm{Im}(\Pi_R^i) = \mathrm{span}(P)$ for all i. Thus the choice of the left inverse does not change the space in which we project. Notice also that Π_R^3 is the only orthogonal projector out of the 3 possible ones.

■

This example make us wonder whether there is a characterization of all left-inverses of P and hence also a characterization of all projectors on the reachable space X^R. Recall that a projector $\Pi = PN$ has to satisfy the following properties:

$$\Pi^2 = \Pi, \qquad \mathrm{Im}(\Pi) = X^R$$

One can verify that any left-inverse N can be parametrized with the help of an arbitrary non-singular square matrix $J \in \mathbb{R}^{n\times n}$ in the form

$$N = (P^\top J P)^{-1} P^\top J$$

while satisfying the above properties. Note that however with this closed form there is not guarantee that N is non-negative. From the monotone matrices characterization we know that when P is monotone a non-negative N exists. However this could be not unique, and even with the latter formulation it could be difficult to find a J such that N is non-negative. In the next section, we provide an intuitive and fast way to compute a non-negative left-inverse.

3.1.3 Computation of the non-negative left-inverse

Coming from this last example, one can intuitively understand how to obtain the non-negative left-inverse. Building on this intuition, we provide a systematic routine to compute it. As it is well known, the computation of the left-inverse of a matrix is a onerous operation, and for this reason the big advantage of this approach is that none or little computational effort is required. Before introducing the method, it is worth remarking, that the left-inverse of a rectangular monotone matrix is not unique. We will show how to compute one particular inverse that may not lead to an orthogonal projector, but still works for our purposes.
The characterization of monotone non-negative matrices of Proposition 3.1.4 is all we need to build this routine. Indeed given a non-negative monotone matrix $A \in \mathbb{R}^{n \times q}$, we know that there are a set of q distinct orthogonal rows. The main idea is to exploit the result of Proposition 3.1.2, that says that these q row vectors have to have disjointed support. Moreover, from the fact that the row vectors live in $\mathbb{R}^q$, it is easy to conclude that these q row vectors have to be proportional to the first q vectors of the canonical base of $\mathbb{R}^q$, namely they will have only one entry different from zero. Let now denote by $a_i \in \mathbb{R}^q$ the i-th row of A, and by a_{ij} the j-th entry of a_i. We will call $B \in \mathbb{R}^{q \times n}$ the non-negative left-inverse of A such that $BA = I_q$. Each of the q orthogonal rows will contribute building one of the columns of B. Let's start from one of the orthogonal rows of A, a_i. As pointed out before, it will have only one entry different from zero. We will denote it as a_{ij}. The corresponding column of B, b_i, will have to have all entries to zero but the j-th element, that has to be equal to $\frac{1}{a_{ij}}$. Repeating this procedure for each of the q distinct orthogonal rows of A we will be left with $n - q$ columns of B to compute. Choosing all of these columns equal to the zero vector will complete our procedure.

3.1.4 Extension of robust positive reachable spaces through QR decomposition

Although we have found sufficient conditions to robustly reduce a positive system, monotonicity requires a very stringent form of the reachability matrix.

We try to extend our approach to a broader family of matrices through the known QR decomposition. The aforementioned factorization decompose a general matrix into a product of an orthogonal matrix and an upper-triangular matrix. However, from the proof of Proposition 3.1.4 one can observe that for monotonicity we actually need a particular lower-triangular matrix. Consequently, we will use the similar LQ decomposition. The LQ decomposition is a factorization of a matrix into a product of a lower triangular matrix and a orthogonal matrix. We can then decompose $P \in \mathbb{R}^{n \times q}$ as

$$P = JQ$$

Notice that $\mathrm{span}(P) = \mathrm{span}(J)$, hence J still spans the reachable subspace of the system. If $J = PQ^\top$ is monotone, then it admits a non-negative inverse. However, this decomposition does not induce positivity on J, as one can see through the following example.

Example 3.1.4.

$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Its LQ decomposition is the following:

$$L = \begin{bmatrix} -\sqrt{2} & 0 \\ -\sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2} & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

■

Recall that how it has been shown in Example 3.1.3, the choice of the left inverse do not change the subspace in which we project. Summarizing, when the reachability matrix P can be decomposed into a non-negative monotone matrix and an orthogonal matrix as $J = JQ$, we can pick E non-negative such that $EJ = I_q$ by monotonicity of J. What follows is that

$$\Pi_R = JE$$

is a projector onto the reachable space. Moreover, the non-negativity constraints on J imply that the QR decomposition (and hence the LQ decomposition) is unique.

Proposition 3.1.5. *Let $A \in \mathbb{R}^{q\times n}$ be a rectangular non-negative matrix with $q < n$. Consider its QR decomposition $A = QR$. If A is invertible and it is required that the diagonal elements of R are positive, such decomposition is unique.*

Proof. Suppose there exist two distinct QR decomposition $A = Q_1[R_1\ N_1]$ and $A = Q_2[R_2\ N_2]$ where R_i are invertible upper triangular matrices, N_i are $q\times(n-q)$ matrices. Note that this implies $Q_1R_1 = Q_2R_2$ where $Q_1^\top Q_1 = Q_2^\top Q_2 = I_q$. Then

$$R_1^\top R_1 = R_1^\top(Q_1^\top Q_1)R_1 = A^\top A$$

$$A^\top A = R_2^\top(Q_2^\top Q_2)R_2 = R_2^\top R_2$$

Hence $(R_2^{-1})^\top R_1^\top = R_2R_1^{-1}$. In this equation, the left hand-side is a lower-triangular matrix while the right-hand side is a upper-triangular matrix. Hence both must be diagonal. Let α_i and β_j be the diagonal entries of R_1 and R_2 respectively. Then it follows that

$$\frac{\alpha_i}{\beta_j} = \frac{\beta_j}{\alpha_i}$$

Thus, $\alpha_i = \beta_i\ \ \forall i,j$ and $(R_2^{-1})^\top R_1^\top = R_2R_1^{-1} = I_q$. This proves that $R_1 = R_2$. By Theorem A.0.1, since $R_2 = SR_1$, it holds that $S = I_q$. Moreover, the theorem also proves that $Q_1 = Q_2S$ and $N_1 = SN_2$ and hence the decomposition is unique. □

Notice that in our case we will apply the LQ decomposition to our truncated reachability matrix P that is, by construction, invertible. Therefore also J will be invertible by the trivial fact that multiplication (left or right) by a full-rank matrix (Q) does not change the rank. Hence, taking also into account the requirement of non-negativity of J, uniqueness of the decomposition is insured.
We can now use this tool to extend the family of matrices for which a RPMR exists.

Theorem 3.1.6. *Consider (Σ) and its truncated reachability matrix P. Let $P = JQ$ be its unique LQ decomposition requiring the diagonal element of J to be positive. If J is a non-negative monotone matrix, then it admits a non-negative left-inverse E and hence*

$$\Sigma_+ : \quad A_+ = EAJ \qquad B_+ = EB \qquad C_+ = CJ \tag{3.1}$$

is a positive realization of dimension $q < n$.

Proof. This directly follows from Theorem 2.4.4. Indeed $\Pi = JE$ is a projector onto the reachable space, since $X^R = \mathrm{Im}(P) = \mathrm{Im}(PQ^\top) = \mathrm{Im}(J)$. □

This theorem gives us a sufficient condition for the robust positive model reduction. We can formulate a systematic approach to RPMR (Algorithm 1).

Algorithm 1 RPMR through LQ

Require: $\Sigma = (A, B, C)$
  $n \leftarrow \dim\Sigma$
  $\mathcal{R} \leftarrow \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$
  if $\mathrm{rank}\mathcal{R} = n$ **then**
    Stop: the system is reachable and hence cannot be reduced.
  else if $\mathrm{rank}\mathcal{R} = q < n$ **then**
    $P \leftarrow \begin{bmatrix} B & AB & \dots & A^{q-1}B \end{bmatrix}$
    if P monotone **then**
      $\Sigma_+ = (P^\dagger AP, P^\dagger B, CP)$ where $P^\dagger \geq 0$ left-inverse of P
    else if P is not monotone **then**
      $[J, Q] \leftarrow lq(P)$
      if J is non-negative monotone **then**
        $\Sigma_+ = (EAJ, EB, CJ)$ through (E, J), where $E \geq 0$ left-inverse of J
      else
        Stop: Inconclusive. A reachable RPMR may still exists.
      end if
    end if
  end if

The algorithm can be understood as the following: the first step is clearly requiring a unreachable system. If now the truncated reachability matrix is monotone, we can perform a RPMR through $(P^\dagger, P)$, if not we can try applying the LQ decomposition. If J is monotone, a RPMR exists through (E, J), E being a non-negative pseudo-inverse of J. If neither of these two conditions are verified, the algorithm is inconclusive and we can say that a reachable RPMR does not exist.

We now make some considerations on whether the Theorem 3.1.6 gives us necessary conditions for a RPMR. Unfortunately this is not the case. To understand why, we have to do a step backward and dive deeper on what monotonicity means to our problem and how we can manipulate the truncated reachability matrix P to get a monotone matrix. What is clear is that monotonicity is a necessary and sufficient condition for a RPMR: indeed, by the definition of RPMR we require a non-negative inverse, and that is the definition of monotonicity. What is less clear is what are the operations on P that are permitted to get to a monotone matrix. What we want to preserve is not the matrix P itself, but its image, that correspond to the reachable space. For this reason, the question becomes what are the operations that preserve the column space, i.e. the image, of P. The answer that column operations do the trick is equivalent to saying that right multiplication of P by invertible matrices preserves the column space of P. This is enough to say that the LQ decomposition consider only a part of the allowed operations, since it consists in right multiplying by an orthogonal matrix, that is by definition invertible. However, for some particular matrix there may be also some non-invertible transformations that preserve the column space, but this happens only if we consider singular matrices, that is not our case since P is non-singular by definition. The next section is then devoted to take into considerations all linear transformations that can be applied to P preserving the reachable space.

3.1.5 Monotonicity through linear transformations

In this section we exploit the fact that right-multiplying the truncated reachability matrix P by a non-singular matrix T does not alter its column space, i.e. the reachable space. Moreover, we now show that the transformed reachability matrix still grants a projector onto the reachable space. Let T be an invertible matrix in $\mathbb{R}^{q\times q}$. Now let

$$J = PT \in \mathbb{R}^{n\times q}$$

be the transformed reachability matrix. By choosing the left-inverse of J as $E = T^{-1}P^{\dagger}$, this is equivalent to show that $\Pi_R = PP^{\dagger} = \Pi = JE$.

$$\Pi = PTT^{-1}P^{\dagger} = PP^{\dagger} = \Pi_{\mathcal{R}}$$

Moreover, $\Pi^2 = \Pi$ by the simple fact that E is a left-inverse for J. This is trivial since we have chosen E as the transformation of the left-inverse of P. However this is not the only choice and it may happen that E has negative entries, violating the non-negativity constraint.

As mentioned in Example 3.1.3, we now show more formally that not only we can freely chose the left-inverse of P, but also the left inverse of J. Suppose that J is monotone and choose $K \neq T^{-1}P^{\dagger}$ as a left-inverse. The projector becomes

$\Pi = JK = PTK$ and it is idempotent

$$\Pi^2 = PTKPTK = PTK = \Pi$$

Moreover, recall that the choice of the left-inverse do not alter the projection space, hence $\mathrm{Im}(\Pi) = \mathrm{Im}(\Pi_R) = X^R$.
In conclusion, we have shown that if we can find *any* invertible matrix $T \in \mathbb{R}^{q\times q}$ that makes $J = PT$ monotone, any non-negative choice of the left-inverse E grants (E, J) to be a RPMR, thanks to Theorem 2.4.4. The next question is what are the conditions on the existence of such T.

For now, suppose to have a truncated reachability matrix that has q linearly independent rows stacked up on the top.

$$P = \left[\begin{array}{c} P_0 \\ \hline P_1 \end{array}\right]$$

Observe that P_0 is a invertible non-negative square matrix in $\mathbb{R}^{q\times q}$. Right-multiplying P by P_0^{-1} would result in

$$J = PP_0^{-1} = \left[\begin{array}{c} P_0 \\ \hline P_1 \end{array}\right] P_0^{-1} = \left[\begin{array}{c} I_q \\ \hline P_1 P_0^{-1} \end{array}\right]$$

that is monotone thanks to the identity in the upper block (Proposition 3.1.4). However, we must not forget the non-negativity constraint on J, hence using P_0^{-1} to diagonalize P_0 is a good choice only if $P_1 P_0^{-1}$ is non-negative.

Remark. Supposing that the linearly independent rows are stacked up top is actually only useful for the sake of visualization. If this is not the case, the rows of P_0 would be scattered throughout the matrix, and this would just imply that the identity block of J would be also scattered. However, as one could remember from Theorem 3.1.1, this would not ruin the monotonicity property of J. Notice instead that rearranging the rows of the reachability matrix is not allowed, as it would result in a change of the space we project into that is not admissible to our aims, as the reduction works only if we project on the reachable space.

Another observation can be made. Recall that by Proposition 3.1.4, J is non-negative monotone if and only if it has q distinct rows that are proportional to the canonical vectors, so we do not require the identity block. However, requiring the rows to be proportional to the canonical vectors is equivalent to say that the upper block has to be a matrix of the form $D_+O \in \mathbb{R}^{q\times q}$, where $D_+ \in \mathbb{R}^{q\times q}$ is a diagonal matrix with positive entry on the diagonal, whereas $O \in \mathbb{R}^{q\times q}$ is a permutation matrix. This would result in

$$J = PP_0^{-1}D_+O = \left[\frac{P_0}{P_1}\right] P_0^{-1}D_+O = \left[\frac{D_+O}{P_1P_0^{-1}D_+O}\right]$$

and since by the upper block this matrix is monotone, we would still require that $P_1P_0^{-1}D_+O \geq 0$. Being D_+ and O non-negative matrix by definition, the condition would still be $P_1P_0^{-1} \geq 0$. This shows that D_+ and O do not give us any further degrees of freedom, hence it is sufficient to focus on obtaining $P_0P_0^{-1} = I$. Before presenting the next theorem, we give a definition that will be useful in the following.

Definition 3.1.4. Let $M \in \mathbb{R}^{n\times q}$ be a matrix. A row sub-matrix is a matrix $S \in \mathbb{R}^{k\times q}$ composed of $k < n$ rows of M.

We are now ready to give the most general theorem of this work regarding RPMR.

Theorem 3.1.7. *Consider an unreachable system* (Σ) *with* $\mathrm{rank}\mathcal{R} = q$ *where* $\mathcal{R}$ *is the reachability matrix.* (Σ) *admits an RPMR on* X^R *if and only if for all* P *made of* q *linearly independent columns of* $\mathcal{R}$ *there exists a full-rank row sub-matrix* $P_0 \in \mathbb{R}^{q\times q}$ *of* P *such that* $P_1P_0^{-1} \geq 0$ *where* $P_1 \in \mathbb{R}^{(n-q)\times q}$ *is the sub-matrix made of the rows not included in* P_0.

Proof. ($\Leftarrow$) Suppose that, having fixed P, such P_0 exists and hence $P_1P_0^{-1} \geq 0$. If this is the case then

$$J = PP_0^{-1} = \left[\frac{I_q}{P_1P_0^{-1}}\right] \geq 0,$$

where we have assumed as before that the linearly independent rows are stacked on top for the sake of visualization, as in general might be distributed differently. This is equivalent to say that J is monotone and non-negative. Therefore (E, J) is an RPMR where $E \geq 0$ exists and is a non-negative left-inverse of J. In particular one can choose $E = [I_q|0]$.

($\Rightarrow$) First notice that the choice of P is indeed arbitrary since any set of q linearly independent column of $\mathcal{R}$ span the reachable space, and any other $\tilde{P}$ takes the form $\tilde{P} = PT$ for some invertible T. It follows that all projectors onto the reachable space are of the form

$$\Pi_R = \tilde{P}\tilde{P}^{\dagger} = PTT^{-1}P^{\dagger}$$

Let $J = PT$. In order to have an RPMR, J has to be non-negative and admit a non-negative left-inverse E. To admit a non-negative left-inverse, by Theorem 3.1.1 J must be monotone. To satisfy the latter condition, by Proposition 3.1.4 J must have q distinct rows proportional to the canonical vectors. Without loss of generality, we can suppose that these q rows are stacked on top. Hence J has to be of the form

$$J = PT = \left[\begin{array}{c} P_0 \\ \hline P_1 \end{array}\right] T = \left[\begin{array}{c} D_+O \\ \hline \tilde{P}_1 \end{array}\right]$$

where $D_+ \in \mathbb{R}^{q\times q}$ is a positive diagonal matrix, $O \in \mathbb{R}^{q\times q}$ is a permutation matrix and $T = P_0^{-1}D_+O$. Notice that if the q rows are not stacked on top, the rows of D_+O would be scattered throughout J and the conditions of Theorem 3.1.4 would still hold. We can now choose

$$E = T^{-1}P^{\dagger} = \left[\begin{array}{c|c} O^T D_+^{-1} & 0 \end{array}\right] \geq 0$$

and observe that J is non-negative (and hence (J, E) gives an RPMR) if and only if $\tilde{P}_1 = P_1T = P_1P_0^{-1}D_+O \geq 0$ that is equivalent to $P_1P_0^{-1} \geq 0$ since D_+ and O are non-negative by construction. □

From the computational point-of-view, a brute force approach can be undertaken. Indeed the choices of P_0 are finite, more precisely $\frac{n!}{(n-q)!q!}$. Notice however that the dimension of P_0 does not depend on n, and hence when $n \to \infty$ only the choices of P_0 grow. On the other hand, the dimension grows proportionally with q. Recall that the computation of an inverse is a very onerous operation, and thus q plays a key role on the algorithmic complexity. We can now propose an algorithm to check whether a RPMR exists (Algorithm 2).

Algorithm 2 Existence of an RPMR

Require: $\Sigma = (A, B, C)$
  $n \leftarrow \dim\Sigma$
  $\mathcal{R} \leftarrow \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$
  if $\operatorname{rank}\mathcal{R} = n$ **then**
    Stop: the system is reachable and hence cannot be reduced.
  else if $\operatorname{rank}\mathcal{R} = q < n$ **then**
    $P \leftarrow \begin{bmatrix} B & AB & \dots & A^{q-1}B \end{bmatrix}$
    if P already monotone **then**
      $\Sigma_+ = (P^\dagger AP, P^\dagger B, CP)$ where $P^\dagger \geq 0$ left-inverse of P
    else if P is not monotone **then**
      for any choice of $P_0 \in \mathbb{R}^{q\times q}$ non-singular matrix **do**
        $P_1 \in \mathbb{R}^{(n-q)\times q}$ composed by the rows not included in P_0
        if $P_1 P_0^{-1} \geq 0$ **then**
          $J = PP_0^{-1}$ monotone
          $\Sigma_+ = (EAJ, EB, CJ)$ through (E, J), where $E \geq 0$ left-inverse of J
          Stop: Σ_+ is the reduced positive model.
        end if
      end for
      Stop: Inconclusive. A reachable RPMR do not exists.
    end if
  end if

3.1.6 Example: Leslie Model

We propose a physical example in which the monotone matrices approach grants an RPMR. [1] The Leslie model is a dynamical model which describes the time evolution of populations in which fertility and survival rates of individuals strongly depend on their age. In the Leslie model, the time t is discrete and denotes the year (or the reproduction season), while the state variables $x_1(t), x_2(t), \dots, x_n(t)$ represent the number of females (or individuals or couples) of age $1, 2, \dots, n$ at the beginning of year t. Assuming that there are no differences in the survival rates of males and females and that the sex ratio is balanced, one can describe the "aging" process by means of the equations:

$$x_{i+1}(t+1) = s_i x(t) \qquad i = 1, \dots, n-1 \tag{3.2}$$

where s_i, is the survival coefficient at age i, that is, the fraction of females of age i that survive at least for 1 year. The first state equations take into account the reproduction process, and are therefore

$$x_1(t+1) = s_0(f_1 x_1(t) + f_2 x_2(t) + \dots + f_n x_n(t)) \tag{3.3}$$

where s_0 is the survival coefficient during the first year of life and f_i is the fertility rate of females of age i, that is, the mean number of females born from each female of age i. These equations are a positive linear autonomous model $x(t+1) = Ax(t)$ where A is the Leslie matrix

$$A = \begin{bmatrix} s_0 f_1 & s_0 f_2 & \dots & s_0 f_{n-1} & s_0 f_n \\ s_1 & 0 & \dots & 0 & 0 \\ 0 & s_2 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & s_{n-1} & 0 \end{bmatrix}$$

Consider now a Leslie model of dimension $n = 4$. We can introduce an input $u(t)$ that enter the system through the matrix B, making the system no longer autonomous

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

that means that at the beginning of year t, $u(t)$ females individuals of age 1 are introduced in the population. The system equation then becomes

$$x(t+1) = Ax(t) + Bu(t)$$

Let now $s_3 = 0$. The reachability matrix is

$$\mathcal{R} = \begin{bmatrix} 0 & s_0 f_2 & s_0 f_1 f_2 + s_0 s_2 f_3 & * \\ 1 & 0 & s_0 f_2 s_1 & * \\ 0 & s_2 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The truncated reachability matrix is

$$P = \begin{bmatrix} 0 & s_0 f_2 & s_0 f_1 f_2 + s_0 s_2 f_3 \\ 1 & 0 & s_0 f_2 s_1 \\ 0 & s_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that it is not monotone. However a trivial RPMR exists. Indeed

$$P_0 = \begin{bmatrix} 0 & s_0 f_2 & s_0 f_1 f_2 + s_0 s_2 f_3 \\ 1 & 0 & s_0 f_2 s_1 \\ 0 & s_2 & 0 \end{bmatrix}$$

and clearly $P_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} P_0^{-1} \geq 0$. It result that

$$J = PP_0^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and hence (EAJ, EB) is positive. This is easily interpreted, and could have been predicted, since setting $s_3 = 0$ means that we have set the survival rate of female individuals of age 3 to zero, and hence there will never be an individual of age 4. It makes sense then to consider the reduced system made of only the individuals of age $1, 2, 3$.

3.2 Minimal positive realizations

The problem that will now be addressed is the following: when can we reduce a positive system such that the reduced system is a minimal positive realization? This question has great interest in the literature, since often the dimension of the minimal positive realization does not coincide with the dimension of the minimal realization. The interested reader can find more about this topic in the survey paper [4]. For low dimensional systems of dimension one or two, the non-negativity of the impulse response is a necessary and sufficient condition for the existence of a positive minimal realization [15]. For greater dimension, the minimum dimension of a positive realization is still an open problem. Note that the minimum dimension of a positive realization may be greater than the dimension of the minimal realization. Indeed, non-negativity of the system matrices impose some limitations on the locations of their eigenvalues. For example, when the eigenvalues of a minimal realization of dimension n are outside of the Karpelevič region, that consist in the region where an $n \times n$ non-negative matrix must have its eigenvalues within, a positive realization of dimension n does not exists. However, our approach enable us to have a minimal positive realization through robust reduction when it is possible.

Theorem 3.2.1. *Consider a positive system $\Sigma = (A, B, C)$. Suppose that conditions of Theorem 3.1.7 holds. Compute the reduced system $\tilde{\Sigma}$ through Algorithm 2. Suppose now that the same conditions holds for $\tilde{\Sigma}_d$, that is the dual system of $\tilde{\Sigma}$. Then, reducing in the same way the dual system to Σ_d^{min} , it holds that Σ^{min} is a minimal positive realization for Σ.*

In words, this theorem says that if we can find a minimal realization of a system using only RPMR, then we can say that it is the positive system of minimum dimension. However, the conditions for this to happen are stringent. As said before, the minimal dimension of a positive realization is still an open problem, and to say something stronger we would need to further develop the non-robust positive model reduction techniques. Indeed, if we cannot find an RPMR to positively reduce our system, there could still be a non-robust reduction that admits a positive reduced system of smaller dimension. However, this work will not study in detail this problem, but we continue investigating into the robust positive reduction, giving an overview over a different approach that has been developed to reduce positive systems through the enclosure of the reachable space.

4

Algebraic approach

In this section we will review an approach to model reduction that exploits algebraic methods to reduce positive systems. We resort to the work of Ticozzi and Grigoletto [8], where the algebraic approach has been used to reduce Hidden Markov Models in a probabilistic setup, and on [9] where they extended the method from a classical point of view to a quantum one. The idea is to close the reachable (or observable) space to an algebra $\mathscr{A}$ so that a RPMR is possible. We prove that it is always possible to project onto an algebra and obtain a positive model. Moreover, we show that the reachable algebra has crucial properties for our approach. For this purpose, we make some considerations comparing this approach to our approach based on monotone matrices, highlighting the parallelism and the differences between them. In this chapter, we use and thus denote the vectors of all ones, denoted by $\mathbf{1}$, and the vector of all zeros, denoted by $\mathbf{0}$. We also use the concept of support defined in Section 3.1.1.

4.1 Definitions and structure

We start from the definition of a vector algebra equipped with the element-wise multiplication.

Definition 4.1.1 (Element-wise multiplication operator)**.** Let $x, y \in \mathbb{R}^n$. The element-wise multiplication operator is defined as following:

$$[x \wedge y]_i = [x]_i[y]_i$$

where with $[x]_i$ we denote the i-th entry of the vector x, and with $[x \wedge y]_i$ we denote the i-th entry of the vector $x \wedge y$.

Definition 4.1.2 (Vector algebra)**.** We define a vector algebra $\mathscr{A}$ over $\mathbb{R}^n$ as a vector space that is closed under the element-wise multiplication operation and such that the element-wise multiplication operation is bilinear and associative. More precisely, the following identities hold for all element $x, y, z \in \mathbb{R}^n$ and all scalars $a, b \in \mathbb{R}$:

- $(x + y) \wedge z = x \wedge z + y \wedge z$
- $z \wedge (x + y) = z \wedge x + z \wedge y$
- $ax \wedge by = (ab) \wedge (x \wedge y)$
- $(x \wedge y) \wedge z = x \wedge (y \wedge z)$

Note that being a vector space is also hold that $\alpha v + \beta w \in \mathscr{A}$, with $\alpha, \beta \in \mathbb{R}$.

More precisely, by closeness under element wise multiplication we mean that if $v, w \in \mathscr{A}$ then $v \wedge w \in \mathscr{A}$. Clearly $\mathbb{R}^n$ is a vector algebra. Moreover, it is unital, since it contains $\mathbf{1}$. A non-unital algebra still contains the vector $\mathbf{1}_{\mathcal{A}}$, which has entries 1 on the support of $\mathscr{A}$ and 0 otherwise and acts as the product identity in $\mathscr{A}$. Without further ado, we can understand better how a vector space can be extended to an algebra, and then present the main relevant properties that such algebra exhibit. Given a vector space $\mathcal{V} \supseteq \mathbb{R}^n$, its closure to algebra, i.e. $\mathscr{A}_{\mathcal{V}} = \mathrm{alg}(\mathcal{V})$ is the span of all the vectors contained in $\mathcal{V}$ plus all the possible vectors generated by the element wise operation between two vectors in the algebra itself. This indeed assures that the algebra is closed under element wise multiplication.

Proposition 4.1.1. *Given a vector subspace $\mathcal{V} \supseteq \mathbb{R}^n$, its closure to algebra $\mathscr{A}_{\mathcal{V}} = \mathrm{alg}(\mathcal{V})$ is given by*

$$\mathscr{A}_{\mathcal{V}} = \mathrm{span}\{v_i, v_j, \ldots, v_i \wedge v_j, \ldots, (v_i \wedge v_j) \wedge v_k, \ldots\}$$

One can understand that it could happen that the algebra is equal to the whole $\mathbb{R}^n$. It can be shown through examples that the algebra has dimension smaller than n when there exists at least 2 indices $\{i, j\}$ for which the following holds for every generator v_k of the subspace:

$$[v_k]_i = [v_t]_i, \quad [v_k]_j = [v_t]_j \quad \forall k, t$$

The main relevant property of these algebras is that they are generated by vectors that have a specific form. Indeed, it has been proved that an algebra can always be generated by special and convenient vectors of zeros and ones with disjointed supports. We call them idempotent generators, since they are idempotent with respect to the element wise multiplication.

Proposition 4.1.2. *Given a vector subspace $\mathcal{V} \supseteq \mathbb{R}^n$, there exists a set of idempotent generators f_i , $i = 1, \ldots, q$ such that its closure $\mathscr{A}_{\mathcal{V}}$ can be written as*

$$\mathscr{A}_{\mathcal{V}} = span\{f_i\} \quad , \quad \sum_i f_i = \mathbf{1}$$

Moreover, this generators are orthogonal and they are composed only of zeros and ones.

The reader may now see the direction we are going, but before reaching the focal point of this chapter we need to make some considerations. We want to close the reachable (or observable) space to an algebra $\mathscr{A}_{\mathcal{R}}$ and then see if projecting onto this algebra we obtain a positive model reduction. As anticipated and as proved in [8], this holds. We will now set up the problem and show that it is possible to prove this also through the non-negative monotone matrices theory.

4.2 Positive reduction onto the reachable (or observable) algebra

Consider now the positive system (Σ) of dimension n. We again suppose that the system is not reachable, hence the reachability matrix is singular and P is its truncated version. We can now define the reachable subspace X^R that is generated by the columns vectors of P, denoted by p_i. Namely,

$$X^R = \text{span}\{p_1, p_2, \ldots, p_q\}$$

The algebra containing X^R is $\mathscr{A}_{\mathcal{R}}$. As pointed out before, it could happen that $\mathscr{A}_{\mathcal{R}} = \mathbb{R}^n$, and in this case projecting into the algebra would not provide a system of smaller dimension. This is why we suppose that $\mathscr{A}_{\mathcal{R}}$ is generated by $q < n$ idempotent vectors. Recall that, by definition, the idempotent generators are non-negative. Let now group the generators of $\mathscr{A}_{\mathcal{R}}$ as columns of a matrix J. The following result highlight the main parallelism between the algebraic approach and the monotone matrices approach.

Proposition 4.2.1. *Let $J \in \mathbb{R}^{n \times q}$, $q < n$ be the matrix with the idempotent generators of an algebra as columns. Then* $\mathrm{cone}(J) = \mathbb{R}^q_+$, *i.e. J is monotone.*

Proof. Call f_i the columns of J. Recall that the f_i , the columns of J, are orthogonal vectors of only ones and zeros and that $\sum_i f_i = \mathbf{1}$, namely $J\mathbf{1} = \mathbf{1}$. By the orthogonality of the non-negative vectors $\{f_i\}$, their supports have to be pair-wise disjointed, and hence the rows of J have to be canonical vectors in $\mathbb{R}^q$. Moreover, by the linear independence of the columns, q rows of J are distinct canonical vectors. By Proposition 3.1.4, it holds that J is monotone. □

Finally, we can now apply the same procedure for RPMR as in the previous chapters to get the positive reduction. We formalize the procedure into a theorem.

Theorem 4.2.2. *Consider a positive system $\Sigma = (A, B, C)$. Let its truncated reachability matrix be defines as*

$$P = \begin{bmatrix} B \; AB \; \dots \; A^{q-1}B \end{bmatrix}$$

The reachable subspace is $X^R = \mathrm{span}\{P\}$, i.e. the span of the columns of P. Consider now the closure of the reachable subspace to an algebra $\mathscr{A}_{\mathfrak{R}} = \mathrm{alg}(X^R)$. Let now $J \in \mathbb{R}^{n \times k}$ be the matrix with the idempotent generators of the algebra as columns and E be a non-negative left inverse. Then,

$$\Sigma_+ : \quad A_+ = EAJ \qquad B_+ = EB \qquad C_+ = CJ \tag{4.1}$$

is a positive realization of dimension k. Moreover, if $k = q$, the pair (E, J) is a RPMR.

Proof. By 4.2.1 it holds that the matrix J, composed by the generators of $\mathscr{A}_{\mathfrak{R}}$ is a monotone matrix, hence it admits a non-negative left inverse E. It immediately follows that (E, J) is a non-negative linear change of basis, since both E and J are non-negative matrices. Thus, (Σ_+) still is a positive realization. Clearly if the dimension of the algebra $\mathscr{A}_{\mathfrak{R}}$ is less than the dimension of the system (Σ), we obtain a positive system of dimension less than n, and therefore by definition (E, J) is a RPMR. What is left to prove is showing that projecting onto an algebra preserve the dynamics of the system. Recall that J is the matrix constructed with the idempotent generators as columns. Firstly build the projector $\Pi_{\mathscr{A}} = JE$.

Consider now also the projector into the reachable space $\Pi_{\mathcal{R}}$ as defined in the proof of Proposition 2.4.4. Since $\mathrm{alg}\{\mathcal{R}\} \supseteq X^R$, namely the algebra contains the reachable space, it also holds that $\mathrm{alg}\{X^R\}$ contains B, hence $\Pi_{\mathscr{A}} B = B$ and

$$\Pi_{\mathscr{A}}\Pi_{\mathcal{R}} = \Pi_{\mathcal{R}}$$

The Markov's coefficient of the system are

$$M_{k-1} = CA^kB = CA^k\Pi_{\mathscr{A}}B = CA^k\Pi_{\mathcal{R}}\Pi_{\mathscr{A}}B = C\Pi_{\mathcal{R}}A^k\Pi_{\mathcal{R}}\Pi_{\mathscr{A}}B = C\Pi_{\mathscr{A}}\Pi_{\mathcal{R}}A^k\Pi_{\mathcal{R}}\Pi_{\mathscr{A}}B$$

Hence $\Pi_{\mathscr{A}}\Pi_{\mathcal{R}} = \Pi_{\mathcal{R}}\Pi_{\mathscr{A}} = \Pi_{\mathcal{R}}$, concluding that

$$= C\Pi_{\mathcal{R}}A^k\Pi_{\mathcal{R}}B = C\Pi_{\mathscr{A}}A^k\Pi_{\mathscr{A}}B = (CJ)(EA^kJ)(EB)$$

and thus the systems Σ and Σ_+ are equivalent. □

This result show that it is always possible to project onto the reachable algebra and robustly obtain a positive model. Indeed, we have shown that closing the reachable space to an algebra always grant that the matrix with the generator of such algebra as columns is monotone. By duality, it is easy to extend this result to the projection onto the observable algebra, and for this reason we will not present it.

4.3 Distorted algebras

As presented in [8], sometimes it may be more efficient to project onto a more restrictive space rather than the algebra itself. For this reason we introduce the concept of *distorted algebra*, that is similar to the algebra but with a more general definition, as it allows for different multiplicative operations. Firstly, we define the element-wise multiplication operator with respect to a vector.

Definition 4.3.1. Let $v \in \mathbb{R}^n$. The element-wise multiplication operator with respect to v denoted as $\wedge_v$ is defined such that taking two vectors $x, y \in \mathbb{R}^n$

$$[x \wedge_v y]_i = \frac{[x]_i[y]_i}{[v]_i}$$

Clearly, the standard element-wise multiplication operator can be seen as $\wedge_1$.

Definition 4.3.2 (Distorted algebra)**.** A distorted algebra is an algebra (4.1.2) that is closed under a element-wise multiplication operator with respect to a vector v. More precisely, let $v \in \mathbb{R}^n$ be a vector and $\mathcal{X} \subseteq \mathbb{R}^n$ be a subspace. Let X be the matrix with the generators of $\mathcal{V}$ as columns. The closure of $\mathcal{X}$ to a distorted algebra $\mathscr{D}_{\mathcal{X}} = \text{alg}_v(\mathcal{X})$ is given by

$$\mathscr{D}_{\mathcal{X}} = \text{span}\{x_i, x_j, \dots, x_i \wedge_v x_j, \dots, (x_i \wedge_v x_j) \wedge_v x_k, \dots\}$$

Moreover, a relation between distorted and "standard" algebras can be obtained [8]:

$$\mathscr{D}_{\mathcal{X}} = v \wedge \text{alg}(v^{-1} \wedge \mathrm{X}) = \text{diag}(v)\ \text{alg}(\text{diag}(v^{-1})X)$$

Although the standard algebra and the distorted algebra are similar concepts, we think it is worth to understand more deeply the key aspects and differences between the two. To this purpose, we provide a brief section and a meaningful example.

4.3.1 Visualizing distorted algebras

We devote this section of this chapter to highlight in a more intuitive way the role of the distorted algebras with respect to the standard algebra. We start by defining what a subvector is.

Definition 4.3.3. Let $v \in \mathbb{R}^n$ be a vector and v_i its i-th entry. Let $\mathcal{S}$ be a subset of indices between 1 and n. Then

$$[v]_{\mathcal{S}} = \begin{bmatrix} v_{i_1} \\ v_{i_2} \\ \vdots \\ v_{i_k} \end{bmatrix}$$

is a subvector of v with respect to the subset of indices $\mathcal{S} = \{i_1, i_2, \dots, i_k\}$ where $k \leq n$ is the cardinality of $\mathcal{S}$. (Note that if $k = n$ the subvector would coincide with the vector itself)

With this definition, we are ready to show how the distorted algebra is a more general definition with respect to the standard algebra that we are considering. As we have previously noticed, the closure to algebra could result in the algebra being the whole space $\mathbb{R}^n$. We now show when this is not the case.

Proposition 4.3.1. *Let* $\mathcal{X} \subsetneq \mathbb{R}^n$ *be a subspace spanned by a* q *vectors, i.e.* $\mathrm{span}\{x_1, \ldots, x_q\} = \mathcal{X}$*. Let* $\mathscr{A} = \mathrm{alg}(\mathcal{R})$ *be its closure to algebra. Then* $\mathscr{A} \subsetneq \mathbb{R}^n$ *if there exists a subset of indices* $\mathcal{S} = \{i_1, i_2, \ldots, i_k\}$ *with* $k \geq 2$*, where for all generators* x_j *of* $\mathcal{X}$ *the following holds*

$$\lambda_j[\mathbf{1}_n]_{\mathcal{S}} = [x_j]_{\mathcal{S}} \quad \forall j = 1, \ldots, q \ , \lambda_j \in \mathbb{R}$$

Proof. Note that if such $\mathcal{S}$ exists, the element-wise multiplication between any of the x_i's would still result in a vector that has 1's in the entries of indices in $\mathcal{S}$. This means that closing to algebra the subspace, those components can be generated only by one vector that has ones in the k entries of indices in $\mathcal{S}$ and zero elsewhere. Take now this vector as one generator of the algebra. Since $k \geq 2$, taking the other generators as canonical vectors orthogonal between each other, we would have $n - k + 1$ generators and hence $\mathscr{A} \subsetneq \mathbb{R}^n$. □

In other words, the algebra has not maximum dimension n when all of its generators have subvectors that are proportional to the vector of all ones. As we will show, it is in this aspect that the distorted algebra is by definition more general with respect to the standard algebra.

Proposition 4.3.2. *Let* $\mathcal{X} \subsetneq \mathbb{R}^n$ *be a subspace spanned by a* q *vectors, i.e.* $\mathrm{span}\{x_1, \ldots, x_q\} = \mathcal{X}$*. If there exist a subset of indices* $\mathcal{S} = \{i_1, i_2, \ldots, i_k\}$ *with* $k \geq 2$ *and a vector* $v \in \mathbb{R}^n$ *where for all generators* x_j *of* $\mathcal{X}$ *the following holds*

$$\lambda_j[v]_{\mathcal{S}} = [x_j]_{\mathcal{S}} \quad \forall j = 1, \ldots, q \ , \lambda_j \in \mathbb{R}$$

we will have that $\mathscr{D} \subsetneq \mathbb{R}^n$*, where* $\mathscr{D} = \mathrm{alg}_v(\mathcal{X})$ *is the closure to distorted algebra of the reachable space equipped with the* $\wedge_v$ *operator.*

Proof. Using the same method as in the proof of Proposition 4.3.1, we can take as generator the vector that has the same components of v in the entries of indices in $\mathcal{S}$ and zero elsewhere. Again, taking all the other generators as canonical vectors orthogonal between each other, we would have $n - k + 1$ generators with $k \geq 2$ and hence $\mathscr{D} \subsetneq \mathbb{R}^n$. □

We now present an example where the distorted algebra approach provides an optimal RPMR, while the standard algebra approach provides a reduced model with a bigger dimension than the minimal.

Example 4.3.1 ([8])**.** Let us consider the following system:

$$A = \begin{bmatrix} 2/5 & 0 & 1/5 \\ 0 & 2/5 & 1/5 \\ 3/5 & 3/5 & 3/5 \end{bmatrix}, \qquad b = \begin{bmatrix} 1/5 \\ 1/5 \\ 3/5 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can compute the following:

$$X^R = \text{span}\left\{ \begin{bmatrix} 1/5 \\ 1/5 \\ 3/5 \end{bmatrix} \right\}$$

Then it is easy to compute its closure to algebra

$$\mathscr{A} = \text{alg}(X^R) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The reduction matrices are then

$$J = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The reduced system is

$$A_+ = EAJ = \begin{bmatrix} 2/5 & 1/5 \\ 6/5 & 3/5 \end{bmatrix}, \quad b_+ = Eb = \begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}, \quad C_+ = CJ = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

and has dimension 2. However, notice that b is an equilibrium, i.e. $Ab = b$. Hence, if we consider the algebra with respect to the vector $v = b$, we get

$$\mathscr{D}_v = v \wedge \text{alg}(v^{-1} \wedge X^R) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Thus the reduction matrices are

$$J = \begin{bmatrix} 1/5 \\ 1/5 \\ 3/5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

and hence the reduced systems is

$$A_+ = 1, \qquad b_+ = 1, \qquad C_+ = \begin{bmatrix} 2/5 & 3/5 \end{bmatrix}$$

and has dimension 1, that is the optimal solution. This example shows that in some cases projecting onto a distorted algebra can give better results then projecting onto the standard one.

What is left to understand is how to choose v such that the choice is optimal with respect to the dimension of the algebra that we obtain. We resort to [8], where the proofs of the following Lemma and Theorem can be found, to prove the optimal choice of v.

Lemma 4.3.3 ([8]). *Given a vector space $\mathcal{X} \subseteq \mathbb{R}^n$ with generators $\{x_i\}$, $\mathcal{X} = \mathrm{span}\{x_i\}$ there exists a vector $v := \sum_i \lambda_i x_i$ with $\lambda_i \neq 0$ for all i and such that $\mathrm{supp}(v) = \mathrm{supp}(\mathcal{X})$.*

Remark. Notice that in our case the vector v will be always non-negative. Indeed since all generators of the reachable space X^R are non-negative, to have that $\mathrm{supp}(v) = \mathrm{supp}(X^R)$ we can choose any $\lambda_i > 0$ for all i. This comes from the fact that positive combinations of non-negative vectors can only increase the dimension of the support. The conclusion is that v can always be chosen non-negative.

The following theorem proves the optimality of the choice of this choice of v.

Theorem 4.3.4 ([8]). *Consider a vector space $\mathcal{X} \subseteq \mathbb{R}^n$ and a vector v as in Lemma 4.3.3. Then there exists a unique algebra $\mathscr{A}$ of minimal dimension such that $\mathcal{X} \subseteq w \wedge \mathscr{A}$ for some $w \in \mathbb{R}^n$. Moreover, $\mathscr{A} = \mathrm{alg}\{v^{-1} \wedge \mathcal{X}\}$ and its unital over the support of $\mathcal{X}$, i.e. $\mathbf{1}_{\mathrm{supp}(\mathcal{X})} \in \mathscr{A}$.*

So far we have understood that projecting onto the reachable algebra is always possible, but often it does not grant a model reduction. We have also seen the natural extension of the algebras to distorted algebras, that enable us to perform an RPMR to a broader family of systems. However, we still have to understand which is the correlation between the algebraic approach and the monotone matrices approach. The rest of this chapter is devoted to understand better whether a RPMR is possible if and only if we project onto an algebra, namely if the reachable (or observable) subspace is already an algebra. If this is the case, the two approaches are equivalent and one can immediately see if a RPMR is possible on the system.

4.4 Algebraic approach vs Monotone matrices approach

In order to develop some intuition, we try to find examples where the monotone matrices approach could lead to a reduction while the algebraic approach does not. Recall that the idempotent generators of a vector algebra $\{f_i\}$ satisfy the following equality

$$\sum_i f_i = \mathbf{1}$$

This is a condition that is not required by the columns of a monotone matrix, hence we construct an ad hoc reachability matrix that do not satisfy that equality. Consider the system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

The truncated reachability matrix is

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

that is trivially monotone by the first two rows and hence admits a non-negative left-inverse

$$P^{\dagger} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The reduced system would be

$$\tilde{A} = P^{\dagger}AP = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \tilde{B} = P^{\dagger}B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

However, if we try to close the reachable space $X^R = \mathrm{span}(P)$ to an algebra we would get

$$\mathscr{A} = \mathrm{alg}(X^R) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

that has dimension equal to 3 and hence would result in a reduced system of that dimension.

This simple example proves that for some systems the monotone approach could be more promising than the algebraic approach. This also imply that the monotone matrices approach could be the optimal approach to RPMR, since in the best case it provides a reduced system of lower dimension then the reduced system obtained through the algebraic approach. However, the algebraic approach is more flexible since we can try to enclose the reachable space in a bigger space until we take $\mathbb{R}^n$, for which we have seen that it does not grant a reduction. The proposed monotone approach do not have this freedom. Indeed, either an RPMR exists and we reduce the system to a reachable system of dimension q, or we don't. Future works will try to find a method, if it exists, to extend the monotone matrices approach with the aim of enclosing the reachable space until an RPMR exists.

5 Conclusions and Future Works

The theory of monotone matrices has been specialized to non-negative monotone matrices, finding interesting properties and characterizations that proved useful in the study of Robust Positive Model Reduction (RPMR). Towards RPMR, it has been first proposed a solution of low computational effort, that is the reduction through the LQ decomposition, but it has also been shown that it could be generalized removing the orthogonality constraint on the Q matrix. In the end, a necessary and sufficient condition has been found to prove the existence of an RPMR. The latter exploits the freedom of choice of the projection onto the reachable space, finding, if it exists, a non-negative monotone matrix that spans the reachable space. The next steps will be taken with the aim of generalizing this approach so that if the reachable space do not admits an RPMR, it can be enclosed as little as it needs to satisfy the properties required by the monotone matrices approach. The monotone matrices approach has proved to be very effective, but still leaves open questions. In particular, the (non-robust) positive model reduction problem still remains open. To this aim, a further effort could be made to understand better the cases in which a non-robust positive reduction still grant a positive reduced system. Finally, we adapted the existing algebraic approach for hidden markov model reduction to general linear systems with inputs, highlighting remarkable similarities to the monotone matrices approach. We thus think it is worth further investigating how to adjust it in order to be also optimal in the cases in which it could not provide a reduced model of the same dimension of the monotone matrices approach.

A

Useful theorems

Theorem A.0.1 (QR decomposition for rectangular matrices). *Let $A \in \mathbb{R}^{q \times n}$ be a full rank rectangular matrix with $q < n$. If $A = Q_1[R_1\ N_1]$ and $A = Q_2[R_2\ N_2]$ are two QR decomposition, then*

$$Q_1 = Q_2 S \quad , \quad R_2 = S R_1 \quad , \quad N_1 = S N_2$$

where $S \in \mathbb{R}^{q \times q}$ is a diagonal matrix with entries ± 1.

Proof. Consider now $Q_1 R_1$ and $Q_2 R_2$. These are two QR decomposition of a square matrix, hence $Q_1 R_1 = Q_2 R_2$ implies $Q_2^{-1} Q_1 = R_2 R_1^{-1}$. The left hand side is an orthogonal matrix, whereas the right hand side is upper triangular. A matrix that is orthogonal and upper triangular can only be a diagonal matrix with entries ± 1. Then by $Q_2^{-1} Q_1 = S$ it follows that $Q_1 = Q_2 S$ and from $R_2 R_1^{-1} = S$ it follows that $R_2 = S R_1$. To prove the last part, we use the proven fact $Q_1 = Q_2 S$ on the equation $Q_1 N_1 = Q_2 N_2$, that becomes $Q_2 S N_1 = Q_2 N_2$. Left-multiplying it by $Q_2^\top$ and then by S proves that $N_1 = S N_2$. □

B

MATLAB code

```
clear
clc
%% Robust Positive Model Reduction

%% System matrices definition
A = [1 1 0 0; 1 0 2 0; 0 0 1 2; 0 0 3 1];
B = [1; 1; 0; 0];
C = [1 0 0 0];


%% Reachability matrix and monotonicity check
R = ctrb(A,B);
q = rank(R);
n = size(A,1);

%% Build P as the first rankR=q columns of R
P = [];
for i=1:q
    P = [P R(:, i)];
end

%% build indices to build all possible P_0
indices = nchoosek(1:n, q);

%% Build P_0, P_1 and check for the existence of a RPMR
```

```
n_iterations = size(indices, 1); %n!/(n-q)!*q! are the number of all
    possible P_0 (taking into account also the singular ones)
P_0 = [];
RPMR_found = false;

for i=1:n_iterations
    for j=1:q
        P_0 = [P_0; P(indices(i,j), :)];
    end
    if rank(P_0) ~= q   %sanity check
        continue
    end
% building P_1 as rows of P not included in P_0
    P_1 = [];
    for j=1:n
        if ~ismember(j, indices(i, :))
            P_1 = [P_1; P(j, :)];
        end
    end
    if isNonnegative(P_1/P_0) %checks if P_1 * P_0^{-1} >= 0
        RPMR_found = true;
        break   %RPMR found
    end

    % clearing the matrices P_0 and P_1
    P_0 = [];
    P_1 = [];

end

if RPMR_found
    J = P/P_0;
    E = leftInverse(J); %find the trivial non-negative left inverse
    if isnan(E)
        disp('Left Inverse failed');
    end

    % computing the reduced positive system
    A_r = E*A*J;
    B_r = E*B;
    C_r = C*J;

    if ~check_equivalence(ss(A,B,C,0), ss(A_r,B_r,C_r, 0), 50)
```

```
        disp('Original system and Reduced system seems to not have the same Markov coefficients');
    end

    disp('P:'); disp(P);
    disp('P_0:'); disp(P_0);
    disp('J:'); disp(J);
    disp('E:'); disp(E);
else
    disp('An RPMR does not exists')
end
```

Code B.1: Main script

```
function [flag, indices] = isMonotone(A)
% isMonotone returns flag=true if J is left-monotone.
    flag = false;
    n = size(A,1);
    q = size(A,2);
    indices = [];
    % if A is singular it cannot be monotone.
    if rank(A) ~= q
        return
    end
    id = eye(q);

    for i=1:n
        e_neq_zero = 0;
        for j=1:q
            if A(i,j) ~= 0
                e_neq_zero = e_neq_zero + 1;
            end
        end
        if e_neq_zero == 1
            [isValid, idx] = ismember(A(i,:)/norm(A(i,:)), id, 'rows');
            if isValid
                id(idx, :) = [];
                indices = [indices, i];
            end
        end
    end
    if isempty(id)
        flag = true;
```

```
    end
end
```

Code B.2: Checking monotonicity

```
function flag = isNonnegative(A)
% isNonnegative   Check if a matrix or vector A is non-negative entry
   -wise

    flag = true;
    for i=1:size(A,1)
        for j=1:size(A,2)
            if A(i,j) < 0
                flag = false;
            end
        end
    end
end
```

Code B.3: Checking non-negativity

```
function E = leftInverse(J)
% leftInverse compute, if it exists, the simplest non-negative left
% Inverse of the matrix J
    n = size(J,1);
    q = size(J,2);
    E = NaN;

    [isMon, indx] = isMonotone(J);
    if ~isMon
        return
    end
    E = zeros(q,n);
    for i=1:q
        E(:,indx(i)) = pinv(J(indx(i), :));
    end

end
```

Code B.4: Computing a left inverse

References

[1] “Age-Structured Population Models”. In: *Positive Linear Systems*. John Wiley Sons, Ltd, 2000. Chap. 13, pp. 117–129. ISBN: 9781118033029. DOI: https://doi.org/10.1002/9781118033029.ch13. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/9781118033029.ch13. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/9781118033029.ch13.

[2] Roy M Anderson and Robert M May. *Infectious diseases of humans: dynamics and control*. Oxford university press, 1991.

[3] Panos Antsaklis and Anthony Michel. *A Linear Systems Primer*. Jan. 2007. ISBN: 978-0-8176-4460-4. DOI: 10.1007/978-0-8176-4661-5.

[4] Luca Benvenuti. “Minimal positive realizations: A survey”. In: *Automatica* 143 (2022), p. 110422. ISSN: 0005-1098. DOI: https://doi.org/10.1016/j.automatica.2022.110422. URL: https://www.sciencedirect.com/science/article/pii/S0005109822002758.

[5] U Narayan Bhat and Gregory K Miller. *Elements of applied stochastic processes*. Vol. 3. Wiley-Interscience Hoboken^ eN. JNJ, 2002.

[6] L. Collatz. “Functional analysis and numerical mathematics”. In: (1910). URL: https://doi.org/10.1016/C2013-0-12430-0.

[7] Ettore Fornasini and Giacomo Marchesini. “Appunti di Teoria dei Sistemi”. In: 1992. URL: https://api.semanticscholar.org/CorpusID:122466043.

[8] Tommaso Grigoletto and Francesco Ticozzi. “Algebraic Reduction of Hidden Markov Models”. In: *IEEE Transactions on Automatic Control* (2023), pp. 1–16. DOI: 10.1109/TAC.2023.3279209.

[9] Tommaso Grigoletto and Francesco Ticozzi. “Model Reduction for Quantum Systems: Discrete-time Quantum Walks and Open Markov Dynamics”. In: (2023). arXiv: 2307.06319 [quant-ph].

[10] John Alfred Jacquez. *Compartmental analysis in biology and medicine: kinetics of distribution of tracer-labeled materials*. Elsevier Amsterdam, 1972.

[11] WW Leontieff. *The Structure of American Economy 1919-1939, deuxième édition élargie*. 1953.

[12] P. H. LESLIE. "ON THE USE OF MATRICES IN CERTAIN POPULATION MATHEMATICS". In: *Biometrika* 33.3 (Nov. 1945), pp. 183–212. ISSN: 0006-3444. DOI: 10.1093/biomet/33.3.183. eprint: https://academic.oup.com/biomet/article-pdf/33/3/183/572559/33-3-183.pdf. URL: https://doi.org/10.1093/biomet/33.3.183.

[13] O. L. Mangasarian. "Characterizations of Real Matrices of Monotone Kind". In: *SIAM Review* 10.4 (1968), pp. 439–441. DOI: 10.1137/1010095. eprint: https://doi.org/10.1137/1010095. URL: https://doi.org/10.1137/1010095.

[14] Colm Art O'Cinneide. "Characterization of phase-type distributions". In: *Communications in Statistics. Stochastic Models* 6.1 (1990), pp. 1–57. DOI: 10.1080/15326349908807134. eprint: https://doi.org/10.1080/15326349908807134. URL: https://doi.org/10.1080/15326349908807134.

[15] Yoshito Ohta, Hajime Maeda, and Shinzo Kodama. "Reachability, Observability, and Realizability of Continuous-Time Positive Systems". In: *SIAM Journal on Control and Optimization* 22.2 (1984), pp. 171–180. DOI: 10.1137/0322013. eprint: https://doi.org/10.1137/0322013. URL: https://doi.org/10.1137/0322013.

[16] L.R. Rabiner. "A tutorial on hidden Markov models and selected applications in speech recognition". In: *Proceedings of the IEEE* 77.2 (1989), pp. 257–286. DOI: 10.1109/5.18626.

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