Università degli Studi di Padova

Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA “TULLIO LEVI-CIVITA”

Corso di Laurea Triennale in Matematica

Random walks on graphs and electric networks

Relatore: **Prof. Marco Formentin**
Laureanda: **Francesca Florio**
Matricola: **2016109**

15 dicembre 2023

Anno Accademico 2022/2023

First I would like to thank my supervisor, Prof. Marco Formentin for his invaluable advice and immense helpfulness.
I would like to express my deepest gratitude to my parents, whom without this would have not been possible.
My sister Federica deserves my wholehearted appreciations, as well, for having always believed in me.
A special thanks finally goes to my boyfriend Galliano, for providing me with unfailing support and standing by my side.

Contents

Introduction

Since Antiquity, the relationship between mathematics and physics has been object of study. Generally considered a relationship of great intimacy, many branches of mathematics, like the probability theory, is indebted to physics as a rich source of problems, inspiration and insight for solving these problems. In particular, in this thesis we will look at the connection between random walks on graphs and electric network quantities (i.e. capacities, resistances and potentials), which has actually been recognized for some time. Our aim will be to prove Pólya's theorem. It states that a random walker on the d-dimensional lattice is bound to return to the starting point when $d = 1, 2$, but has a positive probability of escaping to infinity without returning to the starting point when $d \geq 3$. The key to prove this resault is to give an equivalent version of the theorem in terms of elementary electric network theory and then use some techniques from the same theory, which we owe to Lord Rayleigh, a physicist that made many contributions to science. His studies also included the acoustics, so much that he introduced those techniques in connection with an investigation of musical instruments. A very nice and simple example we could think of to formalize the topic is the following: wind instruments are possible in our 3-dimensional world, but are not possible in Flatland [1], a 2-dimensional world.
Here is the plan of the thesis. Chapter 1 contains some fundamental notions in order to talk about Pólya's recurrence problem. We regard simple random walks as finite state Markov chain, that is why these topics are linked together. This leads to Chapter 2, where we first analyze how Markov chains and resistor networks, one of the most simple electrical circuits, are involved together. We proceed with the treatement of the essential notions of potential, currents and energy, even integrating with some laws and intermediate theorems and thence we have Thomson's principle and Rayleigh's Monotonicity law. The chapter ends with a crucial corollary which formalizes the question about recurrence and transience. The final Chapter 3 is devoted to the proof of Pólya's theorem starting from the easiest case, in one dimension, up to the one in three dimensions, that also generalizes greater dimensions.

Chapter 1

Random walks and Markov chains

The concept of *random walk*, sometimes known as drunkard's walk, may be formalized with the idea of taking one step at the time in random directions. In order to prove the resault we are interested in, we will refer to infinite graphs as lattices. It is important to explain what do we mean by the term *lattice*, so we bring back the construction in [2]. A d-dimensional lattice is constructed by taking as vertices those points $(x_1, \ldots, x_d)$ of $\mathbb{R}^d$ all of whose coordinates are integers and we join each vertex by an undirected line segment - which is parallel to one of the coordinate axes of $\mathbb{R}^d$ - to each of its $2d$ nearest neighbours. In the sequel, we will denote this d-dimensional lattice by $\mathbb{Z}^d$. Moreover, if the jumps are chosen uniformly from the set of available neighbours, the random walk is *symmetric*.

In one dimension, our random walk is just an infinite line divided into segments of length one; in two dimensions, our lattice looks like an infinite network of streets and avenues. The primary question read: "Is a wandering man, starting at some given point, certain to return to its starting point?" The answer is yes and if so, we say that the walk is *recurrent*, in either case. In three dimensions, to fix the ideas, our lattice turns to a sky where a bird is flying in. So, in this case (and more in general, in every higher dimension), there is a positive probability that the bird will never return to its starting point, and we say that the walk is *transient*.

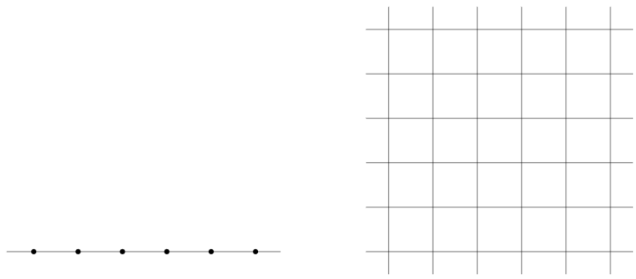

Figure 1.1: The two figures right above show us how a 1-dimensional lattice and a 2-dimensional lattice are done, respectively. In both cases, the walk is said to be recurrent, since there is a certain probability of return to some given starting point which we can choose freely.

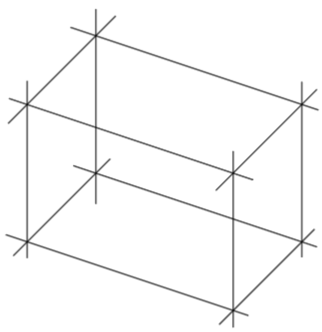

Figure 1.2: The picture gives us a partial view of what a 3-dimensional lattice is. Unlike before, here the walk is transient because we cannot be sure that there will be a return to the starting point eventually.

The solution to the problem of transience and recurrence is given by Pólya's theorem and to be able to tell about it we need to give some fundamental notions.

1.1 Basics on graphs

With the aim of clarifying the notation used in the succeeding paragraphs, we recall the definition of graph.

Definition 1.1. *A graph G consists of a set of vertices, indicated with $V(G)$, a set of edges, indicated with $E(G)$, and a relation called incidence so that each edge is incident with either one or two vertices, its ends.*

Two distinct vertices u, v are adjacent if there is an edge with ends u, v and in this case we will write $u \sim v$. We let uv denote such an edge, but sometimes we may prefer to denote some given edge with e. Finally, we will indicate with $\deg(u)$ the *degree* of a vertex u, that is the number of edges incident to that vertex.

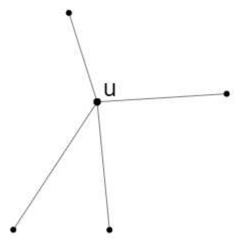

Figure 1.3: In this example, we highlight that $\deg(u) = 4$, because we can count exactly 4 segments outgoing from the vertex u.

We will work under two conditions. The first one is that our graph is *simple*: that means it has no loops or parallel edges. The second one is that our graph is *connected*.

Definition 1.2. *A graph G is connected if for every $u, v \in V(G)$ there is a walk from u to v.*

1.2 Basics on Markov chains

Markov chains represent an essential part of the theory we are dealing with; for this reason let us start by giving its formal definition. For this first part, we will rely on [3].
Let $E \neq \emptyset$ be at most countable set and $M = (M(x, y))_{x,y \in E}$ a stochastic matrix, i.e., a matrix for which each row $M(x, \cdot))$ is a probability density on E. Let's consider a random process on E that at each time step moves from x to y with probability $M(x, y)$.

Definition 1.3. *A sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$ and taking values in E is called a Markov chain with state space E and transition matrix M, if* $\forall n \geq 0, \forall x_0, \ldots, x_{n+1} \in E$

$$P(X_{n+1} = x_{n+1} | X_0 = x_0, \ldots, X_n = x_n) = M(x_n, x_{n+1}) \tag{1.1}$$

with $P(X_0 = x_0, \ldots, X_n = x_n) > 0$.

Basically, a Markov chain is a sequence of random variables with a short memory span, because the behaviour at the next point in time depends only on the current value and not on what happened before (*Markov property*).
We will consider Markov chains whose transition matrix M is irreducible, maybe better known as *ergodic* Markov chains.
Let us assume that $G = (V, E)$ with V state space.
In order to lighten notations, we set $f(x) = f_x$ for all f functions.

Definition 1.4. *A Markov chain on V is reversible if there is some positive function $\pi : V \to (0, +\infty)$ such that the transition probabilities satisfy for $u, v \in V$*

$$\pi_u p_{uv} = \pi_v p_{vu}. \tag{1.2}$$

Now, assume we have a reversible Markov chain $\{X_n\}_{n=0,\ldots,\infty}$ on V.
With each distinct pair $u, v \in V$, we associate the weight

$$w_{uv} = \pi_u p_{uv}, \tag{1.3}$$

noting by (1.2) that $w_{uv} = w_{vu}$.
Then, for $u, v \in V$

$$p_{uv} = \frac{w_{uv}}{W_u}, \tag{1.4}$$

where, for $u \in V$

$$W_u = \sum_{v \in V} w_{uv}. \tag{1.5}$$

That means that, given that $X_n = u$, the chain jumps to a new vertex v with probability proportional to w_{uv}.
One important functions' property we will need in the circuits theory is the following one.

Definition 1.5. *Let $U \subseteq V$, and let X be a Markov chain on V with transition matrix M, that is reversible with respect to the positive function π. The function $f : V \to \mathbb{R}$ is harmonic on U (with respect to the transition matrix M) if for $u \in U$*

$$f(u) = \sum_{v \in V} p_{uv} f(v). \tag{1.6}$$

We need two ingredients in order to describe a Markov chain completely: the transition matrix M and a method for starting the process. That is possible by specifing a state in which the process starts and the strategy consists in comparing the Markov chain states with the vertices of a graph G. So, from now on, S_n will be the position of the random walk at time n and, with a small abuse of notation, the Markov chain considered will be denoted by S.

Let us conclude this introductory part with a simple theorem concerning the harmonic functions. It will come in useful.
We assume $U \subset V$, $W = V \setminus U$ and $s \in U$. For $u \in U$, let $h(u)$ be the probability, starting at u, that the chain hits s before W. That is

$$h(u) = P_u(S_n = s \text{ for some } n < T_W)$$

where

$$T_W = \inf\{n \geq 0 : S_n \in W\}$$

and

$$P_u(\cdot) = P(\cdot\,|S_0 = u).$$

Clearly, $h(s) = 1$ and $h(v) = 0$ for $v \in W$. This fact suggests us to take $u \notin W \cup \{s\}$.

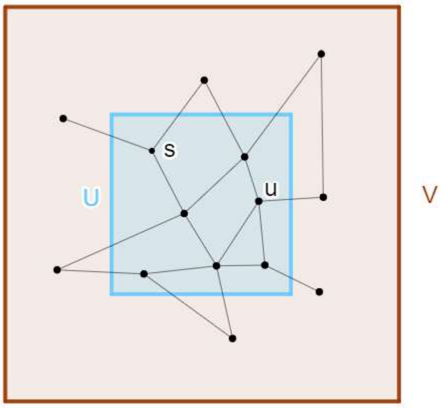

Figure 1.4: Above we are representing the non-trivial case $u \in U \setminus \{s\}$; in fact, if not, $h(u) = 0$ since we would be starting from the outside of U already. Basically, $h(u)$ is the probability that, starting from the inside of U, we reach a point still located in the inside of U without ever touching W.

Theorem 1.1. *The function h is harmonic on $U \setminus \{s\}$.*

Proof. By the Markov property, for $u \notin W \cup \{s\}$

$$\begin{aligned} h(u) &= \sum_{v \in V} p_{uv} P_u(S_n = s \text{ for some } n < T_W | S_1 = v) \\ &= \sum_{v \in V} p_{uv} h(v). \end{aligned}$$

□

Chapter 2

Electrical networks

Having explained what we needed about Markov theory, we will now discuss about the connection between the electrical concepts of current and voltage and corresponding descriptive quantities of random walks regarded as finite state Markov chains.

2.1 Resistor networks and reversible Markov chains

From now on, we will focus on a specific cathegory of electric networks, that is general resistors networks. This kind of electrical circuits is very elementary, because it is based on only two electrical components: voltage generators and resistors. We will assume that $G = (V, E)$ is a simple, finite and connected graph. We will assign to each edge uv a *resistance* R_{uv}; the *conductance* of an edge uv is $C_{uv} = 1/R_{uv}$. Below we give an example of such a graph in which are shown the resistances and the conductances.

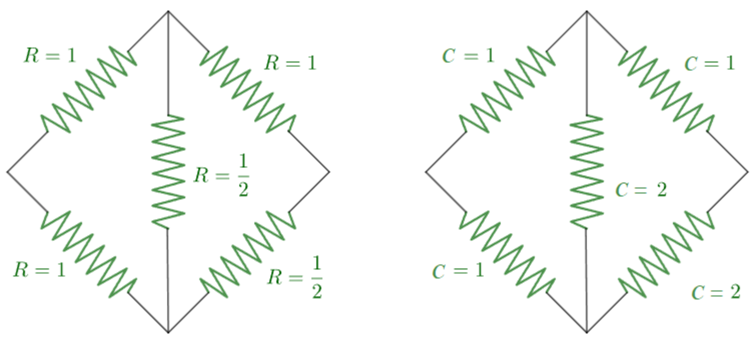

Figure 2.1: Given a connected graph G, we can see the resistances associated to each edge on the left, and the corresponding conductances obtained by $C = 1/R$ on the right.

Now, by combining the notions of graph given above and Markov chains, we are ready to define what a random walk on G is.

Definition 2.1. *A random walk on G is a Markov chain with transition matrix M given by*

$$p_{uv} = \frac{C_{uv}}{C_u}, \tag{2.1}$$

with $C_u = \sum_v C_{uv}$.

For our future reasonings, $C_{uv} = 1$ and $C_u = \deg(u)$.

2.2 Potential and currents

We now introduce some helpful notation. To each edge $e = uv$ of the graph G, it is possible to associate other two directed quantities: the *current* from vertex u to v, i_{uv} and the *potential* from u to v, ϕ_{uv}, which are both anti-symmetric:

$$\phi_{uv} = -\phi_{vu}, \qquad i_{uv} = i_{vu}.$$

The best way to proceed with the discussion is to recall the Kirchhoff's laws.

Kirchhoff's potential law. *The cumulative potential difference around any cycle $v_1, v_2, \ldots, v_n, v_{n+1} = v_1$ of G is zero, that is,*

$$\sum_{j=1}^{n} \phi_{v_j v_{j+1}} = 0. \tag{2.2}$$

Kirchhoff's potential law is equivalent to the statement that there exists a function $\phi : V \to \mathbb{R}$, called a *potential function* or sometimes *voltage*, such that

$$\phi_{uv} = \phi(v) - \phi(u) \equiv \phi_v - \phi_u, \qquad u, v \in E.$$

Since ϕ is determined up to an additive constant, we are free to pick the potential of any single vertex. Notice the convention that current flows uphill: i_{uv} has the same sign as ϕ_{uv}.

Kirchhoff's current law. *The total current flowing out of any vertex $u \in V$ other than the source set is zero, that is,*

$$\sum_{v \in V} i_{uv} = 0. \tag{2.3}$$

Remark 2.1. *Let $s, t \in V$ be distinct vertices termed* sources*; theese differ from other vertices beacause we could suppose to connect a battery across the pair s, t. $S = \{s, t\}$ is the* source set*.*

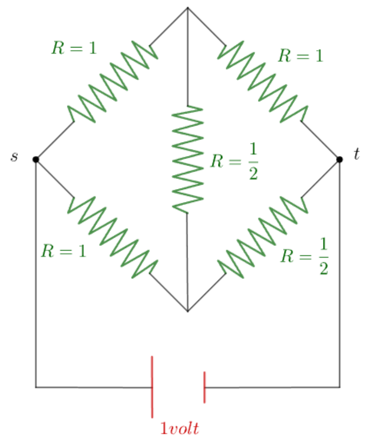

Figure 2.2: In this example, we assume we have a network of resistors assigned to the edges of a connected graph. We choose two points s and t and put a one-volt battery across these points establishing a potential $\phi_s = 1$ and $\phi_t = 0$.

Another essential law is the following one, which describes how potential, current and resistance are involved together.

Ohm's law. *For any edge $e = uv$*

$$\phi_{uv} = R_{uv} i_{uv}.$$

2.3 Probabilistic interpretation of the potential

Our purpose now is to give a probabilistic interpretation to the potential. To make it possible, for one thing we will prove the following resault.

Theorem 2.1. *A potential function is harmonic on the set of vertices other than the source set.*

Proof. Let $U = V \setminus S$. By Ohm's law, the currents through the resistors are determined by the voltages by

$$i_{uv} = \frac{\phi_{uv}}{R_{uv}} = (\phi_v - \phi_u) C_{uv}.$$

Kirchhoff's law tells us, by replacing the expression of the current above, that for $u \in U$

$$\sum_{v \in V} (\phi_v - \phi_u) C_{uv} = 0$$

or

$$\sum_{v \in V} \phi_v C_{uv} = \phi_u \sum_{v \in V} C_{uv}.$$

That means that

$$\phi_u = \frac{\sum_{v \in V} \phi_v C_{uv}}{\sum_{v \in V} C_{uv}}$$

and by (2.1)

$$\phi_u = \sum_{v \in V} p_{uv} \phi_v.$$

That is, by (1.6), ϕ is harmonic on U. □

On the other side, the hitting probabilities are the basic examples of harmonic functions for the chain, using the argument given by Theorem 1.1. To sum up, ϕ and h are both solutions to the problem of finding a harmonic function given its boundary values. Such a problem is called the *Dirichlet problem* and it is well known, by the Uniqueness Principle, that there cannot be two different harmonic functions having the same boundary values. Hence $\phi = h$.

In conclusion, we have the following interpretation of voltage: when a unit voltage is applied between s and t, making $\phi_s = 1$ and $\phi_t = 0$, the voltage ϕ_u at any point u represents the probability that a walker starting from u will return to s before reaching t.

2.4 Currents and energy

In the following, it becomes crucial to give the current a potrait of flow. As such, a flow j from s to t satisfies the following properties:

(a) $j_{uv} = -j_{vu}$;

(b) $j_{uv} = 0$ if u and v are not adjacent;

(c) $\sum_{v \in V} j_{uv} = 0$ for any $u \neq s, t$.

We denote by $J_u = \sum_{v \in V} j_{uv}$ the flow into u from the outside. It follows immediately, by (c), that $J_u = 0$ for any $u \neq s$. Thus,

$$J_s + J_t = \sum_{u \in V} J_u = \sum_{u,v \in V} j_{uv} = \frac{1}{2} \sum_{u,v \in V} (j_{uv} + j_{vu}) = 0.$$

Therefore, $J_s = -J_t$, and we call $|J_s|$ the *size* of the flow j. In particular, if $|J_s| = 1$, we call j a *unit flow*. With this terminology, we can now formulate a useful version of the principle of conservation of energy.

Theorem 2.2. *Let $\psi : V \to \mathbb{R}$ be a function and j an s/t flow. Then*

$$[\psi(t) - \psi(s)]J_s = \frac{1}{2} \sum_{u,v \in V} [\psi(v) - \psi(u)]j_{uv}. \tag{2.4}$$

Proof.

$$\begin{aligned}
\sum_{u,v \in V} [\psi(v) - \psi(u)]j_{uv} &= \sum_u \sum_v \psi(v)j_{uv} - \psi(u)j_{uv} \\
&= \sum_v \psi(v) \sum_u (-j_{vu}) - \sum_u \psi(u) \sum_v j_{uv} \\
&= \sum_v \psi(v)(-J_v) - \sum_u \psi(u)J_u \\
&= \psi(s)(-J_s) + \psi(t)(-J_t) - \psi(s)J_s - \psi(t)J_t \\
&= -2[\psi(s)J_s + \psi(t)J_t] \\
&= 2[\psi(t) - \psi(s)]J_s.
\end{aligned}$$

If we divide both members by 2, the thesis is proven. □

The just-obtained resault is useful to see how currents minimize a quantity called *energy dissipation*, which is given by

$$E(j) = \frac{1}{2} \sum_{u,v \in V} j_{uv}^2 / C_{uv}.$$

Now, if ϕ and i both satisfy the Kirchhoff's laws and we apply (2.4) with $\psi = \phi$ and $j = i$, it follows by Ohm's law that

$$E(i) = [\phi(t) - \phi(s)]I_s.$$

The meaning is that the energy of the true current-flow i between s to t equals the energy dissipated in a single st edge carrying the same potential difference and the total current. The conductance C_{eff} of such an edge would satisfy Ohm's law, that is

$$I_s = C_{\text{eff}}[\phi(t) - \phi(s)], \tag{2.5}$$

and we define the *effective conductance* C_{eff} by this equation. Evidently, the *effective resistance* is

$$\begin{aligned} R_{\text{eff}} &= \frac{1}{C_{\text{eff}}} \\ &= E(i)/I_s^2. \end{aligned}$$

Let us state this as a theorem.

Theorem 2.3. *The effective resistance R_{eff} of the network between vertices s and t equals the dissipated energy when a unit flow passes from s to t.*

Since it is useful to be able to do calculations, we close this section with a formulaic method for calculating the effective resistance of a network.

Series law. *Two resistors of size R_1 and R_2 in series may be replaced by a single resistor of size $R_1 + R_2$.*

Parallel law. *Two resistors of size R_1 and R_2 in parallel may be replaced by a single resistor of size R where $R^{-1} = R_1^{-1} + R_2^{-1}$.*

Figure 2.3: On the left we give an example of two edges e_1 and e_2 in parallel; on the right two edges e_3 and e_4 in series. Note that both the figures can rapresent a part of some given resistor network in which, as we said before, we are assuming that every edge has an associated quantity called resistance.

2.5 Thomson's principle

Now, we want to prove that the effective resistance is smaller than the energy dissipated by any other unit flow from s to t. This resault goes under the name of *Thomson's principle*.

Theorem 2.4 (**Thomson's principle**). *Let $G = (V, E)$ be a connected graph and C_{uv} strictly positive conductances. Let $s, t \in V$, $s \neq t$. Then the flow that satisfies the Kirchhoff's laws, amongst all unit flows through G from s to t, is the unique s/t flow i that minimizes the dissipated energy. That is:*

$$E(i) = \inf \{E(j) : j \text{ a unit flow from } s \text{ to } t\}.$$

Proof. Let j be any unit flow from s to t and let $k = j - i$ where i is the unique unit flow solution to the Kirchoff's laws. Then k is a flow from s to t with size $|K_s| = |\sum_{v \in V} k_{sv}| = 1 - 1 = 0$. On the other hand

$$\begin{aligned} 2E(j) &= \sum_{u,v \in V} j_{uv}^2 R_{uv} \\ &= \sum_{u,v \in V} (k_{uv} + i_{uv})^2 R_{uv} \\ &= \sum_{u,v \in V} k_{uv}^2 R_{uv} + \sum_{u,v \in V} i_{uv}^2 R_{uv} + 2 \sum_{u,v \in V} i_{uv} k_{uv} R_{uv}. \end{aligned}$$

Let ϕ be the potential function corresponding to i. By Ohm's law and (2.4),

$$\sum_{u,v \in V} i_{uv} k_{uv} R_{uv} = \sum_{u,v \in V} [\phi(v) - \phi(u)] k_{uv} = 2[\phi(t) - \phi(s)] K_s = 0.$$

Therefore, $E(j) \geq E(i)$ with equality if and only if $j = i$. $\square$

2.6 Rayleigh's Monotonicity law

All we need now is a law from electric network theory that will be an important tool in our future study of random walks. It may be seen as a consequence of Thomson's principle.

Theorem 2.5 (**Rayleigh's Monotonicity law**). *The effective resistance R_{eff} of the network is a non-decreasing function of the resistances of individual edges.*

Proof. Let i be the unit current flow from s to t with the resistors R_{uv} and j the unit current flow from s to t, with the resistors $\overline{R}_{uv}$. Let $\overline{R}_{uv} \geq R_{uv}$.

By Theorem 2.3

$$\begin{aligned}
R_{\text{eff}} &= \frac{1}{2} \sum_{\substack{u,v \in V \\ u \sim v}} i_{uv}^2 R_{uv} \\
&\leq \frac{1}{2} \sum_{\substack{u,v \in V \\ u \sim v}} j_{uv}^2 R_{uv} \\
&\leq \frac{1}{2} \sum_{\substack{u,v \in V \\ u \sim v}} j_{uv}^2 \overline{R}_{uv} \\
&= \overline{R}_{\text{eff}}.
\end{aligned}$$

□

We owe Rayleigh also the credit of two brief laws which are both equivalent to Rayleigh's Monotonicity law: the Shorting law and the Cutting law.

Shorting law. *Shorting certain sets of nodes together can only decrease the effective resistance of the network between two given nodes.*

Cutting law. *Cutting certain branches can only increase the effective resistance between two given nodes.*

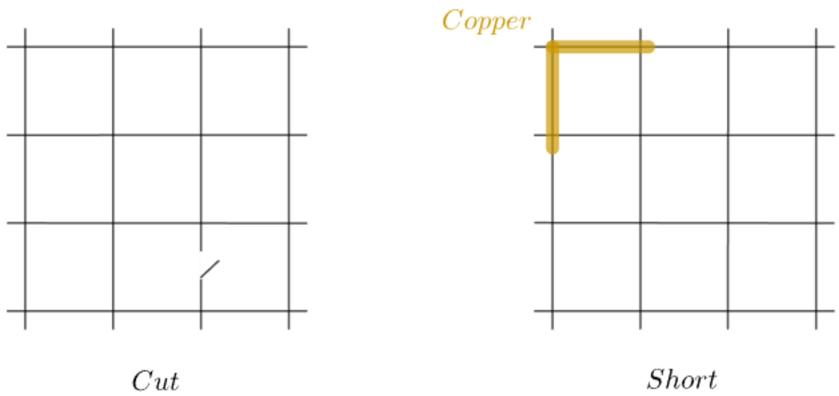

Figure 2.4: In its simplest form, Rayleigh's method consists in modifying the network in order to get a simpler network from the point of view of the resistances. Specifically, we consider two kind of modifications: cutting, which involves nothing more than deleting some branches from the network, and shorting, which means connecting a given set of nodes together with perfectly conducting wires (like copper) so that the nodes shorted together behave as if they were a single node.

Rayleigh's idea was to use these two laws respectively to get lower and upper bounds for the resistance of a network. In fact, in the last chapter, we will apply this method to solve the recurrence problem for simple random walk in three dimensions.

2.7 Recurrence and transience

In this last section we want to establish a condition on the pair composed by an infinite connected graph with finite vertex-degrees $G = (V, E)$, and the conductances C_{uv}, that is equivalent to the recurrence of a reversible Markov chain S on the state space V with transition probabilities given by (2.1). For a better understanding of the examples we will consider, let us proceed in six stages as follows.

1. We fix a vertex of G and call it 0 which is the origin, supposing that $S_0 = 0$.

2. Given $\delta(u, v)$ the minimum distance in terms of edges between two generic vertices u and v, let

$$\Lambda_n = \{u \in V : \delta(0, v) \leq n\}$$

such that

$$\partial\Lambda_n = \Lambda_n \setminus \Lambda_{n-1} = \{u \in V : \delta(0, v) = n\}.$$

3. Let G_n be the subgraph of G comprising the vertex-set Λ_n, together with all edges between them.

4. We let $\overline{G}_n$ be the graph obtained from G_n by identifying all vertices in $\partial\Lambda_n$, and we indicate the identified vertex with I_n, so the resulting finite graph $\overline{G}_n$ may be considered an electrical network with sources 0 and I_n. Let $R_{\text{eff}}(n)$ be the effective resistance of this network.

5. The graph $\overline{G}_n$ may be obtained from $\overline{G}_{n+1}$ by identifying all vertices lying in $\partial\Lambda_n \cup \{I_{n+1}\}$, and thus, by the Rayleigh's Monotonicity law, $R_{\text{eff}}(n)$ is non-decreasing in n.

6. By the theorem about the existence of the limit for monotone functions

$$R_{\text{eff}} = \lim_{n\to\infty} R_{\text{eff}}(n).$$

Now we are able to prove the following theorem whose importance is mainly related to a corollary that we will state and proof in a while.

Theorem 2.6. *The probability of ultimate return by S to the origin 0 is given by*

$$P(S_n = 0 \text{ for some } n \geq 1 | S_0 = 0) = 1 - \frac{1}{C_0 R_{\mathit{eff}}}$$

where $C_0 = \sum_{v:v\sim 0} C_{0v}$.

Proof. Let

$$h_n(v) = P(S \text{ hits } \partial\Lambda_n \text{ before } 0 | S_0 = v), \qquad v \in \Lambda_n.$$

As we have seen in Section 2.4, h_n is the unique harmonic function on G_n with boundary conditions

$$\begin{aligned} h_n(0) &= 0, \\ h_n(v) &= 1 \quad \text{for } v \in \partial\Lambda_n. \end{aligned}$$

Therefore, h_n is a potential function on $\overline{G}_n$ viewed as an electrical network with source 0 and sink I_n. So now we can think about h_n as ϕ and use Ohm's law after having conditioning on the first step of the walk:

$$\begin{aligned} P(S \text{ returns to 0 before reaching } \partial\Lambda_n | S_0 = 0) &= 1 - \sum_{v:v\sim 0} p_{0v} h_n(v) \\ &= 1 - \sum_{v:v\sim 0} \frac{C_{ov}}{C_0} [h_n(v) - h_n(0)] \\ &= 1 - \frac{|i(n)|}{C_0}, \end{aligned}$$

where $i(n)$ is the flow of currents in $\overline{G}_n$ with size $|i(n)|$. $|i(n)| = 1/R_{\text{eff}}$ by (2.5). Finally, it is enough to observe, by the continuity of probability measures that

$$P_0(S \text{ returns to 0 before reaching } \partial\Lambda_n) \to P_0(S_n = 0 \text{ for some } n \geq 1)$$

as $n \to \infty$. □

Let us conclude this section with an essential corollary that give us a characterization of the recurrence and transience of Markov chains.

Corollary 2.1. (Recurrence and transience of S)

(a) *The chain S is recurrent iff $R_{\mathit{eff}} = \infty$.*

(b) *The chain S is transient iff there exists a non-zero $0/\infty$ flow j on G whose energy $E(j) = \sum_{uv} j_{uv}^2 / C_{uv}$ satisfies $E(j) < \infty$.*

Proof. (a) follows immediately by Theorem 2.6, in fact it is sufficient to notice that if $R_{\text{eff}} = \infty$, then

$$1 - \frac{1}{C_0 R_{\text{eff}}} = 1,$$

which means that the probability, starting at 0, that the walk returns to 0 before reaching I_n is certain, a more laborious way to say that the chain S is recurrent.

(b) is equivalent to state that the probability that the walk, starting at 0, returns to the origin is positive. By Theorem 2.3, there exists a unit flow $i(n)$ in $\overline{G}_n$ with sources 0 and I_n such that $E(i(n)) = R_{\text{eff}}(n)$. Now we make this choice: we suppose that i is a non-zero flow so, by dividing by its size, we may take i to be a unit flow. When restricted to the edge-set E_n of $\overline{G}_n$, i forms a unit flow from 0 to I_n, and so, by Thomson's principle,

$$E(i(n)) \leq \sum_{uv} i_{uv}^2 / C_{uv} \leq E(i)$$

whence

$$E(i) = \lim_{n\to\infty} E(i(n)) = R_{\text{eff}}.$$

If the chain is recurrent, by (a) we have $E(i) = \infty$.

Conversely let us suppose that the chain is transient. By Cantor diagonal selection method, there exists a subsequence (n_k) along which $i(n_k)$ converges to some limit j, that is

$$i(n_k)_{uv} \to j_{uv} \qquad \text{for every } u, v \in V.$$

And since $i(n_k)$ is a unit flow from the origin, so j it is. Now

$$\begin{aligned} E(i(n_k)) &= \sum_{uv\in E} i(n_k)_{uv}^2 / C_{uv} \\ &\geq \sum_{uv\in E_m} i(n_k)_{uv}^2 / C_{uv} \\ &\to \sum_{uv\in E_m} j(uv)^2 / C_{uv} \qquad && \text{as } k \to \infty \\ &\to E(j) && \text{as } m \to \infty. \end{aligned}$$

Therefore,

$$E(j) \leq \lim_{k\to\infty} R_{\text{eff}}(n_k) = R_{\text{eff}} < \infty$$

and j is a flow with the required properties. $\square$

Chapter 3

Pólya's theorem

We have come to the core issue: Pólya's theorem, a resault which is as simple as surprising.

Theorem 3.1 (Pólya's theorem). *Symmetric random walk on the d-dimensional lattice $\mathbb{Z}^d$ is recurrent for $d = 1, 2$ and transient for $d \geq 3$.*

The proof of the theorem shall consist, at least at the beginning, of a mere application of the Corollary 2.1. To start with, let us consider the case of one dimension, $\mathbb{Z}$.

3.1 Random walk on $\mathbb{Z}$

This case is almost trivial to prove, given that an infinite line of resistors obviously has infinite resistance by the series law. It follows by Corollary 2.1 (a) that this simple and symmetric random walk is recurrent.

Figure 3.1: If we imagine we have an infinite network-line composed by an infinite number of resitors, this particular kind of circuit has clearly effective resistance equal to ∞, due to the series law.

3.2 Random walk on $\mathbb{Z}^2$

Now we assume that $d = 2$. By retracing the six steps exposed in Section 2.8, we obtain the graph showed below.

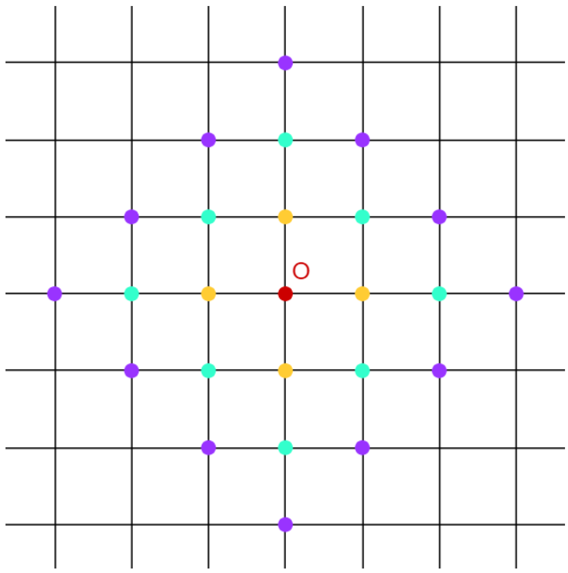

Figure 3.2: The vertices represented with the same colour have in common the same distance from the origin. We aim at shorting together the nodes with the same colour to get a simplified version of the network.

This graph is equivalent to the network showed below.

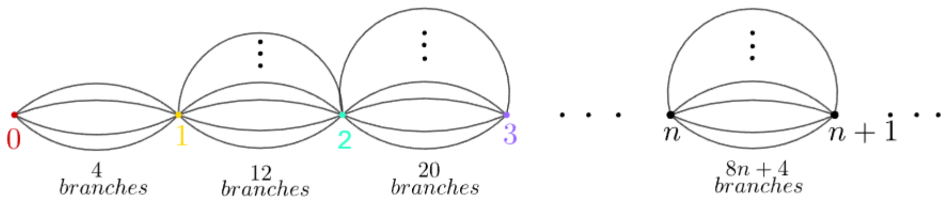

Figure 3.3: The vertex labelled $i \in \{1, 2, \ldots, n, n+1, \ldots\}$ is obtained by identifying all vertices with distance i from 0. The number of branches between each pair of successive nodes is obtained in the following way. We fix one vertex and count the the different total possibilities to get that vertex from all the possible erlier with only one step. We repeat the count for all the vertices with the same colour and finally we take the sum of all these numbers.

Notice that there are $8i-4$ edges of $\mathbb{Z}^2$ joining vertices $i-1$ and i. Figure 3.3 shows a network composed by resistors in parallel, so the modified network is equivalent to the one in Figure 3.4, thanks to the parallel law.

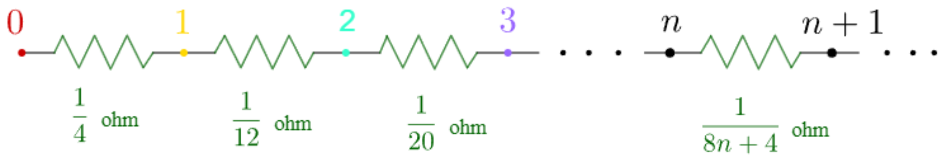

Figure 3.4: By assuming that every branch of the network showed in Figure 3.3 has resistance equal to 1, we can apply the parallel law to obtain a network-line with a sequence of decreasing resistances like given above.

The resistance of this new network out to infinity is clearly given by

$$\sum_{n=0}^{\infty} \frac{1}{8n+4} = \infty$$

thanks to the series law, instead. Since the resistance of the old network can only be bigger, by Rayleigh's Monotonicity law, we conclude it too must be infinite, that is the walk is recurrent when $d = 2$.

3.3 Random walk on binary trees

Going into detail, one more bidimensional case of great interest and especially easy is the binary tree, shown in the figure below.

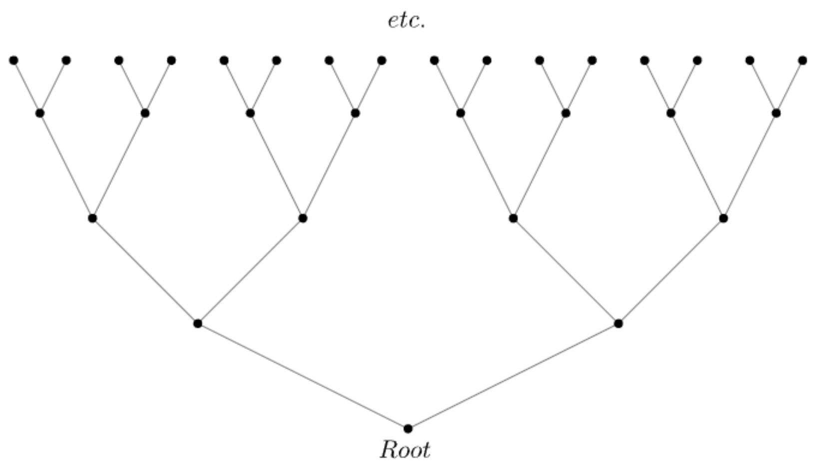

Figure 3.5: By binary tree we mean the tree in which every node has exactly two children, starting from the root.

Let us compute its effective resistance R_∞ from the root out to infinity. If we ground the set of branch points and hook the root up to a 1-volt battery (see Figure 3.6), by Ohm's law we have

$$R_n = \frac{1}{\text{current through the battery}}.$$

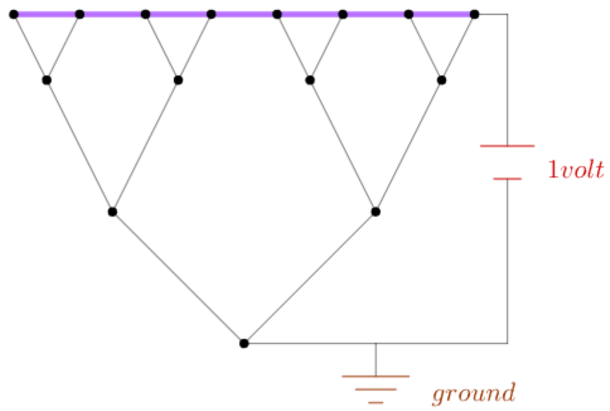

Figure 3.6: In this picture we show a modified circuit built starting from the original one, which is the binary tree truncated at level $n = 3$.

By observing that, by simmetry, all branch points of the same generation are at the same voltage, we are allowed to short together nodes that are at the same pontetial and this still would not affect the distribution of currents in the branches, and so the current through the battery too. Consequently

$$R_n = \frac{1}{\text{current in original circuit}} = \frac{1}{\text{current in modified circuit}}.$$

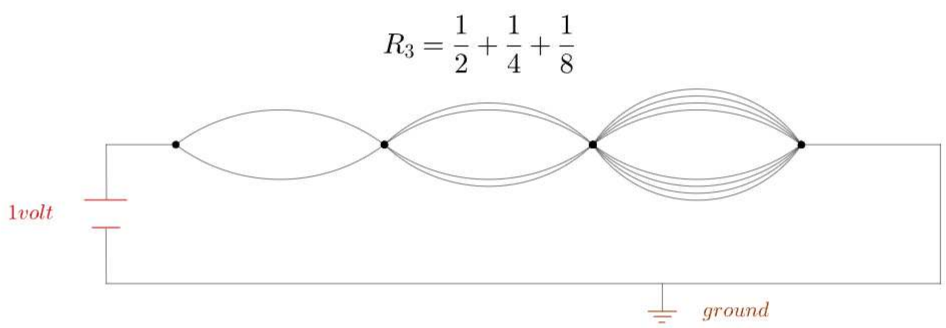

Figure 3.7: Taking the example showed in Figure 3.6 and by reasoning in the same way we did to find the equivalent network in Figure 3.3 starting from the graph, we obtain this modified circuit. For every level of the binary tree, we can count exactly the number of branches, identifying nodes situated at the same level. Remember it goes without saying that, for every branch, we can associate a resistance that amounts to 1 ohm. So, it is easy to apply first the parallel law to find the resistance for every group of branches between two consecutive nodes, and then sum them together to find the equivalent resistance of the network, applying the series law.

If we generalize the figure above, we get

$$R_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.$$

Finally, letting $n \to \infty$

$$R_\infty = \lim_{n\to\infty} R_n = \lim_{n\to\infty} 1 - \frac{1}{2^n} = 1.$$

It turns that the binary tree has always finite resistance equal to 1.

3.4 Random walk on b-ary trees

Following the same line as reasoning, we can show that more in general, for $b \geq 2$, the b-ary tree has finite resistance too. Indeed, in this situation

$$\begin{aligned} R_n &= \frac{1}{b} + \frac{1}{b^2} + \cdots + \frac{1}{b^n} \\ &= \sum_{i=1}^{n} \frac{1}{b^i} \\ &= \sum_{i=0}^{n} \frac{1}{b^i} - 1. \end{aligned}$$

Then, letting $n \to \infty$, we recognize the geometric series $\sum_{i=0}^{n} \frac{1}{b^i}$ converging to $\frac{1}{1-\frac{1}{b}}$, since $|\frac{1}{b}| < 1$ for all $b \geq 2$. Thus,

$$R_\infty = \frac{1}{1-\frac{1}{b}} - 1 = \frac{1}{b-1}.$$

So, in the case of b-ary trees, it turns that the resistance is always finite and equal to $\frac{1}{b-1}$.

The first part of Pólya's theorem is proven.

3.5 Random walk on $\mathbb{Z}^3$

Let us recall the second part of Pólya's theorem: the random walk is transient when $d \geq 3$. It suffices to show that $R_{\text{eff}} < \infty$ when $d = 3$, by Rayleigh's Monotonicity law. There are at least two ways of proceeding, as shown in [4]. As approach to this problem, we will try and find a finite upper bound for R_{eff} and as previously said in Section 2.6, there are two ways to do that: by increasing individual edge-resistances or by cutting edges. Even if the b-ary tree is a good prototype of networks having manifestly finite resistance to infinity, we can't find it as a subgraph of $\mathbb{Z}^3$ since the number of nodes in a ball of radius r grows exponentially with r, whereas in $\mathbb{Z}^3$, it grows like r, i.e. much slower. Therefore, we attempt to build a subgraph T_ρ of the lattice $\mathbb{Z}^3$ such that

$$R_{\text{eff}} \leq R_{T_\rho} < \infty,$$

where R_{T_ρ} is the effective resistance associated to the subgraph T_ρ. For us, T_ρ is a binary tree in which each connection between generation $n-1$ and generation n has resistance ρ^n, where ρ is a strictly positive parameter we will choose appropiately in a while.

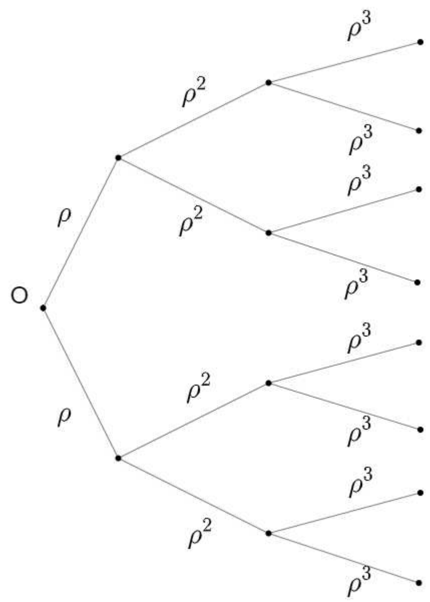

Figure 3.8: Still taking back the previous examples where we had a binary tree truncated to the third level, we represent the tree T_ρ, which is nothing more than a binary tree where the resistances are defined as a function of the level.

Let us compute R_{T_ρ}. For $i = 1, \ldots, n$ $R_i = \left(\frac{\rho}{2}\right)^i$, so

$$R_{T_\rho} = \sum_{n=1}^{\infty} \left(\frac{\rho}{2}\right)^n$$

which we make finite by choosing $\rho < 2$. It is reasonable, then, that our aim is to embed T_ρ in $\mathbb{Z}^3$. In order to do that, we need to cut some excess edges in $\mathbb{Z}^3$.

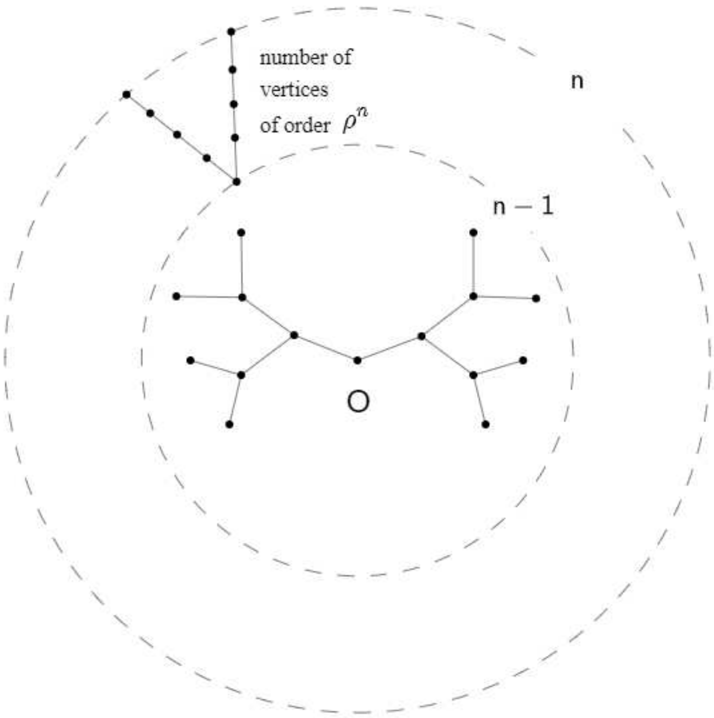

Figure 3.9: The picture gives an idea of how to embed T_ρ in $\mathbb{Z}^3$. We must proceed in such a way that a connection between generation $n-1$ and generation n is a lattice-path of order ρ^n. Then we will have to compare the number of vertices of T_ρ in generation n, whose lattice-distance from the origin has order $\sum_{i=1}^{n} \rho^i$, that is, order ρ^n, with the ones at the same distance in $\mathbb{Z}^3$.

The key lies in figuring out that this construction is geometrically possible.

Remark 3.1. *In T_ρ there are 2^n vertices of generation n.*

Since the surface of a ball of radius r in $\mathbb{R}^3$ has order r^2, we have the following remark.

Remark 3.2. *In $\mathbb{Z}^3$, at the distance $\sum_{i=1}^{n} \rho^i$, that is, order ρ^n, the number of vertices is of order $c(\rho^n)^2$.*

Considering that the vertices of T_ρ are selected among the ones of $\mathbb{Z}^3$, it is necessary that

$$c(\rho^n)^2 \geq 2^n$$

which is true if $\rho > \sqrt{2}$.
To conclude, if we choose such a tree T_ρ where $\sqrt{2} < \rho < 2$, it is trivial that $R_{\text{eff}} < c' R_{T_\rho}$.
The proof in the 3-dimensional lattice can be considered concluded.

Conclusions

We have come to the end of the thesis. Let us remind where we started. We have seen how interpreting a mathematical question in physical terms, that not only allows us to develop ways of thoughts and practice methods, but also leads us to the answers to those questions. In particular, we have seen the utility of involving energy. In took hundreds of years for the concept of energy to emerge and take its rightful place in physical theory, but it is now recognized as perhaps the most fundamental concept in all of physics. As far as concerned Pólya's theorem, the proof is simple and accessible to anyone understands the gradual treatment of the examples: the random walk in one and two dimensions seems almost to constitute a separate demonstration. Then we apparently change course going to talk about binary and b-ary trees but there is a good reason for that, and the reason is that even if the b-ary tree is a good prototype of networks having manifestly finite resistance to infinity, we can't find it as a subgraph of $\mathbb{Z}^3$. There is simply no room for these trees in any finite-dimensional lattice. So we come to the construction of a particular finite-resistance tree that can be embed in $\mathbb{Z}^3$. The argument is not so detailed since it suffices to understand that this construction is geometrically possible.

Bibliography

[1] Abbott, E. (1884). Flatland: A Romance of Many Dimensions. *Seeley & Co.*

[2] Doyle, P. G., & Snell, J.L. (2006). Random walks and electric networks. *Mathematical Association of America.*

[3] Georgii, H.-O. (2008). Stochastics. Introduction to Probability and Statistics. *De Gruyter*, Berlin.

[4] Grimmet, G. (2010). Probability on Graphs. Random Processes on Graphs and Lattices. *Cambridge University Press.*