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Rectifiability of Sets of Finite Perimeter in Carnot Groups

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Introduction

The rectifiability properties of sets is a classical theme of Real Analysis and Geometric Measure Theory. It has been widely studied in the Euclidean case and one of the most important results is De Giorgi's rectifiability theorem, which says that, given a set of locally finite perimeter in $\mathbb{R}^n$, at any point of its reduced boundary the tangent set is unique and is a half-space (see [2], Theorem 3.59). Roughly speaking, we can imagine to take a domain with a nice boundary and zoom in around any point of the boundary; at the limit, we expect to see "something straight" (the tangent hyperplane). As often happens in Mathematical Analysis, given a result in the Euclidean case, the next question is whether the same result holds in a more general space. This is exactly the purpose of this thesis, that is to understand if De Giorgi's rectifiability theorem still holds in Carnot groups. This is a natural question for this ambient space because there is a notion of dilation on it, so a way of zooming. More precisely, we present the proof of the following theorem about the structure of tangent sets to a set of locally finite perimeter in Carnot groups and proved by L. Ambrosio, B. Kleiner and E. Le Donne in [3]:

Theorem. *Let $\mathbb{G}$ be a Carnot group and $E \subseteq \mathbb{G}$ a set of locally finite perimeter. Then at $|D\chi_E|$-a.e. $x \in \partial^* E$ there exists a tangent set which is a half-space.*

In simple terms, if we zoom in around almost every point of the reduced boundary we could see different objects at the limit, but at least one of them is a hyperplane.

In [13] B. Franchi, R. Serapioni and F. Serra Cassano proved that an analogous theorem to De Giorgi's one holds if the Carnot group has step 2. This fact leads to a complete classification of tangent sets and has relevant consequences, as in the classical theory, on the representation of the perimeter in terms of the (spherical) Hausdorff measure and on the rectifiability, in a suitable intrinsic sense, of the measure-theoretic boundary (see [13] for more precise information). However, they also provided a counterexample showing that in a Carnot group of step 3, called Engel group, there is a set of locally finite perimeter with a point in the reduced boundary at which the tangent set is a cone, but not a half-space. Actually, one expects that a similar result holds for a general Carnot group (up to consider almost every point of the reduced boundary and not all) because vectors of the horizontal layer generate the whole Lie algebra, but what is still missing is some monotonicity/stability argument that singles out half-spaces as the only possible tangent sets, making this question completely open.

The result of this work is a text divided into five chapters. In the first one, we review some basic notions and facts from Differential Geometry. Above all, we recall everything we need about vector fields, which play a fundamental role in our arguments, and examine them also as operators on non-smooth functions.

The second chapter is devoted to introduce the ambient space of the thesis, that is Carnot groups. In particular, we show that they can be identified with some $\mathbb{R}^n$ endowed with a suitable group law and a geometry different from the Euclidean one, induced by an intrinsic notion of dilation. Furthermore, we define a specific distance (the Carnot-Carathéodory distance) which derives from the fact that Carnot groups are special examples of sub-Riemannian manifolds.

The third chapter begins with the classical measure theory in metric spaces; we define Radon and Hausdorff measures (other essential concepts for this work) and recall the main related results. Then we introduce Haar measures, which are a natural type of measure to consider when one deals with a group structure, and see how all these notions can be specialized to Carnot groups.

In the fourth chapter, the geometric objects involved in our discussions are presented, above all sets of locally finite perimeter and half-spaces. Moreover, we take a decisive intermediate step towards the announced theorem by proving that if we iterate sufficiently many times the tangent sets to a set of locally finite perimeter we obtain a half-space.

Finally, in the fifth chapter we provide the conclusion of the proof by showing that iterated tangent sets are tangent sets to the initial set. After this, we review the counterexample in [13] of the cone in the Engel group. This chapter is also the occasion to highlight the differences between the step 2 case and the general one.

Here we give a sketch of the proof: let us call regular directions of E the vector fields Z in the Lie algebra $\mathfrak{g}$ such that $Z\chi_E$ is representable by a Radon measure and invariant directions those for which the measure is 0. As in the proof of De Giorgi's rectifiability theorem, the crucial step is the study of the blow-up of E at any point $\bar{x}$ of the reduced boundary $\partial^* E$, i.e. the limits

$$\lim_{i\to\infty} \delta_{\frac{1}{r_i}}(\bar{x}^{-1}E),$$

where $\{r_i\}_{i\in\mathbb{N}}$ is a sequence of positive real numbers which decreases to 0 and $\{\delta_\lambda\}_{\lambda\geq 0}$ are the intrinsic dilations of $\mathbb{G}$. Proceeding as in the proof in [13] for the step 2 case, we can prove that any scaling sequence leads to a tangent set E^1 (which depends on the sequence) with constant horizontal normal equal to the horizontal normal $\nu_E(\bar{x})$ to E at $\bar{x}$. Given now an orthonormal basis $X_1, \ldots, X_m$ of the horizontal layer and the direction

$$X := \sum_{i=1}^{m} \nu_{E,i}(\bar{x}) X_i,$$

we look at the vector space spanned by $\mathrm{Ad}_{\exp(Y)}(X)$ as Y varies among the invariant directions of E^1, since the adjoint operator $\mathrm{Ad}_{\exp(Y)} : \mathfrak{g} \longrightarrow \mathfrak{g}$ has the property of mapping regular directions into regular directions whenever Y is an invariant direction. This fact allows to find a new regular direction Z with the property of having no components in the horizontal layer. We can then consider the last component of Z and it turns out to be an invariant direction of a tangent set E^2 to E^1 at a suitable point in $\partial^* E^1$. Having gained this new direction, this procedure can be restarted: the adjoint operator can be used to generate a new regular direction, so that a new tangent set with an invariant direction appears, and so on. What is then proved is that after a finite number k of iterations (with k depending only on the stratification of $\mathfrak{g}$) we have that E^k is a half-space. At this point, it only remains to prove that this half-space is tangent to E at $\bar{x}$. The proof of this fact is inspired by a similar result in the context of tangent measures in $\mathbb{R}^n$ which appears in [25]; adapting its proof to the setting of Carnot groups, we achieve our goal.

Contents

Chapter 1

Vector fields and related notions

The aim of this chapter is to review some facts regarding vector fields, introduce some related definitions that will be needed and fix the notation which will be used. In the first two sections we will not provide all the details to avoid discussions that are too long and far from our purposes: refer, for example, to [20] to see them.

1.1 Vector fields

From now on, whenever we say "smooth" we mean "of class C^∞". Furthermore, we point out that we will only consider real manifolds (without boundary) of finite dimension.

Definition 1.1. Let M be a smooth manifold. A *tangent vector* v at a point $x \in M$ is a map $v : C^\infty(M) \longrightarrow \mathbb{R}$ which is a derivation, i.e. it is linear and satisfies the Leibniz rule

$$v(fg) = v(f)g(x) + f(x)v(g)$$

for every $f, g \in C^\infty(M)$. The *tangent space* T_xM is the set of all tangent vectors at x.

Clearly the tangent space has the natural structure of vector space. Moreover, it turns out that it has the same dimension of the manifold and a basis is given by partial derivative operators. Precisely, if n is the dimension of the manifold M and $(x_1, \ldots, x_n)$ is a system of local coordinates around a point $\bar{x} \in M$, $\left\{\frac{\partial}{\partial x_i}\Big|_{\bar{x}}\right\}_{i=1,\ldots,n}$ is a basis of $T_{\bar{x}}M$. Sometimes we will simplify the notation with the symbol ∂_{x_i} or ∂_i.

Definition 1.2. Let $F : M \longrightarrow N$ be a smooth map between two smooth manifolds M, N and let $x \in M$. The *differential* of F at x is the map $dF_x : T_xM \longrightarrow T_{F(x)}N$ that assigns to any $v \in T_xM$ the tangent vector $dF_x(v)$ that acts as follows:

$$dF_x(v)(g) = v(g \circ F) \qquad \text{for every } g \in C^\infty(N).$$

It is easy to observe that the differential is a linear map and satisfies the chain rule: given two smooth maps $F : M \longrightarrow N$, $G : N \longrightarrow P$ between manifolds,

$$d(G \circ F)_x = dG_{F(x)} \circ dF_x \qquad \text{for every } x \in M. \tag{1.1}$$

Given a smooth manifold M, the *tangent bundle* TM of M is the disjoint union

$$TM := \bigsqcup_{x \in M} T_xM$$

endowed with the natural projection $\pi : TM \longrightarrow M$, $\pi(v) = x$ if $v \in T_xM$. It can be shown that TM can be equipped with a structure of smooth manifold, so the following definition is well posed.

Definition 1.3. A (*smooth*) *vector field* is a smooth section of TM, i.e. a smooth map $X : M \longrightarrow TM$ such that $\pi \circ X = \mathrm{id}_M$.

We denote by $\mathfrak{X}(M)$ the set of all vector fields on M and, clearly, it is a vector space.
Vector fields can be used as operators that act on smooth functions. Precisely, given a vector field $X \in \mathfrak{X}(M)$, let us indicate by X_x the value $X(x)$ of X at $x \in M$; for any $f \in C^\infty(M)$, we set

$$Xf(x) := X_x(f).$$

Then, by definition of tangent vector, every vector field X induces a linear map $X : C^\infty(M) \longrightarrow C^\infty(M)$ which satisfies

$$X(fg) = (Xf)g + f(Xg). \tag{1.2}$$

Suppose that $\gamma : I \longrightarrow M$ is a smooth curve on the manifold M ($I \subseteq \mathbb{R}$ is an interval). For any $t_0 \in I$, we denote by $\dot{\gamma}(t_0) := d\gamma(\frac{d}{dt}\big|_{t_0})$ the *tangent vector* to γ at the point $\gamma(t_0)$, so that $\dot{\gamma}$ is the *tangent vector field* along γ (sometimes we will indicate it by $\frac{d}{dt}\gamma$). If $F : M \longrightarrow N$ is a smooth map, by the chain rule (1.1) we have

$$\frac{d}{dt}(F \circ \gamma)(t) = dF_{\gamma(t)}(\dot{\gamma}(t)). \tag{1.3}$$

In particular, we can make explicit the action of the differential of a smooth function $f \in C^\infty(M)$ on a vector field $X \in \mathfrak{X}(M)$:

$$df_x(X) = X_x(f) \qquad \text{for every } x \in M.$$

Let us remember that a map between two smooth manifolds is a diffeomorphism if it is smooth, invertible and its inverse is smooth.

Definition 1.4. Given two smooth manifolds M, N and a diffeomorphism $F : M \longrightarrow N$, the *pushforward* of a vector field $X \in \mathfrak{X}(M)$ by F is the vector field $F_*X \in \mathfrak{X}(N)$ defined by the identity

$$(F_*X)_{F(x)} = dF_x(X_x) \qquad \text{for every } x \in M$$

or, equivalently,

$$(F_*X)g = X(g \circ F) \circ F^{-1} \qquad \text{for every } g \in C^\infty(N).$$

The *pullback* of a vector field $Y \in \mathfrak{X}(N)$ by F is the vector field $F^*Y \in \mathfrak{X}(M)$ defined as the pushforward of Y by F^{-1}, i.e. $F^*Y = (F^{-1})_*Y$.

Now let us define an important binary operation on $\mathfrak{X}(M)$:

Definition 1.5. The *Lie bracket* of two vector fields X, $Y \in \mathfrak{X}(M)$ is the vector field $[X,Y] := XY - YX$, i.e. the vector field that acts in the following way: $[X,Y]f = X(Yf) - Y(Xf)$ for every $f \in C^\infty(M)$.

Proposition 1.6. *For any X,Y,$Z \in \mathfrak{X}(M)$ the Lie bracket satisfies the following properties:*

(i) $[X+Y,Z] = [X,Z] + [Y,Z]$,

(ii) $[X,Y] = -[Y,X]$,

(iii) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

Proof. *(i)* and *(ii)* are direct consequences of the definition. *(iii)* is also a simple consequence of the definition (just expand the left-hand side and all terms simplify). □

The next proposition describes the pushforward of Lie bracket, indeed it says that pushforward and Lie bracket commute.

Proposition 1.7. *If $F : M \longrightarrow N$ is a diffeomorphism, then for any X, $Y \in \mathfrak{X}(M)$ we have*

$$F_*[X,Y] = [F_*X, F_*Y].$$

1.2 Flows and Lie derivatives

Given a smooth manifold M and a vector field $X \in \mathfrak{X}(M)$, we can consider the Cauchy problem

$$\begin{cases} \frac{d}{dt}\Phi^X(x,t) = X(\Phi^X(x,t)) \\ \Phi^X(x,0) = x, \end{cases}$$

for $x \in M$ and $t \in \mathbb{R}$. X is smooth, so the classical theory of ODEs ensures local existence, uniqueness and smoothness of the solution $\Phi^X : M \times I \longrightarrow M$ (for some open interval $I \subseteq \mathbb{R}$ containing 0). The map Φ^X is called *flow* of the vector field X. If the solution exists globally ($I = \mathbb{R}$), X and Φ^X are said to be *complete*. For the sake of simplicity, let us assume from now on that any considered vector field is complete (however, this is a property we will have from some point on).
Setting $\Phi^X_t := \Phi_X(\cdot, t)$, we have $\Phi^X_0 = \mathrm{id}_M$ and the semigroup property

$$\Phi^X_{s+t} = \Phi^X_s \circ \Phi^X_t \qquad \text{for every } s,t \in \mathbb{R}. \tag{1.4}$$

In particular, $(\Phi^X_t)^{-1} = \Phi^X_{-t}$ and $\Phi^X_t : M \longrightarrow M$ is a diffeomorphism.
Suppose $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. By definition, X is the tangent vector field along its flow, hence if we consider the curve $\Phi^X(x,\cdot) : \mathbb{R} \longrightarrow M$, for some fixed $x \in M$, we get from (1.3) the following formula:

$$\frac{d}{dt}(f \circ \Phi^X)(x,t) = Xf(\Phi^X(x,t)). \tag{1.5}$$

Remark 1.8. (1.5) says that applying a vector field to a smooth function means to differentiate the function along it (or its flow). As a consequence, if $Xf = 0$, then f is constant along the flow of X, that is $f = f \circ \Phi^X_t$ for every (fixed) $t \in \mathbb{R}$.

The next result provides a useful formula for the flow of the pushforward of a vector field.

Proposition 1.9. *Let $F : M \longrightarrow N$ be a diffeomorphism and $X \in \mathfrak{X}(M)$. Then*

$$\Phi^{F_*X}_t = F \circ \Phi^X_t \circ F^{-1}. \tag{1.6}$$

The flow allows to differentiate differential forms along a vector field. Let us denote by $F^*\omega$ the pullback by a diffeomorphism $F : M \longrightarrow N$ of a differential form ω on N.

Definition 1.10. Let $X \in \mathfrak{X}(M)$ be a vector field and $\omega \in \Omega^k(M)$ a k-form on a smooth manifold M (for some integer $k \geq 0$). The *Lie derivative* of ω along X is the k-form $L_X\omega$ defined by

$$L_X\omega(x) := \left.\frac{d}{dt}\right|_{t=0} (\Phi^X_t)^*\omega(x) \qquad \text{for every } x \in M.$$

If we indicate by $\wedge$ the wedge product between differential forms, from the definition it can be easily shown that the operator L_X enjoys the property

$$L_X(\omega_1 \wedge \omega_2) = (L_X\omega_1) \wedge \omega_2 + \omega_1 \wedge (L_X\omega_2)$$

for every couple of differential forms ω_1, ω_2; in particular, for any $\omega \in \Omega^k(M)$ and $f \in C^\infty(M) = \Omega^0(M)$ we have

$$L_X(f\omega) = (Xf)\omega + fL_X\omega. \tag{1.7}$$

Furthermore, it follows almost immediately from the definition

$$\frac{d}{dt}(\Phi_t^X)^*\omega = (\Phi_t^X)^* L_X\omega. \tag{1.8}$$

In practice the definition of Lie derivative can be difficult to deal with, but the next theorem provides a very practical formula. Let us denote by $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ the exterior derivative operator and by $i_X : \Omega^k(M) \longrightarrow \Omega^{k-1}(M)$ the interior multiplication by a vector field $X \in \mathfrak{X}(M)$, which is defined by $(i_X\omega)(Y_1, \ldots, Y_{k-1}) := \omega(X, Y_1, \ldots, Y_{k-1})$ for every $Y_1, \ldots, Y_{k-1} \in \mathfrak{X}(M)$.

Theorem 1.11 (Cartan's magic formula)**.** *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. For any integer $k \geq 0$ and k-form $\omega \in \Omega^k(M)$*

$$L_X\omega = d(i_X\omega) + i_X(d\omega). \tag{1.9}$$

1.3 Divergence and X-distributional derivative

On the way to our goal we will almost always have to deal with functions that are not smooth in general. In the previous sections we saw how a vector field acts on smooth functions, so now our aim is to expand this action on locally integrable ones.
Let M be an n-dimensional smooth manifold which is orientable, so that we can fix a volume form vol_M, namely a never vanishing top-degree form (in this case, an n-form).

Definition 1.12. The *divergence* of a vector field $X \in \mathfrak{X}(M)$ is the function $\mathrm{div}X \in C^\infty(M)$ such that

$$\int_M Xu \, d\mathrm{vol}_M = -\int_M u \, \mathrm{div}X \, d\mathrm{vol}_M \qquad \text{for every } u \in C_c^\infty(M). \tag{1.10}$$

X is said to be *divergence-free* if $\mathrm{div}X \equiv 0$.

Remark 1.13. If (M, g) is a Riemannian manifold and vol_M the volume form induced by the metric g, one can find an explicit expression for $\mathrm{div}X$ in terms of the components of X and recognize that (1.10) is the divergence theorem on manifolds. Anyhow, we will not need a Riemannian structure and we will take (1.10) as the definition of divergence. Notice that it depends only on the volume form.

Let us immediately state and prove a useful characterization for the divergence of a vector field, which is usually the definition used in Differential Geometry:

Proposition 1.14. *Given a vector field $X \in \mathfrak{X}(M)$, a function $f \in C^\infty(M)$ is equal to* $\mathrm{div}X$ *if and only if $L_X(\mathrm{vol}_M) = f\,\mathrm{vol}_M$.*

Proof. For any $u \in C_c^\infty(M)$ we know by (1.7)

$$L_X(u\,\mathrm{vol}_M) = (Xu)\,\mathrm{vol}_M + uL_X(\mathrm{vol}_M). \tag{1.11}$$

Integrating both sides on M and using (1.10) we get

$$\int_M L_X(u\,\mathrm{vol}_M) = -\int_M u\,\mathrm{div}X\,d\mathrm{vol}_M + \int_M uL_X(\mathrm{vol}_M).$$

The integral on the left-hand side is 0 by Cartan's formula (1.9), the fact that the exterior derivative of a top-degree form is 0, Stokes' theorem and the fact that the boundary of a manifold is empty:

$$\int_M L_X(u\,\mathrm{vol}_M) = \int_M d(i_X(u\,\mathrm{vol}_M)) = \int_{\partial M} i_X(u\,\mathrm{vol}_M) = 0.$$

Therefore we have

$$\int_M u\,\mathrm{div}X\,d\mathrm{vol}_M = \int_M u\,L_X(\mathrm{vol}_M) \qquad \text{for every } u \in C_c^\infty(M)$$

and this implies $L_X(\mathrm{vol}_M) = (\mathrm{div}X)\mathrm{vol}_M$ by the fundamental lemma of calculus of variations. Viceversa, using the hypothesis on (1.11) and integrating we have

$$\int_M L_X(u\,\mathrm{vol}_M) = \int_M Xu\,d\mathrm{vol}_M + \int_M fu\,d\mathrm{vol}_M.$$

As observed before, the integral on the left-hand side is 0 and so f satisfies (1.10). □

Suppose now that $X \in \mathfrak{X}(M)$ is divergence-free. Combining (1.2) and (1.10) we deduce

$$\int_M fXg\,d\mathrm{vol}_M = -\int_M gXf\,d\mathrm{vol}_M \qquad \text{for every } f, g \in C_c^\infty(M). \tag{1.12}$$

This motivates the next definition.

Definition 1.15. Let $f \in L^1_{\mathrm{loc}}(M)$ and $X \in \mathfrak{X}(M)$ be divergence-free. The *X-distributional derivative* of f is the distribution

$$\langle Xf, u\rangle := -\int_M fXu\,d\mathrm{vol}_M\,, \qquad u \in C_c^\infty(M).$$

If $g \in L^1_{\mathrm{loc}}(M)$, we write $Xf = g$ if $\langle Xf, u\rangle = \int_M gu\,d\mathrm{vol}_M$ for all $u \in C_c^\infty(M)$. If μ is a Radon measure (see Definition 3.1) on M, we write $Xf = \mu$ if $\langle Xf, u\rangle = \int_M u\,d\mu$ for all $u \in C_c^\infty(M)$.

This definition is reasonable: if $f \in C^1(M)$, by (1.12) (which is still valid) we obtain that the X-distributional derivative of f coincides with the classical action of X on f defined in Section 1.1 (actually, in Section 1.1 we defined the action of a vector field on smooth functions in such a way that the result is still a smooth function, but in order to be well defined only C^1-regularity is needed).

Example 1.16. Let us compute the X-distributional derivative of the characteristic function of a "nice" domain in $\mathbb{R}^n$. Let $E \subseteq \mathbb{R}^n$ be the sub-level set of a C^1-function, i.e. there exist $f \in C^1(\mathbb{R}^n)$ and $c \in \mathbb{R}$ such that $E = \{x \in \mathbb{R}^n : f(x) \le c\}$ and $\nabla f(x) \neq 0$ for every $x \in \partial E$. Let $X \in \mathfrak{X}(\mathbb{R}^n)$ be divergence-free. If we denote by $\langle \cdot, \cdot \rangle$ the Euclidean scalar product, we have for any $u \in C_c^\infty(\mathbb{R}^n)$

$$\operatorname{div}(uX) = \langle X, \nabla u \rangle + u \operatorname{div} X = Xu,$$

so by the divergence theorem

$$\int_E Xu \, dx = \int_{\partial E} u \langle X, \nu_E^{eu} \rangle \, d\mathcal{H}^{n-1},$$

where ν_E^{eu} is the unit outer normal to E. This proves

$$X\chi_E = -\langle X, \nu_E^{eu} \rangle \mathcal{H}^{n-1}_{\llcorner \partial E}.$$

But from our hypothesis we know $\nu_E^{eu}(x) = \nabla f(x)/|\nabla f(x)|$, so

$$\langle X, \nu_E^{eu} \rangle = \langle X, \frac{\nabla f}{|\nabla f|} \rangle = \frac{\langle X, \nabla f \rangle}{|\nabla f|} = \frac{Xf}{|\nabla f|}$$

and substituting we finally get

$$X\chi_E = -\frac{Xf}{|\nabla f|} \mathcal{H}^{n-1}_{\llcorner \partial E}. \tag{1.13}$$

An important property of divergence-free vector fields is the fact that their flows are volume-preserving:

Proposition 1.17. *Let $X \in \mathfrak{X}(M)$. The flow Φ_t^X is vol_M-measure preserving (i.e. for any $t \in \mathbb{R}$ and $A \subseteq M$ Borel set $\mathrm{vol}_M((\Phi_t^X)^{-1}(A)) = \mathrm{vol}_M(A)$) if and only if X is divergence-free.*

Proof. If $u \in C_c^\infty(M)$, the measure preserving property implies

$$\int_M u \, d\mathrm{vol}_M = \int_M u \circ \Phi_t^X \, d\mathrm{vol}_M \qquad \text{for every } t \in \mathbb{R}.$$

Differentiating both sides with respect to t, moving the derivative inside the integral sign (which can be done because the integrand is smooth) and using (1.5), we have

$$0 = \int_M \frac{d}{dt} u(\Phi^X(x,t)) \, d\mathrm{vol}_M(x) = \int_M Xu(\Phi^X(x,t)) \, d\mathrm{vol}_M(x).$$

Then, again by the measure preserving property and (1.10),

$$0 = \int_M Xu \, d\mathrm{vol}_M = -\int_M u \, \mathrm{div}X \, d\mathrm{vol}_M$$

for all $u \in C_c^\infty(M)$, which implies $\mathrm{div}X = 0$. For the converse implication, let us observe that

$$\mathrm{vol}_M(\Phi_t^X(A)) = \int_{\Phi_t^X(A)} d\mathrm{vol}_M = \int_A (\Phi_t^X)^*(\mathrm{vol}_M),$$

therefore by (1.8) and Proposition 1.14

$$\begin{aligned}\frac{d}{dt}\mathrm{vol}_M(\Phi_t^X(A)) &= \int_A \frac{d}{dt}(\Phi_t^X)^*(\mathrm{vol}_M) \\ &= \int_A (\Phi_t^X)^* L_X(\mathrm{vol}_M) \\ &= \int_{\Phi_t^X(A)} \mathrm{div}X \, d\mathrm{vol}_M = 0\end{aligned}$$

This yields that $\mathrm{vol}_M(\Phi_t^X(A))$ does not depend on t and so we have the thesis. □

In Remark 1.8 we saw that if the derivative of a smooth function along a vector field is 0 then the function is constant along the flow. Let us prove now that this result holds even for X-distributional derivatives.

Proposition 1.18. *Let $f \in L^1_{\mathrm{loc}}(M)$ and $X \in \mathfrak{X}(M)$ be divergence-free. If $Xf = 0$ in the sense of distributions, then, for any $t \in \mathbb{R}$, $f = f \circ \Phi_t^X$ vol_M-a.e. in M.*

Proof. Thanks to the fundamental lemma of calculus of variations, it is enough to show that for any $u \in C_c^\infty(M)$ the quantity $\int_M (f \circ \Phi_t^X) u \, d\mathrm{vol}_M$ is independent of t. For any $s, t \in \mathbb{R}$, from Proposition 1.17 and the semigroup property (1.4) we get

$$\begin{aligned}&\int_M (f \circ \Phi_{t+s}^X) u \, d\mathrm{vol}_M - \int_M (f \circ \Phi_t^X) u \, d\mathrm{vol}_M \\ &\quad = \int_M f(u \circ \Phi_{-t-s}^X) \, d\mathrm{vol}_M - \int_M f(u \circ \Phi_{-t}^X) \, d\mathrm{vol}_M \\ &\quad = \int_M f(u \circ \Phi_{-t}^X \circ \Phi_{-s}^X) \, d\mathrm{vol}_M - \int_M f(u \circ \Phi_{-t}^X) \, d\mathrm{vol}_M.\end{aligned}$$

A Taylor expansion around $s = 0$, combined with (1.5), ensures

$$\begin{aligned}(u \circ \Phi_{-t}^X \circ \Phi_{-s}^X)(x) &= (u \circ \Phi_{-t}^X \circ \Phi^X)(x, -s) \\ &= (u \circ \Phi_{-t}^X)(x) - sX(u \circ \Phi_{-t}^X)(x) + o(s),\end{aligned}$$

so substituting

$$\begin{aligned}\int_M (f \circ \Phi^X_{t+s}) u \, d\mathrm{vol}_M - \int_M (f \circ \Phi^X_t) u \, d\mathrm{vol}_M \\ = -s \int_M f X(u \circ \Phi^X_{-t}) \, d\mathrm{vol}_M + o(s) = o(s),\end{aligned}$$

where the last equality holds by virtue of the hypothesis $Xf = 0$ in the sense of distributions. This concludes the proof. □

Chapter 2

Carnot groups

Carnot groups are the ambient space in which we place ourselves for this thesis, so in this chapter we introduce them with properties that we will need.

2.1 Lie groups and algebras

Definition 2.1. A group $\mathbb{G}$ is said to be a *Lie group* if it is also a smooth manifold in which the group operations of multiplication and inversion, i.e. the maps

$$\mu : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}, \quad \mu(g, h) = gh,$$
$$\iota : \mathbb{G} \longrightarrow \mathbb{G}, \quad \iota(g) = g^{-1},$$

are smooth.

Let us denote by e the identity element of the group.

Definition 2.2. A vector space $\mathfrak{g}$ is a *Lie algebra* if it is equipped with an operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which is bilinear, skew-symmetric and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \qquad \text{for every } X, Y, Z \in \mathfrak{g}.$$

By Proposition 1.6, an example of Lie algebra is the set $\mathfrak{X}(M)$ of vector fields on a smooth manifold M endowed with the Lie brackets $[X, Y] = XY - YX$.
Let $\mathbb{G}$ be a Lie group. For any $g \in \mathbb{G}$, let us denote by L_g the left translation by g, that is the map

$$L_g : \mathbb{G} \longrightarrow \mathbb{G}, \quad L_g(h) = gh.$$

It is the composition of the smooth maps $h \longmapsto (g, h)$ and μ and clearly its inverse is $L_g^{-1} = L_{g^{-1}}$, so it is a diffeomorphism. The same thing applies if we consider the right translation by g

$$R_g : \mathbb{G} \longrightarrow \mathbb{G}, \quad R_g(h) = hg.$$

Definition 2.3. A vector field $X \in \mathfrak{X}(\mathbb{G})$ is *left-invariant* if for any $g \in \mathbb{G}$ $(L_g)_* X = X$, i.e.

$$d(L_g)_h(X_h) = X_{gh} \qquad \text{for every } h \in \mathbb{G}.$$

Remark 2.4. Actually, it is equivalent to say that the property only holds for $h = e$, that is $X_g = d(L_g)_e(X_e)$ for every $g \in \mathbb{G}$. Indeed, if this is true we have by the chain rule (1.1)

$$\begin{aligned} d(L_g)_h(X_h) &= d(L_g)_h d(L_h)_e(X_e) \\ &= d(L_g \circ L_h)_e(X_e) \\ &= d(L_{gh})_e(X_e) = X_{gh} \end{aligned}$$

for all $h \in \mathbb{G}$. In other words, left-invariant vector fields are characterized by their value at one point, for example the identity.

In differential terms, X is left-invariant if

$$X(f \circ L_g)(x) = Xf(L_g(x)) \tag{2.1}$$

for every $x, g \in \mathbb{G}$, $f \in C^\infty(\mathbb{G})$. Let us denote by $\mathfrak{X}_L(\mathbb{G})$ the linear subspace of $\mathfrak{X}(\mathbb{G})$ of left-invariant vector fields. Thanks to Proposition 1.7, it is immediate to see that the Lie bracket of two left-invariant vector fields is still left-invariant, so $\mathfrak{X}_L(\mathbb{G})$ is a Lie subalgebra of $\mathfrak{X}(\mathbb{G})$. Moreover, Remark 2.4 says that it is isomorphic to $T_e\mathbb{G}$ by the Lie algebra isomorphism

$$\begin{aligned} \mathfrak{X}_L(\mathbb{G}) &\longrightarrow T_e\mathbb{G} \\ X &\longmapsto X_e. \end{aligned}$$

In particular, if $\dim \mathbb{G} = n$, $\mathfrak{X}_L(\mathbb{G})$ is an n-dimensional subspace of $\mathfrak{X}(\mathbb{G})$. We will often identify $\mathfrak{X}_L(\mathbb{G})$ and $T_e\mathbb{G}$ and indicate them by $\mathfrak{g}$. $\mathfrak{g}$ is called the *Lie algebra* of $\mathbb{G}$.

Remark 2.5. Analogously we can introduce *right-invariant* vector fields $\mathfrak{X}_R(\mathbb{G})$ and show that they can be identified with $T_e\mathbb{G}$. It is a convention to choose left-invariant ones as the Lie algebra of a Lie group. We just stress the fact that in general a right-invariant vector field is not left-invariant and viceversa.

Proposition 2.6. *For any left-invariant vector field $X \in \mathfrak{g}$ the following properties hold:*

(i) $\Phi_t^X(g) = g\Phi_t^X(e)$ *for every $g \in \mathbb{G}$, $t \in \mathbb{R}$. In other words,* $\Phi_t^X = R_{\Phi_t^X(e)}$.

(ii) $\Phi_{s+t}^X(e) = \Phi_s^X(e)\Phi_t^X(e) = \Phi_t^X(e)\Phi_s^X(e)$ *for every $s, t \in \mathbb{R}$.*

Proof. By left-invariance of X, we have from (1.6) $\Phi_t^X \circ L_g = L_g \circ \Phi_t^X$. Evaluating this equality at e gives *(i)*. Using the semigroup property (1.4) and *(i)* we get $\Phi_{s+t}^X(e) = \Phi_t^X(\Phi_s^X(e)) = \Phi_s^X(e)\Phi_t^X(e)$ and exchanging the order of s and t we also have the other equality of *(ii)*. □

Another important property of left-invariant vector fields is that they are complete (see [20], Theorem 9.18), therefore from now on the assumption of completeness will no longer be necessary. Furthermore, for this reason we can set $\exp(X)(g) \coloneqq \Phi_1^X(g)$, for $g \in \mathbb{G}$ and $X \in \mathfrak{g}$, and the following definition is well posed.

Definition 2.7. The *exponential map* $\exp : \mathfrak{g} \longrightarrow \mathbb{G}$ is defined by $\exp(X) \coloneqq \exp(X)(e)$.

In this way we have $\exp(tX) = \Phi_1^{tX}(e) = \Phi_t^X(e)$ for every $t \in \mathbb{R}$. Moreover, $\exp(X)(g) = g\exp(X)$ by Proposition 2.6, so for any $X, Y \in \mathfrak{g}$

$$\exp(Y)(\exp(X)) = \exp(X)\exp(Y). \tag{2.2}$$

We can further make explicit (2.2) by the so-called *Baker-Campbell-Hausdorff formula*: for two multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{N}^k$ such that $\alpha_k + \beta_k \geq 1$ and two vector fields X and Y, let us define

$$C_{\alpha\beta}(X,Y) \coloneqq \begin{cases} (\mathrm{ad}_X)^{\alpha_1}(\mathrm{ad}_Y)^{\beta_1}\ldots(\mathrm{ad}_X)^{\alpha_k}(\mathrm{ad}_Y)^{\beta_k - 1}Y & \text{if } \beta_k \geq 1 \\ (\mathrm{ad}_X)^{\alpha_1}(\mathrm{ad}_Y)^{\beta_1}\ldots(\mathrm{ad}_X)^{\alpha_k - 1}X & \text{if } \beta_k = 0, \end{cases}$$

where $(\mathrm{ad}_X)^n Y \coloneqq [\underbrace{X, [X, [\ldots [X}_{n}, Y]]\ldots]$ and $(\mathrm{ad}_X)^0 Y \coloneqq Y$. Then, setting $|\alpha| \coloneqq \alpha_1 + \ldots + \alpha_k$ and $\alpha! \coloneqq \alpha_1! \ldots \alpha_k!$, the following formula holds (see [27], Section 2.15):

Theorem 2.8 (Baker-Campbell-Hausdorff formula). *Given two vector fields $X, Y \in \mathfrak{X}(\mathbb{G})$, the vector field*

$$C(X,Y) \coloneqq \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{\substack{\alpha,\beta \in \mathbb{N}^k \\ \alpha_i + \beta_i \geq 1\, \forall i}} \frac{1}{\alpha!\beta!|\alpha+\beta|} C_{\alpha\beta}(X,Y) \tag{2.3}$$

is such that $\exp(X)\exp(Y) = \exp(C(X,Y))$ when the series on the right-hand side is converging.

Remark 2.9. The series in (2.3) is not convergent in general, but it can be treated as a formal series and substituted in (2.2). In any case, we will work with the hypothesis that the group is nilpotent (see below) which reduces the series to a finite sum, so we will not have to worry about this.

In what follows, given two linear subspaces V and W of a Lie algebra, we set $[V, W] \coloneqq \mathrm{span}\{[X, Y] : X \in V, Y \in W\}$. Let $\mathfrak{g}$ be a Lie algebra. Let us define $\mathfrak{g}_1 \coloneqq \mathfrak{g}$ and by induction $\mathfrak{g}_{i+1} \coloneqq [\mathfrak{g}, \mathfrak{g}_i]$.

Definition 2.10. We say that $\mathfrak{g}$ is *s-step nilpotent* if $\mathfrak{g}_s \neq \{0\}$ and $\mathfrak{g}_{s+1} = \{0\}$. We say that $\mathbb{G}$ is *s-step nilpotent* if its Lie algebra is.

The following result is fundamental for the sequel (see [27], Theorem 3.6.2):

Theorem 2.11. *Let $\mathbb{G}$ be a connected, simply connected and nilpotent Lie group. Then the exponential map* $\exp : \mathfrak{g} \longrightarrow \mathbb{G}$ *is a diffeomorphism.*

For a Lie group $\mathbb{G}$ and $k \in \mathbb{G}$, let us denote by $C_k := R_{k^{-1}} \circ L_k$ the conjugation map

$$C_k : \mathbb{G} \longrightarrow \mathbb{G}, \quad C_k(g) = kgk^{-1}.$$

Clearly it is a diffeomorphism, thus we can set $\mathrm{Ad}_k(X) := (C_k)_* X$ for every $X \in \mathfrak{g}$ and so Ad_k is an invertible linear map and

$$\mathrm{Ad}_k(X)f = X(f \circ C_k) \circ C_{k^{-1}} \tag{2.4}$$

for every $f \in C^\infty(\mathbb{G})$. Moreover, $\mathrm{Ad}_k(X)$ is left-invariant by (2.1):

$$\begin{aligned} \mathrm{Ad}_k(X)(f \circ L_g)(x) &= X(f \circ L_g \circ C_k)(k^{-1}xk) \\ &= X(f \circ R_{k^{-1}} \circ L_{gk})(k^{-1}xk) \\ &= X(f \circ R_{k^{-1}})(gxk) \\ &= X(f \circ R_{k^{-1}} \circ L_k)(C_{k^{-1}}(gx)) \\ &= \mathrm{Ad}_k(X)f(L_g(x)). \end{aligned}$$

Then the following definition is well posed.

Definition 2.12. The *adjoint operator* is the map $\mathrm{Ad} : \mathbb{G} \longrightarrow \mathrm{GL}(\mathfrak{g})$, $\mathrm{Ad}(k) = \mathrm{Ad}_k$.

Let us denote by $L(\mathfrak{g})$ the set of all linear maps from $\mathfrak{g}$ to itself and remember that, for $X \in \mathfrak{g}$, $\mathrm{ad}_X : \mathfrak{g} \longrightarrow L(\mathfrak{g})$ indicates the operator $\mathrm{ad}_X Y = [X, Y]$. Then we have a useful expression for the composition of the adjoint operator and the exponential map (see [17], Proposition 1.91):

$$\mathrm{Ad}_{\exp(X)} = e^{\mathrm{ad}_X}, \tag{2.5}$$

where $e^A := \sum_{i=0}^{\infty} A^i/i! \in L(\mathfrak{g})$ for every $A \in L(\mathfrak{g})$.
Let us prove a characterization of the vector space spanned by $\mathrm{Ad}_{\exp(Y)}(X)$ for Y varying in a Lie subalgebra:

Proposition 2.13. *Let $\mathbb{G}$ be an s-step nilpotent Lie group, $\mathfrak{g}' \subseteq \mathfrak{g}$ a Lie subalgebra and $X \in \mathfrak{g}$. Then*

$$\begin{aligned} \mathrm{span}\{\mathrm{Ad}_{\exp(Y)}(X) : Y \in \mathfrak{g}'\} = X &+ [\mathfrak{g}', X] + [\mathfrak{g}', [\mathfrak{g}', X]] + \dots \\ &+ [\underbrace{\mathfrak{g}', [\mathfrak{g}', [\dots [\mathfrak{g}'}_{s-1}, X]] \dots]. \end{aligned}$$

Proof. Let us denote by S the space $\operatorname{span}\{\mathrm{Ad}_{\exp(Y)}(X) : Y \in \mathfrak{g}'\}$. Obviously S contains X (just take $Y = 0$) and all vector fields $\mathrm{Ad}_{\exp(rY)}(X)$ with $r \in \mathbb{R}$ and $Y \in \mathfrak{g}'$. Moreover, we get from (2.5)

$$\begin{aligned}\mathrm{Ad}_{\exp(Y)}(X) = X + [Y, X] + \frac{1}{2}[Y, [Y, X]] + \ldots \\ + \frac{1}{(s-1)!}\underbrace{[Y, [Y, [\ldots [Y}_{s-1}, X]] \ldots].\end{aligned} \tag{2.6}$$

If ν is the dimension of $\mathfrak{g}'$ and $Y_1, \ldots, Y_\nu$ a basis of $\mathfrak{g}'$, taking into account (2.6) we define for any $Y = \sum_{j=1}^{\nu} r_j Y_j \in \mathfrak{g}'$ the function $\Psi : \mathbb{R}^\nu \longrightarrow S$,

$$\begin{aligned}\Psi(r_1, \ldots, r_\nu) &:= \mathrm{Ad}_{\exp(\sum_{j=1}^{\nu} r_j Y_j)}(X) - X \\ &= \sum_{k=1}^{s-1} \frac{1}{k!} \left(\sum_{j=1}^{\nu} r_j \mathrm{ad}_{Y_j} \right)^k X \\ &= \sum_{k=1}^{s-1} \frac{1}{k!} \sum_{j_1, \ldots, j_k = 1}^{\nu} r_{j_1} \ldots r_{j_k} (\mathrm{ad}_{Y_{j_1}} \ldots \mathrm{ad}_{Y_{j_k}}) X.\end{aligned}$$

Since this polynomial takes its values in S, choosing $r_{j_1} = \ldots = r_{j_k} = 1$ and $r_i = 0$ for $i \neq j_1, \ldots, j_k$ we deduce that all its coefficients belong to S. In particular,

$$\mathrm{ad}_{Y_i} X \in S \qquad \text{and} \qquad (\mathrm{ad}_{Y_i}\mathrm{ad}_{Y_j} + \mathrm{ad}_{Y_j}\mathrm{ad}_{Y_i})X \in S.$$

By the Jacobi identity we have $\mathrm{ad}_{[V,W]} = \mathrm{ad}_V \mathrm{ad}_W + \mathrm{ad}_W \mathrm{ad}_V$ for every $V, W \in \mathfrak{X}(\mathbb{G})$, so

$$(\mathrm{ad}_{Y_i}\mathrm{ad}_{Y_j} + \mathrm{ad}_{Y_j}\mathrm{ad}_{Y_i})X = 2\,\mathrm{ad}_{Y_j}\mathrm{ad}_{Y_i}X + \mathrm{ad}_{[Y_j, Y_i]}X.$$

This implies $\mathrm{ad}_{Y_i}\mathrm{ad}_{Y_j}X \in S$, so $X + [\mathfrak{g}', X] + [\mathfrak{g}', [\mathfrak{g}', X]] \subseteq S$. By induction, let us suppose that for some $k \geq 3$

$$\mathfrak{u}_{k-1} := X + [\mathfrak{g}', X] + [\mathfrak{g}', [\mathfrak{g}', X]] + \ldots + \underbrace{[\mathfrak{g}', [\mathfrak{g}', [\ldots [\mathfrak{g}'}_{k-1}, X]] \ldots] \subseteq S.$$

We know

$$\sum_{\sigma \in S_k} (\mathrm{ad}_{Y_{j_{\sigma(1)}}} \ldots \mathrm{ad}_{Y_{j_{\sigma(k)}}})X \in S, \tag{2.7}$$

where S_k is the set of all permutations of k elements. Applying the Jacobi identity on $(\mathrm{ad}_{Y_{j_{\sigma(1)}}} \ldots \mathrm{ad}_{Y_{j_{\sigma(k)}}})X$ sufficiently many times, one can show that we can exchange two indices and fix the other ones up to terms that belong to $\mathfrak{u}_{k-1}$. More precisely, we have

$$(\mathrm{ad}_{Y_{j_{\sigma(1)}}} \ldots \mathrm{ad}_{Y_{j_{\sigma(k)}}})X - (\mathrm{ad}_{Y_{j_{\eta(1)}}} \ldots \mathrm{ad}_{Y_{j_{\eta(k)}}})X \in \mathfrak{u}_{k-1}$$

for every $\sigma, \eta \in S_k$ such that $\sigma^{-1} \circ \eta$ is a transposition. Hence for any $\sigma \in S_k$ we can iterate transpositions and write $(\mathrm{ad}_{Y_{j_{\sigma(1)}}} \dots \mathrm{ad}_{Y_{j_{\sigma(k)}}})X = (\mathrm{ad}_{Y_{j_1}} \dots \mathrm{ad}_{Y_{j_k}})X + W_\sigma$ for a suitable $W_\sigma \in \mathfrak{u}_{k-1}$. In particular $W_\sigma \in S$ by inductive hypothesis, so from (2.7) we get $(\mathrm{ad}_{Y_{j_1}} \dots \mathrm{ad}_{Y_{j_k}})X \in S$ and therefore also $(\mathrm{ad}_{Y_{j_{\sigma(1)}}} \dots \mathrm{ad}_{Y_{j_{\sigma(k)}}})X \in S$. This proves $\mathfrak{u}_k \subseteq S$ and so the inclusion

$$\begin{aligned}\mathrm{span}\{\mathrm{Ad}_{\exp(Y)}(X) : Y \in \mathfrak{g}'\} \supseteq X + [\mathfrak{g}', X] + [\mathfrak{g}', [\mathfrak{g}', X]] + \dots \\ + \underbrace{[\mathfrak{g}', [\mathfrak{g}', [\dots [\mathfrak{g}'}_{s-1}, X]] \dots]\end{aligned}$$

since $\mathfrak{u}_k = \mathfrak{u}_s$ for every $k \geq s$. The opposite inclusion is trivial thanks to (2.6). $\square$

We conclude the section with a result which will be decisive in addressing our problem.

Proposition 2.14. *Let $\mathbb{G}$ be a connected, simply connected, nilpotent Lie group and $\mathfrak{g}'$ be a Lie subalgebra of $\mathfrak{g}$ such that* $\dim \mathfrak{g}' + 2 \leq \dim \mathfrak{g}$. *Assume that there exists $X \notin \mathfrak{g}'$ such that $W := \mathfrak{g}' \oplus \{\mathbb{R}X\}$ generates the whole Lie algebra $\mathfrak{g}$. Then there exists $k \in \exp(\mathfrak{g}')$ such that $\mathrm{Ad}_k(X) \notin W$.*

Proof. The exponential map exp is a diffeomorphism by Theorem 2.11 and $\mathfrak{g}'$ is strictly contained in $\mathfrak{g}$, so $\mathbb{K} := \exp(\mathfrak{g}')$ is a closed proper Lie subgroup of $\mathbb{G}$ and we can consider the left coset space $\mathbb{G}/\mathbb{K}$, i.e. the set of all equivalence classes induced by the relation

$$x \sim y \iff x^{-1}y \in \mathbb{K}.$$

Let $\pi : \mathbb{G} \longrightarrow \mathbb{G}/\mathbb{K}$, $\pi(x) = x\mathbb{K}$, be the quotient map. By Theorem 21.17 of [20], $\mathbb{G}/\mathbb{K}$ is a topological manifold of dimension $\dim \mathbb{G}/\mathbb{K} = \dim \mathbb{G} - \dim \mathbb{K} = \dim \mathfrak{g} - \dim \mathfrak{g}' \geq 2$ and it can be endowed with a structure of smooth manifold such that π is a (smooth) submersion. In particular, if we call $\mathfrak{m}$ the subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}$, by the rank theorem for manifolds $d\pi(X) \neq 0$ because the projection of X on $\mathfrak{m}$ is nonzero.
By the way of contradiction, assume that the statement is false, so that $\mathrm{Ad}_k(X) \in W$ for every $k \in \mathbb{K}$. Since $\mathrm{Ad}_k(\mathfrak{g}') = \mathfrak{g}'$, we have $\mathrm{Ad}_k(W) \subseteq W$, which is equivalent to

$$(R_k)_*(L_{k^{-1}})_*Y \in W \qquad \text{for every } Y \in W,\ k \in \mathbb{K}.$$

But vector fields in W are left-invariant, so $(L_{k^{-1}})_*Y = Y$ and W is $\mathbb{K}$-right-invariant:

$$d(R_k)_x(W_x) \subseteq W_{xk} \qquad \text{for every } x \in \mathbb{G},\ k \in \mathbb{K}.$$

Now let us consider the subspaces $d\pi_x(W_x)$ of $T_{\pi(x)}\mathbb{G}/\mathbb{K}$: they are all 1-dimensional (because $\dim W = \dim \mathfrak{g}' + 1$ and $d\pi(X) \neq 0$) and they depend only on $\pi(x) = x\mathbb{K}$ (and not on x) thanks to the identity $\pi \circ R_k = \pi$, the chain rule (1.1) and $\mathbb{K}$-right-invariance of W:

$$d\pi_x(Y_x) = d\pi_{xk}(d(R_k)_x(Y_x)) \in d\pi_{xk}(W_{xk})$$

for all $Y \in W$ and $k \in \mathbb{K}$. Hence we can define a smooth 1-dimensional distribution $W/\mathbb{K}$ in $\mathbb{G}/\mathbb{K}$ by $(W/\mathbb{K})_y := d\pi_x(W_x)$, where x is any element of $\pi^{-1}(y)$. Any smooth 1-dimensional distribution is involutive and therefore integrable by Frobenius' theorem, so $W/\mathbb{K}$ is an integrable distribution, i.e. there exists a 1-dimensional foliation $\mathcal{F}$ of $\mathbb{G}/\mathbb{K}$ such that $W/\mathbb{K}$ is tangent to $\mathcal{F}$. Furthermore, $\mathcal{F}$ has at least codimension 1 since $\mathbb{G}/\mathbb{K}$ has at least dimension 2. π is a submersion, so we can consider the pullback $\mathcal{F}' := \pi^*\mathcal{F}$ of $\mathcal{F}$ by π, which is a foliation of same codimension as $\mathcal{F}$, so at least 1, whose leaves are the inverse images via π of leaves of $\mathcal{F}$ (see [6], Theorem 3.2.2). If $L' = \pi^{-1}(L)$ is a leaf of $\mathcal{F}'$, for some leaf L of $\mathcal{F}$, we have

$$\begin{aligned} T_x L' &= T_x(\pi^{-1}(L)) \\ &= (d\pi_x)^{-1}(T_{\pi(x)}L) \\ &= (d\pi_x)^{-1}(W/\mathbb{K})_{\pi(x)} \\ &= (d\pi_x)^{-1}(d\pi_x(W_x)) \supseteq W_x \end{aligned}$$

for all $x \in L'$. However, the fact that $\mathcal{F}$ and $\mathcal{F}'$ have same codimension implies $\dim L' = \dim \mathfrak{g}' + \dim L = \dim \mathfrak{g}' + 1 = \dim W$. Therefore, $T_x L' = W_x$ for all $x \in L'$, which means that W is tangent to the leaves of $\mathcal{F}'$, namely W is an integrable distribution. By Frobenius' theorem, this implies that W is a Lie subalgebra of $\mathfrak{g}$, so $W = \mathfrak{g}$ because W generates $\mathfrak{g}$ by hypothesis and this is a contradiction: $\mathfrak{g}$ has codimension 0 and cannot be tangent to the foliation $\mathcal{F}'$, which has at least codimension 1. □

2.2 Carnot groups

Definition 2.15. A Lie algebra $\mathfrak{g}$ is said to be *stratified* if it admits a *stratification*, i.e. there exist nontrivial linear subspaces $V_1, \dots, V_s$ of $\mathfrak{g}$ such that

$$\begin{aligned} &\mathfrak{g} = V_1 \oplus \ldots \oplus V_s, \\ &V_{i+1} = [V_1, V_i] \qquad \text{for } i = 1, \dots, s-1, \\ &[V_1, V_s] = \{0\}. \end{aligned}$$

Setting $V_k = \{0\}$ if $k > s$, it is easy to see that Jacobi identity implies $[V_i, V_j] \subseteq V_{i+j}$ for every $i \neq j$.
A stratification of a Lie algebra is unique up to automorphisms. More precisely, it can be proved the following result (see [19], Proposition 6.2.10):

Theorem 2.16 (Uniqueness of stratifications). *Let $\mathfrak{g}$ be a stratifiable Lie algebra with two stratifications,*

$$V_1 \oplus \ldots \oplus V_s = \mathfrak{g} = W_1 \oplus \ldots \oplus W_m.$$

Then

(i) $s = m$,

(ii) $V_j \oplus \ldots \oplus V_s = W_j \oplus \ldots \oplus W_s$ *for every* $j = 1, \ldots, s$,

(iii) there exists a Lie algebra automorphism $A : \mathfrak{g} \longrightarrow \mathfrak{g}$ *such that* $A(V_i) = W_i$ *for every* $i = 1, \ldots, s$.

A Lie group $\mathbb{G}$ is said to be *stratified* if its Lie algebra is. Clearly a stratified Lie group $\mathbb{G}$ is s-step nilpotent, where s is the number of needed subspaces for the stratification.
A stratification $V_1, \ldots, V_s$ allows to define a one-parameter group of inhomogeneous dilations on a Lie algebra $\mathfrak{g}$: for any $\lambda \geq 0$, let us define $\widetilde{\delta}_\lambda X := \lambda^i X$ if $X \in V_i$ and extend it by linearity on the whole algebra. It is easy to prove that these dilations enjoy the following properties:

- $\widetilde{\delta}_{\lambda\mu} = \widetilde{\delta}_\lambda \circ \widetilde{\delta}_\mu$,
- $\widetilde{\delta}_\lambda([X,Y]) = [\widetilde{\delta}_\lambda X, \widetilde{\delta}_\lambda Y]$,
- $\widetilde{\delta}_\lambda(C(X,Y)) = C(\widetilde{\delta}_\lambda X, \widetilde{\delta}_\lambda Y)$ (where $C(X,Y)$ is defined in (2.3)).

Definition 2.17. A *Carnot group* $\mathbb{G}$ is a connected, simply connected and stratified Lie group.

If $\mathbb{G}$ is a Carnot group and $\mathfrak{g}$ its Lie algebra, by Theorem 2.11 the exponential map $\exp : \mathfrak{g} \longrightarrow \mathbb{G}$ is a diffeomorphism, so it induces a one-parameter group of intrinsic dilations on $\mathbb{G}$: for any $\lambda \geq 0$ and $g \in \mathbb{G}$ let

$$\delta_\lambda(g) := \exp(\widetilde{\delta}_\lambda(\exp^{-1}(g))).$$

The following properties of $\{\delta_\lambda\}_{\lambda \geq 0}$ derive from those of dilations on a Lie algebra:

- $\delta_{\lambda\mu} = \delta_\lambda \circ \delta_\mu$, indeed

$$\begin{aligned}
\delta_{\lambda\mu}(g) &= \exp(\widetilde{\delta}_{\lambda\mu}(\exp^{-1}(g))) \\
&= \exp((\widetilde{\delta}_\lambda \circ \widetilde{\delta}_\mu)(\exp^{-1}(g))) \\
&= \exp((\widetilde{\delta}_\lambda \circ \exp^{-1})(\exp(\widetilde{\delta}_\mu(\exp^{-1}(g))))) \\
&= \exp((\widetilde{\delta}_\lambda \circ \exp^{-1})(\delta_\mu(g))) \\
&= (\delta_\lambda \circ \delta_\mu)(g),
\end{aligned}$$

- $\delta_\lambda(gh) = \delta_\lambda(g)\delta_\lambda(h)$, indeed

$$\begin{aligned}
\delta_\lambda(gh) &= \exp(\widetilde{\delta}_\lambda(\exp^{-1}(gh))) \\
&= \exp(\widetilde{\delta}_\lambda(C(\exp^{-1}(g), \exp^{-1}(h)))) \\
&= \exp(C(\widetilde{\delta}_\lambda(\exp^{-1}(g)), \widetilde{\delta}_\lambda(\exp^{-1}(h)))) \\
&= \exp(\widetilde{\delta}_\lambda(\exp^{-1}(g)))\exp(\widetilde{\delta}_\lambda(\exp^{-1}(h))) \\
&= \delta_\lambda(g)\delta_\lambda(h).
\end{aligned}$$

We have a useful relation between dilations on $\mathbb{G}$ and dilations on $\mathfrak{g}$:

Proposition 2.18. *For any $X \in \mathfrak{g}$, $(\delta_\lambda)_* X = \widetilde{\delta}_\lambda X$, namely*

$$X(f \circ \delta_\lambda)(g) = (\widetilde{\delta}_\lambda X)f(\delta_\lambda(g)) \tag{2.8}$$

for every $f \in C^\infty(\mathbb{G})$, $g \in \mathbb{G}$.

Proof. Since $\exp(tX) = \Phi_t^X(e)$, the thesis follows from (1.5), Proposition 2.6 and properties of dilations:

$$\begin{aligned}
X(f \circ \delta_\lambda)(g) &= \left.\frac{d}{dt}\right|_{t=0} (f \circ \delta_\lambda)(g\exp(tX)) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(\delta_\lambda(g)\delta_\lambda(\exp(tX))) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(\delta_\lambda(g)\exp(t\widetilde{\delta}_\lambda X)) \\
&= (\widetilde{\delta}_\lambda X)f(\delta_\lambda(g)).
\end{aligned}$$

□

2.3 Exponential and graded coordinates

Let $\mathbb{G}$ be a Carnot group of dimension n and $\mathfrak{g}$ its Lie algebra. We want to show that $\mathbb{G}$ is diffeomorphic to $\mathbb{R}^n$ equipped with a suitable operation. Let $X_1, \dots, X_n$ be a basis of left-invariant vector fields of $\mathfrak{g}$, so that all $X, Y \in \mathfrak{g}$ can be expressed as $X = \sum_{i=1}^n x_i X_i$, $Y = \sum_{i=1}^n y_i X_i$ for some $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$.

Definition 2.19. The system of *exponential coordinates* associated with the basis $X_1, \dots, X_n$ is the map

$$\begin{aligned}
E : \mathbb{R}^n &\longrightarrow \mathbb{G} \\
x &\longmapsto \exp\left(\sum_{i=1}^n x_i X_i\right).
\end{aligned}$$

We can equip $\mathbb{R}^n$ with a group law, denoted by $\cdot$, which makes E a group isomorphism:

$$x \cdot y = z \quad \Longleftrightarrow \quad \exp(X)\exp(Y) = \exp\left(\sum_{i=1}^{n} z_i X_i\right),$$

which is equivalent to require $C(X,Y) = \sum_{i=1}^{n} z_i X_i$. This equality tells us that in general this group law can be written as

$$x \cdot y = P(x,y) = x + y + Q(x,y), \tag{2.9}$$

where $P = (P_1, \dots, P_n)$ and $Q = (Q_1, \dots, Q_n)$ are polynomial functions which can be derived from (2.3). Moreover, one can easily see that the identity element is $e = 0$ and the inverse element of $x \in \mathbb{R}^n$ is $x^{-1} = -x$. With this product, $\mathbb{R}^n$ is a Lie group and its Lie algebra is isomorphic to $\mathfrak{g}$, so in particular it is nilpotent. Since it is also connected and simply connected, by Theorem 2.11 $\mathbb{R}^n$ is diffeomorphic to $\mathfrak{g}$, just like $\mathbb{G}$, therefore $\mathbb{G}$ and $(\mathbb{R}^n, \cdot)$ are diffeomorphic.
In the previous argument we never exploited the existence of a stratification for $\mathbb{G}$, so let us show now that, thanks to it, we can make a "privileged" choice of the basis $X_1, \dots, X_n$ which permits to express the dilations in a very intuitive way. Let $\mathfrak{g} = V_1 \oplus \dots \oplus V_s$ be a stratification and set $n_i := \dim V_i$ and $m_i := n_1 + \dots + n_i$ for every $i = 1, \dots, s$. Furthermore, let $m_0 := 0$. We say that a basis $X_1, \dots, X_n$ of $\mathfrak{g}$ is *adapted* to the stratification $V_1, \dots, V_s$ if for any $i = 1, \dots, s$ the n_i vectors $X_{m_{i-1}+1}, \dots, X_{m_i}$ form a basis for V_i.

Definition 2.20. A system of exponential coordinates is said to be a system of *graded coordinates* if it is associated with an adapted basis to a stratification.

We will call the *degree* of the coordinate x_i the unique natural number d_i such that $m_{d_i - 1} < i \leq m_{d_i}$. If $E : \mathbb{R}^n \longrightarrow \mathbb{G}$ is a system of graded coordinates, let $\{\widehat{\delta}_\lambda\}_{\lambda \geq 0}$ be the one-parameter group of dilations induced by E on $\mathbb{R}^n$, i.e. $\widehat{\delta}_\lambda := E^{-1} \circ \delta_\lambda \circ E$. Then we have

$$\begin{aligned} \widehat{\delta}_\lambda : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x &\longmapsto (\lambda x_1, \dots, \lambda x_{m_1}, \lambda^2 x_{m_1+1}, \dots, \lambda^2 x_{m_2}, \dots, \lambda^s x_{m_{s-1}+1}, \dots, \lambda^s x_n). \end{aligned}$$

Observe that, if we set $Q := \sum_{i=1}^{s} i n_i$, λ^Q is the Jacobian of $\widehat{\delta}_\lambda$.
In what follows, we will identify the dilations δ_λ on $\mathbb{G}$ with their expression in coordinates $\widehat{\delta}_\lambda$ and we will denote them simply by δ_λ.

2.4 Heisenberg and Engel groups

Heisenberg and Engel groups are the most classic and studied examples of Carnot groups, so in this section we present them.

Definition 2.21. A Lie algebra $\mathfrak{h}_n$ with a basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ such that the only non vanishing commutations between elements in the basis are

$$[X_i, Y_i] = -4T \qquad \text{for } i = 1, \ldots, n$$

is called a *Heisenberg algebra.*

A Heisenberg algebra $\mathfrak{h}_n$ can be stratified as $\mathfrak{h}_n = V_1 \oplus V_2$, where $V_1 = \text{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ and $V_2 = \text{span}\{T\} = [V_1, V_1]$. In particular, $\mathfrak{h}_n$ is 2-step nilpotent.

Definition 2.22. The *Heisenberg group* $\mathbb{H}^n$ is the Carnot group associated with $\mathfrak{h}_n$.

This definition is well posed thanks to Lie's third theorem, which says that there exists a connected and simply connected Lie group associated with every finite-dimensional real Lie algebra. Let us compute what the Heisenberg group looks like in graded coordinates. Since $\mathfrak{h}_n$ is 2-step nilpotent, by the Baker-Campbell-Hausdorff formula (2.3)

$$C(X, Y) = X + Y + \frac{1}{2}[X, Y].$$

If X and Y are expressed as

$$X = \sum_{i=1}^{n} x_i X_i + \sum_{i=1}^{n} y_i Y_i + tT,$$
$$Y = \sum_{i=1}^{n} x'_i X_i + \sum_{i=1}^{n} y'_i Y_i + t'T$$

for some $(x, y, t), (x', y', t') \in \mathbb{R}^{2n+1}$, we have

$$[X, Y] = 4(\langle x', y\rangle - \langle x, y'\rangle)T,$$

so that

$$C(X, Y) = \sum_{i=1}^{n}(x_i + x'_i)X_i + \sum_{i=1}^{n}(y_i + y'_i)Y_i + (t + t' + 2\langle x', y\rangle - 2\langle x, y'\rangle)T.$$

Therefore, $\mathbb{H}^n$ is isomorphic to $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with group law

$$\begin{pmatrix} x \\ y \\ t \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = \begin{pmatrix} x + x' \\ y + y' \\ t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle) \end{pmatrix}.$$

In this structure, the identity element is 0 and the inverse of an element (x, y, t) is $-(x, y, t) = (-x, -y, -t)$. Furthermore, for any $\lambda \geq 0$ the intrinsic

dilations are $\delta_\lambda(x,y,t) = (\lambda x, \lambda y, \lambda^2 t)$.
Let us find explicitly a basis of left-invariant vector fields: every left-invariant vector field X is characterized by $X(g) = d(L_g)_e(X_e)$ for every element g of the group (Remark 2.4). In our case, the differential of the left traslation $L_{(x,y,t)}$ in 0 is given by the $(2n+1)\times(2n+1)$-matrix

$$d(L_{(x,y,t)})_0 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 2y & -2x & 1 \end{pmatrix},$$

where I is the identity $n \times n$-matrix. Hence, if $\partial_{x_1}, \dots, \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}, \partial_t$ is the canonical basis of $\mathbb{R}^{2n+1}$, a basis of left-invariant vector fields is given by

$$\begin{aligned} X_i(x,y,t) &= d(L_{(x,y,t)})_0(\partial_{x_i}) = \partial_{x_i} + 2y_i\partial_t, \\ Y_i(x,y,t) &= d(L_{(x,y,t)})_0(\partial_{y_i}) = \partial_{y_i} - 2x_i\partial_t, \\ T(x,y,t) &= d(L_{(x,y,t)})_0(\partial_t) = \partial_t. \end{aligned}$$

Definition 2.23. A 4-dimensional Lie algebra $\mathfrak{e}$ with a basis X_1, X_2, X_3, X_4 such that the only non vanishing commutations between elements in the basis are

$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = -X_4$$

is called an *Engel algebra*.

An Engel algebra $\mathfrak{e}$ can be stratified as $\mathfrak{e} = V_1 \oplus V_2 \oplus V_3$, where $V_1 = \mathrm{span}\{X_1, X_2\}$, $V_2 = \mathrm{span}\{X_3\} = [V_1, V_2]$ and $V_3 = \mathrm{span}\{X_4\} = [V_1, V_3]$. In particular, $\mathfrak{e}$ is 3-step nilpotent. Again Lie's third theorem ensures that we can give the next definition.

Definition 2.24. The *Engel group* $\mathbb{E}$ is the Carnot group associated with $\mathfrak{e}$.

As we did for the Heisenberg group, let us find an explicit representation of the Engel group in graded coordinates. Since $\mathfrak{e}$ is 3-step nilpotent, by the Baker-Campbell-Hausdorff formula (2.3)

$$C(X,Y) = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]].$$

If

$$X = \sum_{i=1}^4 x_i X_i \qquad \text{and} \qquad Y = \sum_{i=1}^4 x_i' X_i$$

for some $x, x' \in \mathbb{R}^4$, we obtain

$$\begin{aligned} [X,Y] &= (x_1x_2' - x_2x_1')[X_1,X_2] + (x_1x_3' - x_3x_1')[X_1,X_3] \\ &= -(x_1x_2' - x_2x_1')X_3 - (x_1x_3' - x_3x_1')X_4, \\ [X,[X,Y]] &= -x_1(x_1x_2' - x_2x_1')[X_1,X_3] = x_1(x_1x_2' - x_2x_1')X_4, \\ [Y,[Y,X]] &= x_1'(x_1x_2' - x_2x_1')[X_1,X_3] = -x_1'(x_1x_2' - x_2x_1')X_4. \end{aligned}$$

Therefore, $\mathbb{E}$ is isomorphic to $\mathbb{R}^4$ with group law

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' - \frac{1}{2}(x_1 x_2' - x_2 x_1') \\ x_4 + x_4' - \frac{1}{2}(x_1 x_3' - x_3 x_1') + \frac{1}{12}(x_1 - x_1')(x_1 x_2' - x_2 x_1') \end{pmatrix}.$$

Again, the identity element is 0, the inverse of an element (x_1, x_2, x_3, x_4) is $-(x_1, x_2, x_3, x_4) = (-x_1, -x_2, -x_3, -x_4)$ and for any $\lambda \geq 0$ the intrinsic dilations are $\delta_\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4)$. The differential of the left traslation L_x in 0 is given by the matrix

$$d(L_x)_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{x_2}{2} & -\frac{x_1}{2} & 1 & 0 \\ \frac{x_3}{2} - \frac{x_1 x_2}{12} & \frac{x_1^2}{12} & -\frac{x_1}{2} & 1 \end{pmatrix},$$

so, if $\partial_{x_1}, \partial_{x_2}, \partial_{x_3}, \partial_{x_4}$ is the canonical basis of $\mathbb{R}^4$, a basis of left-invariant vector fields is given by

$$\begin{aligned} X_1(x) &= d(L_x)_0(\partial_{x_1}) = \partial_{x_1} + \frac{x_2}{2}\partial_{x_3} + \left(\frac{x_3}{2} - \frac{x_1 x_2}{12}\right)\partial_{x_4}, \\ X_2(x) &= d(L_x)_0(\partial_{x_2}) = \partial_{x_2} - \frac{x_1}{2}\partial_{x_3} + \frac{x_1^2}{12}\partial_{x_4}, \\ X_3(x) &= d(L_x)_0(\partial_{x_3}) = \partial_{x_3} - \frac{x_1}{2}\partial_{x_4}, \\ X_4(x) &= d(L_x)_0(\partial_{x_4}) = \partial_{x_4}. \end{aligned}$$

Now we provide another basis for $\mathfrak{e}$, which is simpler than the one before. Let us consider a different coordinate system associated with X_1, X_2, X_3, X_4 on $\mathbb{E}$, which is called *strong Malcev coordinates* (or *canonical coordinates of the second kind*):

$$\begin{aligned} &\mathbb{R}^4 \longrightarrow \mathbb{E} \\ &y \longmapsto \exp(y_4 X_4)\exp(y_3 X_3)\exp(y_2 X_2)\exp(y_1 X_1). \end{aligned}$$

Using the Baker-Campbell-Hausdorff formula (2.3) we have

$$\begin{aligned} &\exp(y_4 X_4)\exp(y_3 X_3)\exp(y_2 X_2)\exp(y_1 X_1) \\ &= \exp\left(y_1 X_1 + y_2 X_2 + \left(\frac{1}{2}y_1 y_2 + y_3\right)X_3 + \left(\frac{1}{12}y_1^2 y_2 + \frac{1}{2}y_1 y_3 + y_4\right)X_4\right), \end{aligned}$$

hence setting it equal to $\exp(x_1 X_1 + x_2 X_2 + x_3 X_3 + x_4 X_4)$ we get y in terms of x and so the map to pass from graded coordinates to strong Malcev ones, that is

$$\begin{aligned} H : \mathbb{R}^4 &\longrightarrow \mathbb{R}^4 \\ x &\longmapsto \left(x_1, x_2, x_3 - \frac{x_1 x_2}{2}, x_4 + \frac{x_1^2 x_2}{6} - \frac{x_1 x_3}{2} \right). \end{aligned}$$

Its differential is

$$dH_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{x_2}{2} & -\frac{x_1}{2} & 1 & 0 \\ \frac{x_1 x_2}{3} - \frac{x_3}{2} & \frac{x_1^2}{6} & -\frac{x_1}{2} & 1 \end{pmatrix},$$

so taking the image of X_1, X_2, X_3, X_4 we obtain a new basis Y_1, Y_2, Y_3, Y_4 of left-invariant (with respect to the group law $\cdot$ transformed by the diffeomorphism H) vector fields for $\mathfrak{e}$:

$$\begin{aligned} Y_1(x) &= dH_x(X_1(x)) = \partial_{x_1}, \\ Y_2(x) &= dH_x(X_2(x)) = \partial_{x_2} - x_1 \partial_{x_3} + \frac{x_1^2}{2} \partial_{x_4}, \\ Y_3(x) &= dH_x(X_3(x)) = \partial_{x_3} - x_1 \partial_{x_4}, \\ Y_4(x) &= dH_x(X_4(x)) = \partial_{x_4}. \end{aligned}$$

2.5 Left-invariant vector fields

Let $\mathbb{G}$ be a Carnot group of dimension n and $E : \mathbb{R}^n \longrightarrow \mathbb{G}$ a system of graded coordinates associated with an adapted basis $X_1, \dots, X_n$ of $\mathfrak{g}$. Our aim is to find an explicit formula for left-invariant vector fields.

Definition 2.25. A function $P : \mathbb{G} \longrightarrow \mathbb{R}$ is called a *polynomial* on $\mathbb{G}$ if the composition $P \circ E$ is a polynomial on $\mathbb{R}^n$.

This definition is well posed: if F is another system of graded coordinates, the map $E^{-1} \circ F$ is a change of basis for $\mathbb{R}^n$, so it is linear and therefore $P \circ E$ is a polynomial if and only if $P \circ F = (P \circ E) \circ (E^{-1} \circ F)$ is.
Let $p_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the canonical projection on the i-th coordinate and $\pi_i \coloneqq p_i \circ E^{-1} : \mathbb{G} \longrightarrow \mathbb{R}$. For a multi-index $\alpha \in \mathbb{N}^n$ let us define

$$\begin{aligned} \pi^\alpha : \mathbb{G} &\longrightarrow \mathbb{R} \\ g &\longmapsto \prod_{i=1}^n (\pi_i(g))^{\alpha_i}. \end{aligned}$$

Clearly any such π^α is a polynomial on $\mathbb{G}$ and one can observe that any polynomial on $\mathbb{G}$ can be written as a finite linear combination of suitable π^α's. We will call the *homogeneous degree of* π^α the natural number $\deg_H(\pi^\alpha) \coloneqq \sum_{i=1}^n d_i \alpha_i$.

Definition 2.26. The *homogeneous degree* of a polynomial $P = \sum_\alpha c_\alpha \pi^\alpha$ is the natural number

$$\deg_H(P) := \max\{\deg_H(\pi^\alpha) : c_\alpha \neq 0\}.$$

As one might expect, the homogenous degree of a polynomial does not change if the system of graded coordinates changes:

Proposition 2.27. *The homogeneous degree of a polynomial P is independent of the system of graded coordinates.*

Proof. Let $E : \mathbb{R}^n_x \longrightarrow \mathbb{G}$ and $F : \mathbb{R}^n_y \longrightarrow \mathbb{G}$ be two system of graded coordinates associated respectively with the basis $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ adapted to the stratification $\mathfrak{g} = V_1 \oplus \dots \oplus V_s$. If $Y_j = \sum_{i=1}^n A_{ij} X_i$ is the expression of the vector field Y_j in the basis $X_1, \dots, X_n$, , for any $1 \leq j \leq n$, we have

$$(E^{-1} \circ F)(y) = \left(\sum_{j=1}^n A_{1j} y_j, \dots, \sum_{j=1}^n A_{nj} y_j \right).$$

Since the two basis are adapted, $A_{ij} \neq 0$ if and only if $m_{d_j - 1} < i \leq m_{d_j}$, so the matrix $A = (A_{ij})$ is block diagonal, i.e. of the form

$$A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & A_s \end{pmatrix},$$

where A_k is an $n_k \times n_k$-matrix.

In order to get the thesis, it suffices to prove that for any $\alpha \in \mathbb{N}^n$ the map $\pi^\alpha \circ F : \mathbb{R}^n_y \longrightarrow \mathbb{R}$ has the same homogeneous degree of the polynomial $(\pi^\alpha \circ E)(x) = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. We have

$$(\pi^\alpha \circ F)(y) = (\pi^\alpha \circ E)((E^{-1} \circ F)(y)) = \prod_{i=1}^n \left(\sum_{j=1}^n A_{ij} y_j \right)^{\alpha_i}.$$

This polynomial is not identically 0 because A is invertible, which means that none of its columns is null and so for any $1 \leq i \leq n$ there is an index j_i such that $A_{ij_i} \neq 0$. Since $A_{ij} = 0$ if $d_i \neq d_j$, this implies

$$\deg_H \left(\left(\sum_{j=1}^n A_{ij} y_j \right)^{\alpha_i} \right) = d_i \alpha_i.$$

Observing that the homogeneous degree of a product of polynomials is the sum of the homogeneous degrees, we finally have

$$\begin{aligned}\deg(\pi^\alpha \circ E) &= \sum_{i=1}^n d_i\alpha_i = \sum_{i=1}^n \deg_H\left(\left(\sum_{j=1}^n A_{ij}y_j\right)^{\alpha_i}\right)\\ &= \deg_H\left(\prod_{i=1}^n\left(\sum_{j=1}^n A_{ij}y_j\right)^{\alpha_i}\right)\\ &= \deg_H(\pi^\alpha \circ F).\end{aligned}$$

□

Example 2.28. The polynomial $P(x, y, t) = xy - t^2$ on the first Heisenberg group $\mathbb{H}^1$ has homogeneous degree $\deg_H(P) = 4$ because the coordinate t has degree 2 and power 2.

Definition 2.29. A polynomial P is said to be *homogeneous of degree* $d > 0$ if $P(\delta_\lambda(g)) = \lambda^d P(g)$ for every $g \in \mathbb{G}$ and $\lambda > 0$.

It is not difficult to check that a polynomial P is homogeneous of degree d if and only if it is a linear combination of polynomials π^α with $\deg_H(\pi^\alpha) = d$.

Example 2.30. The polynomial $P(x, y, t) = xy^3 + t^2$ on the first Heisenberg group $\mathbb{H}^1$ is homogeneous of degree 4:

$$\begin{aligned}P(\lambda x, \lambda y, \lambda^2 t) &= \lambda x\lambda^3 y^3 + \lambda^4 t^2\\ &= \lambda^4(xy^3 + t^2)\\ &= \lambda^4 P(x, y, t).\end{aligned}$$

We know from (2.9) that in graded coordinates (identifying $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$) the left translation L_x by an element $x \in \mathbb{R}^n$ can be written as

$$L_x(y) = E^{-1}(E(x) \cdot E(y)) = (P_1(x, y), \dots, P_n(x, y)), \tag{2.10}$$

where $P_i(x, y)$ are polynomials which can be derived from the Baker-Campbell-Hausdorff formula (2.3). They are homogeneous of degree d_i:

$$\begin{aligned}\lambda^{d_i} P_i(x, y) &= (\pi_i \circ \delta_\lambda)(P_1(x, y), \dots, P_n(x, y))\\ &= (\pi_i \circ \delta_\lambda)(E^{-1}(E(x) \cdot E(y)))\\ &= (\pi_i \circ E^{-1})(\delta_\lambda(E(x)) \cdot \delta_\lambda(E(y)))\\ &= (\pi_i \circ E^{-1})(E(\delta_\lambda(x)) \cdot E(\delta_\lambda(y)))\\ &= P_i(\delta_\lambda(x), \delta_\lambda(y)).\end{aligned}$$

We are ready to prove how $X_1, \dots, X_n$ (and therefore all left-invariant vector fields) are represented in graded coordinates:

Proposition 2.31. *Let $\mathbb{G}$ be a Carnot group identified with $\mathbb{R}^n$ through a system of graded coordinates associated with an adapted basis $X_1, \ldots, X_n$; let $\{\partial_i\}_{i=1,\ldots,n}$ be the canonical basis of $\mathbb{R}^n$ and $X_j(x) = \sum_{i=1}^n a_{ij}(x)\partial_i$. Then*

(i) $a_{ij}(x) = \left.\dfrac{\partial P_i(x,y)}{\partial y_j}\right|_{y=0}$ *is a homogeneous polynomial of degree $d_i - d_j$,*

(ii) $X_j(x) = \partial_j + \sum_{\{i\,:\,d_i>d_j\}} a_{ij}(x)\partial_i = \partial_j + \sum_{i=m_{d_j}+1}^{n} a_{ij}(x)\partial_i,$

(iii) a_{ij} *depends only on the coordinates x_r with $d_r < d_i$.*

In particular, $a_{ij}(x) = a_{ij}(x_1, \ldots, x_{i-1})$.

Proof. By (2.1) and the fact that $X_j(0) = \partial_j$, we have for any $f \in C^\infty(\mathbb{G})$

$$\begin{aligned} X_j f(x) &= X_j(f \circ L_x)(0) \\ &= \partial_j(f \circ L_x)(0) \\ &= \sum_{i=1}^n \partial_j L_x^i(0)\partial_i f(x), \end{aligned}$$

where L_x^i denotes the i-th component of L_x, and so from (2.10) we deduce

$$a_{ij}(x) = \partial_j L_x^i(0) = \left.\frac{\partial P_i(x,y)}{\partial y_j}\right|_{y=0}.$$

By the homogeneity of P_i we get

$$\begin{aligned} \lambda^{d_i} a_{ij}(x) = \lambda^{d_i} \left.\frac{\partial P_i(x,y)}{\partial y_j}\right|_{y=0} &= \left.\frac{\partial P_i(\delta_\lambda(x), \delta_\lambda(y))}{\partial y_j}\right|_{y=0} \\ = \lambda^{d_j} \left.\frac{\partial P_i(\delta_\lambda(x), y)}{\partial y_j}\right|_{y=0} &= \lambda^{d_j} a_{ij}(\delta_\lambda(x)), \end{aligned}$$

that is the a_{ij}'s are homogeneous polynomials of degree $d_i - d_j$. This implies $a_{ij} \equiv 0$ if $d_j > d_i$; moreover, since a 0-homogeneous polynomial is constant, we have

$$X_j(x) = \sum_{\{i\,:\,d_i=d_j\}} c_{ij}\partial_j + \sum_{\{i\,:\,d_i>d_j\}} a_{ij}(x)\partial_i$$

for suitable constants $c_{ij} \in \mathbb{R}$. We obtain $c_{ij} = \delta_{ij}$ thanks to the fact that $X_j(0) = \partial_j$, so

$$X_j(x) = \partial_j + \sum_{i=m_{d_j}+1}^{n} a_{ij}(x)\partial_i.$$

Since each a_{ij} is homogeneous of degree $d_i - d_j$, the coordinates x_r with $d_r > d_i - d_j$ cannot appear in the polynomial structure of a_{ij}, therefore a_{ij} cannot depend on the coordinates x_r with $d_r \geq d_i$, namely

$$a_{ij}(x) = a_{ij}(x_1, \dots, x_{m_{d_i - 1}})$$

and, in particular, $a_{ij}(x) = a_{ij}(x_1, \dots, x_{i-1})$. □

2.6 Carnot groups as Carnot-Carathéodory spaces

Let $\Omega \subseteq \mathbb{R}^n$ be an open connected set and $X = \{X_1, \dots, X_m\}$ a family of locally Lipschitz vector fields on Ω, say

$$X_j(x) = \sum_{i=1}^{n} a_{ij}(x)\partial_i$$

with $a_{ij} \in \mathrm{Lip}_{\mathrm{loc}}(\Omega)$, $j = 1, \dots, m$ and $i = 1, \dots, n$. The subspace of $\mathbb{R}^n \cong T_x\Omega$ generated by $X_1(x), \dots, X_m(x)$ is called the *horizontal subspace* at the point x, it will be denoted by $H_x\Omega$ and its elements are called *horizontal vectors*. Let us consider the $n \times m$-matrix A whose j-column is made of the components of X_j, i.e.

$$A(x) = \begin{pmatrix} a_{11}(x) & \dots & a_{1m}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \dots & a_{nm}(x) \end{pmatrix}.$$

Definition 2.32. A Lipschitz curve $\gamma : [0, T] \longrightarrow \Omega$, $T \geq 0$, is *X-admissible* if there exists a measurable function $h = (h_1, \dots, h_m) : [0, T] \longrightarrow \mathbb{R}^m$ such that

(i) $\dot{\gamma}(t) = A(\gamma(t))h(t) = \sum_{j=1}^{m} h_j(t)X_j(\gamma(t))$ for a.e. $t \in [0, T]$,

(ii) $|h| \in L^\infty(0, T)$.

The curve γ is *X-subunit* if it is X-admissible and $\|h\|_\infty \leq 1$.

In other words, an X-admissible curve is a Lipschitz curve such that its tangent vector field is a.e. horizontal (and indeed they are also called *horizontal curves*).

Definition 2.33. The *Carnot-Carathéodory (C-C) distance* between two points $x, y \in \Omega$ is defined as

$$d(x, y) := \inf\{T \geq 0 : \text{there exists an } X\text{-subunit curve } \gamma : [0, T] \longrightarrow \Omega \text{ such that } \gamma(0) = x \text{ and } \gamma(T) = y\},$$

setting $d(x, y) := +\infty$ if $d(x, y) = \inf \emptyset$.

Let us prove that if all points of Ω can be connected via an X-subunit curve then d is a distance:

Proposition 2.34. *If $d(x,y) < +\infty$ for every $x, y \in \Omega$, then (Ω, d) is a metric space.*

Proof. Obviously $d(x,x) = 0$, whereas the symmetry property $d(x,y) = d(y,x)$ follows from the easy observation that if $\gamma : [0,T] \longrightarrow \Omega$ is X-subunit then $\bar{\gamma}(t) := \gamma(T-t)$ is X-subunit too. Furthermore, if $\gamma_1 : [0,T_1] \longrightarrow \Omega$ and $\gamma_2 : [0,T_2] \longrightarrow \Omega$ are X-subunit curves such that $\gamma_1(0) = x$, $\gamma_1(T_1) = z$, $\gamma_2(0) = z$ and $\gamma_2(T_2) = y$ then clearly

$$\gamma(t) := \begin{cases} \gamma_1(t) & \text{if } t \in [0,T_1] \\ \gamma_2(T_2 - t) & \text{if } t \in [T_1, T_1 + T_2] \end{cases}$$

is an X-subunit curve such that $\gamma(0) = x$ and $\gamma_1(T_1+T_2) = y$, so passing to the infimum we get the triangle inequality $d(x,y) \leq d(x,z) + d(z,y)$.
It remains to prove $d(x,y) > 0$ if $x \neq y$ and we will do it by showing that for any $K \Subset \Omega$ there exists $C_K > 0$ such that $d(x,y) \geq C_K|x-y|$ for every $x, y \in K$: let $\varepsilon > 0$ be sufficiently small so that

$$K_\varepsilon := \left\{ z \in \Omega : \min_{w \in K} |z - w| < \varepsilon \right\} \Subset \Omega$$

and set $M := \sup_{K_\varepsilon} \|A\|$ ($\|\cdot\|$ indicates the natural matrix norm), $r := \min\{\varepsilon, |x-y|\}$. Let us consider an X-subunit curve $\gamma : [0,T] \longrightarrow \Omega$ such that $\gamma(0) = x$ and $\gamma(T) = y$ and set

$$\tau := \inf\{t \in [0,T] : |\gamma(t) - x| \geq r\}.$$

Since $|x-y| \geq r$, the definition of τ is well posed, $\tau \leq T$ and by continuity $|\gamma(\tau) - x| = r$. Hence

$$\begin{aligned} r = |\gamma(\tau) - \gamma(0)| &= \left| \int_0^\tau \dot{\gamma}(s)\, ds \right| \\ &= \left| \int_0^\tau A(\gamma(s)) h(s)\, ds \right| \\ &\leq \int_0^\tau \|A(\gamma(s))\| |h(s)|\, ds \leq M\tau \leq MT. \end{aligned}$$

Then $T \geq r/M$ and we have two cases:

(1) if $r = \varepsilon$, $T \geq \dfrac{\varepsilon}{M} \geq \dfrac{\varepsilon}{MD}|x-y|$, where $D := \sup\limits_{x,y \in K} |x-y|$;

(2) if $r = |x-y|$, $T \geq \dfrac{1}{M}|x-y|$.

Therefore it is enough to take $C_K \coloneqq \min\left\{\dfrac{\varepsilon}{MD}, \dfrac{1}{M}\right\}$ to conclude. $\square$

The metric space (Ω, d) is called *Carnot-Carathéodory space*.
The C-C distance could be defined in a different way: let $\gamma : [0,1] \longrightarrow \Omega$ be an X-admissible curve with coordinates $h = (h_1, \dots, h_m) \in L^\infty(0,1)$. For $1 \le p \le \infty$ let

$$\ell_p(\gamma) \coloneqq \|h\|_p = \begin{cases} \left(\displaystyle\int_0^1 |h(t)|^p \, dt\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty \\ \operatorname*{ess\,sup}_{t \in [0,1]} |h(t)| & \text{if } p = \infty \end{cases}$$

and

$$\begin{aligned} d_p(x,y) \coloneqq \inf\{\ell_p(\gamma) : \gamma : [0,1] \longrightarrow \Omega \text{ is an } X\text{-admissible curve} \\ \text{such that } \gamma(0) = x \text{ and } \gamma(1) = y\}, \end{aligned}$$

setting $d_p(x,y) \coloneqq +\infty$ if $d_p(x,y) = \inf \emptyset$. Then it turns out that the C-C distance d is equal to d_p for every $1 \le p \le \infty$. Precisely, the following theorem holds (see [23], Theorem 1.1.6, or [28], Proposition 4.2):

Theorem 2.35. *For any $x, y \in \Omega$ and $1 \le p \le \infty$, $d(x,y) = d_p(x,y)$.*

Now we want to give a sufficient condition for connectivity via horizontal curves. Let $X_1, \dots, X_m \in \mathfrak{X}(\Omega)$ be smooth vector fields on Ω and $\mathfrak{L}(X_1, \dots, X_m)(x)$ the Lie algebra generated by them at the point $x \in \Omega$. We say that $X_1, \dots, X_m$ satisfy the *Chow-Hörmander condition* if, for any $x \in \Omega$, $\mathfrak{L}(X_1, \dots, X_m)(x)$ has maximum rank, namely

$$\operatorname{rank} \mathfrak{L}(X_1, \dots, X_m)(x) = n \qquad \text{for every } x \in \Omega. \tag{2.11}$$

Then we have the following result, whose proof can be found in [7], [23] (Chapter 1, Section 5) or [28] (Theorem 4.2).

Theorem 2.36 (Chow's theorem)**.** *If the vector fields $X_1, \dots, X_m$ satisfy the Chow-Hörmander condition* (2.11)*, then the C-C distance d is finite. In particular, there is always a subunit curve connecting any two points $x, y \in \Omega$ and the topology induced by d coincides with the Euclidean one.*

Let us turn back to Carnot groups and see how they can be seen as Carnot-Carathéodory spaces. Let $\mathbb{G}$ be a Carnot group and let us identify it with $(\mathbb{R}^n, \cdot)$ through a system of graded coordinates associated with a basis adapted to the stratification $\mathfrak{g} = V_1 \oplus \ldots \oplus V_s$. Let $m \coloneqq n_1 = \dim V_1$ and $X = \{X_1, \dots, X_m\}$ be the adapted basis of V_1. By definition of stratification, X generates the whole Lie algebra $\mathfrak{g}$, so it satisfies the Chow-Hörmander condition (2.11) and therefore Theorem 2.36 says that X induces the C-C

distance d on $\mathbb{G}$. Moreover, $(V_1)_x$ takes the role of $H_x\mathbb{G}$ and for this reason we will refer to it as the *horizontal layer*.
The distance d "performs well" with respect to left translations and dilations in the sense of the next proposition.

Proposition 2.37. *For any $x, y, z \in \mathbb{R}^n$ and $\lambda > 0$*

(i) $d(z \cdot x, z \cdot y) = d(x, y)$,

(ii) $d(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d(x, y)$.

Proof. *(i)* is a consequence of the fact that a curve $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ from x to y is X-subunit if and only if the curve $\widetilde{\gamma} := L_z \circ \gamma$ from $z \cdot x$ to $z \cdot y$ is X-subunit. Indeed, if $\dot{\gamma}(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t))$ for a.e. $t \in [0, T]$, we have by (1.3) (which can be used because actually it only requires differentiability of the curve) and left-invariance of X_j's

$$\begin{aligned}\dot{\widetilde{\gamma}}(t) &= d(L_z)_{\gamma(t)}(\dot{\gamma}(t)) = d(L_z)_{\gamma(t)}\left(\sum_{j=1}^m h_j(t) X_j(\gamma(t))\right) \\ &= \sum_{j=1}^m h_j(t) d(L_z)_{\gamma(t)}(X_j(\gamma(t))) \\ &= \sum_{j=1}^m h_j(t) X_j(z \cdot \gamma(t)) \\ &= \sum_{j=1}^m h_j(t) X_j(\widetilde{\gamma}(t))\end{aligned}$$

for a.e. $t \in [0, T]$.
Instead, *(ii)* follows from the fact that a curve $\gamma : [0, T] \longrightarrow \mathbb{R}^n$ from x to y is X-subunit if and only if the curve $\gamma_\lambda : [0, \lambda T] \longrightarrow \mathbb{R}^n$, $\gamma_\lambda := \delta_\lambda(\gamma(t/\lambda))$, from $\delta_\lambda(x)$ to $\delta_\lambda(y)$ is X-subunit. In fact, if for a.e. $t \in [0, T]$

$$\dot{\gamma}(t) = \sum_{j=1}^m h_j(t) X_j(\gamma(t)) = \sum_{i=1}^n \left(\sum_{j=1}^m h_j(t) a_{ij}(\gamma(t))\right) \partial_i,$$

since $d_j = 1$ for every $j = 1, \ldots, m$, by Proposition 2.31 the appearing a_{ij}'s are $(d_i - 1)$-homogeneous polynomials and so

$$\begin{aligned}\dot{\gamma}_\lambda(t) &= \frac{1}{\lambda}\widetilde{\delta}_\lambda\left(\dot{\gamma}\left(\frac{t}{\lambda}\right)\right) = \sum_{i=1}^n \lambda^{d_i - 1}\left(\sum_{j=1}^m h_j\left(\frac{t}{\lambda}\right) a_{ij}\left(\gamma\left(\frac{t}{\lambda}\right)\right)\right)\partial_i \\ &= \sum_{i=1}^n \left(\sum_{j=1}^m h_j\left(\frac{t}{\lambda}\right) a_{ij}(\gamma_\lambda(t))\right)\partial_i \\ &= \sum_{j=1}^m h_j\left(\frac{t}{\lambda}\right) X_j(\gamma_\lambda(t))\end{aligned}$$

for a.e. $t \in [0, \lambda T]$. □

A metric on a Carnot group which enjoys properties of Proposition 2.37 is called *homogeneous distance*. We denote by

$$B_r(x) \coloneqq \{y \in E : d(x, y) < r\}$$

the C-C open ball of radius $r > 0$ and center $x \in E$.

Corollary 2.38. *For any $x, y \in \mathbb{R}^n$ and $r, \lambda > 0$ we have $L_y(B_r(x)) = B_r(L_y(x))$ and $\delta_\lambda(B_r(x)) = B_{\lambda r}(\delta_\lambda(x))$.*

Proof. It is an easy consequence of Proposition 2.37. □

We lastly state an important theorem which says how C-C balls behave in Carnot groups. Set for any $x \in \mathbb{R}^n$ and $\lambda > 0$

$$\mathrm{Box}(x, \lambda) \coloneqq \{x \cdot z \in \mathbb{R}^n : |z_i| < \lambda^{d_i}, 1 \leq i \leq n\}.$$

Theorem 2.39 (Ball-box theorem for Carnot groups)**.** *There exists a constant $C > 1$ such that for all $x \in \mathbb{R}^n$ and $r > 0$*

$$\mathrm{Box}\left(x, \frac{r}{C}\right) \subseteq B_r(x) \subseteq \mathrm{Box}(x, Cr).$$

See [19] (Theorem 8.2.8) for the proof.

Remark 2.40. Observe that Theorem 2.36 and Theorem 2.39 imply that C-C closed balls are compact in Carnot groups.

Chapter 3

Analysis in Carnot groups

In Section 2.6 we saw that Carnot groups can be equipped with a distance. This fact motivates the presence of this chapter, in which we first introduce some notions and results of measure theory and then we specialize them to Carnot groups. Furthermore, we dedicate the last section to the extension of the concept of mollification to the case of Carnot groups.

3.1 Measures in metric spaces

We point out right away that when we use the expressions "nonnegative", "real" or "vector" referring to set functions (therefore also to measures) we mean that they have codomain in $[0, +\infty]$, $\mathbb{R}$ and $\mathbb{R}^m$ with $m \geq 1$ respectively.

Definition 3.1. Let E be a locally compact and separable (LCS) metric space, $\mathcal{B}(E)$ its Borel σ-algebra and consider the measurable space $(E, \mathcal{B}(E))$.

(i) A (nonnegative) Borel measure which is finite on compact sets is called a *nonnegative Radon measure.*

(ii) A real or vector set function which is defined on relatively compact Borel subsets of E that is a measure on $(K, \mathcal{B}(K))$ for every compact set $K \subseteq E$ is called a *real* or *vector Radon measure.* If $\mu : \mathcal{B}(E) \longrightarrow \mathbb{R}^m$ is a vector measure, we say that it is a *finite Radon measure.*

We will denote by $\mathcal{M}^m_{\text{loc}}(E) = [\mathcal{M}_{\text{loc}}(E)]^m$ (resp. $\mathcal{M}^m(E) = [\mathcal{M}(E)]^m$) the set of $\mathbb{R}^m$-valued Radon (resp. finite $\mathbb{R}^m$-valued Radon) measures on E. Moreover, if d is the distance on E, $B_r(x) := \{y \in E : d(x, y) < r\}$ indicates the open ball of radius $r > 0$ and center $x \in E$.

Remark 3.2. In locally compact Hausdorff spaces nonnegative Radon measures are defined with the additional condition to be regular, i.e. outer and inner regular on all Borel sets, but in a LCS metric space it is a consequence of our definition (see [2], Proposition 1.43).

Let us recall that, given an $\mathbb{R}^m$-valued measure μ on a measurable space $(E, \mathcal{E})$, the *total variation* of μ is the set function $|\mu| : \mathcal{E} \longrightarrow [0, +\infty[$,

$$|\mu|(B) := \sup\left\{\sum_{i=0}^{\infty}|\mu(B_i)| : B_i \in \mathcal{E} \text{ pairwise disjoint, } B = \bigcup_{i=0}^{\infty} B_i\right\}$$

and it turns out to be a nonnegative finite measure (see [2], Theorem 1.6). Moreover, if E is a LCS metric space and $\mu \in \mathcal{M}^m_{\text{loc}}(E)$, $|\mu|$ can be extended to a nonnegative Radon measure in the following way: taken an increasing sequence $\{E_i\}_{i\in\mathbb{N}}$ of relatively compact open subsets of E such that $E = \bigcup_{i=0}^{\infty} E_i$, for any $B \in \mathcal{B}(E)$ set

$$|\mu|(B) := \lim_{i\to\infty} |\mu|(B \cap E_i).$$

Lemma 3.3. *For any $\mathbb{R}^m$-valued measure μ on a measurable space $(E, \mathcal{E})$ there exists a unique $\mathbb{S}^{m-1}$-valued function $f \in L^1(E, |\mu|; \mathbb{R}^m)$ such that $\mu = f|\mu|$.*

Proof. μ is absolutely continuous with respect to $|\mu|$, so by Radon-Nikodym theorem there exists a unique function $f \in L^1(E, |\mu|; \mathbb{R}^m)$ such that $\mu = f|\mu|$ and, thanks to the identity $|f\mu| = |f||\mu|$, $|f(x)| = 1$ for $|\mu|$-a.e. $x \in E$. □

Radon measures can be described in a "functional" way. The fundamental result for doing this is the Riesz representation theorem. Let us denote by $C_0(E; \mathbb{R}^m)$ the space of continuous functions on E with values in $\mathbb{R}^m$ which vanish at infinity.

Theorem 3.4 (Riesz representation theorem)**.** *Suppose E is a LCS metric space and $L : C_0(E; \mathbb{R}^m) \longrightarrow \mathbb{R}$ a linear and continuous operator, i.e.*

$$\|L\| := \sup\{L(u) : u \in C_0(E; \mathbb{R}^m), \|u\|_\infty \leq 1\} < \infty.$$

Then there exists a unique $\mathbb{R}^m$-valued finite Radon measure μ on E such that

$$L(u) = \sum_{i=1}^{m} \int_E u_i \, d\mu_i \qquad \textit{for every } u \in C_0(E; \mathbb{R}^m).$$

Moreover, $\|L\| = |\mu|(E)$.

The proof of this result can be found in [26] (Theorem 6.19). Actually, in [26] it is proved in a more general setting and for an operator on $C_0(E)$, but arguing componentwise we can find the measure μ of the statement, whereas the last part is a consequence of Lemma 3.3 and Lusin's theorem:

$$\begin{aligned}\|L\| &= \sup\left\{\sum_{i=1}^m \int_E u_i\, d\mu_i : u \in C_c(E;\mathbb{R}^m), \|u\|_\infty \leq 1\right\}\\ &= \sup\left\{\sum_{i=1}^m \int_E u_i f_i\, d|\mu| : u \in C_c(E;\mathbb{R}^m), \|u\|_\infty \leq 1\right\}\\ &= \sup\left\{\int_E \langle u, f\rangle\, d|\mu| : u \in C_c(E;\mathbb{R}^m), \|u\|_\infty \leq 1\right\}\\ &= \int_E \langle f, f\rangle\, d|\mu| = |\mu|(E).\end{aligned}$$

The next result is a local version of Riesz representation theorem and follows directly from it. Let us remember that $C_c(E;\mathbb{R}^m)$ is the space of continuous functions with compact support on E with values in $\mathbb{R}^m$.

Corollary 3.5. *Suppose E is a LCS metric space and $L : C_c(E;\mathbb{R}^m) \longrightarrow \mathbb{R}$ a linear operator which is continuous in the following sense: for any compact $K \subseteq E$ there exists $C_K > 0$ such that*

$$\sup\{L(u) : u \in C_c(E;\mathbb{R}^m), \|u\|_\infty \leq 1, \operatorname{supp} u \subseteq K\} < C_K.$$

Then there exists a unique $\mathbb{R}^m$-valued Radon measure μ on E such that

$$L(u) = \sum_{i=1}^m \int_E u_i\, d\mu_i \qquad \text{for every } u \in C_c(E;\mathbb{R}^m).$$

Remark 3.6. We can restate Theorem 3.4 by saying that $\mathcal{M}^m(E)$ can be identified with the dual space of $(C_0(E;\mathbb{R}^m), \|\cdot\|_\infty)$ and the dual norm is $|\mu|(E)$, so in particular it is a Banach space. Corollary 3.5 says a similar thing about $\mathcal{M}^m_{\text{loc}}(E)$, but we have to be careful: we made the choice to give the notion of continuity by hiding the topology on $C_c(E;\mathbb{R}^m)$ from which it derives. To see how this fact is better formalized, refer to [2] (Remark 1.44). However, aware of this omission, we say that $\mathcal{M}^m_{\text{loc}}(E)$ identifies with the dual space of $C_c(E;\mathbb{R}^m)$.

Accordingly with Remark 3.6, we have two different notions of weak* convergence for Radon measures:

Definition 3.7. Let E be a LCS metric space, $\{\mu_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}^m_{\text{loc}}(E)$ and $\mu \in \mathcal{M}^m_{\text{loc}}(E)$; we say that $\{\mu_i\}_{i\in\mathbb{N}}$ *locally weakly* converges to* μ, and we write $\mu_i \overset{*}{\rightharpoonup} \mu$, if

$$\lim_{i\to\infty} \int_E u\, d\mu_i = \int_E u\, d\mu \qquad \text{for every } u \in C_c(E).$$

If μ_i's and μ are finite, we say that $\{\mu_i\}_{i\in\mathbb{N}}$ *weakly* converges to* μ if

$$\lim_{i\to\infty} \int_E u\, d\mu_i = \int_E u\, d\mu \qquad \text{for every } u \in C_0(E).$$

Remark 3.8. In general, a dual norm is lower semicontinuous with respect to the weak* convergence, so by Theorem 3.4 the map $\mu \longmapsto |\mu|(E)$ is weak* lower semicontinuous, that is

$$|\mu|(E) \leq \liminf_{i\to\infty} |\mu_i|(E)$$

whenever $\{\mu_i\}_{i\in\mathbb{N}}$ weakly* converges to μ.

The next proposition provides an equivalence between these two types of convergence.

Proposition 3.9. *If E is a LCS metric space, $\{\mu_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}^m_{\text{loc}}(E)$ and $\mu \in \mathcal{M}^m_{\text{loc}}(E)$, then the following statements are equivalent:*

(i) $\mu_i \overset{*}{\rightharpoonup} \mu$ *and* $\sup_{i\in\mathbb{N}} |\mu_i|(E) < \infty$*;*

(ii) $\{\mu_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}^m(E)$, $\mu \in \mathcal{M}^m(E)$ *and* $\{\mu_i\}_{i\in\mathbb{N}}$ *weakly* converges to* μ.

Proof. Let us first assume *(ii)*. Clearly weak* convergence implies local weak* convergence, whereas the sequence $\{|\mu_i|(E)\}_{i\in\mathbb{N}}$ is bounded thanks to Banach-Steinhaus theorem, so we have *(i)*. Viceversa, since $\sup_{i\in\mathbb{N}} |\mu_i|(E)$ is finite, each μ_i belongs to $\mathcal{M}^m(E)$ and by Banach-Alaoglu theorem there exist a subsequence $\{\mu_{i_k}\}_{k\in\mathbb{N}}$ and $\bar{\mu} \in \mathcal{M}^m(E)$ such that $\{\mu_{i_k}\}_{k\in\mathbb{N}}$ weakly* converges to $\bar{\mu}$. If L_μ and $L_{\bar{\mu}}$ are the operators corresponding to μ and $\bar{\mu}$, by hypothesis $L_\mu(u) = L_{\bar{\mu}}(u)$ for every $u \in C_c(E;\mathbb{R}^m)$, but since $C_c(E;\mathbb{R}^m)$ is dense in $(C_0(E;\mathbb{R}^m), \|\cdot\|_\infty)$ we have $\mu = \bar{\mu}$ and the whole sequence $\{\mu_i\}_{i\in\mathbb{N}}$ weakly* converges to μ. □

Let us see now some important properties of local weak* convergence related to the total variation.

Proposition 3.10. *Let E be a LCS metric space, $\{\mu_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}^m_{\text{loc}}(E)$ and $\mu \in \mathcal{M}^m_{\text{loc}}(E)$. Suppose $\mu_i \overset{*}{\rightharpoonup} \mu$. Then we have*

(i) for any $U \subseteq E$ open

$$|\mu|(U) \leq \liminf_{i\to\infty} |\mu_i|(U); \tag{3.1}$$

(ii) for any $K \subseteq E$ compact

$$\sup_{i\in\mathbb{N}} |\mu_i|(K) < \infty. \tag{3.2}$$

Proof. (i). Consider an increasing sequence $\{E_j\}_{j\in\mathbb{N}}$ of relatively compact open subsets of E such that $E = \bigcup_{j=0}^{\infty} E_j$. Since $C_0(U \cap E_j) \subseteq C_c(E)$, by hypothesis $\{\mu_i\}_{i\in\mathbb{N}}$ weakly* converges to μ in $\mathcal{M}^m(U \cap E_j)$ for every $j \in \mathbb{N}$, so by Remark 3.8

$$|\mu|(U) = \lim_{j\to\infty} |\mu|(U \cap E_j) \leq \liminf_{j\to\infty} \liminf_{i\to\infty} |\mu_i|(U \cap E_j) \leq \liminf_{i\to\infty} |\mu_i|(U).$$

(ii). Let U be a relatively compact open subset of E such that $K \subseteq U$. Since $C_0(U) \subseteq C_c(E)$, by hypothesis $\{\mu_i\}_{i\in\mathbb{N}}$ weakly* converges to μ in $\mathcal{M}^m(U)$, hence by Proposition 3.9

$$\sup_{i\in\mathbb{N}} |\mu_i|(K) \leq \sup_{i\in\mathbb{N}} |\mu_i|(U) < \infty. \qquad \square$$

Definition 3.11. Let μ be a nonnegative measure on a LCS metric space E. The *support* of μ $\operatorname{supp}\mu$ is the closure of the set of all points $x \in E$ such that $\mu(U) > 0$ for every neighborhood U of x. If μ is a real or vector measure, the *support* of μ $\operatorname{supp}\mu$ is the support of $|\mu|$.

For a measure μ on a measurable space $(E, \mathcal{E})$, we say that μ is *concentrated* on $S \subseteq E$ if $S \in \mathcal{E}$ and $|\mu|(E \setminus S) = 0$. Therefore, if E is a LCS metric space, $\operatorname{supp}\mu$ is the smallest closed set where μ is concentrated. Moreover, it is not difficult to prove

$$\operatorname{supp}\mu = \{x \in E : \mu(B_r(x)) > 0 \ \ \forall r > 0\}.$$

We conclude the section recalling a useful notion which allows to “carry” a measure from a measurable space to another.

Definition 3.12. Let $(E, \mathcal{E})$, $(F, \mathcal{F})$ be measurable spaces and $f : E \longrightarrow F$ a measurable function. The *pushforward* (or *image*) *measure* of a nonnegative, real or vector measure μ on $(E, \mathcal{E})$ is the measure $f_\sharp\mu$ on $(F, \mathcal{F})$ defined by

$$f_\sharp\mu(A) := \mu(f^{-1}(A)) \qquad \text{for every } A \in \mathcal{F}.$$

Using the classical standard machine argument, one can easily prove that, for any function $u \in L^1(F, \mathcal{F}, f_\sharp\mu)$, $u \circ f \in L^1(E, \mathcal{E}, \mu)$ and the change of variables formula

$$\int_F u \, df_\sharp\mu = \int_E u \circ f \, d\mu \tag{3.3}$$

holds.

3.2 Differentiation of measures

In this section we want to review some results of Lebesgue differentiation theory in metric spaces with hypothesis different from standard ones.

Definition 3.13. Let E be a metric space. A locally finite Borel measure μ is said to be *doubling* if there exists $M > 0$ such that

$$\mu(B_{2r}(x)) \leq M\mu(B_r(x)) \qquad \text{for every } x \in E,\ r > 0.$$

μ is called *asymptotically doubling* if

$$\limsup_{r\searrow 0} \frac{\mu(B_{2r}(x))}{\mu(B_r(x))} < \infty \qquad \text{for } \mu\text{-a.e. } x \in \operatorname{supp}\mu.$$

Obviously a doubling measure is also asymptotically doubling. Furthermore, it is clear by definition that if μ is doubling then every ball has finite measure and if there exists a ball B such that $\mu(B) = 0$ then $\mu \equiv 0$.

Lemma 3.14. *If E is a metric space and μ a Borel measure, then for any $r > 0$ the map $x \longmapsto \mu(B_r(x))$ is lower semicontinuous.*

Proof. Consider any sequence $\{y_i\}_{i\in\mathbb{N}} \subseteq E$ which converges to $x \in E$ and set $f_i \coloneqq \chi_{B_r(y_i)}$, $f \coloneqq \chi_{B_r(x)}$. Then

$$f \le \liminf_{i\to\infty} f_i$$

and so by Fatou's lemma

$$\int_{B_r(x)} f \, d\mu \le \int_{B_{2r}(x)} f \, d\mu \le \int_{B_{2r}(x)} \liminf_{i\to\infty} f_i \, d\mu$$
$$\le \liminf_{i\to\infty} \int_{B_{2r}(x)} f_i \, d\mu,$$

that is (for i large enough)

$$\mu(B_r(x)) \le \liminf_{i\to\infty} \mu(B_r(y_i)). \qquad \square$$

We want to present the previously announced results for asymptotically doubling measures, but we mention that one way of proving them is to show them by initially assuming that the measure μ is doubling and then cover $\operatorname{supp}\mu$ with Borel sets in each of which μ satisfies a doubling condition. Precisely, if we set for any $j, k \in \mathbb{N}$

$$E_{j,k} \coloneqq \left\{ x \in \operatorname{supp}\mu : \mu(B_{2r}(x)) \le j\mu(B_r(x)) \text{ for every } 0 < r < \frac{1}{k} \right\},$$

then $\operatorname{supp}\mu = \bigcup_{j,k=1}^{\infty} E_{j,k}$ and all $E_{j,k}$'s are Borel because

$$E_{j,k} = \bigcap_{r\in\mathbb{Q}\cap\left]0,\frac{1}{k}\right[} \{x \in \operatorname{supp}\mu : \mu(B_{2r}(x)) \le j\mu(B_r(x))\}$$

and each set involved in the intersection is Borel by Lemma 3.14.

Theorem 3.15 (Lebesgue's differentiation theorem)**.** *Let E be a separable metric space, μ an asymptotically doubling measure and $f \in L^1_{\mathrm{loc}}(E, \mu)$. Then μ-a.e. $x \in E$ is a Lebesgue point of f, that is*

$$\lim_{r\searrow 0} ⨍_{B_r(x)} |f(y) - f(x)| \, d\mu(y) = 0 \qquad \textit{for } \mu\textit{-a.e. } x \in E.$$

The proof of the previous theorem can be found in [15] (pages 77-81 and Remark 3.4.29) or [4] (Theorem 5.2.6) for the doubling case.

Theorem 3.16 (Lebesgue's density theorem)**.** *Suppose E is a separable metric space and μ an asymptotically doubling measure. For any Borel set $A \subseteq E$, μ-a.e. $x \in A$ is a density point, i.e.*

$$\lim_{r \searrow 0} \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))} = 1.$$

Proof. The characteristic function $\chi_A \in L^1_{\text{loc}}(E, \mu)$ because A is Borel, so by Theorem 3.15 we have for μ-a.e. $x \in A$

$$\lim_{r \searrow 0} \fint_{B_r(x)} \chi_A \, d\mu = \chi_A(x) = 1$$

and the equality

$$\fint_{B_r(x)} \chi_A \, d\mu = \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))}$$

concludes the proof. □

Remark 3.17. Theorem 3.16 still holds if we replace μ by its associated outer measure μ^*, which is defined for any $A \subseteq E$ by

$$\mu^*(A) \coloneqq \inf\{\mu(B) : B \text{ Borel, } B \supseteq A\}.$$

Indeed, let $\{B_i\}_{i \in \mathbb{N}}$ be a minimizing sequence and $B \coloneqq \bigcap_{i=0}^\infty B_i$; then B is a Borel set and $\mu^*(A) = \mu(B)$. Furthermore, for all Borel sets C we have $\mu^*(A \cap C) = \mu(B \cap C)$ (if not, adding $\mu^*(A \cap C) < \mu(B \cap C)$ to $\mu^*(A \setminus C) \leq \mu(B \setminus C)$ we would get $\mu^*(A) < \mu(B)$ since μ^* is subadditive and μ additive). Hence we can take a density point x of B and $C = B_r(x)$ to deduce from Theorem 3.16

$$\lim_{r \searrow 0} \frac{\mu^*(A \cap B_r(x))}{\mu^*(B_r(x))} = \lim_{r \searrow 0} \frac{\mu(B \cap B_r(x))}{\mu(B_r(x))} = 1.$$

In particular, this proves that the set of points of A that are not density points is contained in a μ-negligible Borel set.

Theorem 3.18 (Radon-Nikodym theorem)**.** *Let E be a separable metric space, μ an asymptotically doubling measure and ν a locally finite Borel regular measure and assume that ν is absolutely continuous with respect to μ. Then the limit*

$$f(x) \coloneqq \lim_{r \searrow 0} \frac{\nu(B_r(x))}{\mu(B_r(x))}$$

exists and is finite for μ-a.e. $x \in \operatorname{supp} \mu$. Moreover, $f \in L^1_{\text{loc}}(E, \mu)$ and $\nu = f\mu$.

The proof can be found in [10] (Section 2.9) in a much more general situation or [15] (pages 82-86).

3.3 Hausdorff measures

Hausdorff measures will be very important for our purposes, so we want briefly to recall them. We refer to [4] for all the details and proofs not provided. Throughout this section we assume that (E, d) is a metric space.

Definition 3.19. Let $k \geq 0$ and set

$$\omega_k := \frac{\pi^{\frac{k}{2}}}{\Gamma\big(1 + \frac{k}{2}\big)},$$

where $\Gamma(t) := \int_0^\infty x^{t-1} e^{-x}\, dx$ $(t > 0)$ is the Euler's Gamma function. For any $\delta > 0$ and $A \subseteq E$ let

$$\mathcal{H}^k_\delta(A) := \frac{\omega_k}{2^k} \inf\left\{ \sum_{i=0}^\infty (\operatorname{diam} A_i)^k : A \subseteq \bigcup_{i=0}^\infty A_i,\ \operatorname{diam} A_i < \delta \right\}.$$

The *k-dimensional Hausdorff measure* of A is

$$\mathcal{H}^k(A) := \lim_{\delta \searrow 0} \mathcal{H}^k_\delta(A) = \sup_{\delta > 0} \mathcal{H}^k_\delta(A).$$

The *k-dimensional spherical Hausdorff measure* $\mathcal{S}^k(A)$ of A is defined in the same way, but with the constraint that the A_i's can only be balls.

From the definition one can easily check that $\mathcal{H}^k$ and $\mathcal{S}^k$ satisfy

$$\mathcal{H}^k \leq \mathcal{S}^k \leq 2^k \mathcal{H}^k. \tag{3.4}$$

The next proposition lists some basic properties of Hausdorff measures.

Proposition 3.20. *For any $k \geq 0$ and $\delta > 0$ we have*

(i) $\mathcal{H}^k_\delta$ and $\mathcal{H}^k$ are outer measures. Moreover, $\mathcal{H}^k$ (restricted to $\mathcal{B}(E)$) is a Borel measure;

(ii) $\mathcal{H}^k$ is Borel regular;

(iii) if $A \subseteq E$ is such that $\mathcal{H}^k(A) > 0$, then $\mathcal{H}^{k'}(A) = +\infty$ for every $k > k' \geq 0$.

Hausdorff measures allow to define a fundamental notion of dimension:

Definition 3.21. For any $A \subseteq E$ we define the *Hausdorff dimension* of A as

$$\dim_{\mathcal{H}}(A) := \inf\{k \geq 0 : \mathcal{H}^k(A) = 0\}.$$

Remark 3.22. It is trivial by Proposition 3.20 that if $k > \dim_{\mathcal{H}}(A)$ then $\mathcal{H}^k(A) = 0$ and if $k < \dim_{\mathcal{H}}(A)$ then $\mathcal{H}^k(A) = +\infty$, but nothing can be said in general when $k = \dim_{\mathcal{H}}(A)$.

Theorem 3.23. *Let μ be a Borel measure on E and assume that there exist $Q > 0$, $C \geq 1$ and $R > 0$ such that*

$$\frac{1}{C} r^Q \leq \mu(B_r(x)) \leq C r^Q \qquad \text{for every } x \in E,\ r \in]0, R].$$

Then for any $x \in E$

(i) $\mathcal{H}^Q(B_R(x)) \in]0, \infty[$,

(ii) $\dim_{\mathcal{H}}(B_R(x)) = Q$

and, if in addition E admits a countable cover of balls of radius R, then $\dim_{\mathcal{H}}(E) = Q$.

Proof. (i). We first show $\mathcal{H}^Q(B_R(x)) < \infty$. Fix $r \in]0, R[$ and $0 < \delta < R - r$. Let us notice that if $x_1, \ldots, x_k \in B_r(x)$ are a certain quantity of points such that $d(x_i, x_j) > \delta$ for all $i, j = 1, \ldots, k$, $i \neq j$, then the balls $B_{\delta/2}(x_i)$ are disjoint and contained in $B_R(x)$, so

$$\begin{aligned} k \frac{\delta^Q}{2^Q C} &= \frac{1}{C} \sum_{i=1}^{k} \left(\frac{\delta}{2} \right)^Q \leq \sum_{i=1}^{k} \mu\Big(B_{\frac{\delta}{2}}(x_i) \Big) \\ &= \mu\left(\bigcup_{i=1}^{k} B_{\frac{\delta}{2}}(x_i) \right) \leq \mu(B_R(x)) \leq C R^Q. \end{aligned}$$

Therefore k cannot be infinite and we can consider the maximal natural number N with the property that there exist $x_1, \ldots, x_N \in B_r(x)$ such that $d(x_i, x_j) > \delta$ for all $i, j = 1, \ldots, N$, $i \neq j$. This maximality implies

$$B_r(x) \subseteq \bigcup_{i=1}^{N} B_\delta(x_i),$$

hence

$$\begin{aligned} \mathcal{H}^Q_{2\delta}(B_r(x)) &= \frac{\omega_Q}{2^Q} \sum_{i=1}^{N} (\operatorname{diam} B_\delta(x_i))^Q \\ &\leq \frac{\omega_Q}{2^Q} N (2\delta)^Q \\ &= 2^Q \omega_Q C N \frac{1}{C} \left(\frac{\delta}{2} \right)^Q \\ &= 2^Q \omega_Q C \sum_{i=1}^{N} \mu\Big(B_{\frac{\delta}{2}}(x_i) \Big) \\ &\leq 2^Q \omega_Q C \mu(B_R(x)). \end{aligned}$$

By letting $\delta \searrow 0$ we get

$$\mathcal{H}^Q(B_r(x)) \leq 2^Q \omega_Q C \mu(B_R(x))$$

for every $r \in]0, R[$, therefore

$$\begin{aligned}\mathcal{H}^Q(B_R(x)) = \mathcal{H}^Q\left(\bigcup_{r<R} B_r(x)\right) &= \lim_{r \nearrow R} \mathcal{H}^Q(B_r(x)) \\ &\leq 2^Q \omega_Q C \mu(B_R(x)) < \infty.\end{aligned}$$

We now prove $\mathcal{H}^Q(B_R(x)) > 0$. Fix $\delta \in]0, R[$. Since $\mathcal{H}^Q(B_R(x))$ is finite, for any $\varepsilon > 0$ we can take countably many subsets $\{A_i\}_{i \in \mathbb{N}}$ of E such that $B_R(x) \subseteq \bigcup_{i=0}^{\infty} A_i$, $\operatorname{diam} A_i < \delta$ and

$$\mathcal{H}^Q_\delta(B_R(x)) \geq \frac{\omega_Q}{2^Q} \sum_{i=0}^{\infty} (\operatorname{diam} A_i)^Q - \varepsilon.$$

Let $x_i \in A_i$, so that $A_i \subseteq B_{\operatorname{diam} A_i}(x_i)$. Hence we obtain

$$\begin{aligned}\mathcal{H}^Q_\delta(B_R(x)) &\geq \frac{\omega_Q}{2^Q} \frac{1}{C} \sum_{i=0}^{\infty} \mu(B_{\operatorname{diam} A_i}(x_i)) - \varepsilon \\ &\geq \frac{\omega_Q}{2^Q C} \mu\left(\bigcup_{i=0}^{\infty} B_{\operatorname{diam} A_i}(x_i)\right) - \varepsilon \\ &\geq \frac{\omega_Q}{2^Q C} \mu(B_R(x)) - \varepsilon \\ &\geq \frac{\omega_Q}{2^Q C^2} R^Q - \varepsilon.\end{aligned}$$

Finally, by the arbitrariness of ε and letting $\delta \searrow 0$ we get

$$\mathcal{H}^Q(B_R(x)) \geq \frac{\omega_Q}{2^Q C^2} R^Q > 0.$$

(ii) and the last part of the statement are immediate consequences of *(i)*. □

We recall the following useful relation between Hausdorff measures and measures on metric spaces.

Theorem 3.24. *Let μ be a locally finite Borel measure on E and $B \subseteq E$ a Borel set.*

(i) If there exist $k \geq 0$ and $t > 0$ such that

$$\limsup_{r \searrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \geq t \qquad \text{for every } x \in B,$$

then

$$\mu(B) \geq t \mathcal{S}^k(B) \geq t \mathcal{H}^k(B).$$

(ii) If there exist $k \geq 0$ and $t > 0$ such that

$$\limsup_{r \searrow 0} \frac{\mu(B_r(x))}{\omega_k r^k} \leq t \qquad \text{for every } x \in B,$$

then

$$\mu(B) \leq 2^k t \mathcal{H}^k(B).$$

Corollary 3.25. *If E is LCS and μ a nonnegative Radon measure, then for any $k \geq 0$ the set*

$$\left\{ x \in E : \limsup_{r \searrow 0} \frac{\mu(B_r(x))}{r^k} > 0 \right\}$$

is σ-finite with respect to $\mathcal{S}^k$.

Proof. Consider an increasing sequence $\{E_i\}_{i \in \mathbb{N}}$ of relatively compact open subsets of E such that $E = \bigcup_{i=1}^{\infty} E_i$. Moreover, for any $j \in \mathbb{N}$ set

$$B_j := \left\{ x \in E : \limsup_{r \searrow 0} \frac{\mu(B_r(x))}{r^k} \geq \frac{1}{j} \right\}.$$

The map

$$x \longmapsto \limsup_{r \searrow 0} \frac{\mu(B_r(x))}{r^k}$$

is Borel measurable by virtue of Lemma 3.14, so each B_j is a Borel set. Therefore, if we let $U_{i,j} := E_i \cap B_j$, $U_{i,j}$ is a Borel set,

$$\left\{ x \in E : \limsup_{r \searrow 0} \frac{\mu(B_r(x))}{r^k} > 0 \right\} = \bigcup_{i,j=1}^{\infty} U_{i,j}$$

and thanks to Theorem 3.24

$$\mathcal{S}^k(U_{i,j}) \leq \frac{j}{\omega_k} \mu(E_i) < \infty$$

for every (fixed) $i, j \in \mathbb{N}$. □

3.4 Haar measures

Let $\mathbb{G}$ be a locally compact group. We continue to denote by L_g and R_g the left and right translations by an element $g \in \mathbb{G}$.

Definition 3.26. A Borel measure μ on $\mathbb{G}$ is said to be *left-invariant* if for any $g \in \mathbb{G}$ $(L_g)_\sharp \mu = \mu$, i.e. $\mu(gB) = \mu(B)$ for every $B \in \mathcal{B}(\mathbb{G})$. Similarly, μ is said to be *right-invariant* if for any $g \in \mathbb{G}$ $(R_g)_\sharp \mu = \mu$, i.e. $\mu(Bg) = \mu(B)$ for every $B \in \mathcal{B}(\mathbb{G})$.

Any locally compact group (and in particular any Lie group) has a natural class of measures which we define now (keeping in mind Remark 3.2).

Definition 3.27. Let $\mathbb{G}$ be a locally compact group. A *left* (resp. *right*) *Haar measure* is a nonzero left (resp. right)-invariant Radon measure on $\mathbb{G}$. If a measure is both a left Haar measure and a right Haar measure, it is called *Haar measure*.

We want to state an existence and uniqueness theorem for Haar measures. Actually, we will do it for left Haar measures, but the same result holds for right ones by virtue of the following easy observation: let μ be a Radon measure on a locally compact group and define the Radon measure $\widetilde{\mu}(B) := \mu(B^{-1})$; then μ is a left Haar measure if and only if $\widetilde{\mu}$ is a right Haar measure. The proof of this result can be found in [11] (Theorem 2.10 and Theorem 2.20).

Theorem 3.28. *Any locally compact group $\mathbb{G}$ possesses a left Haar measure. Moreover, if μ and ν are two left Haar measures on $\mathbb{G}$, then there exists $c > 0$ such that $\nu = c\mu$.*

An important consequence of the existence of a right Haar measure is the following result.

Proposition 3.29. *Let $\mathbb{G}$ be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathrm{vol}_{\mathbb{G}}$ a right Haar measure on $\mathbb{G}$. Then any $X \in \mathfrak{g}$ is divergence-free. Moreover, if $f \in L^1_{\mathrm{loc}}(\mathbb{G})$ then $Xf = 0$ (in the sense of distributions) if and only if $f \circ R_{\exp(tX)} = f$ $\mathrm{vol}_{\mathbb{G}}$-a.e. for every $t \in \mathbb{R}$.*

Proof. Since $\Phi_t^X = R_{\exp(tX)}$ by Proposition 2.6, we have $(\Phi_t^X)_\sharp \mathrm{vol}_{\mathbb{G}} = \mathrm{vol}_{\mathbb{G}}$ and so X is divergence-free by Proposition 1.17.
If $Xf = 0$, we immediately get $f \circ R_{\exp(tX)} = f$ $\mathrm{vol}_{\mathbb{G}}$-a.e. by Proposition 1.18. Conversely, from the proof of Proposition 1.18 we deduce

$$\frac{d}{dt} \int_{\mathbb{G}} (f \circ R_{\exp(tX)}) u \, d\mathrm{vol}_{\mathbb{G}} = - \int_{\mathbb{G}} f X(u \circ R_{\exp(-tX)}) \, d\mathrm{vol}_{\mathbb{G}}$$

for every $u \in C_c^\infty(\mathbb{G})$. The left-hand side is 0 by hypothesis, so choosing $t = 0$ we obtain

$$\langle Xf, u \rangle = - \int_{\mathbb{G}} f X u \, d\mathrm{vol}_{\mathbb{G}} = 0. \qquad \square$$

In general it is not true that left Haar measures are also right Haar measures (and viceversa); a locally compact group which satisfies such property is called *unimodular*. However, the next theorem says that nilpotency is a sufficient condition for a locally compact group to be unimodular. The proof derives from Corollary of Proposition II.25 of [24], the "algebraic" definition of nilpotent group (which is equivalent to ours, see Theorem XII.3.1 of [16]) and the fact that abelian groups are unimodular.

Theorem 3.30. *Let $\mathbb{G}$ be a locally compact group and suppose that $\mathbb{G}$ is nilpotent. Then $\mathbb{G}$ is unimodular.*

In Section 2.3 we defined exponential coordinates for connected, simply connected and nilpotent Lie groups. Let us prove now that these coordinates take the Lebesgue measure to a Haar measure.

Lemma 3.31. *Suppose $\mathbb{G}$ is an n-dimensional connected, simply connected and nilpotent Lie group. Then, for any $g \in \mathbb{G}$, L_g and R_g have Jacobian identically equal to* 1 *in exponential coordinates.*

Proof. We will prove the thesis only for left translations because the case of right ones is completely analogous. In exponential coordinates the group law becomes of the form (2.9) and $L_x(y) = x + y + Q(x, y)$ for every $x, y \in \mathbb{R}^n$. Using the Baker-Campbell-Hausdorff formula (2.3) one can compute Q and check that the differential of L_x is a lower triangular matrix with entries on the diagonal equal to 1 (see also [23], Lemma 1.7.2), hence the proof is concluded. □

Theorem 3.32. *Let $\mathbb{G}$ be an n-dimensional connected, simply connected and nilpotent Lie group. Any system of exponential coordinates pushes forward Lebesgue measure on $\mathbb{R}^n$ to a Haar measure on $\mathbb{G}$.*

Proof. Consider a system of exponential coordinates E. Since E is a diffeomorphism and $\mathcal{L}^n$ a nonzero Radon measure on $\mathbb{R}^n$, $E_\sharp\mathcal{L}^n$ is also a nonzero Radon measure on $\mathbb{G}$, therefore we only need to show that it is left and right-invariant. This is true thanks to Lemma 3.31: for any $B \in \mathcal{B}(\mathbb{G})$ and $g \in \mathbb{G}$,

$$\begin{aligned} E_\sharp\mathcal{L}^n(gB) &= \mathcal{L}^n(E^{-1}(L_g(B))) \\ &= \mathcal{L}^n(L_{E^{-1}(g)}(E^{-1}(B))) \\ &= \mathrm{Jac}(L_{E^{-1}(g)})\mathcal{L}^n(E^{-1}(B)) \\ &= E_\sharp\mathcal{L}^n(B) \end{aligned}$$

and analogously $E_\sharp\mathcal{L}^n(Bg) = E_\sharp\mathcal{L}^n(B)$. □

3.5 Measures in Carnot groups

From now on, whenever we will talk about a Carnot group, we will implicitly identify it with some $(\mathbb{R}^n, \cdot)$ via a system of graded coordinates E.
Consider a Carnot group $\mathbb{G}$ endowed with the C-C distance d. By virtue of Theorem 2.36, $(\mathbb{G}, d)$ is a LCS metric space, therefore Corollary 3.5 ensures that each volume form $\mathrm{vol}_{\mathbb{G}}$ on $\mathbb{G}$ defines a nonzero Radon measure, in particular a left or right Haar measure if $\mathrm{vol}_{\mathbb{G}}$ is respectively left or right-invariant. But since $\mathbb{G}$ is nilpotent, by Theorem 3.30 it is unimodular, so left

and right-invariant volume forms coincide and define Haar measures, which are positively proportional by Theorem 3.28. Henceforth we will identify volume forms with the associated Haar measures.

Definition 3.33. Let $\mathbb{G}$ be a Carnot group with stratification of its Lie algebra $\mathfrak{g} = V_1 \oplus \ldots \oplus V_s$. The *homogeneous dimension* of $\mathbb{G}$ is the natural number

$$Q := \sum_{i=1}^{s} i \dim V_i.$$

Let us show a useful relation between Haar measures and dilations:

Proposition 3.34. *If $\mathbb{G}$ is a Carnot group of homogeneous dimension Q and $\mathrm{vol}_{\mathbb{G}}$ a Haar measure, then*

$$\mathrm{vol}_{\mathbb{G}}(\delta_\lambda(B)) = \lambda^Q \mathrm{vol}_{\mathbb{G}}(B) \tag{3.5}$$

for every $B \in \mathcal{B}(\mathbb{G})$, $\lambda \geq 0$.

Proof. From Theorem 3.32 and Theorem 3.28 we know that there exists $c > 0$ such that for any $A \in \mathcal{B}(\mathbb{G})$

$$\mathrm{vol}_{\mathbb{G}}(A) = c\mathcal{L}^n(E^{-1}(A)).$$

Therefore, recalling that $\mathrm{Jac}(\delta_\lambda) = \lambda^Q$, we obtain

$$\begin{aligned}
\mathrm{vol}_{\mathbb{G}}(\delta_\lambda(B)) &= c\mathcal{L}^n(E^{-1}(\delta_\lambda(B))) \\
&= c\mathcal{L}^n(\delta_\lambda(E^{-1}(B))) \\
&= c\lambda^Q\mathcal{L}^n(E^{-1}(B)) \\
&= \lambda^Q \mathrm{vol}_{\mathbb{G}}(B).
\end{aligned}$$

□

As one might expect from the previous proposition, Q is the Hausdorff dimension of $\mathbb{G}$ with respect to the C-C distance. This is a well known result in a more general context (see [22], Theorem 2), but in our setting we can prove it:

Theorem 3.35. *Let $\mathbb{G}$ be a Carnot group of homogeneous dimension Q. Then $\dim_{\mathcal{H}}(\mathbb{G}) = Q$.*

Proof. Let us take a Haar measure $\mathrm{vol}_{\mathbb{G}}$ (which exists by Theorem 3.28). Corollary 2.38 gives for any $x \in \mathbb{G}$ and $\lambda > 0$

$$L_{x^{-1}}(B_\lambda(x)) = B_\lambda(e) = \delta_\lambda(B_1(e)),$$

hence by Lemma 3.31 and (3.5)

$$\mathrm{vol}_{\mathbb{G}}(B_\lambda(x)) = \mathrm{vol}_{\mathbb{G}}(L_{x^{-1}}(B_\lambda(x))) = \mathrm{vol}_{\mathbb{G}}(\delta_\lambda(B_1(e))) = \lambda^Q \mathrm{vol}_{\mathbb{G}}(B_1(e)).$$

Therefore the proof is concluded by Theorem 3.23. □

With this fact in mind, let us prove that on a Carnot group there are few natural choices of measures (Haar, Hausdorff, Lebesgue) and, actually, they are the same up to a positive factor.

Proposition 3.36. *Let $\mathbb{G}$ be a Carnot group of homogeneous dimension Q. Then the Q-dimensional Hausdorff measure and the Q-dimensional spherical Hausdorff measure (induced by the C-C distance) are Haar measures on $\mathbb{G}$. In particular, they coincide with the Lebesgue measure up to a positive factor.*

Proof. The C-C distance d is homogeneous (Proposition 2.37), so observing that for any $A \subseteq \mathbb{G}$ and $x \in \mathbb{G}$

$$\operatorname{diam}(A) = \sup_{y,z\in\mathbb{G}} d(y,z) = \sup_{y,z\in\mathbb{G}} d(x\cdot y, x\cdot z) = \operatorname{diam}(x\cdot A),$$

we have for any $B \in \mathcal{B}(\mathbb{G})$

$$\mathcal{H}^Q(x\cdot B) = \mathcal{H}^Q(B),$$
$$\mathcal{S}^Q(x\cdot B) = \mathcal{S}^Q(B),$$

namely $\mathcal{H}^Q$ and $\mathcal{S}^Q$ are left-invariant. Analogously, they are right-invariant. Furthermore, since any compact set is contained in a ball, by Theorem 3.23 (which can be applied as observed in the proof of Theorem 3.35) and (3.4) $\mathcal{H}^Q$ and $\mathcal{S}^Q$ are finite on compact sets, so they are Haar measures. The last part of the statement follows from the fact that $\mathcal{L}^n$ is a Haar measure on $(\mathbb{R}^n, \cdot)$ (easy consequence of Lemma 3.31) and Theorem 3.28. □

3.6 Group convolutions

Let us fix for this section an n-dimensional Carnot group $\mathbb{G}$ with its Lie algebra $\mathfrak{g}$, homogeneous dimension Q and equipped with a homogeneous distance.

Definition 3.37. Given two Borel functions f and g on $\mathbb{G}$, their *convolution* $f * g$ is defined by

$$f * g(x) := \int_{\mathbb{G}} f(y)g(y^{-1}\cdot x)\,d\mathcal{L}^n(y) = \int_{\mathbb{G}} f(x\cdot y^{-1})g(y)\,d\mathcal{L}^n(y)$$

for every $x \in \mathbb{G}$, provided that the integrals converge.

We want to prove that, as in the Euclidean case, if we convolve a function with nice functions we get nice approximations. Consider a function $\rho \in C_c^\infty(\mathbb{G})$ such that

$$0 \le \rho \le 1, \quad \int_{\mathbb{G}} \rho\,d\mathcal{L}^n = 1 \quad \text{and} \quad \operatorname{supp}\rho \subseteq B_1(e)$$

and set for any $\varepsilon > 0$

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^Q}\rho\Big(\delta_{\frac{1}{\varepsilon}}(x)\Big), \qquad x \in \mathbb{G}.$$

The functions $\{\rho_\varepsilon\}_{\varepsilon>0}$ are called *mollifiers.*

Theorem 3.38. *The following statements hold:*

(i) if $X \in \mathfrak{g}$ and $g \in C_c^1(\mathbb{G})$, then $X(f * g) = f * (Xg)$;

(ii) if $f \in C(\mathbb{G})$, then $f * \rho_\varepsilon \longrightarrow f$ uniformly as $\varepsilon \searrow 0$ on compact subsets of $\mathbb{G}$;

(iii) if $f \in L^1_{\mathrm{loc}}(\mathbb{G})$, then $f * \rho_\varepsilon \in C^\infty(\mathbb{G})$ for every $\varepsilon > 0$ and $f * \rho_\varepsilon \longrightarrow f$ in $L^1_{\mathrm{loc}}(\mathbb{G})$ as $\varepsilon \searrow 0$.

Proof. (i). Differentiating under the integral sign (it can be done because g is compactly supported and f is integrable on $\operatorname{supp} g$) and using (2.1) we get

$$\begin{aligned} X(f * g)(x) &= X\bigg(\int_{\mathbb{G}} f(y)g(y^{-1}\cdot x)\, d\mathcal{L}^n(y)\bigg) \\ &= \int_{\mathbb{G}} X(f(y)g(y^{-1}\cdot x))\, d\mathcal{L}^n(y) \\ &= \int_{\mathbb{G}} f(y)(Xg)(y^{-1}\cdot x)\, d\mathcal{L}^n(y) \\ &= f * (Xg)(x) \end{aligned}$$

for every $x \in \mathbb{G}$.
(ii). Let $K \subseteq \mathbb{G}$ be compact; f is uniformly continuous on K, so for any $\bar\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in K$

$$|f(x\cdot y^{-1}) - f(x)| \le \bar\varepsilon \qquad \text{for every } y \in B_\delta(e).$$

Observing that

$$f(x) = f(x)\int_{\mathbb{G}} \rho_\varepsilon(y)\, d\mathcal{L}^n(y) = \int_{\mathbb{G}} f(x)\rho_\varepsilon(y)\, d\mathcal{L}^n(y),$$

we have

$$|f * \rho_\varepsilon(x) - f(x)| \le \int_{\mathbb{G}} |f(x\cdot y^{-1}) - f(x)|\rho_\varepsilon(y)\, d\mathcal{L}^n(y).$$

We can choose $\varepsilon < \delta$ so that $\operatorname{supp}\rho_\varepsilon \subseteq B_\delta(e)$ and we obtain

$$\begin{aligned} |f * \rho_\varepsilon(x) - f(x)| &\le \int_{B_\delta(e)} |f(x\cdot y^{-1}) - f(x)|\rho_\varepsilon(y)\, d\mathcal{L}^n(y) \\ &\le \bar\varepsilon \int_{B_\delta(e)} \rho_\varepsilon(y)\, d\mathcal{L}^n(y) = \bar\varepsilon. \end{aligned}$$

(iii). From *(i)* we know $f * \rho_\varepsilon \in C^\infty(\mathbb{G})$. If $K \subseteq \mathbb{G}$ is compact, by the density of $C(K)$ in $L^1(K)$ for any $\bar{\varepsilon} > 0$ there exists $g_{\bar{\varepsilon}} \in C(K)$ such that $\|f - g_{\bar{\varepsilon}}\|_{L^1(K)} \leq \bar{\varepsilon}$. From *(ii)* we know that $g_{\bar{\varepsilon}} * \rho_\varepsilon \longrightarrow g_{\bar{\varepsilon}}$ uniformly as $\varepsilon \searrow 0$ on K, so in particular in $L^1(K)$. Therefore we finally have for $\varepsilon > 0$ small enough

$$\begin{aligned}\|f * \rho_\varepsilon - f\|_{L^1(K)} &\leq \|(f - g_{\bar{\varepsilon}}) * \rho_\varepsilon\|_{L^1(K)} + \|g_{\bar{\varepsilon}} * \rho_\varepsilon - g_{\bar{\varepsilon}}\|_{L^1(K)} \\ &\quad + \|g_{\bar{\varepsilon}} - f\|_{L^1(K)} \\ &\leq \|f - g_{\bar{\varepsilon}}\|_{L^1(K)} + \bar{\varepsilon} + \bar{\varepsilon} \leq 3\bar{\varepsilon}.\end{aligned}$$

□

Chapter 4

Geometric measure theory in Carnot groups

This chapter is devoted to introduce the main object of investigation of this thesis, that is sets of locally finite perimeter, but it is also an opportunity to see some geometric notions and results which will be fundamental in the sequel.
We continue to denote by $\mathbb{G}$ an n-dimensional Carnot group endowed with the C-C distance d, $\mathfrak{g}$ its Lie algebra with stratification $\mathfrak{g} = V_1 \oplus \ldots \oplus V_s$ and Q its homogeneous dimension. Moreover, we fix a Haar measure $\mathrm{vol}_{\mathbb{G}}$ on $\mathbb{G}$.
We also point out that for our purposes it will be important to have a notion of orthogonality on the horizontal layer, so we fix a scalar product $\langle \cdot, \cdot \rangle_x$ on $(V_1)_x$ that makes its adapted basis $X_1(x), \ldots, X_m(x)$ an orthonormal basis for every $x \in \mathbb{G}$. More precisely, if $v = \sum_{i=1}^m v_i X_i(x) = (v_1, \ldots, v_m)$ and $w = \sum_{i=1}^m w_i X_i(x) = (w_1, \ldots, w_m)$ are in $(V_1)_x$, we set $\langle v, w \rangle_x \coloneqq \sum_{i=1}^m v_i w_i$.

4.1 Regular directions and vertical half-spaces

Recall that any $X \in \mathfrak{g}$ is divergence-free (Proposition 3.29), so the next definition is well posed.

Definition 4.1. Let $f \in L^1_{\mathrm{loc}}(\mathbb{G})$. A *regular direction* of f is a vector $X \in \mathfrak{g}$ such that Xf is representable by a Radon measure. An *invariant direction* of f is a regular direction $X \in \mathfrak{g}$ of f such that $Xf = 0$.

We will denote by $\mathrm{Reg}(f)$ the linear subspace of $\mathfrak{g}$ made of the regular directions of f and by $\mathrm{Inv}(f)$ the linear subspace of $\mathrm{Reg}(f)$ made of the invariant directions of f. Moreover, we will denote by $\mathrm{Inv}_0(f)$ the subset of $\mathrm{Inv}(f)$ made of the *homogeneous directions* of f, i.e.

$$\mathrm{Inv}_0(f) \coloneqq \mathrm{Inv}(f) \cap \bigcup_{i=1}^s V_i.$$

Notice that by Proposition 3.29 we have

$$f \circ R_{\exp(tX)} = f \qquad \text{for every } t \in \mathbb{R},\ X \in \operatorname{Inv}(f). \tag{4.1}$$

If $f = \chi_E$ for some $E \subseteq \mathbb{G}$, we set

$$\operatorname{Reg}(E) \coloneqq \operatorname{Reg}(\chi_E), \qquad \operatorname{Inv}(E) \coloneqq \operatorname{Inv}(\chi_E), \qquad \operatorname{Inv}_0(E) \coloneqq \operatorname{Inv}_0(\chi_E).$$

Let us prove the basic properties of regular and invariant directions:

Proposition 4.2. *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{G})$*. Then* $\operatorname{Reg}(f)$*,* $\operatorname{Inv}(f)$ *and* $\operatorname{Inv}_0(f)$ *are invariant under left translations and* $\operatorname{Inv}_0(f)$ *is invariant under intrinsic dilations. Moreover:*

(i) $\operatorname{Inv}(f)$ *is a Lie subalgebra of* $\mathfrak{g}$ *and* $[\operatorname{Inv}_0(f), \operatorname{Inv}_0(f)] \subseteq \operatorname{Inv}_0(f)$*;*

(ii) if $X \in \operatorname{Inv}(f)$ *and* $k = \exp(X)$*, then*

$$\operatorname{Ad}_k(Y) f = (R_{k^{-1}})_\sharp (Y f) \qquad \textit{for every } Y \in \operatorname{Reg}(f), \tag{4.2}$$

namely Ad_k *maps* $\operatorname{Reg}(f)$ *into* $\operatorname{Reg}(f)$ *and* $\operatorname{Inv}(f)$ *into* $\operatorname{Inv}(f)$*.*

Proof. Clearly $\operatorname{Reg}(f)$, $\operatorname{Inv}(f)$ and $\operatorname{Inv}_0(f)$ are invariant under left translations because their elements are left-invariant. Furthermore, the invariance under intrinsic dilations of $\operatorname{Inv}_0(f)$ easily follows from Proposition 2.18.
(i). Observe that for any $X, Y \in \operatorname{Inv}(f)$ and $u \in C_c^\infty(\mathbb{G})$ we have

$$\begin{aligned}
\langle [X,Y] f, u\rangle &= -\int_{\mathbb{G}} f [X,Y] u \, d\mathrm{vol}_{\mathbb{G}} \\
&= -\int_{\mathbb{G}} f X(Yu)\, d\mathrm{vol}_{\mathbb{G}} + \int_{\mathbb{G}} f Y(Xu)\, d\mathrm{vol}_{\mathbb{G}} \\
&= \langle Xf, Yu\rangle - \langle Yf, Xu\rangle = 0,
\end{aligned}$$

therefore $[X,Y] \in \operatorname{Inv}(f)$ and $\operatorname{Inv}(f)$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, this fact and the property $[V_i, V_j] \subseteq V_{i+j}$ for every $i, j = 1, \ldots, s$, $i \neq j$, imply $[\operatorname{Inv}_0(f), \operatorname{Inv}_0(f)] \subseteq \operatorname{Inv}_0(f)$.
(ii). Let $Y \in \operatorname{Reg}(f)$ and $Z = \operatorname{Ad}_k(Y)$. For any $u \in C_c^\infty(\mathbb{G})$ we have by (2.4) and (2.1)

$$\begin{aligned}
Zu(x) &= Y(u \circ C_k)(C_{k^{-1}}(x)) \\
&= Y(u \circ R_{k^{-1}})((L_k \circ C_{k^{-1}})(x)) \\
&= Y(u \circ R_{k^{-1}})(R_k(x))
\end{aligned}$$

for every $x \in \mathbb{G}$, thus $Zu \circ R_{k^{-1}} = Y(u \circ R_{k^{-1}})$. Therefore by a change of variables, (4.1) and the regularity of Y we obtain

$$\begin{aligned}
\langle Zf, u\rangle &= -\int_{\mathbb{G}} fZu \, d\mathrm{vol}_{\mathbb{G}} \\
&= -\int_{\mathbb{G}} (f \circ R_{k^{-1}}) Zu \circ R_{k^{-1}} \, d\mathrm{vol}_{\mathbb{G}} \\
&= -\int_{\mathbb{G}} fY(u \circ R_{k^{-1}}) \, d\mathrm{vol}_{\mathbb{G}} \\
&= \int_{\mathbb{G}} u \circ R_{k^{-1}} \, d(Yf) \\
&= \int_{\mathbb{G}} u \, d((R_{k^{-1}})_\sharp(Yf)) = \langle (R_{k^{-1}})_\sharp(Yf), u\rangle. \qquad \square
\end{aligned}$$

Remark 4.3. Let $X \in \mathrm{Reg}(f)$ and assume that $Xf \geq 0$; for any $t \in \mathbb{R}$ and $Y \in \mathrm{Inv}(f)$ we know from (2.6)

$$\mathrm{Ad}_{\exp(tY)}(X) = Xf + \sum_{i=1}^{s-1} \frac{t^i}{i!} \mathrm{ad}_Y^i(X)f,$$

hence by (4.2) we get

$$Xf + \sum_{i=1}^{s-1} \frac{t^i}{i!} \mathrm{ad}_Y^i(X)f \geq 0.$$

Since $t \in \mathbb{R}$ is arbitrary, this implies

$$\mathrm{ad}_Y^{s-1}(X)f \geq 0$$

for every $Y \in \mathrm{Inv}(f)$. In particular, if s is even, by applying the same inequality with $-Y$ in place of Y we obtain $\mathrm{ad}_Y^{s-1}(X)f = 0$, namely

$$\mathrm{ad}_Y^{s-1}(X) \in \mathrm{Inv}(f). \tag{4.3}$$

Let us also see how regular directions behave with respect to dilations.

Proposition 4.4. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{G})$ and $X \in \mathrm{Reg}(f)$. Then for all $r > 0$ the identity*

$$\widetilde{\delta}_{\frac{1}{r}} X(f \circ \delta_r) = \frac{1}{r^Q} \left(\delta_{\frac{1}{r}}\right)_\sharp (Xf) \tag{4.4}$$

holds.

Proof. Set $X_r := \widetilde{\delta}_{1/r} X$; from (2.8) we have for any $u \in C_c^\infty(\mathbb{G})$

$$X_r(u \circ \delta_r) = Xu \circ \delta_r,$$

therefore by (3.5) we obtain

$$\begin{aligned}
\langle X_r(f\circ\delta_r),u\rangle &= -\int_{\mathbb{G}}(f\circ\delta_r)X_r u\,d\mathrm{vol}_{\mathbb{G}}\\
&= -\frac{1}{r^Q}\int_{\mathbb{G}} f(X_r u)\circ\delta_{\frac{1}{r}}\,d\mathrm{vol}_{\mathbb{G}}\\
&= -\frac{1}{r^Q}\int_{\mathbb{G}} fX\Big(u\circ\delta_{\frac{1}{r}}\Big)\,d\mathrm{vol}_{\mathbb{G}}\\
&= -\frac{1}{r^Q}\int_{\mathbb{G}} u\circ\delta_{\frac{1}{r}}\,d(Xf)\\
&= -\frac{1}{r^Q}\int_{\mathbb{G}} u\,d\Big(\Big(\delta_{\frac{1}{r}}\Big)_\sharp(Xf)\Big) = \langle\frac{1}{r^Q}\Big(\delta_{\frac{1}{r}}\Big)_\sharp(Xf),u\rangle. \qquad \square
\end{aligned}$$

We can now naturally define half-spaces by requiring invariance along a codimension 1 space of directions and monotonicity along the remaining direction; if this direction is horizontal, we call these sets vertical half-spaces. Precisely, we have the following definition.

Definition 4.5. A Borel set $H\subseteq\mathbb{G}$ is a *vertical half-space* if $\mathrm{Inv}_0(H)\supseteq\bigcup_{i=2}^s V_i$, $V_1\cap\mathrm{Inv}_0(H)$ is a codimension 1 subspace of V_1 and there exists $X\in V_1$ such that $X\chi_H\geq 0$ and $X\chi_H\neq 0$.

We can equivalently say that H is a vertical half-space if $\mathrm{span}(\mathrm{Inv}_0(H))$ is a codimension 1 subspace of $\mathfrak{g}$, $V_1\cap\mathrm{span}(\mathrm{Inv}_0(H))$ is a codimension 1 subspace of V_1 and $X\chi_H\geq 0$, $X\chi_H\neq 0$, for some $X\in V_1$: indeed, since

$$\mathrm{span}(\mathrm{Inv}_0(H))=\bigoplus_{i=1}^s(V_i\cap\mathrm{Inv}(H)),$$

requiring that $V_1\cap\mathrm{span}(\mathrm{Inv}_0(H))$ has codimension 1 and there is a direction $X\in V_1$ which is not invariant implies that $V_i\cap\mathrm{Inv}_0(H)=V_i$ for every $i=2,\dots,s$.

Remark 4.6. In the previous definition, X can be chosen to be orthogonal to $V_1\cap\mathrm{Inv}_0(H)$. Indeed, if X were not, it would be a linear combination of invariant directions and an orthogonal one $X^\perp$ and it would be enough to replace X by $X^\perp$.

The next result provides a characterization of vertical half-spaces and says that they are exactly the images of "half-spaces" in $\mathfrak{g}$ via the exponential map. Recall that in our notation $m=\dim V_1$.

Proposition 4.7. *A subset $H\subseteq\mathbb{G}$ is a vertical half-space if and only if there exist $c\in\mathbb{R}$ and $\nu\in\mathbb{S}^{m-1}$ such that $H=H_{c,\nu}$ a.e., where*

$$H_{c,\nu}\coloneqq\exp\left(\left\{\sum_{i=1}^m a_iX_i+\sum_{i=2}^s v_i : v_i\in V_i,\ a\in\mathbb{R}^m,\ \sum_{i=1}^m a_i\nu_i\geq c\right\}\right).$$

Proof. Let us work in the graded coordinates associated with the adapted basis $X_1, \ldots, X_n$ of $\mathfrak{g}$. In these coordinates the sets $H_{c,\nu}$ are the usual half-spaces in $\mathbb{R}^n$ with normal vector in $\mathbb{R}^m \times \{0\}^{n-m}$, so it is easy to realize that they are vertical half-spaces and we only need to prove that H has that form. Let us denote by $\nu \in \mathbb{S}^{m-1}$ the unique vector such that the vector $X \coloneqq \sum_{i=1}^m \nu_i X_i$ is orthogonal to all invariant directions of H in V_1 and $X\chi_H \geq 0$; let $\widetilde{H}$ also be the expression of H in graded coordinates. If we set $n_i \coloneqq \dim V_i$, $m_0 = 0$ and $m_i \coloneqq n_1 + \ldots + n_i$ for every $i = 1, \ldots, s$, we have to show

$$\widetilde{H} = \left\{ (a, v_2, \ldots, v_s) \in \mathbb{R}^n : v_i \in \mathbb{R}^{n_i},\, a \in \mathbb{R}^m,\, \sum_{i=1}^m a_i \nu_i \geq c \right\}$$

for some $c \in \mathbb{R}$. By Proposition 2.31 we know $X_j = \partial_{x_j}$ for every $m_{s-1}+1 \leq j \leq m_s = n$, therefore by Proposition 1.18 and the hypothesis that V_s is contained in $\mathrm{Inv}_0(H)$ we have that $\chi_{\widetilde{H}}$ does not depend on $x_{m_{s-1}+1}, \ldots, x_n$. Now, if $m_{s-2}+1 \leq j \leq m_{s-1}$, again by Proposition 2.31 the vector fields $X_j - \partial_{x_j}$ are given by the sum of polynomials multiplied by ∂_{x_i} with $m_{s-1}+1 \leq i \leq m_s$. As a consequence, by the previous observation and the hypothesis that V_{s-1} is contained in $\mathrm{Inv}_0(H)$ we get $\partial_{x_j}\chi_{\widetilde{H}} = 0$ and again by Proposition 1.18 $\chi_{\widetilde{H}}$ does not depend on $x_{m_{s-2}+1}, \ldots, x_{m_{s-1}}$ either. By iterating this argument we obtain that $\chi_{\widetilde{H}}$ depends only on $x_1, \ldots, x_m$. Furthermore, by our choice of ν we have

$$\begin{cases} \displaystyle\sum_{i=1}^m \xi_i \partial_{x_i} \chi_{\widetilde{H}} = 0 & \text{if } \xi \perp \nu, \\ \displaystyle\sum_{i=1}^m \xi_i \partial_{x_i} \chi_{\widetilde{H}} \geq 0 & \text{if } \xi = \nu. \end{cases}$$

If instead of $\chi_{\widetilde{H}}$ there were a smooth function $u \in C^\infty(\mathbb{G})$ under the previous conditions, it would be easy to check that u depends only on $\langle x, \nu \rangle = \sum_{i=1}^m \nu_i x_i$ and is increasing with respect to it. However, after mollifying $\chi_{\widetilde{H}}$ and applying Theorem 3.38, we find that $\chi_{\widetilde{H}}$ satisfies the same property, namely there exists an increasing function $\gamma : \mathbb{R} \longrightarrow [0,1]$ such that $\chi_{\widetilde{H}}(x) = \gamma(\langle x, \nu \rangle)$ for a.e. $x \in \mathbb{G}$. Since $\chi_{\widetilde{H}} \in \{0,1\}$, this implies that γ is (equivalent to) a characteristic function χ_L for some $L \subseteq \mathbb{R}$. Actually, L must be (equivalent to) a half-line because γ is increasing and L cannot be $\emptyset$ or $\mathbb{R}$ by virtue of the hypothesis $X\chi_H \neq 0$, so there exists $c \in \mathbb{R}$ such that $L = [c, \infty[$. This is what we wanted to prove. □

Remark 4.8. Notice that a function $f \in L^1_{\mathrm{loc}}(\mathbb{G})$ is equivalent to a constant if and only if $\mathrm{Inv}(f) = \mathfrak{g}$. Indeed, if f is equivalent to a constant, then $Xf = 0$ for every $X \in \mathfrak{g}$ because X is divergence-free. Conversely, a computation in graded coordinates similar to that in the previous proof shows that f does not depend on any x_i, $1 \leq i \leq n$, so it must be equivalent to a constant.

Up to modifications on negligible sets, from now on we will identify vertical half-spaces with those of the form $H_{c,\nu}$.

4.2 Sets of locally finite perimeter

Regular directions allow to define the class of sets of locally finite perimeter.

Definition 4.9. A Borel set $E \subseteq \mathbb{G}$ is said to be *of locally finite perimeter* if $X\chi_E$ is a Radon measure for every $X \in V_1$.

If $X_1, \dots, X_m$ is the adapted basis of V_1, for any $f \in L^1_{\mathrm{loc}}(\mathbb{G})$ such that $X_1, \dots, X_m \in \mathrm{Reg}(f)$ we can define the *horizontal gradient* as the $\mathbb{R}^m$-valued Radon measure

$$Df \coloneqq (X_1 f, \dots, X_m f).$$

If $E \subseteq \mathbb{G}$ is a set of locally finite perimeter, we call the *perimeter of* E the measure $|D\chi_E|$; thanks to (4.4), it is homogeneous of degree $Q-1$ with respect to intrinsic dilations, that is

$$|D\chi_{\delta_\lambda(E)}| = \lambda^{Q-1}(\delta_\lambda)_\sharp |D\chi_E|.$$

Similarly to the Euclidean case, the horizontal gradient enjoys the following property:

Proposition 4.10. *If $Df = 0$, then f is (equivalent to) a constant.*

Proof. We have $\mathrm{Inv}(f) = \mathfrak{g}$ by Proposition 4.2 because $X_1, \dots, X_m$ generates the whole Lie algebra $\mathfrak{g}$, so the proof concludes by Remark 4.8. □

Now we just state an important result which will be used later; its proof can be found in [14] (Theorem 1.28).

Theorem 4.11. *Let $\{f_i\}_{i\in\mathbb{N}} \subseteq L^1_{\mathrm{loc}}(\mathbb{G})$ be such that*

$$\sup_{i\in\mathbb{N}} \int_K |f_i| \, d\mathrm{vol}_{\mathbb{G}} + |Df|(K) < \infty \qquad \text{for every } K \Subset \mathbb{G}.$$

Then $\{f_i\}_{i\in\mathbb{N}}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{G})$.

Sets of locally finite perimeter could be very bad, so they are accompained by a new concept of boundary which plays the same role as the topological boundary for smooth sets.

Definition 4.12. Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. The *reduced boundary* of E is the set $\partial^* E$ made of the points $x \in \mathrm{supp}\,|D\chi_E|$ such that

(i) the limit $\nu_E(x) = (\nu_{E,1}(x), \dots, \nu_{E,m}(x)) \coloneqq \lim_{r\searrow 0} \dfrac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}$ exists,

(ii) $|\nu_E(x)| = 1$.

The vector ν_E is called *horizontal normal*.

By Lemma 3.14 it is easily checked that the reduced boundary is a Borel set. Furthermore, it satisfies the following invariance property:

Proposition 4.13. *If $E \subseteq \mathbb{G}$ is a set of locally finite perimeter, then $\partial^* E$ is invariant under left translations, i.e. $y \in \partial^* E$ if and only if $L_x(y) \in \partial^*(L_x(E))$ and $\nu_E(y) = \nu_{L_x(E)}(L_x(y))$.*

Proof. For any $1 \leq i \leq m$ and $u \in C_c^\infty(\mathbb{G})$ we have by (2.1)

$$\begin{aligned} \langle X_i \chi_{L_x(E)}, u \rangle &= - \int_{L_x(E)} X_i u \, d\mathrm{vol}_{\mathbb{G}} = - \int_E X_i(u \circ L_x) \, d\mathrm{vol}_{\mathbb{G}} \\ &= \int_{\mathbb{G}} u \, d((L_x)_\sharp (X_i \chi_E)) = \langle (L_x)_\sharp (X_i \chi_E), u \rangle, \end{aligned}$$

thus

$$D\chi_{L_x(E)} = (L_x)_\sharp D\chi_E.$$

From this equality the thesis easily follows. □

We now present two results that are satisfied by sets of locally finite perimeter in Carnot groups, analogous to well known results in the Euclidean context. The proofs can be found respectively in [14] (Theorem 1.18) and [13] (Lemma 2.31).

Theorem 4.14 (Relative isoperimetric inequality)**.** *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then there exists a constant $\bar{c} > 0$ such that for all $x \in \mathbb{G}$ and $r > 0$*

$$\min\{\mathcal{L}^n(E \cap B_r(x)), \mathcal{L}^n(E^c \cap B_r(x))\}^{\frac{Q-1}{Q}} \leq \bar{c}\, |D\chi_E|(B_r(x)). \tag{4.5}$$

Lemma 4.15 (Density estimates)**.** *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and $x \in \partial^* E$. Then there exist three constants $c_i = c_i(\mathbb{G}) > 0$, $i = 1, 2$, and $r_0 = r_0(x) > 0$ such that*

$$|D\chi_E|(B_r(x)) \leq c_1 r^{Q-1}, \qquad 0 < r < r_0, \tag{4.6}$$

and

$$\mathcal{L}^n(E \cap B_r(x)) \geq c_2 r^Q, \qquad 0 < r < r_0. \tag{4.7}$$

The next theorem is a consequence of these results and states a fundamental property of the perimeter.

Theorem 4.16. *If $E \subseteq \mathbb{G}$ is a set of locally finite perimeter, then there exist three constants $\bar{c}_i = \bar{c}_i(\mathbb{G}) > 0$, $i = 1, 2$, and $\bar{r}_0 = \bar{r}_0(x) > 0$ such that for any $x \in \partial^* E$*

$$\bar{c}_1 r^{Q-1} \leq |D\chi_E|(B_r(x)) \leq \bar{c}_2 r^{Q-1}, \qquad 0 < r < \bar{r}_0. \tag{4.8}$$

Proof. The second inequality is exactly (4.6) with $\bar{c}_2 = c_1$ and $\bar{r}_0 = r_0$. For the first one, let us observe that, since $D\chi_E = -D\chi_{E^c}$, we have $|D\chi_E| = |D\chi_{E^c}|$ and $\partial^* E = \partial^*(E^c)$, so we can apply Lemma 4.15 with E^c in place of E and we get

$$\mathcal{L}^n(E^c \cap B_r(x)) \geq c_2 r^Q, \qquad 0 < r < r_0 = \bar{r}_0.$$

Therefore we deduce from (4.5) and (4.7)

$$\begin{aligned} |D\chi_E|(B_r(x)) &\geq \frac{1}{\bar{c}} \min\{\mathcal{L}^n(E \cap B_r(x)), \mathcal{L}^n(E^c \cap B_r(x))\}^{\frac{Q-1}{Q}} \\ &\geq \frac{1}{\bar{c}} (c_2 r^Q)^{\frac{Q-1}{Q}} = \frac{c_2^{\frac{Q-1}{Q}}}{\bar{c}} r^{Q-1} \end{aligned}$$

for every $0 < r < \bar{r}_0$, which is the first inequality with $\bar{c}_1 = c_2^{\frac{Q-1}{Q}} / \bar{c}$. □

Corollary 4.17. *If $E \subseteq \mathbb{G}$ is a set of locally finite perimeter, then $|D\chi_E|$ is asymptotically doubling and concentrated on $\partial^* E$.*

Proof. We have that $|D\chi_E|$ is asymptotically doubling directly from (4.8). Moreover, this implies by Theorem 3.18 that $\nu_E(x)$ exists and belongs to $\mathbb{R}^m$ for $|D\chi_E|$-a.e. $x \in \mathbb{G}$ and is the function such that $D\chi_E = \nu_E \, |D\chi_E|$. In particular, $|\nu_E| = 1$ $|D\chi_E|$-a.e., thus

$$|D\chi_E|(\mathbb{G} \setminus \partial^* E) = 0. \qquad \square$$

Combining Theorem 3.24, (4.8) and (3.4), we also have that the perimeter $|D\chi_E|$ can be bounded from above and below by the spherical Hausdorff measure $\mathcal{S}^{Q-1}$, namely

$$\frac{\bar{c}_1}{\omega_{Q-1}} \mathcal{S}^{Q-1}(B \cap \partial^* E) \leq |D\chi_E|(B) \leq 2^{Q-1} \frac{\bar{c}_2}{\omega_{Q-1}} \mathcal{S}^{Q-1}(B \cap \partial^* E) \tag{4.9}$$

for every Borel set $B \subseteq \mathbb{G}$, and similar inequalities hold with $\mathcal{H}^{Q-1}$.

4.3 Tangent sets

Studying the rectifiability of a set of locally finite perimeter means to understand the structure of tangent sets, so let us define them. Recall that in a general measure space $(E, \mathcal{B}(E), \mu)$ a sequence of Borel sets $\{B_i\}_{i \in \mathbb{N}} \subseteq \mathcal{B}(E)$ locally converges in measure to a Borel set $B \in \mathcal{B}(E)$ if

$$\mu(K \cap (B_i \,\Delta\, B)) \xrightarrow[i \to \infty]{} 0 \qquad \text{for every } K \subseteq E \text{ compact.}$$

This is equivalent to say that $\{\chi_{B_i}\}_{i \in \mathbb{N}}$ converges to χ_B in $L^1_{\text{loc}}(E)$.

Definition 4.18. Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and $x \in \partial^* E$. We denote by $\mathrm{Tan}(E,x)$ all limit points, in the topology of local convergence in measure, of the sets $\{\delta_{1/r}(x^{-1}E)\}_{r>0}$ as $r \searrow 0$. If $F \in \mathrm{Tan}(E,x)$ we say that F is *tangent* to E at x.

We denote by

$$\mathrm{Tan}(E) := \bigcup_{x \in \partial^* E} \mathrm{Tan}(E,x)$$

the set of all tangent sets to E. We also define the *iterated tangent sets* to E in the following way: for any $x \in \partial^* E$ we set $\mathrm{Tan}^1(E,x) := \mathrm{Tan}(E,x)$ and

$$\mathrm{Tan}^{k+1}(E,x) := \bigcup \{\mathrm{Tan}(F) : F \in \mathrm{Tan}^k(E,x)\}.$$

We want to prove that if we iterate sufficiently many times the tangent operator we get a vertical half-space. To do this, we start by stating a result obtained in [13] (Claim 3.3, Claim 3.4 and beginning of Claim 3.5).

Proposition 4.19. *If $E \subseteq \mathbb{G}$ is a set of locally finite perimeter, then for any $\bar{x} \in \partial^* E$ the following properties hold:*

(i) $\mathrm{Tan}(E,\bar{x}) \neq \emptyset$*;*

(ii) each $F \in \mathrm{Tan}(E,\bar{x})$ is a set of locally finite perimeter and

$$\nu_F(x) = \nu_E(\bar{x}) \qquad \text{for } |D\chi_F|\text{-a.e. } x \in \mathbb{G};$$

(iii) $V_1 \cap \mathrm{Inv}_0(F)$ coincides with the codimension 1 subspace of V_1

$$\left\{ \sum_{i=1}^m a_i X_i : \sum_{i=1}^m a_i \nu_{E,i}(\bar{x}) = 0 \right\}$$

and, setting $X := \sum_{i=1}^m \nu_{E,i}(\bar{x}) X_i \in V_1$, $X \in \mathrm{Reg}(F) \setminus \mathrm{Inv}(F)$ and $X\chi_F \geq 0$.

Furthermore, we have $e \in \operatorname{supp} |D\chi_F|$: indeed, by the way of contradiction, if $e \notin \operatorname{supp} |D\chi_F|$ there would be some $\rho > 0$ such that $|D\chi_F|(B_\rho(e)) = 0$; thus, setting $E_{\bar{x},r} := \delta_{1/r}(\bar{x}^{-1}E)$, we would have by (4.5) (possibly taking a smaller $\rho > 0$)

$$\mathcal{L}^n(F \cap \overline{B_\rho(e)}) = 0 \qquad \text{or} \qquad \mathcal{L}^n(F^c \cap \overline{B_\rho(e)}) = 0.$$

If $\mathcal{L}^n(F \cap \overline{B_\rho(e)}) = 0$, by the fact that $\{\chi_{E_{\bar{x},r_i}}\}_{i\in\mathbb{N}}$ converges to χ_F in $L^1(\overline{B_\rho(e)})$ along a suitable sequence $r_i \searrow 0$ and (4.7)

$$\begin{aligned}
0 = \mathcal{L}^n(F \cap \overline{B_\rho(e)}) &= \lim_{i\to\infty} \mathcal{L}^n(E_{\bar{x},r_i} \cap \overline{B_\rho(e)}) \\
&= \lim_{i\to\infty} \mathcal{L}^n\left(\delta_{\frac{1}{r_i}}(\bar{x}^{-1}E \cap \overline{B_{\rho r_i}(e)})\right) \\
&= \lim_{i\to\infty} \frac{\mathcal{L}^n(\bar{x}^{-1}E \cap \overline{B_{\rho r_i}(e)})}{r_i^Q} \geq c_2 > 0,
\end{aligned}$$

which is a contradiction. If instead $\mathcal{L}^n(F^c \cap \overline{B_\rho(e)}) = 0$, a similar argument leads to another contradiction.
We also need two crucial lemmas. The first one shows that if $X \in \mathrm{Reg}(E)$ belongs to $\bigoplus_{i=2}^s V_i$, then the tangent sets to E at $|D\chi_E|$-a.e. $x \in \partial^* E$ are invariant under the "higher degree component" of X. The underlying reason for this fact is that the intrinsic dilations behave quite differently in the X-direction and in the horizontal directions.

Lemma 4.20. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter, $X \in \mathrm{Reg}(E)$ and assume that $X = \sum_{i=2}^{\ell} v_i$ with $v_i \in V_i$ and $\ell \leq s$. Then, for $|D\chi_E|$-a.e. $x \in \partial^* E$, $v_\ell \in \mathrm{Inv}_0(F)$ for all $F \in \mathrm{Tan}(E, x)$.*

Proof. Let $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G})$ be the Radon measure such that $\mu = X\chi_E$. If $Q = 1$, then $s = 1$ and there is nothing to prove, so we can assume $Q \geq 2$. From Corollary 3.25 we know that the set N of points $x \in \mathbb{G}$ such that

$$\limsup_{r \searrow 0} \frac{|\mu|(B_r(x))}{r^{Q-2}} > 0$$

is σ-finite with respect to $\mathcal{S}^{Q-2}$, in particular $\mathcal{S}^{Q-1}$-negligible and by (4.9) $|D\chi_E|$-negligible. We will prove that the statement holds for every $x \in \partial^* E \setminus N$ and actually, by virtue of Proposition 4.13, we can assume $x = e$ up to a left translation. Given any $u \in C_c^\infty(\mathbb{G})$, let $R > 0$ be such that $\mathrm{supp}\, u \subseteq B_R(e)$; (4.4) with $f = \chi_E$ gives

$$\int_{\mathbb{G}} \chi_{\delta_{\frac{1}{r}}(E)} X_r u \, d\mathrm{vol}_{\mathbb{G}} = \frac{1}{r^{Q-\ell}} \int_{\mathbb{G}} u \circ \delta_{\frac{1}{r}} \, d\mu,$$

where $X_r := r^\ell \widetilde{\delta}_{1/r} X$, so that X_r tends to v_ℓ as $r \searrow 0$. We can bound the right-hand side with

$$\frac{\sup |u|}{r^{Q-\ell}} |\mu|(B_{rR}(e)) = O(r^{\ell-Q}) o(r^{Q-2}) = o(1),$$

where the last equality holds because $\ell \geq 2$, so passing to the limit as $r \searrow 0$ along a suitable sequence and moving the limit inside the integral sign on the left-hand side (it can be done by the dominated convergence theorem since $X_r u \in C_c^\infty(\mathbb{G})$) we get $v_\ell \chi_F = 0$ for every $F \in \mathrm{Tan}(E, e)$. $\square$

The invariance of Inv_0 under left translations and scaling shows that $\mathrm{Inv}_0(F)$ contains $\mathrm{Inv}_0(E)$ for all $F \in \mathrm{Tan}(E)$. Moreover, we know that the codimension of $\mathrm{span}(\mathrm{Inv}_0(E))$ in $\mathfrak{g}$ is at least 1 (because it has codimension 1 in V_1 by Proposition 4.19) and is exactly 1 if E is a vertical half-space by Proposition 4.7. In this setting, the second lemma says that if this codimension is at least 2 then a double tangent strictly increases the set $\mathrm{Inv}_0(E)$ at $|D\chi_E|$-a.e. point.

Lemma 4.21. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and assume that*

$$\dim(\mathrm{span}(\mathrm{Inv}_0(E))) \leq n-2.$$

Then, for any $x \in \partial^* E$, $\mathrm{Inv}_0(L) \supsetneq \mathrm{Inv}_0(E)$ for some $L \in \mathrm{Tan}^2(E,x)$.

Proof. Let $\mathfrak{g}' \coloneqq \mathrm{span}(\mathrm{Inv}_0(E))$ and $X \coloneqq \sum_{i=1}^m \nu_{E,i}(x) X_i$. We first show the existence of $Z \in \mathfrak{g} \setminus (\mathfrak{g}' + V_1)$ such that $Z \in \mathrm{Reg}(F)$ for every $F \in \mathrm{Tan}(E,x)$. Observe that $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$ by Proposition 4.2 and $X \notin \mathfrak{g}'$ by Proposition 4.19, so we can apply Proposition 2.14: there exists $Y \in \mathfrak{g}'$ such that

$$Z \coloneqq \mathrm{Ad}_{\exp(Y)}(X) \notin \mathfrak{g}' \oplus \{\mathbb{R}X\} = \mathfrak{g}' + V_1.$$

In particular, since $\mathrm{Inv}_0(E) \subseteq \mathrm{Inv}_0(F)$ for every $F \in \mathrm{Tan}(E,x)$, $Y \in \mathrm{Inv}(F)$ and so $Z \in \mathrm{Reg}(F)$ for every $F \in \mathrm{Tan}(E,x)$ by Proposition 4.2.
Now, let us fix $F \in \mathrm{Tan}(E,x)$ and consider $\mu \in \mathcal{M}_{\mathrm{loc}}(\mathbb{G})$ the Radon measure such that $\mu = Z\chi_F$. Possibly removing from Z its horizontal component (which can be done because F has locally finite perimeter by Proposition 4.19), we can write $Z = v_{i_1} + \ldots + v_{i_\ell}$ with $i_j \geq 2$ and $v_{i_j} \in V_{i_j} \setminus \{0\}$ for every $j = 1, \ldots, \ell$. Then $v_{i_k} \notin \mathrm{Inv}_0(E)$ for at least one $k \in \{1, \ldots, \ell\}$, so let us choose the largest one with this property. Hence, setting $Z' = v_{i_1} + \ldots + v_{i_k}$, since $v_{i_j} \in \mathrm{Inv}_0(E) \subseteq \mathrm{Inv}_0(F)$ for all $k < j \leq \ell$, we still have $Z'\chi_F = \mu$. Therefore by Lemma 4.20 we can find $L \in \mathrm{Tan}(F)$ such that $v_{i_k} \in \mathrm{Inv}_0(L)$; this concludes the proof because $v_{i_k} \notin \mathrm{Inv}_0(E)$. □

We are now ready to prove the previously announced result. Recall again that $m = \dim V_1$.

Theorem 4.22. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then, for any $x \in \partial^* E$, there exists a natural number $k \leq 2(n-m)+1$ such that $H_{0,\nu_E(x)} \in \mathrm{Tan}^k(E,x)$.*

Proof. Let us define

$$i_k \coloneqq \max\{\dim(\mathrm{span}(\mathrm{Inv}_0(F))) : F \in \mathrm{Tan}^k(E,x)\}.$$

Thanks to Proposition 4.19, we know that sets in $\mathrm{Tan}(E,x)$ are invariant in at least $m-1$ directions, so that $i_1 \geq m-1$. Furthermore, we proved in Lemma 4.21 that $i_{k+2} > i_k$ as long as there exists $F \in \mathrm{Tan}^k(E,x)$ with $\dim(\mathrm{span}(\mathrm{Inv}_0(F))) \leq n-2$, hence, up to apply the tangent operator at most $2(n-m)$ times, we find for some natural number $k \leq 2(n-m)+1$ an iterated tangent set $F \in \mathrm{Tan}^k(E,x)$ with $\dim(\mathrm{span}(\mathrm{Inv}_0(F))) \geq n-1$. Moreover, from Proposition 4.19 we know that $V_1 \cap \mathrm{Inv}_0(F)$ has codimension 1 in V_1 and $\sum_{i=1}^m \nu_{E,i}(x) X_i \chi_F \geq 0$, so F is a half-space. Therefore, by Proposition 4.7 and since $e \in \mathrm{supp}\,|D\chi_F|$ we have $F = H_{0,\nu_E(x)}$. □

Chapter 5

The problem of rectifiability

We have all the tools to present the proof of the main result currently known about rectifiability of sets of finite perimeter in Carnot groups, which is the following theorem.

Theorem 5.1. *Let $\mathbb{G}$ be a Carnot group and $E \subseteq \mathbb{G}$ a set of locally finite perimeter. Then for $|D\chi_E|$-a.e. $x \in \partial^* E$ there exists a vertical half-space H such that $H \in \mathrm{Tan}(E, x)$.*

Moreover, we will provide a counterexample showing that the expression "$|D\chi_E|$-a.e." cannot be avoided and that in general this theorem cannot hold in any point of the reduced boundary $\partial^* E$. All these results can also be found in [3]. We will continue to use the notation introduced at the beginning of Chapter 4.

5.1 Rectifiability in step 2 Carnot groups

In [13] the following theorem is proved:

Theorem 5.2. *Let $\mathbb{G}$ be a step 2 Carnot group. If $E \subseteq \mathbb{G}$ is a set of locally finite perimeter, then for any $x \in \partial^* E$ we have*

$$\mathrm{Tan}(E, x) = \{H_{0,\nu_E(x)}\}.$$

In other words, in step 2 Carnot groups the analogue of De Giorgi's rectifiability theorem holds and at any point of the reduced boundary the tangent set exists, is unique and is a vertical half-space. Observe that Theorem 5.1 is very different from it because it does not hold at every point of the reduced boundary and it does not say anything about uniqueness of the tangent half-space. In this section we explain the idea regarding why there is this big difference.

The proof of Theorem 5.2 proceeds in this way: the starting point is Proposition 4.19, whose proof shows that the blow-up along any scaling sequence

$r_i \searrow 0$ leads to a tangent set F with constant horizontal normal. If the step of $\mathbb{G}$ is 2, this is equivalent to say that F is a vertical half-space. This fact is proved in [13] (Lemma 3.6), but here we provide a different proof, based on Remark 4.3.

Proposition 5.3. *Let $\mathbb{G}$ be a step 2 Carnot group. If $E \subseteq \mathbb{G}$ is a set of locally finite perimeter such that ν_E is (equivalent to) a constant, then E is a vertical half-space.*

Proof. Let $\xi \in \mathbb{R}^m$ be the constant value of ν_E and set $X := \sum_{i=1}^m \xi_i X_i$. By Proposition 4.19 $X\chi_E \geq 0$ and $\mathrm{Inv}(E)$ contains all vectors $Y = \sum_{i=1}^m \eta_i X_i$ with η perpendicular to ξ. Then, since $s = 2$, from (4.3) we get $[X, Y]\chi_E = 0$ for every $Y \in \mathrm{Inv}(E) \cap V_1$; but these commutators, together with the commutators $\{[Y_1, Y_2] : Y_i \in \mathrm{Inv}(E) \cap V_1\}$, span the whole V_2, hence the proof is achieved. $\square$

Here is the discrepancy with the general case because this result does not hold in a Carnot group of step larger than 2 (see Section 5.3 and Section 5.4 for some counterexamples) and this is the difficulty to bypass in order to prove Theorem 5.1.

Remark 5.4. The sign condition $Xf \geq 0$ of Remark 4.3 is essential for the validity of Proposition 5.3. For example, consider the first Heisenberg group $\mathbb{H}^1$ with the representation in graded coordinates $(x, y, t) \in \mathbb{R}^3$ presented in Section 2.4, so that the vectors of the basis of the horizontal layer are $X = \partial_x + 2y\partial_t$ and $Y = \partial_y - 2x\partial_t$; take a smooth function $g \in C^\infty(\mathbb{H}^1)$ and define the function

$$f(x, y, t) := g(t + 2xy).$$

We have $Xf = 4yg'(t + 2xy)$ and $Yf = 0$, therefore for any $c \in \mathbb{R}$ the sets $E_c := \{f \leq c\}$ are Y-invariant, but they are not half-spaces if g is not constant and, in fact, $X\chi_{E_c}$ changes sign because Xf does so and by (1.13). Furthermore, the same example shows that there is no local version of Proposition 5.3 because the sets E_c locally may satisfy $X\chi_{E_c} \geq 0$ or $X\chi_{E_c} \leq 0$ (depending on the sign of g' and y), but, choosing a suitable g, they are not locally half-spaces.
In our proof of Proposition 5.3 the non-locality appears when we use formula (4.3): indeed, its proof depends on the sign condition $\mathrm{Ad}_{\exp(tY)}(X)\chi_E \geq 0$ with t arbitrarily large and by (4.2) this is the right translation of $X\chi_E$ by $\exp(tY)$. Instead, the proof given in [13] relies on the possibility of joining two different points in $\mathbb{H}^1$ by following integral lines of Y in both directions and integral lines of X in just one direction. This is proved by showing that both initial and final points can be joined with the identity element. In particular, this fact reveals that these paths cannot be confined to a bounded region, even if the points are confined within a small region.

5.2 Existence of a tangent half-space

In this section we prove Theorem 5.1. Thanks to Theorem 4.22, we only need to show that iterated tangent sets are tangent sets. Precisely, the following result holds.

Theorem 5.5. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter. Then, for $|D\chi_E|$-a.e. $x \in \partial^* E$ we have*

$$\bigcup_{k=2}^{\infty} \operatorname{Tan}^k(E,x) \subseteq \operatorname{Tan}(E,x).$$

In turn, this theorem is a consequence of an analogous one involving tangent measures, proved in [25] (Theorem 2.12) in the Euclidean case. We will adapt it to our setting after defining what tangent measures are.
From now on, for any $x \in \mathbb{G}$ and $r > 0$ we will set

$$I_{x,r} := \delta_{\frac{1}{r}} \circ L_{x^{-1}}.$$

Recall that, for a general LCS metric space E and $q \geq 0$, a measure $\mu \in \mathcal{M}^m_{\mathrm{loc}}(E)$ is asymptotically q-regular if

$$0 < \liminf_{r \searrow 0} \frac{|\mu|(B_r(x))}{r^q} \leq \limsup_{r \searrow 0} \frac{|\mu|(B_r(x))}{r^q} < \infty \qquad \text{for } |\mu|\text{-a.e. } x \in E.$$

In particular, any asymptotically q-regular measure is asymptotically doubling. Moreover, if $E \subseteq \mathbb{G}$ is a set of locally finite perimeter, then the perimeter measure $|D\chi_E|$ is asymptotically $(Q-1)$-regular thanks to (4.8).

Definition 5.6. Let $\mu \in \mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$ be asymptotically q-regular and $x \in \mathbb{G}$. We denote by $\operatorname{Tan}(\mu,x)$ the set of all local weak* limit points of the family of measures $\{r^{-q}(I_{x,r})_\sharp \mu\}_{r>0}$ as $r \searrow 0$. If $\nu \in \operatorname{Tan}(\mu,x)$ we say that ν is *tangent* to μ at x.

Theorem 5.7. *Let $\mu \in \mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$ be asymptotically q-regular. Then for $|\mu|$-a.e. $x \in \mathbb{G}$ we have*

$$\operatorname{Tan}(\nu,y) \subseteq \operatorname{Tan}(\mu,x) \qquad \textit{for every } \nu \in \operatorname{Tan}(\mu,x),\ y \in \operatorname{supp}|\nu|.$$

The connection between Theorem 5.5 and Theorem 5.7 rests on the following proposition:

Proposition 5.8. *Let $E \subseteq \mathbb{G}$ be a set of locally finite perimeter and $x \in \partial^* E$. Then $F \in \operatorname{Tan}(E,x)$ if and only if $D\chi_F \in \operatorname{Tan}(D\chi_E,x) \setminus \{0\}$.*

Proof. Since $F \in \operatorname{Tan}(E,x)$, there exists a sequence $\{r_i\}_{i\in\mathbb{N}}$ such that $r_i \searrow 0$ and

$$F = \lim_{i\to\infty} \delta_{\frac{1}{r_i}}(x^{-1}E)).$$

For any $1 \le j \le m$ and $u \in C_c^\infty(\mathbb{G})$ we have by (2.1), (2.8) and (3.5)

$$\begin{aligned}
\int_{\mathbb{G}} u \, d((I_{x,r_i})_\sharp (X_j \chi_E)) &= -\int_E X_j \Big(u \circ \delta_{\frac{1}{r_i}} \circ L_{x^{-1}} \Big) \, d\mathrm{vol}_{\mathbb{G}} \\
&= -\int_{L_{x^{-1}}(E)} X_j \Big(u \circ \delta_{\frac{1}{r_i}} \Big) \, d\mathrm{vol}_{\mathbb{G}} \\
&= -\frac{1}{r_i} \int_{L_{x^{-1}}(E)} X_j u \circ \delta_{\frac{1}{r_i}} \, d\mathrm{vol}_{\mathbb{G}} \\
&= -r_i^{Q-1} \int_{I_{x,r_i}(E)} X_j u \, d\mathrm{vol}_{\mathbb{G}},
\end{aligned}$$

thus by the dominated convergence theorem

$$\frac{1}{r_i^{Q-1}} \int_{\mathbb{G}} u \, d((I_{x,r_i})_\sharp (X_j \chi_E)) \xrightarrow[i \to \infty]{} -\int_F X_j u \, d\mathrm{vol}_{\mathbb{G}} = \int_{\mathbb{G}} u \, d(X_j \chi_F).$$

If now $u \in C_c(\mathbb{G})$, by the density of $C_c^\infty(\mathbb{G})$ in $C_c(\mathbb{G})$ (Theorem 3.38) for any $\varepsilon > 0$ there exists $u_\varepsilon \in C_c^\infty(\mathbb{G})$ such that $\|u - u_\varepsilon\|_\infty \le \varepsilon$, hence by setting $E_{x,r_i} \coloneqq I_{x,r_i}(E)$ and (4.6) we have for some $R > 0$ large enough

$$\begin{aligned}
\left| \int_{\mathbb{G}} u \, d(D\chi_{E_{x,r_i}}) - \int_{\mathbb{G}} u \, d(D\chi_F) \right| &\le \frac{1}{r_i^{Q-1}} \int_{B_{Rr_i}(x)} |u - u_\varepsilon| \, d|D\chi_E| \\
&\quad + \left| \int_{\mathbb{G}} u_\varepsilon \, d(D\chi_{E_{x,r_i}}) - \int_{\mathbb{G}} u_\varepsilon \, d(D\chi_F) \right| \\
&\quad + \int_{B_R(e)} |u - u_\varepsilon| \, d|D\chi_F| \\
&\le (c_1 + 1 + |D\chi_F|(B_R(e)))\varepsilon,
\end{aligned}$$

namely $\{r_i^{1-Q} (I_{x,r_i})_\sharp D\chi_E\}_{i \in \mathbb{N}}$ locally weak* converges to $D\chi_F$. Finally, $D\chi_F \neq 0$ because $e \in \operatorname{supp} |D\chi_F|$ (remember the observation after Proposition 4.19).

Conversely, up to a left translation, we can assume $x = e$ thanks to Proposition 4.13. Let $\{r_i\}_{i \in \mathbb{N}}$ be such that $r_i \searrow 0$ and $D\chi_F$ is the local weak* limit of $\{r_i^{1-Q} (I_{e,r_i})_\sharp D\chi_E\}_{i \in \mathbb{N}}$; moreover, set $E_i \coloneqq \delta_{1/r_i}(E)$, so that by the same computation as before we have

$$D\chi_{E_i} = \frac{1}{r_i^{Q-1}} (I_{e,r_i})_\sharp D\chi_E.$$

Thanks to (3.2) and Theorem 4.11, without loss of generality we can assume that $\{E_i\}_{i \in \mathbb{N}}$ locally converges in measure to some $F' \subseteq \mathbb{G}$, so that $F' \in \operatorname{Tan}(E, e)$ and by a density argument analogous to the previous one $\{D\chi_{E_i}\}_{i \in \mathbb{N}}$ locally weak* converges to $D\chi_{F'}$. It follows $D\chi_F = D\chi_{F'}$, therefore by Proposition 4.10 $\chi_F - \chi_{F'}$ must be (equivalent to) a constant; this can happen only when either $F = F'$ or $F = \mathbb{G} \setminus F'$, but the second possibility is ruled out because it would imply $D\chi_F = -D\chi_{F'}$ and so $D\chi_F = 0$. This proves $F = F' \in \mathrm{Tan}(E, e)$. □

Proof of Theorem 5.5. At any point $x \in \partial^* E$ where the property stated in Theorem 5.7 holds with $\mu = D\chi_E$ we can consider any $F \in \mathrm{Tan}(E, x)$ and $L \in \mathrm{Tan}(F, y)$ for some $y \in \partial^* F$; then, by Proposition 5.8 we know $D\chi_F \in \mathrm{Tan}(D\chi_E, x)$ and $D\chi_L \in \mathrm{Tan}(D\chi_F, y) \setminus \{0\}$. As a consequence, Theorem 5.7 gives $D\chi_L \in \mathrm{Tan}(D\chi_E, x) \setminus \{0\}$, hence Proposition 5.8 again yields $L \in \mathrm{Tan}(E, x)$. This proves $\mathrm{Tan}^2(E, x) \subseteq \mathrm{Tan}(E, x)$ and therefore also $\mathrm{Tan}^3(E, x) \subseteq \mathrm{Tan}^2(E, x)$ and so on. □

It only remains to prove Theorem 5.7. We will follow the proof given in [21] (Theorem 14.16) with some adaptations. We start with a lemma.

Lemma 5.9. *Let $A \subseteq \mathbb{G}$, μ be a vector measure on $\mathbb{G}$ and $a \in A$ be a density point for A relative to $|\mu|^*$, i.e.*

$$\lim_{r \searrow 0} \frac{|\mu|^*(B_r(a) \cap A)}{|\mu|(B_r(a))} = 1.$$

If for some $r_i \searrow 0$ and $\lambda_i \geq 0$ the measures $\lambda_i (I_{a,r_i})_\sharp \mu$ locally weak converge to ν, then*

$$\lim_{i \to \infty} \frac{d(a\delta_{r_i}(y), A)}{r_i} = 0 \qquad \text{for every } y \in \operatorname{supp} |\nu|.$$

Proof. Let $\tau := d(y, e)$ and let us argue by contradiction. If the statement were false, τ would be positive (otherwise $y = e$ and the limit would be equal to 0) and there would exist $\varepsilon \in]0, \tau[$ such that $d(a\delta_{r_i}(y), A) > \varepsilon r_i$ for infinitely many values of i. Possibly extracting a subsequence, we can assume without loss of generality that this happens for every $i \in \mathbb{N}$, so that

$$B_{\varepsilon r_i}(a\delta_{r_i}(y)) \subseteq \mathbb{G} \setminus A.$$

Furthermore, since $\varepsilon < \tau$ and d is homogeneous (Proposition 2.37) we have

$$B_{\varepsilon r_i}(a\delta_{r_i}(y)) \subseteq B_{\tau r_i}(a\delta_{r_i}(y)) \subseteq B_{2\tau r_i}(a).$$

Thus combining these two pieces of information with (3.1) we get

$$\begin{aligned}
1 &= \lim_{i \to \infty} \frac{|\mu|^*(B_{2\tau r_i}(a) \cap A)}{|\mu|(B_{2\tau r_i}(a))} \leq \limsup_{i \to \infty} \frac{|\mu|(B_{2\tau r_i}(a) \setminus B_{\varepsilon r_i}(a\delta_{r_i}(y)))}{|\mu|(B_{2\tau r_i}(a))} \\
&= \limsup_{i \to \infty} \frac{|\mu|(B_{2\tau r_i}(a)) - |\mu|(B_{\varepsilon r_i}(a\delta_{r_i}(y)))}{|\mu|(B_{2\tau r_i}(a))} \\
&= 1 - \liminf_{i \to \infty} \frac{|\mu|(B_{\varepsilon r_i}(a\delta_{r_i}(y)))}{|\mu|(B_{2\tau r_i}(a))} = 1 - \liminf_{i \to \infty} \frac{(I_{a,r_i})_\sharp |\mu|(B_\varepsilon(y))}{(I_{a,r_i})_\sharp |\mu|(B_{2\tau}(e))} \\
&\leq 1 - \frac{\liminf_{i \to \infty} |\lambda_i (I_{a,r_i})_\sharp \mu|(B_\varepsilon(y))}{\limsup_{i \to \infty} |\lambda_i (I_{a,r_i})_\sharp \mu|(B_{2\tau}(e))}
\end{aligned}$$

$$\leq 1 - \frac{|\nu|(B_\varepsilon(y))}{\limsup_{i\to\infty}|\lambda_i(I_{a,r_i})_\sharp\mu|(B_{2\tau}(e))}.$$

However, $|\nu|(B_\varepsilon(y)) > 0$ because $y \in \operatorname{supp}|\nu|$ and the denominator is finite by (3.2), so this is a contradiction. □

We will also need a specific distance on $\mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$, which now we present. For any $R > 0$ set

$$\mathcal{D}_R := \{u \in C_c(\mathbb{G}) : \operatorname{supp} u \subseteq B_R(e),\ \sup|u| \leq 1 \text{ and } |u(x) - u(y)| \leq d(x,y) \text{ for every } x, y \in \mathbb{G}\}$$

and for any $\mu, \nu \in \mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$

$$d_R(\mu,\nu) := \sup\left\{\left|\int_{\mathbb{G}} u\,d\mu - \int_{\mathbb{G}} u\,d\nu\right| : u \in \mathcal{D}_R\right\}.$$

It is easy to see that d_R is a distance on the linear subspace of $\mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$ made of Radon measures with support contained in $B_R(e)$. Moreover, d_R induces the local weak* convergence in all bounded sets (with respect to the total variation) of such subspace:

Lemma 5.10. *Let* $\{\mu_i\}_{i\in\mathbb{N}} \subseteq \mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$ *be such that* $\operatorname{supp}\mu_i \subseteq B_R(e)$ *and* $\sup_{i\in\mathbb{N}}|\mu_i|(B_R(e)) < \infty$. *Then* $\mu_i \overset{*}{\rightharpoonup} \mu$ *in* $\mathcal{M}^m_{\mathrm{loc}}(B_R(e))$ *if and only if* $\lim_{i\to\infty} d_R(\mu_i,\mu) = 0$.

Proof. By the way of contradiction, suppose that $\{d_R(\mu_i,\mu)\}_{i\in\mathbb{N}}$ does not converge to 0. Then we can find a sequence $\{u_i\}_{i\in\mathbb{N}} \subseteq \mathcal{D}_R$ and $\varepsilon > 0$ such that

$$\left|\int_{\mathbb{G}} u_i\,d\mu_i - \int_{\mathbb{G}} u_i\,d\mu\right| \geq \varepsilon.$$

On the other hand, possibly extracting a subsequence, by Ascoli-Arzelà theorem we have that $\{u_i\}_{i\in\mathbb{N}}$ converges uniformly to some $u \in \mathcal{D}_R$; since by hypothesis $\sup_{i\in\mathbb{N}}|\mu_i|(B_R(e))$ is finite, we get

$$\lim_{i\to\infty}\int_{\mathbb{G}} u_i\,d\mu_i = \int_{\mathbb{G}} u\,d\mu = \lim_{i\to\infty}\int_{\mathbb{G}} u_i\,d\mu,$$

which gives a contradiction.
Viceversa, it is easily observed that $d_R(\mu_i,\mu) \xrightarrow[i\to\infty]{} 0$ implies

$$\int_{\mathbb{G}} u\,d\mu_i \xrightarrow[i\to\infty]{} \int_{\mathbb{G}} u\,d\mu$$

for every $u \in \mathrm{Lip}(\mathbb{G})$ with $\operatorname{supp} u \subseteq B_R(e)$ (just multiply and divide by $\mathrm{Lip}(u)\sup|u|$). If now $u \in C_c(\mathbb{G})$ with $\operatorname{supp} u \subseteq B_R(e)$, there exists $\bar\varepsilon > 0$ such that $\operatorname{supp} u \subseteq B_{R-\bar\varepsilon}(e)$; take a family of mollifiers $\{\rho_\varepsilon\}_{\varepsilon\in]0,\bar\varepsilon[}$ and observe that $u_\varepsilon \coloneqq u * \rho_\varepsilon$ is a Lipschitz function and $\operatorname{supp} u_\varepsilon \subseteq B_R(e)$. Therefore by Theorem 3.38 we obtain (for any $\varepsilon > 0$ small enough)

$$\begin{aligned}\left| \int_{\mathbb{G}} u \, d\mu_i - \int_{\mathbb{G}} u \, d\mu \right| &\leq \int_{B_R(e)} |u - u_\varepsilon| \, d|\mu_i| + \left| \int_{\mathbb{G}} u_\varepsilon \, d\mu_i - \int_{\mathbb{G}} u_\varepsilon \, d\mu \right| \\ &\quad + \int_{B_R(e)} |u - u_\varepsilon| \, d|\mu| \\ &\leq \left(\sup_{i \in \mathbb{N}} |\mu_i|(B_R(e)) + 1 + |\mu|(B_R(e)) \right) \varepsilon,\end{aligned}$$

namely $\mu_i \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}^m_{\mathrm{loc}}(B_R(e))$. □

We also observe that d_R is separable:

Lemma 5.11. *There is a countable dense set $\mathcal{Q}$ of Radon measures on $\mathbb{G}$ with respect to d_R, that is for any $\mu \in \mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$ and $\varepsilon > 0$ we can find $\nu \in \mathcal{Q}$ for which $d_R(\mu, \nu) < \varepsilon$.*

Proof. Without loss of generality, we can assume $m = 1$ (otherwise just argue componentwise) and that μ is a nonnegative measure (otherwise consider its positive and negative parts $\mu^+ \coloneqq (|\mu| + \mu)/2$, $\mu^- \coloneqq (|\mu| - \mu)/2$). Let us work in graded coordinates; for any $i \in \mathbb{N}$, let $Q_{i,1}, \ldots, Q_{i,m_i}$ be the half-open dyadic cubes of side-length 2^{-i}, i.e. cubes of the form

$$\{x \in \mathbb{R}^n : k_j 2^{-i} \leq x_j < (k_j + 1) 2^{-i}, \, k_j \in \mathbb{Z}, \, j = 1, \ldots, n\},$$

which have nonempty intersection with $B_R(e)$. Denote by $x_{i,j}$ the center of $Q_{i,j}$ and consider the Radon measures

$$\mu_i \coloneqq \sum_{j=1}^{m_i} \mu(Q_{i,j}) \delta_{x_{i,j}},$$

where δ_x is the Dirac measure concentrated at x. Then for any $u \in C_c(\mathbb{G})$ with $\operatorname{supp} u \subseteq B_R(e)$ we have by the dominated convergence theorem

$$\int_{\mathbb{G}} u \, d\mu_i = \int_{\mathbb{G}} \sum_{j=1}^{m_i} u(x_{i,j}) \chi_{Q_{i,j}} \, d\mu \xrightarrow[i \to \infty]{} \int_{\mathbb{G}} u \, d\mu,$$

hence $\lim_{i \to \infty} d_R(\mu_i, \mu) = 0$ by Lemma 5.10. This implies that the family of Radon measures

$$\mathcal{Q} \coloneqq \left\{ \sum_{j=1}^{m_i} q_{i,j} \delta_{x_{i,j}} : q_{i,j} \in \mathbb{Q}^+, i \in \mathbb{N} \right\}$$

satisfies the thesis. □

If now we define for any $\mu, \nu \in \mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$

$$\bar{d}(\mu,\nu) := \sum_{R=1}^{\infty} \frac{1}{2^R} \min\{1, d_R(\mu,\nu)\},$$

we have that $\bar{d}$ is a distance on $\mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$ and by virtue of Lemma 5.10 and Lemma 5.11 it inherits the properties of d_R, namely $\bar{d}$ induces the local weak* convergence in all bounded subsets of $\mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$ made of Radon measures with support contained in some bounded subset of $\mathbb{G}$ and is separable.

Proof of Theorem 5.7. Let $x \in \mathbb{G}$ be such that

$$\ell_x := \limsup_{r \searrow 0} \frac{|\mu|(B_r(x))}{r^q} < \infty.$$

First of all, we check that for any sequence $\{r_i\}_{i\in\mathbb{N}}$ such that $r_i \searrow 0$ we have

$$r_i^{-q}(I_{x,r_i})_\sharp \mu \overset{*}{\rightharpoonup} \nu \iff \lim_{i\to\infty} \bar{d}(\nu, r_i^{-q}(I_{x,r_i})_\sharp \mu) = 0. \tag{5.1}$$

The first implication is a direct consequence of Lemma 5.10 because $\bar{d}$-convergence is equivalent to d_R-convergence for all $R > 0$ and by (3.2) the sequence $\{r_i^{-q}(I_{x,r_i})_\sharp\mu\}_{i\in\mathbb{N}}$ is bounded in total variation on every ball. The converse one is also a consequence of Lemma 5.10, but we have to deduce the boundedness in total variation by our choice of x:

$$\begin{aligned}\sup_{i\in\mathbb{N}} \frac{1}{r_i^q}|(I_{x,r_i})_\sharp\mu|(B_R(e)) &= \sup_{i\in\mathbb{N}} \frac{1}{r_i^q}|\mu|(B_{Rr_i}(x))\\ &\le \max\left\{\max_{1\le i\le N}\left\{\frac{1}{r_i^q}|\mu|(B_{Rr_i}(x))\right\}, \ell_x + 1\right\} < \infty,\end{aligned}$$

where $N \in \mathbb{N}$ is such that $r_i^{-q}|\mu|(B_{Rr_i}(x)) \le \ell_x + 1$ for every $i \ge N$.
Let us suppose that the following statement holds:

(A) for any $\nu \in \mathrm{Tan}(\mu, x)$, $y \in \operatorname{supp}|\nu|$ and $r > 0$ we have $r^{-q}(I_{y,r})_\sharp\nu \in \mathrm{Tan}(\mu, x)$.

With this assumption, the thesis follows by a diagonal argument: let

$$\eta = \lim_{i\to\infty} r_i^{-q}(I_{y,r_i})_\sharp\nu \in \mathrm{Tan}(\nu, y);$$

thanks to (5.1), for each $k \in \mathbb{N}$ we can choose $i_k \in \mathbb{N}$ such that

$$\bar{d}(r_{i_k}^{-q}(I_{y,r_{i_k}})_\sharp\nu, \eta) < \frac{1}{k}.$$

Furthermore, by (A) we know $r_{i_k}^{-q}(I_{y,r_{i_k}})_\sharp\nu \in \mathrm{Tan}(\mu, x)$, so that again by (5.1) there exists a sequence $\{s_k\}_{k\in\mathbb{N}}$ such that $s_k \searrow 0$ and

$$\bar{d}(r_{i_k}^{-q}(I_{y,r_{i_k}})_\sharp\nu, s_k^{-q}(I_{x,s_k})_\sharp\mu) < \frac{1}{k}.$$

This gives $\lim_{k\to\infty} \bar{d}(s_k^{-q}(I_{x,s_k})_\sharp\mu, \eta) = 0$, namely $\eta \in \mathrm{Tan}(\mu, x)$ once again by (5.1).
It only remains to prove (A); however, since $I_{y,r} = I_{e,r} \circ I_{y,1}$ and the operator $\eta \longmapsto r^{-q}(I_{e,r})_\sharp\eta$ is readily seen to map $\mathrm{Tan}(\mu, x)$ into $\mathrm{Tan}(\mu, x)$, we just need to show the following property:

(B) for $|\mu|$-a.e. $x \in \mathbb{G}$ we have $(I_{y,1})_\sharp\nu \in \mathrm{Tan}(\mu, x)$ for every $\nu \in \mathrm{Tan}(\mu, x)$ and $y \in \mathrm{supp}\,|\nu|$.

Let us consider the set S of points where (B) fails: for any $x \in S$ there exist $\nu \in \mathrm{Tan}(\mu, x)$ and $y \in \mathrm{supp}\,|\nu|$ such that $(I_{y,1})_\sharp\nu \notin \mathrm{Tan}(\mu, x)$. Thanks to (5.1), this implies that there exist two integers $k, z \geq 1$ such that

$$\bar{d}((I_{y,1})_\sharp\nu, r^{-q}(I_{x,r})_\sharp\mu) > \frac{1}{k} \qquad \text{for every } r \in \left]0, \frac{1}{z}\right[.$$

If we set

$$A_{k,z} \coloneqq \{x \in \mathbb{G} : \text{there exist } \nu \in \mathrm{Tan}(\mu, x) \text{ and } y \in \mathrm{supp}\,|\nu| \text{ such that } \bar{d}((I_{y,1})_\sharp\nu, r^{-q}(I_{x,r})_\sharp\mu) > 1/k \text{ for every } r \in]0, 1/z[\},$$

we have that S is contained in the union of these sets, so it suffices to show $|\mu|^*(A_{k,z}) = 0$ for every integers $k, z \geq 1$.
Suppose by contradiction $|\mu|^*(A_{k,z}) \neq 0$ for some integers $k, z \geq 1$. Since $\bar{d}$ is separable, we can cover $\mathcal{M}^m_{\mathrm{loc}}(\mathbb{G})$ with a countable family of sets $\{B_\ell\}_{\ell\in\mathbb{N}}$ satisfying

$$\bar{d}(\eta, \eta') < \frac{1}{2k} \qquad \text{for every } \eta, \eta' \in B_\ell. \tag{5.2}$$

Let us now consider the sets

$$A_{k,z,\ell} \coloneqq \{x \in \mathbb{G} : \text{there exist } \nu \in \mathrm{Tan}(\mu, x) \text{ and } y \in \mathrm{supp}\,|\nu| \text{ such that } (I_{y,1})_\sharp\nu \in B_\ell,\ \bar{d}((I_{y,1})_\sharp\nu, r^{-q}(I_{x,r})_\sharp\mu) > 1/k \text{ for every } r \in]0, 1/z[\}.$$

Since $A_{k,z} \subseteq \bigcup_{\ell=0}^\infty A_{k,z,\ell}$ and $|\mu|^*$ is countably subadditive, at least one of these sets satisfies $|\mu|^*(A_{k,z,\ell}) > 0$; let us fix ℓ with this property and set $A \coloneqq A_{k,z,\ell}$. Since $|\mu|^*(A) > 0$ and $|\mu|$ is asymptotically doubling, by Remark 3.17 we can find $a \in A$ which is a density point of A relative to $|\mu|^*$; moreover, by definition of A we have a measure $\nu_a \in \mathrm{Tan}(\mu, a)$ and a point $y_a \in \mathrm{supp}\,|\nu_a|$ associated with a such that $(I_{y_a,1})_\sharp\nu_a \in B_\ell$ and

$$\bar{d}((I_{y_a,1})_\sharp\nu_a, r^{-q}(I_{a,r})_\sharp\mu) > \frac{1}{k} \qquad \text{for every } r \in \left]0, \frac{1}{z}\right[.$$

In particular, $y_a \neq e$; we can also write $\nu_a = \lim_{i\to\infty} r_i^{-q}(I_{a,r_i})_\sharp\mu$ for suitable $r_i \searrow 0$. Now, let us take $a_i \in A$ such that

$$d(a\delta_{r_i}(y_a), a_i) \leq d(a\delta_{r_i}(y_a), A) + \frac{r_i}{i}$$

for every $i \in \mathbb{N}$; Lemma 5.9 ensures $d(a\delta_{r_i}(y_a), a_i) = o(r_i)$ as $i \to \infty$, so $\delta_{1/r_i}(a^{-1}a_i) \xrightarrow[i\to\infty]{} y_a$. Since $I_{\delta_{1/r_i}(a^{-1}a_i),1} \circ I_{a,r_i} = I_{a_i,r_i}$, this gives

$$\begin{aligned}\lim_{i\to\infty} r_i^{-q}(I_{a_i,r_i})_\sharp\mu &= \lim_{i\to\infty} r_i^{-q}(I_{\delta_{1/r_i}(a^{-1}a_i),1})_\sharp(I_{a,r_i})_\sharp\mu \\ &= \lim_{i\to\infty}(I_{\delta_{1/r_i}(a^{-1}a_i),1})_\sharp(r_i^{-q}(I_{a,r_i})_\sharp\mu) = (I_{y_a,1})_\sharp\nu_a.\end{aligned}$$

Therefore, by (5.1) we can fix i sufficiently large such that $r_i < 1/z$ and

$$\bar{d}((r_i^{-q}(I_{a_i,r_i})_\sharp\mu), (I_{y_a,1})_\sharp\nu_a) < \frac{1}{2k}. \tag{5.3}$$

On the other hand, since $a_i \in A = A_{k,z,\ell}$, we can find some $\nu' \in \operatorname{Tan}(\mu, a_i)$ and $y' \in \operatorname{supp}|\nu'|$ with $(I_{y',1})_\sharp\nu' \in B_\ell$ such that

$$\bar{d}((r_i^{-q}(I_{a_i,r_i})_\sharp\mu), (I_{y',1})_\sharp\nu') > \frac{1}{k};$$

thus by applying the triangle inequality and combining (5.2) and (5.3) we finally obtain

$$\frac{1}{k} < \bar{d}((r_i^{-q}(I_{a_i,r_i})_\sharp\mu), (I_{y_a,1})_\sharp\nu_a) + \bar{d}((I_{y_a,1})_\sharp\nu_a, (I_{y',1})_\sharp\nu') < \frac{1}{2k} + \frac{1}{2k}.$$

This contradiction concludes the proof. □

5.3 A cone in the Engel group

In this section we provide a counterexample showing that Theorem 5.2 does not hold if the Carnot group has step greater than 2. More precisely, we will find a set of locally finite perimeter in the Engel group with a tangent set which is a cone, but not a half-space, and explain what does not work in the proof of Theorem 5.1 in this case. From now on, $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure induced by the Euclidean distance.

Let us consider the Engel group $\mathbb{E}$ with the representation in strong Malcev coordinates $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ presented in Section 2.4, so that the vectors of the basis of the horizontal layer are

$$\begin{aligned}X_1 &= \partial_{x_1}, \\ X_2 &= \partial_{x_2} - x_1\partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4}, \\ X_3 &= \partial_{x_3} - x_1\partial_{x_4}, \\ X_4 &= \partial_{x_4}.\end{aligned}$$

Recall that $\mathbb{E}$ is a Carnot group of step $s = 3$ with dimension of the horizontal layer $m = 2$; moreover, it has homogeneous dimension

$$Q = 2 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 7.$$

For any $\alpha > 0$, let $P = P_\alpha : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be the polynomial

$$P(x) = \alpha x_2^3 + 2x_4,$$

whose gradient is

$$\nabla P(x) = (0, 3\alpha x_2^2, 0, 2).$$

In particular, all level sets $\{P = c\}$, $c \in \mathbb{R}$, of P are obviously graphs of smooth functions depending on (x_1, x_2, x_3). Notice that

$$|\nabla P(x)| = \sqrt{4 + 9\alpha^2 x_2^4} = 2 + \frac{9}{4}\alpha^2 x_2^4 + O(|x|^5).$$

We also have

$$\begin{aligned} X_1 P(x) &= \partial_{x_1}(\alpha x_2^3 + 2x_4) = 0, \\ X_2 P(x) &= \left[\partial_{x_2} - x_1 \partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4}\right](\alpha x_2^3 + 2x_4) \\ &= 3\alpha x_2^2 + x_1^2 \geq 0. \end{aligned} \tag{5.4}$$

We define

$$C := \{x \in \mathbb{R}^4 : P(x) \leq 0\},$$

whose boundary is the set $\partial C = \{P = 0\}$. Observe that the polynomial P is homogeneous of degree 3: for any $\lambda > 0$

$$P(\delta_\lambda(x)) = P(\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4) = \alpha \lambda^3 x_2^3 + 2\lambda^3 x_4 = \lambda^3 P(x);$$

this implies $\delta_\lambda C = C$ for every $\lambda > 0$, namely C is a cone.
From (1.13) we know

$$Z\chi_C = -\frac{ZP}{|\nabla P|}\mathcal{H}^3_{\llcorner \partial C} \qquad \text{for every } Z \in \mathfrak{c};$$

hence by (5.4) we have that C is a set of locally finite perimeter and

$$D\chi_C = (X_1\chi_C, X_2\chi_C) = (0,1)X_2\chi_C = -\frac{3\alpha x_2^2 + x_1^2}{|\nabla P(x)|}(0,1)\mathcal{H}^3_{\llcorner \partial C}.$$

It follows

$$|D\chi_C| = \frac{3\alpha x_2^2 + x_1^2}{|\nabla P(x)|}\mathcal{H}^3_{\llcorner \partial C}$$

and so that the horizontal normal

$$\nu_C(x) = \lim_{r \searrow 0} \frac{D\chi_C(B_r(x))}{|D\chi_C|(B_r(x))} = (0, -1)$$

is constant. Therefore all points of $\operatorname{supp}|D\chi_C|$ belong to $\partial^* C$, in particular $0 \in \partial^* C$; thus by the invariance of C under intrinsic dilations we have

$\mathrm{Tan}(C,0) = \{C\}$, but clearly C is not a half-space.
Going through the proof of Theorem 5.1 again, one may realize that the step that cannot hold in this case is when we use Lemma 4.20 to prove Lemma 4.21. It says that non-horizontal regular directions Z for E give rise, after blow-up, to invariant directions, at least at points $\bar{x} \in \partial^* E$ where $|Z\chi_E|(B_r(\bar{x}))/r^{Q-2}$ is infinitesimal as $r \searrow 0$. In our case, C is self-similar under blow-up at $\bar{x} = 0$, so it must happen that $|Z\chi_C|(B_r(0))/r^{Q-2}$ is not infinitesimal as $r \searrow 0$ for every non-horizontal regular direction Z (actually, all directions are regular for C). Let us show explicitly this fact for the direction $Z := \mathrm{Ad}_{\exp(X_1)}(X_2)$: by (2.6) and the commutator relations in $\mathbb{E}$ we get

$$\begin{aligned}
Z &= X_2 + [X_1, X_2] + \frac{1}{2}[X_1, [X_1, X_2]] \\
&= X_2 - X_3 + \frac{1}{2} X_4 \\
&= \partial_{x_2} - x_1 \partial_{x_3} + \frac{x_1^2}{2}\partial_{x_4} - \partial_{x_3} + x_1 \partial_{x_4} + \frac{1}{2}\partial_{x_4} \\
&= \partial_{x_2} - (x_1+1)\partial_{x_3} + \frac{(x_1+1)^2}{2}\partial_{x_4},
\end{aligned}$$

so that

$$\begin{aligned}
ZP(x) &= \left[\partial_{x_2} - (x_1+1)\partial_{x_3} + \frac{(x_1+1)^2}{2}\partial_{x_4}\right](\alpha x_2^3 + 2x_4) \\
&= 3\alpha x_2^2 + (x_1+1)^2 = 1 + O(|x|).
\end{aligned}$$

According to Theorem 2.39, balls $B_r(0)$ are comparable to the boxes

$$Q_r := \mathrm{Box}(0,r) = [-r,r]^2 \times [-r^2, r^2] \times [-r^3, r^3],$$

so we can consider them in place of balls for our computations. We want to investigate the order of $|Z\chi_C|(Q_r)$ as $r \searrow 0$; for the sake of simplicity, let us assume $\alpha \in]0,2]$, so that the function

$$g(x_1, x_2, x_3) := -\frac{\alpha}{2} x_2^3,$$

whose graph is ∂C, has absolute value less or equal than r^3 if $(x_1, x_2, x_3) \in [-r,r]^2 \times [-r^2, r^2]$. Hence $Q_r \cap \partial C$ is the graph of g on the "basis" $[-r,r]^2 \times [-r^2, r^2]$ of the box Q_r and this fact, together with $\nabla g(0) = 0$, gives

$$\begin{aligned}
\mathcal{H}^3_{\llcorner \partial C}(Q_r) = \mathcal{H}^3(Q_r \cap \partial C) &= \int_{[-r,r]^2 \times [-r^2, r^2]} \sqrt{1 + |\nabla g|^2}\, d\mathcal{L}^3 \\
&\approx \int_{[-r,r]^2 \times [-r^2, r^2]} 1\, d\mathcal{L}^3 = \mathcal{L}^3([-r,r]^2 \times [-r^2, r^2]) = 8r^4.
\end{aligned}$$

This implies

$$|Z\chi_C|(Q_r) = \left(\frac{1}{2} + O(|x|^4)\right)\mathcal{H}^3_{\llcorner \partial C}(Q_r) = 4r^4 + O(r^5),$$

which finally gives

$$\frac{|Z\chi_C|(Q_r)}{r^5} = O(r^{-1}) \qquad \text{as } r \searrow 0.$$

5.4 Some sets with constant normal

As mentioned in Section 5.1, in contrast with the Euclidean and step 2 Carnot groups cases, it is not true that in a general Carnot group sets with constant horizontal normal must be vertical half-spaces. We want to provide another couple of examples of this fact. We will consider again the Engel group $\mathbb{E} \cong (\mathbb{R}^4, \cdot)$ in strong Malcev coordinates with the notation introduced in the previous section.

Example 5.12. For $a, b \in \mathbb{R}$, let $P = P_{a,b} : \mathbb{R}^4 \longrightarrow \mathbb{R}$ be the polynomial

$$P_{a,b}(x) = 2ax_4 - bx_3 + x_2.$$

Set $E_c := \{P \leq c\}$ for every $c \in \mathbb{R}$; since $\nabla P \neq 0$, by the regular level set theorem all level sets $\partial E_c = \{P = c\}$ of P are smooth embedded submanifolds. Note that when both a and b are zero, the E_c's are vertical half-spaces. The derivatives along the vector fields of the horizontal layer are

$$\begin{aligned} X_1P(x) &= 0, \\ X_2P(x) &= ax_1^2 + bx_1 + 1, \end{aligned}$$

hence $X_2P(x) > 0$ for every (a, b) in a suitable neighborhood $U \subseteq \mathbb{R}^2$ of $(1, 0)$. Then we have by (1.13)

$$D\chi_{E_c} = -\frac{ax_1^2 + bx_1 + 1}{|\nabla P(x)|}(0,1)\mathcal{H}^3_{\llcorner \partial E_c}$$

and so for any $(a, b) \in U$ the horizontal normal $\nu_{E_c} = (0, -1)$ is constant. However, these sets are not cones, except when they are vertical half-spaces.

Example 5.13. Consider the smooth function $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ defined by $f(x) = x_2 + \arctan(x_4)$ and set

$$E := \{x \in \mathbb{R}^4 : f(x) \geq 0\}.$$

We have by (1.13)

$$\begin{aligned} X_1\chi_E &= 0, \\ X_2\chi_E &= \frac{1}{|\nabla f(x)|}\left(1 + \frac{x_1^2}{2(1 + x_4^2)}\right)\mathcal{H}^3_{\llcorner \partial E} \geq 0, \end{aligned}$$

so that the horizontal normal $\nu_E = (0, 1)$ is constant, but clearly E is not a vertical half-space.

Example 5.13 is important because it gives a negative answer to the asymptotic stability of half-spaces. More precisely, let $E \subseteq \mathbb{G}$ be a set with constant horizontal normal and H the vertical half-space with the same horizontal normal; if

$$\liminf_{R \to +\infty} \frac{\mathrm{vol}_{\mathbb{G}}((E \,\Delta\, H) \cap B_R(e))}{\mathrm{vol}_{\mathbb{G}}(B_R(e))} = 0,$$

one might ask if E is a vertical half-space to get an idea to prove uniqueness for Theorem 5.1. However, the example says that this is not true: the set E satisfies the inclusions

$$\left\{ x_2 \geq \frac{\pi}{2} \right\} \subseteq E \subseteq \left\{ x_2 \geq -\frac{\pi}{2} \right\},$$

so that, if we set $H := \{x_2 \geq 0\}$, $E \,\Delta\, H \subseteq \{-\pi/2 \leq x_2 \leq 0\}$ and by Theorem 2.39 (for R sufficiently large)

$$(E \,\Delta\, H) \cap B_R(e) \subseteq [-CR, CR] \times \left[-\frac{\pi}{2}, 0 \right] \times [-(CR)^2, (CR)^2] \times [-(CR)^3, (CR)^3].$$

Therefore $\mathrm{vol}_{\mathbb{E}}((E \,\Delta\, H) \cap B_R(e)) \leq cR^6$ for a suitable $c > 0$. On the other hand, $\mathrm{vol}_{\mathbb{E}}(B_R(e)) = O(R^7)$ thanks to (3.5), hence we finally get

$$\lim_{R \to +\infty} \frac{\mathrm{vol}_{\mathbb{E}}((E \,\Delta\, H) \cap B_R(e))}{\mathrm{vol}_{\mathbb{E}}(B_R(e))} = 0.$$

In other words, E is asymptotic at infinity to the half-space $H = \{x_2 \geq 0\}$, but we have already observed that E is not a half-space.

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