UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

Master Degree in Physics

Final Dissertation

Worldsheet description of black hole solutions in String Theory

Thesis supervisor

Prof. Stefano Massai

Candidate

Luca Novelli

Academic Year 2023/2024

The goal of the thesis is to investigate exact worldsheet descriptions of some black hole solutions in string theory and to analyse their properties. In particular, recently a worldsheet description has been found for a class of microstate solutions (describing some of the microstates of the black hole) in terms of null-gauged Wess-Zumino-Witten (WZW) models. In such cases, the consistency of the worldsheet theory imposes the absence of horizons, as expected from a solution describing a particular microstate. The thesis will explore more general worldsheet models of this kind with the aim of understanding whether they can describe string dynamics on black hole solutions, and to study their properties. First, we will focus on a worldsheet description of Witten's two-dimensional black hole in both Euclidean and Lorentzian signatures, and we will show that such solutions can be obtained in the null-gauging formalism. This investigation is then extended to a more general class of models that describe non-extremal three-charge black holes. The spectrum of such models will be studied.

Contents

Introduction

In the field of black hole physics, the pursuit of reconciling quantum field theory with general relativity highlights some of the most profound and challenging aspects of modern theoretical physics. Black holes are not only fascinating objects due to their extreme gravitational fields but also serve as crucial arenas for testing and refining our understanding of the universe's fundamental laws. Their study bridges the gap between the macroscopic description of spacetime and the microscopic realm of quantum mechanics, presenting a unique opportunity to explore the frontiers of both theories.

One of the most compelling puzzles that arises at this intersection is the black hole information paradox. This paradox presents a profound challenge to our understanding of information conservation in the context of black holes. Quantum mechanics asserts that information about a system must be preserved, even when the system undergoes transformations or interactions. Moreover, general relativity, particularly through the lens of black hole physics, suggests that information falling into a black hole may be irretrievably lost beyond the event horizon. However, in 1974, Hawking argued that black holes emit thermal radiation [1], and the resulting evaporation leads to contradictions that raise fundamental questions about the nature of information and the structure of black holes.

To tackle these critical issues, it is essential to engage with various aspects of quantum gravity. Quantum gravity seeks to merge the principles of quantum mechanics with those of general relativity into a unified framework. Although no complete theory of quantum gravity exists yet, exploring different theoretical approaches can provide insights into how black holes behave and how spacetime operates on the smallest scales. String theory emerges as a prominent candidate for such a framework. By proposing that fundamental entities are not point-like particles but one-dimensional "strings", string theory aims to provide a comprehensive description of gravity alongside quantum phenomena. This approach opens new avenues for understanding black holes and their complex interactions with fundamental forces.

Over the past few decades, researchers have made substantial progress in understanding the properties of black hole solutions within string theory. This endeavor has involved studying various models, including seminal work by Strominger and Vafa [2], which demonstrated how string theory provides a statistical description of black hole entropy. These contributions have expanded to include a wide variety of black hole solutions in different dimensions. Despite these developments, many approaches rely on effective field theories, which, while useful for approximating certain aspects of string theory, do not fully capture its intricacies.

The primary focus of this thesis is to delve into exact worldsheet descriptions of black hole solutions within string theory. This exploration involves extending beyond effective field theories to understand how black hole solutions can be described in a true string-theoretic framework. In this context, Wess-Zumino-Witten (WZW) models have emerged as a significant tool. Through these models, it is possible to reproduce known supergravity solutions and further investigate potential string effects by gauging certain currents and analyzing their quantum properties.

The structure of this thesis is designed to guide the reader through a comprehensive exploration of the key concepts related to black holes and string theory.

Chapter 1 sets the stage by delving into the black hole information paradox, a fundamental issue that bridges quantum mechanics and general relativity. This chapter provides a detailed examination of the paradox, exploring its implications for information conservation and the theoretical challenges it presents. Although it is not the central focus of the thesis, this chapter establishes the necessary context and some motivations for the more advanced discussions that follow.

Chapter 2 offers a thorough overview of string theory, which serves as the foundational framework for the analysis presented in this thesis. Here, the fundamental principles of string theory are explored, including the notion that particles emerge as vibrational states of one-dimensional strings rather than being mere point-like entities. Additionally, this chapter introduces conformal field theory techniques, which are essential for analyzing the worldsheet descriptions of black hole solutions. By laying this groundwork, Chapter 2 equips the reader with the technical tools needed for the subsequent sections of the thesis.

Chapter 3 shifts focus to known results concerning black hole solutions within the realm of supergravity. This chapter delves into the properties of black holes with one, two, and three charges and examines Mathur's fuzzball proposal, which offers a novel perspective on black hole structure. The fuzzball models aim to describe solutions that are smooth and horizonless, challenging the traditional view of a singularity surrounded by an event horizon.

Chapter 4 constitutes the core of the thesis, presenting the detailed analysis of Wess-Zumino-Witten (WZW) models. This chapter establishes the essential components of these models and explores the null-gauging procedure in depth. The idea is to investigate in what sense a null-gauged WZW model can describe a black hole solution. To address this goal, Witten's two-dimensional black hole in both Euclidean and Lorentzian signatures is considered. Subsequently, other more advanced solutions are analyzed. Furthermore, aspects related to the spectrum of Witten's black hole are studied.

Hence, this thesis aims to identify Wess-Zumino-Witten models that describe black hole solutions, with the goal of studying their properties.

Chapter 1

The Black Hole Information Paradox

It is well-known that black holes emit thermal radiation, commonly referred to as Hawking radiation. This process leads to one of the most influential puzzles in theoretical physics, known as the Black Hole Information Paradox.

Initially, the state of the Hawking radiation is accurately described as a thermal mixed state. Therefore, if a black hole is formed by the collapse of a shell of matter in a pure state $|\psi\rangle$, the state that remains after the black hole has evaporated is represented by a density matrix. Mathematically, if the initial pure state is denoted by $|\psi\rangle$, the final state after evaporation is described by a density matrix ρ given by

$$\rho = |\psi\rangle\langle\psi| \longrightarrow \rho' \coloneqq U|\psi\rangle\langle\psi| U^{\dagger}, \tag{1.1}$$

where U is a unitary operator. This notation indicates the time evolution of the pure state under the action of the unitary operator. In this context, ρ' represents the state after evolution.

One can easily observe that if $\rho = |\psi\rangle\langle\psi|$ is a pure state, then $\rho'^2 = \rho'$ must hold true, implying that the final state ρ' is also a pure state. This follows from the property of unitary operators that preserve the purity of states.

The term "information paradox" arises from the apparent contradiction between this theoretical framework and quantum mechanics. According to quantum mechanics, unitary evolution should preserve information; that is, the evolution of pure states via a unitary operator U should always result in a pure state. In other words, the unitary evolution maps pure states to pure states. Thus, the evaporation of a black hole, which transforms an initial pure state into a thermal mixed state, seems to violate this principle, suggesting that information about the matter that collapsed to form the black hole is lost.

To better understand this paradox, we now proceed with a brief discussion on Hawking radiation and its implications for the black hole information problem. The information and discussions that follow are based on [3].

1.1 Hawking radiation

Let's consider the Schwarzschild black hole metric

$$ds^2 = -\left(1 - \frac{r_{\mathrm{s}}}{r}\right) dt^2 + \left(1 - \frac{r_{\mathrm{s}}}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.2}$$

where the Schwarzschild radius is $r_{\mathrm{s}} = 2GM$.
Natural units will be used here, where $\hbar = c = k_{\mathrm{B}} = 1$, with Newton's constant expressed in

terms of the Planck length and mass as $G = l_{\mathrm{P}}^2 = m_{\mathrm{P}}^{-2}$.

The previous coordinates are smoothly defined only in the exterior region corresponding to $r > r_s$; however, one can extend the validity of the description to the interior area by using the so-called Kruscal extension of the Schwarzschild metric, namely

$$\begin{aligned} ds^2 &= -\frac{4r_{\mathrm{s}}}{r}e^{-r/r_{\mathrm{s}}}dUdV + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \\ UV &= r_{\mathrm{s}}\left(r_{\mathrm{s}} - r\right)e^{r/r_{\mathrm{s}}}, \quad \frac{U}{V} = -e^{-t/r_{\mathrm{s}}}. \end{aligned} \tag{1.3}$$

The expression 1.3 is well-defined for $r \to r_s$ and in the interior region while the value $r = 0$ represent a true physical singularity since the scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges at this point.
Now, let's consider an inertial observer falling through the black hole future horizon: it experiences the whole process in a finite time τ in its own frame, while an asymptotic observer never sees it on the horizon for a finite time t.
The relation betweem these two times is $d\tau \propto e^{-t/r_{\mathrm{s}}}dt$ and it is the origin of all the discussion. In fact, a quantum field can be expanded in modes with a given τ-frequency ν, related to the infalling observer, as well as in terms of the asymptotic modes with t-frequency ω.
The basic point is that the frequencies are not the same: their negative and positive parts are mixed such that the mode blue-shifts as we follow it backward, or red-shifts as we follow it forward.
For simplicity, the angular directions are ignored, treating the system as 1+1 dimensional. Additionally, a massless scalar field is considered here, but analogous results can also be obtained using other types of fields. The Klein-Gordon equation reads as

$$\partial_U\partial_V\phi = \partial_u\partial_v\phi = 0, \tag{1.4}$$

where new coordinates u, v were introduced as

$$u = t - r_* = -2r_{\mathrm{s}}\ln\left(-\frac{U}{r_{\mathrm{s}}}\right), \quad v = t + r_* = 2r_{\mathrm{s}}\ln\left(\frac{V}{r_{\mathrm{s}}}\right), \quad r_* = r + r_s\ln(r - r_s). \tag{1.5}$$

The general solution is given by a superposition of waves

$$\phi = \phi_L(v) + \phi_R(u) = \phi_L(V) + \phi_R(U), \tag{1.6}$$

where the left movers describe the ingoing solutions and the right movers the ougoing ones.
Let's expand the right part of ϕ in modes by using the two different reference frames

$$\begin{aligned} \phi_R &= \int_0^\infty \frac{d\nu}{2\pi(2\nu)^{1/2}}\left(a_\nu e^{-i\nu U} + a_\nu^\dagger e^{i\nu U}\right), \\ &= \int_0^\infty \frac{d\omega}{2\pi(2\omega)^{1/2}}\left(b_\omega e^{-i\omega u} + b_\omega^\dagger e^{i\omega u}\right). \end{aligned} \tag{1.7}$$

After the usual canonical quantization procedure, the bosonic ladder operators satisfy the algebra

$$\left[a_\nu, a_{\nu'}^\dagger\right] = 2\pi\delta\left(\nu - \nu'\right), \quad \left[b_\omega, b_{\omega'}^\dagger\right] = 2\pi\delta\left(\omega - \omega'\right). \tag{1.8}$$

The ω-frequencies are related to the ν-frequencies by

$$b_\omega = \int_0^\infty \frac{d\nu}{2\pi}\left(\alpha_{\omega\nu}a_\nu + \beta_{\omega\nu}a_\nu^\dagger\right), \tag{1.9}$$

where the coefficients can be expressed as

$$\begin{aligned}\alpha_{\omega\nu} &= 2r_\mathrm{s}(\omega/\nu)^{1/2}\,(r_\mathrm{s}\nu)^{2ir_\mathrm{s}\omega}\,e^{\pi r_\mathrm{s}\omega}\Gamma\left(-2ir_\mathrm{s}\omega\right),\\ \beta_{\omega\nu} &= 2r_\mathrm{s}(\omega/\nu)^{1/2}\,(r_\mathrm{s}\nu)^{2ir_\mathrm{s}\omega}\,e^{-\pi r_\mathrm{s}\omega}\Gamma\left(-2ir_\mathrm{s}\omega\right).\end{aligned} \tag{1.10}$$

The strategy now consists of taking the average of the number density of particles associated with the b-modes in terms of the state $|0_a\rangle$, where this state satisfies

$$a_\nu\,|0_a\rangle = 0, \tag{1.11}$$

due to the adiabatic principle.
In other words,

$$\begin{aligned}\langle 0_a|\,b^\dagger_\omega b_{\omega'}\,|0_a\rangle &= 2\left(\omega\omega'\right)^{1/2}\int_0^\infty \frac{d\nu}{2\pi(2\nu)^{1/2}}\frac{d\nu'}{2\pi\left(2\nu'\right)^{1/2}}\beta^*_{\omega\nu}\beta_{\omega'\nu'}\,\langle\psi|\,a_\nu a^\dagger_{\nu'}\,|\psi\rangle =\\ &= 2\left(\omega\omega'\right)^{1/2}\int_0^\infty \frac{d\nu}{4\pi\nu}\beta^*_{\omega\nu}\beta_{\omega'\nu} = \frac{2\pi\delta\left(\omega-\omega'\right)}{e^{4\pi r_s\omega}-1} = \frac{2\pi\delta\left(\omega-\omega'\right)}{e^{\omega/T_\mathrm{H}}-1}.\end{aligned} \tag{1.12}$$

One determines that the asymptotic observer measures a black body radiation with $T_H = \frac{1}{4\pi r_s}$ that is the expected temperature of a Schwarzschild black hole.
The computation in the full 3+1-dimensional spacetime gives the same results up to a positive factor that appears and that represents the trasmission probability related to the scattering process between the right-moving and left-moving modes of the scalar field.

1.2 Where is the paradox?

Only the exterior of the black hole was directly involved in the derivation of the Hawking flux. However, to understand the state related to the black hole, one cannot neglect the interior details.
Here, the modes a are defined in both the exterior and interior regions, while the modes b are defined only in the exterior region. In particular, to correctly invert the expression 1.9, additional modes, denoted as $\tilde{b}$, need to be added.
Moreover, let's defined the state $|0\rangle_{b,\tilde{b}} \in H_{out}\otimes H_{int}$ to be the vacuum related to b and $\tilde{b}$ modes (respectively, the exterior and interior modes).
The decomposition now is

$$a_\nu = \int_0^\infty \frac{d\omega}{2\pi}\left(\alpha^*_{\omega\nu}b_\omega - \beta^*_{\omega\nu}b^\dagger_\omega + \tilde{\alpha}^*_{\omega\nu}\tilde{b}_\omega - \tilde{\beta}^*_{\omega\nu}\tilde{b}^\dagger_\omega\right), \tag{1.13}$$

from which one can deduce that

$$|0\rangle_a \propto \exp\left[\int d\omega e^{-\frac{\omega}{2T_H}}b^+_\omega\tilde{b}^+_\omega\right]|0\rangle_{b,\tilde{b}}. \tag{1.14}$$

The asymptotic observer is not able to detect the internal $\tilde{b}$-modes; therefore, it is necessary to trace over the H_{int} factor,

$$\rho_{out} = \mathrm{Tr}_\mathrm{int}\,|0_a\rangle\langle 0_a| = \prod_\omega |N_\omega|^2\sum_\omega e^{-\frac{n_\omega\omega}{T_H}}\,|n_\omega\rangle\langle n_\omega|, \tag{1.15}$$

where N_ω is a normalization constant.
As briefly explained before, the expression 1.15 consists of a thermal density matrix and it contradicts quantum mechanics since the starting state is a pure one.

Furthermore, one could argue that $[H, {b_\omega}^\dagger] = \omega {b_\omega}^\dagger$ and, since $[H, {b_\omega}^\dagger \tilde{b}_\omega^\dagger] = 0$, the conclusion is that $[H, \tilde{b}_\omega^\dagger] = -\omega \tilde{b}_\omega^\dagger$. The last result seems odd because of the negative energy but it is correct: here, the energy is defined as the conserved charge associated to the Killing vector that looks like the time translation outside the black hole. This Killing vector changes signature at the horizon and thus acts as a momentum for the interior modes, allowing either sign.
Moreover, the equation 1.15 simply indicates that the interior and exterior modes are entangled with each other.
A possible naive interpretation is a pair production process at the horizon, where the particle with negative energy falls into the black hole and the positive energy particle escapes to spatial infinity.

1.3 The three standard alternatives

In addressing the black hole information paradox, three main alternative approaches have been proposed over the past decades. Each approach presents its own advantages and challenges, reflecting the complexity of the issue. Here, we will briefly discuss the key features of each approach, focusing on their main ideas while omitting some of the finer details.

- *Information loss.*
  One significant proposal is the concept of information loss, acknowledged by prominent physicists such as Stephen Hawking, Robert Wald, and Ted Jacobson. This perspective suggests that information that falls into a black hole is irretrievably lost, implying a fundamental reassessment of quantum mechanics as we understand it. The idea of information loss posits that once information crosses the event horizon, it is destroyed and cannot be recovered, leading to a conflict with the principle of unitarity in quantum mechanics. This view is generally considered incompatible with the framework of String Theory and the AdS/CFT correspondence, both of which imply that information should be preserved. Indeed, String Theory and the AdS/CFT correspondence, which provide a duality between a gravity theory in a higher-dimensional space and a conformal field theory in lower dimensions, challenge the feasibility of information loss. Consequently, this perspective is often seen as less satisfactory within the context of contemporary theoretical physics.

- *Remnants.*
  The concept of black hole remnants offers an alternative approach, based on the idea that black hole evaporation does not result in complete information loss. According to this view, black holes might leave behind a remnant after they have evaporated through Hawking radiation. The standard derivation of Hawking radiation, which is rooted in quantum field theory on curved spacetime, breaks down when the event horizon shrinks to the Planck scale during the final stages of evaporation. At this critical point, the current theoretical framework is insufficient, and a theory of quantum gravity becomes necessary to predict the outcome. In this scenario, one might envisage that evaporation ceases and the black hole stabilizes into a remnant, preserving information in some form. However, this proposal faces challenges, notably the requirement for an arbitrarily large number of internal states to account for the information content. Both String Theory and holography struggle to accommodate such a vast collection of states. This difficulty casts doubt on the viability of the remnant hypothesis as a resolution to the information paradox.

- *Corrections.*
  Another approach involves correcting the leading-order computations of Hawking radiation to restore unitarity. Hawking's original derivation, being a leading-order calculation, is expected to be subject to corrections that could potentially address the information loss issue. According to statistical mechanics principles, exponentially suppressed corrections might be sufficient to purify the mixed state of the radiation, thereby preserving information. However, these corrections have not proven effective in fully resolving the paradox. Samir Mathur [4] demonstrated that even with small corrections, the von Neumann entropy of the black hole's final state would not decrease, indicating that the corrections alone cannot purify the final state. Furthermore, in 2013, Almheiri, Marolf, Polchinski, and Sully (AMPS) [5] introduced the concept of a "firewall," a highly energetic radiation zone at the event horizon, which infalling observers would encounter. Their analysis suggested that adherence to general relativity and the nature of the horizon state implies the presence of such a firewall, challenging the notion of a smooth event horizon. This conclusion suggests that some form of horizon-scale structure might be necessary to reconcile Hawking radiation with unitarity. While this notion appears to contradict the "no-hair theorem"—which asserts that black holes are characterized only by their mass, charge, and angular momentum—String Theory offers a potential workaround. By introducing explicit solutions with horizon-scale structures, String Theory provides a framework in which such contradictions can be addressed. The proposal to replace black holes with an ensemble of horizon-scale objects, known as the "fuzzball proposal", will be explored further in Chapter 3 and 4.

The black hole information paradox remains a profound challenge at the intersection of quantum mechanics and general relativity. Its resolution is crucial not only for our understanding of black holes but also for the foundational principles of physics. The exploration of alternative approaches, including the fuzzball proposal within String Theory, highlights the dynamic interplay between theoretical insights and emerging solutions. By offering potential resolutions to the paradox, String Theory and its fuzzball model contribute significantly to the ongoing quest for a unified description of the fundamental nature of black holes and quantum gravity.

Chapter 2

Introduction to String Theory and Conformal Field Theory

2.1 String Theory as a theory of quantum gravity

The main lessons of general relativity teach us that gravity is not merely a conventional force, as in Newtonian mechanics, but rather a profound manifestation of spacetime—a 4-dimensional Lorentzian manifold—that curves in response to any form of energy and matter. The degrees of freedom of the theory are described by the metric, and its dynamics are governed by the Einstein-Hilbert action

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\mathcal{R}. \tag{2.1}$$

From the second part of the last century, many attemps were dedicated to understand how it is possible to quantize the previous action.
Here, the aim is to briefly provide insight into some aspects related to the motivations for a theory of quantum gravity: identifying its main issues, areas where such a theory is crucial (and where it is not), and why String Theory could hold the key to solving these problems.
At first, the relevant coupling in general relativity is the Newton's constant that can be written in terms of the Planck mass m_{P} as $8\pi G_N = \frac{\hbar c}{m_{\mathrm{P}}^2}$, where $m_{\mathrm{P}} \approx 2\times 10^{18}$GeV.
By linearising the metric as $g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{m_{\mathrm{P}}}$, then the action reads as

$$S_{EH} = \int d^4x\left[(\partial h)^2 + \frac{1}{m_{\mathrm{P}}}h(\partial h)^2 + \frac{1}{{m_{\mathrm{P}}}^2}h^2(\partial h)^2 + \dots\right]. \tag{2.2}$$

Hence, there is an infinite bunch of interactions where each term is accompanied by a power of E/m_{P}: therefore, one can treat a quantum gravity as an effective field theory that breaks down when the energy E associated with the process of interest reaches the Planck scale.
It is currently uknown the UV-completion of general relativity at quantum level since the gravitational interactions become strong as the energy involved approaches the Planck scale and a bunch of an infinite number of irrelevant couplings appears: in other words, this theory is non-renormalizable.
One the most prominent solutions is given by String Theory. As explained in the following sections, this theory, which describes the dynamics of a string, naturally incorporates the gravitational interaction. Furthermore, the concept of replacing fundamental objects from points to strings allows for overcoming the problem of renormalizability and constructing a theory that removes UV-divergences. At the beginning of this chapter, it is important to acknowledge that much of the material has been drawn from [6]. In the subsequent sections, I will reference

additional secondary sources that have been utilized to complement and support the discussions and analyses presented.

2.2 The classical string

A particle traces out a worldline in Minkowski as a target spacetime, while a string sweeps out a worldsheet $\mathcal{M}_2$ that can be parametrised by a timelike coordinate τ and a spatial one σ. Here, the worldsheet is embedded into a D-dimensional minkowski spacetime through the embedding $X^\mu(\tau, \sigma)$ that, for closed string, satisfy the periodicity condition $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$. The action that describes the dynamics of a string depends only on its tension T and it is given by

$$S_{NG} = -T \int d^2\sigma \sqrt{-\gamma}. \tag{2.3}$$

It is called Nambu-Goto action and it geometrically represents the area of the worldsheet since γ is the determinant of the metric induced from the target space to $\mathcal{M}_2$.
By varying the action, one can obtain as equations of motion

$$\partial_\alpha \left(\sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\beta X^\mu \right) = 0. \tag{2.4}$$

The same equations are equivalently reproduced by using the Polyakov action

$$S_P = -\frac{T}{2} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}, \tag{2.5}$$

where a new degree of freedom, that is the worldsheet metric $g_{\alpha\beta}$, was introduced in order to get rid of the square root in the action. By using the previous action with the constraint $\delta_{g_{\alpha\beta}} S = 0$, one finds exactly the same equations as before.
In addition to Poincaré and reparameterization invariance, the Polyakov Action, compared with that of Nambu-Goto, presents a new symmetry called Weyl symmetry. It consists of the transformations

$$\begin{aligned} &X^\mu(\tau, \sigma) \to X^\mu(\tau, \sigma), \\ &g_{\alpha\beta}(\tau, \sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\tau, \sigma). \end{aligned} \tag{2.6}$$

Notice that the invariance of the Polyakov action is guaranteed because of the one-dimensional nature of the fundamental object — the string.
One could also simplify the action: indeed, by using the diffeomorphic invariance, the resulting metric is flat up to a conformal factor that can be removed by exploiting Weyl invariance too, finally obtaining $g_{\alpha\beta} = \eta_{\alpha\beta}$.
After the previous gauge choice, called conformal gauge, the expressions 2.5 and 2.4 simply read as

$$\begin{aligned} &S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X \cdot \partial^\alpha X. \\ &\partial_\alpha \partial^\alpha X^\mu = 0, \end{aligned} \tag{2.7}$$

while the constraint equations are given by

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2} \eta_{\alpha\beta} \eta^{\rho\sigma} \partial_\rho X \cdot \partial_\sigma X = 0, \tag{2.8}$$

where the string constant α' is defined by $T = \frac{1}{2\pi\alpha'}$.
The first aim is to solve the theory at the classical level. Achieving this goal is not completely

trivial because of the constraints.
By introducing the lightcone coordinates $\sigma^{\pm} = \tau \pm \sigma$, the equations 2.4 become

$$\partial_+ \partial_- X^\mu = 0. \tag{2.9}$$

Hence, analogously to 1.6, the general solution is given by a superposition, namely

$$X^\mu(\sigma, \tau) = X^\mu_L\left(\sigma^+\right) + X^\mu_R\left(\sigma^-\right). \tag{2.10}$$

The most general periodic solution can be expanded in Fourier modes as

$$\begin{aligned} X^\mu_L\left(\sigma^+\right) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-in\sigma^+}, \\ X^\mu_R\left(\sigma^-\right) &= \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha^\mu_n e^{-in\sigma^-}. \end{aligned} \tag{2.11}$$

where x^μ and p^μ represent the position and momentum of the centre of mass of the string.
The reality condition imposes that

$$\alpha^\mu_n = \left(\alpha^\mu_{-n}\right)^\star, \quad \tilde{\alpha}^\mu_n = \left(\tilde{\alpha}^\mu_{-n}\right)^\star, \tag{2.12}$$

while the constraints are

$$\left(\partial_+ X\right)^2 = \left(\partial_- X\right)^2 = 0, \tag{2.13}$$

and, in Fourier space, they reduce to

$$\begin{aligned} L_n &= \frac{1}{2}\sum_m \alpha_{n-m} \cdot \alpha_m = 0, \\ \tilde{L}_n &= \frac{1}{2}\sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m = 0, \end{aligned} \tag{2.14}$$

with $n \in \mathbf{Z}$ and the zero mode is defined to be $\alpha^\mu_0 = \tilde{\alpha}^\mu_0 \equiv \sqrt{\frac{\alpha'}{2}} p^\mu$.
The zero modes are more relevant since they are directly connected with the square of the momentum p^μ: in other words, with the mass of the string.
In fact, from $L_0 = \tilde{L}_0 = 0$, one can easily find the classical mass spectrum of the string,

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}, \tag{2.15}$$

since $p_\mu p^\mu = -M^2$.
The previous equation is known as level matching, as it involves an equality between right-movers and left-movers oscillators at the same time.

2.3 Open strings and D-Branes

In the closed case the spatial coordinate σ takes values in the compact set $[0, 2\pi]$; in contrast, in the open string case, $\sigma \in [0, \pi]$ and the string dynamics are governed by the same action and equations of motion as before.
The basic differences are related to the boundary conditions; specifically,

$$\begin{aligned} \delta S &= -\frac{1}{2\pi\alpha'} \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \partial_\alpha X \cdot \partial^\alpha \delta X = \\ &= \frac{1}{2\pi\alpha'} \int d^2\sigma \left(\partial^\alpha \partial_\alpha X\right) \cdot \delta X - \frac{1}{2\pi\alpha'} \int d^2\sigma \partial^\alpha \left(\partial_\alpha X \cdot \delta X\right). \end{aligned} \tag{2.16}$$

For the open string, the total derivative picks up the boundary contributions

$$\frac{1}{2\pi\alpha'}\left[\int_0^\pi d\sigma \dot{X}\cdot\delta X\right]_{\tau=\tau_i}^{\tau=\tau_f} - \frac{1}{2\pi\alpha'}\left[\int_{\tau_i}^{\tau_f} d\tau X'\cdot\delta X\right]_{\sigma=0}^{\sigma=\pi}, \tag{2.17}$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial\tau}$ and $X'^\mu = \frac{\partial X^\mu}{\partial\sigma}$.
The first term is always zero by the principle of least action while the second one needs to be put to zero.
This means that the possible boundary conditions are the following:

- Neumann boundary conditions:

$$\partial_\sigma X^\mu = 0 \quad \text{at } \sigma = 0, \pi. \tag{2.18}$$

  Here, the ending points of the string are allowed to move freely and at the speed of light.

- Dirichlet boundary conditions:

$$\delta X^\mu = 0 \quad \text{at } \sigma = 0, \pi. \tag{2.19}$$

  Now, the end points of the string lie at some constant position, $X^\mu = c^\mu$.

By reviewing the calculations previously done in the closed case, one can extract the mass spectrum for the open sector which, apart from a different factor in front, is the same as seen before.
In order to understand the consequences of the two different possibilities for the boundary conditions, let's consider Dirichlet boundary conditions for some coordinates and Neumann boundary conditions for others, such as

$$\begin{aligned} \partial_\sigma X^a = 0 &\quad \text{for } a = 0, \ldots, p, \\ X^I = c^I &\quad \text{for } I = p+1, \ldots, D-1. \end{aligned} \tag{2.20}$$

Therefore, in this situation the end points of the string are fixed to lie in a $(p+1)$-dimensional hypersurface in the target spacetime such that the $SO(1, D-1)$ Lorentz group is broken to $SO(1,p) \times SO(D-p-1)$.
The previous hypersurface is called Dp-brane: D stands for Dirichlet and p is the number of spatial dimensions of the brane.
Such a hypersurface is a new dynamical degree of freedom that appears in String Theory from its open sector.
In order to promote a Dp-brane to be dynamical, one simply can generalize the Nambu-Goto action from 2 to $p+1$-dimensions as

$$S_{Dp} = -T_p \int d^{p+1}\xi\sqrt{-\gamma}, \tag{2.21}$$

where T_p is the tension of the brane and $\{\xi^a\}_{a=0,\ldots,p}$ are the worldvolume coordinates swept out by the p-brane.
This and other related information can also be found in [7].

2.4 The Lightcone Quantization of a string

There are two standard methods to canonically quantize a gauge theory such as string theory.
1) The covariant quantization:

first, quantize the system, and subsequently impose the constraints arising from gauge fixing as operator equations on the physical states of the system.
2) Lightcone quantization:
the alternative method is to first solve all the constraints of the system to determine the space of physically distinct classical solutions, and then to quantize these physical solutions.
Here, the section is dedicated just to the second method since it is more convenient and easier.

2.4.1 Classical degrees of freedom

At first, one has to notice that there is a residual gauge symmetry described by

$$\sigma^+ \to \tilde{\sigma}^+ \left(\sigma^+\right), \quad \sigma^- \to \tilde{\sigma}^- \left(\sigma^-\right), \tag{2.22}$$

that is given by diffeomorphisms that can be undone by a compensating Weyl transformation. In lightcone coordinates on the worldsheet, the flat metric is given by $ds^2 = -d\sigma^+ d\sigma^-$.
A convenient way to fix the residual gauge freedom is to use lightcone coordinates in the target space as well, namely $X^\pm = \sqrt{\frac{1}{2}} \left(X^0 \pm X^{D-1}\right)$.
As always, the solution of the equation of motion for X^+ is the superposition of the left part $X_L^+ \left(\sigma^+\right)$ and of the right part $X_R^+ \left(\sigma^-\right)$. The residual gauge freedom is used to impose

$$X_L^+ = \frac{1}{2} x^+ + \frac{1}{2} \alpha' p^+ \sigma^+, \quad X_R^+ = \frac{1}{2} x^+ + \frac{1}{2} \alpha' p^+ \sigma^-, \tag{2.23}$$

namely,

$$X^+ = x^+ + \alpha' p^+ \tau, \tag{2.24}$$

that is called lightcone gauge.
The function $X^-(\sigma^+, \sigma^-)$ is completely determined in terms of the other fields and one can solve the constraint equation for X^-, going in Fourier space and obtain the mass spectrum by looking at the zero mode,

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i, \tag{2.25}$$

where only the physical and trasversal $D-2$ modes in the oscillators $\{\alpha^i, \tilde{\alpha}^i\}_{i=1,\dots,D-2}$ are taken into account.
In conclusion, the most general classical solution is entirely captured by $2(D-2)$ transverse oscillator modes, together with a number of zero modes describing the center of mass and momentum of the string $x^i, p^i, p^+$and x^-.

2.4.2 Canonical quantization

Now, let's canonically quantize the physical degrees of fredome by imposing

$$\begin{gathered} \left[x^i, p^j\right] = i\delta^{ij}, \quad \left[x^-, p^+\right] = -i, \\ \left[\alpha_n^i, \alpha_m^j\right] = \left[\tilde{\alpha}_n^i, \tilde{\alpha}_m^j\right] = n\delta^{ij}\delta_{n+m,0}. \end{gathered} \tag{2.26}$$

The Hilbert space is determined by defining a vacuum state $|0;p\rangle$ that corresponds to the state with no oscillations and momentum p for the center of mass of the string.
Hence, it satisfies the conditions

$$\hat{p}^\mu |0;p\rangle = p^\mu |0;p\rangle, \quad \alpha_n^i |0;p\rangle = \tilde{\alpha}_n^i |0;p\rangle = 0 \text{ for } n > 0. \tag{2.27}$$

The Fock space is built by acting with the creation operators α^i_{-n} and $\tilde{\alpha}^i_{-n}$, with $n > 0$.
In the operator setting, there is an ambiguity in the summation over oscillator modes on the right-hand side of 2.25: this issue can be addressed by introducing a constant term in the mass spectrum formula, which will be subsequently constrained to a specific value using some consistency conditions. As presented in [8] , the final result is

$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n - a\right). \tag{2.28}$$

After a regularization procedure that consists in an analytic continuation of the sum of all the natural number by using the Riemann Zeta function, the mass spectrum for the closed string results to be

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\tilde{N} - \frac{D-2}{24}\right), \tag{2.29}$$

where $N = \sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n$ and $\tilde{N} = \sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n$.

2.4.3 Why is String Theory a theory of quantum gravity?

The ground state $|0;p\rangle$ represents a tachyon since $N = \tilde{N} = 0$ implies $M^2 = -\frac{1}{\alpha'}\frac{D-2}{6}$ and D must be greater than 2 because it is the dimension of the target space where the two-dimensional worldsheet is embedded.
Even today, tachyons in bosonic String Theory are not fully understood, but they are not relevant to our discussion, since they are projected out by introducing fermionic degrees of freedom in a supersymmetric scenario.
Conversely, the first excited states are the most importart sector since they contain the gravity interaction. In fact, $N = 1$ means that there are $(D-2)^2$ particle states

$$\tilde{\alpha}^i_{-1}\alpha^j_{-1}|0;p\rangle, \tag{2.30}$$

each of which with mass squared

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right). \tag{2.31}$$

These states transform under the vector representation of $SO(D-2)$ but, in order to guarantee the Lorentz invariance to be respected at full quantum level, one has to impose that the previous states need to fit into some representation of the full Lorentz group $SO(1, D-1)$. Recalling the Wigner's classification of representations of the Poincarè group, this is possible only if the previous states are massless, namely if the target spacetime dimension of the bosonic string is set to be

$$D = 26. \tag{2.32}$$

As explained in [9], the proof can also be established by imposing the Lorentz algebra commutation relations on the Lorentz generators. By adding fermions in a supersymmetric framework, the critical dimension becomes $D = 10$ instead of 26.
So, the quantization procedure for a string is able, in order to avoid anomalies in the Lorentz symmetry, to fix the dimension of the target space.
By using $D = 26$, the states 2.30 transform into a $24 \otimes 24$ representation of $SO(24)$ that can be decomposed as follows

$$\text{traceless symmetric} \ \oplus \ \text{anti-symmetric} \ \oplus \ \text{singlet}\ . \tag{2.33}$$

Hence, each of these modes corresponds to a massless field in spacetime, such that the string oscillation can be identified with a quantum of these fields.
The fields are a symmetric tensor $G_{\mu\nu}(X)$, a differential 2-form $B_{\mu\nu}(X)$ and a scalar field $\Phi(X)$ called dilaton.
The particle associated with the symmetric traceless representation of $SO(24)$ is a massless spin 2 particle. Nevertheless, based on general arguments initially proposed by Feynman and Weinberg, any theory involving interacting massless spin-two particles must be equivalent to General Relativity. A discussion of this topic can be found in [10]. Therefore, the tensor $G_{\mu\nu}(X)$ is identified with the spacetime metric.
The previous strategy is the standard way to motivate in which sense String Theory is a theory of quantum gravity. In a supersymmetric scenario, the string spectrum still includes the spacetime metric, the B-field, and the dilaton; consequently, the analysis remains essentially valid.

2.5 Introduction to Conformal Field Theory

The transformations 2.22 are a specific case of what, in general, are called "conformal transformations" and they show that string theory is a conformal field theory on the worldsheet.
This section is dedicated to concisely describing some relevant features of two dimensional Conformal Field Theory at both the classical and quantum levels.
A conformal transformation is a change of coordinates $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ such that the metric changes by a positive function

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma). \tag{2.34}$$

A field theory that is invariant under the previous transformations is called Conformal Field Theory (CFT), namely it is a theory that looks the same at all length scales.

To make it more comfortable, let's have a look at a theory on Euclidean space, where the Euclidean worldsheet coordinates are defined from the Lorentzian ones by a Wick rotation in time $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$. It is easier using complex coordinate: hence, let's introduce z and $\bar{z}$ such that $z = \sigma^1 + i\sigma^2$ and $\bar{z} = \sigma^1 - i\sigma^2$.
These coordinates are completely analogous to the left-moving and right-moving coordinates introduced before in the lightcone quantization of a string.
By using the usual definitions for the holomorphic and antiholomorphic derivatives given by

$$\partial_z \equiv \partial = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} \equiv \bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2), \tag{2.35}$$

the flat Euclidean metric reads as

$$ds^2 = \left(d\sigma^1\right)^2 + \left(d\sigma^2\right)^2 = dz d\bar{z}, \tag{2.36}$$

where the components are $g_{zz} = g_{\bar{z}\bar{z}} = 0$, $g_{z\bar{z}} = \frac{1}{2}$ and the measure factor becomes $dzd\bar{z} = 2d\sigma^1 d\sigma^2$.
In this fashion, the conformal transformations in the 2-dimensional flat Euclidean space are simply given by any holomorphic change of coordinates

$$z \to z' = f(z) \quad \text{and} \quad \bar{z} \to \bar{z}' = \bar{f}(\bar{z}), \tag{2.37}$$

since, under this transformation, the line element transforms as $ds^2 = dzd\bar{z} \to |\frac{df}{dz}|^2 dzd\bar{z}$.
Finally, I would like to emphasize that this introduction to conformal field theories is primarily based on [11] and [12].

2.5.1 Classical theory

Here, the stress-energy tensor is defined to be

$$T_{\alpha\beta} = -\frac{4\pi}{\sqrt{g}}\frac{\partial S}{\partial g^{\alpha\beta}}. \tag{2.38}$$

In any conformal theory the stress-energy tensor is trace-less $T^\alpha{}_\alpha = 0$. Indeed, by taking a rescaling (that is a special case of conformal transformation, just for simplicity) $\delta g_{\alpha\beta} = \epsilon g_{\alpha\beta}$, then

$$\delta S = \int d^2\sigma \frac{\partial S}{\partial g_{\alpha\beta}}\delta g_{\alpha\beta} = -\frac{1}{4\pi}\int d^2\sigma\sqrt{g} \in T^\alpha, \tag{2.39}$$

that must vanish since scale transformations are symmetries of the theory.
The trace-less condition in complex coordinates reads as $T_{z\bar{z}} = 0$.
In addition, the conservation equation in flat space-time $\partial_\alpha T^{\alpha\beta} = 0$ simply becomes

$$\bar{\partial} T_{zz} = 0 \quad \text{and} \quad \partial T_{\bar{z}\bar{z}} = 0. \tag{2.40}$$

In other words, $T_{zz} = T_{zz}(z)$ is a holomorphic function while $T_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}}(\bar{z})$ is an antiholomorphic function.
For instance, let's consider the stress-energy tensor of a free scalar field theory defined as

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X \partial^\alpha X. \tag{2.41}$$

This is the most physically relevant theory for our treatment since the action is essentially the Polyakov one in conformal gauge.
By varying the action with respect to the metric, the stress-energy tensor follows as

$$T = -\frac{1}{\alpha'}\partial X \partial X \quad \text{and} \quad \bar{T} = -\frac{1}{\alpha'}\bar{\partial} X \bar{\partial} X. \tag{2.42}$$

2.5.2 Quantum theory

One of the key ingredients to discuss a conformal field theory at quantum level is the operator product expansion (OPE), where the product of two local operators can be written as a linear combination of all the operators of the theory. More precisely,

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) = \sum_k C^k_{ij}(z-w,\bar{z}-\bar{w})\mathcal{O}_k(w,\bar{w}), \tag{2.43}$$

where by translational invariance the coefficients $C^k_{ij}(z-w,\bar{z}-\bar{w})$ depend only on the separation between the two operators.
To be more specific, the right-hand side of equation 2.43 converges to the left-hand side within a region defined by a radius of convergence. This radius is less than the distance from ω to the nearest insertion point of another operator. Notice that in 2.43 an abuse of notation is used: properly, it is not an operator equality but between time-ordered correlation functions that are neglected in the notation for simplicity.
In other words, any OPE needs to be interpreted as

$$\langle\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w})\rangle = \sum_k C^k_{ij}(z-w,\bar{z}-\bar{w})\langle\mathcal{O}_k(w,\bar{w})\rangle. \tag{2.44}$$

The OPEs have singular behaviour as $z \to w$: the singular terms are the only relevant for the current discussion.

Ward Identity and Primary operators

From the path integral formulation of the correlators

$$\langle \mathcal{O}_1(\sigma_1)\dots\mathcal{O}_n(\sigma_n)\rangle = \frac{1}{Z}\int \mathcal{D}\phi e^{-S[\phi]}\mathcal{O}_1(\sigma_1)\dots\mathcal{O}_n(\sigma_n), \tag{2.45}$$

where ϕ schematically represents all the fields in the theory and the partition function is $Z = \int \mathcal{D}\phi e^{-S[\phi]}$, one can easily derive the generalization at quantum level of the Noether current

$$-\frac{1}{2\pi}\int_\epsilon \partial_\alpha \langle J^\alpha(\sigma)\mathcal{O}_1(\sigma_1)\dots\rangle = \langle \delta\mathcal{O}_1(\sigma_1)\dots\rangle, \tag{2.46}$$

known as Ward identity, where the integral on the left-hand-side is performed over the region of non-zero ϵ parameter.
In any two dimensional quantum field theory, the previous result can be recast as

$$\frac{i}{2\pi}\oint_{\partial\epsilon} dz \langle J_z(z,\bar{z})\mathcal{O}_1(\sigma_1)\dots\rangle - \frac{i}{2\pi}\oint_{\partial\epsilon} d\bar{z} \langle J_{\bar{z}}(z,\bar{z})\mathcal{O}_1(\sigma_1)\dots\rangle = \langle \delta\mathcal{O}_1(\sigma_1)\dots\rangle, \tag{2.47}$$

where the Stokes' theorem has been used.
Let us focus on the shift transformations in z or in $\bar{z}$: indeed, since J_z is holomorphic and $J_{\bar{z}}$ is antiholomorphic, the contour integral is described by a residue.
More precisely, from the change $\delta z = \epsilon(z)$,

$$\delta\mathcal{O}_1(\sigma_1) = -\operatorname{Res}\left[J_z(z)\mathcal{O}_1(\sigma_1)\right] = -\operatorname{Res}\left[\epsilon(z)T(z)\mathcal{O}_1(\sigma_1)\right], \tag{2.48}$$

while from the shift $\delta\bar{z} = \bar{\epsilon}(\bar{z})$,

$$\delta\mathcal{O}_1(\sigma_1) = -\operatorname{Res}\left[\bar{J}_{\bar{z}}(\bar{z})\mathcal{O}_1(\sigma_1)\right] = -\operatorname{Res}\left[\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})\mathcal{O}_1(\sigma_1)\right]. \tag{2.49}$$

Hence, the previous analysis leads to the conclusion that the OPE involving an operator and the components of the stress-energy tensor, $T(z)$ and $\bar{T}(\bar{z})$, reveals how the operator transforms under conformal transformations.

Moreover, to understand the spectrum of a two dimensional CFT, one can introduce a relevant class of operators, known as primary operators.
At first, an operator $\mathcal{O}$ is said to have weights $(h,\tilde{h})$ if, under $\delta z = \epsilon z$ and $\delta\bar{z} = \bar{\epsilon}\bar{z}$, $\mathcal{O}$ transforms as

$$\delta\mathcal{O} = -\epsilon(h\mathcal{O} + z\partial\mathcal{O}) - \bar{\epsilon}(\tilde{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}), \tag{2.50}$$

where $s = h - \tilde{h}$ is the spin and $\Delta = h + \tilde{h}$ is the scaling dimension of the operator.
One can show that for an operator with weights $(h,\tilde{h})$, the OPEs with T and $\bar{T}$ are

$$\begin{aligned} T(z)\mathcal{O}(w,\bar{w}) &= \dots + h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \dots, \\ \bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) &= \dots + \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \dots. \end{aligned} \tag{2.51}$$

Now, it was introduced enough technology to define what a primary operator is.
An operator $\mathcal{O}$ with weights $(h,\tilde{h})$ is said to be primary if its OPEs with the energy-momentum tensor look like

$$\begin{aligned} T(z)\mathcal{O}(w,\bar{w}) &= h\frac{\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \text{ non-singular terms }, \\ \bar{T}(\bar{z})\mathcal{O}(w,\bar{w}) &= \tilde{h}\frac{\mathcal{O}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \text{ non-singular terms}. \end{aligned} \tag{2.52}$$

In other words, the singular terms with the maximum power are $(z-w)^{-2}$ and $(\bar{z}-\bar{w})^{-2}$.
It is a quite important class of operators and they have a simple transformation rule,

$$\mathcal{O}(z,\bar{z}) \to \tilde{\mathcal{O}}(\tilde{z},\bar{\tilde{z}}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-h} \left(\frac{\partial \bar{\tilde{z}}}{\partial \bar{z}}\right)^{-\tilde{h}} \mathcal{O}(z,\bar{z}) \tag{2.53}$$

under

$$z \to \tilde{z}(z) \quad \text{and} \quad \bar{z} \to \bar{\tilde{z}}(\bar{z}). \tag{2.54}$$

Because it is the more influential for string theory, let's revisite some of the features discussed above for the free scalar field described by the action 2.41 and with 2.42 to be the stress-energy tensor of the theory.
At first, let's have a look at the propagator, namely

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X(\sigma)} \left[e^{-S} X(\sigma')\right] = \int \mathcal{D}X e^{-S} \left[\frac{1}{2\pi\alpha'} \partial^2 X(\sigma) X(\sigma') + \delta(\sigma - \sigma')\right]. \tag{2.55}$$

From which, one can extract,

$$\left\langle \partial^2 X(\sigma) X(\sigma') \right\rangle = -2\pi\alpha' \delta(\sigma - \sigma'). \tag{2.56}$$

Then, after some computations involving the Stokes' theorem and by using the polar coordinates,

$$\langle X(\sigma) X(\sigma') \rangle = -\frac{\alpha'}{2} \ln (\sigma - \sigma')^2 \tag{2.57}$$

that is the final expression for the propagator.
Moreover, one can analogously derive some other OPEs starting from the propagator: for instance, some correlators involving derivative terms such as

$$\partial X(z) \partial X(w) = -\frac{\alpha'}{2} \frac{1}{(z-w)^2} + \text{ non-singular terms.} \tag{2.58}$$

In order to compute the OPE with the stress-energy tensor T, it has to be taken normal ordered as always in any QFT. For instance, the correlator

$$T(z)\partial X(w) = -\frac{1}{\alpha'} : \partial X(z) \partial X(z) : \partial X(w) \tag{2.59}$$

simply becomes

$$T(z)\partial X(w) = \frac{\partial X(z)}{(z-w)^2} + \ldots = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} + \ldots, \tag{2.60}$$

by using the Wick's theorem.
Another example that will appear in section 2.6 is $: e^{ikX} :$ which is a primary operator with weights $h = \tilde{h} = \alpha' k^2/4$. The previous operator can also be involved in the derivation of the OPE between the stress energy-tensor components that reads as

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots. \tag{2.61}$$

In conclusion, the stress-energy tensor is not a primary operator in a conformal field theory.

The previous result can be generalized and is valid for any CFT, being essential for defining a crucial quantity known as the central charge.
The central charge is defined as the coefficient (up to a factor of 2) of the term proportional to $(z-w)^{-4}$ in the OPE with the stress-energy tensor.
Hence, as shown in 2.61, in a CFT describing a free scalar field, the central charge is $c = 1$.

The Virasoro Algebra and the State-Operator Map

After having completed the discussion about the operators in a CFT, it remains to discuss what the states of the theory are.
In order to describe the worldsheet theory, that for closed string is topologically a cylinder, by using the tools of the previously introduced conformal field theory on the 2-dimensional plane, one can construct the following map

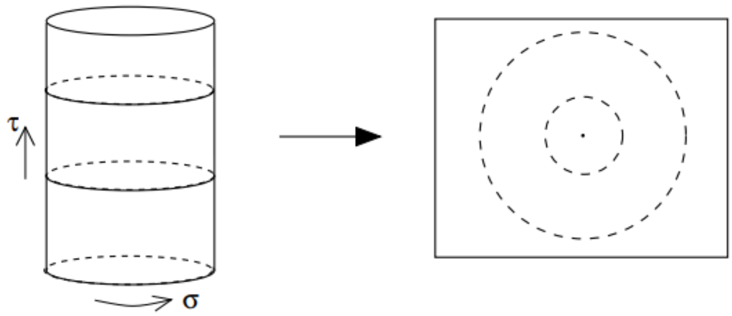

Figure 2.1: The map between the cylinder and the plane

where the complex coordinate ω of the cylinder is related to the plane coordinate z by

$$\omega = \sigma + i\tau, \quad z = e^{-i\omega}. \tag{2.62}$$

On the cylinder, states exist on spatial slices of constant σ and evolve according to the Hamiltonian $H = \partial_\tau$. After mapping to the plane, the Hamiltonian transforms into the dilatation operator $D = z\partial + \bar{z}\bar{\partial}$. To maintain the cylindrical origins of the states on the plane, they should reside on circles with constant radius. Their evolution is dictated by the dilatation operator D. Hence, the time-ordered correlator functions are replaced by some sort of radial-ordered correalator functions by using the previous map (see Figure 2.1).
At first, let's expand the stress-energy tensor components in Laurent series as

$$\begin{aligned} T(z) &= \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}, \\ \bar{T}(\bar{z}) &= \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}, \end{aligned} \tag{2.63}$$

so that, the inverse relations are

$$\begin{aligned} L_n &= \frac{1}{2\pi i} \oint dz z^{n+1} T(z), \\ \tilde{L}_n &= \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}). \end{aligned} \tag{2.64}$$

In radial quantization, L_n is the conserved charge associated to the conformal transformation $\delta z = z^{n+1}$ and, similarly, $\tilde{L}_n$ is the conserved charge associated to $\delta\bar{z} = \bar{z}^{n+1}$.
Now, let's compute the algebra of the generators

$$[L_m, L_n] = \left(\oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \right) z^{m+1} w^{n+1} T(z) T(w), \tag{2.65}$$

where here, z and w are both coordinates on the complex plane and, implicitly, the previous equation contains a radial-ordering.

Considering this, the outcome involves the OPE between the components of the stress-energy tensor, which can be computed as previously described to yield the final result:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0}, \tag{2.66}$$

which is known as the Virasoro algebra. Here, c represents the central charge introduced earlier. Similarly, the operators $\tilde{L}_m$ satisfy the same algebra, but with c replaced by $\tilde{c}$. Instead, the commutation relations between L_m and $\tilde{L}_n$ are trivial, specifically

$$\left[L_m, \tilde{L}_n\right] = 0. \tag{2.67}$$

From this perspective, one can intuitively introduce the State-Operator Map, a distinctive feature of two dimensional conformal field theories that sets it apart from ordinary quantum field theories. In conventional quantum field theories, there typically exists no direct correspondence between states in the Hilbert space and operators within the theory. However, this distinction is fundamentally different in the context of conformal field theories.
Let's consider, for simplicity, a conformal field theory living on a Riemann sphere with three punctures, where each puncture represents the insertion of a local operator.

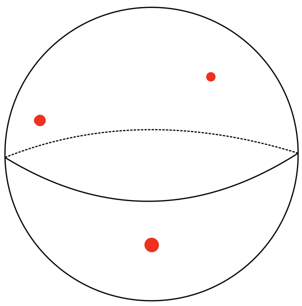

Figure 2.2: The Riemann sphere with three punctures

By leveraging the conformal invariance, it becomes possible to elongate each puncture into a thin, elongated tube. One can further contemplate the scenario where each tube extends infinitely, thereby transforming each puncture (or local operator) into an asymptotic state of the theory.

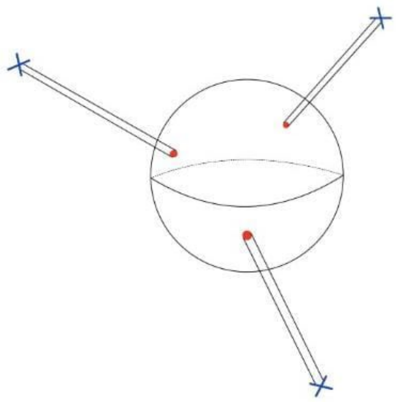

Figure 2.3: The State-Operator Map

In the illustration above, the punctures are represented as red points connected by thin black tubes to the states, which are depicted as crosses in the image.
Obviously, one can reverse the procedure by starting with a state and obtaining a local operator represented as a puncture.

2.6 The BRST quantization

The lightcone quantization of a string can provide all the main results, but in a not rigorous way.
Here, it was discussed a more solid method through which adresses all the question related to string quantization by making use of the results derived in the previous section related to conformal field theory.
Let's consider the Euclidean version of the Polyakov action that is given by

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \delta_{\mu\nu}. \tag{2.68}$$

Instead of the canonical quantization procedure, here the path-integral is involved: the partition function

$$Z = \frac{1}{\text{Vol}} \int \mathcal{D}g \mathcal{D}X e^{-S_{\text{Poly}}\,[X,g]} \tag{2.69}$$

is obtained by integrating over all the possible embedding coordinates and metric configurations. As for any gauge theory, it is not relevant the integration over all the space of gauge fields but just on the space of inequivalent gauge fields, that is the set of gauge fields that are not linked to each others via a gauge transformation (here, diffeomorphisms and Weyl symmetries): for this reason, the necessity to divide by Vol, which is the volume of the gauge action in the field space.
A convenient way of doing so, consists in using the Fadeev-Popov procedure.
At first, in this case the metric changes under a general gauge transformation ζ as

$$g_{\alpha\beta}(\sigma) \longrightarrow g^{\zeta}_{\alpha\beta}(\sigma') = e^{2\omega(\sigma)} \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha} \frac{\partial\sigma^\delta}{\partial\sigma'^\beta} g_{\gamma\delta}(\sigma), \tag{2.70}$$

and, as explained in section 2.2, in two dimensions these gauge symmetries are enough to fix the metric in a specific form $g^{\zeta}_{\alpha\beta}(\sigma')$ called fiducial metric, that for example can be the flat one. After having choosen the fiducial metric, one can compute the partition function related to this choice as

$$\begin{aligned} Z[\hat{g}] &= \frac{1}{\text{Vol}} \int \mathcal{D}\zeta \mathcal{D}X \mathcal{D}g \Delta_{FP}(g) \delta\left(g - \hat{g}^\zeta\right) e^{-S_{\text{P}}\,[X,g]} = \\ &= \frac{1}{\text{Vol}} \int \mathcal{D}\zeta \mathcal{D}X \Delta_{FP}(\hat{g}^\zeta) e^{-S_{\text{P}}[X,\hat{g}^\zeta]} = \frac{1}{\text{Vol}} \int \mathcal{D}\zeta \mathcal{D}X \Delta_{FP}(\hat{g}) e^{-S_{\text{P}}\,[X,\hat{g}]}. \end{aligned} \tag{2.71}$$

where the definition of the Fadeev-Popov determinant $\Delta_{FP}(g)$, that is $1 = \Delta_{FP}(g) \int \mathcal{D}\zeta \delta\left(g - \hat{g}^\zeta\right)$, has been used.
Notice that in the second step, the integration over the metric disappeared because of the Dirac delta function while in the third one the gauge invariance is involved.
Hence, in the integral nothing depends on the gauge transformation ζ: so, the gauge volume factor can be extracted in the numerator that cancels the one present in the denominator,

$$Z[\hat{g}] = \int \mathcal{D}X \Delta_{FP}(\hat{g}) e^{-S_{\text{Poly}}\,[X,\hat{g}]}, \tag{2.72}$$

resulting the final expression of the integral over all the distinct physical solutions.
All the problems converges in the computation of the Fadeev-Popov determinant that is a standard result in gauge theories,

$$\Delta_{FP}^{-1}(\hat{g}) = \int \mathcal{D}v\mathcal{D}\beta \exp\left(4\pi i \int d^2\sigma\sqrt{\hat{g}}\beta^{\alpha\beta}\nabla_\alpha v_\beta\right). \tag{2.73}$$

The steps to do in order to check the previous outcome consists in concentrating around the identity of the gauge group such that the integration over ζ reduces to an integrataion over $\omega(\sigma)$ and $v^\alpha(\sigma)$ that are elements of the Lie algebra of the Weyl group and of the diffeomorphism group respectively.
Then, one has to deal with an integral of a Dirac delta that can be written as an integral of an exponential by introducing an ausiliary field $\beta^{\alpha\beta}$ that is a symmetric 2-tensor on the worldsheet. Then, by treating $\beta^{\alpha\beta}$ as a Lagrange multiplier, one can integrate over $\omega(\sigma)$ and find the result written above.

Introduction of ghosts

The further step consists in computing the inverse of the expression 2.73 to obtain the Fadeev-Popov determinant.
A canonical way is to replace the commuting integration variables with anti-commuting fields as

$$\begin{aligned} \beta_{\alpha\beta} &\longrightarrow b_{\alpha\beta}, \\ v^\alpha &\longrightarrow c^\alpha, \end{aligned} \tag{2.74}$$

where b, c are called ghosts fields since they are Grassman-valued functions.
The final upshot is

$$\Delta_{FP}[g] = \int \mathcal{D}b\mathcal{D}c \exp\left(iS_{\text{ghost}}\right), \tag{2.75}$$

where the ghost action is defined to be $S_{\text{ghost}} = \frac{1}{2\pi}\int d^2\sigma\sqrt{g}b_{\alpha\beta}\nabla^\alpha c^\beta$.
In conclusion, the partition function reads as

$$Z[\hat{g}] = \int \mathcal{D}X\mathcal{D}b\mathcal{D}c \exp\left(-S_{\text{P}}[X,\hat{g}] - S_{\text{ghost}}[b,c,\hat{g}]\right). \tag{2.76}$$

Furthermore, the ghost action can be simplified by employing the conformal gauge, where the flat metric is taken as the fiducial metric. This results in

$$S_{\text{ghost}} = \frac{1}{2\pi}\int d^2z(b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}). \tag{2.77}$$

A more rigourous proof of D=26

The stress-energy tensor of the ghost CFT is

$$T = 2(\partial c)b + c\partial b, \quad \bar{T} = 2(\bar{\partial}\bar{c})\bar{b} + \bar{c}\bar{\partial}\bar{b}. \tag{2.78}$$

Similarly to what was done in the section 2.5.2 for the bosonic scalar free field theory, one can compute some relevant OPEs.
For instance, by computing the OPE with the stress-energy tensor, one can conclude that the ghosts b and c are primary fields with weight $h = 2$ and $h = -1$ respectively.
Instead, the central charge of the ghosts theory is $c = -26$, since

$$T(z)T(w) = \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots. \tag{2.79}$$

A well-known fact is that the Weyl gauge symmetry is anomolous unless $c = 0$ because at quantum level the trace of the stress-energy tensor is given by

$$\langle T^\alpha{}_\alpha \rangle = -\frac{c}{12} R. \tag{2.80}$$

Hence, one needs to add exactly the right degrees of freedom to the string to cancel the contribution from the ghosts.
The simplest way consists of adding D free scalar fields: indeed, the expression 2.61 shows that each of these contributes $c = 1$ to the total central charge, so the entire procedure is consistent if $D = 26$.

Vertex Operators and string spectrum

At first, let's define a special class of states, known as primary states.
A state $|\Psi\rangle$ is said to be a primary state of weights $(a, \tilde{a})$ if

$$\begin{aligned} L_n \, |\Psi\rangle &= 0 \quad \text{ for } n > 0, \\ L_0 \, |\Psi\rangle &= a|\Psi\rangle, \\ \tilde{L}_n \, |\Psi\rangle &= 0 \quad \text{ for } n > 0, \\ \tilde{L}_0 \, |\Psi\rangle &= \tilde{a}|\Psi\rangle, \end{aligned} \tag{2.81}$$

where L_n and $\tilde{L}_n$ are the Virasoro generators defined in 2.64.
The question is: what are the values of a and $\tilde{a}$? By using the State-Operator map, each state is associated to a local operator but only those that are gauge invariant are acceptable: hence, not all the states are physical.
Let's begin by examining reparameterization invariance. In the previous section, operators have been positioned at specific locations on the worldsheet. However, in a theory where the metric is a dynamical degree of freedom, this approach does not produce a diffeomorphism invariant operator. To create an object that remains invariant under reparameterizations of the worldsheet coordinates, one needs to integrate over the entire worldsheet. Consequently, the operator insertions (in conformal gauge) take the following form:

$$V \sim \int d^2 z \mathcal{O}. \tag{2.82}$$

Now, let's consider also the Weyl invariance: the measure d^2z has weight $(-1, -1)$ under rescaling and, in order to to compensate it, the operator must have weight $(1, 1)$.
So that, the previous constants are fixed as $a = \tilde{a} = 1$ and this kind of operators are called Vertex operators.
As an application, let's briefly discuss the closed string spectrum by using the formalism introduced so far.
In general, the vacuum of a two dimensional CFT is linked to the identity operator but the string vacuum has also a centre of mass momentum p^μ that can be generated by acting through the operator $e^{ip\dot{X}}$. Therefore, the Vertex operator associated to the tachyon is

$$V_0 \sim \int d^2 z : e^{ip\cdot X} : . \tag{2.83}$$

Note that in Section 2.5.2, it was stated that the exponential operator is primary with weights $h = \tilde{h} = \frac{\alpha' p^2}{4}$. However, since Weyl invariance requires $h = \tilde{h} = 1$, the ground state must have the well-known mass squared

$$M^2 \equiv -p^2 = -\frac{4}{\alpha'}. \tag{2.84}$$

An analogous procedure can be applied to the excited states, confirming that the first level consists of massless particles as expected.
For a more detailed discussion, I recommend referring to [13] and [14] where I drew part of the material for this section.

2.7 Low-Energy Effective action

Up to now, a flat target space is taken into account: in this section, a more general background describing where the strings propagate is provided. The Polyakov action is simply

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X), \tag{2.85}$$

where $g_{\alpha\beta}$ is the worldsheet metric, while $G_{\mu\nu}(X)$ is the target metric constructed from the quantized gravitons discussed in Section 2.4.3.
In the same section, other fields appeared as the 2-form $B_{\mu\nu}(X)$ and the dilaton $\Phi(X)$ and one needs to understand how a string can couple to them.
In particular, the Vertex operator related to the first excited states of the closed string is

$$V_{\text{excited}} \sim \int d^2z : e^{ip\cdot X} \partial X^\mu \bar{\partial} X^\nu : \zeta_{\mu\nu}, \tag{2.86}$$

where $\zeta_{\mu\nu}$ is the polarization tensor of the state, that is traceless symmetric for the graviton, antysimmetric for the B-field and the trace of $\zeta_{\mu\nu}$ describes the dilaton.
Hence, by exponentiating the operator, one can uniquely determine the interaction terms, resulting in the following action

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \left(G_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu g^{\alpha\beta} + iB_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta} + \alpha' \Phi(X) R^{(2)} \right), \tag{2.87}$$

where $R^{(2)}$ is the worldsheet Ricci scalar. As explained in [15], the specific form of the term involving the dilaton depends on the fact that the dilaton is related to the string coupling as $g_s = e^{\Phi_0}$, where $\Phi_0 = \text{limit}_{X\to\infty} \Phi(X)$. Instead, one can understand the physical interpretation of the B-term by using an analogy with the electromagnetism. Indeed, an eletrically charged point-particle couples to a background gauge 1-form A_μ on its worldline as

$$\int d\tau A_\mu(X) \dot{X}^\mu. \tag{2.88}$$

The analogous coupling on the worldsheet involving a background 2-form is

$$\int d^2\sigma B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \epsilon^{\alpha\beta}, \tag{2.89}$$

where the derivatives appear since the 2-form is pull-backed on the worldsheet from the target space.
Moreover, the analogy can be carried forward by considering the gauge transformation.
In the case of an electrically charged point-particle it is $A_\mu \to A_\mu + \partial_\mu \alpha$, that is generalizable as

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu. \tag{2.90}$$

In addition, the field strength in electromagnetism is a 2-form $F = dA$, while here a 3-form H has to be defined such that $H = dB$.

Another aspect, which is only briefly mentioned here, is the derivation of the Einstein equations in the low-energy limit.
At first, let's consider just the part of the action 2.87 related to the metric. Certainly, one has to demand that at full quantum level the conformal invariance of the theory is preserved, that means

$$\beta_{\mu\nu}(G) \sim \mu \frac{\partial G_{\mu\nu}(X;\mu)}{\partial \mu} = 0, \tag{2.91}$$

with μ an arbitrary energy scale.
At 1-loop level, some UV divergences appear, which can be cured by adding counterterms. By doing so, the wave function renormalization of the embedding coordinates is accompanied by the renormalization involving the metric $G_{\mu\nu}$, such that

$$G_{\mu\nu} \to G_{\mu\nu} + \frac{\alpha'}{\epsilon}\mathcal{R}_{\mu\nu}. \tag{2.92}$$

Hence, the condition for the conformal invariance simply reads as

$$\beta_{\mu\nu}(G) = \alpha'\mathcal{R}_{\mu\nu} = 0, \tag{2.93}$$

namely the target space needs to be Ricci flat.
In other words, General Relativity is recovered from String Theory by imposing its conformal invariance at the quantum level.
In general, one needs to take into account the full bosonic action 2.87 and a generalization is expected because of the presence of new fields. Specifically, the results are given by

$$\begin{aligned}
\beta_{\mu\nu}(G) &= \alpha'\mathcal{R}_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\lambda\kappa}H_\nu^{\lambda\kappa},\\
\beta_{\mu\nu}(B) &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\Phi H_{\lambda\mu\nu},\\
\beta(\Phi) &= -\frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\lambda}H^{\mu\nu\lambda}.
\end{aligned} \tag{2.94}$$

The preservation of conformal invariance means that $\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0$, which can be seen as the equation of motion for the background in which the strings propagate.

One can also focus on the low-energy effective theory of the bosonic sector from the string theory, obtaining

$$S_{EFT} = \frac{1}{2\kappa_0^2}\int d^{26}X\sqrt{-G}e^{-2\Phi}\left(\mathcal{R} - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\partial_\mu\Phi\partial^\mu\Phi\right), \tag{2.95}$$

with k_0 a normalization constant. Indeed, the corresponding equations of motion at leading order are given exactly by 2.94.

2.8 A gentle introduction to Superstring Theory

In this section, only a glimpse of how adding fermionic degrees of freedom on the worldsheet is provided to the reader.
Bosonic string theory encounters notable challenges, including the presence of a tachyon in the spectrum and the lack of fermionic degrees of freedom. These issues lead to instability and incomplete descriptions of fundamental particles. Superstring theory addresses these problems by incorporating fermions and supersymmetry. The addition of fermions helps to project out the tachyonic state, resolving one of the primary issues of bosonic string theory.

Moreover, while bosonic string theory lacks protection against quantum anomalies, supersymmetry ensures anomaly cancellation. This is a crucial aspect that makes superstring theory consistent and well-defined.
Another significant advancement with superstrings is the introduction of dualities such as T-duality and S-duality. These dualities reveal profound relationships between seemingly distinct theories and help in understanding the theory across different coupling regimes.
In the bosonic string theory the mass-shell condition $M^2 = -p^2$ derives from the zero mode constraint $L_0 |\Psi\rangle = 0$ and also $\tilde{L_0} |\Psi\rangle = 0$ for the closed string.
Since the previous mass-shell equation is the Klein-Gordon equation in momentum space, in order to have some fermions in the spectrum the Dirac equation

$$ip_\mu \gamma^\mu + m = 0 \tag{2.96}$$

is needed.
Since L_0 and $\tilde{L_0}$ are the center of mass modes of the bosonic worldsheet energy-momentum tensor $(T_B, \bar{T}_B)$, one can introduce some new fermionic energy-momentum tensor $(T_F, \bar{T}_F)$ such that the corresponding zero modes give the Dirac equation.
One can guess that the gamma matrices γ_μ satisfying the algebra $\{\gamma^\mu, \gamma^v\} = 2\eta^{\mu\nu}$ are the center of mass modes of an anticommuting worldsheet field ψ^μ. In this perspective, one could consider the following action as the generalization of the Polyakov one,

$$S = \frac{1}{4\pi} \int d^2z \left(\frac{2}{\alpha'} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right). \tag{2.97}$$

From here, one can conduct similar analysis to what was previously done. One can compute the spectrum and by using the supersymmetric framework, the critical dimension is found to be

$$D = 10. \tag{2.98}$$

The massless bosonic fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ are always part of the superstring spectrum.
Additionally, there exist massless spacetime fermions and other massless bosonic fields. The specific nature of these additional bosonic fields varies depending on the particular superstring theory under consideration.
Indeed, with respect to bosonic string theory, the supestring one is not unique.
The main different theories are

- Type II: it possess both left and right moving fermions. This yields a spacetime theory in 10 dimensions with $\mathcal{N} = 2$ supersymmetry, indicating the presence of 32 supercharges. Here, the additional degrees of freedom of the string are known as Ramond-Ramond fields.
  - Type IIA: here, the new degrees of freedom include some gauge fields as a 1-form C_μ and a 3-form $C_{\mu\nu\rho}$.
  - Type IIB: the Ramond-Ramond gauge fields include a scalar C, a 2-form $C_{\mu\nu}$ and a 4-form $C_{\mu\nu\rho\sigma}$.
- Heterotic: it features only right-moving fermions on the worldsheet. Each heterotic string includes a non-Abelian gauge field in spacetime, with gauge groups $SO(32)$ and $E_8 \times E_8$.
- Type I: it encompasses open strings propagating in a flat ten-dimensional space, along with closed strings.

These diverse theories offer a broader and richer framework compared to bosonic string theory and have led to significant advancements such as the concept of branes and the formulation of M-theory or F-theory.

M-theory represents an even more comprehensive framework by proposing an eleven-dimensional universe where various superstring theories are unified as different aspects of a single underlying theory. It introduces higher-dimensional objects, such as two-branes and five-branes, expanding our understanding of the fundamental structure of spacetime.

F-theory, on the other hand, extends superstring theory by incorporating additional dimensions and allowing for more generalized solutions. It is particularly adept at describing gauge theories and various compactification scenarios, providing deeper insights into the geometric and physical properties of higher-dimensional spaces.

In addition to these advancements, two notable concepts in the realm of superstring theory are compactifications and the AdS/CFT correspondence.

Compactification involves reducing the higher-dimensional space of superstring theory to the four-dimensional spacetime we observe by factoring out the internal degrees of freedom of the theory into certain manifolds. These geometrical structures are essential for preserving supersymmetry in the lower-dimensional effective theory and play a crucial role in determining the physical properties of the observed universe.

The AdS/CFT correspondence is another pivotal development, positing a duality between a superstring theory or supergravity in Anti-de Sitter space and a conformal field theory on the boundary of that space. This correspondence has provided profound insights into quantum gravity, gauge theories, and the dynamics of strongly-coupled systems, showcasing the deep connections between different theoretical frameworks. The content and analyses introduced previously are based on [16].

Chapter 3

Black holes in Supergravity

In the 1970s, a remarkable parallel was discovered between black hole dynamics and thermodynamics, with Bekenstein-Hawking entropy [17], which is one-quarter of the event horizon's area, behaving like thermodynamic entropy. The challenge has been to provide a precise statistical mechanical interpretation of this entropy. A famous paper written by Strominger and Vafa [2] advanced this effort by examining black holes within five-dimensional string theory that has $\mathcal{N} = 4$ supersymmetry. They showed that the degeneracy of extremal BPS states, which can be computed in these theories, agrees with the Bekenstein-Hawking entropy for large charges. In this chapter, general results will be provided on how to construct black hole solutions from string theory and how to calculate their microstates. The methods for obtaining black hole solutions using D-branes wrapped on compact spaces are discussed. Initially, the procedure for creating an extremal and supersymmetric black hole is demonstrated. Subsequently, the focus shifts to non-extremal solutions for completeness. Furthermore, the chapter includes an introduction to special solutions known as fuzzballs, which have no horizon and represent a single microstate solution. These will also be discussed in Chapter 4, where some Wess-Zumino-Witten models capable of describing fuzzballs will be considered.
Specifically, in order to study black holes in string theory, Susskind proposed considering a highly excited string state with a mass $M \gg \alpha'^{-1/2}$. If the string coupling g is assumed to be small, meaning the string behaves almost like a free one, the left and right oscillator levels are $N_L, N_R \sim \sqrt{\alpha' M} \gg 1$, resulting in a large number of degenerate states at fixed mass. Furthermore, using the statistical mechanics framework, the degeneracy translates to an entropy of

$$S_{\text{micro}} = \ln[N] \sim \sqrt{\alpha' M}.$$

However, to describe a black hole, one needs to consider a regime where g is no longer perturbative, since the Newton constant is $G \sim g^2$.

In particular, it was expected that if M is sufficiently large, then we get a black hole of mass M with its entropy given by

$$S_{\text{BH}} = \frac{A}{4G},$$

which is the standard Bekenstein-Hawking formula.

The microscopic computation of the entropy, namely S_{micro}, is not trivial at large coupling constant g_s, and the matching with the Bekenstein-Hawking entropy (which is expected to be valid) is not easy to prove.

In this scenario, the importance of BPS states emerges. In this context, the microstate counting—and therefore the entropy—does not depend on the coupling, as it is protected by supersymmetry. The microstate counting S_{micro} can be carried out in string perturbation theory at small coupling, and although the number of microstates would generally change for non-BPS states as the coupling increases, this does not happen for BPS states. Thus, for BPS states, one can compare the microstate counting done in string theory at small coupling with the entropy computed using the Bekenstein-Hawking formula.

To provide a more detailed explanation, in the next paragraphs some standard BPS solutions will be considered, which include the one-charge, two-charge, and three-charge cases. For the three-charge case, a computation of the microscopic entropy will be explained and the matching

$$S_{\text{micro}} = S_{\text{BH}}$$

will be verified.
It is essential to mention that a significant portion of this chapter's content is based on [18] and [19].

3.1 One-charge solution

The easiest BPS state is certaintly the one-charge solution.
Let's begin by recalling that the Type IIA string theory can be obtained as a one dimensional reduction of the eleven dimensional M-theory. In particular, let's compactify on a circle S^1 with radius R parametrized by a coordinate y with a periodicity of $2\pi R$. Let's suppose to wrap n_1 times the circle with a $NS1$ brane.
Such a string configuration produces the following supergravity solution,

$$\begin{aligned}
&ds^2 = Z_1^{-1}\left[-dt^2 + dy^2\right] + \sum_{i=1}^{8} dx_i dx_i,\\
&e^{2\phi} = Z_1^{-1},\\
&Z_1 = 1 + \frac{Q_1}{r^6},
\end{aligned} \tag{3.1}$$

where the metric refers to the 10-dimensional one with x_i as transverse coordinates. The Bekenstein-Hawking entropy is trivially zero because the area corresponding to the horizon $r = 0$ vanishes. The fact that $S_{\text{BH}} = 0$ is actually in line with the microscopic counting. The $NS1$ brane is in its ground state, so its only source of degeneracy comes from the zero modes of the string, providing 128 bosonic and 128 fermionic states. Therefore, $S_{\text{micro}} = \ln[256]$, which remains constant and does not increase with n_1. Consequently, in the macroscopic limit as $n_1 \to \infty$, we would find $S_{\text{micro}} = 0$ to leading order, which matches S_{BH}. The same result can be computed in the full 11-dimensional M-theory by treating the $NS1$ of the Type IIA theory as an emergent $M2$ brane in M-theory that wraps the directions x_{11}, y.
In this perspective, the vanishing of the lengths of x_{11} and y leads to the vanishing of the area of the horizon at $r = 0$, confirming the Type IIA result.

3.2 Two-charge solution

The previous problem of the shrinking of the coordinate x_{11} can be overcome by using $M5$ branes oriented transverse to x_{11}: namely, $NS5$ branes in Type IIA theory. This means that additional compact dimensions are required to fully wrap around the five spatial directions of the $NS5$ branes: for instance, let's consider $T^4 \times S^1$.

The supergravity solutions of the previous configuration are the following ones,

$$
\begin{aligned}
ds^2 &= Z_1^{-1}\left[-dt^2+dy^2\right]+Z_5\sum_{i=1}^{4}dx_idx_i+\sum_{a=1}^{4}dz_adz_a,\\
e^{2\phi} &= \frac{Z_5}{Z_1},\\
Z_1 &= 1+\frac{Q_1}{r^2},\quad Z_5=1+\frac{Q_5}{r^2}.
\end{aligned}
\tag{3.2}
$$

The coordinates $\{z^a\}_{a=\{1,\dots,4\}}$ parametrizes the T^4, while the charge Q_5 is proportional to the number n_5 of $NS5$ branes.
However, the horizon area remains zero. This is evident from the M-theory perspective, where both the $NS1$ (obtained from $M2$) and $NS5$ (from $M5$) branes wrap around the y circle, causing it to shrink to zero as $r\to 0$.

3.3 Three-charge solution

As discussed previously, both the one-charge and two-charge BPS models have zero horizon area: this is no longer the case for the three-charge solution.
Indeed, let's add a momentum charge P along the $y-S^1$ manifold. In particular, let's consider N_p units of momentum along the circle, so that these modes have an energy of $\frac{N_p}{R}$: hence, their energy decreases as R increases.
The supergravity solutions

$$
\begin{aligned}
ds^2 &= Z_1^{-1}\left[-dt^2+dy^2+Z_p(dt+dy)^2\right]+Z_5\sum_{i=1}^{4}dx_idx_i+\sum_{a=1}^{4}dz_adz_a,\\
e^{2\phi} &= \frac{Z_5}{Z_1},\\
Z_1 &= 1+\frac{Q_1}{r^2},\quad Z_5=1+\frac{Q_5}{r^2},\quad Z_p=\frac{Q_p}{r^2},
\end{aligned}
\tag{3.3}
$$

display an horizon at $r=0$.
A deeper analysis related to this background was considered in [20], and it will be discussed further when the Wess-Zumino-Witten models are introduced in Chapter 4.
As discussed before, the horizon area no longer vanishes for this configuration due to the presence of momentum.
Let us check this fact by computing the area of the horizon in the Type IIA setup and let's convert this area in the Einstein frame.
In the near horizon limit, the part proportional to Z_5 in the 10-dimensional metric can be written as

$$
Z_5\sum dx_idx_i = Z_5\left(dr^2+r^2d\Omega_3^2\right)\approx Q_5\left[\frac{dr^2}{r^2}+d\Omega_3^2\right]. \tag{3.4}
$$

Hence, approaching the horizon the length of the $y-S^1$ manifold is given by

$$
L_y = 2\pi R\frac{Q_p^{\frac{1}{2}}}{Q_1^{\frac{1}{2}}}, \tag{3.5}
$$

while the area of the transverse three-sphere becomes

$$
A_{S^3}=\left(2\pi^2\right)Q_5^{\frac{3}{2}}. \tag{3.6}
$$

Furthermore, the volume of the torus T^4 in the limit $r \to 0$ simply reads as

$$V_{T^4}^{\text{string}} = (2\pi)^4 V. \tag{3.7}$$

So, the horizon area of the considered $S^3 \times S^1 \times T^4$ geometry is the product

$$A^S = A_{S^3} L_y V_{T^4} = \left(2\pi^2\right)(2\pi R)\left((2\pi)^4 V\right) Q_1^{-\frac{1}{2}} Q_5^{\frac{3}{2}} Q_P^{\frac{1}{2}}. \tag{3.8}$$

In conclusion, by transitioning from the String frame (with 10-dimensional metric g_{ab}^S) to the Einstein frame (with metric g_{ab}^E) using

$$g_{ab}^E = e^{-\frac{\phi}{2}} g_{ab}^S = \frac{Z_1^{\frac{1}{4}}}{Z_5^{\frac{1}{4}}} g_{ab}^S, \tag{3.9}$$

finally, the Beckenstein-Hawking entropy is given by

$$S_{BH} = \frac{A^E}{4G_{10}} = 2\pi\sqrt{N_1 N_5 N_p}, \tag{3.10}$$

where G_{10} is the 10-dimensional Newton constant and the horizon area in Einstein frame A^E is related to the string one A^S by

$$A^E = \left(\frac{g_{ab}^E}{g_{ab}^S}\right)^4 A^S. \tag{3.11}$$

3.3.1 Computation of black hole entropy

After computing the black hole entropy for the three-charge solution using the standard Bekenstein-Hawking formula, one could attempt to perform the same calculation through microscopic counting. This microscopic counting was first carried out by Strominger and Vafa [2].
Instead of using the configuration explained before, known as the $NS1 - NS5 - P$ system, one can compute the microscopic entropy by using a dual solution in the $D1 - D5 - P$ frame, where the computation is easier, as presented in [18].
The two configurations are essentially the same since they are related through a sequence of T and S dualities. In other words, by performing a T-duality in one of the directions of T^4, all the charges remain unchanged, but the theory transitions from Type IIA to Type IIB. Moreover, by using an S-duality transformation

$$NS1NS5P \xrightarrow{S} D1D5P, \tag{3.12}$$

the configuration $D1 - D5 - P$ is reached.
The $D1 - D5 - P$ geometry is described by the following metric

$$ds^2 = -\left(Z_1 Z_5\right)^{-1/2} dt^2 + \left(Z_1 Z_5\right)^{-1/2} Z_p^{-1} dx_5^2 + \left(Z_1 Z_5\right)^{1/2} dx_{78910}^2 + Z_1^{1/2} Z_5^{-1/2} dx_{1234}^2, \tag{3.13}$$

where the coordinates x_{1234} parametrize the torus T^4, x_5 the circle and the others the transverse direction to the D5 brane. Moreover, the three harmonic functions Z_1, Z_5, Z_p have been defined in 3.3.
Hence, by making use of the S and T dualities, one can realize an equivalence between the $D1 - D5 - P$ configuration and the $NS5 - F1 - P$ one. For this section, it was preferred to use the former system. The branes are placed as shown in the picture below.

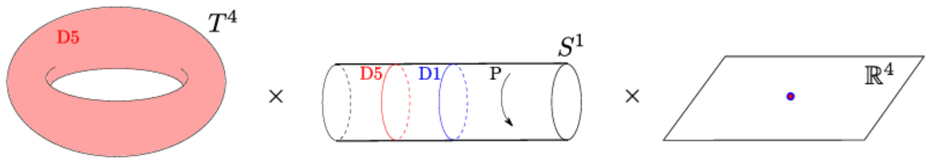

Figure 3.1: Diagram showing the placement of the branes. Image taken from [18].

The supergravity charges are presented below, excluding any additional factors

$$Q_1 \sim g_s \left(\ell_s\right)^2 N_1, \quad Q_5 \sim g_s \left(\ell_s\right)^2 N_5, \quad Q_p \sim g_s^2 N_p. \tag{3.14}$$

The horizon area depends on the string length and the string coupling as

$$A_H \sim \sqrt{Q_1 Q_5 Q_p} \sim g_s^2 \left(\ell_s\right)^3 \sqrt{N_1 N_5 N_p}.$$

Conversely, the Bekenstein-Hawking entropy is independent of the coupling and length scales as can be noticed from 3.10.
The goal now is to reproduce this entropy using string theory through a microstate counting: for simplicity, the limit of small T^4 volume is considered, namely $\frac{Q_p}{Q_1 Q_5} \gg 1$.
Since we are working in the limit $g_s N << 1$, the gauge theory describing the open strings is a free theory. In other words, one can express the wave functions describing strings stretching between $D1$ and $D5$ in the following form

$$\psi\left(x_5\right) \sim e^{2\pi \frac{n}{N_1 N_5 R} x_5}, \tag{3.15}$$

where n denotes the number of momentum units.
Hence, in order to compute the microstates, one has to calculate in how many ways we can get the momentum $p = \frac{N_p}{R}$ from partitioning the momentum over the $D1 - D5$ open strings with the previous partition function.
This statement is the same of counting the number of partitionings

$$\sum_{m=1}^{\infty} \frac{n_m m}{N_1 N_5 R} = \frac{N_p}{R}, \tag{3.16}$$

where m counts the momentum in unit $1/(N_1 N_5 R)$ added by n_m strings of this type.
This kind of counting can be performed by the following partition function:

$$Z = \sum_{n=0}^{\infty} d_n q^n, \tag{3.17}$$

where d_n counts the number of partitionings of the integer n.
In order to compute Z, one can make use of the geometric series for $q < 1$, the canonical ensemble and a high temperature limit.
In particular, let's set $q = e^{-\beta}$ and take the logaritm of the partition function as follows

$$\begin{aligned} \log Z &= -\sum_{n=1}^{\infty} \log\left(1 - q^n\right) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\left(q^n\right)^m}{m} = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} \left(q^m\right)^n = \\ &= \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=1}^{\infty} \left(\frac{1}{1-q^m} - 1\right) = \sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{1-q^m}. \end{aligned} \tag{3.18}$$

Furthermore, the average occupation number is

$$\langle n \rangle = \frac{1}{Z} \sum n d_n e^{-\beta n} = \frac{\partial}{\partial \beta} \log Z. \tag{3.19}$$

The high temperature regime is imposed through $\beta << 1$, namely $q \lesssim 1$, and the obtained result is simply given by

$$\log Z = \frac{1}{\beta}\sum_m m^{-2} + \mathcal{O}\left(\beta^0\right), \tag{3.20}$$

as $\langle n\rangle$ becomes large.
In terms of the Riemann zeta function,

$$\langle n\rangle = \frac{\zeta(2)}{\beta^2}, \tag{3.21}$$

so that, by using the standard formula $S = \log Z + \beta\langle n\rangle$ from thermodynamics, the final result reads as

$$S_{micro} = 2\pi\sqrt{\frac{N_1 N_5 N_p}{6}}, \tag{3.22}$$

since $\langle n\rangle = N_1 N_5 N_p$.
One may wonder about the factor of 6 that appears incorrectly in the denominator, but it is not a problem: indeed, the previous computation was done considering only bosons, and in a supersymmetric scenario, one must account for an equal number of fermionic and bosonic degrees of freedom. By doing so, one can recover

$$S_{\text{micro}} = 2\pi\sqrt{N_1 N_5 N_p}, \tag{3.23}$$

which is the expected result.
In conclusion, the microscopic computation of the entropy for the BPS three-charge black hole solution is in complete agreement with the Bekenstein-Hawking computation, namely

$$S_{micro} = S_{BH}, \tag{3.24}$$

as expected.

3.4 Non extremal black holes

This paragraph provides a brief introduction to the non-extremal generalization as presented in [18].
The easiest solution with $M > Q$ is the so-called black $D3$-brane, namely

$$ds^2 = -Z^{-1/2}\left(-f(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + Z^{1/2}\left(\frac{dr^2}{f(r)} + r^2 d\Omega_5^2\right), \tag{3.25}$$

where the gauge field takes the form $C_{0123} = Z^{-1}$ and both the functions f and Z are harmonic with respect to the transverse directions.
Indeed, the previous solution is a non-extremal generalization of the usual supersymmetric $D3$-brane that can be recovered by setting f to 1.
In this more general framework, f can be expanded around the supersymmetric solution as

$$f(r) = 1 - \frac{\Delta M}{r^4}. \tag{3.26}$$

The charge for this solution is given by

$$Q = \int F_5 = N, \tag{3.27}$$

where F_5 is the self-dual five-form field strength that couples to the brane.
Instead, the mass of the black $D3$-brane can be found by considering the g_{tt} component of the metric. More precisely, by using the previous expansion for f, the resulting mass is

$$M = Q + \Delta M, \tag{3.28}$$

where ΔM must be positive to avoid a naked singularity. If $M > Q$ (non-extremal regime), the solution is not stable and two such branes can eventually collapse to a single object, because the gravitational attraction is larger than the electrostatic repulsion. Furthemore, it has a non-zero temperature.
An extremal solution is defined by the condition $M = Q$. Such a black object has zero Hawking temperature and does not emit radiation. For small ΔM, the temperature is approximately proportional to the excess mass: $T \sim \Delta M$.
Furthermore, one can consider other examples, such as the non-extremal generalization of the results regarding the BPS three-charge black hole solution [21], which is given by:

$$\begin{aligned} ds^2 &= \frac{1}{f_1}\left[-\frac{f}{f_n}dt^2 + f_n\left(dx - \frac{r_0^2 \sinh 2\alpha_n}{2f_n r^2}dt\right)^2\right] + f_5\left(\frac{1}{f}dr^2 + r^2 d\Omega_3^2\right) + ds^2_{T^4}, \\ H &= dx \wedge dt \wedge d\left(\frac{r_0^2 \sinh 2\alpha_1}{2f_1 r^2}\right) + r_0^2 \sinh 2\alpha_5 d\Omega_3, \\ e^{2\Phi} &= g^2 \frac{f_5}{f_1}. \end{aligned} \tag{3.29}$$

The harmonic functions in 3.29 are given by

$$f = 1 - \frac{r_0^2}{r^2}, \quad f_{1,5,n} = 1 + \frac{r_{1,5,n}^2}{r^2}, \quad r_{1,5,n}^2 = r_0^2 \sinh^2 \alpha_{1,5,n}. \tag{3.30}$$

and the parameters $\alpha_{1,5,n}$ are related to k, p, n by

$$\sinh 2\alpha_1 = \frac{2l_s^2 g^2 p}{r_0^2 v}, \quad \sinh 2\alpha_5 = \frac{2l_s^2 k}{r_0^2}, \quad \sinh 2\alpha_n = \frac{2l_s^4 g^2 n}{r_0^2 R^2 v}, \quad v = \frac{V_{T^4}}{(2\pi l_s)^4}, \tag{3.31}$$

where V_{T^4} represents the volume of the 4-torus.
This configuration is crucial since it will appear again in Chapter 4 where a formulation through the Wess-Zumino-Witten model $\frac{SL(2,\mathbb{R})\times U(1)}{U(1)}$ will be discussed.

Let's take the near horizon limit of the $NS5$ branes and, by excluding for simplicity the $S^3 \times T^4$ factors from the metric, one can find

$$\begin{aligned} ds^2 &= \frac{1}{f_1}\left[-\frac{f}{f_n}dt^2 + f_n\left(dx - \frac{r_0^2 \sinh 2\alpha_n}{2f_n r^2}dt\right)^2\right] + \frac{kl_s^2}{fr^2}dr^2, \\ H &= dx \wedge dt \wedge d\left(\frac{r_0^2 \sinh 2\alpha_1}{2f_1 r^2}\right) + r_0^2 \sinh 2\alpha_5 d\Omega_3, \\ e^{2\Phi} &= g^2 \frac{kl_s^2}{f_1 r^2}. \end{aligned} \tag{3.32}$$

The previous formula can be easily obtained from 3.29 since, in the decoupling limit previously mentioned, f_5 takes the form

$$f_5 = \frac{kl_s^2}{r^2} \Longrightarrow r_5^2 = kl_s^2. \tag{3.33}$$

Hence, one can perform the limit by dropping the 1 from the harmonic function f_5, obtaining equation 3.32.
Moreover, for future purposes that will be explored in detail in Chapter 4, let's perform a change of coordinates on the latter solution, namely

$$\tilde{\rho}^2 = r^2 + r_0^2 \sinh^2 \alpha_n. \tag{3.34}$$

Under the previous transformation, 3.32 can be equivalently recast as

$$\begin{aligned} ds^2 &= -\frac{\left(\tilde{\rho}^2 - \tilde{\rho}_-^2\right)\left(\tilde{\rho}^2 - \tilde{\rho}_-^2\right)}{\ell^2 \tilde{\rho}^2} dt^2 + \frac{k l_s^2 \tilde{\rho}^2}{\left(\tilde{\rho}^2 - \tilde{\rho}_-^2\right)\left(\tilde{\rho}^2 - \tilde{\rho}_-^2\right)} d\tilde{\rho}^2 + \frac{\tilde{\rho}^2}{\ell^2}\left(dx - \frac{\tilde{\rho}_+ \tilde{\rho}_-}{\tilde{\rho}^2} dt\right)^2, \\ H &= dx \wedge dt \wedge d\left(\frac{r_0^2 \sinh 2\alpha_1}{2\ell^2}\right), \\ e^{2\Phi} &= g^2 \frac{k l_s^2}{\ell^2}, \end{aligned} \tag{3.35}$$

where

$$\tilde{\rho}_+^2 = r_0^2 \cosh^2 \alpha_n, \quad \tilde{\rho}_-^2 = r_0^2 \sinh^2 \alpha_n$$

and

$$\ell^2 = \tilde{\rho}^2 - r_0^2 \sinh^2 \alpha_n + r_0^2 \sinh^2 \alpha_1.$$

3.5 A gentle introduction to fuzzball solutions

In theoretical physics, the concept of fuzzballs arises in black hole physics, specifically within the framework of string theory. This hypothesis presents a novel approach to understanding black holes by addressing the limitations of classical descriptions provided by general relativity and quantum mechanics.

The primary challenge in black hole physics has been reconciling the classical notion of a singularity—a point of infinite curvature at the heart of a black hole—with the principles of quantum mechanics. Traditional approaches often depict black holes as objects with an event horizon surrounding a singularity. This picture, however, leads to several unresolved issues, including the black hole information paradox.

Fuzzballs offer a different perspective. Instead of a singularity, the fuzzball hypothesis suggests that the entire region within the event horizon is filled with a complex, extended structure. According to this model, the "fuzzball" is a conglomeration of fundamental strings and branes, which avoids the problematic singularity predicted by classical theory.

Mathematically, some fuzzballs are described by solutions to the equations of motion in supergravity that lack an event horizon in the classical sense. These solutions have zero entropy because they correspond to a single microstate rather than an ensemble of states. The previous ones are considered to describe atypical microstate configurations, while the typical ones are expected to be solutions that go beyond the supergravity limit as discussed in [22].

Once again, a key point of discussion is the black hole information paradox. Indeed, fuzzball models can provide a framework to explore how these microstates can contribute to understanding information preservation. As the coupling constant g_s increases, fuzzball models suggest that the microstates themselves become larger and effectively fill the space where the horizon would classically be. This means that fuzzball solutions help illustrate how information might be preserved and not lost as black holes evaporate.

To fully understand the fuzzball solutions in a string-theoretic regime, one can consider specific models, such as Wess-Zumino-Witten (WZW) models, which provide a more comprehensive description of these solutions.
The information provided here and in the following sections is primarily sourced from [19]. More details will be provided in the following chapter, where a specific fuzzball solution will be discussed using a class of Wess-Zumino-Witten models.

3.5.1 D1-D5 fuzzball solution

Here, the idea is to consider a specific model to demonstrate how a configuration of branes and strings can form a solution that is smooth and horizonless, namely a fuzzball. The explicit example that will be considered is the two-charge D1-D5 configuration. Such a configuration can be described as the following supergravity solution

$$\begin{aligned} ds^2 = &-\frac{1}{h}\left(dt^2 - dy^2\right) + hf\left(d\theta^2 + \frac{dr^2}{r^2+a^2}\right) - \frac{2a\sqrt{Q_1Q_5}}{hf}\left(\cos^2\theta dy d\psi + \sin^2\theta dt d\phi\right) + \\ &+ h\left[\left(r^2 + \frac{a^2Q_1Q_5\cos^2\theta}{h^2f^2}\right)\cos^2\theta d\psi^2 + \left(r^2+a^2-\frac{a^2Q_1Q_5\sin^2\theta}{h^2f^2}\right)\sin^2\theta d\phi^2\right] + \\ &+ \sqrt{\frac{Q_1+f}{Q_5+f}}dz_a dz_a, \end{aligned} \tag{3.36}$$

where

$$f = r^2 + a^2\cos^2\theta, \quad h = \left[\left(1+\frac{Q_1}{f}\right)\left(1+\frac{Q_5}{f}\right)\right]^{1/2}. \tag{3.37}$$

In particular, the previous model is specified by the $D1$-charge Q_1, the $D5$-charge Q_5 and $a = \frac{\sqrt{Q_1Q_5}}{R}$ where R the radius of one of the compactified directions.

Furthermore, in the limit $r \ll (Q_1Q_5)^{1/4}$, one can show that the previous metric reduces to

$$\begin{aligned} ds^2 = &\sqrt{Q_1Q_5}\left[-\left(r'^2+1\right)\frac{dt^2}{R^2} + r'^2\frac{dy^2}{R^2} + \frac{dr'^2}{r'^2+1}\right] + \\ &+ \sqrt{Q_1Q_5}\left[d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\right] + \sqrt{\frac{Q_1}{Q_5}}dz_a dz_a, \end{aligned} \tag{3.38}$$

where $r' = \frac{r}{a}$. Namely, the spacetime geometry is given by $AdS_3 \times S^3 \times T^4$ with a flat geometry at infinity. Moreover, such a spacetime structure features a "throat"-like region at smaller r, and instead of ending in a singularity at $r = 0$, it smoothly closes off in a “cap”. Hence, there is no horizon or singularity as expected for a fuzzball .
One could also investigate other fuzzball models such as the three-charge one, but the discussion of the supergravity solution will be avoided here and will be addressed later by reproducing this noted solution using a particular class of Wess-Zumino-Witten models.

Chapter 4

Worldsheet description for black hole solutions

As mentioned in Chapter 2, the quantization and some relevant physical aspects related to the string action for a generic background $G_{\mu\nu}$ are quite difficult topics. However, this is certainly crucial in trying to address some problems related to black hole physics. In order to derive some interesting properties for black holes, one can also proceed in a different fashion: the idea is to use a class of sigma models, known as Wess-Zumino-Witten models.
In particular, the structure of the chapter is as follows. First, a general introduction to Wess-Zumino-Witten (WZW) models will be provided. Subsequently, the null-gauging procedure for WZW models will be presented, with a specific focus on the work of [23], which describes a fuzzball solution. Following this reference, an analysis of the worldsheet consistency of this coset theory will be provided to show that it is possible to reproduce a fuzzball solution through a WZW model. Then, the perspective changes, and some black hole solutions will be examined: first, Witten's black hole solution (in both Euclidean and Lorentzian signatures) will be discussed, demonstrating that the same results can be achieved using the null-gauging procedure; second, a more general coset model will be studied with the aim of understanding whether it is possible to reproduce the metric of the extremal three-charge black hole solution introduced in Section 3.3. Some comments on the spectrum of Witten's black hole and its null-gauged model will conclude the chapter.

4.1 Wess-Zumino-Witten models

Given the string worldsheet M_2, a non-linear sigma model on M_2 is given by an action describing the dynamics of an embedding map g into a pseudo-Riemmanian manifold N that is endowed with a metric $G_{\mu\nu}$. Namely,

$$g : M_2 \longrightarrow N. \tag{4.1}$$

Let's suppose that the sigma model has a compact Lie group G as target space N. Furthermore, G is assumed to be semi-simple that is equivalent to the existence of a non-degenerate inner product on the associated Lie algebra $\mathfrak{g}$ given by

$$\langle X, Y \rangle = \mathrm{tr}(XY), \tag{4.2}$$

normalized such that it acts on the generators as $\mathrm{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$.
For simplicity, let's focus on the Riemann sphere $M_2 = S^2$, so that the group element is represented as the image of the map $g : \mathrm{S}^2 \longrightarrow \mathrm{G}$. It is important to emphasize that this introduction to the basic concepts related to the Wess-Zumino-Witten models is primarily based on [24] and [25].

4.1.1 Classical theory

The most general global $G \times G$-invariant action describing the dynamics of the embedding map g is given by

$$S[g] \equiv S_0[g] + k\Gamma[g], \tag{4.3}$$

where the kinetic part is

$$S_0 = \frac{1}{4\lambda^2} \int_{S^2} \mathrm{d}^2 z \,\mathrm{tr}\left(g^{-1}\partial_\mu g g^{-1}\partial^\mu g\right), \tag{4.4}$$

while the second one, known as Wess-Zumino term, is an integral over a filling region B of the sphere,

$$\begin{aligned}\Gamma[g] &\equiv -\frac{i}{12\pi}\int_B \mathrm{d}^3 y \epsilon_{\alpha\beta\gamma}\,\mathrm{tr}\left(g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\right) = \\ &= -\frac{i}{2\pi}\int_B \mathrm{tr}\left(g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\right).\end{aligned} \tag{4.5}$$

The previous model is called the Wess-Zumino-Witten (WZW) model on the Riemann sphere. The main advantage of the WZW models is that they are a completely solvable class of models with the conformal invariance that is preserved at quantum level.
In fact, if one uses only the first term in the action, the conformal invariance that is manifest at the classical level is spoiled at the quantum one since a negative β-function appears: hence, one can impose $\beta = 0$ by adding the Wess-Zumino term in the classical action.
It is easy to see the global $G \times G$-invariance by using

$$g(z,\bar{z}) \longmapsto g_{\mathrm{L}} g(z,\bar{z}) g_{\mathrm{R}}^{-1}, \tag{4.6}$$

where $g_{\mathrm{L}}, g_{\mathrm{R}} \in \mathrm{G}$ are constant elements.
Furthermore, the classical equations of motion in the usual $(z, \bar{z})$ coordinates are given by

$$\left(1 + \frac{\lambda^2 k}{\pi}\right)\partial\left(g^{-1}\bar{\partial}g\right) + \left(1 - \frac{\lambda^2 k}{\pi}\right)\bar{\partial}\left(g^{-1}\partial g\right) = 0 \tag{4.7}$$

and they are directly derived by varying the action with respect to g.
Moreoever, by computing the Noether currents related to the previous global and continuous transformation, one can find

$$\begin{aligned}\partial\left(g^{-1}\bar{\partial}g\right) &= 0, \\ \bar{\partial}\left(\partial g g^{-1}\right) &= 0,\end{aligned} \tag{4.8}$$

that means that each term of the equation 4.7 vanishes separately. Hence, the theory possesses the desired holomorphic J and antiholomorphic currents $\bar{J}$ defined respectively as $J = k\partial g g^{-1}$ and $\bar{J} = k g^{-1}\bar{\partial} g$, with $k = \frac{\pi}{\lambda^2}$.
In fact, the previous statement is equivalent to the tracelessness of the stress-energy tensor, which must be verified in any conformal field theory as discussed in Chapter 2.
Someone might be concerned by the fact that the Wess-Zumino term required an extension of the embedding g to the interior of the sphere, since in the integral appears B.
As it should be, the action is completely independent of the choice of the extension, and now the aim is to verify it.
The maps of the form $g : \mathrm{S}^2 \longrightarrow \mathrm{G}$ are classified by the second homotopy group $\pi_2(\mathrm{G})$ that, for any Lie group, is well-known to vanish: in other words, every map g is homotopic to the identity that certainly can be extendend on the interior.
However, the extension is not unique and different extensions are related to each other by

$$g : (B \sqcup B)/\partial B \approx \mathrm{S}^3 \longrightarrow \mathrm{G}, \tag{4.9}$$

since one can glue them together along the common boundary.
A mathematical fact is that $\pi_3(\mathrm{G}) \cong \mathbb{Z}$: hence, $\mathrm{G} = \mathrm{SU}(2) \cong \mathrm{S}^3$ since any continuous map from S^3 to G is hopotopic to a map into $SU(2)$.
Therefore, under a small variation δg of the extension, the variation of the Γ-term is just given by

$$\begin{aligned}\delta\Gamma &= -\frac{i}{4\pi}\int_B \mathrm{tr}\left(\left(-g^{-1}\delta g g^{-1}\,\mathrm{d}g + g^{-1}\,\mathrm{d}\delta g\right)\wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\right) = \\ &= -\frac{i}{4\pi}\int_B \mathrm{d}\,\mathrm{tr}\left(g^{-1}\delta g g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\right) = -\frac{i}{4\pi}\int_{\mathrm{S}^2} \mathrm{tr}\left(g^{-1}\delta g g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\right) = \\ &= -\frac{i}{4\pi}\int_{\mathrm{S}^2} \mathrm{tr}\left(g^{-1}\delta g\,\mathrm{d}\left(g^{-1}\,\mathrm{d}g\right)\right),\end{aligned} \tag{4.10}$$

where the Stokes' theorem has been involved.
Hence, by requiring that the variation δg vanishes on S^2, Γ is invariant.
Then, one has to check what happens under considering topologically different estensions: namely, under change of the homotopy class.
By using an expansion around the identity $g(y) = y^0 - iy^k\sigma_k, \quad y \in \mathrm{S}^3 \subset \mathbb{R}^4$, the difference of the Wess-Zumino terms related to two different extension with neighbouring homotopy classes can be computed as follows

$$\begin{aligned}\Delta\Gamma &= -\frac{i}{12\pi}\int_{S^3} \mathrm{d}^3y\epsilon_{\alpha\beta\gamma}\,\mathrm{tr}\left(g^{-1}\partial^\alpha g g^{-1}\partial^\beta g g^{-1}\partial^\gamma g\right) = -\frac{i(-i)^3}{12\pi}2\pi^2\sum_{i,j,k}\epsilon_{ijk}\,\mathrm{tr}\left(\sigma^i\sigma^j\sigma^k\right) = \\ &= \frac{\pi}{12}\sum_{i,j,k}\epsilon_{ijk}\,\mathrm{tr}\left(\left[\sigma^i,\sigma^j\right]\sigma^k\right) = \frac{\pi}{12}2i\sum_{i,j,k,\ell}\epsilon_{ijk}\,\mathrm{tr}\left(\epsilon^{ij\ell}\sigma_\ell\sigma_k\right) = \frac{i\pi}{6}\sum_{k,\ell}2\,\mathrm{tr}\left(\sigma^k\sigma_k\right) = \\ &= \frac{i\pi}{6}12 = 2\pi i,\end{aligned} \tag{4.11}$$

where the volume of the unit 3-sphere, that is $2\pi^2$, was used.
The previous fact can imply some ill-definedness of the theory at the quantum level. At classical level, the expression 4.3 is completely well-defined for any real value of k while at quantum level the situation changes.
Let's consider the partition function in Lorentzian signature

$$Z = \int \mathcal{D}g e^{iS[g]}. \tag{4.12}$$

In order to have a well-defined Z, the exponential of the action has to be single-valued, namely S must be a periodic function with a period of 2π.
However, since $\Delta\Gamma = 2\pi i$, the previous requirement is the same as

$$k \in \mathbb{Z}. \tag{4.13}$$

In conclusion, for topological reasons, in any compact Lie group the number k, known as the level of the model, has to be an integer.

4.1.2 Quantum theory

As in any conformal field theory, at quantum level the main focus is on the OPEs between the several operators of the theory. Here, the aim is to discuss the principal features of current algebra in order to derive the so-called Affine Kac-Moody algebra. Furthermore, some comments will relate to the usual OPE concerning the stress-energy tensor by using the famous Sugawara

construction.
At first, let's start with the holomorphic current $J(z) = k\partial g g^{-1}$: it has a conformal dimension (1,0) because of the partial derivative operator.
This means that the OPE looks like

$$J^a(z)J^b(w) \sim \sum_p \frac{X_p(w)}{(z-w)^p}, \tag{4.14}$$

where $X_p(w)$ is an holomorphic field with conformal weigh $2-p$ and the adjoint indices are keept explicit.
By unitarity, the field $X_p(w)$ has conformal dimension one and is therefore a current itself. In fact, unitarity imposes that there are no operators with negative conformal dimensions; hence, the second-order pole is the highest possible. Therefore, the only possibility (aside from the trivial one, the identity) is to have a current. The previous sentence constrains the OPE to satisfy the so-called current algebra that is

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{ab}_c J^c(w)}{z-w}, \tag{4.15}$$

and similarly for the anti-holomorphic current

$$\bar{J}^a(\bar{z})\bar{J}^b(\bar{w}) \sim \frac{k\delta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{if^{ab}_c J^c(\bar{w})}{\bar{z}-\bar{w}}. \tag{4.16}$$

Analogously to what was done in 2.5.2 of Chapter 2 for the stress-energy tensor, one can formulate the previous equations also in terms of modes by expanding the current as

$$J^a(z) \equiv \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}. \tag{4.17}$$

The commutation relations by contour deformations are

$$\begin{aligned}
\left[J^a_m, J^b_n\right] &= \frac{1}{(2\pi i)^2}\left(\oint \mathrm{d}z \oint_{|z|>|w|} \mathrm{d}w - \oint \mathrm{d}z \oint_{|z|<|w|} \mathrm{d}w\right) z^m w^n J^a(z)J^b(w) = \\
&= \frac{1}{(2\pi i)^2}\oint_0 \mathrm{d}w \oint_w \mathrm{d}z z^m w^n \left(\frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{abc}{}_c J^c(z)}{z-w}\right) = \\
&= \frac{1}{2\pi i}\oint_0 \mathrm{d}w \left(km\delta^{ab}w^{m+n-1} + if^{ab}{}_c J^c(w) w^{m+n}\right) = \\
&= km\delta^{ab}\delta_{m+n,0} + if^{ab}{}_c J^c_{m+n}.
\end{aligned} \tag{4.18}$$

The previous Lie algebra is called Kac-Moody algebra.
Notice that the conserved quantities are given by the zero modes in the expansion

$$J^a_0 = \oint \mathrm{d}z J^a(z), \tag{4.19}$$

and they satisfy the original Lie algebra $\mathfrak{g}$.
A question that remains open in the discussion is: is the theory conformally invariant at the quantum level? The answer that was given is yes, but without giving a solid proof: now, let's delve into it.
The aim is to show that the energy-momentum tensor satisfies the Virasoro algebra.
Let's start by considering the classical stress-energy tensor that is

$$T(z) = \frac{1}{2k}\delta_{ab}J^a(z)J^b(z) = \frac{1}{2k}J^a(z)J^a(z). \tag{4.20}$$

A quantized version can be given after having performed the normal ordered product since the vacuum expectation value is expected to be finite, namely

$$T(z) = \gamma \left(J^a J^a\right)(z). \tag{4.21}$$

The prefactor can be corrected at quantum level, for this reason a coefficient γ was introduced in the definition. Specifically, in this scenario the normal ordering is defined as the constant part of the product of the operators involved, in other words as

$$\left(J^a J^a\right)(z) \equiv \frac{1}{2\pi i} \oint_z \frac{\mathrm{d}x}{x-z} J^a(x) J^a(z). \tag{4.22}$$

At first, one can compute the singular part of the OPE of $T(z)$ with $J^a(w)$ and the result is given by

$$\left[J^a(z)\, T(w)\right]_{\text{sing}} = \gamma \left(\frac{2k\delta^{ab} J^b(w)}{(w-z)^2} - \frac{f^{ab}_{\ c} f^{cb}_{\ d} J^d(w)}{(w-z)^2} \right). \tag{4.23}$$

Moreoever, by using the dual Coxeter number $h^\vee$ defined as $f^{ab}{}_c f^{bc}{}_d = 2h^\vee \delta^{ad}$, one can easily prove that

$$J^a(z) T(w) = 2\gamma \left(k + h^\vee\right) \left(\frac{J^a(z)}{(w-z)^2} + \frac{\partial J^a(z)}{w-z} \right). \tag{4.24}$$

Hence, in order to have for $J^a(z)$ a conformal weight one, one obtains

$$T(z) J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}, \tag{4.25}$$

after having defined the prefactor as $\gamma \equiv \frac{1}{2(k+h^\vee)}$. The previous relation can be used to compute the OPE between the components of the stress-energy tensor. Indeed, starting from

$$[T(z)T(w)]_{sing} = \frac{1}{4\pi i \left(k + h^\vee\right)} \oint \frac{\mathrm{d}x}{x-w} \left([T(z)J^a(x)]_{sing} J^a(w) + J^a(x) [T(z)J^a(w)]_{sing} \right), \tag{4.26}$$

one exactly finds the expected behaviour for a conformal quantum field theory,

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{4.27}$$

where

$$c \equiv \frac{k \dim \mathfrak{g}}{k + h^\vee}. \tag{4.28}$$

This is the so-called Sugawara construction for the energy-momentum tensor.

4.2 Null-gauged Wess-Zumino-Witten models

In the previous chapter, the Wess-Zumino-Witten models were introduced because they are able to reproduce string effects on a curved background, which would otherwise be difficult to treat. The aim here is to provide motivation for this statement by explicitly showing all the details in some specific models that illustrate the key ideas of the so-called null-gauging procedure for a WZW model, as presented in [23], [26] and [27]. In general, given a Wess-Zumino-Witten model defined on an upstairs group G, one can quotient away some specific generators related to a subgroup H of G. The resulting G/H model is said to be gauged because for each current that is quotiented away, there is a corresponding auxiliary gauge field that is introduced and then integrated out. Moreover, the word "null" is related to the fact that the gauged currents are taken to be null with respect to the space-time causality of the upstairs group G (i.e., with respect to the Cartan-Killing form defined on the Lie algebra of G); this condition fixes some parameters of the theory in a natural way. First, before delving into a class of interesting WZW models, the null-gauging procedure on general sigma models will be introduced.

4.2.1 Null-gauging formalism for general sigma models

Let's consider a sigma model describing the dynamics of the embedding field

$$\phi : \mathrm{M}_2 \longrightarrow \mathrm{N}, \tag{4.29}$$

where N is a pseudo-Riemmanian manifold equipped with a metric $G_{\mu\nu}$.
Furthermore, let's suppose that in the model there exists a set of two Killing vectors $\{\xi_a\}_{a=1,2}$ generating isomotries of N.
In order to gauge them, one has to introduce independent worldsheet gauge fields $\mathcal{A}^a$, one corresponding to each Killing vector.
Hence, the kinetic part $\partial\varphi^i G_{ij}\bar{\partial}\varphi^j$ becomes

$$\mathcal{L}_{\mathrm{K}} = \mathcal{D}\varphi^i G_{ij}\bar{\mathcal{D}}\varphi^j, \tag{4.30}$$

where $\mathcal{D}\varphi^i = \partial\varphi^i - \mathcal{A}^a\xi_a^i$ is the covariant derivative.
On the contrast, the WZ-term reads as

$$\mathcal{L}_{\mathrm{WZ}} = B_{ij}\partial\varphi^i\bar{\partial}\varphi^j + \mathcal{A}^a\theta_{a,i}\bar{\partial}\varphi^i - \bar{\mathcal{A}}^a\theta_{a,i}\partial\varphi^i + \xi^i_{[a}\theta_{b],i}\mathcal{A}^a\bar{\mathcal{A}}^b, \tag{4.31}$$

where $\theta_a = (-1)^{a+1}\xi_a \cdot d\varphi \equiv (-1)^{a+1}\xi_a^i G_{ij} d\varphi^j$ are target-space one-forms pulled back to the worldsheet.
The introduction of such one-forms is aimed at obtaining a chiral model, as it allows some components of the gauge field to decouple.
Furthermore, a well-defined gauging is reached by imposing

$$\imath_a H = d\theta_a, \quad \imath_a\theta_b = -\imath_b\theta_a, \tag{4.32}$$

as contraints on $H = dB$.
Hence, in this $U(1)\times U(1)$ gauged models, the gauged part of the lagrangian becomes

$$\mathcal{L}_A = -2\mathcal{A}\xi_2^i G_{ij}\bar{\partial}\varphi^j - 2\bar{\mathcal{A}}\xi_1^i G_{ij}\partial\varphi^j - 4\mathcal{A}\bar{\mathcal{A}}\Sigma, \tag{4.33}$$

where $\Sigma \equiv -\frac{1}{2}\xi_1^i G_{ij}\xi_2^j$ and the notation used is $\mathcal{A} \equiv \mathcal{A}^2, \bar{\mathcal{A}} \equiv \bar{\mathcal{A}}^1$.

Let us define the worldsheet currents $\mathcal{J}$ and $\bar{\mathcal{J}}$ as the pull-backs of the target-space one-forms in the following manner,

$$\mathcal{J} \equiv -\theta_1 \cdot \partial\varphi \equiv -\theta_{1,i}\partial\varphi^i, \quad \bar{\mathcal{J}} \equiv \theta_2 \cdot \bar{\partial}\varphi \equiv \theta_{2,i}\bar{\partial}\varphi^i, \tag{4.34}$$

so that, the lagrangian is now

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_A = \mathcal{L}_K + 2\mathcal{A}\bar{\mathcal{J}} + 2\bar{\mathcal{A}}\mathcal{J} - 4\mathcal{A}\bar{\mathcal{A}}\Sigma. \tag{4.35}$$

Hence, after having integrated out the gauge fields, the result follows as

$$\mathcal{L}_{EFT} = \mathcal{L}_{\text{ungauged}} + \frac{\mathcal{J}\bar{\mathcal{J}}}{\Sigma}. \tag{4.36}$$

In conclusion, the null-gauging procedure acts by adding the term $\frac{\mathcal{J}\bar{\mathcal{J}}}{\Sigma}$ to the ungauged lagrangian.

4.2.2 A class of interesting WZW models

Now, let's consider a Wess-Zumino-Witten model defined on the upstair group G describing the embedding $g : \mathcal{M}_2 \to G$. In particular,

$$G/H \times \mathbb{T}^4 = \frac{SL(2,\mathbb{R}) \times SU(2) \times \mathbb{R}_t \times U(1)_y}{U(1)_L \times U(1)_R} \times \mathbb{T}^4. \tag{4.37}$$

The action of $H = U(1)_L \times U(1)_R$ on the upstair group elements is described through the following group homomorphisms

$$\begin{aligned} l &: H \to G_L, \\ r &: H \to G_R, \end{aligned} \tag{4.38}$$

where the global $G_L \times G_R$ symmetry given by $g \mapsto \ell(h) g r(h)^{-1}, \quad h \in H$ will be gauged. Here and in what follows, the l and r are used also as the induced maps at Lie algebra level, namely $\ell : \mathfrak{h} \to \mathfrak{g}, \quad r : \mathfrak{h} \to \mathfrak{g}$.
Furthermore, $\ell(h)$ and $r(h)$ will be denoted as h_L and h_R respectively.
The choosen parametrization for the upstairs group is

$$g = (g_{\mathrm{sl}}, g_{\mathrm{su}}, g_t, g_y) = \left(e^{\frac{i}{2}(\tau-\sigma)\sigma_3} e^{\rho\sigma_1} e^{\frac{i}{2}(\tau+\sigma)\sigma_3}, e^{\frac{i}{2}(\psi-\phi)\sigma_3} e^{i\theta\sigma_1} e^{\frac{i}{2}(\psi+\phi)\sigma_3}, e^t, e^{iy} \right), \tag{4.39}$$

where σ_i denotes the i^{th} Pauli matrix and $y \in [0, 2\pi R_y)$. Here, by looking at the previous parametrization, $SL(2,\mathbb{R})$ is treated as $SU(1,1)$.
At the Lie algebra level, the chiral embeddings related to the subgroup H are specified by the real parameters $\{l_i, r_i\}_{i=1,2,3,4}$ and act as

$$\begin{aligned} \ell(\alpha) &= \left(i\mathrm{l}_1 \alpha\sigma_3, -i\mathrm{l}_2 \alpha\sigma_3, \mathrm{l}_3\alpha, -i\tfrac{\mathrm{l}_4}{R_y}\alpha \right), \qquad r(\alpha) = 0, \\ r(\beta) &= -\left(i\mathrm{r}_1\beta\sigma_3, -i\mathrm{r}_2\beta\sigma_3, \mathrm{r}_3\beta, -i\tfrac{\mathrm{r}_4}{R_y}\beta \right), \quad \ell(\beta) = 0. \end{aligned} \tag{4.40}$$

Equivalently, the previous transformation acting on the upstair group is the same as

$$g \mapsto h_L g h_R^{-1} = \left(e^{i\mathrm{l}_1\alpha\sigma_3} g_{\mathrm{sl}} e^{i\mathrm{r}_1\beta\sigma_3}, e^{-i\mathrm{l}_2\alpha\sigma_3} g_{\mathrm{su}} e^{-i\mathrm{r}_2\beta\sigma_3}, e^{\mathrm{l}_3\alpha} g_t e^{\mathrm{r}_3\beta}, e^{-i\frac{l_4}{R_y}\alpha} g_y e^{-i\frac{r_4}{R_y}\beta} \right). \tag{4.41}$$

In order to gauge the previous one, let's introduced two independent $\mathfrak{h}$-valued worldsheet gauge fields $(\mathcal{A}_1, \bar{\mathcal{A}}_1)$ and $(\mathcal{A}_2, \bar{\mathcal{A}}_2)$ that are imposed to be null and chiral: hence, one of their components drops out and one can set $\mathcal{A}_1$ and $\bar{\mathcal{A}}_2$ to be zero.
In order to define a Lagrangian that includes the remaining components of the gauge fields, one needs to define similar embeddings for them such as

$$\begin{aligned} \ell\left(\bar{\mathcal{A}}_1\right) &= \left(i1_1\bar{\mathcal{A}}_1\sigma_3, -i1_2\bar{\mathcal{A}}_1\sigma_3, \mathrm{l}_3\bar{\mathcal{A}}_1, -i\tfrac{l_4}{R_y}\bar{\mathcal{A}}_1 \right), \qquad \ell\left(\mathcal{A}_2\right) = 0, \\ r\left(\mathcal{A}_2\right) &= -\left(i\mathrm{r}_1\mathcal{A}_2\sigma_3, -i\mathrm{r}_2\mathcal{A}_2\sigma_3, \mathrm{r}_3\mathcal{A}_2, -i\tfrac{\mathrm{r}_4}{R_y}\mathcal{A}_2 \right), \quad r\left(\bar{\mathcal{A}}_1\right) = 0, \end{aligned} \tag{4.42}$$

that are coherent with the previous ones 4.40.
A well-known fact concerns the expression of the general gauged invariant action of this model [28], which is given by

$$\begin{aligned} S = \sum_j \mathrm{sgn}\left(\kappa_j\right) \frac{k_j}{\pi} \Bigg[\int_{\mathcal{M}_2} \frac{1}{2} \mathrm{Tr}\left[g^{-1}\partial g g^{-1}\bar{\partial} g\right]_j d^2z + \int_{\mathcal{M}_3} \frac{1}{3!} \mathrm{Tr}\left[g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right]_j + \\ + \int_{\mathcal{M}_2} \mathrm{Tr}\left(-\sum_{a=1}^{2} \left[\ell\left(\bar{\mathcal{A}}_a\right)\partial g g^{-1}\right]_j + \sum_{a=1}^{2} \left[r\left(\mathcal{A}_a\right) g^{-1}\bar{\partial} g\right]_j - \sum_{a,b=1}^{2} \left[g^{-1}\ell\left(\bar{\mathcal{A}}_a\right) g r\left(\mathcal{A}_b\right)\right]_j \right) d^2z \Bigg]. \end{aligned} \tag{4.43}$$

Hence, under 4.41, the gauge fields transform as

$$\mathcal{A}_2 \to h_L \mathcal{A}_2 h_L^{-1} + \partial h_L h_L^{-1}, \bar{\mathcal{A}}_1 \to h_R \bar{\mathcal{A}}_1 h_R^{-1} + \bar{\partial} h_R h_R^{-1}. \tag{4.44}$$

Then, let's introduce the currents related to $SL(2,\mathbb{R})$

$$\mathrm{j}^3_{\mathrm{sl}} = k_{\mathrm{sl}} \operatorname{Tr}\left(-i\frac{\sigma_3}{2}\partial g_{\mathrm{sl}} g_{\mathrm{sl}}^{-1}\right), \qquad \bar{\jmath}^3_{\mathrm{sl}} = k_{\mathrm{sl}} \operatorname{Tr}\left(-i\frac{\sigma_3}{2} g_{\mathrm{sl}}^{-1}\bar{\partial} g_{\mathrm{sl}}\right), \tag{4.45}$$

and similarly for $SU(2)$,

$$\mathrm{j}^3_{\mathrm{su}} = k_{\mathrm{su}} \operatorname{Tr}\left(-i\frac{\sigma_3}{2}\partial g_{\mathrm{su}} g_{\mathrm{su}}^{-1}\right), \qquad \bar{\jmath}^3_{\mathrm{su}} = k_{\mathrm{su}} \operatorname{Tr}\left(-i\frac{\sigma_3}{2} g_{\mathrm{su}}^{-1}\bar{\partial} g_{\mathrm{su}}\right). \tag{4.46}$$

One can easily compute them explicitely, finding

$$\begin{aligned} \mathrm{j}^3_{\mathrm{sl}} &= n_5\left(\cosh^2\rho\partial\tau + \sinh^2\rho\partial\sigma\right), \quad \bar{\mathrm{j}}^3_{\mathrm{sl}} = n_5\left(\cosh^2\rho\bar{\partial}\tau - \sinh^2\rho\bar{\partial}\sigma\right), \\ \mathrm{j}^3_{\mathrm{su}} &= n_5\left(\cos^2\theta\partial\psi - \sin^2\theta\partial\phi\right), \quad \bar{\mathrm{j}}^3_{\mathrm{su}} = n_5\left(\cos^2\theta\bar{\partial}\psi + \sin^2\theta\bar{\partial}\phi\right). \end{aligned} \tag{4.47}$$

For the abelian groups, the currents are simply defined as

$$\mathrm{P}^t_L = \partial t, \quad \mathrm{P}^t_R = \bar{\partial} t, \quad \mathrm{P}^y_L = \partial y, \quad \mathrm{P}^y_R = \bar{\partial} y. \tag{4.48}$$

Gauging the transformation 4.41 is the same of gauging the currents

$$\begin{aligned} \mathcal{J} &= \mathrm{l}_1\mathrm{j}^3_{\mathrm{sl}} + \mathrm{l}_2\mathrm{j}^3_{\mathrm{su}} + \mathrm{l}_3\mathrm{P}^t_{\mathrm{L}} + \mathrm{l}_4\mathrm{P}^y_{\mathrm{L}}, \\ \bar{\mathcal{J}} &= \mathrm{r}_1\bar{\mathrm{j}}^3_{\mathrm{sl}} + \mathrm{r}_2\bar{\mathrm{j}}^3_{\mathrm{su}} + \mathrm{r}_3\mathrm{P}^t_{\mathrm{R}} + \mathrm{r}_4\mathrm{P}^y_{\mathrm{R}}, \end{aligned} \tag{4.49}$$

constrained to satisfy the null-condition, namely

$$n_5\left(\mathrm{l}_1^2 - \mathrm{l}_2^2\right) + \mathrm{l}_3^2 - \mathrm{l}_4^2 = 0, \quad n_5\left(\mathrm{r}_1^2 - \mathrm{r}_2^2\right) + \mathrm{r}_3^2 - \mathrm{r}_4^2 = 0. \tag{4.50}$$

With great patience, the previous action 4.43 can be made explicit and simplified, resulting in

$$S = S_0^{\mathrm{sl}} + S_{\mathcal{A}}^{\mathrm{sl}} + S_0^{\mathrm{su}} + S_{\mathcal{A}}^{\mathrm{su}} + S_0^{t,y} + S_{\mathcal{A}}^{t,y}, \tag{4.51}$$

where

$$\begin{aligned} S_0^{\mathrm{sl}} &= \frac{n_5}{\pi}\int\left[\partial\rho\bar{\partial}\rho + \mathrm{sh}^2\rho\partial\sigma\bar{\partial}\sigma - \mathrm{ch}^2\rho\partial\tau\bar{\partial}\tau - \mathrm{sh}^2\rho(\partial\sigma\bar{\partial}\tau - \partial\tau\bar{\partial}\sigma)\right]d^2z, \\ S_{\mathcal{A}}^{\mathrm{sl}} &= \frac{2n_5}{\pi}\int\left[\bar{\mathcal{A}}_1\left(\mathrm{sh}^2\rho\partial\sigma + \mathrm{ch}^2\rho\partial\tau\right) + \mathcal{A}_2\left(\mathrm{ch}^2\rho\bar{\partial}\tau - \mathrm{sh}^2\rho\bar{\partial}\sigma\right) - \mathcal{A}_2\bar{\mathcal{A}}_1\,\mathrm{ch}(2\rho)\right]d^2z, \\ S_0^{\mathrm{su}} &= \frac{n_5}{\pi}\int\left[\partial\theta\bar{\partial}\theta + c_\theta^2\partial\psi\bar{\partial}\psi + s_\theta^2\partial\phi\bar{\partial}\phi + c_\theta^2(\partial\phi\bar{\partial}\psi - \bar{\partial}\phi\partial\psi)\right]d^2z, \\ S_{\mathcal{A}}^{\mathrm{su}} &= \frac{2n_5}{\pi}\int\left[l_2\bar{\mathcal{A}}_1\left(c_\theta^2\partial\psi - s_\theta^2\partial\phi\right) + r_2\mathcal{A}_2\left(c_\theta^2\bar{\partial}\psi + s_\theta^2\bar{\partial}\phi\right) + l_2r_2\mathcal{A}_2\bar{\mathcal{A}}_1\cos(2\theta)\right]d^2z, \\ S_0^{t,y} &= \frac{1}{\pi}\int[-\partial t\bar{\partial}t + \partial y\bar{\partial}y]d^2z, \\ S_{\mathcal{A}}^{t,y} &= \frac{2}{\pi}\int\left[l_3\bar{\mathcal{A}}_1\partial t + r_3\mathcal{A}_2\bar{\partial}t + l_4\bar{\mathcal{A}}_1\partial y + r_4\mathcal{A}_2\bar{\partial}y - (l_3r_3 - l_4r_4)\,\mathcal{A}_2\bar{\mathcal{A}}_1\right]d^2z. \end{aligned} \tag{4.52}$$

For simplicity, the notations $c_\theta = \cos\theta, s_\theta = \sin\theta, \mathrm{ch}\,\rho = \cosh\rho, \mathrm{sh}\,\rho = \sinh\rho$ have been used. Once reached this point, what remains is to integrate out the gauge fields and to perform a gauge choice. In this specific class of models the null-gauging procedure requires quotienting out two abelian groups. Hence, one can fully fix the gauge by fixing two parameters in accordance with the gauge transformation given in equation 4.41. For instance, after having imposed the gauge $\sigma = \tau = 0$, one can easily read the metric and the B-field by looking at the symmetric and antisymmetric parts of the action. More explicitly,

$$\begin{aligned} ds^2 =& -\frac{h_t}{\Sigma_0}dt^2 + \frac{h_y}{\Sigma_0}dy^2 + \frac{(l_3r_4 + l_4r_3)}{n_5\Sigma_0}dtdy + n_5\left(d\theta^2 + d\rho^2\right) + n_5\frac{h_\phi}{\Sigma_0}\sin^2\theta d\phi^2 + n_5\frac{h_\psi}{\Sigma_0}\cos^2\theta d\psi^2 + \\ & -\frac{1}{\Sigma_0}\left[(l_2r_3 - l_3r_2)\,dt + (l_2r_4 - l_4r_2)\,dy\right]\sin^2\theta d\phi + \frac{1}{\Sigma_0}\left[(l_2r_3 + l_3r_2)\,dt + (l_2r_4 + l_4r_2)\,dy\right]\cos^2\theta d\psi, \\ B =& \frac{(l_3r_4 - l_4r_3)}{2n_5\Sigma_0}dt\wedge dy + n_5\frac{h_\phi}{\Sigma_0}\cos^2\theta d\phi\wedge d\psi + \frac{1}{2\Sigma_0}\left[(l_2r_3 + l_3r_2)\,dt + (l_2r_4 + l_4r_2)\,dy\right]\wedge\sin^2\theta d\phi + \\ & -\frac{1}{2\Sigma_0}\left[(l_2r_3 - l_3r_2)\,dt + (l_2r_4 - l_4r_2)\,dy\right]\wedge\cos^2\theta d\psi, \end{aligned} \tag{4.53}$$

where

$$\begin{aligned} \Sigma_0 &= \frac{1}{n_5}\Sigma = \sinh^2\rho - l_2r_2\cos^2\theta + \frac{1 + l_2r_2}{2} + \frac{l_3r_3 - l_4r_4}{2n_5}, \\ h_t &= \sinh^2\rho - l_2r_2\cos^2\theta + \frac{1 + l_2r_2}{2} - \frac{l_3r_3 + l_4r_4}{2n_5}, \\ h_y &= \sinh^2\rho - l_2r_2\cos^2\theta + \frac{1 + l_2r_2}{2} + \frac{l_3r_3 + l_4r_4}{2n_5}, \\ h_\phi &= \sinh^2\rho + \frac{1 - l_2r_2}{2} + \frac{l_3r_3 - l_4r_4}{2n_5}, \\ h_\psi &= \sinh^2\rho + \frac{1 + l_2r_2}{2} + \frac{l_3r_3 - l_4r_4}{2n_5}. \end{aligned} \tag{4.54}$$

Notice that, without loss of generality, the parameters l_1 and r_1 were set to 1.
The above metric describes a background that is the same as the well-known JMaRT solutions [20].
The full results for the null-gauged model can be compared with the upstairs model that is described by

$$\begin{aligned} ds^2 &= n_5\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\sigma^2 + d\theta^2 + \cos^2\theta d\psi^2 + \sin^2\theta d\phi^2\right) - dt^2 + dy^2, \\ H &= n_5(\sinh 2\rho d\rho\wedge d\tau\wedge d\sigma + \sin 2\theta d\theta\wedge d\psi\wedge d\phi). \end{aligned} \tag{4.55}$$

Furthermore, it is possible to extract the expression of the dilaton by a 1-loop computation or, analogously, by imposing that the previous B-field and metric satisfies the supergravity equation of motion: one can solve them for the dilaton, obtaining the following expression,

$$e^{2\Phi} \sim \frac{1}{\Sigma_0}. \tag{4.56}$$

4.3 Worldsheet consistency and horizonless condition

In this paragraph, the aim is to work directly at the coset CFT level, and some constraints will be derived. Specifically, the values of the parameters related to the embedding of the abelian subgroups being gauged cannot be chosen arbitrarily in order to ensure a well-defined gauge invariant quantum theory.
The previous model 4.43 is invariant under the gauge transformation 4.41 and 4.44.
For simplicity, the notation

$$\mathcal{A}_2 = \mathcal{A}, \quad \bar{\mathcal{A}}_1 = \bar{\mathcal{A}}, \tag{4.57}$$

will be used again.
In Chapter 2, we learned that the spectrum of a two dimensional CFT is related to the vertex operators of the theory. It is the requirement of gauge invariance at the operator level that brings forth some constraints, which will be derived here. To begin, a brief review is given of the construction of superstring theory on the $AdS_3 \times S^3 \times T^4$. Next, a discussion about null-gauged model will be done, including some conditions for supersymmetry. The third part is dedicated to constraints for a gauge-invariant spectrum, while the final part focuses on a brief discussion in the supergravity limit, where some CFT conditions will be derived by imposing some constraints on the metric.

4.3.1 A brief review of superstring in $AdS_3 \times S^3 \times T^4$

As discussed in Chapter 2, the OPEs for the quantum $SL(2,\mathbb{R})$ currents are

$$j^a(z)j^b(w) \sim \frac{\eta^{ab}k/2}{(z-w)^2} + \frac{f^{ab}{}_c j^c(w)}{z-w}, \tag{4.58}$$

where k is the level of the affine algebra and η^{ab} is the Killing form.
By using the Sugawara construction, as explained in 4.1.2, one can compute the stress-energy tensor OPEs, finding

$$c_{\rm sl} = \frac{3k}{k-2}. \tag{4.59}$$

The canonical spectrum of the model is generated by lowest and highest weight and continuous representations of the zero-mode algebra. Indeed, from the state $|j,j\rangle$ by acting with j_0^+, one can span the discrete part of the spectrum that is

$$\mathcal{D}_j^+ = \langle |j,m\rangle\,, m = j, j+1, j+2, \cdots \rangle\,, \tag{4.60}$$

where $j_0^3\,|j,m\rangle = m\,|j,m\rangle$. All the discrete states, and also the continuous ones that are given by

$$\mathcal{C}_j^\alpha = \left\langle |j,m,\alpha\rangle\,, 0 \le \alpha < 1, j = \frac{1}{2} + is, s \in \mathbb{R}, m = \alpha, \alpha \pm 1, \alpha \pm 2, \cdots \right\rangle, \tag{4.61}$$

give rise to primary fields with conformal weights

$$\Delta = -\frac{j(j-1)}{k-2}. \tag{4.62}$$

A similar discussion applies to the $SU(2)$ group, as detailed in [23]. Here, k^a is chosen for the current generator to distinguish it from the generators of $SL(2,\mathbb{R})$.
The relevant OPEs are

$$k^a(z)k^b(w) \sim \frac{\delta^{ab}k'/2}{(z-w)^2} + \frac{f'^{ab}_c k^c(w)}{z-w}. \tag{4.63}$$

Analogously, the associated primary operators have weights

$$\Delta' = \frac{j'\,(j'+1)}{k+2}, \tag{4.64}$$

while the central charge is

$$c_{\rm su} = \frac{3k'}{k'+2}. \tag{4.65}$$

In order to encode the supersymmetry, one has to consider the supersymmetric version of the

previous algebra, i.e. $\widehat{\mathrm{sl}(2,\mathbb{R})}_{n_5}$ and $\widehat{\mathrm{su}(2)}_{n_5}$. The first one is generated by linear combinations $\psi^a + \theta J^a$, with θ a Grassmann variable and J^a that satisfied the bosonic $SL(2,\mathbb{R})$ with level n_5.
Hence, the other significant OPEs are

$$\begin{aligned} J^a(z)\psi^b(w) &\sim \frac{f^{ab}{}_c\psi^c(w)}{(z-w)}, \\ \psi^a(z)\psi^b(w) &\sim \frac{\frac{n_5}{2}\eta^{ab}}{(z-w)}. \end{aligned} \tag{4.66}$$

Notice that one can also decompose the current J^a as $J^a = j^a - \frac{1}{n_5} f^a{}_{bc}\psi^b\psi^c$, where the first bosonic current j^a generates an $\widehat{\mathrm{sl}(2,\mathbb{R})}_k$ algebra with level $k = n_5 + 2$. A similar discussion can be done for $\widehat{\mathrm{su}(2)}_{n_5}$: the generators can be expressed as combinations of $\chi^a + \theta K^a$. Just as J^a was defined in terms of j^a, a similar procedure can be followed to define a current k^a from K^a.
All the essential ingredients of the conformal field theory on the worldsheet are encoded in

$$\begin{aligned} T &= \frac{1}{n_5}\left(j^a j_a - \psi^a\partial\psi_a + k^a k_a - \chi^a\partial\chi_a\right) + \frac{1}{2}\left(\partial Y^i\partial Y_i - \lambda^i\partial\lambda_j\right), \\ G &= \frac{2}{n_5}\left(\psi^a j_a - \frac{1}{3n_5} f_{abc}\psi^a\psi^b\psi^c + \chi^a k_a - \frac{1}{3n_5} f'_{abc}\chi^a\chi^b\chi^c\right) + \lambda^i\partial Y_i, \end{aligned} \tag{4.67}$$

where T is the energy-momentum tensor while G is the supercurrent of the WZW model.
Hence, the central charge reads as

$$c = \frac{3\,(n_5+2)}{n_5} + \frac{3}{2} + \frac{3\,(n_5-2)}{n_5} + \frac{3}{2} + 6 = 15, \tag{4.68}$$

that, in order to avoid Weyl anomaly, needs to be cancel out by the central charge of the ghosts CFT that has the bc-sector and the $\beta\gamma$ one.
As always, the BRST charge of the theory is obtained by the "matter" theory and the ghosts by

$$\mathcal{Q} = \oint dz : c\,(T + T_{\beta\gamma}) - \gamma G + c(\partial c)b - \frac{1}{4}b\gamma^2, \tag{4.69}$$

where $T_{\beta\gamma}$ is the stress-energy tensor related to the $\beta\gamma$ ghost system.
Now, it is necessary to better understand which kind of physical states arise in such a model: let's focus only on the NSNS sector.
Here, it is crucial to understand all the possible inequivalent representations. To do that, the following definition is necessary. As detailed in [29], the so-called spectral flow automorphism of the current algebra is defined as

$$j^{\pm}(z) \to \tilde{j}^{\pm}(z) = z^{\pm w} j^{\pm}(z), \quad j^3(z) \to \tilde{j}^3(z) = j^3(z) - \frac{k\omega}{2} z^{-1}. \tag{4.70}$$

The previous transformation induces an automorphism of the Virasoro algebra that is

$$L_n \to \tilde{L}_n = L_n + \omega j^3_n - \frac{k}{4}\omega^2\delta_{n,0}, \tag{4.71}$$

where ω is an integer and it is called spectral flow charge.
A well-known result is that the previous transformation acts on the canonical affine representations typically results in inequivalent representations. These must be taken into account to

ensure the generation of a consistent spectrum. In this perspective, the Virasoro primary operators possess a weight that consists of the usual value plus a correction term dependent on the spectral flow charge, namely

$$\Delta = -\frac{j(j-1)}{k-2} - m\omega - \frac{k}{4}\omega^2. \tag{4.72}$$

A similar analysis holds for the $SU(2)$ group and for the fermionic sectors.
By considering both $SL(2,\mathbb{R})$ and $SU(2)$, the Virasoro condition is given by

$$\frac{1}{2} + \frac{1}{2} - \frac{j(j-1)}{n_5} - m\omega - \frac{n_5}{4}\omega^2 + \frac{j'(j'+1)}{n_5} + m'\omega' + \frac{n_5}{4}\omega'^2 = 1, \tag{4.73}$$

that is satisfies only by specific highest/lowest-weight operators.

4.3.2 Null-gauging procedure of coset CFTs at quantum level

Here, the aim is to learn which kind of constraints arise from the null-gauging procedure of a coset CFT by focusing on the WZW model 4.43. An essential aspect is related to the ghost sector: indeed, by using the following parametrization for the gauge fields

$$\mathcal{A} = \partial H_L H_L^{-1}, \bar{\mathcal{A}} = \bar{\partial} H_R H_R^{-1}, \tag{4.74}$$

it can be demonstrated that the path integral of the gauged theory is equivalently viewed as the path integral of the original WZW model on the upstairs group with an added ghost contribution. The main consequence is directly related to the physical spectrum of the theory. In fact, the presence of ghosts necessitates the inclusion of new chiral terms of the form

$$\oint dz : \tilde{c}J :, \quad \oint dz : \bar{\tilde{c}}\bar{J} :, \tag{4.75}$$

in addition to the usual contributions to the BRST charges Q and $\bar{Q}$. This leads to the previous statement: physical operators must be gauge invariant. Indeed, under the gauging procedure outlined above, the spectrum of the coset model is built simply out of the vertex operators of the upstairs theory that are BRST-closed.
At first, from the usual coset construction given by

$$\frac{SL(2,\mathbb{R}) \times SU(2) \times \mathbb{R}_t \times U(1)_y}{U(1)_L \times U(1)_R} \times U(1)^4, \tag{4.76}$$

let's analyze only the upstairs theory.
The null currents being gauged are

$$\begin{aligned} J = i\mathcal{J} = J^3 + l_2 K^3 + l_3 i\partial t + l_4 i\partial y, \\ \bar{J} = i\bar{\mathcal{J}} = \bar{J}^3 + r_2 \bar{K}^3 + r_3 i\bar{\partial} t + r_4 i\bar{\partial} y, \end{aligned} \tag{4.77}$$

where the notation is slightly changed respect to 4.49.
In order to have a supersymmetric theory describing a BPS background, the superparters of the null-currents have to be defined as

$$\boldsymbol{\lambda} = \psi^3 + l_2\chi^3 + l_3\lambda^t + l_4\lambda^y, \quad \bar{\boldsymbol{\lambda}} = \bar{\psi}^3 + r_2\bar{\chi}^3 + r_3\bar{\lambda}^t + r_4\bar{\lambda}^y. \tag{4.78}$$

In this scenario, the BRST charge reads as

$$\mathcal{Q} = \oint dz : \left[c\left(T + T_{\beta\gamma\tilde{\beta}\tilde{\gamma}}\right) + \gamma G + \tilde{c}J + \tilde{\gamma}\boldsymbol{\lambda} + \text{ ghosts } \right]. \tag{4.79}$$

Hence, only operators which meet the standard Virasoro and γG-invariance conditions, and are also neutral with respect to the bosonic currents J, $\bar{J}$ and annihilated by $\tilde{\gamma}_\lambda$ and $\bar{\tilde{\gamma}}_{\bar{\lambda}}$, are considered physical: these statements are translated into constraint equations that have to be imposed.
Finally, it is possible to determine all the constraints that need to be imposed on the vertex operators to ensure a well-defined gauge-invariant spectrum.
The lightest physical states (without winding) are represented by unflowed operators with a single fermionic excitation. All such operators must fulfill the Virasoro condition

$$-\frac{j(j-1)}{n_5}+\frac{j'(j'+1)}{n_5}-\frac{1}{4}E^2+\frac{1}{4}P_y^2=0, \tag{4.80}$$

and they are invariant under the action of $\boldsymbol{\lambda}$ by construction.
The null-gauge condition reads as

$$m+l_2m'+\frac{l_3}{2}E+\frac{l_4}{2}P_y=0, \quad \bar{m}+r_2\bar{m}'+\frac{r_3}{2}E+\frac{r_4}{2}P_y=0. \tag{4.81}$$

Let's now examine states that are spectrally flowed: here, the null-gauging constraints are given by

$$\begin{aligned} 0&=m+\frac{n_5}{2}\omega+l_2\left(m'+\frac{n_5}{2}\omega'\right)+\frac{l_3}{2}E+\frac{l_4}{2}P_{y,L},\\ 0&=\bar{m}+\frac{n_5}{2}\omega+r_2\left(\bar{m}'+\frac{n_5}{2}\bar{\omega}'\right)+\frac{r_3}{2}E+\frac{r_4}{2}P_{y,R}, \end{aligned} \tag{4.82}$$

where ω the spectral flow charge related to $SL(2,\mathbb{R})$, while $(\omega',\bar{\omega}')$ on $SU(2)$ and the momenta are

$$P_{y,L/R}=\left(\frac{n_y}{R_y}\pm\omega_yR_y\right), n_y,\omega_y\in\mathbb{Z}. \tag{4.83}$$

In this context, the Virasoro conditions become more intricate

$$\begin{aligned} \frac{1}{2}&=\frac{j'(j'+1)-j(j-1)}{n_5}-m\omega+m'\omega'+\frac{n_5}{4}\left(\omega'^2-\omega^2\right)-\frac{1}{4}\left(E^2-P_{y,L}^2\right)+N,\\ \frac{1}{2}&=\frac{j'(j'+1)-j(j-1)}{n_5}-\bar{m}\omega+\bar{m}'\bar{\omega}'+\frac{n_5}{4}\left(\bar{\omega}'^2-\omega^2\right)-\frac{1}{4}\left(E^2-P_{y,R}^2\right)+\bar{N}, \end{aligned} \tag{4.84}$$

with N and N' defining the excitation numbers as always.
Globally, the analysis has to be conducted using the universal cover of $SL(2,\mathbb{R})$, where the left and right spectral flow parameters must be identical, namely $\omega=\bar{\omega}$. Additionally, after gauging the (1+1)-dimensional cylinder $\mathbb{R}\times U(1)$, the resulting model has a single non-compact timelike direction that enforces $l_3=r_3$. There is also a more direct way to understand this constraint. For instance, let's substract the two equations in 4.84 and consider integer shifts given by q of the form

$$(\omega',\bar{\omega}',E,P_{y,L},P_{y,R})\to(\omega'-a_2q,\bar{\omega}'-b_2q,E+a_3q,P_{y,L}-a_4q,P_{y,R}-b_4q)\,. \tag{4.85}$$

After performing a compensating shift and imposing the null-gauge conditions, the weights 4.84 are unchanged under the shifts for some specific choice of the parameters

$$\begin{aligned} &l_2=\mathrm{m}+\mathrm{n}\in2\mathbb{Z}+1, \quad r_2=-(\mathrm{m}-\mathrm{n})\in2\mathbb{Z}+1, \quad \mathrm{m},\mathrm{n}\in\mathbb{Z},\\ &l_4=-\left(\mathrm{k}R_y-\frac{\mathrm{p}}{R_y}\right), \quad r_4=\mathrm{k}R_y+\frac{\mathrm{p}}{R_y}. \end{aligned} \tag{4.86}$$

By plugging the previous parameters into the null-gauging conditions one can easily find that

$$l_3=r_3=-\sqrt{\mathrm{k}^2R_y^2+\frac{\mathrm{p}^2}{R_y^2}+n_5\left(\,\mathrm{m}^2+\mathrm{n}^2-1\right)}, \tag{4.87}$$

as discussed before.
To summarize the results so far: starting with a class of generic null-gauged models, the gauged currents are defined in terms of eight embedding parameters, namely l_i and r_i for $i = 1, 2, 3, 4$ where, since the overall scale becomes irrelevant, one can impose $l_1 = r_1 = 1$. The remaining six parameters must satisfy the two null conditions and also $l_3 = r_3$ due to the non-compactness of t. Finally, by examining the worldsheet CFT, the theory is consistent only if the remaining three parameters can be expressed in terms of three integers denoted as m, n, and k.

4.3.3 Supergravity background and fuzzball solutions

The aim of this section is to briefly explain what can be said about the previous conditions found from the worldsheet analysis at the supergravity level, effectively finding that the consistency of the worldsheet imposes a condition on the absence of horizons for the background metric.
Let's consider the determinant of the metric 4.53, which, after imposing the usual null-gauging conditions, can be recasted as

$$\det g = -\left(\frac{n_5^2 \sin(2\theta)\sinh(2\rho)}{4\Sigma_0}\right)^2. \tag{4.88}$$

From the previous formula, it is easy to see that the determinant only vanishes at $\rho = 0$.
To determine whether this point corresponds to a singularity or an event horizon, one can compute the determinant of the induced metric on surfaces with constant ρ and t, and evaluate it at $\rho = 0$, namely

$$\lim_{\rho\to 0} \det g\Big|_{(y,\theta,\phi,\psi)} = -\left(\frac{n_5\,(l_3 - r_3)\sin(2\theta)}{4\Sigma_0(0,\theta)}\right)^2. \tag{4.89}$$

Hence, the consistency condition $l_3 = r_3$ found above is precisely the condition at the supergravity level that imposes the absence of both horizons and singularities in the background metric.
In other words, the WZW model 4.2.2 describes a consistent worldsheet theory only if, in the supergravity limit, the metric has no horizon: thus, it is precisely a fuzzball solution that was introduced at the end of Chapter 3.

4.4 Witten's black hole

Since, as seen above, WZW models are capable of reproducing fuzzball solutions, the question is: is it possible to also reproduce solutions with horizons? To understand this, the null-gauging of suitable models with horizons will be carried out, starting from Witten's black hole.
The Witten's black hole can be described as a $SL(2,\mathbb{R})$ coset model given by $G/H = \frac{SL(2,\mathbb{R})}{U(1)}$.
In this paragraph the Euclidean form will be presented at first and then, it will show how to obtain the correspondent Lorentzian version in a non trivial way. The main reference for Witten's black hole is certainly given by [30].

4.4.1 Euclidean cigar

Let's consider the following action

$$S_0 = \frac{k}{8\pi}\int_{M_2} \sqrt{h}h^{ij}\,\mathrm{Tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g\right) + k\frac{1}{12\pi}\int_{M_3} \mathrm{Tr}\, g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg. \tag{4.90}$$

Here, h is used to denote the determinant of the worldsheet metric, whereas g denotes the group element.

In order to build a gauge abelian model, one can consider the $U(1)$ subgroup generated infinitesimally by

$$\delta g=\epsilon\cdot\left\{\begin{pmatrix}0&1\\-1&0\end{pmatrix}g+g\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\},\tag{4.91}$$

that is associated to the abelian gauge field

$$\delta\mathcal{A}_i=\partial_i\alpha.\tag{4.92}$$

By using the complex coordinates $(z,\bar{z})$, the action of the model is

$$\begin{aligned}S=S_0+\frac{k}{2\pi}\int d^2z\Bigg\{&\mathcal{A}_{\bar{z}}\,\mathrm{Tr}\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix}g^{-1}\partial_z g\right]+\mathcal{A}_z\,\mathrm{Tr}\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix}\partial_{\bar{z}}gg^{-1}\right]+\\&+\mathcal{A}_z\mathcal{A}_{\bar{z}}\left(-2+\mathrm{Tr}\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix}g\begin{pmatrix}0&1\\-1&0\end{pmatrix}g^{-1}\right]\right)\Bigg\}.\end{aligned}\tag{4.93}$$

The chosen parameterization for the upstairs group is given by

$$g=e^{i\theta_L\sigma_3}e^{\rho\sigma_1}e^{i\theta_R\sigma_3},\tag{4.94}$$

where the parameters above are defined as

$$\theta_L=\frac{\tau-\sigma}{2},\quad\theta_R=\frac{\tau+\sigma}{2}.\tag{4.95}$$

One can easily integrate out the gauge field, finding the following effective action

$$S_{EFT}=\frac{k}{2\pi}\left[\partial\rho\bar{\partial}\rho+\tanh^2\rho\;\partial\sigma\bar{\partial}\sigma\right].\tag{4.96}$$

Hence, by comparing the final action with the string one, one finds a cigar form for the metric

$$ds^2=\frac{k}{2}(d\rho^2+\tanh^2\rho d\sigma^2),\tag{4.97}$$

while the B-field is exactly zero.
In order to obtain the physical results, one needs to remove the gauge redundancies. To do this, a gauge fixing is generally required. Considering that the gauge transformation is given by equations 4.91 and 4.92, one can certainly choose $\tau=0$. However, this gauge choice has no influence on the metric and the B-field since they do not explicitly depend on τ.
Furthermore, one can rewrite the previous model in a more suitable form. In particular, the action $S=S_0+S_A$ can be written as follows

$$S_0=\frac{k}{\pi}\left[\int_{\mathcal{M}_2}\frac{1}{2}\mathrm{Tr}(g_{SL}^{-1}\partial gg^{-1}\bar{\partial}g)+\int_{\mathcal{M}_3}\frac{1}{6}\mathrm{Tr}(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg)\right],\tag{4.98}$$

while the gauged component can be identified as

$$S_A=\frac{k}{\pi}\int_{\mathcal{M}_2}\mathrm{Tr}\left[-\left[\ell\left(\bar{\mathcal{A}}\right)\partial gg^{-1}\right]+\left[r\left(\mathcal{A}\right)g^{-1}\bar{\partial}g\right]-\left[g^{-1}\ell\left(\bar{\mathcal{A}}\right)gr\left(\mathcal{A}\right)\right]-\frac{1}{2}\ell\left(\bar{\mathcal{A}}\right)\ell\left(\mathcal{A}\right)-\frac{1}{2}r(\bar{\mathcal{A}})r(\mathcal{A})\right]d^2z,\tag{4.99}$$

where the Lie algebra embeddings are defined to be

$$l(\bar{\mathcal{A}})=il\bar{\mathcal{A}}\sigma_3,\quad l(\mathcal{A})=il\mathcal{A}\sigma_3,\quad r(\bar{\mathcal{A}})=-ir\bar{\mathcal{A}}\sigma_3,\quad r(\mathcal{A})=-ir\mathcal{A}\sigma_3.\tag{4.100}$$

In this scenario, the gauge transformation is simply given by

$$\mathcal{A} \to \mathcal{A} + \partial\alpha, \quad \bar{\mathcal{A}} \to \bar{\mathcal{A}} + \bar{\partial}\alpha, \quad g \to e^{il\alpha\sigma_3} g e^{ir\alpha\sigma_3}. \tag{4.101}$$

By using the parametrization 4.94 and the definitions 4.95, the last transformation is the same of

$$\begin{cases} \tau \to \tau + \alpha(l + r) \\ \sigma \to \sigma + \alpha(-l + r) \\ \rho \to \rho. \end{cases} \tag{4.102}$$

The gauge transformed action S' is different from the starting one S by

$$\delta S = S' - S = \frac{k}{\pi}(l - r) \int_{\mathcal{M}_2} d^2z[(-\mathcal{A}(l + r) + \partial\tau)\bar{\partial}\alpha + (\bar{\mathcal{A}}(l + r) - \bar{\partial}\tau)\partial\alpha]. \tag{4.103}$$

Hence, the model described by equations 4.98-4.100 is certaintly gauge invariant if $l = r$. This identification (which can be normalized to 1) is the same condition under which the action 4.98-4.100 is the equal to the Witten's action 4.93 (up to an irrelevant factor of 2).

4.4.2 Lorentzian version

In order to obtain the Lorentzian version, one could take the previous metric and just perform a Wick rotation in the angle variable,

$$ds^2 = \frac{k}{2}(d\rho^2 - \tanh^2\rho d\theta^2). \tag{4.104}$$

Witten showed that it is possible to obtain the same result by slightly changing the action of the gauged group [30].
Let's take some convenient different coordinates for the upstairs group, namely

$$g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix}, \text{ with } ab + uv = 1. \tag{4.105}$$

Now, the gauged current is taken to be proportional to the third pauli matrix

$$\delta g = \epsilon \cdot \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g + g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{4.106}$$

The action of the model is straightforward to obtain and reads as follows:

$$\begin{aligned} S = &-\frac{k}{4\pi} \int d^2z\, (\partial_z u \partial_z v + \partial_z u \partial_x v + \partial_x a \partial_\Sigma b + \partial_1 a \partial_z b) + \\ &+ \frac{k}{2\pi} \int d^2z \left(\bar{\mathcal{A}}\, (b\partial a - a\partial b + u\partial v - v\partial u) + \mathcal{A} \left(b\bar{\partial} a - a\bar{\partial} b - u\bar{\partial} v + v\bar{\partial} u \right) + \right. \\ &\left. + \mathcal{A}\bar{\mathcal{A}}(4 - 4uv) + \ln a \left(\partial u \bar{\partial} v - \bar{\partial} u \partial v \right) \right), \end{aligned} \tag{4.107}$$

with the gauge symmetry that simply acts as the following translations

$$\delta a = 2\epsilon a, \quad \delta b = -2\epsilon b, \quad \delta u = \delta v = 0, \quad \delta \mathcal{A}_i = \partial_i \alpha. \tag{4.108}$$

After employing the same technology explained so far and imposing $a = b$ as a gauge choice, one can obtain a vanishing B-field and a metric that is

$$ds^2 = -\frac{k}{2} \frac{dudv}{1 - uv}. \tag{4.109}$$

The previous metric may seem unusual, but it is an extension of the usual Lorentzian Witten's black hole. Indeed, let's consider the proper Lorentzian Witten's black hole metric 4.104 and let's perform a radial redefinition

$$\rho' = \rho + \ln\left(1 - e^{-2\rho}\right). \tag{4.110}$$

In this fashion, the same line element just reads as

$$ds^2 = \frac{k}{2}\left[\tanh^2\rho \cdot \left(d\rho'^2 - dt^2\right)\right]. \tag{4.111}$$

Moreover, by performing the following change of coordinates

$$\begin{aligned} &2v = e^{\rho'+t}, \quad 2u = -e^{\rho'-t} \\ &\cosh^2\rho = 1 - uv, \quad \sinh^2\rho = -uv, \end{aligned} \tag{4.112}$$

finally, the Lorentzian Witten's black hole metric in these coordinates is just

$$ds^2 = -\frac{k}{2}\frac{dudv}{1-uv}, \tag{4.113}$$

that is exactly the metric found above.
In this coordinates, it is clear that the real singularity is $uv = 1$ since the scalar curvature

$$R \sim \frac{1}{\cosh^2\rho} = \frac{1}{1-uv} \tag{4.114}$$

diverges as it approaches the singularity, whereas $\rho = 0$ (namely, $uv = 0$) corresponds to a coordinate singularity.
The utility of the model in equations 4.98-4.100, used in the Euclidean case, can also be realized in the Lorentzian case.
Moreover, one can consider the group parametrization

$$g = e^{\theta_L\sigma_3}e^{\rho\sigma_1}e^{\theta_R\sigma_3} \tag{4.115}$$

obtained by performing an analytic continuations of the parameters in equation 4.95.
Then, by using exactly the same action described by equations 4.98-4.99 and by modifying the embeddings as follows

$$l(\bar{\mathcal{A}}) = l\bar{\mathcal{A}}\sigma_3, \quad l(\mathcal{A}) = l\mathcal{A}\sigma_3, \quad r(\bar{\mathcal{A}}) = -r\bar{\mathcal{A}}\sigma_3, \quad r(\mathcal{A}) = -r\mathcal{A}\sigma_3, \tag{4.116}$$

the effective action, obtained after integrating out the gauge fields and by imposing again the choice $l = r = 1$, is

$$S_{EFT} = \frac{k}{\pi}\left[\partial\rho\bar{\partial}\rho - \tanh^2\rho\,\partial\sigma\bar{\partial}\sigma\right], \tag{4.117}$$

leading to a vanishing B-field and the following spacetime metric

$$ds^2 = k\left[d\rho^2 - \tanh^2\rho\, d\sigma^2\right], \tag{4.118}$$

as expected for the Lorentzian version of Witten's black hole.
In the previous derivation, the fact that both parameters l and r need to have the same sign is essential for two different reasons. The first reason is that if the parameters are set, for instance, to $l = -r = 1$, then the metric contains the following cotangent term:

$$ds^2 = k\left[d\rho^2 - \coth^2\rho\, d\sigma^2\right], \tag{4.119}$$

resulting in a different model.
A more fundamental reason is given by gauge invariance. Indeed, in this Lorentzian model, the gauge transformations are essentially the same as in the Euclidean case, namely they are described by the shifts in equation 4.102.

An analogous computation of the gauge-transformed action can be performed, and due to the absence of the i factors in the group parametrization and in the Lie algebra embeddings, the result has an opposite sign respect to the Euclidean one, namely

$$\delta S = S' - S = -\frac{k}{\pi}(l-r)\int_{\mathcal{M}_2} d^2z \left[(-\mathcal{A}(l+r)+\partial\tau)\bar{\partial}\alpha + (\bar{\mathcal{A}}(l+r)-\bar{\partial}\tau)\partial\alpha\right]. \tag{4.120}$$

Hence, exactly as before, the gauge invariance condition leads to $l = r$, which can be normalized to 1.

4.5 Null-gauged model for Witten's black hole

The aim is to formulate the previous Witten's black hole in an equivalent way by using the null-gauging procedure for both the Euclidean and Lorentzian signatures.

4.5.1 Euclidean version

Let's take the following coset group

$$G/H = \frac{SL(2,\mathbb{R})\times U(1)_y}{U(1)_L \times U(1)_R}, \tag{4.121}$$

where the general group element is taken to be

$$g = (g_{\text{sl}}, g_y) \quad = \left(e^{\frac{i}{2}(\tau-\sigma)\sigma_3}e^{\rho\sigma_1}e^{\frac{i}{2}(\tau+\sigma)\sigma_3}, e^{iy}\right). \tag{4.122}$$

Since this model is a specific case of the null-gauged model discussed in Section 4.2.2, the action is essentially the same as that in equation 4.43, except that terms related to $SU(2)$ and $\mathbb{R}_t$ are neglected, and the Wess-Zumino term is zero in the Abelian case. Moreover, the embeddings of the Lie algebras and the general conventions used are as follows:

$$\begin{aligned}
&\text{sgn}(\kappa_{SL})\kappa_{SL} = k, \quad \text{sgn}(\kappa_y)\kappa_y = -2, \quad \mathcal{A}_1 = 0, \quad \bar{\mathcal{A}}_2 = 0,\\
&l\left(\bar{\mathcal{A}}_1\right)_{SL} = il_1\bar{\mathcal{A}}_1\sigma_3, \quad r\left(\mathcal{A}_2\right)_{SL} = -ir_1\mathcal{A}_2\sigma_3, \quad l\left(\bar{\mathcal{A}}_1\right)_y = -il_4\bar{\mathcal{A}}_1, \quad r\left(\mathcal{A}_2\right)_y = ir_4\mathcal{A}_2.
\end{aligned} \tag{4.123}$$

So that, the ungauged part becomes

$$S_0 = \int_{\mathcal{M}_2} d^2z\frac{1}{\pi}\left[k\left(\partial\rho\bar{\partial}\rho + \text{sh}^2\rho\partial\sigma\bar{\partial}\sigma - \text{ch}^2\rho\partial\tau\bar{\partial}\tau - \text{sh}^2\rho(\partial\sigma\bar{\partial}\tau - \partial\tau\bar{\partial}\sigma\right) + \partial y\bar{\partial}y\right], \tag{4.124}$$

as expected.
On the contrary, by taking the symmetric and antysimmetric part of the action, one can easily find

$$\begin{aligned}
ds^2 &= k\left(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\sigma^2\right) + dy^2,\\
H &= k(sinh(2\rho)d\rho\wedge d\tau\wedge d\sigma).
\end{aligned} \tag{4.125}$$

Instead, by using the notations 4.123, the gauged part of the action is given by

$$S_A = \frac{2}{\pi}\int_{\mathcal{M}_2}\left[\bar{\mathcal{A}}_1\left(l_1\text{j}^3_{\text{sl}} + l_4\partial y\right) + \mathcal{A}_2\left(r_1\bar{\text{j}}^3_{\text{sl}} + r_4\bar{\partial}y\right) - \mathcal{A}_2\bar{\mathcal{A}}_1(kl_1r_1\cosh(2\rho) - l_4r_4\right] d^2z, \tag{4.126}$$

where the currents are defined to be

$$\mathrm{j}^3_{\mathrm{sl}} = k\,\mathrm{Tr}\left(-i\frac{\sigma_3}{2}\partial g_{\mathrm{sl}} g_{\mathrm{sl}}^{-1}\right), \quad \bar{\mathrm{j}}^3_{\mathrm{sl}} = k\,\mathrm{Tr}\left(-i\frac{\sigma_3}{2} g_{\mathrm{sl}}^{-1}\bar{\partial} g_{\mathrm{sl}}\right). \tag{4.127}$$

One can also simplify their expression, finding

$$j^3_{\mathrm{sl}} = k\left(\cosh^2\rho\partial\tau + \sinh^2\rho\partial\sigma\right), \quad \bar{j}^3_{\mathrm{sl}} = k\left(\cosh^2\rho\bar{\partial}\tau - \sinh^2\rho\bar{\partial}\sigma\right) \tag{4.128}$$

.

By integrating out of the gauge fields via their equation of motion, they can be expressed as

$$\begin{aligned} \bar{\mathcal{A}}_1 &= \frac{r_1\bar{j}^3_{sl} + r_4\bar{\partial}y}{l_1r_1k\cosh(2\rho) - l_4r_4}, \\ \mathcal{A}_2 &= \frac{l_1j^3_{sl} + l_4\partial y}{l_1r_1k\cosh(2\rho) - l_4r_4}. \end{aligned} \tag{4.129}$$

Hence, the effective action becomes

$$\begin{aligned} S = S_0 + \int_{\mathcal{M}_2} d^2z\frac{2}{\pi}\Bigg\{\frac{1}{k\cosh^2\rho - l_4r_4}\Big[&k^2\cosh^4\rho\,\partial\tau\,\bar{\partial}\tau - k^2\cosh^2\rho\sinh^2\rho\,\partial\tau\,\bar{\partial}\sigma+ \\ + k^2\cosh^2\rho\sinh^2\rho\,\partial\sigma\,\bar{\partial}\tau - k^2\sinh^4\rho\,\partial\sigma\,\bar{\partial}\sigma + r_4k\,\bar{\partial}y&\left(\cosh^2\rho\,\partial\tau + \sinh^2\rho\,\partial\sigma\right)+ \\ + l_4k\,\partial y\left(\cosh^2\rho\,\bar{\partial}\tau - \sinh^2\rho\,\bar{\partial}\sigma\right) + l_4r_4\,\partial y\,\bar{\partial}y\Big]\Bigg\}&, \end{aligned} \tag{4.130}$$

where l_1 and r_1 were set to 1.

Furthermore, the null conditions are given by

$$\begin{aligned} &-kl_1^2 + l_4^2 = 0, \\ &-kr_1^2 + r_4^2 = 0, \end{aligned} \tag{4.131}$$

and, by using $l_1 = r_1 = 1$, they can be used to express l_4 and r_4 as

$$l_4 = -r_4 = \sqrt{k}. \tag{4.132}$$

Then, after having performed the gauge choice $\sigma = \tau = 0$, the effective action reads as

$$S_{EFT} = \frac{1}{\pi}\int_{\mathcal{M}_2} d^2z[(1+\frac{2l_4r_4}{-l_4r_4 + kl_1r_1\cos(2\rho)})\partial y\bar{\partial}y + k\partial\rho\bar{\partial}\rho] = \frac{1}{\pi}\int_{\mathcal{M}_2} d^2z[\tanh^2\rho\partial y\bar{\partial}y + k\partial\rho\bar{\partial}\rho], \tag{4.133}$$

from which one can extract the metric and the B-field

$$\begin{aligned} &ds^2 = kd\rho^2 + \tanh^2\rho dy^2, \\ &B = 0. \end{aligned} \tag{4.134}$$

The previous gauge choice was performed since, coherently with the previous conventions, the gauge transformation acts as

$$\begin{cases} g \to \left(e^{i\alpha\sigma_3}g_{\mathrm{sl}}e^{i\beta\sigma_3}, e^{-i\sqrt{k}\alpha}g_y e^{i\sqrt{k}\beta}\right) \\ \mathcal{A} \to \mathcal{A} + \partial\beta \\ \bar{\mathcal{A}} \to \bar{\mathcal{A}} + \bar{\partial}\alpha. \end{cases} \tag{4.135}$$

The first line can be expressed as (by using $k = 1$, for simplicity)

$$\begin{cases} \tau \mapsto \tau + \alpha + \beta \\ \sigma \mapsto \sigma + \beta - \alpha \\ y \mapsto y + \beta - \alpha \\ \rho \mapsto \rho. \end{cases} \tag{4.136}$$

Furthermore, one needs to verify that the previous model is exactly gauge invariant with respect to 4.135. By using the null conditions 4.131, the transformed action is different from the starting one by

$$\begin{aligned} \delta S = S' - S = \frac{1}{\pi} \int_{\mathcal{M}_2} d^2 z [\sqrt{k}(\partial y \bar{\partial} \beta - \bar{\partial} y \partial \beta) + \sqrt{k}(\partial y \bar{\partial} \alpha - \bar{\partial} y \partial \alpha) + 2k(\partial \beta \bar{\partial} \alpha - \bar{\partial} \beta \partial \alpha) + \\ k(\partial \tau \bar{\partial} \alpha - \bar{\partial} \tau \partial \alpha) + k(\partial \beta \bar{\partial} \tau - \bar{\partial} \beta \partial \tau)]. \end{aligned} \tag{4.137}$$

Gauge invariance is guaranteed as $\delta S = 0$ and this is exactly the case. In fact, each term in the previous equation is a boundary term as

$$\int_{\mathcal{M}_2} d^2 z [\partial \omega \bar{\partial} \gamma - \partial \gamma \bar{\partial} \omega] = \int_{\mathcal{M}_2} d\omega \wedge d\gamma = \int_{\mathcal{M}_3} d(d\omega \wedge d\gamma) = 0, \tag{4.138}$$

where Stokes' theorem and the nilpotency of the exterior derivative are involved.
In other words, the previous model is gauge invariant as expected.
Let's notice that by employing the following different parametrization for the upstairs group

$$g = (g_{\text{sl}}, g_y) \quad = \left(e^{\frac{i}{2}(\tau - \sigma)\sigma_2} e^{\rho \sigma_1} e^{\frac{i}{2}(\tau + \sigma)\sigma_2}, e^{iy} \right), \tag{4.139}$$

and by gauging the currents j^2_{sl}, $\bar{j}^2_{sl}$ instead of the third ones, the expressions for the metric and the B-field remain the same.

4.5.2 Lorentzian version

Analogously to what was done previously for the Lorentzian non-null case, let us change the parametrization by performing an analytic continuation on the parameters of the group parametrization. Hence, the convenient parametrization is

$$g = (g_{\text{sl}}, g_t) \quad = (e^{\frac{(\tau - \sigma)}{2}\sigma_3} e^{\rho \sigma_1} e^{\frac{(\tau + \sigma)}{2}\sigma_3}, e^t), \tag{4.140}$$

where the coset group involved is $G/H = \frac{SL(2,\mathbb{R}) \times \mathbb{R}_t}{U(1)_L \times U(1)_R}$.
Referring again to 4.43, the null-gauged action is

$$\begin{aligned} S = \frac{k}{\pi} \Bigg[\int_{M_2} d^2 z \left(\frac{1}{2} \operatorname{tr} \left(g_{SL}^{-1} \partial g_{SL} g_{SL}^{-1} \bar{\partial} g_{SL} \right) \right) + \int_{M_3} \frac{1}{6} \operatorname{tr} \left(g_{SL}^{-1} dg_{SL} \wedge g_{SL}^{-1} dg_{SL} \wedge g_{SL}^{-1} dg_{SL} \right) + \\ + \int_{M_2} d^2 z \operatorname{tr} \left[\left(-l \left(\bar{\mathcal{A}}_1 \right)_{SL} \partial g_{SL} {g_{SL}}^{-1} \right) + \left(r \left(\mathcal{A}_2 \right)_{SL} {g_{SL}}^{-1} \bar{\partial} g_{SL} \right) - \operatorname{tr} \left(g_{SL}^{-1} l \left(\bar{\mathcal{A}}_1 \right)_{SL} g_{SL} r \left(\mathcal{A}_2 \right)_{SL} \right] \Bigg] + \\ - \frac{1}{\pi} \int d^2 z \partial t \bar{\partial} t - \frac{2}{\pi} \int d^2 z \left(-l_t \left(\bar{\mathcal{A}}_1 \right) \partial t + r_t \left(\mathcal{A}_2 \right) \bar{\partial} t - l_t \left(\bar{\mathcal{A}}_1 \right) r_t \left(\mathcal{A}_2 \right) \right), \end{aligned}$$

where here the embedding homomorphisms are slightly changed as follows

$$l\left(\bar{\mathcal{A}}_1\right)_{SL} = l_1\bar{\mathcal{A}}_1\sigma_3, \quad r\left(\mathcal{A}_2\right)_{SL} = -r_1\mathcal{A}_2\sigma_3, \quad l\left(\bar{\mathcal{A}}_1\right)_t = l_3\bar{\mathcal{A}}_1, \quad r\left(\mathcal{A}_2\right)_t = -r_3\mathcal{A}_2. \tag{4.141}$$

After having integrated out the gauged fields and under the gauge choice $\sigma = \tau = 0$, the action is

$$S_{EFT} = \frac{1}{\pi}\int_{\mathcal{M}_2}\left[k\partial\rho\bar{\partial}\rho + (-1 + \frac{2l_3r_3}{l_3r_3 - kl_1r_1\cosh(2\rho)})\partial t\bar{\partial}t\right], \tag{4.142}$$

and by imposing the null conditions as $l_3 = -r_3 = \sqrt{k}$, $l_1 = r_1 = 1$, the resulting metric and B-field are

$$\begin{aligned} &ds^2 = kd\rho^2 - \tanh^2\rho dt^2, \\ &B = 0, \end{aligned} \tag{4.143}$$

as expected for the Witten's black hole in Lorentzian signature.

Let us mention that the general gauge transformations here are slightly different, as follows:

$$\begin{cases} \tau \mapsto \tau + l_1\alpha + r_1\beta \\ \sigma \mapsto \sigma + r_1\beta - l_1\alpha \\ t \mapsto t + l_3\alpha + r_3\beta \\ \rho \mapsto \rho, \end{cases} \tag{4.144}$$

while the transformations related to the gauge fields are the same of equation 4.135.
One could prove gauge invariance, in a manner similar to the Euclidean case. Indeed, by applying the null conditions, gauge invariance holds also for such a Lorentzian null-gauged model.

4.6 $\frac{SL(2,\mathbb{R})_k\times U(1)}{U(1)}$ WZW model

Now, the idea is to extend the Witten's previous black hole model to be able to describe more general black hole solutions in null-gauged WZW models. Similar to the approach taken for the cigar, one needs to start with some known non-null gauged WZW models to describe certain black hole solutions and then consider the corresponding null-gauged model. The simplest generalization involves adding an external abelian group $U(1)$, as discussed in [21].
In particular, the goal is to demonstrate the equivalence between the weakly coupled worldsheet string theory described by the coset sigma model $\frac{SL(2,\mathbb{R})_k\times U(1)}{U(1)} \times S^3 \times T^4$ with $SL(2,\mathbb{R})$ WZW level $k \geq 2$ and the non-extremal three-charge black hole in the NS5 decoupling limit.
This decoupling limit theory involves k NS5 branes wrapping $T^4\times S^1, p \gg 1$ F1 strings wrapping S^1 and n units of momentum along the S^1.

4.6.1 Time-like coset formulation

Here, we are using the following parametrization for the upstairs group

$$G = (g, g_x) = (e^{\alpha\sigma_3}e^{\rho\sigma_1}e^{\beta\sigma_3}, e^x), \tag{4.145}$$

that is the usual one up to the identification $\theta_L = \alpha$ and $\theta_R = \beta$.
As mentioned before a lot of times, again the WZW action is the sum of the two terms $S = S_0 + S_A$, where the first piece is

$$
\begin{aligned}
S_0 = \frac{k}{\pi}\left[\int_{\mathcal{M}_2}\frac{1}{2}\,\mathrm{Tr}(g_{SL}^{-1}\partial g_{SL}g_{SL}^{-1}\bar{\partial}g_{SL}) + \int_{\mathcal{M}_3}\frac{1}{6}\,\mathrm{Tr}(g_{SL}^{-1}dg_{SL}\wedge g_{SL}^{-1}dg_{SL}\wedge g_{SL}^{-1}dg_{SL})\right] + \\
+\frac{2}{\pi}\left[\int_{\mathcal{M}_2}\frac{1}{2}\,\mathrm{Tr}(g_x^{-1}\partial g_x g_x^{-1}\bar{\partial}g_x),\right.
\end{aligned}
\tag{4.146}
$$

while the gauged component is given by

$$
\begin{aligned}
S_A = \frac{k}{\pi}\int_{\mathcal{M}_2}\mathrm{Tr}\Bigg[&-\left[\ell\left(\bar{\mathcal{A}}\right)_{SL}\partial g_{SL}g_{SL}{}^{-1}\right] + \left[r\left(\mathcal{A}\right)_{SL}g_{SL}{}^{-1}\bar{\partial}g_{SL}\right] - \left[g_{SL}{}^{-1}\ell\left(\bar{\mathcal{A}}\right)_{SL}g_{SL}r\left(\mathcal{A}\right)_{SL}\right] \\
&-\frac{1}{2}\ell\left(\bar{\mathcal{A}}\right)_{SL}\ell\left(\mathcal{A}\right)_{SL} - \frac{1}{2}r\left(\bar{\mathcal{A}}\right)_{SL}r\left(\mathcal{A}\right)_{SL}\Bigg]d^2z + \frac{2}{\pi}\int_{\mathcal{M}_2}\Bigg[-\left[\ell\left(\bar{\mathcal{A}}\right)_x\partial g_x g_x{}^{-1}\right] \\
&+\left[r\left(\mathcal{A}\right)_x g_x{}^{-1}\bar{\partial}g_x\right] - \left[g_x{}^{-1}\ell\left(\bar{\mathcal{A}}\right)_x g_x r\left(\mathcal{A}\right)_x\right] - \frac{1}{2}r\left(\bar{\mathcal{A}}\right)_x r\left(\mathcal{A}\right)_x - \frac{1}{2}\ell\left(\bar{\mathcal{A}}\right)_x\ell\left(\mathcal{A}\right)_x\Bigg]d^2z,
\end{aligned}
\tag{4.147}
$$

Here, the gauge embeddings are defined by

$$
\begin{aligned}
&l(\bar{\mathcal{A}})_{SL} = \frac{-l_1\bar{\mathcal{A}}}{\sqrt{k}}\sigma_3, \quad l(\mathcal{A})_{SL} = \frac{-l_1\mathcal{A}}{\sqrt{k}}\sigma_3, \quad r(\mathcal{A})_{SL} = \frac{r_1\mathcal{A}}{\sqrt{k}}\sigma_3, \quad r(\bar{\mathcal{A}})_{SL} = \frac{r_1\bar{\mathcal{A}}}{\sqrt{k}}\sigma_3, \\
&l(\bar{\mathcal{A}})_x = -l_4\bar{\mathcal{A}}, \quad l(\mathcal{A})_x = -l_4\mathcal{A}, \quad r(\mathcal{A})_x = r_4\mathcal{A}, \quad r(\bar{\mathcal{A}})_x = r_4\bar{\mathcal{A}},
\end{aligned}
\tag{4.148}
$$

where k is the level of the $SL(2,\mathbb{R})$ part.

After integrating out the gauge fields and imposing the convenient gauge fixing condition $\alpha = -\beta = y/2$, one can find

$$
\begin{aligned}
ds^2 &= kd\rho^2 + \left(\frac{1}{1-\frac{4l_4r_4}{l_1^2+r_1^2+(l_4+r_4)^2-2l_1r_1\cosh(2\rho)}}\right)dx^2 - \left(\frac{k\sinh^2(\rho)[(l_1-r_1)^2+(l_4-r_4)^2]}{l_1^2+r_1^2+(l_4-r_4)^2-2l_1r_1\cosh(2\rho)}\right)dy^2 + \\
&+ \left(\frac{4\sqrt{k}\sinh^2(\rho)(l_4r_1-l_1r_4)}{l_1^2+r_1^2+(l_4-r_4)^2-2l_1r_1\cosh(2\rho)}\right)dxdy, \\
B &= \frac{2\sqrt{k}[l_4r_1+l_1r_4]}{l_1^2+r_1^2+(l_4-r_4)^2-2l_1r_1\cosh(2\rho)}dx\wedge dy.
\end{aligned}
\tag{4.149}
$$

The previous complicated formulas can be simplified by imposing the so-called timelike condition on the currents, namely

$$
\begin{cases} l_1^2 + l_4^2 = 1 \\ r_1^2 + r_4^2 = 1. \end{cases}
\tag{4.150}
$$

Hence, in this timelike coset formulation, the results simply read as

$$
\begin{aligned}
ds^2 &= (1-\frac{2l_4r_4}{\Delta})\,dx^2 - \frac{(-1+l_1r_1+l_4r_4)\sinh^2(\rho)}{\Delta}\,dy^2 \quad + \frac{2\,(-l_4r_1+l_1r_4)\sinh^2(\rho)}{\Delta}\,dx\,dy + k\,d\rho^2, \\
B &= -\frac{(l_4r_1+l_1r_4)\sinh^2(\rho)}{\Delta}\,dx\wedge dy,
\end{aligned}
\tag{4.151}
$$

where

$$\Delta = -1 + l_4 r_4 + l_1 r_1 \cosh(2\rho). \tag{4.152}$$

One can also check that the model is invariant under gauge transformations that here are given by

$$\begin{cases} g = (g_{SL}, g_x) \to \left(e^{-\frac{l_1}{\sqrt{k}}\gamma\sigma_3} g_{\mathrm{SL}} e^{-\frac{r_1}{\sqrt{k}}\gamma\sigma_3}, e^{-l_4\gamma} g_x e^{-r_4\gamma}\right) \\ \mathcal{A} \to \mathcal{A} + \partial\gamma \\ \bar{\mathcal{A}} \to \bar{\mathcal{A}} + \bar{\partial}\gamma, \end{cases} \tag{4.153}$$

or, equivalently,

$$\begin{cases} \alpha \to \alpha - \frac{l_1}{\sqrt{k}}\gamma \\ \beta \to \beta - \frac{r_1}{\sqrt{k}}\gamma \\ x \to x - (l_4 + r_4))\gamma \\ \mathcal{A} \to \mathcal{A} + \partial\gamma \\ \bar{\mathcal{A}} \to \bar{\mathcal{A}} + \bar{\partial}\gamma. \end{cases} \tag{4.154}$$

The timelike conditions can be implemented by defining some χ, ψ angles such that

$$\begin{aligned} l_1 &= \cos(\chi - \psi), \\ l_4 &= \sin(\chi - \psi), \\ r_1 &= -\cos(\chi), \\ r_3 &= -\sin(\chi). \end{aligned} \tag{4.155}$$

It might seem simpler to define the parameters l_i and r_i without considering the difference $\chi - \psi$. However, this approach is taken because it precisely reproduces [21] as will be discussed in the following paragraph.

4.6.2 An alternative formulation

Let's follow the notations used in [21]. In particular, the parametrization of the group element is simply given by $G = (g, g_x) = (e^{\alpha\sigma_3} e^{\rho\sigma_1} e^{\beta\sigma_3}, e^{\sqrt{2/k}x})$.
The action is given by

$$S_0 = \frac{k}{4\pi}\left[\int_\Sigma d^2 z \,\mathrm{tr}\left(g^{-1}\partial g g^{-1}\bar{\partial} g\right) - \frac{1}{3}\int_B d^3 X \,\mathrm{tr}\left(\left(g^{-1}dg\right)^3\right)\right] + \frac{1}{2\pi}\int_\Sigma d^2 z \partial x \bar{\partial} x, \tag{4.156}$$

while the gauged action reads as

$$S_A = \frac{1}{2\pi}\int_\Sigma d^2 z \left(\mathcal{A}\bar{\mathbf{J}} + \bar{\mathcal{A}}\mathbf{J} + 2\mathcal{A}\bar{\mathcal{A}}(R\vec{u})^T M\vec{u}\right), \tag{4.157}$$

with $\vec{u}$ is a unit vector and R, M are two 2×2 matrices defined as

$$R = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix}, \tag{4.158}$$

$$\vec{u} = \begin{pmatrix} \cos\chi \\ \sin\chi \end{pmatrix}, \tag{4.159}$$

$$M = \begin{pmatrix} \frac{1}{2}\operatorname{tr}\left(g^{-1}\sigma_3 g \sigma_3\right) & 0 \\ 0 & 1 \end{pmatrix} + R. \tag{4.160}$$

The action is constructed to be gauge invariant, and additionally, the terms involving R and M are required to be present since they ensure that the theory respects a condition involving the gauge generators T_L and T_R, namely

$$\operatorname{tr}\left(T_L^2\right) = \operatorname{tr}\left(T_R^2\right), \tag{4.161}$$

that is necessary for having an anomaly free gauging.
Up to a factor of 2, the action above has essentially the same structure of 4.146, 4.147. Specifically, by comparing the two actions, there is a clear dictionary that is specified by equation 4.155.
One might wonder about the flip of the sign in the WZ term: the sign (or the presence of an i factor) is strongly dependent on conventions, but in this case, it does not have any effects here since the chosen gauge fixing $\alpha = -\beta = y/2$ is such that the WZ term vanishes.
By performing these subtitutions, one can be easily able to pass from the the initial formulation to this alternative's one.
In this scenario, the currents that need to be gauged are the following

$$\begin{aligned} \mathbf{J} &= i\left(\sqrt{k}\operatorname{tr}\left(\partial g g^{-1}\sigma_3\right), 2\partial x\right)(R\vec{u}) = \left(\frac{2i}{\sqrt{k}}J^3, 2J_x\right)(R\vec{u}) = \frac{2i}{\sqrt{k}}\cos(\chi-\psi)J^3 + 2\sin(\chi-\psi)J_x, \\ \bar{\mathbf{J}} &= -i\left(\sqrt{k}\operatorname{tr}\left(g^{-1}\bar{\partial} g \sigma_3\right), 2\bar{\partial} x\right)\vec{u} = \left(\frac{2i}{\sqrt{k}}\bar{J}^3, -2\bar{J}_x\right)\vec{u} = \frac{2i}{\sqrt{k}}\cos\chi \bar{J}^3 - 2\sin\chi \bar{J}_x. \end{aligned} \tag{4.162}$$

By considering the previous action, one can analogously integrate out the gauge fields and, finally, the gauged action is reduced to

$$S_A = -\frac{1}{4\pi}\int_\Sigma d^2z \left[\frac{\mathbf{J}\bar{\mathbf{J}}}{(R\vec{u})^T(M\vec{u})}\right]. \tag{4.163}$$

Hence, after have performed the gauge choice $\alpha = -\beta = y/2$, the final action is

$$S = \frac{1}{2\pi}\int d^2z \partial x \bar{\partial} x + \frac{k}{2\pi}\int d^2z\left(\partial\rho\bar{\partial}\rho - \sinh^2\rho \partial y \bar{\partial} y\right) - \frac{1}{4\pi}\int d^2z \frac{\mathbf{J}\bar{\mathbf{J}}}{\Delta_\rho}, \tag{4.164}$$

where $\Delta_\rho = (R\vec{u})^T(M\vec{u}) = 1 + \cosh^2\rho\cos\psi + \sinh^2\rho\cos(2\chi-\psi)$.
Hence, the supergravity fields are expressed as

$$\begin{aligned} ds^2 =& \frac{k}{2}d\rho^2 + \frac{1}{2}\left(1 - \frac{2\sin\chi\sin(\chi-\psi)}{\Delta_\rho}\right)dx^2 - \frac{k}{2}\left(\frac{2\cos(\psi/2)\sinh^2\rho}{\Delta_\rho}\right)dy^2 + \\ &+ \frac{\sqrt{k}}{2}\left(\frac{2\sinh^2\rho\sin\psi}{\Delta_\rho}\right)dxdy, \\ B =& -\frac{\sqrt{k}}{2}\frac{\sin(2\chi-\psi)\sinh^2\rho}{\Delta_\rho}dx\wedge dy. \end{aligned} \tag{4.165}$$

Now, we want to argue that this solution is the near-horizon limit of the three-charge black hole given in equation 3.32. Indeed, let's perform the change of coordinates given by

$$\begin{aligned}
\tilde{\rho}^2 &= kl_s^2 \left(\Delta_\rho - 2\sin\chi \sin(\chi-\psi)\right), \\
t &= \sqrt{k} l_s \frac{\cos(\psi/2)\cos(\chi-\psi/2)}{\cos\chi\cos(\chi-\psi)} y, \\
x &\to \sqrt{k} l_s x - \frac{\sin\psi}{2\cos\chi\cos(\chi-\psi)} t.
\end{aligned} \tag{4.166}$$

The resulting theory is defined as

$$\begin{aligned}
ds^2 &= -\frac{\left(\tilde{\rho}^2-\tilde{\rho}_-^2\right)\left(\tilde{\rho}^2-\tilde{\rho}_-^2\right)}{\ell^2\tilde{\rho}^2} dt^2 + \frac{kl_s^2\tilde{\rho}^2}{\left(\tilde{\rho}^2-\tilde{\rho}_-^2\right)\left(\tilde{\rho}^2-\tilde{\rho}_-^2\right)} d\tilde{\rho}^2 + \frac{\tilde{\rho}^2}{\ell^2}\left(dx - \frac{\tilde{\rho}_+\tilde{\rho}_-}{\tilde{\rho}^2} dt\right)^2, \\
H &= dx \wedge dt \wedge d\left(\frac{2kl_s^2\sin(\chi-\psi/2)\cos(\psi/2)}{\ell^2}\right), \\
e^{2\Phi} &= e^{2\Phi_0}\frac{kl_s^2}{\ell^2},
\end{aligned} \tag{4.167}$$

where

$$\begin{aligned}
\tilde{\rho}_+^2 &= kl_s^2(1+\cos\psi - 2\sin\chi\sin(\chi-\psi)), \\
\tilde{\rho}_-^2 &= kl_s^2(1-\cos(\psi)), \\
\ell^2 &= \tilde{\rho}^2 + 2kl_s^2\sin\chi\sin(\chi-\psi).
\end{aligned} \tag{4.168}$$

The two backgrounds described by equations 4.167 and 3.35 are equivalent when considering the following identifications:

$$\begin{aligned}
r_0^2 &= 2kl_s^2\cos\chi\cos(\chi-\psi), \\
\sinh^2\alpha_1 - \sinh^2\alpha_n &= \tan\chi\tan(\chi-\psi), \\
\sinh 2\alpha_1 &= \frac{2\sin(\chi-\psi/2)\cos(\psi/2)}{\cos\chi\cos(\chi-\psi)}.
\end{aligned} \tag{4.169}$$

In addition, let's notice that by neglecting the abelian group in the upstairs group, namely by using $x \to 0$,$\psi \to 0$,$\chi \to 0$, the previous metric is reduced to be

$$\begin{aligned}
ds^2 &= \frac{k}{2}(d\theta^2 - \tanh^2(\rho)dy^2), \\
B &= 0,
\end{aligned} \tag{4.170}$$

that is the Lorentzian Witten's black hole as expected.

Failure in the extremal limit

In the previous paragraph, it was proven that the timelike coset formulation given by $\frac{SL(2,\mathbb{R})_k \times U(1)}{U(1)}$ was able to describe the near horizon limit of the three charge black solution 3.32.
However, it is crucial to stress that this is valid only in the non-extremal scenario and that in the extremal limit the previous timelike formulation is no longer able to describe the black hole solution.
The equivalence of the models 4.165 and 3.32 is based on the change of coordinates 4.166. In fact, the extremal limit is achieved by imposing

$$l_1 r_1 = 0, \tag{4.171}$$

or, equivalently,

$$\cos(\chi-\psi)\cos(\chi)=0. \tag{4.172}$$

This condition means that

$$\begin{aligned} &\chi=(2n+1)\tfrac{\pi}{2}, && \forall n\in\mathbb{Z},\\ &\chi=(2n+1)\tfrac{\pi}{2}+\psi, && \forall n\in\mathbb{Z}. \end{aligned} \tag{4.173}$$

Hence, one can notice that the change of coordinates 4.166 is not defined and the models cannot be matched to be same.

4.7 Null-gauged $\frac{SL(2,\mathbb{R})\times U(1)_x\times U(1)_t}{U(1)_L\times U(1)_R}$ WZW model

After discussing the timelike coset construction, we now aim to formulate the associated null-gauged model.
The group element of the model is taken to be

$$g=(g_{SL},g_x,g_t)=\left(e^{\alpha\sigma_3}e^{\rho\sigma_1}e^{\beta\sigma_3},e^x,e^t\right), \tag{4.174}$$

since in the null-gauging procedure one needs to add an abelian upstairs group, namely $U(1)_t$.
The action of the null-gauged model is

$$\begin{aligned} S_0=&\frac{k}{\pi}\left[\int_{\mathcal{M}_2}\frac{1}{2}\operatorname{Tr}(g_{SL}^{-1}\partial g_{SL}g_{SL}^{-1}\bar\partial g_{SL})+\int_{\mathcal{M}_3}\frac{1}{6}\operatorname{Tr}(g_{SL}^{-1}dg_{SL}\wedge g_{SL}^{-1}dg_{SL}\wedge g_{SL}^{-1}dg_{SL})\right]+\\ &+\frac{2}{\pi}\left[\int_{\mathcal{M}_2}\frac{1}{2}\operatorname{Tr}(g_x^{-1}\partial g_x g_x^{-1}\bar\partial g_x)-\frac{2}{\pi}\left[\int_{\mathcal{M}_2}\frac{1}{2}\operatorname{Tr}(g_t^{-1}\partial g_t g_t^{-1}\bar\partial g_t)\right],\right. \end{aligned} \tag{4.175}$$

where the WZ-term is zero for the abelian group.
Conversely, the gauged part of the action is

$$\begin{aligned} S_A=&\frac{k}{\pi}\int_{\mathcal{M}_2}\operatorname{Tr}\left(-\left[\ell\left(\bar{\mathcal{A}}\right)_{SL}\partial g_{SL}{g_{SL}}^{-1}\right]+\left[r\left(\mathcal{A}\right)_{SL}{g_{SL}}^{-1}\bar\partial g_{SL}\right]-\left[{g_{SL}}^{-1}\ell\left(\bar{\mathcal{A}}\right)_{SL}g_{SL}r\left(\mathcal{A}\right)_{SL}\right]\right)d^2z+\\ &-\frac{2}{\pi}\int_{\mathcal{M}_2}\left[-\left[\ell\left(\bar{\mathcal{A}}\right)_t\partial g_t{g_t}^{-1}\right]+\left[r\left(\mathcal{A}\right)_t{g_t}^{-1}\bar\partial g_t\right]-\left[{g_t}^{-1}\ell\left(\bar{\mathcal{A}}\right)_t g_t r\left(\mathcal{A}\right)_t\right]\right]d^2z+\\ &+\frac{2}{\pi}\int_{\mathcal{M}_2}\left[-\left[\ell\left(\bar{\mathcal{A}}\right)_x\partial g_x{g_x}^{-1}\right]+\left[r\left(\mathcal{A}\right)_x{g_x}^{-1}\bar\partial g_x\right]-\left[{g_x}^{-1}\ell\left(\bar{\mathcal{A}}\right)_x g_x r\left(\mathcal{A}\right)_t\right]\right]d^2z, \end{aligned} \tag{4.176}$$

where

$$\begin{aligned} &l(\bar{\mathcal{A}}_1)_{SL}=-l_1\bar{\mathcal{A}}_1\sigma_3,\quad r(\mathcal{A}_2)_{SL}=r_1\mathcal{A}_2\sigma_3,\quad l(\bar{\mathcal{A}}_1)_x=-l_4\bar{\mathcal{A}}_1,\quad r(\mathcal{A}_2)_x=r_4\mathcal{A}_2,\\ &l(\bar{\mathcal{A}}_1)_t=l_3\bar{\mathcal{A}}_1,\quad r(\mathcal{A}_2)_t=-r_3\mathcal{A}_2. \end{aligned} \tag{4.177}$$

By integrating out the gauge fields and by imposing the gauge choice $\sigma=\tau=0$ (namely, $\alpha=\beta=0$), one finds

$$S_{EFT}=\frac{1}{\pi}\int_{\mathcal{M}_2}d^2z[k\partial\rho\bar\partial\rho-\partial t\bar\partial t+\partial x\bar\partial x-2\frac{(l_3\partial t+l_4\partial x)(r_3\bar\partial t+r_4\bar\partial x)}{\Delta_{\text{NULL}}}], \tag{4.178}$$

where

$$\Delta_{\text{NULL}}=-l_3r_3+l_4r_4+kl_1r_1\cosh(2\rho). \tag{4.179}$$

The effective action described above leads to the following supergravity fields

$$
\begin{aligned}
ds^2 &= kd\rho^2 - \left(\frac{l_3r_3 + l_4r_4 + l_1r_1\cosh(2\rho)}{\Delta_{\text{NULL}}}\right)dt^2 - \left(\frac{2(l_4r_3 + l_3r_4)}{\Delta_{\text{NULL}}}\right)dtdx + \left(1 - \frac{2l_4r_4}{\Delta_{\text{NULL}}}\right)dx^2,\\
B &= \frac{l_3r_4 - l_4r_3}{\Delta_{\text{NULL}}}dx \wedge dt,
\end{aligned}
\tag{4.180}
$$

where the null-constraints are the following

$$
\begin{aligned}
&-kl_1^2 + l_3^2 - l_4^2 = 0,\\
&-kr_1^2 + r_3^2 - r_4^2 = 0.
\end{aligned}
\tag{4.181}
$$

Moreover, the gauge transformations of the model are given by

$$
\begin{cases}
g = (g_{SL}, g_x, g_t) \to \left(e^{-l_1\gamma\sigma_3}g_{\text{SL}}e^{-r_1\eta\sigma_3}, e^{-l_4\gamma}g_xe^{-r_4\eta}, e^{l_3\gamma}g_te^{r_3\eta}\right)\\
\mathcal{A} \to \mathcal{A} + \partial\eta\\
\bar{\mathcal{A}} \to \bar{\mathcal{A}} + \bar{\partial}\gamma,
\end{cases}
\tag{4.182}
$$

and they imply the following shifts

$$
\begin{cases}
\alpha \to \alpha - l_1\gamma\\
\beta \to \beta - r_1\eta\\
x \to x - l_4\gamma - r_4\eta\\
t \to t + l_3\gamma + r_3\eta\\
\mathcal{A} \to \mathcal{A} + \partial\eta\\
\bar{\mathcal{A}} \to \bar{\mathcal{A}} + \bar{\partial}\gamma
\end{cases}
\tag{4.183}
$$

under which the action is invariant since its variation is a sum of boundary terms, analogously to equation 4.137.

Moreover, one can notice that the expressions for the B-field and the metric in equation 4.180 are parameterized in the same way as the ones considered in equation 4.53, after removing the $SU(2)$ component. This does not mean that the solutions are the same, since the null-conditions in equation 4.181 are different from those in equation 4.50; hence, there are two distinct parameter spaces.

As a consistency check, one can consider the limit $x \to 0$, $r_4 \to 0$, $l_4 \to 0$ where the model reduces to $\frac{SL(2)\times U(1)_t}{U(1)_L\times U(1)_R}$.

Indeed, after having integrated out the gauge fields, the effective action becomes

$$
S_{cigar} = \lim_{x\to 0, r_4\to 0, l_4\to 0} S = \frac{1}{\pi}\int_{\mathcal{M}_2}\left[-\partial t\bar{\partial}t - \frac{2l_3r - 3\partial t\bar{\partial}t}{-l_3r_3 + kl_1r_1\cosh(2\rho)}\right],
\tag{4.184}
$$

that is the usual Lorentzian Witten's black hole action by imposing $l_1 = r_1 = 1$ and $l_3 = -r_3 = \sqrt{k}$. Hence, both the timelike coset model previously discussed and the current null-gauged model are able to reproduce the two dimensional Witten's black hole in some certain limits.

4.8 Comments on the spectrum of the two dimensional Witten's black hole and null-gauging procedure

In light of the BRST quantization procedure (see [31], [32] and [21]) introduced in Chapter 2, we now want to briefly analyze some properties related to the spectrum of the worldsheet CFT

defined for the Witten's black hole in Lorentzian signature (i.e., paragraph 4.4.2) and compare it with respect to the null-gauged case (i.e., section 4.5).
Hence, let's consider the ten dimensional critical string theory on a geometry given by $\frac{SL(2,\mathbb{R})_k}{U(1)} \times \mathcal{N}_8$, where $\mathcal{N}_8$ is an eight-dimensional spacelike compact manifold.
A primary vertex operator $V(z,\bar{z})$ can be written as

$$V(z,\bar{z}) = V_t(z,\bar{z})V_N, \tag{4.185}$$

where V_N is a primary vertex operator on $\mathcal{N}_8$ while $V_t(z,\bar{z})$ a primary vertex operator for the coset $\frac{SL(2,\mathbb{R})_k}{U(1)}$.
A standard ansatz is to factorize $V_t(z,\bar{z})$ into a contribution related to the upstairs group and one related to the downstairs group, namely

$$V_t(z,\bar{z}) = \Psi(z,\bar{z})e^{il_s(\kappa_L w_L+\kappa_R w_R)}, \tag{4.186}$$

where the coefficient $w(z,\bar{z}) = w_L(z) + w_R(\bar{z})$ was introduced to parametrize $U(1)$ at the denominator of the coset. The absence of winding can be traced to the condition $\kappa_L = \kappa_R = \kappa$.
A well-known fact is that a vertex operator for $SL(2,\mathbb{R})_k$ can be written in terms of the in the continuous representation of Euclidean $SL(2,\mathbb{R})$ defined to be the space generated by the common eigenstates of J_3 and $\bar{J}_3$ with eigenvalues im, $-i\bar{m}$. This fact can be expressed by writing

$$\Psi(z,\bar{z}) = V^j_{m,\bar{m}}. \tag{4.187}$$

By introducing the ghosts b, c and performing the BRST quantization method through the path integral, one can check that the BRST charges of the theory can be written as

$$Q_{BRST} = \frac{1}{2\pi i}\oint dz J_{BRST}, \quad \bar{Q}_{BRST} = \frac{1}{2\pi i}\oint d\bar{z}\bar{J}_{BRST}, \tag{4.188}$$

where

$$J_{BRST} = c\left(\mathbf{J} + 2J_w\right), \tag{4.189}$$

and similarly for $\bar{J_{BRST}}$.
The current above is defined to be $\mathbf{J} = l \cdot \mathrm{j}^3_{\mathrm{sl}}$, namely the usual $SL(2,\mathbb{R})_k$ current defined in the paragraph 4.4.2 and in section 4.5 with the coefficient l that was set to be 1.
By imposing that the state dual to the operator $V_t(z,\bar{z})$ must be BRST closed, one can derive the following constraints

$$\begin{aligned} l\cdot m + \frac{1}{2}l_s\kappa = 0,\\ r\cdot\bar{m} + \frac{1}{2}l_s\kappa = 0. \end{aligned} \tag{4.190}$$

The previous results that has been derived in the timelike Witten's model are in agreement with the one that can be found from the analysis of the corresponding null-gauged model.
Indeed, one can consider the limit of 4.81 where the upstairs $SU(2)$ and $U(1)_y$ are removed, namely

$$\begin{aligned} m\cdot\tilde{l}_1 + \frac{\tilde{l}_3}{2}E = 0,\\ \bar{m}\cdot\tilde{r}_1 + +\frac{\tilde{r}_3}{2}E = 0, \end{aligned} \tag{4.191}$$

with the parameters of the null model denoted by $\tilde{l}$ and $\tilde{r}$ to distinguish them from the timelike ones.
One can notice that there is a dictionary from the 4.190 to 4.191 that is given by

$$\begin{aligned} \tilde{l}_1 = l_1, \quad \tilde{r}_1 = r_1,\\ \tilde{l}_3 = 1, \quad \tilde{r}_3 = 1, \end{aligned} \tag{4.192}$$

where units such that $l_s = 1$ has been taken.
Therefore, the models are once again tested for equivalence with each other.
More specifically related to the spectrum of the theory, by considering Witten's model as a limit of the $\frac{SL(2,\mathbb{R})_k \times U(1)}{U(1)}$ model, as presented in [21], one can derive the Virasoro constraints. The results are given by

$$-\frac{j(j+1)}{k} - \frac{l_s^2 \kappa}{4} = \frac{1}{2} \tag{4.193}$$

in agreement with the Witten black hole's limit of the null-gauged model analyzed in Section 4.7.

One could repeat an analogous procedure for the timelike model discussed in Section 4.6 and its null-gauged variant analyzed in Section 4.7; however, the situation appears to be more complicated. Indeed, adding an external $U(1)$ appears to be non-trivial. Some efforts have been made to demonstrate the equivalence of these two models at the supergravity level, but it appears that there is no coordinate transformation that makes the metrics in equations 4.151 and 4.180 identical. Despite considering different gauge choices consistent with the gauge transformations given in equations 4.154 and 4.183, the equivalence has not been achieved. Hence, a more detailed analysis will be needed.

Conclusions

In this thesis, we have thoroughly investigated black hole solutions within the framework of string theory, with a particular focus on worldsheet descriptions using Wess-Zumino-Witten (WZW) models.

Chapter 1, which addresses the black hole information paradox, was dedicated to providing motivations and interests for the subsequent studies.

Chapter 2 covered the essential aspects of string theory necessary to understand the following sections.

Moreover, some well-known black hole and fuzzball solutions and their properties were briefly summarized in Chapter 3.

In Chapter 4, we discussed the null-gauging procedure for WZW models and demonstrated how a consistent fuzzball solution can be reproduced by analyzing the properties of a coset theory at the quantum level. We addressed the challenge of formulating a null-gauged WZW model capable of describing a black hole solution, starting with an analysis of Witten's two-dimensional black hole in both Euclidean and Lorentzian signatures and treating this solution in a null-gauged framework. We also considered other solutions, namely those obtained by extending Witten's black hole and its null-gauged version with an additional $U(1)$ group. In particular, it has been established that, in the extremal case, the timelike coset construction is not able to reproduce the decoupling limit of the three-charge black hole solution. Additionally, Witten's black hole is a consistent limit of this model. Moreover, some aspects related to the spectrum of these models were studied.

Looking ahead, it would be particularly interesting to analyze further the seeming differences between the timelike model discussed in Section 4.6 and its null-gauged variant analyzed in Section 4.7. It might be useful to delve deeper into the worldsheet consistency of the timelike model and study some strings effects on this background. Future work could also involve examining other groups, such as $SU(2)$, to describe black holes with angular momentum or more sophisticated scenarios involving non-Abelian groups in the gauging. Additionally, the consistency of the worldsheet CFT at the quantum level was only briefly introduced; further investigation could provide a deeper understanding of the differences compared to fuzzball solutions. An intriguing future direction would be to find a worldsheet description of black hole entropy. Given that the entropy, according to the Bekenstein-Hawking formula, is related to the black hole's area, it would be valuable to identify a CFT operator on the worldsheet that can reproduce the black hole entropy at leading order in α', thereby extending the results of [33], which are valid for the BTZ black hole solution.

Bibliography

[1] Stephen W. Hawking. "Black holes and thermodynamics". In: *Physical Review D* 13.2 (1976), pp. 191–197. DOI: `10.1103/PhysRevD.13.191`. URL: `https://doi.org/10.1103/PhysRevD.13.191`.

[2] Andrew Strominger and Cumrun Vafa. "Microscopic Origin of the Bekenstein-Hawking Entropy". In: *Physical Review Letters* 77.7 (1996), pp. 1471–1474. DOI: `10.1103/PhysRevLett.77.1471`. arXiv: `hep-th/9601029 [hep-th]`.

[3] Joseph Polchinski. "The Black Hole Information Problem". In: *Preprint* (Sept. 2016). Submitted on 13 Sep 2016. arXiv: `1609.04036 [hep-th]`.

[4] Samir D. Mathur. *The Information Paradox: A Pedagogical Introduction*. Tech. rep. Available at `arXiv:1104.2057 [hep-th]`. Columbus, OH 43210, USA: Department of Physics, The Ohio State University, Jan. 2011.

[5] A. Almheiri et al. "An Apologia for Firewalls". In: *Journal of High Energy Physics (JHEP)* 2013.09 (2013), p. 018. DOI: `10.1007/JHEP09(2013)018`. arXiv: `1304.6483 [hep-th]`.

[6] Joseph Polchinski. *String Theory, Volume 1*. Cambridge University Press, 2005.

[7] Thomas Weigand. "String Theory I and II". Lecture Notes, Institut für Theoretische Physik, Universität Heidelberg. 2011.

[8] Nikolay Beisert and Joerg Brodel. "Introduction to String Theory". Lecture Notes, ETH Zurich, HS13. 2014.

[9] Andreas Braun. "Introduction to String Theory". Lecture Notes, Durham. 2022.

[10] R.P. Feynman et al. *Feynman Lectures on Gravitation*. CRC Press, 1996, p. 275. ISBN: 9780429502859. DOI: `10.1201/9780429502859`.

[11] Pascal Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal Field Theory*. 1997.

[12] K. Wray. "An Introduction to String Theory". Lecture Notes. May 2011.

[13] Klaus Becker, Melanie Becker, and John H. Schwarz. *String Theory and M-theory*. 2007.

[14] Ralf Blumenhagen, Stefan Lüst, and Dieter Theisen. *Basic Concepts of String Theory*. 2012.

[15] Oliver Schlotterer. "Introduction to String Theory". Lecture Notes, Uppsala University. 2023.

[16] Joseph Polchinski. *String Theory, Volume 2*. Cambridge University Press, 2005.

[17] Jacob D. Bekenstein. "Black Holes and Entropy". In: *Physical Review D* 7.15 (Apr. 1973). Published 15 April 1973, p. 2333.

[18] I. Bena, S. El-Showk, and B. Vercnocke. "Cours de Physique Théorique: Black Holes in String Theory". 2012.

[19] Samir D. Mathur. "The Fuzzball Proposal for Black Holes: An Elementary Review". In: *Fortschritte der Physik* (2009).

[20] Emil J. Martinec, Stefano Massai, and David Turton. "String dynamics in NS5-F1-P geometries". In: *Journal of High Energy Physics (JHEP)* 2022.03 (2022), p. 085. DOI: 10.1007/JHEP03(2022)085. arXiv: 1803.08505 [hep-th].

[21] Soumangsu Chakraborty. "$SL(2,\mathbb{R}) \times U(1)/U(1)$ CFT, NS5 + **F1** system and single trace $T\bar{T}$". In: *Preprint* (Feb. 2021). arXiv: 2012.03995 [hep-th].

[22] Emil J. Martinec, Stefano Massai, and David Turton. "Little Strings, Long Strings, and Fuzzballs". In: *Journal of High Energy Physics (JHEP)* (May 2020). arXiv: 1906.11473 [hep-th].

[23] Davide Bufalini et al. "Black Hole Microstates from the Worldsheet". In: *Journal of High Energy Physics (JHEP)* (Oct. 2022). arXiv: 2105.02255 [hep-th].

[24] L. Eberhardt. "Wess-Zumino-Witten Models". Lecture Notes, YRISW PhD School in Vienna. Feb. 2019.

[25] José Figueroa-O'Farrill and Noureddine Mohammedi. "Gauging the Wess-Zumino Term of a Sigma Model with Boundary". In: *Journal of High Energy Physics (JHEP)* 0508 (2005), p. 086. DOI: 10.1088/1126-6708/2005/08/086. arXiv: hep-th/0506049 [hep-th].

[26] Emil J. Martinec, Stefano Massai, and David Turton. "Stringy Structure at the BPS Bound". In: *Journal of High Energy Physics (JHEP)* 2012.135 (2020). DOI: 10.1007/JHEP12(2020)135. arXiv: 2005.12344 [hep-th].

[27] Emil J. Martinec and Stefano Massai. "String Theory of Supertubes". In: *Journal of High Energy Physics (JHEP)* 07 (2018), p. 163. DOI: 10.1007/JHEP07(2018)163. arXiv: 1705.10844 [hep-th].

[28] T. Quella and V. Schomerus. "Asymmetric Cosets". In: *Journal of High Energy Physics (JHEP)* 02 (2003), p. 030. DOI: 10.1088/1126-6708/2003/02/030. eprint: hep-th/0212119.

[29] Juan Maldacena and Hirosi Ooguri. "Strings in AdS_3 and the SL(2,R) WZW Model. Part 1: The Spectrum". In: *Journal of Mathematical Physics* 42 (2001), pp. 2929–2960. DOI: 10.1063/1.1377273. arXiv: hep-th/0001053 [hep-th].

[30] Edward Witten. "On String Theory and Black Holes". In: *Preprint* (Mar. 1991). arXiv: hep-th/9108052 [hep-th].

[31] Andrea Dei and Emil J. Martinec. "NS5-brane backgrounds and coset CFT partition functions". In: *Preprint* (Mar. 2024). DOI: 10.48550/arXiv.2403.17258. arXiv: 2403.17258 [hep-th].

[32] Robbert Dijkgraaf, Herman L. Verlinde, and Erik P. Verlinde. "String propagation in a black hole geometry". In: *Nuclear Physics B* 371 (1992), pp. 269–314. DOI: 10.1016/0550-3213(92)90237-6.

[33] Indranil Halder and Daniel L. Jafferis. "Thermal Bekenstein-Hawking entropy from the worldsheet". In: *Preprint* (Feb. 2024). Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138, USA.

Acknowledgements

At the conclusion of this thesis, I feel it is necessary to express my gratitude to everyone who has supported me throughout this journey of study and life.

A special thank you goes to my supervisor, Stefano Massai, who guided me with his valuable insights during the writing of this thesis. I would also like to thank the other professors who, like Stefano, have been sources of inspiration and supported me in the difficult decision regarding my PhD: Alessandro Sfondrini, Paolo Rossi, Marco Matone and Pieralberto Marchetti.

A heartfelt thanks goes to all my friends, colleagues, and roommates, who have been essential during these wonderful years.

To my friends from Salò (especially Simone and Antonio) for always being there: from them, I have learned the value of a friendship that, when true, is not diminished by the passage of time or by different life choices.

To my friends from Trieste (especially Defi and Luca) for the great moments we spent together and for sharing countless discussions and reflections with me, even from afar. A special thanks to Defi for the hospitality he has shown me on numerous occasions (yes, I've slept at his place in Trieste more than once).

To my friends from Padua (like Chicca, Angela, Davide, Erica, Lucrezia, Mattia, Luca, Luigi, Nicolò, and many others) for making every day special with their friendship and for encouraging me to always give my best. A particular thanks to Davide for being my biggest supporter, for always believing in me, even when I had doubts myself.

A sincere thank you also goes to Gaetano, who, like no one else, managed to combine a deep friendship with a strong professional and academic collaboration. He is someone I hold in high esteem and from whom I have learned a lot, both on a personal and academic level.

I cannot forget to mention one of the people who has been by my side the most during the difficult moments of the past years: Riccardo, thank you for your endless patience, wise advice, and constant presence, even on the darkest days.

And, of course, I cannot omit a section dedicated to my girlfriend, Carolina, who has stood by my side with love, understanding, and strength. Thank you for giving me the motivation and serenity needed to face this journey.

Finally, a special thank you goes to my parents, who have always supported the realization of my projects and dreams. To them, I extend my deepest gratitude for their unconditional love and constant support.

In conclusion, a piece of advice for those about to embark on this journey: do not be afraid to ask for help, to seek guidance from those with more experience, and, most importantly, never lose the curiosity and passion that drove you to start this journey.