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BENFORD’S LAW IN DETERMINISTIC PROCESSES

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Introduction

Benford's Law describes how leading digits of numbers are distributed among a large amount of datasets. Intuitively, it seems reasonable to assume that numbers follow a uniform distribution, instead they follow a particular logarithmic distribution. For example, the number 1 appears as the first digit approximately 30% of the times, while the number 9 occurs less than 5% of the times.
This phenomenon was first discovered by the astronomer Simon Newcomb in 1881. He noticed that the the first pages of the logarithmic tables, those with the logarithms of numbers beginning with the first digits, were more dirty than the last ones, i.e. the scientists used numbers with the lowest digits more than expected. So he wrote a law describing the expected distribution of the digits but his article was not successful and was forgotten. About fifty years later, the physicist Frank Benford, regardless of Newcomb, rediscovered the phenomenon with the same observations on his log tables. He supported his thesis collecting more than 20000 numbers from different sources such as square roots of integers, physics constants, sizes of populations, street addresses of *"American Men of Science"* and many others [5]. Benford noticed that the more random the numbers were, the more they fitted a law that later discovered to be logarithmic. In particular, he asserted that the proportion (or relative frequency) of times for which d is the first significant digit of a number taken in a big dataset is equal to $\log_{10}(1 + 1/d)$.
Then, the interest also shifted to the occurrence of the second and higher significant digits, leading to a more complete form of the law: the frequency with which $d_1, d_2, ...d_m$ appear as the first, second,...,m-th significant digits of a number taken in a big dataset is equal to $\log_{10}(1 + (\sum_{i=1}^{n} 10^{n-i} d_i)^{-1})$ [1].
At first this law may seem only theoretical without any applications and also scientists thought that. Then, in the 70s, some applications started to be studied even if it was in the 90s that the most important studies took hold. The mathematician Carslaw suggested that Benford's law could be used as a test for falsification in data collections based on the assumption that human mind prefers to round up or down a number with the result of a non-natural

distribution of digits. Since then, many steps forward have been made and the law is applied in many areas such as computer design, modelling and fraud detection [5].
In this thesis the discussion is focused on the *Real-Valued Deterministic Processes* and their connection with Benford's law. A one-dimensional deterministic process is a system in which there is no randomness that affects the future states of the system itself and for that is one of the simplest models for evolving processes. In mathematics these models are described by one-dimensional iterated maps or sequences and the purpose of this thesis is to study under what conditions the frequencies of visits to a state conform to Benford's law. Most of the results of this thesis are taken from [1].
The work is divided as follows:

Chapter 1: Preliminary notions In this chapter a brief introduction of Benford's law is given, then, the principal tools to describe the Benford behavior of sequences are studied. In particular, the whole theory is based on the concepts of significand and of the uniform distribution modulo one which are going to be recall throughout the thesis.

Chapter 2: Autonomous Systems and the Benford's Law This chapter is dedicated to present Autonomous Systems, i.e. systems in which there is no time dependence. In particular, sequences are going to be defined by iterations of maps. There will be also a distinction between the types of growth of sequences that are polynomial, exponential or super-exponential. With the help of examples, it will be shown the different behavior of these sequences with respect to Benford's law.

Chapter 3: Non-Autonomous Systems and the Benford's Law This chapter contains a generalization of the results of the previous chapter by introducing maps that are time dependent. Here, the types of growth of sequences are exponential and super-exponential and as it will be shown the differences compared to the autonomous case are few.

Chapter 1

Preliminary notions

In this first chapter some definitions and theorems that will be useful throughout the thesis will be recalled. For simplicity denote with *log* the logarithm to base 10, $\log_{10}$.
As already mentioned, Benford's law characterizes the distribution of leading digits that, in particular, it's not uniform. In fact, the probability of finding numbers with first digit 1, in a large dataset, is greater than that of finding numbers that start with 9. More specifically, the first significant digit D_1 of a number conforms to the following law:

$$Prob(D_1 = d_1) = \log\left(1 + \frac{1}{d_1}\right) \quad \text{for all } d_1 = 1, 2, ..., 9$$

For example, $Prob(D_1 = 1) = \log 2 = 0.3010$ and $Prob(D_1 = 9) = \log(10/9)$ $= 0.04575$ and, in fact, number 9 has a less probability to occur than number 1.
The law can also be extended to the distribution of the other significant digits $D_2, D_3, ..., D_m$, $m \in \mathbb{N}$:

$$Prob((D_1, D_2, ..., D_m) = (d_1, d_2, ..., d_m)) = \log\left(1 + \left(\sum_{j=1}^{m} 10^{m-j} d_j\right)^{-1}\right)$$

for all $d_1 \in \{1, 2, ..., 9\}$ and $d_i \in \{0, 1, ..., 9\}$, $i \geq 2$.
An aspect that should be further explored is how to define *Prob*. Since this thesis focused on Deterministic Systems and the more simple way to describe them is through sequences (x_n), then *Prob* refers to the proportion (or relative frequency) of times n for which an event, for example $D_1 = d_1$, occurs. This means that the limiting proportion, as $N \to \infty$, of times $n \leq N$ such that the first significant digit of an entry of a sequence is d_1, is $Prob(D_1 = d_1)$ [2].

Contrary to what one might expect, the general form of the law leads to the fact that the significant digits are dependent. For example,

$$Prob(D_2 = 1) = \sum_{d_1=1}^{9} \log(1 + (10d_1 + 1)^{-1}) = \sum_{d_1=1}^{9} \log(1 + \frac{1}{10d_1 + 1}) = 0.1138$$

whereas the conditional probability that $D_2 = 1$ given that $D_1 = 1$ is

$$Prob(D_2 = 1|D_1 = 1) = \frac{Prob(D_1 = 1, D_2 = 1)}{Prob(D_1 = 1)} = \frac{\log(1 + 1/11)}{\log 2} = 0.1255$$

which is different from the other one.
An important concept for the study of Benford' law is the significand. In fact, this law concerns the statistical distribution of significant digits or, more generally, the significands (fraction parts in floating-point arithmetic). Informally, the *significand* of a real number is its first left part expressed in floating-point notation. The formal definition is as follows:

Definition 1.1. *For $x \in \mathbb{R}^+$ the* decimal significand *of x, denoted* S(x)*, is the function $S : \mathbb{R} \to$[1,10) given by*

$$S(x) = \begin{cases} t & \textit{if } x \neq 0 \\ 0 & \textit{if } x = 0 \end{cases}$$

where t is the unique number in [1,10) with $x = 10^k t$ for some (necessarily unique) $k \in \mathbb{Z}$. For $x \in \mathbb{R}^-, S(x) = S(-x)$.

For example $S(\sqrt{2}) = S(-\sqrt{2}) = S(10\sqrt{2}) = \sqrt{2} = 1.414$ and $S(2019) = S(0.02019) = S(-20.19) = 2.019$.
Starting from this definition it is obvious that the first significant(decimal) digit of a number $x \in \mathbb{R}$, denoted $D_1(x)$, is the first left digit of $S(x)$. More formally, the *first significant decimal digit* of x is the only integer $j \in \{1, 2, ..., 9\}$ that satisfies $10^k j \leq |x| \leq 10^k(j + 1)$ for some (necessarily unique) $k \in \mathbb{Z}$.
For example $D_1(\sqrt{2}) = D_1(-\sqrt{2}) = D_1(10\sqrt{2}) = 1$.
As mentioned before, a sequence of real numbers (x_n) is considered a (base-10) Benford sequence if, as $N \to \infty$, the proportion of indices $n \leq N$ where x_n has the first significant digit d exists and is equal to $\log(1 + d^{-1})$ for all $d \in \{1, 2, .., 9\}$ (and similarly for the other blocks of significant digits). It is now useful to give a more rigorous definition in relation to when a sequence is Benford:

Definition 1.2. *A sequence $(x_n) = (x_1, x_2, \dots)$ of real numbers is a* (base-10) Benford sequence *if*

$$\lim_{N\to\infty} \frac{|\{1 \leq n \leq N : S(x_n) \leq t\}|}{N} = \log t \quad \textit{for all}\, t \in [1, 10)$$

or if for all $m \in \mathbb{N}$, all $d_1 \in$ {1,2,...,9} and all $d_i \in$ {0,1,...,9}, $i \geq$ 2,

$$\lim_{N\to\infty} \frac{|\{1 \leq n \leq N : D_i(x_n) = d_i \, for \, i = 1, 2, ..., m\}|}{N}$$
$$= \log\left(1 + \left(\sum_{i=1}^{m} 10^{m-i} d_i\right)^{-1}\right)$$

The meaning of this definition is that, as $N \to \infty$, the limiting proportion of indices $n \leq N$ for which x_n has significand less than or equal to t is exactly $\log t$. To give an interpretation in probabilistic language, a sequence of real numbers is Benford if, chosen uniformly at random one of the first N entries, the probability that its first significant digit is d converges to the Benford probability $\log(1 + d^{-1})$ as $N \to \infty$ for every $d \in \{1, 2, .., 9\}$ and likewise for all other blocks of significant digits.

Example 1.1. *The sequence of positive integers* $(n) = (1, 2, 3, 4, ...)$ *is not Benford. In fact, more than half of the entries that are less than* $2 \cdot 10^m$ *have first digit 1 for every* $m > 0$ *and the reason of this is almost intuitive. For example, if* $m = 1$ *we are considering the numbers from* 1 *to* $2 \cdot 10 - 1 = 19$. *Among these, numbers starting with* 1 *are* 11 *and this is more than half of the numbers considered. More generally, the numbers starting with* 1 *between* 1 *and* $2 \cdot 10^m - 1$ *are* $10^m + 10^{m-1} + ... + 1$ *and this is more than* 10^m. *Surely, the limiting proportion of entries with significand* ≤ 2 *is at least 0.5, although Benford's law states that it should be approximately* $\log 2 < 0.5$.

As one can imagine, it is not always so simple to determine whether a sequence is Benford or not. So, it is useful to give some fundamental results about Benford sequences. One of the most important properties is the uniform distribution modulo 1, which is widely used to study the Benford behavior of sequences.

Definition 1.3. *A sequence* $(x_n) = (x_1, x_2, \dots)$ *of real numbers is* uniformly distributed modulo one (u.d. mod 1) *if*

$$\lim_{N\to\infty} \frac{|\{1 \leq n \leq N : \langle x_n \rangle \leq s\}|}{N} = s \quad for \, all \, s \in [0, 1) \tag{1.1}$$

where $\langle x_n \rangle$ *denotes the fractional part of* x_n, $n \in \mathbb{N}$.

The next result is a powerful tool in Benford theory, in particular for the characterization of Benford sequences.

Theorem 1.1. *A sequence of real numbers is Benford if and only if the sequence* $(\log |x_n|) = (\log |x_1|, \log |x_2|, ...)$ *is uniformly distributed mod 1.*

Proof. Let (x_n) be a sequence of real numbers.
($\Rightarrow$) If (x_n) is a Benford sequence then: $\lim_{N\to\infty} \frac{|\{1\le n\le N: S(x_n)\le t\}|}{N} = \log t$ for all t $\in$ [1,10).
There is the relation $S(x_n) = 10^{\langle \log|x_n|\rangle}$. In fact, for the definition of significand: $|x_n| = S(x_n)\cdot 10^k$ for k $\in \mathbb{Z}$ and so $\log|x_n| = \log S(x_n) + k$. Now note that k is the integer part of $\log|x_n|$. So we can decompose $\log|x_n|$ in its integer part and its fractional part: $\log|x_n| = k + \langle \log|x_n|\rangle$. Thus, $\langle \log|x_n|\rangle = \log S(x_n)$ and so $10^{\langle \log|x_n|\rangle} = S(x_n)$.
Hence, $S(x_n) = 10^{\langle \log|x_n|\rangle} \le t$ is equal to $\langle \log|x_n|\rangle \le \log t$. Putting this inequality in the limiting proportion and calling $\log t = s$:

$$\lim_{N\to\infty} \frac{|\{1\le n\le N: \langle \log|x_n|\rangle \le \log t\}|}{N} = \log t$$

and this is equivalent to

$$\lim_{N\to\infty} \frac{|\{1\le n\le N: \langle \log|x_n|\rangle \le s\}|}{N} = s$$

In particular, if $t\in[1,10), s\in[0,1)$. So $\log|x_n|$ is u.d. mod 1.
($\Leftarrow$) Let's assume that $\log|x_n|$ is u.d. mod 1. So $\lim_{N\to\infty} \frac{|\{1\le n\le N: \langle \log|x_n|\rangle\le s\}|}{N}$ $= s$ for all s $\in$ [0,1). Like before, $S(x_n) = 10^{\langle \log|x_n|\rangle}$, so $\langle \log|x_n|\rangle = \log S(x_n)$ $\le s$ is equal to $S(x_n) \le 10^s$. Putting again into the limiting proportion and calling $10^s = t$:

$$\lim_{N\to\infty} \frac{|\{1\le n\le N: S(x_n)\le 10^s\}|}{N} = s$$

which is equivalent to

$$\lim_{N\to\infty} \frac{|\{1\le n\le N: S(x_n)\le t\}|}{N} = \log t$$

As said above, if $s\in[0,1), t\in[1,10)$. So (x_n) is Benford.

$\square$

Others important results which help to determine the Benford property for many sequences and in particular the u.d mod 1 are the following propositions:

Proposition 1.1. *The sequence (x_n) is u.d. mod 1 if and only if (kx_n+b) is u.d. mod 1 for every $k\in Z\setminus\{0\}$ and every $b\in\mathbb{R}$.*
Also, if $\lim_{n\to\infty}|y_n - x_n| = 0$, (x_n) is u.d. mod 1 if and only if (y_n) is u.d. mod 1.

As a consequence of the latter proposition, if a sequence is Benford also its powers and reciprocal are Benford:

Theorem 1.2. *Let (x_n) be a Benford sequence. Then, for all $a \in \mathbb{R}$ and $k \in \mathbb{Z}$ with $ak \neq 0$, the sequence (ax_n^k) is also Benford.*

Proposition 1.2. *Let (x_n) be a sequence of real numbers.*

(i) If $\lim_{n\to\infty}(x_{n+1} - x_n) = \theta$ for some irrational θ, then (x_n) is u.d. mod 1.

(ii) If (x_n) is periodic, i.e. $x_{n+p} = x_n$ for some $p \in \mathbb{N}$ and for all n, then $(n\theta + x_n)$ is u.d. mod 1 if and only if θ is irrational.

(iii) The sequence (x_n) is u.d. mod 1 if and only if $(x_n + a \log n)$ is u.d. mod 1 for all $a \in \mathbb{R}$.

(iv) If (x_n) is u.d. mod 1 and non-decreasing, then $(x_n / \log n)$ is unbounded.

(v) If $\lim_{n\to\infty} n(y_{n+1} - y_n) = 0$ for the sequence of real numbers (y_n), (x_n) is u.d. mod 1 if and only if $(x_n + y_n)$ is u.d. mod 1.

Lemma 1.1. *The sequence $(na) = (a, 2a, ...)$ is u.d. mod 1 if and only if a is irrational.*

Proof. $(\Rightarrow)$ It follows by Proposition 1.2(i) that $\lim_{n\to\infty}(n+1)a - na = a$ and if a is irrational, then (na) is u.d. mod 1.
$(\Leftarrow)$ Assume that (na) is u.d. mod 1. If $a \in \mathbb{Z}$, then the fractional part of the sequence is clearly equal to 0 and the sequence cannot be u.d. mod 1. If $a \in \mathbb{Q}$, (na) can be rewritten as $\left(n\frac{p}{q}\right)$ with $p, q \in \mathbb{Z}, q \neq 0$, so the fractional part is periodic with period q (for example $\left(\langle n\frac{1}{3}\rangle\right) = \left(0, \frac{1}{3}, \frac{2}{3}, 0, \frac{1}{3}, ...\right)$. Also in this case the sequence cannot be u.d. mod 1 because a periodic sequence only assumes a finite number of distinct values in the interval [0,1), while a u.d. mod 1 sequence must "cover" [0,1) in a uniform way. It follows that a must be irrational.

□

Example 1.2. *(i) By Theorem 1.1 and Lemma 1.1, the sequence (2^n) is Benford. In fact, $(\log 2^n) = (n \log 2)$ and $\log 2$ is irrational. Conversely, (10^n) is not Benford: we have that $(\log 10^n) = (n \log 10)$ and $\log 10 = 1$ is not irrational.*

(ii) The sequence $(\log n)$ is not u.d. mod 1. In fact, $(\log n)$ is non-decreasing but $(x_n / \log n) = 1$ is not unbounded. So, by Proposition 1.2(iv), this sequence is not u.d. mod 1 and, in particular, the sequence (n) is not Benford.

(iii) The sequence of prime numbers $(p_n) = (2, 3, 5, 7, 11, ...)$ *is not Benford: by the Prime Number Theorem* $\lim_{n\to\infty} p_n = n \ln n$, *so*

$$\lim_{n\to\infty} \frac{\log p_n}{\log n} = \lim_{n\to\infty} \frac{\log (n \ln n)}{\log n} = \lim_{n\to\infty} 1 + \frac{\log (\ln n)}{\log n} = 1$$

As before, by proposition 1.2(iv), the sequence $(\log p_n)$ *is not u.d. mod 1 and so* (p_n) *is not Benford.*

The following theorem gives a necessary and sufficient condition for an asymptotically exponential sequence to be Benford and is a generalization of Lemma 1.1.

Theorem 1.3. *Let* b_n *be a sequence of real numbers such that* $\lim_{n\to\infty} |b_n/a^n|$ *exists and is positive for some* $a > 0$. *Then* (b_n) *is Benford if and only if* $\log a$ *is irrational.*

The proof is almost obvious but needs another results about the Benford property of sequences:

Theorem 1.4. *(i) Let* (a_n) *and* (b_n) *be sequences of real numbers with* $\lim_{n\to\infty} |a_n| = +\infty$ *and such that* $\lim_{n\to\infty} |a_n/b_n|$ *exists and is positive. Then* (a_n) *is Benford if and only if* (b_n) *is Benford.*

(ii) Let (a_n) *and* (b_n) *be sequences of real numbers with* $\lim_{n\to\infty} |a_n| = +\infty$ *and* $\sup_{n\in\mathbb{N}} |a_n - b_n| < +\infty$. *Then* (b_n) *is Benford if and only if* (a_n) *is Benford. [2]*

Proof. Let (a_n) and (b_n) be sequences of real numbers with the hypothesis of the statements.

(i) Suppose without loss of generality that $\lim_{n\to\infty} |a_n/b_n| = 1$. This implies that $|b_n| \to \infty$ and $\log |a_n| - \log |b_n| = \log |a_n/b_n| \to 0$.
($\Rightarrow$) If (a_n) is Benford, $(\log |a_n|)$ is u.d. mod 1 (Theorem 1.1). So, by Proposition 1.1 also $(\log |b_n|)$ is u.d. mod 1. This is equivalent to the fact that (b_n) is Benford.
($\Leftarrow$) The proof of this implication is equivalent to the other's, simply exchanging (a_n) with (b_n).

(ii) Let $c = \sup |a_n - b_n| + 1 > \sup |a_n - b_n|$. Without loss of generality, it can be assumed that $|a_n|, |b_n| \geq 2c$ for all $n \in \mathbb{N}$. In fact, $|a_n| \to +\infty$ so there always exists an index n_0 such that $|a_n| \geq 2c$ for every $n \geq n_0$

while $|b_n|$ must be sufficiently large since $\sup |a_n - b_n| < +\infty$. The following inequalities apply:

$$\begin{aligned} |b_n - a_n| &\leq c \\ |a_n| - c \leq & |b_n| \leq |a_n| + c \\ 1 - \frac{c}{|a_n|} \leq & \frac{|b_n|}{|a_n|} \leq 1 + \frac{c}{|a_n|} \end{aligned}$$

Applying the logarithm:

$$\log \frac{|b_n|}{|a_n|} \leq \log \left(1 + \frac{c}{|a_n|} \right) \leq \log \left(1 + \underbrace{\frac{c}{|a_n| - c}}_{>0} \right)$$

and

$$\log \frac{|b_n|}{|a_n|} \geq \log \left(1 - \frac{c}{|a_n|} \right) = -\log \left(\frac{|a_n|}{|a_n| - c} \right) = -\log \left(1 + \frac{c}{|a_n| - c} \right)$$

The latter two inequalities imply that

$$-\log \left(1 + \frac{c}{|a_n| - c} \right) \leq \log \frac{|b_n|}{|a_n|} \leq \log \left(1 + \underbrace{\frac{c}{|a_n| - c}}_{>0} \right)$$

It follows that

$$\Big| \log |b_n| - \log |a_n| \Big| = \left| \log \frac{|b_n|}{|a_n|} \right| \leq \log \left(1 + \frac{c}{|a_n| - c} \right) \to 0 \quad as\ n \to \infty.$$

By Proposition 1.1, $(\log |b_n|)$ is u.d. mod 1 if and only if $(\log |a_n|)$ is, so (b_n) is Benford if and only if (a_n) is Benford.

□

Proof of Theorem 1.3. Let (b_n) be a sequence of real numbers such that $\lim_{n\to\infty} |b_n/a^n|$ exists and is positive for some $a > 0$. By Theorem 1.4(i) (b_n) is Benford if and only if (a^n) is Benford. Then, by Theorem 1.1, (a^n) is Benford if and only if $(\log a^n) = (n \log a)$ is u.d. mod 1 and this is true if and only if $\log a$ is irrational (Lemma 1.1).

□

Chapter 2

Autonomous Systems and the Benford's Law

The aim of this chapter is to describe the Benford behavior of deterministic sequences. As said in the introduction, the focus will be on one-dimensional deterministic processes which are described by one-dimensional difference equations. These equations lead to sequences (of numbers) which are generated recursively through iterations of a single function, i.e. the same function is applied over and over again. This means that the state of a process at a certain time $n \in \mathbb{N}$ is a function of the previous one (or ones), i.e. in the terminology of sequences, $x_n = f(x_{n-1})$, where $f : C \to \mathbb{R}$ is a function such that $f(C) \subset C$. In this study the set C is often equal to $\mathbb{R}^+$ or $[a, +\infty)$ for some (large) $a \geq 0$ and the function f is usually referred to as a *map*.
In this chapter the attention will be on *autonomous systems*, i.e. systems in which maps do not depend explicitly on n, while in the next chapter there will be some examples of *nonautonomous systems*, i.e. systems in which maps explicitly depend on n.
Here the terminology used throughout this thesis is given.
For any $x_0 \in C$, the difference equation

$$x_n = f(x_{n-1}), \qquad n \in \mathbb{N} \tag{2.1}$$

recursively defines a sequence x_n in C called the *orbit* of x_0 (under f).
The n^{th} iterate of f is denoted by f^n and is defined as $f^n = f^{n-1} \circ f = f(f(f(...)))$, in particular $f^0 = id_C$. So the orbit of x_0 is

$$(x_n) = (f^n(x_0)) = (f(x_0), f(f(x_0)), f(f(f(x_0))), ...)$$

and the interpretation of the orbit as a sequence is evident.
An orbit is a *periodic sequence of period p* if $f^p(x_0) = x_0$ (or equivalently

$x_{n+p} = x_n$ for all $n \in \mathbb{N}$) for some $p \in \mathbb{N}$:

$$(x_n) = (f(x_0), f^2(x_0), ..., f^{p-1}(x_0), x_0, f(x_0), ...)$$

A point x_0 is a *periodic* or *fixed* point if $p = 1$. Furthermore, x_0 and its orbit are *attracting* (x_0 is an *attracting fixed point*) if, starting with a number sufficiently close to x_0, the iterations converge to x_0, i.e. $\lim_{n\to\infty} |f^{np+j}(x) - f^j(x_0)| = 0$ for every j=1,2,...,p whenever $|x - x_0|$ is sufficiently small. Obviously, in this case a sequence (x_n) cannot be Benford. Instead, a *repelling fixed point* shows the opposite behavior: if the starting point is close to it, the iterations tend to move away from it. In this case, the sequence may be Benford because the sequence may cover a wide range of values.

Observation 2.1. *It can happen that* (x_n) *is not periodic, but the sequence* $(S(x_n))$ *is. For example* $f(x) = x\sqrt{10}$ *with the starting point* $x_0 = 1$ *generates the orbit* $(x_n) = (\sqrt{10}, 10, 10\sqrt{10}, 10^2, ...) = (10^{n/2})$ *which tends to* $+\infty$. *On the other hand, the sequence of significands is* $(S(x_n)) = (\sqrt{10}, 10, \sqrt{10}, 10, ...)$ $= (10^{\langle n/2 \rangle})$ *and it is periodic with period 2.*

As will be seen here, there are three types of growth of a sequence and each of them has a different behavior in relation to Benford property:

(i) Polynomial increasing or decreasing sequences are not Benford. In this group there are, for example, (n^2) or (p_n) (see Example 1.2(iii)).

(ii) Exponential increasing or decreasing sequences are usually Benford for all starting points in a region, but their conformity to Benford's Law is given by the specific map, as will be seen. Here some examples are the sequences (2^n) and $(n!)$ (these are Benford and, in particular, $(n!)$ will be studied in the next chapter).

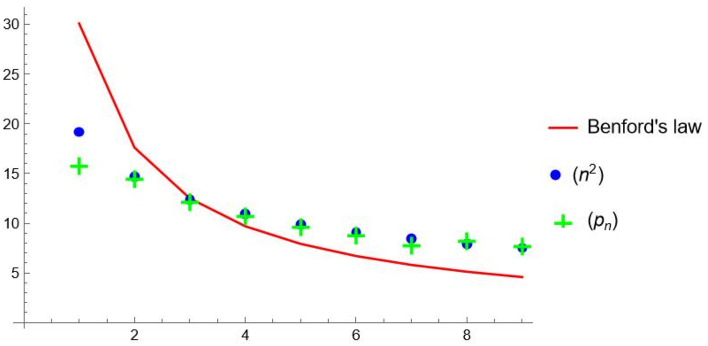
(iii) Super-exponential increasing or decreasing sequences are usually Benford for almost all, but not all, starting point (i.e. there are some sets of measure zero[1] in which points generate orbits that are not Benford). An example is the map $f(x) = 10x^2$ (with the starting point x_0=2 is Benford).

The next table shows whether the sequences mentioned above follow Benford's Law or not. It is evident the Benford behavior of the last 3 sequences (remind that if a sequence is Benford, then the number 1 appears as the first digit approximately 30% of the times and the number 9 less than 5%), see the graphics below.

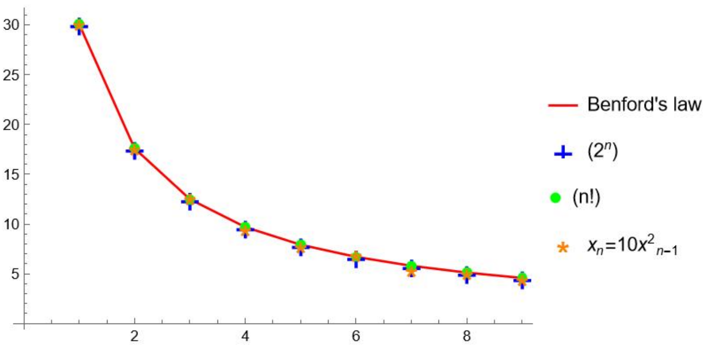
[1]In the measure theory, a set $A \in \mathbb{R}^n$ that is measurable, has measure zero if $\mu(A) = 0$ where $\mu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ is an outer measure.

Digit / $f^n(x)$	1	2	3	4	5	6	7	8	9
(n^2)	19.19	14.69	12.37	10.95	9.84	9.08	8.45	7.91	7.52
(p_n)	16.01	11.29	10.97	10.55	10.13	10.13	10.27	10.03	10.06
(2^n)	30.10	17.61	12.49	9.70	7.91	6.70	5.79	5.12	4.58
$(n!)$	29.56	17.89	12.76	9.63	7.94	7.15	5.71	5.10	4.26
$x_n = 10x_{n-1}^2$, $x_0 = 2$	30.19	17.66	12.68	9.56	7.83	6.97	5.45	5.13	4.53

Table 2.1: Relative frequencies (in percentage) of the leading significant digit for the first 10^4 terms of the five sequences nominated just before.



(a)



(b)

Figure 2.1: In red it is represented how the digits from 1 to 9 occur (in percentage) as first significant digit in accordance with Benford's Law. The points represent the occurrence (in percentage) of these digits as first significant digit among the entries of some sequences. In (a) it is shown the non-Benford behavior of two polynomial sequences, (b) shows how some exponential and super-exponential sequences tend to follow Benford's Law.

2.1 Sequences with polynomial growth

Linear sequences are the orbits generated by maps like $f(x) = x + g(x)$ and, in general, are not Benford for any x_0 as the following theorem shows.

Theorem 2.1. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a map such that $f(x) = x + g(x)$ where $g(x) \geq 0$ for all sufficiently large x. If $g = o(x^{1-\varepsilon})$ as $x \to +\infty$ [2] with some $\varepsilon > 0$, then for all sufficiently large x_0, the orbit $(f^n(x_0))$ is not Benford.*

Proof. Pick $\xi > 0$ such that $g(x) \geq 0$ for all $x \geq \xi$. If $x_0 \geq \xi$, the non-decreasing sequence $(x_n) = (f^n(x_0))$ is bounded and then it is not Benford. In fact, if the sequence is non-decreasing and there exists an upper limit that cannot be exceeded, from a certain $n \in \mathbb{N}$ the iterates will grow less and less and the leading digits tend to remain the same. So, assume that $\lim_{n\to\infty} x_n = +\infty$ and let $y_n = \log x_n$ for every $n \in \mathbb{N}$ (note that this is equivalent to saying that $x_n = 10^{y_n}$). Obviously, (y_n) is non-decreasing, $\lim_{n\to\infty} y_n = +\infty$ and it can be rewritten as

$$y_n = \log x_n = \log x_{n-1} + \log\left(1 + \frac{g(x_{n-1})}{x_{n-1}}\right) = y_{n-1} + h(y_{n-1})$$

with $h(y) = \log\left(1 + \frac{g(10^y)}{10^y}\right)$.
Now fix $0 < \delta < \varepsilon$ and consider the auxiliary function $H : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$H(y) = \frac{1}{\delta}\log(1 + 10^{-\delta y})$$

and note that it is a decreasing function. From

$$h(y) \approx \frac{g(10^y)}{10^y} = \frac{o(10^{y(1-\varepsilon)})}{10^y} = o(10^{-\varepsilon y}) \quad as\ y \to +\infty$$

and

$$\lim_{y\to+\infty} \frac{H(y)}{10^{-\delta y}} = \lim_{y\to+\infty} \frac{1}{\delta}\frac{\log(1 + 10^{-\delta y})}{10^{-\delta y}} = \lim_{y\to+\infty} \frac{1}{\delta}\frac{\log e \ln(1 + 10^{-\delta y})}{10^{-\delta y}} = \frac{1}{\delta}\log e$$

It follows that $H(y) = \mathcal{O}(10^{-\delta y})$ and, in particular, that, since $\varepsilon > \delta$, $h(y) \leq H(y)$ for all $y \geq \eta$ for an appropriate $\eta > 0$. Now, fix any $Y_0 \geq \eta$ and define

[2] A function $f : \mathbb{R}^+ \to \mathbb{R}$ is said to be $o(g)$ as $x \to \infty$ (for any $g : \mathbb{R}^+ \to \mathbb{R}$) if $\lim_{x\to\infty} f(x)/g(x) = 0$, whereas $f = \mathcal{O}(g)$ as $x \to \infty$ if $\limsup_{x\to\infty} |f(x)/g(x)| < \infty$. An important fact that will be used in the examples is that for every $a \in \mathbb{R}$, $f = o(x^a) \Rightarrow f = \mathcal{O}(x^a) \Rightarrow f = o(x^{a+\varepsilon})$ for every $\varepsilon > 0$.

the sequence $Y_n = Y_{n-1} + H(Y_{n-1})$, $n \geq 1$. Given any $y_0 \geq \max\{\eta, \log \xi\}$, note that

$$y_1 - Y_1 = y_0 + h(y_0) - Y_0 - H(Y_0) \leq y_0 - Y_0 + H(y_0) - H(Y_0)$$

If $y_0 \leq Y_0$, then $y_1 - Y_1 \leq H(y_0) \leq H(\eta)$ since H is decreasing. If, instead, $y_0 > Y_0$, then $y_1 - Y_1 \leq y_0 - Y_0$. In both cases,

$$\eta \leq y_0 \leq y_1 \leq Y_1 + \max\{H(\eta), |y_0 - Y_0|\}$$

The second iterate is such that

$$y_2 - Y_2 = y_1 + h(y_1) - Y_1 - H(Y_1) \leq y_1 - Y_1 + h(y_1) - H(Y_1) \leq y_1 - Y_1$$

and with the same argument as before, the conclusion is that

$$y_1 \leq y_2 \leq Y_2 + \max\{H(\eta), |y_0 - Y_0|\}$$

Iterating for all n, it is obtained that

$$\eta \leq y_n \leq Y_n + \max\{H(\eta), |y_0 - Y_0|\} \quad for\ all\ n.$$

Then rewrite Y_n as

$$\begin{aligned}
Y_n &= Y_{n-1} + H(Y_{n-1}) \\
&= Y_{n-1} + \frac{1}{\delta} \log\left(1 + 10^{-\delta Y_{n-1}}\right) \\
&= \frac{\log\left(10^{\delta Y_{n-1}}\right)}{\delta} + \frac{1}{\delta}\log\left(1 + 10^{-\delta Y_{n-1}}\right) \\
&= \frac{1}{\delta}\log\left(10^{\delta Y_{n-1}} + 1\right) \\
&= \frac{1}{\delta}\log\left(10^{\delta\left(\frac{1}{\delta}\log\left(10^{\delta Y_{n-2}}+1\right)\right)} + 1\right) \\
&= \frac{1}{\delta}\log\left(10^{\delta Y_{n-2}} + 2\right) \\
&= \frac{1}{\delta}\log\left(n + 10^{\delta Y_0}\right)
\end{aligned}$$

Putting the equations together, it follows that

$$\eta \leq y_n \leq \frac{1}{\delta}\log(n + 10^{\delta Y_0}) + \max\{H(\eta), |y_0 - Y_0|\}$$

and

$$\frac{y_n}{\log n} \leq \frac{1}{\delta}\frac{\log(n + 10^{\delta Y_0})}{\log n} + \frac{\max\{H(\eta), |y_0 - Y_0|\}}{\log n} \xrightarrow{n\to\infty} \frac{1}{\delta}$$

So, $\left(\frac{y_n}{\log n}\right)$ is bounded and, by Proposition 1.2(iv), the sequence (y_n) is not u.d. mod 1 for $y_0 \geq \max\{\eta, \log \xi\}$. Then it follows by Theorem 1.1 that for every $x_0 \geq \max\{10^\eta, \xi\}$ the orbit $(f^n(x_0))$ is not Benford.

$\square$

Example 2.1. *(i) Consider the map $f(x) = x + 2\sqrt{x} + 1$. For $x_0 \geq 0$ the first iterate of the orbit is $x_0 + 2\sqrt{x_0} + 1 = (\sqrt{x_0} + 1)^2$ and so the second one is $(\sqrt{x_0}+1)^2 + 2(\sqrt{x_0}+1) + 1 = (\sqrt{x_0}+2)^2$. Using the same reasoning, we obtain that the map f generates the orbit $((\sqrt{x_0} + n)^2)$ for any $x_0 \geq 0$. In this case $g(x) = 2\sqrt{x} + 1 = \mathcal{O}(x^{1/2})$ as $x \to \infty$, i.e., for example, $g(x) = o(x^{2/3})$ (see footnote 2) and so the sequence is not Benford by Theorem 2.1. Note that if $x_0 = 0$, then $(x_n) = n^2$ and so this sequence is not Benford, as seen in the table before.*

(ii) Consider the map

$$f(x) = \begin{cases} x+1 & \text{if } x \leq p_1 = 2 \\ \frac{p_{n+2}-p_{n+1}}{p_{n+1}-p_n}(x - p_n) + p_{n+1} & \text{if } p_n \leq x < p_{n+1} \end{cases}$$

where (p_n) is the sequence of prime numbers (see Example 1.2(iii)). This function is defined so that $f(p_n) = p_{n+1}$, $f(p_{n+1}) = p_{n+2}$ and $(f^n(1)) = (p_n)$. In fact, if $x_0 = 1 \leq 2$ then $x_1 = f(x_0) = 1 + 1 = 2 = p_1$ and $x_2 = f(x_1) = 2 + 1 = 3 = p_2$ because $x_0, x_1 \leq p_1$. The other iterates are generated by the second part of the function by the relation $f(p_n) = p_{n+1}$ (in fact, $x_n = p_n$) and so the sequence of prime numbers can be written in the form of an orbit.

Calling $g(x) = f(x) - x$, we can observe that $g(x) \geq 1$ and that $g(x)$ can be rewritten as $\left(\frac{p_{n+2}-p_{n+1}}{p_{n+1}-p_n} - 1\right)(x - p_n) + p_{n+1} - p_n$. This function is linear in x, increasing or decreasing whether $\frac{p_{n+2}-p_{n+1}}{p_{n+1}-p_n}$ is more or less than 1. So, in particular, the maximum value of g over the interval $[p_n, p_{n+1}]$ occurs at one of the endpoints:

$$g(x) = \begin{cases} p_{n+1} - p_n & \text{if } x = p_n \\ p_{n+2} - p_{n+1} & \text{if } x = p_{n+1} \end{cases}$$

Therefore, for any $x \in [p_n, p_{n+1}]$ the difference $f(x) - x$ is bounded by the larger of these values: $g(x) = f(x) - x \leq \max\{p_{n+1} - p_n, p_{n+2} - p_{n+1}\}$. A result of G.Hoheisl says that, for some $\varepsilon > 0$,

$$p_{n+1} - p_n = o(p_n^{1-\varepsilon}) \quad as\, n \to \infty \tag{2.2}$$

(in [3] it is shown that ε can be chosen as large as $\varepsilon = \frac{19}{40}$). From 2.2 it follows that the difference between two consecutive prime numbers

grows slower than $p_n^{1-\varepsilon}$ as $n \to \infty$ and this implies that $g = o(x^{1-\varepsilon})$ as $x \to +\infty$ since $g(x)$ is less than or equal to the difference of two primes. As a result of Theorem 2.1, the sequence $(f^n(x_0))$ is not Benford for any $x_0 \in \mathbb{R}$. In particular, $(f^n(1)) = (p_n)$ is not Benford, as already seen in Example 1.2(iii) and in the previous table.

Until now, only maps with $+\infty$ as an attracting fixed point have been considered but the results can be extended to include maps that have 0 as an attracting fixed point: it is sufficient to consider reciprocals and the fact that a real sequence (x_n) is Benford if and only if the sequence of its reciprocals (x_n^{-1}) is Benford (Theorem 1.2).

Corollary 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and assume that $|f(x)| \leq |x|$ for some $\delta > 0$ and all $|x| \leq \delta$. If $|f'(0)| = 1$, then $(f^n(x_0))$ is not Benford whenever $|x_0| \leq \delta$.*

Example 2.2. *The smooth map $f(x) = \sin x$ satisfies $|f(x)| < |x|$ for all $x \neq 0$. In fact, the Taylor expansion of the function about* 0 *is* $\sin x = x - \frac{x^3}{3} + o(x^3)$*, so for $|x| \leq \delta$, $\delta > 0$, surely $|f(x)| < |x|$. Since $|f'(0)| = |\cos(0)| = 1$, $(f^n(x_0))$ is not Benford for any $|x_0| \leq \delta$ by Corollary 2.1. Furthermore, $\lim_{n\to\infty} f^n(x_0) = 0$ for every $x_0 \in \mathbb{R}$: the reason is that in each iteration the values decrease because $|\sin x| < |x|$ even if the initial point x_0 is large. The conclusion is that this map is not Benford for any $x_0 \in \mathbb{R}$.*

2.2 Sequences with exponential growth

The maps in this section are like $f(x) = ax$ or, more generally, $f(x) = ax + g(x)$ with g that is small in some sense but not identically zero, and they are usually all Benford or none, depending on whether $\log |a|$ is irrational or not. Recall that the sequence $(a^n x_0)$, which is generated by the map ax, is Benford if and only if $\log |a|$ is irrational for any $a \in \mathbb{R}$ and any $x_0 \neq 0$ (Theorem 1.3).

Proposition 2.1 ([2])**.** *Let $f(x) = ax$ with $a \in \mathbb{R}$. Then $(f^n(x_0))$ is Benford for every $x_0 \neq 0$ or for no x_0 at all, depending on whether* $\log |a|$ *is irrational or rational, respectively.*

Now, the maps of this proposition will be slightly modified to study the Benford properties of more general maps.

Theorem 2.2. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a map such that $f(x) = ax + g(x)$ with $a > 1$ and $g = o(x)$ as $x \to +\infty$. Then:*

(i) If $\log a \in \mathbb{R} \setminus \mathbb{Q}$ then $(f^n(x_0))$ is Benford for every sufficiently large x_0.

(ii) If $\log a \in \mathbb{Q}$ *and* $g = o(x/\log x)$ *as* $x \to \infty$ *then, for every sufficiently large* x_0*,* $(f^n(x_0))$ *is not Benford.*

Proof. (i) Since $g(x) = o(x)$ as $x \to +\infty$, there exists $\xi > 0$ such that $|g(x)| \leq \frac{1}{2}(a-1)x$ for all $x \geq \xi$ and so

$$x_n = ax_{n-1} + g(x_{n-1}) \geq ax_{n-1} - |g(x_{n-1})| \geq \frac{1}{2}(a+1)x_{n-1}$$

for $x_{n-1} \geq \xi$. This implies that

$$\begin{aligned} x_n = f^n(x_0) &\geq \frac{1}{2}(a+1)x_{n-1} \\ &\geq \frac{1}{2}(a+1)(ax_{n-2} + g(x_{n-2})) \\ &\geq \frac{1}{4}(a+1)^2 x_{n-2} \\ &\geq \frac{1}{2^n}(a+1)^n x_0 \geq \xi \end{aligned}$$

and so $f^n(x_0) \xrightarrow{n\to\infty} +\infty$ whenever $x_0 \geq \xi$. Now let $y_n = \log x_n$ for every n, then:

$$\begin{aligned} y_n - y_{n-1} &= \log x_n - \log x_{n-1} \\ &= \log\left(\frac{ax_{n-1} + g(x_{n-1})}{x_{n-1}}\right) \\ &= \log a \, + \, \log\left(1 + \frac{g(x_{n-1})}{ax_{n-1}}\right) \xrightarrow{n\to\infty} \log a \end{aligned}$$

By Proposition 1.2(i), it follows that (y_n) is u.d. mod 1 if $\log a$ is irrational. In this case, so, $(f^n(x_0))$ is Benford for all $x_0 \geq \xi$.

(ii) Since $g(x) = o\left(\frac{x}{\log x}\right)$ as $x \to +\infty$, with the same argument used in (i), there exists $\xi > 0$ such that $x_n = f^n(x_0) \geq \xi$ for every n and $f^n(x_0) \xrightarrow{n\to\infty} +\infty$ whenever $x_0 \geq \xi$. Given any $\varepsilon > 0$, the following inequality holds:

$$\left|\log\left(1 + \frac{g(x_n)}{ax_n}\right)\right| \leq \frac{|g(x_n)|}{2ax_n} \leq \frac{\varepsilon x_n}{\log x_n}\frac{1}{2ax_n} < \frac{\varepsilon \log a}{\log x_n}$$

for all sufficiently large n. Note that the first inequality holds since $|\log(1+x)| \leq \frac{1}{2}|x|$ for all $x \geq -\frac{1}{4}$. Now let $y_n = \log x_n$, which has the same properties as in point (i), and define the sequence $z_n = y_n - n\log a$. Note that

$$\begin{aligned}
\frac{z_n}{n} &= \frac{\log x_n - n\log a}{n} \\
&\approx \frac{\log(a^n x_0 + \tilde{g}(x_0)) - n\log a}{n} \\
&= \frac{\log(a^n x_0) + \log\left(1 + \frac{\tilde{g}(x_0)}{a^n x_0}\right) - n\log a}{n} \\
&= \frac{\log x_0 + \log\left(1 + \frac{\tilde{g}(x_0)}{a^n x_0}\right)}{n} \xrightarrow{n\to\infty} 0
\end{aligned}$$

where $\tilde{g}$ is a small correction given by g after n iterations. It follows that

$$\begin{aligned}
n|z_{n+1} - z_n| &= n|\log x_{n+1} - (n+1)\log a - \log x_n + n\log a| \\
&= n\left|\log\left(\frac{x_{n+1}}{a x_n}\right)\right| \\
&= n\left|\log\left(\frac{a x_n + g(x_n)}{a x_n}\right)\right| \\
&\le \frac{n\varepsilon \log a}{\log x_n} \\
&= \frac{n\varepsilon \log a}{z_n + n\log a} \\
&= \frac{\varepsilon \log a}{\log a + z_n/n}
\end{aligned}$$

So, $\limsup_{n\to\infty} n|z_{n+1} - z_n| \le \varepsilon$ and in fact $\lim_{n\to\infty} n(z_{n+1} - z_n) = 0$ since ε is arbitrary. By Proposition 1.2(v), the sequence (y_n) is u.d. mod 1 if and only if $(z_n + y_n) = (n\log a)$ is and this sequence is u.d. mod 1 if and only if $\log a$ is irrational according to Lemma 1.1. So, by the fact that $\log a \in \mathbb{Q}$, $(f^n(x_0))$ is not Benford for all $x_0 \ge \xi$.

□

Corollary 2.2. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a map such that, with some $a > 1$, $f(x) - ax = o(x/\log x)$ as $x \to +\infty$. Then, for every sufficiently large x_0, $(f^n(x_0))$ is Benford if and only if $\log a$ is irrational.*

Example 2.3. *(i) A simple example is the map $f(x) = 2x + e^{-x}$. This map is such that*

$$(f(x) - ax)\frac{\log x}{x} = (2x + e^{-x} - 2x)\frac{\log x}{x} = e^{-x}\frac{\log x}{x} \xrightarrow{x\to+\infty} 0$$

so $f(x) - 2x = o(x/\log x)$*. By Corollary* 2.2 *every orbit under this map is Benford since* $\log 2$ *is irrational.*

(ii) Now consider the map $f(x) = 2x - e^{-x}$*. It has a unique repelling fixed point that is* $\overline{x} = 0.5671$*. If* $x_0 > \overline{x}$*, then* $f^n(x_0) \to +\infty$ *as* $n \to \infty$ *and so Corollary* 2.2 *implies that* $f^n(x_0)$ *is Benford for the same reason as Before. On the other hand, if* $x_0 < \overline{x}$*, then* $f^n(x_0) \to -\infty$ *super-exponentially fast, and this scenario is not covered by any result so far.*

Another tool that is useful to establish the behavior of maps like these (and for maps that will be seen in the next section) is a simple version of *Shadowing Lemma.*

Lemma 2.1 (Shadowing Lemma)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a map such that* $f(x) = bx + \Gamma(x)$ *with* $b > 1$*. If* $\limsup_{x\to+\infty} |\Gamma(x)| < +\infty$*, then there exist* $\eta \in \mathbb{R}$ *and a function* $\hat{s} : [\eta, +\infty) \to \mathbb{R}$ *such that the sequence* $(f^n(x) - b^n\hat{s}(x))$ *is bounded for every* $y \geq \eta$*. The function* $\hat{s}$ *is continuous whenever* f *is continuous. Furthermore, if* $\lim_{x\to+\infty} \Gamma(x) = 0$*, then* $\lim_{x\to+\infty}(\hat{s}(x) - x) = 0$ *and, for every* $y \geq \eta$*,* $\lim_{n\to\infty}(f^n(x) - b^n\hat{s}(x)) = 0$*.*

In many situations, $(f^n(x_0) - b^n x_0)$ is unbounded for every x_0, but there exists a unique point $\hat{s}(x_0) \in \mathbb{R}$ with the property that $(f^n(x_0) - b^n\hat{s}(x_0))$ remains bounded. The point $\hat{s}(x_0)$ is called the *shadow* of x_0.

Proof. Since $\limsup_{n\to\infty} |\Gamma(x)| < +\infty$, there exists $\eta_0 \in \mathbb{R}$ and $\gamma > 0$ such that $|\Gamma(x)| \leq \gamma$ for all $x \geq \eta_0$. So, if $x \geq \eta := \max\{\eta_0, 2\gamma/(b-1)\}$, then

$$f(x) = bx + \Gamma(x) \geq bx - \gamma = (b-1)x + x - \gamma \geq 2\gamma + x - \gamma = \gamma + x$$

and so

$$\begin{aligned} f^n(x) &= bf^{n-1}(x) + \Gamma \circ f^{n-1}(x) \\ &= b^2 f^{n-2}(x) + b\Gamma \circ f^{n-2}(x) + \Gamma \circ f^{n-1}(x) \\ &= b^n x + \sum_{j=1}^{n} b^{n-j}\Gamma \circ f^{j-1}(x) \geq x \geq \eta \quad \text{for all } n \end{aligned}$$

This means that $f^n(x) \to +\infty$ as $n \to \infty$. The number

$$\hat{s}(x) = x + \sum_{j=1}^{+\infty} b^{-j}\Gamma \circ f^{j-1}(x)$$

is well defined and is finite since $b > 1$. Recall that $|\Gamma(x)| \leq \gamma$, it follows that, for every n,

$$\begin{aligned}
|f^n(x) - b^n\hat{s}(x)| &= \left| b^n x + \sum_{j=1}^{n} b^{n-j}\Gamma \circ f^{j-1}(x) - b^n x - b^n \sum_{j=1}^{+\infty} b^{-j}\Gamma \circ f^{j-1}(x) \right| \\
&= \left| \sum_{j=n+1}^{+\infty} b^{n-j}\Gamma \circ f^{j-1}(x) \right| \\
&\underset{\substack{\uparrow \\ k=j-n}}{=} \left| \sum_{k=1}^{+\infty} b^{-k}\Gamma \circ f^{k+n-1}(x) \right| \\
&\leq \sum_{k=1}^{+\infty} b^{-k}\gamma = \frac{\gamma}{b-1}
\end{aligned}$$

This implies that the sequence $(f^n(x) - b^n\hat{s}(x))$ is bounded for every $x \geq \eta$. Furthermore, if there exists another point $\tilde{s}(x)$ such that $|f^n(x) - b^n\tilde{s}(x)|$ is bounded, then $|f^n(x) - b^n\tilde{s}(x)| = |f^n(x) - b^n\hat{s}(x) - b^n(\tilde{s}(x) - \hat{s}(x)|$ is bounded if and only if $\tilde{s}(x) = \hat{s}(x)$ due to the fact that b^n grows really fast. So, $\hat{s}(x)$ is unique for any $x \geq \eta$. If f is continuous on $[\eta, +\infty)$, then $\Gamma \circ f^{j-1}$ is continuous for every $j \in \mathbb{N}$. In this case, according to the Weierstrass criterion, $\sum_{j=1}^{+\infty} b^{-j}\Gamma \circ f^{j-1}(x)$ converges uniformly and so $\hat{s}$ is continuous.
Now assume that $\Gamma(x) \xrightarrow{x \to +\infty} 0$. From this, given any $\varepsilon > 0$,there exists $x_\varepsilon \geq \eta$ such that $|\Gamma(x)| \leq \varepsilon(b-1)$ for all $x \geq x_\varepsilon$. Hence, if $x \geq x_\varepsilon$, then $f^{j-1}(x) \geq x \geq x_\varepsilon \geq \eta$ for all $j \in \mathbb{N}$ and so

$$|\hat{s}(x) - x| = \left| \sum_{j=1}^{+\infty} b^{-j}\Gamma \circ f^{j-1}(x) \right| \leq \sum_{j=1}^{+\infty} b^{-j}\varepsilon(b-1) = \varepsilon$$

Then $\lim_{x\to+\infty}(\hat{s}(x) - x) = 0$ since ε is arbitrary. In the same way, if $y \geq \eta$ then $|\Gamma \circ h^{j+n-1}(y)| \leq \gamma$ for all j and all n and $\lim_{n\to\infty} \Gamma \circ h^{j+n-1}(y) = 0$. This implies that $\lim_{n\to\infty}(h^n(y) - b^n\hat{s}(y)) = 0$ by the above inequality, where the limit can be taken inside the sum by the Dominated Convergence Theorem.

$\square$

Reconsider the map $f(x) = 2x + e^{-x}$ and note that $f^n(x_0) \to +\infty$ for every x_0. The basic idea of shadowing is that for large n the orbit generated by f is similar to the orbit generated by $2x$, with some initial point $\bar{x}_0$, which is $2^n\bar{x}_0$. In fact, $f^n(x_0)$ can be rewritten as

$$f^n(x_0) = 2^n x_0 + \sum_{j=1}^{n} 2^{n-j} e^{-f^{j-1}(x_0)} \geq 0$$

for every $x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$. Let's now consider the number

$$\bar{x}_0 = x_0 + \sum_{j=1}^{+\infty} 2^{-j} e^{-f^{j-1}(x_0)}$$

It is well-defined and positive because $f^n(x_0) \geq 0$. Note also that $f(x) \geq \max\{0, x+1\}$ for all x and, so, $f^n(x) \geq \max\{0, x+n\}$. Using this fact, we can apply the Shadowing Lemma:

$$\begin{aligned}
|f^n(x_0) - 2^n \bar{x}_0| &= \left| 2^n x_0 + \sum_{j=1}^{n} 2^{n-j} e^{-f^{j-1}(x_0)} - 2^n x_0 - 2^n \sum_{j=1}^{+\infty} 2^{-j} e^{-f^{j-1}(x_0)} \right| \\
&= \left| - \sum_{j=n+1}^{+\infty} 2^{n-j} e^{-f^{j-1}(x_0)} \right| \\
&\underset{\substack{\uparrow \\ k=j-n}}{=} \sum_{k=1}^{+\infty} 2^{-k} e^{-f^{k+n-1}(x_0)} \\
&\leq \sum_{k=1}^{+\infty} 2^{-k} e^{-(x_0+k++n-1)} \\
&= e^{-(x_0+n-1)} \sum_{k=1}^{+\infty} (2e)^{-k} \\
&= e^{-(x_0+n-1)} \cdot \frac{1}{2e-1} \xrightarrow{n\to\infty} 0.
\end{aligned}$$

This proves that $f^n(x_0)$ is similar to $2^n \bar{x}_0$ and this implies that $f^n(x_0)$ is Benford for all $x_0 \in \mathbb{R}$ according to Theorem 1.4(ii). Note that the conclusion of example 2.3 above is correct because, even if e^{-x} is not bounded for $x \to -\infty$, f maps $\mathbb{R}$ into $\mathbb{R}^+$ and so $\bar{x}_0$ is well-defined for every $x_0 \in \mathbb{R}$.
Like in the previous section, there is also a result for 0 instead of $+\infty$ as attracting fixed point. Again, it is sufficient to consider reciprocals.

Corollary 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 with $f(0) = 0$ and $0 < |f'(0)| < 1$. Then, for every $x_0 \neq 0$ sufficiently close to 0, $(f^n(x_0))$ is Benford if and only if $\log |f'(0)|$ is irrational.*

Example 2.4. *(i) The map $f(x) = x + \frac{1}{3}e^{-x} - \frac{1}{3}$ is smooth, with $f(0) = 0$ and $0 < f'(0) = 1 - \frac{1}{3} = \frac{2}{3} < 1$. This implies that the sequence $(f^n(x_0))$ is Benford for every $x_0 \neq 0$ sufficiently close to 0 because $\log f'(0) = \log \frac{2}{3}$ is irrational. Another way to see the Benford property of this sequence is to note that $f^n(x_0) \xrightarrow{n\to\infty} 0$ [3] for every $x_0 \in \mathbb{R}$ with an*

[3] A fixed point x_0 is attractive if $|f'(x_0)| < 1$ and is repulsive if $|f'(x_0)| > 1$. If $|f'(x_0)| = 1$, nothing can be said.

exponential decay, thereby covering several orders of magnitude, which means it is Benford unless $f^n(x_0) = 0$ for some $n \in \mathbb{N}$. Since f is strictly convex ($f''(x) = \frac{1}{3}e^{-x} > 0$ for all $x \in \mathbb{R}$), $f^n(x_0) = 0$ only if $f(x_0) = 0$ and this happens when $x_0 = 0$ or $x_0 = -1.903$. So, $(f^n(x_0))$ is Benford for every $x_0 \in \mathbb{R} \setminus \{0, -1.903\}$.

(ii) *Consider again the map $f(x) = 2x + e^{-x}$. To see that Corollary 2.3 applies to this map, it is useful the following tool: let*

$$\tilde{f}(x) = f(x^{-2})^{-1/2} = \begin{cases} \frac{|x|}{\sqrt{2+x^2e^{-1/x^2}}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

The map $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is smooth and $0 < \tilde{f}'(0) = \frac{1}{\sqrt{2}} < 1$. Since $\log\left(\frac{1}{\sqrt{2}}\right)$ is irrational, $(\tilde{f}^n(x_0))$ is Benford for every $x_0 \neq 0$ and hence $(f^n(x_0))$ is Benford for all $x_0 \neq 0$ by Theorem 1.2 because $f^n(x) = \tilde{f}^n(|x|^{-1/2})^{-2}$ for all n.

2.3 Sequences with super-exponential growth

In this section, maps such as $f(x) = ax^b$, $a > 0$, $b > 1$, are going to be studied. To observe how substantially different these maps are from those of the maps in the previous two sections, consider this example: the map $f(x) = \sqrt{x^4 + 12x^2 + 30}$. Obviously, $\lim_{n\to\infty} f^n(x) = +\infty$ for every $x \in \mathbb{R}$ but it follows from $f(x)^2 + 6 = (x^2 + 6)^2$ that

$$f^n(x) = \sqrt{(x^2+6)^{2^n} - 6} = (x^2+6)^{2^{n-1}} + \mathcal{O}(6^{-2^{n-1}}) \quad as\, n \to \infty$$

and this shows that every orbit has a double-exponential growth. Clearly, this map has a faster growth than, for example, $f^n(x) = 2^n x$. The purpose of this section is to give some results about the Benford behavior of maps like this and it will be shown that they are Benford for Lebesgue almost every $x_0 \in \mathbb{R}$ (but not all)[4]. Unfortunately, it is more simple to find points that create orbits that are not Benford. In the previous map, for example, if $x_0 = 2$ the associated orbit is $(f^n(2)) = (2, \sqrt{94}, \sqrt{9994}, \sqrt{999994}, ...)$ and it is evident that $D_1(f^{n-1}(x_0)) = 9$ for every $n \in \mathbb{N}$ and clearly $f^n(x_0)$ is not Benford. The following proposition is an analog of Proposition 2.1 in the super-exponential setting.

[4] A statement holds for *almost every* x if there is a set of Lebesgue measure zero that contains all x for which the statement does not hold.

Proposition 2.2. *Let $f(x) = ax^b$ with $a > 0, b > 1$. Then $(f^n(x_0))$ is Benford for almost $x_0 > 0$, but every non-empty open interval in $\mathbb{R}^+$ contains uncountably* exceptional points, *i.e. points $x_0 > 0$ for which $(f^n(x_0))$ is not Benford.*

Example 2.5. *Consider the map $f(x) = 10x^2$. By Proposition 2.2, $(f^n(x_0))$ is Benford for almost all but not all $x_0 > 0$ and, in fact, the statement can be extended for $x_0 \in \mathbb{R}$ because $f(x) \geq 0$ for every $x \in \mathbb{R}$. The associated sequence is $f^n(x_0) = 10^{2^n-1}x_0^{2^n}$ and, for example, if $x_0 = 10^k$ for some $k \in \mathbb{Z}$, then $f^n(x_0)$ always has first significant digit 1 and so for these initial points the sequence is not Benford.*

To better understand the statement of the previous proposition, given any map $f : \mathbb{R}^+ \to \mathbb{R}^+$, let

$$B = \{x \in \mathbb{R}^+ : (f^n(x)) \text{ is Benford}\}.$$

Proposition 2.2 says that, if $f(x) = ax^b$, then $\mathbb{R}^+ \setminus B = \{x \in \mathbb{R}^+ : x \notin B\}$ has measure zero. At the same time, however, $\mathbb{R}^+ \setminus B$ is also uncountable and everywhere dense in $\mathbb{R}^+$. So, technically, also almost all $x \notin B$. This may explain why it is not so simple to find even a single point x_0 for which $(f^n(x_0))$ is Benford despite Proposition 2.2 asserts that there exists a lot of such points.
The next step is to study more general maps, i.e. maps with the property that $f(x) - ax^b = o(x^b)$ as $x \to +\infty$ for some $a > 0$ and $b > 1$. The problems that occur here are that, even if $f(x) - ax^b$ may decay very rapidly, the Benford properties of the orbits may be quite different to those in Proposition 2.2.

Theorem 2.3. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a map such that $f(x) = ax^b + g(x)$ with $a > 0$, $b > 1$ and $g(x) = o(x^b)$ as $x \to +\infty$.*

(i) If f is continuous then, for every $c > 0$, there exist uncountably many $x_0 \geq c$ for which $(f^n(x_0))$ is Benford, but also uncountably many $x_0 \geq c$ for which $(f^n(x_0))$ is not Benford, i.e. $[c, +\infty) \cap B$ and $[c, +\infty) \setminus B$ are both uncountable.

(ii) If f is differentiable with continuity and $g' = o(x^{b-1}/\log x)$ as $x \to +\infty$, then there exists $c > 0$ such that $(f^n(x_0))$ is Benford for almost all $x_0 \geq c$, i.e. $[c, +\infty) \setminus B$ has (Lebesgue) measure zero.

Proof. To demonstrate this theorem it will be useful the Shadowing Lemma (Lemma 2.1). Note first that $\tilde{f}(x) = \alpha f(x/\alpha)$ for any $\alpha > 0$ is such that

$f^n(x) = \frac{1}{\alpha}\tilde{f}^n(\alpha x)$ and putting $\alpha = a^{(b-1)^{-1}}$ it follows that

$$\tilde{f}(x) = a^{(b-1)^{-1}} f\left(x \Big/ \left(a^{(b-1)^{-1}}\right)\right) = a^{(b-1)^{-1}} \cdot a \left(\frac{x}{a^{(b-1)^{-1}}}\right)^b + g\left(\frac{x}{(a^{(b-1)^{-1}})}\right)$$
$$= x^b + g\left(\frac{x}{(a^{(b-1)^{-1}})}\right)$$

So, without loss of generality, it can be assumed that $a = 1$. Now, given any $x_0 > 0$, define $y_n = \log x_n = \log f^n(x_0)$ and rewrite it as

$$\begin{aligned} y_n &= \log(x_{n-1}^b + g(x_{n-1})) \\ &= b \log x_{n-1} + \log\left(1 + \frac{g(x_{n-1})}{x_{n-1}^b}\right) \\ &= by_{n-1} + \log\left(1 + \frac{g(10^{y_{n-1}})}{10^{by_{n-1}}}\right) = h(y_{n-1}) \end{aligned}$$

where $h : \mathbb{R} \to \mathbb{R}$ is given by $h(y) = by + \log\left(1 + \frac{g(10^y)}{10^{by}}\right)$. Since $g = o(x^b)$ as $x \to +\infty$, then

$$\lim_{y\to+\infty}(h(y) - by) = \lim_{y\to+\infty}\left(by + \log\left(1 + \frac{g(10^y)}{10^{by}}\right) - by\right) = 0$$

After these considerations, the theorem can be proven.

(i) Since f is continuous, also h is continuous and by the Shadowing Lemma, there exist $\eta \in \mathbb{R}$ and a continuous function $\hat{s} : [\eta, +\infty) \to \mathbb{R}$ with $\hat{s}(y) - y \to 0$ as $y \to +\infty$ such that $\lim_{n\to\infty}(h^n(y) - b^n\hat{s}(y)) = 0$ for all $y \geq \eta$. By Proposition 1.1, $(h^n(y))$ is u.d. mod 1 if and only if $(b^n\hat{s}(y))$ is for $y \geq \eta$. By the Intermediate Value Theorem, $[\hat{s}(\log c), +\infty) \subset \hat{s}([\log c, +\infty))$ and by Proposition 2.2 the set that contains the points $y \geq \hat{s}(\log c)$ such that $(b^n y)$ is u.d. mod 1 and the set that contains the points $y \geq \hat{s}(\log c)$ such that $(b^n y)$ is not u.d. mod 1 are both uncountable. So, the set

$$\begin{aligned} U_c &= \{y \geq \log c : (h^n(y))\text{is u.d mod } 1\} \\ &= \{y \geq \log c : (b^n\hat{s}(y))\text{is u.d. mod } 1\} \end{aligned}$$

is uncountable and so is $[\log c, +\infty) \setminus U_c$. This implies that the sets $[c, +\infty) \cap B = \{x_0 \geq c : (f^n(x_0)) \text{ is Benford}\}$ and $[c. +\infty) \setminus B$ are both uncountable.

(ii) Take $\hat{s}(y) = y + \sum_{j=1}^{+\infty} b^{-j}\Gamma \circ h^{j-1}(y)$ as in Lemma 2.1, where

$$\Gamma(y) = \log\left(1 + \frac{g(10^y)}{10^{by}}\right), \quad y \in \mathbb{R}$$

Now, let's derive Γ:

$$\begin{aligned}\Gamma'(y) &= \frac{\Big(b10^{by} + g'(10^y)10^y\Big) - \Big(10^{by} + g(10^y)\Big)b}{\Big(10^{by} + g(10^y)\Big)} \\ &= \frac{10^y g'(10^y) - bg(10^y)}{10^{by} + g(10^y)}\end{aligned}$$

and note that

$$\lim_{y\to+\infty} y\Gamma'(y) = \lim_{y\to+\infty} \frac{y10^y g'(10^y)}{10^{by} + g(10^y)} - \frac{byg(10^y)}{10^{by} + g(10^y)} = 0$$

since $g'(x) = o(x^{b-1}/\log x)$ and $g(x) = o(x)$ as $x \to +\infty$ implies that $g'(10^y) = o\Big(\frac{10^{y(b-1)}}{y}\Big)$ and $g(10^y) = o(10^{by})$ as $y \to +\infty$. This means that $\Gamma'(y) = o(1/y)$. Then note that

$$\begin{aligned}(b^{-j}\Gamma \circ h^{j-1}(y))' &= b^{-j}\Gamma'(h^{j-1}(y)) \cdot (h^{j-1})'(y) \\ &= b^{-j}\Gamma' \circ h^{j-1}(y) \prod_{k=1}^{j-1}(b + \Gamma' \circ h^{k-1}(y))\end{aligned}$$

Since $\Gamma' = o(1/y)$, it follows that $\lim_{y\to+\infty}(b^{-j}\Gamma \circ h^{j-1}(y))' = 0$ as $y \to +\infty$ (in fact, Γ' tends really quickly to 0 as $y \to +\infty$) and in particular $|(b^{-j}\Gamma \circ h^{j-1}(y))'| \leq \gamma_0 b^{-j}$ for all $j \in \mathbb{N}, y \geq \eta$ and for some appropriate constant $\gamma_0 > 0$. Thus, the function $\hat{s}$ is C^1 with its derivative

$$\hat{s}'(y) = 1 + \sum_{j=1}^{+\infty}(b^{-j}\Gamma \circ h^{j-1}(y))' \xrightarrow{y\to+\infty} 1$$

where the limit can be taken due to the Dominated Convergence Theorem. Since the derivative is strictly positive near $+\infty$, this implies that, by the sign-preserving Theorem and by the inverse function Theorem, for every sufficiently large $c > 0$, in this interval $\hat{s}$ is a diffeomorphism of $[\log c, +\infty)$ onto $[\hat{s}(\log c), +\infty)$ and in particular maps nullsets onto nullsets. This means that the set $[\log c, +\infty) \setminus U_c$ is a nullset and so $[c, +\infty) \setminus B$ is.

□

Example 2.6. *(i) Let* $f(x) = a_p x^p + a_{p-1}x^{p-1} + ... + a_1 x + a_0$ *where* $p \in \mathbb{N}$, $p \neq 1$ *and* $a_0, a_1, ...a_p \in \mathbb{R}$ *with* $a_p \neq 0$. *Assume without loss of generality that* $a_p > 0$. *In fact, if* $a_p < 0$, f *can be replaced by* $-f(-x)$ *if* p *is even or by* $f(x)^2$ *if* p *is odd and by Theorem 1.2 the conclusions are the same. The map* f *is differentiable with continuity and*

$$g(x) = f(x) - a_p x^p = a_{p-1}x^{p-1} + ... + a_1 x + a_0 = o(x^p)$$

is such that

$$g'(x) = (p-1)a_{p-1}x^{p-2} + (p-2)a_{p-2}x^{p-3} + ... + a_1 = o(x^{p-1}/\log x)$$

as $x \to +\infty$. *By Theorem 2.3(ii),* $(f^n(x_0))$ *is Benford for almost all* x_0 *with* $|x_0|$ *sufficiently large. For example, consider* $f(x) = x^2 + 1$. *It is such that* $f^n(x_0) \to +\infty$ *as* $n \to \infty$ *for every* $x_0 \in \mathbb{R}$ *so, by the previous considerations,* $(f^n(x_0))$ *is Benford for almost all* x_0 *(in this case,* $x_0 \in \mathbb{R}$*). As also said by the theorem, there are many exceptional points: for example, it can be shown that with the choice of* $x_0 = \lim_{n\to\infty} \sqrt{...\sqrt{\sqrt{10^{2^n}-1}-1}} = 9.949...$, *the first significant digit of* $f^n(x_0)$ *is always* 9.

(ii) Let $h : \mathbb{R} \to \mathbb{R}$ *be the continuous function*

$$h(y) = \begin{cases} 2y - \frac{\sin(2\pi y^2)}{2\pi y} & y \neq 0 \\ 0 & y = 0 \end{cases}$$

Define the countable union of intervals

$$J = \bigcup_{k \in \mathbb{Z}\setminus\{0\}} \left[k - \frac{1}{25|k|}, k + \frac{1}{25|k|}\right]$$

With a short calculation, or via graphic, it can be shown that $h(J) \subset J$ *and furthermore that* $(h^n(y) - 2^n k) \to 0$ *as* $n \to \infty$ *for every* $y \in J$ *and the proper* $k \in Z$. *For this reason,* $(h^n(y))$ *is not u.d. mod 1 for all* $y \in J$. *In fact, by Proposition 1.1,* $(h^n(y))$ *is u.d. mod 1 if and only if* $(2^n k)$ *is, but this sequence cannot be u.d. mod 1 because* $\langle 2^n \rangle \equiv 0$ [5]. *Now define the map* $f : \mathbb{R}^+ \to \mathbb{R}^+$

$$f(x) = 10^{h(\log x)} = 10^{2\log x}10^{-\frac{\sin(2\pi(\log x)^2)}{2\pi \log x}} = x^2 10^{-\frac{\sin(2\pi(\log x)^2)}{2\pi \log x}} = x^2 + g(x)$$

[5] It is interesting to know that it is not so simple to determine if sequences like (a^n) are u.d. mod 1. Clearly, as in this example, if $a \in \mathbb{Z}$ the fractional part is equal to zero for all entries and the sequence cannot be u.d. mod 1. Instead, even if a is rational, there are no results that can state whether a^n is u.d. mod 1 or not. For instance, to establish if $((3/2)^n)$ is u.d. mod 1 is a famous open problem. However, there is a statement that asserts that $(a^n x)$ is u.d. mod 1 for almost all $x \in \mathbb{R}$(see [1]).

where $g(x)$ is given by

$$g(x) = f(x) - x^2 = x^2 \left(10^{-\frac{\sin(2\pi(\log x)^2)}{2\pi \log x}} - 1\right)$$

In particular,

$$\frac{g(x)}{x^2} = 10^{-\frac{\sin(2\pi(\log x)^2)}{2\pi \log x}} - 1 \to 0 \quad as\, x \to +\infty$$

so $g(x) = o(x^2)$ and by Theorem 2.3(i) the map is Benford for almost every x such that $|x|$ is sufficiently large. However, from the previous observation, it is simple to find some points x_0 such that $(f^n(x_0))$ is not Benford. In fact, if $\log x_0 \in J$, the sequence generated by f is not Benford because $\log f(x) = h(\log x)$ is not u.d. mod 1 for every $y = \log x \in J$. Furthermore, $f^n(x_0)$ is not Benford also if $\log x_0 \in 10^J = \{10^y : y \in J\} = \bigcup_{k\in\mathbb{Z}\setminus 0} 10^k \left[10^{-1/(25|k|)}, 10^{1/(25|k|)}\right]$ that, in fact, has infinite Lebesgue measure. This means that, effectively, there are uncountable many x_0 for which $(f^n(x_0))$ is not Benford.

As in the previous sections, Theorem 2.3 yields a corollary that consider the reciprocals.

Corollary 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth map with $f(0) = 0$, $f'(0) = 0$ and $f^{(p)} \neq 0$ for some $p \in \mathbb{N} \setminus \{1\}$. Then $(f^n(x_0))$ is Benford for almost every x_0 sufficiently close to 0, but there are also many uncountable exceptional points.*

Example 2.7. *(i) The map $f(x) = x - 1 + e^{-x}$ is such that $f(0) = 0$, $f'(0) = 0$ and $f''(0) = 1$. Then, by Corollary 2.4, $(f^n(x_0))$ is Benford for almost all x_0 near 0. Since $f^n(x_0) \xrightarrow{n\to\infty} 0$ for every x_0, in fact $(f^n(x_0))$ is Benford for almost all $x_0 \in \mathbb{R}$.*

(ii) Consider the map $f(x) = \frac{1}{2}(x^2 + x^4)$ and note that $\lim_{n\to\infty} f^n(x_0) = 0$ if and only if $|x_0| < 1$. By the latter corollary, $(f^n(x_0))$ is Benford for almost all $x_0 \in (-1, 1)$. If one wants to study the map also for $|x_0| > 1$, i.e. for initial points such that $\lim_{n\to\infty} f^n(x_0) = +\infty$, just consider

$$\tilde{f}(x) = f(x^{-1})^{-1} = \left(\frac{1}{2}\left(x^{-2} + x^{-4}\right)\right)^{-1} = \frac{2x^4}{1 + x^2}$$

This map is such that $\tilde{f}(0) = 0$, $\tilde{f}'(0) = 0 = \tilde{f}''(0) = \tilde{f}'''(0)$ and $\tilde{f}^{iv}(0) = 48$ and this implies that $(\tilde{f}^n(x_0))$ is Benford for almost all $|x_0| > 1$ and so even $(f^n(x_0))$ is. The conclusion is that $(f^n(x_0))$ is Benford for almost all $x_0 \in \mathbb{R}$.

Since maps like $f(x) = e^x$, which have a faster growth than the maps studied in this section, have no result that can give some tool to analyze the Benford behavior, here it follows another proposition.

Proposition 2.3. *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a map such that, for some $c \geq 0$, both of the following conditions hold:*

(i) The function $\log f(10^x)$ is convex on $(c, +\infty)$

(ii) $\frac{\log f(10^x) - \log f(10^c)}{x-c} > 1$ for all $x > c$

then $(f^n(x_0))$ is Benford for almost all sufficiently large x_0, but there also exist uncountably many $x_0 > c$ for which $(f^n(x_0))$ is not Benford.

Example 2.8. *Consider $f(x) = e^x$ and let's verify the conditions. The function $h(x) = \log f(10^x) = \log(e^{10^x}) = 10^x \log e$ is such that $h''(x) = 10^x \ln(10) > 0$ for all $x \in \mathbb{R}$ and so $h(x)$ is convex on $\mathbb{R}$. To demonstrate the second point, some additional calculations are necessary:*

$$\frac{h(x) - h(c)}{x - c} = \log e \frac{10^x - 10^c}{x - c} \overset{?}{>} 1$$

If $c = 0$, the inequality becomes

$$\log e \frac{10^x - 1}{x} > 1$$

and this is true for all $x > 0$. In fact, the left-hand side of the inequality is equal to 1 only for $x = 0$. So, Proposition 2.3 applies with $c = 0$ and this means that $(f^n(x_0))$ is Benford for almost all sufficiently large x_0. Furthermore, it can be shown that $f^n(x_0) > 2^{n-2}$ for every $x_0 \in \mathbb{R}$ and $n \geq 2$ (even if $x_0 << 0$, x_1 will be approximately 1 and so the next iterations will increase more and more). So, the sequence is Benford for almost all $x_0 \in \mathbb{R}$.

Chapter 3

Nonautonomous Systems and the Benford's Law

The sequences that have been considered so far were not explicitly dependent on n. The aim of this chapter is to give some results about nonautonomous systems or *time-dependent systems*. These systems have been known a recent interest in many fields of study because of their important practical applications as well as purely mathematical questions. Here, the focus will be mainly on extending the theory seen in the previous chapter. The notation will be slightly different since the nonautonomous maps change with n. The maps are replaced by sequences of functions (f_n) that map $C \subset \mathbb{R}$ into itself. As before, the set often is equal to $\mathbb{R}^+$, but can be adapted to other sets. This means also a little change in the sequences for the fact that now there is not the iteration of a single function but at every iteration the map is modified: so, given any $x_0 \in C$, the *nonautonomous orbit* of x_0 under $f_1, f_2, ...$ is defined by

$$(x_n) = (f_1(x_0), f_2(f_1(x_0)), ...) = (f_n \circ ... \circ f_1(x_0))$$

Equivalently, to write it in the form of difference equations, such as (2.1),

$$x_n = f_n(x_{n-1}), \quad n \in \mathbb{N}$$

where x_n is the unique solution.
In this chapter only sequences with exponential and super-exponential growth are going to be study. It is interesting to note that most of the results in this chapter have as special cases the results of their autonomous counterparts.

3.1 Sequences with exponential growth

Let $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ be a sequence of maps of the form $f_n(x) = a_n x + g_n(x)$, $n \in \mathbb{N}$, with $a_n > 0$ and g_n that is small in some sense that will be specified later. The following theorem is a generalization of Corollary 2.2 with the difference that a_n is not required to be strictly greater than 1.

Theorem 3.1. *For every $n \in \mathbb{N}$, let $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ be a sequence of maps such that $f_n(x) = a_n x + g_n(x)$ with $a_n > 0$. Suppose that $\liminf_{n\to\infty} a_n > 1$ and that $g_n = o(x/\log x)$ uniformly[1] as $x \to +\infty$. Then, for every sufficiently large x_0, $(f_n \circ ... \circ f_1(x_0))$ is Benford if and only if $\left(\prod_{j=1}^n a_j\right) = (a_1, a_1a_2, a_1a_2a_3, ...)$ is Benford.*

Proof. Pick $\delta > 0$ and $N \in \mathbb{N}$ such that $a_n \geq 1 + 2\delta$ for all $n \geq N$. Then pick ξ_1 large enough such that $|g_n(x)| < \delta x/\log x$ for all $x \geq \xi_1$ and $n \in \mathbb{N}$. So,

$$f_n(x) = x\left(a_n + \frac{g_n(x)}{x}\right) > x\left(a_n - \frac{\delta}{\log x}\right) > x(a_n - \delta) \geq x(1+\delta)$$

for all $x \geq \xi_1$, $n \geq N$. Note that $\lim_{x\to+\infty} f_N \circ ... \circ f_1(x) = +\infty$ for all $x \geq \xi_1$ and so there exists $\xi_2 > 0$ such that $f_N \circ ... \circ f_1(x) \geq \xi_1$ and so there exists $\xi_2 > 0$ such that $f_N \circ ... \circ f_1(x) \geq \xi_1$ for all $x \geq \xi_2$. Hence, for all $x \geq \xi_2$ and $n \geq N$,

$$f_n \circ ... \circ f_1(x) = f_n \circ ... f_{N+1}(f_N \circ ... \circ f_1(x)) \geq (1+\delta)^{n-N}\xi_1$$

Now take $x_0 \geq \xi_2$ and put $y_n = \log x_n = \log f_n \circ ... \circ f_1(x_0)$. Then define

[1] A sequence of functions $f_n : \mathbb{R}^+ \to \setminus$ is said to be $o(g)$ *uniformly* as $x \to +\infty$ (for $g : \mathbb{R}^+ \to \mathbb{R}$) if, for every $\varepsilon > 0$, there exists $c \geq 0$ such that $|f_n(x)/g(x)| < \varepsilon$ for all $n \in \mathbb{N}$ and for all $x \geq c$. Furthermore, $f_n = \mathcal{O}(g)$ *uniformly* as $x \to +\infty$ if there exists $c > 0$ such that $|f_n(x)/g(x)| < c$ for every $n \in \mathbb{N}$ and for all x sufficiently large.

$z_n = y_n - \log \prod_{j=1}^{n} a_j$ and observe that, picking an arbitrary $\varepsilon > 0$:

$$\begin{aligned}
n|z_{n+1} - z_n| &= n\left| y_{n+1} - \log \prod_{j=1}^{n+1} a_j - y_n + \log \prod_{j=1}^{n} a_j \right| \\
&= n\left| \log\left(\frac{x_{n+1} \prod_{j=1}^{n} a_j}{x_n \prod_{j=1}^{n+1} a_j} \right) \right| \\
&= n\left| \log\left(\frac{x_{n+1}}{x_n a_{n+1}} \right) \right| \\
&= n\left| \log\left(1 + \frac{g_{n+1}(x_n)}{x_n a_{n+1}} \right) \right| \\
&\leq \frac{n|g_{n+1}(x_n)|}{2x_n a_{n+1}} \\
&\leq \frac{n\delta}{2a_{n+1} \log x_n} \\
&\leq \frac{n\varepsilon}{\log x_n} \\
&\leq \frac{n\varepsilon}{\log((1+\delta)^{n-N}\xi_1)} = \frac{n\varepsilon}{(n-N)\log(1+\delta) + \log \xi_1}
\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} n|z_n - z_{n+1}| \leq \frac{\varepsilon}{\log(1+\delta)}$$

and so

$$\lim_{n\to\infty} n(z_{n+1} - z_n) = 0$$

from the fact that $\varepsilon > 0$ was arbitrary. So, the sequence (y_n) is u.d. mod 1 if and only if $\left(\log \prod_{j=1}^{n} a_j \right)$ is, by Proposition 1.2(v). This implies that (x_n) is Benford if and only if $\left(\prod_{j=1}^{n} a_j \right)$ is Benford.

$\square$

Maybe it is not that simple to prove that $\left(\prod_{j=1}^{n} a_j \right)$ is Benford, so there is another result that will help to determine it.

Lemma 3.1. *Let (a_n) be a sequence of positive real numbers. Then the sequence $\left(\prod_{j=1}^{n} a_j \right) = (a_1, a_1 a_2, a_1 a_2 a_3, ...)$ is Benford if one of the following statements holds:*

(i) $\lim_{n\to\infty} a_n = a_\infty$ *exists and is such that* $a_\infty > 0$ *and* $\log a_\infty$ *is irrational*

(ii) $a_n = g(n)$ *for all* $n \in \mathbb{N}$*, where* g *is any non-constant polynomial*

Proof. (i) Note that $\log\prod_{j=1}^{n+1} a_j - \log\prod_{j=1}^{n} = \log a_{n+1} \xrightarrow{n\to\infty} \log a_\infty$ that exists by assumption. In particular, by Proposition 1.2(i), $(\prod_{j=1}^{n} a_j)$ is Benford if $\log a_\infty$ is irrational, which is by hypothesis. So, the first point is proved.

(ii) Since g is a non-constant polynomial, it can be written as $g(t) = c_p t^p + c_{p-1}t^{p-1} + ... + c_0$ where $p \in \mathbb{N}$ and $c_0, ...c_p \in \mathbb{R}$ with $c_p > 0$. Observe that

$$\begin{aligned}\log\prod_{j=1}^{n} a_j &= \log\prod_{j=1}^{n} g(j)\\ &= \sum_{j=1}^{n}\log\left(c_p j^p + c_{p-1}j^{p-1} + ... + c_0\right)\\ &= \sum_{j=1}^{n}\left(\log\left(c_p j^p\right) + \log\left(1 + \frac{c_{p-1}}{c_p}j^{-1} + ... + \frac{c_0}{c_p}j^{-p}\right)\right)\\ &= \sum_{j=1}^{n}\left(p\log(j) + \log(c_p)\right) + \sum_{j=1}^{n}\log e\left(\frac{c_{p-1}}{c_p}j^{-1} + ... + \frac{c_0}{c_p}j^{-p}\right)\\ &= \sum_{j=1}^{n} p\log(j) + n\log(c_p) + \frac{c_{p-1}}{c_p}\log e\sum_{j=1}^{n}\frac{1}{j} + \beta_n\end{aligned}$$

where (β_n) is a convergent sequence. Then, it is known that

$$\log e\sum_{j=1}^{n}\frac{1}{j} \approx \log n + \gamma$$

where γ is the Eulero-Mascheroni constant and, by the Eulero-Maclaurin formula

$$\sum_{j=1}^{n}\log(j) \approx \frac{1}{2}\log(n) + n\log(n) - n\log(e) + C$$

where C is a constant. So, putting them together, it follows that

$$\log\prod_{j=1}^{n} a_j - pn\log n - n(\log c_p - p\log e) - \left(\frac{p}{2} + \frac{c_{p-1}}{c_p}\right)\log n$$

is convergent. By Proposition 1.2(iii), the sequence $(\prod_{j=1}^{n} a_j)$ is Benford if and only if $\left(pn\log n + n\left(\log c_p - p\log e\right) + \left(\frac{p}{2} + \frac{c_{p-1}}{c_p}\right)\log n\right)$ is u.d mod 1 and in fact it is u.d. mod 1(see [4],Exc.2.26).

□

Example 3.1. *(i) Consider the linear maps $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) = (2 + \frac{1}{n})x$. Here $a_n = 2 + \frac{1}{n} > 2$ and $g_n \equiv 0$. The aim is to determine whether $\left(\prod_{j=1}^{n} a_j\right)$ is Benford or not:*

$$\log \prod_{j=1}^{n} a_j - \log \prod_{j=1}^{n-1} a_j = \log a_n \xrightarrow{n\to\infty} \log 2 \notin \mathbb{Q}$$

and this implies that $\left(\log \prod_{j=1}^{n} a_j\right)$ is u.d. mod 1 by Proposition 1.2(i) and so $\left(\prod_{j=1}^{n} a_j\right)$ is Benford. So, by Theorem 3.1, the nonautonomous orbit $(f_n \circ ... \circ f_1(x_0))$ is Benford for all sufficiently large x_0

(ii) Consider $f_n(x) = nx$ where $a_n = n$. Then, Lemma 3.1(ii) applies and so $(f_n \circ ... \circ f_1(x_0)) = (n!x_0)$ is Benford for every $x_0 \neq 0$, as already seen in Table 2.1.

(iii) The Fibonacci sequence is defined by $F_{n+2} = F_{n+1} + F_n$ for all $n \in \mathbb{N}$ and the first iterates are equals to $(F_n) = (1, 1, 2, 3, 5, 8, 13, ...)$. In particular the sequence is also given by the formula

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^n - \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n\right) = \frac{\phi^n - (-\phi^{-1})^n}{\sqrt{5}}, \quad n \in \mathbb{N}$$

where $\phi = \frac{1}{2}(1 + \sqrt{5}) = 1.618$.
Consider now the maps $f_n(x) = \frac{F_{n+1}}{F_n}x$, $n \in \mathbb{N}$, where $a_n = \frac{F_{n+1}}{F_n} > 0$. The sequence a_n has the well-known property that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$ and, by Proposition 1.2(i), the sequence $\left(\prod_{j=1}^{n} a_j\right)$ is Benford, since

$$\log \prod_{j=1}^{n} \frac{F_{j+1}}{F_j} - \log \prod_{j=1}^{n-1} \frac{F_{j+1}}{F_j} = \log \frac{F_{n+1}}{F_n} \xrightarrow{n\to\infty} \log \phi$$

which is irrational. This implies that $(f_n \circ ... \circ f_1(x_0)) = (F_n x_0)$ is Benford for all $x_0 \neq 0$ by Theorem 3.1. In particular, with $x_0 = F_1 = 1$, the maps generated the Fibonacci sequence and this means that the Fibonacci sequence is Benford.
The same conclusion of points (i) and (iii) could be found applying Lemma 3.1(i).

As in the sections of the previous chapter, even here there is a result in which the reciprocals are taken into account.

Theorem 3.2 ([2]). *Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of C^2-maps with $f_n(0) = 0$ and $f_n' = a_n \neq 0$ for all $n \in \mathbb{N}$. Assume that $\sup_n \max_{|x|\leq 1} |f_n''(x)|$ and $\sum_{n=1}^{\infty} \prod_{j=1}^{n} |a_j|$ are both finite. If $\lim_{n\to\infty} \log |a_n|$ exists and is irrational, then $(f_n \circ ... \circ f_1(x_0))$ is Benford for all $x_0 \neq 0$ sufficiently close to 0.*

Example 3.2. *(i) Reconsider the maps $f_n(x) = \left(2 + \frac{1}{n}\right) x$. The same conclusion of exercise 3.1(i) can be given by applying Theorem 3.2 just considering $F_n(x) = f_n\left(x^{-1}\right)^{-1} = \left(\left(2 + \frac{1}{n}\right) x^{-1}\right)^{-1} = \frac{nx}{2n+1}$. These maps are such that $F_n(0) = 0$ and $F_n'(0) = \frac{n}{2n+1} \neq 0$. Furthermore, $F_n''(x) = 0$ and*

$$\sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{j}{2j+1} = \sum_{n=1}^{\infty} \frac{n!}{(2n+1)!!}$$

is such that $\lim_{n\to\infty} \frac{(n+1)!}{(2(n+1)+1)!!} \frac{(2n+1)!!}{n!} = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2} < 1$ so, by the ratio asymptotic test, the sum is convergent and so finite. Then,

$$\lim_{n\to\infty} \log \left| \frac{n}{2n+1} \right| = \log(1/2) = -\log 2$$

that is irrational. Hence, $F_n(x_0)$ (and so $f_n(x_0)$) is Benford for all $x_0 \neq 0$.

(ii) Let $f_n(x) = 10^{-1+\sqrt{n+1}-\sqrt{n}} x$, $n \in \mathbb{N}$, be a sequence of linear maps. Here $a_n = 10^{-1+\sqrt{n+1}-\sqrt{n}}$, $f_n(0) = 0$ and $f_n'(0) = 10^{-1+\sqrt{n+1}-\sqrt{n}} \neq 0$. Again $f_n''(x) = 0$ and

$$\sum_{n=1}^{\infty} \prod_{j=1}^{n} 10^{-1+\sqrt{j+1}-\sqrt{j}} = \sum_{n=1}^{\infty} 10^{-(n+1)+\sqrt{n+1}}$$

which is convergent by the ratio test since $\lim_{n\to\infty} \frac{10^{-(n+2)+\sqrt{n+2}}}{10^{-(n+1)+\sqrt{n+1}}} = 1/10$. Now the limit

$$\lim_{n\to\infty} \log \left| 10^{-1+\sqrt{n+1}-\sqrt{n}} \right| = \log 10^{-1} = -1$$

that is not irrational, so Theorem 3.2 can't be used to determine the Benford behaviour of the sequence. However, $f^n(x) = 10^{-(n+1)+\sqrt{n+1}}$ and the logarithm $\log |f^n(x)| = -(n+1) + \sqrt{n+1} + \log |x|$ is u.d. mod 1. So, the sequence $(f_n \circ ... \circ f_1(x_0))$ is Benford for all $x_0 \neq 0$.

3.2 Sequences with super-exponential growth

Let $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ be a sequence of maps such that $f_n(x) = a_n x^{b_n}$, $n \in \mathbb{N}$, where (a_n) is a sequence of positive real numbers and (b_n) is a sequence of non-zero real numbers. This section has some results that recall the results of the analogue autonomous section. The following theorem has as special case Proposition 2.2 and, again, the difference holds in the fact that b_n may be not strictly greater than 1.

Theorem 3.3. *Let $f_n(x) = a_n x^{b_n}$ be a sequence of maps with $a_n > 0$ and $b_n \neq 0$ for every $n \in \mathbb{N}$. If $\liminf_{n\to\infty} |b_n| > 1$, then $(f_n \circ ... \circ f_1(x_0))$ is Benford for almost all $x_0 > 0$, but every non-empty open interval in $\mathbb{R}^+$ contains uncountably many x_0 for which $(f_n \circ ... \circ f_1(x_0))$ is not Benford.*

Example 3.3. *(i) Consider the maps $f_n(x) = 2^n x^2$, $n \in \mathbb{N}$, where $a_n = 2^n$ and $b_n \equiv 2$. Since $\liminf_{n\to\infty} |b_n| = \liminf_{n\to\infty} |2| = 2 > 1$, the sequence $(f_n \circ ... \circ f_1(x_0))$ is Benford for almost all $x_0 > 0$. In fact, it is Benford for almost all $x_0 \in \mathbb{R}$ since $f_n(x) > 0$ for all $x \neq 0$.*

(ii) Let $f_n(x) = x^{3^{1/n^2}}$ where $a_n = 1$ and $b_n = 3^{1/n^2}$ which is such that $b_n > 1 + 1/n^2 > 1$. In particular, note that $\liminf_{n\to\infty} |b_n| = 1$. The orbit generated by these maps is

$$(x_n) = (f_n \circ ... \circ f_1(x_0)) = x_0^{\prod_{j=1}^{n} 3^{1/j^2}} = x_0^{3^{\sum_{j=1}^{n} 1/j^2}}, \quad n \in \mathbb{N}$$

and $\lim_{n\to\infty} x_n = x_0^{3^{\sum_{j=1}^{\infty} 1/j^2}} = x_0^{3^{\pi^2/6}}$. So, the sequence cannot be Benford for any $x_0 > 0$.
This example shows that it is not sufficient to assume only $b_n > 1$ since the orbit generated by a sequence of functions may not be Benford for any x_0 (or, on the contrary, may be Benford for every $x_0 > 0$: see [1]).

If now the attention is moved on more general maps, it is obvious that these sequences may have a really fast growth. So, a more general statement of Proposition 2.3 is given.

Theorem 3.4. *Let $c \geq 0$ and let $f_n : \mathbb{R}^+ \to \mathbb{R}^+$, $n \in \mathbb{N}$, be a sequence of maps such that both of the following conditions hold:*

(i) the function $\log f_n(10^x)$ is convex on $(c, +\infty)$

(ii) $\frac{\log f_n(10^x) - \log f_n(10^c)}{x - c} \geq b_n > 0$ for all $x > c$

If $\liminf_{n\to\infty} b_n > 1$, then $(f_n \circ ... \circ f_1(x_0))$ is Benford for almost all sufficiently large x_0, but there also exist uncountably many $x_0 > c$ for which $(f_n \circ ... \circ f_1(x_0))$ is not Benford.

Example 3.4. *(i) Consider*

$$f_n = \begin{cases} x^2 & \textit{if } n \textit{ is even} \\ 2^x & \textit{if } n \textit{ is odd} \end{cases}$$

The function $\log f_n(10^x) = \begin{cases} \log(10^{2x}) \\ \log(2^{10^x}) \end{cases} = \begin{cases} 2x & \textit{if } n \textit{ is even} \\ 10^x \log 2 & \textit{if } n \textit{ is odd} \end{cases}$ *is convex on* $(0, +\infty)$ *.*

Then consider

$$\frac{\log f_n(10^x) - \log f_n(1)}{x} = \begin{cases} \frac{2x}{x} \\ \frac{10^x \log 2 - \log 2}{x} \end{cases} = \begin{cases} 2 > 0 \\ \frac{\log 2(10^x - 1)}{x} > 0 \end{cases}$$

for all $x > 0$*. So, by Theorem 3.4 with* $c = 0$*, the sequence* $(f_n \circ ... \circ f_1(x_0))$ *is Benford for almost all, but not all, sufficiently large* x_0 *and, in fact, for almost all* $x_0 \in \mathbb{R}$ *since clearly* $\lim_{n\to\infty} x_n = +\infty$*.*

(ii) Consider

$$f_n(x) = \begin{cases} 10 & \textit{if } n = 1 \\ x^2 & \textit{if } n \geq 2 \end{cases}$$

By point (i) it is known that the function satisfies the assumptions of the theorem for $n > 1$*. Instead, for* $n = 1$*, the second statement does not hold since*

$$\frac{\log f_1(10^x) - \log f_1(10^c)}{x - c} = 0$$

for all x *and* c*. So, Theorem 3.4 cannot be applied because one statement does not hold, even if only for an index. In fact,* $D_1(x_n) = 1$ *for every* $n \geq 1$*, so, in any case, the sequence* $(f_n \circ ... \circ f_1(x_0))$ *could not be Benford.*

To conclude the section, it can be interesting to give another result about polynomial maps that follows from Theorem 3.4.

Corollary 3.1 ([2])**.** *Let the sequence of maps* f_n *be polynomials, i.e.* $f_n(x) = x^{p_n} + a_{n,p_n-1}x^{p_n-1} + ... + a_{n,1}x + a_{n,0}$ *where* $p_n \in \mathbb{N} \setminus \{1\}$ *and* $a_{n,l} \in \mathbb{R}$ *for all* $n \in \mathbb{N}$ *and* $0 \leq l \leq p_n$*. If* $\sup_{n\in\mathbb{N}} \max_{l=0}^{p_n-1} |a_{n,l}| < +\infty$*, then* $(f_n \circ ... \circ f_1(x_0))$ *is Benford for almost every* $x_0 \in \mathbb{R} \setminus [-c, c]$ *with some* $c \geq 0$*. However,* $\mathbb{R} \setminus [-c, c]$ *also contains uncountable exceptional points.*

Example 3.5. *Consider* $f_n(x) = x^n - 1$*,* $n \in \mathbb{N}$*. Note that this map does not satisfy assumption (i) of Theorem 3.4 since* $\log f_n(10^x) = \log(10^{nx} - 1)$ *always has the second derivative negative. Despite this, by the latter corollary,*

$(f_n \circ ... \circ f_1(x_0))$ is Benford for almost every x_0, but there exist also many uncountable points for which the orbit is not Benford.

Conclusions

Benford's law is one of the most interesting results about the distribution of significant digits. In this thesis the focus is on one-dimensional Deterministic Systems and how they conform to this law, however, there also exist Random Systems which are not taken into account (see [1]). The obvious difference between them is that, in a deterministic system, the evolution of its next states are completely determined by its initial conditions and the rules that govern the system, while in a random system, the future states are influenced by stochastic variables or probabilistic processes. In particular, Deterministic Systems give the simplest way to describe processes that evolve over time and these processes are described by one-dimensional difference equations which take the form of sequences given by iterations of a single function or map f.
The first subdivision to be made is between autonomous and nonautonomous systems, i.e. systems in which maps do or do not depend explicitly on time respectively. Among them, the distinction is given by the growth of the sequences. In the autonomous case, it has been shown that sequences with polynomial growth are not Benford for any initial point x_0. These latter sequences are generated by maps of the form $f(x) = x + g(x)$, with g that is somehow small. Then, exponential and super-exponential increasing or decreasing sequences are taken into account. The difference of these two types of sequences may seem very little but the super-exponential ones, besides the fact that they have a clearly faster growth, they also conform to Benford's law without any specific request on the nature of the coefficients, contrary to the exponential ones. In fact, sequences with exponential growth are generated by maps of the form $f(x) = ax + g(x)$ and their Benford behavior is given by the nature of the logarithm of a. Sequences that have a super-exponential growth, instead, are given by maps written in the form $f(x) = ax^b + g(x)$ and are usually Benford for almost all, but not all, initial point x_0. The interesting fact is that for many maps of this type the set of the initial points for which $(f^n(x_0))$ is Benford and also its complementary are both uncountable.
In the nonautonomous case there is not so much difference in the results compared to the previous case. Here, maps that lead to an exponential growth

are in the form of $f_n(x) = a_n x + g_n(x)$ and even here the fact that a sequence is Benford or not depends on how a_n is constructed (the distinction is that now a_n is a sequence and not a number). Then, super-exponential growth sequences, are studied mainly in the form of maps like $f_n(x) = a_n x^{b_n}$ which have the same properties of those of autonomous case, i.e. they are Benford for almost all $x_0 > 0$ and there also exists uncountable exceptional points.

Even though it might not be immediately apparent, the applications of this law are numerous. As mentioned in the introduction, this law is used mostly to test falsifications in data collections and this includes a lot of areas, such as natural science, medicine, economics... The most famous application is fraud detection. One of the main works in this field was given by Nigrini [6] that noticed that some tax data adhere to Benford's law but fraudulent data do not. In the medical field the law is used, for example, to evaluate data for new drugs or to identify possibly falsified scientific publication. Another interesting fact is that Benford's law can be used to find changes in natural phenomena such as earthquakes. In fact, at the beginning of an earthquake, the set of values of the ground movements tend to conform to this law and this may help to detect the start of a seism even without reading the seismogram [7].
In conclusion, Benford's Law offers an unexpected way to look at the distribution of numbers in real life datasets and, since the interest in this law is increasing, in the future there will probably emerge new applications and discoveries.

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