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Exponent growth in groups acting on regular rooted trees

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Introduction

There is a famous problem in group theory, whose formulation is as simple as its solution is interesting and elaborate. From the early 20th century, the so-called Burnside Problem has been worked on and formulated in different versions. In one of its forms, which is known as the General Burnside Problem, the problem is posed via the following question.

Question 1 (General Burnside Problem). Is it true that every finitely generated periodic group must be finite?

Here we recall that a group is said to be periodic if all its elements have finite order. In 1964, Golod and Shafarevich were able to construct the first examples of infinite finitely generated periodic groups ([7][8]), giving a negative answer to Question 1. Since then, many other examples of infinite finitely generated periodic groups have been found: among them, some of the easiest turned out to belong to the class of groups acting on regular rooted trees, which will be the true protagonist of this dissertation. By way of example, we mention the well-known Grigorchuk groups (defined in [9]) and Gupta-Sidki groups (defined in [10]).

Another version of the problem raised by Burnside, known as the Burnside Problem, can be formulated as follows.

Question 2 (Burnside Problem). Is it true that every finitely generated group with finite exponent must be finite?

The answer is still negative. Indeed, in 1968 Adian and Novikov proved that whenever m and n are integers such that $m \geq 2$, $n \geq 4381$ and n is odd, there must exist a group which is generated by m elements, has exponent n and is infinite (see [1][2][3]). In fact, the same holds if n is odd and $n \geq 665$, as evidenced in [4].

As often happens in mathematics, it is not only interesting to find objects which give a negative answer to both Question 1 and Question 2, but also to find objects having a kind of "intermediate behaviour", i.e. giving a negative answer to Question 1 but not to Question 2. In that regard, we point out that all the examples of groups acting on the regular rooted tree $\mathcal{T}$ which answer negatively Question 1 are not eligible to answer Question 2. To motivate this, we introduce a third version of the Burnside Problem, which goes by the name of Restricted Burnside Problem.

Question 3 (Restricted Burnside Problem). Given a finite group G with exponent n and m generators, is it possible to find a bound for the order of G which only depends on n and m?

In the early 1990s, Zelmanov showed, in two remarkable papers ([15] and [16]), that the answer to Question 3 is affirmative. Now, the negative answer to Question 2 makes it clear that a finitely generated group with finite exponent may not be finite. But what can we say about groups that, besides being finitely generated and with finite exponent, are residually finite? We recall that a group is said to be residually finite if the intersection of all its finite-index normal subgroups is 1. Therefore, if a group G is residually finite, there exists a decreasing sequence

$$G \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \ldots \trianglerighteq N_i \trianglerighteq N_{i+1} \trianglerighteq \ldots \qquad (0.1)$$

of finite-index normal subgroups of G such that N_{i+1} is strictly contained in N_i whenever $N_i \neq 1$. If moreover G has m generators and finite exponent equal to n, $\{G/N_i\}_{i\in\mathbb{N}}$ is a sequence of finite groups with m generators and exponent dividing n. It follows from Zelmanov's results that the increasing sequence of orders $\{|G/N_i|\}_{i\in\mathbb{N}}$ is bounded. Thus the chain of subgroups (0.1) is stationary, which implies that $N_k = 1$ for some k, and G is finite. All in all, the following question is answered in the affirmative.

Question 4. Is it true that every finitely generated residually finite group with finite exponent must be finite?

In the special case that G is a group acting on the regular rooted tree $\mathcal{T}$, the sequence (0.1) can be chosen in a canonical way, naming N_i the ith level stabilizer in G and $G/\operatorname{St}_G(i)$ the ith congruence quotient of G. Such sequence is natural and has a very concrete meaning: as we will see in Chapter 1, the ith congruence quotient $G/\operatorname{St}_G(i)$ is the group which encodes the action of G on the first i levels of the infinite tree $\mathcal{T}$. Furthermore, the intersection of all the level stabilizers in G is 1. We have then that every group acting on $\mathcal{T}$ is residually finite and, by the affirmative answer to Question 4, a finitely generated infinite group acting on $\mathcal{T}$ must have infinite exponent. In other words, as claimed above, a group acting on $\mathcal{T}$ is never a suitable object to answer Question 2.

Morally, the common aim of the different versions of the Burnside Problem is to compare, given a finitely generated group, the number of its elements and the order of its elements. When G is a finitely generated infinite group acting on $\mathcal{T}$, according to the previous considerations, the order and the exponent of G are two infinite quantities. To go one step further in the comparison between them, a typical technique consists in considering the action of G on the first i levels of $\mathcal{T}$, computing the order and the exponent of the group encoding such action, which is $G/\operatorname{St}_G(i)$, and letting i go to infinity. In fact, it is easy to prove that

$$\lim_{i\to\infty}\left|\frac{G}{\operatorname{St}_G(i)}\right| = |G| = +\infty$$

$$\lim_{i\to\infty}\exp\left(\frac{G}{\operatorname{St}_G(i)}\right) = \exp(G) = +\infty$$

and it is of a great interest to compute and compare $|G/\operatorname{St}_G(i)|$ and $\exp(G/\operatorname{St}_G(i))$ as i varies. We will refer to the problem of determining how fast $\exp(G/\operatorname{St}_G(i))$ goes to infinity (when i goes to infinity) as the exponent growth problem for G. The final goal of this thesis will be to solve the exponent growth problem for an important family of finitely generated infinite subgroups of $\operatorname{Aut}\mathcal{T}$, which take the name of multi-EGS-groups. For some of them, we will also be able to compare the two quantities $|G/\operatorname{St}_G(i)|$ and $\exp(G/\operatorname{St}_G(i))$.

Layout of the thesis.

In the first chapter, we will give an introduction to the general theory of groups acting on regular rooted trees. If we fix a natural number $d \geq 1$, the d-adic regular rooted tree $\mathcal{T}$ is the rooted tree where every vertex has exactly d children. By saying that a group G acts on the d-adic tree $\mathcal{T}$, we mean that the elements of G are bijections that move the vertices of $\mathcal{T}$ preserving its tree structure. Such bijections are called automorphisms of $\mathcal{T}$, and the set of all the automorphisms, which will be denoted by $\operatorname{Aut}\mathcal{T}$, is a group under composition. We will introduce all the needed technical tools to work with automorphisms of $\mathcal{T}$, and we will dwell on relevant types of subgroups of $\operatorname{Aut}\mathcal{T}$, such as self-similar groups and fractal groups, analyzing their properties and giving examples and counterexamples.

In Chapter 2 we will focus on the family of multi-EGS-groups. The multi-EGS-groups generalize the aforementioned Gupta-Sidki groups and give many examples that answer in the negative the General Burnside Problem. We will see that every multi-EGS-group is finitely generated and self-similar, and we will prove that, when a multi-EGS-group G acts on the p-adic tree $\mathcal{T}$ and p is prime, G is infinite and fractal. Finally, we will provide a characterization for periodic multi-EGS-groups over the p-adic tree. Let us notice that, if p is a prime number, a multi-EGS-group G acting on the p-adic tree is finitely generated and infinite, and then the sequence $\exp(G/\operatorname{St}_G(i))$ converges to infinity as i goes to infinity.

The third chapter will be devoted to the main result of this thesis, which is the solution of the exponent growth problem for every multi-EGS-group G acting on the p-adic regular rooted tree, when p is prime. More precisely, we will prove that

$$\exp\left(\frac{G}{\operatorname{St}_G(i)}\right) = \begin{cases} p^{\lfloor\frac{i+1}{2}\rfloor} & \text{if } G \text{ is periodic} \\ p^i & \text{otherwise} \end{cases} \tag{0.2}$$

where $\lfloor\frac{i+1}{2}\rfloor$ is the greatest integer number less than or equal to $\frac{i+1}{2}$ (see [12]). This expression for $\exp(G/\operatorname{St}_G(i))$ is very simple, and it only depends on p, i and on the periodicity of G.

On the other hand, computing the order of $G/\operatorname{St}_G(i)$ for a multi-EGS-group G acting on the p-adic tree is not an easy task. However, when G belongs to the family of GGS-groups, which is a significant subfamily of multi-EGS-groups, it is possible to provide a formula for $|G/\operatorname{St}_G(i)|$. This has been done in [6] in 2014. Using such result and (0.2), we will be able to compare the two quantities $|G/\operatorname{St}_G(i)|$ and $\exp(G/\operatorname{St}_G(i))$, in the event that G is a GGS-group acting on the p-adic tree and p is prime.

The first step in our strategy to prove (0.2) will be to realize that there is a special subgroup of $\operatorname{Aut}\mathcal{T}$ which contains all the multi-EGS-groups. Such group will be denoted by Γ, it will prove to be self-similar, fractal and infinite, but not finitely generated and not periodic. Moreover we will see that the ith congruence quotient of Γ has exponent p^i, and

$$\exp\left(\frac{G}{\operatorname{St}_G(i)}\right) \quad \text{divides} \quad \exp\left(\frac{\Gamma}{\operatorname{St}_\Gamma(i)}\right) = p^i. \tag{0.3}$$

This already tells us that the growth of $\exp(G/\operatorname{St}_G(i))$ to infinity cannot be more than exponential in i. In the case that G is non-periodic, we will explicitly find elements of order p^i inside $G/\operatorname{St}_G(i)$ and this, together with (0.3), will prove (0.2). In the case that G is periodic, we will use the following property which holds for every self-similar subgroup S of $\operatorname{Aut}\mathcal{T}$:

$$\exp\left(\frac{S}{\operatorname{St}_S(i+2)}\right) \leq \exp\left(\frac{S}{\operatorname{St}_S(i)}\right) \cdot \exp\left(\frac{S}{\operatorname{St}_S(2)}\right). \tag{0.4}$$

In such a way, it will be enough to deal with the "small" group $G/\operatorname{St}_G(2)$ in order to prove that $\exp(G/\operatorname{St}_G(2)) = p$, and then to conclude by (0.4) that the quantity $\exp(G/\operatorname{St}_G(i))$ cannot grow more than p when i increases by 2. This will yield that $\exp(G/\operatorname{St}_G(i)) \leq p^{\lfloor \frac{i+1}{2} \rfloor}$. Proving that also the converse inequality holds will be the most delicate part of our proof; it will proceed by induction on i and it will rely on results in [6].

The group Γ is not finitely generated and contains the whole family of multi-EGS-groups, so it is large, and much larger than every multi-EGS-group G. This is the reason why the information given by (0.2) is surprising, since it tells us that the two sequences $\exp(G/\operatorname{St}_G(i))$ and $\exp(\Gamma/\operatorname{St}_\Gamma(i))$ grow to infinity essentially with the same speed.

Chapter 1

Preliminaries

In this chapter, we give an overview of the main objects and tools in the theory of groups acting on trees, underlining the properties which will be relevant for our work and introducing the notation which will be used throughout this thesis.

1.1 The d-adic rooted tree

The elements of the groups that we will have to do with are bijections that "move" the vertices of a regular rooted tree in a suitable way. That is why we first need to define what a regular rooted tree is.

Let $d \geq 1$ be an integer. If $X = \{1, \ldots, d\}$, we denote with X^* the free monoid generated by X. In other words, X^* is the monoid whose elements are all the words of finite length in the alphabet X, with word concatenation as the monoid operation, and with the unique word of length zero (called the *empty word* and denoted by $\emptyset$) as the identity element.

Definition 1.1.1. Let $u = x_1 \ldots x_i$ and $v = y_1 \ldots y_j$ be elements of X^* (that is, $i, j \geq 0$ are integers and $x_1, \ldots, x_i, y_1, \ldots, y_j \in X$). We say that v is a *child* of u (and u is a *parent* of v) if $j = i + 1$ and $x_k = y_k$ for any $k \leq i$.

Note. We write $u = x_1 \ldots x_i$ meaning that, if $i = 0$, u is the empty word.

Definition 1.1.2. Let $d \geq 1$. The *d-adic (regular) rooted tree* $\mathcal{T}$ is the undirected graph that has X^* as set of vertices and has an edge between two vertices if and only if one of the two is a child of the other.

With this notation, we can classify the vertices of $\mathcal{T}$ according to the length of the word they are represented by: we will denote with X^n the subset of X^* whose elements are all the words of length n in the alphabet X, and we will say that a vertex belongs to the nth level of $\mathcal{T}$ if it belongs to X^n. Clearly $|X^n| = d^n$. The root of the tree corresponds to the empty word $\emptyset$ and it is the only vertex belonging to level 0 (see figure 1.1).

For any $n \geq 0$, we can consider the undirected graph that has $\bigcup_{i=0}^{n} X^i$ as set of vertices and has an edge between two of its vertices if and only if $\mathcal{T}$ has an edge

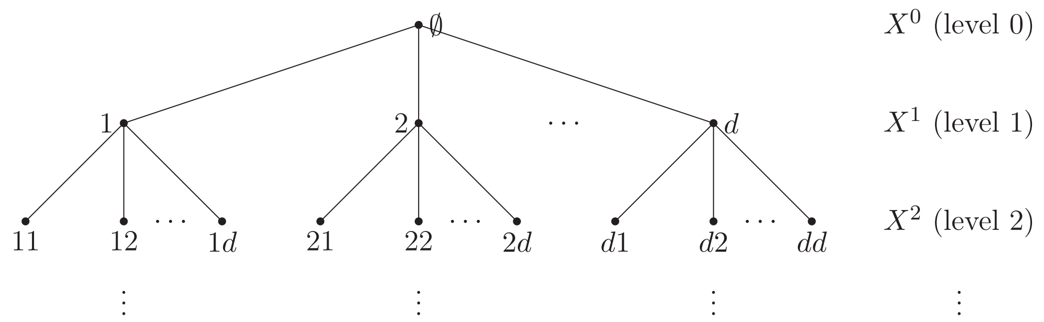


Figure 1.1: The d-adic rooted tree $\mathcal{T}$.

between the two of them. Such graph, which turns out to be a finite truncated tree with root $\emptyset$ and n further levels of vertices, will be denoted by $\mathcal{T}_n$.

1.2 Automorphisms of $\mathcal{T}$

Before giving the definition of automorphism of $\mathcal{T}$, we point out that, throughout this thesis, we will indicate with fg the composition of two maps f and g where we apply first f and then g, and we will write $(u)f$ for the image of u under f (then $(u)fg = ((u)f)g$ for any u lying in the domain of f).

Definition 1.2.1. An *automorphism of* $\mathcal{T}$ is a bijection $f : X^* \longrightarrow X^*$ that preserves incidence, that is, if u and v are connected by an edge of $\mathcal{T}$ then $(u)f$ and $(v)f$ are connected by an edge of $\mathcal{T}$.

Observe that, if f is an automorphism and we denote by D_u the set of vertices connected to u (i.e. $D_u = \{v \in X^* : v \text{ is a child of } u \text{ or } u \text{ is a child of } v\}$), the restriction map $f_{|D_u} : D_u \longrightarrow D_{(u)f}$ is well-defined and injective by definition of automorphism. Then $|D_u| \leq |D_{(u)f}|$ and since

$$|D_u| = \begin{cases} d & \text{if } u = \emptyset \\ d+1 & \text{if } u \neq \emptyset \end{cases}$$

we get that

$$\text{if } u \neq \emptyset\,, \quad d+1 = |D_u| \leq |D_{(u)f}| \implies (u)f \neq \emptyset. \tag{1.1}$$

This leads to proving a first basic property of automorphisms of $\mathcal{T}$, which is stated in the following proposition.

Proposition 1.2.2. *Let f be an automorphism of $\mathcal{T}$. Then:*

(i) $(\emptyset)f = \emptyset$.

(ii) $(X^n)f = X^n$ *for any* $n \geq 0$.

Proof. *(i)* Since $\emptyset = ((\emptyset)f^{-1})f$, (1.1) implies that $(\emptyset)f^{-1} = \emptyset$ and then $\emptyset = (\emptyset)f$.

(ii) We argue by induction on n. For $n = 0$, *(ii)* coincides with *(i)*. If $n = 1$, we have $X^1 = D_\emptyset = D_{(\emptyset)f}$ by *(i)* and $f_{|D_\emptyset} : D_\emptyset = X^1 \longrightarrow D_{(\emptyset)f} = X^1$ is a bijection, which yields $(X^1)f = X^1$.

Let us assume that $n \geq 2$. Since f is a bijection and X^n is finite, it suffices to prove that $(X^n)f \subseteq X^n$. If $v = x_1 \dots x_n \in X^n$, v is a child of $u = x_1 \dots x_{n-1} \in X^{n-1}$ and $(u)f \in X^{n-1}$ by induction hypothesis. By definition of automorphism, $(v)f$ must be either a child or a parent of $(u)f$. Hence $(v)f \in X^n$ or $(v)f \in X^{n-2}$. Since $(X^{n-2})f = X^{n-2}$ by induction hypothesis, if $(v)f$ belonged to X^{n-2} then $(v)f = (w)f$ for some $w \in X^{n-2}$ and, by the injectivity of f, $v = x_1 \dots x_n$ would belong to X^{n-2}, which is a contradiction. Thus the only possibility is that $(v)f \in X^n$. □

We know that, whenever $v \in X^{n+1}$ is a child of $u \in X^n$, an automorphism f sends v either to a child or to a parent of $(u)f$. Thanks to Proposition 1.2.2 we have, moreover, that $(v)f \in X^{n+1}$ and $(u)f \in X^n$, and then $(v)f$ must be a child of $(u)f$. If we denote by C_u the set of children of the vertex u, it follows that $(C_u)f \subseteq C_{(u)f}$ and, since $|C_u| = |C_{(u)f}| = d$, the map $f_{|C_u} : C_u \longrightarrow C_{(u)f}$ is a bijection. Hence for every automorphism f and every $u, v \in X^*$,

$$v \text{ is a child of } u \iff (v)f \text{ is a child of } (u)f. \tag{1.2}$$

Of course a bijection of X^* which sends children of a vertex to children of its image is an automorphism, but moreover (1.2) ensures that every automorphism sends children of a vertex to children of its image (notice that to prove (1.2) we heavily exploited the regularity of the tree structure of $\mathcal{T}$). Therefore

$$\text{a bijection } f : X^* \longrightarrow X^* \text{ is an automorphism of } \mathcal{T} \iff (C_u)f \subseteq C_{(u)f}\ \forall u \in X^*$$

and we will often use this as a more operative definition of automorphism.

We will write $\operatorname{Aut}\mathcal{T}$ for the set of all the automorphisms of $\mathcal{T}$. The set $\operatorname{Aut}\mathcal{T}$, endowed with the composition between automorphisms, turns out to be a group. Indeed the identity map on X^* is clearly an automorphism of $\mathcal{T}$ and composition is an associative operation. If $f, g \in \operatorname{Aut}\mathcal{T}$ and v is a child of u then $(v)f$ is a child of $(u)f$ and as a consequence $((u)f)g$ is a child of $((v)f)g$, that is, $fg \in \operatorname{Aut}\mathcal{T}$. Finally, if $f \in \operatorname{Aut}\mathcal{T}$ and $v = ((v)f^{-1})f$ is a child of $u = ((u)f^{-1})f$ then, by (1.2), $(v)f^{-1}$ is a child of $(u)f^{-1}$. Hence f^{-1} is an automorphism and $\operatorname{Aut}\mathcal{T}$ is a group.

In a completely analogous way, we can define the group of automorphisms of the finite rooted tree $\mathcal{T}_n$, for every $n \geq 0$.

Definition 1.2.3. Let $n \geq 0$ be an integer. An *automorphism of* $\mathcal{T}_n$ is a bijection $g : \bigcup_{i=0}^n X^i \longrightarrow \bigcup_{i=0}^n X^i$ that preserves incidence, that is, if u and v are connected by an edge of $\mathcal{T}_n$ then $(u)g$ and $(v)g$ are connected by an edge of $\mathcal{T}_n$.

We will write $\operatorname{Aut}\mathcal{T}_n$ for the set of all the automorphisms of $\mathcal{T}_n$. With the same argument used to show that $\operatorname{Aut}\mathcal{T}$ is a group, it can be proved that the set $\operatorname{Aut}\mathcal{T}_n$, endowed with composition, is a finite group.

Let us notice that $\operatorname{Aut}\mathcal{T}_n$ coincides with the set of restrictions $\{f_{|\bigcup_{i=0}^n X^i} : f \in \operatorname{Aut}\mathcal{T}\}$. Indeed, if $f \in \operatorname{Aut}\mathcal{T}$, the map $f_{|\bigcup_{i=0}^n X^i} : \bigcup_{i=0}^n X^i \longrightarrow \bigcup_{i=0}^n X^i$ is a bijection by Proposition 1.2.2 and preserves incidence by definition of automorphism of $\mathcal{T}$. Conversely, if $g \in \operatorname{Aut}\mathcal{T}_n$, the map $g^* : X^* \longrightarrow X^*$ given by

$$(x_1 \dots x_i)g^* = \begin{cases} (x_1 \dots x_i)g & \text{if } i \le n \\ (x_1 \dots x_n)g\ x_{n+1} \dots x_i & \text{if } i > n \end{cases}$$

is such that $g^*_{|\bigcup_{i=0}^n X^i} = g$, it is invertible with inverse $(g^{-1})^*$ and it sends children of a vertex u to children of $(u)g^*$. This means that $g^* \in \operatorname{Aut}\mathcal{T}$ and any automorphism of $\mathcal{T}_n$ can be extended to an automorphism of $\mathcal{T}$.

We observe, in addition, that the finite group $\operatorname{Aut}\mathcal{T}_n$ can be embedded in the symmetric group S_{d^n}. This follows from the fact that, if $g \in \operatorname{Aut}\mathcal{T}_n$, knowing the image under g of all the vertices in X^n is enough to know the image under g of all the vertices in $\bigcup_{i=0}^n X^i$. In other words, if $g_1, g_2 \in \operatorname{Aut}\mathcal{T}_n$ and $g_{1|X^n} = g_{2|X^n}$, then $g_1 = g_2$. Indeed, if $u = x_1 \dots x_k \in \bigcup_{i=0}^n X^i$ and $x \in X$, the vertex $ux \overset{n-k}{\dots} x$ belongs to X^n and then

$$(ux \overset{n-k}{\dots} x)g_1 = (ux \overset{n-k}{\dots} x)g_2.$$

Since $ux \overset{n-k}{\dots} x$ is a child of $ux \overset{n-k-1}{\dots} x$, $(ux \overset{n-k-1}{\dots} x)g_1$ and $(ux \overset{n-k-1}{\dots} x)g_2$ must be equal to the (unique) parent of $(ux \overset{n-k}{\dots} x)g_1 = (ux \overset{n-k}{\dots} x)g_2$, and then

$$(ux \overset{n-k-1}{\dots} x)g_1 = (ux \overset{n-k-1}{\dots} x)g_2.$$

Iterating this argument we get $(ux \overset{n-k-2}{\dots} x)g_1 = (ux \overset{n-k-2}{\dots} x)g_2$, $(ux \overset{n-k-3}{\dots} x)g_1 = (ux \overset{n-k-3}{\dots} x)g_2$ and so on. Finally $(u)g_1 = (u)g_2$ and then g_1 and g_2 coincide on $\bigcup_{i=0}^n X^i$. As a consequence, the map

$$\begin{aligned} \operatorname{Aut}\mathcal{T}_n &\longrightarrow S_{d^n} \\ g &\longmapsto g_{|X^n} \end{aligned} \tag{1.3}$$

is an injective group homomorphism.

1.3 Portraits

The aim of this section is introducing the notion of portrait of an automorphism of $\mathcal{T}$, which will give us a very direct way to represent an automorphism by "drawing" it. As we already observed in Section 1.2, if $f \in \operatorname{Aut}\mathcal{T}$, u is a vertex of $\mathcal{T}$ and $C_u = \{ux : x \in X\} = \{u1, u2, \dots, ud\}$ is the set of its children, then $f_{|C_u} : C_u \longrightarrow C_{(u)f}$ is a bijection. This means that for any $y \in X$ there is a unique $x \in X$ such that $(ux)f = (u)f\ y$ (as shown in Figure 1.2) and allows to give the following definition.

Definition 1.3.1. Let $f \in \operatorname{Aut}\mathcal{T}$. We call the *label of f at the vertex u* and we denote with $f_{(u)}$ the bijection of $X = \{1, \dots, d\}$ which satisfies

$$(ux)f = (u)f\ (x)f_{(u)}$$

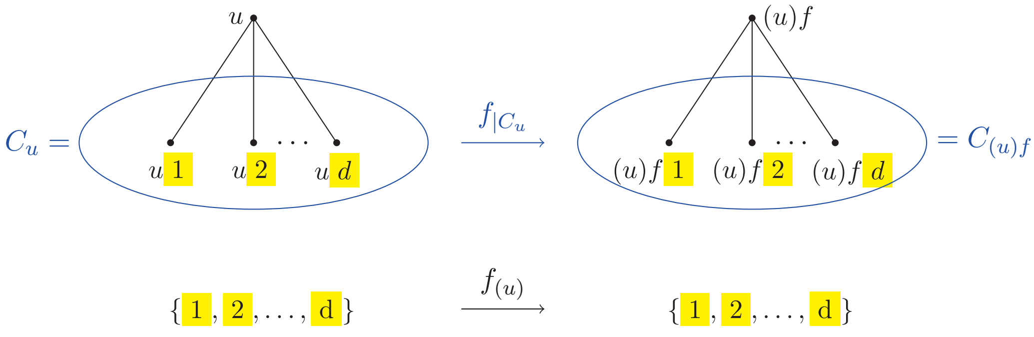


Figure 1.2: The label of f at u.

for every $x \in X$.

Definition 1.3.2. The collection of all the labels of an automorphism f

$$\mathcal{P}_f = \{f_{(u)} : u \in X^*\}$$

is called the *portrait* of f.

We will draw the portrait of an automorphism as in Figure 1.3. The next proposition ensures that an automorphism of $\operatorname{Aut}\mathcal{T}$ is uniquely determined by its portrait, and any collection of bijections of $X = \{1, \ldots, d\}$ (i.e. elements of the symmetric group S_d) labelled by the elements of X^* is the portrait of some automorphism.

Proposition 1.3.3. *There is a one-to-one correspondence between automorphisms of $\mathcal{T}$ and collections of the form $\{\phi_{(u)} \in S_d\}_{u \in X^*}$, which assigns to any automorphism its portrait.*

Proof. First, we show by induction on $i \geq 0$ that

$$(x_1 \ldots x_i)f = (x_1)f_{(\emptyset)}\,(x_2)f_{(x_1)}\,(x_3)f_{(x_1x_2)} \cdots (x_i)f_{(x_1 \ldots x_{i-1})} \tag{1.4}$$

for every $f \in \operatorname{Aut}\mathcal{T}$ and $x_1, \ldots, x_i \in X$. When $i = 0$, the above equality becomes $(\emptyset)f = \emptyset$ and it holds by Proposition 1.2.2. When $i = 1$, $(x_1)f = (\emptyset x_1)f = (\emptyset)f\,(x_1)f_{(\emptyset)} = \emptyset\,(x_1)f_{(\emptyset)} = (x_1)f_{(\emptyset)}$ by definition of label. Assume then $i \geq 2$. We have

$$\begin{aligned}(x_1 \ldots x_{i-1}x_i)f &= (x_1 \ldots x_{i-1})f\,(x_i)f_{(x_1 \ldots x_{i-1})} \\ &= (x_1)f_{(\emptyset)}\,(x_2)f_{(x_1)}\,(x_3)f_{(x_1x_2)} \cdots (x_{i-1})f_{(x_1 \ldots x_{i-2})}\,(x_i)f_{(x_1 \ldots x_{i-1})}\end{aligned}$$

where the first equality holds by definition of label, the second one by induction hypothesis. The equation (1.4) guarantees that two automorphisms with the same portrait are equal.

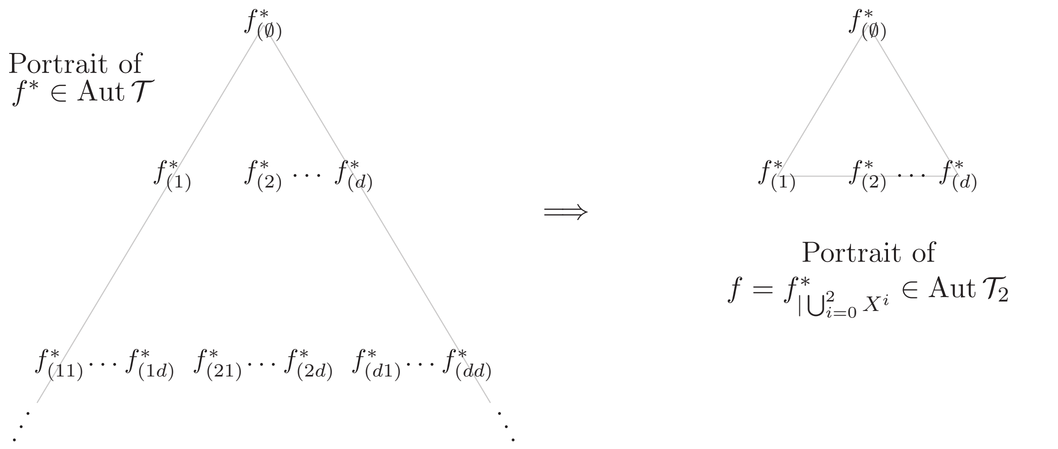


Figure 1.3: Portraits in $\operatorname{Aut}\mathcal{T}$ and in $\operatorname{Aut}\mathcal{T}_2$.

On the other hand, we want to show that, given $\mathcal{P} = \{\phi_{(u)} \in S_d\}_{u \in X^*}$ a collection of bijections of $X = \{1, \ldots, d\}$, the map $\phi : X^* \longrightarrow X^*$ defined by

$$(x_1 \ldots x_i)\phi = (x_1)\phi_{(\emptyset)}\,(x_2)\phi_{(x_1)}\,(x_3)\phi_{(x_1x_2)} \ldots (x_i)\phi_{(x_1 \ldots x_{i-1})}$$

is an automorphism and has portrait $\mathcal{P}$ (i.e. the label of ϕ at u is $\phi_{(u)}$). It preserves incidence because, for every $i \geq 1$,

$$(x_1 \ldots x_i)\phi = (x_1)\phi_{(\emptyset)} \ldots (x_i)\phi_{(x_1 \ldots x_{i-1})} = (x_1 \ldots x_{i-1})\phi\,(x_i)\phi_{(x_1 \ldots x_{i-1})} \tag{1.5}$$

is a child of $(x_1 \ldots x_{i-1})\phi$. The injectivity of ϕ follows from the injectivity of the elements of $\mathcal{P}$. To see that ϕ is surjective, for $y_1 \ldots y_i \in X^*$, we recursively define

$$\begin{aligned} x_1 &= (y_1)\phi^{-1}_{(\emptyset)} \\ x_k &= (y_k)\phi^{-1}_{(x_1 \ldots x_{k-1})} \qquad \text{for } 2 \leq k \leq i \end{aligned}$$

and we observe that $(x_1 \ldots x_i)\phi = y_1 \ldots y_i$. Then ϕ is an automorphism and, by (1.5), it has portrait $\mathcal{P}$. □

Thanks to this result, any assignment of an element of S_d to each vertex of the tree is the portrait of a unique automorphism.

Example 1.3.4. Let τ be a permutation in the symmetric group S_d. We will call the *rooted automorphism associated with the permutation* τ the automorphism $f \in \operatorname{Aut}\mathcal{T}$ that has portrait

$$f_{(u)} = \begin{cases} \tau & \text{if } u = \emptyset \\ 1 & \text{if } u \neq \emptyset. \end{cases}$$

The rooted automorphism associated with the permutation $(12 \ldots d) \in S_d$ will have a special relevance in the next chapters and for this reason, from now onwards, it will

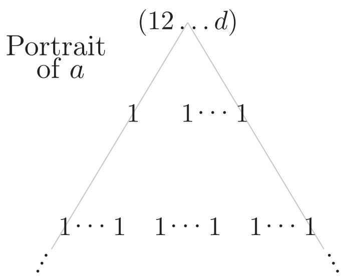


Figure 1.4: The portrait of the rooted automorphism a.

be indicated with the letter a. Let us try to understand how a moves the vertices of $\mathcal{T}$. If $x_1x_2\dots x_i \in X^*$ and $i \geq 1$, the equation (1.4) yields that

$$\begin{aligned}(x_1x_2\dots x_i)a &= (x_1)a_{(\emptyset)}\,(x_2)a_{(x_1)}\dots(x_i)a_{(x_1\dots x_{i-1})}\\ &= (x_1)(12\dots d)\,x_2\dots x_i\\ &= x_1+1\;\;x_2\dots x_i\end{aligned}$$

where x_1+1 is understood to be an integer modulo d (i.e. $x_1+1=1$ if $x_1=d$). Therefore a is the automorphism that fixes the root (as any other automorphism does, by Proposition 1.2.2) and rigidly permutes the d subtrees hanging from the root according to the permutation $(12\dots d)$ (see Figure 1.5). It readily follows that the order of a (as an element of the group $\operatorname{Aut}\mathcal{T}$) is d.

Since we will often define automorphisms of the d-adic rooted tree via their portrait, we will happen to use the next result, which gives us an easy way to compute the portrait of a composition of automorphisms.

Proposition 1.3.5. *Let $r \geq 2$, $f_1,\dots,f_r \in \operatorname{Aut}\mathcal{T}$ and $u \in X^*$. Then*

$$(f_1\dots f_r)_{(u)} = (f_1)_{(u)}(f_2)_{((u)f_1)}(f_3)_{((u)f_1f_2)}\cdots(f_r)_{((u)f_1\dots f_{r-1})}.$$

Proof. We argue by induction on r. For $r=2$, $u \in X^*$ and $x \in X$, we have

$$\begin{aligned}(u)(f_1f_2)\,(x)(f_1f_2)_{(u)} &= (ux)(f_1f_2)\\ &= ((ux)f_1)f_2\\ &= ((u)f_1\,(x)(f_1)_{(u)})f_2\\ &= ((u)f_1)f_2\,((x)(f_1)_{(u)})(f_2)_{((u)f_1)}\\ &= (u)(f_1f_2)\,(x)((f_1)_{(u)}(f_2)_{((u)f_1)})\end{aligned}$$

by definition of label. The equality between these two words in the alphabet X yields that

$$(x)(f_1f_2)_{(u)} = (x)((f_1)_{(u)}(f_2)_{((u)f_1)})$$

for all $x \in X$, which is our claim when $r=2$. If $r \geq 3$, by what we just proved

$$(f_1\dots f_r)_{(u)} = (f_1\dots f_{r-1})_{(u)}(f_r)_{((u)(f_1\dots f_{r-1}))}$$

and this concludes the proof by induction hypothesis. □

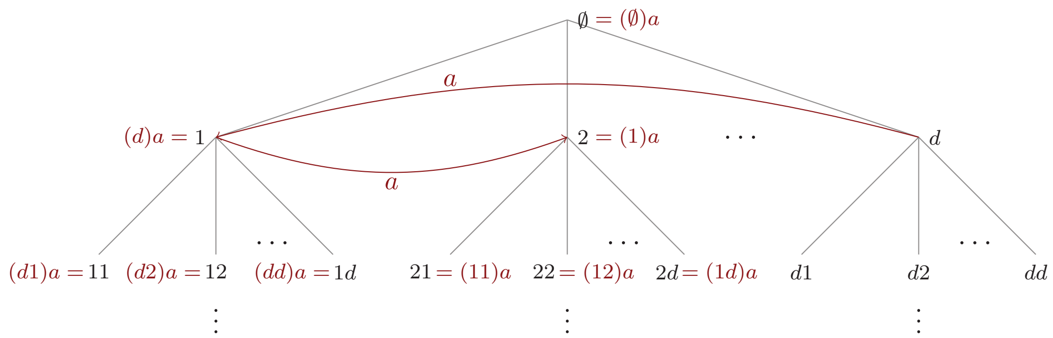


Figure 1.5: The action of a.

Corollary 1.3.6. *Let $f \in \operatorname{Aut}\mathcal{T}$ and $u \in X^*$. Then*

$$(f^{-1})_{(u)} = (f_{((u)f^{-1})})^{-1}.$$

Proof. Since the identity has only trivial labels, we have by the previous result that

$$1 = 1_{(u)} = (f^{-1}f)_{(u)} = (f^{-1})_{(u)}(f)_{((u)f^{-1})}$$

and the result follows. □

Let us end this section by observing that the notion of portrait can be adapted to the elements of $\operatorname{Aut}\mathcal{T}_n$, for any fixed $n \geq 1$. Since for any $f \in \operatorname{Aut}\mathcal{T}_n$ there is $f^* \in \operatorname{Aut}\mathcal{T}$ such that $f^*|_{\bigcup_{i=0}^n X^i} = f$, the equation (1.4) guarantees that f is completely determined by the bijections $\{f^*{}_{(u)} : u \in \bigcup_{i=0}^{n-1} X^i\}$. In other words, in order to describe f, we just have to focus on the labels of f^* at the vertices lying between level 0 and level $n-1$. We will say that $f^*_{(u)}$ is the *label* of f at u (whenever $u \in \bigcup_{i=0}^{n-1} X^i$) and that $\{f^*{}_{(u)} : u \in \bigcup_{i=0}^{n-1} X^i\}$ is the *portrait* of f (see Figure 1.3). Recalling that the elements of $\operatorname{Aut}\mathcal{T}_n$ are exactly the restrictions of elements of $\operatorname{Aut}\mathcal{T}$ to $\bigcup_{i=0}^n X^i$ and using Proposition 1.3.3, we get a one-to-one correspondence between elements of $\operatorname{Aut}\mathcal{T}_n$ and collections of the form $\{\phi_{(u)} \in S_d\}_{u \in \bigcup_{i=0}^{n-1} X^i}$, which assigns to every element of $\operatorname{Aut}\mathcal{T}_n$ its portrait. To compute the order of the finite group $\operatorname{Aut}\mathcal{T}_n$ is then enough to count the number of such collections, that is

$$|\operatorname{Aut}\mathcal{T}_n| = |S_d|^{|\bigcup_{i=0}^{n-1} X^i|} = (d!)^{\sum_{i=0}^{n-1} d^i} = (d!)^{\frac{d^n-1}{d-1}}. \tag{1.6}$$

1.4 The nth level stabilizer

We learned from the previous section that an automorphism f can be defined via its portrait, and focusing on the labels in the first levels of $\mathcal{T}$ means looking at some element of $\operatorname{Aut}\mathcal{T}_n$ that is a restriction of f. The next step in our trip to understanding the structure of the group $\operatorname{Aut}\mathcal{T}$ is to find out what are the relations between $\operatorname{Aut}\mathcal{T}$ and the sequence of finite groups $\{\operatorname{Aut}\mathcal{T}_n\}_{n\geq 0}$. We will see that, for any $n \geq 0$, $\operatorname{Aut}\mathcal{T}_n$ can be realised both as a finite quotient (Proposition 1.4.1) and as a subgroup (Proposition 1.4.2) of $\operatorname{Aut}\mathcal{T}$.

First of all, we fix $n \geq 0$ and we observe that the natural restriction map

$$\begin{aligned} \operatorname{Aut}\mathcal{T} &\longrightarrow \operatorname{Aut}\mathcal{T}_n \\ f &\longmapsto f_{|\bigcup_{i=0}^{n} X^i} \end{aligned}$$

is a surjective group homomorphism. Its kernel, which gives the name to this section, is called the *nth level stabilizer* of $\mathcal{T}$, it is denoted with $\operatorname{St}(n)$ and given by

$$\begin{aligned} \operatorname{St}(n) &= \{f \in \operatorname{Aut}\mathcal{T} : (u)f = u \text{ for every } u \in \cup_{i=0}^{n} X^i\} \\ &= \{f \in \operatorname{Aut}\mathcal{T} : (u)f = u \text{ for every } u \in X^n\} \end{aligned}$$

(where the last equality follows from the injectivity of (1.3) and justifies the use of the term *nth level stabilizer*). Then the following result is clear.

Proposition 1.4.1. *For every $n \geq 0$, the nth level stabilizer $\operatorname{St}(n)$ is normal in $\operatorname{Aut}\mathcal{T}$ and we have*

$$\frac{\operatorname{Aut}\mathcal{T}}{\operatorname{St}(n)} \simeq \operatorname{Aut}\mathcal{T}_n.$$

Moreover, by (1.6), $|\operatorname{Aut}\mathcal{T} : \operatorname{St}(n)| = (d!)^{\frac{d^n-1}{d-1}}$.

We remark that, by (1.4), an automorphism leaves fixed all vertices in the nth level of $\mathcal{T}$ if and only if all its labels at vertices lying between level 0 and level $n-1$ are trivial. Then we have the following alternative definition for $\operatorname{St}(n)$:

$$\operatorname{St}(n) = \{f \in \operatorname{Aut}\mathcal{T} : f_{(u)} = 1 \text{ for every } u \in \cup_{i=0}^{n-1} X^i\}.$$

Namely, the elements of $\operatorname{St}(n)$ are exactly the automorphisms with portrait trivial up to level $n-1$. In a complementary way, we define the subgroup of $\operatorname{Aut}\mathcal{T}$ given by

$$H_n = \{f \in \operatorname{Aut}\mathcal{T} : f_{(u)} = 1 \text{ for every } u \in \cup_{i \geq n} X^i\}.$$

We have, for instance, that H_1 is the set of all rooted automorphisms (defined in Example 1.3.4) and the map

$$\begin{aligned} H_1 &\longrightarrow S_d \\ f &\longmapsto f_{(\emptyset)} \end{aligned} \tag{1.7}$$

is an isomorphism (indeed it is a morphism by Proposition 1.3.5, it has kernel equal to $\{f \in H_1 : f_{(\emptyset)} = 1\} = 1$ and it is surjective because a permutation in S_d is the image of the rooted automorphism associated with such permutation). For any $n \geq 0$, H_n is a subgroup because the identity has only trivial labels and if $f, g \in H_n$, $u \in \cup_{i \geq n} X^i$ then $(u)f^{-1} \in \cup_{i \geq n} X^i$ by Proposition 1.2.2 and $(f^{-1}g)_{(u)} = (f_{((u)f^{-1})})^{-1} g_{((u)f^{-1})} = 1$ by Proposition 1.3.5 and Corollary 1.3.6.

As the following result states, $\operatorname{Aut}\mathcal{T}$ can be decomposed as the semidirect product of H_n and $\operatorname{St}(n)$, however we fix $n \geq 0$.

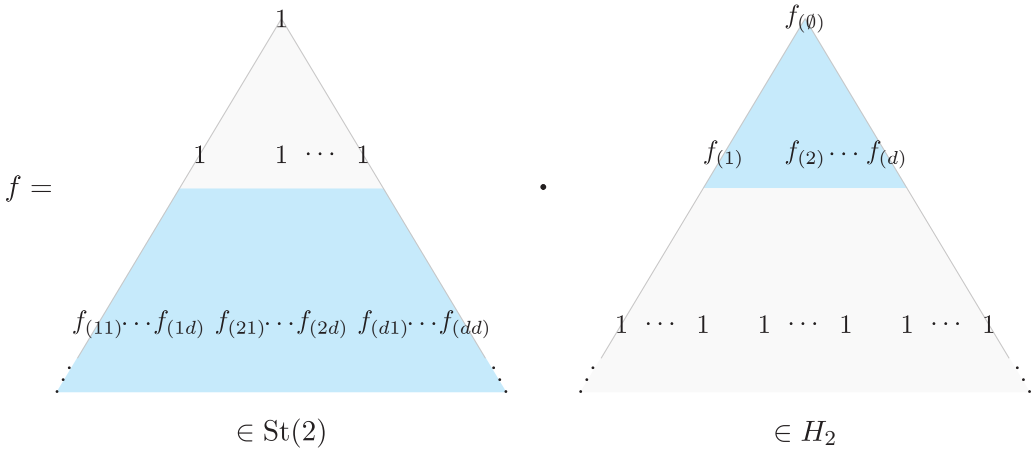


Figure 1.6: Decomposition of $\operatorname{Aut}\mathcal{T}$ as $\mathrm{St}(2) \rtimes H_2$.

Proposition 1.4.2. *For every $n \geq 0$*

$$\operatorname{Aut}\mathcal{T} = \mathrm{St}(n) \rtimes H_n.$$

As a consequence, $H_n \simeq \operatorname{Aut}\mathcal{T}/\mathrm{St}(n) \simeq \operatorname{Aut}\mathcal{T}_n$.

Proof. We already know that $\mathrm{St}(n) \trianglelefteq \operatorname{Aut}\mathcal{T}$ and $\mathrm{St}(n) \cap H_n = 1$ (indeed, by Proposition 1.3.3, the identity is the only automorphism that has all its labels equal to 1). Then we only have to show that $\operatorname{Aut}\mathcal{T} = \mathrm{St}(n) \cdot H_n$, i.e. every element of $\operatorname{Aut}\mathcal{T}$ can be written as the product of an element in $\mathrm{St}(n)$ and an element in H_n. Let then $f \in \operatorname{Aut}\mathcal{T}$. We define the automorphism g with portrait

$$g_{(u)} = \begin{cases} 1 & \text{if } u \in \cup_{i=0}^{n-1} X^i \\ f_{(u)} & \text{otherwise,} \end{cases}$$

the automorphism h with portrait

$$h_{(u)} = \begin{cases} f_{(u)} & \text{if } u \in \cup_{i=0}^{n-1} X^i \\ 1 & \text{otherwise} \end{cases}$$

and we claim that $f = gh$, as illustrated in Figure 1.6. This will conclude the proof, since $g \in \mathrm{St}(n)$ and $h \in H_n$.

By Proposition 1.3.5, $(gh)_{(u)} = g_{(u)}h_{((u)g)}$. If $u \in \cup_{i=0}^{n-1} X^i$, since $g \in \mathrm{St}(n)$, $g_{(u)} = 1$ and $(u)g = u$, which implies $(gh)_{(u)} = h_{(u)} = f_{(u)}$. If $u \in \cup_{i \geq n} X^i$, by Proposition 1.2.2 $(u)g \in \cup_{i \geq n} X^i$ and, since $h \in H_n$, $h_{((u)g)} = 1$. It follows that gh has the same portrait as f, and then they are equal by Proposition 1.3.3.

Now by the second isomorphism theorem

$$H_n \simeq \frac{H_n}{H_n \cap \mathrm{St}(n)} \simeq \frac{\mathrm{St}(n) \cdot H_n}{\mathrm{St}(n)} = \frac{\operatorname{Aut}\mathcal{T}}{\mathrm{St}(n)}$$

and $\operatorname{Aut}\mathcal{T}/\mathrm{St}(n) \simeq \operatorname{Aut}\mathcal{T}_n$ by Proposition 1.4.1. □

Since this section is devoted to the group $\mathrm{St}(n)$, this seems to be a good point to introduce some related notations. If $G \leq \mathrm{Aut}\,\mathcal{T}$, we set

$$\mathrm{St}_G(n) = \mathrm{St}(n) \cap G$$

and we observe that $\mathrm{St}_G(n) \trianglelefteq G$. Then we can consider the quotient

$$G_n = \frac{G}{\mathrm{St}_G(n)} \tag{1.8}$$

which is called the *nth congruence quotient* of G (notice that, according to (1.8), the nth congruence quotient of $\mathrm{Aut}\,\mathcal{T}$ will be denoted by the symbol $\mathrm{Aut}\,\mathcal{T}_n$, which already has a meaning for us, but this does not cause ambiguity because the nth congruence quotient of $\mathrm{Aut}\,\mathcal{T}$ is isomorphic to $\mathrm{Aut}\,\mathcal{T}_n$ by Proposition 1.4.1). We will say that an element $g \in G$ *has order t in G_n* if the element of G_n represented by g (i.e. the coset $g\,\mathrm{St}_G(n)$) has order t in G_n. Clearly, $\mathrm{St}_G(n) \trianglelefteq \mathrm{St}_G(k)$ whenever k is an integer less than or equal to n. In this case we will write

$$\mathrm{St}_{G_n}(k) = \frac{\mathrm{St}_G(k)}{\mathrm{St}_G(n)}.$$

Even though, thanks to Proposition 1.4.1, we have an explicit formula for the order of $\mathrm{Aut}\,\mathcal{T}/\,\mathrm{St}(n)$, sometimes computing the order of G_n can prove to be a hard task. We end this section with the following lemma, which ensures that, however we choose $G \leq \mathrm{Aut}\,\mathcal{T}$, the order of G_n divides the order of $\mathrm{Aut}\,\mathcal{T}/\,\mathrm{St}(n)$.

Lemma 1.4.3. *Let H, K be subgroups of $\mathrm{Aut}\,\mathcal{T}$ such that $H \leq K$. Then, for $n \geq 0$, the natural map*

$$\begin{aligned} H_n &\longrightarrow K_n \\ f\,\mathrm{St}_H(n) &\longmapsto f\,\mathrm{St}_K(n) \end{aligned}$$

is an injective group homomorphism. In particular, for every $G \leq \mathrm{Aut}\,\mathcal{T}$, G_n can be embedded in $\mathrm{Aut}\,\mathcal{T}/\,\mathrm{St}(n)$.

Proof. The map in the statement is well defined because, if $f\,\mathrm{St}_H(n) = g\,\mathrm{St}_H(n)$, then $f^{-1}g \in \mathrm{St}_H(n) \leq \mathrm{St}_K(n)$ and $f\,\mathrm{St}_K(n) = g\,\mathrm{St}_K(n)$. It is obviously a homomorphism and, if $f\,\mathrm{St}_H(n)$ belongs to its kernel, we have $f \in H \cap \mathrm{St}_K(n) = \mathrm{St}_H(n)$, that is, $f\,\mathrm{St}_H(n)$ is the identity element of H_n. □

1.5 Sections and self-similarity

One of the reasons why the d-adic rooted tree $\mathcal{T}$ is a very special graph is that, whenever we fix a vertex u, the set X_u of all descendants of u (i.e. vertices of the form uv with $v \in X^*$) gives rise again to a d-adic tree with root u. In the same spirit, if we have the portrait $\{f_{(v)} : v \in X^*\}$ of an automorphism f, we can focus on the labels at vertices lying in X_u to get the collection of permutations $\{f_{(uv)} : v \in X^*\}$ which, according to Proposition 1.3.3, defines a new automorphism of $\mathcal{T}$ (Figure 1.7 shows this phenomenon when u belongs to X^1). We give a name to such automorphism via the following definition.

Definition 1.5.1. Let $f \in \operatorname{Aut}\mathcal{T}$ and $u \in X^*$. We denote by f_u the automorphism with portrait
$$(f_u)_{(v)} = f_{(uv)} \qquad \text{for every } v \in X^*$$
and we say that f_u is the *section* of f at u.

It is useful to observe that, for every $f \in \operatorname{Aut}\mathcal{T}$ and every $u, v, w \in X^*$,
$$(f_{uv})_{(w)} = f_{(uvw)} = (f_u)_{(vw)} = ((f_u)_v)_{(w)}.$$
Then f_{uv} and $(f_u)_v$ have the same portrait and
$$f_{uv} = (f_u)_v.$$
Furthermore, if $f \in \operatorname{Aut}\mathcal{T}$ and $u = x_1 \dots x_i, v = y_1 \dots y_j \in X^*$, Equation (1.4) implies that
$$\begin{aligned}(uv)f &= (x_1)f_{(\emptyset)} \dots (x_i)f_{(x_1\dots x_{i-1})}\,(y_1)f_{(x_1\dots x_i)} \dots (y_j)f_{(x_1\dots x_i y_1\dots y_{j-1})}\\ &= (u)f\ (y_1)f_{(u)} \dots (y_j)f_{(uy_1\dots y_{j-1})}\\ &= (u)f\ (y_1)(f_u)_{(\emptyset)} \dots (y_j)(f_u)_{(y_1\dots y_{j-1})}\\ &= (u)f\ (v)f_u.\end{aligned} \tag{1.9}$$
This allows us to prove the next result, which is the analogous, for sections, of the property stated in Proposition 1.3.5 for labels.

Proposition 1.5.2. *Let $r \geq 2$, $f_1, \dots, f_r \in \operatorname{Aut}\mathcal{T}$ and $u \in X^*$. Then*
$$(f_1 \dots f_r)_u = (f_1)_u (f_2)_{(u)f_1} (f_3)_{(u)f_1f_2} \dots (f_r)_{(u)f_1\dots f_{r-1}}.$$

Proof. As in the proof of Proposition 1.3.5, it is enough to deal with the case $r = 2$ and the result will follow by induction. Namely, it suffices to show that, given $f, g \in \operatorname{Aut}\mathcal{T}$ and $u \in X^*$, we have $(fg)_u = f_u g_{(u)f}$. For every $v \in X^*$, the label of $(fg)_u$ at v is
$$((fg)_u)_{(v)} = (fg)_{(uv)} = f_{(uv)} g_{((uv)f)}$$
(where the last equality holds by Proposition 1.3.5) and then, using (1.9) and again Proposition 1.3.5, we get
$$\begin{aligned}((fg)_u)_{(v)} &= f_{(uv)} g_{((u)f\ (v)f_u)}\\ &= (f_u)_{(v)} (g_{(u)f})_{((v)f_u)}\\ &= (f_u g_{(u)f})_{(v)}.\end{aligned}$$
Thus $(fg)_u$ and $f_u g_{(u)f}$ have the same portrait. □

Corollary 1.5.3. *Let $f \in \operatorname{Aut}\mathcal{T}$ and $u \in X^*$. Then*
$$(f^{-1})_u = (f_{(u)f^{-1}})^{-1}.$$

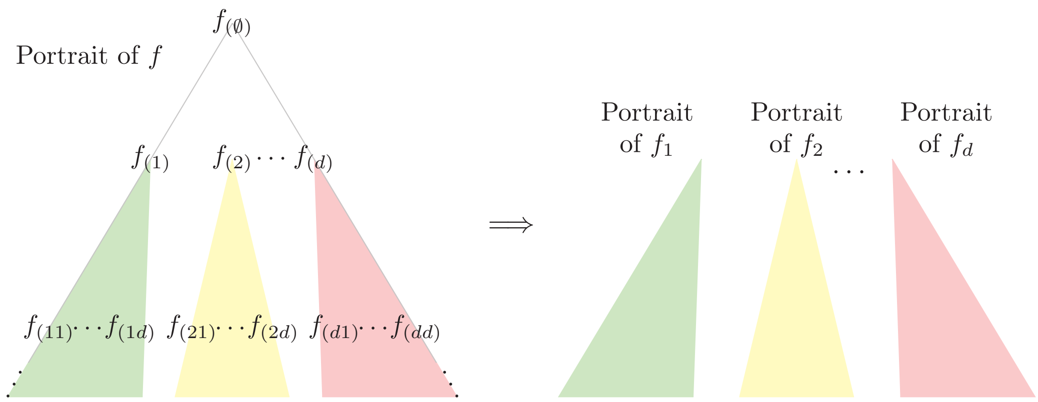


Figure 1.7: Portraits of sections at vertices in level 1.

Proof. The result follows from Proposition 1.5.2 just as Corollary 1.3.6 follows from Proposition 1.3.5. □

Roughly speaking, the portrait of an automorphism "contains" the portraits of other automorphisms, which are its sections. We will see that sometimes it is convenient to define an automorphism starting from portions of its portrait, i.e. starting by defining some of its sections. We know, for instance, from Section 1.4, that an automorphism in the first level stabilizer St(1) has trivial label at the root, and then one could imagine to define it by defining its sections at vertices lying in level 1 (see Figure 1.7). To translate this into mathematical terms, we introduce the map

$$\begin{aligned} \psi \ : \ \mathrm{St}(1) &\longrightarrow \mathrm{Aut}\,\mathcal{T} \times \overset{d}{\cdots} \times \mathrm{Aut}\,\mathcal{T} \\ f \ &\longmapsto \ (f_1, f_2 \ldots, f_d) \end{aligned}$$

which will be denoted with ψ throughout this thesis and is a group isomorphism. Indeed, if $f, g \in \mathrm{St}(1)$ and $u \in X^1 = \{1, \ldots, d\}$ then $(u)f = u$ and

$$(fg)_u = f_u g_{(u)f} = f_u g_u$$

by Proposition 1.5.2. Hence $(fg)\psi = (f_1 g_1, \ldots, f_d g_d)$ and ψ is a homomorphism. If $f \in \ker \psi$, we have $f_1 = \ldots = f_d = 1$ and

$$f_{(x_1 \ldots x_i)} = \begin{cases} 1 & \text{if } x_1 \ldots x_i = \emptyset \\ (f_{x_1})_{(x_2 \ldots x_i)} = 1 & \text{otherwise,} \end{cases}$$

which guarantees the injectivity of ψ. Finally ψ is surjective because, given arbitrarily $g^{(1)}, \ldots, g^{(d)} \in \mathrm{Aut}\,\mathcal{T}$, we can define the automorphism $g \in \mathrm{St}(1)$ given by

$$g_{(x_1 \ldots x_i)} = \begin{cases} 1 & \text{if } x_1 \ldots x_i = \emptyset \\ (g^{(x_1)})_{(x_2 \ldots x_i)} & \text{otherwise,} \end{cases}$$

which clearly satisfies $(g)\psi = (g^{(1)}, \dots, g^{(d)})$. The invertibility of ψ tells us that, given d automorphisms $g^{(1)}, \dots, g^{(d)} \in \operatorname{Aut}\mathcal{T}$, there is a unique automorphism in $\mathrm{St}(1)$ that has $g^{(1)}, \dots, g^{(d)}$ as sections at $1, \dots, d$, respectively. We will exploit this property in the following to define special automorphisms belonging to the first level stabilizer.

As already observed, the d-adic rooted tree contains d-adic rooted trees and the portrait of an automorphism contains portraits of other automorphisms. Our interest in dealing with objects that somehow "contain objects similar to themselves" leads to give the following definition.

Definition 1.5.4. Let G be a subgroup of $\operatorname{Aut}\mathcal{T}$. Then G is said to be *self-similar* if for every $f \in G$ and $u \in X^*$ we have that $f_u \in G$.

Thereby a self-similar group G contains all the sections of its elements, i.e. if f belongs to G then all the automorphisms that have portrait contained in the portrait of f still belong to G.

Example 1.5.5. Let K be a subgroup of the symmetric group S_d. Then we define

$$\mathrm{Lab}(K) = \{f \in \operatorname{Aut}\mathcal{T} : f_{(u)} \in K \text{ for every } u \in X^*\}.$$

In other words, $\mathrm{Lab}(K)$ is the set of all the automorphisms having all labels in K. We have that, for every $K \le S_d$, $\mathrm{Lab}(K)$ is a self-similar subgroup of $\operatorname{Aut}\mathcal{T}$. Indeed, the identity belongs to $\mathrm{Lab}(K)$ because its labels are all equal to 1. If $f, g \in \mathrm{Lab}(K)$, $fg \in \mathrm{Lab}(K)$ by Proposition 1.3.5 and $f^{-1} \in \mathrm{Lab}(K)$ by Corollary 1.3.6. Thus $\mathrm{Lab}(K)$ is a subgroup of $\operatorname{Aut}\mathcal{T}$. The self-similarity follows immediately from the definition of section and the definition of $\mathrm{Lab}(K)$.

We observe that $\mathrm{Lab}(S_d) = \operatorname{Aut}\mathcal{T}$ and $\mathrm{Lab}(1) = 1$. The case where K is the cyclic group generated by $(12 \dots d) \in S_d$ will turn out to be important in the next chapters. We then set

$$\Gamma = \mathrm{Lab}(\langle (12 \dots d) \rangle)$$

and we will always use this notation. We will also denote by σ the d-cycle $(12 \dots d) \in S_d$.

As we will see in the following, the groups $\mathrm{Lab}(K)$ are a very special kind of self-similar group. Determining the order and the exponent of G_n when G is any self-similar group, for example, is in general quite difficult, but in the case that $G = \mathrm{Lab}(K)$ for some $K \le S_d$ an elementary argument is enough to compute $|\mathrm{Lab}(K)_n|$. Indeed, the map β given by the composition

$$\begin{aligned} \mathrm{Lab}(K)_n &\longrightarrow \frac{\operatorname{Aut}\mathcal{T}}{\mathrm{St}(n)} \longrightarrow \operatorname{Aut}\mathcal{T}_n \\ f\,\mathrm{St}_{\mathrm{Lab}(K)}(n) &\longmapsto f\,\mathrm{St}(n) \longmapsto f_{\big| \bigcup_{i=0}^{n} X^i} \end{aligned}$$

is an injective homomorphism by Lemma 1.4.3 and Proposition 1.4.1. The one-to-one correspondence, highlighted in Section 1.3, between elements of $\operatorname{Aut}\mathcal{T}_n$ and collections of the form $\{\phi_{(u)} \in S_d\}_{u \in \bigcup_{i=0}^{n-1} X^i}$ induces then a one-to-one correspondence between elements of $(\mathrm{Lab}(K)_n)\beta$ and collections of the form $\{\phi_{(u)} \in K\}_{u\in\bigcup_{i=0}^{n-1} X^i}$. Therefore we get the formula

$$|\,\mathrm{Lab}(K)_n| = |(\mathrm{Lab}(K)_n)\beta| = |K|^{\frac{d^n-1}{d-1}}, \tag{1.10}$$

which generalizes (1.6).

In the even more special case that $G = \Gamma$ (i.e. $G = \mathrm{Lab}(\langle\sigma\rangle)$ with $\sigma = (12\ldots d)$) and $d = p$ is a prime number, we can easily compute the exponent of Γ_n.

Proposition 1.5.6. *Assume that $d = p$ is a prime number (i.e. we are working on the p-adic rooted tree). Then, for every $n \geq 1$, Γ_n is a Sylow p-subgroup of the symmetric group S_{p^n}. As a consequence,*

$$\exp(\Gamma_n) = p^n$$

for all $n \geq 1$.

Proof. By Lemma 1.4.3 and by (1.3) we know that Γ_n can be embedded in S_{p^n}. Moreover, (1.10) ensures that $|\Gamma_n| = |\langle\sigma\rangle|^{\frac{p^n-1}{p-1}} = p^{\frac{p^n-1}{p-1}}$. Then, to prove that Γ_n is a Sylow p-subgroup of S_{p^n}, it is enough to show that $|S_{p^n}| = p^{\frac{p^n-1}{p-1}} \cdot m$, with m a natural number not divisible by p. In other words, we want to show that $p^{\frac{p^n-1}{p-1}}$ is the highest power of p which divides $|S_{p^n}| = (p^n)!$. If for any natural number x we indicate with x_p the highest power of p which divides x (and we refer to x_p as the p-part of x), we get

$$|S_{p^n}|_p = ((p^n)!)_p = \prod_{i=1}^{p^n} i_p.$$

Now, among the natural numbers $\{1, 2, \ldots, p^n\}$ there are exactly

$$\begin{array}{cl} p^{n-1} & \text{multiples of } p \\ p^{n-2} & \text{multiples of } p^2 \\ \vdots & \\ p & \text{multiples of } p^{n-1} \\ 1 & \text{multiple of } p^n. \end{array}$$

Then among $\{1, 2, \ldots, p^n\}$ there are

$$\begin{array}{cl} p^{n-1} - p^{n-2} & \text{numbers with } p\text{-part } p \\ p^{n-2} - p^{n-3} & \text{numbers with } p\text{-part } p^2 \\ \vdots & \\ p-1 & \text{numbers with } p\text{-part } p^{n-1} \\ 1 & \text{number with } p\text{-part } p^n. \end{array}$$

It follows that

$$\begin{aligned} |S_{p^n}|_p &= (p)^{(p^{n-1}-p^{n-2})}(p^2)^{(p^{n-2}-p^{n-3})}\dots(p^{n-1})^{(p-1)}(p^n)^{(1)} \\ &= p^{(p^{n-1}-p^{n-2}+2p^{n-2}-2p^{n-3}+\dots+(n-1)p-(n-1)1+n\cdot 1)} \\ &= p^{(p^{n-1}+p^{n-2}+\dots+p+1)} \\ &= p^{\frac{p^n-1}{p-1}}. \end{aligned}$$

Therefore Γ_n is a Sylow p-subgroup of S_{p^n}.

Now, we recall that any permutation in S_{p^n} can be written as a product of disjoint cycles of length $\le p^n$, and its order is the least common multiple between the lengths of such cycles. Since Γ_n is a p-group, its elements are products of cycles whose length must belong to $\{1, p, p^2, \dots, p^n\}$, and then the exponent of Γ_n divides p^n.

On the other hand, we have that the permutation $\rho = (12\dots p^n) \in S_{p^n}$ is a cycle of length p^n and then it has order p^n. By the Sylow theorems, we know that ρ must belong to some Sylow p-subgroup P of S_{p^n}. Moreover, since all Sylow p-subgroups of S_{p^n} are conjugate, there is $g \in S_{p^n}$ such that $\Gamma_n = P^g$. Thus $\rho^g \in \Gamma_n$ and ρ^g is still a cycle of length p^n, which implies that p^n divides the exponent of Γ_n. □

It is clear that, if G is self-similar, $(\mathrm{St}_G(1))\psi \subseteq G \times \overset{d}{\cdots} \times G$. In fact, whenever $f \in G$ (even in the case that $f \notin \mathrm{St}(1)$) the sections $f_1, \dots, f_d$ belong to G. The following lemma ensures that this condition is sufficient to have self-similarity of G.

Lemma 1.5.7. *Let G be a subgroup of $\operatorname{Aut}\mathcal{T}$ and let S be a generating set for G. Assume that for every $f \in S$ and every $u \in X^1 = \{1, \dots, d\}$ we have $f_u \in G$. Then G is self-similar.*

Proof. We set $S^{-1} = \{g^{-1} : g \in S\}$ and we observe that, by Corollary 1.5.3 and by our assumption,

$$g_u \in G \qquad \text{for every } g \in S \cup S^{-1} \text{ and every } u \in X^1. \tag{1.11}$$

We want to show, by induction on i, that $f_v \in G$ for any $f \in G$ and $v \in X^i$. If $i = 0$, the only vertex in X^0 is the root and, for every $f \in G$, $f_\emptyset = f \in G$. Assume then $i = 1$ and let $f \in G$, $v \in X^1$. Since $G = \langle S \rangle$, $f = f^{(1)} \dots f^{(k)}$ for some $f^{(1)}, \dots, f^{(k)} \in S \cup S^{-1}$. We have $v \in X^1$ and $f_v = (f^{(1)})_v \dots (f^{(k)})_{(v)f^{(1)}\dots f^{(k-1)}}$ belongs to G due to (1.11). Finally, if $i \ge 2$, f is any element of G and $v = x_1 \dots x_i$ any vertex in X^i, $f_{x_1} \in G$ by what we have just proved and $f_v = (f_{x_1})_{x_2 \dots x_i}$ belongs to G by induction hypothesis. □

Since we will mainly work with subgroups of $\operatorname{Aut}\mathcal{T}$ that are self-similar, it is now convenient to present some of their properties.

Lemma 1.5.8. *Let G be a self-similar subgroup of $\operatorname{Aut}\mathcal{T}$. Then, for any natural numbers k, n such that $k \le n$, the map*

$$\begin{aligned} \mathrm{St}_{G_n}(k) &\longrightarrow G_{n-k} \times \overset{d^k}{\cdots} \times G_{n-k} \\ f\,\mathrm{St}_G(n) &\longmapsto \big(f_u\,\mathrm{St}_G(n-k)\big)_{u \in X^k} \end{aligned}$$

is an injective group homomorphism.

Proof. We first show that the map in the statement, which in the course of this proof will be denoted by ϕ, is well defined. If $f \operatorname{St}_G(n)$ is an element of $\operatorname{St}_{G_n}(k)$ then $f \in G$ and, since G is self-similar, $f_u \in G$ for any $u \in X^k$. Thus $f_u \operatorname{St}_G(n-k) \in G_{n-k}$ for any $u \in X^k$ and the image of ϕ is indeed contained in $G_{n-k} \times \overset{d^k}{\cdots} \times G_{n-k}$. To show that ϕ is well defined we also need to check that, given two elements $f \operatorname{St}_G(n)$ and $g \operatorname{St}_G(n)$ which are equal in $\operatorname{St}_{G_n}(k)$, the two tuples $\big(f_u \operatorname{St}_G(n-k)\big)_{u\in X^k}$ and $\big(g_u \operatorname{St}_G(n-k)\big)_{u\in X^k}$ are the same element of $G_{n-k} \times \overset{d^k}{\cdots} \times G_{n-k}$. The equality $f \operatorname{St}_G(n) = g \operatorname{St}_G(n)$ yields that $f = hg$ for some $h \in \operatorname{St}_G(n)$. Since $h \in \operatorname{St}_G(n)$, the labels of h at vertices in $\cup_{i=0}^{n-1} X^i$ are trivial. Hence, by definition of section, for every $u \in X^k$ the labels of h_u at vertices in $\cup_{i=0}^{n-k-1} X^i$ are trivial. It follows that $h_u \in \operatorname{St}_G(n-k)$ whenever $u \in X^k$. Furthermore, if $u \in X^k$, $(u)h = u$ because $h \in \operatorname{St}_G(n)$ and we have

$$f_u = (hg)_u = h_u g_{(u)h} = h_u g_u.$$

This, together with the fact that $h_u \in \operatorname{St}_G(n-k)$, yields that $f_u \operatorname{St}_G(n-k) = g_u \operatorname{St}_G(n-k)$ for every $u \in X^k$.

Now, ϕ is a morphism because for any $f \operatorname{St}_G(n), g \operatorname{St}_G(n) \in \operatorname{St}_{G_n}(k)$ we have $f \in \operatorname{St}_G(k)$ and then $(fg)_u = f_u g_{(u)f} = f_u g_u$ for all $u \in X^k$.

The kernel of ϕ is trivial because, if $f \operatorname{St}_G(n) \in \operatorname{St}_{G_n}(k)$ and $f_u \in \operatorname{St}_G(n-k)$ for all $u \in X^k$, then the label of f at a vertex $v = x_1 \dots x_j \in \cup_{i=0}^{n-1} X^i$ is

$$f_{(v)} = \begin{cases} 1 & \text{if } v \in \cup_{i=0}^{k-1} X^i \text{ (because } f \in \operatorname{St}_G(k)) \\ (f_{x_1 \dots x_k})_{(x_{k+1}\dots x_j)} = 1 & \text{if } v \in \cup_{i=k}^{n-1} X^i \text{ (because } f_{x_1\dots x_k} \in \operatorname{St}_G(n-k)). \end{cases}$$

Thus $f \in \operatorname{St}_G(n)$ and $\ker \phi = 1$. □

We observe that, if $0 \lneqq k \lneqq n$, the injective morphism given by Lemma 1.5.8 may not be surjective (notice that the morphism is trivially surjective when $k = 0$, because it is the identity, and when $k = n$, because $G_{n-k} = G_0 = G/\operatorname{St}_G(0) = 1$). If, for instance, $G = \langle a \rangle$ (where a is the automorphism defined in Example 1.3.4), we have that G is self-similar by Lemma 1.5.7 (because $a_u = 1$ whenever $u \in X^1$) but, provided $0 \lneqq k \lneqq n$, the (injective) morphism

$$\begin{aligned} \operatorname{St}_{G_n}(k) &\longrightarrow G_{n-k} \times \overset{d^k}{\cdots} \times G_{n-k} \\ f \operatorname{St}_G(n) &\longmapsto \big(f_u \operatorname{St}_G(n-k)\big)_{u \in X^k} \end{aligned}$$

is not surjective. Indeed, since $a \in H_1$, $G = \langle a \rangle \le H_1$ and $\operatorname{St}_G(k) \le \operatorname{St}_G(1) = 1$ (because $k \ge 1$). Hence $\operatorname{St}_{G_n}(k) = 1$ and the image of our morphism is 1, but $G_{n-k} \times \overset{d^k}{\cdots} \times G_{n-k} \neq 1$ (because $n - k \ge 1$ and then $a \operatorname{St}_G(n-k)$ is a non trivial element of G_{n-k}).

Nevertheless, in the special case that $G = \operatorname{Lab}(K)$ for some $K \le S_d$, the surjectivity is guaranteed by the following lemma.

Lemma 1.5.9. *Let K be a subgroup of S_d and set $G = \operatorname{Lab}(K)$. Then, for any natural numbers k, n such that $k \leq n$, the map*

$$\begin{aligned} \operatorname{St}_{G_n}(k) &\longrightarrow G_{n-k} \times \overset{d^k}{\cdots} \times G_{n-k} \\ f \operatorname{St}_G(n) &\longmapsto \big(f_u \operatorname{St}_G(n-k)\big)_{u \in X^k} \end{aligned}$$

is a group isomorphism. In particular, this holds when $G = \operatorname{Lab}(S_d) = \operatorname{Aut} \mathcal{T}$. Thus

$$\begin{aligned} \operatorname{St}_{\operatorname{Aut} \mathcal{T}_n}(k) &\longrightarrow \operatorname{Aut} \mathcal{T}_{n-k} \times \overset{d^k}{\cdots} \times \operatorname{Aut} \mathcal{T}_{n-k} \\ f \operatorname{St}(n) &\longmapsto \big(f_u \operatorname{St}(n-k)\big)_{u \in X^k} \end{aligned}$$

is a group isomorphism.

Proof. The map in the statement is an injective homomorphism by Lemma 1.5.8. If $\big(f^{(u)} \operatorname{St}_G(n-k)\big)_{u \in X^k}$ is an element of $G_{n-k} \times \overset{d^k}{\cdots} \times G_{n-k}$, we define the automorphism g whose portrait is

$$g_{(x_1 \ldots x_i)} = \begin{cases} 1 & \text{if } i \leq k-1 \\ (f^{(x_1 \ldots x_k)})_{(x_{k+1} \ldots x_i)} & \text{if } i \geq k. \end{cases}$$

Then $g \in G = \operatorname{Lab}(K)$ since its labels are either 1 or labels of some $f^{(u)} \in G = \operatorname{Lab}(K)$. Moreover $g \in \operatorname{St}_G(k)$ by its definition and our morphism sends $g \operatorname{St}_G(n)$ to the tuple $\big(f^{(u)} \operatorname{St}_G(n-k)\big)_{u \in X^k}$. □

As mentioned above, it can be challenging to understand properties of the finite group G_n, such as its order and its exponent, as n varies. When G is self-similar, the injective homomorphism given by Lemma 1.5.8 can be a good tool to deal with such a problem. Indeed, it relates a subgroup of G_n with a cartesian product of copies of G_{n-k}, for any $k \leq n$, and sometimes this allows to reduce the problem of studying G_n for all n to the problem of studying G_n for small n.

The following two consequences of Lemma 1.5.8 go in that direction, giving examples of properties of G_n that can be deduced from properties of $G_1, \ldots, G_{n-1}$.

Lemma 1.5.10. *Let G be a self-similar subgroup of $\operatorname{Aut} \mathcal{T}$ and let g be an element of $\operatorname{St}_G(1)$. If a component of $(g)\psi$ has order t in G_n, then the order of g in G_{n+1} is a multiple of t.*

Proof. By Lemma 1.5.8, the map

$$\begin{aligned} \operatorname{St}_{G_{n+1}}(1) &\longrightarrow G_n \times \overset{d}{\cdots} \times G_n \\ f \operatorname{St}_G(n+1) &\longmapsto \big(f_u \operatorname{St}_G(n)\big)_{u \in X^1} = (f_1 \operatorname{St}_G(n), \ldots, f_d \operatorname{St}_G(n)) \end{aligned}$$

is an injective group homomorphism. Then the order of $g \operatorname{St}_G(n+1)$ in $\operatorname{St}_{G_{n+1}}(1)$ (which is the order of g in G_{n+1}) is equal to the order of $(g_1 \operatorname{St}_G(n), \ldots, g_d \operatorname{St}_G(n))$ in $G_n \times \overset{d}{\cdots} \times G_n$, i.e. it is equal to the least common multiple of the orders of $g_1, \ldots, g_d$ in G_n. This proves the claim, since $g_1, \ldots, g_d$ are the components of $(g)\psi$. □

Lemma 1.5.11. *Let $G \leq \operatorname{Aut}\mathcal{T}$ be a self-similar group. Then*

$$\exp(G_{n+k}) \leq \exp(G_n) \cdot \exp(G_k)$$

for every $n, k \geq 0$.

Proof. Let us observe that for any $n, k \geq 0$

$$G_n = \frac{G}{\mathrm{St}_G(n)} \simeq \frac{\frac{G}{\mathrm{St}_G(n+k)}}{\frac{\mathrm{St}_G(n)}{\mathrm{St}_G(n+k)}} = \frac{G_{n+k}}{\mathrm{St}_{G_{n+k}}(n)}$$

which yields that $\exp(G_{n+k})$ divides $\exp(G_n) \cdot \exp(\mathrm{St}_{G_{n+k}}(n))$ (indeed, whenever A, B, C are groups such that $A \simeq B/C$, for any $b \in B$ we have $b^{\exp(A)} \in C$, hence $b^{\exp(A)\exp(C)} = 1$). Then it suffices to show that $\exp(\mathrm{St}_{G_{n+k}}(n))$ divides $\exp(G_k)$. This follows from the fact that $\mathrm{St}_{G_{n+k}}(n)$ can be embedded in $G_k \times \overset{p^n}{\cdots} \times G_k$ by Lemma 1.5.8 (and clearly $\exp(G_k \times \overset{p^n}{\cdots} \times G_k) = \exp(G_k)$). □

1.6 Fractal groups

When G is a self-similar subgroup of $\operatorname{Aut}\mathcal{T}$, $f \in G$ and $u \in X^*$, the section of f at u belongs to G. So one could wonder whether we can go in the opposite direction, i.e. whether, given $f \in G$ and $u \in X^*$, it is possible to write f as the section at u of some automorphism which belongs to G. This idea is the starting point to define an important subclass of self-similar subgroups of $\operatorname{Aut}\mathcal{T}$, whose elements take the name of fractal groups.

Definition 1.6.1. Let G be a subgroup of $\operatorname{Aut}\mathcal{T}$ and $u \in X^*$. The *stabilizer of u in G* is the subgroup of G defined by

$$\mathrm{St}_G(u) = \{f \in G : (u)f = u\}.$$

Definition 1.6.2. Let G be a self-similar subgroup of $\operatorname{Aut}\mathcal{T}$. We say that G is *fractal* if, whenever $f \in G$ and $u \in X^*$, there exists $g \in \mathrm{St}_G(u)$ such that $g_u = f$.

Let us observe that, for every $K \leq S_d$, the group $\mathrm{Lab}(K)$ is fractal. Indeed, given $f \in \mathrm{Lab}(K)$ and $u \in X^*$, the automorphism g with portrait

$$g_{(v)} = \begin{cases} f_{(w)} & \text{if } v = uw \text{ for some } w \in X^* \\ 1 & \text{otherwise} \end{cases}$$

lies in $\mathrm{Lab}(K)$ as f lies in $\mathrm{Lab}(K)$. Moreover $g_u = f$ and, if $u \in X^n$, then $g \in \mathrm{St}_{\mathrm{Lab}(K)}(n) \leq \mathrm{St}_{\mathrm{Lab}(K)}(u)$.

Not every self-similar subgroup of $\operatorname{Aut}\mathcal{T}$ is fractal. Take for example $G = \langle a \rangle$, with a denoting, as usual, the rooted automorphism associated to $\sigma = (12\ldots d)$. As remarked in Section 1.5, G is self-similar and $G \leq H_1$. Therefore $g_1 = 1$ whenever

$g \in G$, and there is no element of G whose section at 1 can be equal to a. Other more complicated examples of self-similar groups that are not fractal will be provided in Chapter 2.

The next lemma shows that, to check that a group G is fractal, we can reduce to checking that the condition in Definition 1.6.2 holds for a set of generators of G and for vertices in X^1. This is the analogous, for fractal groups, of what is stated in Lemma 1.5.7 for self-similar groups.

Lemma 1.6.3. *Let G be a subgroup of $\operatorname{Aut}\mathcal{T}$ and let S be a generating set for G. Assume that for every $f \in S$ and every $u \in X^1 = \{1, \dots, d\}$ there exists $g \in \mathrm{St}_G(u)$ such that $g_u = f$. Then G is fractal.*

Proof. We set $S^{-1} = \{g^{-1} : g \in S\}$ and we observe that, if $f \in S^{-1}$ and $u \in X^1$, there is $g \in \mathrm{St}_G(u)$ with $g_u = f^{-1}$ as $f^{-1} \in S$. We have $g^{-1} \in \mathrm{St}_G(u)$ and, by Corollary 1.5.3, $(g^{-1})_u = (g_{(u)g^{-1}})^{-1} = (g_u)^{-1} = f$. Hence

$$\text{for every } f \in S \cup S^{-1} \text{ and every } u \in X^1, \text{ there is } g \in \mathrm{St}_G(u) \text{ such that } g_u = f. \tag{1.12}$$

We want to show, by induction on i, that, for every $f \in G$ and $v \in X^i$, there exists $g \in \mathrm{St}_G(v)$ with $g_v = f$. If $i = 0$, the only vertex in X^0 is the root and, for every $f \in G$, $f_\emptyset = f \in G = \mathrm{St}_G(\emptyset)$. Assume then $i = 1$ and let $f \in G$, $v \in X^1$. Since $G = \langle S \rangle$, $f = f^{(1)} \dots f^{(k)}$ for some $f^{(1)}, \dots, f^{(k)} \in S \cup S^{-1}$. By (1.12), there exist $g^{(1)}, \dots, g^{(k)} \in \mathrm{St}_G(v)$ satisfying $(g^{(j)})_v = f^{(j)}$ for all $j = 1, \dots, k$, and

$$(g^{(1)} \dots g^{(k)})_v = (g^{(1)})_v \dots (g^{(k)})_{(v)g^{(1)}\dots g^{(k-1)}} = (g^{(1)})_v \dots (g^{(k)})_v = f^{(1)} \dots f^{(k)} = f.$$

Finally, if $i \geq 2$, f is any element of G and $v = x_1 \dots x_i$ any vertex in X^i, by what we just proved there is $g \in \mathrm{St}_G(x_i)$ such that $g_{x_i} = f$ and, by induction hypothesis, there is $h \in \mathrm{St}_G(x_1 \dots x_{i-1})$ such that $h_{x_1 \dots x_{i-1}} = g$. It follows that

$$h_v = (h_{x_1 \dots x_{i-1}})_{x_i} = g_{x_i} = f.$$

It is then enough to check that $h \in \mathrm{St}_G(v)$. This follows from the fact that $h \in \mathrm{St}_G(x_1 \dots x_{i-1})$, $g \in \mathrm{St}_G(x_i)$ and by (1.9)

$$(v)h = (x_1 \dots x_{i-1})h \, (x_i)h_{x_1 \dots x_{i-1}} = x_1 \dots x_{i-1} \, (x_i)g = v.$$

□

1.7 Wreath products inside $\operatorname{Aut}\mathcal{T}_n$

We recall that the wreath product between a group K and a group $H \leq S_m$ is the group

$$K \wr H = (K \times \overset{m}{\cdots} \times K) \rtimes H$$

where the semidirect product is defined by the following action of H on $K \times \overset{m}{\cdots} \times K$:

$$(k_1, k_2, \dots, k_m)^h = (k_{(1)h^{-1}}, k_{(2)h^{-1}}, \dots, k_{(m)h^{-1}}) \tag{1.13}$$

for every $k_1, k_2, \ldots, k_m \in K, h \in H$. In other words h, which is a permutation in the symmetric group S_m, acts on the tuple $(k_1, k_2, \ldots, k_m)$ by permuting its components. To understand the link between wreath products and the group of automorphisms of the d-adic rooted tree, let us start with an example. Let a be the automorphism defined in Example 1.3.4, namely, the rooted automorphism associated to $\sigma = (12\ldots d) \in S_d$. If $f \in \mathrm{St}(1)$ and $x \in X^1 = \{1, \ldots, d\}$, we have

$$(f^a)_x = (a^{-1}fa)_x = (a^{-1})_x\, f_{(x)a^{-1}}\, a_{(x)a^{-1}f}$$

by Proposition 1.5.2. Since $a \in H_1$ (and $a^{-1} \in H_1$), the sections of a (and the ones of a^{-1}) at vertices lying in level 1 are trivial. Hence

$$(f^a)_x = f_{(x)a^{-1}} = f_{(x)\sigma^{-1}},$$

where the last equality holds because $(x)a^{-1} = (x)(a^{-1})_{(\emptyset)} = (x)\sigma^{-1}$ by Corollary 1.3.6. It follows that

$$(f^a)\psi = (f_{(1)\sigma^{-1}}, f_{(2)\sigma^{-1}}, \ldots, f_{(d)\sigma^{-1}}) = (f_d, f_1, \ldots, f_{d-1}) \tag{1.14}$$

(notice that, since $\mathrm{St}(1) \trianglelefteq \mathrm{Aut}\,\mathcal{T}$, $f^a \in \mathrm{St}(1)$ and then it makes sense to write $(f^a)\psi$). Therefore, if we identify elements of $\mathrm{St}(1)$ with tuples in $\mathrm{Aut}\,\mathcal{T} \times \overset{d}{\cdots} \times \mathrm{Aut}\,\mathcal{T}$ via the isomorphism ψ, the action of a on f by conjugation corresponds to the action of the permutation σ on $(f_1, f_2, \ldots, f_d)$ considered in (1.13). The same argument works if we replace a by any rooted automorphism, i.e. by any element of H_1.

If we look at this phenomenon modulo $\mathrm{St}(2)$ (i.e. we consider the action of $a\,\mathrm{St}(2)$ on $f\,\mathrm{St}(2)$ by conjugation), since elements of $\mathrm{St}_{\mathrm{Aut}\,\mathcal{T}_2}(1)$ can be seen as tuples in $\mathrm{Aut}\,\mathcal{T}_1 \times \overset{d}{\cdots} \times \mathrm{Aut}\,\mathcal{T}_1$ by Lemma 1.5.9, we realise that the action of $a\,\mathrm{St}(2)$ turns the tuple $(f_1\,\mathrm{St}(1), f_2\,\mathrm{St}(1) \ldots, f_d\,\mathrm{St}(1))$ into the tuple

$$(f_{(1)\sigma^{-1}}\,\mathrm{St}(1), f_{(2)\sigma^{-1}}\,\mathrm{St}(1), \ldots, f_{(d)\sigma^{-1}}\,\mathrm{St}(1)) = (f_d\,\mathrm{St}(1), f_1\,\mathrm{St}(1), \ldots, f_{d-1}\,\mathrm{St}(1)).$$

Then we will not be surprised to find some kind of wreath products inside $\mathrm{Aut}\,\mathcal{T}_2$, and in fact it will turn out that

$$\mathrm{Aut}\,\mathcal{T}_2 \simeq \mathrm{St}_{\mathrm{Aut}\,\mathcal{T}_2}(1) \rtimes H_1 \simeq \mathrm{Aut}\,\mathcal{T}_1 \wr H_1 \simeq S_d \wr S_d$$

(notice that $H_1 \simeq \mathrm{Aut}\,\mathcal{T}_1$ by Proposition 1.4.2 and $H_1 \simeq S_d$ by (1.7)). Since $\mathrm{Aut}\,\mathcal{T}_2 \leq S_{d^2}$ by (1.3) and $\mathrm{St}_{\mathrm{Aut}\,\mathcal{T}_3}(2)$ is isomorphic to $\mathrm{Aut}\,\mathcal{T}_1 \times \overset{d^2}{\cdots} \times \mathrm{Aut}\,\mathcal{T}_1$ by Lemma 1.5.9, we can define the wreath product $\mathrm{St}_{\mathrm{Aut}\,\mathcal{T}_3}(2) \rtimes \mathrm{Aut}\,\mathcal{T}_2 \simeq \mathrm{Aut}\,\mathcal{T}_1 \wr \mathrm{Aut}\,\mathcal{T}_2 \simeq S_d \wr (S_d \wr S_d)$ which can be proved to be isomorphic to $\mathrm{Aut}\,\mathcal{T}_3$. Iterating this reasoning, we get that $\mathrm{Aut}\,\mathcal{T}_n$ is isomorphic to the iterated wreath product $S_d \wr \overset{n}{\cdots} \wr S_d$ (see the statement of Proposition 1.7.1 for a formal definition of that) for every $n \geq 1$, as the next result guarantees.

Proposition 1.7.1. *Let K be a subgroup of S_d. Then, for every $n \geq 1$, $\mathrm{Lab}(K)_n \leq S_{d^n}$ and*

$$\mathrm{Lab}(K)_n \simeq K \wr \overset{n}{\cdots} \wr K, \tag{1.15}$$

where the iterated wreath product $K \wr \overset{n}{\cdots} \wr K$ *is recursively defined as follows. We put* $K \wr \overset{1}{\cdots} \wr K = K$ *and, once we defined* $K \wr \overset{i}{\cdots} \wr K$ *for all* $i \in \{1, \dots, n-1\}$ *and we proved that* $K \wr \overset{i}{\cdots} \wr K \simeq \mathrm{Lab}(K)_i$*, we set*

$$K \wr \overset{n}{\cdots} \wr K = K \wr (K \wr \overset{n-1}{\cdots} \wr K)$$

(where the wreath product $K \wr (K \wr \overset{n-1}{\cdots} \wr K)$ *is defined by the action of* $K \wr \overset{n-1}{\cdots} \wr K \simeq \mathrm{Lab}(K)_{n-1} \leq S_{d^{n-1}}$ *on* $K \times \overset{d^{n-1}}{\cdots} \times K$ *as in (1.13)).*

In particular, with the notation of Example 1.5.5, we have

$$\begin{aligned} \mathrm{Aut}\,\mathcal{T}_n &\simeq S_d \wr \overset{n}{\cdots} \wr S_d \\ \Gamma_n &\simeq \langle \sigma \rangle \wr \overset{n}{\cdots} \wr \langle \sigma \rangle \simeq C_d \wr \overset{n}{\cdots} \wr C_d \end{aligned}$$

where σ *is the d-cycle* $(12\dots d) \in S_d$*.*

Proof. First, we observe that $\mathrm{Lab}(K) \leq \mathrm{Aut}\,\mathcal{T}$ and, by Lemma 1.4.3, $\mathrm{Lab}(K)_n \leq \mathrm{Aut}\,\mathcal{T}_n \leq S_{d^n}$.

Let us notice that an arbitrary element of $K \wr \overset{n}{\cdots} \wr K$ has form

$$(x_{(u)})_{u \in X^{n-1}} (x_{(u)})_{u \in X^{n-2}} \dots (x_{(1)}, \dots, x_{(d)}) x_{(\emptyset)}$$

with $x_{(u)} \in K$ for all $u \in \cup_{i=0}^{n-1} X^i$. Indeed this holds trivially for $n = 1$ and, if by induction hypothesis the elements of $K \wr \overset{n-1}{\cdots} \wr K$ have form

$$(x_{(u)})_{u \in X^{n-2}} \dots (x_{(1)}, \dots, x_{(d)}) x_{(\emptyset)}$$

with $x_{(u)} \in K$ for all $u \in \cup_{i=0}^{n-2} X^i$, the shape of the elements in $K \wr \overset{n}{\cdots} \wr K = K \wr (K \wr \overset{n-1}{\cdots} \wr K) = (K \times \overset{d^{n-1}}{\cdots} \times K) \rtimes (K \wr \overset{n-1}{\cdots} \wr K)$ will be as claimed above. Moreover it can be proved, again by induction on n, that two elements

$$(x_{(u)})_{u \in X^{n-1}} (x_{(u)})_{u \in X^{n-2}} \dots (x_{(1)}, \dots, x_{(d)}) x_{(\emptyset)}$$

and

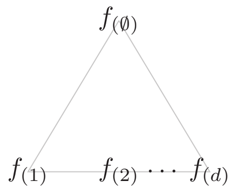
$$(y_{(u)})_{u \in X^{n-1}} (y_{(u)})_{u \in X^{n-2}} \dots (y_{(1)}, \dots, y_{(d)}) y_{(\emptyset)}$$

are equal in $K \wr \overset{n}{\cdots} \wr K$ if and only if $x_{(u)} = y_{(u)}$ for all $u \in \cup_{i=0}^{n-1} X^i$. Now, to show (1.15), we prove that the map

$$\begin{aligned} \mathrm{Lab}_n(K) &\longrightarrow K \wr \overset{n}{\cdots} \wr K \\ f\,\mathrm{St}_{\mathrm{Lab}(K)}(n) &\longmapsto (f_{(u)})_{u \in X^{n-1}} (f_{(u)})_{u \in X^{n-2}} \dots (f_{(1)}, \dots, f_{(d)}) f_{(\emptyset)} \end{aligned}$$

is an isomorphism for every $n \geq 1$. It is well defined because $f\,\mathrm{St}_{\mathrm{Lab}(K)}(n) = g\,\mathrm{St}_{\mathrm{Lab}(K)}(n)$ implies $f^{-1}g \in \mathrm{St}_{\mathrm{Lab}(K)}(n)$ and $f_{|\cup_{i=0}^n X^i} = g_{|\cup_{i=0}^n X^i}$. Hence, by (1.4), $f_{(u)} = g_{(u)}$ for all $u \in \cup_{i=0}^{n-1} X^i$.

$$\mathrm{Lab}(K)_2 \quad \xrightarrow{\sim} \quad K \wr K = (K \times \overset{d}{\cdots} \times K) \rtimes K$$



$$\longmapsto \quad (f_{(1)}, \dots, f_{(d)}) f_{(\emptyset)}$$

Figure 1.8: Isomorphism between $\mathrm{Lab}(K)_2$ and $K \wr K$.

We prove, by induction on $n \geq 1$, that the image of $fg\,\mathrm{St}_{\mathrm{Lab}(K)}(n)$ is equal to the product between the image of $f\,\mathrm{St}_{\mathrm{Lab}(K)}(n)$ and the image of $g\,\mathrm{St}_{\mathrm{Lab}(K)}(n)$. For $n = 1$, this is true because $(fg)_{(\emptyset)} = f_{(\emptyset)} g_{(\emptyset)}$ by Proposition 1.3.5. Assume $n \geq 2$. We want to show that, if we set

$$\begin{aligned}
F &= (f_{(u)})_{u \in X^{n-2}} \dots (f_{(1)}, \dots, f_{(d)}) f_{(\emptyset)} \\
G &= (g_{(u)})_{u \in X^{n-2}} \dots (g_{(1)}, \dots, g_{(d)}) g_{(\emptyset)} \\
H &= ((fg)_{(u)})_{u \in X^{n-2}} \dots ((fg)_{(1)}, \dots, (fg)_{(d)}) (fg)_{(\emptyset)}
\end{aligned}$$

we have

$$((fg)_{(u)})_{u \in X^{n-1}} H = (f_{(u)})_{u \in X^{n-1}} F \cdot (g_{(u)})_{u \in X^{n-1}} G.$$

These are elements in $K \wr \overset{n}{\cdots} \wr K = (K \times \overset{d^{n-1}}{\cdots} \times K) \rtimes (K \wr \overset{n-1}{\cdots} \wr K)$, $F, G, H \in K \wr \overset{n-1}{\cdots} \wr K$ and $H = FG$ by induction hypothesis. Thus

$$\begin{aligned}
(f_{(u)})_{u \in X^{n-1}} F \cdot (g_{(u)})_{u \in X^{n-1}} G &= (f_{(u)})_{u \in X^{n-1}} ((g_{(u)})_{u \in X^{n-1}})^{F^{-1}} FG \\
&= (f_{(u)})_{u \in X^{n-1}} ((g_{(u)})_{u \in X^{n-1}})^{F^{-1}} H.
\end{aligned}$$

Now, by induction hypothesis

$$\begin{aligned}
\mathrm{Lab}_{n-1}(K) &\longrightarrow K \wr \overset{n-1}{\cdots} \wr K \\
f\,\mathrm{St}_{\mathrm{Lab}(K)}(n-1) &\longmapsto (f_{(u)})_{u \in X^{n-2}} \dots (f_{(1)}, \dots, f_{(d)}) f_{(\emptyset)}
\end{aligned}$$

is an isomorphism. Then F^{-1} is the image of $f^{-1}\,\mathrm{St}_{\mathrm{Lab}(K)}(n-1)$ under such isomorphism, and $f^{-1}\,\mathrm{St}_{\mathrm{Lab}(K)}(n-1)$ is associated to the permutation $f^{-1}_{|X^{n-1}}$ by the embedding $\mathrm{Lab}_{n-1}(K) \hookrightarrow \mathrm{Aut}\,\mathcal{T}_{n-1} \hookrightarrow S_{d^{n-1}}$ (this follows from (1.3) and Lemma 1.4.3). Hence by definition of wreath product

$$\begin{aligned}
((g_{(u)})_{u \in X^{n-1}})^{F^{-1}} &= ((g_{(u)})_{u \in X^{n-1}})^{\left(f^{-1}_{|X^{n-1}}\right)} \\
&= (g_{((u)f)})_{u \in X^{n-1}}.
\end{aligned}$$

It follows that

$$\begin{aligned}(f_{(u)})_{u\in X^{n-1}}F\cdot(g_{(u)})_{u\in X^{n-1}}G &= (f_{(u)})_{u\in X^{n-1}}(g_{((u)f)})_{u\in X^{n-1}}H\\ &= (f_{(u)}g_{((u)f)})_{u\in X^{n-1}}H\\ &= ((fg)_{(u)})_{u\in X^{n-1}}H\end{aligned}$$

where the last equality holds by Proposition 1.3.5.

Therefore our map is a morphism. It is injective because if $f\,\mathrm{St}_{\mathrm{Lab}(K)}(n)$ belongs to the kernel, then

$$(f_{(u)})_{u\in X^{n-1}}(f_{(u)})_{u\in X^{n-2}}\dots(f_{(1)},\dots,f_{(d)})f_{(\emptyset)} = (1)_{u\in X^{n-1}}(1)_{u\in X^{n-2}}\dots(1,\dots,1)1$$

(indeed it is easy to prove by induction on $n\geq 1$ that

$$(1)_{u\in X^{n-1}}(1)_{u\in X^{n-2}}\dots(1,\dots,1)1$$

is the identity element of $K\wr\overset{n}{\cdots}\wr K$). Thus $f_{(u)}=1$ for any $u\in\cup_{i=0}^{n-1}X^i$, that is, $f\in\mathrm{St}_{\mathrm{Lab}(K)}(n)$.

Finally, it is surjective because $(x_{(u)})_{u\in X^{n-1}}(x_{(u)})_{u\in X^{n-2}}\dots(x_{(1)},\dots,x_{(d)})x_{(\emptyset)}$ is the image of the coset (in $\mathrm{Lab}(K)_n$) represented by the automorphism h with portrait

$$h_{(u)}=\begin{cases}x_{(u)} & \text{if } u\in\cup_{i=0}^{n-1}X^i\\ 1 & \text{otherwise.}\end{cases}$$

□

The iterated wreath product $K\wr\overset{n}{\cdots}\wr K$ is a mathematical construction which, according to (1.15), gives a way to understand the structure of the finite group $\mathrm{Lab}(K)_n$, but at the same time the elements of $\mathrm{Lab}(K)_n$ (and their representation by portraits) give a way to imagine what $K\wr\overset{n}{\cdots}\wr K$ truly is. For example, when $n=2$, the product of two elements $(f_{(1)},\dots,f_{(d)})f_{(\emptyset)}$ and $(g_{(1)},\dots,g_{(d)})g_{(\emptyset)}$ inside $K\wr K$ is given by a very formal computation, but the isomorphism exhibited in the proof of Proposition 1.7.1 (and illustrated in Figure 1.8) allows to compute the composition of two automorphisms f and g in place of performing such computation. Also, it might not be clear in principle why an element $(f_{(1)},\dots,f_{(d)})f_{(\emptyset)}\in K\wr K$ should be an element of S_{d^2}, while this is immediate if we consider the automorphism f and its restriction to X^2. Therefore, in some situation, working with the abstract object $K\wr\overset{n}{\cdots}\wr K$ is easier if we take into account (1.15), i.e. if we immerse $K\wr\overset{n}{\cdots}\wr K$ in the framework of automorphisms of $\mathcal{T}$, where it finds a more concrete form.

1.8 Lab(K) as a profinite group

We start by recalling the definition of profinite group. Let $(I,\leq)$ be a directed set (i.e. $(I,\leq)$ is a partially ordered set such that for every $a,b\in I$ there is $c\in I$ with

$a \leq c, b \leq c$). An *inverse system* of groups is given by a family of groups $\{A_i\}_{i \in I}$ and a family of group homomorphisms $\{\varphi_{ji} : A_j \longrightarrow A_i\}_{\substack{i,j \in I \\ i \leq j}}$ such that

$$\begin{aligned} &\varphi_{ii} = id_{A_i} && \text{for all } i \in I \\ &\varphi_{kj}\varphi_{ji} = \varphi_{ki} && \text{for all } i, j, k \in I \text{ such that } i \leq j \leq k. \end{aligned} \tag{1.16}$$

Then we consider the subgroup of the cartesian product $\prod_{i \in I} A_i$ defined by

$$\varprojlim_{i \in I} A_i := \{(a_i)_{i \in I} \in \prod_{i \in I} A_i : (a_j)\varphi_{ji} = a_i \text{ for any } i \leq j\}$$

and we say that an *inverse limit* of the inverse system $\big(\{A_i\}_{i \in I}, \{\varphi_{ji} : A_j \longrightarrow A_i\}_{\substack{i,j \in I \\ i \leq j}}\big)$ is a group isomorphic to $\varprojlim_{i \in I} A_i$.

Now, a group G is said to be *profinite* if it is an inverse limit of finite groups, i.e. if there exist an inverse system $\big(\{A_i\}_{i \in I}, \{\varphi_{ji} : A_j \longrightarrow A_i\}_{\substack{i,j \in I \\ i \leq j}}\big)$ (with A_i finite for all $i \in I$) and a group isomorphism $\Phi : (\varprojlim_{i \in I} A_i) \longrightarrow G$. Moreover, if we equip every finite group A_i with the discrete topology, the corresponding product topology on $\prod_{i \in I} A_i$ induces a topology on $(\varprojlim_{i \in I} A_i) \leq \prod_{i \in I} A_i$, which we denote by T. Then $(T)\Phi$ is a topology on G, which turns out to be compatible with the group structure of G (meaning that the map

$$\begin{aligned} G \times G &\longrightarrow G \\ (x, y) &\longmapsto xy^{-1} \end{aligned}$$

is continuous with respect to $(T)\Phi$). This is the most natural and meaningful way to endow a profinite group with a topology, and actually profinite groups are usually thought of as groups equipped just with this topology.

The point of this section is to show that Lab(K) is a profinite group whenever $K \leq S_d$. We want to show, in particular, that Lab(K) is an inverse limit of the inverse system $\big(\{\mathrm{Lab}(K)_n\}_{n \in \mathbb{N}}, \{\varphi_{mn} : \mathrm{Lab}(K)_m \longrightarrow \mathrm{Lab}(K)_n\}_{\substack{n,m \in \mathbb{N} \\ n \leq m}}\big)$, where

$$\begin{aligned} \varphi_{mn} \ : \quad \mathrm{Lab}(K)_m \quad &\longrightarrow \mathrm{Lab}(K)_n \\ f\,\mathrm{St}_{\mathrm{Lab}(K)}(m) &\longmapsto f\,\mathrm{St}_{\mathrm{Lab}(K)}(n) \end{aligned}$$

are well defined homomorphisms because $\mathrm{St}_{\mathrm{Lab}(K)}(m) \leq \mathrm{St}_{\mathrm{Lab}(K)}(n)$ when $n \leq m$ (notice that this is an inverse system since the φ_{mn}'s satisfy (1.16) and $\mathbb{N}$ is a directed set with its natural order). For this purpose, we consider the map

$$\begin{aligned} \varphi \ : \ \mathrm{Lab}(K) &\longrightarrow \prod_{n \in \mathbb{N}} \mathrm{Lab}(K)_n \\ f \quad &\longmapsto (f\,\mathrm{St}_{\mathrm{Lab}(K)}(n))_{n \in \mathbb{N}} \end{aligned}$$

which is obviously a group homomorphism and it is injective as

$$\ker(\varphi) = \{f \in \mathrm{Lab}(K) : f \in \mathrm{St}_{\mathrm{Lab}(K)}(n) \text{ for all } n \in \mathbb{N}\} = 1.$$

Then $\mathrm{Lab}(K) \simeq \mathrm{Im}(\varphi)$ and it suffices to prove that $\mathrm{Im}(\varphi) = \lim_{\longleftarrow n\in\mathbb{N}} \mathrm{Lab}(K)_n$. Since an element in the image of φ is of the form $(f\,\mathrm{St}_{\mathrm{Lab}(K)}(n))_{n\in\mathbb{N}}$ with $f \in \mathrm{Lab}(K)$, it is obvious that $\mathrm{Im}(\varphi) \subseteq \lim_{\longleftarrow n\in\mathbb{N}} \mathrm{Lab}(K)_n$. Conversely, let $(g_n\,\mathrm{St}_{\mathrm{Lab}(K)}(n))_{n\in\mathbb{N}}$ be an element of $\lim_{\longleftarrow n\in\mathbb{N}} \mathrm{Lab}(K)_n$. This means that $g_n \in \mathrm{Lab}(K)$ for every n, and

$$(g_m\,\mathrm{St}_{\mathrm{Lab}(K)}(m))\varphi_{mn} = g_m\,\mathrm{St}_{\mathrm{Lab}(K)}(n) = g_n\,\mathrm{St}_{\mathrm{Lab}(K)}(n)$$

for all $n \leq m$. Hence $g_m^{-1}g_n \in \mathrm{St}_{\mathrm{Lab}(K)}(n)$ and then, by our first definition of nth level stabilizer, $g_m^{-1}g_n$ is the identity when restricted to $\cup_{i=0}^{n}X^i$. It follows that

$$(g_m)_{|\cup_{i=0}^{n}X^i} = (g_n)_{|\cup_{i=0}^{n}X^i} \text{ for all } n \leq m$$

which yields, by (1.4),

$$(g_m)_{(u)} = (g_n)_{(u)} \text{ for every } u \in \cup_{i=0}^{n-1}X^i \text{ and } n \leq m. \tag{1.17}$$

Now, we denote by g the only automorphism of $\mathcal{T}$ whose portrait satisfies

$$g_{(u)} = (g_{m+1})_{(u)} \text{ for every } u \in X^m, m \geq 0.$$

Clearly $g \in \mathrm{Lab}(K)$, since $g_n \in \mathrm{Lab}(K)$ for any $n \in \mathbb{N}$. Moreover, if $m \geq 0$ and $u \in X^n$ for some $n \in \{0, \ldots, m\}$ we have

$$g_{(u)} = (g_{n+1})_{(u)} = (g_{m+1})_{(u)},$$

where the last equality holds by (1.17). Equivalently, for every fixed $m \geq 0$, $g_{(u)} = (g_{m+1})_{(u)}$ for all $u \in \cup_{i=0}^{m}X^i$. Thus, again by (1.4), $(g)_{|\cup_{i=0}^{m+1}X^i} = (g_{m+1})_{|\cup_{i=0}^{m+1}X^i}$ and $g\,\mathrm{St}_{\mathrm{Lab}(K)}(m+1) = g_{m+1}\,\mathrm{St}_{\mathrm{Lab}(K)}(m+1)$ for every $m \geq 0$. This allows to conclude that $(g)\varphi = (g_n\,\mathrm{St}_{\mathrm{Lab}(K)}(n))_{n\in\mathbb{N}}$ and that $\mathrm{Im}(\varphi) \supseteq \lim_{\longleftarrow n\in\mathbb{N}} \mathrm{Lab}(K)_n$.

Summing up, $\mathrm{Lab}(K)$ is a profinite group whenever $K \leq S_d$, and

$$\mathrm{Lab}(K) \simeq \varprojlim_{n\in\mathbb{N}} \mathrm{Lab}(K)_n.$$

In particular, $\mathrm{Aut}\,\mathcal{T}$ and Γ are profinite groups with

$$\mathrm{Aut}\,\mathcal{T} \simeq \varprojlim_{n\in\mathbb{N}} \mathrm{Aut}\,\mathcal{T}_n$$

$$\Gamma \simeq \varprojlim_{n\in\mathbb{N}} \Gamma_n.$$

A reason why this is relevant is that knowing the family of finite groups $\{\mathrm{Lab}(K)_n\}_n$ and the family of epimorphisms $\{\varphi_{mn} : \mathrm{Lab}(K)_m \longrightarrow \mathrm{Lab}(K)_n\}_{\substack{n,m\in\mathbb{N}\\ n\leq m}}$ allows, simply computing their inverse limit, to fully determine the structure of the group $\mathrm{Lab}(K)$. Then a property of $\mathrm{Lab}(K)$ can always be described in terms of $\mathrm{Lab}(K)_n$ and φ_{mn}. This is not true for every subgroup of $\mathrm{Aut}\,\mathcal{T}$. In general, if $G \leq \mathrm{Aut}\,\mathcal{T}$, there is no guarantee that G is an inverse limit of the inverse system

$$\mathcal{S}_G = \big(\{G_n\}_{n\in\mathbb{N}}, \{\alpha_{mn} : G_m \longrightarrow G_n\}_{\substack{n,m\in\mathbb{N}\\ n\leq m}}\big),$$

where the morphisms α_{mn} are defined by

$$\begin{aligned} \alpha_{mn} \ : \quad G_m \quad &\longrightarrow G_n \\ f\,\mathrm{St}_G(m) &\longmapsto f\,\mathrm{St}_G(n) \end{aligned}$$

for all $n \leq m$. If for instance $G \leq \mathrm{Aut}\,\mathcal{T}$ is infinite and finitely generated, it must be infinite and countable, and then it cannot be profinite (indeed it is well known that infinite profinite groups are uncountable). Therefore an infinite finitely generated subgroup of $\mathrm{Aut}\,\mathcal{T}$ cannot be an inverse limit of finite groups, and in particular, it cannot be the inverse limit of its congruence quotients. What happens in this case is that the map

$$\begin{aligned} G &\longrightarrow \prod_{n\in\mathbb{N}} G_n \\ f &\longmapsto (f\,\mathrm{St}_G(n))_{n\in\mathbb{N}} \end{aligned}$$

is still an injective homomorphism, but its image is strictly contained in $\varprojlim_{n\in\mathbb{N}} G_n$. Hence recovering the structure of G starting from $\mathcal{S}_G$ will not be as simple as computing an inverse limit. Nevertheless, as we will see in Chapter 3, there are properties of G which can always be described in terms of $\{G_n\}_{n\in\mathbb{N}}$, even in the event that G is not profinite.

Chapter 2

Multi-EGS-groups

It is now time to introduce an important family of subgroups of $\mathrm{Aut}\,\mathcal{T}$, which take the name of multi-EGS-groups and will be the subject of our main result (Theorem 3.1.1). In this chapter, we intend to describe in detail what a multi-EGS-group is and describe some of its key features.

2.1 Definition and initial remarks

Multi-EGS-groups were first defined in [11] (even though there they were referred to as generalised multi-edge spinal groups) and they are a broad generalisation of the well-known GGS-groups (here the acronym GGS stands for Grigorchuk-Gupta-Sidki and EGS stands for extended Gupta-Sidki). Since a multi-EGS-group is defined via its generators, we first need to give a name to a special kind of elements of $\mathrm{Aut}\,\mathcal{T}$, which will contribute to the generation of our groups. Such automorphisms lie in $\mathrm{St}(1)$ and, as brought up in Section 1.5, they can be defined by specifying their sections at vertices in X^1, thanks to the isomorphism ψ.

Let a be the rooted automorphism introduced in Example 1.3.4, i.e. the one associated to $\sigma = (12\dots d) \in S_d$. Given $x, y \in \mathbb{Z}$, we have

$$a^x = a^y \iff x \equiv y \pmod d$$

due to the fact that a has order d. Then we will write a^e, with e belonging to the ring $\mathbb{Z}/d\mathbb{Z}$, meaning that $a^e = a^x$ for every integer x which represents the equivalence class e in $\mathbb{Z}/d\mathbb{Z}$. Now, given $j \in \{1, \dots, d\} = X^1$ and a tuple $\mathbf{e} = (e_1, \dots, e_{d-1})$ in the $\mathbb{Z}/d\mathbb{Z}$-module $(\mathbb{Z}/d\mathbb{Z})^{d-1}$, we will prove that there exists a unique element b of $\mathrm{St}(1)$ satisfying

$$(b)\psi = (a^{e_{d-j+1}}, \dots, a^{e_{d-1}}, b, a^{e_1}, \dots, a^{e_{d-j}}). \tag{2.1}$$

In particular, we show that the only automorphism which solves Equation (2.1) is

Let b be the only automorphism of St(1) such that $(b)\psi = (a^{e_{d-1}}, b, a^{e_1}, \dots, a^{e_{d-2}})$.

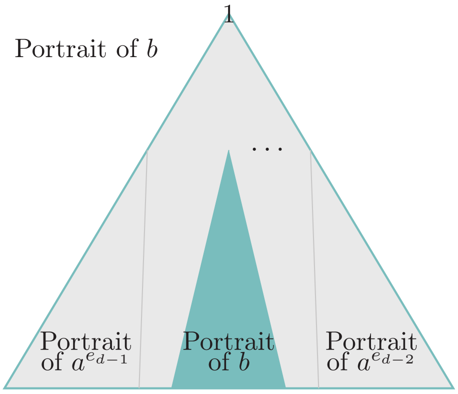


Figure 2.1: The directed automorphism along P_2 with defining tuple $\mathbf{e}$.

the automorphism $\tilde{b}$ whose portrait is recursively defined by

$$\tilde{b}_{(\emptyset)} = 1$$

$$\tilde{b}_{(x_1 \dots x_i)} = \begin{cases} (a^{e_{d-j+x_1}})_{(x_2 \dots x_i)} & \text{if } 1 \leq x_1 \leq j-1 \\ (a^{e_{x_1-j}})_{(x_2 \dots x_i)} & \text{if } j+1 \leq x_1 \leq d \\ \tilde{b}_{(x_2 \dots x_i)} & \text{if } x_1 = j \end{cases} \qquad \text{for } i \geq 1.$$

We have, for all $x \in \{1, \dots, d\} = X^1$ and all $u \in X^*$, that

$$(\tilde{b}_x)_{(u)} = \tilde{b}_{(xu)} = \begin{cases} (a^{e_{d-j+x}})_{(u)} & \text{if } 1 \leq x \leq j-1 \\ (a^{e_{x-j}})_{(u)} & \text{if } j+1 \leq x \leq d \\ \tilde{b}_{(u)} & \text{if } x = j, \end{cases}$$

i.e. the section of $\tilde{b}$ at x is $a^{e_{d-j+x}}$ when $1 \leq x \leq j-1$, it is $a^{e_{x-j}}$ when $j+1 \leq x \leq d$ and $\tilde{b}$ when $x = j$. In other words, $(\tilde{b})\psi = (a^{e_{d-j+1}}, \dots, a^{e_{d-1}}, \tilde{b}, a^{e_1}, \dots, a^{e_{d-j}})$ and $\tilde{b}$ solves (2.1). On the other hand, if an automorphism b satisfies (2.1), its sections at the vertices $1, \dots, d$ must be $a^{e_{d-j+1}}, \dots, a^{e_{d-1}}, b, a^{e_1}, \dots, a^{e_{d-j}}$, respectively. Thus b satisfies the relations that define $\tilde{b}$, and then $\tilde{b} = b$.

Definition 2.1.1. We will refer to the unique automorphism satisfying (2.1) as the *directed automorphism along the path* $P_j = (\emptyset, j, jj, \dots)$ *with defining tuple* $\mathbf{e}$ (such automorphism, in the case that $j = 2$, is portrayed in Figure 2.1).

The use of this term is justified by the fact that the label of $\tilde{b}$ at a vertex $u \neq \emptyset$ can be non trivial only when the parent of u lies in P_j. This can be proved by induction on the length of u as a word in the alphabet X. If $u \in X^1$, the parent of u is the root, which belongs to P_j. If $u = x_1 \dots x_i$ with $i \geq 2$,

$$\tilde{b}_{(u)} = \begin{cases} (a^{e_{d-j+x_1}})_{(x_2 \dots x_i)} & \text{if } 1 \leq x_1 \leq j-1 \\ (a^{e_{x_1-j}})_{(x_2 \dots x_i)} & \text{if } j+1 \leq x_1 \leq d \\ \tilde{b}_{(x_2 \dots x_i)} & \text{if } x_1 = j. \end{cases}$$

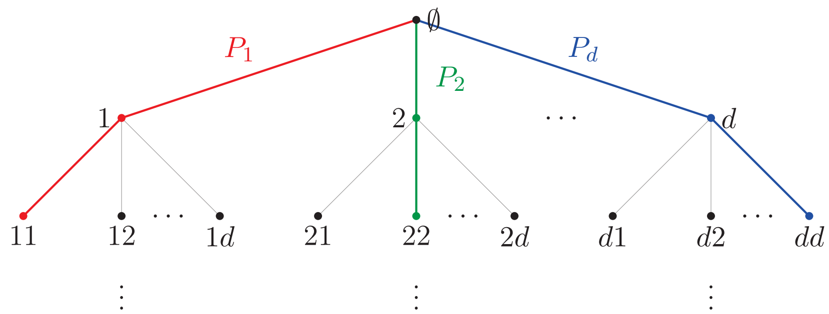


Figure 2.2: The directed paths $P_1, P_2, \ldots, P_d$.

Since every power of a belongs to H_1 and $i \geq 2$, $\tilde{b}_{(u)} = 1$ when $x_1 \neq j$. If $x_1 = j$ and the parent of u, which is $x_1 \ldots x_{i-1}$, does not belong to the path P_j, we have $x_k \neq j$ for some $2 \leq k \leq i-1$. Thus the parent of $x_2 \ldots x_i$ is not in P_j either, and $\tilde{b}_{(u)} = \tilde{b}_{(x_2 \ldots x_i)} = 1$ by induction hypothesis.

Since directed automorphisms are defined in terms of a and in terms of themselves, and $a \in \mathrm{Lab}(\langle \sigma \rangle) = \Gamma$, one can guess that directed automorphisms are contained in Γ. In fact, we prove that every label $\tilde{b}_{(u)}$ of the directed automorphism $\tilde{b}$ belongs to $\langle \sigma \rangle$, again by induction on the length of u. If $u = \emptyset$, $\tilde{b}_{(u)} = 1 \in \langle \sigma \rangle$. If $u = x_1 \ldots x_i$ and $i \geq 1$, by definition of $\tilde{b}$, $\tilde{b}_{(u)}$ can be either equal to $\tilde{b}_{(x_2 \ldots x_i)}$ (which belongs to $\langle \sigma \rangle$ by induction) or equal to a label of some power of a (which belongs to $\langle \sigma \rangle$ because $a \in \Gamma$). Thus a directed automorphism is always contained in Γ.

We are then ready to say what we mean with the term multi-EGS-group.

Definition 2.1.2. Let $E^{(1)}, \ldots, E^{(d)}$ be submodules of $(\mathbb{Z}/d\mathbb{Z})^{d-1}$ not all equal to the null submodule. For every $j \in \{1, \ldots, d\}$ we denote with $B^{(j)}$ the set of all directed automorphisms along P_j with defining tuple in $E^{(j)}$. Then we say that $K = \langle a, B^{(1)}, \ldots, B^{(d)} \rangle$ is the *multi-EGS-group associated to the tuple* $E = (E^{(1)}, \ldots, E^{(d)})$.

It is clear that every multi-EGS-group is finitely generated, as $(\mathbb{Z}/d\mathbb{Z})^{d-1}$ is finite.

Lemma 2.1.3. *Every multi-EGS-group is finitely generated.*

We point out that, since $a \in \mathrm{Lab}(\langle \sigma \rangle) = \Gamma$ and every directed automorphism lies in Γ, a multi-EGS-group is always contained in Γ.

Lemma 2.1.4. *Every multi-EGS-group is contained in* Γ.

Another meaningful property that multi-EGS-groups are endowed with is self-similarity. Indeed, if $K = \langle a, B^{(1)}, \ldots, B^{(d)} \rangle$ is a multi-EGS-group as in the previous definition, a section of $b \in B^{(j)}$ at a vertex in X^1 is either b (which belongs to K) or a power of a (which belongs to K); a section of a at a vertex in X^1 is trivial (and then it belongs to K). Therefore K is self-similar by Lemma 1.5.7.

Lemma 2.1.5. *Every multi-EGS-group is self-similar.*

It is also worthy of note that, with the notation of Definition 2.1.2, for every $j \in \{1, \dots, d\}$ there is a natural group isomorphism from $(E^{(j)}, +) \leq ((\mathbb{Z}/d\mathbb{Z})^{d-1}, +)$ to $(B^{(j)}, \cdot) \subseteq (\operatorname{Aut} \mathcal{T}, \cdot)$, which makes $B^{(j)}$ into a finite abelian subgroup of $\operatorname{Aut} \mathcal{T}$ with exponent that divides d. Such isomorphism is the map sending a tuple $\mathbf{e} \in E^{(j)}$ to the directed automorphism (along P_j) defined by $\mathbf{e}$. It is a group homomorphism because, if $(e_1, \dots, e_{d-1}), (f_1, \dots, f_{d-1}) \in E^{(j)}$ and

$$(b)\psi = (a^{e_{d-j+1}}, \dots, a^{e_{d-1}}, b, a^{e_1}, \dots, a^{e_{d-j}})$$
$$(c)\psi = (a^{f_{d-j+1}}, \dots, a^{f_{d-1}}, c, a^{f_1}, \dots, a^{f_{d-j}}),$$

we get

$$(bc)\psi = (b)\psi \cdot (c)\psi = (a^{e_{d-j+1}+f_{d-j+1}}, \dots, a^{e_{d-1}+f_{d-1}}, bc, a^{e_1+f_1}, \dots, a^{e_{d-j}+f_{d-j}}),$$

that is, bc is the directed automorphism along P_j with defining tuple $(e_1, \dots, e_{d-1}) + (f_1, \dots, f_{d-1})$. It is surjective by definition of $B^{(j)}$ and its kernel is trivial, because if the directed automorphism along P_j defined by $(e_1, \dots, e_{d-1})$ is 1, then

$$(a^{e_{d-j+1}}, \dots, a^{e_{d-1}}, 1, a^{e_1}, \dots, a^{e_{d-j}}) = (1)\psi = (1, \dots, 1),$$

which implies that $(e_1, \dots, e_{d-1}) = (0, \dots, 0)$ in $(\mathbb{Z}/d\mathbb{Z})^{d-1}$. It follows that every directed automorphism has order that divides d.

Let us remark, in addition, that

$$K = \langle a, \mathcal{B}^{(1)}, \dots, \mathcal{B}^{(d)} \rangle \tag{2.2}$$

whenever $\mathcal{E}^{(j)}$ is a generating set for $E^{(j)}$ and $\mathcal{B}^{(j)}$ is the set of all directed automorphisms along P_j with defining tuple in $\mathcal{E}^{(j)}$. Indeed in this case $\mathcal{B}^{(j)}$ is the image of $\mathcal{E}^{(j)}$ under the above isomorphism. Thus $\mathcal{B}^{(j)}$ is a generating set for $B^{(j)}$ and $\langle a, B^{(1)}, \dots, B^{(d)} \rangle = \langle a, \mathcal{B}^{(1)}, \dots, \mathcal{B}^{(d)} \rangle$.

As an easy consequence of the fact that $\operatorname{Aut} \mathcal{T}$ is equal to the semidirect product $\mathrm{St}(1) \rtimes H_1$ (by Proposition 1.4.2), every multi-EGS-group can be decomposed as a semidirect product of the following form.

Lemma 2.1.6. *Let K be a multi-EGS-group. Then*

$$K = \mathrm{St}_K(1) \rtimes \langle a \rangle$$

and, with the same notation as in Definition 2.1.2, $\mathrm{St}_K(1)$ is generated by the set $\Delta = \{b^{(a^l)} : b \in \bigcup_{j=1}^{d} B^{(j)}$ and $0 \leq l \leq d-1\}$.

Proof. We recall that $\operatorname{Aut} \mathcal{T} = \mathrm{St}(1) \rtimes H_1$. Then, since $\langle a \rangle \leq H_1$ and $\mathrm{St}_K(1) \leq \mathrm{St}(1)$, we have $\langle a \rangle \cap \mathrm{St}_K(1) \leq H_1 \cap \mathrm{St}(1) = 1$. Moreover $\mathrm{St}_K(1) \trianglelefteq K$ and $\langle a \rangle \cdot \mathrm{St}_K(1) \geq \langle a, B^{(1)}, \dots, B^{(d)} \rangle = K$. It follows that $K = \mathrm{St}_K(1) \rtimes \langle a \rangle$.

Now, since $B^{(1)}, \dots, B^{(d)} \leq \mathrm{St}_K(1)$ and $\mathrm{St}_K(1) \trianglelefteq K$, $\langle \Delta \rangle$ is contained in $\mathrm{St}_K(1)$. Moreover if $b^{(a^l)} \in \Delta$ (i.e. $b \in \bigcup_{j=1}^d B^{(j)}$ and $0 \leq l \leq d-1$) we have

$$(b^{(a^l)})^a = b^{(a^{l+1})} \in \Delta$$
$$(b^{(a^l)})^c \in \langle \Delta \rangle \quad \text{for all } c \in \bigcup_{j=1}^{d} B^{(j)} \quad (\text{because } c \in \Delta).$$

Hence $\langle \Delta \rangle$ is normal in K and the quotients

$$\frac{K}{\langle \Delta \rangle} = \frac{\langle a, B^{(1)}, \dots, B^{(d)} \rangle}{\langle \Delta \rangle} = \langle a \langle \Delta \rangle \rangle$$
$$\frac{K}{\mathrm{St}_K(1)} = \frac{\langle a, B^{(1)}, \dots, B^{(d)} \rangle}{\mathrm{St}_K(1)} = \langle a \, \mathrm{St}_K(1) \rangle$$

are both cyclic of order d (because both a and $a\,\mathrm{St}_K(1)$ have order d). We get from $\langle \Delta \rangle \trianglelefteq \mathrm{St}_K(1) \trianglelefteq K$ that $d = |K : \langle \Delta \rangle| = |K : \mathrm{St}_K(1)| \cdot |\mathrm{St}_K(1) : \langle \Delta \rangle| = d \cdot |\mathrm{St}_K(1) : \langle \Delta \rangle|$, which yields $\mathrm{St}_K(1) = \langle \Delta \rangle$. □

Since every directed automorphism b has finite order, Lemma 2.1.6 yields that every element in the multi-EGS-group K is of the form

$$a^i b_1 \dots b_r, \tag{2.3}$$

with $0 \leq i \leq d-1$, $r \geq 0$ and $b_1, \dots, b_r \in \Delta = \{b^{(a^l)} : b \in \bigcup_{j=1}^d B^{(j)} \text{ and } 0 \leq l \leq d-1\}$. To compute products of elements of this form, we will use several times the following calculation rule. For $i, j \in \{0, \dots, d-1\}$, $r, s \geq 0$ and $b_1, \dots, b_r, c_1, \dots, c_s \in \Delta$, we have

$$\begin{aligned}(a^i b_1 \dots b_r)(a^j c_1 \dots c_s) &= a^{i+j} (b_1 \dots b_r)^{(a^j)} c_1 \dots c_s \\ &= a^{i+j} b_1^{(a^j)} \dots b_r^{(a^j)} c_1 \dots c_s.\end{aligned}$$

More generally, for $k \geq 3$,

$$\begin{aligned}&(a^{i_1} b_{11} \dots b_{1r_1})(a^{i_2} b_{21} \dots b_{2r_2}) \dots (a^{i_k} b_{k1} \dots b_{kr_k}) = \\ &= a^{i_1+i_2+\dots+i_k} b_{11}^{(a^{i_2+\dots+i_k})} \dots b_{1r_1}^{(a^{i_2+\dots+i_k})} b_{21}^{(a^{i_3+\dots+i_k})} \dots b_{2r_2}^{(a^{i_3+\dots+i_k})} \dots b_{k1} \dots b_{kr_k}.\end{aligned} \tag{2.4}$$

Some subfamilies of multi-EGS-groups have played an important role in modern group theory and deserve a different name.

Definition 2.1.7. Let K be a multi-EGS-group as in Definition 2.1.2.

If $E^{(1)}, \dots, E^{(d-1)}$ are all equal to the null submodule, we say that K is the *multi-GGS-group with defining module* $E^{(d)}$.

If $E^{(1)}, \dots, E^{(d-1)}$ are all equal to the null submodule and $E^{(d)}$ is cyclic, we say that K is the *GGS-group with defining module* $E^{(d)}$ or the *GGS-group with defining tuple* $\mathbf{e}$, where $\mathbf{e}$ is any generator of $E^{(d)}$. Indeed, once $E^{(d)}$ is cyclic, the GGS-group with defining module $E^{(d)}$ is equal, by (2.2), to $\langle a, b \rangle$ for every directed automorphism b defined along P_d by a generator of $E^{(d)}$.

Among the most famous examples of multi-EGS-groups are the so-called second Grigorchuk group (defined in [9]) and the Gupta-Sidki group (defined in [10]). The second Grigorchuk group is the GGS-group obtained by choosing $d = 4$ and $E^{(4)} = \langle(1,0,1)\rangle \leq (\mathbb{Z}/4\mathbb{Z})^3$. Thus the second Grigorchuk group is, in line with Definition 2.1.7, the GGS-group with defining tuple $(1,0,1)$. The Gupta-Sidki group is defined when $d = p$ is an odd prime, and it is the GGS-group with defining tuple $(1,-1,0,\ldots,0) \in (\mathbb{Z}/p\mathbb{Z})^{p-1}$.

Notice that, when, as in the case of the Gupta-Sidki group, $d = p$ is a prime number, $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$ is the field of p elements and $(\mathbb{F}_p)^{p-1}$ is a vector space. Therefore, with the notation of Definition 2.1.2, $E^{(1)},\ldots,E^{(p)}$ are vector subspaces of $(\mathbb{F}_p)^{p-1}$ and the aforementioned isomorphism between $B^{(j)}$ and $E^{(j)}$ guarantees that $B^{(j)}$ is an abelian subgroup of $\operatorname{Aut}\mathcal{T}$ with exponent that divides p. It follows that $B^{(j)}$ is an elementary abelian p-subgroup of $\operatorname{Aut}\mathcal{T}$ and, in particular, every directed automorphism defined over the p-adic rooted tree by a non-zero vector of $(\mathbb{F}_p)^{p-1}$ has order p. We will happen to use the word "space" in place of "module" and the word "vector" in lieu of "tuple".

In the case where $d = 2$, there exists only one GGS-group defined over the d-adic rooted tree, and such GGS-group is isomorphic to the infinite dihedral group.

Example 2.1.8. Let $d = 2$ and let G be a GGS-group defined over the 2-adic rooted tree $\mathcal{T}$. By definition of GGS-group, the defining vector (tuple) of G is a non zero vector in $(\mathbb{F}_2)^1$, i.e. the defining vector of G must be $1 \in \mathbb{F}_2$. Thereby

$$G = \langle a, b\rangle$$

with

$$(b)\psi = (a, b).$$

The GGS-group G is in particular a multi-EGS-group and then it is contained in Γ. By Lemma 1.4.3, $G_n \leq \Gamma_n$ for all $n \geq 0$, and then the exponent of G_n divides the exponent of Γ_n, which is 2^n by Proposition 1.5.6. It follows that the power

$$(ab)^{(2^n)} \in \mathrm{St}_G(n) \quad \text{for all } n \geq 0. \tag{2.5}$$

Moreover, we show that

$$((ab)^{(2^{n+1})})\psi = ((ba)^{(2^n)}, (ab)^{(2^n)}) \quad \text{for all } n \geq 0 \tag{2.6}$$

(notice that we are allowed to write $((ab)^{(2^{n+1})})\psi$ because $(ab)^{(2^{n+1})} \in \mathrm{St}(1)$ by (2.5)). For $n = 0$, $(ab)^2 = abab = b^a b$ (because a has order 2) and, since $b \in \mathrm{St}_G(1)$ and $\mathrm{St}_G(1) \trianglelefteq G$, $b^a \in \mathrm{St}_G(1)$. Thus

$$((ab)^2)\psi = (b^a b)\psi = (b^a)\psi \cdot (b)\psi = (b, a) \cdot (a, b) = (ba, ab),$$

where the third equality holds by (1.14). We then have that

$$((ab)^{(2^{n+1})})\psi = (((ab)^2)\psi)^{(2^n)} = (ba, ab)^{(2^n)} = ((ba)^{(2^n)}, (ab)^{(2^n)}),$$

as desired. It is now easy to prove that

$$(ab)^{(2^n)} \notin \mathrm{St}_G(n+1) \quad \text{for all } n \geq 0. \tag{2.7}$$

Indeed, $ab \notin \mathrm{St}_G(1)$ (because $a \notin \mathrm{St}_G(1)$ and $b \in \mathrm{St}_G(1)$). For $n \geq 1$, $((ab)^{(2^n)})\psi = ((ba)^{(2^{n-1})}, (ab)^{(2^{n-1})})$ by (2.6) and $(ab)^{(2^{n-1})} \notin \mathrm{St}_G(n)$ by induction hypothesis. Since G is self-similar (because it is a multi-EGS-group), Lemma 1.5.8 yields that

$$\begin{array}{rcc} \mathrm{St}_{G_{n+1}}(1) & \longrightarrow & G_n \times G_n \\ f\,\mathrm{St}_G(n+1) & \longmapsto & (f_1 \,\mathrm{St}_G(n), f_2 \,\mathrm{St}_G(n)) \end{array}$$

is a homomorphism. Hence the fact that $((ba)^{(2^{n-1})} \mathrm{St}_G(n), (ab)^{(2^{n-1})} \mathrm{St}_G(n))$ is not trivial in $G_n \times G_n$ implies that $(ab)^{(2^n)} \notin \mathrm{St}_G(n+1)$. All in all, (2.5) and (2.7) guarantee that ab has order 2^n in G_n for any $n \geq 0$. Remembering that $G_n \leq \Gamma_n$ and $\exp(\Gamma_n) = 2^n$, we get

$$\exp(G_n) = 2^n \quad \text{for all } n \geq 0.$$

Now, if by contradiction the order of ab in G were finite and equal to t, we would have $(ab)^t = 1 \in \mathrm{St}_G(n)$ for all $n \geq 0$, that is, the order of ab in G_n divides t for all $n \geq 0$. This would imply that 2^n divides t for all $n \geq 0$ and in particular that 2^t divides t, which is a contradiction.

Hence ab has infinite order in G. The intersection $\langle a \rangle \cap \langle ab \rangle$ is trivial, because every non trivial element of $\langle ab \rangle$ has infinite order, while $\langle a \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ cannot contain infinite-order elements. The group $\langle ab \rangle$ is normal in G: indeed b has order 2 for being a non trivial directed automorphism over the 2-adic tree, and

$$\begin{aligned} (ab)^a &= ba = b^{-1}a^{-1} = (ab)^{-1} \in \langle ab \rangle \\ (ab)^b &= b^{-1}ab^2 = ba = (ab)^{-1} \in \langle ab \rangle. \end{aligned}$$

Finally, since

$$G = \langle a, b \rangle = \langle a, ab \rangle = \langle a \rangle \cdot \langle ab \rangle,$$

G is the semidirect product

$$\langle ab \rangle \rtimes \langle a \rangle \simeq \mathbb{Z} \rtimes \frac{\mathbb{Z}}{2\mathbb{Z}}$$

defined by the action $(ab)^a = (ab)^{-1}$. In other words, G is the infinite dihedral group D_∞.

We also notice that, for all $n \geq 1$, $\langle ab \,\mathrm{St}_G(n) \rangle \trianglelefteq G_n$ and

$$G_n = \langle a \,\mathrm{St}_G(n), b \,\mathrm{St}_G(n) \rangle = \langle a \,\mathrm{St}_G(n) \rangle \cdot \langle ab \,\mathrm{St}_G(n) \rangle.$$

Moreover, if $n \geq 2$, $\langle a \,\mathrm{St}_G(n) \rangle \cap \langle ab \,\mathrm{St}_G(n) \rangle = 1$. Indeed, if this were not true, $a \,\mathrm{St}_G(n)$, which has order 2, should belong to $\langle ab \,\mathrm{St}_G(n) \rangle$, which is isomorphic to $\mathbb{Z}/2^n\mathbb{Z}$ by the previous considerations. Thus $a \,\mathrm{St}_G(n)$ should be equal to the unique element of order 2 in $\langle ab\,\mathrm{St}_G(n)\rangle$, which is $(ab)^{(2^{n-1})}\,\mathrm{St}_G(n)$ (because $n \geq 2$). Since $n \geq 2$, $\mathrm{St}_G(n) \leq \mathrm{St}_G(1)$ and $(ab)^{(2^{n-1})} \in \mathrm{St}_G(1)$ by (2.5). The equality $a\,\mathrm{St}_G(n) = (ab)^{(2^{n-1})}\,\mathrm{St}_G(n)$ would then imply

$$a\,\mathrm{St}_G(1) = (ab)^{(2^{n-1})}\,\mathrm{St}_G(1) \subseteq \mathrm{St}_G(1)$$

and $a \in \mathrm{St}_G(1)$, which is a contradiction. It follows that, when $n \geq 2$, G_n is the semidirect product

$$\langle ab\,\mathrm{St}_G(n)\rangle \rtimes \langle a\,\mathrm{St}_G(n)\rangle \simeq \frac{\mathbb{Z}}{2^n\mathbb{Z}} \rtimes \frac{\mathbb{Z}}{2\mathbb{Z}}$$

defined by the action $(ab\,\mathrm{St}_G(n))^{(a\,\mathrm{St}_G(n))} = (ab\,\mathrm{St}_G(n))^{-1}$. Therefore, for all $n \geq 2$, G_n is the dihedral group $D_{2^{n+1}}$ of order

$$|G_n| = 2^{n+1}.$$

The case $n = 1$ is an exception, since $G_1 = \langle a\,\mathrm{St}_G(1), b\,\mathrm{St}_G(1)\rangle = \langle a\,\mathrm{St}_G(1)\rangle \simeq \mathbb{Z}/2\mathbb{Z}$ and $|G_1| = 2$.

Summarizing, Example 2.1.8 shows that the only GGS-group G that can be defined over the 2-adic rooted tree is the infinite dihedral group D_∞. For $n \geq 2$, its nth congruence quotient G_n is a finite dihedral group, has order 2^{n+1} and exponent 2^n.

When p is an odd prime number and G is any GGS-group defined over the p-adic rooted tree, it is still possible to determine the order and the exponent of all congruence quotients of G, but that is far from being easy. An explicit formula for the order of such congruence quotients is provided by Theorem A in [6]; an explicit formula for their exponent will be given by the central result of this thesis (Theorem 3.1.1), whose proof will rely on strong results from [6].

2.2 Main properties

Section 2.1 made us come in contact with the class of multi-EGS-groups: a family of finitely generated and self-similar subgroups of $\mathrm{Aut}\,\mathcal{T}$ which are all contained in Γ. We now present some further properties of multi-EGS-groups which are interesting in themselves and will be helpful in the following.

We start by proving a result that concerns the cardinality of GGS-groups, and will be extended to multi-EGS-groups in Proposition 2.2.3. The proof of Proposition 2.2.1 is taken from [14].

Proposition 2.2.1. *Let p be a prime number. Then every GGS-group defined over the p-adic rooted tree is infinite.*

Proof. Let G be the GGS-group with defining vector $\mathbf{e} = (e_1, \ldots, e_{p-1}) \in (\mathbb{F}_p)^{p-1}$ and let b be the directed automorphism along P_p defined by $\mathbf{e}$. The self-similarity of G ensures that $(\mathrm{St}_G(1))\psi \leq G \times \overset{p}{\cdots} \times G$ and for every $i \in \{1, \ldots, p\}$ we can consider the composition map $\psi\pi_i\colon \mathrm{St}_G(1) \longrightarrow G$, where the symbol π_i, from now on, will denote the canonical projection

$$\begin{aligned}\pi_i\colon G \times \overset{p}{\cdots} \times G &\longrightarrow G\\ (g_1, \dots, g_p) &\longmapsto g_i\end{aligned}$$

on the ith component.

If we are able to prove that the morphism $\psi\pi_i\colon \mathrm{St}_G(1) \longrightarrow G$ is surjective for some i, it will follow that G is infinite. Indeed $b \in \mathrm{St}_G(1)$ and

$$G_1 = \frac{\langle a, b\rangle}{\mathrm{St}_G(1)} = \langle a\,\mathrm{St}_G(1)\rangle$$

is the cyclic group of order p. Assuming by contradiction G to be finite, we would get

$$|G| = |G_1| \cdot |\,\mathrm{St}_G(1)| = p \cdot |\,\mathrm{St}_G(1)|$$

and

$$|\,\mathrm{St}_G(1)| \geq |G|$$

by the surjectivity of $\psi\pi_i$, which cannot hold together.

We are then left to show that $\psi\pi_i\colon \mathrm{St}_G(1) \longrightarrow G$ is surjective for some $i \in \{1, \dots, p\}$. By definition of GGS-group, $\mathbf{e}$ is not the zero vector and then there is $i \in \{1, \dots, p-1\}$ such that $e_i \neq 0$ in $\mathbb{F}_p$. We aim to prove that, for this choice of i, the morphism $\psi\pi_i\colon \mathrm{St}_G(1) \longrightarrow G$ is surjective. Since $G = \langle a, b\rangle$ and $\psi\pi_i$ is a morphism, it suffices to show that a and b belong to $(\mathrm{St}_G(1))\psi\pi_i$. Since e_i is a non zero element in the field $\mathbb{F}_p$, there is $x \in \mathbb{F}_p$ with $e_i x = 1$ and

$$(b)\psi\pi_i = (a^{e_1}, \dots, a^{e_{p-1}}, b)\pi_i = a^{e_i} \in (\mathrm{St}_G(1))\psi\pi_i$$

implies that $a = (a^{e_i})^x \in (\mathrm{St}_G(1))\psi\pi_i$. Moreover, using repeatedly (1.14), we get

$$(b^{(a^i)})\psi = (a^{e_{p-i+1}}, \dots, a^{e_{p-1}}, b, a^{e_1}, \dots, a^{e_{p-i}})$$

and

$$(b^{(a^i)})\psi\pi_i = b.$$

It follows that a and b are in $(\mathrm{St}_G(1))\psi\pi_i$, as desired. □

To extend this result to every multi-EGS-group, we need the following lemma, which guarantees that every multi-EGS-group contains an isomorphic copy of some GGS-group.

Lemma 2.2.2. *Let $d \geq 1$, $j \in \{1, \dots, d\}$ and let $\mathbf{e}$ be a non zero tuple in $(\mathbb{Z}/d\mathbb{Z})^{d-1}$. Then, if we denote by b the directed automorphism along P_j with defining tuple $\mathbf{e}$, we have that the group*

$$H = \langle a, b\rangle$$

is conjugate to the GGS-group (over the d-adic rooted tree) with defining tuple $\mathbf{e}$.

In particular, if K is the multi-EGS-group associated to $(E^{(1)}, \dots, E^{(d)})$ and we fix a tuple $\mathbf{f} \in (\bigcup_{i=0}^{d} E^{(i)}) \setminus \{\mathbf{0}\}$, K contains an isomorphic copy of the GGS-group with defining tuple $\mathbf{f}$.

Proof. We set $G = \langle a, g\rangle$, where g is the directed automorphism along P_d with defining tuple $\mathbf{e}$ (i.e. G is the GGS-group with defining tuple $\mathbf{e}$). We denote with f the automorphism of $\mathcal{T}$ that has all the labels equal to $\sigma^{d-j} = (12\dots d)^{d-j}$. By what is shown in the proof of Proposition 1.4.2 and illustrated in Figure 1.6, we can decompose f as a product $f = xy$, where $x \in \mathrm{St}(1)$, $y \in H_1$ and

$$x_{(u)} = \begin{cases} 1 & \text{if } u = \emptyset \\ \sigma^{d-j} & \text{otherwise} \end{cases}$$

$$y_{(u)} = \begin{cases} \sigma^{d-j} & \text{if } u = \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

We observe that $y = a^{d-j}$ (indeed a^{d-j} belongs to H_1 and $(a^{d-j})_{(\emptyset)} = (a_{(\emptyset)})^{d-j} = \sigma^{d-j}$ by Proposition 1.3.5). The automorphism x belongs to $\mathrm{St}(1)$ and all its sections at vertices in X^1 have all the labels equal to σ^{d-j}, namely

$$(x)\psi = (f, \dots, f).$$

By (1.14) we have $(x^a)\psi = (f, \dots, f) = (x)\psi$. The injectivity of ψ yields that $x^a = x$, then x commutes with a and $f = xa^{d-j}$ commutes with a. It follows that

$$\begin{aligned}
(b^x)\psi &= ((b)\psi)^{(x)\psi} \\
&= (a^{e_{d-j+1}}, \dots, a^{e_{d-1}}, b, a^{e_1}, \dots, a^{e_{d-j}})^{(f,\dots,f)} \\
&= ((a^{e_{d-j+1}})^f, \dots, (a^{e_{d-1}})^f, b^f, (a^{e_1})^f, \dots, (a^{e_{d-j}})^f) \\
&= (a^{e_{d-j+1}}, \dots, a^{e_{d-1}}, b^f, a^{e_1}, \dots, a^{e_{d-j}})
\end{aligned}$$

and by (1.14)

$$(b^f)\psi = ((b^x)^{a^{d-j}})\psi = (a^{e_1}, \dots, a^{e_{d-1}}, b^f),$$

which yields that b^f is the directed automorphism along P_d with defining tuple $\mathbf{e}$, i.e. $b^f = g$. Therefore $H^f = (\langle a, b\rangle)^f = \langle a^f, b^f\rangle = \langle a, b^f\rangle = \langle a, g\rangle = G$, as wanted.

Finally, if K is a multi-EGS-group as in the statement, $\mathbf{f} \in E^{(j)} \setminus \{\mathbf{0}\}$ and c is the directed automorphism along P_j with defining tuple $\mathbf{f}$, K contains the group $\langle a, c\rangle$, which is isomorphic to the GGS-group with defining tuple $\mathbf{f}$ by what we just proved. □

It is now easy to prove that a multi-EGS-group over the p-adic rooted tree (with p prime) is always infinite.

Proposition 2.2.3. *Let p be a prime number. Then every multi-EGS-group defined over the p-adic rooted tree is infinite.*

Proof. Let K be the multi-EGS-group (defined over the p-adic rooted tree) associated to the tuple $(E^{(1)}, \dots, E^{(p)})$ and let $\mathbf{f} \in (\bigcup_{i=0}^{p} E^{(i)}) \setminus \{\mathbf{0}\}$. By Lemma 2.2.2, K contains an isomorphic copy of the GGS-group (over the p-adic rooted tree) with defining vector $\mathbf{f}$, which is infinite by Proposition 2.2.1. Thus K is infinite as well. □

Let us observe that Proposition 2.2.3 does not hold anymore when multi-EGS-groups are defined over the d-adic rooted tree and d is not prime. This is shown in the following example.

Example 2.2.4. Let G be the GGS-group, defined over the 4-adic rooted tree, with defining tuple $(2,0,0)$. We want to show that G is finite.

As in the proof of Proposition 2.2.1, we have that $G_1 = \langle a\,\mathrm{St}_G(1)\rangle$ is the cyclic group of order 4. However, since $d = 4$ is not prime, we are now not able to prove that $\psi\pi_i$ is surjective. If we denote by b the directed automorphism along P_4 with defining tuple $(2,0,0)$, we have

$$\begin{aligned}(b)\psi &= (a^2, 1, 1, b)\\ (b^a)\psi &= (b, a^2, 1, 1)\\ (b^{(a^2)})\psi &= (1, b, a^2, 1)\\ (b^{(a^3)})\psi &= (1, 1, b, a^2)\end{aligned}$$

by (1.14). Moreover b has order 2 (indeed $(b^2)\psi = (a^2,1,1,b)^2 = (a^4,1,1,b^2) = (1,1,1,b^2)$ and then b^2 is the directed automorphism along P_4 defined by the zero tuple, i.e. $b^2 = 1$). With the notation of Definition 2.1.2, $E^{(4)} = \langle(2,0,0)\rangle$ and the isomorphism between $E^{(4)}$ and $B^{(4)}$ yields $B^{(4)} = \langle b\rangle$. Then, by Lemma 2.1.6, $\mathrm{St}_G(1)$ is generated by

$$\Delta = \{c^{(a^l)} : c \in \langle b\rangle \text{ and } 0 \le l \le 3\} = \{(b^i)^{(a^l)} : 0 \le i \le 1 \text{ and } 0 \le l \le 3\}$$

and, since ψ is an isomorphism, $(\mathrm{St}_G(1))\psi$ is generated by

$$\{(a^2,1,1,b), (b,a^2,1,1), (1,b,a^2,1), (1,1,b,a^2)\}.$$

It follows that $(\mathrm{St}_G(1))\psi\pi_i \le \langle a^2, b\rangle =: H$ for any $1 \le i \le 4$. Then $(\mathrm{St}_G(1))\psi \le H \times \overset{4}{\cdots} \times H$. If we are able to prove that H is finite, $|\mathrm{St}_G(1)| = |(\mathrm{St}_G(1))\psi| \le |H|^4$ will be finite and, since

$$G = \mathrm{St}_G(1) \rtimes \langle a\rangle$$

by Lemma 2.1.6, $|G| = |\langle a\rangle| \cdot |\mathrm{St}_G(1)| = 4 \cdot |\mathrm{St}_G(1)|$ will be finite as well.

We are then only left to show that H is finite. We have

$$((a^2b)^2)\psi = (a^4 b^{(a^2)} b)\psi = (b^{(a^2)} b)\psi = (a^2, b, a^2, b).$$

Since a^2 and b have order 2, (a^2,b,a^2,b) has order 2 in $G \times \overset{4}{\cdots} \times G$ and a^2b has order 4 as ψ is an isomorphism. Furthermore

$$(a^2b)^{(a^2)} = ba^2 = (a^2b)^{-1}.$$

Hence $H = \langle a^2b\rangle \rtimes \langle a^2\rangle \simeq D_8$ and $|H| = 8 \lneqq \infty$.

Another question that one might ask about multi-EGS-groups is whether they are fractal. Even though every multi-EGS-group is self-similar, not every multi-EGS-group is fractal. There is a nice and explicit characterization for fractal multi-EGS-groups, which is given by the following proposition.

Proposition 2.2.5. *Let K be a multi-EGS-group as in Definition 2.1.2 and let $\mathcal{C}$ be the set of all integers in $\{0,\ldots,d-1\}$ representing some component of some tuple lying in $\bigcup_{j=1}^{d} E^{(j)}$, i.e.*

$$\mathcal{C} = \left\{ \begin{array}{r} x \in \{0,\ldots,d-1\} : \text{ there is } (e_1,\ldots,e_{d-1}) \in \bigcup_{j=1}^{d} E^{(j)} \text{ such that} \\ e_i = x + d\mathbb{Z} \text{ for some } i \in \{1,\ldots,d-1\} \end{array} \right\}.$$

Then

$$K \text{ is fractal} \quad \text{if and only if} \quad \gcd(\mathcal{C} \cup \{d\}) = 1.$$

Proof. Suppose first that the greatest common divisor of all the elements of $\mathcal{C} \cup \{d\}$ is 1. To show that $K = \langle a, B^{(1)}, \ldots, B^{(d)} \rangle$ is fractal, we want to use Lemma 1.6.3, i.e. we want to prove that for every $s \in S := \{a\} \cup \bigcup_{j=1}^{d} B^{(j)}$ and for every $u \in X^1$, there exists $g \in \mathrm{St}_K(u)$ with $g_u = s$. Even more, we will prove that such a g can be chosen inside the first level stabilizer $\mathrm{St}_K(1)$, which is equal to $\bigcap_{u \in X^1} \mathrm{St}_K(u)$.

If $s \in B^{(j)}$ for some $j \in \{1,\ldots,d\}$, then

$$(s)\psi = (a^{e_{d-j+1}}, \ldots, a^{e_{d-1}}, s, a^{e_1}, \ldots, a^{e_{d-j}})$$

for some $(e_1,\ldots,e_{d-1}) \in E^{(j)}$. Hence for every $u \in X^1 = \{1,\ldots,d\}$ we have

$$(s^{(a^{u-j})})\psi = (a^{e_{d-u+1}}, \ldots, a^{e_{d-1}}, s, a^{e_1}, \ldots, a^{e_{d-u}})$$

by (1.14). We have that $s^{(a^{u-j})} \in \mathrm{St}_K(1)$ and the section of $s^{(a^{u-j})}$ at u is equal to s, as desired.

We are then left to show that whenever $u \in X^1$, there is $g \in \mathrm{St}_K(1)$ such that $g_u = a$. Recall that we are assuming $\gcd(\mathcal{C} \cup \{d\}) = 1$. Then there are $c_1,\ldots,c_k \in \mathcal{C}$ such that $\gcd(c_1,\ldots,c_k,d) = 1$. If we denote by t the greatest common divisor of $c_1,\ldots,c_k$, there must exist $\alpha_1,\ldots,\alpha_k \in \mathbb{Z}$ with

$$\alpha_1 c_1 + \ldots + \alpha_k c_k = t.$$

Moreover t and d are coprime, that is,

$$ty \equiv 1 (\mathrm{mod}\ d) \tag{2.8}$$

for some $y \in \mathbb{Z}$. Now, for every $m \in \{1,\ldots,k\}$, since $c_m \in \mathcal{C}$, there are $\mathbf{e_m} = (e_{m,1},\ldots,e_{m,d-1}) \in E^{(j_m)}$ and $i_m \in \{1,\ldots,d-1\}$ such that $e_{m,i_m} = c_m + d\mathbb{Z}$. Thus the directed automorphism along P_{j_m} with defining tuple $\mathbf{e_m}$, which in this proof will be indicated with b_m, belongs to K. We have

$$(b_m)\psi = (a^{e_{m,d-j_m+1}}, \ldots, a^{e_{m,d-1}}, b_m, a^{e_{m,1}}, \ldots, a^{e_{m,d-j_m}})$$

and therefore, again by (1.14),

$$(b_m^{(a^{d-i_m-j_m+1})})\psi = (a^{e_{m,i_m}}, \dots, a^{e_{m,d-1}}, b_m, a^{e_{m,1}}, \dots, a^{e_{m,i_m-1}}).$$

If we set $f_m = d - i_m - j_m + 1$, we get $(b_m^{(a^{f_m})})\psi\pi_1 = a^{e_{m,i_m}} = a^{c_m}$, and

$$(((b_1^{(a^{f_1})})^{\alpha_1} \dots (b_k^{(a^{f_k})})^{\alpha_k})^y)\psi\pi_1 = a^{(\alpha_1 c_1 + \dots + \alpha_k c_k)y} = a^{ty} = a$$

by (2.8). All in all, there exists $B \in \mathrm{St}_K(1)$ whose section at the vertex 1 is equal to a (just choose $B = ((b_1^{(a^{f_1})})^{\alpha_1} \dots (b_k^{(a^{f_k})})^{\alpha_k})^y)$. Now, if $u \in X^1 = \{1, \dots, d\}$, thanks to (1.14) we have

$$(B^{(a^{u-1})})\psi\pi_u = a,$$

that is, the section of $B^{(a^{u-1})}$ at u is a. Since $B^{(a^{u-1})} \in \mathrm{St}_K(1)$, this guarantees that K is fractal.

Let us prove the converse implication. We assume that $\gcd(\mathcal{C} \cup \{d\}) = z \neq 1$. By definition of multi-EGS-group the modules $E^{(1)}, \dots, E^{(d)}$ are not all equal to the null submodule, hence $d \geq 2$ and $\mathcal{C} \subseteq \{0, \dots, d-1\}$ contains some non-zero element. It follows that $z \in \{2, \dots, d-1\}$.

Suppose by contradiction that K is fractal. Thus there exists $g \in K$ such that the section g_1 is equal to a. By (2.3) we have

$$g = a^i h_1 \dots h_r$$

for some $h_1, \dots, h_r \in \Delta = \{b^{(a^l)} : b \in \bigcup_{j=1}^d B^{(j)} \text{ and } 0 \leq l \leq d-1\}$. By Proposition 1.5.2 the section of g at 1 can be computed as

$$\begin{aligned} g_1 &= (a^i h_1 \dots h_r)_1 \\ &= (a^i)_1 (h_1)_{(1)a^i} (h_2)_{(1)a^i h_1} \dots (h_r)_{(1)a^i h_1 \dots h_{r-1}} \\ &= (h_1)_{(1)a^i} (h_2)_{(1)a^i h_1} \dots (h_r)_{(1)a^i h_1 \dots h_{r-1}}. \end{aligned}$$

This tells us that g_1 is a product of sections at vertices in X^1 of elements of Δ. Now, if $h \in \Delta$, $h = b^{(a^l)}$ where b is some directed automorphism with defining tuple $(e_1, \dots, e_{d-1}) \in \bigcup_{j=1}^d E^{(j)}$ and, by (1.14), the components of $(h)\psi$ are $b, a^{e_1}, \dots, a^{e_{d-1}}$. Since $z = \gcd(\mathcal{C} \cup \{d\})$, $a^{e_1}, \dots, a^{e_{d-1}}$ are powers of a^z and then the components of $(h)\psi$ (i.e. the sections of h at vertices in X^1) lie in $\langle \mathrm{St}_K(1), a^z \rangle$. It follows that $a = g_1$ belongs to $\langle \mathrm{St}_K(1), a^z \rangle$, which yields

$$K = \langle \mathrm{St}_K(1), a \rangle = \langle \mathrm{St}_K(1), a^z \rangle$$

(here the first equality follows from Lemma 2.1.6). Thus

$$\frac{K}{\mathrm{St}_K(1)} = \langle a \, \mathrm{St}_K(1) \rangle = \langle a^z \, \mathrm{St}_K(1) \rangle,$$

which is a contradiction because $a \, \mathrm{St}_K(1)$ has order d and $z = \gcd(\mathcal{C} \cup \{d\}) \in \{2, \dots, d-1\}$ is a proper divisor of d. $\square$

The last feature of multi-EGS-groups that we wish to investigate is their periodicity. The periodicity of a multi-EGS-group can be explicitly characterized when the group acts over the p-adic tree, with p prime, but once again such characterization (stated in Proposition 2.2.8) does not hold when the groups are defined over the d-adic tree and d is not prime.

We first need to prove a partial result which concerns GGS-groups.

Lemma 2.2.6. *Let p be a prime number and let $G = \langle a, b\rangle$ be the GGS-group defined over the p-adic tree with defining vector $\mathbf{e} = (e_1, \dots, e_{p-1})$ (here, as usual, b denotes the directed automorphism along P_p with defining vector $\mathbf{e}$). Suppose further that $x = e_1 + \cdots + e_{p-1}$ is not zero in $\mathbb{F}_p$. Then, for every $n \geq 1$, any element of form $a^x b^{(a^j)}$ with $j \in \{0, \dots, p-1\}$ has order $\geq p^n$ in G_n.*

Proof. We argue by induction on n. For $n = 1$, $G_1 = \langle a, b\rangle / \mathrm{St}_G(1) = \langle a\,\mathrm{St}_G(1)\rangle$ is the cyclic group of order p. Since $x \neq 0$ in $\mathbb{F}_p$, for any $j \in \{0, \dots, p-1\}$, $a^x b^{(a^j)} \notin \mathrm{St}_G(1)$, which yields that $a^x b^{(a^j)}$ has order p in G_1.

Assume now $n \geq 2$ and let $j \in \{0, \dots, p-1\}$. By the calculation rule (2.4) we have

$$\begin{aligned}(a^x b^{(a^j)})^p &= a^{xp} b^{(a^{j+x(p-1)})} b^{(a^{j+x(p-2)})} \dots b^{(a^{j+x})} b^{(a^j)} \\ &= b^{(a^{j+x(p-1)})} b^{(a^{j+x(p-2)})} \dots b^{(a^{j+x})} b^{(a^j)}\end{aligned}$$

and, since $x \neq 0$ in $\mathbb{F}_p$, $\{j+x(p-1), j+x(p-2), \dots, j+x, j\}$ is a set of representatives for the congruence classes modulo p . In other words, $(a^x b^{(a^j)})^p$ is the product, in some order, of the p factors $b^{(a^0)}, b^{(a^1)}, \dots, b^{(a^{p-1})}$. Then, since by (1.14)

$$\begin{aligned}(b^{(a^0)})\psi &= (a^{e_1}, a^{e_2}, \dots, a^{e_{p-1}}, b) \\ (b^{(a^1)})\psi &= (b, a^{e_1}, \dots, a^{e_{p-2}}, a^{e_{p-1}}) \\ &\vdots \\ (b^{(a^{p-1})})\psi &= (a^{e_2}, a^{e_3}, \dots, b, a^{e_1}),\end{aligned}$$

the first component of $((a^x b^{(a^j)})^p)\psi$ is the product, in some order, of the p factors $a^{e_1}, \dots, a^{e_{p-1}}, b$. This means that there are $0 \leq k \leq p-1$ and $f_1, \dots, f_{p-1} \in \mathbb{F}_p$ such that $f_1 + \cdots + f_{p-1} = e_1 + \cdots + e_{p-1} = x$ and

$$\begin{aligned}((a^x b^{(a^j)})^p)\psi\pi_1 &= a^{f_1} \dots a^{f_k} b a^{f_{k+1}} \dots a^{f_{p-1}} \\ &= a^{f_1+\cdots+f_{p-1}} b^{(a^{f_{k+1}+\cdots+f_{p-1}})}\end{aligned}$$

(where the last equality follows from (2.4)). It follows that $((a^x b^{(a^j)})^p)\psi\pi_1$ is of the form $a^x b^{(a^l)}$ for some $l \in \{0, \dots, p-1\}$ and by induction hypothesis it has order $\geq p^{n-1}$ in G_{n-1}. This yields, by Lemma 1.5.10, that $(a^x b^{(a^j)})^p$ has order $\geq p^{n-1}$ in G_n and $a^x b^{(a^j)}$ has order $\geq p^n$ in G_n. □

Now the following result is straightforward.

Proposition 2.2.7. *Let p be a prime number and let G be a GGS-group defined over the p-adic rooted tree. If $\mathbf{e} = (e_1, \dots, e_{p-1})$ is the defining vector of G and $x = e_1 + \dots + e_{p-1}$ is not zero in $\mathbb{F}_p$, then G is not periodic.*

Proof. Let us denote with b the directed automorphism along P_p with defining vector $\mathbf{e}$. If by contradiction G were periodic, the order of the element $a^x b \in G$ would be equal to a finite number t and $(a^x b)^t = 1 \in \mathrm{St}_G(n)$ for all $n \geq 1$. This would imply that the order of $a^x b$ in G_n is $\leq t$ for any $n \geq 1$. However we also know, by Lemma 2.2.6, that the order of $a^x b = a^x b^{(a^0)}$ in G_n must be $\geq p^n$ for any $n \geq 1$. We would then get $p^n \leq t$ for any $n \geq 1$ and in particular $p^t \leq t$, which is a contradiction. □

From now onwards, we will denote with V the submodule of $(\mathbb{Z}/d\mathbb{Z})^{d-1}$ defined by

$$V = \{(f_1, \dots, f_{d-1}) \in (\mathbb{Z}/d\mathbb{Z})^{d-1} : f_1 + \dots + f_{d-1} = 0\}.$$

Notice that, when $d = p$ is prime, V is a hyperplane of $(\mathbb{F}_p)^{p-1}$. The next statement extends Proposition 2.2.7 to every multi-EGS-group, and gives some converse implication. The proof can also be found in [13] (Lemma 3.13).

Proposition 2.2.8. *Let p be a prime and let K be the multi-EGS-group, defined over the p-adic rooted tree, associated to the tuple $(E^{(1)}, \dots, E^{(p)})$. Then*

$$K \text{ is periodic} \quad \text{if and only if} \quad E^{(1)}, \dots, E^{(p)} \text{ are all contained in } V.$$

Moreover, in this case K is a p-group (i.e. all its elements have as order a power of p).

Proof. One implication can be proved with the same philosophy used to extend Proposition 2.2.1 to Proposition 2.2.3, that is, by exploiting the fact that a multi-EGS-group always contains an isomorphic copy of some GGS-group. Indeed, if $E^{(1)}, \dots, E^{(p)}$ are not all contained in V, there is a vector $\mathbf{f} = (f_1, \dots, f_{p-1}) \in \bigcup_{i=0}^{p} E^{(i)} \setminus \{\mathbf{0}\}$ which does not belong to V, i.e. $f_1 + \ldots + f_{p-1} \neq 0$ in $\mathbb{F}_p$. Then K contains an isomorphic copy of the GGS-group with defining vector $\mathbf{f}$ by Lemma 2.2.2, and such GGS-group is not periodic by Proposition 2.2.7. Thus K cannot be periodic.

Conversely, we suppose that $\bigcup_{i=0}^{p} E^{(i)} \subseteq V$ and we aim to prove that K is periodic. As we pointed out in (2.3), any element of K has form

$$a^i b_1 \dots b_r$$

with $0 \leq i \leq p-1, r \geq 0$ and $b_1, \dots, b_r \in \Delta = \{b^{(a^l)} : b \in \bigcup_{j=1}^{p} B^{(j)} \text{ and } 0 \leq l \leq p-1\}$ (where, as usual, $B^{(j)}$ is defined as in Definition 2.1.2). For $g \in K$, we will say that the *length* of g is the smallest natural number r such that g can be written as

$$g = a^i b_1 \dots b_r$$

for some $0 \leq i \leq p-1$ and $b_1, \dots, b_r \in \Delta$. Observe that, by the calculation rule (2.4),

$$\text{the length of the product of } k \text{ elements whose lengths are } r_1, r_2, \dots, r_k \text{ is at most } r_1 + r_2 + \ldots + r_k. \tag{2.9}$$

We prove that every element g of K has finite order and its order is a power of p, by induction on the length of g. If the length of an element g is 0, then $g = a^i$ for some $i \in \{0, \dots, p-1\}$ and then the order of g divides the order of a, which is p. Let now g be an element of K with length $n \geq 1$. To prove that its order is a power of p, we consider two cases separately.

Case 1: Suppose $g \in \mathrm{St}_K(1)$. If we are able to prove that the orders of the sections $g_1, \dots, g_p$ are powers of p, since $(g)\psi = (g_1, \dots, g_p)$ and ψ is an isomorphism, it will follow that the order of g is a power of p. So we fix $z \in \{1, \dots, p\}$ and we want to prove that the order of g_z is a power of p. If g_z has length $\leq n-1$, its order is a power of p by induction hypothesis. Assume that g_z has length $\geq n$. Since g has length n and $g \in \mathrm{St}_K(1)$, $g = b_1 \dots b_n$ for some $b_1, \dots, b_n \in \Delta$. Thus

$$(g)\psi = (b_1)\psi \dots (b_n)\psi$$

and $(b_1)\psi, \dots, (b_n)\psi$ belong to

$$(\Delta)\psi = \left\{ (b^{(a^l)})\psi : 0 \leq l \leq p-1 \text{ and } b \in \bigcup_{j=1}^{p} B^{(j)} \right\}$$

$$= \left\{ (a^{e_{p-t+1}}, \dots, a^{e_{p-1}}, b, a^{e_1}, \dots, a^{e_{p-t}}) : 1 \leq t \leq p, \ b \in \bigcup_{j=1}^{p} B^{(j)} \text{ and } b \text{ has defining vector } (e_1, \dots, e_{p-1}) \right\}$$

(here the last equality holds by (1.14)). It follows that

$$g_z = (g)\psi\pi_z = (b_1)\psi\pi_z \dots (b_n)\psi\pi_z$$

is the product of n elements, which can be powers of a (which have length 0) or elements of $\bigcup_{j=1}^{p} B^{(j)} \subseteq \Delta$ (which have length 0 or 1). Since we are assuming the length of g_z to be $\geq n$, we deduce from (2.9) that the n elements $(b_1)\psi\pi_j =: c_1, \dots, (b_n)\psi\pi_j =: c_n$ are all inside $\bigcup_{j=1}^{p} B^{(j)}$.

If all the components of $(c_1 \dots c_n)\psi$ have length $\leq n-1$, their orders are powers of p by induction, and then also the order of $c_1 \dots c_n = g_z$ is a power of p. Otherwise, there is $t \in \{1, \dots, p\}$ such that $(c_1 \dots c_n)\psi\pi_t = (c_1)\psi\pi_t \dots (c_n)\psi\pi_t$ has length $\geq n$. Since $c_1, \dots, c_n \in \bigcup_{j=1}^{p} B^{(j)}$ are directed automorphisms, for $m \in \{1, \dots, n\}$, $(c_m)\psi\pi_t$ is either c_m (which has length ≤ 1) or a power of a (which has length 0). Then the fact that $(c_1 \dots c_n)\psi\pi_t$ has length $\geq n$ implies, by (2.9), that $(c_m)\psi\pi_t = c_m$ for every m. Hence $c_1, \dots, c_n$ are directed automorphisms along the path P_t, i.e.

$c_1, \ldots, c_n$ belong to $B^{(t)}$. As observed in Section 2.1, $B^{(t)}$ is an elementary abelian p-subgroup of $\operatorname{Aut}\mathcal{T}$, and then the order of $g_z = c_1 \ldots c_n \in B^{(t)}$ must divide p.

Case 2: Suppose $g \notin \operatorname{St}_K(1)$. We have $K_1 = (\langle\Delta\rangle \rtimes \langle a\rangle)/\operatorname{St}_K(1) = \langle a \operatorname{St}_K(1)\rangle \simeq \mathbb{Z}/p\mathbb{Z}$ by Lemma 2.1.6. Then K_1 has order p and $g^p \in \operatorname{St}_K(1)$. If we are able to prove that the orders of the components of $(g^p)\psi$ are powers of p, it will follow that the order of g^p is a power of p, and as a consequence the order of g will be a power of p. So we fix $z \in \{1, \ldots, p\}$ and we want to show that the order of $(g^p)\psi\pi_z$ is a power of p.

Since g has length n and $g \notin \operatorname{St}_K(1)$, $g = a^i b_1 \ldots b_n$ for some $i \in \mathbb{F}_p \setminus \{0\}$ and some $b_1, \ldots, b_n \in \Delta$. Then, for every $m \in \{1, \ldots, n\}$, there exist $0 \le l_m \le p-1$ and a directed automorphism $d_m \in \bigcup_{j=1}^p B^{(j)}$ defined by $(e_{m,1}, \ldots, e_{m,p-1}) \in \bigcup_{j=1}^p E^{(j)}$ such that $b_m = d_m^{(a^{l_m})}$. By the calculation rule (2.4) we have

$$\begin{aligned} g^p &= (a^i b_1 \ldots b_n)^p \\ &= a^{ip} b_1^{(a^{i(p-1)})} \ldots b_n^{(a^{i(p-1)})} b_1^{(a^{i(p-2)})} \ldots b_n^{(a^{i(p-2)})} \ldots b_1 \ldots b_n \\ &= d_1^{(a^{l_1+i(p-1)})} \ldots d_n^{(a^{l_n+i(p-1)})} d_1^{(a^{l_1+i(p-2)})} \ldots d_n^{(a^{l_n+i(p-2)})} \ldots d_1^{(a^{l_1})} \ldots d_n^{(a^{l_n})}. \end{aligned} \tag{2.10}$$

Now, for every $m \in \{1, \ldots, n\}$, as $i \neq 0$ in $\mathbb{F}_p$ the set $\{l_m + i(p-1), l_m + i(p-2), \ldots, l_m + i, l_m\}$ is a set of representatives for the congruence classes modulo p. Therefore, for every $m \in \{1, \ldots, n\}$,

$$\{d_m^{(a^{l_m+i(p-1)})}, d_m^{(a^{l_m+i(p-2)})}, \ldots, d_m^{(a^{l_m})}\} = \{d_m^{(a^{p-1})}, d_m^{(a^{p-2})}, \ldots, d_m\}$$

and then

$$\begin{aligned} \{d_m^{(a^{l_m+i(p-1)})}, d_m^{(a^{l_m+i(p-2)})}, \ldots, d_m^{(a^{l_m})}\}\psi = \{ &(a^{e_{m,1}}, a^{e_{m,2}}, \ldots, a^{e_{m,p-1}}, d_m), \\ &(d_m, a^{e_{m,1}}, \ldots, a^{e_{m,p-2}}, a^{e_{m,p-1}}), \\ &\vdots \\ &(a^{e_{m,2}}, a^{e_{m,3}}, \ldots, d_m, a^{e_{m,1}}) \}. \end{aligned}$$

It follows that

$$\{d_m^{(a^{l_m+i(p-1)})}, d_m^{(a^{l_m+i(p-2)})}, \ldots, d_m^{(a^{l_m})}\}\psi\pi_z = \{a^{e_{m,1}}, a^{e_{m,2}}, \ldots, a^{e_{m,p-1}}, d_m\}$$

and, from (2.10), we deduce that $(g^p)\psi\pi_z$ is the product, in some order, of the $n \cdot p$ elements

$$\begin{gathered} a^{e_{1,1}}, a^{e_{1,2}}, \ldots, a^{e_{1,p-1}}, d_1 \\ a^{e_{2,1}}, a^{e_{2,2}}, \ldots, a^{e_{2,p-1}}, d_2 \\ \vdots \\ a^{e_{n,1}}, a^{e_{n,2}}, \ldots, a^{e_{n,p-1}}, d_n. \end{gathered}$$

The calculation rule (2.4) yields then that $(g^p)\psi\pi_z$ is of the form

$$a^{\sum_{m=1}^n (e_{m,1}+e_{m,2}+\ldots+e_{m,p-1})} f_1 \ldots f_n,$$

for some $f_1, \ldots, f_n \in \Delta$. Since $(e_{m,1}, e_{m,2}, \ldots, e_{m,p-1}) \in \bigcup_{j=1}^{p} E^{(j)} \subseteq V$ by assumption, $e_{m,1} + e_{m,2} + \ldots + e_{m,p-1} = 0$ for all m. Thus $(g^p)\psi\pi_z$ is equal to $f_1 \ldots f_n$, it belongs to $\mathrm{St}_K(1)$ and it has length $\leq n$. We conclude that the order of $(g^p)\psi\pi_z$ is a power of p by Case 1. □

As reported above, Proposition 2.2.8 does not hold when our groups act on the d-adic tree and d is not prime. Indeed, take for example $d = 4$. The GGS-group defined in Example 2.2.4 acts on the 4-adic rooted tree, it is finite, hence periodic, but its defining tuple $(2, 0, 0)$ does not belong to $V = \{(f_1, f_2, f_3) \in (\mathbb{Z}/4\mathbb{Z})^3 : f_1 + f_2 + f_3 = 0\}$. Furthermore, there are also infinite periodic GGS-groups that have defining tuple out of V. The aforementioned second Grigorchuk group, for instance, is known to be infinite and periodic, but its defining tuple is $(1, 0, 1) \in (\mathbb{Z}/4\mathbb{Z})^3 \setminus V$.

Chapter 3

The exponent growth problem

As mentioned in the introduction, the exponent growth problem is a natural question which arises from the results available in literature about the Burnside Problem.

3.1 Solution for multi-EGS-groups

In general, if G is residually finite and we assume that there exists a chain

$$G \trianglerighteq N_1 \trianglerighteq N_2 \trianglerighteq \cdots \trianglerighteq N_i \trianglerighteq N_{i+1} \trianglerighteq \ldots$$

of finite-index normal subgroups of G such that $\bigcap_{i\geq 1} N_i = 1$, we have

$$\lim_{i\to\infty} \left|\frac{G}{N_i}\right| = |G|$$
$$\lim_{i\to\infty} \exp\left(\frac{G}{N_i}\right) = \exp(G).$$

This applies in particular when $G \leq \operatorname{Aut}\mathcal{T}$ and $N_i = \mathrm{St}_G(i)$. In general one could wonder how the quantities $|G/N_i|$ and $\exp(G/N_i)$ compare. Clearly $\exp(G/N_i)$ divides $|G/N_i|$, hence the second sequence will always grow more slowly than the first sequence does.

If for example p is a prime, $G = \mathbb{Z}$ and $N_i = p^i\mathbb{Z}$, we have

$$\left|\frac{\mathbb{Z}}{p^i\mathbb{Z}}\right| = p^i$$
$$\exp\left(\frac{\mathbb{Z}}{p^i\mathbb{Z}}\right) = p^i.$$

If $G = \Gamma = \mathrm{Lab}(\langle\sigma\rangle)$ is defined over the p-adic rooted tree, where p is a prime number, and $N_i = \mathrm{St}_\Gamma(i)$, we have

$$\left|\frac{\Gamma}{\mathrm{St}_\Gamma(i)}\right| = |\Gamma_i| = p^{\frac{p^i-1}{p-1}} = p^{1+p+\ldots+p^{i-1}}$$
$$\exp\left(\frac{\Gamma}{\mathrm{St}_\Gamma(i)}\right) = \exp(\Gamma_i) = p^i$$

by Equation (1.10) and Lemma 1.5.6. Note that Γ is a profinite group, it is infinite but not finitely generated as an abstract group (because it is infinite and profinite, hence it cannot be countable) and not periodic (because it has non-periodic subgroups, such as all the non-periodic multi-EGS-groups). The group $\mathbb{Z}$ is infinite and finitely generated but not periodic. Now, if G is a multi-EGS-group defined over the p-adic rooted tree and p is prime, G is finitely generated and infinite (by Proposition 2.2.3), and in some cases it can be periodic (according to Proposition 2.2.8). But what can we say, in this case, about $|G_i|$ and $\exp(G_i)$?

As pointed out in the introduction, the solution to the Restricted Burnside Problem guarantees that $\lim_{i\to\infty}\exp(G_i) = \infty$ when G is an infinite finitely generated subgroup of $\operatorname{Aut}\mathcal{T}$. We learned from Section 1.8 that infinite finitely generated subgroups of $\operatorname{Aut}\mathcal{T}$ are not profinite, hence they are not the inverse limit of their congruence quotients, but still their order and exponent can be computed as limits of the orders and exponents of their congruence quotients. Solving the exponent growth problem for an infinite finitely generated subgroup G of $\operatorname{Aut}\mathcal{T}$ means determining how quick the sequence $\exp(G_i)$ goes to infinity as i goes to infinity. In the special case that $G = K$ is a multi-EGS-group, K is contained in Γ, hence K_i can be embedded in Γ_i and

$$|K_i| \quad \text{divides} \quad |\Gamma_i| = p^{1+p+\ldots+p^{i-1}} \tag{3.1}$$

$$\exp(K_i) \quad \text{divides} \quad \exp(\Gamma_i) = p^i. \tag{3.2}$$

If moreover K is a periodic multi-EGS-group, K is periodic and finitely generated. These are two properties "close to finiteness" that Γ is not endowed with. Then one could expect that the exponent of the ith congruence quotient of a multi-EGS-group K, at least in the case where K is periodic, is much smaller than $\exp(\Gamma_i) = p^i$ (notice that the multi-EGS-groups giving a negative answer to the General Burnside Problem are exactly the periodic ones). Theorem 3.1.1 below shows that this intuition is wrong, since the growth of $\exp(K_i)$ to infinity is exponential in i, both in the case that K is a non-periodic multi-EGS-group and in the case that K is a periodic multi-EGS-group.

Theorem 3.1.1. *Let K be a multi-EGS-group defined over the p-adic rooted tree, with p prime. Then for every $i \geq 1$*

$$\exp(K_i) = \begin{cases} p^{\lfloor \frac{i+1}{2} \rfloor} & \text{if } K \text{ is periodic} \\ p^i & \text{otherwise} \end{cases}$$

where $\lfloor \frac{i+1}{2} \rfloor$ is the greatest integer number less than or equal to $\frac{i+1}{2}$.

The proof of Theorem 3.1.1 can be found in [12] and will be discussed in Section 3.2. Clearly, we already have, thanks to (3.2), a good upper bound for $\exp(K_i)$. What is surprising and more delicate to be proved is that, starting from $\exp(K_1) = p$, the exponent of K_i does grow by p whenever i increases by 1 (if K is non-periodic) or whenever i increases by 2 (if K is periodic).

To make a comparison between $|K_i|$ and $\exp(K_i)$, we exploit the following result from [6].

Theorem 3.1.2. *Let G be a GGS-group over the p-adic rooted tree, where p is an odd prime, and let $\mathbf{e} = (e_1, \dots, e_{p-1})$ be the defining vector of G. Then, for every $i \geq 2$, we have*

$$\log_p |G_i| = tp^{i-2} + 1 - \delta \, \frac{p^{i-2} - 1}{p - 1} - \varepsilon \, \frac{p^{i-2} - (i-2)p + i - 3}{(p-1)^2},$$

where t is the rank of the circulant matrix

$$C(\mathbf{e}) = \begin{pmatrix} e_1 & e_2 & \cdots & e_{p-1} & 0 \\ 0 & e_1 & \cdots & e_{p-2} & e_{p-1} \\ \vdots & \vdots & & \vdots & \vdots \\ e_2 & e_3 & \cdots & 0 & e_1 \end{pmatrix},$$

$$\delta = \begin{cases} 1 & \text{if } \mathbf{e} \text{ is symmetric,} \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \varepsilon = \begin{cases} 1 & \text{if } \mathbf{e} \text{ is constant,} \\ 0 & \text{otherwise.} \end{cases}$$

Here a vector $(e_1, \dots, e_{p-1})$ is symmetric if $e_i = e_{p-i}$ for all $i = 1, \dots, p-1$, and a vector $(f_1, \dots, f_{p-1})$ is said to be constant if $f_1 = \dots = f_{p-1}$. Recall that, even if Theorem 3.1.2 is stated in the case that p is an odd prime, the order of congruence quotients is known also when $p = 2$ (see Example 2.1.8).

For example, if K is the Gupta-Sidki group over the p-adic tree, with p odd prime, the defining vector $\mathbf{e} = (1, -1, 0, \dots, 0)$ is not symmetric and the matrix $C(\mathbf{e})$ has rank $p - 1$. Therefore

$$|K_i| = p^{(p-1)p^{i-2}+1}$$

by Theorem 3.1.2. Furthermore K is periodic by Proposition 2.2.8, and then

$$\exp(K_i) = p^{\lfloor \frac{i+1}{2} \rfloor}$$

by Theorem 3.1.1. One can have fun computing and comparing $|K_i|$ and $\exp(K_i)$ whenever K is a GGS-group defined over the p-adic tree and $p \geq 3$ is prime, using Theorem 3.1.2, Theorem 3.1.1 and Proposition 2.2.8.

3.2 Proof of the main result

The point of this last section is to give a complete proof of Theorem 3.1.1. We fix a prime number p and we agree that every multi-EGS-group mentioned in this section will be understood to be defined over the p-adic rooted tree.

We start by proving Theorem 3.1.1 for non-periodic multi-EGS-groups.

Theorem 3.2.1. *Let K be a non-periodic multi-EGS-group. Then*

$$\exp(K_n) = p^n$$

for every $n \geq 1$.

Proof. Take K a multi-EGS-group as in Definition 2.1.2. By (3.2), $\exp(K_n) \leq p^n$ and then it is enough to show that $\exp(K_n) \geq p^n$.

Since K is not periodic, by Proposition 2.2.8 there is $\mathbf{e}$ which lies in $\bigcup_{j=1}^{p} E^{(j)}$ but not in $V = \{(f_1, \dots, f_{p-1}) \in (\mathbb{F}_p)^{p-1} : f_1 + \cdots + f_{p-1} = 0\}$. Therefore, by Lemma 2.2.2, K has a subgroup H which is conjugate to the GGS-group G with defining vector $\mathbf{e}$. Namely, there is $g \in \operatorname{Aut} \mathcal{T}$ such that $G^g = H$. We have

$$G_n = \frac{G}{\operatorname{St}_G(n)} \simeq \frac{G^g}{G^g \cap (\operatorname{St}(n))^g} \overset{(*)}{=} \frac{H}{\operatorname{St}_H(n)} = H_n$$

where the equality (*) holds because $\operatorname{St}(n)$ is normal in $\operatorname{Aut} \mathcal{T}$. Moreover H_n can be embedded in K_n by Lemma 1.4.3. All in all, K_n contains an isomorphic copy of G_n. Hence, to show that $\exp(K_n) \geq p^n$ it suffices to show that $\exp(G_n) \geq p^n$.

As G is the GGS-group with defining vector $\mathbf{e}$ and $\mathbf{e} \notin V = \{(f_1, \dots, f_{p-1}) \in (\mathbb{F}_p)^{p-1} : f_1 + \cdots + f_{p-1} = 0\}$, Lemma 2.2.6 ensures that, for every $n \geq 1$, there are elements of order p^n in G_n. In other words $\exp(G_n)$ is at least p^n, as desired. □

We now prove Theorem 3.1.1 for a periodic multi-EGS-group K. As for the non-periodic case, it will be easy to show that the expression provided by Theorem 3.1.1 is an upper bound for $\exp(K_n)$. More efforts will be needed to prove that such expression is a lower bound too.

We first compute the exponent of K_2. For such purpose, we need to fix the following notation: for every $W \subseteq (\mathbb{F}_p)^{p-1}$, we indicate with $\overline{W}$ the subset of $(\mathbb{F}_p)^p$ given by

$$\overline{W} = \{(e_{p-j+1}, \dots, e_{p-1}, 0, e_1, \dots, e_{p-j}) : 1 \leq j \leq p \text{ and } (e_1, \dots, e_{p-1}) \in W\}.$$

Lemma 3.2.2. *Let K be a multi-EGS-group as in Definition 2.1.2. Then*

$$|K_2| = p^{t+1},$$

where t is the dimension of the $\mathbb{F}_p$-vector space $\langle \overline{E^{(j)}} : j = 1, \dots, p \rangle$.

Proof. Recall that $K = \operatorname{St}_K(1) \rtimes \langle a \rangle$ and $K_2 = \frac{\operatorname{St}_K(1) \rtimes \langle a \rangle}{\operatorname{St}_K(2)} = \frac{\operatorname{St}_K(1)}{\operatorname{St}_K(2)} \rtimes \langle a \operatorname{St}_K(2) \rangle$ has order $|\frac{\operatorname{St}_K(1)}{\operatorname{St}_K(2)}| \cdot |\langle a \operatorname{St}_K(2) \rangle| = |\operatorname{St}_{K_2}(1)| \cdot p$. Then it suffices to show that $|\operatorname{St}_{K_2}(1)| = p^t$, with t as in the statement. Now, the composition φ between the map

$$\begin{aligned} \operatorname{St}_{K_2}(1) &\longrightarrow K_1 \times \overset{p}{\cdots} \times K_1 \\ g \operatorname{St}_K(2) &\longmapsto (g_1 \operatorname{St}_K(1), \dots, g_p \operatorname{St}_K(1)) \end{aligned}$$

and the map

$$\begin{aligned} K_1 \times \overset{p}{\cdots} \times K_1 &\longrightarrow (\mathbb{F}_p)^p \\ (a^{f_1} \operatorname{St}_K(1), \dots, a^{f_p} \operatorname{St}_K(1)) &\longmapsto (f_1, \dots, f_p), \end{aligned}$$

is an injective homomorphism. Indeed, the first map is an injective morphism by Lemma 1.5.8 and the second map is an isomorphism because $K_1 = \frac{\mathrm{St}_K(1)\rtimes\langle a\rangle}{\mathrm{St}_K(1)} \simeq \langle a\rangle \simeq \mathbb{F}_p$. If we are able to prove that

$$(\{g\,\mathrm{St}_K(2) : g \in \Delta\})\varphi = \bigcup_{j=1}^{p} \overline{E^{(j)}}, \tag{3.3}$$

where, as in Lemma 2.1.6, $\Delta = \{b^{(a^l)} : b \in \bigcup_{j=1}^{p} B^{(j)} \text{ and } 0 \leq l \leq p-1\}$, we will get that the $\mathbb{F}_p$-vector space generated by $\bigcup_{j=1}^{p} \overline{E^{(j)}}$ has the same cardinality as the group $\langle\{g\,\mathrm{St}_K(2) : g \in \Delta\}\rangle$, which is equal to $\mathrm{St}_K(1)/\,\mathrm{St}_K(2) = \mathrm{St}_{K_2}(1)$ by Lemma 2.1.6. Such cardinality is p^t, hence this will conclude the proof.

We are then only left to prove (3.3). If $g \in \Delta$, there exist $b \in B^{(j)}$ and $l \in \{0,\dots,p-1\}$ such that $g = b^{(a^l)}$. Then there is $(e_1,\dots,e_{p-1}) \in E^{(j)}$ such that b is the directed automorphism along P_j with defining vector $(e_1,\dots,e_{p-1})$, and Equation (1.14) implies that

$$(b^{(a^l)})\psi = (a^{e_{p-l-j+1}},\dots,a^{e_{p-1}}, b, a^{e_1},\dots,a^{e_{p-l-j}}).$$

It follows that

$$(b^{(a^l)}\,\mathrm{St}_K(2))\varphi = (e_{p-l-j+1},\dots,e_{p-1},0,e_1,\dots,e_{p-l-j})$$

and this vector belongs to $\overline{E^{(j)}}$ because $(e_1,\dots,e_{p-1}) \in E^{(j)}$. Conversely, if a vector $(x_1,\dots,x_p)$ is in $\overline{E^{(j)}}$, there exist $(e_1,\dots,e_{p-1}) \in E^{(j)}$ and $l \in \{1,\dots,p\}$ such that

$$(x_1,\dots,x_p) = (e_{p-l+1},\dots,e_{p-1},0,e_1,\dots,e_{p-l}).$$

Hence $(x_1,\dots,x_p) = (b^{(a^{l-j})}\,\mathrm{St}_K(2))\varphi$, where b is the directed automorphism along P_j with defining vector $(e_1,\dots,e_{p-1})$. Since $(e_1,\dots,e_{p-1})$ lies in $E^{(j)}$, $b \in B^{(j)}$ and $b^{(a^{l-j})} \in \Delta$.

□

Lemma 3.2.3. *Let K be a periodic multi-EGS-group. Then the exponent of K_2 is p.*

Proof. Since K is periodic, keeping as usual the notation of Definition 2.1.2, we have that $\bigcup_{j=1}^{p} \overline{E^{(j)}}$ is contained in $U = \{(f_1,\dots,f_p) \in (\mathbb{F}_p)^p : f_1 + \dots + f_p = 0\}$ by Proposition 2.2.8. Since U is a vector space of dimension $p-1$, Lemma 3.2.2 yields that $|K_2| \leq p^p$. This implies, by Theorem 2.8 of [5], that K_2 is a regular p-group. By a classical result concerning regular p-groups, a regular p-group generated by elements of order p must have exponent p (see [5], Corollary 2.11). The generators of $K = \langle a, B^{(1)},\dots,B^{(p)}\rangle$ have order p, hence the same holds for the generators of the regular p-group K_2, therefore $\exp(K_2) = p$. □

As a matter of fact, there is another way to prove that the exponent of K_2 is p when K is periodic. Indeed, repeating the same steps that we did in the proof of Proposition 2.2.8 to treat Case 1, we can prove that, for any $g \in K$, $g^p \in \mathrm{St}_K(1)$ and all the components of $(g^p)\psi$ belong to $\mathrm{St}_K(1)$. Hence, the injectivity of the morphism

$$\begin{aligned} \mathrm{St}_{K_2}(1) &\longrightarrow K_1 \times \overset{p}{\cdots} \times K_1 \\ f\,\mathrm{St}_K(2) &\longmapsto (f_1\,\mathrm{St}_K(1), \dots, f_p\,\mathrm{St}_K(1)), \end{aligned}$$

which is given by Lemma 1.5.8, yields that the class of g^p in $\mathrm{St}_{K_2}(1)$ is trivial. In other words $g^p \in \mathrm{St}_K(2)$. For the arbitrariness of $g \in K$ we get that K_2 has exponent p.

It was deemed worthy to present also the first procedure (the one that goes through Lemma 3.2.2 and Lemma 3.2.3) because it has some interesting elements. Lemma 3.2.2 is a nice property that has some overlap with Theorem 3.1.2. Regular p-groups, which appear in the proof of Lemma 3.2.3, are a relevant class of p-groups, whose properties save from doing some of the computation.

Since $\exp(K_2) = p$ and K is self-similar, Lemma 1.5.11 yields that $\exp(K_n)$ can grow at most by p whenever n increases by 2. This allows us to bound from above $\exp(K_n)$.

Proposition 3.2.4. *Let K be a periodic multi-EGS-group. Then*

$$\exp(K_n) \leq p^{\lfloor \frac{n+1}{2} \rfloor}$$

for every $n \geq 1$.

Proof. We argue by induction on $n \geq 1$. The result holds for $n = 1$ and $n = 2$ thanks to Lemma 3.2.3. We assume then $n \geq 3$. We have, by Lemma 1.5.11, that $\exp(K_n) \leq \exp(K_{n-2}) \cdot \exp(K_2) = \exp(K_{n-2}) \cdot p$. Thus by induction hypothesis

$$\exp(K_n) \leq p^{\lfloor \frac{n-1}{2} \rfloor} \cdot p = p^{\lfloor \frac{n+1}{2} \rfloor},$$

which concludes the proof. □

Analogously to the non-periodic case, our strategy to bound $\exp(K_n)$ from below will be to prove a bound for GGS-groups and then to look for a suitable GGS-group inside any multi-EGS-group. The most noteworthy part of our proof comes with the next lemma.

Notation. We will write $[x, y]$ for the commutator of x and y, i.e. $[x, y] = x^{-1}y^{-1}xy$.

Lemma 3.2.5. *Assume G to be a periodic GGS-group and let $n \geq 1$ be an integer. Then*

$$\exp(G_{n+2}) \geq \exp(G_n) \cdot p$$

Proof. We deduce from (3.2) that $\exp(G_n) = p^k$ for some $k \geq 0$; then there is $g \in G$ which has order p^k in G_n. In order to prove the claim, it is enough to find an element of G that has order $\geq p^{k+1}$ in G_{n+2}.

By Proposition 2.2 in [6], the composition $\psi\pi_i : \mathrm{St}_G(1) \longrightarrow G$ is a surjective homomorphism. Hence there is $h \in \mathrm{St}_G(1)$ such that $(h)\psi\pi_1 = g$ and, by Lemma 1.5.10, the order of h in G_{n+1} is $\geq p^k$.

From now until the end of this proof, we will denote with $\mathbf{e} = (e_1, \ldots, e_{p-1})$ the defining vector of G and we will indicate with c the directed automorphism along P_p defined by $\mathbf{e}$. If we set $c_i := c^{(a^i)}$ for any integer i, we have $c_i = c_j$ whenever $i \equiv j (\mathrm{mod}\ p)$ and we get, by (1.14), that

$$\begin{aligned} (c_0)\psi &= (a^{e_1}, a^{e_2}, \ldots, a^{e_{p-1}}, c) \\ (c_1)\psi &= (c, a^{e_1}, \ldots, a^{e_{p-2}}, a^{e_{p-1}}) \\ &\vdots \\ (c_{p-1})\psi &= (a^{e_2}, a^{e_3}, \ldots, c, a^{e_1}). \end{aligned} \tag{3.4}$$

Since $h \in \mathrm{St}_G(1)$, $\mathrm{St}_G(1) = \langle c_0, \ldots, c_{p-1} \rangle$ (by Lemma 2.1.6) and $c_0, \ldots, c_{p-1}$ have order p, we can write

$$h = c_{j_1} \ldots c_{j_r}$$

for some $r \geq 1$ and $j_1, \ldots, j_r \in \{0, \ldots, p-1\}$.

By Theorem 2.16 in [6], we may assume $e_1 = 1$, up to replacing G with a GGS-group that is conjugate to G. Indeed, the periodicity of the group does not change if we replace G with a conjugate of G. Moreover, with the same argument used in the proof of Theorem 3.2.1, one can prove that the nth congruence quotient of a conjugate of G is isomorphic to the nth congruence quotient of G. Hence we can assume that $e_1 = 1$ and $(c_1)\psi\pi_2 = a$, as far as our goal is to prove this lemma.

We define the element

$$f_1 = {c_1}^{(c^{j_1})} {c_1}^{(c^{j_2})} \ldots {c_1}^{(c^{j_r})}.$$

Observe that for any integer λ

$$\begin{aligned} ({c_1}^{(c^\lambda)})\psi &= ((c_1)\psi)^{((c^\lambda)\psi)} \\ &= (c, a, \ldots, a^{e_{p-2}}, a^{e_{p-1}})^{(a^\lambda, a^{\lambda e_2}, \ldots, a^{\lambda e_{p-1}}, c^\lambda)} \\ &= (c_\lambda, a, \ldots, a^{e_{p-2}}, (a^{e_{p-1}})^{(c^\lambda)}) \end{aligned}$$

and then

$$\begin{aligned} (f_1)\psi &= \prod_{\beta=1}^{r} (c_{j_\beta}, a, \ldots, a^{e_{p-2}}, (a^{e_{p-1}})^{(c^{j_\beta})}) \\ &= (h, a^r, \ldots, a^{re_{p-2}}, F) \end{aligned}$$

where

$$\begin{aligned}F &= (a^{e_{p-1}})^{(c^{j_1})} \dots (a^{e_{p-1}})^{(c^{j_r})} \\ &= a^{e_{p-1}}[a^{e_{p-1}}, c^{j_1}] \dots a^{e_{p-1}}[a^{e_{p-1}}, c^{j_r}] \\ &= a^{re_{p-1}}[a^{e_{p-1}}, c^{j_1}]^{(a^{(r-1)e_{p-1}})}[a^{e_{p-1}}, c^{j_2}]^{(a^{(r-2)e_{p-1}})} \dots [a^{e_{p-1}}, c^{j_r}]\end{aligned}$$

(here the last equality follows from the calculation rule (2.4)). We set

$$y = [a^{e_{p-1}}, c^{j_1}]^{(a^{(r-1)e_{p-1}})}[a^{e_{p-1}}, c^{j_2}]^{(a^{(r-2)e_{p-1}})} \dots [a^{e_{p-1}}, c^{j_r}],$$

so that $F = a^{re_{p-1}}y$. Clearly y belongs to $\gamma_2(G)$, where, as it is customary, we write $\gamma_i(G)$ for the ith term in the lower central series of G.

At this point, it is convenient to consider the case in which $\mathbf{e}$ is non-symmetric and the case in which $\mathbf{e}$ is symmetric separately. In both cases $\mathbf{e} \in (\mathbb{F}_p)^{p-1}$ cannot be constant. Otherwise we would have $e_1 + \ldots + e_{p-1} = (p-1)e_1 = p-1$ and $p-1 \equiv 0 \pmod p$ by the periodicity of G, which is not possible. Hence $\mathbf{e}$ is not constant and in particular $p \neq 2$.

Assume first $\mathbf{e}$ non-symmetric. In this case, Lemma 3.4 of [6] yields that $\gamma_2(G) \times \overset{p}{\ldots} \times \gamma_2(G) \subseteq (\mathrm{St}_G(1))\psi$. Then there exists $f_2 \in \mathrm{St}_G(1)$ which satisfies $(f_2)\psi = (1, \dots, 1, y^{-1})$ and we get

$$\begin{aligned}(f_1 f_2)\psi &= (h, a^r, a^{re_2}, \dots, a^{re_{p-2}}, F) \cdot (1, \dots, 1, y^{-1}) \\ &= (h, a^{re_1}, a^{re_2}, \dots, a^{re_{p-2}}, a^{re_{p-1}}).\end{aligned}$$

In general, if $(f)\psi = (f_1, \dots, f_p)$, we have by (2.4) and (1.14) that

$$\begin{aligned}((af)^p)\psi &= (f^{(a^{p-1})})\psi(f^{(a^{p-2})})\psi \dots (f^{(a)})\psi(f)\psi \\ &= (\ f_2 \dots f_{p-1} f_p f_1\ ,\ f_3 \dots f_p f_1 f_2\ ,\ \dots\ ,\ f_1 \dots f_{p-2} f_{p-1} f_p\ ).\end{aligned} \tag{3.5}$$

Since G is periodic we have $e_1 + e_2 + \cdots + e_{p-1} = 0$ and (3.5) yields that

$$((af_1f_2)^p)\psi\pi_p = h \cdot a^{r(e_1+e_2+\cdots+e_{p-1})} = h.$$

This implies, by Lemma 1.5.10, that $(af_1f_2)^p$ has order $\geq p^k$ in G_{n+2} and af_1f_2 has order $\geq p^{k+1}$ in G_{n+2}, as desired.

Let us now consider the case in which $\mathbf{e}$ is symmetric. Since we are assuming $e_1 = 1$, we have $e_{p-1} = 1$.

Recall that $h = c_{j_1} \dots c_{j_r}$ and the only assumption we made on h is $(h)\psi\pi_1 = g$. We want to show that, thanks to the symmetry of $\mathbf{e}$, we may assume $r \equiv j_1 + \cdots + j_r \pmod p$. Since 2 is invertible in $\mathbb{F}_p$ (as $p \neq 2$), there is an integer α such that $2\alpha \equiv r - (j_1 + \cdots + j_r) \pmod p$. Then the element

$$h' = ({c_0}^{p-1}c_2)^\alpha h$$

can be written in the form $h' = c_{l_1} \dots c_{l_s}$, with $s = p\alpha + r \equiv r$ and $l_1 + \cdots + l_s \equiv 2\alpha + j_1 + \cdots + j_r \equiv r \pmod p$. Moreover, recalling that $e_1 = e_{p-1} = 1$ and using (3.4), we have

$$(h')\psi\pi_1 = (({c_0}^{p-1}c_2)^\alpha)\psi\pi_1\ (h)\psi\pi_1 = (a^{p-1}a)^\alpha\ (h)\psi\pi_1 = g.$$

Then $h' = c_{l_1} \dots c_{l_s}$ satisfies $s \equiv l_1 + \cdots + l_s \pmod p$ and $(h')\psi\pi_1 = g$. So, we can assume that our element h has the property that $r \equiv j_1 + \cdots + j_r \pmod p$.

Now, by using the commutator identity

$$[\tau, \rho_1 \dots \rho_n] = [\tau, \rho_n][\tau, \rho_{n-1}]^{\rho_n}[\tau, \rho_{n-2}]^{\rho_{n-1}\rho_n} \dots [\tau, \rho_1]^{\rho_2 \cdots \rho_n}$$

and the fact that $\frac{\gamma_2(G)}{\gamma_3(G)} \leq Z\left(\frac{G}{\gamma_3(G)}\right)$, we get

$$\begin{aligned} y &= [a, c^{j_1}]^{(a^{r-1})}[a, c^{j_2}]^{(a^{r-2})} \dots [a, c^{j_r}] \\ &\equiv [a, c^{j_1}][a, c^{j_2}] \dots [a, c^{j_r}] \\ &\equiv [a, c]^{j_1 + \cdots + j_r} \\ &\equiv [a, c]^r \pmod{\gamma_3(G)}, \end{aligned}$$

where the last congruence holds because $[a, c]$ has order p in the quotient $\gamma_2(G)/\gamma_3(G)$ (see Theorem 2.1 in [6]). This yields $y = z \cdot [a, c]^r$ for some $z \in \gamma_3(G)$. Working again modulo $\gamma_3(G)$, we get

$$\begin{aligned} [a, h]^{(a^{-1})} &\equiv [a, c_{j_1} \dots c_{j_r}] \\ &\equiv [a, c_{j_1}] \dots [a, c_{j_r}] \\ &\equiv [a, c]^{(a^{j_1})} \dots [a, c]^{(a^{j_r})} \\ &\equiv [a, c]^r \pmod{\gamma_3(G)} \end{aligned}$$

and $[a, c]^r = [a, h]^{(a^{-1})} \cdot w$ for some $w \in \gamma_3(G)$. Now, since $\mathbf{e}$ is non-constant, Lemma 3.2 of [6] ensures that $\gamma_3(G) \times \overset{p}{\dots} \times \gamma_3(G) \subseteq (\mathrm{St}_G(1))\psi$. Then there must be $f_3 \in \mathrm{St}_G(1)$ with $(f_3)\psi = (w^{-1}, 1, \dots, 1, z^{-1})$ and

$$\begin{aligned} (f_1[c_0, c_1]^r f_3)\psi &= (h, a^r, \dots, a^{re_{p-2}}, a^r \cdot y)([a, c]^r, 1, \dots, 1, [c, a]^r)(w^{-1}, 1, \dots, 1, z^{-1}) \\ &= (h \cdot [a, h]^{(a^{-1})}, a^r, \dots, a^{re_{p-2}}, a^r) \\ &= (h^{(a^{-1})}, a^{re_1}, \dots, a^{re_{p-2}}, a^{re_{p-1}}), \end{aligned}$$

where the second equality holds because $[c, a]$ is the inverse of $[a, c]$, and the last equality holds by the commutator identity $[a, h]^{(a^{-1})} = [h, a^{-1}]$. This implies, again by (3.5), that

$$\begin{aligned} ((af_1[c_0, c_1]^r f_3)^p)\, \psi\pi_p &= h^{(a^{-1})} \cdot a^{r(e_1 + e_2 + \cdots + e_{p-1})} \\ &= h^{(a^{-1})} \end{aligned}$$

and, by Lemma 1.5.10, $af_1[c_0, c_1]^r f_3$ has order $\geq p^{k+1}$ in G_{n+2}. □

Now the next proposition follows immediately by induction on n, using Proposition 3.2.4 and Lemma 3.2.5.

Proposition 3.2.6. *Let G be a periodic GGS-group. Then*

$$\exp(G_n) = p^{\lfloor \frac{n+1}{2} \rfloor}$$

for every $n \geq 1$.

To prove Theorem 3.1.1, we are only left to extend Proposition 3.2.6 to any periodic multi-EGS-group.

Theorem 3.2.7. *Let K be a periodic multi-EGS-group. Then*

$$\exp(K_n) = p^{\lfloor \frac{n+1}{2} \rfloor}$$

for every $n \geq 1$.

Proof. The proof follows the same line of the proof of Theorem 3.2.1. The only difference is that now K is periodic, hence $E^{(j)} \subseteq V = \{(f_1, \ldots, f_{p-1}) \in (\mathbb{F}_p)^{p-1} : f_1 + \ldots + f_{p-1} = 0\}$ for every $j \in \{1, \ldots, p\}$ and, since $E^{(1)}, \ldots, E^{(p)}$ are not all equal to the null subspace, we can pick $\mathbf{e} \in \bigcup_{j=1}^{p} E^{(j)} \setminus \{\mathbf{0}\} \subseteq V$. As in the proof of Theorem 3.2.1, K_n contains an isomorphic copy of G_n, where G is the GGS-group with defining vector $\mathbf{e}$. Since $\mathbf{e} \in V$, G is periodic and $\exp(G_n) = p^{\lfloor \frac{n+1}{2} \rfloor}$ by Proposition 3.2.6. All in all

$$p^{\lfloor \frac{n+1}{2} \rfloor} = \exp(G_n) \leq \exp(K_n) \leq p^{\lfloor \frac{n+1}{2} \rfloor}$$

where the last inequality holds by Proposition 3.2.4. □

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