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Varifolds: a Modern Approach to Classical Results

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INTRODUCTION

In the present thesis work, the theory of varifolds and its application to the Plateau problem are discussed.

Chapter 1 provides a concise, non exhaustive, summary of the minimal theory necessary to fully understand the content of the subsequent chapters. We begin by recalling fundamental concepts of measure theory, functional analysis, submanifolds and n-rectifiable sets. Particularly important results are Riesz representation theorem for Radon measure and the divergence theorem for C^2-class n-dimensional manifold, which will play a major role for the definition of generalized mean curvature for varifolds in the following chapter. It is advised that the reader has a solid preliminary knowledge of all the results mentioned in this chapter as, for the sake of readability and practicality, no proofs will be included in Chapter 1. The main references for this chapter are: [Fol99], [BB11],[Sim84],[BR07].

Chapter 2 introduces the definition of n-varifold, which, roughly speaking, one can interpret as a measure-theoretical generalization for the concept of n - dimensional surface. Initially, the most abstract definition of general n-varifold as Radon measures on the space $G_n(\mathbb{R}^{n+k})$, which is the product of $\mathbb{R}^{n+k}$ with the space of n - dimensional planes contained in it, is stated. Later, n-rectifiable varifolds are defined. The chapter proceeds by defining the generalized mean curvature for varifolds which extends the notion of mean curvature for C^2-class manifolds. The integrability properties of this mean curvature are examined, ensuring monotonicity properties of the mass measure of varifolds, by stating various versions of the monotonicity formula. Chapter 2 concludes with the proof of a Sobolev-type inequality for varifolds, known as the Michael-Simon's inequality. The reference for this chapter are: [All72] and [All75] for the definition of n-varifolds and the monotonicity formulae; [DPGS24] for the inspiration of the proof of the weighted monotonicity formula; finally [Sim84] for the Michael-Simon's inequality.

Chapter 4 is centered around proving Allard's regularity theorem for n - rectifiable varifolds. The original proof, provided by Allard in Section 8 of [All72] and subsequently rewritten in Chapter 5 of Simon's book [Sim84], relies on Lipschitz approximation results for the support of varifolds with small curvature in the L^p norm, for some p larger than the dimension of the varifold's support. This approach then uses harmonic approximation to demonstrate rapid decay of the tilt-excess energy (which morally represent the Sobolev $W^{1,2}$ norm of a varifold), leading to the $C^{1,1-n/p}$ regularity for the varifold support. In contrast, we propose a more modern approach based on the ideas present in [DPGS24]. Specifically we establish the same result for varifolds with essentially bounded generalized curvature with respect to the mass measure. While it is reasonable to expect that similar techniques, or slight adaptations, could potentially prove the same regularity theorem in the more general setting of curvature in L^p, this extension is not trivial and warrants further research and efforts (see Section 4 for a more detailed commentary).

Finally, Chapter 5 applies the theory developed in the previous chapters to the Plateau problem, namely the problem of finding the minimal surface spanning a given boundary. The chapter begins with a brief introduction to the problem, followed by the rigorous definition of *surface that spans a given boundary* and a formulation of the Plateau problem in a sufficiently general context. After recalling some technical results, including the deformation theorems present in [DS00], the existence and C^∞ regularity (up to a $\mathscr{H}^n$-negligible set) for solutions of this problem are proved using the arguments presented in [DLGM17] and [DPDRG20].

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CONTENTS

1. PRELIMINARY RESULTS AND NOTATION

1.1. Basic Measure Theory

1.1.1 First Definitions

In the following (X, d) denotes a locally compact, separable metric space. For brevity we will write X in place of (X, d), when the metric d is intended.

We indicate by $\mathcal{P}(X)$ the family of subsets of X, and by $\mathcal{B}(X)$ be the Borel σ-algebra of X (i.e. the smallest σ-algebra containing the open subsets of X).

Whenever we write $(a_j)_j \subseteq P$, where P now is any set, we mean that $a_k, \dots, a_\ell, \dots$ is a countable sequence, starting from a non-negative integer $k \in \mathbb{N}$. The value of k will always be the minimum between 0 and the smallest non-negative integer for which a_k is well defined. We simply write (a_j) if the index j is intended.

Definition 1.1 (Measures). We say that a set function μ is

1. an outer measure, if $\mu : \mathcal{P}(X) \to [0, \infty]$ is increasing (i.e. $\mu(A) \leq \mu(B)$ whenever $A \subseteq B \subseteq X$), countably subadditive (i.e. $\mu(\bigcup_j A_j) \leq \sum_j \mu(A_j)$ for all $(A_j) \subseteq \mathcal{P}(X)$) and $\mu(\varnothing) = 0$;

2. a (positive) measure, if $\mu : \mathcal{B}(X) \to [0, \infty)$ is countably additive (i.e. $\mu(\bigcup_j A_j) = \sum_j \mu(A_j)$ for all $(A_j) \subseteq \mathcal{B}(X)$ pairwise disjoint) and $\mu(\varnothing) = 0$;

3. a Radon measure, if $\mu : \mathcal{B}(X) \to [0, \infty]$ and, for any compactly contained $Y \Subset X$, the restriction

   $$A \mapsto \mu_{\llcorner Y}(A) := \mu(Y \cap A)$$

   is a measure;

4. a vector-valued measure, if $\mu : \mathcal{B}(X) \to \mathbb{R}^N$ is countably additive and $\mu(\varnothing) = 0 \in \mathbb{R}^N$;

5. a vector-valued Radon measure, if $\mu : \mathcal{B}(X) \to [0, \infty]$ and, for any compactly contained $Y \Subset X$, the restriction

   $$A \mapsto \mu_{\llcorner Y}(A) := \mu(Y \cap A)$$

   is a vector-valued measure;

6. a signed-measure, if $\mu : \mathcal{B}(X) \to \mathbb{R}$ is a vector-valued measure.

For the rest of this subsection, μ will always be a vector valued Radon measure, unless we specify otherwise.

Definition 1.2 (Total Variation). The total variation of $\mu : \mathcal{B}(X) \to \mathbb{R}^N$ is the finite set function $\|\mu\| : \mathcal{B}(X) \to [0,\infty)$ defined by the position

$$\|\mu\|(B) := \sup\left\{ \sum_j |\mu(B_j)| : B = \bigcup_j B_j, (B_j) \subseteq \mathcal{B}(X) \text{ disjoint sequence} \right\}$$

We invite the reader to actually check that, if μ is vector-valued measure as in Definition 1.1, then there is no $B \in \mathcal{B}(X)$ such that $\|\mu\|(B) = \infty$.

Definition 1.3 (Set of Concentration). We say that E is the set of concentration of μ, or equivalently that μ is concentrated on E, if

$$\mu(B) = \mu(B \cap E) \quad \forall B \in \mathcal{B}(X).$$

Definition 1.4 (Support). The support of μ is defined to be

$$\text{spt}\mu := \left\{ x \in X : \|\mu\|(B_r(x)) > 0 \quad \forall r > 0 \right\}.$$

One can easily prove that sptμ is characterized by being the smallest closed subset C of X such that $\|\mu\|(X \setminus C) = 0$. Therefore spt$\mu$ is always larger then the set of concentration of μ.

1.1.2 Covering Theorems

In the present subsection we introduce the main covering theorems that will be used in the rest of the work.

The ambient space X will always be the Euclidean N-dimensional space $\mathbb{R}^N$, with N positive integer.

Theorem 1.1.1 (Besicovitch Covering Theorem). *There exists $k = k(N) \in \mathbb{N}$ with the following property. Let $\mathscr{B}$ be a collection of closde ball of $\mathbb{R}^N$ such that*

$$A := \left\{ x \in \mathbb{R}^N : \exists r > 0 \textit{ such that } \overline{B_r(x)} \in \mathscr{B} \right\}$$

is bounded. Then there are $\mathscr{B}_1, ..., \mathscr{B}_k$ subfamilies of $\mathscr{B}$ such that

(i) $\mathscr{B}_j$ *is a disjoint family for each* $j \in \{1, ..., k\}$;

(ii) $A \subseteq \bigcup_{j=1}^k \left(\bigcup \mathscr{B}_j \right)$.

The second result we need to recall is the following.

Theorem 1.1.2 (5-Covering Lemma). *Let $\mathscr{B}$ be a family of closed balls with radii not larger than a constant $R > 0$. Then there exists $\mathscr{B}' \subseteq \mathscr{B}$ disjoint and countable family such that*

$$\bigcup \mathscr{B} \subseteq \bigcup 5\mathscr{B}' := \bigcup_{B \in \mathscr{B}'} 5B.$$

In the statement of (1.1.2) the set $5B$ denotes the ball with center the center of B and radius 5 times the radius of B.

Before stating the last covering theorem, we need to introduce the notion of fine covering.

Definition 1.5 (Fine Covering). Let $\mathscr{B}$ be a family of closed balls and $A \subseteq \mathbb{R}^N$. We say that $\mathscr{B}$ is a fine covering of A if for each $x \in A$ we have

$$\inf\left\{r > 0 : \overline{B_r(x)} \in \mathscr{B}\right\} = 0.$$

Theorem 1.1.3 (Vitali Covering Theorem). *Let $A \subseteq \mathbb{R}^N$ be bounded and Borel. Suppose $\mathscr{B}$ to be a fine covering of A and let μ be a (positive) Radon measure on $\mathbb{R}^N$. Then there exists $\mathscr{B}' \subseteq \mathscr{B}$ at most countable and disjoint subfamily such that*

$$\mu(A \setminus \bigcup \mathscr{B}') = 0.$$

1.1.3 Uniform Integrability and Vitali's Theorem

Definition 1.6 (Uniform Integrability). Let (X, Σ, μ) be any measure space and let $\mathcal{F} \subseteq L^1(X, \Sigma, \mu)$. We say that $\mathcal{F}$ is uniformly integrable if

$$\lim_{M\to\infty} \sup_{f\in\mathcal{F}} \int_{\{|f|>M\}} |f|\, d\mu = 0.$$

Theorem 1.1.4 (Vitali). *Let (X, Σ, μ) be a finite measure space. Suppose that $f : X \to \mathbb{R}$ is measurable and $(f_j)_j \subseteq L^1(X, \Sigma, \mu)$. Then the following are equivalent:*

(i) $f_j \xrightarrow{\mu} f$ *and* $\{f_j : j \in \mathbb{N}\}$ *is uniformly intregrable;*

(ii) $f \in L^1(X, \Sigma, \mu)$ *and* $f_j \xrightarrow{L^1} f$.

1.1.4 Differentiation of Measures

Definition 1.7 (Absolutely Continuous Measure). Let μ be a vector valued measure and ν a positive measure on X. We say that μ is absolutely continuous with respect to ν, and write $\mu \ll \nu$, if the following implication holds for any $B \in \mathcal{B}(X)$:

$$\nu(B) = 0 \implies \mu(B) = 0$$

Definition 1.8 (Singular Measures). Let μ, ν be vector valued measures on X. Then μ and ν are (mutually) singular, and write $\mu \perp \nu$, if there exists a set $B \in \mathcal{B}(X)$ such that

$$\|\mu\|(B) = 0 \quad \text{and} \quad \|\nu\|(X \setminus B) = 0.$$

The following theorem establishes a remarkable decoposition result for measures.

Theorem 1.1.5 (Lebesgue-Radon-Nikodym). *Let $\mu : \mathcal{B}(X) \to \mathbb{R}^N$ be a vector-valued measure on X, and let $\nu : \mathcal{B}(X) \to [0,\infty)$ be a (positive) measure on X. Then there exists a function $f \in L^1(X,\nu,;\mathbb{R}^N)$ and a vector valued measure $\mu^s \perp \nu$ such that*

$$d\mu = f\,d\nu + d\mu^s,$$

meaning that

$$\int_X \varphi\,d\mu = \int_X \varphi f\,d\nu + \int_X \varphi\,d\mu^s$$

for every μ-integrable function $\varphi : X \to \mathbb{R}$.

Definition 1.9 (Lebesgue-Radon-Nikodym Derivative). The function f in the statement of Theorem 1.1.5 is called Lebesgue-Radon-Nikodym derivative of μ with respect to ν, and will be sometimes denoted by one of the following

$$f = \frac{d\mu}{d\nu} = D_\nu \mu$$

As a corollary we give a construction for the Lebesgue-Radon-Nikodym derivatives of measures in a slightly more general setting.

Theorem 1.1.6. *Let ν be a (positve) Radon measure on X, and let μ be a vector valued measure on X. If $\mu \ll \nu$, then for ν-a.e. $x \in X$ we have that*

$$D_\nu \mu(x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))} \in \mathbb{R}^N.$$

Observe that Theorem 1.1.5 implies that $\mu \ll \nu$ if and only if there exists a ν-integrable function f such that $d\mu = f\,d\nu$. In particular, recalling that for any vector-valued measure μ we have $\mu \ll \|\mu\|$, then

$$d\mu = f\,d\|\mu\|. \tag{1.1}$$

Furthermore it is not difficult to prove, using some basic properties of the total variation of a measure, that the function f in (1.1) takes values in $\mathbb{S}^{N-1} := \partial B_1$ $\|\mu\|$-a.e..

Corollary 1.1.1 (Lebesgue Differentiation Theorem). *Let ν be a positive measure on an open subset $\Omega \subseteq \mathbb{R}^N$. If $f \in L^p(\Omega,\nu)$, then*

$$\lim_{r\to 0} \fint_{B_r(x)} |f - f(x)|^p\,d\nu = 0$$

for ν-a.e. $x \in \Omega$.

1.1.5 Lebesgue and Hausdorff Measures

Definition 1.10 (Lebesgue Measure). In $\mathbb{R}^N$ we define the Lebesgue measure as the unique outer measure $\mathscr{L}^N : \mathcal{P}(\mathbb{R}^N) \to [0,\infty]$ such that

$$\mathscr{L}^n([a_1,b_1) \times \ldots \times [a_N,b_N)) = \prod_{j=1}^{N} (b_j - a_j).$$

We refer to [Fol99] for the well-posedness of the above definition, as well as the construction of the Lebesgue measure via Caratheodory's theorem. Furthermore, in the same reference, one is able to find proof for the fact that the restiction of $\mathscr{L}^N$ to the Borel σ-algebra defines a Radon measure.

Fix two parameters $s > 0$ and $\delta > 0$. Then we define the Hausdorff (s,δ)-premeasure to be

$$\mathscr{H}^s_\delta(A) := \frac{\omega_s}{2^s}\inf\left\{\sum_j (\operatorname{diam}(A_j))^s : A \subseteq \bigcup_j A_j, \operatorname{diam}(A_j) < \delta\right\},$$

where

$$\omega_s := \frac{\pi^{s/2}}{\Gamma(1+s/2)}$$

and Γ is the Euler's Gamma function.

Definition 1.11 (Hausdorff s-Measure). We define the Hausdorff s-Measure on $\mathbb{R}^N$ as the outer measure $\mathscr{H}^s : \mathcal{P}(\mathbb{R}^N) \to [0,\infty]$ by the position

$$\mathscr{H}^s(A) := \lim_{\delta\to 0}\mathscr{H}^s_\delta(A).$$

Observe that $\mathscr{H}^s(A)$ exists in $[0,\infty]$ for any $A \in \mathcal{P}(\mathbb{R}^N)$ by monotonicity of the map $\delta \mapsto \mathscr{H}^s_\delta(A)$.

Moreover one can prove the following well-known results about the Lebesgue measure $\mathscr{L}^N$ and the Hausdorff s-measure $\mathscr{H}^s$:

1. both $\mathscr{L}^N$ and $\mathscr{H}^s$ are invariant under translations and under isometries. Moreover, for every $\lambda > 0$ and every $A \in \mathcal{P}(\mathbb{R}^N)$ we have

   $$\mathscr{L}^N(\lambda A) = \lambda^N \mathscr{L}^N(A) \quad \text{and} \quad \mathscr{H}^s(\lambda A) = \lambda^s \mathscr{H}^s(A).$$

   The aforementioned properties, together with some further arguments involving the isodiametric inequality prove that $\mathscr{H}^N = \mathscr{L}^N$ (a detalied argument is given in [Fal85]). Therefore we can continue by stating properties only regarding $\mathscr{H}^s$.

2. $\mathscr{H}^s$ are Borel-regular measures, meaning that

   $$\mathscr{H}^s(A) = \inf\left\{\mathscr{H}^s(B) : B \in \mathcal{B}(\mathbb{R}^N), A \subseteq B\right\} \quad \forall A \in \mathcal{P}(\mathbb{R}^N).$$

   By virtue of this, together with the observation made right after the definition of Lebesgue measure, from now on we say interchangeably (if the context is clear), with a small abuse of notation, that a Borel-regular measure μ is an outer measure, a Radon measure or a measure.

3. $\mathscr{H}^0$ is the counting measure.

4. if $s_1 < s_2$ then the following implications hold for every $A \in \mathcal{P}(\mathbb{R}^N)$:

   $$\mathscr{H}^{s_1}(A) < \infty \implies \mathscr{H}^{s_2}(A) = 0 \quad \text{and} \quad \mathscr{H}^{s_2}(A) > 0 \implies \mathscr{H}^{s_1}(A) = 0.$$

Item 4. of the above list motivates the following

Definition 1.12 (Hausdorff Dimension). The Hausdorff dimension of a set $A \in \mathcal{P}(\mathbb{R}^N)$ is

$$\dim_{\mathscr{H}}(A) := \inf\left\{s > 0 : \mathscr{H}^s(A) = 0\right\} = \sup\left\{s > 0 : \mathscr{H}^s(A) = \infty\right\}.$$

In what follows we will only consider Hausdorff l-measures with $l \in \mathbb{N}$.

1.1.6 Densities

All of the [outer/Radon] measures considered from now on, will be Borel-regular, hence, according to the convention established in the previous subsection, we will call them all *measures* if no confusion is likely to arise.

Definition 1.13 (Lower and Upper l-Densities). Let μ be any measure in $\mathbb{R}^N$, and fix $\ell \in \mathbb{N}$. We define the lower and upper ℓ-densities of μ at a point $x \in \mathbb{R}^N$ as the quantities

$$\Theta^\ell_*(\mu, x) := \liminf_{r\to 0} \frac{\mu(B_r(x))}{\omega_\ell r^\ell} \quad \text{and} \quad \Theta^{\ell*}(\mu, x) := \limsup_{r\to 0} \frac{\mu(B_r(x))}{\omega_\ell r^\ell}$$

respectively. Whenever the lower and upper ℓ-density at a point x of μ coincide, we say that μ has l-density at x and set $\Theta^\ell(\mu, x)$ to be the common value. More in general, we define the [upper/lower] (μ, ℓ)-density of a set $A \in \mathcal{P}(\mathbb{R}^N)$ at a point x, ad write

$$\Theta^\ell_*(\mu, A, x) \quad \text{and} \quad \Theta^{\ell*}(\mu, A, x)$$

respectively, as [upper/lower] ℓ-density of the measure $\mu_{\llcorner A}$, where we recall that $\mu_{\llcorner A}$ is defined by

$$\mu_{\llcorner A}(B) := \mu(A \cap B) \quad \forall B \in \mathcal{P}(\mathbb{R}^N).$$

Theorem 1.1.7 (Upper Density Theorem). *If μ is a measure and A is μ-measurable set with $\mu(A) < \infty$, then*

$$\Theta^{\ell*}(\mu, A, x) = 0 \quad \mathscr{H}^\ell - a.e.\, x \in \mathbb{R}^N \setminus A.$$

Easy consequences of the above theorem are

Corollary 1.1.2 (Lebesgue Density Theorem). *If $A \in \mathcal{P}(\mathbb{R}^N)$ is $\mathscr{L}^N$-measurable, then the density $\Theta^N(\mathscr{L}^N, A, x)$ exists $\mathscr{L}^N$-a.e. on $\mathbb{R}^N$, and*

$$\Theta^N(\mathscr{L}^N, A, x) = \begin{cases} 0 & \mathscr{L}^N - a.e.\, x \in \mathbb{R}^N \setminus A \\ 1 & \mathscr{L}^N - a.e.\, x \in A \end{cases}.$$

and

Corollary 1.1.3. *Let μ be a Radon measure on $\mathbb{R}^N$. We set*

$$\mu_* := \mu_{\llcorner M_*},$$

where M^* denote the set of positive lower ℓ-density for μ, namely

$$M_* := \left\{x \in \mathbb{R}^N : \Theta_*(\mu, x) > 0\right\}.$$

Then

$$\lim_{r \to 0} \frac{\mu_*(B_r(x))}{\mu(B_r(x))} = 1 \quad \mu_* - a.e.\, x.$$

In particular, $\Theta_*^\ell(\mu_*, x) > 0$ for μ_*-a.e. x.

1.2. Miscellanous from Functional Analysis

In this section, Ω will always be an open subset of $\mathbb{R}^N$.

1.2.1 Lipschitz Function

Definition 1.14 (Lipscitz Function). A function $f : \Omega \to \mathbb{R}^\ell$ is said to be a Lipschitz function if there exists a constant $L \in [0, \infty)$ such that

$$|f(x) - f(y)| \leq L|x - y| \quad \forall x, y \in \Omega. \tag{1.2}$$

We collect of of the Lipscitz function defined on Ω with values in $\mathbb{R}^\ell$ in the family $\mathscr{L}\mathrm{ip}(\Omega; \mathbb{R}^\ell)$ (or simply $\mathscr{L}\mathrm{ip}(\Omega)$, if $\ell = 1$). Finally we set $\mathrm{Lip}(f)$ to be the smallest constant L such that (1.2) holds.

Observe that if $f \in \mathscr{L}\mathrm{ip}(\Omega; \mathbb{R}^\ell)$, then we can always write $f = (f_1, \ldots, f_\ell)$, with $f : \Omega \to \mathbb{R}$, and one can readily prove that $f_j \in \mathscr{L}\mathrm{ip}(\Omega; \mathbb{R})$ for all $j \in \{1, \ldots, \ell\}$. Therefore we will only present theorems for functions taking values in $\mathbb{R}$.

We first have the following basic extension theorem:

Theorem 1.2.1. *Let E be any nonempty subset of $\mathbb{R}^N$, and let $f \in \mathscr{L}\mathrm{ip}(E)$. Then there exists a function $\widetilde{f} \in \mathscr{L}\mathrm{ip}(\mathbb{R}^N)$ such*

$$\mathrm{Lip}(\widetilde{f}) = \mathrm{Lip}(f) \quad \text{and} \quad \widetilde{f}\big|_E = f.$$

Next we need the theorem of Rademacher concerning differentiability of Lipschitz functions on $\mathbb{R}^N$.

Theorem 1.2.2 (Rademacher's Theorem). *If $f \in \mathscr{L}\mathrm{ip}(\mathbb{R}^N)$, then f is differentiable $\mathscr{L}^N$-a.e.; that is, the gradient $\nabla f(x) = (\partial_1 f(x), \ldots, \partial_N f(x)) \in \mathbb{R}^N$ exists and*

$$\lim_{y \to x} \frac{f(y) - f(x) - \nabla f(x) \cdot (y - x)}{|y - x|} = 0$$

for $\mathscr{L}^N$-a.e. $x \in \mathbb{R}^N$.

We shall also need the following C^1 approximation theorem for Lipschitz functions in our discussion of n-rectifiable sets

Theorem 1.2.3 (**C^1 Approximation Theorem**). *Suppose $f \in \mathscr{L}\mathrm{ip}(\mathbb{R}^N)$. Then, for each $\varepsilon > 0$ there exists a function $g_\varepsilon \in C^1(\mathbb{R}^N)$ such that*

$$\mathscr{L}^N(\{x : f(x) \neq g(x)\} \cup \{x : \nabla f(x) \neq \nabla g(x)\}) < \varepsilon.$$

Finally we establish some basic facts about Hausdorff measures of Lipschitz images.

Theorem 1.2.4. *Let $A \subseteq \mathbb{R}^N$ be $\mathscr{H}^\ell$-measurable and assume $\mathscr{H}^\ell(A) < \infty$. Fix $f \in \mathscr{L}\mathrm{ip}(A, \mathbb{R}^m)$, and let $N(f, y) := \mathscr{H}^0(f^{-1}(\{y\}))$. Then*

(i) $f(A) \subseteq \mathbb{R}^m$ is $\mathscr{H}^\ell$ measurable, and

$$\mathscr{H}^\ell(f(A)) \leq \mathrm{Lip}(f)^\ell \mathscr{H}^\ell(A);$$

(ii) $y \mapsto N(f, y)$ is $\mathscr{H}^\ell$-measurable function (i.e. the preimages of Borel subsets of $\mathbb{R}$ via $N(f, \cdot)$ are $\mathscr{H}^\ell$-measurable sets) and

$$\int_{\mathbb{R}^m} N(f, y)\, d\mathscr{H}^\ell(y) \leq \mathrm{Lip}(f)^\ell \mathscr{H}^\ell(A).$$

1.2.2 Riesz Representation Theorem

The following theorem will be needed for the definition of generalized mean curvature for n-varidolds in Chapter 2. In particular it gives a representation for the dual of space of test functions $(C_c^0, \|\cdot\|_{C^0})$. For sake of readability we are stating the theorem only in the particular case where the ambient space is Euclidean; however the same result holds in much more general enviroments (cfr. [BB11])

Theorem 1.2.5 (Riesz Representation Theorem). *Let $\Omega \subseteq \mathbb{R}^N$ be open and suppose $\Psi : C_c^0(\Omega; \mathbb{R}^l) \to \mathbb{R}$ to be a linear functional such that*

$$\sup_{\substack{\varphi \in C_c^0(K) \\ \|\varphi\|_{C^0} = 1}} \langle \Psi, \varphi \rangle < \infty \quad \forall K \subseteq X \text{ compact.}$$

Then there exists a Radon measure μ_Ψ on X, called representation of Ψ, and a μ-measurable function $h : \Omega \to \mathbb{R}^l$ with $|h| = 1$, such that

$$\langle \Psi, \varphi \rangle = \int_\Omega \varphi \cdot h\, d\mu_\Psi \quad \forall \varphi \in C_c^0(\Omega).$$

1.2.3 Compactness for Radon Measures

Let us denote by $\mathrm{Rad}(\Omega)$ the set containing all of the positive Radon measures on a fixed open subset $\Omega \subseteq \mathbb{R}^N$. Recalling Theorem 1.2.5, we recover a natural topology for the set $\mathrm{Rad}(\Omega)$, which is given by the following definition.

Definition 1.15 (Weak-* Convergence for Measures). We say that a sequence of Radon measure $(\mu_j) \subseteq \mathrm{Rad}(\Omega)$ converges weakly-* to $\mu \in \mathrm{Rad}(\Omega)$, and write $\mu_j \rightharpoonup^* \mu$, if

$$\lim_{j\to\infty} \int_\Omega \varphi \, d\mu_j = \int_\Omega \varphi \, d\mu \quad \forall \varphi \in C^0_c(\Omega).$$

Observe that, if μ and μ_j are the positive Radon measures associated with the positive positive functionals Ψ and Ψ_j respectively by Theorem 1.2.5, then $\mu_j \rightharpoonup^* \mu$ in the sense of Definition 1.15 if and only if $\Psi_j \rightharpoonup^* \Psi$ in the classical sense for distributions.

The following result is therefore not surprising.

Theorem 1.2.6 (Compactness). *Let $(\mu_j) \subseteq \mathrm{Rad}(\Omega)$ be a sequence with the following property:*

$$\sup_j \mu_j(K) < \infty \quad \forall K \Subset \Omega.$$

Then there is $\mu \in \mathrm{Rad}(\Omega)$ and a subsequence $(\mu_{j_l})_l$ such that $\mu_{j_l} \rightharpoonup^* \mu$.

1.3. Submanifolds

Let $n, k \in \mathbb{N}$ be positive integer. From now on the dimension of the (Euclidean) ambient space will always be $N = n + k$.

1.3.1 Definition and Tangent Spaces

Definition 1.16 (C^r Submanifold). Let $0 < r \in \mathbb{N} \cup \{\infty\}$. We say that M is a n-dimensional C^r submanifold of $\mathbb{R}^{n+k}$ if for each $x \in M$ there are open sets $V \subseteq \mathbb{R}^n, W \subseteq \mathbb{R}^{n+k}$, with $x \in W$, and a bijective proper map $\psi \in C^r(V; W)$ such that

$$\psi(V) = W \cap M.$$

We recall that ψ is proper, by definition, if and only if for each $K \subseteq W$ compact, $\psi^{-1}(K) \subseteq V$ is compact. This requirement on ψ allows us to exclude some pathological examples: for instance

$$M := (\{0\} \times (-1,1)) \cup \{(t, \sin(1/t)) : t > 0\} \subseteq \mathbb{R}^2$$

is not a submanifold of $\mathbb{R}^2$ for any $0 < r \in \mathbb{N} \cup \{\infty\}$.

Moreover, using the inverse function theorem, one proves that every n-dimensional C^r submanifold of $\mathbb{R}^{n+k}$ can be written, up to rotations, as the graph of a C^r function $u : V \subseteq \mathbb{R}^n \to \mathbb{R}^k$.

Definition 1.17 (Tangent Space). Let M be a n-dimensional C^r submanifold of $\mathbb{R}^{n+k}$ and fix a point $x \in M$. The tangent space $T_x M$ of M at x is the collection of all vectors $v \in \mathbb{R}^{n+k}$ such that there exists a C^1 curve $\gamma : (-1,1) \to M$ with the following properties:

$$\gamma(0) = x \quad \text{and} \quad \gamma'(0) = v.$$

One can readily check that T_xM is a n-dimensional vector space (or simply an n-plane) with basis

$$\left\{\partial_1\psi(y),...,\partial_n\psi(y)\right\},$$

where ψ is as in Definition 1.16 and $\psi(y) = x$.

We now discuss some differentiability properties on submanifolds.

Definition 1.18 (C^m Functions). We say that a function $f : M \to \mathbb{R}^\ell$ is of class C^m, and write $f \in C^m(M;\mathbb{R}^\ell)$ (or simply $f \in \mathscr{C}^m(M)$, if $\ell = 1$), if f is the restriction to M of a function $\widetilde{f} \in C^m(U;\mathbb{R}^\ell)$, where U is an open subset of $\mathbb{R}^{n+k}$ containing M.

For a given $v \in T_xM$, we define the directional derivative of f at x as

$$\partial_v f(x) := \frac{d}{dt} f(\gamma(t))\big|_{t=0} \tag{1.3}$$

for any C^1 curve $\gamma : (-1,1) \to M$ with $\gamma(0) = x$ and $\gamma'(0) = v$. Moreover one can easily check that the definition (1.3) does not depend on the choice of the curve γ, and that the position

$$v \mapsto \partial_v f(x) \tag{1.4}$$

defines a linear map on T_xM.

1.3.2 Area and Coarea Formulae

Definition 1.19 (Locally Lipschitz Function). We that a function $f : M \to \mathbb{R}^\ell$ is locally Lipschitz, and write $f \in \mathscr{L}\mathrm{ip}_{\mathrm{loc}}(M;\mathbb{R}^\ell)$ (or simply $f \in \mathscr{L}\mathrm{ip}_{\mathrm{loc}}(M)$, if $\ell = 1$), if the following property holds:

$$\forall x \in M\, \exists \rho > 0, L > 0 \text{ such that } |f(y) - f(z)| \le L|y - z| \quad \forall y,z \in B_\rho(x)$$

We define the derivative of f in direction $v \in T_xM$ as in (1.3), provided the right hand-side is well defined. We claim that in fact there exists a set E oh $\mathscr{H}^n$-measure 0 such that $\partial_v f(x)$ exists for every $x \in M\setminus E$ and for each $v \in T_xM$. The proof indeed follows by Rademacher's theorem (Theorem 1.2.2). Letting ψ be as in Definition 1.16, one considers the Lipschitz map $f \circ \psi \in \mathrm{Lip}(V;M)$; hence there exists a $\mathscr{H}^n$-negligible set $E_0 \subseteq V$ such that $f\circ\psi$ is differentiable at every point of $V\setminus E_0$. The rest of the proof is a simple exercise left for the reader.

Definition 1.20 (n-Jacobian). Let $f \in \mathscr{L}\mathrm{ip}_{\mathrm{loc}}(M;\mathbb{R}^\ell)$. We define the n-Jacobian of f at a point $x \in M$ as

$$J_M f(x) := \sqrt{\det(\nabla^*_M f(x) \circ \nabla_M f(x))},$$

where we set, for $\mathscr{H}^n$-a.e. $x \in M$,

$$\nabla_M f(x) := \sum_{j=1}^n \partial_{\tau_j} f(x)\tau_j, \tag{1.4}$$

$$\nabla_M * f(x) \circ \nabla_M f(x) := (\partial_{\tau_j} f(x) \cdot \partial_{\tau_m} f(x))_{1\le j,m\le n} \in \mathbb{R}^n \otimes \mathbb{R}^n,$$

and $\{\tau_1,...,\tau_n\}$ forms an orthonormal basis for T_xM.

With a small abuse of notation, we use the same symbol $\nabla_M f(x)$ to denote the linear function defined by (1.4), namely we write $\nabla_M f(x) : T_x M \to \mathbb{R}^\ell$ and

$$\nabla_M f(x)\tau := \partial_\tau f(x)$$

In the case $\ell \geq n$ we conclude that the area formula holds for f. More precisely we have the following theorem.

Theorem 1.3.1 (Area Formula). *Let M be a n-dimensional C^r submanifold of $\mathbb{R}^{n+k}$. Fix $f \in \mathscr{L}\mathrm{ip}_{\mathrm{loc}}(M,\mathbb{R}^\ell)$ for some $\ell \geq n$. Then*

(i) if f is bijective and $A \subseteq M$ is $\mathscr{H}^n$-measurable, $f(A) \subseteq \mathbb{R}^\ell$ is $\mathscr{H}^n$ measurable, and

$$\mathscr{H}^n(f(A)) = \int_A J_M f \, d\mathscr{H}^n;$$

(ii) if f is bijective and $\varphi : M \to [0,\infty)$ is any $\mathscr{H}^n$-measurable function, we have

$$\int_{f(M)} \varphi \circ f^{-1} \, d\mathscr{H}^n = \int_M \varphi J_M f \, d\mathscr{H}^n;$$

(iii) if f is not assumed to be bijective, and we set $N(A, f, y) := \mathscr{H}^0(f^{-1}(\{y\}) \cap A)$, then $y \mapsto N(A, f, y)$ is $\mathscr{H}^n$-measurable and

$$\int_{f(A)} N(A, f, y) \, d\mathscr{H}^n(y) = \int_A J_M f \, d\mathscr{H}^n.$$

We conclude this subsection with the coarea formula. Let us first introduce the coarea factor of a function.

Definition 1.21 (Coarea Factor). Let M be a n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$ and let $f \in \mathscr{L}\mathrm{ip}_{\mathrm{loc}}(M;\mathbb{R}^\ell)$, for some $\ell < n$. Then the coarea factor of f at a point $x \in M$ is

$$J_M^* f(x) := \sqrt{\det(\nabla_M f(x) \circ \nabla_M^* f(x))}, \tag{1.5}$$

whenever the right hand-side is well defined. In (1.5), $\nabla f^*(x)$ denotes the transposed linear map of $\nabla f(x) : T_x M \to \mathbb{R}^\ell$.

Theorem 1.3.2 (Coarea Formula). *Let M be a n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$ and let $f \in \mathscr{L}\mathrm{ip}_{\mathrm{loc}}(M;\mathbb{R}^\ell)$, for some $\ell < n$. Then, for any Borel set $A \subseteq M$*

$$\int_A J_M^* f \, d\mathscr{H}^n = \int_{\mathbb{R}^\ell} \mathscr{H}^{n-m}(A \cap f^{-1}(\{y\})) \, d\mathscr{L}^\ell(y).$$

1.3.3 Divergence Theorem and Mean Curvature

In the following, a vector field on M is any function of the form $X = (X_1, ..., X_{n+k}) : M \to \mathbb{R}^{n+k}$. A vector field X on M is tangent to M if $X(x) \in T_x M$ for each $x \in M$. Finally, we say that a vector field X is of class C^m if X is of class C^m in the sense of Definition (1.18).

Definition 1.22 (Divergence). Let X be a vector field on M of class C^1. We define the divergence of X on M as

$$\operatorname{div}_M X := \sum_{j=1}^{n+k} \nabla_M X_j \cdot e_j,$$

with $\{e_1, ..., e_{n+k}\}$ being the canonical basis of $\mathbb{R}^{n+k}$.

Theorem 1.3.3 (Divergence Theorem). *Let M be a n-dimensional C^r submanifold of $\mathbb{R}^{n+k}$, and assume that the clousure $\overline{M}$ of M is a smooth compact submanifold with boundary $\partial M := \overline{M} \setminus M$. If X is a C^1 vector filed on M and X is tangent to M, then*

$$\int_M \operatorname{div}_M X \, d\mathscr{H}^n = -\int_{\partial M} X \cdot \eta \, d\mathscr{H}^{n-1},$$

where $\eta : \partial M \to \mathbb{R}^{n+k}$ is the inward pointing unit co-normal of ∂M; that is $|\eta| = 1$, η is normal to ∂M, tangent to M, and points into M at each point of ∂M.

We remark that M needs not to be orientable in Theorem 1.3.3. Moreover, in general, the clousure $\overline{M}$ of M will not be a nice manifold with bounday; indeed it can certainly happen that $\mathscr{H}^n(\partial M) > 0$. For example consider the case

$$M := \left\{ (t, \sin(1/t)) \in \mathbb{R}^2 : t > 0 \right\} \subseteq \mathbb{R}^2.$$

Then M is a 1-dimensional C^∞ submanifold of $\mathbb{R}^2$ in the sense of Definition 1.16, but ∂M is the interval $\{0\} \times [-1, 1]$. Nevertheless in the general case we still have the following.

Corollary 1.3.1. *Let M be a n-dimensional C^r submanifold of $\mathbb{R}^{n+k}$. If X is a C^1 vector filed on M, X is tangent to M and $\operatorname{spt} X \cap M \Subset M$, then*

$$\int_M \operatorname{div}_M X \, d\mathscr{H}^n = 0.$$

We now provide a construction for the mean curvature of a (at least C^2) submanifold of $\mathbb{R}^{n+k}$, and a generalization of Theorem 1.3.3 in the general case of X C^1 vector field on M (not necessarily tangent).

Definition 1.23 (Second Fundamental Form). Let M be a n-dimensional C^2 submanifold of $\mathbb{R}^{n+k}$. We define the second fundamental form of M at a point $x \in M$ to be the bilinear form

$$B_x : T_x M \times T_x M \to (T_x M)^\perp$$

such that

$$B_x(\tau, \eta) := -\sum_{j=1}^{k} (\eta \cdot \partial_\tau \nu_j(x)) \nu_j(x) \quad \forall \tau, \eta \in T_x M,$$

where $\nu_1, ..., \nu_k$ are (locally defined, near x) vector field such that $\{\nu_1(y), ..., \nu_k(y)\}$ forms an orthonormal basis for $(T_x M)^\perp$ for every y in some neighborhood of x.

Definition 1.24 (Mean Curvature). With the same notation of the above definition, we define the mean curvature of M as the vector field

$$\underline{H}(x) := \mathrm{tr}B_x = \sum_{j=1}^{n} B_x(\tau_j, \tau_j),$$

where $\{\tau_1, ..., \tau_n\}$ is any orthonormal basis for $T_x M$.

Observe that, if $\nu_1, ..., \nu_k$ are vector fields as in Definition 1.23, then

$$\underline{H}(x) = -\sum_{j=1}^{k}\sum_{l=1}^{n}(\tau_l \cdot \partial_{\tau_l}\nu_j(x))\nu_j(y) = -\sum_{j=1}^{k}(\mathrm{div}_M \nu_j)(x)\nu_j(x). \tag{1.6}$$

Let now X be any C^1 vector field on M and write

$$X = X^T + X^\perp,$$

with X^T tangent to M, and $X^\perp(x) \in (T_x M)^\perp$ at all $x \in M$. Then, by virtue of (1.6), near x we have

$$\mathrm{div}_M X^\perp = \sum_{j=1}^{k}(\nu_j \cdot X)\mathrm{div}_M \nu_j = -X \cdot \underline{H}.$$

Therefore the formula

$$\mathrm{div}_M X = \mathrm{div}_M X^T - X \cdot \underline{H}$$

holds at each point of M. Together with 1.3.3, this proves the following more general result.

Theorem 1.3.4. *Let M be a n-dimensional C^r submanifold of $\mathbb{R}^{n+k}$, and assume that the clousure $\overline{M}$ of M is a smooth compact submanifold with boundary $\partial M := \overline{M}\setminus M$. If X is a C^1 vector filed on M, then*

$$\int_M \mathrm{div}_M X \, d\mathcal{H}^n = -\int_M X \cdot \underline{H} \, d\mathcal{H}^n - \int_{\partial M} X \cdot \eta \, d\mathcal{H}^{n-1},$$

where $\eta : \partial M \to \mathbb{R}^{n+k}$ is the inward pointing unit co-normal of ∂M.

1.4. n-Rectifiable Sets and Measures

1.4.1 Tangent Properties

Definition 1.25 (n-Rectifiable Sets). A set $M \subseteq \mathbb{R}^{n+k}$ is said to be n-rectifiable if there exists a $\mathcal{H}^n$-negligible set M_0 and there are functions $F_j \in \mathcal{L}\mathrm{ip}(A_j \subseteq \mathbb{R}^n; \mathbb{R}^{n+k})$ for $j \in \mathbb{N}\setminus\{0\}$ such that

$$M = M_0 \cup \left(\bigcup_{j=1}^{\infty} F_j(A_j)\right)$$

By virtue of the C^1 approximation theorem (Theorem 1.2.3) one easily proves the following proposition.

Proposition 1.4.1. *Let $M \subseteq \mathbb{R}^{n+k}$ be fixed. The following are equivalent:*

(i) M is n-rectifiable;

(ii) $M = \bigcup_{j=0}^{\infty} N_j$, where the union is disjoint, N_0 is $\mathscr{H}^n$-negligible and each N_j, $j \geq 1$, is a n-dimensional C^1 submanifold of $\mathbb{R}^{n+k}$.

We now want to give an important caracterization of n-rectifiable sets in terms of approximate tangent space, which we first define:

Definition 1.26 (Approximate Tangent Space). If $M \subseteq \mathbb{R}^{n+k}$ is a $\mathscr{H}^n$-measurable set and let $\vartheta \in L^1(M, \mathscr{H}^n_{\llcorner M})$. For each $x \in \mathbb{R}^{n+k}$, we say that a n-plane $T \subseteq \mathbb{R}^{n+k}$ is the approximate tangent space of M with respect to ϑ it

$$\lim_{r \to 0} \int_{\varphi_{r,x}(M)} f(y)\vartheta(x + ry)\, d\mathscr{H}^n(y) = \vartheta(x) \int_T f\, d\mathscr{H}^n \quad \forall f \in C_c^0(\mathbb{R}^{n+k}),$$

where $\varphi_{r,x} : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is the function

$$\varphi_{r,x}(y) := \frac{y - x}{r} \quad \forall x, y \in \mathbb{R}^{n+k}, r > 0$$

Of course the approximate tangent space is unique, if it exists; therefore we shall denote it by $T_x M$. Moreover, from Proposition 1.4.1, one deduces that, if $M = \bigcup_j N_j$, with N_0 $\mathscr{H}^n$-negligible and N_j C^1 submanifold, then

$$T_x M = \vartheta(x) T_x N_j \quad \mathscr{H}^n - a.e.\, x \in N_j, \quad j \geq 1,$$

where $T_x M$ denotes the approximate tangent space of M, while $T_x N_j$ is the tangent space as in Definition 1.17.

We are now ready to give the aforementioned characterization.

Theorem 1.4.1. *Let $M \subseteq \mathbb{R}^{n+k}$ be $\mathscr{H}^n$-measurable and $\vartheta \in L^1(M, \mathscr{H}^n_{\llcorner M})$. The following are equivalent:*

(i) M is n-rectifiable;

(ii) the approximate tangent space $T_x M$ of M at x with respect to ϑ exists for $\mathscr{H}^n$-a.e. $x \in M$.

1.4.2 Area and Coarea

Throughout this subsection M is supposed to be n-rectifiable, so that we can express M as the disjoint union $\bigcup_j M_j$ as in item (ii) of Proposition 1.4.1.

Definition 1.27 (Gradient on M). Let $f \in \mathscr{L}\mathrm{ip}_{\mathrm{loc}}(U)$, where $U \subseteq \mathbb{R}^{n+k}$ is an open set containing M. Then define the gradient of f at $\mathscr{H}^n$-a.e. $y \in M$ by

$$\nabla_M f(x) := \nabla_{M_j} f(x) \quad \text{if } y \in M_j,$$

where $\nabla_{M_j} f(x)$ is as in Definition 1.4.

As for the case of submanifolds, with a small abuse of notation we use the same symbol $\nabla_M f(x)$ to the note the linar form $\nabla_M f(x) : T_x M \to \mathbb{R}$

$$\nabla_M f(x)\tau := \partial_\tau f(x) \quad \tau \in T_x M_j.$$

Definition 1.28 (M-Jacobian). If $f = (f_1, ..., f_\ell)$ with f_j locally Lipschitz for all $1 \leq j \leq \ell$, then we define the M-Jacobian (or simply Jacobian, when M is intended) of f at $x \in M$ as

$$J_M f(x) := \sqrt{\det(\nabla_M^* f(x) \otimes \nabla_M f(x))}$$

(cfr. Definition 1.20 for the notation).

In view of the area formula for submanifolds (Theorem 1.3.1) and Proposition 1.4.1, we deduce the more general result:

Theorem 1.4.2 (Area Formula for n-Rectifiable Sets). *Let $M \subseteq \mathbb{R}^{n+k}$ be n-rectifiable and $f \in \mathrm{Lip}_{\mathrm{loc}}(M; \mathbb{R}^\ell)$. Then*

$$\int_A \phi J_M f \, d\mathcal{H}^n = \int_{\mathbb{R}^{n+k}} \phi \circ f \mathcal{H}^0(A \cap f^{-1}(y)) \, d\mathcal{H}^n(y)$$

holds for every $\mathcal{H}^n$-measurable set $A \subseteq M$ and every $\mathcal{H}^n$-measurable function $\phi : M \to [0, \infty)$.

Finally, as in the smooth case we define the coarea factor of a function $f \in \mathrm{Lip}_{\mathrm{loc}}(M; \mathbb{R}^\ell)$, where here $\ell < n$, as

$$J_M^* f(x) := \sqrt{\det(\nabla_M f(x) \circ \nabla_M^* f(x))},$$

so that we can state

Theorem 1.4.3 (Coarea Formula for n-Rectifiable Sets). *Let $M \subseteq \mathbb{R}^{n+k}$ be a n-rectifiable set and let $f \in \mathscr{L}\mathrm{ip}_{\mathrm{loc}}(M; \mathbb{R}^\ell)$, for some $\ell < n$. Then, for any Borel set $A \subseteq M$*

$$\int_A J_M^* f \, d\mathcal{H}^n = \int_{\mathbb{R}^\ell} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\})) \, d\mathcal{L}^\ell(y).$$

2. VARIFOLDS

2.1. General and Rectifiable Varifolds

In this section we introduce the notation and terminology for varifolds that we will use throughout the whole work.

First, let

$$G(n+k,n) := \{S \subseteq \mathbb{R}^{n+k} : S \text{ is a linear subspace of dimension } n\}.$$

We then define the metric $\rho : G(n+k,n) \times G(n+k,n) \to \mathbb{R}_0^+$ defined by

$$\rho(S,T) := |\Pi_S - \Pi_T| := \sqrt{\sum_{i,j=1}^{n+k} (\Pi_S^{i,k} - \Pi_T^{i,j})^2},$$

where, for each $R \in G(n+k,n)$, $\Pi_R : \mathbb{R}^{n+k} \to R$ is the orthogonal projections to R and $(\Pi_R^{i,j})_{i,j=1}^{n+k} \in \mathbb{R}^{n+k} \otimes \mathbb{R}^{n+k}$ is its matrix in the canonical base of $\mathbb{R}^{n+k}$. One may easily prove the compactness of $(G(n+k,n),\rho)$.

Given $A \subseteq \mathbb{R}^{n+k}$ we define

$$G_n(A) := A \times G(n+k,n)$$

and we proceed to endow it with the product metric of the restriction of the Eucledian with the above defined ρ.

Definition 2.1 (Varifold). Given $\Omega \subseteq \mathbb{R}^{n+k}$, an n-varifold in Ω is any Radon measure V defined on $G_n(\Omega)$. When we say that V is an n-varifold without specifying the set Ω we intend that V is a Radon measure on $G_n(\mathbb{R}^{n+k})$. We collect all of the n-varifolds on Ω in the set $\mathbb{V}_n(\Omega)$.

Denoting by $\pi : G_n(\Omega) \to \Omega$ the canonical projection $\pi(y,S) := y$, given $V \in \mathbb{V}_n(\Omega)$, there is a naturally associated Radon measure $\mu_V \in \mathrm{Rad}(\Omega)$ defined by

$$\mu_V(B) := V(\pi^{-1}(B)), \quad \forall B \in \mathcal{B}(\Omega).$$

The measure μ_V is called *mass measure of the varifold V* and the total mass of V is

$$\underline{\mathscr{M}}(V) := \mu_V(\Omega).$$

Given a Borel subset $B \subseteq \Omega$ and a $V \in \mathbb{V}_n(\Omega)$ it is always possible define another n-varifold by restriction on $G_n(B)$ in the following way

$$V_{\llcorner G_n(B)}(A) := V(A \cap G_n(B)), \quad \forall A \in \mathcal{B}(G_n(\Omega)). \tag{2.1}$$

The measure on the left hand side of (2.1) is called restriction varifold of V to B.

The following lemma should help the reader to have a better understanding of the geometry underlying the definition of n-varifolds.

Lemma 2.1.1. *Let $V \in \mathbb{V}_n(\Omega)$ be a given n-varifold. Then, for μ_V-a.e. $x \in \Omega$ there exists a Radon measure η^x on $G(n+k,n)$ such that*

$$\int_{G(n+k,n)} \beta(S)\, d\eta^x(S) = \lim_{\rho \to 0^+} \frac{\int_{G_n(B_\rho(x))} \beta(S)\, dV(y,S)}{\mu_V(B_\rho(x))}$$

for each $\beta \in C^0_c(G(n+k,n))$. Moreover, for any $A \in \mathcal{B}(\Omega)$,

$$\int_{G_n(A)} \beta(S)\, dV(x,S) = \int_A \int_{G(n+k,n)} \beta(S)\, d\eta^{(x)}_V\, d\mu_V(x)$$

holds for every $\beta \in C^0_c(G(n+k,n))$.

Proof. Since $G(n+k,n)$, endowed with the metric defined in the beginning of this section, is compact and separable, then also $\left(C^0_c(G(n+k,n);\mathbb{R}^+_0), \|\cdot\|_{C^0}\right)$ is separable, hence we may fix

$$\left\{\beta_j \in C^0_c(G(n+k,n);\mathbb{R}^+_0) : j \in \mathbb{N}\right\} \subseteq C^0_c(G(n+k,n);\mathbb{R}^+_0)$$

which is dense. Observing that the position

$$B \mapsto \int_{G_n(B)} \beta(S)\, dV(y,S) \quad B \in \mathcal{B}(\Omega)$$

defines, for each $j \in \mathbb{N}$, a measure on Ω which is absolutely continuous with respect to μ_V, by Lebesgue-Radon-Nikodym there are functions μ_V-measurable function $\{f_j : j \in \mathbb{N}\}$ such that

$$f_j(x) = \lim_{\rho \to 0^+} \frac{\int_{G_n(B_\rho(x))} \beta_j(S)\, dV(y,S)}{\mu_V(B_\rho(x))}$$

for $\mu_V - a.e.$ $x \in \Omega$ and for all $j \in \mathbb{N}$. This means that

$$\int_B f_j(x)\, d\mu_V(x) = \int_{G_n(B)} \beta(S)\, dV(x,S) \quad \forall B \in \mathcal{B}(\Omega).$$

Let the Z_j and Z be the sets defined by the positions

$$Z_j := \left\{x \in \Omega : f_j(x) \text{ is not well defined}\right\} \quad \forall j \in \mathbb{N}, \tag{2.2}$$

$$Z := \bigcup_{j \in \mathbb{N}} Z_j, \tag{2.3}$$

so that f_j is well defined on $\Omega\setminus Z$ for each $j \in \mathbb{N}$ and $\mu_V(Z) = 0$. We now claim that the limit

$$\widetilde{\eta}^x(\beta) := \lim_{\rho\to 0^+} \frac{\int_{G_n(B_\rho(x))} \beta(S)\, dV(y,S)}{\mu_V(B_\rho(x))} \tag{2.4}$$

exists for each $x \in \Omega\setminus Z$ and for each $\beta \in C^0_c(G(n+k,n))$; moreover we claim that the position

$$\beta \longmapsto \widetilde{\eta}^x \tag{2.5}$$

defines a continuous linear functional $\left(C^0_c(G(n+k,n);\mathbb{R}^+_0), \|\cdot\|_{C^0}\right) \to \mathbb{R}$ for every $x \in \Omega\setminus Z$. If this was the case, by Riezs representation theorem, there would be a Radon measure η^x on $G(n+k,n)$ such that

$$\widetilde{\eta}^x(\beta) = \int_{G(n+k,n)} \beta(S)\, d\eta^x \quad \forall \beta \in C^0_c(G(n+k,n)), \mu_V - a.e.\ x \in \Omega,$$

and, from (2.4), the first claim would immediately follow. The second claim is an immediate consequence of the first and the definition of Lebesgue-Radon-Nikodym derivative.

Therefore we only have to prove that the limit in (2.4) is well posed and that the map in (2.5) is continuous. Fix $\beta \in C^0_c(G(n+k,n);\mathbb{R})$ and a constant $\varepsilon > 0$. Then we can write $\beta = \beta^+ - \beta^-$, with $\beta^+, \beta^- \geq 0$, hence, by density of $(\beta_j)_j$, we can find $j^+, j^- \in \mathbb{N}$ such that

$$\|\beta^+ - \beta_{j^+}\|_{C^0} < \frac{\varepsilon}{2}, \quad \|\beta^- - \beta_{j^-})\|_{C^0} < \frac{\varepsilon}{2}.$$

Therefore the right hand side of (2.4) can be written as

$$\lim_{\rho\to 0^+} \left(\frac{\int_{G_n(B_\rho(x))} (\beta - (\beta_{j^+} - \beta_{j^-}))\, dV}{\mu_V(B_\rho(x))} + \frac{\int_{G_n(B_\rho(x))} (\beta_{j^+} - \beta_{j^-})\, dV}{\mu_V(B_\rho(x))} \right), \tag{2.6}$$

and since the absolute value of the limit of the first term appearing in (2.6) is arbitrarily small, and the limit of the second exists and it is

$$f_{j^+}(x) - f_{j^-}(x),$$

we have proved that the definition of $\widetilde{\eta}^x$ is well posed. Finally

$$\begin{aligned} |\widetilde{\eta}^x(\beta)| &\leq \lim_{\rho\to 0^+} \frac{\int_{G_n(B_\rho(x))} |\beta(S)|\, dV}{\mu_V(B_\rho(x))} \\ &\leq \lim_{\rho\to 0^+} \frac{\int_{G_n(B_\rho(x))} dV}{\mu_V(B_\rho(x))} \|\beta\|_{C^0} \\ &= \|\beta\|_{C^0}. \end{aligned}$$

The proof is concluded. □

The very last lines of the proof of Lemma 2.1.1 imply that the family of Radon measures $(\eta^x)_x$ is in fact a family of probability measures on $G(n+k,n)$. More precisely η^x is a probability measure for μ_V-a.e x. A n-Varifold on Ω therefore consists of a Radon measure μ together with, at μ_V-a.e. point x, a probability measure. By virtue of this, we shall establish the following notation for a given varifold $V \in \mathbb{V}_n(\Omega)$:

$$V = \mu \otimes (\eta^x), \tag{2.7}$$

meaning

$$\int_{G_n(\mathbb{R}^{n+k})} f\, dV = \int_\Omega \left(\int_{G(n+k,n)} f(x,S)\, d\eta^x(S) \right) d\mu(x) \tag{2.8}$$

for all functions $f \in C^0_c(G_n(\Omega))$. It is also easy to see that, if $V \in \mathbb{V}_n(\Omega)$ is defined by (2.7), then $\mu = \mu_V$.

Vice versa it is clear that all of the measures defined by the position (2.7) (i.e. acting on compactly supported continuous functions as in (2.8)) are n-varifolds. Hence Lemma 2.1.1 gives a complete characterization of the elements of $\mathbb{V}_n(\Omega)$.

We also remark that the class of smooth n-dimensional manifold injects into the one of n-varifolds in the following way. Given M smooth n-dimensional manifold, we consider the mass measure $\mu_M = \mathscr{H}^n_{\llcorner M}$ for the $\mathbb{R}^{n+k}$ part, and the family of Dirac delta measures $(\delta_{T_xM})_x$ for the Grassmaniann part. Then the map

$$M \hookrightarrow V(M) := \mu_M \otimes (\delta_{T_xM})_x$$

is injective, and extends the notions of support and tangent plane in a weaker and more general sense, guaranteeing in fact a reasonable measure-theoretic extension of the concept of n-dimensional surface.

Definition 2.2 (Rectifiable n-Varifold). We say that a n-varifold $V \in \mathbb{V}_n(\Omega)$ is a *rectifiable n-varifold* if there exists a couple (M, ϑ) with $M \subseteq \Omega$ n-recfiable set and $\vartheta \in L^1_{\text{loc}}(M; \mathscr{H}^n)$ positive function which is supported in M such that the decomposition of V induced by Lemma 2.1.1 is

$$V = \vartheta \mathscr{H}^n_{\llcorner M} \otimes (\delta_{T_xM}),$$

where δ_{T_xM} is the approximate tangent space of M at $\mathscr{H}^n$-a.e. point x. In this case we will write $V = \underline{\text{v}}(M, \vartheta)$ and we will say that ϑ is the multiplicity (or multiplicity function) of V.

We will denote by $\mathscr{R}_n(\Omega)$ the set of all n-recrifiable varifolds and by $\mathscr{I}_n(\Omega)$ the subset of n-rectifiable varifolds with integral multiplicity.

From the above remark it is now easy to see how varifolds are a natural measure-theoretical generalization of the concept of manifolds.

We conclude this section with the following definition.

Definition 2.3 (Convergence of Varifold). Let $(V_j)_j$ be a sequence of varifold in $\Omega \subseteq \mathbb{R}^{n+k}$. We say that the sequence $(V_j)_j$ converges to a varifold $V \in \mathbb{V}_n(\Omega)$, and will write $V_j \rightharpoonup^* V$, if

$$\lim_{j\to\infty} \int_{G_n(\Omega)} f(x,S)\, dV_j(x,S) = \int_{G_n(\Omega)} f(x,S)\, dV(x,S)$$

holds for all $f \in C^0_c(G_n(\Omega))$.

In other words the convergence of $(V_j)_j$ is defined to be the weak-* convergence of $(V_j)_j$ as a sequence of Radon measures.

2.2. First Variation and Mean Curvature

Given a n-varifold V on $\Omega \subseteq \mathbb{R}^{n+k}$ we wish to define a notion of *rate of change* of its mass with respect to *compact variation*, in the same spirit of the definition of first variation of a functional in Calculus of Variations. In this case the compact variations will be represented by deformations induced by compactly supported vector field. A compact variation of a n-varifold V will be the n-varifold obtained by transporting V along the flow of a continuous compactly supported (in Ω) vector field. Hence, before giving the definition of first variation, it is necessary to specify what we mean by *transporting V along the flow of a vector field.* More generally we can give the following.

Definition 2.4 (Push-Forward Varifold). Let V be a n-varifold on $\Omega \subseteq \mathbb{R}^{n+k}$ and let $\psi : \Omega \to \widetilde{\Omega}$ be a C^1 map with $\psi_{|\mathrm{spt}\mu_V \cap \Omega}$ proper (i.e. $\psi^{-1}(K) \cap \mathrm{spt}\mu_V$ is compact for every compact subset $K \Subset \widetilde{\Omega}$). Then the push-forward varifold of V through ψ is the n-varifold $\psi^\# V$ on $\widetilde{\Omega}$ defined by

$$\int_{G_n(\widetilde{\Omega})} f(y,T)\, d(\psi^\# V)(y,T) = \int_{G_n(\Omega)} f(\psi(x), \nabla\psi(S))\, J_S\psi(x)\, dV(x,S)$$

for all $f \in C^0_c(G_n(\widetilde{\Omega}))$, where, recalling the notation introduced in Chapter 1, $J_S\psi(x)$ is defined as

$$J_S\psi(x) := \sqrt{\det(\nabla^*_S\psi(x) \circ \nabla_S\psi(x))}. \tag{2.9}$$

Given a vector field $X \in C^1_c(\Omega;\mathbb{R}^{n+k})$, we can always consider the family of Cauchy problems $\{(CP_x)\}_x$

$$\begin{cases} \varphi'_x(t) = X(\varphi_x(t)) \\ \varphi_x(0) = x \end{cases} . \tag{CP_x}$$

Since X is compactly supported, there exists $\varepsilon_0 > 0$ such that (CP_x) admits one unique solution $\varphi_x : (-\varepsilon_0, \varepsilon_0) \to \Omega$ for each $x \in \Omega$. Hence we can consider the flow of X defined as the map

$$\Phi : (-\varepsilon_0, \varepsilon_0) \times \Omega \to \Omega, \quad (t,x) \mapsto \Phi(t,x) := \varphi_x(t).$$

In particular, given a vector field $X \in C_c^1(\Omega;\mathbb{R}^{n+k})$, there is a naturally associated family of maps $\{\Phi_t(\cdot):\Omega\to\Omega\}_{-\varepsilon_0<t<\varepsilon_0}$ with the following three properties:

$$\Phi_0 \equiv \mathrm{id}_\Omega, \tag{2.10}$$

$$\Phi_{t|\Omega\setminus\mathrm{spt}X} \equiv \mathrm{id}_{|\Omega\setminus\mathrm{spt}X} \quad \text{for all } -\varepsilon_0 < t < \varepsilon_0, \tag{2.11}$$

$$(t,x)\mapsto \Phi_t(x) \in C^1((-\varepsilon_0,\varepsilon_0)\times\Omega). \tag{2.12}$$

Definition 2.5 (First Variation). Let $V \in \mathbb{V}_n(\Omega)$ be fixed. We define its first variation to be the linear functional $\delta V : C_c^1(\Omega;\mathbb{R}^{n+k}) \to \mathbb{R}$ by

$$\delta V(X) := \frac{d}{dt}\underline{\mathscr{M}}(\Phi_t^\# V_{\llcorner G_n(\mathrm{spt}X)})\big|_{t=0}, \quad \forall X \in C_c^1(\Omega;\mathbb{R}^{n+k}),$$

where $(t,x)\mapsto\Phi_t(x)$ is the flow of X.

If $X \in C_c^1(\Omega;\mathbb{R}^{n+k})$, $S \in G(n+k,n)$ and $\{\tau_1,\dots,\tau_n\}$ is an orthonormal basis for S, then we have defined the divergence of X on S as

$$\mathrm{div}_S X := \sum_{i=1}^n \tau_i\cdot\partial_{\tau_i}X = S:\nabla X, \tag{2.13}$$

where : is the matrix opration defined by

$$A:B := \mathrm{tr}(A^tB) = \sum_{i,j=1}^{n+k} a_{i,j}b_{i,j} \quad \forall A,B \in \mathbb{R}^{n+k}\otimes\mathbb{R}^{n+k},$$

and in (2.13) we have used the identification $S \equiv \Pi_S$. Therefore we can write

$$\begin{aligned}\frac{d}{dt}\underline{\mathscr{M}}(\Phi_t^\# V_{\llcorner G_n(\mathrm{spt}X)})\big|_{t=0} &= \int \frac{d}{dt}J_S\Phi_t(y)\big|_{t=0}\,dV(y,S)\\ &= \int \mathrm{div}_S\left(\frac{d}{dt}\Phi_t(y)\big|_{t=0}\right)dV(y,S)\\ &= \int \mathrm{div}_S X(y)\,dV(y,S).\end{aligned}$$

Recalling the definition of first variation, we obtain

$$\delta V(X) = \int_{G_n(\Omega)} \mathrm{div}_S X(y)\,dV(y,S), \quad \forall X \in C_c^1(\Omega;\mathbb{R}^{n+k}). \tag{2.14}$$

Recall the general version of the divergence theorem presented in Chapter 1, which states that if $M \subseteq \mathbb{R}^{n+k}$ is a C^2-class n-dimensional submanifold, and $X \in C^1(M;\mathbb{R}^{n+k})$, then

$$\int_M \mathrm{div}_M X\,d\mathscr{H}^n = -\int_M X\cdot\underline{H}\,d\mathscr{H}^n - \int_M X\cdot\nu\,d\mathscr{H}^{n-1}_{\llcorner\partial M}, \tag{2.15}$$

where $\underline{H}$ is the mean curvature vector of M (i.e. the trace of the second fundamental form of M) and $\nu_{|\partial M}$ is the inward pointing unit normal vector defined on the boundary on M. Therefore, given $V := \underline{v}(M,1)$ and $X \in C^1_c(\Omega;\mathbb{R}^{n+k})$ with $\|X\|_{C^0(\Omega;\mathbb{R}^{n+k})} \leq 1$ and $\Omega \Subset \mathbb{R}^{n+k}$, we have, by virtue of the above computations

$$|\delta V(X)| \leq \int_{M\cap \mathrm{spt} X} |X\cdot \underline{H}|\, d\mathscr{H}^n + \int_{\partial M\cap \mathrm{spt} X} |X\cdot \nu|\, d\mathscr{H}^{n-1}.$$

Since $\underline{H} \in L^1_{\mathrm{loc}}(M;\mathbb{R}^{n+k})$,

$$|\delta V(X)| \leq \|\underline{H}\|_{L^1(M\cap\Omega;\mathscr{H}^n)} + \mathscr{H}^{n-1}(\partial M\cap\Omega) < \infty,$$

therefore the map δV is a continuous functional defined on a dense subset of $C^0_c(\Omega;\mathbb{R}^{n+k})$. Using some classical extension arguments one can easily see that δV can be extended as a continuous functional (not relabeled)

$$\delta V : (C^0_c(\Omega;\mathbb{R}^{n+k}), \|\cdot\|_{C^0}) \to \mathbb{R}.$$

Therefore, by Riezs representation theorem, δV is identified by a vector valued Radon measure $\overline{\delta V}$ on Ω and its total variation measure $\|\overline{\delta V}\|$ is the non-negative Radon measure on Ω

$$\|\overline{\delta V}\|(B) := \sup_{\substack{X\in C^0_c(B;\mathbb{R}^{n+k})\\ \|X\|_{C^0}\leq 1}} |\delta V(X)|, \quad \forall B\in \mathcal{B}(\Omega). \tag{2.16}$$

Moreover, form the classical measure-theoretical results (cfr. Chapter 1), there exists a $\|\overline{\delta V}\|$-measurable function $\tilde{\nu} : \Omega \to \mathbb{R}^{n+k}$ such that $|\tilde{\nu}(x)| = 1$ for $\|\delta V\| - a.e.\, x \in \Omega$ and

$$\delta V(X) = -\int_{M\cap\Omega} X\cdot\tilde{\nu}\, d\|\overline{\delta V}\|, \quad \forall X \in C^0_c(\Omega;\mathbb{R}^{n+k}). \tag{2.17}$$

We now set, for each x such that the following limit exists (i.e. for $\mathscr{H}^n$-a.e. set, by virtue of Theorem 1.1.7),

$$g(x) := \lim_{\rho\to 0^+} \frac{\|\delta V\|(B\rho(x))}{\omega_n \rho^n}. \tag{2.18}$$

Subsequently, let $\tilde{H}$ be the vector valued function function

$$\tilde{H}(x) := g(x)\tilde{\nu}(x) \quad \mathscr{H}^n - \text{a.e. } x\in\Omega.$$

Writing the decomposition of $\|\delta V\|$ as

$$d\|\delta V\| = g\, d\mathscr{H}^n + d\sigma \llcorner Z \quad Z := \{x\in\Omega : g(x) = \infty\} \tag{2.19}$$

from (2.17), we obtain

$$\delta V(X) = -\int_M X\cdot\tilde{H}\, d\mathscr{H}^n - \int_Z X\cdot\tilde{\nu}\, d\sigma. \tag{2.20}$$

From (2.14), (2.15), (2.20) and arbitrariness of X and Ω we deduce that

$$\tilde{H} = \underline{H}\ \mathscr{H}^n - \text{a.e}, \quad \tilde{\nu} = \nu\ \mathscr{H}^{n-1} - \text{a.e}, \quad \mathscr{H}^{n-1}(Z\triangle\partial M) = 0.$$

To the aim of obtaining weaker and more general notions of mean curvature and unit normal which coincide with the classical ones on C^2-class manifolds, but extend also to general n-varifolds, one may try to repeat the same process presented in the previous lines, by dropping the restrictive assumption

$$\exists \text{ a } C^2\text{-class } n\text{-dimensional submanifold such that } V = \underline{\mathrm{v}}(M,1),$$

but it will soon realize that (2.16) is, in general, not finite on compact sets. On the other hand, as soon as δV can be extended with continuity to the whole $C_c^0(\Omega;\mathbb{R}^{n+k})$ (endowed with its natural norm), replacing $\mathscr{H}^n$ with μ_V in (2.18) yields the following expression for the first variation of V:

$$\delta V(X) = -\int_\Omega X\cdot \underline{\widetilde{H}}\, d\mu_V - \int_Z X\cdot\widetilde{\nu}\, d\sigma.$$

Being more precise, we can give the next well posed definitions.

Definition 2.6 (n-Varifold of Locally Bounded Total Variation). Let V be a n-varifold in $\Omega\subseteq\mathbb{R}^{n+k}$. We say that V is of locally bounded total variation in Ω, and write $V\in\mathcal{V}_n(\Omega)$, if for each $W\Subset\Omega$, there exists a constant $C\geq 0$ such that

$$\sup_{\substack{X\in C_c^1(W;\mathbb{R}^{n+k})\\ \|X\|_{C^0}\leq 1}} |\delta V(X)|\leq C.$$

If V is of locally bounded total variation then δV can be extended to a map $C_c^0(\Omega;\mathbb{R}^{n+k})\to\mathbb{R}$, which will still be denoted by δV and called first variation of V.

Definition 2.7 (Total Variation of a Varifold). If $V\in\mathcal{V}_n(\Omega)$, we denote by $\|\delta V\|$ the unique non-negative Radon measure on $\mathcal{B}(\Omega)$ (which existence is guaranteed by Riezs representation theorem) such that

$$\|\delta V\|(B) := \sup_{\substack{X\in C_c^0(B;\mathbb{R}^{n+k})\\ \|X\|_{C^0}\leq 1}} |\delta V(X)|$$

for each $B\in\mathcal{B}(\Omega)$.

Definition 2.8 (Generalized Mean Curvature, Inner Unit Normal and Boundary). Given a n-varifold of bounded total variation in Ω and called V, let g be the Lebesgue-Radon-Nikodym derivative of $\|\delta V\|$ with respect to the mass measure μ_V, i.e.

$$g(x) := \lim_{\rho\to 0^+}\frac{\|\delta V\|(B_\rho(x))}{\mu_V(B_\rho(x))}\quad \mu_V - a.e.\, x\in\Omega,$$

and let $\widetilde{\nu}$ be a $\|\delta V\|$-measurable function such that

$$\delta V(X) = -\int_\Omega X\cdot\widetilde{\nu}\, d\|\delta V\|.$$

Then the vector valued functions $\underline{H}$, ν and the set Z defined by

$$\underline{H}(x) := g(x)\widetilde{\nu}(x),\quad \nu(x) := \chi_Z(x)\widetilde{\nu}(x),\quad Z := \{x\in\Omega : g(x)=\infty\}.$$

are respectively called *generalized mean curvature*, *generalized inner unit normal* and *generalized boundary* of the varifold V.

Finally, inspired by the classical definition of stationary n-dimensional submanifolds of $\mathbb{R}^{n+k}$, we give:

Definition 2.9 (Stationary n-Varifold). Let V be a n-varifold in Ω. We say that V is stationary in Ω if $\delta V \equiv 0$.

The first variation $\|\delta V\|$ is therefore a quantifier for the sum of mean curvature and the measure of the boundary (in generalized sense) of a n-varifold. Hence stationary n-varifolds are to be interpreted as those varifolds that have both 0 mean curvature and a 0-measure boundary in Ω.

Ending this section we introduce the notation $\mathscr{V}_n^p(\Omega)$ to indicate the subfamily of $\mathcal{V}_n(\Omega)$ containing varifolds V that have empty generalized boundary and generalized mean curvature $\underline{H}_V$ in $L^p(\mu_V)$. In the case of $p = 1$ we will simply write $\mathscr{V}_n(\Omega)$ instead of $\mathscr{V}_n^1(\Omega)$. Assuming that a n-varifolds $V \in \mathscr{V}_n(\Omega)$, which is equivalent of saying that the first variation δV is absolutely continuous with respect to the mass measure μ_V, then δV writes as

$$\delta V(X) = -\int X \cdot \underline{H}\, d\mu.$$

Therefore, establishing the notation

$$\langle \underline{H}, X\rangle := \int X \cdot \underline{H}\, d\mu \quad \forall X \in C_c^1(\Omega; \mathbb{R}^{n+k}), \tag{2.20}$$

$$\|\underline{H}\|_1(B) := \int_B |\underline{H}|\, d\mu \quad \forall B \in \mathcal{B}(\Omega), \tag{2.20}$$

under the standing assumption $\partial V = \varnothing$, we can write

$$\delta V(X) = -\langle X, \underline{H}\rangle, \quad \|\delta V\|(B) = \|\underline{H}\|_1(B)$$

for all X and B as in (2.20) and (2.20) respectively.

Finally we remark that by Theorem 1.1.5 (Lebesgue-Radon-Nikodym) we have

$$\underline{H}(x) := \lim_{\rho\to 0} -\frac{\delta V(B_\rho(x))}{\mu(B_\rho(x))} \quad \mu - a.e.\, x \tag{2.21}$$

where δV denotes the Radon measure associated with the functional $X \mapsto \delta V(X) = -\langle X, \underline{H}\rangle$.

2.3. Monotonicity Formulae

In this section we are analyzing some important properties of the mass measure of a varifolds with locally bounded first variation (i.e. admitting generalized mean curvature and boundary). More precisely, given $V \in \mathscr{R}_n(\Omega) \cap \mathscr{V}_n(\Omega)$ and $x \in \Omega$, we will prove some meaningful monotonicity properties of function

$$\rho \mapsto \frac{\mu_V(B_\rho(x))}{\omega_n \rho^n} \tag{2.22}$$

defined on a reasonable neighborhood of $0 \in \mathbb{R}$. Therefore, by extension, we will recover salient information about the n-density of the mass measure. For sake of readibility we briefly recall that for a measure $\mu \in \mathrm{Rad}(\Omega)$ the lower and upper n-densities at a point x are defined to be

$$\Theta^n_*(\mu, x) := \liminf_{\rho \to 0} \frac{\mu(B_\rho(x))}{\omega_n \rho^n}, \tag{2.23}$$

$$\Theta^{n*}(\mu, x) := \limsup_{\rho \to 0} \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \tag{2.24}$$

respectively. If the lower and upper n-densities coincide at some point x, we say that the common value is the n-density of μ at x, and we denote it by $\Theta^n(\mu, x)$.

Since we will state different monotnicity formulae, requiring less and less integrability properties of the mean curvature, we have decided to divide this section into subsections. The first two will be dedicated to the monotonicty formula (MF) for bounded mean curvature and L^p_{loc} mean curvature respectively; the third one will contain an important corollary concerning the semi-continuity of the density of a varifold; finally in the last two we will present weaker but more general formulations of the MF which will be useful for proving a Sobolev-type inequality for varifolds in the next section.

Before starting to state and prove the aforementioned results, we need to introduce some useful notations. If $f : G_n(\Omega) \to \mathbb{R}$ is such that the following quantities are well defined, we set

$$\nabla_T f(y) := \Pi_T \nabla f(y) \quad \forall T \in G(n+k, n)$$

and, if $V = \underline{\mathrm{v}}(M, \vartheta)$ is rectifiable, then, according to the notation of Chapter 1

$$\nabla_M f(y) := \Pi_{T_y M} \nabla f(y)$$

for all y such that $T_y M$ is defined. Analogously the differential operator $\nabla^\perp_S$ is defined as $\nabla_{S^\perp}$ and $\nabla^\perp_M$ is defined by projecting the gradient onto $T_y M^\perp$. When we refer about a varifold $V \in \mathscr{R}_n(\Omega)$, unless we explicitly specify otherwise, we denote by M its support and ϑ its multiplicity (i.e. $V = \underline{\mathrm{v}}(M, \vartheta)$).

2.3.1 MF for Bounded Mean Curvature.

Theorem 2.3.1 (Monotonicity Formula). *Let $V \in \mathscr{R}_n(\Omega) \cap \mathscr{V}_n(\Omega)$. Fix $x \in \mathrm{spt}\mu$ with $\rho_0 := \mathrm{dist}(x, \partial\Omega) > 0$ and assume the existence of a constant $\Lambda > 0$ such that $|\underline{H}| < \Lambda$ μ-a.e.. Then we have the monotonicity formula*

$$\begin{aligned} F(\rho)\frac{\mu(B_\rho(x))}{\rho^n} - F(\sigma)\frac{\mu(B_\sigma(x))}{\sigma^n} \\ = G(\sigma, \rho) \int_{B_\rho(x) \setminus B_\sigma(x)} \frac{|\nabla^\perp_M r|^2}{r^n}\, d\mu(y) \end{aligned} \tag{MF_B}$$

for all $0 < \sigma < \rho < \rho_0$, where $r := |y - x|$, $F(\rho) \in [e^{-\Lambda\rho}, e^{\Lambda\rho}]$ and $G(\sigma, \rho) \in [e^{-\Lambda\rho_0}, e^{\Lambda\rho_0}]$

Proof. Let $\varphi \in C^1(\mathbb{R})$ be a function with the following properties

$$\varphi \equiv 1 \quad \text{on } (-\infty, 1], \quad \varphi \equiv 0 \quad \text{on } [1+\varepsilon, \infty), \quad \varphi' \le 0$$

for some $\varepsilon > 0$ to be chosen later. Then, for $\rho \in (0, \rho_0)$ we define the function $\gamma : \mathbb{R} \to \mathbb{R}$ and the vectorfield $X \in C^1_c(B_{\rho_0}(x); \mathbb{R}^{n+k})$

$$\gamma(r) := \varphi(r/\rho), \quad X(y) := \gamma(|y-x|)(y-x).$$

For simplifying the notation, from now on we will use $r := |y-x|$. The gradient of X is therefore written as

$$\begin{aligned} \nabla X(y) &= \gamma'(r)\nabla r \otimes (y-x) + \gamma(r)\nabla(y-x) \\ &= \gamma'(r) r \nabla r \otimes \nabla r + \gamma(r)\mathbb{1}_{n+k}, \end{aligned}$$

hence,

$$\begin{aligned} \langle X, \underline{H} \rangle &= -\int T_y M : \nabla X(y)\, d\mu(y) \\ &= -\int \gamma'(r) r |\nabla_M r|^2\, d\mu - \int \gamma(r) \operatorname{tr}(T_y M)\, d\mu \\ &= \rho \frac{d}{d\rho} \int \varphi(r/\rho)\, d\mu - \rho \frac{d}{d\rho} \int \varphi(r/\rho) |\nabla_M^\perp r|^2\, d\mu - n \int \varphi(r/\rho)\, d\mu. \end{aligned} \tag{2.25}$$

Therefore, making the left hand side of (2.25) explicit, we obtain the expression

$$\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^n}\right) = \frac{J'(\rho)}{\rho^n} + \frac{1}{\rho^{n+1}} \int \varphi(r/\rho)(y-x) \cdot \underline{H}\, d\mu, \tag{2.26}$$

where I and J in (2.26) are defined as

$$I(\rho) := \int \varphi(r/\rho)\, d\mu, \quad J(\rho) := \int \varphi(r/\rho) |\nabla_M^\perp r|^2\, d\mu. \tag{2.27}$$

Since $\varphi(r/\rho) = 0$ for $r > (1-\varepsilon)\rho$ and $|\underline{H}| \le \Lambda$, then

$$-\Lambda_\varepsilon \frac{I(\rho)}{\rho^n} \le \frac{1}{\rho^n} \int \varphi(r/\rho) \frac{(y-x)}{\rho} \cdot \underline{H}\, d\mu \le \Lambda_\varepsilon \frac{I(\rho)}{\rho^n} \tag{2.28}$$

where we defined $\Lambda_\varepsilon := (1+\varepsilon)\Lambda$. Therefore there exists a constant $E(\rho) \in [-\Lambda_\varepsilon, \Lambda_\varepsilon]$ for each $\rho \in (0, \rho_0)$ such that

$$\frac{1}{\rho^n} \int \varphi(r/\rho) \frac{(y-x)}{\rho} \cdot \underline{H}\, d\mu = E(\rho) \frac{I(\rho)}{\rho^n}.$$

Multiplying both sides of (2.26) by the integrating factor

$$F(\rho) := e^{\int_0^\rho E(\tau)\, d\tau} \tag{2.29}$$

we obtain

$$e^{-\Lambda_\varepsilon \rho_0} \frac{J'(\rho)}{\rho^n} \le \frac{d}{d\rho}\left(F(\rho) \frac{I(\rho)}{\rho^n}\right) \le e^{\Lambda_\varepsilon \rho_0} \frac{J'(\rho)}{\rho^n}. \tag{2.30}$$

Recalling the definition of $J(\rho)$ and the fact that $\frac{d}{d\rho}\varphi(r/\rho) = 0$ for $r > (1+\varepsilon)\rho$ and for $r < \rho$, we deduce

$$\frac{J'(\rho)}{((1+\varepsilon)r)^n} \le \frac{J'(\rho)}{\rho^n} \le \frac{J'(\rho)}{r^n}$$

hence, together with (2.30), we have

$$e^{-\Lambda_\varepsilon \rho_0}\frac{J'(\rho)}{((1+\varepsilon)r)^n} \le \frac{d}{d\rho}\left(F(\rho)\frac{I(\rho)}{\rho^n}\right) \le e^{\Lambda_\varepsilon \rho_0}\frac{J'(\rho)}{r^n}. \tag{2.31}$$

Integrating (2.31) in the interval (σ, ρ) gives the two inequalities

$$\begin{aligned}\int \frac{e^{-\Lambda_\varepsilon \rho_0}}{((1+\varepsilon)r)^n}|\nabla_M^\perp r|^2(\varphi(r/\rho)-\varphi(r/\sigma))\,d\mu \\ \le F(\rho)\frac{I(\rho)}{\rho^n} - F(\sigma)\frac{I(\sigma)}{\sigma^n} \\ \le \int \frac{e^{\Lambda_\varepsilon \rho_0}}{r^n}|\nabla_M^\perp r|^2(\varphi(r/\rho)-\varphi(r/\sigma))\,d\mu\end{aligned}$$

hence, letting $\varepsilon \searrow 0$ as well as $\varphi \searrow \chi_{(-\infty,1]}$, by monotone convergence we deduce

$$\begin{aligned}e^{-\Lambda_\varepsilon \rho_0}\int_{B_\rho(x)\setminus B_\sigma(x)} \frac{|\nabla_M^\perp r|^2}{r^n}\,d\mu \\ \le F(\rho)\frac{\mu(B_\rho(x))}{\rho^n} - F(\sigma)\frac{\mu(B_\sigma)}{\sigma^n} \\ \le e^{\Lambda_\varepsilon \rho_0}\int_{B_\rho(x)\setminus B_\sigma(x)} \frac{|\nabla_M^\perp r|^2}{r^n}\,d\mu,\end{aligned}$$

which is exactly (MF_B).

□

As an immediate corollary we establiosh monotonicity properties for the mass measure of stationary varifolds.

Corollary 2.3.1. *Let $V \in \mathscr{R}_n(\Omega)$ be a stationary varifold. Fix $x \in \operatorname{spt}\mu$ with $\rho_0 :=$* $\operatorname{dist}(x, \partial\Omega) > 0$. *Then the function*

$$\rho \mapsto \frac{\mu(B_\rho(x))}{\omega_n \rho^n} \quad \rho \in (0, \operatorname{dist}(x, \rho_0))$$

is non decreasing.

2.3.2 MF for Mean Curvature in L^p_{loc} and $p > n$.

Let us continue to assume $V \in \mathscr{R}_n(\Omega) \cap \mathscr{V}_n(\Omega)$ but assume that $\underline{H}$ is merely an $L^p_{\mathrm{loc}}(\mu)$ function, with $p > n$, rather than than bounded (i.e. $L^\infty(\mu)$). Then we can repeat all of the computations made in the proof of the Monotonicity Formula up until (2.27), then (2.28) will be no longer true. On the other hand, provided

$$\rho_0^{1-\frac{n}{p}}\left(\int_{B_{\rho_0}(x)}|\underline{H}|^p\,d\mu\right)^{\frac{1}{p}}\leq\alpha\Lambda,\quad \alpha:=\frac{p}{4(p-n)}$$

for some constant Λ to be chosen later. Assume $\rho<\rho_0/(1+\varepsilon)$. Recalling both Hölder's inequality and the fact that $\varphi(r/\rho)=0$ for $r>(1+\varepsilon)\rho$, we have

$$\begin{aligned}
&|\frac{1}{\rho^n}\int\varphi(r/\rho)\frac{(y-x)}{\rho}\cdot\underline{H}\,d\mu|\\
&\quad\leq\frac{1+\varepsilon}{\rho^n}\|H\|_{L^p(B_{\rho_0})}(I(\rho))^{1-\frac{1}{p}}\\
&\quad=\frac{1+\varepsilon}{\rho_0}\left(\frac{\rho_0}{\rho}\right)^{\frac{p}{n}}\rho_0^{1-\frac{n}{p}}\left(\int_{B_{\rho_0}(x)}|\underline{H}|^p\,d\mu\right)^{\frac{1}{p}}\left(\frac{I(\rho)}{\rho^n}\right)^{\frac{1}{p}}\\
&\quad\leq\frac{2\alpha\Lambda}{\rho_0}\left(\frac{\rho_0}{\rho}\right)^{\frac{p}{n}}\left(1+\frac{I(\rho)}{\rho^n}\right)
\end{aligned}\tag{2.32}$$

where, in (2.32) clearly $L^p(B_{\rho_0}(x))$ is intended with respect to the mass measure of the varifold, and the for the last inequality we have used the fact that $(1+a)^{1-\frac{n}{p}}\leq 1+a$ for any $a>0$. Therefore we can write (2.27) as

$$\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^n}\right)=\frac{J'(\rho)}{\rho^n}-F_0(\rho)\left(1+\frac{I(\rho)}{\rho^n}\right)\tag{2.33}$$

for every $\rho\in(0,\rho_0/(1+\varepsilon))$, with

$$|F_0(\rho)|\leq\frac{2\alpha\Lambda}{\rho_0}\left(\frac{\rho_0}{\rho}\right)^{\frac{p}{n}}.$$

After multiplying both sides of (2.33) by the integrating factor

$$F(\rho):=e^{\int_0^\rho F(\tau)\,d\tau}\tag{2.34}$$

and defining $E(\rho):=F(\rho)-F(0)$ we then obtain

$$\frac{d}{d\rho}\left(F(\rho)\frac{I(\rho)}{\rho^n}+E(\rho)\right)=F(\rho)\frac{J'(\rho)}{\rho^n}\quad\forall\rho\in(0,\rho_0/(1+\varepsilon)).\tag{2.35}$$

Observing that

$$\int_0^\rho|F_0(\tau)|\,d\tau\leq\frac{1}{2}\Lambda\left(\frac{\rho}{\rho_0}\right)^{1-\frac{n}{p}}\tag{2.36}$$

$$|F(\rho)-F(0)|=|\int_0^\rho F'(\tau)\,d\tau|\leq e^{\frac{1}{2}\Lambda}\int_0^\rho|F_0(\tau)|\,d\tau,\tag{2.37}$$

we recover the bounds

$$e^{-\Lambda} \le e^{-\frac{1}{2}\Lambda(\rho/\rho_0)^{1-n/p}} \le F(\rho) \le e^{\frac{1}{2}\Lambda(\rho/\rho_0)^{1-n/p}} \le e^{\Lambda} \tag{2.38}$$

$$|E(\rho)| \le \frac{1}{2} e^{\frac{1}{2}\Lambda} \Lambda \left(\frac{\rho}{\rho_0}\right)^{1-\frac{n}{p}}, \quad \rho \in (0, \rho_0) \tag{2.39}$$

In particular, if $\Lambda \le 1$, then

$$|E(\rho)| \le \Lambda \left(\frac{\rho}{\rho_0}\right)^{1-\frac{n}{p}}, \quad \rho \in (0, \rho_0),$$

thus, proceding as in the previous proof: integrating (2.35) and passing to the limit as $\varepsilon \searrow 0$ and $\varphi \searrow \chi_{(-\infty,1]}$, we conclude

$$\begin{aligned}\left(F(\rho)\frac{\mu(B_\rho(x))}{\rho^n} + E(\rho)\right) - \left(F(\sigma)\frac{\mu(B_\sigma(x))}{\sigma^n} + E(\sigma)\right) \\ = G(\sigma,\rho) \int_{B\rho(x)\setminus B_\sigma(x)} \frac{|\nabla_M^\perp r|^2}{r^n}\, d\mu,\end{aligned} \tag{MF_{L^p}}$$

for some $G(\sigma,\rho) \in [e^{-\Lambda}, e^{\Lambda}]$ and for all $0 < \sigma < \rho \le \rho_0$.

2.3.3 Semi-Continuity of the Density.

Corollary 2.3.2. *Let $V \in \mathscr{R}_n(\Omega) \cap \mathscr{V}_n(\Omega)$, $x \in \operatorname{spt}\mu$ and assume its mean curvature is either*

1. *bounded by a constant Λ_1;*
2. *in $L^p_{\text{loc}}(\mu)$ and satisfies*

$$\rho_0^{1-\frac{n}{p}} \left(\int_{B_{\rho_0}(x)} |\underline{H}|^p \, d\mu\right)^{\frac{1}{p}} \le \alpha\Lambda_2, \quad \alpha := \frac{p}{4(p-n)}$$

for some constant $\Lambda_2 \in [0,1)$ where $\rho_0 := \min\{1, \operatorname{dist}(x,\partial\Omega)\}$.

Then $\Theta^n(\mu,x)$ exists.

Proof. In the first case we see that (MF_B) guarantees that the function

$$\rho \mapsto F(\rho)\frac{\mu(B_\rho(x))}{\rho^n} \tag{2.40}$$

is increasing in a right neighborhood of 0, where F is the function defined in 2.29, hence it is increasing as well and its limit at 0 is 1. Therefore $\Theta^n(\mu,x)$ coincides with the limit in 0 of the function defined in (2.40).

We repete the same argument for the second case, observing that (MP_{L^p}) gives the monotonicity of

$$\rho \mapsto F(\rho)\frac{\mu(B_\rho(x))}{\rho^n} \tag{2.41}$$

with F defined in (2.34) and $E(\rho) := F(\rho) - 1$.

□

Corollary 2.3.3. *Let* $V \in \mathscr{R}_n(\Omega) \cap \mathscr{V}_n(\Omega)$, $x \in \operatorname{spt}\mu$ *and suppose the existence an open set* $U \Subset \Omega$ *and a constant* $R > 0$ *such that* $U + B_R \subseteq \Omega$. *If* $\underline{H}$ *satisfies either*

1. $|\underline{H}| \le \Lambda_1$ *in* U;
2. $\underline{H} \in L^p_{\text{loc}}(\mu)$ *and*

$$R^{1-\frac{n}{p}} \left(\int_{B_R(x)} |\underline{H}|^p \, d\mu \right)^{\frac{1}{p}} \le \alpha \Lambda_2, \quad \alpha := \frac{p}{4(p-n)}$$

holds for some constant $\Lambda_2 \in [0,1)$ *for all* $x \in U$.

Then the function $x \mapsto \Theta^n(\mu, x)$ *is upper-semicontinuous in* U.

Proof. In any case, thanks to Corollary 2.3.2, the density $\Theta^n(\mu, x)$ is well defined at every point of U.

Assume that 1. is met and let $(x)_j \subseteq U$ be a sequence converging to some $x \in U$. Take $\rho, \varepsilon > 0$ such that $B_{\rho+\varepsilon}(x) \subseteq U$. Then, for $\tilde{j}$ large enough we have

$$B_\rho(x_j) \subseteq B_{\rho+\varepsilon}(x) \quad \forall j \ge \tilde{j},$$

therefore

$$\Theta^n(\mu, x_j) \le F(\rho) \frac{\mu(B_\rho(x_j))}{\omega_n \rho^n} \le F(\rho+\varepsilon) \frac{\mu(B_{\rho+\varepsilon}(x))}{\omega_n \rho^n}. \tag{2.42}$$

Since (2.42) holds for all j large enough

$$\limsup_{j \to \infty} \Theta^n(\mu, x_j) \le F(\rho+\varepsilon) \frac{B_{\rho+\varepsilon}(x)}{\omega_n \rho^n} \tag{2.43}$$

Letting $\varepsilon \searrow 0$ and then $\rho \searrow 0$ in (2.43) we are able to conclude.

The second case is completely analogous: one simply need to add the term $E(\rho)$ as it has been done in the proof of Corollary 2.3.2.

□

2.3.4 Generalized MF for Mean Curvature in L^1_{loc}.

In this subsection we will actually establish a generalization of the statement of the Monotonicity Formula. In particular, inspired by the fact that in the proof of the above-mentioned theorem we can merely require $\underline{H}$ to be L^1_{loc} and also consider functions different from γ to compose the testing vectorfield X as long as they are regular enough, we give

Theorem 2.3.2. *Let* $V \in \mathscr{R}_n(\Omega) \cap \mathscr{V}_n(\Omega)$, $f \in C^1_c(\Omega)$ *be a non-negative function and let* $\overline{x} \in \operatorname{spt}\mu$. *Then there exists a constant* $C = C(n) > 0$ *such that the inequality*

$$\begin{aligned} &\frac{1}{\omega_n \rho^n} \int_{B_\rho(\overline{x})} f \, d\mu \\ &\quad \ge \frac{1}{\omega_n \sigma^n} \int_{B_\sigma(\overline{x})} f \, d\mu - \frac{1}{n\omega_n} \int_{B_\rho(\overline{x})} \frac{|\nabla_M f| + f|\underline{H}|}{r^{n-1}} \, d\mu \end{aligned} \tag{2.44}$$

holds for every $0 \le \sigma \le \rho < \rho_0$, *with* $\rho_0 := \min\{1, \operatorname{dist}(\overline{x}, \partial\Omega)\}$

Proof of Theorem 2.3.2. Without loss of generality, we assume $\overline{x} = 0$. We define the function

$$h(x) := \frac{1}{\omega_n n(n-2)} \begin{cases} \frac{n}{2} - \frac{n-2}{2}|x|^2, & \text{if } |x| \le 1 \\ \frac{1}{|x|^{n-2}}, & \text{if } |x| > 1 \end{cases},$$

so that $h \in C^{1,1}(\mathbb{R}^{n+k}) \cap C^2(\mathbb{R}^{n+k} \setminus \partial B_1)$ and

$$\nabla h(x) = -\frac{1}{\omega_n n} \begin{cases} x, & \text{if } |x| \le 1 \\ \frac{x}{|x|^n}, & \text{if } |x| > 1 \end{cases},$$

$$D^2 h(x) = -\frac{1}{\omega_n} \begin{cases} \frac{1}{n} \mathbb{1}_{n+k}, & \text{if } |x| \le 1 \\ \frac{1}{|x|^n} \left(\frac{1}{n} \mathbb{1}_{n+k} - \frac{x \otimes x}{|x|^2} \right), & \text{if } |x| > 1 \end{cases}.$$

For $0 < \sigma \le \eta < \omega \le \rho < R$ we set

$$g_{\tau,\eta}(x) := \frac{h(x/\eta)}{\eta^{n-2}} - \frac{h(x/\tau)}{\tau^{n-2}}.$$

Then, simple computations give

$$g_{\tau,\eta} \equiv 0 \quad \text{on } \mathbb{R}^{n+k} \setminus B_\eta,$$

$$\nabla g_{\tau,\eta}(x) = -\frac{1}{\omega_n} \begin{cases} \left(\frac{1}{\tau^n} - \frac{1}{\eta^n} \right) x, & \text{if } 0 \le |x| < \tau \\ \left(\frac{1}{|x|^n} - \frac{1}{\eta^n} \right) x, & \text{if } \tau \le |x| < \eta \\ 0, & \text{if } \eta \le |x| < \infty \end{cases}.$$

Moreover, for each $S \in G(n+k, n)$ we have

$$\Delta_S g_{\tau,\eta} = \frac{\chi_{B_\eta}}{\omega_n \eta^n} - \frac{\chi_{B_\tau}}{\omega_n \tau^n} - \frac{\chi_{B_\eta \setminus B_\tau}}{\omega_n |x|^n} \left(1 - \frac{|\Pi_S x|^2}{|x|^2} \right) \le \frac{\chi_{B_\eta}}{\omega_n \eta^n} - \frac{\chi_{B_\tau}}{\omega_n \tau^n}.$$

Thus

$$\operatorname{div}_S(f \nabla g_{\tau,\eta}) \le \left(\frac{\chi_{B_\eta}}{\omega_n \eta^n} - \frac{\chi_{B_\tau}}{\omega_n \tau^n} \right) f + \nabla g_{\tau,\eta} \cdot \nabla_S f.$$

Finally we define

$$I(r) := \frac{1}{\omega_n r^n} \int_{B_r} f \, d\mu \quad \forall 0 < r \le \rho_0. \tag{2.45}$$

Then, recalling the definition of mean curvature and the properties of $g_{\tau,\eta}$ listed above, it follows that

$$\begin{aligned} \frac{I(\eta) - I(\tau)}{\eta - \tau} &= \frac{1}{\eta - \tau} \int \left(\frac{\chi_{B_\eta}}{\omega_n \eta^n} - \frac{\chi_{B_\tau}}{\omega_n \tau^n} \right) f \, d\mu \\ &\ge -\frac{1}{\eta - \tau} \int_{B_\eta} \left(f \nabla g_{\tau,\eta} \cdot \underline{H} + \nabla g_{\tau,\eta} \cdot \nabla_M f \right) d\mu \\ &\ge -\frac{1}{\eta - \tau} \|\nabla g_{\tau,\eta}\|_{L^\infty} \int_{B_\eta} \left(|\nabla_M f| + f |\underline{H}| \right) d\mu \\ &\ge -\frac{1}{\omega_n \tau^n} \int_{B_\eta} \left(|\nabla_M f| + f |\underline{H}| \right) d\mu. \end{aligned}$$

Letting $\eta \searrow \tau$, together with standard approximation arguments (for instance, the one used in the proof of the monotonicity formula in the $h \in \mathscr{L}^\infty(\mu_V)$ case), we deduce

$$I'(\tau) \geq -\frac{1}{\omega_n \tau^n} \int_{B_\tau} \left(|\nabla_M f| + f|\underline{H}|\right) d\mu. \tag{2.46}$$

Integrating (2.46) from $\tau = \sigma$ to $\tau = \rho$ yields, after an exchange of integrals,

$$I(\rho) - I(\sigma) \geq -C \int_{B_\rho} \frac{|\nabla_M f| + f|\underline{H}|}{r^{n-1}} \, d\mu, \tag{2.47}$$

for some constant $C = C(n)$. Recalling the definition of I, (2.47) gives exactly the claim.

□

2.3.5 Weighted MF.

The following result is a natural generalization of [DPGS24] in the case of $\underline{H} \in L^p(\mu)$ for some $p \in (n, \infty]$.

We now show that if, in the statement of the generalized monotonicity formula (Theorem 2.3.2), we require $\underline{H} \in L^p(\mu_V)$ for a sufficiently large p, a slight modification of the previous argument allows us to have a stronger monotonicity formula for convex non-negative functions.

Lemma 2.3.1. *Let $p \in (n, \infty]$ and $V = \underline{\mathrm{v}}(M, \vartheta) \in \mathscr{V}_n^p(\Omega) \cap \mathscr{R}_n(\Omega)$ such that $\vartheta(x) \geq 1$ for μ_V-a.e. x. Fix a point $\overline{x} \in \operatorname{spt}\mu_V$ and let $\rho_0 := \min\{1, \operatorname{dist}(\overline{x}, \partial\Omega)\}$. If $f \in C_c^1(\Omega)$ is a non-negative convex function such that $\|\nabla f\|_{L^\infty} \leq 1$, then the inequality*

$$\frac{1}{\omega_n \rho^n} \int_{B_\rho(0)} f \, d\mu \geq f(0) - C_0 \|\underline{H}\|_{L^p} (\|f\|_{L^\infty} + \rho) \rho^{1-n/p}, \tag{2.48}$$

where $C = C(n, p)$, holds for every $0 < \rho < \rho_0$.

Proof. Without losing generality, we assume $\overline{x} = 0$. Let us fix $0 < \sigma \leq \tau < \eta \leq \rho \leq \rho_0$ and consider the functions h, $g_{\tau,\eta}$ and I defined in the proof of Theorem 2.3.2. Then, recalling that $f \geq 0$ is convex

$$\operatorname{div}_S(f \nabla g_{\tau,\eta} - g_{\tau,\eta} \nabla f) \leq f \left(\frac{\chi_{B_\eta}}{\omega_n \eta^n} - \frac{\chi_{B_\tau}}{\omega_n \tau^n} \right). \tag{2.49}$$

Therefore,

$$\begin{aligned}
\frac{I(\eta) - I(\tau)}{\eta - \tau} &\geq \frac{1}{\eta - \tau} \int \operatorname{div}_M (f \nabla g_{\tau,\eta} - g_{\tau,\eta} \nabla f) \, d\mu \\
&= -\frac{1}{\eta - \tau} \int_{B_\eta} \left(f \nabla g_{\tau,\eta} \cdot \underline{H} - g_{\tau,\eta} \nabla f \cdot \underline{H} \right) d\mu \\
&\geq -\frac{1}{\eta - \tau} \left(\left| \int_{B_\eta} f \nabla g_{\tau,\eta} \cdot \underline{H} \, d\mu \right| + \left| \int_{B_\eta} g_{\tau,\eta} \nabla f \cdot \underline{H} \, d\mu \right| \right).
\end{aligned} \tag{2.50}$$

Denoting by $q \in [1, n/(n-1))$ the Young conjugate of p and using Hölder's inequality we obtain

$$|\int_{B_\eta} f \nabla g_{\tau,\eta} \cdot \underline{H}\, d\mu| \le C \|\nabla g_{\tau,\eta}\|_{L^\infty} \|f\|_{L^\infty} \|\underline{H}\|_{L^p} \eta^{n/q},$$
$$|\int_{B_\eta} g_{\tau,\eta} \nabla f \cdot \underline{H}\, d\mu| \le C \|g_{\tau,\eta}\|_{L^\infty} \|\underline{H}\|_{L^p} \vartheta^{n/q},$$

for some constant $C = C(n, p) > 0$. Since, upon changing C for a large one (still depending only on n and p), we have

$$\|g_{\tau,\eta}\|_{L^\infty} \le C \frac{\eta - \tau}{\tau^n} \quad \text{and} \quad \|\nabla g_{\tau,\eta}\|_{L^\infty} \le C \frac{\eta-\tau}{\tau^{n-1}},$$

then (2.50) implies

$$\frac{I(\eta) - I(\tau)}{\eta - \tau} \ge -\frac{C}{\tau^n} \|\underline{H}\|_{L^p} \left(\|f\|_{L^\infty} + \tau\right) \eta^{n/q}.$$

Using a standard approximation argument, letting η decrease to τ we deduce

$$I'(\tau) \ge -C \|\underline{H}\|_{L^p} \left(\|f\|_{L^\infty} \tau^{-n/p} \tau^{1-n/p}\right). \tag{2.51}$$

Integrating (2.51) $\tau = \sigma$ to $\tau = \rho$, we recover

$$I(\rho) - I(\sigma) \ge -C \|\underline{H}\|_{L^p} \left(\|f\|_{L^\infty} + (\rho^{1-n/p} - \sigma^{1-n/p}) + \rho^{2-n/p} - \sigma^{2-n/p}\right). \tag{2.52}$$

If

$$\vartheta(0) \ge 1, \tag{I}$$
$$0 \text{ is a Lebesgue point of } \vartheta, \tag{II}$$

than, passing to the limit as $\sigma \searrow 0$ in (2.52) gives immediately the claim.

In the general case we repeat the above arguments for a sequence $x_j \to 0$ such that (I) and (II) hold and consider the functions

$$I_j(r) := \frac{1}{\omega_n r^n} \int_{B_r(x_j)} f\, d\mu \quad \forall r \in (0, \mathrm{dist}(x_j, \partial B_1)).$$

We deduce the existence of a constant $C = C(n, p)$ such that

$$\frac{1}{\omega_n \rho^n} \int_{B_\rho(x_j)} f\, d\mu \ge f(x_j) - C \|\underline{H}\|_{L^p} \left(\|f\|_{L^\infty} + \rho\right) \rho^{1-n/p} \quad \forall j \in \mathbb{N}.$$

Finally (2.48) follows by continuity of the functions

$$x \mapsto \int_{B_r(x)} f\, d\mu, \quad x \mapsto f(x).$$

□

2.4. A Sobolev-Type Inequality for Varifolds

It is a well-know that, in a flat Euclidean space of dimension $N \in \mathbb{N}$, the integrability of the (weak) derivatives imply some better integrability the function. More precisely if a function f is integrable and non-negative in the whole space, that is $f \in L^1(\mathbb{R}^N)$ and f admits a (weak) gradient ∇f which is integrable too, then we can deduce the integrability of the function $f^{\frac{n}{n-1}}$; moreover there exists a constant $C = C(N)$ such that

$$\left(\int_{\mathbb{R}^N} f^{\frac{n}{n-1}}\, d\mathscr{L}^N\right)^{\frac{n-1}{n}} \leq C \int_{\mathbb{R}^N} |\nabla f|\, d\mathscr{L}^N. \tag{2.53}$$

In what follows we will prove something very analogous to (2.53) which applies to integrable functions on a rectifiable varifold rather than on a Euclidean space. Clearly we expect that the gradient which apprears in the right hand side of (2.53) will be replaced by the projected gradient onto the support of the varifold, namely ∇_M; moreover it is natural to expect that the mean curvature will play a crucial role as - in general - the geometry of an object usually plays important roles in regards to the generalization of integral relations which hold in the flat spaces (cfr. the generalization of the Divergence Theorem on Riemannian manifolds vs the standard Divergence Theorem). In fact the correct generalization of (2.53) has exactly the shape that we expect, that is

$$\left(\int_M f^{\frac{n}{n-1}}\, d\mu\right)^{\frac{n-1}{n}} \leq C \int_M \left(|\nabla_M f| + f|\underline{H}|\right) d\mu.$$

Before giving the statement of the theorem we prove a rather technical and useful lemma, on which, for simplifying the notation, we use the convention $f(\infty)$ instead of writing $\lim_{\tau\to\infty} f(\tau)$.

Lemma 2.4.1. *Suppose f, g are bounded, non-decreasing functions defined on $(0,\infty)$. If $n \geq 2$ and for any $0 < \sigma < \rho$ we have*

$$1 \leq \frac{f(\sigma)}{\sigma^n} \leq \frac{f(\rho)}{\rho^n} + \int_0^\rho \frac{g(\tau)}{\tau^n}\, d\tau. \tag{2.54}$$

Then there exists $\rho \in (0,\rho_0)$, with $\rho_0 := 2f(\infty)^{\frac{1}{n}}$, such that

$$f(5\rho) \leq \frac{5^n \rho_0}{2} g(\rho) \tag{2.55}$$

Proof. Assume by contradiction (2.55) to be false for all $\rho \in (0, \rho_0)$. Then

$$\begin{aligned}\sup_{0<\sigma<\rho_0} \frac{f(\sigma)}{\sigma^n} &\le \frac{f(\rho_0)}{{\rho_0}^n} + \int_0^{\rho_0} \frac{g(\tau)}{\tau^n}\, d\tau \\ &\le \frac{f(\rho_0)}{{\rho_0}^n} + \frac{2}{5^n \rho_0} \int_0^{\rho_0} \frac{f(5\tau)}{\tau^n}\, d\tau \\ &\le \frac{f(\infty)}{{\rho_0}^n} + \frac{2}{5\rho_0} \int_0^{5\rho_0} \frac{f(t)}{t^n}\, dt \\ &= \frac{f(\infty)}{{\rho_0}^n} + \frac{2}{5\rho_0} \left(\int_0^{\rho_0} \frac{f(t)}{t^n}\, dt + \int_{\rho_0}^{5\rho_0} \frac{f(t)}{t^n}\, dt \right) \\ &\le \frac{f(\infty)}{{\rho_0}^n} + \frac{2}{5} \sup_{0<\sigma<\rho_0} \frac{f(\sigma)}{\sigma^n} + \frac{2}{5(n-1)} \frac{f(\infty)}{{\rho_0}^n}.\end{aligned}$$

Therefore

$$\frac{3}{5} \sup_{0<\sigma<\rho_0} \frac{f(\sigma)}{\sigma^n} \le \frac{5(n-1)+2}{5(n-1)} \frac{f(\infty)}{{\rho_0}^n}. \tag{2.56}$$

Using the first inequality of (2.54), (2.56), the definition of ρ_0 and the elementary inequality

$$\frac{5(n-1)+2}{5(n-1)} \le 2 \quad \forall n \ge 2,$$

we deduce

$$1 \le \frac{1}{2^{n-1}},$$

which is a contradiction.

□

Theorem 2.4.1 (Michael-Simon's Inequality). *Let $V = \underline{\mathrm{v}}(M\vartheta) \in \mathscr{R}_n(\Omega) \cap \mathscr{I}_n(\Omega)$ and assume $\vartheta \ge 1$ $\mathscr{H}^n_{\llcorner M}$-a.e.. Then there exists a constant $C = C(n)$ such that*

$$\left(\int_M f^{\frac{n}{n-1}}\, d\mu \right)^{\frac{n-1}{n}} \le C \int_M \left(|\nabla_M f| + f|\underline{H}| \right) d\mu \tag{2.57}$$

holds for every non-negative function $f \in C^1_c(\Omega)$.

Proof. Fix a non-negative $f \in C^1_c(\Omega)$. Since the support of f is compact in Ω, then we can run through again the proof of Therem 2.3.2 untill we integrate in the interval (σ, ρ) the expression (2.46), so that we can write

$$\begin{aligned}&\frac{1}{\rho^n} \int_{B_\rho(x)} f\, d\mu - \frac{1}{\sigma^n} \int_{B_\sigma(x)} f\, d\mu \\ &\qquad \ge -C \int_0^\rho \frac{1}{\tau^n} \int_{B_\tau(x)} (|\nabla_M f| + f|\underline{H}|)\, d\mu\, dt,\end{aligned} \tag{2.58}$$

for any $x \in \operatorname{spt}\mu$ and for any $0 < \sigma \le \rho \le R_x$ with R_x small enough. From (2.58) we deduce that the functions

$$\varphi(\rho) := \int_{B_\rho(x)} f\, d\mu \quad \psi(\rho) := \int_{B_\rho(x)} (|\nabla_M f| + f|\underline{H}|)\, d\mu$$

satisfy the assumptions of Lemma 2.4.1 as soon as $f(x) \ge 1$, hence, for any element x of the set

$$M_1 := \operatorname{spt}\mu \cap \{y \in \Omega : f(y) \ge 1\}$$

we can find

$$\rho = \rho_x < 2\left(\int f\, d\mu\right)^{\frac{1}{n}}$$

such that

$$\varphi(5\rho) \le 5^n \left(\int f\, d\mu\right)^{\frac{1}{n}} \psi(\rho). \tag{2.59}$$

By virtue of the 5-Covering Lemma we can find countable sequences $(x_j)_j \subseteq M_1$ and $(\rho_j)_j$ such that, defining $B_j := B_{\rho_j}(x_j)$,

$$\begin{gathered} \text{(2.59) holds for every couple } (x_j, \rho_j), \\ B_j \cap B\ell = \varnothing \quad \forall j \ne \ell, \\ M_1 \subseteq \bigcup_{j \ge 0} B_j; \end{gathered}$$

therefore summing among all of the balls (B_j) the expression (2.59), we obtain

$$\int_{M_1} f\, d\mu \le 5^n \left(\int f\, d\mu\right)^{\frac{1}{n}} \int (|\nabla_M f| + f|\underline{H}|)\, d\mu.$$

Let us fix a function $\gamma \in C^1(\mathbb{R})$ with the following properties

$$\gamma \equiv 0 \quad \text{in } (-\infty, 0], \quad \gamma \equiv 1 \quad \text{in } [\varepsilon, \infty), \quad \gamma' \ge 0,$$

with $\varepsilon > 0$ to be chosen later, and let us define the function

$$\widetilde{f}_t(y) := \gamma(f(y) - t) \quad \forall t > 0, y \in \Omega.$$

Defining

$$\begin{aligned} M_\alpha &:= \operatorname{spt}\mu \cap \{y \in \Omega : f(y) \ge \alpha\} \quad \forall \alpha > 0, \\ \widetilde{M}_1 &:= \operatorname{spt}\mu \cap \{y \in \Omega : \widetilde{f}_t(y) \ge 1\} \supseteq M_{t+\varepsilon}, \end{aligned}$$

and repeating all of the above arguments to the function $\widetilde{f}_t$ instead of f we get

$$\begin{aligned} \mu(M_{t+\varepsilon}) &\le \int_{M_1} \widetilde{f}_t\, d\mu \\ &\le 5^n (\mu(M_t))^{\frac{1}{n}} \int (\gamma'(f-t)|\nabla_M f| + \widetilde{f}_t|\underline{H}|)\, d\mu, \end{aligned} \tag{2.60}$$

hence, multiplying by $(t+\varepsilon)^{\frac{1}{n-1}}$ both sides of (2.60), and obseving that

$$\mu(M_t)(t+\varepsilon)^{\frac{n}{n-1}} \le \int_{M_t} (f+\varepsilon)^{\frac{n}{n-1}}\, d\mu \le \int_M (f+\varepsilon)^{\frac{n}{n-1}}\, d\mu,$$

we get

$$\begin{aligned}&(t+\varepsilon)^{\frac{1}{n-1}}\mu(M_{t+\varepsilon})\\ &\qquad \le 5^n \left(\int (f+\varepsilon)^{\frac{n}{n-1}}\right)^{\frac{1}{n}} \int (\gamma'(f-t)|\nabla_M f| + \widetilde{f}_t|\underline{H}|)\, d\mu.\end{aligned} \tag{2.61}$$

Now we integrate (2.61) for $t \in (0,\infty)$. On the left hand side we get, using Fubini-Tonelli,

$$\int_0^\infty (t+\varepsilon)^{\frac{1}{n-1}}\mu(M_{t+\varepsilon})\, dt = \frac{n}{n-1}\int_{M_\varepsilon} (f^{\frac{n}{n-1}} - \varepsilon^{\frac{n}{n-1}})\, d\mu.$$

On the other hand, observing that the Fundamental Theorem of Calculus tells us

$$\int_0^\infty \int (\gamma'(f-t)|\nabla_M f|)\, d\mu\, dt \le \int |\nabla_M f|\, d\mu,$$

and that Fubini-Tonelli again, this gives

$$\begin{aligned}\int_0^\infty \int \widetilde{f}_t|\underline{H}|\, d\mu\, dt &\le \int_0^\infty \int_{M_t} |\underline{H}|\, d\mu\\ &= \int f|\underline{H}|\, d\mu,\end{aligned}$$

we recover

$$\begin{aligned}&\frac{n}{n-1}\int_{M_\varepsilon} (f^{\frac{n}{n-1}} - \varepsilon^{\frac{n}{n-1}})\, d\mu\\ &\qquad \le 5^n \left(\int (f+\varepsilon)^{\frac{n}{n-1}}\right)^{\frac{1}{n}} \int (|\nabla_M f| + f|\underline{H}|)\, d\mu.\end{aligned} \tag{2.62}$$

Passing to the limit as $\varepsilon \searrow 0$ in (2.62) yields (2.57) with the constant $C(n) = 5^n(n-1)/n$.

□

3. Rectifiability and Precompactness of Integral Varifolds

3.1. Introduction

The goal of this chapter is to state and prove two results: a rectifiablity criterion for general varifolds, and the pre-compactness of the class of integer rectifiable varifolds.

First we will prove that as soon as a n-varifold admits a (generalized) mean curvature in L^1_{loc} then its restriction to the set of positive lower n-density of it forms a n-rectifiable varifold. The precise statement is the content of the following theorem, see below for the notation used.

Theorem 3.1.1. *Let $V \in \mathscr{V}_n(\Omega)$. Then V_* is a n-rectifiable varifold.*

For sake of readability, we briefly recall the notation that will be used throughout this chapter: $\Omega \subseteq \mathbb{R}^{n+k}$ is any open subset; $\mathbb{V}_n(\Omega)$ is the class of n-varifolds in Ω; $\mathscr{V}_n(\Omega)$ denotes the class of varifolds having locally bounded first variation (i.e. they have a generalized mean curvature in $L^1_{\mathrm{loc}}(\mu_V)$, with μ_V being the mass measure of a varifold V); by $\mathscr{R}_n(\Omega)$ we denote the classes of n-rectifiable varifolds and by $\mathscr{I}_n(\Omega)$ the subclass of n-rectifabile varifolds having integral multiplicity. Finally, if $V \in \mathbb{V}_n(\Omega)$, then we will write, in virtue of LEMMA(NUMBER),

$$V = \mu \otimes \left(\eta^x\right),$$

with μ Radon measure on Ω, that we will briefly denote by $\mu \in \mathrm{Rad}(\Omega)$, and η^x probability measure on $G(n+k,n)$ for μ–a.e. x and V_* will be the restriction of V to the set

$$M_* := \left\{x \in \mathrm{spt}\mu : \Theta^n_*(\mu,x) > 0\right\}.$$

The second result that will be covered in this chapter is a precompactness theorem for integral varifolds. In particular we will give sufficient conditions to a sequence of integral n-rectifiable varifolds $(V_j)_j$ to admit a subsequence $(V_{j_\ell})_\ell$ such that $V_{j_\ell} \rightharpoonup^* V$, for some $V \in \mathscr{I}_n(\Omega)$.

Theorem 3.1.2. *Let $(V_j)_j \subseteq \mathscr{I}_n(\Omega) \cap \mathscr{V}_n(\Omega)$ such that the following property is satisfied*

$$\sup_{j\in\mathbb{N}} \Big(\mu_j(W) + \|\underline{H}_j\|_1(W) \Big) < \infty \quad \forall W \Subset \Omega,$$

then there exists a subsequence $(V_{j_\ell})_\ell$ of $(V_j)_j$ and a varifold $V \in \mathscr{I}_n(\Omega) \cap \mathscr{V}_n(\Omega)$ such that $V_{j_\ell} \rightharpoonup^* V$.

In order to prove the main theorems, let us introduce some preliminary notions.

Definition 3.1. Let η be any probability measure on $G(n+k,n)$. Then we define $\mathcal{A}(\eta)$ as the matrix

$$\mathcal{A}(\eta) := \int_{G(n+k,n)} S \, d\eta(S),$$

where we have identified as usual the n-plane S with the matrix of the orthogonal projection $\mathbb{R}^{n+k} \to S \subseteq \mathbb{R}^{n+k}$.

By definition of first variation δV of a varifold V, then we see that for any vector field $X \in C^1_c(\mathbb{R}^{n+k};\mathbb{R}^{n+k})$ we have

$$\begin{aligned}
\delta V(X) &= \int_{G_n(\mathbb{R}^{n+k})} \operatorname{div}_S X(y) \, dV(y,S) \\
&= \int_{\mathbb{R}^{n+k}} \left(\int_{G(n+k,n)} S : \nabla X(y) \, d\eta^y(S) \right) d\mu(y) \\
&= \int_{\mathbb{R}^{n+k}} \mathcal{A}(\eta^y) : \nabla X(y) \, d\mu(y).
\end{aligned}$$

Recalling the definition of mean curvature for a varifolds having locally bounded first variation,

$$\begin{aligned}
\langle \underline{H}, X \rangle &:= \int_{\mathbb{R}^{n+k}} \underline{H} \cdot X \, d\mu \\
&= -\delta V(X) \\
&= -\int_{\mathbb{R}^{n+k}} \mathcal{A}(\eta^y) : \nabla X \, d\mu.
\end{aligned}$$

Let us end this section with a key property of $\mathcal{A}(\eta)$.

Proposition 3.1.1. *Let η be a probability measure on $G(n+k,n)$. Then*

$$\dim \ker \mathcal{A}(\eta) \leq k. \tag{3.1}$$

Furthermore the equality in holds in (3.1) *if and only if η is a Dirac delta measure centered at some $S_0 \in G(n+k,n)$.*

Proof. Assume that $v \in \ker \mathcal{A}(\eta) \subseteq \mathbb{R}^{n+k}$, then

$$0 = \langle v, \mathcal{A}(\eta) v \rangle = \int_{G(n+k,n)} \langle v, Sv \rangle \, d\eta(S).$$

Since $S \in \mathrm{Sym}^+_{n+k}(\mathbb{R})$, then this implies

$$|Sv|^2 = \langle v, Sv\rangle = 0 \quad \eta - a.e.\, S \in G(n+k,n),$$

therefore $v \in \ker S$ for η-a.e. S, hence there exists $G \subseteq G(n+k,n)$ of full η-measure such that

$$\ker \mathcal{A}(\eta) \subseteq \bigcap_{S\in G} \ker S. \tag{3.2}$$

Recalling that $\eta(G) = 1$, we deduce that $G \neq \emptyset$, thus the right-hand side of (3.2) has dimension not greater than k.

One implication of the second claim is trivial, so we only need to prove that if η is not a Dirac delta, then the dimension of $\ker \mathcal{A}(\eta)$ is strictly smaller than k. If η is not a Dirac delta, then every η-full measure subsets of $G(n+k,n)$ contains at least two distinct elements P, Q. Thene $\ker P \cap \ker Q$ has dimension strictly smaller than k. The conclusion follows from (3.2).

If η is not a Dirac delta, then every η-full measure subsets of $G(n+k,n)$ contains at least two distinct elements P, Q. Therefore □

3.2. Preliminary Results

Recall that if μ is any Radon measure on $\mathbb{R}^{n+k}$ and $x \in \mathbb{R}^{n+k}$, then the upper and lower n-density of μ at x are denoted by $\Theta^n_*(\mu,x)$ and $\Theta^{n*}(\mu,x)$ respectively. If those two coincide, then we say that μ has n-density at x and denote it by $\Theta^n(\mu,x)$.

For any $x \in \mathbb{R}^{n+k}$ and $r > 0$ we consider the function $\varphi_{r,x}(y) := (y-x)/r$. We define the dilation of a Radon measure μ by a factor of r and center x as the Radon measure

$$\mu_{r,x} := \frac{1}{\mu(B_r(x))}\left(\varphi_{r,x}\right)_\# \mu_{\llcorner B_1(0)}.$$

Observe that, thanks to the normalization factor in the definition of $\mu_{r,x}$ and Banach-Alaoglu theorem, given μ and x for every sequence $r_j \searrow 0$ there exists a subsequence $(r_{j_\ell})_\ell$ and a Radon measure σ_x such that

$$\mu_{x,r_{j_\ell}} \rightharpoonup^* \sigma_x$$

Definition 3.2. Given $\mu \in \mathrm{Rad}(\Omega)$ and $x \in \mathrm{spt}\mu$, we define the set $\mathrm{Tan}(\mu,x)$ as the family of all measures σ such that there exists a sequence $r_j \searrow 0$ with $\mu_{r_j,x} \rightharpoonup^* \sigma$. The elemets of $\mathrm{Tan}(\mu,x)$ are called tangent measures of μ at the point x.

Lemma 3.2.1. *Let $\mu \in \mathrm{Rad}(\Omega)$ and $x \in \Omega$ any point such that $\Theta^n_*(\mu,x) > 0$. Then, for every $t \in (0,1)$ there exists a tangent measure σ_t at x such that*

$$\sigma_t(\overline{B_t}) \geq t^n$$

Proof. <u>Step 1</u>. We claim that for all $x \in \Omega$ such that $\Theta^n_*(\mu,x) > 0$ we have

$$\limsup_{r\to 0} \frac{\mu(B_{tr}(x))}{\mu(B_r(x))} \geq t^n \quad \forall\, t \in (0,1). \tag{3.3}$$

Assume indeed the existence of $x_0 \in \Omega$ of positive lower n-density and of constants $\varepsilon_0 > 0$, $t_0 \in (0,1)$ and $r_0 > 0$ such that

$$\mu(B_{t_0 r}(x_0)) \leq (1-\varepsilon_0)\, t_0^n \mu(B_r(x_0) \quad \forall r \in (0, r_0]. \tag{3.4}$$

Since $t_0^j r_0 < t_0 r_0 < r_0$ for any $j \geq 2$, we can iterate (3.4) in order to obtain

$$\mu(B_{t_0^j r_0}(x_0)) \leq (1-\varepsilon_0)^j\, t_0^{nj} \mu(B_{r_0}(x_0)),$$

and so we deduce

$$0 < \lim_{j\to\infty} \frac{\mu(B_{t_0^j}(x))}{\omega_n t_0^j} \leq \lim_{j\to\infty} (1-\varepsilon_0)^j \mu(B_{r_0}(x)) = 0,$$

a contradiction.

STEP 2. Fix $t \in (0,1)$ and let $r_j \searrow 0$ realizing the limsup corresponding to (3.3), i.e.

$$\limsup_{r\to 0} \frac{\mu(B_{tr}(x))}{\mu(B_r(x))} = \lim_{j\to\infty} \frac{\mu(B_{tr_j}(x))}{\mu(B_{r_j}(x))}. \tag{3.5}$$

Recalling the definition of dilation of a measure we see that the right-hand side of (3.5) consists of the limit as $r_j \searrow 0$ of $\mu_{r_j,x}(B_t)$, and since - up to subsequences - $\mu_{r_j,x}$ converges weakly$-^*$ to some tangent measure σ_t, then, by Fatou,

$$\sigma_t(\overline{B_t}) \geq \limsup_{j\to\infty} \mu_{r_j,x}(\overline{B_t}) \geq t^n.$$

□

The above lemma states that at, as soon as the lower n-density of a measure at a point is positive, then the tangent set is non-trivial. Furthermore it says that there exists at least one measure in $\mathrm{Tan}(\mu, x)$ which, at scale t, looks at most n-dimensional.

The definition of push-forward of a varifold needs to be slightly different from the one of push-forward of a measure. More precisely we know that if τ is any Radon measure defined on an abstract Borel space $(X, \mathcal{B}(X))$, the the push-forward of τ through a function $\psi : X \to X$ is defined by testing against C_c^0 functions as

$$\int_X f(z)\, d(\psi_\# \tau)(z) := \int_X f(\psi(y))\, d\tau(y). \tag{3.6}$$

On the other hand, given a varifold $V \in \mathbb{V}_n(\Omega)$ and $\psi : \Omega \to \mathbb{R}^{n+k}$ a differentiable function, we see that we can not use the formula (3.6) for constructing the push-forward of V through ψ. Indeed the test functions for V are of the type $g \in C_c^0(G_n(\mathbb{R}^{n+k}))$, hence they have a $\mathbb{R}^{n+k}$ part and a Grassmannian part. The most natural thing is to try and replicate (3.6) with the map $(y, T) \mapsto (\psi(y), \nabla\psi(S))$, in order to get a formula of the type

$$\int g(z,S)\, d(\psi^\# V)(z,S) \overset{\text{NON DEF}}{=} \int g(\psi(y), \nabla\psi(T))\, dV(y,T), \tag{3.7}$$

where the notation $\psi^\# V$ denotes the push-forward of V as a varifold on Ω rather than the one of a measure on $G_n(\mathbb{R}^{n+k})$; but (3.7) is not well suited for applying the area formula. A better definition would consider the deformation factor of the measure induced by ψ, i.e. the Jacobian. The definition that we give of push-forward of V with respect to a locally Lipschitz function $\psi : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is given by the position

$$\int g(z,S)\, d(\psi^\# V)(z,S) := \int g(\psi(y), \nabla\psi(T))\, J_T\psi(y)\, dV(y,T) \tag{3.8}$$

for all $g \in C^0_c(G_n(\mathbb{R}^{n+k}))$.

The presence of the Jacobian factor in the definition of push-forward of varifolds require us to make appear a compensation factor in the definition of dilation of a varifold. Observe that the dilation function $\varphi_{r,x}(y) := (y-x)/r$ has n-dimensional Jacobian equal to $1/r^n$ for every n-plane, therefore if we wish to use Banach-Alaoglu Theorem to prove the existence of tangent varifolds at almost every point the definition of dilated varifold needs to be as follows.

Definition 3.3. Let $V \in \mathbb{V}_n(\Omega)$, $x \in \Omega$ and $r > 0$. Denoting by $\varphi_{r,x}$ the map $y \mapsto (y-x)/r$, we define the dilation of V by factor r and of center x as the varifold

$$V_{r,x} := \frac{r^n}{\mu(B_r(x))} \varphi^\#_{r,x} V_{\llcorner G_n(B_1)},$$

with $\varphi^\#_{r,x} V$ defined as in (3.8).

Lemma 3.2.2. *Let $V = \mu \otimes (\eta^x)_x \in \mathbb{V}_n(\Omega)$. For μ-a.e. $x \in \Omega$ and every sequence $r_j \searrow 0$ there exists a subsequence $(r_{j_\ell})_\ell$ and $\sigma \in \operatorname{Tan}(\mu, x)$ such that*

$$V_{r_{j_\ell},x} = \mu_{r_{j_\ell},x} \otimes (\eta^{x+yr_{j_\ell}})_y \rightharpoonup^* \sigma \otimes \eta^x =: V^\infty_x \tag{3.9}$$

Proof. The first equality of (3.9) follows immediately from the definitions of dilations of varifolds and measures.

Fix $x \in \Omega$ such that $\operatorname{Tan}(\mu, x) \neq \varnothing$ (from the above remarks μ-a.e. x satisfies this property) and choose $\sigma \in \operatorname{Tan}(\mu, x)$. We set $(r_{j_\ell})_\ell$ to be a subsequence of $(r_j)_j$ such that

$$\mu_{r_{j_\ell},x} \rightharpoonup^* \sigma \tag{3.10}$$

and fix $f \in C^0_c(G_n(B_1))$ arbitrarily. Then

$$\begin{aligned}
&\int_{B_1} \int f\, d\eta^{x+r_{j_\ell}z}\, d\mu_{r_{j_\ell},x}(z) \\
&= \int_{B_1} \left(\int f\, d\eta^x + \int f\, d\eta^{x+r_{j_\ell}z} - \int f\, d\eta^x \right) d\mu_{r_{j_\ell},x}(z) \\
&= \int f\, d(\mu_{r_{j_\ell},x} \otimes \eta^x) + \fint_{B_r(x)} \left(\int f_{r_{j_\ell},x}\, d\eta^y - \int f_{r_{j_\ell},x}\, d\eta^x \right) d\mu(y)
\end{aligned}$$

where $f_{r_{j_\ell},x}(y,T) := f(\varphi_{r_{j_\ell},x}(y),T)$. From (3.10) and the continuity of the function

$$z \mapsto \int f(z,T)\, d\eta^x(T)$$

we get

$$\int f\, d(\mu_{r_{j_\ell},x} \otimes \eta^x) \xrightarrow[\ell\to\infty]{} \int f\, d(\sigma \otimes \eta^x).$$

We now prove that

$$\fint_{B_r(x)} \left(\int f_{r_{j_\ell},x}\, d\eta^y - \int f_{r_{j_\ell},x}\, d\eta^x \right) d\mu(y) \xrightarrow[\ell\to\infty]{} 0 \tag{3.11}$$

holds for μ-a.e. $x \in \Omega$. Recalling that $C_c^0(G(n+k,n))$ endowed with the topology of the uniform convergence is separable, we define a countable family $\mathscr{S} \subseteq C_c^0(G(n+k,n))$ which is dense with respect to the aforementioned topology. For each $\phi \in \mathscr{S}$ we define the function

$$F_\phi : \Omega \to \mathbb{R}, \quad F_\phi(x) := \int \phi\, d\eta^x.$$

Then $F_\phi \in L^1(\Omega,\mu)$ for all $\phi \in \mathscr{S}$ hence the set S_ϕ of Lebesgue points of F_φ is a full μ-measure subset of Ω. Recalling that $\mathscr{S}$ is countable, and defining $S := \bigcap_{\phi\in\mathscr{S}} S_\phi$ then

$$\mu(\Omega \setminus S) = 0.$$

If $x \in S$, $\psi \in C_c^0(B_1)$ and $\phi \in \mathscr{S}$, then, writing $\psi_{r_{j_\ell},x}(y) := \psi(\varphi_{r_{j_\ell},x}(y))$, we have

$$\int_{B_r(x)} \int \psi_{r_{j_\ell},x}(y)\phi(T)\, d\eta^y(T)\, d\mu(y) = \int_{B_r(x)} \psi_{r_{j_\ell},x}(y) F_\phi(y)\, d\mu(y)$$
$$\int_{B_r(x)} \int \psi_{r_{j_\ell},x}(y)\phi(T)\, d\eta^x(T)\, d\mu(y) = \int_{B_r(x)} \psi_{r_{j_\ell},x}(y) F_\phi(x)\, d\mu(y).$$

Moreover, recalling that x is a Lebesgue point of F_ϕ, then

$$|\fint_{B_r(x)} \psi_{r_{j_\ell},x}(F_\phi - F_\phi(x))\, d\mu| \le \|\psi\|_{C^0} \fint_{B_r(x)} |F_\phi - F_\phi(x)|\, d\mu \xrightarrow[\ell\to\infty]{} 0.$$

This proves (3.11) in the particular case $f = \psi\phi$. By linearity we recover (3.11) for all of the functions of the form

$$(y,T) \mapsto \sum_{m=1}^{N} \psi_m(y)\phi_m(T) \quad \psi_m \in C_c^0(B_1), \phi \in \mathscr{S}, N \in \mathbb{N},$$

which are dense in $C_c^0(G_n(B_1))$ in the topology induced by the norm of the uniform convergence. Via a standard approximation argument we deduce (3.11) for the arbitrarily fixed f.

□

Definition 3.4. We call the varifolds V_x^∞ arising from (3.9) tangent varifolds of V at x, and we collect all of them in the set $\mathrm{Tan}(V,x)$.

An important remark about Lemma 3.2.2 is the fact that the Grasmannian part of the tangent varifolds is independent of both the space variable and of the chosen blow up $(r_j)_j$; hence one could write, with some abuse of notation that

$$\mathrm{Tan}(V,x) = \mathrm{Tan}(\mu,x) \otimes \eta^x.$$

Furthermore, by using standard theorems in differentiation of measures, we have

$$\lim_{r\to 0} \frac{\mu_*(B_r(x))}{\mu(B_r(x))} = 1 \quad \mu_* - a.e\, x,$$

and, recalling Corllary 1.1.3,

$$\Theta_*^n(\mu_*,x) > 0 \quad \text{for } \mu_* - a.e.\, x,$$

and, as a consequence

$$\mathrm{Tan}(\mu,x) = \mathrm{Tan}(\mu_*,x) \quad \text{for } \mu_* - a.e.\, x,. \tag{3.12}$$

Finally Lemma 3.2.2, combined with (3.12), implies

$$\mathrm{Tan}(V,x) = \mathrm{Tan}(V_*,x) \quad \text{for } \mu_* - a.e.\, x. \tag{3.13}$$

Later on we will see how these considerations play a crucial role in the proof of Theorem 3.1.1.

Lemma 3.2.3. *Let $V \in \mathscr{V}_n(\Omega)$. Then*

1. *at μ-a.e. point x, every tangent varifold $W \in \mathrm{Tan}(V,x)$ is stationary (i.e. $\underline{H}_W \equiv 0$);*
2. *if $W \in \mathrm{Tan}(V,x)$ and $W = \sigma \otimes \eta^x$ for some $\sigma \in \mathrm{Tan}(\mu,x)$ (which, by Lemma 3.2.2, happens μ-a.e.) then σ is invariant under translations by directions in $\mathcal{A}(\eta^x)$.*

Before proving the above result, we recall that if $\sigma \in \mathrm{Rad}(\Omega)$, then we have a naturally associated distribution T_σ defined by the position

$$T_\sigma(\varphi) := \int_\Omega \varphi\, d\sigma =: \langle \sigma, \varphi \rangle \quad \forall \varphi \in C_c^\infty(\Omega).$$

Furthermore the distributional derivatives $\partial_i \sigma$ are defined as the distributional derivatives of T_σ, namely

$$\langle \partial_i \sigma, \varphi \rangle := -\langle \sigma, \partial_i \varphi \rangle \quad \forall \varphi \in C_c^\infty(\Omega).$$

The distributional gradient of σ is therefore the (unique) vector valued distribution on Ω such taht

$$\nabla\sigma \cdot X := \langle \nabla \sigma, X \rangle := \sum_{i=1}^n \langle \partial_i \sigma, X_i \rangle = -\int_\Omega \mathrm{div} X\, d\sigma$$

where X is any element of $C^1_c(\Omega, \mathbb{R}^{n+k})$. With these definitions we are able to define consistently the partial derivative of σ in direction v, where v in any unitary vector of $\mathbb{R}^{n+k}$, as

$$\partial_v \sigma = \nabla\sigma \cdot v.$$

Adopting this convention it is not hard to see that a Radon measure σ is invariant under translations in direction v if and only if the partial derivative of σ with respect to v is 0. In fact we have the formal computation

$$\frac{d}{dt}\sigma(B+tv) = \nabla\sigma(B+tv)\cdot v = \partial_v\sigma(B+tv) \quad B \in \mathcal{B}(\Omega), v \in B_1$$

which one can make rigorous by testing against smooth functions. Therefore, proving that a measure σ is invariant under translations in direction lying on a n-plane T is equivalent of proving that

$$T\nabla\sigma = 0$$

in the distributional sense. More explicitly:

$$\int T : \nabla X \, d\sigma = 0 \quad \forall X \in C^1_c(\Omega; \mathbb{R}^{n+k}).$$

Proof of Lemma 3.2.3. Let Ω' be the set of $x \in \Omega$ be such that both the conclusion of Lemma 3.2.2 and the mean curvature of V is well defined and has finite absolute value at x, in the sense that

$$|\underline{H}(x)| = \lim_{r\to 0} \frac{\|\delta V\|(B_r(x))}{\mu(B_r(x))} \in \mathbb{R},$$

i.e. x is a Lebesgue point for $|\underline{H}|$. From Lemma 3.2.2 and Lebesgue differentiation theorem, we deduce that Ω' is a full μ-measure subset of Ω, hence it will be enough to prove the statement for every point of Ω'.

Let $r_j \searrow 0$ be a sequence such that

$$V_{r_j,x} \rightharpoonup^* W = \sigma \otimes \eta^x. \tag{3.14}$$

For an arbitrary vector field $X \in C^1_c(B_1; \mathbb{R}^{n+k})$ we define $X_j := X \circ \varphi_{r_j,x}$, where $\varphi_{r_j,x}$ denotes as usual the dilation of factor r_j with center x. Then, observing that $\nabla X \circ \varphi_{r_j,x} = \nabla X_j / r_j$, and recalling the definition of push-forward of a varifold, we deduce

$$\begin{aligned}
\langle \underline{H}_j, X\rangle &= \int_{G(B_1)} S : \nabla X(y)\, dV_{r_j,x}(y,S) \\
&= r_j \frac{\langle \underline{H}, X_j\rangle}{\mu(B_{r_j}(x))} \\
&\le r_j \left(\fint_{B_{r_j}(x)} \underline{H}\, d\mu \right) \|X_j\|_{C^0} \\
&\xrightarrow[j\to\infty]{} 0.
\end{aligned} \tag{3.15}$$

On the other hand, by (3.14), we deduce that W has a mean curvature $\underline{H}_W$ which is μ_W integrable, and we have

$$\langle \underline{H}_j, X\rangle = \int_{G(B_1)} S : \nabla X(y)\, dV_{r_j,x}(y,S) \xrightarrow[j\to\infty]{} \langle \underline{H}_W, X\rangle. \tag{3.16}$$

From (3.15) and (3.16) we obtain the first claim.

For the second claim we observe that, by simply recalling the definition of mean curvature,

$$0 = \langle \underline{H}_W, X\rangle = \int S : \nabla X(y)\, dV(y,S) = \int_{B_1} \mathcal{A}(\eta^x) : \nabla X\, d\sigma$$

holds true for any $X \in C^1_c(B_1;\mathbb{R}^{n+k})$. By the remark made right before the start of this proof, we deduce that σ is invariant under translations in directions of $\mathcal{A}(\eta^x)$

□

3.3. Further Lemmas

The key step for proving the n-rectifiability theorem is to prove the following. Under the same assumptions of Theorem 3.1.1 then:

1) the Grassmannian part of the varifold concentrated on a single plane at μ_*-almost every point;

2) the mass measure is absolutely continuous with respect to $\mathscr{H}^n$;

3) the tangent measure of μ at μ_*-a.e. every point is one unique plane, which coincides with the plane of 1).

In particular, recalling the definition of V_*, we can also drop the assumption on the lower n-density at the cost of replacing V by V_* in 1), 2) and 3).

Let us begin by proving that the existence of a locally integrable mean curvature forces the Grassmannain part into a single n-plane.

Lemma 3.3.1. *Let $V = \mu \otimes (\eta^x) \in \mathscr{V}_n(\Omega)$. Then for μ_* -a.e. $x \in \Omega$, $\eta^x = \delta_{T_x}$, for some $T_x \in G(n+k,n)$.*

Proof. By virtue of Lemma 3.2.2, we know that for μ-a.e. x we can find a sequence $r_j \searrow 0$ and $\sigma \in \mathrm{Tan}(\mu,x)$ such that

$$\mu_{r_j,x} \rightharpoonup^* \sigma, \quad V_{r_j,x} \rightharpoonup^* \sigma\otimes\eta^x.$$

Therefore, recalling the definition of μ_* and remarks (3.12) and (3.13), then we can repete the above considerations for μ_*-a.e. point replacing μ and V by μ_* and V_* respectively. Hence we suppose

$$(\mu_*)_{r_j,x} \rightharpoonup^* \sigma, \quad (V_*)_{r_j,x} \rightharpoonup^* W := \sigma\otimes\eta^x.$$

Moreover, invoking Lemma 3.2.1, for any $t \in (0, \tau(n,k))$ (with τ to be chosen later) we can - upon changing the sequence r_j and the tangent measure σ - assume that

$$\sigma(\overline{B_t}) \geq t^n; \tag{3.17}$$

and, from Lemma 3.2.3, follows that σ is invariant under translations in direction of $Z := \mathcal{A}(\eta^x)$. Let $N := \dim Z$. Using some elementary theorems of geometric measure theory we deduce the existence of a constant $C = C(n,N) > 0$ and a probability measure γ supported on $Z^\perp$ such that if $\mathcal{C}$ is the cylinder $\mathcal{C} := \Pi_Z(B_{\frac{\sqrt{2}}{2}}) \times \Pi_{Z^\perp}(B_{\frac{\sqrt{2}}{2}})$, we have

$$\sigma_{\llcorner \mathcal{C}} = C\mathscr{H}^N_{\llcorner Z \cap \mathcal{C}} \otimes \gamma_{\llcorner Z^\perp \cap \mathcal{C}}.$$

Assume by contradiction that η^x is not a Dirac delta. Then, from Proposition 3.1.1, it follows that

$$N \geq n+1,$$

and since for any $t \in (0, \sqrt{2}/2)$ we have that $B_t^{n+k} \subseteq \Pi_Z(B_t) \times \Pi_{Z^\perp}(B_t) \subseteq \mathcal{C}$, then

$$\sigma_{\llcorner \mathcal{C}}(B_t) \leq C\mathscr{H}^N(B_t^N)\gamma(B_t^{n+k-N}) \leq C\omega_N t^N. \tag{3.18}$$

If we choose $\tau(n,k)$ smaller the $\min\{(C\omega_{n+k}2^{n+k})^{-1}, \sqrt{2}/4\}$, then clearly (3.18) can not hold for any $t \in (0,\tau)$, hence we contradict (3.17).

□

Before continuing with our Lemmas, we briefly recall the notion of uniform integrability and Vitali convergence theorem (Theorem 1.1.4) established in Chapter 1.

We now provide a technical result, that will be useful it the near future.

Lemma 3.3.2. *Let $(v_j)_j, (a_j)_j, (b_j)_j \subseteq L^1_{\text{loc}}(\mathbb{R}^n)$ and assume*

1. $0 \leq v_j = a_j + b_j$*;*
2. $\{b_j : j \in \mathbb{N}\}$ *precompact in* $L^1_{\text{loc}}(\mathbb{R}^n)$*;*
3. $\sup_{\lambda>0} \lambda \mathscr{L}^n(\{|a_j| > \lambda\}) \to 0$ *as* $k \to \infty$*;*
4. $a_j \rightharpoonup^* 0$ *in* L^1_{loc}*.*

Then $\{v_j : j \in \mathbb{N}\}$ *is precompact in* $L^1_{\text{loc}}(\mathbb{R}^n)$*.*

Proof. Fix any non-negative cut-off function $\chi \in C_c^\infty(\mathbb{R}^n)$. Then, by arbitrariness of our choice of χ, it is enough to show that there exists a sequence $(j_\ell)_\ell \subseteq \mathbb{N}$ such that both (χb_{j_ℓ}) converges in L^1 and $\chi a_{j_\ell} \to 0$ in L^1. Observe that 1. implies $a_j^- \leq |b_j|$ for all $j \in \mathbb{N}$; hence, if $(j_\ell)_\ell$ is such that $(b_{j_\ell})_\ell$ converges in L^1_{loc}, then $\{\chi a_{j_\ell}^- : \ell \in \mathbb{N}\}$ is uniformly integrable as

$$\int_{\{|\chi a_{j_\ell}^-| > M\}} |\chi a_{j_\ell}| \leq \int_{\{|b_j| > M\}} |b_j|,$$

and $\{b_{j_\ell} : \ell \in \mathbb{N}\}$ is uniformly integrable by 2. Theorem 1.1.4. Also, form Theorem 1.1.4 and assumption 3. we obtain that $a^-_{l_j} \xrightarrow{L^1_{loc}} 0$, hence, by 4., we deduce

$$\int \chi |a_{j_\ell}| = \int \chi a_{j_\ell} + \int \chi a^-_{j_\ell} \xrightarrow[\ell\to\infty]{} 0.$$

□

Lemma 3.3.3. *Assume $(V_j)_j \subseteq \mathcal{V}_n(\Omega)$, $V_j := \mu_j \otimes (\eta^x_j)_x$ to be a sequence of varifolds satisfying the following properties:*

1. *the sequence $(V_j)_j$ is equi-compactly supported in B_1, which means that there exists a set $K \Subset B_1$ such that $\operatorname{spt}\mu_j \subseteq K$ for all $j \in \mathbb{N}$;*
2. *$\sup_{j\in\mathbb{N}} \|\underline{H}_j\|_1(B_1) < \infty$;*
3. *there exists a n-plane S such that $\int |T - S|\, dV_j(y,T) \to 0$ as $j \to \infty$.*

Then, up to subsequences, there exists a function $\gamma \in L^1(B^n_1; \mathscr{L}^n_{\llcorner B^n_1})$ such that

$$\|\mu^S_j - \gamma^S \mathscr{H}^n_{\llcorner B^S_1}\|(B^S_t) \xrightarrow[j\to\infty]{} 0 \quad \forall t \in (0,1) \tag{3.19}$$

where $\mu^S_j := (\Pi_S)_\#\mu_j$, $\gamma^S := \gamma\circ\Pi_S$, $B^S_1 := \Pi_S(B^{n+k}_1)$ and $\Pi_S : \mathbb{R}^{n+k} \to S$ is the orthogonal projection onto S.

Proof. <u>STEP 1</u>. We begin by observing that without loss of generality, we can assume $S = \mathbb{R}^n \times \{0\}^k \simeq \mathbb{R}^n$ and so we can simplify the notation by adopting the conventions $\Pi := \Pi_S$ and

$$\mathbb{R}^{n+k} \ni x = (\xi,\zeta) \in \mathbb{R}^n \times \mathbb{R}^k.$$

Let $\nu_j := \Pi_\#\mu_j \in \mathrm{Rad}(B^n_1)$ and let $\psi \in C^\infty_c(B^n_1)$ be a non negative and radial mollifier so that $\{\psi_\varepsilon\}_{\varepsilon>0}$ is a standard approximation of the unity. Then defining $\gamma_j := \nu_j \star \psi_{\varepsilon_j}$ for all $j \in \mathbb{N}$ and $\varepsilon_j \searrow 0$ any arbitrary sequence, we see that (3.19) is proved as soon as we prove

$$\{\gamma_j : j \in \mathbb{N}\} \text{ is precompact in } L^1(B^n_1). \tag{3.20}$$

<u>STEP 2</u>. Let us remark some elementary properties and notations that we will use throughout the whole proof. For a function $\varphi \in C^1_c(B^n_1)$ we will denote by ∇_ξ the gradient with respect to the $\xi \in \mathbb{R}^n$ variables, and by ∇ the standard gradient with respect to the $x \in \mathbb{R}^{n+k}$ variable, so that

$$\Pi^*(\nabla_\xi\varphi(\Pi(x)) = \nabla(\varphi\circ\Pi)(x). \tag{3.21}$$

Let $e \in \mathbb{R}^n$, then we can compute the distributional derivative of ν_j in direction e by testing against functions $\varphi \in C^1_c(B^n_1)$, using (3.21), as

$$-\langle \partial_e \nu_j, \varphi\rangle = \int_{B^{n+k}_1} \partial_e\varphi\circ\Pi\, d\mu_j = \int_{B^{n+k}_1} \nabla(\varphi\circ\Pi)\cdot e\, d\mu_j$$

where, with a small abuse of notation, in the last integral we used (and we will continue to use) the same symbol e to denote both the vectors $e \in \mathbb{R}^n$ and $\widetilde{e} := e \otimes 0 \in \mathbb{R}^{n+k}$. Since

$$\nabla(\varphi \circ \Pi) \cdot e = \sum_{\ell=1}^{n+k} \partial_j(\varphi \circ \Pi) \cdot e_j = S : e \otimes \nabla(\varphi \circ \Pi) = S : \nabla(e(\varphi \circ \Pi)),$$

we can write

$$\begin{aligned} -\langle \partial_e \nu_j, \varphi \rangle &= \int_{G_n(B_1^{n+k})} S : \nabla(e(\varphi \circ \Pi))\, dV_j(x,T) \\ &= \int (S - T) : \nabla(e(\varphi \circ \Pi))\, dV_j + \int T : \nabla(e(\varphi \circ \Pi))\, dV_j \\ &= \int (S - T) : e \otimes \Pi^*(\nabla_\xi \varphi \circ \Pi)\, dV_j - \int (e \cdot \underline{H}_j)\, \varphi \circ \Pi \, d\mu_j, \end{aligned} \tag{3.22}$$

and so, defining the distributions

$$\langle X_j^e, \Phi \rangle := \int (S - T) : e \otimes \Pi^*(\Phi \circ \Pi)\, dV_j \quad \forall \Phi \in C_c^1(B_1^n; \mathbb{R}^n)$$
$$\langle g_j^e, \varphi \rangle := \int (e \cdot \underline{H}_j) \varphi \circ \Pi \, d\mu_j \quad \forall \varphi \in C_c^1(B_1^n),$$

(3.22) reads

$$\langle \partial_e \nu_j, \varphi \rangle = \langle \mathrm{div} X_j^e, \varphi \rangle + \langle g_j^e, \varphi \rangle,$$

hence, in the sense of distributions,

$$\nabla_\xi \nu_j := \begin{pmatrix} \partial_{e_1} \nu_j \\ \vdots \\ \partial_{e_n} \nu_j \end{pmatrix} = \begin{pmatrix} \mathrm{div}_\xi X_j^{e_1} \\ \vdots \\ \mathrm{div}_\xi X_j^{e_n} \end{pmatrix} + \begin{pmatrix} g_j^{e_1} \\ \vdots \\ g_j^{e_n} \end{pmatrix} =: \mathrm{div}_\xi X_j + g_j. \tag{3.23}$$

<u>Step 3</u>. From assumptions 2. and 3. respectively, it follows that $X_j^e \in \mathrm{Rad}(B_1^n; \mathbb{R}^n)$ and $g_j \in \mathrm{Rad}(B_1)$. Recalling that $\gamma_j = \nu_j \star \psi_{\varepsilon_j}$ is a smooth function for every $j \in \mathbb{N}$, then, by convolving both sides of (3.23) with the kernel ψ_{ε_j} we obtain

$$\nabla_\xi \gamma_j = \mathrm{div}_\xi Y_j + h_j \tag{3.24}$$

where $Y_j := X_j \star \psi_{\varepsilon_j}$ and $h_j := g_j \star \psi_{\varepsilon_j}$ and the derivatives are taken in the classical sense. Again by assumptions 2. and 3. respectively, we deduce

$$\sup_{|e|=1} \|X_j^e\|(B_1^n) \xrightarrow[j \to \infty]{} 0 \quad \text{and} \quad \sup_{j \in \mathbb{N}} \sup_{|e|=1} \|g_j^e\|(B_1^n) < \infty,$$

then

$$\gamma_j \geq 0, \quad \int_{B_1^n} |Y_j| \to 0, \quad \{h_j : j \in \mathbb{N}\} \subseteq L^1(B_1) \text{ is bounded.} \tag{3.25}$$

STEP 4. From (3.24) we can obtain an explicit expression for γ_j by taking the divergence w.r.t ξ and applying the inverse of the Laplacian on both sides (note that by assumption 1., all of the functions in (3.24) are equi-compaclty supported on B_1, so the inverse of the Laplacian is well defined). In particular we get

$$\gamma_j = \Delta_\xi^{-1}\mathrm{div}_\xi \mathrm{div}_\xi Y_j + \Delta_\xi^{-1}\mathrm{div}_\xi h_j,$$

where the operator Δ_ξ^{-1} is defined as

$$\Delta_\xi^{-1} f := E \star f, \quad E(y) := \begin{cases} -\frac{c_n}{|y|^{n-2}}, & \text{if } n \geq 2 \\ c_2 \ln(|y|), & \text{if } n = 2 \end{cases} \quad y \in \mathbb{R}^n.$$

Hence

$$\begin{aligned} \Delta_\xi^{-1}\mathrm{div}_\xi \mathrm{div}_\xi Y_j(\xi) &= D^2 E \star Y_j(\xi) \\ &=: K \star Y_j(\xi) \\ &= c_n \mathrm{P.V.} \int \frac{(\xi - y) \otimes (\xi - y) - |\xi - y|^2 \mathbb{1}_n}{|\xi - y|^{n+2}} : Y_j(y)\, dy, \end{aligned} \tag{3.26}$$

where P.V. denotes the principal value of the integral, and

$$\begin{aligned} \Delta_\xi^{-1}\mathrm{div}_\xi h_j &= \nabla_\xi E \star h_j(\xi) \\ &=: G \star h_j(\xi) \\ &= c_n \int \frac{(\xi - y) \cdot h_j(y)}{|\xi - y|^n}\, dy. \end{aligned}$$

By Frechet-Kolmogorov theorem on precompacness in L^p, the operator $h \mapsto G \star h$ is a compact operator $L^1_c(B^n_1) \to L^1_{\mathrm{loc}}(\mathbb{R}^n)$ (here $L^1_c(B^n_1)$ denotes the $L^1(B^n_1)$ functions having compact support in B^n_1). Indeed we have, for some $M \geq 1$,

$$\int_{B^n_1} |G \star h(z+y) - G \star h(y)|\, dy \leq C|z| \ln\left(\frac{M}{|z|}\right) \int_{B^n_1} |h|\, dy$$

for all $z \in B^n_1$. In particular $\{h_j : j \in \mathbb{N}\}$ is precompact in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. On the other hand one can show, using the expression (3.26) that there exists a constant C independent of j such that

$$\sup_{\lambda > 0} \lambda \mathscr{L}^n(\{|K \star Y_j| > \lambda\}) < C \int_{B^n_1} |Y_j|.$$

From (3.25) and Lemma 3.3.2 we are able to recover (3.20), and so the result is proved.

□

We are ready to prove that the existence of a mean curvature in L^1_{loc} implies that the restriction of the mass measure to the set of positive lower n-density is absolutely continuous with respect to the measure $\mathscr{H}^n$.

Lemma 3.3.4. *Assume $V \in \mathcal{V}_n(\Omega)$. Then $\mu_* \ll \mathcal{H}^n$*

Proof. Define the sets M_λ and M_* as

$$M_\lambda := \{x \in \operatorname{spt}\mu : \Theta^n_*(\mu, x) > \lambda\} \quad \forall \lambda > 0$$
$$M_* := \{x \in \operatorname{spt}\mu : \Theta^n_*(\mu, x) > 0\}.$$

Then clearly

$$\mathcal{H}^n_{\llcorner M_\lambda}(B) \leq \frac{1}{\lambda}\mu_*(B)$$

for any Borel set $B \subseteq \mathbb{R}^{n+k}$, therefore $\mathcal{H}^n_{\llcorner M_*}$ is a σ-finite Radon measure. We can apply Lebesgue-Radon-Nikodym differentiation theorem with μ_* and $\mathcal{H}^n_{\llcorner M_*}$ obtaining

$$\mu_* = \vartheta \mathcal{H}^n_{\llcorner M_*} + \rho,$$

where $\vartheta \in L^1_{\mathrm{loc}}(M_*, \mathcal{H}^n_{\llcorner M_*})$ is the derivative of μ_* with respect to $\mathcal{H}^n_{\llcorner M_*}$ and $\rho \perp \mathcal{H}^n_{\llcorner M_*}$.

We now show that ρ is the zero measure. By contradiction assume $\mu \neq 0$. Since ρ and $\mathcal{H}^n_{\llcorner M_*}$ are mutually singular, then ρ is concentrated on a set E such that

$$E \subseteq M_*, \quad \mathcal{H}^n(E) = 0. \tag{3.27}$$

As an immediate consequence of (3.27) and the area formula follows that

$$\mathcal{H}^n(F(E)) = 0 \quad \forall F \in \mathcal{L}\mathrm{ip}(\mathbb{R}^{n+k}; \mathbb{R}^{n+k}). \tag{3.28}$$

Let us fix a point $\overline{x} \in M_*$ and a sequence $r_j \searrow 0$ with the following properties

$$\lim_{r\to 0} \frac{\rho(B_r(\overline{x}))}{\mu_*(B_r(\overline{x}))} = \lim_{r\to 0} \frac{\mu_*(B_r(\overline{x}))}{\mu(B_r(\overline{x}))} = 1 \tag{I}$$

$$\exists \sigma \in \operatorname{Tan}(\mu, \overline{x}) = \operatorname{Tan}(\mu_*, \overline{x}) = \operatorname{Tan}(\rho, \overline{x}) \text{ s.t. } \sigma_{\llcorner B_{\frac{1}{2}}} \neq 0 \tag{II}$$

$$|\underline{H}(\overline{x})| < \infty \tag{III}$$

$$\exists S \in G(n+k, n) \text{ s.t. } V_j := V_{r_j, \overline{x}} \rightharpoonup^* \sigma \otimes \delta_S =: W. \tag{IV}$$

Observe that properties (I),(II) and (III) are satisfied ρ-a.e. by some basic measure theoretic arguments, while property (IV) is also satisfied ρ-a.e. from what we have already remarked about the structure of the set of tangent varifolds together with Lemma 3.3.1. The mass measures of the varifolds V_j can be written as

$$\mu_j := \mu_{V_j} = (\varphi_{r_j, \overline{x},})_\# \left(\vartheta \mathcal{H}^n_{\llcorner M_*}\right) + (\varphi_{r_j, \overline{x},})_\# \rho, \tag{3.29}$$

hence from (3.27) follows that (3.29) is the decomposition of μ_j in the absolutely continuous and singular parts with respect to $\mathcal{H}^n_{\llcorner M_*}$. In particular, denoting for sake of notation $(\varphi_{r_j, \overline{x},})_\# \rho$ as ρ_j, we have that ρ_j is concentrated on the set

$$E_j := \frac{E - \overline{x}}{r_j},$$

therefore, from (3.28), we have

$$\mathscr{H}^n(E_j) = 0 \quad \forall j \geq 1.$$

Setting $E_j^S := \Pi_S(E_j)$, by the same argument we deduce

$$\mathscr{H}^n(E_j^S) = 0 \quad \forall j \geq 1.$$

Fix $X \in C_c^1(B_1^{n+k}; \mathbb{R}^{n+k})$ and let $\underline{H}_j$ be the generalized mean curvature of V_j. Denoting by $X_j := X \circ \varphi_{r_j, \overline{x}}$ we have

$$\begin{aligned}
\langle \underline{H}_j, X \rangle &= \int T : \mathrm{div} X(y)\, dV(y,T) \\
&= \frac{r_j}{\mu(B_{r_j}(\overline{x}))} \int T : \mathrm{div} X_j(y)\, dV(y,T) \\
&= r_j \frac{\langle \underline{H}, X_j \rangle}{\mu(B_r(\overline{x}))} \\
&\leq r_j \left(\fint_{B_{r_j}(\overline{x})} |\underline{H}|\, d\mu \right) \|X\|_{C^0} \\
&\xrightarrow[j\to\infty]{} 0
\end{aligned}$$

where the limit goes to zero thanks to (III). Furthermore, from (IV) we immediately get

$$\int |T - S|\, dV_j(y,T) \xrightarrow[j\to\infty]{} \int |T - S|\, dW = 0.$$

By virtue of Lemma 3.3.3, we can choose a function $\gamma \in L^1(B_1^n)$ such that

$$\|\mu_j^S - \gamma^S \mathscr{H}^S_{\llcorner B_1^S}\|(B_t^S) \to 0 \quad \forall t \in (0,1), \tag{3.30}$$

where in (3.30) we adopted the same convention as in the proof of Lemma 3.3.3, that is

$$\mu_j^S := (\Pi_S)_\# \mu_j, \quad \gamma^S := \Pi_S \circ \gamma, \quad B_t^S := \Pi_S(B_1^{n+k}).$$

Using (I), we are allowed to replace μ_j^S with $\rho_j^S := (\Pi_S)_\# \rho_j$. Since $\rho_j \rightharpoonup^* \sigma$ as $j \to \infty$ and $\sigma_{\llcorner B_{1/2}^{n+k}} \neq 0$ we have

$$\begin{aligned}
0 &< \|\sigma^S\|(B_{\frac{1}{2}}^S) \\
&\leq \liminf_{j\to\infty} \|\rho_j^S\|(B_{\frac{1}{2}}^S) \\
&= \liminf_{j\to\infty} \|\rho_j^S\|(E_j^S \cap B_{\frac{1}{2}}^S),
\end{aligned} \tag{3.31}$$

where $\sigma^S := (\Pi_S)_\#\sigma$ and $E_j^S = \Pi_S(E_j)$, to be coherent with the previous notation. On the other hand $\mathscr{H}^n(E_j^S) = 0$ for all $j \geq 1$, therefore

$$\liminf_{j\to\infty} \|\rho_j^S\|(E_j^S \cap B_{\frac{1}{2}}^S) = \liminf_{j\to\infty} \|\rho_j^S - \gamma^S \mathscr{H}^n_{\llcorner B_1^S}\|(E_j^S \cap B_{\frac{1}{2}}^S). \tag{3.32}$$

The contradiction follows from (3.30), (3.31) and (3.32).

□

Finally we state and prove 3).

Lemma 3.3.5. *Let $V \in \mathcal{V}_n(\Omega)$ and define M to be the set of all points x such that $\Theta^n(\mu_V, x) > 0$. If $\mu_V(\Omega\setminus M) = 0$ then:*

(i) $\mathscr{H}^n_{\llcorner M}$ is σ-finite and there exists a positive function $\vartheta \in L^1(M, \mathscr{H}^n_{\llcorner M})$ such that $V = \vartheta\mathscr{H}^n_{\llcorner M} \otimes (\delta_{T_x})_x$.

(ii) $\mathrm{Tan}(V, x) = \left\{\frac{1}{\omega_n}\mathscr{H}^n_{\llcorner T_x \cap B_1} \otimes \delta_{T_x}\right\}$ *for μ_V-a.e. x.*

Proof. The first statement follows from the proof of Lemma 3.3.4 together with Lemma 3.3.1; therefore only (ii) requires a proof.

Let us simplify the notation writing μ instead of μ_V. Using Lemma 3.2.2 and Lemma 3.2.3 it suffices to prove the result for every x in M such that every tangent varifold $W \in \mathrm{Tan}(V, x)$ can be written as

$$W = \sigma \otimes \delta_{T_x}$$

for some $\sigma \in \mathrm{Tan}(\mu, x)$ which is invariant by translations in directions lying in T_x. This very last property of σ, together with some basic decomposition theorems form measures, allows us to write

$$\sigma = C\mathscr{H}^n_{\llcorner T_x} \otimes \gamma \tag{3.33}$$

for some probability measure γ supported in $T_x^\perp$ and a normalization constant $C > 0$. Fix any such x, W, σ, γ and C and let $r_j \searrow 0$ such that $\mu_{r_j,x} \rightharpoonup^* \sigma$.

<u>Step 1: $\gamma(\{0\}) \neq 0$.</u> Using Fatou's Lemma and recalling that $x \in M$, for any $\rho \in (0, 1)$ we have

$$\begin{aligned}
\sigma(\overline{B_\rho^{n+k}}) &\geq \limsup_{j\to\infty} \frac{\mu(B_{\rho r_j}(x))}{\mu(B_{r_j}(x))} \\
&\geq \liminf_{r\to 0} \frac{\mu(B_{\rho r}(x))}{\mu(B_r(x))} \\
&\geq \left(\liminf_{r\to 0} \frac{\mu(B_{\rho r}(x))}{\omega_n(\rho r)^n}\right)\left(\liminf_{r\to 0} \frac{\omega_n r^n}{\mu(B_r(x))}\right)\rho^n \\
&= \rho^n.
\end{aligned} \tag{3.34}$$

On the other hand decomposition (3.33) gives

$$C\omega_n\rho^n\gamma(\overline{B_\rho^k}) \geq \sigma(\overline{B_\rho^n \times B_\rho^k}) \geq \sigma(\overline{B_\rho^{n+k}}). \tag{3.35}$$

From (3.34), (3.35) and arbitrariness of the choice of $\rho \in (0,1)$ we deduce

$$\gamma(\overline{B_\rho^k}) \geq \frac{1}{C\omega_n} \quad \forall \rho \in (0,1);$$

which in turn implies

$$\alpha := \gamma(\{0\}) \geq \frac{1}{C\omega_n} > 0$$

<u>STEP 2: $\sigma(B_\rho) = \rho^n \sigma(B_1)$ for a.e. $\rho \in (0,1)$</u>. The monotonicity formula for stationary varifolds and our choice of W give

$$\frac{\sigma(B_\rho)}{\rho^n} \leq \frac{\sigma(B_R)}{R^n} \quad \forall 0 < \rho \leq R \leq 1.$$

Choosing $R = 1$ we prove one inequality. The other one follow by choosing $\rho \in (0,1)$ such that $\sigma(\partial B_\rho) = 0$ (which is the case for a.e. $\rho \in (0,1)$), (3.34) and recalling that $\sigma(B_1) = 1$.

<u>STEP 3: conclusion</u>. Using STEP 1 we recover the non-trivial decomposition in mutually singular measures

$$\gamma = \alpha\delta_0 + \widetilde{\gamma}, \tag{3.36}$$

hence

$$\widetilde{\gamma}(\{0\}) = 0. \tag{3.37}$$

Observe that the measure

$$\hat{\sigma} := C\mathcal{H}^n_{T_x} \otimes (\alpha\delta_0)$$

satisfies

$$\hat{\sigma}(B_\lambda) = \lambda^n \hat{\sigma}(B) \quad \forall \lambda \in (0,1).$$

By virtue of STEP 2 and of the fact that $\sigma - \hat{\sigma}$ is a (positive) measure, we deduce that the inequalities

$$0 \leq (\sigma - \hat{\sigma})(B_1) = C\frac{\mathcal{H}^n_{T_x} \otimes \widetilde{\gamma}(B_\rho^{n+k})}{\rho^n} \leq C\omega_n \widetilde{\gamma}(B_\rho^k) \tag{3.38}$$

hold for a.e. $\rho \in (0,1)$. From (3.37) and (3.38) follows $\widetilde{\gamma} = 0$, hence (3.36) gives the conclusion.

□

3.4. The Rectifiability Theorem

This section is fully dedicated to the proof of the rectifiability criterion which we have stated as Theorem 3.1.1 For sake of readibility we write again the statement

Theorem 3.1.2. Let $V \in \mathcal{V}_n(\Omega)$. Then V_* is a n-rectifiable varifold.

Proof of Theorem 3.1.2. Using (i) of Lemma 3.3.5 with

$$M = M_* := \{x \in \Omega : \Theta^n_*(\mu_*, x) > 0\},$$

it follows that $\mathcal{H}^n_{\llcorner M_*}$ is σ-finite and that we can write

$$V_* = \vartheta \mathcal{H}^n_{\llcorner M_*} \otimes (\delta_{T_x})_x$$

for some positive function $\vartheta \in L^1(M_*, \mathcal{H}^n_{\llcorner M_*})$.

Applying (ii) of Lemma 3.3.5 to V_* we deduce that for μ_*-a.e. $x \in \Omega$ and for every $f \in C^0_c(B_1)$ we have

$$\int f\, d(\mu_*)_{r,x} \xrightarrow[r \to 0]{} \fint_{T_x \cap B_1} f\, d\mathcal{H}^n. \tag{3.39}$$

Therefore given any arbitrary $f \in C^0_c(B_1)$ and any Lebesgue point x of ϑ such that (3.39) holds

$$\begin{aligned} \frac{1}{\omega_n r^n} \int f \circ \varphi_{r,x}\, d\mu_* &= \frac{\mu(B_r(x))}{\omega_n r^n} \int f\, d\mu_{r,x} \\ &\xrightarrow[r \to 0]{} \vartheta(x) \fint_{T_x \cap B_1} f\, d\mathcal{H}^n. \end{aligned}$$

Finally using the rectifiability criterion Theorem 1.4.1, we obtain n-rectifiability of the set M_*, which in turn proves that at μ_*-a.e. x the approximate tangent space of M_* coincides with T_x.

□

3.5. Integral Pre-Compactness

As an immediate consequence of Theorem 3.1.1 we deduce an inclomplete precompactness theorem for n-rectifiable varifolds.

Proposition 3.5.1. *Assume $(V_j)_j \subseteq \mathcal{R}_n(\Omega) \cap \mathcal{V}_n(\Omega)$ is a sequence such that*

$$\sup_{j \in \mathbb{N}} \Big(\mu_j(C) + \|\underline{H}_j\|_1(C) \Big) < \infty \quad \forall C \Subset \Omega,$$

then exists a subsequence $(V_{j_\ell})_\ell$ and a varifold $V \in \mathcal{V}_n(\Omega)$ such that $V_{j_\ell} \rightharpoonup^* V$. Moreover, if the limiting varifold V as positive lower n-density at μ_V-a.e. point, then $V \in \mathcal{R}_n(\Omega) \cap \mathcal{V}_n(\Omega)$.

Indeed the existence of a limiting varifold $V \in \mathcal{V}_n(\Omega)$ is an immediate consequence of the definition of total variation for measures, the definition of weak-* convergence and Banach-Alaoglu Theorem. The second part of the statement - i.e. the rectifiability of V - is literally an immediate effect of Theorem 3.1.1.

What about if we replace the assumption $(V_j)_j \subseteq \mathcal{R}_n(\Omega) \cap \mathcal{V}_n(\Omega)$ with $(V_j)_j \mathcal{I}_n(\Omega) \cap \mathcal{V}_n(\Omega)$ in Theorem 3.5.1? Can we still prove that the limiting varifolds V belongs to $\mathcal{I}_n(\Omega)$, provided is lower n-density is positive? The following lemma gives a positive answer to these questions.

Lemma 3.5.1. *Assume* $(V_j)_j \subseteq \mathscr{I}_n(\Omega) \cap \mathscr{V}_n(\Omega)$ *is a sequence such that*

$$\sup_{j\in\mathbb{N}} \Big(\mu_j(K) + \|\underline{H}_j\|_1(K) \Big) < \infty \quad \forall K \Subset \Omega, \tag{3.40}$$

then exists a subsequence $(V_{j_\ell})_\ell$ *and a varifold* $V \in \mathscr{V}_n(\Omega)$ *such that* $V_{j_\ell} \rightharpoonup^* V$. *Moreover, if the limiting varifold* V *as positive lower* n*-density at* μ_V*-a.e. point, then* $V \in \mathscr{I}_n(\Omega) \cap \mathscr{V}_n(\Omega)$.

Proof. The first part of the statement is a particular case of the fist part of Proposition 3.5.1, so we only need to prove the integral-n-rectifiability of the limiting varifold, provided its lower n-density is positive at almost every point of its support. From the same Proposition 3.5.1 it follows that $V \in \mathscr{R}_n(\Omega) \cap \mathscr{V}_n(\Omega)$, hence we can write

$$V = \vartheta \mathscr{H}^n_{\llcorner M} \otimes (\delta_{T_x M}),$$

where - as usual - M is n-rectifiable set, $\vartheta \in L^1(M; \mathscr{H}^n)$ is positive function and $T_x M$ is the approximate tangent space of M at the point x.

From assumption (3.40) follows the existence of $\kappa \in \mathrm{Rad}(\Omega)$ such that - up to subsequences - $(\|\underline{H}_j\|_1)_j$ converges weakly-* to κ. By standard differentiation theorems then we get that the Lebesgue-Radon-Nikodym (LRN) derivative

$$f(x) := \lim_{r\to 0} \frac{\kappa(B_r(x))}{\mu_V(B_r(x)} \in (0,\infty) \quad \mu_V - a.e.\, x \tag{3.41}$$

Let us now fix a point $\overline{x}$ such that

$$\vartheta(\overline{x}) \in (0,\infty) \tag{I}$$

$$T_{\overline{x}}M =: S \text{ exists} \tag{II}$$

$$f(\overline{x}) \in (0,\infty). \tag{III}$$

Without loss of generality we can assume

$$S = \mathbb{R}^n \times \{0\}^k \simeq \mathbb{R}^n \tag{IV}$$

and let us denote by Π and $\Pi^\perp$ the orthogonal projections to S and $S^\perp$ respectively, so that - with a small abuse of notation - we can write $x = (\Pi(x), \Pi^\perp(x))$. Let $r_j \searrow 0$ be a sequence such that $\kappa(\partial B_{r_j}) = 0$ for all $j \geq 1$. Then, by convergence of $(\|\underline{H}_j\|_1)_j$, for any $\ell \in \mathbb{N}$ there exists $j_\ell \in \mathbb{N}$ large enough such that

$$\|\underline{H}_{j_\ell}\|_1(B_{r_\ell}(\overline{x})) = (1 + o_{r_\ell}(\overline{x}))\kappa(B_{r_\ell}(\overline{x})). \tag{3.42}$$

Moreover we can choose j_ℓ such that $(j_\ell)_\ell$ forms an increasing sequence of natural numbers, starting from a ℓ sufficiently large (that, for simplicity we will assume starting from 0). Combining (3.41) and (3.42) we deduce

$$\lim_{\ell\to\infty} \frac{\|\underline{H}_{j_\ell}\|_1(B_{r_\ell}(\overline{x}))}{\mu_V(B_{r_\ell}(\overline{x}))} = \lim_{\ell\to\infty} \frac{\kappa(B_{r_\ell}(\overline{x}))}{\mu_V(B_{r_\ell}(\overline{x}))} = f(\overline{x}) < \infty,$$

and so, for $\ell \in \mathbb{N}$ large enough,

$$\|\underline{H}_{j_\ell}\|_1(B_{r_\ell}(\overline{x})) \le 2f(\overline{x})\mu_V(B_{r_\ell}(\overline{x})). \tag{3.43}$$

Consider now the standard dilations of factor r_ℓ and center $\overline{x}$ (that we will call as usual $\varphi_{r_\ell,\overline{x}}$) and let us define the blow-ups

$$V^\ell := \varphi^\#_{r_\ell,\overline{x}}V, \quad V^\ell_j := \varphi^\#_{r_\ell,\overline{x}}V_j, \quad j,\ell \in \mathbb{N}.$$

Since $V_j \rightharpoonup^* V$ as $j \to \infty$, we deduce that

$$V^\ell_j \rightharpoonup^* V^\ell \quad \text{as } j \to \infty \quad \forall \ell \in \mathbb{N},$$

and because $S = T_{\overline{x}}M$, we also get

$$V^\ell \rightharpoonup^* \vartheta(\overline{x})\mathscr{H}^n_{\llcorner S} \otimes \delta_S \quad \text{as } \ell \to \infty.$$

Via a diagonal argument - upon extracting a subsequence of $(j_\ell)_\ell$ - defining $\widetilde{V}^\ell := V^\ell_{j_\ell}$ we get

$$\mu_{V_{j_\ell}}(B_{r_\ell}(\overline{x})) \le 2\mu_V(B_{r_\ell}(\overline{x})) \le 4\vartheta(\overline{x})\omega_n r^n_\ell \tag{3.44}$$

$$\mu_{\widetilde{V}^\ell}((B^n_1 \times B^k_1)\setminus(B^n_1 \times B^k_{\frac{1}{2}})) = o_{r_\ell}(1) \tag{3.45}$$

$$\mu_{\widetilde{V}^\ell}(B^n_1 \times B^k_1) \le 2\vartheta(\overline{x})\omega_n$$

$$\widetilde{\vartheta}_\ell\mathscr{H}^n_{\llcorner \widetilde{M}_\ell} \otimes (\delta_{T_x\widetilde{M}}) =: \widetilde{V}^\ell \rightharpoonup^* \vartheta(\overline{x})\mathscr{H}^n_{\llcorner S} \otimes \delta_S, \quad \text{as } \ell \to \infty. \tag{3.46}$$

Let us now define the cut-off functions

$$\chi_1 \in C^\infty_c(B^n_{\sqrt{2}/2}) \quad \text{such that } \chi_1 \equiv 1 \text{ in } B^n_{\frac{1}{2}} \tag{3.47}$$

$$\chi_2 \in C^\infty_c(B^k_{\sqrt{2}/2}) \quad \text{such that } \chi_2 \equiv 1 \text{ in } B^k_{\frac{1}{2}} \tag{3.48}$$

$$\chi(x) := \chi_1(\xi)\chi_2(\zeta), \quad x = (\xi,\zeta) = (\Pi(x),\Pi^\perp(x))$$

$$W_\ell := \chi\widetilde{V}^\ell \quad \forall \ell \in \mathbb{N}.$$

The sequence $(W_\ell)_\ell$ is obviously compaclty supported in B_1. Suppose the existence of a function $\gamma \in L^1(B^n_1;\mathscr{H}^n_{\llcorner B^n_1})$ such that

$$\|\mu^S_{W_\ell} - \gamma^S\mathscr{H}^n_{\llcorner B^S_1}\|(B^S_t) \to 0 \quad \forall t \in (0,1), \tag{3.49}$$

where the notation used in (3.49) is the same as the one used in the proof of Lemma 3.3.3. Recalling (3.47) and (3.48) and (3.45),

$$\begin{aligned}\mu^S_{W_\ell}(B^n_{\frac{1}{2}}) &= \mu_{W_\ell}(B^n_{\frac{1}{2}} \times \mathbb{R}^k)\\ &= (\chi_2 \circ \Pi^\perp)\mu_{W_\ell}(B^n_{\frac{1}{2}} \times B^k_1)\\ &= \mu^S_{\widetilde{V}^\ell_{\llcorner(B^n_1 \times B^k_1)}}(B^n_{\frac{1}{2}}) - o_{r_\ell}(1),\end{aligned}$$

therefore

$$\|\mu^S_{\widetilde{V}^\ell \llcorner (B_1^n \times B_1^k)} - \gamma^S \mathcal{H}^n_{\llcorner B_1^S}\|(B^S_{\frac{1}{2}}) \to 0. \tag{3.50}$$

Using the Area Formula we deduce that $\mu^S_{V_\ell}$ is of the form

$$\mu^S_{\widetilde{V}^\ell \llcorner (B_{1/2}^n \times B_1^k)} = \hat{\vartheta}_\ell \mathcal{H}^n_{\llcorner B^S_{1/2}} + \widetilde{\nu}_\ell,$$

with $\widetilde{\nu}_\ell \perp \mathcal{H}^n_{\llcorner B^S_{1/2}}$ and

$$\hat{\vartheta}_\ell(y) := \sum_{\substack{\xi \in \widetilde{M}_\ell \\ \Pi(\xi)=y}} \frac{\widetilde{\vartheta}_\ell(\xi)}{J_{\widetilde{M}_\ell}\Pi(\xi)} =: \sum_{\substack{\xi \in \widetilde{M}_\ell \\ \Pi(\xi)=y}} \widetilde{\vartheta}_\ell(\xi) + \eta_\ell(y)$$

for each $y \in B^S_{1/2}$ such that $J_{\widetilde{M}_\ell}\Pi(\xi) \neq 0$ for all $\xi \in \Pi^{-1}(\{y\})$. Moreover from (3.50) follows that

$$\|\widetilde{\nu}_\ell\|(B^S_{1/2}) \xrightarrow[\ell \to \infty]{} 0,$$
$$\hat{\vartheta}_\ell \xrightarrow[\ell \to \infty]{} \gamma \quad \text{in } L^1(B^S_{1/2}, \mathcal{H}^n),$$

while from (3.46) and the observation $J_S\Pi \equiv 1$ follows

$$\eta_\ell \xrightarrow[\ell \to \infty]{} 0 \quad \text{in } L^1(B^S_{1/2}, \mathcal{H}^n).$$

Therefore - up to subsequences - we also have

$$\mathbb{N} \ni \sum_{\substack{\xi \in \widetilde{M}_\ell \\ \Pi(\xi)=y}} \widetilde{\vartheta}_\ell(\xi) \xrightarrow[\ell \to \infty]{} \gamma(y) \in \mathbb{N} \quad \mathcal{H}^n - a.e.\, y \in B^n_{1/2}.$$

On the other hand, (3.46) implies

$$\mu^S_{\widetilde{V}^\ell \llcorner (B_{1/2}^n \times B_1^k)} \rightharpoonup^* \vartheta(\overline{x}) \mathcal{H}^n_{\llcorner B_1^n}.$$

By uniqueness of the limit and the fact that $\mathcal{H}^n$-a.e. point satisfies (I),(II) and (III), we conclude that

$$\vartheta \in \mathbb{N} \quad \mu_V - a.e.$$

The only thing left to prove is (3.49). Since we have already remarked that $(W_\ell)_\ell$ equi-compacly supported, it is enough to show that it also satisfies assumptions 2. and 3. of Lemma 3.3.4, namely it suffices to prove that

$$\sup_{\ell \in \mathbb{N}} \|\underline{H}_{W_\ell}\|_1(B_1^{n+k}) < \infty, \tag{3.51}$$

$$\lim_{\ell \to \infty} \int_{G_n(B_1^{n+k})} |T - S|\, dW_\ell(y, T) = 0. \tag{3.52}$$

For (3.51) we observe that for any arbitrary $X \in C^1_c(B_1^{n+k};\mathbb{R}^{n+k})$, and defining $X_\ell := X \circ \varphi_{r_\ell,\overline{x}}$ and $\chi_\ell := \chi \circ \varphi_{r_\ell,\overline{x}}$, we have

$$
\begin{aligned}
|\langle \underline{H}_{W_\ell}, X\rangle| &= \left| \int T : \nabla X(z)\chi(z)\, d\widetilde{V}^\ell(z,T) \right| \\
&= \frac{1}{r_\ell{}^{n-1}} \left| \int T : \nabla X_\ell(y)\chi_\ell(y)\, dV_{j_\ell}(y,T) \right| \\
&= \frac{1}{r_\ell{}^{n-1}} \left| \int T : \nabla(X_\ell\chi_\ell)\, dV_{j_\ell} - \int T : X_\ell \otimes \nabla\chi_\ell\, dV_{j_\ell} \right| \\
&\le \frac{1}{r_\ell{}^{n-1}} \Big(|\langle \underline{H}_{V_{j_\ell}}, X_\ell\chi_\ell\rangle| + \mu_{V_{j_\ell}}(B_{r_\ell}(\overline{x})) \|\nabla\chi_\ell\|_{C^0} \|X\|_{C^0} \Big) \\
&\le \frac{2\mu_V(B_{r_\ell}(\overline{x}))}{r_\ell{}^n} \big(f(\overline{x}) r_\ell + 4\vartheta(\overline{x}) \|\nabla\chi\|_{C^0} \big) \|X\|_{C^0}\omega_n,
\end{aligned}
$$

where the last inequality follows from (3.43) and so (3.51) is a consequence of (3.44). Finally (3.52) is trivially implied by the definition of W_ℓ (as a cut-off of $\widetilde{V}_\ell$) and (3.46) □

The previous result (Lemma 3.5.1) is nevertheless not satisfactory as the necessity of the assumption of a positive lower n-density for the limiting varifold makes it only an incomplete pre-compactness theorem. We wish to prove that the aforementioned assumption is redundant and implied by the uniform lower bound for the density of the sequence. In [Sim84] there is a proof that not only this is the case, but of also the stronger statement

$$
V \mapsto \Theta^n(\mu_V, x) \quad \text{is USC under varifold convergence} \tag{3.53}
$$

hold. However the proof of (3.53) presented in Simon's book heavily relies on the Monotonicity Identity (cfr. Chapter 2). With the scope of generalizing this theory in the case of varifolds with locally bounded first variation with respect to some anisotropic functional on which the Monotonicity Identity is no longer applicable, we present a proof of a slightly weaker statement than (3.53) which only relies on the validity of the Michael Simon inequality for varifolds (cfr. Chapter 2).

We commence giving a formal argument. Assume $(V_j)_j$ to be a sequence of varifolds in $\mathscr{V}_n(\Omega)$ having n-density uniformly bounded from below by a positive constant, that - without loss of generality - we can assume to be 1, and mean curvatures $(\underline{H}_j)_j$ which are equi-bounded in L^∞ by a constant $M > 0$. This very last condition is actually equivalent to asking

$$
\exists M > 0 \text{ s.t. } \|\underline{H}_j\|_1(B_\rho(x)) \le M\mu_j(B_\rho(x))\, \mu_j - a.e.\, x.
$$

Then the Michael Simon inequality tested against a positive and smooth function f writes

$$
\left(\int f\, d\mu_j \right)^{\frac{n-1}{n}} \le C \left(\int |\nabla f|\, d\mu_j + M \int f\, d\mu_j \right), \tag{3.54}
$$

If we let $V_j \rightharpoonup^* V$, assume $x \in \mathrm{spt} V$ and formally substitute f by the characteristic function of a ball $B_\rho(x)$ in (3.54) we obtain the expression

$$(\mu(B_\rho(x)))^{\frac{n-1}{n}} \le C\left(\frac{d}{d\rho}\mu(B_\rho(x)) + M\mu(B_\rho(x))\right);$$

hence we deduce that the function $\rho \mapsto \mu(B_\rho(x))$ solves the differential inequality

$$y' \ge \frac{1}{C} y^{1-\frac{1}{n}}(1 - My^{\frac{1}{n}})$$

with the further conditions

$$y(0) = 0, \quad y(\rho) > 0 \quad \forall \rho > 0.$$

Assuming $\rho \mapsto \mu(B_\rho(x))$ to be continuous in 0, we can find ρ_0 sufficiently small such that the function is a solution of the problem

$$\begin{cases} y' \ge \frac{1}{C} y^{1-\frac{1}{n}} & \text{in } (0,\rho_0) \\ y(0) = 0 & \\ y(\rho) > 0 & \forall \rho \in (0,\rho_0) \end{cases}. \tag{3.55}$$

Since the function $y(\rho) := (\rho/(Cn))^n$ is the unique smooth solution of the problem

$$\begin{cases} y' = \frac{1}{C} y^{1-\frac{1}{n}} & \text{in } (0,\rho_0) \\ y(0) = 0 & \\ y(\rho) > 0 & \forall \rho \in (0,\rho_0) \end{cases}, \tag{3.56}$$

we deduce that

$$\frac{\mu(B_\rho(x))}{\omega_n \rho^n} \ge \frac{1}{(Cn)^n} \quad \forall \rho \in (0,\rho),$$

and so we have the lower density bound that we need in order to apply Lemma 3.5.1 and conclude. Before making this argument rigorous, we prove a preliminary result.

Lemma 3.5.2. *Assume that $f : [0,\delta) \to \mathbb{R}$, where $\delta > 0$, is a measurable function satisfying the following properties:*

1. $f(0) = 0$;
2. $f(\rho) > 0$ *for all* $\rho \in (0,\delta)$
3. *there exists a constant $\Lambda > 0$ such that f satisfies the following integral condition for every $0 \le \sigma \le \rho < \delta$*

$$f(\rho) - f(\sigma) \ge \frac{1}{\Lambda}\int_\sigma^\rho (f(\tau))^{\frac{n-1}{n}}\, d\tau.$$

Then $f(\rho) \ge (\rho/(\Lambda n)^n)$ for every $\rho \in [0,\delta)$.

Proof. From 3. we deduce that f is non-decreasing, hence differentiable almost everywhere; furthermore its derivative satisfies

$$f'(\rho) \geq \frac{1}{\Lambda}(f(\rho))^{\frac{n-1}{n}} \quad a.e.\, \rho \in [0,\delta). \tag{3.57}$$

Using 2. and (3.57) we can write

$$f(\rho)^{\frac{1}{n}-1} f'(\rho) \geq \frac{1}{\Lambda} \quad a.e.\, \rho \in (0,\delta). \tag{3.58}$$

Integrating (3.58) in the interval $(0,\rho)$, using again the monotonicity of f and 1. we obtain on the left hand side

$$\begin{aligned}\int_0^\rho f(\tau)^{\frac{1}{n}-1} f'(\tau)\, d\tau &= \int_0^\rho \left(n f(\tau)^{\frac{1}{n}}\right)' d\tau \\ &\leq n f(\rho)^{\frac{1}{n}};\end{aligned}$$

while on the right hand side we obtain ρ/Λ.

□

Lemma 3.5.3. *Let $(V_j)_j \subseteq \mathscr{V}_n(\Omega)$ be a sequence of varifolds and let $(\underline{H}_j)_j$ be the sequence of corresponding mean curvatures. Assume the following properties are satisfied:*

1. $\liminf_{j\to\infty} \|\underline{H}_j\|_1(K) < \infty$ *for every* $K \Subset \Omega$;
2. $\Theta^n(\mu_j, x) \geq 1$ *for* $\mu_j - a.e.\ x$ *(as usual* μ_j *denotes the mass measure of* V_j*);*
3. *there exists a varifold* $V \in \mathbb{V}_n(\Omega)$ *such that* $V_j \rightharpoonup^* V$.

Then $V \in \mathscr{V}_n(\Omega)$ *and*

$$\|\underline{H}\|_1(K) \leq \liminf_{j\to\infty} \|\underline{H}_j\|_1(K) \quad \forall K \Subset \Omega, \tag{3.59}$$

$$\Theta^n_*(\mu, x) > 0 \quad \mu - a.e.\, x \in \Omega, \tag{3.60}$$

where $\underline{H}$ *and* μ *denote the mean curvature and mass measure of* V *respectively.*

We remark that each varifold in the sequence of the statement satisfies the rectifiability criterion Theorem 3.1.1, hence we can write

$$V_j = \vartheta_j \mathscr{H}^n_{\llcorner M_j} \otimes (\delta_{T_x M_j})_x \quad \forall j \in \mathbb{N}$$

where $M_j := \operatorname{spt}\mu_j$ is a n-rectifiable set, ϑ_j is a L^1_{loc} function which is not smaller than 1 on M_j and $T_x M_j$ is - as usual - the approximate tangent space of M_j at x.

Proof. The fact that $V \in \mathscr{V}_n(\Omega)$ and (3.59) are immediate consequences of the definition of total variation of a measure and the definition of weak-* convergence.

Indeed, given $K \Subset \Omega$, and recalling that $(y,T) \mapsto \mathrm{div}_T X(y)$ is continuous for any smooth vetorfield X, then

$$\begin{aligned}
\|\underline{H}\|_1(K) &= \sup_{\substack{X \in C^0_c(K;\mathbb{R}^{n+k}) \\ \|X\|_{C^0} \le 1}} \langle \underline{H}, X\rangle \\
&= \sup_{\substack{X \in C^0_c(K;\mathbb{R}^{n+k}) \\ \|X\|_{C^0} \le 1}} \int \mathrm{div}_T X(y)\, dV(y,T) \\
&= \sup_{\substack{X \in C^0_c(K;\mathbb{R}^{n+k}) \\ \|X\|_{C^0} \le 1}} \lim_{j\to\infty} \int \mathrm{div}_T X(y)\, dV_j(y,T) \\
&\le \liminf_{j\to\infty} \sup_{\substack{X \in C^0_c(K;\mathbb{R}^{n+k}) \\ \|X\|_{C^0} \le 1}} \int \mathrm{div}_T X(y)\, dV_j(y,T) \\
&= \liminf_{j\to\infty} \|\underline{H}_j\|_1(K).
\end{aligned}$$

To the aim of proving (3.60) we establish the following notation. Fix $K \Subset \Omega$. With N_j will denote the points of M_j such that $\Theta^n(\mu_j, x) < 1$, and so

$$\mu_j(N_j) = 0 \quad \forall j \in \mathbb{N}. \tag{3.61}$$

Then we let $\rho^* := \min\{1, \mathrm{dist}(K, \partial\Omega)\}$ and we define the sets

$$E^m_\ell := \left\{ x \in K \setminus N_\ell : \frac{\|\underline{H}_\ell\|_1(B_\rho(x))}{\mu_\ell(B_\rho(x))} \le m \,\forall \rho \in (0,\rho^*) \right\},$$
$$F^m_\ell := K \setminus E^m_\ell$$

for any $\ell, m \in \mathbb{N}$. Therefore $x \in F^m_\ell$ if and only if $\Theta^n(\mu_\ell, x) < 1$ or

$$\exists \sigma_x \in (0,\rho^*) : \mu_\ell(B_{\sigma_x}(x)) < \frac{1}{m} \|\underline{H}_\ell\|_1(B_{\sigma_x}(x)). \tag{3.62}$$

Let $\mathscr{B}$ be the collection of all balls of center $x \in F^m_\ell \setminus N_\ell$ of radius σ_x as in (3.62). Then, using Besicovich Convering lemma we find $N = N(n+k) \in \mathbb{N}$ and subfamilies $\mathscr{B}_1, \dots, \mathscr{B}_N$ of $\mathscr{B}$ such that

$$\mathscr{B}_i \text{ is a disjoint family } \forall i \in \{1,\dots,N\}$$
$$F^m_\ell \setminus N_\ell \subseteq \bigcup_{i=1}^N \left(\bigcup_{B \in \mathscr{B}_i} B \right).$$

Therefore, from (3.61), (3.62), the above mentioned properties of $\mathscr{B}_1, \dots, \mathscr{B}_N$, and 1.

of the statement we deduce

$$\begin{aligned}\mu_\ell(F_\ell^m) &= \mu_\ell(F_\ell^m \setminus N_\ell)\\ &\le \frac{1}{m}\sum_{i=1}^{N}\sum_{B\in\mathscr{B}_i}\|\underline{H}_\ell\|_1(B)\\ &\le \frac{N}{m}\|\underline{H}_\ell\|_1(K+B_{\rho^*})\\ &\le \frac{C}{m}\end{aligned}$$

for some universal constant $C>0$. Hence

$$\begin{aligned}\mu(\mathrm{int}(\bigcap_{i\ge\ell}F_i^m)) &\le \liminf_{j\to\infty}\mu_j(\mathrm{int}(\bigcap_{i\ge\ell}F_i^m))\\ &\le \liminf_{j\to\infty}\mu_j(F_j^m)\\ &\le \frac{C}{m},\end{aligned}\tag{3.63}$$

and since for all $m\ge 1$ we have

$$\mu(\bigcap_{m\ge1}\bigcup_{\ell\ge1}\mathrm{int}(\bigcap_{i\ge\ell}F_i^m)) \le \mu(\bigcup_{\ell\ge1}\mathrm{int}(\bigcap_{i\ge\ell}F_i^m)),$$

from (3.63) and continuouity for measures follows that

$$\mu(\bigcap_{m\ge1}\bigcup_{\ell\ge1}\mathrm{int}(\bigcap_{i\ge\ell}F_i^m)) = 0.$$

Let $F := \bigcap_{m\ge1}\bigcup_{\ell\ge1}\mathrm{int}(\bigcap_{i\ge\ell}F_i^m)$. Then it is enough to prove that

$$\Theta^n(\mu,x) > 0 \quad \forall x\in K\setminus F.$$

Let us fix $x\in K\setminus F$. Then, by the very definition of F, it follows that there exists $m\ge1$ such that

$$x\in\overline{\bigcup_{i\ge\ell}K\setminus F_i^m} \quad \forall \ell\ge1.$$

Therefore, via a diagonal argument, we can find sequences $(y_j)_j$ and $(i_j)_j$ such that

$$|y_j - x| < \frac{1}{j}, \quad \mathbb{N}\ni i_j\nearrow\infty, \quad y_j\in E_{i_j}^m \quad \forall j\in\mathbb{N},$$

hence

$$\|\underline{H}_{i_j}\|_1(B_\rho(y_j)) \le m\mu_{i_j}(B_\rho(y_{i_j})) \quad \forall\rho\in(0,\rho^*).\tag{3.64}$$

From now on, for sake of readability, we relabel $(V_{i_j})_j$ as (V_j). Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a function with the following properties

$$\begin{gathered}\varphi\equiv1 \quad \text{on } (-\infty,1/2],\\ \varphi\equiv0 \quad \text{on } [1,\infty),\\ \varphi\in C^1(\mathbb{R}) \quad \text{and} \quad \varphi'\le0.\end{gathered}$$

and define the functions $\gamma : \mathbb{R} \to \mathbb{R}$, $f_j : \mathbb{R}^{n+k} \to \mathbb{R}$, $f : \mathbb{R}^{n+k} \to \mathbb{R}$ as

$$\gamma(r) := \varphi(r/\rho), \quad f_j(y) := \gamma(|y - y_j|), \quad f(y) := \gamma(|y - x|),$$

where $0 < \rho < \rho^*$ is chosen arbitrarily. For sake of notation we will write $r_j := r_j(y) := |y - y_j|$ and $r := r(y) := |y - x|$. From the definition of f it follows that

$$\nabla f(y) = \gamma'(r)\nabla r = -\frac{d}{d\rho}(\varphi(r/\rho))\frac{\rho}{r}\nabla r,$$

and since $\frac{d}{d\rho}\varphi(r/\rho) \equiv 0$ if $\frac{r}{\rho} \notin [1/2, 1]$, then for any $T \in G(n+k, n)$ we have

$$|\nabla_T f(y)| \leq 2\frac{d}{d\rho}(\varphi(r/\rho)), \tag{3.65}$$

and the same computations made with r_j in place of r prove (3.65) with f replaced by f_j. Since the varifolds $(V_j)_j$ and the functions $(f_j)_j$ satisfy the assumption of the Sobolev Inequality, then we can write

$$\begin{aligned}&\left(\int \varphi(r_j/\rho)^{\frac{n}{n-1}}\, d\mu_j\right)^{\frac{n-1}{n}} \\ &\qquad \leq C\left(2\frac{d}{d\rho}\int \varphi(r_j/\rho)\, d\mu_j + \int \varphi(r_j/\rho)|\underline{H}_j|\, d\mu_j\right),\end{aligned} \tag{3.66}$$

where $C > 0$ depends only on n. Moreover (3.64) implies that

$$\int_{B_\rho(y_j)} g|\underline{H}_j|\, d\mu_j \leq m\int_{B_\rho(y_j)} g\, d\mu_j \tag{3.67}$$

holds for every non-negative simple function g. Using measure-theoretical approximation arguments one proves that (3.67) holds also for every integrable function, thus (3.66) writes

$$\begin{aligned}&\frac{d}{d\rho}\int \varphi(r_j/\rho)\, d\mu_j \\ &\qquad \geq \frac{1}{2C}\left(\int \varphi(r_j/\rho)^{\frac{n}{n-1}}\, d\mu_j\right)^{\frac{n-1}{n}} - \frac{m}{2}\int \varphi(r_j/\rho)\, d\mu_j.\end{aligned} \tag{3.68}$$

Now we integrate both sides of (3.68) in the interval (σ, ρ), with $0 \leq \sigma < \rho < \rho^*$, so that we obtain

$$\begin{aligned}&\int \varphi(r_j/\rho)\, d\mu_j - \int \varphi(r_j/\sigma)\, d\mu_j \\ &\quad \geq \int_{\tau=\sigma}^{\tau=\rho}\left(\frac{1}{2C}\left(\int \varphi(r_j/\rho)^{\frac{n}{n-1}}\, d\mu_j\right)^{\frac{n-1}{n}} - \frac{m}{2}\int \varphi(r_j/\rho)\, d\mu_j\right) d\tau.\end{aligned} \tag{3.69}$$

Since $\|f_j - f\|_{C^0} \to 0$ and $\mu_j \rightharpoonup^* \mu$ as $j \to \infty$ and , passing to the limit in j in (3.69) yields

$$\begin{aligned}&\int \varphi(r/\rho)\, d\mu - \int \varphi(r/\sigma)\, d\mu \\ &\quad \geq \int_{\tau=\sigma}^{\tau=\rho}\left(\frac{1}{2C}\left(\int \varphi(r/\rho)^{\frac{n}{n-1}}\, d\mu\right)^{\frac{n-1}{n}} - \frac{m}{2}\int \varphi(r/\rho)\, d\mu\right) d\tau.\end{aligned}$$

and letting φ increase to the characteristic function of $(-\infty, 1]$ gives, by monotone convergence

$$\begin{aligned}\mu(B_\rho(x)) - \mu(B_\sigma(x)) \\ \geq \int_\sigma^\rho \mu(B_\tau(x))^{\frac{n-1}{n}} \left(\frac{1}{2C} - \frac{m}{2}\mu(B_\tau(x))^{\frac{1}{n}} \right) d\tau \end{aligned} \tag{3.70}$$

By contradiction assume $\Theta^n_*(\mu, x) = 0$. Using this assumption we deduce the existence of a sequence $\rho_j \searrow 0$ such that

$$\lim_{\ell\to\infty} \frac{\mu_{B_{\rho_\ell}}(x)}{\omega_n \rho_\ell^n} = 0,$$

therefore, by monotonicity of $\rho \mapsto \mu(B_\rho(x))$ we deduce the existence of $\tilde{\rho}_x = \tilde{\rho}_x(m, n)$ such that

$$\frac{1}{2C} - \frac{m}{2}\mu(B_\rho(x))^{\frac{1}{n}} > \frac{1}{4C} \quad \forall \rho \in (0, \tilde{\rho}_x),$$

hence (3.70) writes

$$\mu(B_\rho(x)) - \mu(B_\sigma(x)) \geq \frac{1}{4C}\int_\sigma^\tau \mu(B_\tau(x))^{\frac{n-1}{n}} \, d\tau, \tag{3.71}$$

and since (3.71) holds for all $0 < \sigma < \rho < \tilde{\rho}_x$, then, by virtue of Lemma 3.5.2 we deduce

$$\mu(B_\rho(x)) \geq \frac{\rho^n}{(4Cn)^n} \quad \forall \rho \in (0, \tilde{\rho}_x),$$

which contradicts the absurd assumption $\Theta^n_*(\mu, x) = 0$.

□

Now we can finally prove the precompactness result for integral varifolds that we stated as Theorem 3.1.2. As we did for Theorem 3.1.1 we re-write here the statement

Theorem 3.1.2. Let $(V_j)_j \subseteq \mathscr{I}_n(\Omega) \cap \mathscr{V}_n(\Omega)$ be a sequence such that

$$\sup_{j\in\mathbb{N}} \Big(\mu_j(C) + \|\underline{H}_j\|_1(C) \Big) < \infty \quad \forall C \Subset \Omega,$$

then is a subsequence $(V_{j_\ell})_\ell$ and a varifold $V \in \mathscr{I}_n(\Omega) \cap \mathscr{V}_n(\Omega)$ such that $V_{j_\ell} \rightharpoonup^* V$.

Proof of Theorem 3.1.2. The existence of a subsequence $(V_{j_\ell})_\ell$ and a varifold $V \in \mathscr{V}_n(\Omega)$ is trivial (cfr. proof of Lemma 3.5.1), therefore $(V_{j_\ell})_\ell$ and V satisfy all of the assumptions of Lemma 3.5.3, so that V has positive density at μ_V-a.e. point. The conclusion of the proof follows from the second part of Lemma 3.5.1.

□

4. Allard's Regularity Theorem

4.1. Introduction

This chapter is dedicated to the proof of Allard's Regularity Theorem. Let us begin by introducing the following set of assumptions depending on parameters $p \in (n,\infty]$ and $\delta > 0$:

$$\begin{cases} 0 \in M \subseteq B_1 \ n\text{-rectifiable and closed;} \\ V = \underline{\mathrm{v}}(M,\vartheta) = \mu \otimes (\delta_{T_x M})_x \quad \text{and} \quad \vartheta \geq 1 \quad \mu - a.e.; \\ \underline{H} \in L^p(\mu) \quad \text{and} \quad \|\underline{H}\|_{L^p} \leq \delta; \\ \mu(B_1) \leq (1+\delta)\omega_n. \end{cases} \tag{$\bigstar_{p,\delta}$}$$

Before asserting the main theorem we need to introduce some notation. Given a function $u : X \to Y$ between we denote by $\mathrm{Graph}(u)$ its graph, namely

$$\mathrm{Graph}(u) := \{(x, u(x)) : x \in X\} \subseteq X \times Y.$$

Finally, if $f : \Omega \subseteq \mathbb{R}^N \to \mathbb{R}^M$ is a continuous function, we define the α-Hölder seminorm of f as

$$[f]_{C^{0,\alpha}(\Omega)} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

If $[f]_{C^{0,\alpha}(\Omega)}$ is finite, then we say that f is α-Hölder continuous and write $f \in C^{0,\alpha}(\Omega)$. With the notation $C^{j,\alpha}(\Omega)$ we denote the subfamily of $C^j(\Omega)$ containing the functions F such that

$$[\partial_\beta F]_{C^{0,\alpha}(\Omega)} < \infty \quad \forall \beta \in \mathbb{N}^N, |\beta| = j.$$

Theorem 4.1.1 (Allard's Ragularity Theorem). *For any $p \in (n,\infty)$ there are constants $\delta_0 = \delta_0(n,k,p)$ and $\gamma = \gamma(n,k,p)$ in $(0,1)$ such that if $(\bigstar_{p,\delta})$ holds for any $\delta \in (0,\delta_0]$, then there exists a linear isometry $q : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ and a function $u \in C^{1,1-n/p}(B^n_\gamma;\mathbb{R}^k)$ such that $M \cap B_\gamma = q(\mathrm{Graph}(u))$.*

The proof of Theorem 4.1.1 constitutes the content of Chapter 5 of [Sim84]. Furthermore, in the same reference, an estimate for the $C^{1,1-n/p}$-norm of u in terms of δ and a multiplicative constant depending only on n, k and p is established.

We hereby provide a concise overview of the principal concepts delineated in Allard's proof. First, we introduce the *Tilt-Excess* of a varifold $V = \underline{v}(M,\vartheta)$ relative to a n-plane S in a ball $B_r(x_0)$, which is

$$E(V,S,x_0,r) := \int_{B_r(x_0)} |T_x M - S|^2 \, d\mu. \tag{4.1}$$

In can be easily demonstrated that the tilt-excess can be controlled, in a smaller ball, by the sum of L^2-distance from M to the plane and the L^2-norm of the mean curvature $\underline{H}$. More precisely that

$$E(V,S,x_0,r/2) \leq C\left(\int_{B_r(x)} \frac{\operatorname{dist}^2(x,x_0+S)}{\rho^{n+2}}\, d\mu + \int_{B_r(x)} \frac{|\underline{H}|^2}{\rho^{n-2}}\, d\mu\right)$$

holds for some global constant $C = C(n,k,p) > 0$. The key result is the Lipschitz approximation theorem, essentially stating that the support of the varifold coincides, up to a $\mathscr{H}^n$-small set, with the graph of a Lipshitz function f. Subsequently, by an approximation of such an f in L^2 with an harmonic function u, and using standard elliptic estimates, one establishes the existence of global constants η and δ_0, together with a n-plane S such that

$$E(V,S,x_0,\eta r) \leq \eta^{2(1-n/p)} \inf_{T\in G(n+k,n)} E(V,T,x_0,r), \tag{4.2}$$

provided $(\bigstar_{p,\delta})$ is satisfied for some $\delta \in (0,\delta_0]$. This result goes by the name of *Tilt-Excess Decay Lemma.*

Using the Tilt-Excess Decay Lemma, one proves the existence of a global constant δ_0 such that, if $(\bigstar_{p,\delta})$ holds true for some $\delta \in (0,\delta_0]$, then there exists a Lipschitz function f whose graph *coincides* with $M = \operatorname{spt}\mu_V$. Moreover, by plugging an explicit formulation of (4.1) in the case of $M = \operatorname{Graph}(f)$, into (4.2), one proves the $(1-n/p)$-Hölder continuity of the gradient of f.

An immediate consequence of Theorem 4.1.1 is

Theorem 4.1.2 (Allard's Regularity Theorem, $\mathbf{L}^\infty$ case). *For every $\alpha \in (0,1)$ there are constants $\delta_0 = \delta_0(n,k,\alpha)$ and $\gamma = \gamma(n,k,\alpha)$ in $(0,1)$ such that if $(\bigstar_{\infty,\delta})$ holds for any $\delta \in (0,\delta_0]$, then there exists a linear isometry $q : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ and a function $u \in C^{1,\alpha}(B^n_\gamma;\mathbb{R}^k)$ such that $M \cap B_\gamma = q(\operatorname{Graph}(u))$.*

In this chapter we present a proof of Theorem 4.1.2 via a more modern approach. The main reference in [DPGS24], on which all of the results of this chapter have been stated and proved only for the case $p = \infty$.

4.2. Decay of Oscillations

For the whole section, we assume the following:

1. $n < p \leq \infty$, and use the convention $\alpha/\infty := 0$ for all $\alpha \in \mathbb{R}$;

2. $0 \in M$ n-rectifiable closed subset of B_R, with $R \leq 1$;

3. $V = \underline{\mathrm{v}}(M,\vartheta) \in \mathscr{V}_n^p(B_R)$ where $\mathscr{V}_n^p(B_R)$ denotes the family of varifolds having mean curvature in $L^p(\mu_V)$;

4. $\vartheta \geq 1$ μ-a.e.;

5. $\mu(B_r) \leq \frac{3}{2}\omega_n r^n$ for every $0 \leq r \leq R$.

6. the constants denoted by C_j for $j \in \mathbb{N}\setminus\{0\}$ appearing in the proofs will always depend only on n, k, p unless we specify otherwise.

For sake of readability we recall the weighted monotonicity formula for varifolds with mean curvature in L^p that we stated in Chapter 2.

Lemma 4.2.1. *Fix a point $\overline{x} \in \mathrm{spt}\mu_V$ and let $\rho_0 := \min\{1, \mathrm{dist}(\overline{x}, \partial B_R)\}$. If $f \in C_c^1(B_R)$ is a non-negative convex function such that $\|\nabla f\|_{L^\infty} \leq 1$, then the inequality*

$$\frac{1}{\omega_n \rho^n}\int_{B_\rho(0)} f\,d\mu \geq f(0) - C_0\|\underline{H}\|_{L^p}(\|f\|_{L^\infty} + \rho)\rho^{1-n/p},$$

where $C = C(n,p)$, holds for every $0 < \rho < \rho_0$.

The above result allows us to prove a *partial Harnack inequality*. Before doing so we need to introduce the notion of oscillation of a varifold relative to a n-plane.

Definition 4.1 (Oscillation). Let V be any varifold in B_1 such that $x_0 \in \mathrm{spt}\mu_V$ and fix $S \in G(n+k,n)$. Then we set

$$\mathrm{osc}_S(V, x_0, r) := \frac{1}{2}\sup\left\{|\Pi_S^\perp(x-y)| : x, y \in \mathrm{spt}\mu_V \cap B_r(x_0)\right\},$$

where as usual we denote by $\Pi_S^\perp : \mathbb{R}^{n+k} \to S^\perp$ the orthogonal projection onto $S^\perp$. If $x_0 = 0$ we write $\mathrm{osc}_S(V,r)$; furthermore, if V and S are intended, we may use $\mathrm{osc}(r)$ in place of $\mathrm{osc}_S(V,r)$.

Lemma 4.2.2 (Harnack inequality). *Under the standing assumptions for a varifold V, there exists a constant $\eta = \eta(n,k,p) \in (0,1/2)$ with the following property. If $\rho \in (0,1]$ and*

$$\mathrm{osc}_S(\rho) \leq \eta\rho \quad \textit{and} \quad \|\underline{H}\|_{L^p} \leq \frac{\mathrm{osc}_S(\rho)}{\rho^{2-n/p}}$$

for some $S \in G(n+k,n)$, then

$$\mathrm{osc}_S(\eta\rho) \leq (1-\eta)\mathrm{osc}_S(\rho). \tag{4.3}$$

Proof. <u>STEP 1.</u> We can assume, without loss of generality, $\rho = 1$. Indeed suppose the statement to be true for $\rho = 1$ and let $\widetilde{\rho} \in (0,1)$. We define the scaled varifold

$$\widetilde{V} := \underline{\mathrm{v}}(M/\widetilde{\rho}, \vartheta(\widetilde{\rho}\cdot)) =: \underline{\mathrm{v}}(\widetilde{M}, \widetilde{\vartheta}).$$

Then for any $\alpha > 0$ we have

$$\mathrm{osc}(\widetilde{V}, \alpha) = \frac{1}{\widetilde{\rho}}\mathrm{osc}(V, \alpha\widetilde{\rho}); \tag{4.4}$$

and, testing against $C_c^1(\mathbb{R}^{n+k};\mathbb{R}^{n+k})$ vector fields, one deduces

$$\underline{\widetilde{H}} = \widetilde{\rho}\underline{H}(\widetilde{\rho}\cdot), \tag{4.5}$$

where in (4.5), $\underline{\widetilde{H}}$ denotes the mean curvature of $\widetilde{V}$. Using (4.5) to compute the L^p-norm of $\underline{\widetilde{H}}$ and plugging $\alpha = 1$ into (4.4) we deduce

$$\widetilde{\rho}^{1-n/p}\|\underline{H}\|_{L^p(B_{\widetilde{\rho}})} = \|\underline{\widetilde{H}}\|_{L^p(B_1)} \le \mathrm{osc}(\widetilde{V},1) = \widetilde{\rho}^{1-n/p}\frac{\mathrm{osc}(V,\widetilde{\rho})}{\widetilde{\rho}^{2-n/p}}.$$

<u>STEP 2.</u> Assume $\rho = 1$, $S = \mathbb{R}^n \times \{0\}^k$ and write

$$\mathbb{R}^{n+k} \ni x = (x^1, x^2) \in \mathbb{R}^n \times \mathbb{R}^k.$$

Suppose

$$\lambda := \mathrm{osc}(1) \le \eta \quad \text{and} \quad \|\underline{H}\|_{L^p} \le \lambda, \tag{4.6}$$

but (4.3) to be false for some $\eta > 0$ yet to be chosen. From (4.6) we deduce the existence of $y_0 \in \mathrm{spt}\mu \cap B_1$ such that

$$\mathrm{spt}\mu \cap B_1 \subseteq \left\{ y \in \mathbb{R}^{n+k} : |y^2 - y_0^2| \le \lambda \right\}.$$

Since (4.3) does not hold, we can fix $y_1, y_2 \in B_\eta \cap \mathrm{spt}\mu$ such that

$$|y_1^2 - y_2^2| > 2(1-\eta)\lambda.$$

Then we set the unitary vector

$$\kappa := \frac{0 \otimes (y_1^2 - y_2^2)}{|y_1^2 - y_2^2|} \in \partial B_1^{n+k},$$

and the convex functions on B_1

$$f_1(x) := \left((x - y_0)\cdot\kappa - \frac{\lambda}{2} \right)^+, \quad f_2(x) := \left(-(x - y_0)\cdot\kappa - \frac{\lambda}{2} \right)^+.$$

Some rapid computations give

$$0 \le f_i \le \frac{\lambda}{2} \tag{4.7}$$

$$f_i(y_i) \ge \left(\frac{1}{2} - 2\eta \right)\lambda \tag{4.8}$$

for $i \in \{1,2\}$. Let

$$I_i(r) := \frac{1}{\omega_n r^n}\int_{B_r(y_i)} f_i \, d\mu, \quad 0 < r \le 1-\eta, i \in \{1,2\}.$$

Recalling that $\lambda < \eta$, (4.6), (4.7), (4.8) and Lemma 4.2.1, we deduce that for every $r \in [\eta, 1/2]$ and for every $i \in \{1,2\}$ we have

$$\begin{aligned} I_i(r) &\geq \left(\frac{1}{2} - 2\eta\right)\lambda - C_0\lambda\left(\frac{\lambda}{2} + r\right) r^{1-n/p} \\ &\geq \left(\frac{1}{2} - 2\eta - C_0 r^{1-n/p}\right)\lambda \\ &\geq \left(\frac{1}{2} - (C_0+2) r^{1-n/p}\right)\lambda. \end{aligned}$$

Therefore

$$I_1(r) + I_2(r) \geq \left(1 - 2(C_0+2) r^{1-n/p}\right)\lambda \quad \forall r \in [\eta, 1/2]. \tag{4.9}$$

On the other hand, recalling (4.7), item 5. of the assumptions in the beginning of the section and observing that $\operatorname{spt} f_1$ and $\operatorname{spt} f_2$ are disjoint and $y_1, y_2 \in B_\eta$,

$$\begin{aligned} I_1(r) + I_2(r) &= \frac{1}{\omega_n r^n} \int_{B_r(y_1)\cup B_r(y_2)} (f_1 + f_2)\, d\mu \\ &\leq \frac{1}{\omega_n r^n} \mu(B_{r+2\eta}) \left(\|f_1 + f_2\|_{L^\infty}\right) \\ &\leq \frac{3\lambda}{4} \frac{(r+2\eta)^n}{r^n} \\ &\leq \frac{3}{4}\left(1 + C_1 \frac{\eta}{r}\right)\lambda. \end{aligned} \tag{4.10}$$

$$1 - 2(C_0+2) r^{1-n/p} \leq \frac{3}{4}\left(1 + C_1\frac{\eta}{r}\right) \quad \forall r \in (\eta, 1/2). \tag{4.11}$$

Choosing $r^* = r^*(n,k,p)$ and $\eta = \eta(n,k,p)$ such that

$$1 - 2(C_0+2) r^{1-n/p} > \frac{3}{4}(1 + C_1 r) \quad \forall r \in (0, r^*) \tag{4.12}$$

and subsequently $\eta = \eta(n,k,p) < (r^*)^2$, (4.11) and (4.12) yield to a contradiction.

□

The above result also makes clear our need for item 5. of the assumptions stated in the beginning of this section: we need to exclude the possibility of having a varifold which consists of two of two separated sheets.

As a consequence of Lemma 4.2.2 we have the following.

Lemma 4.2.3. *There are positive constants $\beta = \beta(n,k,p)$ and $C = C(n,k,p)$ with the following property. Let $V = \underline{\mathrm{v}}(M,\vartheta)$ be a varifold satisfying the standing assumptions, together with $0 \in M$ Suppose that $V = \underline{\mathrm{v}}(M\vartheta)$. For any $R \in (0,1]$, and $S \in G(n+k,n)$, then*

$$\operatorname{osc}_S(r) \leq C(\operatorname{osc}_S(R) + \|\underline{H}\|_{L^p} R^{2-n/p})\left(\frac{r}{R}\right)^\beta$$

for every $r \in [C(\operatorname{osc}_S(R) + \|\underline{H}\|_{L^p} R^{2-n/p}), R]$.

Proof. By virtue of a rescaling argument (cfr. *Proof of Lemma* 4.2.2) we can assume $R = 1$. Fix $S \in G(n+k, n)$ and let us write osc instead of osc_S in the rest of the proof. Let $K > 0$ be a large number yet to be fixed and define the function

$$F(r) := \mathrm{osc}(r) + K\|\underline{H}\|_{L^p} r^{2-n/p} \leq 1 \quad \forall r \in (0,1).$$

<u>STEP 1.</u> We claim that if $\eta = \eta(n,k,p)$ is the constant determined in Lemma 4.2.2, then

$$F(\eta r) \leq (1-\eta)F(r) \quad \forall r : \mathrm{osc}(r) \leq \eta r. \tag{4.13}$$

Fix indeed r such that $\mathrm{osc}(r) \leq \eta r$. We have two cases:

1. If $\|\underline{H}\|_{L^p} r^{2-n/p} \leq \mathrm{osc}(r)$, then Lemma 4.2.2 gives

$$\begin{aligned} F(\eta r) &\leq (1-\eta)\mathrm{osc}(r) + \eta^{2-n/p} K \|\underline{H}\|_{L^p} r^{2-n/p} \\ &\leq (1-\eta)F(r) \end{aligned}$$

   Since η is much smaller than $\frac{1}{2}$.

2. If $\mathrm{osc}(r) < \|\underline{H}\|_{L^p} r^{2-n/p}$, then a straightforward computation gives

$$\begin{aligned} F(\eta r) &\leq \mathrm{osc}(r) + \eta^{2-n/p} K \|\underline{H}\|_{L^p} r^{2-n/p} \\ &\leq (1-\eta) K \|\underline{H}\|_{L^p} r^{2-n/p} \\ &\leq (1-\eta)F(r), \end{aligned}$$

   provided $K = K(n,k,p) > 1/(1-\eta-\eta^{2-n/p})$

<u>STEP 2.</u> By induction, if $F(1) \leq \eta^j$, then

$$\mathrm{osc}(\eta^j) \leq F(\eta^j) \leq (1-\eta)^j F(1).$$

<u>STEP 3.</u> Fix r such that $F(1) \leq r < 1$ and choose $j \in \mathbb{N}$ such that $\eta^{j+1} \leq r < \eta^j$. Then we have

$$\mathrm{osc}(r) \leq \mathrm{osc}(\eta^j) \leq F(\eta^j) \leq \frac{\eta^{\beta(j+1)}}{1-\eta} F(1) \leq K r^\beta F(1), \tag{4.14}$$

provided $\beta = \beta(n,k,p)$ and $K = K(,k,p)$ are such that

$$\eta^\beta > 1-\eta, \quad K \geq \frac{1}{1-\eta}. \tag{4.15}$$

Finally (4.14), together with the choice $C = \max\{1, K^2\}$ with $K = K(n,k,p)$ as in (4.15), concludes the proof.

□

4.3. Proof of for $p = \infty$ via Improvement of Flatness

Theorem 4.3.1 (Improvement of Flatness, p = ∞). *For every $\alpha \in (0,1)$, there are positive constants c, ε_0, η and C depending only on n and k with the following property. If $V = \underline{\mathrm{v}}(M,\vartheta) \in \mathscr{V}_n^\infty(B_R)$ is n-rectifiable varifold with*

1. *$x_0 \in M$;*
2. *$\vartheta \geq 1$;*
3. *$\mu(B_r) \leq \frac{3}{2}\omega_n r^n$ for each $0 \leq r \leq R$;*

and, for some $\delta \leq \delta_0$ and $S \in G(n+k,n)$,

$$\operatorname{osc}_S(V,x_0,R) \leq \varepsilon R \quad \textit{and} \quad \|\underline{H}\|_{L^\infty} \leq \frac{c\varepsilon}{R}. \tag{$\diamond_{\varepsilon,c}$}$$

Then there exists $T \in G(n+k,n)$ with $|S-T| \leq C\varepsilon$ such that

$$\operatorname{osc}_T(V,x_0,\eta R) \leq \eta^{1+\alpha}\varepsilon R. \tag{4.16}$$

Notice that the conclusion of Theorem 4.3.1 may be iterated at all scales. In particular, if we let

$$E(V,x,r) := \inf_{S\in G(n+k,n)} \frac{\operatorname{osc}_S(V,x,r)}{r} + C\|\underline{H}\|_{L^\infty(V\llcorner B_r)}r,$$

for some C large, then Theorem 4.3.1 yields the existence of global constants η, ε_0 such that

$$E(V,0,\eta R) \leq \eta^\alpha E(V,0,R),$$

provided $E(V,0,R) < \varepsilon_0$. A straightforward induction argument yields, for every $r > 0$,

$$\inf_{S\in G(n+k,n)} \frac{\operatorname{osc}_S(V,0,r)}{r} \leq E(V,0,r) \leq C\left(\frac{r}{R}\right)^\alpha E(V,0,R) \tag{4.17}$$

We now show how Theorem 4.3.1 implies Theorem 4.1.2. For making the reading easier we rewrite the statement of Allard's theorem

> *Theorem 4.1.2 For every $\alpha \in (0,1)$ there are constants $\delta_0 = \delta_0(n,k,\alpha)$ and $\gamma = \gamma(n,k,\alpha)$ in $(0,1)$ such that if $(\bigstar_{\infty,\delta})$ holds for any $\delta \in (0,\delta_0]$, then there exists a linear isometry $q : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ and a function $u \in C^{1,\alpha}(B^n_\gamma;\mathbb{R}^k)$ such that $M \cap B_\gamma = q(\mathrm{Graph}(u))$.*

Proof of Theorem 4.1.2. Assume the result is false. Then there exists a sequence $\delta_j \searrow 0$ and a sequence of varifolds $V_j = \underline{\mathrm{v}}(M_j,\vartheta_j)$ satisfying the assumptions of Theorem 4.1.2 with δ_0 replaced by δ_j, for which the conclusion of the theorem does not hold.

We claim that, for j large enough, the assumptions of Theroem 4.3.1 are in place for $R = 1/2$ and any $x_0 \in M_j \cap B_{1/2}$. Then the fact that M_j can be written as the graph of a $C^{1,\alpha}$ function follows from (4.17).

In the direction of proving the above claim, we first remark that, up to subsequences, there exists a stationary varifold $V = \underline{\mathbf{v}}(M,\vartheta)$ with $M \subseteq B_1$, $\mu_V(B_1) \leq \omega_n$ and $\vartheta \geq 1$ μ_V-a.e.. By virtue of Theorem 5.3 of [All72] we have $M = \mathscr{H}^n_{\llcorner S}$, for some $S \in G(n+k,n)$.

Given $\alpha \in (0,1)$, let now ε_0 be the constant given in Theorem 4.3.1. Then, for j large enough,

$$M_j \subseteq \left\{ y \in B_1^{n+k} : |\Pi_S^{\perp} y| \leq \frac{\varepsilon_0}{2} \right\}.$$

The only thing left to prove is that

$$\mu_j(B_{1/2}(x)) \leq \left(1 + \frac{1}{4}\right) \frac{\omega_n}{2^n} \tag{4.18}$$

for every j large and every $x \in M_j$. Given the above inequality, by the Monotonicity formula, provided δ_j is smaller than some universal constant, we obtain

$$\mu_j(B_r(x)) \leq \frac{3}{2} \omega_n r^n \quad \forall 0 \leq r \leq \frac{1}{2},$$

as required in Theorem 4.3.1.

We now prove (4.18). If the result is false, then there exists a subsequence $(j_\ell)_\ell$ and points $x_\ell \in M_{j_\ell} \cap B_{1/2}$ such that

$$\mu_{j_\ell}(B_{1/2}(x_{j_\ell})) \geq \left(1 + \frac{1}{4}\right) \frac{\omega_n}{2^n}. \tag{4.19}$$

Up to extracting a further subsequence, we can assume $x_\ell \to x \in \overline{B_{1/2}}$ and, by monotonicity, $x \in M$. Therefore, for any $\varepsilon > 0$,

$$\begin{aligned} \omega_n \left(\frac{1}{2} + \varepsilon \right)^n &= \mu(B_{\varepsilon + 1/2}(x)) \\ &\geq \limsup_{\ell \to \infty} \mu_{j_\ell}(B_{\varepsilon+1/2}(x)) \\ &\geq \limsup_{\ell \to \infty} \mu_{j_\ell}(B_{1/2}(x_\ell)) \end{aligned}$$

contradicting (4.19) and thus concluding the proof.

□

The rest of this section is dedicated to the proof of Theorem 4.3.1. We begin with a key preliminary result.

Lemma 4.3.1. *Let $\Omega \subseteq \mathbb{R}^{n+k}$ be open, and fix a varifold $V = \underline{\mathbf{v}}(M,\vartheta) \in \mathscr{V}_n^\infty(\Omega)$ with $\vartheta \geq 1$ μ_V-a.e. . If a function $f \in C^2(\Omega)$ is such that $f\big|_{\operatorname{spt}\mu_V}$ achieves a local maximum at $x_0 \in \operatorname{spt}\mu_V \cap \Omega$, then*

$$\operatorname{tr}_n D^2 f(x_0) \leq \|\underline{H}\|_{L^\infty} |\nabla f(x_0)|;$$

where we set $\operatorname{tr}_n A$ to be the n smallest eigenvalues of $A \in \mathscr{M}_{n+k}$.

Proof. Without loss of generality, we assume $x_0 = 0$ and that f has a global strict maximum at 0. By contradiction, assume that there are $\delta > 0$ and $r > 0$ such that

$$\operatorname{tr}_n D^2 f(x_0) - \|\underline{H}\|_{L^\infty} \geq \delta \tag{4.20}$$

for every $x \in B_r$. Up to choosing a smaller r and adding a constant to f, we also assume that $f(0) > 0$ and that $\{f > 0\} \cap \operatorname{spt}\mu_V \subseteq B_r$. By mollification and the standing assumptions we obtain

$$\begin{aligned} \|\underline{H}\|_{L^\infty} \int_{\{f>0\}} f|\nabla f|\, d\mu_V &\geq \int_{\{f>0\}} \operatorname{div}_M(f\nabla f)\, d\mu_V \\ &= \int_{\{f>0\}} \left(|\nabla_M f|^2 + f \operatorname{div}_M \nabla f\right) d\mu_V \\ &\geq \int_{\{f>0\}} f \operatorname{div}_M \nabla f\, d\mu_V. \end{aligned} \tag{4.21}$$

Combining (4.20) and (4.21) we deduce

$$\delta \int_{\{f>0\}} f\, d\mu_V \leq \int_{\{f>0\}} f\left(\operatorname{div}_M \nabla f - \|\underline{H}\|_{L^\infty} |\nabla f|\right) d\mu_V,$$

which contradicts the fact that $0 \in \operatorname{spt}\mu_V$.

□

Proof of Theorem 4.3.1. Without loss of generality we may assume $R = 1$. By contradiction, assume there are sequences $\varepsilon_j \searrow 0$, $c_j \searrow 0$ and $(V_j)_j$ such that V_j satisfies all of the assumptions of the statement for some fixed $S \in G(n+k, n)$ (that we assume to be $\mathbb{R}^n \times \{0\}^k$) with $(\diamond_{\varepsilon,c})$ replaced by $(\diamond_{\varepsilon_j,c_j})$, for which however (4.16) fails for every choice of $\eta > 0$ and $T \in G(n+k, n)$.

Using the compactness result Theorem 3.1.2, up to subsequences, there exists a n-rectifiable varifold $V = \underline{\mathrm{v}}(M, \vartheta)$ with the following properties

$$V_j \rightharpoonup^* V, \quad \vartheta \geq 1 \quad \mu_V - a.e., \quad \underline{H}_V \equiv 0.$$

To the aim of making the notation lighter, from now on we write μ_j instead of μ_{V_j} and μ instead of μ_V; moreover . Recalling that $(\diamond_{\varepsilon_j,c_j})$ implies

$$\operatorname{spt}\mu_j \subseteq \left\{x = \left(x^1, x^2\right) \in \mathbb{R}^n \times \mathbb{R}^k : |x^2| \leq \varepsilon_j\right\} =: S_{\delta_j},$$

therefore

$$\mu = \vartheta \mathscr{H}^n_{\llcorner B_1^n}.$$

Moreover, since $x \mapsto \vartheta(x)$ has vanishing distributional gradient,

$$\vartheta \equiv \vartheta_0 \geq 1$$

for some constant ϑ_0.

For each $j \in \mathbb{N}$ we set

$$F_j : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}, \quad F_j(x^1, x^2) := \left(x^1, \frac{x^2}{\varepsilon_j}\right) \quad \forall x \in \mathbb{R}^{n+k},$$

and consider the blow-up varifolds

$$\widetilde{V}_j := F_j^{\#} V_j = \underline{\mathrm{v}}\left(F_j(M_j), \vartheta_j \circ F_j^{-1}\right) =: \underline{\mathrm{v}}(\widetilde{M}_j, \widetilde{\vartheta}_j).$$

Clearly

$$\widetilde{M}_j \subseteq B_1^n \times \overline{B_1^k} \quad \forall j \in \mathbb{N}. \tag{4.22}$$

Moreover, since F_j is an homeomorphism for each $j \in \mathbb{N}$, $\widetilde{M}_j$ relatively is closed and non-empty. Up to considering a subsequence, there exists a closed non-empty set $\widetilde{M}$ such that

$$\lim_{j\to\infty} \operatorname{dist}_{\mathscr{H}}(\widetilde{M}_j, \widetilde{M}) = 0, \tag{4.23}$$

where $\operatorname{dist}_{\mathscr{H}}$ is the Hausdorff distance defined on relatively compact subsets of $B_1^n \times \overline{B_1^k}$.

We now prove that M is the graph of an harmonic function. In order to do so, we set

$$\mathfrak{U}(y) := \left\{z \in \mathbb{R}^k : (y, z) \in \widetilde{M}\right\},$$

and claim

$$\mathfrak{U}(y) \neq \varnothing \quad \forall y \in B_1^n, \tag{$\mathfrak{A}$}$$

$$\exists u \in C^0(B_{1/2}^n; \overline{B_1^k}) \quad \text{such that} \quad \mathfrak{U}(y) = \{u(y)\}, \tag{$\mathfrak{B}$}$$

$$u \quad \text{is harmonic.} \tag{$\mathfrak{C}$}$$

Proof of ($\mathfrak{A}$). By contradiction, assume there exists $y \in B_1^n$ such that $\mathfrak{U}(y) = \varnothing$. Since $\widetilde{M}$ is relatively closed, this implies

$$\widetilde{M} \cap \left(B_{2r}^n(y) \times \overline{B_1}\right) = \varnothing,$$

for some $r > 0$ small enough. By (4.23), then, for j large enough,

$$\widetilde{M}_j \cap \left(B_r^n(y) \times \overline{B_1}\right) = \varnothing.$$

Therefore

$$\begin{aligned} 0 &= \liminf_{j\to\infty} \mu_j(B_r^n(y) \times \overline{B_1}) \\ &\geq \vartheta_0 \mathscr{H}^n_{\llcorner \mathbb{R}^n}(B_r^n(y) \times \overline{B_1}) \\ &= \vartheta_0 \omega_n r^n, \end{aligned}$$

which is false.

Proof of (𝔅). By virtue of Lemma 4.2.3 we can find constants $C = C(n,k)$ and $\beta = \beta(n,k)$ such that, if $x = (y,z) \in M_j \cap B_{1/2}$ and $r \in (C\varepsilon_j, 1/2)$, then

$$\operatorname{osc}(V_j, x, r) \le C\varepsilon_j r^\beta \quad \forall j \in \mathbb{N}.$$

In particular

$$|z_1 - z_2| \le 2C\varepsilon_j r^\beta \quad \forall x_i = (y_i, z_i) \in M_j \cap B_r(x), i = 1,2.$$

Therefore

$$|\widetilde{z}_1 - \widetilde{z}_2| \le 2Cr^\beta \quad \forall \widetilde{x}_i = (y_i, \widetilde{z}_i) \in \widetilde{M}_j \cap \left(B^n_r(y) \times B^k_1\right), i = 1,2.$$

Hence, by (4.23), passing to the limit as $j \to \infty$ we deduce

$$|\widetilde{z}_1 - \widetilde{z}_2| \le 2C|y_1 - y_2|^\beta \quad \forall \widetilde{x}_i = (y_i, \widetilde{z}_i) \in \widetilde{M} \cap \left(B^n_{1/2} \times B^k_1\right), i = 1,2,$$

which implies (B).

Proof of (ℭ). Let $h : B^n_{1/4} \to \mathbb{R}^k$ be the harmonic function determined by the condition

$$(h-u)\big|_{\partial B^n_{1/4}} = 0.$$

If $u \ne h$, then there is $\delta \in (0,1/2)$ small such that, for all $j \in \mathbb{N}$ sufficiently large, the function

$$G(x) := \frac{1}{2}\left|\frac{z}{\varepsilon_j} - h(y)\right| + \frac{\delta}{2}|y|^2, \quad x = (y,z) \in B^n_1 \times B^k_1$$

is such that $G\big|_{M_j}$ achieves its maximum at some point $x_1 = (y_1, z_1)$ with

$$|y_1| \le \frac{1}{4} - \delta.$$

We claim that j can be chosen so large that, for every $T \in G(n+k,n)$, it holds

$$\operatorname{div}_T \nabla G(x_1) > \|\underline{H}_{V_j}\|_{L^\infty}, \tag{4.24}$$

which would contradict Lemma 4.3.1. In the rest of the proof of (C), C will be some constants depending only on n, k and δ, which may be different every time, but are independent of j.

By standard elliptic estimates,

$$\max\left\{|\nabla h(y_1)|, |D^2 h(y_1)\right\} \le C.$$

Therefore

$$|\nabla G(x_1)| \le \frac{C}{\varepsilon_j} \quad \forall j \text{ large enough},$$

and

$$\|\underline{H}_{V_j}\|_{L^\infty}|\nabla G(x_1)| \le Cc_j \xrightarrow[j\to\infty]{} 0.$$

Thus, in order to prove (4.24), it is sufficient to prove that, for j sufficiently large,

$$\inf_{T\in G(n+k,n)} \mathrm{div}_T \nabla G(x_1) \ge \delta. \tag{4.25}$$

Using our usual convention, we define the function

$$f(x) := \frac{1}{\varepsilon_j} z - h(y).$$

Then

$$\begin{aligned} D^2G(x_1) &= D^2 f(x_1)\cdot f(x_1) + \nabla f(x_1)\left(\nabla f(1)\right)^* + \delta\Pi_S \\ &\ge D^2 f(x_1)\cdot f(x_1) + \delta\Pi_S \\ &=: \Lambda. \end{aligned}$$

In particular, since $\Delta h = 0$,

$$\begin{aligned} \mathrm{div}_S \nabla G(x_1) &\ge \mathrm{tr}_S \Lambda \\ &\quad \Delta h(y_1)\cdot\left(\frac{z_1}{\varepsilon_j} - h(y_1)\right) + n\delta \\ &\quad = n\delta, \end{aligned}$$

where $\mathrm{tr}_S B = \sum_{i=1}^n A\xi_i\cdot\xi_i$, for any orthonormal basis $\{\xi_1,\dots,\xi_n\}$ of S. Since $|\Lambda|\le C$, by continuity there exists $\gamma>0$ such that

$$\mathrm{div}_T\nabla G(x_1) \ge \delta \quad \forall T : |T-S| \le \gamma;$$

proving (4.25) in the case $|T-S|\le\gamma$.

On the other hand, if $|T-S|>\gamma$, then there is a unit vector $\nu = (\nu_1,\nu_2) \in T \subseteq \mathbb{R}^n\times\mathbb{R}^k$ and a constant $\kappa=\kappa(n)$ such that

$$|\nu_2| \ge \kappa\gamma.$$

Then

$$\mathrm{div}_T\nabla G(x_1) \ge \mathrm{tr}_T\Lambda + |(\nabla f(x_1)^*\eta)|^2.$$

We have $|\mathrm{tr}_T A| \le |A| \le C$ and

$$\begin{aligned} |(\nabla f(x_1)^*\eta)| &= |-\nabla h(y_1)\nu_1 + \frac{1}{\varepsilon_j}\nu_2| \\ &\ge \frac{1}{\varepsilon_j}|\nu_2| - |\nabla h(y_1)| \\ &\ge \frac{\kappa\gamma}{\varepsilon_j} - C. \end{aligned}$$

Thus

$$\mathrm{div}_T \nabla G(x_1) \geq \left(\frac{\kappa\gamma}{\varepsilon_j} - C\right)^C - C \geq \delta$$

for j large enough. This proves (4.25) in the case $|T - S| \geq \gamma$.

Conclusion of the proof of Theorem 4.3.1. From (4.22) follows that

$$\sup_{B^n_{1/4}} |u| \leq 1;$$

and from (C) and classical elliptic estimates, we deduce the existence of a constant $C = C(n,k)$ such that

$$\sup_{B^n_{1/8}} \left(|\nabla u| + |D^2 u|\right) \leq C.$$

We can now choose $\eta = \eta(n,k) > 0$ small enough such that

$$|u(y) - u(0) - \nabla u(0)y| \leq C|y|^2 \leq \frac{\eta^{1+\alpha}}{2} \quad \forall y \in B^n_{2\eta}. \tag{4.26}$$

Recalling that $0 \in \mathrm{spt}\mu_j$ for each $j \in \mathbb{N}$, (4.23) and multiplying (4.26) by ε_j, for $j \in \mathbb{N}$ large enough we have

$$|z - \delta_j \nabla u(0) y| \leq \varepsilon_j \eta^{1+\alpha} \quad \forall x = (y,z) \in M_j \cap B_\eta.$$

We have therefore proved that

$$\mathrm{osc}(V_j, 0, \eta) \leq \varepsilon_j \eta^{1+\alpha}$$

for all $j \in \mathbb{N}$ sufficiently large: a contradiction.

□

An important corollary, which will play an important role in Chapter 5 for proving the regularity theorem for solutions of the Plateau problem is the following.

Corollary 4.3.1. *Let $\Omega \subseteq \mathbb{R}^{n+k}$ be open and $V = \underline{\mathrm{v}}(M,\vartheta) \in \mathscr{R}_n(\Omega)$ and let $\underline{H} \in L^\infty(\mu)$ be its generalized mean curvature. Assume V to satisfy $(\bigstar_{\infty,\delta})$ for $\delta \in (0,\delta_0)$. Futthermore suppose $\vartheta(x) = 1$ for μ-a.e. $x \in \Omega$ and $\underline{H} = h\big|_M$, where $h \in C^{q,\alpha}(U;\mathbb{R}^{n+k})$ for some open set $U \subseteq \Omega$ containing M and $q \in \mathbb{N}$. Then, for any $x \in M$ and $\alpha \in (0,1)$, up to rotations of $\mathbb{R}^{n+k}$, there exists an open subset $W \subseteq \mathbb{R}^n$, a function $u \in C^{q+2,\alpha}(W;\mathbb{R}^k)$ and $\rho > 0$ such that*

$$M \cap B_\rho(x) = \mathrm{Graph}(u) \quad \textit{and} \quad \Theta^n(\mu,\cdot) \equiv 1 \textit{ on } \mathrm{Graph}(u).$$

Proof. Fix $\alpha > 0$. Under the standing assumption, we can choose $\rho > 0$ such that, after applying the appropriate translation and rotation, there exists a function $u \in C^{1,\alpha}(W;\mathbb{R}^k)$ with the property

$$M \cap B_\sigma(x) = \mathrm{Graph}(u),$$

where $\sigma = \rho/2$. Since $\vartheta = 1$ $\mathscr{H}^n$-a.e. on Graph(u), and Graph(u) is a C^1 submanifold, then $\Theta^n(\mu, \cdot) = 1$ at every point of Graph(u).

Let $\varepsilon_0 \in (0,1)$. Since, without loss of generality, $\nabla u(x) = 0$, by choosing a smaller σ if necessary, we can assume that $|\nabla u| \le \varepsilon_0$ on $B_\sigma(x)$ and so it is easy to prove that u satisfies the following system of equations

$$\Delta u_i = \sum_{j=1}^{n} \partial_j (A_{i,j}(\nabla u)) + h_{n+i} \quad \forall i \in \{1, \dots, k\},$$

with $A_{i,j}(\cdot) \in C^\infty(\mathbb{R}^n \otimes \mathbb{R}^k)$ such that

$$|A_{i,j}(P)| \le C|P|^2 \quad \text{and} \quad |\nabla A_{i,j}(P)| \le C|P|$$

for some constant $C = C(n)$. Then by the Schauder theory for elliptic equations we see that $h \in C^{q,\alpha}(B_\sigma(x); \mathbb{R}^{n+k})$ implies $u \in C^{q+2,\alpha}(B_\sigma(x); \mathbb{R}^k)$.

□

4.4. Ideas for a Generalization to $n < p < \infty$

All of the arguments presented in Section 4.2 and in Section 4.3 work also in L^p for $p \in (n, \infty)$; the only exceptions are Lemma 4.3.1 and, as a consequence, the harmonicity of the function u constructed in the proof of Theorem 4.3.1. On the other hand, if one is able to prove that the aforementioned function u is harmonic where defined, then the very same arguments used in the proof of Theorem 4.1.2 yield to the proof of Theorem 4.1.1 for $p \in (n, \infty)$.

In this section we will only focus in the case of codimension $k = 1$. To the aim of proving the harmonicity of the function u of the proof of Theorem 4.3.1, we begin with a formal argument.

Let us recall how the function u was constructed. We started with a sequence of varifolds $(V_j)_j$ for each $j \in \mathbb{N}$, $M_j := \operatorname{spt}\mu_j$ is contained in the strip $B_1^n \times [-\varepsilon_j, \varepsilon_j]$ and the mean curvature $\underline{H}_j$ had $L^p(\mu_j)$-norm not greater than $\varepsilon_j c_j$, for some $\varepsilon_j \searrow 0$ and $c_j \searrow 0$. Subsequently, we considered the sequence of functions $(F_j)_j$ defined as

$$F(y,z) := \left(y, \frac{z}{\varepsilon_j}\right) \quad \forall (y,z) \in \mathbb{R}^n \times \mathbb{R}^k$$

and the sequence of rescaled supports $(\widetilde{M}_j)_j$,

$$\widetilde{M}_j := F_j(M_j) \subseteq B_1^n \times \overline{B_1^k} \quad \forall j \in \mathbb{N}.$$

By virtue of classical results on precompactness on the metric space of of compacts sets endowed with the Hausdorff distance $\operatorname{dist}_{\mathscr{H}}$, we deduce that, up to a not relabelled subsequence, there exists a relatively closed set $\widetilde{M} \in B_1^n \times \overline{B_1^k}$ such that

$$\lim_{j \to \infty} \operatorname{dist}_{\mathscr{H}}(\widetilde{M}_j, \widetilde{M}) = 0.$$

Finally, using Lemma 4.2.3 we are able to prove that $\widetilde{M} \cap B^n_{1/2} \times \mathbb{R}^k$ is actually the graph of a function $u \in C^0(B_1)$.

Assuming $(M_j)_j$ to be a C^2-graph-like sequence the result that we need to prove can be written as

Proposition 4.4.1. *Let $p \in (n,\infty)$ and $(v_j)_j \subseteq C^1(B^n_1)$. Assume the existence of sequences $\varepsilon_j \searrow 0$, $c_j \searrow 0$ and of a function $u \in C^0(B_1)$ such that:*

1. $\|v_j\|_{L^\infty} \le \varepsilon_j$;
2. $\|H_j\|_{L^p} \le c_j \varepsilon_j$, *where H_j is defined as*

$$H_j := \operatorname{div}\left(\frac{\nabla v_j}{\sqrt{1+|\nabla v_j|^2}}\right) \quad \forall j \in \mathbb{N}$$

   we have ;

3. *setting $u_j := v_j/\varepsilon_j$, we have that $(u_j)_j$ converges uniformly to u;*

then u is harmonic on B_1.

The proof of Proposition 4.4.1 is greatly inspired by the proof of the Alexandroff-Bakelman-Pucci principle presented in [CC95].

Proof. We prove that u is harmonic via a viscosity approach. For simplifying the notation, in this proof we shall write B_ρ instead of B^n_ρ. We argue by contradiction. Suppose there exists $x_0 \in B_1$ and $\varphi \in C^2_c(B_\rho(y_0))$ such that

$$\begin{cases} u - \varphi \text{ has a strict maximum at } x_0, \\ \Delta\varphi(y_0) < 0 \end{cases}.$$

Without losing of generality we can assume $y_0 = 0$ and the existence of constants $\lambda > 0$ and $r > 0$ such that

$$\begin{aligned} u(0) - \varphi(0) &\le -2\lambda, \\ u(y) - \varphi(y) &> 2\lambda \quad \text{on } \partial B_r, \\ \Delta\varphi(y) &\ge \lambda \quad \text{on } B_r. \end{aligned}$$

Let $\varphi_j := \varepsilon_j \varphi$. By uniform convergence of $(u_j)_j$ we can write, for $j \in \mathbb{N}$ large enough,

$$v_j(0) - \varphi_j(0) \le -\varepsilon_j \lambda, \tag{4.27}$$

$$v_j(y) - \varphi_j(y) > \varepsilon_j \lambda \quad \text{on } \partial B_r, \tag{4.28}$$

$$\Delta\varphi_j(y) \ge \varepsilon_j \lambda \quad \text{on } B_r. \tag{4.29}$$

Moreover we can find a sequence $(y_j)_j$ such that

$$y_j \text{ global minimum point of } \chi_{B_r}(v_j - \varphi_j), \quad y_j \xrightarrow[j\to\infty]{} 0.$$

Let us denote by K_j the convex envelope of the function $\chi_{B_r}(v_j-\varphi_j)$ in B_{2r}. From now on we only consider the truncation to B_r of the function v_j and φ_j (i.e. $\chi_{B_r}v_j$ and $\chi_{B_r}\varphi_j$), which will still be denoted by v_j and φ_j respectively. For all $j\in\mathbb{N}$ such that (4.27) holds, we have

$$B_{\frac{\varepsilon_j\lambda}{2\rho}} \subseteq B_{-\frac{(u_j-\varphi_j)(y_j)}{2\rho}} \subseteq \nabla K_j(B_{2\rho}).$$

Thus, recalling the area formula and that

$$\det D^2K_j(y)\equiv 0 \quad \text{in } \{y\in B_r : K_j(y)\neq(v_j-\varphi_j)(y)\},$$

for the values of j such that (4.27), (4.28) and (4.29) hold, there exists a constant $C_1=C_1(n)$ such that

$$\begin{aligned}\frac{1}{C_1}\left(\frac{\varepsilon_j\lambda}{4\rho}\right)^n &\leq \int_{B_{\nabla K_j(B_{2\rho})}} \frac{1}{\left(1+|p|^2\right)^{\frac{n+2}{2}}}\,dp\\ &= \int_{B_{2\rho}} \frac{\det D^2K_j}{\left(1+|\nabla K_j|^2\right)\frac{n+2}{2}}\,dy\\ &\leq \int_{\{v_j-\varphi_j=K_j\}} \mathcal{G}[v_j-\varphi_j]\,dy,\end{aligned} \tag{4.30}$$

where $\mathcal{G}[v_j-\varphi_j]$ denotes the Gaussian curvature of the graph of the function $v_j-\varphi_j$, hence the determinant of the second fundamental form of Graph$(v_j-\varphi_j)$. Denoting by $\mathcal{H}[v_j-\psi_j]$ the mean curvature of Graph$(v_j-\varphi_j)$ (i.e. the trace of the second fundamental form of Graph$(v_j-\varphi_j)$), and applying the the AM-GM inequality, (4.30) writes

$$\frac{1}{C_1}\left(\frac{\lambda n}{4\rho}\right)^n\varepsilon_j^n \leq \int_{\{v_j-\varphi_j=K_j\}} \left(\mathcal{H}[v_j-\varphi_j]\right)^n\,dy. \tag{4.31}$$

Suppose

$$\mathcal{H}[v_j-\varphi_j]=(1+O(\varepsilon_j))H_j+O(\varepsilon_j^3) \quad \text{on} \quad \left\{K_j=v_j-\varphi_j\right\}. \tag{$*$}$$

By $(*)$, and (4.31), we deduce existence of constants $C_2, C_3>0$ which are independent of j such that

$$\begin{aligned}C_2 &\leq C_3\int_{\{v_j-\varphi_j=K_j\}}\left(\frac{H_j}{\varepsilon_j}\right)^n dy+O(\varepsilon_j^{2n})\\ &\leq C_3\omega^{1-n/p}\left(\frac{\|H_j\|_{L^p}}{\varepsilon_j}\right)^n+O(\varepsilon_j^{2n})\\ &= C_3\omega^{1-n/p}c_j^n+O(\varepsilon_j^{2n})\\ &\xrightarrow[j\to\infty]{} 0.\end{aligned}$$

Which is a contradiction.

This proves that u is super-harmonic in B_1.

The proof of the harmonicity of u in B_1 follows from a specular argument, using $\psi \in C_c^2(B_\rho(x_0))$ such that

$$\begin{cases} u - \psi \text{ has a strict maximum at } x_0, \\ \Delta\psi(x_0) < 0 \end{cases}$$

instead of φ.

To the aim of proving $(*)$, we recall that

$$\|\nabla\varphi_j\|_{L^\infty(B_r)} + \|D^2\varphi_j\|_{L^\infty(B_r)} = O(\varepsilon_j),$$
$$|\nabla v_j| = O(\varepsilon_j) \quad \text{on} \quad \{K_j = v_j - \varphi_j\}.$$

Where (4.4) follows from the fact that

$$\nabla K_j = \nabla v_j - \nabla\varphi_j \quad \mathscr{L}^n - a.e. \text{ on } \{K_j = v_j - \varphi_j\}. \tag{4.32}$$

Since by convexity of K_j we have

$$|\nabla K_j| \le \frac{\lambda\varepsilon_j}{2\rho},$$

then, invoking (4.4), (4.32), and continuity of ∇v_j, (4.4) follows. Recalling the expression for the mean curvature of a smooth graph, on $\{K_j = v_j - \varphi_j\}$ we write

$$\begin{aligned} H[v_j - \varphi_j] &= \operatorname{div}\left(\frac{\nabla v_j - \nabla\varphi_j}{\sqrt{1 + |\nabla v_j - \nabla\varphi_j|^2}}\right) \\ &= \frac{1}{\sqrt{1 + |\nabla v_j - \nabla\varphi_j||^2}}\left(\mathbb{1}_n - L[v_j - \varphi_j]\right) : \left(D^2 v_j - D^2\varphi_j\right); \end{aligned} \tag{4.33}$$

where in (4.33) we defined, for $f \in C^2$,

$$L[f] := \frac{\nabla f \otimes \nabla f}{1 + |\nabla f|^2} \in \mathscr{M}_n(\mathbb{R}).$$

Therefore $|L[f]| \le 1$, hence the matrix $\mathbb{1}_n - L[f]$ is invertible for every choice of $f \in C^2$. Let $R[f]$ be the inverse matrix of $\mathbb{1}_n - L[f]$. Then, by the previous estimates (4.4), (4.4),

$$\begin{aligned} &H[v_j - \varphi_j] \\ &= \frac{\sqrt{1 + |\nabla v_j|^2}}{\sqrt{1 + |\nabla(v_j - \varphi_j)|^2}}\left(\mathbb{1}_n - L[v_j - \varphi_j]\right) R[v_j] \frac{\left(\mathbb{1}_n - L[v_j]\right)}{\sqrt{1 + |\nabla v_j|^2}} : \left(D^2(v_j - \varphi_j)\right) \\ &= \left(1 + O(\varepsilon_j)\right) \frac{\left(\mathbb{1}_n - L[v_j]\right)}{\sqrt{1 + |\nabla v_j|^2}} : \left(D^2(v_j - \varphi_j)\right) \\ &= \left(1 + O(\varepsilon_j)\right)\left(H_j - \frac{\Delta\varphi_j}{\sqrt{1 + |\nabla v_j|^2}} + \frac{\langle\nabla v_j, D^2\varphi_j \nabla v_j\rangle}{\left(1 + |\nabla v_j|^2\right)^{3/2}}\right). \end{aligned} \tag{4.34}$$

Using (4.29) and (4.4) in (4.34), we conclude the proof of both (∗) and of Proposition 4.4.1.

□

Generalizing the arguments presented in the proof of Proposition 4.4.1 is however not easy. The first difficulty is the fact that we do not know *a priori* that M_j is a graph-type set. The second and more subtle obstacle is that the standard notion of viscosity solutions is not well suited for the environment of L^p functions, with $p \in (n,\infty)$, on wich we can no longer rely on pointwise estimates. What seems the correct notion is the one of of $W^{2,p}$-viscosity solutions. We refer to [Sch04] for the aforementioned definition.

In [Sch04] the author addresses both of the issues considering the upper and lower functions of a varifold V, i.e. the functions

$$v^+(y) := \sup\left\{z \in \mathbb{R}^k : (y,z) \in \operatorname{spt}\mu_V\right\},$$
$$v^-(y) := \inf\left\{z \in \mathbb{R}^k : (y,z) \in \operatorname{spt}\mu_V\right\}$$

respectively, and proves:

1. The function $v^\pm$ are twice approximately differentiable $\mathscr{L}^n$-a.e. and their approximate differentials satisfy

$$\underline{H}(y, v^\pm(y)) = \operatorname{div}\left(\frac{\nabla v^\pm}{\sqrt{1+|\nabla v^\pm|^2}}\right)\frac{(-\nabla v^\pm(y),1)}{\sqrt{1+|\nabla v^\pm(y)}}$$

   for $\mathscr{L}^n$-a.e. y such that $v^\pm(y) \in \mathbb{R}$.

2. The functions upper and lower functions v^+ and v^- are $W^{2,p}$-viscosity sub and super solutions respectively of the partial differential equation

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = \underline{H}_j(\cdot, v)\frac{(-\nabla v,1)}{\sqrt{1+|\nabla v|^2}}.$$

3. A strong maximum principle holds for $v^\pm$. More precisely there are no $W^{2,p}$-class functions ψ that both touch from above v^+ (i.e. $v^+ - \psi$ has a maximum point) and make

$$\operatorname{div}\left(\frac{\nabla \psi}{\sqrt{1+|\nabla \psi|^2}}\right) \le \underline{H}_j(\cdot, v^+)\frac{(-\nabla v^+,1)}{\sqrt{1+|\nabla v^+|^2}}$$

   hold in an open set U, unless their graph is contained in $\operatorname{spt}\mu_V$. A symmetrical result holds true for v^-.

These three result should therefore guarantee an extension of Proposition 4.4.1 to the general case in codimension $k = 1$. However all of the results proved in [Sch04] are based on the classical proof of Theorem 4.1.1 as in [Sim84]; making therefore pointless the whole discussion. Further research is thus needed in order to give an analogous of Theorem 4.1.2 in the case $p \in (n,\infty)$ even in the simpler case of codimension $k = 1$.

5. Application to the Plateau Problem

5.1. Introducing the Plateau Problem

The Plateau problem, named after Belgian physicist Joseph Plateau, emerged as a challenge in both mathematics and physics during the 19th century. At its core, the problem revolves around finding the surface of minimal area spanning a given boundary curve. While the question seems simple, not only its solution involves intricate mathematical concepts, but the very formulation of the problem together with a reasonable notion of solution has puzzled mathematicians and physicists for decades.

The origins of the problem are found in the observation of soap film bubbles. Particularly, it stems from the investigation into how soap bubbles minimize the surface area while spanning a given boundary.

Mathematically, the Plateau problem can be framed within the realm of calculus of variations. In this case, the functional to be minimized is the surface area functional $\mathscr{H}^n(\cdot)$, subject to certain boundary conditions. However, unlike classical problems in calculus of variations, the Plateau problem presents unique challenges due to the nonlinearity and nonconvexity of the associated Euler-Lagrange equations.

Throughout history, mathematicians have made significant strides in addressing the Plateau problem. Prominent figures such as Bernhard Riemann, David Hilbert, and Jesse Douglas have contributed important insights, leading to the development of powerful mathematical tools such as geometric measure theory and the theory of partial differential equations.

Despite these advancements, many aspects of the Plateau problem remain unresolved. Challenges persist in extending solutions to higher dimensions, understanding the behavior of minimal surfaces in non-Euclidean spaces, and characterizing the global structure of minimal surfaces.

5.2. Formulation of the Problem

In this section, we give a rigorous formulation of the problem withing the framework of calculus of variation. We will only that assume the boundary to be spanned is any closed subset of the ambient space and we will guarantee existence of a manifold solution which is C^∞ smooth up to a $\mathscr{H}^n$-negligible set. First and foremost we introduce the main notions for the Chapter.

Definition 5.1. Let $\Phi : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ be a function and $B_r(x) \subseteq \mathbb{R}^{n+k}$ a given ball. For $\ell \in \mathbb{N} \cup \{\infty\}$ we say that Φ is C^ℓ-isotopic to the identity in $B_r(x)$ if there exists a function $\Lambda \in C^\ell([0,1] \times \mathbb{R}^{n+k}; \mathbb{R}^{n+k})$ such that:

1. $\Lambda(0,\cdot) \equiv \mathrm{id}_{\mathbb{R}^{n+k}}$;
2. $\Lambda(1,\cdot) \equiv \Phi$;
3. $\Lambda(t,\cdot) \equiv \mathrm{id}_{\mathbb{R}^{n+k} \setminus B_r(x)}$ in $\mathbb{R}^{n+k} \setminus B_r(x)$ for every $t \in [0,1]$;
4. $\Lambda(t,\cdot) : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ is a C^ℓ-class diffeomorphism for every $t \in [0,1]$.

We collect all of the functions which are C^ℓ-isotopic to the identity in $B_r(x)$ into the family $\mathbb{D}^\ell(x,r)$. When $\ell = \infty$ we will simply write $\mathbb{D}(x,r)$.

Observe that properties 3. and 4. respectively of the above definition imply that if $\Phi \in \mathbb{D}^\ell(x,r)$, then $\Phi \equiv \mathrm{id}$ outside of $B_r(x)$ and Φ is a C^ℓ diffeomorphism, in particular it is bijective.

Definition 5.2 (Lipschitz Deformations). We define the class $\mathscr{D}(x,r)$ of Lipschitz deformations on the ball $B_r(x)$ as the family of Lipschitz functions $\Phi : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ for which there exists a sequence $(\Phi_j)_j \subseteq \mathbb{D}(x,r)$ uniformly converging to Φ; i.e.

$$\mathscr{D}(x,r) := \overline{\mathbb{D}(x,r)}^{\|\cdot\|_{C^0}} \cap \mathscr{L}\mathrm{ip}(\mathbb{R}^{n+k}; \mathbb{R}^{n+k}).$$

Definition 5.3 (Deformed Competitor). Fix a closed subset $\Gamma \subseteq \mathbb{R}^{n+k}$ and let $M \subseteq \mathbb{R}^{n+k} \setminus \Gamma$ be relatively closed and n-rectifiable. If $B_r(x) \Subset \mathbb{R}^{n+k} \setminus \Gamma$ we say that a deformed competitor of M in $B_r(x)$ is any element of the family

$$\mathrm{DComp}(M,x,r) := \left\{ \Phi(M) : \Phi \in \mathscr{D}(x,r) \right\}.$$

When dealing with the Plateau problem we want to work in class $\mathcal{F}$ of sets which is large enough so that we can exclude the possibility that a Lipshitz deformation of a set in $\mathcal{F}$ outperforms the infimum in $\mathcal{F}$, hence the following definition.

Definition 5.4 (Good Class). Given $\Gamma \subseteq \mathbb{R}^{n+k}$ closed, we say that a family $\mathcal{G}(\Gamma)$ of n-rectifiable and relatively closed subsets of $\mathbb{R}^{n+k} \setminus \Gamma$ is a good class if for any $M \in \mathcal{G}(\Gamma)$, and for all balls $B_r(x) \Subset \mathbb{R}^{n+k} \setminus \Gamma$ with center $x \in M$ we have

$$\inf_{\substack{N \in \mathcal{G}(\Gamma) \\ N \setminus \overline{B_r(x)} = M \setminus \overline{B_r(x)}}} \mathscr{H}^n(N) \le \inf_{P \in \mathrm{DComp}(M,x,r)} \mathscr{H}^n(P)$$

We will show how the notion of good class gives the correct environment on which to consider the problem of minimizing the measure $\mathscr{H}^n$, namely the problem

$$\texttt{minimize } \mathscr{H}^n \texttt{ among } \mathcal{G}(\Gamma). \tag{5.1}$$

In order to transform (5.1) into a Plateau problem we need to define a reasonable notion of *boundary*, that will be embodied by Γ, and of *spanning the given boundary* on which we can use the theory developed in the previous chapters. To this aim we give the following definitions.

Definition 5.5. Let $\Gamma \subseteq \mathbb{R}^{n+k}$ be closed. We set $\mathbb{S}^k$ to be the k-dimensional sphere and

$$\mathscr{C}_\Gamma := \left\{ \gamma : \mathbb{S}^k \to \mathbb{R}^{n+k} \text{ smooth embedding} \right\}.$$

We say that a subfamily $\mathscr{C} \subseteq \mathscr{C}_\Gamma$ is closed by isotopy with respect to Γ if $\mathscr{C}$ satisfies the implication

$$\gamma \in \mathscr{C} \implies [\gamma]_\Gamma \subseteq \mathscr{C},$$

where $[\gamma]_\Gamma$ is the isotopy class of γ with respect to Γ.

Definition 5.6 ($\mathscr{C}$-spanning). Let $\Gamma \subseteq \mathbb{R}^{n+k}$ be a closed set and let $\mathscr{C}$ be closed by isotopy w.r.t. Γ. We say that a n-rectifiable relativley closed subset $M \subseteq \mathbb{R}^{n+k} \setminus \Gamma$ is C-spanning Γ if

$$M \cap \gamma(\mathbb{S}^k) \neq \varnothing \quad \forall \gamma \in \mathscr{C},$$

and we collect them all into the family $\mathcal{F}(\Gamma, \mathscr{C})$.

We will prove that $\mathcal{F}(\Gamma, \mathscr{C})$ is a good class in the sense of Definition 5.4, hence, with these notions established, we can formulate the Plateau problem as

$$\texttt{minimize } \mathscr{H}^n \texttt{ among } \mathcal{F}(\Gamma, \mathscr{C}). \tag{$\mathfrak{P}$}$$

In this chapter we establish existence for solutions of ($\mathfrak{P}$), provided $\mathscr{H}^n$ is finite for at least one $M \in \mathcal{F}(\Gamma, \mathscr{C})$. Moreover we prove that solutions of ($\mathfrak{P}$) are analytic except on a $\mathscr{H}^n$-negligible subset.

We briefly discuss the idea we are presenting. We have already remarked how n-rectifiable sets inject naturally into the class of n-rectifiable measures; in particular, for any n-rectifiable set M we have a canonically associated n-rectifiable Radon measure $\mu = \mathscr{H}^n_{\llcorner M}$, therefore we can embed the problem ($\mathfrak{P}$) into the larger class of measures. The latter , together with the notion of weak-* convergence, being a suitable environment on which to apply the direct method of Calculus of Variations. We therefore commence by finding a measure solution μ of ($\mathfrak{P}$). The critical part will be proving that the measure solution μ is actually a set solution in the sense that we can find M spanning Γ such that $\mu = \mathscr{H}^n_{\llcorner M}$. In order to show that μ is obtained by the injection of a rectifiable set M we use a rectifiability criterion (which is due to Preiss) hence we reduce the problem into proving that the n-density of μ is 1 almost everywhere, while showing M is $\mathscr{C}$-spanning Γ will follow by a closure property of $\mathcal{F}(\Gamma, \mathscr{C})$.

Once a minimizing set M is obtained the we will be able to prove that M is a local minimum of the area functional $\mathscr{H}^n(\cdot)$, hence that the n-rectifiable varifold $V := \underline{\mathrm{v}}(M, 1)$ is stationary. Applying then Allard's Regularity Theorem (cfr Chapter 4) we prove that M is a C^1-class manifold, which in turn implies that M is analytic up to a $\mathscr{H}^n$-negligible subset.

5.3. Technical Lemmas

Throughout the whole section, Γ will always denote a closed subset of $\mathbb{R}^{n+k}$, and $\mathscr{C}$ will always be a family of embeddings $\mathbb{S}^k \hookrightarrow \mathbb{R}^{n+k}\setminus\Gamma$ closed under isotopy with respect to Γ. Furthermore, given $x \in \mathbb{R}^{n+k}\setminus\Gamma$ we set $d_x := \mathrm{dist}(x,\Gamma)$.

Lemma 5.3.1. *Under the standing assumptions on Γ and $\mathscr{C}$, the family $\mathcal{F}(\Gamma,\mathscr{C})$ is a good class in the sense of Definition 5.4.*

Proof. We prove that $\mathcal{F}(\Gamma,\mathscr{C})$ is a good class by proving the stronger statement that for any $M \in \mathcal{F}(\Gamma,\mathscr{C})$, $x \in M$, $r \in (0,d_x)$ and any $\Phi \in \mathscr{D}(x,r)$, we have $\Phi(M) \in \mathcal{F}(\Gamma,\mathscr{C})$. Let us fix any M,x, r, Φ as above, and assume $\Phi(M) \notin \mathcal{F}(\Gamma,\mathscr{C})$. Then let $\gamma \in \mathscr{C}$ be such that

$$\gamma(\mathbb{S}^k) \cap \Phi(M) = \varnothing.$$

Without loss of generality, we can suppose

$$\gamma(\mathbb{S}^k) \cap \Phi(M\setminus B_r(x)) = \varnothing.$$

By definition of $\mathscr{D}(x,r)$, there exists a sequence $(\Phi_j)_j \subseteq \mathbb{D}(x,r)$ such that

$$\lim_{j\to\infty} \|\Phi_j - \Phi\|_{C^0} = 0.$$

Recalling that $\gamma(\mathbb{S}^k)$ is compact, and $\Phi_j \equiv \mathrm{id}$ outside of $B_r(x)$, then

$$\gamma(\mathbb{S}^k) \cap \Phi_j(M) = \varnothing \tag{5.2}$$

for any j sufficiently large.

On the other hand $\Phi_j \in \mathbb{D}(x,r)$, hence Φ_j is invertible. Therefore

$$\Phi_j^{-1}\left(\gamma(\mathbb{S}^k) \cap \Phi_j(M)\right) = \left(\Phi_j^{-1} \circ \gamma(\mathbb{S}^k)\right) \cap M.$$

Recalling that Φ_j is isotropic to the identity in $B_r(x)$, thus $\Phi_j^{-1} \circ \gamma \in \mathscr{C}$, and $M \in \mathcal{F}(\Gamma,\mathscr{C})$, follows that

$$\left(\Phi_j^{-1} \circ \gamma(\mathbb{S}^k)\right) \cap M \neq \varnothing. \tag{5.3}$$

The contradiction follows by applying Φ_j to (5.3) and comparing it to (5.2). □

We introduce a notion of convergence for n-rectifiable sets.

Definition 5.7 (Weak-* Convergence of Sets). Let $\Omega \subseteq \mathbb{R}^{n+k}$ be open. We say that a sequence of relatively closed and n-rectifiable sets $(M_j) \subseteq \Omega$ weakly-* converges to a set M, if the sequence of measures $(\mathscr{H}^n_{\llcorner M_j})_j$ weakly-* converges to the measure $\mathscr{H}^n_{\llcorner M}$.

The following lemma established a closing property of the class $\mathcal{F}(\Gamma,\mathscr{C})$ under weak-* convergence of sets.

Lemma 5.3.2. *Assume* $(M_j)_j \subseteq \mathcal{F}(\Gamma,\mathscr{C})$ *to be weakly-** *converging to a relatively closed, n-rectifiable set* $M \subseteq \mathbb{R}^{n+k}\backslash\Gamma$. *Then* $M \in \mathcal{F}(\Gamma,\mathscr{C})$.

Proof. We argue by contradiction. If $M \notin \mathcal{F}(\Gamma,\mathscr{C})$, then there exists $\gamma \in \mathscr{C}$ such that

$$\gamma(\mathbb{S}^k) \cap M = \emptyset.$$

Since $\gamma(\mathbb{S}^k)$ is compact and M is relatively closed in $\mathbb{R}^{n+k}\backslash\Gamma$, there exists $\varepsilon > 0$ such that the tubular neighborhood of radius 2ε of $\gamma(\mathbb{S}^k)$ (denoted by $U_{2\varepsilon}(\gamma)$) has empty intersection with M. Let $\mu := \mathscr{H}^n_{\llcorner M}$ and $\mu_j := \mathscr{H}^n_{\llcorner M_j}$ for each $j \in \mathbb{N}$. By weak-* convergence of $(M_j)_j$ to M, we deduce

$$\lim_{j\to\infty} \mu_j(U_{2\varepsilon}(\gamma)) = 0. \tag{5.4}$$

Let $\Psi : \mathbb{S}^k \times B^n_\varepsilon \to U_\varepsilon(\gamma)$ be a diffeomorphims such that $\Psi(\cdot,0) = \gamma$ and $\Psi(\cdot,y)$ is parallel to γ and passes through the point $\gamma((1,0,...,0)) + y$. Then $\gamma_y := \Psi(\cdot,y)$ is isotopic to γ for each $y \in B^n_\varepsilon$, hence

$$M_j \cap \gamma_y(\mathbb{S}^k) \neq \emptyset \quad \forall y \in B^n_\varepsilon, j \in \mathbb{N}. \tag{5.5}$$

Finally we define $\widetilde{\Pi} : \mathbb{S}^k \times B^n_\varepsilon \to B^n_\varepsilon$ to be the projection $(y,z) \mapsto z$, and set

$$\Pi : U_\varepsilon(\gamma) \to B_\varepsilon, \quad \Pi := \widetilde{\Pi} \circ \Psi^{-1}.$$

It is easy to check that Π is Lipschitz, hence the coarea formula and (5.5) imply

$$\mathscr{H}^n(M_j \cap U_\varepsilon(\gamma)) \geq \frac{\omega_n \varepsilon^n}{(\mathrm{Lip}(\Pi))^n} > 0. \tag{5.6}$$

The contradiction is given by (5.4) and (5.6).

□

A key result we are going to use, is a deformation theorem for closed set due to David and Semmes [DS00].

Before stating the theorem, let us introduce some further notation. Given a closed cube $Q = Q_\ell(x)$ of center $x \in \mathbb{R}^{n+k}$ and side length $\ell > 0$, we cover it with a grid of closed cubes with side length ε, for some $0 < \varepsilon \ll \ell$, each having non-empty intersection with $\mathrm{Int}(Q)$; moreover we assume such a covering to be centered at x (i.e. one of the cubes is centered at x). The family of smaller cubes will be denoted by $\Lambda_\varepsilon(Q)$. Then we define

$$C_1 := \bigcup \left\{T \cap Q : T \in \Lambda_\varepsilon(Q), T \cap \partial Q \neq \emptyset\right\},$$
$$C_2 := \bigcup \left\{T \in \Lambda_\varepsilon(Q) : (T \cap Q) \not\subseteq C_1, T \cap \partial C_1 \neq \emptyset\right\},$$
$$Q^1 := \overline{Q\backslash(C_1 \cup C_2)}.$$

As a consequence

$$\Lambda_\varepsilon(Q^1 \cup C_2) = \left\{T \in \Lambda_\varepsilon(Q) : T \subseteq (Q^1 \cup C_2)\right\}.$$

For each non-negative integer $j \le n+k$, we let $\Lambda_{\varepsilon,j}(Q^1 \cup C_2)$ denote the collection of all j-dimensional faces of cubes in $\Lambda_\varepsilon(Q^1 \cup C_2)$, and $\Lambda^*_{\varepsilon,j}(Q^1 \cup C_2)$ be the subfamily of $\Lambda_{\varepsilon,j}(Q^1 \cup C_2)$ of elements which are not contained in $\partial(Q^1 \cup C_2)$.

Finally we define the j-skeleton of order ε in $Q^1 \cup C_2$ as

$$S_{\varepsilon,j}(Q^1 \cup C_2) := \bigcup \Lambda_{\varepsilon,j}(Q^1 \cup C_2).$$

Theorem 5.3.1 (Deformation Theorem for Cubes). *Let $r > 0$ and $\varepsilon > 0$ be fixed constants, and assume E to be a compact subset of a cube $Q \subseteq B_r(x_0)$, such that $\mathscr{H}^n(E) < \infty$. Then there exists $\Phi_{\varepsilon,E} \in \mathscr{D}(x_0, r)$ satisfying the following properties:*

(i) $\Phi_{\varepsilon,E}(x) = x$ *for all* $x \in \mathbb{R}^{n+k} \setminus (Q^1 \cup C_2)$*;*

(ii) $\Phi_{\varepsilon,E}(x) = x$ *for all* $x \in S_{\varepsilon,n-1}(Q^2 \cup C_2)$*;*

(iii) $\Phi_{\varepsilon,E}(E \cap (Q^1 \cup C_2)) \subseteq S_{\varepsilon,n}(Q^1 \cup C_2) \cup \partial(Q^1 \cup C_2)$*;*

(iv) $\Phi_{\varepsilon,E}(T) \subseteq T$ *for every* $T \in \Lambda_{\varepsilon,j}(Q^1 \cup C_2)$*, for all* $j \in \{n, ..., n+k\}$*;*

(v) either $\mathscr{H}^n(\Phi_{\varepsilon,E}(E) \cap T) = 0$ *or* $\mathscr{H}^n(\Phi_{\varepsilon,E}(E) \cap T) = \mathscr{H}^n(T)$*, for every* $T \in \Lambda^*_{\varepsilon,n}(Q^1)$*;*

(vi) $\mathscr{H}^n(\Phi_{\varepsilon,E}(E \cap T)) < L_1 \mathscr{H}^n(E \cap T)$ *for every* $T \in \Lambda_\varepsilon(Q^1 \cup C_2)$*, for some constant* L_1 *which depends only on n and k (but not on* ε*).*

Later we will need to implement the above deformation theorem for a closed set E on rectangles rather than cubes. More precisely let us consider a closed rectangle

$$R := [0, \ell_1] \times ... \times [0, \ell_{n+k}], \quad \ell_1 \le ... \le \ell_{n+k},$$

and a tiling of $\mathbb{R}^{n+k}$ made of rectangles which are ε-homothetic to R. Let $\Lambda^R_\varepsilon(R)$ denote the family of translalted and ε-scaled copies of R, and let us set

$$C_1^R := \bigcup \{T \cap Q : T \in \Lambda^R_\varepsilon(Q), T \cap \partial R \neq \varnothing\},$$
$$C_2^R := \bigcup \{T \in \Lambda^R_\varepsilon(R) : (T \cap R) \not\subseteq C_1^R, T \cap \partial C_1^R \neq \varnothing\},$$
$$R^1 := \overline{R \setminus (C_1^R \cup C_2^R)}.$$

As before, for any integer $j \le n+k$ we let $\Lambda^R_{\varepsilon,j}(R^1 \cup C_2^R)$ denote the collection of j-dimensional faces of rectangles in $\Lambda^R_\varepsilon(R^1 \cup C_2^R)$, and $\Lambda^{R*}_{\varepsilon,j}(R^1 \cup C_2^R)$ will be the set of elements of $\Lambda^R_\varepsilon(R^1 \cup C_2^R)$ which are not contained in $\partial(R^1 \cup C_2^R)$. We also define the j-skeleton of order ε in $R^1 \cup C_2^R$ as

$$S^R_{\varepsilon,j}(R^1 \cup C_2^R) := \bigcup \Lambda_{\varepsilon,j}(Q^1 \cup C_2).$$

The following theorem is an immediate consequence of Theorem 5.3.1

Theorem 5.3.2 (Deformation Theorem for Rectangles). *Let $r > 0$ and $\varepsilon > 0$ be fixed constants, and assume E to be a compact subset of a rectangle $R \subseteq B_r(x_0)$, such that $\mathscr{H}^n(E) < \infty$. Then there exists $\Phi_{\varepsilon,E} \in \mathscr{D}(x_0, r)$ satisfying the following properties:*

(i) $\Phi_{\varepsilon,E}(x) = x$ *for all* $x \in \mathbb{R}^{n+k} \setminus (R^1 \cup C_2)$*;*

(ii) $\Phi_{\varepsilon,E}(x) = x$ *for all* $x \in S_{\varepsilon,n-1}(R^2 \cup C_2)$;

(iii) $\Phi_{\varepsilon,E}(E \cap (R^1 \cup C_2)) \subseteq S_{\varepsilon,n}(R^1 \cup C_2) \cup \partial(R^1 \cup C_2)$;

(iv) $\Phi_{\varepsilon,E}(T) \subseteq T$ *for every* $T \in \Lambda_{\varepsilon,j}(R^1 \cup C_2)$*, for all* $j \in \{n,...,n+k\}$;

(v) *either* $\mathscr{H}^n(\Phi_{\varepsilon,E}(E) \cap T) = 0$ *or* $\mathscr{H}^n(\Phi_{\varepsilon,E}(E) \cap T) = \mathscr{H}^n(T)$*, for every* $T \in \Lambda^*_{\varepsilon,n}(R^1)$;

(vi) $\mathscr{H}^n(\Phi_{\varepsilon,E}(E \cap T)) < L_2 \mathscr{H}^n(E \cap T)$ *for every* $T \in \Lambda_\varepsilon(R^1 \cup C_2)$*, for some constant* L_2 *which depends only on* n*,* k *and* ℓ_n/ℓ_2 *(but not on* ε*).*

5.4. Existence and Regularity Theorems

Theorem 5.4.1 (Existence). *Let* $\Gamma \subseteq \mathbb{R}^{n+k}$ *be a closed set, and let* $\mathscr{C}$ *be a family of embeddings* $\mathbb{S}^k \hookrightarrow \mathbb{R}^{n+k} \setminus \Gamma$ *closed under isotopy w.r.t.* Γ*. If*

$$\inf_{\mathcal{F}(\Gamma,\mathscr{C})} \mathscr{H}^n < \infty,$$

and $(M_j) \subseteq \mathcal{F}(\Gamma,\mathscr{C})$ *is a minimizing sequence, then there exists* $M \in \mathcal{F}(\Gamma,\mathscr{C})$ *such that* $M_j \rightharpoonup^* M$*. In particular* M *is a solution of the Plateau problem* ($\mathfrak{P}$)*.*

Proof. For each $j \in \mathbb{N}$ we set

$$\mu_j := \mathscr{H}^n_{\llcorner M_j}.$$

Under the standing assumptions we can suppose without loss of generality that $\mu_j \rightharpoonup^* \mu$, for some $\mu \in \mathrm{Rad}(\mathbb{R}^{n+k} \setminus \Gamma)$. We then define $M := \mathrm{spt}\mu \setminus \Gamma$ and also consider the canonical density one rectifiable varifolds

$$V_j := \mathscr{H}^n_{\llcorner M_j} \otimes (\delta_{T_x M_j})_x = \mu_j \otimes (\delta_{T_x M_j}). \tag{5.7}$$

Since (M_j) is a minimizing sequence, we can assume the bound

$$\mu_j(\mathbb{R}^{n+k} \setminus \Gamma) \leq 2\mu_0(\mathbb{R}^{n+k} \setminus \Gamma) < \infty \quad \forall j \in \mathbb{N},$$

and therefore we can assume that

$$V_j \rightharpoonup^* V = \mu \otimes (\eta^x)_x$$

in the sense of varifolds.

By virtue of Lemma 5.3.2, proving that V is n-rectifiable and with density constantly equal to 1 will be sufficient to prove the theorem. In particular, doing so will prove that M is a solution of the Plateau problem ($\mathfrak{P}$).

<u>STEP1.</u> The limiting varifold V is stationary in $\mathbb{R}^{n+k} \setminus \Gamma$. Assume indeed the existence of a smooth vector field $X \in C^1_c(\mathbb{R}^{n+k} \setminus \Gamma; \mathbb{R}^{n+k})$ such that

$$\delta V(X) < 0$$

where δV is the first variation of the varifold V (cfr. Chapter 2). By a standard partition of unity argument for the compact set $\mathrm{spt}(X)$ in the open $\mathbb{R}^{n+k} \setminus \Gamma$, we get existence of a ball $B_r(x) \Subset \mathbb{R}^{n+k}\setminus\Gamma$ and a vector field (not relabeled) $X \in C^1_c(B_r(x))$ such that the first variation (cfr. Chapter 2 for the definition and the notation)

$$\delta V(X) < -2c \quad \text{for some } c > 0.$$

For arbitrarily small $t > 0$, we have $\Phi := \mathrm{id} + tX \in \mathscr{D}(x,r)$. Moreover, there exists an open set $A \subseteq \mathbb{R}^{n+k}$ containing $B_r(x)$ such that

$$\mu_{\Phi^\# V}(\partial A) = 0. \tag{5.8}$$

Recalling the definition of first variation of a varifold, we have

$$\mu_{\Phi^\# V}(A) \le -ct + \mu(A).$$

By lower semicontinuity and (5.8), for $j \in \mathbb{N}$ large enough

$$\mu_{\Phi^\# V_j}(A) - \frac{1}{j} \le -ct + \mu_j(A).$$

Therefore

$$\mathscr{H}^n(A \cap \mathrm{spt}\mu_{\Phi^\# V_j}) \le -ct + \mathscr{H}^n(A \cap \mathrm{spt}\mu_j) + \frac{2}{j}. \tag{5.9}$$

Recalling (5.7), adding to both sides of (5.9) the term $\mathscr{H}^n((\mathbb{R}^{n+k}\setminus A) \cap M_j)$ and noting that $\Phi(M_j)\setminus A = M_j \setminus A$, then

$$\mathscr{H}^n(\Phi(M_j)) \le \mathscr{H}^n(M_j) - ct + \frac{2}{j}.$$

Since $\Phi \in \mathscr{D}(x,r)$ and $B_r(x) \Subset \mathbb{R}^{n+k}\setminus\Gamma$, this is a contradiction with the minimizing property of the sequence $(M_j) \in \mathcal{F}(\Gamma,\mathscr{C})$. Therefore V is stationary, namely $V \in \mathscr{V}_n(\mathbb{R}^{n+k}\setminus\Gamma)$ has mean curvature $\underline{H}_V \equiv 0$.

<u>STEP 2.</u> We claim that

$$\exists \vartheta_0 = \vartheta_0(n,k) : \frac{\mu(B_r(x_0))}{\omega_n x^n} \ge \vartheta_0 \quad \forall x \in M, r \in (0, d_x).$$

Equivalently, we prove that

$$\exists \beta = \beta(n,k) > 0 : \frac{\big(\mu(Q_\ell(x))\big)^{1/n}}{\ell} \ge \beta \quad \forall x \in M, \ell \in (0, 2d_x/\sqrt{n}).$$

Fix $x \in M$ and $0 < \ell < 2d_x/\sqrt{n}$ such that $\mu(\partial Q_\ell(x)) = 0$, and assume

$$\mu(Q_\ell(x)) < \beta^n \ell^n, \tag{5.10}$$

for some constant $\beta > 0$ that will be chosen later. We show that we can select $\beta = \beta(n,k)$ such that, if (5.10) holds, then x can not belong to $\mathrm{spt}\mu$, as we are able to construct a sequence of cubes $Q_i := Q_{\ell_i}(x)$ such that:

(I) $Q_0 := Q_\ell(x)$;

(II) $\mu(\partial Q_i) = 0$ for all $i \in \mathbb{N}$;

(III) Setting $m_i := \mu(Q_i)$, then

$$\frac{m_i}{\ell_i^n} < \beta^n;$$

(IV) letting L_1 be the constant of Theorem 5.3.1, then

$$m_{i+1} < \left(1 - \frac{1}{L_1}\right) m_i$$

for all $i \in \mathbb{N}$.

(V) Defining

$$L := \max\left\{6, \frac{6}{1-(1-1/L_1)^{1/n}}\right\} \quad \text{and} \quad \varepsilon_i := \frac{m_i^{1/n}}{L\beta\ell_i}, \tag{5.11}$$

we have

$$(1-6\varepsilon_i)\ell_i \le \ell_{i+1} \le (1-4\varepsilon_i)\ell_i;$$

(VI) the following properties are satisfied

$$\lim_{i\to\infty} m_i = 0 \quad \text{and} \quad \liminf_{i\to\infty} \ell_i > 0.$$

The cube Q_0 satisfies by construction (I), (II) and (III). Suppose that the cubes $Q_0, \dots, Q_i$ have already been defined, and set

$$m_i^j := \mathscr{H}^n(M_j \cap Q_i).$$

We now cover Q_i with the family $\Lambda_{\varepsilon_i\ell_i}(Q_i)$ of closed cubes of side length $\varepsilon_i\ell_i$, and let C_1^i, C_2^i and Q_i^1 be the corresponding sets defined at the end of the previous section. We then set $Q_{i+1} := Q_i^1$. Note that (5.11) implies that both C_2^i and Q_{i+1} are non-empty; moreover $C_i^1 \cup C_i^2$ is a strip of width not larger than $2\varepsilon_i\ell_i$ around ∂Q_i, hence the side length ℓ_{i+1} of Q_{i+1} satisfies

$$(1-4\varepsilon_i)\ell_i \le \ell_{i+1} \le (1-2\varepsilon_i)\ell_i.$$

We now apply, for each $i, j \in \mathbb{N}$, Theorem 5.3.1 with $E = M_j$, $Q = Q_i$ and $\varepsilon = \varepsilon_i\ell_i$, obtaining maps $\Phi^{i,j} := \Phi_{\varepsilon_i\ell_i,K_j}$.

We claim that, for j sufficiently large, we have

$$m_i^j \le L_1(m_i^j - m_{i+1}^j) + o_j(1). \tag{5.12}$$

To the aim of proving (5.12), we first observe that if there exists some $T \in \Lambda^*_{\varepsilon_i\ell_i,n}(Q_{i+1})$ such that

$$\mathscr{H}^n(\Phi^{i,j}(M_j \cap T)) = \mathscr{H}^n(T),$$

then

$$\begin{aligned}(\varepsilon_i \ell_i)^n &= \mathscr{H}^n(T)\\ &\le \mathscr{H}^n(\Phi^{i,j}(M_j)\cap Q_{i+1})\\ &\le L_1 \mathscr{H}^n(M_j \cap Q_i)\\ &= L_1 m_i^j\\ &\xrightarrow[j\to\infty]{} L_1 m_i.\end{aligned}$$

Therefore

$$m_i \ge \frac{(\varepsilon_i \ell_i)^n}{L_1}, \tag{5.13}$$

and plugging (5.13) into (5.11), yields

$$\beta \ge \frac{1}{L_1 L}.$$

Therefore, recalling item (v) of Theorem 5.3.1, we can choose $\beta = \beta(n,k) > 0$ small enough such that

$$\mathscr{H}^n(\Phi^{i,j}(M_j \cap Q_{i+1})) = 0 \quad \forall j \text{ large enough.} \tag{5.14}$$

Finally, since $(M_j)_j$ is a minimizing sequence and, by Lemma 5.3.1, $\mathcal{F}(\Gamma,\mathscr{C})$ is a good class, (5.14) gives

$$\begin{aligned}m_i^j &\le m_i + o_j(1)\\ &\le \mathscr{H}^n(\Phi^{i,j}(M_j\cap Q_i)) + o_j(1)\\ &= \mathscr{H}^n(\Phi^{i,j}(M_j \cap Q_{i+1})) + \mathscr{H}^n(\Phi^{i,j}(M_j \cap (C_i^1 \cup C_i^2))) + o_j(1)\\ &\le L_1 \mathscr{H}^n(M_j \cap (C_1^i \cup C_2^i)) + o_j(1)\\ &= L_1(m_i^j - m_{i+1}^j) + o_j(1).\end{aligned}$$

Passing to the limit as $j \to \infty$ in (5.12), we obtain

$$m_{i+1} \le \left(1 - \frac{1}{L_1}\right) m_i, \tag{5.15}$$

that is property (IV). Since $\ell_{i+1} \ge (1-4\varepsilon_i)\ell_i$, we can slightly shrink the cube Q_{i+1} to a concentric cube Q'_{i+1} with side length $\ell'_{i+1} \ge (1-4\varepsilon_j)\ell_j$ and $\mu(\partial Q'_{i+1}) = 0$ for which (IV) still holds, just getting a lower value for m_{i+1}. With a slight abuse of notation, we relabel these all of these cubes Q'_{i+1} as Q_{i+1}.

Using (5.15) and (III) for Q_i, we deduce

$$m_{i+1} \le \left(1-\frac{1}{L_1}\right) m_i < \left(1 - \frac{1}{L_1}\right)\beta^n \frac{\ell_{i+1}^n}{(1-6\varepsilon_i)^n}.$$

Therefore (5.11) gives (III) for Q_{i+1}. Furthermore, observing that (III) and (V) for Q_0 imply $\varepsilon_0 < 1/L$, we also obtain

$$\varepsilon_{i+1} \leq \varepsilon_i. \tag{5.16}$$

We finally prove (VI). From (IV), immediately follows

$$\lim_{i\to\infty} m_i = 0.$$

On the other hand, recalling (5.16) and our definitions of ε_i and L, we have

$$\begin{aligned}
\liminf_{i\to\infty} \frac{\ell_i}{\ell_0} &\geq \prod_{i=0}^{\infty}(1-6\varepsilon_i) \\
&\geq \prod_{i=0}^{\infty}\left(1 - \frac{6m_0^{1/n}}{L\beta\ell_0\prod_{h=0}^{j-1}(1-6\varepsilon_h)}\left(1-\frac{1}{L_1}\right)^{1/n}\right) \\
&\geq \prod_{i=0}^{\infty}\left(1 - \frac{6}{L(1-6\varepsilon_0)^i}\left(1-\frac{1}{L_1}\right)^{1/n}\right) \\
&> 0.
\end{aligned}$$

Therefore

$$\liminf_{i\to\infty} \ell_i > 0$$

is proved.

<u>STEP 3.</u> From STEP 1 and STEP 2, together with Theorem 3.1.1 we deduce that V is n-rectifiable, hence it is of the form

$$V = \vartheta \mathcal{H}^n_{\llcorner \widetilde{M}} \otimes (\delta_{T_x\widetilde{M}}),$$

for some n-rectifiable set $\widetilde{M} \subseteq \mathbb{R}^{n+k}$. On the other hand, being M the support of μ, we deduce $\mu(M \triangle \widetilde{M}) = 0$. Hence, for the rest of the proof we write

$$\mu = \vartheta \mathcal{H}^n_{\llcorner M} \quad \text{and} \quad V = \vartheta \mathcal{H}^n_{\llcorner M} \otimes (\delta_{T_xM}).$$

Furthermore, Corollary 2.3.1 guarantees that, for any $x \in M$, the function

$$r \mapsto \frac{\mu(B_r(x))}{\omega_n r^n} \quad r \in (0, d_x)$$

is increasing.

<u>STEP 4.</u> We prove that

$$\vartheta(x) \geq 1 \quad \forall x \in M : T_xM \text{ exists.}$$

Actually we prove something slightly more general, that is

$$\vartheta \geq 1 \quad \forall x \in M : \exists r_j \searrow 0 \text{ s.t. } \frac{(\varphi_{r_j,x})_\# \mu}{r_j^n} \rightharpoonup^* \vartheta(x)\mathcal{H}^n_{\llcorner S} \tag{5.17}$$

where T is any element of $G(n+k,n)$.

Fix any point x, sequence $(r_j)_j$ and $S \in G(n+k,n)$, as in (5.17). We can suppose, without loss of generality, $x = 0$ and $T = \mathbb{R}^n \times \{0\}^k$. Let us set

$$\mu^{r_j,x} := \frac{(\varphi_{r_j,x})_\# \mu}{r_j^n}$$

(which is a different definition from the one of $\mu_{r_j,x}$ given in Chapter 3). Observe that

$$\mathrm{spt}(\mu^{r_j,x}) \subseteq \frac{M-x}{r_j},$$

and that the weak-* convergence, together with the lower density estimates of STEP 1, imply the Kuratowsky convergence of $\mathrm{spt}\mu^{r_j,x}$ to S. In particular, for any $\varepsilon > 0$, there are infinitely many $\rho > 0$ such that

$$M \cap B_\rho \subseteq \left\{ y \in \mathbb{R}^{n+k} : |y_n|, \dots, |y_{n+k}| < \frac{\varepsilon\rho}{100} \right\}. \tag{5.18}$$

By contradiction, assume $\vartheta(0) < 1$. Thanks to the monotonicity of the function $r \mapsto \mu(B_r(x))/(\omega_n r^n)$, (5.18) implies the existence of $r > 0$ and $\alpha < 1$ such that the following properties hold

$$\mu(\partial Q_r) = 0, \quad \frac{\mu(Q_r)}{r^n} \le \alpha, \quad M \cap (Q_r \setminus R_{r,\varepsilon r}) = \emptyset, \tag{5.19}$$

where $R_{r,\varepsilon r}$ is the rectangle $[-r/2, r/2]^n \times [-\varepsilon r/2, \varepsilon r/2]^k$. In particular, thanks to the weak-* convergence $\mu_j \rightharpoonup^* \mu$, we have

$$\frac{\mu_j(Q_r)}{r^n} \le \alpha \quad \text{and} \quad \mu_j(Q_r \setminus B_{r,r\varepsilon}) = o_j(1). \tag{5.20}$$

We now wish to clear the small amount of mass appearing in the complement of $R_{r,\varepsilon r}$: we achieve this by repeatedly applying Theorem 5.3.2. We set

$$R := Q_r \cap \left\{ x \in \mathbb{R}^{n+k} : x_{n+1} \ge \frac{\varepsilon}{2} r \right\}, \quad E := M_j^0 := M_j$$

and we apply Theorem 5.3.2 to this rectangle with these choices of R, ε and E, obtaining the map $\Psi^{1,j} := \Phi_{\varepsilon, M_j}$. We recall that the obtained constant L_2 for the area bound is universal, since it depends on the side ration of R, which is bounded from below by 1 and from above by 4, provided ε small enough. We then set $M_j^1 := \Psi^{1,j}(M_j^0)$ and repeat the argument with

$$R := Q_r \cap \left\{ x \in \mathbb{R}^{n+k} : x_{n+1} \le -\frac{\varepsilon}{2} r \right\}, \quad E := M_j^1,$$

obtaining a map $\Psi^{2,j} := \Phi_{\varepsilon, M_j^1}$. We set again $M_j^2 := \Psi^{2,j}(M_j^1)$ and iterate this procedures to the rectangles

$$Q_r \cap \left\{ x \in \mathbb{R}^{n+k} : x_{n+2} \ge \frac{\varepsilon}{2} r \right\}, \dots, Q_r \cap \left\{ x \in \mathbb{R}^{n+k} : x_{n+k} \le -\frac{\varepsilon}{2} r \right\}.$$

After $2k$ iteration, we set

$$M_j^{2k} := \Psi^{2k,j} \circ \ldots \circ \Psi^{1,j}(M_j).$$

We are going to use the cube $Q_{r(1-\sqrt{\varepsilon})}$ because, taking ε small enough, then

$$\sqrt{\varepsilon} > 4\overline{C}\varepsilon,$$

where $\overline{C} > 1$ is the side ration considered before. This allows us to claim that

$$\mathscr{H}^n(M_j^{2k} \cap (Q_{r(1-\sqrt{\varepsilon})} \setminus R_{r(1-\sqrt{\varepsilon}),6\varepsilon r})) = 0. \tag{5.21}$$

Otherwise there would exists a n-face of a smaller rectangle $T \subseteq (Q_r \setminus R_{r,\varepsilon r})$ such that

$$\mathscr{H}^n(M_j^{2k} \cap T) = \mathscr{H}^n(T) \geq (\varepsilon r)^n,$$

which would lead to the following contradiction for j large:

$$\begin{aligned}(\varepsilon r)^n &\leq \mathscr{H}^n(T)\\ &\leq \mathscr{H}^n(M_j^{2k} \cap (Q_r \setminus R_{r,\varepsilon r}))\\ &\leq L_2^{2k}\mathscr{H}^n(M_j \cap (Q_r \setminus R_{r,\varepsilon r}))\\ &\xrightarrow[j\to\infty]{} 0.\end{aligned}$$

In particular, we cleared any measure on every slab

$$\bigcup_{i=n+1}^{n+k} \left\{x \in \mathbb{R}^{n+k} : 3\varepsilon r < |x_i| < (1-\sqrt{\varepsilon})\frac{r}{2}\right\} \cap Q_{r(1-\sqrt{\varepsilon})}.$$

Now we want to construct a map $P \in \mathscr{D}(0,r)$, collapsing $R_{r(1-\sqrt{\varepsilon}),6\varepsilon r}$ onto the tangent plane $S = \mathbb{R}^n \times \{0\}^k$. We adopt the notation

$$\mathbb{R}^{n+k} \ni x = (x', x'') \in \mathbb{R}^n \times \mathbb{R}^{n+k}.$$

Then we set

$$\|x'\| := \max\{|x_1|,\ldots,|x_n|\} \quad \text{and} \|x''\| := \max\{|x_{n+1}|,\ldots,|x_{n+k}|\},$$

and we define P as follows:

$$P(x) := \begin{cases}\left(x', \frac{g(\|x'\|)(\|x''\|-3\varepsilon r)^+}{1-6\varepsilon}\frac{x''}{\|x''\|} + (1-g(\|x'\|)x''\right) & \text{if } \max\{\|x'\|,\|x''\|\} \leq \frac{r}{2}\\ \mathrm{id} & \text{otherwise,}\end{cases}$$

where $(\cdot)^+$ denotes the positive part, and $g : [0, r/2] \to [0,1]$ is a compacly supported cut off function such that

$$g \equiv 1 \quad \text{on} \quad [0, r(1-\sqrt{\varepsilon})/2] \quad \text{and} \quad |g'| \leq \frac{10}{r\sqrt{\varepsilon}}.$$

Then $P \in \mathscr{D}(0,r)$ and it is not difficult to show that there exists a constant $C = C(n,k) > 0$ such that

$$\mathrm{Lip}(P) \leq 1 + C\sqrt{\varepsilon}.$$

We now set $\widetilde{M}_j := P(M_j^{2k})$, which verifies , thanks to (5.21),

$$\mathscr{H}^n(\widetilde{M}_j \cap (Q_{(1-\sqrt{\varepsilon})r} \setminus Q^n_{(1-\sqrt{\varepsilon})r})) = 0, \tag{5.22}$$

and

$$\begin{aligned}\mathscr{H}^n(\widetilde{M}_j \cap (Q_r \setminus Q_{(1-\sqrt{\varepsilon})r})) &\leq (1+C\sqrt{\varepsilon})^n \mathscr{H}^n(M_j^{2k} \cap (Q_r \setminus Q_{(1-\sqrt{\varepsilon})r}))\\ &\leq (1+C\sqrt{\varepsilon}) L_2^{2k} \mathscr{H}^n(M_j \cap (Q_r \setminus (Q_{(1-\sqrt{\varepsilon})} \cup R_{r,\varepsilon r})))\\ &\quad + (1+C\sqrt{\varepsilon})\mathscr{H}^n(M_j \cap (R_{r,\varepsilon r \setminus Q_{(1-\sqrt{\varepsilon})r}}))\\ &\leq (1+C\sqrt{\varepsilon})\mathscr{H}^n(M_j \cap (R_{r,\varepsilon r} \setminus Q_{(1-\sqrt{\varepsilon})r})) + o_j(1),\end{aligned} \tag{5.23}$$

where in the last inequality we have used (5.20). Moreover, by using (5.19), (5.20) and (5.22), we also have, for j large

$$\begin{aligned}\frac{\mathscr{H}^n(\widetilde{M}_j \cap Q^n_{(1-\sqrt{\varepsilon})})}{((1-\sqrt{\varepsilon})r)^n} &= \frac{\mathscr{H}^n(\widetilde{M}_j \cap Q_{(1-\sqrt{\varepsilon})r})}{((1-\sqrt{\varepsilon})r)^n}\\ &\leq (1+C\sqrt{\varepsilon})\frac{\mathscr{H}^n(M_j^{2k} \cap Q_r)}{r^n}\\ &\leq (1+C\sqrt{\varepsilon})\frac{\mathscr{H}^n(M_j \cap Q_r) + o_j(1)}{r^n}\\ &\leq \alpha + o_j(1)\\ &< 1.\end{aligned} \tag{5.24}$$

As a consequence of (5.24) and compactness of $\widetilde{M}_j$, there exists $y'_j \in Q^n_{(1-\sqrt{\varepsilon})r}$ and $\delta_j > 0$ such that, if we set $y_j := (y'_j, 0)$, then

$$\widetilde{M}_j \cap B^n_{\delta_j}(y_j) = \varnothing \quad \text{and} \quad B^n_{\delta_j}(y_j) \Subset Q^n_{(1-\sqrt{\varepsilon})r}. \tag{5.25}$$

After the last deformation, our set $\widetilde{M}_j \cap Q_{(1-\sqrt{\varepsilon})r}$ is contained in the tangent plane $S = \mathbb{R}^n \times \{0\}^k$ and we want to use the property (5.25) to collapse $\widetilde{M}_j \cap Q_{(1-\sqrt{\varepsilon})r}$ into $(\partial Q^n_{(1-\sqrt{\varepsilon})r}) \times 0^k$. To this end, for every $j \in \mathbb{N}$ let us define the following Lipschitz map:

$$\Upsilon_j(x) := \begin{cases} \left(x' + z'_{j,x}, x''\right) & \text{if } x \in R_{(1-\sqrt{\varepsilon})r,r} \\ x & \text{otherwise} \end{cases},$$

with

$$z'_{j,x} := \min\left\{1, \frac{|x' - y'_j|}{\delta_j}\right\} \frac{(r - 4\|x''\|)^+}{r} \gamma_{j,x}(x' - y'_j),$$

where $\gamma_{j,x} > 0$ is such that

$$x' + \gamma_{j,x}(x' - y'_j) \in \partial Q^n_{(1-\sqrt{\varepsilon})r} \times 0^k.$$

One can easily check that $\Upsilon_j \in \mathscr{D}(0,r)$. Moreover, setting

$$M'_j := \Upsilon_j(\widetilde{M}_j),$$

we have that

$$M'_j \setminus Q_r = M_j \setminus Q_r \quad \text{and} \mathscr{H}^n(M'_j \cap Q_{(1-\sqrt{\varepsilon})r}) = 0, \tag{5.26}$$

thanks to (5.22), since

$$\mathscr{H}^n((\partial Q^n_{(1-\sqrt{\varepsilon})r}) \times \{0\}^k) = 0.$$

Since $\mathcal{F}(\Gamma, \mathscr{C})$ is a good class by virtue of Lemma 5.3.2, by Definition 5.4 there exists a sequence of competitors $(N_j)_j \subseteq \mathcal{F}(\Gamma, \mathscr{C})$ such that

$$N_j \setminus \overline{B_r} = M_j \setminus \overline{B_r} \quad \text{and} \quad \mathscr{H}^n(N_j) = \mathscr{H}^n(M'_j) + o_j(1).$$

Hence, thanks to (5.23) and (5.26), we get

$$\begin{aligned}
\mathscr{H}^n(M_j) - \mathscr{H}^n(N_j) &\geq \mathscr{H}^n(M'_j) - \mathscr{H}^n(N_j) + o_j(1) \\
&= \mathscr{H}^n(M_j \cap Q_r) - \mathscr{H}^n(M'_j \cap Q_r) - o_j(1) \\
&\geq \mathscr{H}^n(M_j \cap Q_{(1-\sqrt{\varepsilon})r}) + \mathscr{H}^n(M_j \cap (R_{r,\varepsilon r} \setminus Q_{(1-\sqrt{\varepsilon})r})) + \\
&\quad - o_j(1) - (1 + C\sqrt{\varepsilon})\mathscr{H}^n(M_j \cap (R_{r,\varepsilon r} \setminus Q_{(1-\sqrt{\varepsilon})r})) \\
&\geq \mathscr{H}^n(M_j \cap Q_{(1-\sqrt{\varepsilon})r}) - C\sqrt{\varepsilon}\mathscr{H}^n(M_j \cap (R_{r,\varepsilon r} \setminus Q_{(1-\sqrt{\varepsilon})r})) - o_j(1).
\end{aligned}$$

Passing to the limit as $j \to \infty$ and using the weak-* convergence $\mu_j \rightharpoonup^* \mu$, STEP 1 and (5.19), we deduce

$$\begin{aligned}
\liminf_{j\to\infty} \mathscr{H}^n(M_j) &\geq \liminf_{j\to\infty} \mathscr{H}^n(N_j) + \mu(Q_{(1-\sqrt{\varepsilon})r}) - C\sqrt{\varepsilon} r^n \\
&\geq \liminf_{j\to\infty} \mathscr{H}^n(N_j) + (\vartheta_0(1-\sqrt{\varepsilon})^n - C\sqrt{\varepsilon}) r^n.
\end{aligned}$$

Since, for $\varepsilon > 0$ small, this is a contradiction with the fact that $(M_j)_j$ is a minimizing sequence, we finally conclude that $\vartheta(0) \geq 1$.

STEP 5. We prove that

$$\vartheta(x) \leq 1 \quad \forall x \in M : T_x M \text{ exists}.$$

As we did in STEP 4, we prove the stronger claim

$$\vartheta \leq 1 \quad \forall x \in M : \exists r_j \searrow 0 \text{ s.t. } \frac{(\varphi_{r_j,x})_\# \mu}{r_j^n} \rightharpoonup^* \vartheta(x) \mathscr{H}^n_{\llcorner T}, \tag{5.27}$$

for some $T \in G(n+k, n)$.

Indeed assume wlog $x=0$, $(r_j)_j$ and $\mathbb{R}^n\times\{0\}^k$ to satisfy (5.27), but $\vartheta(0)=1+\sigma>0$. By the monotonicity result established in STEP 2, for every $\varepsilon>0$ there exists $r>0$ such that

$$M\cap Q_r\subseteq R_{r,\varepsilon r},\quad 1+\sigma<\frac{\mu(Q_r)}{r^n}\le 1+(1+\varepsilon)\sigma. \tag{5.28}$$

Recalling that $\mathscr{H}^n_{\llcorner M_j}=\mu_j\rightharpoonup^*\mu$, we have

$$\begin{aligned}\mathscr{H}^n(M_j\cap Q_r)&>\left(1+\frac{\sigma}{2}\right)r^n\\ \mathscr{H}^n((M_j\cap Q_r)\setminus R_{r,\varepsilon r})&<\frac{\sigma}{4}r^n\quad\forall j\ge j_0(r).\end{aligned}\tag{5.29}$$

Consider the map $P\in\mathscr{D}(0,r)$ with $\mathrm{Lip}(P)<1+C\sqrt{\varepsilon}$ defined in STEP 4, which collapses $R_{r(1-\sqrt{\varepsilon}),\varepsilon r}$ onto T. Since $\mathcal{F}(\Gamma,\mathscr{C})$ is a good class,

$$\mathscr{H}^n(M_j\cap Q_r)\le\mathfrak{A}_j+\mathfrak{B}_j+\mathfrak{C}_j,$$

where we defined

$$\begin{aligned}\mathfrak{A}_j&:=\mathscr{H}^n(P(M_j\cap R_{r(1-\sqrt{\varepsilon}),\varepsilon r})),\\ \mathfrak{B}_j&:=\mathscr{H}^n(P(M_j\cap(R_{r,\varepsilon r}\setminus R_{r(1-\sqrt{\varepsilon}),\varepsilon r})),\\ \mathfrak{C}_j&:=\mathscr{H}^n(P(M_j\cap(Q_r\setminus R_{r,\varepsilon r}))).\end{aligned}$$

By construction

$$\mathfrak{A}_j\le r^n,$$

while, by virtue of (5.29), we have

$$\mathfrak{C}_j\le\big(\mathrm{Lip}(P)\big)^n\mathscr{H}^n(M_j\cap(Q_r\setminus R_{r,\varepsilon r}))\le(1+C\sqrt{\varepsilon})^n\frac{\sigma}{4}r^n.$$

Therefore, passing to the limit as $j\to\infty$, we obtain

$$\left(1-\frac{\sigma}{2}\right)r^n\le r^n+\liminf_{j\to\infty}\mathfrak{B}_j+(1+C\sqrt{\varepsilon})^n\frac{\sigma}{4}r^n.$$

Which in turn implies

$$\left(\frac{1}{2}-\frac{(1+C\sqrt{\varepsilon})}{4}\right)\sigma\le\liminf_{j\to\infty}\frac{\mathfrak{B}_j}{r^n}. \tag{5.30}$$

On the other hand, from (5.28), it follows

$$\limsup_{j\to\infty}\frac{\mathfrak{B}_j}{r^n}\le(1+C\sqrt{\varepsilon})^n\mu(Q_r\setminus Q_{(1-\sqrt{\varepsilon})r}) \tag{5.31}$$

$$\le(1+C\sqrt{\varepsilon})^n\left((1+(1+\varepsilon)\sigma)r^n-(1+\sigma)(1-\sqrt{\varepsilon})^n r^n\right). \tag{5.32}$$

Choosing ε sufficiently small, (5.30) and (5.31) provide a contradiction.

Together with the previous steps, we have proved $\mu=\mathscr{H}^n_{\llcorner M}$, with M relatively closed and n-rectifiable. Recalling Lemma 5.3.2, we deduce that $M\in\mathcal{F}(\Gamma,\mathscr{C})$ is a solution of the Plateau problem ($\mathfrak{P}$).

□

Theorem 5.4.2 (Regularity). *Let $M \in \mathcal{F}(\Gamma, \mathscr{C})$ be a solution of the Plateau problem ($\mathfrak{P}$). Then there exists a $\mathscr{H}^n$-negligible set $\Sigma \subseteq M$ such that $M \setminus \Sigma$ is a C^∞ sub manifold of $\mathbb{R}^{n+k}$.*

Proof. By virtue of Theorem 5.4.1, there exists a solution $M \in \mathcal{F}(\Gamma, \mathscr{C})$ of ($\mathfrak{P}$). Moreover, recalling STEP 1 of *Proof of Theorem 5.4.1*, the varifold $V := \underline{\mathrm{v}}(M, 1)$ is stationary. Applying Corollary 4.3.1 we deduce the existence of a $\mathscr{H}^n$-negligible set Σ such that $M \setminus \Sigma$ is a C^∞ submanifold of $\mathbb{R}^{n+k} \setminus \Gamma$.

□

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