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Exact penalty functions for optimal control problems

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Chapter 1

Introduction

In control theory literature, *optimal control* problems have been vastly investigated, see e.g. [1]–[7], and have gained a central role in the solution of many theoretical problems, see e.g. [8]–[11], and practical applications, see e.g. [12]–[14]. Given a state space model of a control system[1], optimal control deals with the problem of steering the state of the system in an optimal way, according to some prescribed cost criterion, which is typically an integral that weights conflicting requirements on the state evolution and the control effort. On the other hand, optimization, namely the quest of obtaining the most efficient solution (according to some prescribed criterion) to a given problem, is a key step in a wide variety of engineering contexts. Optimal control can be regarded as a dynamic optimization problem[2].

Two classes of optimal control problems can be distinguished in continuous time, depending on the time interval in which the cost is evaluated. In particular, in *infinite horizon* optimal control problems the integral cost depends on the evolution of the underlying system in an unbounded time interval, whereas in *finite horizon* optimal control problems the integral cost depends on the evolution of the underlying system in a bounded time interval. Infinite horizon problems, therefore, are usually related with asymptotic properties of the trajectory of the underlying dynamic, such as stability. In particular, for many problems in which the objective is to minimize a positive semidefinite cost (e.g., LQR problems), the strong relation between stabilizability and the existence of an opti-

[1] In this thesis we consider continuous time state space models, described by a set of differential equations.

[2] The main difference with respect to traditional "static" optimization (i.e. mathematical programming) problems is that, in optimal control, one seeks optimal solutions under dynamic constraints on the state of an underlying system, and the cost is usually an integral functional that depends on the whole evolution of the state and the control over time (i.e., the cost is defined over a functional space). Technically, optimization deals with problems in $\mathbb{R}^n$ (or in any finite dimensional space), whereas optimal control deals with problems in a functional space.

mal time-invariant feedback solution is well established [6]. Moreover, in infinite horizon problems, the cost evaluated along bounded state evolutions with bounded controls may be divergent (or, in general, even not defined). Therefore, they must be treated with proper care. In this thesis, the focus is on infinite horizon problems, in which the cost is positive definite.

In the last century, two main approaches have been introduced to treat optimal control problems in which the dynamics are governed by systems of ordinary differential equations (in the case where no *pure* static constraints[3] are given on the state variables), namely *Pontryagin's Minimum Principle (PMP)* [15] and *Dynamic Programming (DP)* [16].

PMP has its roots in the *Calculus of Variations* [17] and provides necessary conditions that an *open loop* optimal control signal must satisfy, identifying in such a way candidate solutions called *extremals*. Its standard formulation deals uniquely with continuous time state space systems. PMP, in the finite horizon setting, relies on the solution of a two points boundary value problem[4] (TPBVP) for an extended (Hamiltonian) dynamical system of ordinary differential equations, in which the set of variables is extended with the so-called *costate* variables. In the infinite horizon setting (which is the focus of this thesis) the solution relies on an initial value problem[5] (IVP) for the same extended system, in which the initial condition of the costate variable is determined by the solution for an *Hamilton-Jacobi-Bellman (HJB)* partial differential equation (PDE). The notion of Hamiltonian system has its roots in Mathematical Physics and is related to Mechanics and the Calculus of Variations. In [17] an account of the history of optimal control is given, tracing its origins back to the Brachistocrone problem, and the main connections with Hamiltonian Mechanics are outlined. A detailed treatment of the PMP is outside the scope of this thesis.

On the other hand, DP is based on Bellman's *optimality principle* [16]. The basic assumption underlying the optimality principle is that the system is in state space form. The state, at some specific time t, summarizes completely the effects of the control actions up to time t. In this setting one can expect that any optimal control signal in a time interval $[t, T]$ starting at some state $x(t)$, should also be an optimal signal in the subinterval $[t + \delta t, T]$ starting at the updated state $x(t + \delta t)$, for $\delta t > 0$. This intuitive result is formalized in Chapter 2. Given the nature of the underlying hypotheses, DP is a very general paradigm, that can be applied equally to continuous and discrete time state

[3] A formal definition of the *pure* static constraints that we consider is given in Chapter 4 and Chapter 5. Informally, these constraints can be understood as describing a fixed set in the state space that the state cannot leave throughout the entire time horizon.

[4] A TPBVP requires finding a solution of a differential equation in which both the initial and the final condition are only partially known.

[5] An IVP requires finding a solution of a differential equation in which the initial condition is known.

space models. In continuous time DP characterizes, under mild regularity conditions [6], optimal *(state) feedback* solutions for optimal control problems via the solution of an HJB PDE, which is, in general, a nonlinear partial differential equation, and is obtained as the differential formulation of the optimality principle (i.e. taking time derivatives in its standard (*integral*) formulation). In the particular case of the *Linear-Quadratic Regulator (LQR)*, the HJB PDE can be reduced to the well known (continuous time) *Algebraic Riccati Equation (ARE)*. For general nonlinear systems instead, finding solutions to the HJB PDE is an hard computational problem.
Both PMP and DP, in their simplest formulations, study problems with no constraints on the state. Nevertheless, many control and optimization applications, for example in the areas of robotics [18], [19], electrical networks [20], and in a wide variety of other contexts, are naturally formulated with the use of *pure*, *mixed*[6] or a broad spectrum of other static and dynamic constraints. Consequently, significant effort has been dedicated to develop systematic tools to solve constrained optimal control problems.
One possible strategy to tackle problems with static pure state constraints is to draw analogies from mathematical programming (as optimal control problems can be seen as *infinite dimensional* mathematical programming problems). For example in [22] the notion of *exact penalty functions* was introduced as a tool to transform constrained nonlinear programming problems to equivalent unconstrained nonlinear programming problems without resorting to *nonsmooth* techniques. In this spirit, the objective of this thesis is to develop a similar tool which allows transforming a state constrained optimal control problem to an equivalent[7] *exact penalization* problem without state constraints.
The thesis is organized as follows. In Chapter 2 and Chapter 3 a brief review of the main tools of optimal control and nonlinear geometric control which are used in the thesis, is given. Subsequently, the novel contributions of this work, which consist of the notion of exact penalization and its solution, given in Proposition 4, Corollary 1, Proposition 5, and Corollary 2, are discussed in Chapter 4 and Chapter 5, respectively. Specifically, the following is a detailed description of the contents of the thesis.

- In Chapter 2, fundamental definitions related to optimal control are discussed, such as the notion of local and global solutions and the notion of feedback solution. Subsequently, the main tools of Dynamic Programming, such as the HJB PDE and the Algebraic Riccati Equation (ARE) are reviewed, with particular focus toward their application in the solution of infinite horizon optimal control problems.

[6]That is problems in which the constraint involves both the control and the state, such as the ones in which the control set depends on the state, see, e.g., [21].

[7]In a sense to be defined.

- In Chapter 3, fundamental tools from Geometric Nonlinear Control Theory are briefly reviewed, in a local setting, following the presentation in [23]. In particular, the notions of Lie brackets, Lie derivatives and distributions, which allow to extend fundamental notions of linear systems theory to nonlinear systems, are recalled.

- In Chapter 4, the notion of *exact penalization* of a linear quadratic optimal control problem is introduced, and a systematic way to obtain such a penalization in the simple case of fully actuated control systems with only linear equality state constraints is proposed. In particular, both the cases in which the structure of the control system can be left unchanged in the penalized problem (for a particular class of control systems) and the more general case in which the structure of the control system is modified in the penalized problem are discussed. The differences, advantages, and drawbacks of such approaches are outlined.

- In Chapter 5, the notion of *local exact penalization* of an optimal control problem associated to a control affine system is introduced, and a systematic way to obtain such a penalization in the simple case of fully actuated control systems with only *regular* equality state constraints is proposed. As in Chapter 4, both the cases in which the structure of the control system can be left unchanged in the penalized problem (for a particular class of control systems) and the more general case in which the structure of the control system is modified in the penalized problem are discussed. The differences, advantages, and drawbacks of such approaches are outlined.

- In Chapter 6, numerical examples that illustrates the effectiveness, the applicability, and the advantages and drawbacks of the exact penalization tools introduced in Chapter 4 and Chapter 5 are given.

- In Chapter 7 a summary of the novel results presented in this work is given. Moreover, advantages of the proposed exact penalizations with respect to other techniques are discussed, and future research directions are identified.

Chapter 2

Elements of Optimal Control via Dynamic Programming

In this chapter the main tools to solve optimal control problems, with a focus towards Dynamic Programming, for infinite horizon problems are reviewed. The main reference for this discussion is [6].

2.1 Optimal control problems

Consider the optimal control problem

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad && J(x_0,u) = \int_0^{+\infty} l(x,u,t)\, dt, \\ &\text{subject to} \quad && \dot{x} = f(x,u,t), \qquad x(0) = x_0, \\ & && x(t) \in \mathcal{X}, \qquad \forall t \geq 0, \end{aligned} \tag{2.1}$$

where $u : [0,+\infty) \to \mathbb{R}^m$ is a control signal; $x : [0,+\infty) \to \mathbb{R}^n$ represents the state of the system; $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is a $\mathcal{C}^\infty$ function; $l : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function, also called *loss function* or *running cost*; $\mathcal{X} \subseteq \mathbb{R}^n$ is the state constraint; and $x_0 \in \mathbb{R}^n$ is the initial condition.
We first introduce notions related to *open loop* optimal solutions.

Definition 1 (Feasible Input)**.** $u : [0,+\infty) \to \mathbb{R}^m$ is a *feasible* input for problem (2.1) if it is bounded and such that the solution $x(\cdot;u) : [0,+\infty) \to \mathbb{R}^n$ of the initial value problem $\dot{x} = f(x,u(t),t)$ with $x(0) = x_0$ exists, is bounded, and is such that $x(t;u) \in \mathcal{X}$ for all $t \geq 0$.

Definition 2 (Global Minimizer)**.** $u^* : [0,+\infty) \to \mathbb{R}^m$ feasible is a global *minimizer* of problem (2.1) if it is such that

$$J(x_0,u^*) = \int_0^{+\infty} l(x(t;u^*),u^*(t),t)\,dt \le \int_0^{+\infty} l(x(t;u),u(t),t)\,dt = J(x_0,u)$$

for all feasible u.

Given the relevance of feedback in control theory, a natural problem in optimal control is to obtain a *feedback synthesis* of the optimal minimizers, of the form $u = \phi(x,t)$. In this spirit, we define notions of optimal (state) feedback solution.

Definition 3 (Feedback Law and Closed Loop System)**.** Consider problem (2.1). A continuous map $\phi : \Omega \times [0, +\infty) \to \mathbb{R}^m$, with $\Omega \subseteq \mathbb{R}^n$, is a feedback law, and $\dot{x} = f(x,\phi(x,t),t)$ is the associated *closed loop system.* For all $x_0 \in \Omega$, the associated state trajectory is the solution[1] $x : [0, +\infty) \to \mathbb{R}^n$ of the closed loop system $\dot{x} = f(x,\phi(x,t),t)$ with $x(0) = x_0$, and $u(t) = \phi(x(t),t)$ is the corresponding open loop control.

Definition 4 (Global Optimal Feedback)**.** The mapping $\phi : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^m$ is an *optimal feedback solution* of problem (2.1) if the corresponding open loop controls u are feasible and (global) minimizers, for all $x_0 \in \mathcal{X}$.

In particular, for regulation problems, i.e. problems in which the objective is to drive the state to the origin, we also define the notion of local optimal feedback around the regulation point, as in [24].

Definition 5 (Local Optimal Feedback)**.** The mapping $\phi : \Omega \times [0, +\infty) \to \mathbb{R}^m$, with Ω a neighborhood of 0, is a *local optimal feedback solution* of problem (2.1) for the set of initial condition Ω if the corresponding open loop controls u are optimal in the constraint set $\mathcal{X} \cap \Omega$, for all $x_0 \in \mathcal{X} \cap \Omega$.

Remark 1. Definition 5 does not require any notion of distance, and is therefore consistent with the geometric analysis (which relies on change of coordinates that do not preserve the Euclidean distance) given in the following chapters. ■

Remark 2. Under the mild assumption that a continuous (local) optimal feedback law ϕ exists, an optimal open loop solution u^* that is continuous exists for all initial conditions $x_0 \in \Omega$. Therefore, in this thesis, the focus is on continuous control inputs, since hypotheses which guarantee the existence of a continuous (local) optimal feedback law are always assumed. ■

[1] If it exists.

2.2 Solutions via Dynamic Programming

Given the notions of optimal solution and optimal feedback solution, we investigate how the Dynamic Programming principle can be applied to characterize such solutions. In particular, the focus of this section is on problems without state constraints of the form

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad J(x,u) = \int_t^{+\infty} l(x,u,\tau)\, d\tau, \\ &\text{subject to} \quad \dot{x} = f(x,u,\tau), \qquad x(t) = x, \end{aligned} \tag{2.2}$$

where $u : [0, +\infty) \to \mathbb{R}^m$ is a control signal; $x : [0, +\infty) \to \mathbb{R}^n$ represents the state of the system; $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n$ is a $\mathcal{C}^\infty$ function; the loss $l : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function. In the following, it is always assumed that an optimal input u^* exists for all initial conditions $x \in \mathbb{R}^n$ and initial times $t \geq 0$.
First, we review the derivation of an optimization equation that provides necessary conditions for the optimality of a control input. Subsequently, we state sufficient conditions which guarantee that the solution to this equation yields an optimal feedback. To this end, define the *value function* of problem (2.2) as the function V that maps every pair of state $x \in \mathbb{R}^n$ and time $t \geq 0$ to the optimal cost with initial state $x(t) = x$ at time t, namely

$$V(x,t) = \min_{u[t,+\infty)} \left\{ \int_t^{+\infty} l(x(\tau),u(\tau),\tau)\, d\tau \right\}, \tag{2.3}$$

where $u[t, +\infty)$ denotes an input signal restricted to the time interval $[t, +\infty)$.

2.2.1 Necessary conditions

To derive the optimization equation we use the *optimality principle* (or Dynamic Programming principle) introduced by Bellman [16], which gives a property that any optimal control input necessarily satisfies.

Theorem 1 (Optimality Principle [6]). *Consider the optimal control problem (2.2). If u^* is an optimal input over the time interval $[t, +\infty)$ starting at the state $x = x(t)$, then u^* is necessarily optimal over the subinterval $[t+\delta t, +\infty)$ starting at the state $x(t+\delta t)$*[2]*, for all $x \in \mathbb{R}^n$, $t \geq 0$, and $\delta t > 0$.*

An application of the optimality principle, by the additive properties of integrals, leads to the *integral* formulation

$$V(x,t) = \min_{u[t,t+\delta t)} \left\{ \int_t^{t+\delta t} l(x(\tau),u(\tau),\tau)\, d\tau + V(x(t+\delta t), t+\delta t) \right\},$$

[2] $x(t+\delta t)$ is the state at time $t+\delta t$ obtained via the optimal input over the interval $[t,t+\delta t]$.

for all $x \in \mathbb{R}^n$ and $t \geq 0$. Using the integral formulation of the optimality principle, assuming the knowledge of the value function V, the problem to find the optimal input over $[t, +\infty)$ has been reduced to finding an optimal control over the reduced interval $[t,t+\delta t]$. Under mild regularity conditions, the corresponding *differential* formulation, namely the Hamilton-Jacobi-Bellman (HJB) equation for the value function V of problem (2.2), takes the form

$$-\frac{\partial V}{\partial t} = \min_u \left\{ l + \frac{\partial V}{\partial x} f \right\}, \tag{2.4}$$

for all $x \in \mathbb{R}^n$ and $t \geq 0$. Moreover, provided that the minimization in equation (2.4) has a unique solution, gives the necessary condition

$$\phi(x,t) = \arg\min_u \left\{ l(x,u,t) + \frac{\partial V}{\partial x}(x,t) f(x,u,t) \right\}$$

that the optimal feedback has to satisfy for all $x \in \mathbb{R}^n$ and $t \geq 0$.

2.2.2 Sufficient conditions

As already discussed, a sufficient condition for the optimality of a feedback law can be stated via the HJB equation. We specialize it to problems for time-invariant control-affine systems of the form

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad J(x,u) = \int_t^{+\infty} l(x,u)\, d\tau, \\ &\text{subject to} \quad \dot{x} = f(x) + g(x)u, \qquad x(t) = x, \end{aligned} \tag{2.5}$$

where $l(x,u) = \frac{1}{2}q(x) + \frac{1}{2}||u||^2$, with $q : \mathbb{R}^n \to \mathbb{R}$ a positive definite $\mathcal{C}^\infty$ function; $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class $\mathcal{C}^\infty$, with $f(0) = 0$; and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is of class $\mathcal{C}^\infty$. In this case the value function $V : \mathbb{R}^n \to \mathbb{R}$ does not depend on time, hence the HJB equation takes the form

$$0 = \min_u \left\{ l + \frac{\partial V}{\partial x}(f + gu) \right\}, \tag{2.6}$$

for all $x \in \mathbb{R}^n$, and consequently any optimal feedback is a map $\phi : \mathbb{R}^n \to \mathbb{R}^m$ that satisfies

$$\phi(x) = \arg\min_u \left\{ l(x,u) + \frac{\partial V}{\partial x}(x)(f(x) + g(x)u) \right\} \tag{2.7}$$

for all $x \in \mathbb{R}^n$.

Theorem 2. *Consider the optimal control problem (2.5). Let $\phi : \mathbb{R}^n \to \mathbb{R}^m$ be a feedback law, and $V : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ positive definite and radially unbounded function, which* *satisfies (2.6) and is such that ϕ satisfies (2.7), i.e. such that*

$$0 = l(x,\phi(x)) + \frac{\partial V}{\partial x}(x)(f(x) + g(x)\phi(x)) \leq l(x,u) + \frac{\partial V}{\partial x}(x)(f(x) + g(x)u), \qquad (2.8)$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Then ϕ is an optimal feedback law, and V is the value function of the problem.

We give a version of the proof in the spirit of the one given in [6], adapted to infinite horizon problems with the structure in (2.5). See Remark 3 for a discussion on the differences with respect to finite horizon problems.

Proof. Let $x_0 \in \mathbb{R}^n$ be a fixed initial condition. Consider the trajectory $x^*(\cdot; u^*) : [0, +\infty) \to \mathbb{R}^n$ of the closed loop system $\dot{x} = f(x) + g(x)\phi(x)$ given by the feedback ϕ and from the initial condition $x^*(0; u^*) = x_0$, where $u^* : [0, +\infty) \to \mathbb{R}^m$ is the corresponding open loop control. Observe that V is a $\mathcal{C}^1$ positive definite radially unbounded function such that

$$\frac{\partial V}{\partial x}(x)\,(f(x) + g(x)\phi(x)) = -l(x,\phi(x)),$$

and $l(x,\phi(x))$ is positive definite, for all $x \in \mathbb{R}^n$. This shows that V is a Lyapunov function for the closed loop system $\dot{x} = f(x) + g(x)\phi(x)$, and hence $x^*(\cdot; u^*)$ is well defined, bounded, and such that $x^*(t; u^*) \to 0$ as $t \to +\infty$. Integrating the equality in (2.8) over the interval $[0,T]$ yields

$$\int_0^T l(x^*(t; u^*),\phi(x^*(t; u^*)))\,dt = \int_0^T l(x^*(t; u^*),u^*(t))\,dt = V(x_0) - V(x^*(T; u^*)).$$

Taking the limit as $T \to +\infty$, it follows that $V(x^*(T; u^*)) \to 0$, by continuity of V and the fact that $x^*(T; u^*) \to 0$. Therefore,

$$\int_0^{+\infty} l(x^*(t; u^*),u^*(t))\,dt = V(x_0),$$

and hence the cost associated to the input u^* is finite and equal to $J(x_0,u^*) = V(x_0)$. We now show that, for any other feasible input u, the inequality $J(x_0,u^*) \leq J(x_0,u)$ holds. Clearly, without loss of generality, it is possible to restrict the analysis to inputs u with finite cost. Let $x(\cdot; u) : [0, +\infty) \to \mathbb{R}^n$ be the corresponding trajectory, starting at x_0. First, observe that, since both $x(\cdot; u)$ and u are bounded by hypothesis, it follows that the norm of the derivative of the state trajectory

$$||\dot{x}|| \leq ||f(x)|| + ||g(x)||||u||$$

is bounded, and hence $x(\cdot;u)$ is uniformly continuous. Now, since the cost $J(x_0,u)$ is finite, it follows that

$$\int_0^{+\infty} q(x(t;u))\,dt \leq J(x_0,u) < +\infty.$$

Observe that, since $x(\cdot;u)$ is bounded, $q(x(\cdot;u))$ is uniformly continuous. As a consequence, $q(x(t;u)) \to 0$ as $t \to +\infty$, by Barbalat's lemma. Since q is positive definite by hypothesis, and $x(\cdot;u)$ is bounded, it follows that $x(t;u) \to 0$. Integrating the inequality in (2.8) over the interval $[0,T]$ yields

$$\int_0^{T} l(x(t;u),u(t))\,dt \geq V(x_0) - V(x(T;u)).$$

Now, taking the limit as $T \to +\infty$, it follows that $V(x(T;u)) \to 0$, by continuity of V and the fact that $x(T;u) \to 0$. Hence,

$$J(x_0,u) = \int_0^{+\infty} l(x(t;u),u(t))\,dt \geq V(x_0) = J(x_0,u^*).$$

This shows that u^* is a minimizer for problem (2.5). □

Remark 3. For finite horizon problems the hypotheses of positive definiteness and radially unboundedness of the function V (and the other hypotheses on the structure of the cost and the time invariance of the system) are not required (while other boundary conditions are necessary). In the infinite horizon case, the given hypotheses, together with the structure of problem (2.5) guarantee that:

- V acts as a Lyapunov function for the closed loop system obtained using the feedback law ϕ, and therefore 0 is a globally asymptotically stable equilibrium;
- q is positive definite and, hence, for any feasible open loop input $u : [0,+\infty) \to \mathbb{R}^m$ that corresponds to a trajectory $x : [0,+\infty) \to \mathbb{R}^n$ with finite cost, the state converges to 0.

A similar argument, that is applied in Chapter 5, guarantees that if V and q are locally positive definite, then ϕ is a local feedback minimizer for problem (2.5). ■

A direct application of Theorem 2 shows that, if V is a $\mathcal{C}^1$ radially unbounded positive definite (respectively, locally positive definite) solution of the equation

$$\frac{1}{2}q - \frac{1}{2}\frac{\partial V}{\partial x}gg^T\frac{\partial V}{\partial x}^T + \frac{\partial V}{\partial x}f = 0, \tag{2.9}$$

then the feedback law

$$\phi(x) = -g^T \frac{\partial V}{\partial x}^T$$

is globally (respectively, locally) optimal.

2.2.3 Solution to the LQR problem

As an application of Theorem 2, consider a special case of the problems in (2.5), namely LQR problems, described by equations of the form

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad J(x,u) = \int_t^{+\infty} \left(\frac{1}{2} x^T Q x + \frac{1}{2} u^T u \right) d\tau, \\ &\text{subject to} \quad \dot{x} = Ax + Bu, \qquad x(t) = x, \end{aligned} \tag{2.10}$$

where $Q \in \mathbb{R}^{n\times n}$ is a positive definite matrix; $A \in \mathbb{R}^{n\times n}$; and $B \in \mathbb{R}^{n\times m}$. Consider the candidate value function $V(x) = x^T P x$, with $P \in \mathbb{R}^{n\times n}$ positive definite. Then the HJB equation (2.9) takes the form

$$A^T P + PA + Q - PBB^T P = 0, \tag{2.11}$$

known as *Algebraic Riccati Equation (ARE)*. An application of Theorem 2, together with linear algebraic considerations, gives the following result.

Theorem 3 (Linear Quadratic Regulator)**.** *Consider problem (2.10). Assume that the pair (A,B) is stabilizable. Then there exists a unique positive definite solution* $P^* \in \mathbb{R}^{n\times n}$ *to the ARE (2.11), which is such that* $V(x) = x^T P x$ *is the value function of the problem and* $\phi(x) = -B^T P x$ *is the unique optimal feedback.*

Remark 4. Through a minor modification of the argument given in Theorem 2, it is possible to replace the hypothesis that Q is positive definite with the weaker hypothesis that (A,Q) is observable. ■

Chapter 3

Elements of Nonlinear Geometric Control

In this chapter, fundamental tools from geometric nonlinear control, which we will exploit to construct *exact penalizations*, are introduced. The discussion is given in a *local* setting, i.e. on open subsets of $\mathbb{R}^n$, in which all the notions are introduced using local coordinates. Nevertheless, this treatment is suitable for a coordinate free analysis, as all the given notions are coordinate invariant. The main reference for this chapter is [23], in which both a local and a global (i.e., on manifolds, which are locally identifiable with open subsets of $\mathbb{R}^n$, but may have a different global structure) treatment of geometric nonlinear control are given. In this context, by *smooth* function we always mean that the function is sufficiently regular for our purpose, i.e. of class $\mathcal{C}^r$, with r sufficiently large so that all the derivatives considered are well defined and continuous.

3.1 Diffeomorphisms, vector fields and Lie brackets

3.1.1 Diffeomorphisms

Consider the space $\mathbb{R}^n$ endowed with the canonical coordinates system $x_1, \ldots, x_n$, and let $U \subseteq \mathbb{R}^n$ be an open set. The generalization of the notion of a linear change of variables (or change of basis) in the nonlinear setting is the notion of *diffeomorphism*.

Definition 6 (Diffeomorphism)**.** A mapping $\Phi : U \to V$ with $V \subseteq \mathbb{R}^n$ an open set is a *diffeomorphism* if:

- Φ is invertible, i.e. its inverse $\Phi^{-1} : V \to U$ is well defined;
- Φ and Φ^{-1} are smooth.

A diffeomorphism can be interpreted as a representation of the points in U in the canonical coordinates system in V, via the map $y = \Phi(x)$, and sometimes it is also called a *local chart* or a *local coordinates system.* Informally, a notion is *coordinate free* if its coordinates description has the same form in all the coordinates systems obtained via diffeomorphism. A detailed introduction to coordinate free objects, given in an intrinsic way, is given in [25].

3.1.2 Vector fields

Vectors and covectors, in geometric control, are usually defined on the *tangent space* and *cotangent space* of a manifold. It is well known that, using the canonical coordinates in $\mathbb{R}^n$, one can identify the tangent space T_xU at any $x \in U$ with the space $\mathbb{R}^n$ of column vectors. Analogously, the dual space of T_xU, denoted as T_x^*U, is identified with the dual space $(\mathbb{R}^n)^*$ of $\mathbb{R}^n$, with $(\mathbb{R}^n)^*$ the vector space of row vectors. For a general definition of tangent and cotangent spaces to a point on a manifold see [25] and [23]. In this setting, it is possible to define the notions of vector and covector fields.

Definition 7 (Vector and covector fields)**.** A map $f : U \to \mathbb{R}^n$ which is smooth in all its components is called a vector field. A map $\omega : U \to (\mathbb{R}^n)^*$ which is smooth in all its components is called a covector field. $V(U)$ and $V^*(U)$ are the set of all smooth vector and covector fields, respectively, on U.

Given a vector field f and a covector field ω on U, the row-by-column product ωf is a smooth scalar function on U, called the *pairing* of ω and f.
Given a scalar function, it is of particular interest to define its *differential.*

Definition 8 (Differential)**.** Let $h : U \to \mathbb{R}$ be a smooth scalar function. Its *differential* $\dfrac{\partial h}{\partial x}$ (or dh) is the covector field

$$\frac{\partial h}{\partial x} = \left(\frac{\partial h}{\partial x_1}, \ldots, \frac{\partial h}{\partial x_n}\right).$$

Definition 9 (Exact differential)**.** A covector field $\omega : U \to (\mathbb{R}^n)^*$ is an *exact differential* if there exists a smooth map $h : U \to \mathbb{R}$ such that $\omega = \dfrac{\partial h}{\partial x}$.

A necessary and sufficient condition for ω to be an exact differential is given as follows.

Theorem 4. *The covector field $\omega : U \to (\mathbb{R}^n)^*$, with components $\omega = (\omega_1, \ldots, \omega_n)$, and U simply connected, is an exact differential if and only if the relationship on the partial derivatives*

$$\frac{\partial \omega_i}{\partial x_j} = \frac{\partial \omega_j}{\partial x_i}$$

holds for all pairs of indeces (i,j).

3.1.3 Lie derivatives and Lie brackets

We are now in the position to define three types of differential operations, involving vector fields and covector fields.

Definition 10 (Lie derivative of a scalar function)**.** Let $f : U \to \mathbb{R}^n$ be a vector field, and $h : U \to \mathbb{R}$ be a scalar function. The *Lie derivative* of λ along f is the scalar function $L_f h = \dfrac{\partial h}{\partial x} f$.

In the following, given a vector field $f : U \to \mathbb{R}^n$, with components $f = (f_1, \ldots, f_n)$ in canonical coordinates, we use its Jacobian (in canonical coordinates), denoted as

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \cdots & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Definition 11 (Lie derivative of a covector field)**.** Let $f : U \to \mathbb{R}^n$ be a vector field, and $\omega : U \to (\mathbb{R}^n)^*$ be a covector field. The *Lie derivative* of ω along f is the covector field

$$L_f \omega = f^T \left(\frac{\partial \omega^T}{\partial x} \right)^T + \omega \frac{\partial f}{\partial x}.$$

Definition 12 (Lie bracket)**.** Let $f : U \to \mathbb{R}^n$ and $g : U \to \mathbb{R}^n$ be two vector fields. The *Lie bracket* of f and g is the vector field

$$[f,g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g.$$

The Lie bracket operator has the properties given in Proposition 1.

Proposition 1. *The Lie bracket has the following properties.*

- *It is bilinear over* $\mathbb{R}$*, i.e. if* f_1, f_2, g_1, $g_2 \in V(U)$ *and* r_1, r_2 *are real numbers then*

$$[r_1 g_1 + r_2 g_2, f_1] = r_1 [g_1, f_1] + r_2 [g_2, f_1],$$
$$[g_1, r_1 f_1 + r_2 f_2] = r_1 [g_1, f_1] + r_2 [g_1, f_2].$$

- *It is skew-commutative, i.e.*

$$[f,g] = -[g,f].$$

- *It satisfies the Jacobi identity, i.e. if f, g, $p \in V(U)$, then*

$$[f,[g,p]] + [g,[p,f]] + [p,[f,g]] = 0.$$

3.1.4 Flows of vector fields

The first step to study control systems in a geometric setting is to study the evolution of ordinary differential equations (ODEs). In particular, given a vector field $f \in V(U)$, consider the associated ODE

$$\dot{x} = f(x), \tag{3.1}$$

as the initial condition varies in U. In this context, it is possible to define the *flow* associated to f.

Definition 13 (Flow of a vector field)**.** Let $f \in V(U)$ be a vector field and consider the associated ODE (3.1). The mapping $\Phi_t^f(x)$ (depending on both $x \in U$ and $t \in \mathbb{R}^n$) is the *flow* of f if it is defined for all $x \in U$ and for all $t \in (-\bar{t}(x),\bar{t}(x))$ with[1] $\bar{t}(x)$ sufficiently small, and satisfies

$$\frac{\partial \Phi_t^f}{\partial t}(x) = f(\Phi_t^f), \quad \Phi_0^f(x) = x,$$

for all $x \in U$ and all $t \in (-\bar{t}(x),\bar{t}(x))$.

For any $x_0 \in U$, the local existence and uniqueness theorem for ODEs guarantees that there exists a sufficiently small neighborhood $U_0 \subseteq U$ of x_0, in which the flow exists for sufficiently small times, and the properties in Proposition 2 hold.

Proposition 2. *Let $x_0 \in U$. Then there exists a sufficiently small neighborhood $U_0 \subseteq U$ of x_0, in which the flow $\Phi_t^f : (-\bar{t},\bar{t}) \times U_0 \to V_0$ exists, with $\bar{t}$ sufficiently small, and $V_0 \subseteq U$ is open. Moreover,*

- *for $t \in (-\bar{t},\bar{t})$ fixed, the flow Φ_t^f is a diffeomorphism onto its image;*
- *for $t \in (-\bar{t},\bar{t})$ fixed, the inverse of the flow satisfies[2] $(\Phi_t^f)^{-1} = \Phi_{-t}^f$;*
- *for all t, $s \in (-\bar{t},\bar{t})$ such that Φ_{t+s}^f is defined, and for all $x \in U_0$, the flow is such that*

$$\Phi_{t+s}^f(x) = \Phi_t^f(\Phi_s^f(x)).$$

[1] In general, the time interval $(-\bar{t}(x),\bar{t}(x))$ in which the flow is defined depends on x. Nevertheless, in Proposition 2 it is shown that, in a sufficiently small neighbourhood, the time $\bar{t}$ can be selected uniformly with respect to x.

[2] Note that Φ_{-t}^f is well defined on the image of Φ_t^f.

A vector field $f \in V(U)$ is *complete* if its flow exists on U for all $t \in \mathbb{R}$.
To conclude, consider a local change of variable as in (6), i.e. a diffeomorphism $\Phi : U \to V$, with $V \subseteq \mathbb{R}^n$ open set, and the new coordinates system $y = \Phi(x)$. A vector field $f : U \to \mathbb{R}^n$, in the new coordinates system y, is represented by the map $\bar{f}(y) = \left(\frac{\partial \Phi}{\partial x} f\right)\Big|_{x=\Phi^{-1}(y)}$ defined on V. This representation is consistent with our identification between ODEs and vector fields, since in the new coordinates the ODE (3.1) takes the form

$$\dot{y} = \frac{d}{dt}\Phi(x) = \frac{\partial \Phi}{\partial x}\dot{x} = \left(\frac{\partial \Phi}{\partial x}(x) f(x)\right)\Big|_{x=\Phi^{-1}(y)} = \bar{f}(y).$$

3.2 Distributions and Frobenius theorem

A *distribution* is a geometric notion which generalizes the notion of linear subspace to the nonlinear setting. In particular, it is a mapping that assigns a subspace of the tangent space $T_x U$ (here identified with $\mathbb{R}^n$) to every $x \in U$.

Definition 14 (Distribution and codistribution)**.** A *distribution* (respectively, *codistribution*) Δ defined on U is a map of the form $\Delta : x \in U \mapsto \Delta(x)$, with $\Delta(x)$ a subspace of $\mathbb{R}^n$ (respectively, $(\mathbb{R}^n)^*$). A distribution (respectively, codistribution) is *smooth* if there exists a set of vector fields (respectively, covector fields) $\{f_i : i \in I\}$, with I a set of indeces, such that

$$\Delta(x) = \operatorname{span}\{f_i(x) : i \in I\},$$

for all $x \in U$.

If $F : U \to \mathbb{R}^{n \times d}$ is a matrix having smooth entries, then $\Delta = \operatorname{Im}(F)$ denotes the smooth distribution spanned by the d columns of F. As in the case of vector subspaces, it is possible to define operations with distributions.

Definition 15 (Operations on distributions)**.** Let Δ_1 and Δ_2 be two distributions (respectively, codistributions) on U. Then

- the *sum* of Δ_1 and Δ_2 is the pointwise sum of the vector subspaces, i.e.

$$(\Delta_1 + \Delta_2)(x) = \Delta_1(x) + \Delta_2(x),$$

  for all $x \in U$;

- the *intersection* of Δ_1 and Δ_2 is the pointwise intersection of the vector subspaces, i.e.

$$(\Delta_1 \cap \Delta_2)(x) = \Delta_1(x) \cap \Delta_2(x),$$

for all $x \in U$.

It is possible to prove that the sum of two smooth distributions is smooth, while the intersection may fail to be smooth.

Definition 16 (Nonsingular distributions and regular points)**.** A distribution (respectively, codistribution) Δ on U is *nonsingular* if there exists an integer $d \leq n$ such that $\dim(\Delta(x)) = d$, for all $x \in U$. A distribution (respectively, codistribution) Δ on U is *regular* in $x_0 \in U$ if there exists a neighborhood of x_0 where Δ is nonsingular.

It is possible to show that, given a smooth distribution Δ, and a regular point $x_0 \in U$ in which Δ has dimension d, there exists a neighborhood $U_0 \subseteq U$ of x_0 in which Δ is *locally finitely generated*, i.e. there exist a set of d vector fields $f_1, \ldots, f_d$ such that

$$\Delta(x) = \operatorname{span}\{f_1(x), \ldots, f_d(x)\},$$

for all $x \in U_0$.
Many times, given a distribution Δ, it is convenient to construct the associated *annihilating* codistribution.

Definition 17 (Annihilating codistribution)**.** Let Δ be a distribution on U. Its *annihilating codistribution* is the codistribution $\Delta^{\perp}$ such that

$$\Delta^{\perp}(x) = \{w \in (\mathbb{R}^n)^* : wv = 0 \quad \forall v \in \Delta(x)\},$$

for all $x \in U$.

Definition 18 (Completely integrable distribution)**.** Let Δ be a nonsingular distribution on U, of dimension d. Δ is *completely integrable* if, for all $x_0 \in U$, there exist a neighborhood $U_0 \subseteq U$ of x_0 and $n-d$ scalar smooth functions $h_1, \ldots, h_{n-d}$ such that $\Delta^{\perp}$ is locally spanned by the differentials of $h_1, \ldots, h_{n-d}$, i.e.

$$\Delta^{\perp}(x) = \operatorname{span}\left\{\frac{\partial h_1}{\partial x}(x), \ldots, \frac{\partial h_{n-d}}{\partial x}(x)\right\},$$

for all $x \in U_0$.

Given a completely integrable distribution on U, it is possible to identify a *foliation*, on the neighborhood U_0 of x_0, given by the functions $h_1, \ldots, h_{n-d}$, i.e. a partition of U_0 in sets, called *leaves*, in which $h_1, \ldots, h_{n-d}$ are constant, which have a submanifold structure by the implicit function theorem. Moreover, it is always possible to extend the set of functions $h_1, \ldots, h_{n-d}$ (which clearly have linearly independent differentials in

every $x \in U_0$), with d conveniently selected smooth functions $h_{n-d+1}, \ldots, h_n$ such that the mapping $\Phi = (h_1, \ldots, h_n)^T$ is a diffeomorphism.
A vector field $f \in U$ belongs to a smooth distribution Δ on U if $f(x) \in \Delta(x)$ for all $x \in U$.

Definition 19 (Involutive distribution)**.** A smooth distribution Δ on U is *involutive* if

$$\tau_1 \in \Delta, \tau_2 \in \Delta \implies [\tau_1, \tau_2] \in \Delta.$$

We are now in a position to state Frobenius theorem, one of the most important results in the field of geometric control, which relates Lie brackets to the property of complete integrability.

Theorem 5 (Frobenius)**.** *A smooth regular distribution is completely integrable if and only if it is involutive.*

3.2.1 Invariant distributions

The notion of *invariant distribution* under a vector field generalizes the notion of invariant subspace under a linear map, and leads to a *triangular* form analogous to the block triangular structure of linear systems with an invariant subspace.

Definition 20 (Invariant distribution)**.** A distribution Δ on U is *invariant* under the vector field $f \in V(U)$ if

$$\tau \in \Delta \implies [f, \tau] \in \Delta,$$

or, with a more convenient notation, $[f, \Delta] \subseteq \Delta$.

Proposition 3 relates the notion of invariance with block triangular decompositions.

Proposition 3. *Let $f \in V(U)$ be a vector field, and Δ on U be a smooth nonsingular and involutive distribution on U of dimension $d \leq n$ invariant under f. Then, given $x_0 \in U$, there exist a neighborhood $U_0 \subseteq U$ of x_0 and a diffeomorphism $\Phi : U_0 \to V_0$, with $V_0 \subseteq \mathbb{R}^n$ an open set, such that the representation of f in the coordinates system $y = (y_1, \ldots, y_d, y_{d+1}, \ldots, y_n)^T = \Phi(x)$, denoted as $\bar{f}$, is given by the equation*

$$\bar{f}(y) = \begin{bmatrix} f_1(y_1, \ldots, y_d, y_{d+1}, \ldots, y_n) \\ \cdots \\ f_d(y_1, \ldots, y_d, y_{d+1}, \ldots, y_n) \\ f_{d+1}(y_{d+1}, \ldots, y_n) \\ \cdots \\ f_n(y_{d+1}, \ldots, y_n) \end{bmatrix},$$

for all $y \in V_0$.

Chapter 4

The Linear Quadratic Case

4.1 Problem formulation

Consider the optimal control problem

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J(x_0,u) = \int_0^{+\infty} \left(x^T Q x + u^T u\right)\, dt, \\ \text{subject to} \quad & \dot{x} = Ax + Bu, \qquad x(0) = x_0, \\ & \tilde{H}^T x(t) = 0, \qquad \forall t \geq 0, \end{aligned} \tag{4.1}$$

where $u : [0, +\infty) \to \mathbb{R}^n$ is a continuous input signal; $x : [0, +\infty) \to \mathbb{R}^n$ represents the state of the system; $A \in \mathbb{R}^{n\times n}$; $B \in \mathbb{R}^{n\times n}$, with $\det(B) \neq 0$; and $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix. Moreover $\tilde{H} \in \mathbb{R}^{n\times p}$ (full rank, without loss of generality) is a matrix describing p linear equality constraints which are active for all $t \geq 0$.
Note that the state constraints can be rewritten as $H^T \left(BB^T\right)^{-1} x = 0$, with $H = \left(BB^T\right) \tilde{H}$, and that $\left(BB^T\right)^{-1}$ is symmetric and positive definite, inducing therefore an inner product and a notion of orthogonality. In what follows, we use the subspaces $V^\perp = \mathrm{Im}(H)$ and $V = \mathrm{Im}(H^\perp)$, where $H^\perp \in \mathbb{R}^{n\times(n-p)}$ is a full rank matrix such that $H^T \left(BB^T\right)^{-1} H^\perp = 0$ which spans the orthogonal of $\mathrm{Im}(H)$, consistent with our orthogonality notion; we use the manifold $\mathcal{V} = \left\{x \in \mathbb{R}^n : \tilde{H}x = 0\right\}$ associated to the subspace V; and we define the projection matrices

$$\Pi_H = H \left(H^T \left(BB^T\right)^{-1} H\right)^{-1} H^T \left(BB^T\right)^{-1},$$

and $\Pi_{H^\perp} = I - \Pi_H$.

Remark 5. The assumption that $B \in \mathbb{R}^{n\times n}$, with $\det(B) \neq 0$, simplifies the problem, in the sense that B does not impose any limitation to the "control authority". Technically, this hypothesis guarantees that the subspace V is always a *controlled invariant subspace* for the dynamic in (4.1), i.e., given two states x_0 and x_1 in the manifold $\mathcal{V}$ it is always possible to find a control signal that steers the state from x_0 to x_1 without leaving $\mathcal{V}$. However, while it is essential to guarantee the existence of an optimal solution, the controlled invariant condition may still hold under weaker hypotheses. Possible generalizations of the given results to problems with an arbitrary number of inputs are discussed in Chapter 7. ■

The objective of this work is to define a new optimal control problem which is an *exact penalization* (in a sense to be defined) of the problem in (4.1) and does not have any constraint on the state.

Definition 21 (Exact penalization)**.** An infinite horizon optimal control problem without state constraints is an *exact penalization* of the optimal control problem (4.1) if the optimal feedback solution[1] u^* of the penalized problem is unique and is an optimal feedback solution of (4.1).

In the spirit of Definition 21, we formulate the following problems.

Problem LQEP (Linear Quadratic Exact Penalization)**.** Consider the infinite horizon optimal control problem (4.1). Find symmetric matrices $L \in \mathbb{R}^{n\times n}$ and $M \in \mathbb{R}^{n\times n}$, and a constant $\tilde{k} > 0$ such that:

- $x^T Lx = x^T Mx = 0$, for all $x \in V$;
- for all $k > \tilde{k}$ the problem

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad \int_0^{+\infty} \left(x^T Qx + x^T Lx + k^2 x^T Mx + u^T u\right) \, dt, \\ &\text{subject to} \quad \dot{x} = Ax + Bu, \quad x(0) = x_0, \end{aligned} \tag{4.2}$$

  is an *exact penalization* of the optimal control problem (4.1).

Problem LQEPM (Linear Quadratic Exact Penalization with modification of the dynamic)**.** Consider the infinite horizon optimal control problem (4.1). Find a matrix $A' \in \mathbb{R}^{n\times n}$, symmetric matrices $L \in \mathbb{R}^{n\times n}$ and $M \in \mathbb{R}^{n\times n}$, and a constant $\tilde{k} > 0$ such that:

- $x^T Lx = x^T Mx = 0$ for all $x \in V$;

[1] If it exists.

- for all $k > \tilde{k}$ the problem

$$\begin{aligned} &\underset{u}{\text{minimize}} \quad \int_0^{+\infty} \left(x^T Q x + x^T L x + k^2 x^T M x + u^T u\right) \, dt, \\ &\text{subject to} \quad \dot{x} = A' x + B u, \quad x(0) = x_0, \end{aligned} \tag{4.3}$$

  is an *exact penalization* of the optimal control problem (4.1).

Remark 6. Problem LQEPM is more general than Problem LQEP, since it has an additional degree of freedom, hence its solution requires fewer assumptions. Nevertheless, the feedback solution of Problem LQEPM is stabilizing only for the modified dynamic in equation (4.3). Therefore, contrary to the case of Problem LQEP, the solution of Problem LQEPM does not yield, in general, a stabilizing feedback for the original dynamic in (4.1) (even if, clearly, with this feedback, $\mathcal{V}$ is a stable invariant manifold for the original dynamic): there may be initial conditions $x_0 \notin \mathcal{V}$ such that the corresponding closed loop trajectory does not converge to 0. This can be seen via simple examples. Consequently, both problems are worth investigating. ■

4.2 Main results

To solve the Problem LQEP we represent the system in a set of convenient coordinates $x \mapsto T^{-1} x$, where $T \in \mathbb{R}^{n \times n}$ is nonsingular and of the form

$$T = \left[\begin{array}{c|c} H & H^{\perp} \end{array} \right].$$

This yields

$$\left(BB^T\right)^{-1} \mapsto T^T \left(BB^T\right)^{-1} T = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array} \right], \tag{4.4}$$

with the four blocks of dimensions $(n-p) \times (n-p)$, $(n-p) \times p$, $p \times (n-p)$, and $p \times p$, respectively. Inspecting equation (4.4) we conclude that

$$B \mapsto T^{-1} B = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array} \right],$$

where $B_1 \in \mathbb{R}^{(n-p) \times n}$ and $B_2 \in \mathbb{R}^{p \times n}$ are such that $B_1 B_2^T = 0$, and that

$$A \mapsto T^{-1} A T = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right],$$

with $A_{11} \in \mathbb{R}^{(n-p)\times(n-p)}$, $A_{12} \in \mathbb{R}^{(n-p)\times p}$, $A_{21} \in \mathbb{R}^{p\times(n-p)}$, $A_{22} \in \mathbb{R}^{p\times p}$. It is also useful to define $\tilde{A}_{11} = \left(B_1B_1^T\right)^{-1} A_{11}$, $\tilde{A}_{12} = \left(B_1B_1^T\right)^{-1} A_{12}$, $\tilde{A}_{21} = \left(B_2B_2^T\right)^{-1} A_{21}$, $\tilde{A}_{22} = \left(B_2B_2^T\right)^{-1} A_{22}$, and $\tilde{A} = \left(BB^T\right)^{-1} A$. In what follows we decompose every matrix and vector consistent with the dimensions of the matrix T. In addition, with some abuse of notation, we denote the new representation of the system with the same symbols as the original one. In the new coordinates the matrix BB^T takes the block diagonal form

$$BB^T = \left[\begin{array}{c|c} B_1B_1^T & 0 \\ \hline 0 & B_2B_2^T \end{array}\right],$$

and the projections take the form

$$\Pi_H = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I \end{array}\right], \quad \Pi_{H^\perp} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right],$$

respectively. We decompose Q in the same way as

$$Q = \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{12}^T & Q_{22} \end{array}\right].$$

To conclude note also that the constraint is now described as $\mathcal{V} = \left\{x = [x_1, x_2]^T \in \mathbb{R}^n : x_2 = 0\right\}$.

To solve Problem LQEP we proceed in steps. First we characterize the unique optimal control of the original problem (4.1). Then we construct a class of feedback matrices $P_k \in \mathbb{R}^{n\times n}$, parametrized by $k > 0$, such that:

- P_k is the positive definite solution of the Algebraic Riccati Equation (ARE) associated to a penalized problem of the form (4.2) for some L,M to be defined;
- the optimal feedback control $u^* = -B^T P_k x$ resulting from the solution of the Problem LQEP coincides with the optimal control of the original problem for all $x_0 \in \mathcal{V}$.

Step 1 (Characterizing the solution of problem (4.1)) Observe that the input vector can always be uniquely decomposed in the form $u = B_1^T v_1 + B_2^T v_2$, with vectors v_1 and v_2 such that $\dim(v_1) = n - p$ and $\dim(v_2) = p$.
Given an initial condition $x_0 = [x_{01}, 0]^T \in \mathcal{V}$, in order to fulfill the condition $x_2(t) = 0$, for all $t \geq 0$, we select[2] $v_2 = -\left(B_2B_2^T\right)^{-1} A_{21}x_1$. It follows that any optimal control for

[2]See Remark 7.

the original problem (4.1) must take the form

$$u = B_1^T v_1 - B_2^T \left(B_2 B_2^T\right)^{-1} A_{21} x_1,$$

and any control of this form is such that the constraint holds for all $t \geq 0$. Thanks to this decomposition we have decoupled the variables of the original problem (4.1) to obtain the dynamical equations $\dot{x}_2 = A_{22} x_2$ (and hence $x_2(t) = 0$, for all $t \geq 0$) and

$$\dot{x}_1 = A_{11} x_1 + \left(B_1 B_1^T\right) v_1, \tag{4.5}$$

with the associated cost

$$\begin{aligned} J(x_0, v_1) &= \int_0^{+\infty} x^T Q x + u^T u \, dt \\ &= \int_0^{+\infty} x_1^T Q_{11} x_1 + \left(B_1^T v_1 - B_2^T \left(B_2 B_2^T\right)^{-1} A_{21} x_1\right)^T \left(B_1^T v_1 - B_2^T \left(B_2 B_2^T\right)^{-1} A_{21} x_1\right) dt \\ &= \int_0^{+\infty} x_1^T Q_{11} x_1 + x_1^T A_{21}^T \left(B_2 B_2^T\right)^{-1} A_{21} x_1 + v_1^T \left(B_1 B_1^T\right) v_1 \, dt. \end{aligned}$$

These equations describe a classical LQR problem for a reduced system. Given the fact that $Q_{11} + A_{21}^T \left(B_2 B_2^T\right)^{-1} A_{21}$ is positive definite and $(A_{11}, \left(B_1 B_1^T\right))$ is reachable, this problem admits a unique optimal solution given by $v_1 = -\left(B_1 B_1^T\right)^{-1} \left(B_1 B_1^T\right) P_c x_1 = -P_c x_1$, where $P_c \in \mathbb{R}^{n \times n}$ is the unique positive definite solution of the ARE

$$A_{11}^T P_c + P_c A_{11} + Q_{11} + \tilde{A}_{21}^T \left(B_2 B_2^T\right) \tilde{A}_{21} - P_c \left(B_1 B_1^T\right) P_c = 0. \tag{4.6}$$

Therefore the optimal solution of problem (4.1) is given by the feedback

$$u = -B_1^T P_c x_1 - B_2^T \left(B_2 B_2^T\right)^{-1} A_{21} x_1, \tag{4.7}$$

and this guarantees $x_2(t) = 0$, for all $t \geq 0$.

Remark 7. The selected vector v_2 is unique for all $x \in \mathcal{V}$. Specifically, the state constraint imposes the condition $Ax + Bu \in V$, leading to $Ax + \begin{bmatrix} 0 \\ B_2 B_2^T \end{bmatrix} v_2 \in V$. Since $\operatorname{Im}\left(\begin{bmatrix} 0 \\ B_2 B_2^T \end{bmatrix}\right) = V^\perp$, the feedback vector $-\begin{bmatrix} 0 \\ B_2 B_2^T \end{bmatrix} v_2$ must be the unique projection of the vector Ax onto $V^\perp$, thus identifying v_2 uniquely for all $x \in \mathcal{V}$. Any other possible choice, e.g. of the form $v_2 = -\left(B_2 B_2^T\right)^{-1} A_{21} x_1 + \Gamma x_2$, for some $\Gamma \in \mathbb{R}^{p \times p}$, coincides on $\mathcal{V}$. Consequently, the reduced dynamics (4.5) and the cost, along with the subsequent arguments, are unaffected by the selection of Γ, which can be set to zero with loss of generality. ■

Step 2 (Obtaining a solution of the penalized problem (4.2) that matches the solution of the original problem (4.1)) Consider the ARE

$$A^T P_k + P_k A + Q + L + k^2 M - PBB^T P = 0, \tag{4.8}$$

associated to the penalized problem (4.2). We consider a class of candidate solutions P_k, parametrized by $k > 0$, of the ARE (4.8), defined as[3]

$$\begin{aligned} P_k &= \left[\begin{array}{c|c} P_c & A_{21}^T \left(B_2 B_2^T\right)^{-1} \\ \hline \left(B_2 B_2^T\right)^{-1} A_{21} & k \left(B_2 B_2^T\right)^{-1} + \left(B_2 B_2^T\right)^{-1} A_{22} \end{array}\right] \\ &= \left[\begin{array}{c|c} P_c & \tilde{A}_{21}^T \\ \hline \tilde{A}_{21} & k \left(B_2 B_2^T\right)^{-1} + \tilde{A}_{22} \end{array}\right]. \end{aligned} \tag{4.9}$$

Assuming symmetry of P_k, we conclude that the resulting feedback control is

$$u = -B^T P_k x \overset{x_2=0}{=} -B_1^T P_c x_1 - B_2^T \left(B_2 B_2^T\right)^{-1} A_{21} x_1, \tag{4.10}$$

which is identical to the control (4.7) whenever $x_2 = 0$. It is therefore immediate to observe that the closed loop trajectory that solves the original problem (4.1) given by the optimal control (4.7) is also the closed loop trajectory that solves the system (4.2) with the control (4.10) for any initial condition $x_0 \in \mathcal{V}$. Hence, if (4.10) were actually the optimal control for the penalized problem (4.2) for k sufficiently large, the two problems would have the same optimal feedback solution.
The argument (described in the next statement) then shows that L and M that make P_k a solution to the ARE can always be found. Note that, to obtain an explicit form not depending on P_c it is convenient to assume that $\tilde{A}_{12} = \tilde{A}_{21}^T$ and $\tilde{A}_{22} = \tilde{A}_{22}^T$.
We summarize the above discussion in the following statement which is given in a coordinate free form.

Proposition 4 (Solution of the Problem LQEP)**.** *Consider the optimal control problem (4.1) and the penalized optimal control problem (4.2). Assume that the matrix*

$$\left(BB^T\right)^{-1} A - \Pi_{H^\perp}^T \left(BB^T\right)^{-1} A \Pi_{H^\perp} \tag{4.11}$$

[3]Under suitable hypotheses, such as the ones given in Proposition 4, the matrix P_k is symmetric.

is symmetric. Then there exists $\tilde{k} > 0$ such that the selection

$$L = -\left(Q - \Pi_{H^\perp}^T Q \Pi_{H^\perp}\right) - \left(A^T \left(BB^T\right)^{-1} A - \Pi_{H^\perp}^T A^T \left(BB^T\right)^{-1} A \Pi_{H^\perp}\right),$$

and

$$M = \Pi_H^T \left(BB^T\right)^{-1} \Pi_H,$$

solves the Problem LQEP, i.e. it is such that problem (4.2) is an exact penalization of problem (4.1).

Proof. In the coordinates system (4.4) the condition (4.11) in the proposition reads $\tilde{A}_{21} = \tilde{A}_{12}^T$ and $\tilde{A}_{22} = \tilde{A}_{22}^T$. The other matrices are given by

$$M = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \left(B_2 B_2^T\right)^{-1} \end{array}\right],$$

and

$$\begin{aligned} L = &- \left[\begin{array}{c|c} 0 & Q_{12} \\ \hline * & Q_{22} \end{array}\right] - A^T \tilde{A} \\ &+ \left[\begin{array}{c|c} A_{11}^T & A_{21}^T \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} \tilde{A}_{11} & 0 \\ \hline \tilde{A}_{21} & 0 \end{array}\right] \\ = &- \left[\begin{array}{c|c} 0 & Q_{12} \\ \hline * & Q_{22} \end{array}\right] - \left[\begin{array}{c|c} 0 & A_{11}^T \tilde{A}_{12} + A_{21}^T \tilde{A}_{22} \\ \hline * & A_{12}^T \tilde{A}_{12} + A_{22}^T \tilde{A}_{22} \end{array}\right] \\ = &\left[\begin{array}{c|c} 0 & -Q_{12} - A_{11}^T \tilde{A}_{12} - A_{21}^T \tilde{A}_{22} \\ \hline * & -Q_{22} - \tilde{A}_{12}^T A_{12} - A_{22}^T \tilde{A}_{22} \end{array}\right], \end{aligned}$$

where the lower left term is disregarded due to symmetry considerations. We verify that the expression (4.9) of P_k is a solution of the ARE (4.8). Define $K = k\left(B_2 B_2^T\right)^{-1} + \tilde{A}_{22}$.

The resulting ARE takes the form

$$
\begin{aligned}
& A^T P_k + P_k A + Q + k^2 M + L - P_k B B^T P_k \\
& = \left[\begin{array}{c|c} A_{11}^T P_c + \tilde{A}_{21}^T \left(B_2 B_2^T\right) \tilde{A}_{21} & A_{11}^T \tilde{A}_{21} + A_{21}^T K \\ \hline * & A_{12}^T \tilde{A}_{21}^T + A_{22}^T K \end{array}\right] \\
& + \left[\begin{array}{c|c} P_c A_{11} + \tilde{A}_{21}^T \left(B_2 B_2^T\right) \tilde{A}_{21} & P_c A_{12} + \tilde{A}_{21}^T A_{22} \\ \hline * & \tilde{A}_{21} A_{12} + K A_{22} \end{array}\right] \\
& + \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline * & Q_{22} + k^2 \left(B_2 B_2^T\right)^{-1} \end{array}\right] + \left[\begin{array}{c|c} 0 & -Q_{12} - A_{11}^T \tilde{A}_{21} - \tilde{A}_{21}^T A_{22} \\ \hline * & -Q_{22} - \tilde{A}_{12}^T A_{12} - A_{22}^T \tilde{A}_{22} \end{array}\right] \\
& - \left[\begin{array}{c|c} P_c B_1 B_1^T P_c + \tilde{A}_{21}^T \left(B_2 B_2^T\right) \tilde{A}_{21} & P_c \left(B_1 B_1^T\right) \tilde{A}_{21}^T + A_{21}^T K \\ \hline * & \tilde{A}_{21} \left(B_1 B_1^T\right) \tilde{A}_{21}^T + K \left(B_2 B_2^T\right) K \end{array}\right].
\end{aligned}
\tag{4.12}
$$

The term in the upper left block of the ARE (4.12) is 0 thanks to the reduced ARE (4.6). The remaining term in the upper right block is such that

$$
P_c A_{12} - P_c \left(B_1 B_1^T\right) \tilde{A}_{21}^T = P_c \left(B_1 B_1^T\right) \left(\tilde{A}_{12} - \tilde{A}_{21}^T\right) = 0,
$$

by condition (4.11). In the lower right block, applying again condition (4.11), one has

$$
A_{22}^T K + K A_{22} - A_{22}^T \tilde{A}_{22} - K \left(B_2 B_2^T\right) K + k^2 \left(B_2 B_2^T\right)^{-1} = 0,
$$

where we have used the substitution $K = k \left(B_2 B_2^T\right)^{-1} + \tilde{A}_{22}$. This proves that P_k solves the ARE (4.8).

We now show that there exists a constant $\tilde{k}_1 > 0$ such that P_k is positive definite for all $k > \tilde{k}_1$. This is a consequence of the structure of P_k: indeed, by applying the Schur complement, we conclude that P_k is positive definite if and only if P_c is positive definite and $k \left(B_2 B_2^T\right)^{-1} + \tilde{A}_{22} - \tilde{A}_{21} P_c \tilde{A}_{21}^T$ is positive definite. The first condition is always satisfied in accordance with the definition of P_c, and the second condition is satisfied for all $k > \tilde{k}_1$, with $\tilde{k}_1$ a sufficiently large constant (recall that $B_2 B_2^T$ is full rank).

It remains to show that the ARE (4.8) has a unique positive definite solution. To this end, observe that the matrix

$$
Q + L + k^2 M = \left[\begin{array}{c|c} Q_{11} & * \\ \hline * & * + k^2 \left(B_2 B_2^T\right)^{-1} \end{array}\right]
$$

is positive definite for all $k > \tilde{k}_2$, with $\tilde{k}_2$ a sufficiently large constant. We select $\tilde{k} = \max\{\tilde{k}_1, \tilde{k}_2\}$. This guarantees that the ARE (4.8) has the unique positive definite solution

P_k, corresponding to the unique optimal control.
We have proved that (4.10) is the unique optimal solution of the penalized problem (4.2) and of the original problem (4.1), for all $k > \tilde{k}$, and therefore (4.2) is an *exact penalization* of problem (4.1). □

The solution of Problem LQEPM is obtained as a direct consequence of Proposition 4, and it is summarized in the following corollary.

Corollary 1 (Solution of the Problem LQEPM)**.** *Consider the optimal control problem (4.1) and the penalized optimal control problem (4.3). There exists $\tilde{k} > 0$ such that the selection*

$$A' = A\Pi_{H^\perp} + \left(BB^T\right)\Pi_{H^\perp}^T A^T \left(BB^T\right)^{-1}\Pi_H,$$

$$L = -\left(Q - \Pi_{H^\perp}^T Q \Pi_{H^\perp}\right) - \left(A'^T \left(BB^T\right)^{-1} A' - \Pi_{H^\perp}^T A'^T \left(BB^T\right)^{-1} A'\Pi_{H^\perp}\right),$$

and

$$M = \Pi_H^T \left(BB^T\right)^{-1}\Pi_H,$$

solves the Problem LQEPM, i.e. it is such that problem (4.3) is an exact penalization of problem (4.1).

Proof. Observe that $Ax = A'x$ for all $x \in \mathcal{V}$, since $\Pi_H x = 0$ and $\Pi_{H^\perp} x = x$. Consider now the optimal control problem

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J(x_0,u) = \int_0^{+\infty} \left(x^T Q x + u^T u\right) dt, \\ \text{subject to} \quad & \dot{x} = A'x + Bu, \qquad x(0) = x_0, \\ & \tilde{H}^T x(t) = 0, \qquad \forall t \geq 0, \end{aligned}$$

which has the same solution as problem (4.1), since the dynamical equations coincide for all $x \in \mathcal{V}$. This new problem satisfy condition (4.11) of Proposition 4, therefore (4.3) is an *exact penalization* of the new problem, and consequently of (4.1). □

Remark 8. Corollary 1 has weaker hypotheses than Proposition 4, as no condition is given on the dynamic matrix A, thanks to the additional degree of freedom in the Problem LQEPM. As discussed in Remark 6, the major drawback of this approach is that the feedback law obtained by the selection proposed in Corollary 1 may not be, in general, stabilizing for the original system (4.1), even if $\mathcal{V}$ is guaranteed to be a stable invariant manifold for the closed loop system. As a consequence, even with an initial condition $x_0 \in \mathcal{V}$, small numerical errors in the integration of the system can lead the trajectory to leave the manifold and not to converge to 0. Examples of this behavior are given in Chapter 6. ■

Chapter 5

The Nonlinear Case

5.1 Problem formulation

Consider the optimal control problem

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J(x_0,u) = \int_0^{+\infty} \left(\frac{1}{2} q(x) + \frac{1}{2} ||u||^2 \right) dt, \\ \text{subject to} \quad & \dot{x} = f(x) + g(x)u, \qquad x(0) = x_0, \\ & h(x(t)) = 0, \qquad \forall t \geq 0, \end{aligned} \tag{5.1}$$

where $u : [0, +\infty) \to \mathbb{R}^n$ is a continuous input signal; $x : [0, +\infty) \to \mathbb{R}^n$ represents the state of the system; $f : \mathbb{R}^n \to \mathbb{R}^n$ is of class $\mathcal{C}^\infty$ and $f(0) = 0$; $g : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is[1] of class $\mathcal{C}^\infty$ with $g(x)$ nonsingular for all $x \in \mathbb{R}^n$; and the function $q : \mathbb{R}^n \to \mathbb{R}$ is positive definite and $\mathcal{C}^\infty$, with $\frac{\partial^2 q}{\partial x^2}(0)$ positive definite and $q(0) = 0$. Moreover $h : \mathbb{R}^n \to \mathbb{R}^p$ is a $\mathcal{C}^\infty$ and regular (i.e. $\dfrac{\partial h}{\partial x}$ is full rank for all $x \in \mathbb{R}^n$) function with $h(0) = 0$, and describes p equality constraints which are active for all $t \geq 0$. In what follows, we define the manifold $\mathcal{M} = \{x \in \mathbb{R}^n : h(x) = 0\}$, and the projection matrices

$$\Pi_h = \left(gg^T\right) \frac{\partial h}{\partial x}^T \left(\frac{\partial h}{\partial x} \left(gg^T\right) \frac{\partial h}{\partial x}^T \right)^{-1} \frac{\partial h}{\partial x},$$

and $\Pi_{h^\perp} = I - \Pi_h$, where $\operatorname{Im}\left(\Pi_{h^\perp}(x)\right) = T_x\mathcal{M}$ for all $x \in \mathcal{M}$. In addition we define the tangent vector fields

$$v_{h^\perp} = \left(gg^T\right) \frac{\partial h^\perp}{\partial x}^T \left(\frac{\partial h^\perp}{\partial x} \left(gg^T\right) \frac{\partial h^\perp}{\partial x}^T \right)^{-1},$$

[1]We consider problems with n inputs for the same reasons as in Remark 5.

and

$$v_h = \left(gg^T\right)\frac{\partial h}{\partial x}^T\left(\frac{\partial h}{\partial x}\left(gg^T\right)\frac{\partial h}{\partial x}^T\right)^{-1}.$$

As in the linear case, the objective is to define a new optimal control problem which is an *exact penalization* (in a sense to be defined) of problem (5.1).

Definition 22 (Local exact penalization)**.** An infinite horizon optimal control problem without state constraints is a *local exact penalization* of problem (5.1) if, for all initial conditions x_0 in a neighborhood of the origin, a (local) optimal feedback solution u^* of problem (5.1) is a (local) optimal feedback solution of the penalized problem.

In the spirit of Definition 22, we formulate the following problems.

Problem NLEP (Nonlinear Exact Penalization)**.** Consider the optimal control problem (4.1). Find functions $l : \mathbb{R}^n \to \mathbb{R}$ and $m : \mathbb{R}^n \to \mathbb{R}$ and a constant $\tilde{k} > 0$ such that:

- $l(x) = m(x) = 0$ for all $x \in \mathcal{M}$;
- for all $k > \tilde{k}$ the problem

$$\begin{aligned} &\underset{u}{\text{minimize}} && \int_0^{+\infty}\left(\frac{1}{2}q(x)+\frac{1}{2}l(x)+\frac{1}{2}k^2m(x)+\frac{1}{2}||u||^2\right)dt, \\ &\text{subject to} && \dot{x} = f(x)+g(x)u, \quad x(0)=x_0, \end{aligned} \tag{5.2}$$

  is a *local exact penalization* of problem (5.1).

Problem NLEPM (Nonlinear Exact Penalization with modified dynamic[2])**.** Consider the optimal control problem (4.1). Find functions $f' : \mathbb{R}^n \to \mathbb{R}^n$ and $g' : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, functions $l : \mathbb{R}^n \to \mathbb{R}$ and $m : \mathbb{R}^n \to \mathbb{R}$, and a constant $\tilde{k} > 0$ such that

- $l(x) = m(x) = 0$ for all $x \in \mathcal{M}$;
- for all $k > \tilde{k}$ the problem

$$\begin{aligned} &\underset{u}{\text{minimize}} && \int_0^{+\infty}\left(\frac{1}{2}q(x)+\frac{1}{2}l(x)+\frac{1}{2}k^2m(x)+\frac{1}{2}||u||^2\right)dt, \\ &\text{subject to} && \dot{x} = f'(x)+g'(x)u, \quad x(0)=x_0, \end{aligned} \tag{5.3}$$

  is a *local exact penalization* of problem (5.1).

[2] We study both Problems NLEP and NLEPM for the same reasons as in Remark 6 and Remark 8: the feedback law given by a selection that solves Problem NLEPM is not necessarily stabilizing for the original dynamics (5.1), even if it is such that $\mathcal{M}$ is a locally stable invariant manifold.

5.2 Main results

To formulate our main result, we rely on the following assumption.

Assumption 1. The smooth distribution $\Delta = \text{span}\left\{gg^T \dfrac{\partial h}{\partial x}^T\right\}$ is nonsingular and involutive.

Under Assumption 1, as a consequence of Frobenius Theorem [23], there exists (locally) a $\mathcal{C}^\infty$ mapping $h^\perp$ that takes values in $\mathbb{R}^{n-p}$ and such that $\dfrac{\partial h^\perp}{\partial x} gg^T \dfrac{\partial h}{\partial x}^T = 0$. The matrices $\dfrac{\partial h^\perp}{\partial x}$ and $\dfrac{\partial h}{\partial x}^T$ together are formed by n linearly independent rows. Therefore

$$y = T(x) = \left[\begin{array}{c} h^\perp(x) \\ \hline h(x) \end{array}\right] = \left[\begin{array}{c} y_1 \\ \hline y_2 \end{array}\right] \tag{5.4}$$

is a (local) diffeomorphism. We additionally assume that T is a global diffeomorphism.

Remark 9. The assumption that T is a global diffeomorphism is convenient to define the penalized cost globally. Analogous results can be obtained even if T is only defined in a neighborhood Ω_0 of the origin, provided that the optimal trajectory of the original system (5.1) is such that $x(t) \in \Omega_0$ for all $t \geq 0$. In this case it is sufficient to restrict the admissible controls of the penalized problems to those such that $x(t) \in \Omega_0$ for all $t \geq 0$, and define the penalized cost on Ω_0. ■

To solve the Problem NLEP we represent a system in a convenient set of coordinates $x \mapsto y = T(x)$; and we decompose every matrix and vector consistent with the structure of T.

In the new coordinates gg^T has a block diagonal representation, which we write as

$$gg^T \mapsto \left(\frac{\partial T}{\partial x} gg^T \frac{\partial T}{\partial x}^T\right)\Bigg|_{x=T^{-1}(y)} = \left[\begin{array}{c|c} g_1 g_1^T & 0 \\ \hline 0 & g_2 g_2^T \end{array}\right],$$

where $g_1 : \mathbb{R}^n \to \mathbb{R}^{(n-p)\times n}$ and $g_2 : \mathbb{R}^n \to \mathbb{R}^{p\times n}$ are of class $\mathcal{C}^\infty$ and such that $g_1^T g_2 = g_2^T g_1 = 0$, with

$$g \mapsto \left(\frac{\partial T}{\partial x} g\right)\Bigg|_{x=T^{-1}(y)} = \left[\begin{array}{c} g_1 \\ \hline g_2 \end{array}\right].$$

We represent f as

$$f \mapsto \left(\frac{\partial T}{\partial x} f\right)\Bigg|_{x=T^{-1}(y)} = \left[\begin{array}{c} f_1 \\ \hline f_2 \end{array}\right],$$

with $f_1 : \mathbb{R}^n \to \mathbb{R}^{n-p}$ and $f_2 : \mathbb{R}^n \to \mathbb{R}^p$. It is also useful to define $\tilde{f}_1 = (g_1 g_1^T)^{-1} f_1$,

$\tilde{f}_2 = (g_2 g_2^T)^{-1} f_2$, and $\tilde{f} = (gg^T)^{-1} f$. With some abuse of notation we denote the new representations of the system and the cost with f, g, and q, respectively. The projections take the form

$$\Pi_h(y) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & I \end{array}\right], \quad \Pi_{h^\perp}(y) = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array}\right],$$

respectively. To conclude note also that the constraint is described as $\mathcal{M} = \{y = [y_1, y_2]^T \in \mathbb{R}^n : y_2 = 0\}$.

To solve the Problem NLEP, as in the linear case, we proceed in steps. First we characterize the unique optimal control of the original problem (5.1). Then we construct a class of functions $V_k : \mathbb{R}^n \to \mathbb{R}$, parametrized by $k > 0$, such that:

- V_k is a solution of the Hamilton-Jacobi-Bellman (HJB) equation associated to a penalized problem of the form (5.2) for some l and m to be defined;
- the optimal feedback control $u^* = -g^T \dfrac{\partial V}{\partial y}^T$ resulting from the solution of the Problem NLEP coincides with the optimal control of the original problem (5.1) for all $x_0 \in \mathcal{M}$.

Step 1 (Characterizing the optimal solution of problem (5.1)) Observe that the input vector can always be uniquely decomposed in the form

$$u = g_1^T v_1 + g_2^T v_2, \tag{5.5}$$

with vectors v_1 ans v_2 such that $\dim(v_1) = n - p$ and $\dim(v_2) = p$.

Given an initial condition $y_0 = [y_{01}, 0]^T \in \mathcal{M}$, in order to fulfill the condition $y_2(t) = 0$, for all $t \geq 0$, we select[3] $v_2 = -\left(g_2 g_2^T\right)^{-1} f_2$. It follows that any optimal control of the original problem (5.1) takes, without loss of generality, the form

$$u = g_1^T v_1 - g_2^T \left(g_2 g_2^T\right)^{-1} f_2,$$

and any control of this form is such that the constraint holds for all $t \geq 0$. Thanks to this decomposition we have decoupled the variables of the original problem (5.1) to obtain the equations $\dot{y}_2 = 0$ (and hence $y_2(t) = 0$, for all $t \geq 0$) and $\dot{y}_1 = f_1 + g_1 g_1^T v_1$, with the

[3]The use of this choice follows from the same argument as in Remark 7: any other choice of the form $v_2 = -(g_2 g_2^T)^{-1} f_2 + \gamma$, where $\gamma : \mathbb{R}^n \to \mathbb{R}^p$ is a function such that $\gamma|_{y_2=0}$, coincides with (5.5) on $\mathcal{M}$, and hence leaves the cost and the reduced dynamic unaffected. We can therefore set $\gamma = 0$ without loss of generality. This argument holds, in particular, for the choice $v_2 = -(g_2 g_2^T)^{-1} f_2|_{y_2=0}$, analogous to the one chosen in the linear case.

associated cost

$$\begin{aligned} J(y_0,v_1) &= \int_0^{+\infty} \frac{1}{2}q + \frac{1}{2}u^T u\, dt \\ &= \int_0^{+\infty} \frac{1}{2}q + \frac{1}{2}\left(g_1^T v_1 - g_2^T \left(g_2 g_2^T\right)^{-1} f_2\right)^T \left(g_1^T v_1 - g_2^T \left(g_2 g_2^T\right)^{-1} f_2\right) dt \\ &= \int_0^{+\infty} \frac{1}{2}q + \frac{1}{2}f_2^T \left(g_2 g_2^T\right)^{-1} f_2 + \frac{1}{2}v_1^T \left(g_1 g_1^T\right) v_1\, dt. \end{aligned}$$

These equations describe now an unconstrained optimal control problem for a reduced system.

Assumption 2. The value function $V_c : \Omega' \subset \mathbb{R}^{n-p} \to \mathbb{R}$ of the reduced problem is defined on a neighborhood Ω' of 0, $\mathcal{C}^2$, and it is such that $\dfrac{\partial^2 V_c}{\partial y_1^2}(0)$ is positive definite.

Under Assumption 2, the reduced system admits an optimal control given by $v_1 = -(g_1 g_1^T)^{-1}(g_1 g_1^T)\dfrac{\partial V_c}{\partial y_1}^T = -\dfrac{\partial V_c}{\partial y_1}^T$, with V_c solution of the HJB equation

$$\left.\left(\frac{1}{2}q + \frac{1}{2}f_2^T(g_2 g_2^T)^{-1} f_2 - \frac{1}{2}\frac{\partial V_c}{\partial y_1}(g_1 g_1^T)\frac{\partial V_c}{\partial y_1}^T + \frac{\partial V_c}{\partial y_1} f_1\right)\right|_{y_2=0} = 0. \tag{5.6}$$

Therefore the optimal solution of problem (5.1) is given by the feedback

$$u = -g_1^T \frac{\partial V_c}{\partial y_1}^T - g_2^T \tilde{f}_2, \tag{5.7}$$

and this guarantees $y_2(t) = 0$, for all $t \geq 0$.

Step 2 (Obtaining a solution of the penalized problem (5.2) that matches the solution of the original problem (5.1)) Consider the HJB equation

$$\frac{1}{2}q - \frac{1}{2}\frac{\partial V_k}{\partial y} g g^T \frac{\partial V_k}{\partial y}^T + \frac{\partial V}{\partial y} f + \frac{1}{2}l + \frac{1}{2}k^2 m = 0, \tag{5.8}$$

associated to the penalized problem (5.2).

Assume that $\left[\tilde{f}_1^T - \tilde{f}_1^T|_{y_2=0}, \tilde{f}_2^T\right]$ is an exact differential, and hence there exists a C^∞ function $F : \mathbb{R}^n \to \mathbb{R}$ such that $\dfrac{\partial F}{\partial y} = \left[\tilde{f}_1^T - \tilde{f}_1^T|_{y_2=0}, \tilde{f}_2^T\right]$, with $F(0) = 0$.

We consider a class of candidate solutions V_k, parametrized by $k > 0$, to the HJB equation (5.8), defined as

$$V_k = V_c + F + \frac{1}{2}k||y_2||^2. \tag{5.9}$$

We conclude that the resulting feedback control is

$$u = -g^T \frac{\partial V_k}{\partial y}^T \overset{y_2=0}{=} -g_1^T \frac{\partial V_c}{\partial y_1}^T - g_2^T \tilde{f}_2, \tag{5.10}$$

which is identical to the control (5.7) whenever $y_2 = 0$. It is immediate to observe that a closed loop trajectory for the original problem (5.1) resulting from the use of the optimal control (5.7) is also the closed loop trajectory that solves the system (5.2) resulting from the use of the control (5.10), for any initial condition $y_0 \in \mathcal{M}$. Hence, if (5.10) were actually the optimal control for the penalized problem (5.2), for k sufficiently large, the two problems would be equivalent.
We summarize the above discussion in the following statement, which is given in a coordinate free form.

Proposition 5 (Solution of the Problem NLEP)**.** *Consider the optimal control problem (5.1) and the penalized optimal control problem (5.2). Assume that Δ is invariant under g and that*

$$\left[f^T(gg^T)^{-1} - \left(f^T(gg^T)^{-1}\Pi_{h^\perp}\right)\big|_{\mathcal{M}}\right] dx \tag{5.11}$$

is an exact differential. Then there exists $\tilde{k} > 0$ such that the selection

$$l = -\left(q - q\big|_{\mathcal{M}}\right) - \left(f^T(gg^T)^{-1}f - \left(f^T\left(gg^T\right)^{-1}f\right)\big|_{\mathcal{M}}\right),$$

and

$$m = h^T \frac{\partial h}{\partial x} g g^T \frac{\partial h}{\partial x}^T h,$$

solves the Problem NLEP, i.e. it is such that the optimal control problem (5.2) is a local exact penalization of the optimal control problem (5.1).

Proof. In the coordinates system (5.4) the hypotheses imply that $g_1 = g_1|_{y_2=0}$, and that $\left[\tilde{f}_1^T - \tilde{f}_1^T|_{y_2=0}, \tilde{f}_2^T\right]$ is an exact differential, which guarantees the existence of a $\mathcal{C}^\infty$ function $F : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial F}{\partial y}^T = \left[\frac{\tilde{f}_1 - \tilde{f}_1\big|_{y_2=0}}{\tilde{f}_2}\right],$$

with $F(0) = 0$. Observe that $\frac{\partial F}{\partial y_1}^T\big|_{y_2=0} = \tilde{f}_1|_{y_2=0} - \tilde{f}_1|_{y_2=0} = 0$, and hence $F|_{y_2=0} = 0$. Additionally, $m = y_2^T(g_1 g_1^T)y_2$.
We now verify that V_k, as expressed in (5.9), solves the HJB equation (5.8). To this end,

observe that

$$\begin{aligned}\frac{\partial V_k}{\partial y_1} &= \frac{\partial V_c}{\partial y_1} + \left(\tilde{f}_1 - \tilde{f}_1\big|_{y_2=0}\right)^T,\\ \frac{\partial V_k}{\partial y_2} &= \tilde{f}_2^T + k y_2^T,\end{aligned}$$

and hence (since g_1 does not depend on y_2, and so the identity $\tilde{f}_1|_{y_2=0} = (g_1 g_1^T)^{-1} f_1|_{y_2=0}$ holds)

$$\begin{aligned}&-\frac{1}{2}\frac{\partial V_k}{\partial y} g g^T \frac{\partial V_k}{\partial y}^T + \frac{\partial V_k}{\partial y} f\\ &= -\frac{1}{2}\frac{\partial V_k}{\partial y_1} g_1 g_1^T \frac{\partial V_k}{\partial y_1}^T - \frac{1}{2}\frac{\partial V_k}{\partial y_2} g_2 g_2^T \frac{\partial V_k}{\partial y_2}^T + \frac{\partial V_k}{\partial y} f\\ &= -\frac{1}{2}\frac{\partial V_c}{\partial y_1} g_1 g_1^T \frac{\partial V_c}{\partial y_1}^T - \frac{\partial V_c}{\partial y_1}\left(f_1 - f_1\big|_{y_2=0}\right)\\ &- \frac{1}{2}\left(\tilde{f}_1 - \tilde{f}_1\big|_{y_2=0}\right)^T\left(f_1 - f_1\big|_{y_2=0}\right) - \frac{1}{2}\tilde{f}_2^T g_2 g_2^T \tilde{f}_2 - k y_2^T f_2 - k^2\frac{1}{2} y_2^T g_2 g_2^T y_2\\ &+ \frac{\partial V_c}{\partial y_1} f_1 + \left(\tilde{f}_1 - \tilde{f}_1\big|_{y_2=0}\right)^T f_1 + f_2^T (g_2 g_2^T)^{-1} f_2 + k y_2^T f_2.\end{aligned}$$

Collecting $\left(\tilde{f}_1 - \tilde{f}_1\big|_{y_2=0}\right)^T$ we obtain

$$\begin{aligned}&-\frac{1}{2}\frac{\partial V_k}{\partial y} g g^T \frac{\partial V_k}{\partial y}^T + \frac{\partial V_k}{\partial y} f\\ &= -\frac{1}{2}\frac{\partial V_c}{\partial y_1} g_1 g_1^T \frac{\partial V_c}{\partial y_1}^T + \frac{\partial V_c}{\partial y_1} f_1\big|_{y_2=0} + \frac{1}{2}\left(\tilde{f}_1 - \tilde{f}_1\big|_{y_2=0}\right)^T\left(f_1 + f_1\big|_{y_2=0}\right)\\ &+ \frac{1}{2} f_2^T \left(g_2 g_2^T\right)^{-1} f_2 - k^2\frac{1}{2} y_2^T g_2 g_2^T y_2.\end{aligned}$$

Subtracting the HJB equation (5.6) of the reduced problem yields

$$\begin{aligned}&-\frac{1}{2}\frac{\partial V_k}{\partial y} g g^T \frac{\partial V_k}{\partial y}^T + \frac{\partial V_k}{\partial y} f\\ &= +\frac{1}{2}\left(\tilde{f}_1 - \tilde{f}_1\big|_{y_2=0}\right)^T\left(f_1 + f_1\big|_{y_2=0}\right) + \frac{1}{2}\left(f_2^T \left(g_2 g_2^T\right)^{-1} f_2 - \left(f_2^T \left(g_2 g_2^T\right)^{-1} f_2\right)\big|_{y_2=0}\right)\\ &- k^2\frac{1}{2} y_2^T g_2 g_2^T y_2 - \frac{1}{2} q\big|_{y_2=0}\\ &= \frac{1}{2}\left(f^T (g g^T)^{-1} f - \left(f^T \left(g g^T\right)^{-1} f\right)\big|_{y_2=0}\right) - k^2\frac{1}{2} y_2^T g_2 g_2^T y_2 - \frac{1}{2} q\big|_{y_2=0}.\end{aligned}$$

Substituting this expression in the HJB equation (5.8) of the penalized problem (5.2) yields

$$\frac{1}{2}\left(f^T(gg^T)^{-1}f-\left(f^T\left(gg^T\right)^{-1}f\right)|_{y_2=0}\right)-k^2\frac{1}{2}y_2^Tg_2g_2^Ty_2+\frac{1}{2}\left(q-q\big|_{y_2=0}\right)+\frac{1}{2}l+\frac{1}{2}k^2m=0,$$

which holds because of the selection of l and m. This proves that V_k in (5.9) satisfies the HJB equation (5.8).

We now prove that there exists a constant $\tilde{k}_1 > 0$ such that, for all $k > \tilde{k}_1$, there exists a neighborhood Ω'' of 0 in which V_k is positive definite. To this end, observe that $V_k(0) = 0$,

$$\frac{\partial V_k}{\partial y}(0) = \frac{\partial V_c}{\partial y_1}(0) + \left[\tilde{f}_1^T(0) - \tilde{f}_1^T(0)|_{y_2=0}, \tilde{f}_2(0)^T\right] = 0,$$

and the Hessian has the form

$$\frac{\partial^2 V_k}{\partial y^2}(0) = \left[\begin{array}{c|c} \frac{\partial^2 V_c}{\partial y_1^2}(0) & * \\ \hline * & kI + \frac{\partial^2 F}{\partial y_2^2}(0) \end{array}\right].$$

Applying the Schur complement as in Proposition 4, since $\dfrac{\partial^2 V_c}{\partial y_1^2}(0)$ is positive definite, it follows that $\dfrac{\partial^2 V_k}{\partial y_1^2}(0)$ is positive definite for all $k > \tilde{k}_1$, with $\tilde{k}_1$ a sufficiently large constant. Since $\dfrac{\partial V_k}{\partial x}(0) = 0$ and $\dfrac{\partial^2 V_k}{\partial y_1^2}(0)$ is positive definite, it follows that 0 is a strict local minimizer of V_k [26], and since $V_k(0) = 0$, there exists a neighborhood Ω'' of 0 in which V_k is positive definite.

It remains to show that there exists a constant $\tilde{k}_2 > 0$ such that, for all $k > \tilde{k}_2$, there exists a neighborhood Ω''' of zero in which the modified cost $q+l+k^2m$ is positive definite. To this end, observe that $q(0) + l(0) + k^2m(0) = 0$ and, since $f(0) = 0$ and q is positive definite, the stationarity condition $\dfrac{\partial q}{\partial y}(0) + \dfrac{\partial l}{\partial y}(0) + k^2\dfrac{\partial m}{\partial y}(0) = 0$ follows. Moreover, the matrix

$$\frac{\partial^2 q}{\partial y^2}(0) + \frac{\partial^2 l}{\partial y^2}(0) + k^2\frac{\partial^2 m}{\partial y^2}(0) = \left[\begin{array}{c|c} \frac{\partial^2 q}{\partial y_1^2} & * \\ \hline * & * + k^2 I \end{array}\right]$$

is positive definite for all $k > \tilde{k}_2$, with $\tilde{k}_2$ a sufficiently large constant, applying again the Schur complement. It follows that 0 is a strict local minimizer of $q + l + k^2m$ [26], and, since $q(0) + l(0) + k^2m(0) = 0$, there exist a neighborhood Ω''' of 0 in which $q + l + k^2m$ is positive definite. We select $\tilde{k} = \max\{\tilde{k}_1, \tilde{k}_2\}$, and $\Omega = \Omega'' \cap \Omega'''$.

Using standard arguments from the theory of HJB equations[4], this implies that the control

[4] See Chapter 2.

(5.10) is a local minimizer of problem (5.2), for all initial conditions x_0 such that the corresponding closed loop trajectory $x(t) \in \Omega$, for all $t \geq 0$[5]; hence the penalized problem (5.2) is a *local exact penalization* of the original problem (5.1) for $k > \tilde{k}$ and for x_0 in a neighborhood of 0. □

The solution of Problem NLEPM is obtained as a direct consequence of Proposition 5, and it is summarized in the following corollary.

Corollary 2 (Solution of the Problem NLEPM)**.** *Consider the optimal control problem (5.1) and the penalized optimal control problem (5.3). There exists $\tilde{k} > 0$ such that the selection $g' = g\big|_{\mathcal{M}}$ and*

$$f' = f\big|_{\mathcal{M}} + \left((gg^T) \frac{\partial h^{\perp}}{\partial x}^T \left(L_{v_{h^\perp}} \left(f^T \left(g' g'^T \right)^{-1} \right) \right) v_h \right) \Big|_{\mathcal{M}} h,$$

together with

$$l = -\left(q - q|_{\mathcal{M}}\right) - \left(\left(f'^T \left(g' g'^T \right)^{-1} f' \right) - \left(f'^T \left(g' g'^T \right)^{-1} f' \right) |_{\mathcal{M}} \right),$$

and

$$m = h^T \frac{\partial h}{\partial x} g' g'^T \frac{\partial h}{\partial x}^T h,$$

solves the Problem NLEPM, i.e. it is such that the optimal control problem (5.3) is a local exact penalization of the optimal control problem (5.1).

Proof. The optimal control problem

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & J(x_0,u) = \int_0^{+\infty} \left(\frac{1}{2} q(x) + \frac{1}{2} ||u||^2 \right) dt, \\ \text{subject to} \quad & \dot{x} = f'(x) + g'(x) u, \qquad x(0) = x_0, \\ & h(x) = 0, \qquad \forall t \geq 0, \end{aligned} \tag{5.12}$$

has the same solutions as the original problem (5.1); indeed, $f' = f$ and $g' = g$ on the constraint $\mathcal{M}$, and hence the dynamical equations coincide on $\mathcal{M}$.
In the coordinates system (5.4), $g' = g|_{y_2=0}$ and

$$\tilde{f}' = (g' g'^T)^{-1} f' = \begin{bmatrix} \tilde{f}_1' \\ \hline \tilde{f}_2' \end{bmatrix} = \begin{bmatrix} \tilde{f}_1|_{y_2=0} + \frac{\partial \tilde{f}_2}{\partial y_1}^T \big|_{y_2=0} y_2 \\ \hline \tilde{f}_2|_{y_2=0} \end{bmatrix}.$$

[5]This can be guaranteed, e.g., for initial conditions x_0 contained in the nonempty interior of a conveniently selected compact invariant set contained in Ω, by standard Lyapunov arguments [27].

As a consequence

$$\left[\frac{\tilde{f}_1' - \tilde{f}_1'|_{y_2=0}}{\tilde{f}_2'}\right] = \left[\frac{\frac{\partial \tilde{f}_2}{\partial y_1}^T\Big|_{y_2=0} y_2}{\tilde{f}_2|_{y_2=0}}\right],$$

which has a symmetric Jacobian and is therefore an exact differential. The new problem (5.12) satisfy the hypotheses of Proposition 5; therefore problem (5.3) is a *local exact penalization* of problem (5.12), which, as discussed above, has the same solutions as the original problem (5.1). □

Remark 10. The hypotheses $\dfrac{\partial^2 V_c}{\partial y_1^2} > 0$ and $\dfrac{\partial^2 q}{\partial x^2} > 0$ could be replaced with any hypothesis which guarantees that the modified cost $q + l + k^2 m$ and the candidate value function V_k are locally positive definite (for all $k > \tilde{k}$, with $\tilde{k}$ a sufficiently large constant), and both Proposition 5 and Corollary 2 would still hold. An example in which weaker hypotheses are considered is Example 6.3. ■

Chapter 6

Numerical Examples

In this chapter, some numerical examples that illustrate the main results given in Chapter 4 and Chapter 5 are discussed. In particular three examples are considered.

- In Example 6.1 an the application of Corollary 1 is given. This shows that the applied numerical procedure, which involves the computation of the proposed *exact penalization* and subsequently its solution, never requires to find an explicit basis for $\ker(\tilde{H}^T)$ (i.e. it is not necessary to "make the constraint explicit").
- In Example 6.2 it is shown, in a context in which Proposition 4 cannot be applied, that the application of Corollary 1 can lead to the undesired numerical instabilities discussed in Remark 6 and Remark 8.
- Example 6.3 shows how the *local exact penalization* proposed in Proposition 5 leads to a feedback control law that makes the desired constraint set $\mathcal{M}$ invariant for the corresponding closed loop dynamic. Due to the difficulty that arises in the numerical solution of the HJB partial differential equation (and the computation of the desired change of coordinates), a simple example, for which the solution can be computed explicitly, is given.

6.1 A three dimensional linear system

Consider an optimal control problem of the form (4.1), with $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$, and

$$A = \begin{bmatrix} -5 & 2 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 7 & 1 \\ 1 & 13 & 2 \\ 5 & 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

together with

$$\tilde{H}^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Let V be the subspace identified by the constraint and $\mathcal{V}$ be the corresponding manifold, defined as in Chapter 4. This problem, by Corollary 1, admits an *exact penalization*, which has the form (4.3), with

$$L = \begin{bmatrix} 0.2817 & -4.2470 & 2.1984 \\ -4.2470 & -8.7758 & -2.3304 \\ 2.1984 & -2.3304 & 4.1150 \end{bmatrix}, \quad M = \begin{bmatrix} 0.0020 & 0.0020 & 0.0020 \\ 0.0020 & 0.0020 & 0.0020 \\ 0.0020 & 0.0020 & 0.0020 \end{bmatrix},$$

and

$$A' = \begin{bmatrix} -4.6279 & 2.3721 & 0.3721 \\ -3.3261 & 1.6739 & -0.2361 \\ -0.4176 & -0.4176 & 4.5824 \end{bmatrix},$$

and value of the parameter $k = 30$. This *exact penalization* is now an unconstrained minimization problem, that admits a feedback solution, which can be computed through the solution of the ARE associated to the problem. This procedure yields a closed loop systems, the trajectories of which, corresponding to different initial conditions x_0 that satisfy the constraint $\tilde{H}^T x_0 = 0$, are integrated numerically via the software MATLAB and displayed in Figure 6.1. The off-the-manifold error, computed as $err(t) = ||\tilde{H}^T x(t)||$, is plotted in Figure 6.2.

6.2 A three dimensional linear system in which the closed loop solution of the *exact penalization* is unstable

The objective of this example is to show that the application of Corollary 1 can lead to the unstable closed loop behavior associated to the optimal feedback solution discussed in Remark 6 and Remark 8.

Consider an optimal control problem of the form (4.1), with $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$, and

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 7 & 1 \\ 1 & 13 & 2 \\ 5 & 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

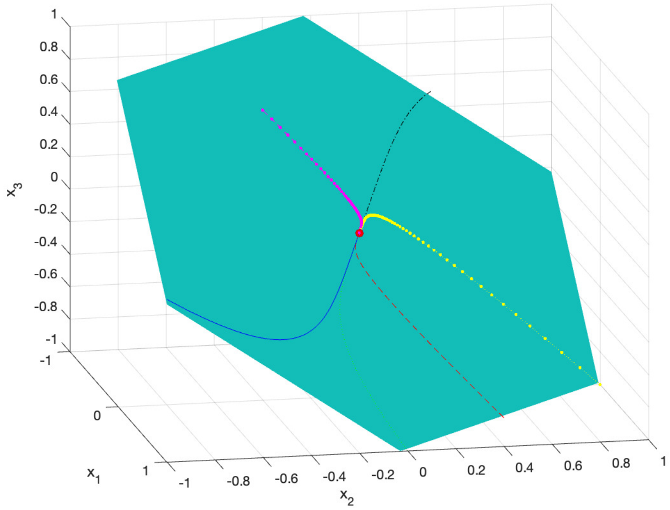

Figure 6.1: Trajectories of the closed loop system described in Example 6.1 with initial conditions: $x_0 = [1, -1, 0]^T$ (blue, solid); $x_0 = [0.5, 0.5, -1]^T$ (red, dashed); $x_0 = [1, 0, -1]^T$ (green, dotted); $x_0 = [-1, 0.5, 0.5]^T$ (black, dashed-dotted); $x_0 = [0, 1, -1]^T$ (yellow, dotted); $x_0 = [-1/3, -1/3, 2/3]^T$ (magenta, dashed-dotted). The manifold $\mathcal{V}$ is colored in light green, and the origin is the solid red dot.

together with

$$\tilde{H}^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Let V be the subspace identified by the constraint and $\mathcal{V}$ be the corresponding manifold, defined as in Chapter 4. Observe that $V = \text{span}\left\{[0, -1, -1]^T\right\}$.

In this example, Proposition 4 cannot be applied, since

$$(BB^T)^{-1}A - \Pi_{H^\perp}^T (BB^T)^{-1} A \Pi_{H^\perp} = \begin{bmatrix} 346.7335 & -97.8896 & -98.8433 \\ -183.8496 & 51.9546 & 52.3905 \\ -30.5033 & 8.5105 & 8.9464 \end{bmatrix}$$

is not symmetric, and hence condition (4.11) is not satisfied. Therefore, we resort to the *exact penalization* result given by Corollary 1. The procedure for the construction of an *exact penalization* described in Example 6.1 yields the feedback matrix

$$P = \begin{bmatrix} 8.4683 & -4.5154 & -0.4972 \\ -4.5154 & 2.4421 & 0.2399 \\ -0.4972 & 0.2399 & 0.2457 \end{bmatrix},$$

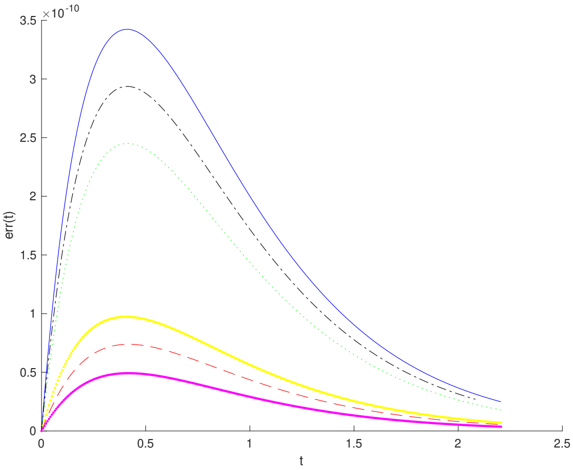

Figure 6.2: Off-the-manifold error for the trajectories in Figure 6.1, with the same color coding. The error is computed as $err(t) = ||\tilde{H}^T x(t)||$.

which gives the closed loop dynamic

$$\dot{x} = A_{cl} x, \tag{6.1}$$

with

$$A_{cl} = A - BB^T P = \begin{bmatrix} -4.5036 & 2.3895 & 0.3895 \\ -4.3903 & 2.3378 & 0.3378 \\ 11.1456 & -5.9221 & -0.9221 \end{bmatrix}.$$

The spectral analysis of the matrix A_{cl} shows that $[0, -1, -1]^T$, which spans the subspace V, is an eigenvector of the matrix A_{cl}, associated with the stable eigenvalue -3.3501. As a consequence, $\mathcal{V}$ is a stable manifold for the closed loop dynamics (6.1). Nevertheless, the other two eigenvalues of A_{cl}, namely 1.8807 and 5.2528, are unstable. Therefore, even if $\mathcal{V}$ is a stable manifold for the closed loop dynamics (6.1), the system is unstable, as discussed in Remark 8. Indeed, the trajectory of the system (6.1) with the initial condition $x_0 = [0, -1, 1]^T$, computed via numerical integration with the software MATLAB, diverges, as shown in Figure 6.3. This is due to numerical errors that drive the trajectory outside the manifold $\mathcal{V}$.

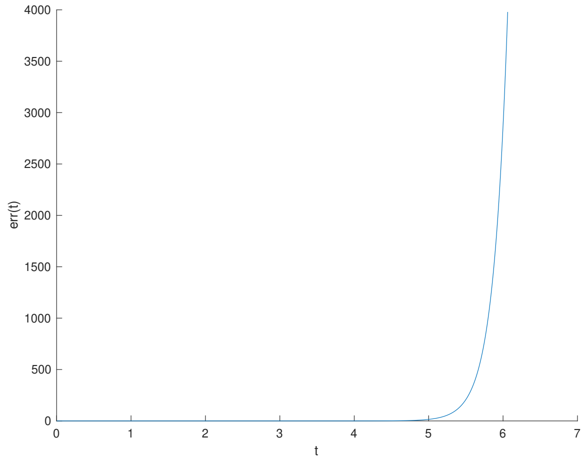

Figure 6.3: Off-the-manifold error of the trajectory of system (6.1) with initial condition $x_0 = [0, -1, 1]^T$, integrated numerically. The error is computed as $err(t) = ||\tilde{H}^T x(t)||$.

6.3 A two dimensional nonlinear system

Consider an optimal control problem of the form (5.1), with $x = [x_1, x_2]^T \in \mathbb{R}^2$, and

$$f(x) = \begin{bmatrix} -x_1^3 \\ -2x_1^4 + x_1^2 - x_2 \end{bmatrix}, \quad g(x) = \begin{bmatrix} 1 & 0 \\ 2x_1 & 1 \end{bmatrix}, \quad q(x) = 3x_1^6 + (x_2 - x_1^2)^2,$$

together with

$$h(x) = x_2 - x_1^2.$$

Let the distribution Δ, the manifold $\mathcal{M}$, and the covector field $\tilde{f}^T$ be defined as in Chapter 5. Observe that $\Delta(x) = \text{span}\left\{ g(x)g(x)^T \frac{\partial h}{\partial x}(x)^T \right\} = \text{span}\left\{ [0, 1]^T \right\}$ is a one dimensional and regular distribution (and hence involutive), and its kernel is spanned by the exact differential $\frac{\partial h^\perp}{\partial x}$ of the globally defined mapping $h^\perp(x) = x_1$. Note also that the covector field

$$f(x)^T (g(x)g(x)^T)^{-1} = \begin{bmatrix} -x_1^3 + 2x_1x_2 & x_1^2 - x_2 \end{bmatrix}$$

is an exact differential, since $\bar{F}(x) = -\frac{1}{4}x_1^4 - \frac{1}{2}x_2^2 + x_1^2 x_2$ is such that

$$\frac{\partial \bar{F}}{\partial x}(x) = \begin{bmatrix} -x_1^3 + 2x_1x_2 & x_1^2 - x_2 \end{bmatrix} = f(x)^T (g(x)g(x)^T)^{-1}.$$

This, in particular, implies that the condition (5.11) is verified, and it is therefore possible to apply Proposition 5 to find a *local exact penalization* of the problem.

The objective of this Example is to verify, via numerical computations, a consequence of the Proposition 5, namely that the optimal feedback solution of the *local exact penalization* proposed in Proposition 5 is such that the manifold $\mathcal{M}$ is locally invariant for the corresponding closed loop dynamic. To this end, we first compute the candidate value functions V_k, defined as in (5.9), of the *local exact penalization*; we then show that they solve the corresponding HJB equation (5.8) and are positive definite (for sufficiently large values of the parameter k), and therefore characterize the optimal feedback solution $u^* = -g^T \frac{\partial V_k}{\partial x}^T$ of the *local exact penalization.* To conclude, we show that $\mathcal{M}$ is locally invariant for the associated closed loop system

$$\dot{x} = f - gg^T \frac{\partial V_k}{\partial x}^T. \tag{6.2}$$

Observe that the global diffeomorphism (5.4), described in Chapter 5, has the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} h^{\perp}(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1^2 \end{bmatrix}, \tag{6.3}$$

and the associated vector fields of the nonlinear system and the cost, in the coordinates system (6.3), read as

$$f(y) = \begin{bmatrix} -y_1^3 \\ -y_2 \end{bmatrix}, \quad g(y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad q(y) = 3y_1^6 + y_2^2.$$

The constraint, in the coordinates (6.3), has the form $\mathcal{M} = \{y \in \mathbb{R}^2 : y_2 = 0\}$. The reduced system on the manifold $\mathcal{M}$ is therefore $\dot{y}_1 = -y_1^3 + v_1$, with reduced cost $q'(y) = 3y_1^6$, and the corresponding value function is[1] $V_c(y_1) = \frac{1}{4}y_1^4$. Moreover, the covector field $\tilde{f}^T - \tilde{f}^T|_{y_2=0}$ admits the primitive $F(y) = -\frac{y_2^2}{2}$, with $F(0) = 0$. Hence, the candidate value functions are $V_k(y) = \frac{1}{4}y_1^4 - \frac{1}{2}y_2^2 + \frac{1}{2}ky_2^2$, which are positive definite for $k > 1$, and their differentials are $\frac{\partial V_k}{\partial y} = \left[y_1^3, (k-1)y_2\right]$. We verify that these functions satisfy the HJB equation (5.8) of the *local exact penalization* in question: indeed, with the selection

$$l(y) = -2y_2^2, \quad m(y) = y_2^2,$$

[1] Assumption 2 does not hold for this value function. Nevertheless, the results given in Proposition 5 hold, as discussed in Remark 11.

given by the Proposition 5, the HJB equation (5.8) of the *local exact penalization* becomes

$$\frac{1}{2}\begin{bmatrix} y_1^3 & (k-1)y_2 \end{bmatrix}\begin{bmatrix} y_1^3 \\ (k-1)y_2 \end{bmatrix} - \begin{bmatrix} y_1^3 & (k-1)y_2 \end{bmatrix}\begin{bmatrix} -y_1^3 \\ -y_2 \end{bmatrix} - \frac{3}{2}y_1^6 - \frac{1}{2}y_2^2 + y_2^2 - \frac{1}{2}k^2 y_2^2 = 0,$$

and is therefore verified. To conclude, the closed loop system (6.2), in the coordinates (6.3), is

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = f(y) - g(y)g(y)^T \left(\frac{\partial V_k}{\partial y}(y)\right)^T = \begin{bmatrix} -2y_1^3 \\ -ky_2 \end{bmatrix},$$

where the variables y_1 and y_2 are decoupled and, for any initial condition $y_0 \in \mathcal{M}$, it is immediate to verify that the closed loop trajectory satisfies $y(t) \in \mathcal{M}$ for all $t \geq 0$.

Remark 11. In this example, the candidate value functions V_k and the modified cost $q + l + k^2 m$ are (globally) positive definite for $k > 1$, even if the assumptions $\frac{\partial^2 V_c}{\partial y_1^2}(0) > 0$ and $\frac{\partial^2 q}{\partial x^2}(0) > 0$ are not satisfied. Consequently, the arguments given in the proof of Proposition 5 hold regardless, as only the positive definiteness of V_k and the modified cost are essential (as discussed in Remark 10). Moreover, the feedback law $u^* = -g^T \frac{\partial V_k}{\partial x}^T$ is globally optimal for the *local exact penalization*, and the manifold $\mathcal{M}$ is globally invariant for the closed loop system (6.2). In general, the optimal solution of the *local exact penalization* given by Proposition 5 exists and coincides with $u^* = -g^T \frac{\partial V_k}{\partial x}^T$ only in a neighborhood of the origin. ■

Chapter 7

Conclusions

In this work, the novel notion of *exact penalization* for infinite horizon optimal control problems has been introduced, and explicit constructions of such penalizations have been proposed. These constructions enable, in the infinite horizon case, to solve state constrained optimal control problems without resorting to the sophisticated analytical tools which are commonly used to deal with finite horizon state constrained problems (see e.g, [5]). In particular, such penalizations (for regular constraints) have been constructed without employing nonsmooth techniques. This suggests that these new constructions may be more tractable than their nonsmooth counterpart, both from the analytical and the numerical point of view, as in the case of smooth exact penalty functions in nonlinear programming. For linear quadratic problems, in particular, it has been shown that the exact penalization yields a standard LQR problem, which can be constructed via a projection matrix.

The technical proofs reveal that the combined use of tools from Geometric Control and Dynamic Programming, described in Chapter 2 and Chapter 3, together with ideas borrowed from smooth nonlinear and quadratic programming, can be fruitful to deal with such problems, even in the more general cases of underactuated systems and inequality constraints. Therefore, the research carried out in this thesis leads naturally to several open questions and possible future developments. In particular, the following lines of research are promising.

- **Underactuated systems.** The given results suggest that, under the hypothesis of controlled invariance of the constraint subspace, or the generalizes hypothesis of controlled invariance of the constraint distribution, see [23], an exact penalization can be constructed with a similar approach as the one proposed in this work.
- **Inequality constraints.** Drawing again analogies with nonlinear programming,

it is possible to consider inequality constraints as equality constraints in an extended space through auxiliary variables. Similar approaches could be applied to deal with inequality constraints in optimal control problems, formulated as equality constraints in an extended state space.

- **Relaxing the hypotheses on the constraint.** In the nonlinear case discussed in Chapter 5, Assumption 1 is essential to determine a valid change of variable to tackle the problem. It is therefore natural to investigate if this hypothesis is strictly necessary to construct an exact penalization along the lines of the ones proposed in this work.

- **Modification of the dynamic.** In the exact penalizations with modification of the dynamic, it has been shown that the constraint is always a locally asymptotically stable manifold for the closed loop solution, but the origin may not be asymptotically stable (trajectories outside of the manifold may diverge, see Remark 6, Remark 8 and Example 6.2). To guarantee robustness of the solutions of the penalizations it is therefore essential to understand how to determine a modification of the dynamic that guarantees asymptotic stability (if possible).

- **Problems with semidefinite cost.** In the theory of LQR problems it is well known that an optimal control synthesis can be obtained via the ARE also when the matrix Q is positive semidefinite, if observability is assumed. It is therefore natural to investigate how to construct an exact penalization under an observability hypothesis, both in the linear and in the nonlinear case (using the notion of nonlinear observability, given e.g. in [23]).

- **Finite horizon optimal control problems.** In finite horizon optimal control, the value function depends on time, and the structure of the optimal feedback is generally different from the infinite horizon case (e.g. for linear quadratic finite horizon problems the value function is obtained as a solution of a *Differential Riccati Equation*). A natural question is to investigate if an exact penalization can still be constructed along the lines of the ones proposed in this work.

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