Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

Corso di Laurea Magistrale in Matematica

On the local Langlands correspondence for $\mathbf{GL}_n(F)$

Relatore:
Prof. Francesco Esposito

Laureando: Mario Falciatore
Matricola: 2062874

Anno Accademico 2023/2024

data 20/09/2024

Introduction

In this work, we will analyse the local Langlands correspondence, studying the necessary concepts to understand what it says and trying to explain the main concepts behind it.

Let F be a non Archimedean local field of characteristic zero, i.e. a finite extension of $\mathbb{Q}_p$ for some prime number p. All the representations are to be intended on complex vector spaces.
The local Langlands correspondence for $\mathrm{GL}_n(F)$ associates some type of representations of the general linear group $\mathrm{GL}_n(F)$ of F with some representations related to the absolute Galois group of F.
Firstly, since the groups involved in this correspondence are topological, it is natural to require that the used representations preserve, in some ways, the topology of these groups. For this reason, our focus will be on a class of representations called **smooth representations**. The main property of these representations is that the stabiliser of a vector is open.

From the side of $\mathrm{GL}_n(F)$, the core of the correspondence involves in the **supercuspidal representations**. We can define them for a bigger class of groups: p-adic reductive groups, i.e. reductive groups on a non-Archimedean local field. For any parabolic subgroup P of G with Levi decomposition $P = MN$, we can construct a representation of G starting from an irreducible representation σ of M. Indeed, it is enough to extend the representation σ to a representation $\mathrm{Inf}_P^M \sigma$ of P defining it to be trivial on N and inducing this representation on G. The representation so defined is called parabolically induced from σ. We are interested in the irreducible representations of G which are not subrepresentations of parabolically induced representations from some proper parabolic group, these are the supercuspidal representations which are the building blocks of this theory for different reasons. In particular, they control the irreducible representations of G and they allow us to classify them with the Bernstein-Zelevinsky classification.

On the other side, the central role is taken by a subgroup of the absolute Galois group: the **Weil group**. It is defined starting from the canonical exact sequence of Galois theory

$$1 \to \mathrm{Gal}(\overline{F}/F^{nr}) \to \mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(F^{nr}/F) \to 1.$$

where $\overline{F}$ and F^{nr} are the closure and the maximal unramified extension of F respectively.
Since the group $\mathrm{Gal}(F^{nr}/F)$ is isomorphic to $\hat{\mathbb{Z}} := \prod_{l\in\mathbb{N}} \mathbb{Z}_l \supseteq \mathbb{Z}$, we can define the Weil group W_F of F by the exact sequence

$$1 \to I_F \to W_F \to \mathbb{Z} \to 1.$$

Moreover, we are not interested in the topology that W_F inherits from the absolute Galois group, but we endow W_F with a finer topology in which the inertia subgroup I_F of F is open and it has the same topology that it has in the absolute Galois group of F.
We are interested in the semisimple representations (V, π) of W_F with an attached endomorphism $\mathfrak{n}$ such that

$$r(\sigma)\,\mathfrak{n}\,r(\sigma)^{-1} = |Art_F^{-1}(\sigma)|_F\,\mathfrak{n},$$

where $\mathrm{Art}_F : F^\times \to W_F^{ab}$ is the isomorphism of local class field theory. It allows to study all the abelian extensions of F studying the property of the multiplicative group of F. In particular the attached endomorphism $\mathfrak{n}$ contains a lot of information about the representation.

In both the cases, we can define some invariants. The so called L-factor and ϵ-factor that are complex functions constructed from the representations. Despite having the same name, the constructions for the two sides are different. In particular, the role of supercuspidal representations is crucial for the GL_n-side. We expect a sort of compatibility of the invariants with respect to the correspondence.

With local Langlands correspondence we mean a series of bijections between the set of equivalence classes of representations of $\mathrm{GL}_n(F)$ and n-dimensional F-semisimple Weil-Deligne representations

$$\mathrm{rec}_F : \mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C}) \to \text{WD-Rep}_n^{ss}(W_F/\mathbb{C})$$

such that

1. $\mathrm{rec}_F(\pi) = \pi \circ \mathrm{Art}^{-1}$ for any π in $\mathrm{IrrRep}(\mathrm{GL}_1(F)/\mathbb{C}) \cong \mathrm{IrrRep}(F^\times/\mathbb{C})$.
2. For any π_1 in $\mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C})$ and π_2 in $\mathrm{IrrRep}(\mathrm{GL}_m(F)/\mathbb{C})$

   $$L(\pi_1 \times \pi_2, s) = L(\mathrm{rec}_F(\pi) \otimes \mathrm{rec}_F(\pi_2), s)$$

   and

   $$\varepsilon(\pi_1 \times \pi_2, s, \psi) = \varepsilon(\mathrm{rec}_F(\pi) \otimes \mathrm{rec}_F(\pi_2), s, \psi).$$

3. For any π in $\mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C})$ and $\mathrm{IrrRep}(\mathrm{GL}_1(F)/\mathbb{C})$

$$\mathrm{rec}_F(\pi \otimes (\chi \circ det)) = \mathrm{rec}_F(\pi) \otimes \mathrm{rec}_F(\chi).$$

4. For any π in $\mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C})$ with central character χ

$$\det \mathrm{rec}_F(\pi) = \mathrm{rec}_F(\chi).$$

5. For any π in $\mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C})$

$$\mathrm{rec}_F(\tilde{\pi}) = \widetilde{\mathrm{rec}_F(\pi)}.$$

The work is organized as follows: In the first chapter we will analise the property of smooth representations with a focus on the difference between the representations of finite groups.
In the second and third chapters we will analise, respectively, the two sides of the correspondence.
In the fourth chapter, we will construct the invariants using the previously cited Berstein-Zelevinsky classification and we will expose the correspondence.

Contents

Chapter 1

Representations of Locally Profinite Groups

Definition 1.1. A **locally profinite group** G is a topological group G that satisfies one of the two equivalent properties

1. Any open neighbourhood of the identity in G contains an open compact subgroup of G.
2. G is Hausdorff, locally compact and totally disconnected.

Remark 1.2. Clearly, a compact locally profinite group is profinite.

We always assume that G is a locally compact group such that any open compact subgroup has countable index and V is a vector space over $\mathbb{C}$. We want to combine the classical notion of group representation with the topology of a locally compact group. To do this

Definition 1.3. Let (V, π) be a representation of G. We say that V is **smooth** if

$$V = \bigcup_{\substack{K \subseteq G \\ K \text{ open compact}}} V^K,$$

where $V^K = \{v \in V : \pi(g)v = v \quad \forall g \in K\}$.

Remark 1.4. Clearly, if G is finite and it is endowed with the discrete topology, then any representation is smooth.

The terminology "smooth" makes us think about continuity, indeed there is a deep connection that we clarify in the next proposition.

Proposition 1.5. *Let (V, π) be a representation of G. The following are equivalent*

(1) (V, π) is smooth.

(2) The ***stabilizer*** $\mathrm{Stab}_G(v) := \{g \in G : \pi(g)v = v\}$ *of v in G is open for any vector v in V.*

(3) The map

$$\begin{aligned}\Pi :&G \times V \to V\\ &(g, v) \to \pi(g)v\end{aligned}$$

is continuous, where V is endowed with the discrete topology and $G\times V$ is endowed with the product topology.

Proof. (1) $\implies$ (2): Assume (V, π) be smooth and let v in V. Then there exists K open compact subgroup of G such that v is in V^K and then

$$K \subseteq \mathrm{Stab}_K(v) \subseteq \mathrm{Stab}_G(v).$$

So $\mathrm{Stab}_G(v)$ contains an open subgroup and then it is itself open.
(2) $\implies$ (3): Assume $\mathrm{Stab}_G(v)$ be open for any vector v in V. Since V is endowed with the discrete topology, it is enough to prove that the preimage of $\{\pi(g)v\}$ is open. Clearly, the preimage of $\{\pi(g)v\}$ is $\bigcup_{x\in G} g\,\mathrm{Stab}_G(v)x^{-1}\times \{\pi(x)v\}$ that is open by hypothesis.
(3) $\implies$ (1): Assume Π is continuous and take a generic vector v in V, then $\Pi^{-1}(v)$ is open in $G \times V$. So, there exist L open in G and U open in V such that $L \times U \subseteq \Pi^{-1}(v)$. Furthermore, since

$$(1, v) \mapsto v,$$

we can suppose $1 \in L$ and so L is an open neigherbood of 1. At this point, since G is locally profinite there exists K open compact in G such that $K \subseteq G$. So

$$K \times U \subseteq \Pi^{-1}(v)$$

and we can conclude that K stabilizes v.

□

We can note that there is no condition on the dimension of V. In general for finite groups it is enough to work with finite dimensional representations since irreducible representations of finite groups are of finite dimension. This is not true for representations of locally profinite groups. For this reason we introduce a new concept.

Definition 1.6. Let V be a smooth representation of G. We say that V is **admissible** if the vector space V^K has finite dimension for any K open compact subgroup of G.

Definition 1.7. A **morphism** f between smooth representations (V_1, π_1), (V_2, π_2) is a linear map $f : V_1 \to V_2$ such that the diagram

$$\begin{tikzcd}
V_1 \arrow[r, "\pi_1(g)"] \arrow[d, "f"'] & V_1 \arrow[d, "f"] \\
V_2 \arrow[r, "\pi_2(g)"'] & V_2
\end{tikzcd}$$

commutes for any g. We denote by $\mathrm{Hom}_G(V_1, V_2)$ the space of morphisms of smooth representation which are also called G-morphisms.

So we can define the category $\mathrm{Rep}(G)$ of smooth representations of a locally profinite group G.

Theorem 1.8. *The category* $\mathrm{Rep}(G)$ *is abelian.*

Definition 1.9. A representation (V, π) is said to be **irreducible** if $V \neq 0$ and V has no G-stable subspaces different from 0 and V.

1.1 Semisimplicity

In representation theory of finite groups we are used to work with representations that are direct sums of irreducible G-subspaces.[1] In general this is not true for a generic representation of a locally profinite group. For this reason we introduce the following definition.

Definition 1.10. We say that (V, π) is **G-semisimple**[2] if V is the sum of its irreducible G-subspaces.[3]

Proposition 1.11. *For a representation* (V, π) *the following are equivalent*

(1) V is the sum of its irreducible subspaces.

(2) V is the direct sum of a family of irreducible G-subspaces.

(3) Any G-subspace has a G-complement in V.

Proof. (1) $\implies$ (2): Assume that $V = \sum_{i \in I} U_i$, with $\{U_i : i \in I\}$ family of irreducible G-subspaces of V. Let

$$\mathcal{I} = \{J \subseteq I : \sum_{i \in J} U_i = \bigoplus_{i \in J} U_i\}$$

Our aim is to use Zorn's lemma on $\mathcal{I}$. For this aim, take $\{J_a : a \in A\}$ a totally ordered set in $\mathcal{I}$ and $J := \bigcup_{a \in A} J_a$. We want to prove that J is an element of $\mathcal{I}$. Suppose by contradiction that J is not in $\mathcal{I}$, i.e. $\sum_{j \in J} U_j$

---

[1] Remember that we are working with vector spaces over $\mathbb{C}$ and so Maschke's theorem holds.

[2] This is clearly compatible with the theory of finite groups.

[3] Notice that we do not require V to be the direct sum, but only the sum.

is not direct, so there exists $S \subseteq J$ finite such that the sum $\sum_{j \in S} U_j$ is not direct. But $S \subseteq J_a$ for some $a \in A$, so the $\sum_{j \in J_a} U_j$ is not direct. This is a contradiction, hence J is an element of $\mathcal{I}$. In this way we have a maximal element $\mathcal{J}$ of $\mathcal{I}$. Let $V' = \sum_{j \in \mathcal{J}} U_j \subseteq V$. For any $i \in I \setminus \mathcal{J}$, let us consider the intersection $U_i \cap V'$. Since W_i is irreducible, we have only two possibilities:

1. $U_i \cap V' = 0$. In this case, the sum $U_i \oplus V'$ is direct, but this is a contradiction since $V' = \sum_{j \in \mathcal{J}} U_i$ and $\mathcal{J}$ is maximal in $\mathcal{I}$.
2. $U_i \cap V' = U_i$, so $U_i \subseteq V'$.

Therefore $V = \sum_{i \in I} U_i \subseteq V'$ and then the equality holds.
(2) $\implies$ (3): Assume that $V = \bigoplus_{i \in I} U_i$, with $\{U_i : i \in I\}$ a family of irreducible G-subspaces of V. Let

$$\mathcal{I} = \{J \subseteq I : W \cap \oplus_{i \in J} U_i = 0\}.$$

As in the previous point, $\mathcal{I}$ is inductively ordered and, by Zorn's lemma, it admits a maximal element $\mathcal{J}$. Let $U = \oplus_{i \in \mathcal{J}} U_i$. U_i is the G-subspace that we are looking for. Indeed

$$W + U = W \oplus U$$

by hypothesis and if, by contradiction, $V \neq W \oplus U$ then there would exist $U_j \not\subseteq W \oplus U$ such that

$$W + (U + U_j) = W + (\oplus_{i \in \mathcal{I}} U_i \oplus U_j) = W + \oplus_{i \in \mathcal{I} \cup \{j\}} U_i = W \oplus (\oplus_{i \in \mathcal{I} \cup \{j\}} U_i).$$

This is a contradiction for the maximality of $\mathcal{J}$. Hence $V = X \oplus U$.
(3) $\implies$ (1): Assume that any G-subspace of V has a G-complement in V. Let $U = \sum_i U_i$ be the sum of all the irreducible representations of V. By hypothesis, there exists a G-complement W of U in V. In other word, $V = U \oplus W$. Our intent is to prove that $W = 0$. If W has an irreducible G-subspace, it is an irreducible G-subspace of V and so it is in U. This is a contradiction for the hypothesis of direct sum. So W has no irreducible G-subspaces. In the same way, W is not irreducible. Let w be in W and consider the G-stable subspace W' of W defined as

$$W' = \mathrm{Span}\{\pi(g)w : g \in G\}.$$

W' is a finitely generated $\mathbb{C}[G]$-module, so we have a maximal subrepresentation W'' of W' [4]. In particular W'/W'' is irreducible. Moreover, since W'' is a vector subspace of W, then U and W'' are in direct sum.
By hypothesis, there exists a G-stable subspace of V, say W_0, such that

$$U \oplus W'' \oplus W_0.$$

---

[4] That is a maximal $\mathbb{C}[G]$-submodule of W'.

Moreover, the image of W' under the projection in W_0 is isomorphic to W'/W''. So W_0 contains an irreducible representation of V, that is not possible since all irreducible representations of V are in U. Hence $W = 0$. ☐

In the above proof we have dealt with vector spaces which can have infinite dimension. There are cases in which we are sure of working with finite dimensional representations. The next lemma provides an important example.

Lemma 1.12. *Let G be a compact locally profinite group (i.e. a profinite group) and let (V, π) be an irreducible representation of G. Then V is finite dimensional.*

Proof. Let v be a vector in V and suppose v in V^K for some open compact subgroup K of G. Since G is compact, the index $|G : K|$ is finite and the set $\{\pi(g)v : g \in G/K\}$ is finite. But it spans a G-stable subspace that has to be V since it is irreducible. So $\dim V \leq |\{\pi(g)v : g \in G/K\}| < \infty$. ☐

Proposition 1.13. *Let (V, π) be a smooth representation of G. If G is compact, then V is G-semisimple.*

Proof. Since (V, π) is smooth, we have

$$V = \bigcup_{\substack{K \subseteq G \\ K \text{ open compact}}} V^K.$$

Substituting K with its normal core $K_G = \cap_{g \in G} gKg^{-1}$ we can suppose that any K is normal. Moreover the action of G on V^K is the same of G/K since V^K is fixed by K. Since G is compact, any index $|G : K|$ is finite and hence V^K is G-semisimple. Therefore V is G-semisimple. ☐

Corollary 1.14. *Let (V, π) be a representation of G. Then V is K-semisimple for any K open compact subgroup of G.*

We conclude this section with a generalization of Maschke's theorem.

Theorem 1.15. *Let H be an open subgroup of G of finite index and let (V, π) be a smooth representation of G. The following are equivalent*

(1) V is G-semisimple,

(2) V is H-semisimple.

Proof. (2) $\implies$ (1): Suppose that V is H-semisimple. By Proposition 1.11 it is enough to prove that any G-subspace has a G-complement in V. For this aim, take U a G-subspace that is clearly an H-subspace. By hypothesis,

V is H-semisimple and so U has an H-complement in V: $V = U \oplus W$. Take the projection $f_U : V \to U$ and let

$$\begin{aligned} f : V &\to U \\ v &\mapsto \frac{1}{|G:H|} \sum_{g \in G/H} \pi(g) f_U(\pi(g^{-1})v), \end{aligned}$$

so that

$$f = \frac{1}{|G:H|} \sum_{g \in G/H} \pi(g) \circ f_U \circ \pi(g)^{-1}.$$

For all u in U, $f(u) = u$ and so $V = U \oplus \mathrm{Ker}(f)$. Now it is enough to prove that $\mathrm{Ker}(f)$ is a G-subspace. It is easy to compute that

$$\pi(x) \circ f \circ \pi(x)^{-1} = f$$

and then

$$f \circ \pi(x) = \pi(x) \circ f$$

for all x in G. In particular, for all x in G and for all v in $\mathrm{Ker}(f)$

$$f(\pi(x))(v) = \pi(x)(f(v)) = \pi(x)(0) = 0.$$

Therefore, $\mathrm{Ker}(f)$ is G-stable and (1) holds.
In order to prove the other implication we need other two important tools in representation theory: the **induced representation** and **Shur's lemma** □

Corollary 1.16. ***(Maschke's Theorem)*** *If G is a finite group then every representation is G-semisimple.*

Proof. Let (V, π) be a representation of G. By the previous proposition, it is enough to prove that G has a subgroup H for which V is H-semisimple. $H = \{1\}$ satisfies the condition and so the theorem holds. □

1.2 Induced Representations

In this section we want to describe a way to construct representations of a locally profinite group G starting with a smooth representation (W, σ) of a closed subgroup H of G. For this aim, consider the $\mathbb{C}$-vector space

$$X = \{f : G \to W : f(hg) = \sigma(h) f(g), h \in H, g \in G\}.$$

We define an action of G on X as

$$(g \cdot f)(g') := f(g'g) \text{ for all } g, g' \text{ in } G$$

i.e. a representation of G in X. This is the classical induced representation that does not take smoothness into account and it is, in general, not a smooth representation of G. In order to obtain a smooth representation we introduce the following definition.

Definition 1.17. Let $G, H, (W, \sigma)$ and X as before. We define the **representation of G smoothly induced by** σ as

$$\begin{aligned}\operatorname{Ind}_H^G \sigma = \{f \in X : {} & \text{there is a compact open subgroup } K \text{ of } G \\ & \text{such that } f(gx) = f(g), \text{ for } g \in G \text{ and } x \in K\}.\end{aligned}$$

So we are taking the smooth part of the induced representation. This technique can be used for all non smooth representations of a group G.
Let (W, σ) be a representation (not necessarily smooth) of G and consider

$$W^\infty := \bigcup_{\substack{K \subseteq G \\ K \text{ open compact}}} W^K$$

$$\sigma^\infty(g) = \sigma(g)_{W^\infty}$$

$(W^\infty, \sigma^\infty)$ is a smooth representation[5] of G and it is constructed starting with (W, σ) in the same way in which we constructed the representation smoothly induced. We can say more: if (V, π) is a smooth representation of G and f is a G-homomorphism from V in W, for all v in V there exists an open compact subgroup K of G that fixes v. Then

$$(\pi(k)f)(v) = f(\sigma(k)(v)) = f(v) \quad \text{for all } k \in K$$

and so $f(V) \subseteq W$. In particular, $\operatorname{Hom}_G(V, W) = \operatorname{Hom}_G(V, W^\infty)$.
This would be enough to prove the **Frobenius reciprocity** for smooth representations starting with the classical version. Nevertheless, we will give a direct proof.
Notice that now we have a functor

$$\begin{aligned}\operatorname{Ind}_H^G : \operatorname{Rep}(H) &\to \operatorname{Rep}(G). \\ (W, \sigma) &\mapsto (\operatorname{Ind}_H^G W, \operatorname{Ind}_H^G \sigma).\end{aligned}$$

This is not the only functor that we have constructed, indeed there is another one defined by

$$\begin{aligned}\operatorname{Res}_H^G : \operatorname{Rep}(G) &\to \operatorname{Rep}(H) \\ (V, \pi) &\mapsto (V, \pi_{|H})\end{aligned}$$

The connection between these two is expressed by the next theorem.

---

[5] If w is fixed by an open compact K, then $\pi(g)w$ is fixed by the open compact gKg^{-1} for any g in G. So W^∞ is a representation.

Theorem 1.18. *With the same notation as before, we have*

$$\operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G \pi, \sigma).$$

This isomorphism is called ***Frobenius Reciprocity****.*

Proof. Consider the homomorphism

$$\begin{aligned} \alpha_\sigma : \operatorname{Ind}_H^G W &\to W \\ f &\mapsto f(1). \end{aligned}$$

α_σ is an H-homomorphism. Indeed, for any h in H and f in $\operatorname{Ind}_H^G W$ we have

$$\alpha_\sigma(hf) = h(f(1)) = f(h) = \sigma(h)f(1) = \sigma(h)\alpha_\sigma(f).$$

So the map

$$\begin{aligned} \operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma) &\to \operatorname{Hom}_H(\operatorname{Res}_H^G \pi, \sigma) \\ \phi &\mapsto \alpha_\sigma \circ \phi \end{aligned}$$

is well-defined since composition of H-morphisms is again an H-morphism. Now, for a G-homomorphism $f : V \to W$, we define

$$\begin{aligned} f_* : V &\to \operatorname{Ind}_H^G W \\ v &\mapsto f_*(v) : G \to W \\ &\qquad\qquad g \mapsto f((\pi(g)v). \end{aligned}$$

Our claim is that $f \mapsto f_*$ is the inverse of $\phi \mapsto \alpha_\sigma \circ \phi$.

- $f \mapsto f_* \mapsto \alpha_\sigma \circ f_* = f$
  For all v in V we have

  $$(\alpha_\sigma \circ f_*)(v) = \alpha_\sigma(f_*(v)) = f_*(v)(1)) = f(\pi(1)(v)) = f(v)$$

  and so $\alpha_\sigma \circ f_* = f$.

- $\phi \to \alpha_\sigma \circ \phi \to (\alpha_\sigma \circ \phi)_* = \phi$
  For all v in V and g in G we have

  $$\begin{aligned} (\alpha_\sigma \circ \phi)_*(v)(g) =& (\alpha_\sigma \circ \phi)(\pi(g)(v)) = \alpha_\sigma(\pi(g)(v)) = \\ =& \alpha_\sigma(\operatorname{Ind}_H^G \sigma(g)(\phi(v)) = \alpha_\sigma(\operatorname{Ind}_H^G \sigma(g)(\phi(v))) = \\ =& \alpha_\sigma(\operatorname{Ind}_H^G \sigma(g) \circ \phi)(v) = \operatorname{Ind}_H^G \sigma(g)(\phi(v))(1) = \\ =& \phi(v)(g) \end{aligned}$$

  and so $(\alpha_\sigma \circ \phi)_* = \phi$.

These equalities complete the proof. $\square$

Remark 1.19. With the same notation as before we have that Res_H^G is the left adjoint of Ind_H^G. There are several important consequences, e.g. Res_H^G preserves colimits and Ind_H^G preserves limits.

Since we are working with a topological group we can construct another induced representation related with compactness.

Definition 1.20. With the same notation as before, we define the **compact induction** of (W, σ) as

$$\text{c-}\operatorname{Ind}_H^G W = \{f \in \operatorname{Ind}_H^G W : \ \operatorname{Supp}(f) \subseteq CH \text{ for some compact } C \subseteq G\},$$

that is a G-subspace of $\operatorname{Ind}_H^G W$.

For this type of induced representation, we have an analog of the Frobenius reciprocity.

Proposition 1.21. *With the same notation as before, we have*

$$\operatorname{Hom}_G(\text{c-}\operatorname{Ind}_H^G, \pi) \cong \operatorname{Hom}_H(\sigma, \operatorname{Res}_H^G \pi).$$

Proof. Consider the map

$$\begin{aligned} G \times W &\to \text{c-}\operatorname{Ind}_H^G W \\ (g, w) &\mapsto f_{(g,w)} : x \to \begin{cases} \sigma(xg)w \quad if \quad xg \in H. \\ 0 \quad otherwise. \end{cases} \end{aligned}$$

It will be fundamental that functions of the form $f_{(g,w)}$ generate $\text{c-}\operatorname{Ind}_H^G W$. Take $\psi \in \text{c-}\operatorname{Ind}_H^G$. For any x in G, let us consider a trasversal R of H in G such that x is in R, i.e. the lateral of H that contains x is Hx. So we obtain

$$\sum_{g \in R} f_{(g^{-1}, \psi(g))}(x) = \sigma(xx^{-1})\psi(x) = \psi(x).$$

Now we consider the morphism

$$\begin{aligned} \alpha_\sigma^c : W &\to \text{c-}\operatorname{Ind}_H^G W \\ w &\mapsto f_{(1,w)} =: f_w \end{aligned}$$

that is an H-homomorphism. Indeed for any h, g [6] in H and w in W we have

$$\begin{aligned} \text{c-}\operatorname{Ind}_H^G \sigma(h)\alpha_\sigma^c(w)(g) =& \text{c-}\operatorname{Ind}_H^G \sigma(h) f_w(g) = f_w(gh) = \\ =& \sigma(g)\sigma(h)w = f_{\sigma(h)w}\sigma(g) = \\ =& \alpha_\sigma^c(\sigma(h)(w))(g) \end{aligned}$$

---

[6] We can suppose $g \in H$ since $gh \in H$ if and only if $g \in G$ and in the other case $f_w(g) = 0$.

and then

$$\operatorname{c-Ind}_H^G \sigma(h) f_w = \alpha_\sigma^c(\sigma(h)(w)).$$

So the map

$$\begin{aligned} \operatorname{Hom}_G(\operatorname{c-Ind}_H^G \sigma, \pi) &\to \operatorname{Hom}_H(\sigma, \operatorname{Res}_H^G \pi) \\ \phi &\mapsto \phi \circ \alpha_\sigma^c \end{aligned}$$

is well-defined since composition of H-morphisms is again an H-morphism. Now, for all $\psi : W \to V$ H-homomorphism we define

$$\begin{aligned} \psi_* : \operatorname{c-Ind}_H^G W &\to V \\ f_{g,w} &\mapsto \pi(g)\psi(w). \end{aligned}$$

Our claim is that $\psi \mapsto \psi_*$ is the inverse of $\phi \mapsto \phi \circ \alpha_\sigma^c$.

- $\psi \mapsto \psi_* \mapsto \psi_* \circ \alpha_\sigma^c = \psi$
  For all w in W we have

  $$(\psi_* \alpha_\sigma)(w) = \psi_*(f_w) = \psi_*(f_{(1,w)}) = \pi(1)\psi(w) = \psi(w)$$

  and so $\psi_* \circ \alpha_\sigma = \psi$.

- $\phi \mapsto \phi \circ \alpha_\sigma^c \mapsto (\phi \circ \alpha_\sigma^c)_* = \phi$
  For all w in W and g in G we have

  $$\begin{aligned} (\phi \circ \alpha_\sigma^c)_*(f_{(g,w)}) =& \pi(g)\phi(\alpha_\sigma^c(w)) = \phi(\operatorname{c-Ind}_H^G \sigma(g)(f_{(1,w)}) = \\ =& \phi(f_{(g,w)}) \end{aligned}$$

  and so $(\phi \circ \alpha_\sigma^c)_* = \phi$.

These equalities complete the proof. □

Theorem 1.22. ***(Mackey decomposition)*** *Let H, K be two subgroups of a locally profinite group G respectively closed and open. Let (W, σ) be a smooth representation of H. Then we have an isomorphism of representations of K given by*

$$\begin{aligned} \operatorname{Res}_K^G \operatorname{Ind}_H^G (W, \sigma) &\to \Big(\prod_{g \in H\backslash G/K} \operatorname{Ind}_{K \cap g^{-1}Hg}^K \operatorname{Res}_{K \cap g^{-1}Hg}^{g^{-1}Hg} (W, g_*^{-1}\sigma) \Big)^\infty \\ f &\mapsto (f_g)_{g \in H\backslash G/K} \end{aligned}$$

where $(W, g_*^{-1}\sigma)$ is the representation of $g^{-1}Hg$ such that $g_*^{-1}\sigma(g^{-1}hg) = \sigma(h)$ and $f_g(x) = f(gx)$.

Proof. For any g in the double quotient $H\backslash G/K$ Consider the K-representation

$$V_{HgK} = \{f \in X : f(x) = 0 \text{ if } x \notin HgK\}$$

and its smooth part

$$\begin{aligned} V_{HgK}^{\infty} = \{f \in V_{HgK} : &\text{ there is a compact open subgroup } L \text{ of } K \\ &\text{such that } f(kx) = f(k), \text{ for } k \in K \text{ and } x \in L\}. \end{aligned}$$

Since K is open, the open compact of K are again open compact in G, so

$$\begin{aligned} V_{HgK}^{\infty} &= \operatorname{Ind}_H^G(W) \cap V_{HgK} \\ &= \{f \in \operatorname{Ind}_H^G W : f(x) = 0 \text{ if } x \notin HgK\}. \end{aligned}$$

Moreover, since $G = \bigcup_{g \in HgK} HgK$, the representation $\prod_{g \in H\backslash G/K} V_{HgK}^{\infty}$ of K is the same of $\operatorname{Ind}_K^G W$ [7]. So

$$\begin{aligned} \operatorname{Res}_K^G \operatorname{Ind}_H^G W &\to \prod_{g \in H\backslash G/K} V_{HgK}^{\infty} \\ f &\mapsto (f_g)_{g \in H\backslash G/K} \end{aligned}$$

is an isomorphism.
After this observation, it is enough to note that the K map

$$\begin{aligned} V_{HgK}^{\infty} &\to \operatorname{Ind}_{K \cap g^{-1}Hg}^{K} \operatorname{Res}_{K \cap g^{-1}Hg}^{g^{-1}Hg}(W, g_*^{-1}\sigma) \\ f &\mapsto f_g \end{aligned}$$

has an inverse

$$\begin{aligned} \operatorname{Ind}_{K \cap g^{-1}Hg}^{K} \operatorname{Res}_{K \cap g^{-1}Hg}^{g^{-1}Hg}(W, g_*^{-1}\sigma) &\to V_{HgK}^{\infty} \\ f &\mapsto \tilde{f} : [hgk \mapsto \sigma(h)f(k)]. \end{aligned}$$

Indeed

- $f \mapsto f_g \mapsto \tilde{f_g} = f$
  For all hgk in HgK we have
  $$\tilde{f_g}(hgk) = \sigma(h)f_g(k) = \sigma(h)f(gk) = f(hgk).$$
- $f \mapsto \tilde{f} \mapsto (\tilde{f})_g$
  For all x in G we have
  $$(\tilde{f})_g(x) = \tilde{f}(gx) = f(x).$$

---

[7] Only as representations of K!

So $f \mapsto f_g$ is an isomorphism of K-representations and this conclude the proof. $\square$

Remark 1.23. Let H, K, G and (W, σ) as above. If K is compact, then we have the isomorphism

$$\begin{aligned}(\operatorname{Ind}_H^G W)^K \cong & \Big(\prod_{g \in H\backslash G/K} \operatorname{Ind}_{K\cap g^{-1}Hg}^K \operatorname{Res}_{K\cap g^{-1}Hg}^{g^{-1}Hg} (W, g_*^{-1}\sigma) \Big)^K = \\ &= \prod_{g \in H\backslash G/K} \Big(\operatorname{Ind}_{K\cap g^{-1}Hg}^K \operatorname{Res}_{K\cap g^{-1}Hg}^{g^{-1}Hg} (W, g_*^{-1}\sigma) \Big)^K \cong \\ &\cong \prod_{g \in H\backslash G/K} (g_*^{-1}V)^{g^{-1}Hg\cap K} = \prod_{g \in H\backslash G/K} W^{H\cap gKg^{-1}}\end{aligned}$$

Now we can prove the other implication of 1.15.

Proof. We need to prove that if a representation (V, π) is G-semisimple, then it is H-semisimple for any open subgroup H of G that has finite index. Without losing generality, we can suppose H normal. Indeed it is enough to substitute it with its normal core as done in 1.13.
Since V is G-semisimple, it is direct sum of irreducible G-subspace. We can therefore assume V irreducible that is equivalent to be a cyclic $\mathbb{C}[G]$ module generated by any non zero vector v in V. So, if $\{g_1, \dots, g_n\}$ is a set of representatives of G/H, then V is generated by $\{\pi(g_1)v, \dots, \pi(g_n)v\}$ as $\mathbb{C}[H]$-module. Since it is finitely generated, it has a maximal submodule W and then an irreducible quotient $(U := V/W, \sigma)$ and a non zero map

$$\operatorname{Res}_H^G V \to U.$$

Now, since G/H is finite and hence compact we have a non zero injective map [8]

$$V \to \operatorname{Ind}_H^G U = \text{c-}\operatorname{Ind}_H^G U = \bigoplus_{g_i \in \{g_1, \dots, g_n\}} (U, g_{i*}\sigma)$$

with $g_*\sigma(h)(w) = \sigma(g^{-1}hg)(w)$. Moreover $g_*\sigma$ is clearly irreducible, indeed if W' is an H-subspace of U with respect to $g_*\sigma$, then

$$\sigma(h)(w) = g_*\sigma(ghg^{-1})(w) \in W'.$$

So W' is an H-subspace of U with respect to σ and so it is zero or U itself. So c-$\operatorname{Ind}_H^G U$ is H-semisimple and the same is true for its H-subspace V. $\square$

---

[8] The injectivity is a consequence of the irreducibility of V.

1.3 Schur's Lemma

Theorem 1.24. ***(Schur's Lemma)*** *Let* (V,π),(W,ρ) *be two smooth irreducible representations of* G. *Then*

(1) $\operatorname{End}_G(V)=\mathbb{C}$.

(2) If $V\cong W$ *then* $\operatorname{Hom}_G(V,W)\cong\mathbb{C}$.

(3) If $V\ncong W$ *then* $\operatorname{Hom}_G(V,W)=0$.

Proof. Let ϕ be a G-endomorphism of V. The vector spaces $\operatorname{Ker} f$ and $\operatorname{Im} f$ are G-stable and so they have to be 0 or V. If $\operatorname{Ker} f=V$ or $\operatorname{Im} f=0$ then $f=0$. Hence, if $f\neq 0$ we have $\operatorname{Im} f=V$ and $\operatorname{Ker} f=0$, i.e. f is an isomorphism and $\operatorname{End}_G(V)$ is a division algebra over $\mathbb{C}$.
Now let v be a non zero vector in V and let K be an open compact subgroup of G such that v is in V^K. Since G/K is countable, the set $\{\pi(g)v : g\in G/K\}$ is countable and since V is irreducible this set spans V. In particular, $\dim_{\mathbb{C}} V$ is countable. The value of a function in $\operatorname{End}_G(V)$ is determined by its value in v, hence $\operatorname{End}_G(V)$ is countable. Now suppose that $\phi\neq a\cdot\mathrm{Id}_V$. In particular, $\phi-a\cdot\mathrm{Id}_V$ is non zero and so it is invertible. The set $\left\{\frac{1}{\phi-a\cdot Id_V}:a\in\mathbb{C}\right\}$ is linearly dependent since it is uncountable and inside $\operatorname{End}_G(V)$ that has countable dimension. So there exist $a_1,\dots a_n,b_1,\dots b_n$ in $\mathbb{C}$ such that

$$\sum_i^n\frac{b_1}{\phi-a_i\cdot\mathrm{Id}_V}=0.$$

Multiplying by $\Pi_i^n(\phi-a_i\cdot\mathrm{Id}_V)$ we obtain a polynomial $f(t)\in\mathbb{C}[t]$ such that $f(\phi)=0$. So there exist $c_1,\dots c_m$ such that

$$0=f(\phi)=\phi^{n_0}\Pi_i(\phi-c_i\cdot\mathrm{Id}_V)^{n_j}.$$

Since $\operatorname{End}_G(V)$ has no nilpotent, $\phi^{n_0}\neq 0$. Then we have a contradiction: there exists $a\in\mathbb{C}$ such that $\phi-a\cdot\mathrm{Id}_V$ is not invertible and so $\phi-a\cdot\mathrm{Id}_V=0$. Therefore (1) and (2) hold. For (3), take $f:V\to W$ a G-homomorphism. $\operatorname{Ker} f$ and $\operatorname{Im} f$ are G-subspaces of V and W respectively and since V,W are irreducible the only possibilities are $f=0$ or f is an isomorphism. The latter is not possible and so $f=0$. □

Remark 1.25. This is the first time that we use the assumption that G/K has countable index for any compact open subgroup. Without this assumption we have a less strong statement than the previous one. Indeed, we can only say that $\operatorname{End}_G(V)$ is a division algebra. If, in addition, (V,π) is admissible $\mathrm{End}_G(V) = \mathbb{C}$ as in the version of the previous theorem. For this aim, recall that

$$V = \bigcup_{\substack{K \subseteq G \\ K \text{ open compact}}} V^K.$$

In this case V^K is finite dimensional for all K. Then, for any ϕ in $\mathrm{End}_G(V)$, the restriction $\phi_{|V^K}$ has an eigenvalue that we call λ. Since λ is an eigenvalue for $\phi_{|V^K}$, then $\phi - \lambda \cdot \mathrm{Id}_V$ is not an isomorphism and so $\phi - \lambda \cdot \mathrm{Id}_V = 0$ by the irreducibility of V.

Remark 1.26. Let (V, π) be an irreducible smooth representation of G. Let $Z(G)$ be the center of G. For all z in $Z(G)$ and g in G we have $\pi(g)\pi(z) = \pi(z)\pi(g)$ and so $\pi(z)$ is in $\mathrm{End}_G(V) = \mathbb{C}$. Then we have a map

$$w_z : Z(G) \to \mathbb{C}^\times$$

that satisfies $\pi(z)v = w_\pi(z)v$ for all v in V and z in $Z(G)$. Moreover, for all K open compact subgroup of G, w_z acts trivially on $K \cap Z(G)$ that is still an open compact subgroup of $Z(G)$ [9]. So w_z is a representation of $Z(G)$ of dimension 1. We will call it **central character** of π.

1.4 One-dimensional representations

In this section we want to introduce some tools to study one-dimensional representations: for this aim, one fundamental class is formed by local fields. Let F be a local field with discrete valuation ring $\mathcal{O}_F$, maximal ideal $\underline{m}_F$, uniformizer $\overline{w}$, valuation val, finite residue field k_F. It is well known that

$$\mathcal{O}_F \cong \varprojlim \mathcal{O}_F \,/\underline{m}_F^n$$

with $\mathcal{O}_F \,/\underline{m}_F^n$ finite and so that $\mathcal{O}_F$ is profinite. Moreover, we have a fundamental system of open neighborhoods of 1 for $F^\times$ given by

$$1 + \underline{m}_F \supseteq \cdots \supseteq 1 + \underline{m}_F^n \supseteq \dots$$

Since they are open compact, we conclude that $F^\times$ is a locally profinite group.

Lemma 1.27. *Let (V, π) be an irreducible representation of an abelian locally profinite group G. Then V is one-dimensional.*

---

[9] $\{1\}$ is closed and the map $f : h \to ghgh^{-1}$ is continuous, so $Z(G) = \bigcap_{g \in G} C_G(g)$ is closed.

Proof. Since G is abelian, we have

$$\pi(g)\pi(h) = \pi(h)\pi(g) \text{ for all } g, h \text{ in } G.$$

This implies that $\pi(g)$ is a G-endomorphism of V for any g in G and so, by the Schur's lemma $\pi(g) = z_g \cdot \mathrm{Id}_V$. Then any line in V is G-stable. So V is one dimensional since V is irreducible. □

So all the irreducible representations of $F^\times$ are one-dimensional. These representations take a central role in the representation theory.

Definition 1.28. Let G be a locally profinite group. A representation of dimension 1 is called **smooth character** of G.

Remark 1.29. There is a very important difference between character theory for finite groups and the one for locally profinite groups. Let $\chi : G \to \mathbb{C}^\times$ be a smooth character. If G is finite we have $\chi(g)^{O(G)} = 1$, so the image of G under a character is in the unit circle S^1. This tool is very useful in representation theory of finite groups but, in general, it is not true for infinite groups. For this reason we introduce the next definition.

Definition 1.30. A character of a locally profinite group is said to be **unitary** if its image is contained in the unit circle.

Proposition 1.31. *If G is the union of its compact open subgroups, then any smooth character is unitary.*

Proof. Let χ be a smooth character of G. Since S^1 is the unique maximal compact subgroup of $\mathbb{C}^\times$ and since χ is continuous, any compact subgroup of G is such that $\chi(G) \subseteq S^1$. So $\chi(G)$ is a subset of S^1. □

Another important example is the abelian group F endowed with the sum. with respect to this operation, the following

$$\underline{m}_F \supseteq \cdots \supseteq \underline{m}_F^n \supseteq \ldots$$

is a fundamental system of open neighborhoods of 0 for F, composed by open compact groups. So also F is locally profinite and, with the product topology, the same holds for F^n. A really important example is $GL_n(F) \subseteq M_n(F) \cong F^{n\times n}$.

Moreover $F = \bigcup_{n\in\mathbb{Z}} \underline{m}_F^n$, so F is the union of its compact open subgroups. In particular, its characters are unitary.

Proposition 1.32. $\mathrm{GL}_n(F)$ *is locally profinite and* $\mathrm{GL}_n(\mathcal{O}_F)$ *is profinite with fundamental system of open neighborhoods of* 1 *given by* $K_r := 1 + \overline{w}^r M_n(\mathcal{O}_F)$, *where* $\overline{w}$ *is a uniformizer for* F.

Proof. We have already seen that $M_n(F) \cong F^{n\times n}$ is locally profinite with the product topology. Now consider the map

$$\det : M_n(F) \to F.$$

It is continuous since it is given by a polynomial equation. Hence $\mathrm{GL}_n(F) = \det^{-1}(F^\times)$ is open, closed and in particular it is locally profinite.
In the same way, since $\mathcal{O}_F$ is profinite, $\mathrm{M}_n(\mathcal{O}_F) \cong \mathcal{O}_F^{n\times n}$ is profinite and so $\mathrm{GL}_n(\mathcal{O}_F) = det^{-1}(\mathcal{O}_F^\times) \cap M_n(\mathcal{O}_F)$ [10] is profinite.
For the last sentence, consider for any r the canonical map

$$\mathrm{GL}_n(\mathcal{O}_F) \to \mathrm{GL}_n(\mathcal{O}_F\,/\underline{m}_F^r).$$

Our aim is to prove that the Ker of this map is K_r. But, it is a direct consequence of

$$\det(1 + \overline{w}^r A) \equiv 1 \mod \underline{m}_F^r \quad \forall A \in M_n(F)$$

and hence $1 + \overline{w}^r\,\mathrm{M}_n(F) \subseteq \mathrm{GL}_n(\mathcal{O}_F)$.
In particular, the subgroup K_r are open normal subgroup and so profinite. □

Definition 1.33. Let n be a positive integer. A finitely generated $\mathcal{O}_F$-submodule of F^n is called **lattice** if it spans F^n as vector space.

Lemma 1.34. *Let L be a lattice for F^n. Then there exists a basis $x_1, \dots x_n$ of F^n on F such that*

$$L = \bigoplus_{i=1}^n \mathcal{O}_F\, x_i$$

Proof. Let $\{x_1, \dots, x_m\}$ be a minimal set of generators of L over $\mathcal{O}_F$. Our aim is to prove that this set is indeed a basis. By definition of lattice, they generate F^n and so it is enough to prove that $x_1, \dots, x_m$ are linearly independent. Suppose

$$\sum_{i=1}^m a_i x_i = 0$$

with a_i in F and suppose for absurd that they are not all zero. Let j be such that $\mathrm{val}(a_j) = \min_i \mathrm{val}(a_i)$, so $\mathrm{val}(\frac{a_i}{a_j}) = \mathrm{val}\, a_i - \mathrm{val}\, a_j \geq 0$ for all i and $\mathrm{val}(\frac{a_j}{a_j}) = \mathrm{val}(1) = 0$. In particular we have

$$\sum_{i=1}^m \frac{a_i}{a_j} x_i = 0$$

---

[10] Notice that not only the matrices in $M_n(\mathcal{O}_F)$ can have determinant in $\mathcal{O}_F$.

and in particular

$$x_j = -\sum_{\substack{i=1 \\ i\neq j}}^{m} \frac{a_i}{a_j} x_i.$$

So x_j is a linear combination of the other x_i with coefficients in $\mathcal{O}_F$. That is an absurd for the minimality of $\{x_1, \dots, x_m\}$. Therefore, we have $a_i = 0$ for all i. $\square$

Proposition 1.35. $\mathrm{GL}_n(\mathcal{O}_F)$ *is a maximal compact subgroup of* $\mathrm{GL}_n(F)$ *and any compact subgroup of* $\mathrm{GL}_n(F)$ *is conjugate to a subgroup of* $\mathrm{GL}_n(\mathcal{O}_F)$*.*

Proof. Suppose that $\mathrm{GL}_n(\mathcal{O}_F)$ is not maximal compact. So there exists H subgroup of $\mathrm{GL}_n(F)$, compact and that contains $\mathrm{GL}(\mathcal{O}_F)$. Let $A = (a_{ij})$ be in $H\setminus \mathrm{GL}(\mathcal{O}_F)$ and without losing generality suppose that $\mathrm{val}_F(a_{11}) = min_{i,j}\{\mathrm{val}(a_{ij})\}$, that $a_{1j} = 0$ for all $j \neq 1$ [11] and that $\mathrm{val}(a_{11}) < 0$[12]. This implies that in the entry $1-1$ of A^m we have a_{11}^m. So

$$H = \bigcup_{m\geq 0} \varpi^{-m} M_n(\mathcal{O}_F) \cap H$$

has not a finite subcover. This is an absurd and so $\mathrm{GL}_n(\mathcal{O}_F)$ is a maximal compact subgroup of $\mathrm{GL}_n(F)$.

Let H be a compact subgroup of $\mathrm{GL}_n(F)$, let $\{e_1, \dots e_n\}$ be the canonical basis of F^n over F and $L := \oplus_{i=1}^n \mathcal{O}_F\, e_i$. Consider the map

$$\begin{aligned} \Phi : H \times L &\to F^n \\ (h, v) &\mapsto h(v). \end{aligned}$$

The image of Φ is compact since H is compact. Moreover,

$$\mathrm{Im}(\Phi) = \bigcup_{m\geq 0} \varpi^{-m} L \cap \mathrm{Im}(\Phi)$$

is an open cover and so it admits a finite subcover that has to be of the form $\bigcup_{m=1}^r \varpi^{-m} L \cap \mathrm{Im}(\Phi)$ and in particular $\mathrm{Im}(\Phi) \subseteq \varpi^{-r} L$. Since L is a finitely generated $\mathcal{O}_F$-module we conclude that $\mathrm{Im}(\Phi)$ is a finitely generated $\mathcal{O}_F$-module by the Noetherianity of $\mathcal{O}_F$. Moreover, $\mathrm{Im}(\Phi)$ is a lattice since $L = \mathrm{Id}\,\mathrm{Im}(\Phi)$ and hence we can wrote

$$\mathrm{Im}(\Phi) = \bigoplus_{i=1}^n \mathcal{O}_F\, x_i$$

---

[11] This is not restrictive, indeed we can do it using only row/column operations with matrices of $\mathrm{GL}_n(\mathcal{O}_F)$.

[12] Since A is in H but not in $\mathrm{GL}_n(\mathcal{O}_F)$, also A^{-1} is in $A\setminus \mathrm{GL}_n(\mathcal{O}_F)$ and hence it is enough to substitute A with A^{-1} if necessary.

with $\{x_1, \dots, x_n\}$ basis of F^n over F. Let g be a matrix in $\mathrm{GL}_n(F)$ such that $g(x_i) = e_i$. This implies that gHg^{-1} stabilizes $L = \oplus \mathcal{O}_F \, e_i$ and hence $gHg^{-1} \subseteq \mathrm{M}_n(\mathcal{O}_F)$ and then $gHg^{-1} \subseteq \mathrm{GL}_n(\mathcal{O}_F)$ since

$$\det(gHg^{-1}) = \det(H) \subseteq F^{\times}.$$

□

1.5 Hecke Algebra

It is well known that there is an equivalence between representations of a finite group G on $\mathbb{C}$ and $\mathbb{C}[G]$-modules [13]. It can be used, in general, to study representations of groups endowed with the discrete topology. For example, if H is a subgroup of a discrete group G with a representation W, we have that

$$\mathrm{Ind}_H^G \, W \cong K[G] \otimes_{K[H]} W.$$

Unfortunately, this machinery does not work with smoothness condition. To fix it we introduce another $\mathbb{C}$-algebra constructed starting with G.
Let $C_c^{\infty}(G)$ be the vector space of the locally constant functions $f : G \to \mathbb{C}$ such that the support of f

$$\mathrm{Supp}(f) := \overline{\{g \in G : f(g) \neq 0\}}$$

is compact. We can easily define two representations of G

$$\begin{aligned} \lambda : G &\to \mathrm{End}(C_c^{\infty}(G)) \\ g &\mapsto \lambda_g : x \mapsto f(g^{-1}x) \end{aligned}$$

and

$$\begin{aligned} \rho : G &\to \mathrm{End}(C_c^{\infty}(G)) \\ g &\mapsto \rho_g : x \mapsto f(xg) \end{aligned}$$

respectively called **left translation** λ and **right translation** ρ. Moreover, ρ and λ are smooth representations of G. Now we prove it is true for ρ.
Let $C_c^{\infty}(G/K) := C_c^{\infty}(G)^K$ be the vector subspace of $C_c^{\infty}(G)$ fixed by a compact open subgroup K. So the functions in $C_c^{\infty}(G/K)$ satisfy

$$f(gk) = f(g) \text{ for all } g \text{ in } G \text{ and } k \text{ in } K.$$

The elements of $C_c^{\infty}(G)$ are locally constant, so for any g in $\mathrm{Supp}(f)$ there exists a compact open subgroup K_g of G for which f is constant on gK_g.

---

[13] In this context $\mathbb{C}[G]$ is the group algebra of G over $\mathbb{C}$.

Since gK_g is a subset of Supp(f) that is compact, there exist $g_1, \dots g_n$ such that

$$\mathrm{Supp}(f) = \bigcup_{i=1}^{n} g_i K_g.$$

Now f is in $C_c^\infty(G/K)$ with $K = \cap_i^n K_{g_i}$ and so ρ is a smooth representation. In the same way it is possible to verify that λ is likewise a smooth representation of G.
For this section we suppose that G is n unimodular locally profinite group and μ is a Haar measure on G.

Definition 1.36. Let f_1, f_2 be functions in $C_c^\infty(G)$. The associative operation

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

is called **convolution** and the associative algebra $\mathcal{H}(\mathrm{G}) = (C_c^\infty(G), *)$ is called **Hecke algebra** of G.

Remark 1.37. If G is a discrete group,

$$\int_G f(g) d\mu(g) = \sum_{g \in G} f(g)$$

is a Haar integral. So $\mathcal{H}(\mathrm{G})$ and $\mathbb{C}[G]$ are isomorphic with isomorphism

$$f \mapsto \sum_{g \in G} f(g) g.$$

It is then clear that $\mathcal{H}(\mathrm{G})$ is the algebra that we were searching. Now our aim is to construct a bridge that allows us to work with representations of G as $\mathcal{H}(\mathrm{G})$ modules and viceversa.

Lemma 1.38. *Let K be an open compact subgroup of G. Let $e_K(x) = \frac{1}{\mu(K)} 1_K(x)$ and let f be in $\mathcal{H}(\mathrm{G})$. Then*

1. *e_K is an idempotent element of $\mathcal{H}(\mathrm{G})$.*
2. *$e_K * f = f$ if and only if $f(kg) = f(g)$.*

Proof. (1) $e_K * e_K(g) = \int_G e_K(x) e_K(x^{-1}g) d\mu(x)$. We have two cases, if one between g and x is not in K, the integral is 0. If they are in K we obtain

$$\int_K \mu(K)^{-2} d\mu(x) = \mu(K)^{-1}. \tag{1.1}$$

So the equality holds.

(2) If f is left K-invariant

$$\begin{aligned}(e_K * f)(g) &= \int_G e_K(x) f(x^{-1}g) d\mu(x) \\ &= \int_G e_K(x) f(g) d\mu(x) \\ &= \frac{1}{\mu(K)} \int_K f(g) d\mu(x) = f(g)\end{aligned}$$

On the other hand, $e_K * f$ is left K-invariant. So if $e_K * f = f$, then the same holds for f.

□

Before stating the principal theorem of this section, we need to define what a smooth module is. Let M be a left $\mathcal{H}(\mathrm{G})$-module. By convention we write the action as $f * m$ for all f in $\mathcal{H}(\mathrm{G})$ and m in M.

Definition 1.39. A **smooth left $\mathcal{H}(\mathrm{G})$-module** V is a $\mathcal{H}(\mathrm{G})$-module such that $\mathcal{H}(\mathrm{G}) * V = V$.

Lemma 1.40. *With the same notation as before, the following are equivalent.*

(1) V is smooth as left $\mathcal{H}(\mathrm{G})$-module.

(2) For any v in V there is an open compact subgroup K of G such that $e_K * v = v$.

Proof. (1) $\implies$ (2) Let v be in V. By hypothesis there exist $f_1, \dots, f_r$ in $\mathcal{H}(\mathrm{G})$ and $v_1, \dots, v_r$ in V such that $v = \sum_{i=1}^r f_i * v_i$. For any f_i let K be compact such that f_i lies in $e_{K_i} * \mathcal{H}(\mathrm{G}) * e_{K_i}$[14] and consider $K = \cap K_i$. Now, we obtain

$$e_K v = \sum_{i=1}^r (e_K * f_i) * v_i = \sum_{i=1}^r f_i * v_i = v. \tag{1.2}$$

since the other implication is direct, this conclude the proof. □

In the same way in which we have constructed $C_c^\infty(G)$, we now construct $C_c^\infty(G, V)$, where V is a complex vector space, as the vector space of locally constant functions $f : G \to V$ with compact support. For example, $C_c^\infty(G) = C_c^\infty(G, \mathbb{C})$. [15]

Now we are ready te expose the theorem that we were talking about.

---

[14] Note that by the previous lemma $\mathcal{H}(\mathrm{G}) = \bigcup_K e_K * \mathcal{H}(\mathrm{G}) * e_K$.

[15] It is possible to prove that $C_c^\infty(G, V) \cong C_c^\infty(G) \otimes V$.

Theorem 1.41. *There exists an equivalence of categories between the category of smooth left* $\mathcal{H}(\mathrm{G})$*-modules and the category of smooth representations of* G*.*

Proof. Let (V, π) be a smooth representation of G. We define

$$\pi(f)v = \int_G f(g)\pi(g)v d\mu(g).$$

It is in V since the map $g \mapsto f(g)\pi(x)v$ lies in $C_c^\infty(G, V)$. Let f_1, f_2 in $\mathcal{H}(\mathrm{G})$. We have

$$\begin{aligned}
\pi(f_1 * f_2)v &= \int_G f_1 * f_2(g)\pi(g)v d\mu(g) \\
&= \int_G \int_G f_1(h) f_2(h^{-1}g)\pi(g)v d\mu(h) d\mu(g) \\
&\overset{(*)}{=} \int_G \int_G f_1(s) f_2(t)\pi(st) d\mu(ts) d\mu(s) \qquad\qquad s = h, t = h^{-1}g \\
&= \int_G \int_G f_1(s) f_2(t)\pi(st) d\mu(t) d\mu(s) \\
&= \int_G f_1(s)\pi(s) \int_G f_2(t)\pi(t) d\mu(t) d\mu(s) \\
&= \pi(f_1)\pi(f_2)v.
\end{aligned}$$

So $f \mapsto \pi(f)$ is a homomorphism and then this defines an $\mathcal{H}(\mathrm{G})$-module on V. For the smoothness of V as $\mathcal{H}(\mathrm{G})$-modules, take K an open compact subgroup of G. We get that $\pi(e_K)v = v$, i.e. v is in V^K. So V is smooth as $\mathcal{H}(\mathrm{G})$-module.

Now let $\phi : V \to V'$ be a G-homomorphism between two G-representations (V, π) and (V', π'). In order to complete the definition of the functor, it is enough to show that ϕ is a homomorphism of $\mathcal{H}(\mathrm{G})$-modules. To do this, we can re-define $\pi(f)$. For, we take an open compact subgroup K of G such that $v \in V^K$ and $f(gk) = f(g)$. Then the integral can be re-defined as

$$\pi(f)v = \mu(K) \sum_{g \in G/K} f(g)\pi(g)v.$$

Therefore, since the diagram

$$\begin{tikzcd}
V \arrow[r, "\phi"] \arrow[d, "\pi(g)"'] & V' \arrow[d, "\pi'(g)"] \\
V \arrow[r, "\phi"'] & V'
\end{tikzcd}$$

commutes for any g in G, it is clear that

$$\begin{tikzcd}
V \arrow[r, "\phi"] \arrow[d, "\pi(f)"'] & V' \arrow[d, "\pi'(f)"] \\
V \arrow[r, "\phi"'] & V'
\end{tikzcd}$$

commutes.
So these conditions define a functor from smooth representations of G to smooth left $\mathcal{H}$(G)-modules.
For the inverse direction, take a smooth $\mathcal{H}$(G)-module V and consider the morphism

$$\begin{aligned}\mathcal{H}(\mathrm{G}) \otimes_{\mathcal{H}(\mathrm{G})} V &\to V \\ f \otimes v &\to f * v.\end{aligned}$$

By definition of $\mathcal{H}$(G)-modules this is $\mathcal{H}$(G)-linear and since $\mathcal{H}(\mathrm{G}) * V = V$, by hypothesis of smoothness, it is surjective. This map is also injective. For, take $\sum_{i=1}^{r} f_i * v_i = 0$ so that $\sum_{i=1}^{r} f_i \otimes v_i$ lies in the kernel of the function defined before. So, by the previous lemma

$$\begin{aligned}\sum_{i=1}^{n} f_i \otimes v_i &= \sum_{i=1}^{n}(e_K * f_i \otimes v_i) = \\ &= \sum_{i=1}^{n}(e_K \otimes f_i * v_i) = \\ &= e_K \otimes \sum_{i=1}^{n} f_i * v_i = 0\end{aligned}$$

where K is an open compact subgroup of G that fixes each f_i. So this defines a representation π of G with the action $\pi(g) \cdot (f \otimes v) = (\lambda(g)f) \otimes v$. This representation can be written as

$$\pi(g)v = e_{gK} * v,$$

where K is an open compact subgroup of G with $e_K * m = m$. With this last equation, it is obvious that if $\phi : V \to W$ is a homomorphism of $\mathcal{H}$(G)-modules then ϕ is a morphism between the two associated representations.
The last step is determining the path on the bridge between representations and $\mathcal{H}$(G)-modules, i.e. the two functors are quasi-inverse. Let (V, π) be a representation of G. Let us consider V as $\mathcal{H}$(G)-module, so that we obtain a representation (V, γ) of G. Take v a vector in V fixed by an open compact subgroup K of G, with respect to γ. We obtain

$$\gamma(g)v = \pi(e_{gK})v = \pi(g)v.$$

On the other hand, let $(V, *)$ be a smooth $\mathcal{H}$(G)-module. Considering V as G-representation, we obtain the smooth representation (V, π). It is such that

$$\pi(g)v = \pi(g)(f * v) = (\lambda(g)f) * v.$$

In particular,

$$\pi(f)v = f * v.$$

The smooth $\mathcal{H}(\mathrm{G})$-module associated to (V, π) is defined by $(f, v) \mapsto \pi(f)v$ and the last one is, by construction, $f * v$. So the two modules coincide. $\square$

Chapter 2

Representations of $\mathrm{GL}_n(F)$

2.1 Reductive groups

Now we want to introduce the core of our representations: **supercuspidal representations**. This type of representations make sense in a very rich class of groups: the **reductive groups**.
Now we want to describe their construction in the generic case. Nevertheless, we are interested in $\mathrm{GL}_n(F)$, so our attention will be on the specific situations useful to study $\mathrm{GL}_n(F)$
Let $\overline{F}$ be an algebraic closure of F. From now any extension of F will be intended in $\overline{F}$. Consider the general linear group with coefficient in $\overline{F}$, $\mathrm{GL}_n(\overline{F})$. It is endowed of the Zariski topology thought as

$$\mathrm{GL}_n(\overline{F}) = \{(X,t) \subseteq \overline{F}^{n^2} \times \overline{F} : (\det X)t - 1 = 0\} \subseteq \overline{F}^{n^2+1}.$$

Definition 2.1. A linear algebraic group over $\overline{F}$ is a subgroup of $\mathrm{GL}_n(\overline{F})$, for some n, that is closed with respect to the Zariski topology.

Remark 2.2. A linear algebraic group is an affine variety with the structure of a group. Since we can see an affine variety as an affine scheme, a linear algebraic group can be view as an affine group scheme over $\overline{F}$ (reduced and of finite type).

Now, we need to extend our definition to a subfield of $\overline{F}$, in particular we need to extend it to F.

Definition 2.3. Let L be a field contained in $\overline{F}$ such that $\overline{F}$ is an algebraic closure of L, an affine variety over L is the set of common zeros of a colletion of polynomials with coefficients in L. [1]

---

[1] This is still a closed in the Zariski topology of $\mathrm{GL}_n(\overline{F})$ for some n, but since the coefficients are in a fixed subfield we need to be careful about our consideration.

Remark 2.4. A crucial remark is that, despite having coefficients in L, an affine variety over L lives in $\overline{F}^{n^2+1}$ for some n and, in general, not in L^{n^2+1}. The points that are contained in L^{n^2+1} are called L-rational point. Equivalently, if we denote by V the variety over L, the L-rational points of V are the points in $V \cap L^{n^2+1}$.

Definition 2.5. Let L be a subfield of $\overline{F}$ such that $\overline{F}$ is an algebraic closure of L. A linear algebraic group G over L is an affine variety over L with a structure of group where the maps of product and inverse are regular maps with respect to the structure of variety.

We denote by $G(L)$ the L-rational point of a linear algebraic group G.

Proposition 2.6. *Let G be a reductive group over F. Then the F-points $G(F)$ of F are a locally profinite group.*

Now, consider the subgroup $N_n(\overline{F})$ of $\mathrm{GL}_n(\overline{F})$ formed by the matrix of the form

$$\begin{pmatrix} 1 & x_{12} & \dots & x_{1n} \\ 0 & 1 & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

So, the upper triangular matrices with only 1 on the diagonal. Note that the characteristic polynomial of a matrix of this form is $(x-1)^n$. So it is diagonalizable if and only if the the geometrical dimension of the eigenvalue 1 is n, but this is possible only if the matrix is the identity. So the only diagonalizable matrix in $N_n(\overline{F})$ is the identity.

Definition 2.7. An affine algebraic group in $\mathrm{GL}_n(\overline{F})$ is called **unipotent** if is conjugate to a subgroup of $N_n(\overline{F})$.

Definition 2.8. The **unipotent radical** of a linear algebraic group G is the maximal connected unipotent normal subgroup of G.

Remark 2.9. Without losing generality we can suppose G connected. Indeed, if G^0 is the irreducible component of 1 in G we have that the unipotent radical of G^0 is the same of G. In particular the next proposition explains the role of G^0.

Proposition 2.10. *With the previous notation, the followings hold:*

(1) G^0 is a closed normal subgroup of G.

(2) The index of G^0 in G is finite.

(3) Any closed subgroup of finite index of G contains G^0.

Proof. First of all we have to prove that G^0 is the unique irreducible component of G containing 1. If X and Y are as in the previous, then $X \times Y$ is an irreducible variety and the same for XY that is the image of $X \times Y$ under the product of G. In particular, the closure $\overline{X \times Y}$ of $X \times Y$ is again closed. But X and Y are subvarieties of $\overline{X \times Y}$. So $X = \overline{X \times Y} = Y$. So G^0 is well-defined and it is closed. Moreover, for any xy in G^0, $x^{-1}yG^0$ is again irreducible and it contains 1. So $x^{-1}yG^0 = G^0$ and so $x^{-1}y$ is in G^0. Hence G^0 is closed. In the same way gG^0g^{-1} is irreducible and contains 1 for all g in G. So G^0 is normal. This conclude the proof of (1).
For the point (2), note that the irreducible components of G have to be finite [2]. In particular, take g in G. The unique irreducible component of G that contains g is gG^0, and so there are only finitely many possibilities for gG^0. Therefore, G^0 has finite index in G.
To conclude the proof, let H be a closed subgroup of finite index of G. $G^0/G^0 \cap H$ is a subgroup of G/H, in particular $G^0/G^0 \cap H$ is finite. So $G^0 \cap H$ is open and closed. Since G^0 is irreducible, $G^0 \cap H = G^0$ and hence $G^0 \subseteq H$. □

Definition 2.11. A linear algebraic group is called **reductive** if its unipotent radical is trivial.

Definition 2.12. Let G be a linear algebraic group. A **Borel subgroup** of G is a maximal closed connected solvable subgroup of G.

Definition 2.13. Let G be a linear algebraic group. A **parabolic subgroup** of G is a subgroup of G that contains a Borel subgroup of G.

Now we are ready to state a central theorem in the construction of representations of reductive groups.

Proposition 2.14. *Let G be a reductive group and let P be a parabolic subgroup of G. Then there exists a reductive group M such that*

$$P = M \ltimes N$$

where N is the unipotent radical of P. This decomposition is called ***Levi decomposition.***

2.2 Parabolic induction and supercuspidal representations

Let G be a reductive group with a parabolic subgroup $P = M \ltimes N$. Our intent is to produce new representations of G starting with representations of M. If (W, σ) is an irreducible representation of M, we can easily define it

---

[2] G is a scheme over a Noetherian ring.

on P since $M \cong P/N$. So, the resulting representation on P is an irreducible representation of M which acts trivially on N and, in particular, it is again irreducible. This representation is indicated with $\mathrm{Inf}_P^M \sigma$ and it is called **inflation**. It is better explained in the following diagram.

$$\begin{tikzcd}
P \arrow[r, "\mathrm{Inf}_P^M \sigma"] \arrow[dr, two heads] & GL(V) \\
 & M = P/N \arrow[u, "\sigma"']
\end{tikzcd}$$

Moreover, if (V, π) is a smooth representation of P, M acts on V^N and this defines a smooth representation (V^N, π^N) of M. We have a $\mathbb{C}$-linear map

$$\mathrm{Hom}_P(\mathrm{Inf}_P^M \sigma, \pi) \cong \mathrm{Hom}_M(\sigma, \pi^N)$$
$$f \mapsto f$$

that is well-defined since the representation Inf_P^M doesn't work on N, so the action of P is the action of P/N and hence the image of the map is in V^N. In an analogous way, (V, π) defines an action of M on $V_N := V/V(N)$, where $V(N) = \mathrm{Span}\{v - \pi(n)v : v \in V, n \in N\}$. So we have a representation $(V_N, J_N(\pi))$ of M. As in the previous, we have a $\mathbb{C}$-linear map

$$\mathrm{Hom}_P(\pi, \mathrm{Inf}_P^M \sigma) \cong \mathrm{Hom}_M(J_N(\pi), \sigma)$$
$$f \mapsto f$$

So Inf_P^M has a right adjoint $\pi \mapsto \pi^N$ and a left adjoint J_N.

Definition 2.15. With the notation as before, the functor

$$J_N : \mathrm{Rep}(P) \to \mathrm{Rep}(M)$$

is called **Jacquet Functor**.

Now let μ be a left Haar measure on G. Clearly it is invariant under left translation, i.e.

$$\int_G f(gx)d\mu(x) = \int_G f(x)d\mu(x).$$

But it is not invariant under right translation, so we can define $\delta_G : G \to \mathbb{R}_{>0}$ such that

$$\int_G f(xg)d\mu(x) = \delta_G(g) \int_G f(x)d\mu(x)$$

in particular

$$\delta_G(gg') \int_G f(x)d\mu(x) \int_G f(xgg')d\mu(x) = \delta_G(g)\delta_G(g') \int_G f(x)d\mu(x)$$

and then

$$\delta_G(gg') = \delta_G(g)\delta_G(g').$$

Definition 2.16. With the notation as above

$$\delta_G : G \to \mathbb{R}_{>0}$$

is called **modulus character** of G.

Notice that for any compact subgroup K of G we have $\sigma_G(k) = 1$, then σ_G is clearly smooth.

Definition 2.17. Let G be a reductive group with a parabolic subgroup $P = M \ltimes N$. Let (W, σ) be a representation of M. The representation

$$i_P^G(W, \sigma) := \operatorname{Ind}_P^G(W, \delta_P^{1/2} \otimes \operatorname{Inf}_P^M \sigma)$$

is called **normalized parabolically induced** from (W, σ).

Clearly i_P^G defines a functor $\operatorname{Rep}(M) \to \operatorname{Rep}(G)$. As in the previous chapter, we want to construct a functor $\operatorname{Rep}(G) \to \operatorname{Rep}(M)$ adjoint to i_P^G.

Definition 2.18. Let G be a reductive group with a parabolic subgroup $P = M \ltimes N$. Let (V, π) be a representation of G. The representation

$$r_P^G(V, \pi) := (V_N, J_N(\delta_P^{-1/2} \otimes \pi_{|P}))$$

is called **normalized parabolically restricted** from (V, π).

Proposition 2.19. *With the notation before introduced, the following holds*

$$\operatorname{Hom}_M(r_P^G \pi, \sigma) \cong \operatorname{Hom}_G(\pi, i_P^G \sigma).$$

Proof. Using the left and right adjoint of Inf_P^M and the Frobenius reciprocity, we obtain

$$\begin{aligned}
\operatorname{Hom}_M(r_P^G \pi, \sigma) &= \operatorname{Hom}_M(J_N(\delta_P^{-1/2} \otimes \pi_{|P}), \sigma) \\
&\cong \operatorname{Hom}_P(\delta_P^{-1/2} \otimes \pi_{|P}, \operatorname{Inf}_P^M \sigma) \\
&= \operatorname{Hom}_P(\pi_{|P}, \delta_P^{1/2} \otimes \operatorname{Inf}_P^M \sigma) \\
&\cong \operatorname{Hom}_G(\pi, \operatorname{Ind}_P^G(\delta_P^{1/2} \otimes \operatorname{Inf}_P^M \delta)) \\
&= \operatorname{Hom}_G(\pi, i_P^G \sigma).
\end{aligned}$$

This completes the proof. □

Finally we can define what is a supercuspidal representation.

Definition 2.20. Let G be a reductive group with a parabolic subgroup $P = M \ltimes N$. An irreducible smooth representation (V, π) of G is called **supercuspidal** if $r_P^G(V) = 0$ for all proper parabolic subgroups P of G.

Remark 2.21. The previous definition can be viewed in a different way using 2.19. Indeed being supercuspidal is equivalent to not being a subrepresentation of a normalized parabolically induced from a representation of M.
Suppose (V, π) is a supercuspidal representation of G. By definition $r_P^G(V) = 0$ for any parabolic subgroup $P = MN$ of G and then

$$0 = \mathrm{Hom}_M(r_P^G \pi, \sigma) \cong \mathrm{Hom}_G(\pi, i_P^G \sigma)$$

for any representation σ of M. So π cannot be embedded in $i_P^G \sigma$.
Conversely, if π is a smooth irreducible representation of G that is not a subrepresentation of $i_P^G \sigma$ for some $P = MN$ and σ representation of M, by the irreducibility of π [3], we have

$$\mathrm{Hom}_M(r_P^G \pi, \sigma) \cong \mathrm{Hom}_G(\pi, i_P^G \sigma) = 0$$

and hence $r_P^G \pi$ is zero.

In order to describe the property of the supercuspidal representations in which we are interested , we need some tools.

Definition 2.22. Let G be a locally profinite group [4] and let (V, π) be a representation of G. The action of G on the dual V^* of V define a representation

$$(g\phi)(v) := \phi(\pi(g^{-1})v)$$

for all ϕ in V^* and for all g in G.
The obtained representation is called **dual representation** of π.

Note that if π is smooth, the dual representation of π is not necessarily smooth. For this reason we introduce a smooth one.

Definition 2.23. Let G be a locally profinite group and let (V, π) be a smooth representation of G. The smooth part of the dual representation of π is called **contragredient representation** or **smooth dual representation** of π. It is indicate by $(\tilde{V}, \tilde{\pi})$.

Definition 2.24. Let (V, π) be a smooth representation of a locally profinite group G. π is said to be **compact** if for any non zero vector v in V and for any compact open subgroup K of G, the function

$$\begin{aligned} f_{K,v} : G &\to V \\ g &\mapsto \pi(e_K)\pi(g^{-1})v \end{aligned}$$

lies in $C_c^\infty(G, V)$, i.e. it has compact support.

---

[3] Since π irreducibile, any morphism of representations from π has to be injective or zero.

[4] Not necessarily a reductive group.

Definition 2.25. Let (V, π) be a smooth representation of a locally profinite group G. For any non zero vector v in V and for any non zero morphism f in $\tilde{V}$, the function

$$\begin{aligned} m_{f,v} : G &\to \mathbb{C} \\ g &\mapsto f(\pi(g^{-1})v) \end{aligned}$$

is called a **matrix coefficient** of (V, π).

Lemma 2.26. *Let (V, π) be a smooth representation of G. For any compact open subgroup K of G we have*

$$V \cong V^K \oplus V(K)$$

as smooth representations of K.

Proof. Consider the function $\pi(e_K) : V \to V$. We know that it is an idempotent and clearly its image is V^K. Indeed, for any k in K and v in V we have

$$\begin{aligned} \pi(k)(\pi(e_K)v) &= \pi(k)\frac{1}{\mu(K)}\int_K \pi(g)v d\mu(g) \\ &= \frac{1}{\mu(K)}\int_K \pi(kg)v d\mu(g) \\ &= \frac{1}{\mu(K)}\int_K \pi(h)v d\mu(h) = \pi(e_K)v. \end{aligned}$$

Moreover, the previous calculation shows that $\pi(e_K)$ is the identity on V^K. So it is a projection and

$$V = V^K \oplus \operatorname{Ker}(\pi(e_K)).$$

It remains to prove that $\operatorname{Ker}(\pi(e_K)) = V(K)$. For this aim, let v be in V and H be an open subgroup of K such that $v \in V^H$. Without losing generality we can suppose H normal [5]. Then, for any k in K we have

$$\pi(e_K)\pi(k)v = \frac{1}{|K:H|}\sum_{g\in K/H}\pi(gk)v = \frac{1}{|K:H|}\sum_{g\in K/H}\pi(g)v = \pi(e_K)v.$$

In other words $\pi(e_K)(v - \pi(k)v) = 0$ and so $V(K) \subseteq \operatorname{Ker}(\pi(e_K))$. For the other inclusion, we have that for v in $\operatorname{Ker}(\pi(e_K))$ holds

$$v = v - \pi(e_K)v = v - \frac{1}{|K:H|}\sum_{k\in K/H}\pi(k)v = \frac{1}{|K:H|}\sum_{k\in K/H}(v - \pi(k)v).$$

Then $\operatorname{Ker}(\pi(e_K)) = V(K)$. ☐

---

[5] If H is not normal we can take the normal core of H in K.

Proposition 2.27. *Let (V, π) be a smooth representation of G. Then the following are equivalent*

(1) V is compact.

(2) All matrix coefficients have compact support.

Proof. Let (V, π) be a compact smooth representation of G. Take v in V and ξ in $\tilde{V}^K$ where K is an open compact subgroup of G. We have that

$$\xi(v - \pi(k)v) = \xi(v) - \xi(\pi(k)v) = \xi(v) - \pi(k^{-1})(\xi(v)) = \xi(v) - \xi(v) = 0.$$

So $\xi|_{V(K)} = 0$, and since $\pi(e_K)$ is the projection of V to V^K, we obtain $\xi \circ \pi(e_K) = \xi$. Hence, for all g in G we have

$$\xi(f_{K,v}(g)) = \xi(\pi(e_K)\pi(g^{-1})v) = \xi(\pi(g^{-1})v) = m_{\xi,v}(g)$$

and in particular

$$\operatorname{Supp} m_{\xi,v} \subseteq \operatorname{Supp} f_{\xi,v}{}^{6}$$

Moreover, since $\pi(e_K)$ is the projection of V to V^K, both the functions $f_{K,v}$ and $m_{\xi,v}$ are constant on the cosets of K, that are compact. So they have compact support if and only if their support in G/K is compact, hence finite. But this is true for $f_{K,v}$, and for the previous equation, it is true for $m_{\xi,v}$. So $m_{\xi,v}$ has compact support.

Now suppose that all the matrix coefficients have compact support and let E_v be the subspace of V^K generated by the image of $f_{K,v}$. We can consider a basis of E_v of the form $\{w_i := f_{K,v}(g_i)\}_{i \in I}$ and a ξ_0 in $\tilde{V}^K$ such that $\xi_0(w_i) = 1$ for any i in I. This implies that $\bigcup_{i \in I} g_i K \subseteq \operatorname{Supp} m_{\xi_0,v}$. But, for the same reason of the other implication, $\operatorname{Supp} m_{\xi_0,v}$ modulo K is finite and so the same is for $\bigcup_{i \in I} g_i K$. Hence I has to be finite that is equivalent of E_v being finite dimensional. Moreover, since $(V^K)^* = \tilde{V}^K$, we can take $\xi_1, \dots \xi_n$ in $(V^K)^*$ such that they form a basis for E_v^* [7].

So, for any g in G there is i in I such that $m_{\xi_i,v}(g) = \xi_i(f_{K,v}(g)) \neq 0$. Therefore, we have

$$\operatorname{Supp} f_{K,v} \subseteq \bigcup_{i=1}^{n} \operatorname{Supp} m_{\xi_i,v}.$$

Since the set on the right is finite modulo K, so it is the one on the left. This concludes the proof. □

Proposition 2.28. *Let (V, π) be a smooth representation of G. If (V, π) is compact and finitely generated, then it is admissible.*

---

[6] Notice that this is true without using the hypothesis of compactness of (V, π).

[7] In this case we are considering the restriction of ξ_i on E_v.

Proof. Suppose that (V, π) is compact and finitely generated and let $\{v_1, \dots, v_n\}$ be a set of generators. We have already seen that f_{K,v_i} is constant on the cosets gK for any K open compact subgroup of G. Moreover, since G/K is finite, also $\mathrm{Im}(f_{K,v_i})$ is finite. Now, let v be a generic vector in V^K. By hypothesis $v = \sum_{i,j} a_{ij}\pi(g_{ij})v_i$. Hence we have that

$$v = \pi(e_K)v = \pi(e_K)\sum_{i,j} a_{ij}\pi(g_{ij})v_i = \sum_{i,j} a_{ij}\pi(e_K)\pi(g_{ij})v_i = \sum_{i,j} f_{K,v_i}(g_{ij}^{-1}).$$

So, the finite set $\bigcup_{i=1}^n \mathrm{Im}(f_{K,v_i})$ generates V^K as vector space. □

Finally we can state the theorems that we were looking for. They explain why the supercuspidal representations are the building blocks of this theory. As we said before, the proofs will be done in the next section for the case useful for $\mathrm{GL}_n(F)$.

Theorem 2.29. *Let (V, π) be a smooth irreducible representation of G. Then there exists a parabolic subgroup $P = M \ltimes N$ of G and an irreducible supercuspidal representation of M such that π is a subrepresentation of the parabolically induced of M.*

Theorem 2.30. *Let (V, π) be an irreducible smooth representation of a reductive group G. Then π is admissible.*

2.3 Construction for $\mathrm{GL}_n(F)$

Now we are interested in the representations of $\mathrm{GL}_n(F)$. In particular, we will see that we need to study the representations of $\mathrm{GL}_{n_1}(F) \times \dots \mathrm{GL}_{n_r}(F)$. This is not only a way to extend our studies but it is necessary in order to work in the case described in the previous section.

2.3.1 Levi decomposition

Definition 2.31. Consider the group $\mathrm{GL}_n(F)$, then :

- The **standard Borel subgroup** B of $\mathrm{GL}_n(F)$ is the set of upper triangular matrices in $\mathrm{GL}_n(F)$.
- The **standard maximal torus** T of $\mathrm{GL}_n(F)$ is the set of diagonal matrices in $\mathrm{GL}_n(F)$.
- The **unipotent radical** U of B is the set of upper triangular matrices that have 1 on the diagonal.

By the construction of the previous section, it is clear that we are interested in the parabolic subgroups of $\mathrm{GL}_n(F)$ and in their Levi decompositions.

Definition 2.32. Let $\underline{n} = (n_1, \ldots, n_r)$ be a composition of n.

- The subgroup $P_{\underline{n}}$ of $\mathrm{GL}_n(F)$ formed by the matrices of the form
$$\begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ 0 & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{rr} \end{pmatrix}$$
is called **standard parabolic** subgroup of shape $\underline{n}$ of $\mathrm{GL}_n(F)$.
- The subgroup $U_{\underline{n}}$ of $P_{\underline{n}}$ formed by the matrices of the form.
$$\begin{pmatrix} Id_{11} & A_{12} & \ldots & A_{1n} \\ 0 & Id_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & Id_{rr} \end{pmatrix}$$
is the **unipotent radical** of $P_{\underline{n}}$.
- The subgroup $M_{\underline{n}}$ of $P_{\underline{n}}$ formed by the matrix of the form
$$\begin{pmatrix} A_{11} & 0 & \ldots & 0 \\ 0 & A_{22} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{rr} \end{pmatrix}$$
is the **standard Levi subgroup** of $P_{\underline{n}}$.
- A subgroup P (respectively M) is called **parabolic subgroup** (**Levi subgroup**) if it is conjugate to a standard parabolic (standard Levi) subgroup.

Where the $A_{i,j}$ are matrices of order $n_i \times n_j$ and $A_{i,i}$ is invertible for any i.

Remark 2.33. Clearly we have that $P_{\underline{n}} = M_{\underline{n}} U_{\underline{n}}$ and that $M_{\underline{n}} \cap U_{\underline{n}} = \{1\}$. Moreover if $P = P_{\underline{n}}^g$ is a parabolic subgroup, then $P = MU$ where $M := M_{\underline{n}}^g$ and $U := U_{\underline{n}}^g$. This explains why we are interested in the representations of $M_{\underline{n}} \cong \mathrm{GL}_{n_1}(F) \times \ldots \mathrm{GL}_{n_r}(F)$. Indeed, mixing the notions before met, all the irreducible representations of $\mathrm{GL}_n(F)$ arise from supercuspidal representations of $\mathrm{GL}_{n_1}(F) \times \ldots \mathrm{GL}_{n_r}(F)$ by parabolic induction.

Remark 2.34. The connection between the last two definitions is clear. Indeed we have the obvious equalities:

- $P_{(1,\ldots,1)} = B$

- $U_{(1,\dots,1)} = U$
- $M_{(1,\dots,1)} = T$
- $P_{(n)} = M_{(n)} = G$

In order to study the supercuspidal representations of $M_{\underline{n}}$, we need to introduce the parabolic subgroups of $M_{\underline{n}}$ itself. For this aim, it is necessary to define an order relation on the set of compositions of n.

Definition 2.35. Let $\underline{n}$ and $\underline{n'}$ be two compositions of n. We say that $\underline{n} \leq \underline{n'}$ if

$$n'_i = \sum_{j=r_{i-1}+1}^{r_i} n_j \text{ for any } j$$

where $0 = r_0 < r_1 < \cdots < r_s = r$ are integers.

Remark 2.36. Let $\underline{n}$ and $\underline{n'}$ be compositions of n such that $\underline{n} \leq \underline{n'}$. Then

- $P_{\underline{n}} \subseteq P_{\underline{n'}}$
- $M_{\underline{n}} \subseteq M_{\underline{n'}}$
- $U_{\underline{n'}} \subseteq U_{\underline{n}}$

Definition 2.37. Let $\underline{n}$ and $\underline{n'}$ be as in the previous proposition. Then $P_{\underline{n}} \cap M_{\underline{n'}}$ is a **standard parabolic subgroup** of $M_{\underline{n'}}$ with **standard Levi subgroup** $M_{\underline{n}}$ and **unipotent radical** $U_{\underline{n}} \cap M_{\underline{n'}}$.

In other words, the parabolic subgroups of $M_{\underline{n}}$ arise from parabolic subgroup of GL_n.

2.3.2 Representations of $\mathrm{GL}_n(F)$

Theorem 2.38. *Let (V, π) be a smooth representation of $\mathrm{GL}_n(F)$. Then V is one dimensional or infinite dimensional.*

Proof. Let (V, π) be a smooth representation of $\mathrm{GL}_n(F)$ of finite dimension. Let $\{v_1, \dots, v_m\}$ be a basis of V. Since V is smooth, for any v_i there exists an open compact subgroup K_i of $\mathrm{GL}_n(F)$ that stabilises v_i. So $K = \bigcap K_i$ is again an open compact subgroup of $\mathrm{GL}_n(F)$ and moreover it stabilises all $v_1, \dots, v_n$ and so all V. Hence $\operatorname{Ker} \pi$ has an open compact subgroup. Now, consider the matrices of the form

$$(e_{ij}(x))_{r,s}$$

that have 1 on the diagonal, x in the entry i, j and 0 otherwise. Since $\operatorname{Ker} \pi$ is open and $(e_{ij}(0))_{r,s} = \mathrm{Id}$ is in $\operatorname{Ker} \pi$, that is open, there exists x such that $(e_{ij}(x))_{r,s}$ is in the kernel. So, by conjugation $(e_{ij}(x))_{r,s}$ is in the kernel for any x. In particular $\mathrm{SL}_n(F)$ is generated by the matrices of that form and so we can factorise the representation through the quotient

$$\mathrm{GL}(F)/\,\mathrm{SL}_n(F) \cong F^{\times}$$

and hence V is one dimensional since $F^{\times}$ is abelian. □

Proposition 2.39. *The subgroup* ${}^0G = \det^{-1}(\mathcal{O}_F^{\times})$ *of* $\mathrm{GL}_n(F)$ *contains all the compact subgroups of* $\mathrm{GL}_n(F)$ *and* $\mathrm{GL}_n(F)/{}^0GZ(\mathrm{GL}_n(F))$ *is finite.*

Proof. ${}^0G$ is the kernel of

$$\mathrm{GL}_n(F) \xrightarrow{\det} F^{\times} \xrightarrow{|-|_F} \mathbb{R}_{>0} \leq \mathbb{C}^{\times}$$

so it is a normal subgroup of $\mathrm{GL}_n(F)$. Let H be a compact subgroup of $\mathrm{GL}_n(F)$, it is conjugate with $\mathrm{GL}_n(\mathcal{O}_F)$ by 1.35 moreover $\mathrm{GL}_n(\mathcal{O}_F) \leq^0 G$. So

$$H \leq \mathrm{GL}_n(\mathcal{O}_F)^g \leq ({}^0G)^g =^0 G$$

It remains to prove that $\mathrm{GL}_n(F)/{}^0GZ(\mathrm{GL}_n(F))$ is finite. Since the centre of $\mathrm{GL}_n(F)$ is $\{A \in \mathrm{GL}_n(F) : A = \lambda \,\mathrm{Id} \text{ with } \lambda \in F^{\times}\}$, we obtain that ${}^0GZ(\mathrm{GL}_n(F))$ is the kernel of

$$\mathrm{GL}_n(F) \xrightarrow{\det} F^{\times} \cong \overline{w}^{\mathbb{Z}} \times \mathcal{O}_F \twoheadrightarrow F^{\times}/(\overline{w}^n)^{\mathbb{Z}} \times \mathcal{O}_F \cong \mathbb{Z}/n\mathbb{Z}$$

and this conclude our proof. □

In order to prove the two theorems stated in the previous section, we need some different lemmas. Moreover, from now and until the end of this section, we will assume

$$\begin{gathered} G = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F) \\ {}^0G =^0 \mathrm{GL}_{n_1}(F) \times \cdots \times^0 \mathrm{GL}_n(F) \end{gathered}$$

with $(n_1, \ldots, n_r)$ a composition of n. Notice that, we can take as composition of n, (n) itself, so any statement that we will prove for G is valid also for GL_n.

Lemma 2.40. *Let* $\Lambda^+ := \{A \in T : a_{ii} = w^{m_i}, m_1 \geq m_2 \geq \cdots \geq m_n\}$. *Then*

$$\mathrm{GL}_n(F) = \sqcup_{\lambda \in \Lambda^+} \mathrm{GL}_n(\mathcal{O}_F)\lambda \,\mathrm{GL}_n(\mathcal{O}_F).$$

This result is called ***Cartan decomposition****.*

Lemma 2.41.

$$\mathrm{GL}_n(F) = \mathrm{GL}_n(\mathcal{O}_F)B = B\,\mathrm{GL}_n(\mathcal{O}_F)$$

where B is the set of upper triangular matrices. This result is called ***Iwasawa decomposition****.*

Corollary 2.42. *Since* $G = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$*, it is immediate that*

$$\begin{aligned} G =& \mathrm{GL}_{n_1}(\mathcal{O}_F)B_1 \times \cdots \times \mathrm{GL}_{n_r}(\mathcal{O}_F)B_r \\ =& B_1 \mathrm{GL}_{n_1}(\mathcal{O}_F) \times \cdots \times B_r \mathrm{GL}_{n_r}(\mathcal{O}_F) \end{aligned}$$

where B_i *is the subset of triangular matrices in* $\mathrm{GL}_{n_i}(F)$*.*

Proposition 2.43. *The functors* i_P^G *and* r_P^G *are exact.*

Proof. Since Ind_P^G and Res_P^G are exact, it is enough to prove that Inf_P^M, $\delta_P^{1/2}$, $\delta_P^{-1/2}$ and J_N are exact.
We know that Inf_P^M has a left and a right adjoint, so it is exact.
For any smooth character $\chi : G \to \mathbb{C}$ we have that $\mathbb{C}$ is a $\mathcal{H}$(G)-module with operation

$$f * v = c_f v$$

for any f in $\mathcal{H}$(G) and v in $\mathbb{C}$ and where $c_f := \int_G f(g)\chi(g)d\mu(g)$. Moreover, for any α complex number we have $c_f = \alpha c_f$. So, fixing a c_f not zero, we obtain for any v in $\mathbb{C}$, $v = (\alpha f) \cdot 1$ where $\alpha = \frac{v}{c_f}$. Therefore $\mathbb{C}$ is a free $\mathcal{H}$(G)-module and in particular $\chi \otimes -$ is exact [8]. This prove that $\delta_P^{1/2}$ and $\delta_P^{-1/2}$ are exact.
It remains to prove that J_N is exact. It is right exact since Inf_P^M is the right adjoint of J_N. So we need to prove that if $\phi : (V, \pi) \to (W, \rho)$ is an injective map of P representations then $J_N(\phi) : J_N(V, \pi) = V_N \to J_N(W, \rho) = W_N$ is an injective map of M representations. Take $\phi(v)$ in $W(N)$ and let H be an open compact subgroup of N such that $\phi(v) \in W(H) \subseteq W(N)$. Since H is open and compact we have that

$$V = V^H \oplus V(H) \qquad\qquad W = W^H \oplus W(H)$$

and in particular

$$\phi(V^H) \subseteq W^H \qquad\qquad \phi(V(H)) \subseteq W(H).$$

Therefore, by the injectivity, $\phi^{-1}(W(H)) \subseteq V(H)$. In particular v is in $V(H)$ and so $J_N(\phi)$ is injective. □

Lemma 2.44. *Let* (V, π) *be a finitely generated representation over* $\mathbb{C}[G]$*, then* $r_P^G(V, \pi)$ *is a finitely generated representation over* $\mathbb{C}[M]$*.*

Proof. Suppose that $\{v_1, \cdots, v_m\}$ is a set of generators of V as $\mathbb{C}[G]$-module. By the Iwasawa decomposition $\mathrm{GL}_{n_i}(F) = \mathrm{GL}_{n_i}(\mathcal{O}_F)B_i = B_i \mathrm{GL}_{n_i}(\mathcal{O}_F)$ and in particular $\mathrm{GL}_{n_i}(F)/B_i$ is compact since $\mathrm{GL}_{n_i}(\mathcal{O}_F)$ is compact. The same is true for their product G. Since $B_1 \times \cdots B_r \subseteq P$, G/P is compact.

---

[8] Since free implies flat

Now, take H_i an open compact subgroup of G such that $v_i \in V^{H_i}$ and let $H = \cap H_i$. In particular v_i is in V^H for any i. Since it is open and G/P is compact, the double cosets $P\backslash G/H$ is finite and so it is a set of representatives $\{g_1, \cdots, g_l\}$. The set $\{\pi(g_i)v_j\}_{i,j}$ generates $(V, \mathrm{Res}_P^G \pi)$ as $\mathbb{C}[P]$-module and moreover it generates $(V, \delta^{-1/2} \otimes \mathrm{Res}_P^G \pi)$ [9] and hence $\{\pi(g_i)v_j + V(N)\}_{i,j}$ generate $(V_N, J_N(\delta^{-1/2} \otimes \mathrm{Res}_P^G \pi)) = r_P^G(V, \pi)$ as $\mathbb{C}[M]-$module. □

Lemma 2.45. *The functor $i_P^G(W, \sigma)$ preserves admissible representations.*

Proof. Let (W, τ) be a smooth representation of M. Since $i_P^G(W, \tau) = \mathrm{Ind}_P^G(W, \delta_P^{1/2} \otimes \mathrm{Inf}_P^M)$, it is enough to prove that Ind_P^G, $\delta \otimes -$ and Inf_P^M preserve admissibility.

If (W, τ) is admissible, then it is $W = \bigcup W^K$ where K runs over the open compact subgroup of M. Our aim is to prove that $(W, \mathrm{Inf}_P^M \tau)$ is again admissible. If K is an open compact of P, then $K \cap M$ is an open compact of M and hence $V^{K \cap M}$ is finite dimensional. Moreover, since $\mathrm{Inf}_P^M \tau$ is trivial on $K \setminus M$, we have $V^K = V^{M \cap K}$.

Now suppose (U, σ) is an admissible smooth representation of P and consider $(U, \delta^{1/2} \otimes \sigma)$, then

$$(\delta^{1/2} \otimes \tau)(p)v = \delta^{1/2}(p)\tau(p)v.$$

Since K is compact, $\delta^{1/2}(k) = 1$ and so the subspace of U fixed by K respect to $\delta^{1/2} \otimes \sigma$ is the same subspace fixed by K respect to σ.

Finally it remains to prove that Ind_P^G preserves admissibility.

$P\backslash G/H$ is finite since G/P is compact and H is open. Let W be a representation of P. By 1.23 we have

$$(\mathrm{Ind}_P^G W)^H \cong \Big(\prod_{g \in P\backslash G/H} W^{P \cap gHg^{-1}} \Big)$$

that are finite dimensional since $W^{P \cap gHg^{-1}}$ is finite dimensional from the admissibility of W. This concludes the proof. □

Theorem 2.46. *Let (V, π) be an irreducible representation of G. Then it is supercuspidal if, and only if, $(V, \mathrm{Res}_{{}^0G}^G)$ is compact.*

Proof. (Theorem 2.29) Let (V, π) be an irreducible representation of G. Let $P = MN$ be a minimal parabolic subgroup of G such that $r_P^G(V, \pi) \neq 0$. Since P is minimal, then $r_P^G(V, \pi)$ is supercuspidal. Moreover, it is also finitely generated since (V, π) is irreducible and in particular finitely generated. Hence, by Zorn's lemma, $r_P^G(V, \pi)$ admits a maximal subrepresentation and so an irreducible quotient (W, τ) and in particular

$$\mathrm{Hom}_G(V, i_P^G(W)) \cong \mathrm{Hom}_M(r_P^G(V), W) \neq 0.$$

---

[9] We have already seen that $\delta^{-1/2}$ is exact

Now, since r_P^G is an exact functor, (W, τ) is supercuspidal. In particular we have a G-homomorphism $\phi : V \to i_P^G(W)$ that is injective since V is irreducible and $\mathrm{Ker}(\phi)$ is a subrepresentation. $\square$

Proof. (Theorem 2.30) Let (V, π) be a smooth irreducible representation of G. It is a subrepresentation of a supercuspidal representation (W, τ) of M, where $P = MN$ for a standard parabolic subgroup of G. Since i_P^G preserves admissible representations, we can suppose $G = P$ and (V, π) supercuspidal. By 2.39, $G/{}^0GZ(G)$ is finite and hence

$$G = \bigsqcup_{i=1}^{l} {}^0GZ(G)g_i$$

for some $g_1, \ldots, g_l$ in G.
By Schur's lemma, $Z(G)$ acts as a character on G. Moreover, V is irreducible and hence it is cyclic and generated by any non zero vector in V as $\mathbb{C}[G]$ module. This implies that $\{\pi(g_1)v, \ldots \pi(g_l)v\}$ is a set of generators of V as $\mathbb{C}[{}^0G]$-module. So $(V, \mathrm{Res}_{^0G}^G)$ is compact and finitely generated and hence admissible. Now, if K is a compact subgroup of G, it is a compact subgroup of ${}^0G$ and therefore it is finite dimensional. $\square$

2.4 Compactness and supercuspidality

In order to conclude this chapter we need to prove the theorem 2.46 that allows us to prove the theorem 2.30. Our aim is to prove something stronger. If $\underline{n}$ and $\underline{n}'$ are composition of n such that $\underline{n}' \leq \underline{n}$, we define

$$\Lambda^{++}(M_{\underline{n}}') := \{\mathrm{diag}(w^{m_1}\,\mathrm{Id}_{n_1'}, \ldots, w^{m_s}\,\mathrm{Id}_{n_s'}) \in \Lambda^+ \cap Z(M_{\underline{n}'}) \mid m_1 > m_2 > \cdots > m_s\}$$

and

$$\Lambda^{++}(M_{\underline{n}'}, M_{\underline{n}}) = \{\mathrm{diag}(\lambda_1, \ldots, \lambda_r) \mid \lambda_i \in \Lambda^{++}(M_{\underline{n}_i'}) \text{ for all } 1 \leqslant i \leqslant r\}$$

where diag indicate the elements on the diagonal.

As in the previous section, we use the notation

$$G = \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$$
$${}^0G ={}^0 \mathrm{GL}_{n_1}(F) \times \cdots \times^0 \mathrm{GL}_{n_r}(F)$$

with $n_1 + \cdots + n_r = n$.
Before proving the theorem, we need some tools.

Lemma 2.47. *Let* $\underline{n} = (n_1, \ldots, n_r)$ *be a composition of* n *and consider the group*

$$\begin{aligned} K_m &= 1 + w^m M_n(\mathcal{O}_F) & \qquad K_m^+ &= K_m \cap U_{\underline{n}} \\ K_m^0 &= K_m \cap M_{\underline{n}} & \qquad K_m^- &= K_m \cap \bar{U}_{\underline{n}} \end{aligned}$$

where $\bar{U}_{\underline{n}}$ *is the transpose of* $U_{\underline{n}}$*. Then we have*

1. $K_m = K_m^+ K_m^0 K_m^- = K_m^- K_m^0 K_m^+$*;*
2. *For all* $\lambda \in \Lambda^+$*we have* $\lambda K_m^+ \lambda^{-1} \subseteq K_m^+$*and* $\lambda K_m^- \lambda^{-1} \supseteq K_m^-$*;*
3. *Let* $\underline{n}' = (n_1', \ldots, n_s') \leqslant \underline{n}$ *be a composition of* n *and let*

$$\lambda \in \Lambda^{++}\left(M_{\underline{n}'}\right)$$

then

$$\bigcap_i \lambda^i K_m^+ \lambda^{-i} = \{1\} = \bigcap_i \lambda^{-i} K_m^- \lambda^i$$
$$\bigcup_i \lambda^{-i} K_m^+ \lambda^i = U_{\underline{n}}$$

Lemma 2.48. *Let* λ *be in* Λ^+*. Let* $\underline{n'}$ *be the unique composition of* n *such that* $\underline{n'} \leq \underline{n} = (n_1, \ldots, \mathfrak{n}_r)$ *and* λ *is in* $\Lambda^{++}(M_{\underline{n'}}, G)$*. Now, let* $N = U_{\underline{n'}} \cap G$*. If* (V, π) *is a smooth representation of* G*, then*

$$\bigcup_{l \geqslant 0} \operatorname{Ker} \pi\left(c_{\lambda^l}\right) \cap V^{K_m} = V(N) \cap V^{K_m}$$

with m *positive integer and* $c_\lambda = e_{K_m \lambda K_m}$*.*

Proof. Consider the open compact subgroup $N_l := \lambda^{-l} K_m^+ \lambda^l$ of N. By 2.47, we have

$$N = \bigcup_{l \geqslant 0} N_l$$

and in particular

$$V(N) = \bigcup_{l \geqslant 0} V\left(N_l\right).$$

By 2.26, we have $V\left(N_l\right) = \operatorname{Ker} \pi_{|N}\left(e_{N_l}\right)$. Therefore, our aim is to prove that

$$\pi\left(c_{\lambda^l}\right) v = 0 \Longleftrightarrow \pi_{|N}\left(e_{N_l}\right) v = 0$$

for any v in V^{K_m}.

We can write

$$\lambda^{-l} K_m^+ \lambda^l = \bigsqcup_{i=1}^{d} u_i K_m^+$$

and by 2.47 we have

$$\begin{aligned} K_m &= K_m^+ K_m^0 K_m^- \\ \lambda^{-l} K_m^0 K_m^- \lambda^l &\subseteq K_m. \end{aligned}$$

In particular, we have the disjoint union

$$K_m \lambda^l K_m = \lambda^l \cdot \lambda^{-l} K_m \lambda^l K_m = \lambda^l \cdot \lambda^{-l} K_m^+ \lambda^l K_m = \bigsqcup_{i=1}^{d} \lambda^l u_i K_m.$$

Indeed, if u_i is in $u_j K_m$, then

$$u_j^{-1} u_i \in K_m \cap N = K_m^+$$

hence

$$u_i = u_j.$$

The next computation complete the proof

$$\pi\left(c_{\lambda^l}\right) v = \frac{1}{d} \sum_{i=1}^{d} \pi\left(\lambda^l u_i\right) v = \frac{1}{d} \cdot \pi(\lambda^l) \sum_{i=1}^{d} \pi(u_i) v = \pi(\lambda^l) \pi_{|N}\left(e_{N_l}\right) v.$$

□

Theorem 2.49. *Let (V, π) be a smooth representation of G. Than, the following are equivalent:*

(1) (V, π) is supercuspidal.

(2) The functions $f_{H,v} : G \to V$ have compact support modulo $Z(G)$ for all open compact subgroups H of G and all non zero vector v in V.

(3) The matrix coefficients of (V, π) have compact support modulo $Z(G)$.

(4) $(V, \pi|_{{}^0G})$ is compact.

Proof. (1) $\implies$ (2) Let H and v be as in hypothesis. By 1.32 there exists a positive integer m such that $K_m \subseteq H$ and in particular

$$\operatorname{Supp} f_{H,v} \subseteq \operatorname{Supp} f_{K_m,v}.$$

So, we can suppose $H = K_m$ and v in V^H. Let ϕ_v be the function

$$\begin{aligned} \phi_v : \Lambda^+(G) &\to V^H \\ \lambda &\mapsto \pi(e_{H\lambda H}) v = \pi(e_H)\pi(\lambda) v = f_{H,v}(\lambda^{-1}). \end{aligned}$$

By 2.40, $G = K \Lambda^+ K$ with $K = \mathrm{GL}_n(\mathcal{O}_F) \cap G$. Now, by the normality of H in K, we have

$$\pi(e_H)\pi(k) = \pi(k)\pi(e_K) \quad \forall k \in K.$$

So, if $g = k'\lambda k$ with k, k' in K and λ in Λ we obtain

$$f_{H,v}(g^{-1}) = \pi(k')f_{H,\pi(k)}(\lambda^{-1}) = \pi(k')\phi_{\pi(k)}(\lambda)$$

that led us to the inclusion

$$(\operatorname{Supp} f_{H,v}^{-1})^{-1} \subseteq \bigcup_{k \in K/H} K \operatorname{Supp} \phi_{\pi(k)v} V.$$

Our aim is to prove that for any v in V^H, $\operatorname{Supp} \phi_v$ is finite modulo $Z(G)$. With this aim, let v be in V^H, ν in $\Lambda^+ \setminus Z(G)$ and consider

$$P_\nu = M_\nu U_\nu$$

be the proper parabolic subgroup of G such that ν in $\Lambda^{++}(M_\nu, G)$.
Bu supercuspidality of (V, π) and 2.48, we obtain

$$V^H = V^H \cap V(U_\nu) = V^H \cap \bigcup_{k \geq 0} \ker \pi(c_{\nu^k}).$$

Therefore, there exists $k_\nu \in \mathbb{Z}_{\geq 0}$ such that

$$\phi_\nu(u^k) = 0 \quad \text{for all } k \geq k_\nu.$$

Now, consider

$$\lambda_{s,i} := \{A \in T : a_{jj} = w \text{ for } j \leq i, a_{jj} = 1 \text{ otherwise}\} \in \mathrm{GL}_{n_s}(F) \subseteq \mathrm{GL}_n(F)$$

If λ is in Λ^+, then have

$$\lambda = \prod_{s=1}^{r} \prod_{i_s=1}^{n_s} \lambda_{s,i}^{d_{s,i}(\lambda)}$$

for $d_{s,i}$ non negative integers and d_{s,n_s} integer.
Now consider the set

$$X := \{\lambda \in \Lambda^+ \mid d_{s,i_s}(\lambda) < k_0 \text{ for all } 1 \leq i_s \leq n_s, 1 \leq s \leq t\}.$$

where

$$k_0 := \max\{k_{\lambda_{s,i}}; 1 \leq s \leq t, 1 \leq i_s \leq n_s\}.$$

In particular X has finite image in G/Z since $\lambda_{s,0}$ is in $Z(G)$ for all s. Now, if $\lambda \in \Lambda^+(G) \setminus X$, then it is of the form

$$\lambda = \lambda' \lambda_{s,i}^{d_{s,i}(\lambda)}$$

for some $\lambda' \in \Lambda^+(G)$, $1 \leq s \leq r$ and $1 \leq i_s \leq n_s$ with $d_{s,i}(\lambda) \geq k_0$. Finally

$$\phi_v(\lambda) = \pi(c_{\lambda'})\phi_v {\lambda_{s,i}}^{d_{s,i}(\lambda)} = 0.$$

So $\operatorname{Supp}\phi_v \subseteq X$ and this conclude this implication.

(2) $\implies$ (3) It is enough to prove that

$$\operatorname{Supp} m_{\xi,v} \subseteq \operatorname{Supp} f_{H,v}$$

for any non-zero ξ in $\tilde{V}$ and non zero vector ν in V for a compact subgroup H of G. [10]
Fixed ξ and v, the subgroup H that we are looking for is such that $\xi \in \tilde{V}^H$. Then

$$\langle \xi, f_{H,v}(g)\rangle = \langle \xi, \pi(e_H)\pi(g^{-1})v\rangle = \langle \xi, \pi(g^{-1})v\rangle = m_{\xi,v}(g)$$

for all $g \in G$, since $\xi = \xi\pi(e_H)$.

(3) $\implies$ (4) by 2.27, proving that $(V, \pi|_{{}^0G})$ is compact is equivalent to prove that all the matrices coefficients have compact support.

Let ξ be in $\tilde{V}$ non zero and let v be a non-zero vector in V.
By hypothesis, the matrix coefficient $m_{\xi,v}$ has compact support modulo $Z(G)$. Now, consider an open compact subgroup H of ${}^0G$ such that ξ is fixed by H, i.e. H is in $\tilde{V}^H$.
Therefore

$$\operatorname{Supp} m_{\xi,v} = \bigcup_{i=1}^{d} H g_i Z H$$

with $g_1, \dots, g_d$ in G. At this point, if

$$g_i Z(G) \cap {}^0G \neq \emptyset$$

we can suppose, without losing generality, g_i in ${}^0G$ and so

$$g_i Z(G) \cap {}^0G = g_i(Z(G) \cap {}^0G).$$

Moreover, if

$$g_i Z(G) \cap {}^0G = \emptyset$$

then we obtain

$$H g_i Z(G) H \cap {}^0G = \emptyset$$

since H is a subgroup of ${}^0G$. Hence, we conclude that

$$\operatorname{Supp} m_{\xi,v} \cap {}^0G \subseteq \bigcup_{i=1}^{d} H g_i (Z \cap {}^0G) H$$

which shows that $m_{\xi,v} :{}^0 G \to \mathbb{C}$ has compact support.

(4) $\implies$ (1) We need to prove that (V, π) is supercuspidal, i.e.

$$r_P^G(V) = 0$$

---

[10] The subgroup H depends of the choice of ξ and v.

for all proper parabolic subgroup P of G.
Consider a proper parabolic subgroup

$$P = MN$$

of G and let λ be in $\Lambda^{++}(M,G) \cap {}^0G$. In order to show that $r_P^G(V) = 0$ it is enough to prove that $V_N = V/V(N)$ is zero, or equivalently, that $V = V(N)$. Let v be in V and consider $m > 0$ such that v is fixed by K_m, i.e. v is in V^{K_m}. By hypothesis, the function

$$\begin{aligned} f_{K_m,\nu} : {}^0G &\to V \\ g &\mapsto \pi(e_{K_m})\pi(g^{-1})v = \pi(c_{g^{-1}})\nu \end{aligned}$$

has compact support, where $c_{g^{-1}} := e_{K_m g^{-1} K_m}$. In particular,

$$f_{K_m,v}(\lambda^{-l}) = \pi\left(c_{\lambda^l}\right) v = 0$$

for $l \gg 0$. So, using again 2.48, we have

$$v \in V^{K_m} \cap \bigcup_{l \geqslant 0} \operatorname{Ker} \pi\left(c_{\lambda^l}\right) = V^{K_m} \cap V(N)$$

Hence v is in $V(N)$ and the proof is complete. □

Chapter 3

Weil-Deligne representations

3.1 Weil Groups

3.1.1 Construction of Weil groups

Let F be a non-Archimedian local field with fixed separable closure F^{s}. Let $\mathcal{O}_F$ be its discrete valuation ring with maximal ideal $\underline{m}_F$ and finite residue field $k_F \cong \mathbb{F}_q$ of characteristic p.
It is well known that the absolute Galois group of F, i.e. the Galois group $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F)$ of F^{s} on F, can be viewed as the inverse limit of its intermediate finite fields extensions. So

$$\varprojlim_{E} \mathrm{Gal}(E/F)$$

In particular it is a profinite group.
Let F_m be the unique unramified extension of F in F^{s} of degree m and let F^{nr} be the maximal unramified extension of F in F^{s}. Then we have the canonical short exact sequence

$$1 \to \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F^{nr}) \to \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F) \to \mathrm{Gal}(F^{nr}/F) \to 1.$$

Definition 3.1. The isomorphism

$$\begin{aligned} \phi_m : \mathbb{Z}/m\mathbb{Z} &\to \mathbb{Z}/m\mathbb{Z} \\ x &\mapsto x^q \end{aligned}$$

is called **Frobenius isomorphism**.

Since $\mathrm{Gal}(F_m/F) \cong \mathbb{Z}/m\mathbb{Z}$, then

$$\mathrm{Gal}(F^{nr}/F) \cong \varprojlim_{m} \mathbb{Z}/m\mathbb{Z} := \hat{\mathbb{Z}} \cong \prod_{l \in \mathbb{N}} \mathbb{Z}_l.^{1}$$

---

[1] The last isomorphism is a direct consequences of chinese remainder theorem.

and we can imagine the Frobenius isomorphism, firstly as an element of $\mathrm{Gal}(F_m/F)$. Moreover we can consider

$$\phi_F := (\phi_m)_m$$

and

$$\Phi_F := (\Phi_m)_m$$

where Φ_m is the inverse of ϕ_m.

Definition 3.2. With the notation before introduced, ϕ_F and Φ_F are called, respectively, **arithmetic Frobenius substitution** and **geometric Frobenius substitution**.

Definition 3.3. An element of $\mathrm{Gal}(\mathrm{F^s}/F)$ is called **geometric Frobenius element** if its image in $\mathrm{Gal}(F^{nr}/F)$ by the canonical map

$$\mathrm{Gal}(\mathrm{F^s}/F) \to (F^{nr}/F)$$

is Φ_F.

Definition 3.4. Let L be an extension of F contained in $\mathrm{F^s}$. A **Frobenius morphism** of L over F $\Phi_{L/F}$ is the restriction of a geometric Frobenius Φ_F on L.

Proposition 3.5. *With the notation before introduced,* $< \Phi_F > \cong \mathbb{Z}$.

Proof. Suppose that there exists m in $\mathbb{Z}$ such that $< \Phi_F > \cong \mathbb{Z}/m\mathbb{Z}$. In particular Φ_F has finite order, say n. This implies that $\Phi_m^n = \mathrm{Id}$ on $\mathbb{Z}/m\mathbb{Z}$ for all m in $\mathbb{Z}$ and so $\phi_m^n = Id$ for all m in $\mathbb{Z}$. Notice that $\phi_m^n : x \mapsto x^{qm}$. So we are saying that any element in $\mathbb{Z}/mZ$ has order a divisor of qm for all m in $\mathbb{Z}$. This is a contradiction, so $< \Phi_F > \cong \mathbb{Z}$. □

Definition 3.6. The **Weil Group** W_F is the group algebraically constructed as the pre-image of $< \Phi_F > \cong \mathbb{Z}$ in the canonical map

$$\mathrm{Gal}(\mathrm{F^s}/F) \to \mathrm{Gal}(F^{nr}/F).$$

We are interested in defining a topology on W_F finer with respect to the topology induces on W_F by $\mathrm{Gal}(\mathrm{F^s}/F)$.

Definition 3.7. With the previous notation, the group

$$I_F := \mathrm{Gal}(\mathrm{F^s}/F^{nr})$$

is called the **absolute inertia group** of F.

Remark 3.8. The open subgroups of $\mathrm{Gal}(\mathrm{F^s}/F)$ are of the form $\mathrm{Gal}(E/F)$ where E is a finite extension of F. So, in general, I_F is not open.

Now we define the topology on W_F as the topology in which

- I_F is open in W_F.
- $I_F \subseteq W_F$ has the same topology of $I_F \subseteq \mathrm{Gal}(\mathrm{F^s}/F)$.

In particular W_F is a locally profinite group and so we can use the notion obtained in the first chapter.

Remark 3.9. Since I_F is open in W_F, the topology of I_F gives to us a lot of information about W_F. This is a consequence of the fact that W_F is the union of the set aI_F with a in W_F. So, a subgroup H of W_F is open if, and only if, $H \cap I_F$ is open in I_F.
The open set in I_F are easily found, indeed, by Galois theory, they are of the form $\mathrm{Gal}(\mathrm{F^s}/L)$ with L finite extension of F^{nr}. Moreover, by the primitive element theorem, $L = F^{nr}[\alpha]$ with α in L. So we can define $E = F[\alpha]$ that is a finite extension since α is algebraic over F.[2] So we can bring back an open of W_F to an open of $\mathrm{Gal}(\mathrm{F^s}/F)$.

Proposition 3.10. *With the notation before introduced, the canonical injection*

$$i_F : W_F \to \mathrm{Gal}(\mathrm{F^s}/F)$$

is continuous.

Proof. Let H be open in $\mathrm{Gal}(\mathrm{F^s}/F)$. Our aim is to prove that

$$i_F^{-1}(H) = H \cap W_F$$

is open in W_F. It holds if, and only if, $(H \cap W_F) \cap I_F = H \cap I_F$ is open in I_F that is true since the topology of I_F in W_F is the same of the induced by $\mathrm{Gal}(\mathrm{F^s}/F)$. □

Let E be a finite extension of F contained in $\mathrm{F^s}$. In the same way in which we have constructed W_F, we can construct W_E that can be viewed, by i_F, in W_F [3]. In particular, we have a bijection

$$W_E \to W_F \cap \mathrm{Gal}(\mathrm{F^s}/E)$$

induced by the canonical map

$$\mathrm{Gal}(\mathrm{F^s}/E) \to \mathrm{Gal}(\mathrm{F^s}/F).$$

In other words, we have an embedding of W_E in W_F.
Now, we will formalise the property of this embedding in the next theorem

---

[2] α is in L that is a finite extension of F^{nr}, so α is in $\mathrm{F^s}$.

[3] Here we are fixing $\mathrm{F^s}$ as separable closure of E.

Theorem 3.11. *Let E be a finite extension of F and fix*

$$W_F^E = W_F \cap \mathrm{Gal}(\mathrm{F^s}/E).$$

Then W_F^E is open and of finite index in W_F and it is normal if, and only if, E/F is Galois. Moreover, W_E and W_F^E are topologically isomorphic.

Proof. W_F^E is open in W_F if, and only if, $W_F^E \cap I_F$ is open in I_F. From Galois theory, we know how the open of $I_F \subseteq \mathrm{Gal}(\mathrm{F^s}/F^{nr})$ are made. Now, notice that

$$\begin{aligned} W_F^E \cap I_F =& W_F \cap \mathrm{Gal}(\mathrm{F^s}/E) \cap I_F = \mathrm{Gal}(\mathrm{F^s}/E) \cap I_F \\ =& \mathrm{Gal}(\mathrm{F^s}/E) \cap \mathrm{Gal}(\mathrm{F^s}/F^{nr}) = \mathrm{Gal}(\mathrm{F^s}/EF^{nr}). \end{aligned}$$

Clearly $\mathrm{F^s}/EF^{nr}$ is a finite extension since E/F is so. Hence $W_F^E \cap I_F$ is open in I_F. Moreover it is of finite index in I_F since open in a compact group and so W_F^E is of finite index in W_F.

It remains to prove that W_F and W_F^E are topologically isomorphic, but it is a direct consequence of 3.10 applied on E. Indeed, the continuity of

$$i_E : W_E \to \mathrm{Gal}(\mathrm{F^s}/E)$$

implies that W_E is isomorphic to $i_E(W_E) = W_F \cap \mathrm{Gal}(\mathrm{F^s}/E)$. [4] □

3.1.2 Representations of the Weil Group

Our aim is to study representations of two different classes of groups, $\mathrm{GL}_n(F)$ and W_F. We have already studied the dimension of the representations of $\mathrm{GL}_n(F)$, now, it is the moment to do the same for W_F. Contrary to what we might think, they are completely different from those of $\mathrm{GL}_n(F)$, as we will see. Indeed, despite W_F is not compact, it maintains several properties of $\mathrm{Gal}(\mathrm{F^s}/F)$ since W_F is dense in $\mathrm{Gal}(\mathrm{F^s}/F)$.
Before stating our main theorem about dimension, we need two lemmas.

Lemma 3.12. *Let (V, ρ) be an irreducible representation of a group G. If H is a normal subgroup of G that fixes a non zero vector v of V, then H fixes all V and hence H is in the kernel of the representation.*

Proof. Consider

$$W = \{v \in V : \ v \text{ is fixed by } H\}.$$

By hypothesis it is not zero. Moreover, by the normality of H, W is a subrepresentation of V. Indeed, suppose g is in G and w is in W. Since H is normal, $g^{-1}hg$ is in H for any h in H, so

$$\rho(g^{-1}hg)w = w$$

---

[4] The equality as sets is clear, the interesting part in the theorem is the continuity of the isomorphism.

and hence

$$\rho(hg)w = \rho(g)w.$$

In other words, $\rho(g)w$ is fixed by H and so it is in W. Now, since W is a non zero subrepresentation of the irreducible representation V, we have $V = W$, and therefore H stabilises all V. $\square$

Lemma 3.13. *Let (V, ρ) be an irreducible representation of a group G. If there exists a subgroup H of G of finite index whose elements act as a scalar, then V is finite dimensional.*

Proof. Since H has finite index on G, then we have a finite trasversal of H in G, say $\{g_1, \dots, g_n\}$. Moreover, since V is irreducible, it is a cyclic $\mathbb{C}[G]$-module generated by any non zero vector in V, say v. Now, let g in G. There exist g_i in $\{g_1, \cdot, g_n\}$ and h in H such that $g = g_i h$ and hence

$$\rho(g)v = \rho(g_i h)v = \rho(g_i)\rho(h)v = c_h \rho(g_i)v$$

where the last equality holds since h acts by a scalar c_h.
Therefore, $\{\rho(g_1)v, \dots, \rho(g_n)v\}$ generates V and $\dim_{\mathbb{C}} V \leq |G : H|$. $\square$

Theorem 3.14. *Let (V, π) be a smooth irreducible representation of the Weil group W_F. Then V is finite dimensional.*

Proof. Let v be a non zero vector in V. It is fixed by an open subgroup K of W_F and so by an open subgroup $J = K \cap I_F$ of I_F. Since J is open in I_F, it is of the form $J = \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/L)$ with L finite extension of F^{nr}. Now, by the primitive element theorem, $L = F^{nr}[\alpha]$ with α algebraic over F^{nr} and so over F. Hence $L = F[\alpha]F^{nr}$ and so

$$J = \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/L) = \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F[\alpha]F^{nr}) = \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F^{nr}) \cap \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F[\alpha]).$$

Without losing generality we can suppose $E := F[\alpha]$ a Galois extension of F. Indeed if E' is the Galois closure of the extension E/F, we have $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/E') \subseteq \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/E)$ and so v is stabilised by $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/E') \cap I_F$.
Now, since E/F is Galois, J is normal in $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F)$ and so in W_F. So, J is normal and fixes a non zero vector v, hence it is in the kernel of ρ by 3.12.
Now, let Φ be a Frobenius element in W_F. Since I_F/J is finite [5], there is a power of Φ, say Φ^d, that acts trivially on I_F/J. But ρ factors as in the diagram

$$\begin{tikzcd}
I_F \arrow[rr, "\rho"] \arrow[dr] & & \mathrm{Aut}_{\mathbb{C}} V \\
& I_F/J \arrow[ur] &
\end{tikzcd}$$

---

[5] I_F is compact, so any quotient with an open subgroup is finite.

and hence it acts trivially on the whole I_F. So $\rho(\Phi^d)$ commutes with the elements of $\rho(I_F)$, and since it clearly commutes with $\rho(\Phi)$, we can conclude that $\rho(\Phi^d)$ commutes with the whole $\rho(W_F)$ and hence it is a scalar by Schur's lemma 1.24. Now, the subgroup $< \Phi^d, J >$ has finite index in W_F whose elements act as scalar, so by 3.13, V is finite dimensional. $\square$

Despite W_F is not compact, all its irreducible representations are finite dimensional as the representations of $\mathrm{Gal}(\mathrm{F^s}/F)$.
Since we have a continuous map $i_F : W_F \to \mathrm{Gal}(\mathrm{F^s}/F)$, we can construct smooth representations of W_F starting with smooth representations of $\mathrm{Gal}(\mathrm{F^s}/F)$ as in the next diagram. [6]

$$\begin{tikzcd}
W_F \arrow[rr, "i_F"] \arrow[dr, "\rho\circ i_F"'] & & \mathrm{Gal}(\mathrm{F^s}/F) \arrow[dl, "\rho"] \\
& \mathrm{Aut}_{\mathbb{C}}(V) &
\end{tikzcd}$$

Proposition 3.15. *Let (V,ρ) be an irreducible smooth representation of $\mathrm{Gal}(\mathrm{F^s}/F)$. Then $\rho \circ i_F$ is an irreducible representation of W_F.*

Proof. Consider $\operatorname{Ker}\rho$. It is an open subgroup of $\mathrm{Gal}(\mathrm{F^s}/F)$, so $\operatorname{Ker}\rho \nsubseteq W_F$, otherwise W_F would be open in $\mathrm{Gal}(\mathrm{F^s}/F)$. Moreover, $W_F \nsubseteq \operatorname{Ker}\rho$, otherwise ρ would be the trivial representation since W_F is dense in $\mathrm{Gal}(\mathrm{F^s}/F)$. So, we have

$$\mathrm{Gal}(\mathrm{F^s}/F) = W_F \cdot \operatorname{Ker}\rho$$

and hence

$$\rho(\mathrm{Gal}(\mathrm{F^s}/F)) = \rho(W_F).$$

Therefore, since V is an irreducible $\mathbb{C}[\mathrm{Gal}(\mathrm{F^s}/F)]$-module, it is also an irreducible $\mathbb{C}[W_F]$-module. $\square$

Remark 3.16. Let (V_1,ρ_1) and (V_2,ρ_2) be two irreducible representations of $\mathrm{Gal}(\mathrm{F^s}/F)$. They are isomorphic $\mathrm{Gal}(\mathrm{F^s}/F)$ representations if, and only if, they are isomorphic as W_F representations. Indeed, any isomorphism of W_F representations can be extended to a isomorphism of $\mathrm{Gal}(\mathrm{F^s}/F)$ representation since W_F is dense in $\mathrm{Gal}(\mathrm{F^s}/F)$. [7]

---

[6] In this way, the category of representations of $\mathrm{Gal}(\mathrm{F^s}/F)$ can be viewed as a subcategory of representations of W_F. Moreover, these two categories are different, indeed we can construct representations of W_F that cannot be induced from representations of $\mathrm{Gal}(\mathrm{F^s}/F)$. A central example is taken by the one-dimensional representation $||\,\mathrm{Art}_F^{-1}\,||_F : W_F \to \mathbb{C}$. It cannot be constructed as a representation of $\mathrm{Gal}(\mathrm{F^s}/F)$. Moreover, it is possible to prove that any representations of W_F is a of the form $\rho \otimes ||\,\mathrm{Art}_F^{-1}\,||_F^s$ with s in $\mathbb{C}$ and ρ the restriction of W_F of a representation of $\mathrm{Gal}(\mathrm{F^s}/F)$.

[7] The density of W_F in $\mathrm{Gal}(\mathrm{F^s}/F)$ is a direct consequence of the fact that Φ_F in W_F with the topology of W_F generates $\mathbb{Z}$, while Φ_F in $W_F \subseteq \mathrm{Gal}(\mathrm{F^s}/F)$ generates $\hat{\mathbb{Z}}$.

3.2 Local class field theory

Given a local field F, local class field theory is the branch of mathematics that studies its abelian extension. Our focus is on the main theorem of this theory, which we will not prove.

Theorem 3.17. *Let F be a local field and for any finite extension L of F let $\mathrm{N}_{L/F}$ be its field extension norm.*
Then, there exists one, and only one, morphism of topological groups

$$\mathrm{Art}_F : F^\times \to \mathrm{Gal}(F^{ab}/F)$$

such that

- $\mathrm{Art}_F(\pi)_{|L} = \Phi_{K/L}$ *where π is a uniformizer for F and L/F is an unramified finite extension.*
- $\mathrm{N}_{L/F} \subseteq \mathrm{Ker}(a \mapsto \mathrm{Art}_F(a)_{|L})$ *for any abelian extension L/F and so we have the commutative diagram*

$$\begin{tikzcd}
F^\times \arrow[r, "\mathrm{Art}_F"] \arrow[d, two heads] & \mathrm{Gal}(F^{ab}/F) \arrow[d] \\
F^\times / \mathrm{N}_{L/F}(L^\times) \arrow[r, "\mathrm{Art}_{L/F}"'] & \mathrm{Gal}(L/F)
\end{tikzcd}$$

Moreover, if L is abelian over F, we have

$$[L : F] = |F^\times : \mathrm{N}_{L/F}(L^\times)|.$$

In the future we will indicate both $F^\times \to \mathrm{Gal}(L/F)$ and $F^\times / \mathrm{N}_{L/F}(L^\times) \to \mathrm{Gal}(L/F)$ with $\mathrm{Art}_{L/F}$. Notice that only the second one is bijective, while the first one is only surjective.
Our aim is to construct a more useful version of the Artin map, in particular a version that involves the Weil group of F. Indeed, the version that we are looking for states that the Artin map can be viewed as an isomorphism of topological groups between $F^\times$ and the abelianization of the Weil group. First of all, we need a new point of view for the Galois group of F^{ab} over F.

Definition 3.18. Let G be a group. Given a, b two elements of G, we call commutator of a and b, the element of G

$$[a, b] = a^{-1}b^{-1}ab.$$

Definition 3.19. Let G be a topological group. The closure of the group generated by the commutators of G is called **commutator subgroup** of G.

Definition 3.20. Let G be a topological group and $[G, G]$ its commutator subgroup, the **abelianization** of G is the quotient group $G^{ab} = G/[G, G]$.

Lemma 3.21. *Let G be a topological group. Its abelianization G^{ab} is abelian.*

Proof. Let g, h be in G and consider

$$[g,h] = g^{-1}h^{-1}gh \equiv 1 \mod [G:G]$$

hence

$$gh \equiv hg \mod [G:G].$$

□

Remark 3.22. If $f : G \to H$ is a map of topological groups and H is abelian, the commutator subgroup is in the kernel and hence we can factorise f through G'

$$\begin{tikzcd}
G \arrow[rr, "f"] \arrow[dr] & & H \\
& G' \arrow[ur] &
\end{tikzcd}$$

Proposition 3.23. *With the notation before introduced, the following holds*

$$\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F)^{ab} = \mathrm{Gal}(F^{ab}/F).$$

Proof. Since the commutator subgroup of $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F)$ is by definition closed, it is of the form $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/L)$, with $F \subseteq L \subseteq \mathrm{F}^{\mathrm{s}}$, and then

$$\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F)^{ab} = \mathrm{Gal}(L/F).$$

By Galois theory, it is enough to prove that $L = F^{ab}$.
Since $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F)^{ab}$ is abelian, $\mathrm{Gal}(L/F)$ is so. Hence, L is an abelian extension of F and it is contained in F^{ab}. This gives us the inclusion

$$\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F^{ab}) \subseteq \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/L).$$

For the other one, consider $[\sigma,\tau]$ in $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/L)$ with σ,τ in $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F)$. Clearly $[\sigma_{|F^{ab}}, \tau_{|F^{ab}}] = [\sigma,\tau]_{|F^{ab}}$, so $[\sigma,\tau]_{|F^{ab}}$ is in $\mathrm{Gal}(F^{ab}/F)$ and thus it is the identity map since $\mathrm{Gal}(F^{ab}/F)$ is abelian. Hence, $[\sigma,\tau]_{|F^{ab}}$ stabilises F^{ab} and therefore $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/L) \subseteq \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F^{ab})$. So, $\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/L) = \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F^{ab})$ and then $L = F^{ab}$. □

Remark 3.24. Combining the previous proposition and remark we obtain a map π_{ab} as in the diagram

$$\begin{tikzcd}
\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F) \arrow[rr, "\pi"] \arrow[dr] & & \hat{\mathbb{Z}} = \mathrm{Gal}(F^{nr}/F) \\
& \mathrm{Gal}(F^{ab}/F) = \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F)^{ab} \arrow[ur, "\pi_{ab}"'] &
\end{tikzcd}$$

As done for the construction of the Weil group of F, we can construct $W_{\pi^{ab}}$ as the group that makes this sequence a short exact sequence

$$0 \to \mathrm{Gal}(F^{ab}/F^{nr}) \to W_{\pi^{ab}} \to \mathbb{Z} \to 0.$$

So, we have the commutative diagram with exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F^{nr}) \arrow[r] \arrow[d] & W_F \arrow[r, "\pi"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Gal}(F^{ab}/F^{nr}) \arrow[r] & W_{\pi^{ab}} \arrow[r, "\pi_{ab}"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

and applying the snake lemma we obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\Big(\mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F) \to \mathrm{Gal}(F^{ab}/F)\Big) \arrow[r] & \mathrm{Ker}(W_F \to W_{\pi^{ab}}) \arrow[r] & 0 \arrow[dlll, rounded corners] \\
0 \arrow[r] & W_{\pi^{ab}}/\,\mathrm{Im}\Big(W_F \to W_{\pi^{ab}}\Big) \arrow[r] & 0. &
\end{tikzcd}$$

Hence, we have $\mathrm{Ker}(W_F \to W_{\pi^{ab}}) \cong \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F^{ab})$ and $W_F \to W_{\pi^{ab}}$ surjective, so

$$0 \longrightarrow \mathrm{Gal}(\mathrm{F}^{\mathrm{s}}/F^{ab}) \longrightarrow W_F \longrightarrow W_{\pi^{ab}} \longrightarrow 0$$

and therefore $W_F^{ab} \cong W_{\pi^{ab}}$.

Proposition 3.25. *Let L be an abelian extension of F. Then the Artin map*

$$\mathrm{Art}_{L/F} : F^{\times} \to \mathrm{Gal}(L/F)$$

maps $\mathcal{O}_F^{\times}$ into $\mathrm{Gal}(L/F^{nr} \cap L)$.[8]

Proof. Since $\mathrm{Art}_{L/F}$ is surjective, it is enough to prove that

$$\mathrm{Art}_{L/F}^{-1}(\mathrm{Gal}(L/F^{nr} \cap L)) = \mathcal{O}_F^{\times}.$$

In order to prove it, take $a \in F^{\times}$. Then

$$a = \pi^{\mathrm{val}(a)} x$$

with x in $\mathcal{O}_F^{\times}$, and hence

$$\mathrm{Art}_{L/F}(a)_{|F^{nr} \cap L} = \Phi_{F^{nr} \cap L/F}^{\mathrm{val}(a)} \mathrm{Art}_{L/F}(x)_{|F^{nr} \cap L} = \Phi_{F^{nr} \cap L/F}^{\mathrm{val}(a)}.$$

Therefore

$$\mathrm{Art}_{L/F}(a)_{|F^{nr} \cap L} = \mathrm{Id} \iff \mathrm{val}(a) = 0 \iff a \in \mathcal{O}_F^{\times}.$$

In other words, $\mathrm{Art}_{L/F}(a)$ is in $\mathrm{Gal}(L/F^{nr} \cap L)$ if, and only if, a is in $\mathcal{O}_F^{\times}$ □

---

[8] Notice that $F^{nr} \cap L$ is the maximal unramified extension of F inside L.

Remark 3.26. Notice that, if a is an element of $F^\times$, it can be written as $a = \pi^{\mathrm{val}(a)}x$ with x in $\mathcal{O}_F^\times$. So

$$\mathrm{Art}_F(a)_{|F^{nr}} = \mathrm{Art}_F(\pi^{\mathrm{val}(a)}x)_{|F^{nr}} = \mathrm{Art}_F(\pi)^{\mathrm{val}(a)}_{|F^{nr}} = \Phi_F^{\mathrm{val}(a)}.$$

In particular

$$\mathrm{Art}_F : F^\times \to \mathrm{Gal}(F^{ab}/F) = \mathrm{Gal}(F/F)^{ab}$$

has image in W_F^{ab} and the restriction of Art_F on $\mathcal{O}_F^\times$ has image in $\mathrm{Gal}(F^{ab}/F^{nr})$[9], so

$$\mathrm{Art}_F : \mathcal{O}_F^\times \to \mathrm{Gal}(F^{ab}/F^{nr})$$

is well defined.

Lemma 3.27. *The map*

$$\mathrm{Art}_F : \mathcal{O}_F^\times \to \mathrm{Gal}(F^{ab}/F^{nr})$$

is surjective.

Proof. Consider $\mathrm{Gal}(F^{ab}/F^{nr})$ as subgroup of $\mathrm{Gal}(F^{ab}/F)$,

$$\mathrm{Gal}(F^{ab}/F) \cong \varprojlim_{L} \mathrm{Gal}(L/F)$$

where L runs through the finite Galois extensions of F^{nr} in F^{ab}, that are clearly abelian. This isomorphism induces the isomorphism

$$\mathrm{Gal}(F^{ab}/F^{nr}) \cong \varprojlim_{L} \mathrm{Gal}(L/F^{nr} \cap L)$$

with L abelian and finite over $F^{nr} \cap L$. Moreover, by 3.25, $\mathrm{Art}_{L/F}$ maps $\mathcal{O}_F^\times$ to $\mathrm{Gal}(L/F^{nr} \cap L)$ for any L as before. Hence, $\mathrm{Gal}(L/F^{nr} \cap L)$ is in the image of Art_F for any L and therefore Art_F has dense image. Furthermore, Art_F is continuous and $\mathcal{O}_F$ is compact and hence $\mathrm{Art}_F(\mathcal{O}_F)$ is compact and dense in $\mathrm{Gal}(F^{ab}/F^{nr})$, hence $\mathrm{Art}_F(\mathcal{O}_F) = \mathrm{Gal}(F^{ab}/F^{nr})$. ☐

Proposition 3.28. *The Artin map*

$$\mathrm{Art}_F : F^\times \to \mathrm{Gal}(F^{ab}/F)$$

is injective.

Proof. By 3.17, $\mathrm{Ker}(\mathrm{Art}_F)$ is the intersection of all finite index open subgroups of $F^\times$. Since $\varpi_F^{m\mathbb{Z}} \times (1 + \varpi_F^n \mathcal{O}_F)$ is open and of finite index [10], then

$$\mathrm{Ker}(\mathrm{Art}_F) \subseteq \bigcap_{m,n\geq 1} \varpi_F^{m\mathbb{Z}} \times (1 + \varpi_F^n \mathcal{O}_F) = 1$$

☐

---

[9] In this case it is important to not confuse $\mathrm{Gal}(F^{ab}/F^{nr})$ with $\mathrm{Gal}(\mathrm{F^s}/F^{nr})^{ab}$, indeed in general they are not isomorphic.

[10] It can be viewed as a consequence of 1.32 when $n = 1$.

Now, we are ready to prove the main theorem of this section.

Theorem 3.29. *Let F be a local field with Artin map*

$$\mathrm{Art}_F : F^\times \to \mathrm{Gal}(F^{ab}/F).$$

Then, it induces an isomorphism

$$\mathrm{Art}_F : F^\times \to W_F^{ab}$$

that we will continue to call Artin map.

Proof. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_F^\times \arrow[r] \arrow[d, "\cong"] & F^\times \arrow[r] \arrow[d, "\mathrm{Art}_F"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Gal}(F^{ab}/F^{nr}) \arrow[r] & W_F^{ab} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma we obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}\left(F^\times \to W_F^{ab}\right) \arrow[r] & 0 \arrow[dll] \\
0 \arrow[r] & W_F^{ab}/\,\mathrm{Im}\left(F^\times \to W_F^{ab}\right) \arrow[r] & 0
\end{tikzcd}$$

and hence $\mathrm{Art}_F : F^\times \to W_F^{ab}$ is an isomorphism of groups. It remains to prove that it is an isomorphism of topological groups, so that Art_F and Art_F^{-1} are continuous.

Consider the restriction of Art_F on $\mathcal{O}_F$

$$\mathrm{Art}_F : \mathcal{O}_F^\times \to W_F^{ab}$$

that is continuous since composition of continuous functions.

$$\begin{tikzcd}
\mathcal{O}_F^\times \arrow[r] \arrow[ddr, dashed] & \mathrm{Gal}(F^{ab}/F) \arrow[d] \\
 & \mathrm{Gal}(F^{ab}/F^{nr}) \arrow[d] \\
 & W_F^{ab}
\end{tikzcd}$$

Since $F^\times \cong \mathbb{Z} \times \mathcal{O}_F^\times$ then

$$\mathrm{Art}_F : F^\times \to W_F^{ab}$$

is continuous. The same is true for its inverse, indeed the restriction of Art_F^{-1} on $\mathrm{Gal}(F^{ab}/F^{nr})$ has image in $\mathcal{O}_F^\times$

$$\mathrm{Art}_F^{-1} : \mathrm{Gal}(F^{ab}/F^{nr}) \to \mathcal{O}_F^\times$$

hence, it is continuous since $\mathrm{Gal}(F^{ab}/F^{nr})$ inherits the topology from $\mathrm{Gal}(F^{ab}/F)$. □

3.2.1 Weil-Deligne representations

In this section we will introduce the core of the Galois side: the Weil-Deligne representations.
Let $|| - ||_F = q^{-\operatorname{val}(-)}$ be the norm of F where q is the cardinality of the residue field $k_F = \mathcal{O}_F / \underline{m}_F$ of F.

Definition 3.30. A **Weil-Deligne representation** of the Weil group W_F of F is pair $(\rho, \mathfrak{n})$ where $\rho : W_F \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a smooth representation of W_F and $\mathfrak{n}$ is an endomorphism of V, called **monodromy operator**, such that

- $\mathfrak{n}$ is nilpotent.
- For any x in W_F we have

$$\rho(\sigma)\, \mathfrak{n}\, \rho(\sigma)^{-1} = ||\, \mathrm{Art}_F^{-1}(\sigma)||_F\, \mathfrak{n}\,.$$

Lemma 3.31. *let f be a monic polynomial with coefficients in $\mathbb{C}$. If there exists p in $\mathbb{C}$ such that $f(x) = p^{-n} f(px)$ for any x in $\mathbb{C}$, then $f(x) = x^n$ with $n = \deg(f)$.*

Proof. Let f be as in hypothesis. Then

$$f(x) = \sum_{i=1}^{n} a_i x^i$$

with $a_1 = 1$ and a_i in $\mathbb{C}$ for any other i, hence

$$p^{-n} f(px) = p^{-n} \sum_{i=1}^{n} a_i p^i x^i = \sum_{i=1}^{n} a_i p^{i-n} x^i.$$

Therefore

$$a_i = a_i p^{i-n}$$

and then $a_i = 0$ for any $i \neq 1$. □

Remark 3.32. To be nilpotent is not really a request that we need to do,, it is a consequence of the fact that V is finite dimensional. Indeed, if Φ_F is a Frobenius in W_F, $||\operatorname{Art}^{-1}(\Phi_F)|| = q^{-1}$ and then

$$\rho(\Phi_F)\,\mathfrak{n}\,\rho(\Phi_F)^{-1} = q^{-1}\,\mathfrak{n}\,.$$

Now, since V is finite dimensional we can compute its characteristic polynomial

$$\begin{aligned} f(\lambda) &= \det(\lambda\operatorname{Id}_n - \mathfrak{n}) = \det(\lambda\operatorname{Id}_n - \rho(\Phi_F)\,\mathfrak{n}\,\rho(\Phi_F)^{-1}) \\ &= \det(\lambda\operatorname{Id}_n - p^{-1}\,\mathfrak{n}) = p^{-n} f(p\lambda) \end{aligned}$$

hence $f(\lambda) = \lambda^n$. So, with respect to the Jordan canonical form, $\mathfrak{n}$ is a upper triangular matrix with 0 on the diagonal. So $\mathfrak{n}$ is nilpotent.

Definition 3.33. Let $(\rho_1, \mathfrak{n}_1)$ and $(\rho_2, \mathfrak{n}_2)$ be Weil-Deligne representations of W_F. A morphism $f : V_1 \to V_2$ of Weil-Deligne representations is a morphism of Weil representations such that the diagram

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "\mathfrak{n}_1"'] & V_2 \arrow[d, "\mathfrak{n}_2"] \\ V_1 \arrow[r, "f"] & V_2 \end{tikzcd}$$

commutes.

Definition 3.34. Let $(\rho, \mathfrak{n})$ be a Weil-Deligne representation. It is called irreducible (respectively F-semisimple) if the representation (V, ρ) is irreducible (respectively semisimple).

Remark 3.35. We are not really interested in the irreducible Weil-Deligne representations.
If $(\rho, \mathfrak{n})$ is an irreducible Weil-Deligne representation, then $\operatorname{Ker}\mathfrak{n}$ is stable. Indeed, for any σ in W_F and v in $\operatorname{Ker}\mathfrak{n}$

$$\rho(\sigma^{-1})\,\mathfrak{n}\,\rho(\sigma)v = ||\operatorname{Art}_F^{-1}(\sigma^{-1})||_F\,\mathfrak{n}\,v$$

and hence

$$\mathfrak{n}\,\rho(\sigma)v = ||\operatorname{Art}^{-1}(\sigma^{-1})||_F\rho(\sigma)\,\mathfrak{n}\,v = 0.$$

Therefore, $\operatorname{Ker}\mathfrak{n}$ is a non zero stable subspace of V and hence is the whole V. So, in this case the monodromy operator $\mathfrak{n}$ doesn't give us information. In other word, a Weil-Deligne irreducible representation is "only" a irreducible representation of the Weil group.

There is another way to think about a Weil-Deligne representation. Consider the action of W_F on $\mathbb{C}$

$$\sigma z \sigma^{-1} = ||\operatorname{Art}^{-1}(\sigma)||_F z$$

where $\mathbb{C}$ is considered as additive group and consider the induced semidirect product

$$W_F' = W_F \ltimes \mathbb{C}.$$

Definition 3.36. The group

$$W_F' = W_F \ltimes \mathbb{C}$$

is called **Weil-Deligne group**.

Definition 3.37. A complex finite dimensional **representation** of the Weil-Deligne group is a morphism of groups

$$\rho : W_F' \to \mathrm{GL}_n(V)$$

such that

- ρ restricted to W_F is a smooth representation of W_F.
- ρ restricted to $\mathbb{C}$ is analytic.

Definition 3.38. Let $\mathfrak{n} : V \to V$ be nilpotent. Then we define

$$\exp(\mathfrak{n}) = 1 + \sum_{j \geq 1} \frac{\mathfrak{n}^j}{j!}.$$

Let $\mathfrak{u} : V \to V$ be unipotent. Then we define

$$\log \mathfrak{u} = \sum_{j \geq 1} (-1)^{j-1} \frac{(\mathfrak{u} - 1)^j}{j}.$$

Proposition 3.39. *There is a canonical bijection between the Weil-Deligne representations of W_F and the representations of the Weil-Deligne group.*

Proof. Given a Weil-Deligne representation of the Weil group, we can construct a representation ρ' of the Weil-Deligne group as

$$\rho'(\sigma z) = \rho(\sigma) \exp(z\, \mathfrak{n}).$$

Conversely, let ρ' be a representation of the Weil-Deligne group. By definition

$$\rho := \rho'_{|W_F} : W_F \to \mathrm{GL}_n(V)$$

is a smooth representation of the Weil group. To be a Weil-Deligne representation we need a monodromy operator compatible with the action of W_F on $\mathbb{C}$ that defines W_F'. Our first claim is that $\rho'(z)$ is unipotent for any z in $\mathbb{C}^\times$. Since we have

$$\sigma z \sigma^{-1} = ||\, \mathrm{Art}^{-1}(\sigma)||_F z$$

it is enough to take $\sigma = \Phi_F^{-1}$ inverse of a Frobenius, and since

$$|| \operatorname{Art}^{-1}(\Phi_F^{-1})||_F = q,$$

we have

$$\rho'(\Phi_F)^{-1}\rho'(z)\rho'(\Phi_F) = \rho'(qz) = \rho'(z)^q \quad {}^{11}$$

This implies that $\rho'(z)$ and $\rho'(z)^q$ are similar. In particular, if λ is an eigenvalue for $\rho'(z)$, then λ^q is an eigenvalue for $\rho'(z)^q$ and hence for $\rho'(z)$ by similarity. Moreover, by iteration, λ^{q^n} is an eigenvalue of $\phi'(z)$ for any natural n. Furthermore, the eigenvalues of $\rho'(z)$ are at most $\dim_{\mathbb{C}} V$, thus

$$\lambda^{q^{m_0}} = \lambda^{q^{n_0}}$$

for some $0 \leq m_0 \leq n_0 \leq \dim_{\mathbb{C}} V$. Let

$$r := \prod_{0 \leq m \leq n \leq \dim_{\mathbb{C}} V} (q^n - q^m).$$

It is a positive number such that any eigenvalues of $\rho'(z)$ is a r-th root of unity.
Replaying the same idea on $\rho'(\frac{z}{r})$ and since

$$\rho'(z) = \rho'\left(\frac{z}{r}\right)^r$$

the eigenvalues of $\rho'(z)$ are all 1.
Now, consider

$$\mathfrak{n} := \frac{\log(\rho'(z))}{z}$$

with z a non zero complex number. It is well defined since $\rho'(z)$ is unipotent, and it is independent of z. Indeed, since $\rho'(z)$ is analityc, we have

$$\log \rho'(tz_0) = \log(\rho'(z_0)^t) = t \log \rho'(z_0)$$

for any z_0, t in $\mathbb{C}$. Hence, for $t = z$ and $z_0 = 1$ we have

$$\log \rho'(z) = z \log \rho'(1)$$

and so $\mathfrak{n}$ is independent of z. Therefore we have a Weil-Deligne representation $(\rho, \mathfrak{n})$. □

In order to preserve this property, we need a new definition of contragradient and tensor product of Weil-Deligne representations.

---

[11] Recall that $\mathbb{C}$ is taken as additive group, so the product in C under the representation is equivalent to the power.

Definition 3.40. Let $(\rho, \mathfrak{n})$ be a Weil-Deligne representation of W_F. The **contragradient representation** of $(\rho, \mathfrak{n})$ is the Weil-Deligne representation $(\tilde{\rho}, \tilde{\mathfrak{n}})$ where $\tilde{\rho}$ is the contragradient of the representation ρ and $\tilde{\mathfrak{n}} = -\,\mathfrak{n}^t$ is the opposite of the traspose of $\mathfrak{n}$.

Definition 3.41. Let $(\rho_1, \mathfrak{n}_1)$ and $(\rho_2, \mathfrak{n}_2)$ be two Weil-Deligne representations. The **tensor product** between $(\rho_1, \mathfrak{n}_1)$ and $(\rho_2, \mathfrak{n}_2)$ is the Weil-Deligne representation $(\rho_1 \otimes \rho_2, \mathfrak{n})$ where $\rho_1 \otimes \rho_2$ is the tensor product of the two representations and

$$\mathfrak{n} = \mathrm{Id}_1 \otimes \mathfrak{n}_2 + \mathfrak{n}_1 \otimes \mathrm{Id}_2\,.$$

Clearly, the Weil-Deligne representations are more than "classic representations", indeed, the monodromy operators that match with the representations of the Weil group encode some information which we are interested in. This information leads us to the ℓ-adic representations that we will study later. Before it, we need a digression on some subgroups of I_F.

3.2.2 The wild inertia group.

Now, we come back to our local field F with residue field of characteristic p and consider its maximal unramified extension F^{nr}. In order to continue our path in the study of the ℓ-adic representations of W_F, we need more information about the extensions of F^{nr}.

Lemma 3.42. *Let n be a number that is not divided by the characteristic p of k_F. Then, for any u in $\mathcal{O}_{F^{nr}}^{\times}$, the polynomial*

$$x^n - u$$

has n different solutions in k_F^{nr}.

Proof. By definition F^{nr} is the union of all the unramified extensions of F and hence the residue field $k_{F^{nr}}$ of F^{nr} is the union of all the finite extensions of the residue field k_F of F. [12] So $k_{F^{nr}}$ is an algebraic closure $\overline{k_F}$ of k_F. Now, let u be an invertible element of $\mathcal{O}_{F^{nr}}$ and consider the polynomial

$$x^n - u.$$

Since k_F^{nr} is algebraically closed, it has a solution α and then we can consider the polynomial

$$\left(\frac{x}{\alpha}\right)^n - 1 = y^n - 1 \quad {}^{13}$$

that has n different solutions since the characteristic of $k_{F^{nr}}$ does not divide n. Therefore, $x^n - u$ has n different solutions. $\square$

---

[12] This is a direct consequence of the fact that the unramified extensions of F are one to one with the finite extensions of its residue field.

[13] This is well definied since u is invertible in $\mathcal{O}_F^{nr}$ and hence it is non zero in k_F^{nr}.

Proposition 3.43. *Any invertible element of $\mathcal{O}_{F^{nr}}$ is an nth power for any n non divisible by p.*

Proof. Let u be as in the hypothesis. By the previous lemma

$$x^n - u$$

has n different solution in $k_{F^{nr}}$. Since F^{nr} is not complete with respect to its valuation, we cannot use the Hensel lemma on it. Nevertheless, we can use the Hensel lemma on the smaller field extension L of F such that its residue field k_L contains one of the roots of the polynomial. Hence $x^n - u$ has a solution in $\mathcal{O}_L$ by the Hensel lemma and hence it has a solution in $\mathcal{O}_{F^{nr}}$. Moreover, since the choice of the root is arbitrary, $\mathcal{O}_{F^{nr}}$ contains all the nth roots of u. $\square$

Remark 3.44. A special case of the previous proposition is given for $u = 1$. Indeed, it tells us that for any n that is not divisible by p, F^{nr} contains all the nth roots of unity.

Theorem 3.45. *For any n non divisible by p, F^{nr} has a unique extension E_n of degree n.*

Proof. Let E_n be an extension of degree n. Since F^{nr} is the maximal unramfied extension of F, all its extensions are totally ramified. Indeed if an extension is unramified over F^{nr}, it is unramified over F and hence it is contained in F^{nr}. Now, since E_n is totally ramified over F^{nr}, then it is of the form

$$E_n = F^{nr}[\alpha]$$

with $\alpha^n = \overline{w}$ for a uniformizer $\overline{w}$. In order to prove that this extension is uniquely determined, it is enough to prove that it is independent of the choice of a root of $\overline{w}$ and independent of the choice of a uniformizer.
If β is another nth root of $\overline{w}$, then $\frac{\alpha}{\beta}$ is a n-th root of unity and then it is in F^{nr} by the previous remark. Hence, E_n is independent of the choice of the root of the uniformizer.
Let $\overline{w}'$ be another uniformizer. Since $\overline{w}$, $\overline{w}'$ generate the same ideal, they are of the form

$$\overline{w} = u\overline{w}'$$

with u invertible in $\mathcal{O}_{F^{nr}}$. By the previous proposition, u is an nth power and hence L is independent of the choice of the uniformizer.
Therefore L is uniquely determined and F^{nr} has a unique extension of degree n. $\square$

Remark 3.46. The previous theorem gives us an explicit construction of the unique extension E_n of degree n of F^{nr}. It is of the form $F^{nr}[\alpha]$ with

$\alpha^n = \overline{w}$, and in particular we have an isomorphism

$$\begin{aligned} \mathrm{Gal}(E_n/F^{nr}) &\to \mu_n \\ \sigma &\mapsto \frac{\sigma(\alpha)}{\alpha} \end{aligned}$$

where μ_n is the group of nth roots of unity in F^{nr}.
Moreover, since E_n is unique as extension of degree n and we cannot have other unramified extensions, the composition of tamely ramified extensions of F^{nr} is again a tamely ramified extension of F^{nr} and hence of F.

Definition 3.47. The **maximal tamely ramified extension** E_∞ of F is the field composited by all the extension E_n of F^{nr}.

Definition 3.48. The Galois group $P_F := \mathrm{Gal}(\mathrm{F^s}/E_\infty)$ of $\mathrm{F^s}$ over E_∞ is the **wild inertia group** of F.

By the previous remark we have the isomorphism

$$\mathrm{Gal}(E_\infty/F^{nr}) \cong \prod_{m \neq p} \mathbb{Z}_m.$$

Hence, the wild inertia group P_F of F represent the "p-part" of the inertia group of F. This leads us to the exact sequence

$$1 \longrightarrow \mathrm{Gal}(\mathrm{F^s}/E_\infty) \longrightarrow \mathrm{Gal}(\mathrm{F^s}/F^{nr}) \longrightarrow \mathrm{Gal}(E_\infty/F^{nr}) \longrightarrow 1$$

that is

$$1 \longrightarrow P_F \longrightarrow I_F \longrightarrow \textstyle\prod_{m \neq p} \mathbb{Z}_m \longrightarrow 0.$$

In particular, if $t_0 : I_F/P_F \to \prod_{m \neq p} \mathbb{Z}_m$ is a topological isomorphism as in the exact sequence, it is possible to prove that by the action of $\mathrm{Gal}(F^{nr}/F)$ on $\mathrm{Gal}(E_\infty/F^{nr})$ [14].

$$t_0(\Phi_F \sigma \Phi_F^{-1}) = q^{-1} t_0(\sigma)$$

with σ in I_F/P_F and Φ_F a geometric Frobenius in W_F.
Now, consider a prime ℓ different from p. The map $I_F \to \prod_{m \neq p} \mathbb{Z}_m$ induces a morphism

$$t : I_F \to \mathbb{Z}_\ell$$

and hence we have the exact sequence

$$1 \longrightarrow P_F \longrightarrow \mathrm{Ker}\, t \longrightarrow \textstyle\prod_{m \neq p, \ell} \mathbb{Z}_m \longrightarrow 0.$$

Therefore the idea behind $\mathrm{Ker}\, t$ is to delate the "ℓ-part" of the inertia group of F, moreover we have the analogous property

$$t(g\sigma g^{-1}) = || \, \mathrm{Art}^{-1}(g) ||_F t(\sigma)$$

with x in I_F and g in W_F.

---

[14] The action is given by the quotient $\mathrm{Gal}(E_\infty/F^{nr}) \cong \mathrm{Gal}(E_\infty/F)/\,\mathrm{Gal}(F^{nr}/F)$

3.2.3 ℓ-adic representations

In order to understand the role of the monodromy operator in a complex Weil-Deligne representation, we need to enlarge our focus and to study a different class of representations: the so-called ℓ-adic representations.

Let ℓ be a prime different from p and consider a fixed separable closure $\overline{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. The valuation $\mathrm{val}_{\mathbb{Q}_\ell}$ of $\mathbb{Q}_\ell$ induces a valuation $\mathrm{val}_{\overline{\mathbb{Q}}_\ell}$ on

$$\overline{\mathbb{Q}}_\ell = \bigcup_{\substack{\mathbb{Q}_\ell \subseteq K \subseteq \overline{\mathbb{Q}}_\ell \\ [K:\mathbb{Q}_\ell]<\infty}} K$$

as

$$\begin{aligned} \mathrm{val}_{\overline{\mathbb{Q}}_\ell} : \overline{\mathbb{Q}}_\ell &\to \mathbb{Q} \cup \{\infty\} \\ x &\mapsto \mathrm{val}_{\mathbb{Q}_\ell[x]}(x) \end{aligned}$$

where $\mathrm{val}_{\mathbb{Q}_\ell[x]}$ is the unique valuation of $\mathbb{Q}_\ell[x]$ such that $\mathrm{val}_{\mathbb{Q}_\ell[x]|\mathbb{Q}_\ell} = \mathrm{val}_{\mathbb{Q}_\ell}$. Despite any finite extension of $\mathbb{Q}_\ell$ is complete with respect to the valuation induced by $\mathbb{Q}_\ell$, it is not the same for $\overline{\mathbb{Q}}_\ell$ that is not complete.

Definition 3.49. A finite dimensional representation of a locally profinite group G over a $\overline{\mathbb{Q}}_\ell$-vector space is said to be **smooth** if for any v in V, the stabilizer $\mathrm{Stab}_{W_F}(v)$ of v in W_F is open.

Moreover, if V is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space of $\dim_{\overline{\mathbb{Q}}_\ell} V = d$, then we have an isomorphism [15]

$$V \cong \overline{\mathbb{Q}}_\ell^{\,d}$$

and hence

$$\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell} V \cong \mathrm{GL}_d(\overline{\mathbb{Q}}_\ell).$$

Therefore, we can equip $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell} V$ with the product topology of $\overline{\mathbb{Q}}_\ell^{\,d\times d}$.

Definition 3.50. A finite dimensional representation (V,ρ) of a locally profinite group G over a $\overline{\mathbb{Q}}_\ell$-vector space is said to be **continuous** if the map $\rho : W_F \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$ is continuous with respect to the topology previously defined.

Remark 3.51. As previously said, the topology of $\mathrm{GL}_d(\overline{\mathbb{Q}}_\ell)$ is the topology of $\overline{\mathbb{Q}}_\ell^{\,d\times d}$ and hence a basis for this topology is given by the sets of the form

$$V_{i,j}(U) = \{A \in \mathrm{GL}_d(\overline{\mathbb{Q}}_\ell) : e_i A e_j \in U\}$$

---

[15] This isomoprhism is not canonical and it depends on the choice of a basis of V over $\overline{\mathbb{Q}}_\ell$.

for an open U of $\overline{\mathbb{Q}}_\ell$ and where e_i, e_j are the i-th and j-th vectors in the canonical basis.

Now, let (V, ρ) a smooth representation of a locally profinite group G and let σ be in G such that $\rho(\sigma)$ is in $V_{i,j}(U)$ for some i, j and U and consider

$$S := \mathrm{Stab}_G(\rho(\sigma)e_j).$$

By assumption S is open and thus $S\sigma$ is so. Moreover, if h is in $S\sigma$, we obtain

$$h = s\sigma$$

with s in S and hence

$$\rho(h)e_j = \rho(\sigma)e_j$$

and therefore we have $\rho(S\sigma) \subseteq V_{i,j}(U)$ and $S\sigma \subseteq \rho^{-1}(V_{i,j}(U))$, in particular ρ is a continuous representation.

Theorem 3.52. *Let (V, ρ) be a finite dimensional, continuous representation of the Weil group W_F over $\overline{\mathbb{Q}}_\ell$. Then, there exists one, and only one, nilpotent endomorphism*

$$\mathfrak{n}_\rho : V \to V$$

such that

$$\rho(\sigma) = \exp(t(\sigma)\,\mathfrak{n}_\rho)$$

for all σ in some open subgroup of I_F and where $t : I_F \to \mathbb{Z}_\ell$ is as above described.

Proof. Let $\mathfrak{D}_\ell$ be the integral closure of $\mathbb{Z}_\ell$ [16] in $\overline{\mathbb{Q}}_\ell$ and consider an isomorphism $V \cong \overline{\mathbb{Q}}_\ell^{\,d}$ where d is the dimension of the vector space V. By this isomorphism we have $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell(V)} \cong \mathrm{GL}_d(\overline{\mathbb{Q}}_\ell)$.

For any positive integer m, let K_m be the open subgroup [17] of $\mathrm{GL}_d(\overline{\mathbb{Q}}_\ell)$ defined as

$$K_m = 1 + \ell^m M_d(\mathfrak{D}_\ell).$$

They define a chain

$$\dots K_{m+1} \subseteq K_m \subseteq K_{m-1} \dots$$

such that K_{m+1} is normal in K_m and K_m/K_{m+1} is discrete. [18] Moreover, for any M in $M_d(\mathfrak{D}_\ell)$

$$(1 + \ell^m M)^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} 1^{\ell-j} (\ell^m M)^j$$

---

[16] Although for finite extensions of $\mathbb{Q}_\ell$ the integral closure of $\mathbb{Z}_\ell$ and the ring of integers coincide, we have not prove it for infinite extension. If x is in $\mathcal{O}_{\overline{\mathbb{Q}}_\ell}$ then there exists a finite extension L of $\mathbb{Q}_\ell$ such that x is in L and then x is in $\mathcal{O}_L$ and in particular in the integral closure of $\mathbb{Z}_\ell$ in L and hence in $\mathfrak{D}_\ell$. Conversely, if x is in $\mathfrak{D}_\ell$ it is a root of a polynomial f in $\mathbb{Z}_\ell[x]$. If L is the splitting field of f in $\overline{\mathbb{Q}}_\ell$ then x is in $\mathcal{O}_L \subseteq \mathcal{O}_{\overline{\mathbb{Q}}_\ell}$. Therefore $\mathfrak{D}_\ell = \mathcal{O}_{\overline{\mathbb{Q}}_\ell}$.

[17] By the previous footnote $\mathfrak{D}_\ell = \mathcal{O}_{\overline{\mathbb{Q}}_\ell}$ is open.

[18] Notice that this is a generalization of the proposition 1.32.

and for any j greater than 1

$$(\ell^m M)^j = \ell^{m+j} M^j \in K_{m+1}.$$

Hence

$$(1+\ell^m M)^\ell \equiv 1 \mod K_{m+1}.$$

Therefore, K_m/K_{m+1} has exponent ℓ. Moreover, if M and N are in $M_d(\mathfrak{O}_\ell)$ we have

$$(1+\ell^m M)(1+\ell^m N) = 1+\ell^m M+\ell^m N+\ell^{2m} MN \equiv 1+\ell^m M+\ell^m N$$
$$(1+\ell^m N)(1+\ell^m M) = 1+\ell^m N+\ell^m M+\ell^{2m} NM \equiv 1+\ell^m N+\ell^m M$$

and so K_m/K_{m+1} is abelian.
Now, define

$$J = \operatorname{Ker}(t) \cap \rho^{-1}(K_2).$$

By definition ρ is continuous and hence J is open since intersection of open. Moreover, $\rho(J)/K_3$ is abelian of exponent ℓ since it is inside K_2/K_3. But, $\rho(J) \subseteq \rho(\operatorname{Ker}(t))$ that cannot have exponent ℓ by construction of $\operatorname{Ker}(t)$. Therefore $\rho(J) \subseteq K_3$ and inductively

$$\rho(J) \subseteq K_m \quad \forall m \geq 2$$

and hence

$$\rho(J) \subseteq \bigcap_{m\geq 2} K_m = \{1\}.$$

Since I_F is profinite and J and $\operatorname{Ker}(t)$ are open, then there exists H_0 opens subgroup of I_F such that

$$H_0 \cap \operatorname{Ker}(t) \subseteq J.$$

Moreover we can suppose $\rho(H_0) \subseteq K_2$, shrinking H_0 (if necessary). Moreover, since H_0 is open in $I_F \subseteq W_F$, there there exists a finite index open, normal subgroup H of W_F [19] such that

$$H \cap I_F \subseteq H_0.$$

So, ρ factors through a continuous homomorphism $\phi : t(H \cap I_F) \to K_2$ [20], i.e. we have the commutative diagram

$$\begin{tikzcd}
H \cap I_F \arrow[r, "\rho_{|H\cap I_F}"] \arrow[d, "t"] & \mathrm{GL}_d(\overline{\mathbb{Q}}_\ell) \\
t(H \cap I_F) \arrow[r, "\phi"] & K_2 \arrow[u]
\end{tikzcd}$$

---

[19] We can take an open subgroup H' of W_F such that $H' \cap I_F = H_0$ by definition of induced topology and then it is sufficient to take the normal core of H'.

[20] We are combining $H \cap I_F \subseteq H_0$, $\rho(H_0) \subseteq K_2$, $H_0 \cap \operatorname{Ker}(t) \subseteq J$ and $J \subseteq \operatorname{Ker}\rho$.

and in particular

$$\phi(t(h)) = \rho(h) \quad \forall h \in H \cap I_F.$$

Moreover, since t respects

$$t(g\sigma g^{-1}) = ||\operatorname{Art}^{-1}(g)||_F t(\sigma)$$

for any σ in I_F and g in W_F and $||\operatorname{Art}^{-1}(\Phi_F)|| = q^{-1}$ for any Frobenius in W_F, we obtain

$$\phi(qt(\Phi_F h \Phi_F^{-1})) = \phi(t(h))$$

and hence [21]

$$\rho(\Phi_F h \Phi_F^{-1})^q = \rho(h).$$

With a computation similar to the one done in 3.39, we obtain that $\rho(h)$ is unipotent for any h in $H \cap I_F$ and then

$$\mathfrak{n}_\rho := t(h_0)^{-1} \log \rho(h_0)$$

is nilpotent for any h_0 as above and such that $t^{-1}(h_0) \neq 0$. In particular

$$\rho(h_0) = \exp(t(h_0)\, \mathfrak{n}_\rho).$$

It remains to find an open that satisfies our condition.
Let h_0 be a fixed element as above. The two functions

$$\begin{aligned} \mathbb{Z}_\ell t(h_0) &\to \operatorname{GL}_d(\overline{\mathbb{Q}}_\ell) \\ x &\mapsto \phi(x) \\ x &\mapsto \exp(x\, \mathfrak{n}_\rho) \end{aligned}$$

coincide on $t(h_0)$, so on $\mathbb{Z}t(h_0)$ and then on $\mathbb{Z}_\ell t(h_0)$. Therefore

$$H' := t^{-1}(\mathbb{Z}_\ell t(h_0)).$$

$H' := t^{-1}(\mathbb{Z}_\ell t(h_0))$. This concludes the existence.
For the uniqueness, let $\mathfrak{n} : V \to V$ be nilpotent and such that

$$\rho(y) = \exp(t(y)\, \mathfrak{n})$$

for any y in an open H of I_F. Hence, whenever $t(y)$ is not zero, we get

$$\mathfrak{n} = t(y)^{-1} \log(\rho(y))$$

and hence it is equal to $\mathfrak{n}_\rho$ in an open of I_F. This concludes the proof. □

---

[21] $t(H \cap I_F) \subseteq \mathbb{Z}_l$ is an additive group while the other groups in the diagram are with the composition of function or with the product of matrices.

Existence and uniqueness of the nilpotent endomorphism are not the only interesting things. Indeed, the next corollary is the one we are most interested in. Moreover, it gives a first idea of the connection between ℓ-adic representations and the Weil-Deligne representations over $\overline{\mathbb{Q}}_\ell$.

Corollary 3.53. *Let (V, ρ) be a finite dimensional, continuous representation of W_F over $\overline{\mathbb{Q}}_\ell$ and let $\mathfrak{n}_\rho$ be the unique operator that satisfies the condition of the previous theorem. Then*

$$\rho(g)\,\mathfrak{n}_\rho\,\rho(g)^{-1} = ||\operatorname{Art}^{-1}(g)||_F\,\mathfrak{n}_\rho$$

for any g in W_F.

Proof. Since $\frac{1}{||\operatorname{Art}^{-1}(g)||_F}\rho(g)\,\mathfrak{n}_\rho\,\rho(g)^{-1}$ has the characterising property of $\mathfrak{n}_\rho$, they must coincide for the uniqueness of $\mathfrak{n}_\rho$. ☐

Definition 3.54. A **Weil-Deligne** representation of W_F over $\overline{\mathbb{Q}}_\ell$ is a pair $(\rho, \mathfrak{n})$ where (V, ρ) is a smooth finite dimensional representation of W_F over $\overline{\mathbb{Q}}_\ell$ and $\mathfrak{n} : V \to V$ is a nilpotent map such that

$$\rho(g)\,\mathfrak{n}_\rho\,\rho(g)^{-1} = ||\operatorname{Art}^{-1}(g)||_F\,\mathfrak{n}_\rho$$

for any g in W_F.

Despite the pair $(\rho, \mathfrak{n}_\rho)$ of the previous theorem respects the second condition to be a Weil-Deligne representation, it is not, because in general it is not smooth.

Corollary 3.55. *Let (V, ρ) be a finite dimensional, continuous representation of W_F over $\overline{\mathbb{Q}}_\ell$ and let $\mathfrak{n}_\rho$ the unique operator the satisfies the condition of the previous theorem. Then, ρ is smooth if, and only if, $\mathfrak{n}_\rho = 0$.*

Proof. By hypothesis there exists an open subgroup H of I_F such that

$$\rho(\sigma) = \exp(t(\sigma)\,\mathfrak{n}_\rho)$$

for any σ in H. If $\mathfrak{n}_\rho = 0$ we obtain

$$\rho(\sigma) = \exp(0) = \operatorname{Id}.$$

Hence H, that is open, is contained in the stabiliser of any vector v of V and therefore any stabiliser is open.
Conversely, suppose ρ is smooth. Let $\{v_1, \dots, v_d\}$ a basis of V over $\overline{\mathbb{Q}}_\ell$. For any v_i the stabiliser $\operatorname{Stab}_{W_F}(v_i)$ is open and so it is not trivial [22]. Then their intersection is in the kernel of ρ that, therefore, cannot be trivial. In the

---

[22] Suppose $\operatorname{Stab}_{W_F}(v_i)$ is trivial. It is open in W_F and so open in I_F. Hence $I_F/\operatorname{Stab}_{W_F}(v_i) = I_F/\{1\}$ is finite, that is a contradiction.

intersection $\operatorname{Ker}\rho \cap H$, that it is not trivial since $\operatorname{Ker}\rho$ is open and normal, we have

$$\rho(\sigma) = \exp(t(\sigma)\,\mathfrak{n}_\rho) = \operatorname{Id}$$

then $\mathfrak{n}_\rho$ is zero. □

As already mentioned, the pair $(\rho, \mathfrak{n}_\rho)$, constructed from a continuous representation (V, ρ), is not a Weil-Deligne representation. Nevertheless, fixing a Frobenius element Φ_F in W_F, we have a canonical way to associate a Weil-Deligne representation to the representation (V, ρ).

Lemma 3.56. *Let (V, ρ) be a continous representation of W_F on $\overline{\mathbb{Q}}_\ell$ and let $\mathfrak{n}_\rho$ the nilpotent endomorphism of V associated to (V, ρ) by the theorem 3.52. The pair $(\rho_{\Phi_F}, \mathfrak{n}_\rho)$ with*

$$\begin{aligned} \rho_{\Phi_F} : W_F &\to \operatorname{Aut}_{\overline{\mathbb{Q}}_\ell}(V) \\ \Phi_F^a \sigma &\mapsto \rho(\Phi_F^a \sigma) \exp(-t(\sigma)\,\mathfrak{n}_\rho) \end{aligned}$$

where a in $\mathbb{Z}$ and σ in I_F is a Weil-Deligne representation.

Proof. Since ρ is a representation and $\mathfrak{n}_\rho$ is an endomorphism of V, clearly (V, ρ_{Φ_F}) is a representation of the Weil group W_F. By the theorem 3.52, there exists an open subgroup H of I_F such that

$$\rho(\sigma) = \exp(t(\sigma)\,\mathfrak{n}_\rho).$$

Hence, in H we have

$$\rho_{\Phi_F}(\sigma) = \rho(\sigma)\exp(-t(\sigma)\,\mathfrak{n}_\rho) = \exp(t(\sigma)\,\mathfrak{n}_\rho)\exp(-t(x)\,\mathfrak{n}_\rho) = \operatorname{Id}.$$

Since H fixes any vector v in V, it is contained in any stabiliser. Hence $\operatorname{Stab}_{W_F}(v)$ is open for any v and therefore ρ_{Φ_F} is smooth.
It remains to prove that

$$\rho_{\Phi_F}(g)\,\mathfrak{n}_\rho\,\rho_{\Phi_F}(g)^{-1} = ||\operatorname{Art}^{-1}(g)||_F\,\mathfrak{n}_\rho$$

for any g in W_F, but this is a direct consequence of the fact that $\mathfrak{n}_\rho$ and $\exp(\pm t(\sigma)\,\mathfrak{n})$ commute. So, if $g = \Phi_F^a \sigma$ we have

$$\begin{aligned} &\rho_{\Phi_F}(g)\,\mathfrak{n}_\rho\,\rho_{\Phi_F}(g)^{-1} \\ =&\rho(g)\exp(-t(\sigma)\,\mathfrak{n}_\rho)\,\mathfrak{n}_\rho\exp(t(\sigma)\,\mathfrak{n}_\rho)\rho(g)^{-1} \\ =&\rho(g)\,\mathfrak{n}_\rho\,\rho(g)^{-1} \\ =&||\operatorname{Art}^{-1}(g)||_F\,\mathfrak{n}_\rho\,. \end{aligned}$$

□

Now we have a way of constructing a Weil-Deligne representation from a less sophisticated representation: the continuous ones. But these types of representations of W_F are more related than we have noticed so far.
Let $\mathrm{Rep}(W_F/\overline{\mathbb{Q}}_\ell)$ be the category of representations of the Weil group on $\overline{\mathbb{Q}}_\ell$ and let $\text{WD-Rep}(W_F/\overline{\mathbb{Q}}_\ell)$ be the category of Weil-Deligne representations over $\overline{\mathbb{Q}}_\ell$. The next theorem is the key to understand the importance of the operator $\mathfrak{n}$.

Theorem 3.57. *Let Φ_F be a Frobenius element in W_F. The functor*

$$\begin{aligned} \mathrm{Rep}(W_F/\overline{\mathbb{Q}}_\ell) &\to \text{WD-Rep}(W_F/\overline{\mathbb{Q}}_\ell) \\ (V,\rho) &\mapsto (\rho_{\Phi_F}, \mathfrak{n}_\rho) \end{aligned}$$

is an equivalence of categories.

Proof. First of all we need to prove that this is really a functor. Hence, we need to associate a morphism of Weil-Deligne representations to any morphism of continuous representations of W_F. The key point is that morphism of representations do not require any condition on the continuity or on the smoothness. So, if

$$\phi : (V,\rho) \to (U,\pi)$$

is a morphism of W_F-representations, the commutativity of the diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "\rho(g)"'] & U \arrow[d, "\pi(g)"] \\ V \arrow[r, "\phi"'] & U \end{tikzcd}$$

for any g in W_F, induces a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\phi"] \arrow[d, "t(g)^{-1}\log(\rho(g))"'] & U \arrow[d, "t(g)^{-1}\log(\pi(g))"] \\ V \arrow[r, "\phi"'] & U \end{tikzcd}$$

in the intersection of the two opens associated to $\mathfrak{n}_\rho$ and $\mathfrak{n}_\pi$. The uniqueness property of $\mathfrak{n}_\rho$ and $\mathfrak{n}_\pi$ extends the commutative of the diagram to the whole $\mathfrak{n}_\rho$ and $\mathfrak{n}_\pi$ and hence ϕ is a morphism of Weil-Deligne representations.
Now, consider the functor

$$\begin{aligned} \mathrm{Rep}(W_F/\overline{\mathbb{Q}}_\ell) &\to \text{WD-Rep}(W_F/\overline{\mathbb{Q}}_\ell) \\ (\tau,\mathfrak{n}) &\mapsto (V,\tau^{\Phi_F}) \end{aligned}$$

where

$$\begin{aligned} \tau^{\Phi_F} : W_F &\to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V) \\ \Phi_F^a\sigma &\mapsto \tau(\Phi_F^a\sigma)\exp(t(\sigma)\,\mathfrak{n}) \end{aligned}$$

for any a in $\mathbb{Z}$ and σ in I_F. The continuity of τ^{Φ_F} descends from the smoothness of τ and the continuity of $\sigma \mapsto \exp(t(x)\,\mathfrak{n})$ [23].
Now, if

$$\phi : (\tau, \mathfrak{n}) \to (\delta, \mathfrak{m})$$

is a representation of Weil-Deligne representations, it is obvious a morphism of W_F representations.
It remains to prove that the two functors are quasi-inverse that is clear since

$$\exp(t(\sigma)\,\mathfrak{n}) \exp(-t(\sigma)\,\mathfrak{n}) = \mathrm{Id}\,.$$

this conclude the proof. $\square$

Now, We come back to the field $\mathbb{C}$. In particular, we are interested in the isomorphism

$$i : \mathbb{C} \to \overline{\mathbb{Q}}_\ell$$

with inverse

$$j : \overline{\mathbb{Q}}_\ell \to \mathbb{C}.$$

The idea behind these isomorphisms is that $\mathbb{Q}_p$ and $\mathbb{C}$ have the same cardinality and both contain $\mathbb{Q}$. So, if S and T are trascendente bases respectively of $\mathbb{C}$ and $\mathbb{Q}_p$ over $\mathbb{Q}$, then S and T have same cardinality. So

$$\mathbb{Q}(S) \cong \mathbb{Q}(T)$$

and therefore

$$\mathbb{C} = \overline{\mathbb{Q}(S)} \cong \overline{\mathbb{Q}(T)} = \overline{\mathbb{Q}}_\ell\,.$$

These two maps induce another equivalence of categories between the category of smooth representations of W_F over $\mathbb{C}$ and the category of smooth representations of W_F over $\overline{\mathbb{Q}}_\ell$:

$$\begin{aligned} \mathrm{Rep}(W_F/\mathbb{C}) &\to \mathrm{Rep}(W_F/\,\overline{\mathbb{Q}}_\ell) \\ (V, \rho) &\mapsto (\overline{\mathbb{Q}}_\ell \otimes_j V, \rho_\ell) \end{aligned}$$

where

$$\begin{aligned} \rho_\ell : W_F &\to \mathrm{Aut}_V(\overline{\mathbb{Q}}_\ell \otimes_j V) \\ g &\mapsto \mathrm{Id} \otimes \rho(g) \end{aligned}$$

with quasi-inverse functor:

$$\begin{aligned} \mathrm{Rep}(W_F/\,\overline{\mathbb{Q}}_\ell) &\to \mathrm{Rep}(W_F/\mathbb{C}) \\ (V, \rho) &\mapsto (\mathbb{C} \otimes_i V, \rho_{\mathbb{C}}) \end{aligned}$$

---

[23] In particular, τ^{Φ_F} is a continuous representations but it could be not smooth.

where

$$\begin{aligned} \rho_{\mathbb{C}} : W_F &\to \mathrm{Aut}_V(\mathbb{C} \otimes_i V) \\ g &\mapsto \mathrm{Id} \otimes \rho(g). \end{aligned}$$

Moreover, as consequence, we have the equivalence of categories

$$\text{WD-Rep}(W_F/\,\overline{\mathbb{Q}}_\ell) \cong \text{WD-Rep}(W_F/\mathbb{C}).$$

By 3.57, the category on the left is equivalent to the category $\mathrm{Rep}(W_F/\,\overline{\mathbb{Q}}_\ell)$ and finally we have the equivalence that we were looking for

$$\text{WD-Rep}(W_F/\mathbb{C}) \cong \mathrm{Rep}(W_F/\,\overline{\mathbb{Q}}_\ell).$$

So, speaking about Weil-Deligne representations over $\mathbb{C}$ is equivalent to speak about continuous representations of W_F over $\overline{\mathbb{Q}}_\ell$. Therefore, the monodromy operator $\mathfrak{n}$ of a Weil-Deligne representation over $\mathbb{C}$ encodes the information about the field $\overline{\mathbb{Q}}_\ell$.

Remark 3.58. The central role of the path that we have just concluded is taken by two objects:

1. The map $t : I_F \to \mathbb{Z}_\ell$,
2. The choice of the Frobenius Φ_F in W_F.

In the first one, the choice of a prime number ℓ different from p is crucial to determinate the uniqueness of $\mathfrak{n}_\rho$. Moreover, despite the association of a Weil-Deligne representation to any continuous representation of W_F depends on these two objects it is possible to prove that doing another choice of t and Φ_F the two Weil-Deligne representations constructed are isomorphic.

Chapter 4

The local Langlands correspondence

In the last two chapters we have constructed various tools for the representations of $\mathrm{GL}_n(F)$ and W_F. As we have seen, the irreducible ones behave very differently. The most obvious example is the dimensions. For the first ones, the unique "small" [1] representations are characters, after which we have only infinite dimensional representations. While the latter are not allowed for the Weil group. So it is intuitive that the connection that we want to establish is not between irreducible representations of $\mathrm{GL}_n(F)$ and W_F, but it involves more than irreducibility and more than just these two groups. We are not ready to give a complete version of the correspondence, but we have enough tools to give an incomplete but relevant one.
As in the previous chapter, we will indicate the category of representations of a group G over a field K with $\mathrm{Rep}(G/K)$ and with $\mathrm{IrrRep}(G/K)$ the set of equivalence classes of irreducible ones. Moreover we will indicate $\text{WD-Rep}_n^{ss}(W_F/\mathbb{C})$ the set of equivalnce classe of n-dimensional F-semisimple representations of W_F over $\mathbb{C}$.

The local Langlands correspondence (Incomplete one)

Let p be a prime number and F a finite extension of $\mathbb{Q}_p$. Then there exists one, and only one, collection of bijections

$$\mathrm{rec}_F : \mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C}) \to \text{WD-Rep}_n^{ss}(W_F/\mathbb{C})$$

such that

1. $\mathrm{rec}_F(\pi) = \pi \circ \mathrm{Art}^{-1}$ for any π in $\mathrm{IrrRep}(\mathrm{GL}_1(F)/\mathbb{C}) \cong \mathrm{IrrRep}(F^\times/\mathbb{C})$.

2.

---

[1] In this case, with "small" we mean dimensionally speaking.

3. For any π in IrrRep$(\mathrm{GL}_n(F)/\mathbb{C})$ and χ in IrrRep$(\mathrm{GL}_1(F)/\mathbb{C})$

$$\mathrm{rec}_F(\pi \otimes (\chi \circ det)) = \mathrm{rec}_F(\pi) \otimes \mathrm{rec}_F(\chi).$$

4. For any π in IrrRep$(\mathrm{GL}_n(F)/\mathbb{C})$ with central character χ

$$\det \mathrm{rec}_F(\pi) = \mathrm{rec}_F(\chi).$$

5. For any π in IrrRep$(\mathrm{GL}_n(F)/\mathbb{C})$

$$\mathrm{rec}_F(\tilde{\pi}) = \widetilde{\mathrm{rec}_F(\pi)}.$$

where the last one indicates the contragradient representation. The point two is intentionally left blank and it is the last topic that we need to cover in order to expose the correspondence. It concerns two invariants that we can associate with a representation of $\mathrm{GL}_n(F)$ or a representation of W_F. In both cases, they are called L-factor and ε-factor. Despite the name, the construction for one group is very different respect to the construction for the other one. Nevertheless, it is enough intuitive that the point two requires a "sort of compatibility" with respect to rec_F [2].

4.1 L-factor and ε-factor

As previously mentioned, the paths that we have to follow are different despite the name of the invariant. As done for the representations, we will start with the $\mathrm{GL}_n(F)$ side. A rigorous treatment of the topics in this chapter is somewhat technical, so we present most of the results without proofs.

4.1.1 $\mathrm{GL}_n(F)$ side

Our focus will not be on the construction in all its details, but on the way in which it is possible to generalise the definition of these factors to all representations, starting with a certain class that we will introduce early on.
Let ψ be an additive unitary character [3] of F, i.e. a character of the additive group of F and let U be the unipotent radical of the standard Borel subgroup B of $GL_n(F)$, i.e. the set of upper triangular matrices that have 1 on the

---

[2] Notice that we refer to any bijection of the correspondence with rec_F and not only to one map.

[3] F is the union of its compact open subgroups and hence any additive character is unitary.

diagonal. From ψ we can define a one dimensional representation of U as

$$\begin{aligned} \theta_\psi : U &\to \mathbb{C} \\ (u_{ij})_{1\leq i,j\leq n} &\mapsto \psi(\sum_{i=1}^{n} u_{i,i+1}). \end{aligned}$$

Definition 4.1. A smooth irreducible representation (V,π) of $\mathrm{GL}_n(F)$ is generic if

$$\mathrm{Hom}_U(\mathrm{Res}_U^{\mathrm{GL}_n(F)}\pi, \theta_\psi) \neq 0.$$

It is possible to prove that being generic does not depend on the choice of a character ψ. A proof of this claim can be found in [1].

Definition 4.2. Let (V,π) be an irreducible smooth representation of $\mathrm{GL}_n(F)$. A **Whittaker functional** is a function

$$\Lambda : V \to \mathbb{C}^\times$$

such that

$$\Lambda(\pi(u)v) = \psi(u)\Lambda(v)$$

for any u in U and v in V.

The existence of a Whittaker functional is not guaranteed, but clearly it exists for generic representations. Indeed, if (V,π) is generic, by definition we have

$$\mathrm{Hom}_U(\mathrm{Res}_U^{\mathrm{GL}_n(F)}\pi, \psi) \neq 0.$$

So the Whittaker functional are the elements Λ in $\mathrm{Hom}_U(\mathrm{Res}_U^{\mathrm{GL}_n(F)}\pi, \psi)$.

Definition 4.3. Let (V,π) be a generic representation of $\mathrm{GL}_n(F)$ and let Λ be a Whittaker functional in $\mathrm{Hom}_U(\mathrm{Res}_U^{\mathrm{GL}_n(F)}\pi, \psi)$. The image $\mathcal{W}(\pi,\psi)$ of the map

$$\begin{aligned} F : V &\to \mathrm{Ind}_U^{\mathrm{GL}_n(F)}\psi \\ v &\mapsto (g \mapsto \Lambda(\pi(g))v) \end{aligned}$$

is called **Whittaker model** of π with respect to ψ.

Remark 4.4. Remaining in the above setting we can better understand the meaning of $\mathcal{W}(\pi,\psi)$. If g and g' are in $\mathrm{GL}_n(F)$ and v is in V, we have

$$\begin{aligned} (g'\cdot F(v))(g) =& F(v)(gg') = \\ =&\Lambda(\pi(gg')v) = \\ =&\Lambda(\pi(g)\pi(g')v) = \\ =&F(\pi(g')v)(g) \end{aligned}$$

so F is a morphism of representations between π and $\mathrm{Ind}_U^{\mathrm{GL}_n(F)}(\psi)$. Moreover, since π is irreducible, F is injective and hence it is an injective morphism of representations that induces an isomorphism between

$$V \to \mathcal{W}(\pi,\psi).^4$$

So, we can think at $\mathcal{W}(\pi,\psi)$ as a way to study π as subrepresentation of $\mathrm{Ind}_U^{\mathrm{GL}_n(F)}\psi$. Clearly, in order to do it, π has to be generic, and hence we can think of generic representations as irreducible smooth representations that can be embedded in $\mathrm{Ind}_U^{\mathrm{GL}_n(F)}\psi$.

To define the factors for $\mathrm{GL}_n(F)$, we start with two generic representations π and π' of $\mathrm{GL}_n(F)$ and a measure dx on $\mathrm{GL}_n(F)/U$ invariant under the action of $\mathrm{GL}_n(F)$.
Now, define the set

$$\mathcal{S}(F^n) := \{\phi : F^n \to \mathbb{C} | \phi \text{ locally constant and with compact support}\}.$$

Given a triple $(W, W', \phi) \in \mathcal{W}(\pi,\psi) \times \mathcal{W}(\pi',\overline{\psi}) \times \mathcal{S}(F^n)$, we define the integral

$$Z(W, W', \phi, s) = \int_{\mathrm{GL}_n(F)/U} W(g)W'(g)\phi((0,\dots,0,1)g)|\det(g)|^s dg \quad ^5$$

moreover, it is a rational function of q^{-s}. Finally, consider the fractional ideal generated by

$$\{Z(W, W', \phi, s) : (W, W', \phi) \in \mathcal{W}(\pi,\psi) \times \mathcal{W}(\pi',\overline{\psi}) \times \mathcal{S}(F^n)\}$$

in $\mathbb{C}[q^s, q^{-s}]$. It has a unique generator $L(\pi \times \pi', s)$ of the form $P(q^{-s})^{-1}$ where P is a polynomial with complex coefficients such that $P(0) = 1$.
Notice that the definition made for the moment is very strict and it works only for a pair (π, π') of generic representations of $\mathrm{GL}_n(F)$.

Theorem 4.5. *(Gelfand and Kazhdan) Let (V, π) be a smooth and irreducible representation of $\mathrm{GL}_n(F)$. Then, the smooth dual representation $\tilde{\pi}$ of π is isomorphic to the representation*

$$g \mapsto \pi({}^t g^{-1})$$

where ${}^t g$ is the transpose of g.

---

[4] This implies that $\mathcal{W}(\pi,\psi)$ does not depend on the choice of a Whittaker functional, up to isomorphism.

[5] This integral is not always convergent, but for $\Re(s) >> 0$ it is absolutely convergent.

By the previous theorem, if W is in $\mathcal{W}(\pi, \phi)$ for a generic representation π, then $\tilde{W}$ defined as

$$\tilde{W}(g) = W(w_n {}^t g^{-1})$$

is in $\mathcal{W}(\tilde{\pi}, \overline{\psi})$ where w_n is the permutation matrix in $\mathrm{GL}_n(F)$ associated to the permutation

$$i \to n+1-i.$$

Finally, if

$$\hat{\phi}(x) := \int_{F^n} \phi(y)\psi({}^t yx)dy \quad x \in F^n$$

and $\omega_{\pi'}$ is the central character of π', we define $\varepsilon(\pi \times \pi', s, \psi)$ by the equality

$$\frac{Z(\widetilde{W}, \widetilde{W'}, 1-s, \hat{\phi})}{L(\widetilde{\pi} \times \widetilde{\pi}', 1-s)} = \omega_{\pi'}(-1)^n \varepsilon(\pi \times \pi', s, \psi) \frac{Z(W, W', 1-s, \phi)}{L(\pi \times \pi', 1-s)}.$$

In the same way, we define them for a pair of representations (π, π') of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$ respectively. Without losing generality, we can suppose $m < n$. Let dg a measure on $\mathrm{GL}_m(F)/U$ invariant under the action of $\mathrm{GL}_m(F)$, dx an Haar measure on $M_{j\times m}(F)$ and define the integral

$$Z\left(W, W', j, s\right) = \int_{\mathrm{GL}_m(F)/U} \int_{M_{j\times m}(F)} W(g_j)W'(g) \cdot |\det(g)|^{s-(n-m)/2} dx dg$$ [6]

where W is in $\mathcal{W}(\pi, \psi)$, W' is in $\mathcal{W}(\pi', \overline{\psi})$, j runs through $\{0, 1, \ldots n-m-1\}$ and

$$g_j = \begin{pmatrix} g & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{n-m-j} \end{pmatrix}.$$

As in the previous case, the fractional ideal generated by

$$\{Z(W, W', j, s) : (W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi', \overline{\psi}), 0 \leq j \leq n-m-1\}$$

in $\mathbb{C}[q^s, q^{-s}]$ has a unique generator $L(\pi, \pi', s)$ of the form $P(q^{-s})^{-1}$ where P is a complex coefficient such that $P(0) = 1$. Finally, if

$$w_{n,m} = \begin{pmatrix} I_m & 0 \\ 0 & w_{n-m} \end{pmatrix} \in GL_n(K)$$

we define $\varepsilon(\pi \times \pi', s, \psi)$ by the equality

$$\frac{Z\left(w_{n,m}\widetilde{W}, \widetilde{W'}, n-m-1-j, 1-s\right)}{L\left(\widetilde{\pi} \times \widetilde{\pi}', 1-s\right)} = \omega_{\pi'}(-1)^{n-1} \varepsilon\left(\pi \times \pi', \psi, s\right) \frac{Z\left(W, W', j, s\right)}{L\left(\pi \times \pi', s\right)}.$$

---

[6] As in the previous case, this integral is absolutely convergent for $\Re(s) >> 0$.

The last case that we need to define is the one in which $m > n$. In this case we define

$$L\left(\pi \times \pi', s\right) := L\left(\pi' \times \pi, s\right), \quad \varepsilon\left(\pi \times \pi', \psi, s\right) := \varepsilon\left(\pi' \times \pi, \psi, s\right).$$

For the moment, we assume that these definitions can be extended to any irreducible smooth representation starting with the generic ones. This topic will be discussed in the last section of this chapter.

4.1.2 W_F side

Let $(\rho, \mathfrak{n})$ be a smooth irreducible Weil-Deligne F-semisimple representation. Let $V_{\mathfrak{n}}$ be the kernel of $\mathfrak{n}$ and consider

$$V_{\mathfrak{n}}^{I_F} := \{v \in V_{\mathfrak{n}} : \rho(\sigma)v = v \quad \forall \sigma \in I_F\}.$$

We define the L-factor of $(\rho, \mathfrak{n})$ as

$$L(\rho, s) = \det(1 - q^{-s}\rho(\Phi)_{|V_{\mathfrak{n}}^{I_F}})^{-1}.$$

For the construction of the ε-factor we use the following theorem.

Theorem 4.6. *Let ψ be a non trivial character of $F^{\times}$, and let E range over finite extensions of F. There is a unique family of functions*

$$\begin{aligned} \text{WD-Rep}_n^{ss}(W_E/\mathbb{C}) &\longrightarrow \mathbb{C}\left[q^s, q^{-s}\right]^{\times} \\ \rho &\longmapsto \varepsilon\left(\rho, s, \psi_E\right) \end{aligned}$$

such that:

1. *If χ is a character of $E^{\times}$, then*

$$\varepsilon\left(\chi \circ \mathrm{Art}_E, s, \psi_E\right) = \varepsilon\left(\chi, s, \psi_E\right)$$

2. *If $\rho_1, \rho_2 \in \text{WD-Rep}_n^{ss}(W_E/\mathbb{C})$, then*

$$\varepsilon\left(\rho_1 \oplus \rho_2, s, \psi_E\right) = \varepsilon\left(\rho_1, s, \psi_E\right) \varepsilon\left(\rho_2, s, \psi_E\right)$$

3. *If $\rho \in \text{WD-Rep}_n^{ss}(W_E/\mathbb{C})$ and $F \subseteq E \subseteq K$, then*

$$\frac{\varepsilon\left(\mathrm{Ind}_{W_E}^{W_K} \rho, s, \psi_K\right)}{\varepsilon\left(\rho, s, \psi_E\right)} = \frac{\varepsilon\left(\mathrm{Ind}_{W_E}^{W_K} 1_K, s, \psi_K\right)^n}{\varepsilon\left(1_E, s, \psi_E\right)^n}$$

where $\psi_K = \psi \circ \mathrm{Tr}_{K/F}$, $\psi_E = \psi \circ \mathrm{Tr}_{E/F}$ and where 1_E and 1_F are the trivial characters respectively of E and F.

Proof. A proof can be found in [3] chapter 7 section 29-30. □

Now, we can complete the statement of the local Langlands correspondence before introduced. As previously said, we need a compatibility of the different factors defined for the two cases.

The local Langlands correspondence (Complete one)

Let p be a prime number and F a finite extension of $\mathbb{Q}_p$. Then there exists one, and only one, collection of bijections

$$\mathrm{rec}_F : \mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C}) \to \text{WD-Rep}_n^{ss}(W_F/\mathbb{C})$$

such that

1. $\mathrm{rec}_F(\pi) = \pi \circ \mathrm{Art}^{-1}$ for any π in $\mathrm{IrrRep}(\mathrm{GL}_1(F)/\mathbb{C}) \cong \mathrm{IrrRep}(F^\times/\mathbb{C})$.
2. For any π_1 in $\mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C})$ and π_2 in $\mathrm{IrrRep}(\mathrm{GL}_m(F)/\mathbb{C})$
$$L(\pi_1 \times \pi_2, s) = L(\mathrm{rec}_F(\pi) \otimes \mathrm{rec}_F(\pi_2), s)$$
and
$$\varepsilon(\pi_1 \times \pi_2, s, \psi) = \varepsilon(\mathrm{rec}_F(\pi) \otimes \mathrm{rec}_F(\pi_2), s, \psi).$$
3. For any π in $\mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C})$ and $\mathrm{IrrRep}(\mathrm{GL}_1(F)/\mathbb{C})$
$$\mathrm{rec}_F(\pi \otimes (\chi \circ det)) = \mathrm{rec}_F(\pi) \otimes \mathrm{rec}_F(\chi).$$
4. For any π in $\mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C})$ with central character χ
$$\det \mathrm{rec}_F(\pi) = \mathrm{rec}_F(\chi).$$
5. For any π in $\mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C})$
$$\mathrm{rec}_F(\tilde{\pi}) = \widetilde{\mathrm{rec}_F(\pi)}.$$

4.2 Bernstein-Zelevinsky classification

In order to complete the definition of L and ε-factors for $\mathrm{GL}_n(F)$ we need a way to classify the irreducible representations of $\mathrm{GL}_n(F)$ starting with the generic ones.
Clearly, this aim is not only useful to complete our definitions, but it is more powerful. Indeed, we will see that there is a particular class of representations that are generic and that can classify all the other irreducible representations. This class of representations is the class of supercuspidal representations.

Let $\mathcal{C}$ be the set of equivalence classes of irreducible representations of $\mathrm{GL}_n(F)$ for any n natural number.

Definition 4.7. A **segment** in $\mathcal{C}$ is a subset Δ of $\mathcal{C}$ of the form $\{\rho, \nu\rho, \nu^2\rho, ..., \nu^{k-1}\rho\}$ where ρ is in $\mathcal{C}$ and $\nu(g) := |\det(g)|$.
ρ is called **begininning** of Δ and $\rho' := \nu^{k-1}\rho$ is called **end** of Δ.

Definition 4.8. Let Δ_1 and Δ_2 be segments in $\mathcal{C}$. We say that they are **linked** if

- $\Delta_1 \nsubseteq \Delta_2$,
- $\Delta_1 \nsubseteq \Delta_2$,
- $\Delta_1 \cup \Delta_2$ is a segment.

in particular, if we also have $\Delta_1 \cap \Delta_2 = \emptyset$ we say that they are **juxtaposed**.

Notice that, Δ_1 and Δ_2 can be juxtaposed if, and only if, they are of the form

$$\Delta_1 = [\rho_1, \rho_1'] \qquad\qquad \Delta_2 = [\rho_2, \rho_2']$$

with $\rho_2 = \nu\rho_1'$ or $\rho_1 = \nu\rho_2'$. If $\rho_2 = \nu\rho_1'$ we say that Δ_1 precedes Δ_2.

Remark 4.9. If $\Delta = [\rho, \rho'] = \{\rho, \dots, \nu^{k-1}\rho = \rho'\}$ is a segment in $\mathcal{C}$ with ρ an irreducible representation of $\mathrm{GL}_m(F)$, then we have two representations attached to it.
Consider $n = km$ and the composition $\underline{km} = (m, \dots, m)$ of n. The first representation that we attach to Δ is the representation

$$\rho \otimes \nu\rho \otimes \cdots \otimes \nu^{k-1}\rho$$

of $M_{\underline{km}} = \mathrm{GL}_m(F) \times \cdots \times \mathrm{GL}_m(F)$. At this point, the group $M_{\underline{km}}$ is the standard Levi subgroup of the standard parabolic subgroup $P_{\underline{km}}$ of $\mathrm{GL}_{km}(F)$. So, if

$$\rho \times \nu\rho \times \cdots \times \nu^{k-1}\rho := i_{P_{\underline{km}}}^{\mathrm{GL}_{km}(F)} \rho \otimes \nu\rho \otimes \cdots \otimes \nu^{k-1}\rho$$

where $i_{P_{\underline{km}}}^{\mathrm{GL}_{km}(F)}$ is the parabolically induced functor, we can define a representation $\langle\Delta\rangle$ of $\mathrm{GL}_n(F)$ as the unique irreducible representation of $\rho \times \nu\rho \times \cdots \times \nu^{k-1}\rho$ such that

$$r_{P_{\underline{km}}}^{\mathrm{GL}_{km}(F)}(\langle\Delta\rangle) = \rho \otimes \nu\rho \otimes \cdots \otimes \nu^{k-1}\rho$$

where $r_{P_{\underline{km}}}^{\mathrm{GL}_{km}(F)}$ is the parabolically restricted functor.

The main theorem of this section is the next one.

Theorem 4.10. ***(Bernstein-Zelevinsky classification)***

- *Let* $\Delta = [\rho, \nu^{k-1}\rho]$ *be a segment in* $\mathcal{C}$*. Then* $\rho \times \nu\rho \times \cdots \times \nu^{k-1}\rho$ *has a unique irreducible quotient* $\langle\Delta\rangle^t$*.*
- *Let* $\Delta_1, \ldots, \Delta_r$ *be segments in* $\mathcal{C}$*. Suppose for each pair of indices* i, j *such that* $i < j$*,* Δ_i *does not precede* Δ_j*. Then the representation*

  $$\langle\Delta_1\rangle \times \cdots \times \langle\Delta_r\rangle$$

  has a unique irreducible subrepresentation $\langle\Delta_1, \ldots, \Delta_r\rangle$ *and the representation*

  $$\langle\Delta_1\rangle^t \times \cdots \times \langle\Delta_r\rangle^t$$

  has a unique irreducible quotient $\langle\Delta_1, \ldots, \Delta_r\rangle^t$*.*
- *Any irreducible representation of* $\mathrm{GL}_n(F)$ *is isomorphic to some representation of the form* $\langle\Delta_1, \ldots, \Delta_r\rangle^t$*.*
- *An irreducible representation* $\pi = \langle\Delta_1, \ldots, \Delta_r\rangle^t$ *of* $\mathrm{GL}_n(F)$ *is generic if, and only if, it is of the form*

  $$\pi = \langle\Delta_1\rangle^t \times \cdots \times \langle\Delta_r\rangle^t$$

  with Δ_i, Δ_j *no linked segment for any* $i \neq j$*.*
- *If* π *is supercuspidal it is of the form* $\langle\Delta\rangle^t$*. In particular any supercuspidal representation is generic.*

Proof. The main proof of this result can be found in [14]. □

The previous theorem is central in the study of the representations of $\mathrm{GL}_n(F)$ and it allows us to complete the definitions of L-factor and ε-factor for the $\mathrm{GL}_n(F)$ side. The key point is that the representations of the form $\langle\Delta\rangle^t$ are generic.

1. If π is of the form $\langle\Delta_1, \ldots, \Delta_r\rangle^t$ and if π' is arbitrary, then

   $$L(\pi \times \pi', s) = \prod_{i=1}^{r} L(\langle\Delta_i\rangle^t \times \pi', s)$$

   $$\varepsilon(\pi \times \pi', \psi, s) = \prod_{i=1}^{r} \varepsilon(\langle\Delta_i\rangle^t \times \pi', \psi, s).$$

2. If π is of the form $\langle\Delta\rangle^t$ with $\Delta = [\rho, \nu^{k-1}\rho]$ and $\pi' = \langle\Delta'\rangle^t$, with $\Delta' = [\rho', \nu^{k'-1}\rho']$ with $k' \geq k$, then

   $$L(\pi \times \pi', s) = \prod_{i=1}^{k} L(\rho \times \rho', s + k + k' - 1)$$

$$\varepsilon(\pi \times \pi', \psi, s) = \prod_{i=1}^{k} \left(\prod_{j=0}^{k+k'-2i} \varepsilon(\rho \times \rho', \psi, s+i+j-1) \right)$$

$$\times \left(\prod_{j=0}^{k+k'-2i-1} \frac{L(\tilde{\rho} \times \tilde{\rho}', 1-s-i-j)}{L(\rho \times \rho', s+i+j-1)} \right).$$

3. If π is a smooth irreducible representation of $\mathrm{GL}_n(F)$ we define

$$L(\pi, s) = L(\pi \times 1, s),$$

$$\varepsilon(\pi, \psi, s) = \varepsilon(\pi \times 1, \psi, s).$$

with $1 : F^\times \to \mathbb{C}^\times$ the trivial character of F.

Another important consequence of the Bernstein-Zelevinsky classification is related to the local Langlands correspondence. Indeed supercuspidal representations are sent, via the correspondence, to Weil-Deligne representation of the form $(\rho, \mathfrak{n})$ with ρ irreducible and $\mathfrak{n} = 0$. In other words, supercuspidal representations are sent to irreducible representations of the Weil group. This consideration, mixed with the work of Bernshtein and Zelevinsky, allows to reduce the proof of the existence of the series of bijections

$$\mathrm{rec}_F : \mathrm{IrrRep}(\mathrm{GL}_n(F)/\mathbb{C}) \to \text{WD-}\mathrm{Rep}_n^{ss}(W_K/\mathbb{C})$$

to a series of bijections

$$\mathrm{rec}_F : \mathrm{SCusp}(\mathrm{GL}_n(F)/\mathbb{C}) \to \mathrm{Irr}\,\mathrm{Rep}_n(W_K/\mathbb{C}).$$

where $\mathrm{SCusp}(\mathrm{GL}_n(F)/\mathbb{C})$ is the set of equivalence classes of supercuspidal representation of $\mathrm{GL}_n(F)$.

Bibliography

[1] I.N.Bernstein and A.V.Zelevinsky, *Representations of the group* $\mathrm{GL}(n, F)$ *where* F *is a non-archimedean local field*, Russian Math. Surveys 31:3, 1-68 (1976).

[2] I.N.Bernstein and A.V.Zelevinsky, *Induced representations of reductive* p*-adic groups.I*, Annales scientifiques de l'École Normale Supérieure 10.4: 441-472, (1977).

[3] C.J.Bushnell and G.Henniart, *The Local Langlands Conjecture for* GL(2), Springer-Verlag, Heidelberg, (2006).

[4] M. Harris, R.Taylor, *The geometry and cohomology of some simple Shimura varieties*, Princeton University Press, Annales of Mathematics Studies, (2001).

[5] C.Heyer, *Smooth Representations of* p*-adic Groups*, (2023).

[6] J.S.Milne, *Algebraic Number Theory*, www.jmilne.org/math/ , (2020).

[7] J.S.Milne, *Class Field Theory*, www.jmilne.org/math/ , (2020).

[8] J. Neukirch, *Algebraic Number Theory*, Springer-Verlag, Heidelberg, (1999).

[9] D.Renard and L.Schwartz, *Representations Des Groupes Reductifs p-adiques*, (2008).

[10] D.E.Rohrlich, *Elliptic curves and the Weil-Deligne Group*, Centre de Recherches Mathemariques Proceedings & Lecture notes (1994).

[11] J.P. Serre, *Linear Representations Of Finite Groups*, Springer-Verlag, New York, (1977).

[12] J.Tate, *Number Theoretic Background*, Proceedings of Symposia in Pure Mathematics, (1979).

[13] T.Wedhorn, *The local Langlands correspondence for* $\mathrm{GL}(n)$ *over* p*-adic fields* (2000).

[14] A.V.Zelevinsky, *Induced representations of reductive p-adic groups. II. On irreducible representations of* GL(n). Annales scientifiques de l'École Normale Supérieure 13.2: 165-210, (1980).