Università degli Studi di Padova

DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

Master Degree in Mathematics

An abelian approach to symplectic reduction for a certain class of groups and momentum values.

Supervisor: Prof. Luis García Naranjo
Student: Enzo Rigato
Student number: 1187380

Academic Year 2023–2024

20/09/2024

Acknowledgments.

First of all, I am extremely grateful to Professor Luis García Naranjo for his expertise in the redaction of this thesis. It has been really fascinating to dive into the challenges that the Mathematics offered us, and this is thanks to his professional support and approach.

I am also thankful to Alejandro Bravo-Doddoli and Nicola Paddeu for their interest in my thesis and for their suggestions of specific examples. Their insights were very useful in the process of formulating the theoretical results.

Lastly I would be remiss in not mentioning my parents, Stefano and Roberta, and my girlfriend, Giorgia, whose patience and belief have been essential through my academic journey.

Contents

1 Introduction **7**

2 The Main Theorem **9**

2.1 Geometric setting . . . 9
2.2 Assumptions on the momentum values . . . 11
2.3 A pair of Hamiltonian actions and their reduction . . . 12
2.4 Main Theorem . . . 14
  2.4.1 Proof of the Main Theorem. . . . 14

3 Semidirect product Lie groups **23**

3.1 Preliminaries . . . 23
3.2 Main Theorem in semidirect products . . . 25
3.3 Coadjoint orbits of semidirect products . . . 28

4 Examples **33**

4.1 $SE(2)$ example. . . . 33
  4.1.1 $SE(2)$, $\mathfrak{se}(2)$ and $\mathfrak{se}(2)^*$ preliminaries . . . 34
  4.1.2 Symplectic reduction of $T^*SE(2)$ by $SE(2)$ . . . 36
  4.1.3 Symplectic reduction of $T^*SE(2)$ by $\mathbb{R}^2$ . . . 39
  4.1.4 Equivalence of the symplectic reduction of $T^*SE(2)$ by $SE(2)$ and $\mathbb{R}^2$ at generic momentum values . . . 40
4.2 $SE(3)$ and the $SO(3)$ action on $\mathbb{R}^3 \times \mathbb{R}^3$. . . . 42
4.3 The Heisenberg group . . . 43
  4.3.1 $\mathbb{H}^3$ as a semidirect product . . . 45
  4.3.2 $\mathbb{H}^3$ as a non semidirect product . . . 47

A Preliminaries **49**

A.1 Differential geometry . . . 49
  A.1.1 Lie group actions . . . 49

    - A.1.2 A technical lemma to prove smoothness of a map 51
    - A.1.3 The smooth structure of the orbit of an action 52
- A.2 Standard symplectic reduction 54
- A.3 Principal Bundles Over a Lie Group 57

Bibliography **60**

Chapter 1

Introduction

All the historical information that follows is taken from an introductory preface of Marsden, Ratiu and Scheurle [11] and Marsden, Weinstein [13].

Reduction as a mathematical topic finds its roots in the works in mechanical systems of Euler, Lagrange, Hamilton, Jacobi, Routh, Riemann, Liouville, Lie, Poincaré. For them, reduction was just the tool to simplify computations in their examples by reducing the degrees of freedom. This is by the way the technique encoded in the classical Noether Theorem (which was originally discovered by variational principles and not using symplectic methods). An instance of early reductions is the Routh one (for systems with cycling coordinates).

A more geometric approach to reduction takes place in Poisson and symplectic spaces, and it is due to Lagrange and Jacobi. Nevertheless some objects that modern theory takes into account were first discovered by Lie (for instance, the Lie Poisson brackets of the dual Lie algebra $\mathfrak{g}^*$ of a Lie group G).

Along the 1960's geometric mechanics had been developing considerably and it started to be interesting not only to mathematical physicists such as Arnold, Small and Soriau but also to pure mathematicians such as Kostant and Kirillov, who linked orbit symplectic reduction with group representation theory. The 60's school led to the modern era of reduction theory which, accorded with the community, it was born with the fundamental papers of Arnold [2] and Smale [15]. In particular, Smale set the usual construction of symplectic reduction and he realized the importance of the role of two key objects, namely momentum maps and quotient constructions (for the case of lifted action of a Lie group G on a manifold Q to its cotangent bundle T^*Q).

Standard symplectic reduction is the following: if one has a symplectic, free, proper Lie group action on a symplectic manifold with an equivariant momentum map (with respect to Coadjoint action on the dual Lie algebra of the group) then one can perform a reduction on level sets of the momentum map by a suitable subgroup to form the orbit space which happens to be a symplectic manifold as well. This thesis investigates conditions in which the symplectic reduction by a non-abelian group is equivalent to the symplectic reduction by an abelian subgroup.

Interesting results already appear in literature for compact Lie groups. Without going into details it is relevant to cite Blaom Lemma 7.3 in [3]. In this reference the author proposes restricting to an appropriate invariant symplectic subspace and then perform symplectic reduction by a maximal torus (i.e. a compact, abelian, connected subgroup).

To our knowledge, the possibility of obtaining similar results for non-compact Lie groups has not been investigated, and this was the driving motivation of this work. Interestingly, the symplectic reduction by certain abelian subgroups of metabelian nilpotent groups has been recently used by Bravo-Doddoli, Le Donne and Paddeu to determine properties of their sub-Riemannian geodesic flow [4]. The precise relationship between their results and those contained in this thesis remains to be defined.

The main reference in the developing of the thesis has been the book [10]. In particular, the main results are a consequence of the general theory of symplectic reduction by stages developed in Chapter 5.2 of the book. However, to the best of our knowledge, these results, formulated in this thesis as Theorems 2.4.1 and 3.3.1, are original.

Structure of the thesis.

We begin in Chapter 2 by introducing the objects which lead to the statement and the proof of the main result of the thesis. In Chapter 3 we will review the theorem in the particular case of a group with semidirect structure. In this framework, we give a result on the symplectic structure of Coadjoint orbits. Chapter 4 is devoted to some explicit examples in which we will be able to illustrate elements of the theory developed before. Finally in the appendix we review some basic results in differential geometry and theory of fibrations that turn out to be useful in the thesis and we briefly recall how symplectic reduction works.

Chapter 2

The Main Theorem

This chapter has the goal to state and prove Theorem 2.4.1 which is the main result of the thesis, which will be applied in Chapter 3. The first sections are devoted to the setting of the objects that will appear in the theorem, such as the groups we take into account, their properties and the actions on a symplectic manifold. After that we state and prove the main theorem.

2.1 Geometric setting

Let M be a Lie Group with regular, abelian, normal Lie subgroup $N \trianglelefteq M$. Let $\mathfrak{m} = Lie(M)$ and $\mathfrak{n} = Lie(N)$. Call the immersion map $I : N \hookrightarrow M$. The tangent map at the identity is a natural immersion of the Lie subalgebra $\mathfrak{n}$ in the total one: $i := T_e\, I : \mathfrak{n} \hookrightarrow \mathfrak{m}$. This is a Lie algebra homomorphism and in particular a linear map. We can consider:

$$i^* : \mathfrak{m}^* \longrightarrow \mathfrak{n}^*, \tag{2.1}$$

where i^* is the dual map of i. We will look at the map i^* in two different ways:

- let $\mu \in \mathfrak{m}^*$. This is a functional acting on the vector space $\mathfrak{m}$; this can be restricted, as a map, to the vector subspace $\mathfrak{n}$ so that it has the exact meaning: $\mu|_{\mathfrak{n}} = i^*(\mu)$. We check this relation in $\mathfrak{n}^*$ by testing it on a generic $v \in \mathfrak{n}$:

$$[i^*(\mu)](v) \;=\; \langle i^*(\mu), v\rangle \;=\; \langle \mu, i(v)\rangle = \mu \circ i(v) \;=\; \mu|_{\mathfrak{n}}(v).$$

- i^* is a surjective (linear) map as the dual map of an injective one. We will call the set $(i^*)^{-1}(\nu)$ the *fiber* of ν inside of $\mathfrak{m}^*$, for $\nu \in \mathfrak{n}^*$.

A fundamental role will be played by the Lie group action:

$$\psi : M \times \mathfrak{n}^* \longrightarrow \mathfrak{n}^*,$$

defined by:

$$\psi_m(\nu) \; = \; i^*(Ad^*_{m^{-1}}(\mu)), \quad m \in M, \nu \in \mathfrak{n}^*, \tag{2.2}$$

where μ is any element in the fiber $(i^*)^{-1}(\nu)$.

Lemma 2.1.1. *The action ψ above is a well defined smooth Lie group action.*

Proof. We have to show that $\psi_m(\nu)$ does not depend on the representative in the i^* fiber, i.e. the result:

$$i^*(Ad^*_{m^{-1}}(\mu)) \; = \; i^*(Ad^*_{m^{-1}}(\mu'))$$

holds whenever $i^*(\mu) = i^*(\mu')$.

The core fact we use here is that N is normal in M. This implies that the conjugation action of M on itself leaves N invariant and this also happens at the Lie algebras level, namely:

$$Ad_m(v) \in \mathfrak{n}, \quad \forall\, m \in M,\; v \in \mathfrak{n}. \tag{2.3}$$

Let $\xi \in \mathfrak{n}$. Then:

$$\langle i^*(Ad^*_{m^{-1}}\mu), \xi\rangle \; = \; \langle Ad^*_{m^{-1}}\mu, i(\xi)\rangle \; = \; \langle \mu, (Ad_{m^{-1}}(i(\xi))\rangle.$$

Now $i^*(\mu) = i^*(\mu')$ means in particular that $\mu(v) = \mu'(v)$ for every $v \in \mathfrak{n}$. Also $(Ad_{m^{-1}}(i(\xi)) \in \mathfrak{n}$ by (2.3), so the last term of the above equality equals to:

$$\langle \mu', (Ad_{m^{-1}}(i(\xi))\rangle \; = \; \langle i^*(Ad^*_{m^{-1}}\mu'), \xi\rangle,$$

which shows what we wanted. ☐

Proposition 2.1.2. *The restriction of ψ to $N \times \mathfrak{n}^*$ coincides with the Coadjoint action of N on $\mathfrak{n}^*$ (which is a trivial action on $\mathfrak{n}^*$).*

Proof. First, as a consequence of (2.3), the inclusion map $i : \mathfrak{n} \hookrightarrow \mathfrak{m}$ satisfies:

$$Ad_{n^{-1}} \circ i \; = \; i \circ Ad_{n^{-1}}, \quad \forall n \in N.$$

Dualizing we get:

$$i^* \circ Ad^*_{n^{-1}} \; = \; Ad^*_{n^{-1}} \circ i^*.$$

We apply the last identity to $\nu \in \mathfrak{n}^*$:

$$Ad^*_{n^{-1}}(\nu) \; = \; Ad^*_{n^{-1}} \circ i^*(\mu) \; = \; i^* \circ Ad^*_{n^{-1}}(\mu) \; = \; \psi_n(\nu),$$

where $\mu \in \mathfrak{m}^*$ is any element such that $i^*(\mu) = \nu$. ☐

The action ψ gives rise to a subgroup which will be crucial in the formulation of the main theorem of the thesis, namely the *$\psi-$isotropy subgroup* of $\nu \in \mathfrak{n}^*$:

$$M_\nu \;=\; \{\; m \in M \mid \psi_m(\nu) = \nu \;\} \tag{2.4}$$

The following lemma shows that in our framework the subgroup N is contained in M_ν for any $\nu \in \mathfrak{n}^*$. We will further see that an important focus in the theory is to spot those $\nu \in \mathfrak{n}^*$ such that the converse inclusion holds as well, reaching $M_\nu = N$.

Lemma 2.1.3. *Let $\nu \in \mathfrak{n}^*$. Then $N \subset M_\nu$.*

Proof. Here we use the fact that N is abelian and so the Adjoint action of N on $\mathfrak{n}$ is trivial (see Proposition A.1.2). Let $n \in N$ and $v \in \mathfrak{n}$. Let $\mu \in (i^*)^{-1}(\nu)$. Then:

$$\begin{aligned}\langle \psi_n(\nu), v\rangle \;&=\; \langle i^*(Ad^*_{n^{-1}}\mu), v\rangle \;=\; \langle Ad^*_{n^{-1}}\mu, i(v)\rangle \\ &=\; \langle \mu, Ad_{n^{-1}}(i(v))\rangle \;=\; \langle \mu, i(v)\rangle \\ &=\; \langle i^*(\mu), v\rangle \;=\; \langle \nu, v\rangle,\end{aligned}$$

so that $\psi_n(\nu) = \nu$, hence $n \in M_\nu$. □

Another property of the subgroups we are taking into account (that will be useful later) is the following.

Proposition 2.1.4. *Let $\mu \in \mathfrak{m}^*$ and $\nu = i^*(\mu) \in \mathfrak{n}^*$. The isotropy subgroups M_μ, M_ν (with respect to the Coadjoint Action of M on $\mathfrak{m}^*$ and the ψ-action of M on $\mathfrak{n}^*$) satisfy:*

$$M_\mu \subset M_\nu.$$

Proof. Let $m \in M_\mu$: then $Ad^*_{m^{-1}}\mu = \mu$. As a consequence:

$$\psi_m(\nu) \;=\; i^*(Ad^*_{m^{-1}}\mu) \;=\; i^*(\mu) = \nu.$$

□

2.2 Assumptions on the momentum values

We are now ready to state the conditions on the momentum values $\mu \in \mathfrak{m}^*$ (on which we are going to perform reduction) must respect. As usual, in the following definitions, M_ν denotes the isotropy subgroup of $\nu \in \mathfrak{n}^*$ under the ψ-action of M on $\mathfrak{n}^*$.

Definition 2.2.1 (Hypothesis (A)). *Let* $\mu \in \mathfrak{m}^*$. *We say that* μ *satisfies Hypothesis (A) if:*

$$M_{i^*(\mu)} \; = \; N.$$

The next hypothesis is a specialized version of the Stages Hypothesis of [10] which is simpler to work with and it is convenient to our purposes. In accordance with [10] we term it *SSH* which stands for “Special Stages Hypothesis”.

Definition 2.2.2 (Hypothesis (SSH)). *Let* $\mu \in \mathfrak{m}^*$. *We say that* μ *satisfies Hypothesis (SSH) if for any other* $\mu' \in \mathfrak{m}^*$ *such that* $i^*(\mu') = i^*(\mu)$ *there exists* $n \in N$ *such that:*

$$\mu \; = \; Ad^*_{n^{-1}}(\mu').$$

2.3 A pair of Hamiltonian actions and their reduction

Let us consider now the standard Symplectic reduction framework recalled in Appendix A.2, where the Lie group M that acts on the symplectic manifold (P, ω) has $N \trianglelefteq M$ as above. Namely, the ingredients are:

- (P, ω) a symplectic manifold;
- M a Lie group with N normal and abelian regular Lie subgroup;
- $\chi : M \times P \longrightarrow P$ a free, proper and symplectic Lie group action;
- $J_M : P \longrightarrow \mathfrak{m}^*$ an equivariant *momentum map* for the action χ.

Let $\mu \in J_M(P) \subset \mathfrak{m}^*$. We recall from Appendix A.2 that the reduced space (P_μ, ω_μ) is a symplectic manifold where:

$$P_\mu := J_M^{-1}(\mu)/M_\mu,$$

and the symplectic form $\omega_\mu \in \Lambda^2(P_\mu)$ is characterized by the condition:

$$\pi_\mu^* \omega_\mu \; = \; i_\mu^* \omega, \tag{2.5}$$

where:

$$\pi_\mu : J_M^{-1}(\mu) \longrightarrow P_\mu, \quad i_\mu : J_M^{-1}(\mu) \hookrightarrow P,$$

are the orbit projection and inclusion.

In our context we can also consider the restriction of the χ action to N.

Lemma 2.3.1. *The restriction* $\tilde{\chi} := \chi|_{N\times P} : N \times P \longrightarrow P$ *still is a free, proper and symplectic action and it has an equivariant momentum map given by*

$$J_N := i^* \circ J_M : P \longrightarrow \mathfrak{n}^*. \tag{2.6}$$

Proof. The fact that the restriction is free and symplectic follows by definition. Also, let $K \subset P$ be a compact set. Then:

$$\{n \in N \mid P \,\cap\, \chi_n(P) \neq \emptyset\} = \{m \in M \mid P \,\cap\, \chi_m(P) \neq \emptyset\} \cap N,$$

where both the last sets are closed, the first being a compact set in the manifold M (which is Hausdorff), the second by the assumption that N is a regular Lie subgroup of M. So the intersection is a closed set which lies in a compact one, hence it is compact. This gives properness of $\tilde{\chi}$.
Let us prove now that $J_N := i^* \circ J_M$ is a momentum map for this action. Let $v \in \mathfrak{n}$. Our goal is to prove:

$$d\langle J_N(z), v\rangle \;=\; i_{v_P}\omega(z),$$

for every $z \in P$, i.e. that $J_v = \langle J_N, v\rangle \in C^\infty(P)$ is the Hamiltonian function of $v_P \in \mathfrak{X}(P)$ the *infinitesimal generator* of the $\tilde{\chi}$ action. What we know is that:

$$d\langle J_M(z), \eta\rangle \;=\; i_{\eta_P}\omega(z)$$

holds for every $\eta \in \mathfrak{m}$ since J_M is a momentum map. By basic Lie group theory the exponential maps of the groups N and M are related by $exp_M|_{\mathfrak{n}} \;=\; exp_N$. We use this fact in the computation of the infinitesimal vector fields:

$$v_P(z) \;=\; \left.\frac{d}{dt}\right|_{t=0} \tilde{\chi}_{exp_N(tv)}(z) \;=\; \left.\frac{d}{dt}\right|_{t=0} \chi_{exp_M(i(tv))}(z) \;=\; i(v)_P(z).$$

As a consequence we get that:

$$d\langle J_N(z), v\rangle \;=\; d\langle i^*(J_M(z)), v\rangle \;=\; d\langle J_M(z), i(v)\rangle \;=\; i(v)_P(z) \;=\; v_P(z),$$

which is what we wanted.
Let us finally prove that J_N is equivariant. We are actually able to prove more: χ is M-equivariant with respect to the action ψ. Indeed, for $m \in M$ and $z \in P$:

$$J_N(\chi_m(z)) \;=\; i^*(J_M(\chi_m(z))) \;=\; i^*(Ad^*_{m^{-1}}(J_M(z))) \;=$$

$$\psi_m(i^*(J_M(z))) \;=\; \psi_m(J_N(z)).$$

This gives in particular N-equivariance in view of Proposition 2.1.2. □

Remark 2.3.1. As a matter of fact, J_N is the unique map that makes the following diagram commute:

$$\begin{tikzcd}
P \arrow[r, "J_M"] \arrow[dr, "J_N"'] & \mathfrak{m}^* \arrow[d, "i^*"] \\
 & \mathfrak{n}^*.
\end{tikzcd}$$

The above lemma allows us to consider the symplectic reduction of P by the $\tilde{\chi}$ action of N following the framework of Appendix A.2. Given that N is abelian, the Coadjoint action of N on $\mathfrak{n}^*$ is trivial (see Proposition A.1.2) and hence $N_\nu = N$ for every $\nu \in \mathfrak{n}^*$. Therefore, for $\nu \in J_N(P) \subset \mathfrak{n}^*$, the symplectic reduced space is given by (P_ν, ω_ν) with:

$$P_\nu \;=\; J_N^{-1}(\nu)/N,$$

and the symplectic form $\omega_\nu \in \Lambda^2(P_\nu)$ is characterized by the condition:

$$\pi_\nu^* \, \omega_\nu = i_\nu^* \, \omega, \tag{2.7}$$

where:

$$\pi_\nu : J_N^{-1}(\nu) \longrightarrow P_\nu, \quad i_\nu : J_N^{-1}(\nu) \hookrightarrow P \tag{2.8}$$

are the orbit projection and inclusion map.

In the Main Theorem of the thesis we compare the two reduced spaces P_μ and P_ν for $\mu \in J_M(P)$ and $\nu = i^*(\mu)$. Roughly speaking, whenever such μ satisfies Hypothesis (A) 2.2.1 and (SSH) 2.2.2, then P_μ is symplectomorphic to P_ν.

2.4 Main Theorem

We are now ready to formally state the Main Theorem of the thesis and prove it.

Theorem 2.4.1 (Main Theorem). *Let M be a Lie group with abelian and normal regular Lie subgroup N. Suppose M acts freely, properly and by symplectomorphisms on a symplectic manifold (P, ω) with equivariant momentum map $J_M : P \longrightarrow \mathfrak{m}^*$. Let $\mu \in J_M(P) \subset \mathfrak{m}^*$ and suppose μ satisfies hypotheses (A) and (SSH) given in Section 2.2. Then the symplectic reduced spaces (P_μ, ω_μ) and (P_ν, ω_ν) with $\nu = i^*(\mu)$ are symplectomorphic.*

2.4.1 Proof of the Main Theorem.

The proof of the theorem is organized as follows. We first set what the symplectomorphism F between the two manifolds is and we prove it is a bijective symplectic map. After that we propose what is its inverse ϕ and we prove it is a smooth map. The whole construction is summarized in the following commuting diagram whose undefined maps will be specified as we go along.

$$\begin{tikzcd}
J_M^{-1}(\mu) \arrow[r, "j_\mu"] \arrow[rr, bend left, "i_\mu"] \arrow[d, "\pi_\mu"] \arrow[dr, "\pi_\nu \circ j_\mu"] & J_N^{-1}(\nu) \arrow[r, "i_\nu"] \arrow[d, "\pi_\nu"] & P \arrow[r, "J_M"] \arrow[d, "J_N"] & \mathfrak{m}^* \arrow[dl, "i^*"] \\
P_\mu \arrow[r, shift left, "F"] & P_\nu \arrow[l, shift left, "\phi"] & \mathfrak{n}^* &
\end{tikzcd}$$

Step 1: the inclusion map j_μ.

We recall from Lemma 2.3.1 that the momentum map for the restricted action, $\tilde{\chi}$, of N on P is $J_N = i^* \circ J_M$. As a consequence we have:

$$J_M^{-1}(\mu) \ \subset \ J_N^{-1}(i^*(\mu)) \ = \ J_N^{-1}(\nu).$$

Call $j_\mu : J_M^{-1}(\mu) \longrightarrow J_N^{-1}(\nu)$ the inclusion map. We claim j_μ is a smooth embedding. The technique we use here is to recover $J_M^{-1}(\mu)$ as a level set of a regular value of a smooth function with $J_N^{-1}(\nu)$ as a domain. Denote by $\mathfrak{n}^\circ \subset \mathfrak{m}^*$ the annihilator of $\mathfrak{n} \subset \mathfrak{m}$ and consider the map $C_\mu : J_N^{-1}(\nu) \longrightarrow \mathfrak{n}^\circ$, where:

$$C_\mu(z) := J_M(z) - \mu, \quad z \in J_N^{-1}(\nu).$$

First of all C_μ is well defined as a map into $\mathfrak{n}^\circ$ since if we take $\xi \in \mathfrak{n}$:

$$\begin{aligned}\langle C_\mu(z), i(\xi)\rangle &= \langle J_M(z), i(\xi)\rangle - \langle \mu, i(\xi)\rangle = \langle i^*(J_M(z)), \xi\rangle - \langle i^*(\mu), \xi\rangle \\ &= \langle J_N(z), \xi\rangle - \langle \nu, \xi\rangle = 0.\end{aligned}$$

We trivially have $J_M^{-1}(\mu) = C_\mu^{-1}(0)$ and $T_z C_\mu = T_z J_M|_{J_N^{-1}(\nu)}$. We should check, as we claimed, that $T_z C_\mu$ is surjective for every $z \in J_M^{-1}(\mu)$. Fix $z \in J_M^{-1}(\mu)$ and let $V := Im(T_z C_\mu) \leq \mathfrak{n}^\circ$. We wish to show that $V = \mathfrak{n}^\circ$. We do this as follows. Consider a direct complement $\mathfrak{k}$ of $\mathfrak{n}$ so that $\mathfrak{m} = \mathfrak{n} \oplus \mathfrak{k}$. There is a natural identification $\mathfrak{n}^\circ \simeq \mathfrak{k}^*$ and hence also $(\mathfrak{n}^\circ)^* \simeq (\mathfrak{k}^*)^* \simeq \mathfrak{k}$. In view of this, the subset:

$$W \ := \ \{\xi \in \mathfrak{k} \mid \langle v, \xi\rangle = 0 \quad \forall\, v \in V\},$$

is identified with the annihilator of V within $(\mathfrak{n}^\circ)^*$. Therefore the condition $V = \mathfrak{n}^\circ$ holds if and only if $W = \{0\}$.

Let $a \in T_z J_N^{-1}(\nu)$ and $c : (-\varepsilon, \varepsilon) \longrightarrow J_N^{-1}(\nu)$ a smooth curve such that $c(0) = z$, $c'(0) = a$. Let $\xi \in W$. Then since $T_z C_\mu(a) \in V$:

$$\begin{aligned} 0 \;&=\; \langle T_z C_\mu(a), \xi \rangle = \left\langle \left. \frac{d}{dt} \right|_{t=0} C_\mu(c(t)), \xi \right\rangle \\ &=\; \left. \frac{d}{dt} \right|_{t=0} \langle J_M(c(t)), \xi \rangle \;=\; \left. \frac{d}{dt} \right|_{t=0} (J_M)_\xi(c(t)). \end{aligned}$$

Given that $(J_M)_\xi \in C^\infty(P)$ is the Hamiltonian function of $\xi_P \in \mathfrak{X}(P)$, we conclude that:

$$\omega(z)(\xi_P(z), a) \;=\; 0, \quad \forall a \in T_z J_N^{-1}(\nu).$$

This implies $\xi_P(z) \in (T_z J_N^{-1}(\nu))^\omega$ and therefore by item *(iii)* of the Reduction Lemma A.2.3:

$$\xi_P(z) \in T_z(\chi_N(z)).$$

In summary, we conclude that the infinitesimal generator ξ_P at z is tangent to the N-orbit through z. This implies that $\xi \in \mathfrak{n}$ and therefore we conclude that $\xi \in \mathfrak{k} \cap \mathfrak{n} = \{0\}$, which leads to $W = \{0\}$ as required.

Step 2: the symplectomorphism F.

Let $z \in J_M^{-1}(\mu)$ and $[z]_{M_\mu} = \pi_\mu(z) \in P_\mu$. We define $F : P_\mu \longrightarrow P_\nu$ by:

$$F([z]_{M_\mu}) = \pi_\nu \circ j_\mu(z), \tag{2.9}$$

where j_μ is the inclusion map as in *Step 1.*

We now show that:

- *(a)* F is a well defined map.
- *(b)* F is bijective.
- *(c)* F is smooth.

(a). Let $[z]_{M_\mu} = [z']_{M_\mu}$. In view of Proposition 2.1.4 and Hypothesis (A) we have $M_\mu \subset M_\nu = N$. Hence we may write $z' = \chi_n(z)$ with $n \in N$ which implies:

$$\pi_\nu \circ j_\mu(z) \;=\; \pi_\nu \circ j_\mu(z').$$

In other words, F does not depend on the representative of the class we choose, hence it is well defined.

(b). F is an injective map. Let $z, z' \in J_M^{-1}(\mu)$ and assume $F([z]_{M_\mu}) = F([z']_{M_\mu})$,

i.e. $\pi_\nu(z) = \pi_\nu(z')$. So there is $n \in N \subset M$ such that $\chi_n(z) = z'$. We need to check that $\pi_\mu(z) = \pi_\mu(z')$. By item *(ii)* of the Reduction Lemma A.2.3:

$$z' \in J_M^{-1}(\mu) \cap (M \cdot z) \;=\; M_\mu \cdot z,$$

as desired.

F is surjective. Let $z \in J_N^{-1}(\nu)$ and take for granted that:

$$(N \cdot z) \cap J_M^{-1}(\mu) \neq \emptyset. \tag{2.10}$$

In this case there exists a representative $z' \in J_M^{-1}(\mu)$ of $[z]_N \in P_\nu$ such that:

$$F([z']_{M_\mu}) \;=\; \pi_\nu \circ j_\mu(z') \;=\; [z']_N \;=\; [z]_N.$$

Therefore the surjectivity of F may be established by showing that (2.10) holds for every $z \in J_N^{-1}(\nu)$. To prove this, let $\tilde{\mu} = J_M(z) \in \mathfrak{m}^*$. Notice μ, $\tilde{\mu}$ belong to the same fiber since, by (2.6):

$$\begin{aligned} i^*(\mu) \;&=\; (i^* \circ J_M)(z) \;=\; J_N(z) \\ &=\; \nu \;=\; i^*(\mu), \end{aligned}$$

so that Hypothesis (SSH) 2.2.2 applies: there exists $n \in N$ such that $\mu = Ad^*_{n^{-1}}(\tilde{\mu})$. Set $z' := \chi_n(z)$. Then on the one hand:

$$z' \in N \cdot z.$$

On the other hand, by equivariance of J_M:

$$\begin{aligned} J_M(z') \;&=\; J_M(\chi_n(z)) \;=\; Ad^*_{n^{-1}}(J_M(z)) \\ &=\; Ad^*_{n^{-1}}(\tilde{\mu}) \;=\; \mu, \end{aligned} \tag{2.11}$$

so $z' \in J_M^{-1}(\mu)$. Therefore, $z' \in (N \cdot z) \cap J_M^{-1}(\mu)$, showing that the intersection is non-empty.

(c). The definition (2.9) of F implies:

$$F \circ \pi_\mu \;=\; \pi_\nu \circ j_\mu. \tag{2.12}$$

Now, π_μ is a surjective submersion and $\pi_\nu \circ j_\mu$ is smooth. Using Lemma A.1.3 from differential geometry we conclude that F is smooth.

Step 3: F is symplectic

We now take into account the symplectic structures of P_μ and P_ν given by the *2-forms* ω_μ, ω_ν characterized by (2.5) and (2.7). We claim $F : (P_\mu, \omega_\mu) \longrightarrow (P_\nu, \omega_\nu)$ is symplectic, i.e.

$$F^*\omega_\nu \;=\; \omega_\mu.$$

Given that (2.5) characterises ω_μ it suffices to show:

$$\pi_\mu^* \, F^* \, \omega_\nu \;=\; i_\mu^* \omega. \tag{2.13}$$

Using (2.12):

$$\begin{aligned} \pi_\mu^* \, F^* \, \omega_\nu \;&=\; (F \circ \pi_\mu)^* \omega_\nu \;=\; (\pi_\nu \circ j_\mu)^* \omega_\nu \\ &=\; j_\mu^* \, \pi_\nu^* \, \omega_\nu \;=\; j_\mu^* \, i_\nu^* \, \omega, \end{aligned}$$

where we used (2.7) in the last equality. Now, the following inclusion diagram clearly commutes:

$$\begin{tikzcd} J_M^{-1}(\mu) \arrow[r, "j_\mu"] \arrow[dr, "i_\mu"'] & J_N^{-1}(\nu) \arrow[d, "i_\nu"] \\ & P \end{tikzcd}$$

so that:

$$j_\mu^* \, i_\nu^* \, \omega \;=\; (i_\nu \circ j_\mu)^* \omega \;=\; i_\mu^* \omega,$$

which is right hand side in (2.13).

Step 4: the inverse ϕ.

We first give the definition of what we propose as the inverse of the map F. Let $[z]_N \in P_\nu$. If $\mu' := J_M(z)$, then, applying (2.6) as above, we conclude that $i^*(\mu') = i^*(\mu) = \nu$. By (SSH) Hypothesis (see 2.2.2) there exists $n \in N$ such that $\mu = Ad^*_{n^{-1}}(\mu')$, and by equivariance of J_M we get $J_M(\chi_n(z)) = \mu$. We define $\phi : P_\nu \longrightarrow P_\mu$ by:

$$\phi([z]_N) \;=\; \pi_\mu(\chi_n(z)). \tag{2.14}$$

Similarly to what we did for F, we will prove that:

- *(a)* ϕ is well defined;
- *(b)* ϕ is the inverse map of F;
- *(c)* ϕ is smooth.

(a). Let $z' \in J_N^{-1}(\nu)$ such that $[z']_N = [z]_N$. There exists an $\tilde{n} \in N$ such that $z' = \chi_{\tilde{n}}(z)$. We repeat the same construction for z' as we did for z in the definition of ϕ. Let us consider $\tilde{\mu} = J_M(z')$. Again, since $i^*(\mu) = i^*(\tilde{\mu}) = \nu$, by (SSH) Hypothesis 2.2.2 there exists $\hat{n} \in N$ such that $\mu = Ad^*_{\hat{n}^{-1}}(\tilde{\mu})$; as a consequence:

$$\phi([z']_N) \;=\; \pi_\mu(\chi_{\hat{n}}(z')) \;=\; \pi_\mu(\chi_{\hat{n}} \circ \chi_{\tilde{n}}(z)) \;=\; \pi_\mu(\chi_{\hat{n}\tilde{n}}(z)).$$

We must check that $\phi([z]_N) = \phi([z']_N)$, which is equivalent to the condition:

$$\pi_\mu(\chi_n(z)) \;=\; \pi_\mu(\chi_{\hat{n}\tilde{n}}(z)).$$

In other words, we must prove the existence of $m \in M_\mu$ such that:

$$\chi_n(z) = \chi_{m\hat{n}\tilde{n}}(z).$$

Clearly this holds if and only if $z = \chi_{n^{-1}m\hat{n}\tilde{n}}(z)$. By freeness of χ this is satisfied if and only if $n^{-1}m\hat{n}\tilde{n} = e$, and so in order to conclude the proof we need to check that $m = n\,\tilde{n}^{-1}\,\hat{n}^{-1} \in M_\mu$. But this is true because, by equivariance of J_M and the definition of n and $\hat{n}$:

$$\chi_n(z), \chi_{\hat{n}}(z') \in J_M^{-1}(\mu).$$

On the other hand, we clearly have $\chi_n(z), \chi_{\hat{n}}(z') \;=\; \chi_{\hat{n}\tilde{n}}(z) \;\in\; \chi_N(z) \;\subset\; \chi_M(z)$. Therefore, by item *(ii)* of Reduction Lemma A.2.3:

$$\chi_n(z), \chi_{\hat{n}}(z') = \chi_{n\hat{\tilde{n}}}(z) \in J_M^{-1}(\mu) \cap \chi_M(z) \;=\; \chi_{M_\mu}(z).$$

Hence,

$$n,\ \hat{n}\tilde{n} \in M_\mu \;\;\Longrightarrow\;\; m = n\,\tilde{n}^{-1}\,\hat{n}^{-1} \in M_\mu,$$

as we wanted.

(b). The fact that ϕ is the inverse of F easily follows by the definitions of the two maps. Fix $z \in J_N^{-1}(\nu)$ and consider an element $n \in N$ that works out the definition of $\phi([z]_N)$ as $\pi_\mu(\chi_n(z))$. Then, using (2.12):

$$\begin{aligned} F \circ \phi([z]_N) \;&=\; F(\phi(\pi_\nu(z))) \;=\; F \circ \pi_\mu \circ \chi_n(z) \\ &=\; \pi_\nu \circ j_\mu(\chi_n(z)) \;=\; \pi_\nu \circ \chi_n(z) \\ &=\; [\chi_n(z)]_N \;=\; [z]_N, \end{aligned}$$

whatever $n \in N$ is.

(c). The most delicate part in the proof of the smoothness of ϕ is how to encode the smoothness of the assignment $z \mapsto n$ (see the definition of $\phi[(z)_N]$). In order to do so, we will pass through a section on an N_μ-Bundle (the language of this is developed in Appendix A.3).

Let us take $z_0 \in J_N^{-1}(\nu)$ and consider any local section:

$$\Gamma : U \longrightarrow \pi_\nu^{-1}(U),$$

where $U \subset P_\nu$ is a neighborhood of $[z_0]_N$, i.e. Γ is a smooth map such that $\pi_\nu \circ \Gamma = id_U$. The map:

$$J_M \circ \Gamma : U \longrightarrow \mathfrak{m}^*$$

is smooth and we claim it takes values on the N-Coadjoint Orbit $Ad^*_N(\mu)$. Actually we see that $J_M|_{J_N^{-1}(\nu)}$ does, which is enough. Indeed, let $z \in J_N^{-1}(\nu)$. Then $i^*(J_M(z)) = \nu$ by (2.6) and so by (SSH) Hypothesis 2.2.2 there exists an $n \in N$ such that $Ad^*_{n^{-1}}(J_M(z)) = \mu$. So our claim holds.

Before we continue with the proof we make a remark on the notation. As we see from (A.2), the notations $(Ad)^*$ and Ad^* mean two different things, which are linked by the following equalities:

$$(Ad)^*(n,\mu) \;=\; (Ad)^*_n(\mu) \;=\; Ad^*_{n^{-1}}(\mu) \;=\; Ad^*(n^{-1},\mu).$$

We will use both notations in the following.

By the Orbit identification Theorem A.1.5 we may identify the Coadjoint orbit $Ad^*_N(\mu)$ with the quotient group N/N_μ, where N_μ is the isotropy subgroup of μ in N with respect to the N-Coadjoint action. This identification is done via the diffeomorphism:

$$B_\mu : N/N_\mu \xrightarrow{\sim} Ad^*_N(\mu), \quad B_\mu(nN_\mu) \;=\; Ad^*(n,\mu), \quad \forall\, n \in N.$$

Let us call $\overline{J_M \circ \Gamma} := B_\mu^{-1} \circ (J_M \circ \Gamma)$, which is the map that represents $J_M \circ \Gamma$ in the identification above. Precisely, we have the following diagram that commutes:

$$\begin{tikzcd}
U \arrow[r, "J_M\circ\Gamma"] \arrow[dr, "\overline{J_M\circ\Gamma}"'] & Ad^*_N(\mu) \\
 & N/N_\mu. \arrow[u, "B_\mu"]
\end{tikzcd} \tag{2.15}$$

Now by theory of Principal Bundles we can interpret $\tau : N \longrightarrow N/N_\mu$ as a Principal N_μ-Bundle (see Corollary A.3.2). The local lift existence Theorem A.3.3 then ensures the existence of $V \subset U$, neighborhood of $[z_0]_N$, and $(J_M \circ \Gamma)^\ell : V \longrightarrow N$ smooth lift map of $\overline{J_M \circ \Gamma}$. Precisely, last diagram (2.15) is completed, obtaining:

$$\begin{tikzcd}
V \arrow[r, "J_M\circ\Gamma"] \arrow[d, "(J_M\circ\Gamma)^\ell"'] \arrow[dr, "\overline{J_M\circ\Gamma}"'] & Ad^*_N(\mu) \\
N \arrow[r, "\tau"] & N/N_\mu. \arrow[u, "B_\mu"]
\end{tikzcd} \tag{2.16}$$

Also, notice that if we denote by $((Ad)^*)^\mu : N \longrightarrow Ad^*_N(\mu)$ the orbit map at μ, then $((Ad)^*)^\mu = B_\mu \circ \tau$.

Consider the smooth map:

$$\sigma : V \longrightarrow N, \quad \sigma([z]_N) \;=\; ((J_M \circ \Gamma)^\ell([z]_N))^{-1},$$

where $^{-1}$ is the group inverse. Use σ to construct the map:

$$\alpha : V \longrightarrow P_\mu, \quad \alpha([z]_N) \;=\; \pi_\mu(\chi_{\sigma([z]_N)}(\Gamma([z]_N))),$$

which is clearly smooth being the composition of smooth maps. We claim that:

$$\phi([z]_N) = \alpha([z]_N) \quad \forall [z]_N \in V, \tag{2.17}$$

which implies that ϕ is smooth at $[z_0]_N$ and hence everywhere on P_ν. The rest of the proof consists of showing that (2.17) holds.

On the one hand, given that $J_M \circ \Gamma([z]_N) \in Ad^*_N(\mu)$, there exists an $n \in N$ such that:

$$(Ad)^*(n, J_M \circ \Gamma([z]_N)) \;=\; \mu. \tag{2.18}$$

Also, by commutativity of the diagram (2.16), we have:

$$\begin{aligned} J_M \circ \Gamma([z]_N) \;&=\; B_\mu \circ \tau \circ (J_M \circ \Gamma)^\ell([z]_N) \\ &=\; ((Ad)^*)^\mu \circ (J_M \circ \Gamma)^\ell([z]_N) \\ &=\; (Ad)^*((J_M \circ \Gamma)^\ell([z]_N), \mu). \end{aligned}$$

Therefore:

$$\begin{aligned} \mu \;&=\; (Ad)^*(((J_M \circ \Gamma)^\ell([z]_N))^{-1}, J_M \circ \Gamma([z]_N)) \\ &=\; (Ad)^*(\sigma([z]_N), J_M \circ \Gamma([z]_N)). \end{aligned}$$

Combining the above expression for μ with (2.18) yields:

$$\mu \;=\; (Ad)^*(\sigma([z]_N)n^{-1}, \mu),$$

which implies that:

$$\tilde{n} \;:=\; \sigma([z]_N)n^{-1} \in N_\mu \subset M_\mu. \tag{2.19}$$

As a consequence we may write:

$$\begin{aligned} \alpha([z]_N) \;&=\; \pi_\mu(\chi_{\sigma([z]_N)}(\Gamma([z]_N))) \quad \text{(by definition of } \alpha) \\ &=\; \pi_\mu(\chi_{\tilde{n}n}(\Gamma([z]_N))) \quad \text{(by equation (2.19))} \\ &=\; \pi_\mu(\chi_{\tilde{n}}(\chi_n(\Gamma([z]_N)))) \quad \text{(by action properties)} \\ &=\; \pi_\mu(\chi_n(\Gamma([z]_N))) \quad (\text{ since } \tilde{n} \in M_\mu)). \end{aligned} \tag{2.20}$$

On the other hand, $\Gamma([z]_N)$ is a representative of $[z]_N$ in $J_N^{-1}(\nu)$, since Γ is a local section of π_ν. Also, the choice of n we made before equation (2.18) is exactly the one that realizes the (SSH) Hypothesis 2.2.2 with the pair $\mu, J_M \circ \Gamma([z]_N)$, since (2.18) is equivalently written as:

$$Ad^*_{n^{-1}}(J_M \circ \Gamma([z]_N)) \;=\; \mu.$$

As a consequence, by definition of ϕ, we have:

$$\phi([z]_N) \;=\; \pi_\mu(\chi_n(\Gamma([z]_N))). \tag{2.21}$$

Comparing (2.20), (2.21) we get (2.17), as we wanted.

This finishes the proof of the Main Theorem 2.4.1.

Chapter 3

Semidirect product Lie groups

In this chapter we apply our Main Theorem 2.4.1 to specific cases. In Section 3.1 the goal is to examine the case where the Lie group M is a semidirect product of a Lie group G and a vector space V. In Section 3.2 we characterize the momentum values $\mu \in \mathfrak{m}^*$ that satisfy the Hypotheses (A) 2.2.1 and (SSH) 2.2.2 of the theorem in the semidirect product setting. In Section 3.3 we focus on the case in which the symplectic manifold $P = T^*M$ and the action of M on T^*M is by the cotangent lift of left multiplication, where M is the semidirect product Lie group $M = G \ltimes N$, with N abelian. We will determine conditions ensuring that the Coadjoint orbits in $\mathfrak{m}^*$ are symplectomorphic to the cotangent bundle T^*G.

3.1 Preliminaries

In this section we examine the case where the Lie group M in Theorem 2.4.1 is the semidirect product:

$$M \; = \; G \ltimes V,$$

where G is a Lie group itself acting linearly on the $\mathbb{R}$-vector space V by an action:

$$\phi : G \times V \longrightarrow V.$$

As a consequence we can think of G as a regular matrix group in $GL(V)$, precisely we have a Lie group representation:

$$\alpha : G \longrightarrow GL(V), \quad \alpha(g)(v) \; = \; \phi_g(v). \tag{3.1}$$

This case fits the framework of main theorem because we think of the vector space V as a group with $+$ as the group operation. Also, in a semidirect product, the "acted" subgroup is normal by definition. Thus V plays the role of the normal, abelian, regular subgroup N of M.

We recall that the group structure of M is given by endowing $M = G \times V = \{(g, v) \mid g \in G, v \in V\}$, with the product:

$$(g_1, v_1) \cdot_M (g_2, v_2) \;=\; (g_1 \cdot_G g_2, v_1 + \phi_{g_1} v_2).$$

From now on we will omit the group product notation. Since M coincides with $G \times V$ as a manifold, then every tangent space at a point is the direct sum of the tangent spaces of the two single manifolds. This is true in particular at the identity element:

$$\mathfrak{m} \;=\; T_{e_M} M \;=\; T_{e_G} G \oplus T_0 V \;=\; \mathfrak{g} \oplus V.$$

As a consequence an element α in the Lie algebra $\mathfrak{m}$ is uniquely written as $\alpha = \xi + u$, for some unique $\xi \in \mathfrak{g}$, $u \in V$. We will write $\alpha = (\xi, u)$. Our goal now is to derive a formula for the Adjoint and Coadjoint representations and the Lie brackets in this context.

Let $(g, v), (k, l) \in M$. Then the conjugation of (k, l) by (g, v) is:

$$\begin{aligned} C_{(g,v)}(k, l) \;&=\; (g, v)(k, l)(g, v)^{-1} \;=\; (g, v)(k, l)(g^{-1}, \phi_{g^{-1}}(v^{-1})) \\ &=\; (g, v)(k, l)(g^{-1}, \phi_{g^{-1}}(-v)) \;=\; (g, v)(kg^{-1}, l + \phi_k \phi_{g^{-1}}(-v)) \\ &=\; (gkg^{-1}, v + \phi_g(l + \phi_k \phi_{g^{-1}}(-v))) \;=\; (gkg^{-1}, v + \phi_g(l) - \phi_{gkg^{-1}}(v))), \end{aligned}$$

where in the last equality we used the fact that ϕ is linear as an action. As a consequence the Adjoint action (as defined in Appendix A.1.1) on $(\xi, u) \in \mathfrak{m}$ is given by:

$$Ad_{(g,v)}(\xi, u) \;=\; T_{(e_G,0)} C_{(g,v)}(\xi, u) \;=\; (Ad^G_g(\xi), \phi'_g(u) - \rho_v(Ad^G_g(\xi))), \tag{3.2}$$

where Ad^G is the Adjoint action of the group G, $\phi'_g = T_0 \phi_g : V \longrightarrow V$ and, for $v \in V$, the map:

$$\rho_v : \mathfrak{g} \longrightarrow V \tag{3.3}$$

is the infinitesimal action of ξ on v, namely:

$$\rho_v(\xi) \;=\; \left.\frac{d}{dt}\right|_{t=0} \phi_{\exp(\xi t)}(v). \tag{3.4}$$

We will shortly abbreviate (3.2) as:

$$Ad_{(g,v)}(\xi, u) \;=\; (g\xi, gu - \rho_v(g\xi)).$$

The dual Lie algebra is $\mathfrak{m}^* = \mathfrak{g}^* \times V^*$ (notice we cannot express it as a direct sum since $\mathfrak{g}^*$ and V^* are not subspaces of $\mathfrak{m}^*$). Let $\mu = (\gamma, a) \in \mathfrak{m}^*$. By definition of Coadjoint action (A.2), $Ad^*_{(g,v)^{-1}}(\gamma, a)$ is the element in $\mathfrak{m}^*$ such that for every $(\xi, u) \in \mathfrak{m}$:

$$\begin{aligned} \langle Ad^*_{(g,v)^{-1}}(\gamma, a), (\xi, u)\rangle &= \langle(\gamma, a), Ad_{(g,v)^{-1}}(\xi, u)\rangle \\ &= \langle(\gamma, a), (g^{-1}\xi, g^{-1}u - \rho_{-v}(g^{-1}\xi))\rangle \\ &= \langle(g\gamma, ga), (\xi, u + \rho_v(\xi))\rangle \qquad (3.5) \\ &= \langle(g\gamma, ga), (\xi, u)\rangle + \langle(g\gamma, ga), (0, \rho_v(\xi))\rangle \\ &= \langle(g\gamma, ga), (\xi, u)\rangle + \langle g\gamma, 0\rangle_{\mathfrak{g}} + \langle ga, \rho_v(\xi)\rangle_V. \end{aligned}$$

Denote by $\rho_v^* : V^* \longrightarrow \mathfrak{g}^*$ the dual map of ρ_v defined by (3.3). Then, for every $\xi \in \mathfrak{g}$:

$$\langle ga, \rho_v(\xi)\rangle_V = \langle \rho_v^*(ga), \xi\rangle_{\mathfrak{g}}.$$

Therefore, the last term of (3.5) is equal to:

$$\begin{aligned} &\langle(g\gamma, ga), (\xi, u)\rangle + \langle \rho_v^*(ga), \xi\rangle_{\mathfrak{g}} + \langle 0, u\rangle_V \\ &\qquad = \langle(g\gamma, ga), (\xi, u)\rangle + \langle(\rho_v^*(ga), 0), (\xi, u)\rangle \\ &\qquad = \langle(g\gamma + \rho_v^*(ga), ga), (\xi, u)\rangle. \end{aligned}$$

By arbitrariness of $(\xi, u) \in \mathfrak{m}$ we conclude that:

$$Ad^*_{(g,v)^{-1}}(\gamma, a) = (g\gamma + \rho_v^*(ga), ga). \qquad (3.6)$$

3.2 Main Theorem in semidirect products

Recall that we have the surjective linear map $i^* : \mathfrak{m}^* \longrightarrow \mathfrak{n}^* = V^*$ defined in (2.1). Here it acts precisely as a standard projection:

$$i^* : \mathfrak{g}^* \times V^* \longrightarrow V^*, \quad i^*(\gamma, a) = pr_2(\gamma, a) = a. \qquad (3.7)$$

Proposition 3.2.1. *The action $\psi : M \times V^* \longrightarrow V^*$ defined by* (2.2) *is given by:*

$$\psi_{(g,v)}(a) = ga,$$

where $(g, v) \in G \times V = M$ and $a \in V^*$.

Proof. For an $a \in V^*$, an element μ in the fiber $(i^*)^{-1}(a)$ is $\mu = (\gamma, a)$, for any $\gamma \in \mathfrak{g}^*$. As a consequence:

$$\begin{aligned} \psi_{(g,v)}(a) &= i^*(Ad^*_{(g,v)^{-1}}(\gamma, a)) \\ &= i^*((g\gamma + \rho_v^*(ga), ga)) \\ &= ga. \end{aligned}$$

□

The following theorem considerably simplifies the verification of the Hypotheses (A) 2.2.1 and (SSH) 2.2.2 in the context above by a simple algebraic computation.

Theorem 3.2.2. *Consider the semidirect product Lie group $M = G \ltimes V$, with dual Lie algebra $\mathfrak{m}^* = \mathfrak{g}^* \times V^*$. Let $a \in V^*$. If the isotropy subgroup of a with respect to the induced linear action of G on V^* is trivial ($G_a = \{e_G\}$), then both Hypotheses (A) 2.2.1 and (SSH) 2.2.2 hold for $\mu = (\gamma, a) \in \mathfrak{m}^*$ (for any $\gamma \in \mathfrak{g}^*$). On the other hand, if $G_a \neq \{e_G\}$ then hypothesis (A) does not hold.*

Proof. By Proposition 3.2.1 we have:

$$M_a = \{(g,v) \in G \times V \mid ga = a\}.$$

Hence we have:

$$\begin{aligned} M_{i^*(\gamma,a)} &= M_a \\ &= \{(g,v) \in G \times V \mid g \in G_a\} \\ &= G_a \times V. \end{aligned}$$

In our setting $V = N$, so Hypothesis (A) holds if and only if $G_a = \{e_G\}$.

To complete the proof we shall use the following lemma.

Lemma 3.2.3. *The hypothesis $G_a = \{e_G\}$ implies:*

$$\{\rho_v^*(a) \mid v \in V\} = \mathfrak{g}^*. \tag{3.8}$$

Let $\mu = (\gamma, a)$, $\mu' = (\gamma', a') \in \mathfrak{m}^*$ be such that $i^*(\mu) = i^*(\mu')$. By (3.7) we conclude $a = a'$. Moreover, by (3.6) we have:

$$Ad^*_{(e_G,v)^{-1}}(\gamma, a) = (\gamma + \rho_v^*(a), a).$$

So by the Lemma we conclude the existence of a certain $\tilde{v} \in V$ such that $\gamma + \rho_{\tilde{v}}^*(a) = \gamma'$ and hence:

$$Ad^*_{(e_G,\tilde{v})^{-1}}\mu = \mu',$$

so Hypothesis (SSH) holds. This finishes the proof of the theorem. □

Proof (of Lemma 3.2.3). We take into account the G-representation α from (3.1). α, being a Lie group homomorphism has tangent map:

$$\hat{\alpha} = T_e\alpha : \mathfrak{g} \longrightarrow \mathfrak{gl}(V),$$

which is a Lie algebra homomorphism. Now let $v \in V$. In view of (3.4), for every $\xi \in \mathfrak{g}$:

$$\begin{aligned} \rho_v(\xi) &= \left.\frac{d}{dt}\right|_{t=0} \phi_{\exp(t\xi)}(v) \\ &= \left.\frac{d}{dt}\right|_{t=0} \alpha(\exp(t\xi))(v) \\ &= \hat{\alpha}(\xi)(v), \end{aligned}$$

and so for every $a \in V^*$:

$$\begin{aligned} \langle \rho_v^*(a), \xi \rangle_{\mathfrak{g}} &= \langle a, \rho_v(\xi) \rangle_V \\ &= \langle a, \hat{\alpha}(\xi)(v) \rangle_V \\ &= \langle (\hat{\alpha}(\xi))^*(a), v \rangle_V . \end{aligned} \tag{3.9}$$

The induced linear action on V^*, $\tilde{\phi} : G \times V^* \longrightarrow V^*$, works as follows:

$$\begin{aligned} \langle \tilde{\phi}_g(a), v \rangle_V &= \langle a, \phi_{g^{-1}}(v) \rangle_V \\ &= \langle a, \alpha(g^{-1})(v) \rangle_V \\ &= \langle a, (\alpha(g))^{-1}(v) \rangle_V \\ &= \langle ((\alpha(g))^{-1})^*(a), v \rangle_V , \end{aligned}$$

so we have:

$$\tilde{\phi}_g(a) = ((\alpha(g))^{-1})^*(a). \tag{3.10}$$

Let again $\xi \in \mathfrak{g}$. The infinitesimal generator of $\tilde{\phi}$ associated to ξ at $a \in V^*$ is $\xi_{V^*}(a) \in V^*$ such that $\forall\, v \in V$:

$$\begin{aligned} \langle \xi_{V^*}(a), v \rangle_V &= \left.\frac{d}{dt}\right|_{t=0} \langle \tilde{\phi}_{\exp(t\xi)}(a), v \rangle_V \quad \text{(using (3.10))} \\ &= \left.\frac{d}{dt}\right|_{t=0} \langle \alpha(\exp(-\xi t))^*(a), v \rangle_V \\ &= \langle a, \left.\frac{d}{dt}\right|_{t=0} \alpha(\exp(-t\xi))(v) \rangle_V \\ &= \langle a, \hat{\alpha}(-\xi)(v) \rangle_V \\ &= \langle (\hat{\alpha}(-\xi))^*(a), v \rangle_V , \end{aligned}$$

so we have explicitly:

$$\xi_{V^*}(a) = -(\hat{\alpha}(\xi))^*(a). \tag{3.11}$$

Now consider $a \in V^*$ as in the statement. Suppose $\xi \in \mathfrak{g}$ is such that $\langle \rho_v^*(a), \xi \rangle_{\mathfrak{g}} = 0$ for every $v \in V$. Then by (3.9):

$$\begin{aligned} 0 &= \langle (\hat{\alpha}(\xi))^*(a), v \rangle_V \qquad \text{(using (3.11))} \\ &= \langle \xi_{V^*}(a), v \rangle_V . \end{aligned}$$

Since this holds for every $v \in V$ we conclude $\xi_{V^*}(a) = 0$. Now the hypothesis $G_a = \{e_G\}$ at a Lie algebra level reads $\mathfrak{g}_a = \{0\}$. This implies that the infinitesimal generator at a, $\xi_{V^*}(a) = 0$, which holds if $\xi = 0$.

Therefore we have shown that $\langle \rho_v^*(a), \xi\rangle_{\mathfrak{g}}$ vanishes for all $v \in V$ if and only if $\xi = 0$. In other words $\{\rho_v^*(a) \mid v \in V\} = \mathfrak{g}^*$, as we wanted. □

3.3 Coadjoint orbits of semidirect products

In this section we apply our Main Theorem 2.4.1 to show the following result about the structure of Coadjoint orbits of semidirect product Lie groups.

Theorem 3.3.1 (Coadjoint orbits of semidirect products)**.** *Let $M = G \ltimes N$ be a semidirect product Lie group with N abelian. If $\mu \in \mathfrak{m}^*$ satisfies Hypotheses (A) 2.2.1 and (SSH) 2.2.2 then the Coadjoint orbit $\mathcal{O}_\mu$ equipped with the KKS form is symplectomorphic to T^*G equipped with the canonical symplectic form.*

A particular case of the above theorem, applicable when N is a vector space V (as in the previous section), is the following.

Theorem 3.3.2. *Let $M = G \ltimes V$ be a semidirect product Lie group with V a vector space. Let $\mu = (\gamma, a) \in \mathfrak{m}^* = \mathfrak{g}^* \times V^*$ and suppose that the isotropy subgroup of $a \in V^*$ with respect to the induced linear action of G on V^* is trivial. Then the Coadjoint orbit $\mathcal{O}_\mu$ equipped with the KKS form is symplectomorphic to T^*G equipped with the canonical symplectic form.*

The above theorem is a direct consequence of Theorems 3.3.1 and 3.2.2, so we only give a proof of Theorem 3.3.1.

Proof (of Theorem 3.3.1). M acts on T^*M by $\tilde{L} = L^{T^*M}$ cotangent lift of left multiplication. In particular $M = G \times N$ as a manifold, and so $T^*M = T^*G \times T^*N$.

What we know from Theorem A.2.4 is that in this setting, $J_M : T^*M \longrightarrow \mathfrak{m}^*$ is surjective, coincides with the dual Lie algebra component of the right trivialization $\rho(g, \alpha_g) = T_e^* R_g(\alpha_g)$, and the orbit space $P_\mu = J_M^{-1}(\mu)/M_\mu$ is symplectomorphic to $\mathcal{O}_\mu$ the Coadjoint orbit through $\mu \in \mathfrak{m}^*$. On the other hand we will show below that, for every $\nu \in \mathfrak{n}^*$, the symplectic reduced space $J_N^{-1}(\nu)/N = P_\nu$ is symplectomorphic to the cotangent bundle T^*G equipped with the *canonical symplectic form.* The result is therefore a consequence of our Main Theorem 2.4.1 which guarantees that P_μ and P_ν are symplectomorphic.

Let us prove that P_ν is symplectomorphic to T^*G as we claimed above. Denote by $\bar{L} : N \times (T^*G \times T^*N) \longrightarrow T^*G \times T^*N$ the restriction to $N \times T^*M$ of $\tilde{L}$. We first claim that:

$$\bar{L}_n \; = \; (id_{T^*G}, L_n^{T^*N}), \quad \forall\, n \in N. \tag{3.12}$$

Indeed, for every $g, \tilde{g} \in G$, $n, \tilde{n} \in N$:

$$L_{(g,n)}(\tilde{g}, \tilde{n}) \; = \; (g, n)(\tilde{g}, \tilde{n}) \; = \; (g \cdot_G \tilde{g}, n \cdot_N \phi[g](\tilde{n})),$$

so that, since $L|_N = L_{(e_G, \cdot)}$:

$$\bar{L}_n(\tilde{g}, \tilde{n}) \; = \; (\tilde{g}, n\tilde{n}) \; = \; (id_G, L_n)(\tilde{g}, \tilde{n}),$$

then:

$$\bar{L}_n \; = \; (id_{T^*G}, \tilde{L}_n).$$

Moreover:

$$J_N \; = \; i^* \circ J_M \; = \; i^* \circ \rho : T^*G \times T^*N \longrightarrow \mathfrak{g}^* \times \mathfrak{n}^* \longrightarrow \mathfrak{n}^*,$$

and using (3.12) we could prove in fact that:

$$J_N = \rho_N \circ pr_2 : T^*G \times T^*N \longrightarrow T^*N \longrightarrow \mathfrak{n}^*.$$

Also, if P_N denotes the right trivialization of T^*N, the map:

$$P \; := \; (id_{T^*G}, P_N) : T^*G \times T^*N \longrightarrow T^*G \times (N \times \mathfrak{n}^*)$$

is a diffeomorphism, thus there exists a map $\tilde{J}_N : T^*G \times (N \times \mathfrak{n}^*) \longrightarrow \mathfrak{n}^*$ such that the diagram commutes:

$$\begin{tikzcd}
T^*G \times T^*N \arrow[r, "P"] \arrow[dr, "J_N"'] & T^*G \times (N \times \mathfrak{n}^*) \arrow[d, "\tilde{J}_N"] \\
 & \mathfrak{n}^*.
\end{tikzcd}$$

Since $\tilde{J}_N = J_N \circ P^{-1}$, then for every $\nu \in \mathfrak{n}^*$:

$$\begin{aligned}
\tilde{J}_N^{-1}(\nu) \; &= \; P(J_N^{-1}(\nu)) \; = \; T^*G \times P_N(\rho_N^{-1}(\nu)) \\
&= \; T^*G \times (N \times \{\nu\}) \simeq T^*G \times N.
\end{aligned}$$

We propose as a model for the orbit space:

$$P_\nu = \tilde{J}_N^{-1}(\nu)/N \simeq T^*G,$$

with orbit projection $\tau = pr_1$:

$$\tau : T^*G \times N \longrightarrow T^*G, \quad \tau((g, \alpha_g), n) \mapsto (g, \alpha_g).$$

In order to check this is a correct model we should see that:

1. τ is a surjective submersion, which is trivially true;
2. τ respects the $\tilde{L}$ orbits;
3. $\tau^*\omega_{T^*G} = (\tilde{i_\nu})^*\omega_{T^*M}$, where $i_\nu : T^*G \times N \longrightarrow T^*G \times (N \times \mathfrak{n}^*)$ is the inclusion map and $\tilde{i_\nu} : T^*G \times N \longrightarrow T^*M$ is its representation in the identification by P. In other words the following diagram commutes:

$$\begin{tikzcd}
T^*G \times N \arrow[r, "i_\nu"] \arrow[d, "\tau"] \arrow[dr, "\tilde{i_\nu}"'] & T^*G \times (N \times \mathfrak{n}^*) \arrow[d, "P^{-1}"] \\
T^*G & T^*M.
\end{tikzcd}$$

Let us check *2.* If we take for granted that in the cotangent bundle trivialization $\tilde{L}_n : T^*N \longrightarrow T^*N$ is represented by:

$$(L_n, id_{\mathfrak{n}^*}) : N \times \mathfrak{n}^* \longrightarrow N \times \mathfrak{n}^*, \tag{3.13}$$

2 means that if we take $((g, \alpha_g), n), ((\bar{g}, \beta_{\bar{g}}), \bar{n}) \in T^*G \times N$ then:

$$\tau((g, \alpha_g), n) \ = \ \tau((\bar{g}, \beta_{\bar{g}}), \bar{n}) \tag{3.14}$$

if and only if there exists an $\tilde{n} \in N$ such that:

$$\begin{aligned} P^{-1}((g, \alpha_g), (n, \nu)) \ &= \ \bar{L}_{\tilde{n}}(P^{-1}((\bar{g}, \beta_{\bar{g}}), (\bar{n}, \nu)) \\ &= \ P^{-1}((\bar{g}, \beta_{\bar{g}}), (\tilde{n}\bar{n}, \nu)), \end{aligned}$$

where in the last equality we used (3.13). But this is true if and only if:

$$\begin{cases} (g, \alpha_g) \ = \ (\bar{g}, \beta_{\bar{g}}) \\ \exists \tilde{n} \in N : \ n \ = \ \tilde{n}\bar{n} \end{cases}$$

which is actually true by meaning of (3.14) and transitivity of $L : N \times N \longrightarrow N$. It only remains to check what we claimed in (3.13), precisely that the following diagram commutes:

$$\begin{tikzcd}
T^*N \arrow[r, "\tilde{L}_{\bar{n}}"] \arrow[d, "P_N"] & T^*N \arrow[d, "P_N"] \\
N \times \mathfrak{n}^* \arrow[r, "(L_{\bar{n}}, id_{\mathfrak{n}^*})"] & N \times \mathfrak{n}^*
\end{tikzcd}$$

i.e. that for every (n, α_n):

$$P_N \circ \tilde{L}_{\bar{n}}(n, \alpha_n) \ = \ (L_{\bar{n}}, id_{\mathfrak{n}^*}) \circ P_N(n, \alpha_n). \tag{3.15}$$

Now, left hand side is:

$$\begin{aligned}
P_N \circ \tilde{L}_{\bar{n}}(n, \alpha_n) \;&=\; P_N(\bar{n}n, T^*_{\bar{n}n} L_{\bar{n}^{-1}}(\alpha_n)) \;=\; (\bar{n}n, T^*_e R_{\bar{n}n})(T^*_{\bar{n}n} L_{\bar{n}^{-1}}(\alpha_n)) \\
&=\; (\bar{n}n, (T_{\bar{n}n} L_{\bar{n}^{-1}} \circ T_e R_{\bar{n}n})^*(\alpha_n)) \;=\; (\bar{n}n, T^*_e (L_{\bar{n}^{-1}} \circ R_{\bar{n}n})(\alpha_n)) \\
&=\; (\bar{n}n, T^*_e R_n(\alpha_n)),
\end{aligned}$$

where in the last equality we used the fact that N is abelian. Right hand side of (3.15) is:

$$(L_{\bar{n}}, id_{\mathfrak{n}^*}) \circ P(n, \alpha_n) \;=\; (L_{\bar{n}}, id_{\mathfrak{n}^*})(n, T^*_e R_n(\alpha_n)) \;=\; (\bar{n}n, T^*_e R_n(\alpha_n)),$$

which gives the claim.

In order to prove *3* we need the explicit formula for the canonical symplectic form on a cotangent bundle T^*H of a Lie group H in the trivialized coordinates. This is done in Proposition 4.4.1 of [1].

Proposition 3.3.3. *Let ω_{T^*H} be the canonical symplectic form on T^*H. Let also $\omega^S \in \Lambda^2(H \times \mathfrak{h}^*)$ be defined as:*

$$\omega^S = (P_H)_* \omega_{T^*H}.$$

Then for $(h, \mu) \in H \times \mathfrak{h}^*$ and $(v, \rho), (w, \sigma) \in T_{(h,\mu)}(H \times \mathfrak{h}^*) = T_h H \times \mathfrak{h}^*$ we have:

$$\begin{aligned}
\omega^S(h, \mu)((v, \rho), (w, \sigma)) \;&=\; -\langle \rho, T_h R_{h^{-1}}(w) \rangle + \langle \sigma, T_h R_{h^{-1}}(v) \rangle \\
&\qquad + \langle \mu, [\![T_h R_{h^{-1}}(v), T_h R_{h^{-1}}(w)]\!] \rangle.
\end{aligned}$$

Now we can say more if H is abelian. Fix $\mu \in \mathfrak{h}^*$ and consider the immersion map:

$$j_\mu : H \longrightarrow H \times \mathfrak{h}^*, \quad h \mapsto (h, \mu).$$

If $h \in H$ and $v, w \in T_h H$, by definition of pull back:

$$\begin{aligned}
(j^*_\mu \omega^S)(v, w) \;&=\; \omega^S(h, \mu)(T_h j_\mu(v), T_h j_\mu(w)) \\
&=\; \omega^S(h, \mu)((v, 0), (w, 0)) \\
&=\; \langle \mu, [\![T_h R_{h^{-1}}(v), T_h R_{h^{-1}}(w)]\!] \rangle.
\end{aligned}$$

Now if H is abelian then $\mathfrak{h}$ is abelian, i.e. $[\![\xi, \eta]\!] = 0$ for every $\xi, \eta \in \mathfrak{h}$. Thus:

$$(j_\mu)^* \omega^S \;=\; 0.$$

All the considerations above clearly apply to $H = N$, thus $(j^N_\nu)^* \omega^S_N = 0$. We recall that we need to prove the equality in $\Lambda^2(T^*G \times N)$:

$$\tau^* \omega_{T^*G} \;=\; (\tilde{i}_\nu)^* \omega_{T^*M}. \tag{3.16}$$

With a slight abuse of notation we will denote $\omega_{T^*M} = \omega_{T^*G} + \omega_{T^*N}$. Now, on one hand, left hand side of (3.16) clearly is:

$$\tau^* \omega_{T^*G} \;=\; \omega_{T^*G},$$

as τ is a projection. On the other hand, by construction $\tilde{i}_\nu = (id_{T^*G}, P^{-1} \circ j_\nu^N)$ so right hand side of (3.16) is:

$$\begin{aligned} (\tilde{i}_\nu)^* \omega_{T^*M} &= \omega_{T^*G} + \left(j_\nu^N\right)^* \, (P^{-1})^* \, \omega_{T^*N} \\ &= \omega_{T^*G} + \left(j_\nu^N\right)^* \, P_* \, \omega_{T^*N} \\ &= \omega_{T^*G} + \left(j_\nu^N\right)^* \, \omega_N^S \\ &= \omega_{T^*G}, \end{aligned}$$

as we wanted. This concludes the proof of the theorem. □

Chapter 4

Examples

In this chapter we go deeper on specific examples than we did for the general semidirect product case. Precisely we will recover what is the symplectomorphism between the reduced spaces for the example of the 2-Euclidean Lie group $SE(2) \simeq \mathbb{S}^1 \ltimes \mathbb{R}^2$. After that we compare two similar linear actions on the same vector space, checking that the fact that the non normal part G is abelian is not the key point for the application of our Main Theorem 2.4.1. In the third section we review two different realizations of the Heisenberg group which allow us to touch an explicit non semidirect product example.

4.1 $SE(2)$ example.

We define the *2-Euclidean group* as the matrix group:

$$SE(2) \;:=\; \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & \cos(\theta) & -\sin(\theta) \\ b & \sin(\theta) & \cos(\theta) \end{pmatrix} \;\middle|\; a, b \in \mathbb{R},\ \theta \in \mathbb{S}^1 \right\}, \tag{4.1}$$

with matrix multiplication as the operation. This group is representable as the semidirect product $\mathbb{S}^1 \ltimes \mathbb{R}^2$, with linear action of $\mathbb{S}^1$ on $\mathbb{R}^2$ the rotation of a vector. Precisely the group multiplication is, for $\theta, \theta' \in \mathbb{S}^1$ and $v = (a, b), v' = (a', b') \in \mathbb{R}^2$:

$$(\theta', v') \cdot_{SE(2)} (\theta, v) \;=\; (\theta' + \theta, v' + R_{\theta'} v), \tag{4.2}$$

where R_θ is the 2×2 rotation matrix of angle θ. The group structures in (4.1) and (4.2) are isomorphic via the map:

$$(\theta, a, b) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ a & \cos(\theta) & -\sin(\theta) \\ b & \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{4.3}$$

In order to apply the Main Theorem 2.4.1 we know we have to investigate the induced $\mathbb{S}^1$-action on $(\mathbb{R}^2)^*$ and precisely the related isotropy subgroups (see Theorem 3.2.2). Identifying $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ via the euclidean scalar product, we have that such an action is given by:

$$\theta \cdot \nu \;=\; R_\theta \nu,$$

for $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$, and so:

$$G_\nu \;=\; \begin{cases} \{0\} & \text{if } (\nu_1, \nu_2) = (0,0) \\ \mathbb{S}^1 & \text{if } (\nu_1, \nu_2) \neq (0,0). \end{cases}$$

As a consequence we get the following corollary of Theorem 3.2.2 as an application for an $SE(2)$-action on a symplectic manifold.

Corollary 4.1.1. *Let us consider an* $SE(2) \simeq \mathbb{S}^1 \ltimes \mathbb{R}^2$ *free, proper and symplectic action on a symplectic manifold* (P, ω) *with equivariant momentum map* $J_M : P \longrightarrow \mathfrak{se}(2)^*$. *Let* $\mu = (\gamma, \nu_1, \nu_2) \in J_M(P)$ *be such that* $(\nu_1, \nu_2) \neq (0,0)$. *Then there exists a symplectomorphism between the reduced spaces* (P_μ, ω_μ) *and* (P_ν, ω_ν) *(see Section A.2 for notation. In particular* $P_\mu = J_M^{-1}(\mu)/SE(2)_\mu$ *and* $P_\nu = J_N^{-1}(\nu)/\mathbb{R}^2$*).*

For pedagogical reasons we consider in detail the case of $SE(2)$ acting on its cotangent bundle $T^*SE(2)$ by cotangent lift of left multiplication. We will be able to give an explicit expression of the symplectomorphism F between two models of the orbit spaces.

4.1.1 $SE(2)$, $\mathfrak{se}(2)$ and $\mathfrak{se}(2)^*$ preliminaries

Let us find first the Lie algebra structure. On one hand, using the matrix representation, we have that a vector in $\mathfrak{se}(2)$ is the derivative at $t = 0$ of a matrix curve in $SE(2)$ passing through the identity:

$$\left.\frac{d}{dt}\right|_{t=0} \begin{pmatrix} 1 & 0 & 0 \\ a(t) & \cos(\theta(t)) & -\sin(\theta(t)) \\ b(t) & \sin(\theta(t)) & \cos(\theta(t)) \end{pmatrix} = \left.\begin{pmatrix} 0 & 0 & 0 \\ \dot{a} & -\sin(\theta)\dot{\theta} & -\cos(\theta)\dot{\theta} \\ \dot{b} & \cos(\theta)\dot{\theta} & -\sin(\theta)\dot{\theta} \end{pmatrix}\right|_{t=0}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \dot{a} & 0 & \dot{\theta} \\ \dot{b} & \dot{\theta} & 0 \end{pmatrix} = \overline{(\dot{\theta}, \dot{a}, \dot{b})},$$

where by $\bar{\cdot} : \mathbb{R}^3 \longrightarrow \mathfrak{se}(2)$ we mean the tangent map of the identification (4.3) at the identity, which is the Lie algebras identification. Since in the matrix representation the Lie brackets are given by the commutators, then we can find the Lie brackets in (4.2) version. Precisely, if $\xi = (\omega, v_1, v_2), (\omega', v_1', v_2') \in \mathfrak{se}(2)$:

$$\begin{aligned}
[\![\bar{\xi}, \bar{\eta}]\!] &= \bar{\xi}\bar{\eta} - \bar{\eta}\bar{\xi} = \begin{pmatrix} 0 & 0 & 0 \\ -\omega v_2' & -\omega\omega' & 0 \\ v_1'\omega & 0 & -\omega\omega' \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ -\omega' v_2 & -\omega\omega' & 0 \\ v_2\omega' & 0 & -\omega\omega' \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ \omega' v_2 - \omega v_2' & 0 & 0 \\ v_1'\omega - \omega' v_1 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

As a consequence:

$$[\![\xi, \eta]\!] = (0, \omega' v_2 - \omega v_2', v_1'\omega - \omega' v_1).$$

In the same way we recover the Adjoint action of $SE(2)$ on $\mathfrak{se}(2)$. In the matrix representation, for $A \in SE(2)$, $\xi \in \mathfrak{se}(2)$:

$$Ad_A(\xi) = A\xi A^{-1}.$$

Thus, since:

$$Ad_A(\bar{\xi}) = \left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline v_1\cos(\theta) - v_2\sin(\theta) + \omega b & 0 & -\omega \\ v_1\sin(\theta) + v_2\cos(\theta) - \omega a & \omega & 0 \end{array}\right),$$

then $Ad : (\mathbb{S}^1 \ltimes \mathbb{R}^2) \times (\mathbb{R} \times \mathbb{R}^2) \longrightarrow (\mathbb{R} \times \mathbb{R}^2)$ is:

$$Ad_{(\theta,a,b)}(\omega, v_1, v_2) = (\omega, v_1\cos(\theta) - v_2\sin(\theta) + \omega b, v_1\sin(\theta) + v_2\cos(\theta) - \omega a).$$

The dual Lie algebra $\mathfrak{se}(2)^*$ is again identified with $(\mathbb{R}^3)^*$, which corresponds to $\mathbb{R}^3$ itself if the duality pairing is the scalar product. For $\mu = (\gamma, \nu_1, \nu_2) \in \mathbb{R}^3 = (\mathbb{R}^3)^*$, $Ad^*_{(\theta,a,b)^{-1}}(\mu)$ acts on ξ as:

$$\begin{aligned}
\langle Ad^*_{(\theta,a,b)^{-1}}(\mu), \xi \rangle &= \langle (\gamma, \nu_1, \nu_2), Ad_{(\theta,a,b)^{-1}}(\omega, v_1, v_2) \rangle \\
&= \left\langle \begin{pmatrix} \gamma \\ \nu_1 \\ \nu_2 \end{pmatrix}, \begin{pmatrix} \omega \\ v_1\cos(\theta) + v_2\sin(\theta) + \omega(a\sin(\theta) - b\cos(\theta)) \\ -v_1\sin(\theta) + v_2\cos(\theta) + \omega(a\cos(\theta) + b\sin(\theta)) \end{pmatrix} \right\rangle \\
&= \gamma\omega + \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \cdot R_\theta^{-1} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \omega \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \cdot R_\theta^{-1} \begin{pmatrix} -b \\ a \end{pmatrix} \\
&= \omega\big(\gamma + R_\theta \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix}\big) + \big(R_\theta \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}\big) \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\
&= \Big\langle \Big(\gamma + R_\theta \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix}, R_\theta \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}\Big), (\omega, v_1, v_2) \Big\rangle,
\end{aligned}$$

so that, in the end:

$$Ad^*_{(\theta,a,b)^{-1}}(\mu) \;=\; \left(\gamma + R_\theta \begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix} \cdot \begin{pmatrix}-b\\ a\end{pmatrix}, R_\theta \begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}\right). \tag{4.4}$$

With the above expression, it is easy to see that the Coadjoint orbits in $\mathfrak{se}(2)^*$ are of two types, depending on $\nu = (\nu_1, \nu_2)$:

1. If $\nu = 0$ orbits are singletons: $Ad^*_{SE(2)}(\mu) = \{(\gamma, 0, 0)\}$;
2. If $\nu \neq 0$ orbits are cylinders: $Ad^*_{SE(2)}(\mu) \simeq \mathbb{S}^1 \times \mathbb{R}$.

It will be useful to consider the isotropy subgroup of a value μ with respect to Coadjoint action in view of a symplectic reduction. For generic values (Case 2), writing $v = (a, b)$, we have:

$$\begin{aligned} SE(2)_\mu \;=\; SE(2)_{(\gamma,\nu)} \;&=\; \{(\theta, v) \in SE(2) \mid -\nu_1 b + \nu_2 a = 0\} \\ &=\; \{(\theta, v) \in SE(2) \mid v = \lambda\nu, \text{ for some } \lambda \in \mathbb{R}\}. \end{aligned} \tag{4.5}$$

4.1.2 Symplectic reduction of $T^*SE(2)$ by $SE(2)$

Cotangent lifted action. We now consider the cotangent lift of the action by left multiplication of $SE(2)$ on itself, which we denote as:

$$\tilde{L} : SE(2) \times T^*SE(2) \longrightarrow T^*SE(2).$$

This is a free, proper and symplectic action. With the notation conventions introduced above, let $(\theta', a', b') \in SE(2)$ and $(\theta, a, b, p_\theta, p_a, p_b) \in T^*SE(2)$, where (p_θ, p_a, p_b) are the induced bundle coordinates for covectors on $T^*_{(\theta,a,b)}SE(2)$, with $(\theta, a, b) \in SE(2)$. Then, abbreviating by p the column vector (p_θ, p_a, p_b), we have:

$$\tilde{L}_{(\theta',a',b')}(\theta, a, b, p_\theta, p_a, p_b) \;=\; ((\theta', a, b) \cdot (\theta, a, b), J_L^{-t} p),$$

where J_L is the Jacobian matrix of left multiplication by $(\theta', a', b') \in SE(2)$ evaluated at (θ, a, b) and $^{-t}$ denotes inverse transpose. In view of (4.2) we have:

$$J_L \;=\; \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$R_{\theta'}$}} \\ 0 & & \end{array}\right),$$

and hence, since $R_{\theta'}^{-t} = R_{\theta'}^{-1}$, the matrix J_L^{-t} coincides with J_L. Therefore:

$$\tilde{L}_{(\theta',v')}(\theta, v, p_\theta, p_v) \;=\; (\theta' + \theta, v' + R_{\theta'} v, p_\theta, R_{\theta'} p_v), \tag{4.6}$$

where we have written $v' = (a', b')$, $v = (a, b)$, $p_v = (p_a, p_b)$.

Momentum map. We now compute the momentum map $J_{SE(2)} : T^*SE(2) \longrightarrow \mathbb{R}^3$ with the identification of $\mathfrak{se}(2)^*$ with $\mathbb{R}^3$ of section 4.1.1. In view of Theorem A.2.4, for $(\theta, a, b, p_\theta, p_a, p_b) \in T^*SE(2)$ we have:

$$J_{SE(2)}(\theta, a, b, p_\theta, p_a, p_b) \;=\; J_R^t p,$$

where J_R is the Jacobian matrix of right multiplication by $(\theta, a, b) \in SE(2)$ evaluated at $(0, 0, 0)$ (corresponding to the group identity) and we have again abbreviated by p the column vector $(p_\theta, p_a, p_b) \in \mathbb{R}^3$. In view of (4.2) we have:

$$J_R \;=\; \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -b & 1 & 0 \\ a & 0 & 1 \end{array}\right),$$

and therefore:

$$J_{SE(2)}(\theta, a, b, p_\theta, p_a, p_b) \;=\; (p_\theta - p_a b + p_b a, p_a, p_b). \tag{4.7}$$

Symplectic reduction at generic momentum values. Fix $\mu = (\gamma, \nu_1, \nu_2) \in \mathbb{R}^3 = \mathfrak{se}(2)^*$ satisfying the generality condition of Corollary 4.1.1, namely $\nu_1^2 + \nu_2^2 \neq 0$. In view of (4.7) we have:

$$J_{SE(2)}^{-1}(\mu) \;=\; \{(\theta, a, b, p_\theta, p_a, p_b) \mid p_a = \nu_1, p_b = \nu_2, -b\nu_1 + a\nu_2 = \gamma - p_\theta\}. \tag{4.8}$$

It is clear from the above expression that $J_{SE(2)}^{-1}(\mu)$ may be parametrized with the coordinates (θ, a, b) . This is a consequence of the fact that $J_{SE(2)}^{-1}(\mu)$ is diffeomorphic to $SE(2)$ since the momentum map $J_{SE(2)}$ is a (right) trivialization.

Proposition 4.1.2. *The symplectic reduced space* $P_\mu = J_{SE(2)}^{-1}(\mu)/SE(2)_\mu$ *is diffeomorphic to the cylinder through* μ *in* $\mathbb{R}^3 = \mathfrak{se}(2)^*$*, given by:*

$$C_\mu \;=\; \{(\tilde{\gamma}, \tilde{\nu_1}, \tilde{\nu_2}) \mid \nu_1^2 + \nu_2^2 = \tilde{\nu_1}^2 + \tilde{\nu_2}^2\},$$

with orbit map $\pi_\mu : J_{SE(2)}^{-1}(\mu) \longrightarrow C_\mu$ *given by:*

$$\pi_\mu(\theta, a, b, p_\theta, \nu_1, \nu_2) \;=\; (p_\theta, R_\theta p_v),$$

where $p_v = (p_a, p_b)$*. Moreover, using cylindrical coordinates* (α, h) *on* C_μ *(so that* $\tilde{\gamma} = h$ *and* α *is the positive oriented angle between* ν_1, ν_2 *and* $(\tilde{\nu_1}, \tilde{\nu_2})$*) the symplectic reduced form reads* $\omega_\mu = d\alpha \wedge dh$*.*

Proof. It is easily seen that π_μ is a surjective submersion, so to prove that P_μ is diffeomorphic to C_μ we only need to check that for $z, z' \in J^{-1}_{SE(2)}(\mu)$ the condition $\pi_\mu(z) = \pi_\mu(z')$ holds if and only if z and z' are on the same $SE(2)_\mu$-orbit. First note that, in view of (4.8), the condition $z, z' \in J^{-1}_{SE(2)}(\mu)$ implies that we may write:

$$\begin{aligned} z &= (\theta, a, b, \gamma + b\nu_1 - a\nu_2, \nu), \\ z' &= (\theta', a', b', \gamma + b'\nu_1 - a'\nu_2, \nu), \end{aligned}$$

for some (θ, a, b), $(\theta', a', b') \in SE(2)$, where we have written $\nu = (\nu_1, \nu_2)$. Hence, by definition of π_μ, the condition $\pi_\mu(z) = \pi_\mu(z')$ holds if and only if:

$$-b\nu_1 + a\nu_2 \;=\; -b'\nu_1 + a'\nu_2 \quad \text{and} \quad R_\theta \nu \;=\; R_{\theta'}\nu. \tag{4.9}$$

Considering that $\nu = (\nu_1, \nu_2) \neq 0$, the above conditions hold if and only if

$$\theta \;=\; \theta' \quad \text{and} \quad v' = v + \lambda_0 \nu \tag{4.10}$$

for some $\lambda_0 \in \mathbb{R}$, where $v = (a, b)$, $v' = (a', b')$.

Now recall from (4.5) that:

$$SE(2)_\mu \;=\; \{(0, \lambda\nu) \in SE(2) \mid \lambda \in \mathbb{R}\},$$

and notice that $(0, \lambda_0\nu) \in SE(2)_\mu$. Using (4.6), (4.9) and (4.10) we have:

$$\begin{aligned} \tilde{L}_{(0,\lambda_0\nu)}(z) &= \tilde{L}_{(0,\lambda_0\nu)}(\theta, v, \gamma + b\nu_1 - a\nu_2, \nu) \\ &= (\theta, \lambda_0\nu + v, \gamma + b\nu_1 - a\nu_2, \nu) \\ &= (\theta', v', \gamma + b'\nu_1 - a'\nu_2, \nu) \\ &= z'. \end{aligned}$$

Conversely, using (4.6) and the definition of π_μ, for any $(0, \lambda\nu) \in SE(2)_\mu$ we have:

$$\begin{aligned} \pi_\mu(\tilde{L}_{(0,\lambda\nu)}(z)) &= \pi_\mu(\theta, \lambda\nu + v, \gamma + b\nu_1 - a\nu_2, \nu) \\ &= (\gamma + b\nu_1 - a\nu_2, R_\theta\nu) \\ &= \pi_\mu(z). \end{aligned}$$

This proves that P_μ is diffeomorphic to C_μ as claimed.

The reduced symplectic form $\omega_\mu \in \Lambda^2(C_\mu)$ is characterized by the condition:

$$(\pi_\mu)^*\omega_\mu \;=\; (j_\mu)^*\omega_{T^*SE(2)}, \tag{4.11}$$

where $j_\mu : J^{-1}_{SE(2)}(\mu) \hookrightarrow T^*SE(2)$ is the inclusion and $\omega_{T^*SE(2)} = da \wedge dp_a + db \wedge dp_b + d\theta \wedge dp_\theta$ is the canonical symplectic form on $T^*SE(2)$. Using (θ, a, b) as coordinates on $J^{-1}_{SE(2)}(\mu)$ we have, in view of (4.8):

$$\begin{aligned}(j_\mu)^*\omega_{T^*SE(2)} &= (j_\mu)^*(da \wedge dp_a + db \wedge dp_b + d\theta \wedge dp_\theta)\\ &= da \wedge d(\nu_1) + db \wedge d(\nu_2) + d\theta \wedge d(\gamma + b\nu_1 - a\nu_2)\\ &= \nu_1 d\theta \wedge db - \nu_2 d\theta \wedge da.\end{aligned}$$

On the other hand, the expression for π_μ in the coordinates (θ, a, b) for $J^{-1}_{SE(2)}(\mu)$ is $\pi_\mu(\theta, a, b) = (\theta, \gamma + b\nu_1 - a\nu_2)$. Therefore, the pull-back of the generic 2-form $f(\alpha, h)\, d\alpha \wedge dh$ on C_μ is:

$$\begin{aligned}(\pi_\mu)^*(f(\alpha, h)\, d\alpha \wedge dh) &= f(\theta, \gamma + b\nu_1 - a\nu_2)\,(d\theta \wedge d(\gamma + b\nu_1 - a\nu_2))\\ &= f(\theta, \gamma + b\nu_1 - a\nu_2)\,(\nu_1 d\theta \wedge db - \nu_2 d\theta \wedge da).\end{aligned}$$

Using this formula it is clear that $\omega_\mu = d\alpha \wedge dh$ is the unique 2-form on C_μ satisfying (4.11). □

4.1.3 Symplectic reduction of $T^*SE(2)$ by $\mathbb{R}^2$

We now work with the restricted action of $\mathbb{R}^2$ on $T^*SE(2)$ which, with a slight abuse of notation, we also denote by $\tilde{L} : \mathbb{R}^2 \times T^*SE(2) \longrightarrow T^*SE(2)$. In view of (4.6), for $(a', b') \in \mathbb{R}^2$ and $(\theta, a, b, p_\theta, p_a, p_b) \in T^*SE(2)$ we have:

$$\tilde{L}_{(a',b')}(\theta, a, b, p_\theta, p_a, p_b) = (\theta, a + a', b + b', p_\theta, p_a, p_b). \tag{4.12}$$

The momentum map $J_{\mathbb{R}^2} : T^*SE(2) \longrightarrow \mathbb{R}^2$ is given by:

$$J_{\mathbb{R}^2}(\theta, a, b, p_\theta, p_a, p_b) = (p_a, p_b).$$

It is readily seen that $J_{\mathbb{R}^2} = i^* \circ J_{SE(2)}$, as predicted by Lemma 2.3.1, since, under identifications, the map i^* is:

$$i^* : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad (\gamma, \nu_1, \nu_2) \mapsto (\nu_1, \nu_2).$$

Given $(\nu_1, \nu_2) \in \mathbb{R}^2$ it is clear that:

$$J^{-1}_{\mathbb{R}^2}(\nu_1, \nu_2) = \{(\theta, a, b, p_\theta, p_a, p_b) \in T^*SE(2) \mid p_a = \nu_1,\ p_b = \nu_2\}.$$

In analogy with Proposition 4.1.2, we have the following.

Proposition 4.1.3. *Let* $\nu \in \mathbb{R}^2$*. The symplectic reduced space* $P_\nu = J_{\mathbb{R}^2}^{-1}(\nu)/\mathbb{R}^2$ *is diffeomorphic to* $\mathbb{S}^1 \times \mathbb{R}$ *with projection map* $\pi_\nu : J_{\mathbb{R}^2}^{-1}(\nu) \longrightarrow \mathbb{S}^1 \times \mathbb{R}$ *given by:*

$$\pi_\nu(\theta, a, b, p_\theta, p_a, p_b) \; = \; (\theta, p_\theta).$$

Moreover, in the coordinates (θ, p_θ) *for* $\mathbb{S}^1 \times \mathbb{R}$ *the reduced symplectic form is* $\omega_\nu = d\theta \wedge dp_\theta$*.*

The proof follows from analogous (but simpler) considerations to the ones given in the proof of Proposition 4.1.2. We omit the details.

4.1.4 Equivalence of the symplectic reduction of $T^*SE(2)$ by $SE(2)$ and $\mathbb{R}^2$ at generic momentum values

Fix $\mu = (\gamma, \nu) \in \mathfrak{se}(2)^* = \mathbb{R}^3$ with $\nu \neq 0$ (in view of the choice made in Corollary 4.1.1). We now work out the explicit form of the symplectomorphism F as in the proof of Theorem 2.4.1, with the models for the symplectic reduced spaces P_μ and P_ν given in Propositions 4.1.2 and 4.1.3.

Let $(\tilde{\gamma}, \tilde{\nu}) \in C_\mu$ with cylindrical coordinates (α, h). According to the convention in the statement of Proposition 4.1.2, we have:

$$\tilde{\gamma} \; = \; h, \quad R_\alpha \nu \; = \; \tilde{\nu}. \tag{4.13}$$

Now, according to the definition (2.9) of F, we have:

$$F(\tilde{\gamma}, \tilde{\nu}) \; = \; \pi_\nu \circ j_\mu(z),$$

where $z \in J_{SE(2)}^{-1}(\mu)$ satisfies $\pi_\mu(z) = (\tilde{\gamma}, \tilde{\nu})$. Using (4.8) and the definition of π_μ in the statement of Proposition 4.1.2, we see that z is of the form:

$$z \; = \; (\theta_0, a_0, b_0, \gamma + b_0\nu_1 - a_0\nu_2, \nu_1, \nu_2) \in T^*SE(2),$$

with $(\theta_0, a_0, b_0) \in SE(2)$ such that:

$$\tilde{\gamma} \; = \; \gamma + b_0\nu_1 - a_0\nu_2, \quad R_{\theta_0}\nu \; = \; \tilde{\nu}. \tag{4.14}$$

Comparing (4.13) and (4.14) we see that the angles θ_0 and α coincide and therefore the coordinate representation of F is:

$$\begin{aligned} F(\alpha, h) \; &= \; \pi_\nu \circ j_\mu(\alpha, a_0, b_0, h, \nu_1, \nu_2) \\ &= \; \pi_\nu(\alpha, a_0, b_0, h, \nu_1, \nu_2) \\ &= \; (\alpha, h), \end{aligned} \tag{4.15}$$

where on the right hand side (α, h) is interpreted as an element of $\mathbb{S}^1 \times \mathbb{R}$, which is our model for P_ν. It is clear that F is a symplectomorphism with respect to the symplectic structures specified for P_μ and P_ν in Propositions 4.1.2 and 4.1.3.

Finally, we also give an explicit construction of the mapping $\phi : P_\nu \longrightarrow P_\mu$ as in the proof of Theorem 2.4.1. Let $(\theta, p_\theta) \in \mathbb{S}^1 \times \mathbb{R} = P_\nu$. By (2.14) we have:

$$\phi(\theta, p_\theta) \;=\; \pi_\mu(\tilde{L}_{(a',b')}(z)), \tag{4.16}$$

where $z \in J^{-1}_{\mathbb{R}^2}(\nu)$ is such that:

$$\pi_\nu(z) \;=\; (\theta, p_\theta), \tag{4.17}$$

and $(a', b') \in \mathbb{R}^2$ satisfies:

$$Ad^*_{(0,-a',-b')}(J_{SE(2)}(z)) \;=\; (\gamma, a, b) \;=\; \mu_0. \tag{4.18}$$

Now, an element $z \in J^{-1}_{\mathbb{R}^2}(\nu)$ satisfying (4.17) is of the form:

$$z = (\theta, a, b, p_\theta, \nu_1, \nu_2), \tag{4.19}$$

for arbitrary $(a, b) \in \mathbb{R}^2$, and satisfies:

$$J_{SE(2)}(z) \;=\; (p_\theta - \nu_1 b + \nu_2 a, \nu_1, \nu_2). \tag{4.20}$$

Therefore, using (4.4), we see that (a', b') as defined by (4.18) satisfies:

$$\gamma \;=\; p_\theta - \nu_1 b + \nu_2 a - \nu_1 b' + \nu_2 a'.$$

Without determine explicit values of (a', b') satisfying the above relation, we may compute the value of $\phi(\theta, p_\theta)$ using (4.12) and the definition of π_μ in (4.16). Namely:

$$\begin{aligned}
\phi(\theta, p_\theta) &= \pi_\mu(\tilde{L}_{(a',b')}(z)), \quad \text{(by (4.16))} \\
&= \pi_\mu(\tilde{L}_{(a',b')}(\theta, a, b, p_\theta, \nu_1, \nu_2)), \quad \text{(by (4.19))} \\
&= \pi_\mu(\theta, a + a', b + b', p_\theta, \nu_1, \nu_2), \quad \text{(by (4.12))} \\
&= (p_\theta, R_\theta \nu) \in C_\mu, \quad \text{(by definition of } \pi_\mu).
\end{aligned}$$

It is clear that the cylindrical coordinates of the point $(p_\theta, R_\theta\nu) \in C_\mu$ are $h = p_\theta$, $\alpha = \theta$, so the coordinate representation of ϕ is:

$$\phi(\theta, p_\theta) \;=\; (\theta, p_\theta).$$

This is consistent with the fact that $\phi = F^{-1}$ and the coordinate representation of F is given by (4.15).

4.2 $SE(3)$ and the $SO(3)$ action on $\mathbb{R}^3 \times \mathbb{R}^3$.

We have as always the structure $M = G \ltimes V$ as in Section 3.1. One may think that the existence of non degenerate values $\mu \in \mathfrak{m}^*$ (i.e. values that respect Hypotheses (A), (SSH), in order to apply the Main Theorem), depends on the fact that G is abelian as well. This is not true. The following counterexamples show that this may happen or not, even if G is non-abelian.

We first consider the non abelian group:

$$SO(3) \; = \; \{A \in GL(3) \mid A^t A \; = \; \mathbb{I}_3, \; \det(A) = 1 \; \}.$$

Also, the *3-euclidean* group is:

$$M \; = \; SE(3) \; = \; SO(3) \ltimes \mathbb{R}^3,$$

namely the matrix group of rototraslation linear maps on $\mathbb{R}^3$. The action of $SO(3)$ on $\mathbb{R}^3$ is given by ϕ:

$$\phi_g(v) \; = \; gv, \quad g \in SO(3), \; v \in \mathbb{R}^3. \tag{4.21}$$

We use the characterization of Hypothesis (A) given by Theorem 3.2.2 in order to check that correct values $(\gamma, a) \in \mathfrak{so}(3)^* \times V^* \simeq \mathfrak{so}(3)^* \times \mathbb{R}^3$ exist (here we identify duality in $\mathbb{R}^3$ with the scalar product). The induced linear action of $SO(3)$ on $\mathbb{R}^3 \times \mathbb{R}^3$ exactly replies (4.21): $g \cdot a = ga$ for every $g \in SO(3)$, $a \in \mathbb{R}^3$. We study isotropy subgroups of a, i.e. the equation in g:

$$a \; = \; ga.$$

We have two cases:

1. If $a = 0$ then of course $SO(3)_a = SO(3)$.
2. If $a \neq 0$ then the set of $g \in SO(3)$ that fix a is the one that only rotates the two dimensional plane $\langle a \rangle^\perp$. In this case $SO(3)_a$ is the rotation group of a plane: it is then a copy of $SO(2)$ in $SO(3)$.

Thus, we see there are no choices of (γ, a) for which the hypotheses of the theorem hold, in the sense that $G_a \neq \{\mathbb{I}_3\}$ for every $a \in \mathbb{R}^3$.

Now let us take into account the same $SO(3)$ action on $\mathbb{R}^3$ as in (4.21) but "doubled":

$$\phi_g(u, v) \; = \; (gu, gv), \quad g \in SO(3), \; u, v \in \mathbb{R}^3. \tag{4.22}$$

This gives the semidirect structure of the group $M = SO(3) \ltimes (\mathbb{R}^3 \times \mathbb{R}^3)$, which is the one we care about. Again the induced linear action of $SO(3)$ on $(\mathbb{R}^3 \times \mathbb{R}^3)^* \simeq \mathbb{R}^3 \times \mathbb{R}^3$ replies (4.22), so that if $(a, b) \in (\mathbb{R}^3 \times \mathbb{R}^3)^*$ the following cases arise for $SO(3)_{(a,b)}$:

1. If $(a, b) = (0, 0)$ then $SO(3)_{(a,b)} = SO(3)$.
2. If $a \neq 0$, by the analysis of the previous example we have $ga = a$ if and only if g rotates $\langle a \rangle^{\perp}$. If we want the same g to fix b as well, then we need $\langle a \rangle^{\perp} \leq \langle b \rangle^{\perp}$. But since we are in dimension 3 this is simply obtained by having a parallel to b. In this case $SO(3)_{(a,b)} \simeq SO(2)$. Of course the same considerations hold by considering $b \neq 0$ first.
3. From case 2 we also deduce that if $(a, b) \neq (0, 0)$ and a, b are not parallel to each other, then $SO(3)_{(a.b)} = \{\mathbb{I}_3\}$.

Thus, choices of $(\gamma, (a, b)) \in \mathfrak{so}(3)^* \times (\mathbb{R}^3 \times \mathbb{R}^3)$ with (a, b) as in case 3 are the one we were searching for.

4.3 The Heisenberg group

We define the Heisenberg group as the unique 3 dimensional 2-step nilpotent Lie group (up to isomorphisms) $\mathbb{H}^3$ whose only non trivial bracket relation in the Lie algebra is:

$$[\![X_1, X_2]\!] = X_3,$$

where $\{X_1, X_2, X_3\}$ is a basis of $\mathfrak{h}^3 = Lie(\mathbb{H}^3)$. As a matter of fact, $\exp : \mathfrak{h}^3 \longrightarrow \mathbb{H}^3$ is a global diffeomorphism, $\mathbb{H}^3$ being a nilpotent group (see [6], Theorem 1.2.1). So we have exponential coordinates on the group, in the sense that any element $g \in \mathbb{H}^3$ is written as $g = \exp\left(\sum_{i=1}^{3} x_i X_i\right)$, $(\mathbb{H}^3, \cdot) \simeq (\mathbb{R}^3, *)$, and the product $*$ is recovered using *Baker Campbell Hausdorff* formula (see $\mathbf{N_{3,2}}$ example in [8]):

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} * \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \end{pmatrix}.$$

Here the identity element is $0_{\mathbb{R}^3}$ and the inverse element of a vector $x \in \mathbb{R}^3$ is $-x$.

Also, we could prove that the following map:

$$(x_1, x_2, x_3) \mapsto \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.23}$$

is a Lie group isomorphism. We will refer to the image group as the Matrix Heisenberg group $\tilde{\mathbb{H}}^3$.

We will mainly use the $\mathbb{R}^3$ realization. It is useful to consider the conjugation map:

$$C_x y \;=\; x * y * x^{-1} \;=\; \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ {\scriptstyle x_3 + y_3 + \frac{1}{2}(x_1 y_2 - x_2 y_1)} \end{pmatrix} \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \;=\; \begin{pmatrix} y_1 \\ y_2 \\ {\scriptstyle y_3 + x_1 y_2 - x_2 y_1} \end{pmatrix}. \tag{4.24}$$

The Adjoint action is, as defined in (A.2), $Ad_x\xi = T_e C_x(\xi)$, for every $x = (x_1, x_2, x_3) \in \mathbb{H}^3$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathfrak{h}^3$. Now:

$$C'_x(0) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_2 & x_1 & 1 \end{pmatrix},$$

so that:

$$Ad_x(\xi) \;=\; C'_x \xi \;=\; (\xi_1, \xi_2, \xi_3 - \xi_1 x_2 + \xi_2 x_1).$$

Let us also compute how the Coadjoint action works. As always if the duality pairing coincides with the scalar product in $\mathbb{R}^3$ we have for $\mu = (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 \simeq (\mathbb{R}^3)^*$ and $\xi \in \mathbb{R}^3$:

$$\begin{aligned} \langle Ad^*_{x^{-1}}(\mu), \xi \rangle &= \langle (\mu_1, \mu_2, \mu_3), Ad_{x^{-1}}(\xi) \rangle \\ &= \left\langle \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}, \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 + \xi_1 x_2 - \xi_2 x_1 \end{pmatrix} \right\rangle \\ &= \mu_1 \xi_1 + \mu_2 \xi_2 + \mu_3 (\xi_3 + \xi_1 x_2 - \xi_2 x_1) \\ &= \xi_1(\mu_1 + x_2 \mu_3) + \xi_2(\mu_2 - x_1 \mu_3) + \xi_3 \mu_3. \end{aligned}$$

As a consequence:

$$Ad^*_{x^{-1}}(\mu) \;=\; (\mu_1 + x_2\mu_3,\ \mu_2 - x_1\mu_3,\ \mu_3), \tag{4.25}$$

and so Coadjoint orbits through μ are of two types:

1. If $\mu_3 = 0$ then $\mathcal{O}_\mu = \{\mu\}$;
2. If $\mu_3 \neq 0$ then $\mathcal{O}_\mu = \mu + \langle (x_1, x_2, 0) \rangle \simeq \mathbb{R}^2$.

We will take into account now two different normal and abelian subgroups N of $\mathbb{H}^3$. The first one will lead to a semidirect structure, while the second one will not. Our goal is to compare the chance to apply our Main Theorem in these two realizations: we will see that the existence of momentum values that satisfy the theorem hypotheses is non trivial.

4.3.1 $\mathbb{H}^3$ as a semidirect product

Let us consider the subgroup:

$$N = \exp(\langle X_1, X_3 \rangle).$$

This is an abelian and normal subgroup of $\mathbb{H}^3$, so it fits the setting of Main Theorem 2.4.1. Also one could prove that $(N, *) \simeq (\mathbb{R}^2, +)$. There is more. If we set:

$$G = \exp(\langle X_2 \rangle)$$

this is again a subgroup and $(G, *) \simeq (\mathbb{R}, +)$. We claim that $\mathbb{H}^3$ has the semidirect product structure of $\mathbb{H}^3 = G \ltimes N$.

Let us check this claim. It is convenient to think for the moment to $\mathbb{H}^3$ as the matrix Heisenberg group $\tilde{\mathbb{H}}^3$ as we did in (4.23). We should establish what is the isomorphism that maps an element:

$$\begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (g, n) \in \mathbb{R} \ltimes \mathbb{R}^2,$$

representing it. By group theory the meaning of such a (g, n) is the following. If $n = (x, z) \in \mathbb{R}^2$ and $g = y \in \mathbb{R}$:

$$(g, n) = n \cdot g = \begin{pmatrix} 1 & x & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & z + xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

It is important here to notice that the elements $(y, (x, z))$ and $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$ represent two different points in $\mathbb{R} \ltimes \mathbb{R}^2$. Rather the identification is:

$$(g, (n_1, n_2)) \mapsto \begin{pmatrix} 1 & n_1 & n_2 + g n_1 \\ 0 & 1 & g \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (y, (x, z - xy)).$$

As a consequence we have that $(g,n)\cdot(g',n')$ is, on one hand:

$$\begin{pmatrix}1 & n_1 & {\scriptstyle n_2+gn_1}\\ 0 & 1 & g\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & n'_1 & {\scriptstyle n'_2+g'n'_1}\\ 0 & 1 & g'\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & {\scriptstyle n_1+n'_1} & {\scriptstyle n_2+n'_2+(g+g')n_1+g'n'_1}\\ 0 & 1 & g+g'\\ 0 & 0 & 1\end{pmatrix}. \tag{4.26}$$

On the other hand it is:

$$(g\cdot_G g', n\cdot_N \phi[g](n')) \;=\; \left(g+g', \begin{pmatrix}n_1+\phi_1[g](n'_1,n'_2)\\ n_2+\phi_2[g](n'_1,n'_2)\end{pmatrix}\right) \;=$$

$$\begin{pmatrix}1 & n_1+\phi_1[g](n'_1,n'_2) & n_2+\phi_2[g](n'_1,n'_2)+(g+g')(n_1+\phi_1[g](n'_1,n'_2))\\ 0 & 1 & g+g'\\ 0 & 0 & 1\end{pmatrix} =$$

$$= \begin{pmatrix}1 & n_1+\phi_1[g](n'_1,n'_2) & n_2+(g+g')n_1+(g+g')\phi_1+\phi_2\\ 0 & 1 & g+g'\\ 0 & 0 & 1\end{pmatrix}. \tag{4.27}$$

By comparing (4.26) and (4.27) we deduce:

$$\phi_1[g](n'_1,n'_2) \;=\; n'_1, \quad \phi_2[g](n'_1,n'_2) \;=\; n'_2 - gn'_1.$$

The following proposition summarizes all the computations above.

Proposition 4.3.1. *The Heisenberg group $\mathbb{H}^3$ is isomorphic in the sense of Lie groups to $\mathbb{R}\ltimes\mathbb{R}^2$ with group product:*

$$(g,(n_1,n_2))\cdot(g',(n'_1,n'_2)) \;=\; (g+g',(n_1+n'_1, n_2+n'_2-gn'_1)),$$

and following the notation we use in Section 3.1:

$$\phi_g(n_1,n_2) \;=\; (n_1, n_2-gn_1) \tag{4.28}$$

is the $\mathbb{R}$ action on $\mathbb{R}^2$.

Our focus now is to see that if we consider such N as a first stage reduction group, then there actually is the possibility to satisfy the Main Theorem (differently to what will happen in Section 4.3.2). Relying on the characterization of Hypotheses (A) and (SSH) in semidirect products given by Theorem 3.2.2, a value $(\gamma,a)\in(\mathfrak{h}^3)^*\simeq\mathbb{R}\times\mathbb{R}^2$ satisfies both hypotheses if $G_a=\{e_G\}=\{0_\mathbb{R}\}$. But:

$$G_a \;=\; \{g\in\mathbb{R}\mid g\cdot(a_1,a_2)=(a_1,a_2)\} \;=\; \{g\in\mathbb{R}\mid a_2=a_2-ga_1\}.$$

So we have:

$$G_a = \begin{cases} \{0\} & \text{if } a_1 \neq 0 \\ \mathbb{R} & \text{if } a_1 = 0. \end{cases}$$

So correct choices for $\mu = (\gamma, a)$ in order to satisfy Main Theorem 2.4.1 are such that $a_1 \neq 0$.

4.3.2 $\mathbb{H}^3$ as a non semidirect product

We now consider a different choice for N, namely the 1-dimensional Lie subgroup:

$$N = \exp(\langle X_3 \rangle).$$

N is trivially abelian and normal in $\mathbb{H}^3$ (this is easily proved by applying the conjugation map (4.24) to an element in N).

The immersion map of $\mathfrak{n}$ in the group is $i : \mathfrak{n} \longrightarrow \mathfrak{h}^3$ such that:

$$i(v) = (0, 0, v), \quad v \in \mathbb{R},$$

so that the projection $i^* : (\mathfrak{h}^3)^* \longrightarrow \mathfrak{n}^* \simeq \mathbb{R}$ as in (2.1) is:

$$i^*(\gamma, \nu) = \nu, \quad \gamma \in \mathbb{R}^2, \nu \in \mathbb{R}.$$

The action $\psi : \mathbb{H}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$ from (2.2) works as follows:

$$\psi_x(\nu) = i^*(Ad^*_{x^{-1}}(\mu)) = i^*(\gamma + x_2\nu_2, \nu_1 - x_1\nu_2, \nu_2) = \nu,$$

where we used (4.25) with $x \in \mathbb{H}^3$, $\nu \in \mathbb{R}$ and $\mu = (\mu_1, \mu_2, \mu_3) = (\gamma, \nu_1, \nu_2)$, which is a generic element such that $i^*(\mu) = \nu$. As a consequence we see that $M_\nu = M \neq N$ for every $\nu \in \mathbb{R}$, and so there's no hope for Hypothesis (A) 2.2.1 to hold for any $\nu \in \mathbb{R}$.

Remark 4.3.1. We claim there cannot exists a Lie subgroup $A \leq \mathbb{H}^3$ such that $\mathbb{H}^3 = A \ltimes N$.

By contradiction such A exists. Then by algebraic properties of semidirect products, there must be a Lie group isomorphism $S : A \longrightarrow G/N$ and so its tangent map at the identity is a Lie algebra isomorphism:

$$\mathfrak{s} : \mathfrak{a} \longrightarrow Lie(G/N) \simeq \mathfrak{g}/\mathfrak{m}.$$

In particular this is a linear surjective map. It follows that $\mathfrak{a}$ contains any vector of the form:

$$\xi_1 = X_1 + aX_3, \quad \xi_2 = X_2 + bX_3.$$

Thus, $\mathfrak{a} = \langle X_1, X_2, X_3 \rangle = \mathfrak{h}^3$ ($\mathfrak{a}$ being a Lie algebra), which implies A is 3 dimensional. This is the contradiction.

Appendix A

Preliminaries

In this appendix we review some known results in differential geometry, Lie group (in particular symplectic) reduction, and theory of fibrations we use in the thesis.

A.1 Differential geometry

A.1.1 Lie group actions

Lie groups and Lie algebras are key objects for this work. We now recall them very quickly, but one can read [9] or [7] for more details.

Definition A.1.1 (Lie group)**.** *G is a Lie group if it is both a group and a smooth manifold where the product and the inverse maps:*

$$G \times G \longrightarrow G, \quad (g,h) \mapsto gh$$
$$G \longrightarrow G, \quad g \mapsto g^{-1}$$

are smooth with respect to the differentiable structure.

For a fixed $g \in G$ one can consider the left and right translations:

$$\begin{aligned} L_g : G \longrightarrow G, \quad L_g(h) &:= gh \\ R_g : G \longrightarrow G, \quad R_g(h) &:= hg. \end{aligned}$$

L_g, R_g are diffeomorphisms for every $g \in G$.

The construction of the Lie algebra of a Lie group requires the notion of the so called left invariant vector fields.

Definition A.1.2 (Left invariant vector field)**.** *A vector field X on a Lie group G is left invariant if it is invariant under every left translation:*

$$(L_g)_* X = X, \quad \forall g \in G.$$

We will denote the set of left invariant vector fields on G by $\mathfrak{X}_L(G)$. Now, if we consider the standard Lie brackets $[\cdot,\cdot]$ on $\mathfrak{X}(G)$, then $(\mathfrak{X}_L(G),[\cdot,\cdot])$ happens to be a Lie subalgebra of $\mathfrak{X}(G)$.

Proposition A.1.1. *Let G be a Lie group and $\xi \in T_eG$. Let X_ξ be the vector field such that:*

$$X_\xi(g) \ := \ T_eL_g(\xi), \quad g \in G.$$

Then X_ξ is left invariant and the mapping:

$$\lambda : T_eG \longrightarrow \mathfrak{X}_L(G), \quad \lambda(\xi) \ = \ X_\xi$$

is a linear isomorphism.

Proof. For a proof, see 8.37 of [9]. ☐

As a consequence of the previous proposition we can endow T_eG with a Lie bracket inherited from $(\mathfrak{X}_L(G),[\cdot,\cdot])$, namely:

$$[\![\xi,\eta]\!] \ := \ [X_\xi, X_\eta](e), \quad \forall \xi,\eta \in T_eG. \tag{A.1}$$

Definition A.1.3 (Lie algebra of a Lie group)**.** *Let G be a Lie group. The Lie algebra of G is $\mathfrak{g} := T_eG$ endowed with the Lie algebra structure as in* (A.1)*.*

As a vector space, $\mathfrak{g}$ has a dual space $\mathfrak{g}^*$ which is the space of linear functionals on $\mathfrak{g}$. We will denote the action of a $\mu \in \mathfrak{g}^*$ on a vector $\xi \in \mathfrak{g}$ using the duality pairing:

$$\langle \mu, \xi \rangle \ = \ \mu(\xi).$$

Lie groups are useful because they act on smooth manifolds.

Definition A.1.4. *A (left) action of a Lie group G on a smooth manifold M is a smooth map:*

$$\psi : G \times M \longrightarrow M, \quad (g,m) \mapsto \psi(g,m) \ := \ \psi_g(m)$$

such that:

1. *$\psi_e = id_M$;*
2. *$\psi_{gh} = \psi_g \circ \psi_h$ for every $g,h \in G$.*

It will be crucial to consider the isotropy subgroup of a Lie group action.

Definition A.1.5 (Isotropy subgroup)**.** *Let* $\psi : G \times M \longrightarrow M$ *be a* G*-action on* M*. The isotropy subgroup of an* $m \in M$ *with respect to* ψ *is:*

$$G_m \;:=\; \{g \in G \mid \psi_g(m) \;=\; m\},$$

which is in fact a subgroup of G*.*

We recall in particular the definition of Adjoint and Coadjoint actions. Let G be a Lie group: it acts on itself by conjugation, namely $C : G \times G \longrightarrow G$ such that $\forall\, g \in G$:

$$C_g(h) \;=\; L_g \circ R_{g^{-1}}(h) \;=\; ghg^{-1}.$$

C_g is a Lie group isomorphism and so $Ad_g := T_eC_g$ is a Lie algebra isomorphism from $\mathfrak{g}$ to itself. As a matter of fact the so called Adjoint action (or representation) is the linear action $Ad : G \times \mathfrak{g} \longrightarrow \mathfrak{g}$.
The Coadjoint action of a Lie group G is the (left) action Ad^* of G on the dual Lie algebra $\mathfrak{g}^*$ obtained by dualizing Ad. Precisely:

$$Ad^* :\; G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \qquad (g, \mu) \mapsto (Ad)^*_g(\mu) = Ad^*_{g^{-1}}(\mu), \tag{A.2}$$

where $Ad^*_{g^{-1}}$ is the *dual map* of $Ad_{g^{-1}}$, namely:

$$\langle Ad^*_{g^{-1}}(\mu), \xi\rangle \;=\; \langle \mu, Ad_{g^{-1}}(\xi), \quad \forall\, \mu \in \mathfrak{g}^*,\; \forall\, \xi \in \mathfrak{g}.$$

The following proposition is central in the geometric setting we develop in the thesis since it translates the fact that a Lie group is abelian at the Lie algebra level.

Proposition A.1.2. *If* G *is abelian then both the Adjoint and Coadjoint actions of* G *on* $\mathfrak{g}$, $\mathfrak{g}^*$ *are trivial. In particular* $G_\mu = G$ *for any* $\mu \in \mathfrak{g}^*$, G_μ *being the isotropy subgroup of* μ *with respect to the Coadjoint action.*

Proof. The fact that Ad is trivial follows by definition. We have $C_gh = h$, $\forall\, g, h \in G$, thus $C_g = id_G$. Differentiating it at $h = e$ we obtain:

$$T_eC_g \;=\; Ad_g \;=\; id_{\mathfrak{g}}.$$

Triviality of Ad^* comes from dualizing the maps in the previous line. ☐

A.1.2 A technical lemma to prove smoothness of a map

Lemma A.1.3. *Let* M, N, K *smooth manifolds. Let* $\pi : M \longrightarrow N$ *be a surjective submersion and* $f : N \longrightarrow K$ *be such that* $f \circ \pi : M \longrightarrow K$ *is smooth. Then* f *is smooth.*

Proof. For a proof one can see the one of Theorem 4.29 in [9]. ☐

A.1.3 The smooth structure of the orbit of an action

The following is mainly based on [7], section 1.11 and [14].

Let $\psi : G \times M \longrightarrow M$ be a Lie group action on a manifold M. Let $m \in M$. We know that:

$$G_m \; = \; \{g \in G \mid \psi_g(m) \; = \; m\}$$

is a regular (closed) Lie subgroup of G. G_m acts on G by right multiplication. This is a proper action (Example 2.3.5 in [14]) which is also obviously free. Consequently the quotient space G/G_m has a unique smooth manifold structure such that:

$$\tau : G \longrightarrow G/G_m$$

is a surjective submersion.

Consider the orbit map $\psi^m : G \longrightarrow M$ such that $\psi^m(g) = \psi_g(m)$ for every $g \in G$. It induces a well defined map:

$$B_m : G/G_m \longrightarrow M$$

such that:

$$B_m \circ \tau = \psi^m.$$

Namely, the diagram:

$$\begin{tikzcd}
G \arrow[r, "\tau"] \arrow[dr, "\psi^m"'] & G/G_m \arrow[d, "B_m"] \\
& M
\end{tikzcd}$$

commutes. Moreover, since τ is a surjective submersion, differential geometry Lemma A.1.3 implies that B_m is smooth.

Proposition A.1.4. *We have two properties of the map B_m:*

1. *B_m is a bijection onto the orbit $\mathcal{O}_m$ through m.*

2. *B_m is an injective immersion.*

An important consequence of Proposition A.1.4 (precisely point *2*) is the following theorem that enlightens the goal of the above construction.

Theorem A.1.5 (Orbit identification theorem). *Let $\psi : G \times M \longrightarrow M$ be a Lie group action on a manifold M and let $m \in M$. Then the two manifolds are diffeomorphic:*

$$\mathcal{O}_m \simeq G/G_m,$$

where $\mathcal{O}_m$ has the smooth structure as an immersed submanifold in M.

Proof (of Proposition A.1.4). Let us first prove *1.* B_m is injective: assume $g_1, g_2 \in G$ such that:

$$\psi_{g_1}(m) = B_m(g_1\, G) = B_m(g_2\, G) = \psi_{g_2}(m).$$

Then $\psi_{g_1^{-1} g_2}(m) = m$, thus $g_1\, G_m = g_2\, G_m$.
B_m is surjective on $\mathcal{O}_m$: if $m' \in \mathcal{O}_m$ there exists $g \in G$ such that $\psi_g(m) = m' = B_m(g\, G_m)$.
Now we prove *2.* We show that if $g \in G$, then:

$$\ker\, T_g\psi^m \;=\; \ker\, T_g\tau. \tag{A.3}$$

This is actually enough to get the thesis because chain rule gives:

$$T_g\psi^m \;=\; T_{[g]}B_m \circ T_g\tau$$

and from linear algebra, (A.3) gives necessarily that $\ker\, T_{[g]}B_m = \langle 0 \rangle$.

We claim it suffices to show (A.3) at $g = e_G$. Indeed, on one hand ψ^m is trivially equivariant with respect to L and ψ, in the sense that:

$$\psi^m \circ L_g \;=\; \psi_g \circ \psi^m, \quad \forall\, g \in G.$$

Differentiating the last identity at $h = e_G$:

$$T_g\psi^m \circ T_e L_g \;=\; T_m\psi_g \circ T_e\psi^m.$$

Since L_g, ψ_g are diffeomorphisms, $T_e L_g, T_m\psi_g$ are linear isomorphisms and so $r(T_g\psi^m) = r(T_e\psi^m)$ for every $g \in G$. On the other hand, τ has constant rank because it is a submersion.

As a consequence we should prove the identity:

$$\ker\, T_e\psi^m \;=\; \ker\, T_e\tau, \tag{A.4}$$

and we will do it comparing the two spaces with $\mathfrak{g}_m = Lie(G_m)$. Now $T_e\psi^m : \mathfrak{g} \longrightarrow T_m M$ and for every $\xi \in \mathfrak{g}$:

$$T_e\psi^m(\xi) \;=\; \left.\frac{d}{dt}\right|_{t=0} \psi_{\exp(t\xi)}(m) \;=\; \xi_M(m),$$

where $\xi_M \in \mathfrak{X}(M)$ is the infinitesimal vector field with respect to the action ψ. We first check that $\ker T_e\psi^m = \mathfrak{g}_m$ with double inclusions.
" $\subset$ ". We have $\xi_M(m) = 0$ if and only if $\psi_{\exp(t\xi)}(m) = m$ for every $t \in \mathbb{R}$, if and only if $\exp(t\xi) \in G_m$ for every $t \in \mathbb{R}$, which implies $\xi = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \in \mathfrak{g}_m$.

" $\supset$ ". For a $\xi \in \mathfrak{g}_m$, there exists a local smooth curve $g(t) \subset G_m$ such that $\xi = \frac{d}{dt}\big|_{t=0} g(t)$. Then:

$$T_e\psi^m(\xi) \;=\; \frac{d}{dt}\bigg|_{t=0} \psi^m(g(t)) \;=\; \frac{d}{dt}\bigg|_{t=0} m \;=\; 0.$$

On the other hand we trivially have:

$$\ker T_e\tau \subset \mathfrak{g}_m \tag{A.5}$$

from the fact that $\tau|_{G_m}$ is constantly equal to e. Also (A.5) must be a set equality by a dimension check and the fact that τ is a submersion:

$$\dim(\ker T_e\tau) \;=\; \dim(\mathfrak{g}) - (\dim(\mathfrak{g}) - \dim(\mathfrak{g}_m)) \;=\; \dim(\mathfrak{g}_m).$$

This finally gives (A.4) and concludes the proof of the Proposition. □

A.2 Standard symplectic reduction

Point-wise symplectic reduction is the a central ingredient of the thesis. It involves several objects, so it is useful to fix the notation.
Let (P, ω) be a symplectic manifold and $\chi : M \times P \longrightarrow P$ be a free, proper and symplectic M-action. We recall that if $\xi \in \mathfrak{m}$ and $\xi_P \in \mathfrak{X}(P)$ is its infinitesimal generator, then this is naturally a local Hamiltonian vector field and we shall assume it happens to be global. A Momentum map is a function $J : P \longrightarrow \mathfrak{m}^*$ such that:

$$\langle J(z), \xi \rangle = J_\xi(z),$$

for every $z \in P$, where $J_\xi \in C^\infty(P)$ is a Hamiltonian function of ξ_P. A Momentum map is equivariant if it is with respect to Coadjoint action on $\mathfrak{m}^*$, i.e. $J \circ \chi_m = Ad^*_{m^{-1}} \circ J$.

Proposition A.2.1. *If the Lie group M acts in a free and symplectic way onto a symplectic manifold P and $J : P \longrightarrow \mathfrak{m}^*$ is a momentum map, then J is a submersion onto its image.*

Proof. For a proof one can rely on Proposition 1.1.2 of [10], observing that the symmetry algebra at every point $z \in P$:

$$\mathfrak{g}_z \;=\; \{\xi \in \mathfrak{g} \mid \xi_P(z) = 0\}$$

is trivial by having a free action on P. □

As a consequence of this fact, if we take $\mu \in \mathfrak{m}^*$ in the image of J, then the level set:

$$J^{-1}(\mu) = \{z \in P \mid J(z) = \mu\}$$

is a smooth embedded manifold in P. This is actually the manifold we will reduce. A problem occurs: the restriction of χ to $J^{-1}(\mu)$, in general, is not well defined if we let the whole group act. In any case we can consider the restriction:

$$\bar{\chi} : M_\mu \times J^{-1}(\mu) \longrightarrow J^{-1}(\mu),$$

where M_μ is the isotropy subgroup of μ with respect to Coadjoint action. Assume J to be equivariant. Then $\bar{\chi}$ is well defined because if $z \in J^{-1}(\mu)$ and $m \in M_\mu$:

$$J(\bar{\chi}_m(z)) \ = \ Ad^*_{m^{-1}}(J(z)) \ = \ Ad^*_{m^{-1}}(\mu) \ = \ \mu.$$

Also one could prove $\bar{\chi}$ still is free and proper. This allows to say that the orbit space:

$$P_\mu \ := \ J^{-1}(\mu)/M_\mu$$

has a smooth structure given by the quotient topology, with the orbit projection $\pi_\mu : J^{-1}(\mu) \longrightarrow P_\mu$ a surjective submersion. We are ready to state the *Reduction Theorem*, which states that P_μ has a symplectic structure as well. It is due to Meyer [1973] and Marsden-Weinstein [1974] and it can be found as Theorem 1.1.3 in [10].

Theorem A.2.2 (Symplectic Reduction Theorem). *Let $\chi : M \times P \longrightarrow P$ be a free, proper, symplectic Lie group action with $J : P \longrightarrow \mathfrak{m}^*$ equivariant momentum map. Let $\mu \in \mathfrak{m}^*$ be a momentum value in the image of J. Then, as in the above notation, the reduced space (P_μ, ω_μ) is a symplectic manifold with $\omega_\mu \in \Lambda^2(P_\mu)$ uniquely characterised by the equality:*

$$\pi_\mu^* \omega_\mu \ = \ i_\mu^* \omega,$$

where $i_\mu : J^{-1}(\mu) \hookrightarrow P$ is the natural inclusion map, which is smooth as $J^{-1}(\mu)$ is embedded in P.

We also have some geometrical properties of the spaces we introduced.

Lemma A.2.3 (Reduction Lemma). *Let $M \cdot z$, $M \cdot \mu$ be the orbits of $z \in P$, $\mu \in J(P)$ with respect to the actions of χ and Ad^* resp. and $M_\mu \cdot z$ the one if we only let M_μ act. Then:*

(i) $J^{-1}(M \cdot \mu) \ = \ \{M \cdot z \mid J(z) = \mu\}$;

(ii) $M_\mu \cdot z \;=\; (M \cdot z) \cap J^{-1}(\mu)$*;*

(iii) $T_z\,(M \cdot z)$ *and* $T_z\,(J^{-1}(\mu)$ *are* ω*-orthogonal.*

Proof. We present a proof which is essentially taken from [10].

(i) $z \in J^{-1}(M \cdot \mu)$ if and only if $J(z) = Ad^*_{m^{-1}}\mu$ for some $m \in M$. As a consequence by equivariance:

$$\mu \;=\; Ad^*_m(J(z)) \;=\; J(\chi_{m^{-1}}(z)),$$

so that:

$$\chi_{m^{-1}}(z) \in J^{-1}(\mu) \iff z \in M \cdot J^{-1}(\mu),$$

as we wanted.

(ii) $\chi_m(z) \in J^{-1}(\mu)$ if and only if:

$$\mu = J(\chi_m(z)) \;=\; Ad^*_{m^{-1}}(J(z)) \;=\; Ad^*_{m^{-1}}(\mu)$$

if and only if $m \in M_\mu$.

(iii) By definition $v_z \in (T_z(M \cdot z))^\omega$ means that for every $\xi \in \mathfrak{m}$:

$$\omega_z(\xi_P(z), v_z) \;=\; 0.$$

By definition of J, this is equivalent to:

$$\langle dJ(z)v_z, \xi\rangle \;=\; 0 \quad \forall \xi \in \mathfrak{m} \iff dJ(z)v_z \;=\; 0.$$

Thus:

$$v_z \in (T_z(M \cdot z))^\omega \iff v_z \in Ker(T_zJ) \;=\; T_zJ^{-1}(\mu),$$

as we wanted.

□

The following Theorem applies instead to the particular case where the action is the cotangent lift of a left multiplication of a Lie group G on its cotangent bundle T^*G. Before getting into the statement it is useful to recall the fact that T^*G trivialize by the maps $\Lambda, P : T^*G \longrightarrow G \times \mathfrak{g}^*$ such that for any $(g, \alpha_g) \in T^*G$:

$$\Lambda(g, \alpha_g) \;=\; (g, T^*_eL_g(\alpha_g)), \quad P(g, \alpha_g) \;=\; (g, T^*_eR_g(\alpha_g)), \tag{A.6}$$

in the sense that such maps are (global) diffeomorphisms. Moreover we call $\lambda, \rho : G \longrightarrow \mathfrak{g}^*$ the dual Lie algebra components of Λ, P resp.

Theorem A.2.4. *Let G be a Lie group and $\tilde{L} : G \times T^*G \longrightarrow T^*G$ be the cotangent lifted action of the left multiplication of G on itself. This is a free, proper, and symplectic action on T^*G with the standard symplectic structure of a cotangent bundle. Moreover the map $\rho : T^*G \longrightarrow \mathfrak{g}^*$ as in* (A.6) *is an equivariant and surjective momentum map for $\tilde{L}$. Let us also consider the natural symplectic structure of a coadjoint orbit $\mathcal{O}_\mu$ as the symplectic leaf in the Poisson manifold $\mathfrak{g}^*$ (given by the Kirillov-Kostant-Soriau form ω_{KKS}). Then if we consider the reduced space (P_μ, ω_μ) as in Theorem A.2.2, this is symplectomorphic to $(\mathcal{O}_\mu, \omega_{KKS})$.*

Proof. See Theorem 1.2.3 of [10]. □

A.3 Principal Bundles Over a Lie Group

We give a brief overview of introductory elements of the theory of *Principal Bundles*, in order to fix the notation and gain the (local) existence of the so called *Lift map* of a smooth map. We use this result in the proof of the smoothness of the symplectomorphism inverse in the Main theorem of the thesis. The following is mainly based on section 1.2 of [5].

We start with the definition of a Principal Bundle over a Lie group G.

Definition A.3.1. *Let G be a Lie group. A Principal G-Bundle is a fibration $\pi : E \longrightarrow B$ such that:*

- *G acts freely on the right on E by ψ and the action respects the fibers, i.e. the following diagram commute:*

$$\begin{tikzcd}
G \times E \arrow[r, "\psi"] \arrow[d, "(\pi,\xi)"'] & E \arrow[d, "\pi"] \\
B \times \{e\} \arrow[r, "id_B"] & B.
\end{tikzcd}$$

- *The induced action on the fibers is transitive;*
- *E is locally trivial: for every $b \in B$, $\exists\, U \subset B$ neighborhood of b such that there exists $\chi : \pi^{-1}(U) \longrightarrow U \times G$ diffeomorphism which is G-equivariant:*

$$\begin{tikzcd}
\pi^{-1}(U) \times G \arrow[r, "(\chi{,}id_G)"] \arrow[d, "\psi"'] & U \times G \times G \arrow[d, "(id{,}mult_G)"] \\
\pi^{-1}(U) \arrow[r, "\chi"] & U \times G.
\end{tikzcd}$$

Example A.3.1. *Let us consider a free group action of a Lie group G on a manifold M:*

$$\psi : G \times M \longrightarrow M.$$

Set B to be the orbit space $B = M/G$ and $\pi : M \longrightarrow M/G$ the natural projection. Then the fibers are $\pi^{-1}([y]) = y\,G$ and the action satisfies the first two properties of the previous definition. In order to say $\pi : M \longrightarrow M/G$ is a Principal G-Bundle we should check the local triviality condition. In general for this to hold we need an "extra compactness property" either of the group or the action. The following theorem gives sufficient conditions for this to happen.

Theorem A.3.1. *Let $\psi : G \times M \longrightarrow M$ be a free, proper Lie group action of G on a manifold M. Then the construction in the Example A.3.1 defines a Principal G-Bundle.*

Corollary A.3.2. *Let H be a regular Lie subgroup of a Lie group G (which acts on G by left multiplication). Then:*

$$\pi : G \longrightarrow G/H$$

is a Principal H-Bundle.

Now let us take into account a generic Principal G-Bundle $\pi : E \longrightarrow B$ and a smooth map $\varphi : Z \longrightarrow B$ from a manifold Z to the base space B.

Definition A.3.2. *The map $\varphi^\ell : Z \longrightarrow B$ is a lift of φ if the diagram commutes:*

$$\begin{tikzcd} Z \arrow[r, "\varphi^\ell"] \arrow[dr, "\varphi"'] & E \arrow[d, "\pi"] \\ & B \end{tikzcd}$$

Lift of smooth maps always exists locally (but not necessarily globally).

Theorem A.3.3 (Local lift existence)**.** *Let $z_0 \in Z$. There exists a neighborhood $V \subset Z$ of z_0 and a map $\varphi^\ell_V : V \longrightarrow E$ such that φ^ℓ_V is a lift of $\varphi|_V$.*

Proof. By the Principal Bundle Structure there exists a neighborhood $U \subset B$ of $\varphi(z_0) \in B$ and a diffeomorphism $\chi : \pi^{-1}(U) \longrightarrow U \times G$ such that:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\chi"] \arrow[dr, "\pi"'] & U \times G \arrow[d, "p_1"] \\ & U. \end{tikzcd}$$

Let $V := \varphi^{-1}(U)$: this is an open set in Z containing z_0, hence a neighborhood of it. Set $\hat{\varphi} : V \longrightarrow U \times G$ such that:

$$\hat{\varphi}(z) \;=\; (\varphi(z), e),$$

which is of course smooth. Then the map:

$$\varphi^{\ell}_{V} \;:=\; \chi^{-1} \circ \hat{\varphi}$$

actually is an instance of lift of $\varphi|_V$. $\square$

Bibliography

[1] Ralph Abraham and Jerrold Marsden. *Foundations of mechanics.* American Mathematical Society, 1978.

[2] V. Arnold. Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. In *Annales de l'institut Fourier*, volume 16, 1966.

[3] Anthony Blaom. Hamiltonian G-Spaces with regular momenta. *Control and Dynamical Systems Technical Memorandum CIT-CDS*, 1996.

[4] A. Bravo-Doddoli, E. Le Donne, and N. Paddeu. Symplectic reduction of the sub-Riemannian geodesic flow for metabelian nilpotent groups. *Geometric Mechanics*, 01(01):2450002, 2024.

[5] Ralph L. Cohen. The topology of fiber bundles, lecture notes. *Standford University*, 1998.

[6] L. Corwin and F.P. Greenleaf. *Representations of nilpotent Lie groups and their applications: Volume 1, Part 1, Basic theory and examples*, volume 18. Cambridge university press, 1990.

[7] J.J. Duistermaat and J.A.C. Kolk. *Lie groups.* Springer Science & Business Media, 1999.

[8] Enrico Le Donne and Francesca Tripaldi. A cornucopia of Carnot groups in low dimensions. *Analysis and Geometry in Metric Spaces*, 10(1):155–289, 2022.

[9] John M. Lee. *Smooth manifolds.* Springer, second edition, 2013.

[10] J. Marsden, G. Misiolek, J.P. Ortega, M. Perlmutter, and T.S. Ratiu. *Hamiltonian reduction by stages.* Springer, 2007.

[11] J. Marsden, T. S Ratiu, and J. Scheurle. Reduction theory and the Lagrange-Routh equations. *Journal of mathematical physics*, 41(6), 2000.

[12] J. Marsden and T.S. Ratiu. *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, volume 17. Springer Science & Business Media, second edition, 1999.

[13] J. Marsden and A. Weinstein. Some comments on the history, theory, and applications of symplectic reduction. *in Quantization of singular symplectic quotients, Progress in Math, Eds. N.P. Landsman, M. Pflaum, M. Schlichenmaier*, 198, 2001.

[14] J.P. Ortega and T.S. Ratiu. *Momentum maps and Hamiltonian reduction*, volume 222. Springer Science & Business Media, 2004.

[15] S. Smale. Topology and mechanics. ii: the planar n-body problem. *Inventiones mathematicae*, 11, 1970.