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Day Convolution for (equivariant) ∞-operads

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"You will never know your limits until you learn category theory"

Me, after finishing this thesis

To my little sister

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Contents

Introduction

The starting point for this thesis is the notion of *symmetric monoidal category.* Intuitively, this is a category $\mathcal{C}$ equipped with a bifunctor:

$$\otimes \colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$$

and with a specified object $1 \in \mathcal{C}$, whose behaviour resembles that of a commutative monoid. The tensor product $\otimes$ should be associative, commutative and 1 should act as a unit for it. Of course, in the categorical setting we should not ask for strict equalities in the associative, commutative and unital laws but rather for isomorphisms, which are then part of the data, and must satisfy some further coherence conditions.

Consider now two ordinary categories $\mathcal{C}$ and $\mathcal{D}$ equipped with a symmetric monoidal structure. Under appropriate hypotheses, the Day convolution construction is a natural way of endowing the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with a symmetric monoidal structure that is particularly desirable: for example, its commutative algebra objects correspond to lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

This construction can be generalized to the ∞-categorical setting, as was done by Glasman in [Gla16]. The resulting symmetric monoidal ∞-category structure satisfies properties analogous to the ordinary case. The need for such a generalization arose as many other topics were generalized to ∞-categories, as highlighted in the introduction of [Gla16]. For example, Day convolution can be used to equip G-spectra, when viewed as spectral Mackey functors, with a suitable symmetric monoidal ∞-category structure.

An important example, studied in this thesis, is that of filtered spectra, which are collected in the ∞-category $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$. Day convolution provides a way of endowing this ∞-category with a symmetric monoidal structure, for which a very explicit formula can be obtained at the level of objects. Given two filtered spectra $X, Y : \mathbb{Z} \to \mathrm{Sp}$, one can compute pointwise their Day convolution product as:

$$(X \otimes Y)(n) = \operatorname*{colim}_{i+j \leq n} X(i) \wedge Y(j)$$

for every $n \in \mathbb{Z}$, where the tensor product in Sp is the usual smash product of spectra, denoted by $\wedge$.

A great advantage of Glasman's construction is precisely that it is very explicit: the constructions are done directly step by step, and are therefore easy to work with. This makes it well suited for direct computations, for example as we just highlighted for filtered spectra.

This is however quite limited in its applications. Symmetric monoidal ∞-categories are just a special case of ∞-operads, and one would expect to be able to obtain a Day convolution structure for the latter, at least in some special cases. This is done by Lurie in [Lur17, Section 2.2.6], where he develops a Day convolution tensor product for functor ∞-categories between $\mathcal{O}$-monoidal ∞-categories, where $\mathcal{O}^{\otimes}$ is an ∞-operad. Choosing $\mathcal{O}^{\otimes}$ to be the simplest possible, i.e. the commutative ∞-operad, recovers Glasman's construction.

Lurie's approach, in contrast with Glasman's, has the advantage of being more abstract, as well as more general, and it is therefore easily applicable in a wider range of situations. One can then recover things in a more explicit form when needed.

A further generalization of the theory of ∞-categories is obtained by introducing a "parametrizing category" $\mathcal{T}$, which can be any other ∞-category. Informally, the idea is to work with Cat_∞-valued presheaves on $\mathcal{T}$, that is, functors $\mathcal{T}^{\mathrm{op}} \to \mathrm{Cat}_\infty$; usual ∞-category theory is recovered when $\mathcal{T} \simeq \{*\}$ is contractible. Such an object is called a $\mathcal{T}$-∞-category, and can equivalently be described as a cocartesian fibration over $\mathcal{T}^{\mathrm{op}}$.

Depending on the choice of $\mathcal{T}$, one may obtain different flavours of parametrized higher category theory. A good, and relatively simple, example occurs when $\mathcal{T}$ is the orbit category $\mathcal{O}rb_G$ of a finite abelian group G. Intuitively, a G-∞-category $\underline{\mathcal{C}}$ assigns to every orbit G/H an ∞-category $\underline{\mathcal{C}}(G/H)$ of "H-objects" of $\underline{\mathcal{C}}$, and to every morphism $G/H \to G/K$ a "restriction map" $\underline{\mathcal{C}}(G/K) \to \underline{\mathcal{C}}(G/H)$.

This gives rise to equivariant structures which are well-suited to model typical equivariant behaviours in mathematics with ∞-categories. A simple example coming from 1-categories is given by the complex representations of G. The corresponding G-∞-category, seen as a functor $\mathcal{O}rb_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$, maps G/H to the (nerve of the category of the) ring $R(H)$ of complex representations of $H \leq G$, and for every $H \leq K \leq G$, giving a morphism $G/H \to G/K$, this is mapped to the usual restriction map $\mathrm{Res}_H^K \colon R(K) \to R(H)$.

One can then develop higher algebra in the equivariant setting; this was initially done for more general parametrizing categories $\mathcal{T}$ by Nardin and Shah in [NS22]. In particular, one can define the notion of a G-symmetric monoidal G-∞-category. This structure is the right choice to model mathematical objects of equivariant nature which, in addition to restriction functors that arise contravariantly, also have other natural functors which instead are covariant. Think again at the complex representations of G: for any $H \leq K \leq G$, giving a morphism $G/H \to G/K$, we also have the induction functor $\mathrm{Ind}_H^K \colon R(H) \to R(K)$, which is covariant. Another important example is given by G-spectra, equipped contravariantly with the usual restriction maps, and covariantly with the Hill-Hopkins-Ravenel norm functors. This whole thing may remind the reader of Mackey functors for 1-categories, and in fact there is an equivalence between G-symmetric monoidal G-∞-categories and the ∞-categorical version of Mackey functors.

A natural goal within our interests is then to obtain a theory of Day convolution in this context. This is done by Nardin and Shah in [NS22], where they develop parametrized higher algebra, working with parametrized ∞-operads, and they describe a Day convolution tensor product in this context. In particular, in the equivariant setting we get Day convolution for $\mathcal{O}$-monoidal G-∞-categories, where $\mathcal{O}^{\otimes}$ is a G-∞-operad.

The goal of this thesis is to give a comprehensive overview of all these different flavours of Day convolution:

(1) for symmetric monoidal ∞-categories ([Gla16]);

(2) for $\mathcal{O}$-monoidal ∞-categories, where $\mathcal{O}^{\otimes}$ is an ∞-operad ([Lur17]);

(3) for $\mathcal{O}$-monoidal G-∞-categories, where $\mathcal{O}^{\otimes}$ is a G-∞-operad ([NS22]).

Outline of the thesis

We give here an overview of the contents of the thesis, chapter by chapter.

In Chapter 1, we start by recalling the definition of symmetric monoidal category, and look at an equivalent formulation in terms of fibrations. We then use a generalization of the latter to define symmetric monoidal ∞-categories, and we discuss morphisms between them. We conclude by looking at the ∞-category Sp of spectra, and see two examples of symmetric monoidal structures on it.

Chapter 2 is dedicated to Glasman's Day convolution construction on symmetric monoidal ∞-categories, following and also expanding on [Gla16]. After giving some intuition on Day convolution for 1-categories, we introduce the main technical tools needed: relative colimits and left Kan extensions along inclusions. Then, we go deep into the details of Glasman's construction, proving most of the results. Finally, we apply this to filtered spectra, and we obtain an explicit formula to evaluate on integers the tensor product of two filtered spectra.

In Chapter 3, we give an overview on the extension of Day convolution to $\mathcal{O}$-monoidal ∞-categories, where $\mathcal{O}^{\otimes}$ is an ∞-operad, following [Lur17, Section 2.2.6]. In doing so, we first also discuss colored operads, ∞-operads and maps between them.

In Chapter 4, we move to equivariant higher category theory. We first discuss G-∞-categories, and then we move to G-symmetric monoidal G-∞-categories, giving several examples to motivate them.

Finally, Chapter 5 is dedicated to specializing Nardin and Shah's Day convolution construction, done in [NS22], to the equivariant setting. We discuss G-∞-operads and then we sketch the main features of Day convolution for $\mathcal{O}$-monoidal G-∞-categories, with $\mathcal{O}^{\otimes}$ a G-∞-operad.

Conventions

We generally assume that the reader is familiar with all those definitions, constructions and results that form the basics of the theory of ∞-categories, as developed by Lurie in the first chapters of [Lur09]. A good source for this is also Kerodon [Lur18].

To ease the notation, whenever we are dealing with ordinary 1-categories seen as ∞-categories, we will omit the nerve N from the notation. For example, we will write $\mathbb{Z}$ instead of $\mathrm{N}(\mathbb{Z})$, or $\mathcal{F}_*$ instead of $\mathrm{N}(\mathcal{F}_*)$.

To challenge the reader's knowledge of both higher category theory and the Dutch language, we propose the following question:

Vraag 0.0.1 (Kies een van de volgende twee)**.** Kun je deze grap begrijpen?

$$\Delta^0 \qquad\qquad \Delta^1$$

Chapter 1

Symmetric monoidal ∞-categories

In this first chapter, we define symmetric monoidal ∞-categories and look at some of the main examples. In particular, we will discuss the smash product on the ∞-category Sp of spectra.

We start with a brief discussion on symmetric monoidal categories in the 1-categorical setting, in order to motivate what comes next. The classical definition has an equivalent formulation in terms of certain kinds of fibrations; this latter characterization, although apparently more complicated, is easily generalized to ∞-categories, whereas the former is not. We then therefore turn to the definition of symmetric monoidal ∞-categories and morphisms between them, in the ∞-categorical setting, after recalling some of the main theoretical definitions needed. Finally, we look at some examples, and in particular at the symmetric monoidal structure on Sp given by the smash product.

1.1 Symmetric monoidal categories

We start by briefly recalling the definition of a symmetric monoidal category in the ordinary context, and we spell out an equivalent characterization of such a structure as a cocartesian fibration satisfying the so-called Segal condition. This motivates the definition of a symmetric monoidal ∞-category, as the latter situation is easier to generalize to the higher context. A reference for this material is [Gro20].

Definition 1.1.1. A *symmetric monoidal category* $(\mathcal{C}, \otimes, 1)$ is a category $\mathcal{C}$ together with an object $1 \in \mathcal{C}$, a functor:

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

and natural isomorphisms expressing the associativity, commutativity and unit axioms, that are required to satisfy some coherence conditions.[1]

Example 1.1.2. Let $\mathbb{Z}$ denote the category given by the ordered set of integers with the standard order. That is, the objects are the integers, and between two integers $i, j \in \mathbb{Z}$ there is a unique morphism if and only if $i \leq j$. Then, there is a symmetric monoidal structure on $\mathbb{Z}$ such that the tensor product of two integers i and j is given by the sum $i + j$, and consequently the unit object is 0. This example will come into play later in Chapter 2 when discussing filtered spectra.

[1] For a complete definition, see for example [Lan98, Chapter XI]

Example 1.1.3. Let $\mathcal{C}$ be a category with all finite products, so in particular there exist final objects. Then, there is a symmetric monoidal structure on $\mathcal{C}$, called the *cartesian symmetric monoidal structure*, with tensor product given by the product in $\mathcal{C}$:

$$\otimes_{\mathcal{C}} = \times : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \qquad (C_1, C_2) \mapsto C_1 \times C_2$$

and unit given by a chosen final object of $\mathcal{C}$.

Dually, suppose that $\mathcal{C}$ is a category with all finite coproducts, so in particular there exist initial objects. Then, the *cocartesian symmetric monoidal structure* on $\mathcal{C}$ has as tensor product the coproduct in $\mathcal{C}$:

$$\otimes_{\mathcal{C}} = \amalg : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}, \qquad (C_1, C_2) \mapsto C_1 \amalg C_2$$

with unit given by a chosen initial object.

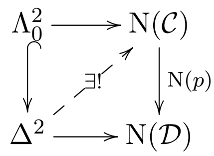
Given two different ways of tensoring the same objects, expressed by different bracketing order, there may be multiple distinct ways of using associativity to go from one expression to the other. We would like all of these to result in the same object. One of the coherence axioms, the commutativity of the pentagon diagram, takes care of this. Intuitively, given objects $X, Y, Z, W \in \mathcal{C}$, it guarantees that the two different ways of using associativity to go from the expression $((X \otimes Y) \otimes Z) \otimes W$ to the expression $X \otimes (Y \otimes (Z \otimes W))$ coincide. Requiring this turns out to be enough to ensure that for any pair of expressions with different bracketing, going from one to the other in any way using associativity gives the same result.

However, this is not enough anymore in the higher categorical context: to generalize the above definition, we would need to add coherence conditions for every possible way of using associativity (see the Stasheff associahedra), as we expect everything to hold only up to homotopy; this would result in an enormous amount of data. We therefore spell out a different characterization of symmetric monoidal categories, which will be easily generalizable for ∞-categories.

We need the following preliminary definitions:

Definition 1.1.4. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let $f\colon c_1 \to c_2$ be a morphism in $\mathcal{C}$. We say that f is *p-cocartesian* if for every $f'\colon c_1 \to c_3$ and every $g\colon p(c_2) \to p(c_3)$ such that $g \circ p(f) = p(f')$, there exists a unique lift $f''\colon c_2 \to c_3$ of g, i.e. such that $p(f'') = g$.

Remark 1.1.5. An intuitive way of thinking of this condition is the following. If we apply the nerve construction, which is fully faithful, to p, we are essentially requiring that there is a unique lift for every diagram:



such that the first edge of Λ^2_0 is mapped to f. This gives a clear intuition on how to generalize this definition to ∞-categories, by requiring the same for higher left horns, and removing unicity, expecting a contractible space of choices of lifts instead; see Definition 1.2.1 and the subsequent.

Remark 1.1.6. Equivalently, with the same notation as above, a morphism f is p-cocartesian if and only if the following square is a pullback:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(c_2, c_3) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(p(c_2), p(c_3)) \\ \downarrow & & \downarrow {\scriptstyle -\circ p(f)} \\ \mathrm{Hom}_{\mathcal{C}}(c_1, c_3) & \xrightarrow[p(-)]{} & \mathrm{Hom}_{\mathcal{D}}(p(c_1), p(c_3)). \end{array}$$

Definition 1.1.7. A functor $p\colon \mathcal{C} \to \mathcal{D}$ is a *cocartesian fibration of categories*, or a *Grothendieck opfibration*, if for every $c \in \mathcal{C}$ and every morphism $g\colon p(c) \to d$ in $\mathcal{D}$, there exists a cocartesian lift of g in $\mathcal{C}$.

Remark 1.1.8. Given a cocartesian fibration, there is a way to obtain a pseudo-functor $\mathcal{D} \to \mathrm{Cat}$ that on objects maps $d \in \mathcal{D}$ to the fiber $\mathcal{C}_d = \mathcal{C} \times_{\mathcal{D}} \{d\}$. That is, given a morphism $g\colon d_1 \to d_2$ in $\mathcal{D}$, there is an essentially unique way (up to natural isomorphism) of obtaining a functor $\mathcal{C}_{d_1} \to \mathcal{C}_{d_2}$ in a covariant way. Such a construction is done using the cocartesian lifts of g; given $c \in \mathcal{C}_{d_1}$, the idea is to pick a cocartesian lift $c \to c'$ of g with source c, and then map c to c'. For a more detailed treatment, see [Gro20, Lemma 4.4] and the following discussion.

Definition 1.1.9. We define $\mathcal{F}_*$ to be the category with objects the finite sets $\langle n \rangle := \{*, 1, \dots, n\}$, for every $n \geq 0$, where $\langle 0 \rangle = \{*\}$, and with morphisms given by maps $\alpha\colon \langle m \rangle \to \langle n \rangle$ such that $\alpha(*) = *$. Denote $\langle m \rangle^\circ = \langle m \rangle \setminus \{*\}$.

Definition 1.1.10. Let $f\colon \langle m \rangle \to \langle n \rangle$ be a morphism in $\mathcal{F}_*$. We will call f:

(1) *inert* if $f^{-1}(\{j\})$ has exactly one element for every $j \in \langle n \rangle^\circ$, that is, f induces an injective map $\langle n \rangle^\circ \to \langle m \rangle^\circ$.

(2) *active* if $f^{-1}(\{*\}) = \{*\}$.

Moreover, for every $1 \leq i \leq n$, denote by $\rho_i\colon \langle n \rangle \to \langle 1 \rangle$ the morphisms in $\mathcal{F}_*$ defined by mapping $i \in \langle n \rangle$ to 1, and all other elements of $\langle n \rangle$ to $*$.

Remark 1.1.11. The category $\mathcal{F}_*$ just defined is equivalent to the category Fin_* of finite sets with a basepoint $*$, and morphisms basepoint-preserving maps of sets. It will play a fundamental role in this first chapter, as it is the right category to encode all the information that we care about in the symmetric context.

Our next goal is to encode all the information of a symmetric monoidal category $\mathcal{C}$ into a "new" category $\mathcal{C}^\otimes$ equipped with a functor $\mathcal{C}^\otimes \to \mathcal{F}_*$. These should remember all the information, such as how to tensor together multiple objects, or the various coherence conditions, and so the functor to $\mathcal{F}_*$ should also satisfy some requirements. The latter however are way more easy to spell out, and so it turns out that this is a compact way to encode all possible coherence conditions, without having to specify them, thus providing a great starting point to generalize to ∞-categories.

Construction 1.1.12. Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category. We construct a new category $\mathcal{C}^\otimes$ in the following way:

(1) the objects of $\mathcal{C}^\otimes$ are finite, possibly empty, sequences $[C_1, \dots, C_n]$ of objects of $\mathcal{C}$;

(2) the morphisms from a sequence $[C_1, \dots, C_n]$ to another sequence $[C'_1, \dots, C'_m]$ are given by a morphism $\alpha\colon \langle n\rangle \to \langle m\rangle$ in $\mathcal{F}_*$ and, for every $1 \leq j \leq m$, by morphisms $f_j\colon \bigotimes_{\alpha(i)=j} C_i \to C'_j$.

Composition is then defined in the obvious way.

There is a forgetful functor $\mathcal{C}^{\otimes} \to \mathcal{F}_*$, which turns out to be a cocartesian fibration. Moreover, it satisfies an additional property, called *Segal condition*:

(S): For every $1 \leq i \leq n$ let $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ be as in 1.1.9. According to 1.1.8, each ρ_i induces a functor $\mathcal{C}^{\otimes}_{\langle n\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{C}$ between fibers. Then, the induced map:

$$\mathcal{C}^{\otimes}_{\langle n\rangle} \longrightarrow \prod_{1\leq i\leq n} \mathcal{C}^{\otimes}_{\langle 1\rangle} \simeq \prod_{1\leq i\leq n} \mathcal{C}$$

is an equivalence of categories.

So, given a symmetric monoidal category $\mathcal{C}$, we have constructed an associated cocartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{F}_*$, satisfying the Segal condition. There is also an inverse construction: a cocartesian fibration $p\colon \mathcal{D} \to \mathcal{F}_*$ satisfying the Segal condition determines, up to canonical equivalence, a symmetric monoidal structure on the fiber $\mathcal{D}_{\langle 1\rangle}$. For example, the tensor product is determined by the cocartesian pushforward of the active morphism $\langle 2\rangle \to \langle 1\rangle$ of $\mathcal{F}_*$:

$$\mathcal{D}_{\langle 2\rangle} \simeq \mathcal{D}_{\langle 1\rangle} \times \mathcal{D}_{\langle 1\rangle} \to \mathcal{D}_{\langle 1\rangle}.$$

For all the details, we refer to [Lur17, Remark 2.0.0.6].

This correspondence between symmetric monoidal categories and cocartesian fibrations satisfying the Segal condition motivates the definition of a symmetric monoidal ∞-category, which will be given in Definition 1.2.7.

1.2 Symmetric monoidal ∞-categories

We now turn to the setting of ∞-categories. We first give the necessary preliminary definitions, which generalize Definitions 1.1.4 and 1.1.7. The goal is to provide a higher categorical setting to encode the structure of the fibration obtained in Construction 1.1.12.

Definition 1.2.1. Let $p\colon X \to S$ be a functor between simplicial sets X and S, and let f be an edge of X. Consider a lifting problem:

$$\begin{array}{ccc} \Lambda^n_0 & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow p \\ \Delta^n & \longrightarrow & S \end{array}$$

such that the first edge $\Delta^{\{0,1\}}$ of Λ^n_0 is mapped to f. We say that f is *p-cocartesian* if every such lifting problem admits a solution.

This is the obvious generalization of Definition 1.1.4. It admits an equivalent formulation, analogously to Remark 1.1.6, via requiring a certain commutative square to be a pullback. For a statement and proof of this, see for example [Lur18, Tag 01TL].

Definition 1.2.2. A functor $p\colon X \to S$ of simplicial sets is a *cocartesian fibration* if

(1) p is an inner fibration;

(2) for every $x \in X_0$ and every edge $g\colon p(x) \to s$ in S, there exists a p-cocartesian lift $f\colon x \to \overline{s}$ of g, that is, $p(f) = g$.

Example 1.2.3. Every left fibration is in particular a cocartesian fibration.

It follows straight from the definition that cocartesian fibrations satisfy the following properties, which we will use later on.

Proposition 1.2.4. *The following hold:*

(1) Any isomorphism of simplicial sets is a cocartesian fibration.

(2) The pullback of a cocartesian fibration is again a cocartesian fibration. That is, the class of cocartesian fibrations is stable under pullbacks.

(3) A composition of cocartesian fibrations is again a cocartesian fibration.

Given a simplicial set S, the Grothendieck construction provides a canonical equivalence between cocartesian fibrations over S and functors from S to Cat_∞, the ∞-category of ∞-categories. This result is among those referred to as "straightening-unstraightening" correspondences; see for example [Lur09, Section 3.2].

Theorem 1.2.5. *Let S be an ∞-category. There is a canonical equivalence:*

$$\mathrm{Cocart}(S) \simeq \mathrm{Fun}(S, \mathrm{Cat}_\infty)$$

where $\mathrm{Cocart}(S)$ *is the full subcategory of the slice category* $(\mathrm{Sset})_{/S}$ *on the cocartesian fibrations.*

Remark 1.2.6. In particular, under the above equivalence, given a functor $F\colon S \to \mathrm{Cat}_\infty$, the values of F on objects of S are precisely the fibers of the associated cocartesian fibration, and viceversa. Therefore, given a cocartesian fibration $p\colon X \to S$ and an edge $g\colon s_1 \to s_2$ in S, there is a functorial way to obtain a map $X_{s_1} \to X_{s_2}$ between the fibers of the fibration over s_1 and s_2. We will call this the *cocartesian pushforward* of g, and denote it as $g_!$. This is the analogue of Remark 1.1.8.

We are now ready to give the definition of a symmetric monoidal ∞-category.

Definition 1.2.7. A *symmetric monoidal ∞-category* is a cocartesian fibration $\mathcal{C}^\otimes \to \mathcal{F}_*$ such that it satisfies the *Segal condition*:

(S): Let $n \geq 0$ and write $\mathcal{C}^\otimes_{\langle n\rangle}$ for the fiber over $\langle n\rangle$. Recall that in $\mathcal{F}_*$ we have morphisms ρ_i for every $1 \leq i \leq n$, and that by Remark 1.2.6 these give rise to cocartesian pushforward functors $\mathcal{C}^\otimes_{\langle n\rangle} \to \mathcal{C}^\otimes_{\langle 1\rangle}$. Then, for every $n \geq 0$, these induce an equivalence of ∞-categories:

$$\mathcal{C}^\otimes_{\langle n\rangle} \simeq \prod_{1\leq i\leq n} \mathcal{C}^\otimes_{\langle 1\rangle}$$

We call $\mathcal{C} := \mathcal{C}^\otimes_{\langle 1\rangle}$ the *underlying ∞-category* of $\mathcal{C}^\otimes$. We will sometimes also say that the map $\mathcal{C}^\otimes \to \mathcal{F}_*$ endows $\mathcal{C}$ with a *symmetric monoidal ∞-category structure.*

In analogy to the situation in the ordinary context, this definition encodes in a compact way all the information that we expect from a symmetric monoidal structure. For example:

(1) the Segal condition for $n = 0$ tells us that $\mathcal{C}^{\otimes}_{\langle 0\rangle}$ is a contractible Kan complex, as it is equivalent to Δ^0, so up to equivalence the unique map $\langle 0\rangle \to \langle 1\rangle$ in $\mathcal{F}_*$ provides a functor $\Delta^0 \to \mathcal{C}$, which we can view as specifying a unit object in $\mathcal{C}$.

(2) The Segal condition for $n = 2$ provides an equivalence $\mathcal{C}^{\otimes}_{\langle 2\rangle} \simeq \mathcal{C} \times \mathcal{C}$. Together with the functor $\mathcal{C}^{\otimes}_{\langle 2\rangle} \to \mathcal{C}$, induced by the unique active morphism $\langle 2\rangle \to \langle 1\rangle$, we obtain a map:

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\simeq} \mathcal{C}^{\otimes}_{\langle 2\rangle} \to \mathcal{C}$$

which we can think of as a tensor product bifunctor on $\mathcal{C}$.

Example 1.2.8. The identity functor $\mathcal{F}_* \to \mathcal{F}_*$ is a symmetric monoidal ∞-category. Its underlying ∞-category is $\{\langle 1\rangle\} \cong \Delta^0$.

Example 1.2.9. Let $\mathcal{C}$ be an ordinary symmetric monoidal category. Consider the associated cocartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{F}_*$, constructed in 1.1.12. Then, applying the nerve construction yields a symmetric monoidal ∞-category $\mathrm{N}(\mathcal{C}^{\otimes}) \to \mathcal{F}_*$ (recall that we still denote by $\mathcal{F}_*$ the nerve of $\mathcal{F}_*$, to lighten the notation).

Example 1.2.10. As a special case of the previous example, starting from the symmetric monoidal category $\mathbb{Z}$ with the sum as tensor product of Example 1.1.2 and passing to its associated cocartesian fibration, we obtain a symmetric monoidal ∞-category. The fiber over an $S \in \mathcal{F}_*$ is given by a product of $|S^{\circ}|$ copies of $\mathbb{Z}$, and the cocartesian pushforwards are given by summation.

We now define the types of morphisms between symmetric monoidal ∞-categories that we want to work with. We want to do this in analogy to the usual (lax) symmetric monoidal functors; the idea is that we want such morphisms to "commute" with the tensor product operation. Before the actual definition, let us give some motivation for it, as it otherwise would seem quite strange.

Consider then a symmetric monoidal ∞-category $\mathcal{C}^{\otimes} \to \mathcal{F}_*$ and write objects C in the fiber $\mathcal{C}^{\otimes}_{\langle n\rangle}$ as $C \simeq C_1 \oplus \cdots \oplus C_n$ with the $C_i \in \mathcal{C}$ using the Segal condition. Then, for a functor $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ to be symmetric monoidal we would want to have equivalences:

$$F(C_1 \otimes \cdots \otimes C_n) \simeq F(C_1) \otimes \cdots \otimes F(C_n)$$

for all $n \geq 0$ and all objects $C \in \mathcal{C}^{\otimes}_{\langle n\rangle}$. That is, the components of $F(C)$ should be given by F applied to the components of C; note that since F is a functor over $\mathcal{F}_*$, both C and $F(C)$ have to be in fibers over the same $\langle n\rangle$. As the equivalence of the Segal condition comes from cocartesian pushforwards, it makes sense to require F to preserve cocartesian edges, and this turns out to be the right choice. For symmetric monoidal functors, we will require F to preserve all cocartesian edges, whereas for lax symmetric monoidal functors it is enough to preserve only the ones lying over inert edges.

Let us look for example at why this is reasonable in the case with two objects X and Y in $\mathcal{C}$. In $\mathcal{C}^{\otimes}$, we have a cocartesian edge $(X, Y) = X \oplus Y \to X \otimes Y$ lying over the active morphism $\langle 2\rangle \to \langle 1\rangle$. Then, F maps this to a cocartesian edge $(F(X), F(Y)) = F(X \oplus Y) \to F(X \otimes Y)$, where the first equality holds as the Segal condition is given by cocartesian pushforwards of inert edges, which are preserved by F. We also naturally have a cocartesian edge $(F(X), F(Y)) \to F(X) \otimes F(Y)$.

Therefore, looking back at the Definition 1.2.1 of cocartesian edges, we obtain arrows in both directions between $F(X) \otimes F(Y)$ and $F(X \otimes Y)$, which give an equivalence between them.

For a lax monoidal functor, it is enough to require F to preserve cocartesian edges only over inert morphisms of $\mathcal{F}_*$. Then, the only thing preserved by F in the situation above is the Segal condition, so the only cocartesian edge is $(F(X), F(Y)) \to F(X) \otimes F(Y)$, whereas the other edge $(F(X), F(Y)) \to F(X \otimes Y)$ is not necessarily cocartesian. Then, we only get an arrow:

$$F(X) \otimes F(Y) \to F(X \otimes Y)$$

again by the property defining cocartesian edges.

We therefore have the following definition.

Definition 1.2.11. Let $p\colon \mathcal{C}^{\otimes} \to \mathcal{F}_*$ and $q\colon \mathcal{D}^{\otimes} \to \mathcal{F}_*$ be symmetric monoidal ∞-categories. Let $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a functor such that the diagram:

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \longrightarrow & \mathcal{D}^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{F}_* & \end{array}$$

commutes. We say that F is:

(1) *symmetric monoidal* if F sends p-cocartesian edges to q-cocartesian edges;

(2) *lax symmetric monoidal* if F sends p-cocartesian edges lying over inert edges of $\mathcal{F}_*$ to q-cocartesian edges.

We let:

$$\mathrm{Fun}^{\mathrm{Mon}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \subseteq \mathrm{Fun}^{\mathrm{Lax}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \subseteq \mathrm{Fun}_{\mathcal{F}_*}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$$

denote the full subcategories of $\mathrm{Fun}_{\mathcal{F}_*}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the symmetric monoidal and lax symmetric monoidal functors, respectively.

The commutative algebra objects of a symmetric monoidal ∞-category are particularly important. In the ordinary situation, a *commutative algebra object* of a symmetric monoidal category $\mathcal{C}$ is an object A of $\mathcal{C}$ together with a multiplication map $A \otimes A \to A$ which has a unit and satisfies the usual associativity and commutativity constraints. These can be identified with lax symmetric monoidal functors from the trivial symmetric monoidal category $\{e\}$ to $\mathcal{C}$; see for example [Lur18, Tag 00DW]. This latter characterization is then easily generalized to ∞-categories.

Definition 1.2.12. Let $\mathcal{C}^{\otimes} \to \mathcal{F}_*$ be a symmetric monoidal ∞-category. A *commutative algebra object* is a lax symmetric monoidal functor from the identity symmetric monoidal structure on $\mathcal{F}_*$ of Example 1.2.8 to $\mathcal{C}^{\otimes} \to \mathcal{F}_*$. The corresponding ∞-category is:

$$\mathrm{CAlg}(\mathcal{C}^{\otimes}) := \mathrm{Fun}^{\mathrm{Lax}}(\mathcal{F}_*, \mathcal{C}^{\otimes}).$$

We conclude this section by sketching another pair of important examples, which generalize the cartesian and cocartesian symmetric monoidal structures of Example 1.1.3 to ∞-categories.

Example 1.2.13. Let $\mathcal{C}$ be an ∞-category with all finite products. Then, in [Lur17, Section 2.4.1], Lurie proves that there is an essentially unique way to endow $\mathcal{C}$ with a symmetric monoidal structure such that the tensor product $\otimes_{\mathcal{C}}$ is given by the product in $\mathcal{C}$, and the unit is a final object. This is made precise in [Lur17, Definition 2.4.0.1].

Dually, if $\mathcal{C}$ is an ∞-category with all finite coproducts, there is again an essentially unique way to endow it with a symmetric monoidal structure such that the tensor product $\otimes_{\mathcal{C}}$ is given by the coproduct in $\mathcal{C}$, and the unit is an initial object. This is done in [Lur17, Section 2.4.3].

1.3 The ∞-category of spectra

In this section, we briefly recall the definition of the ∞-category Sp of spectra, following parts of [Lur17, Section 1.4]. We assume the reader has some familiarity with [Lur17, Chapter 1], and so we recall only the details and results relevant to our purposes.

The aim is to endow Sp with a symmetric monoidal ∞-category structure, called smash product, which descends to the usual smash product of spectra on the homotopy category. In particular, we are looking for a symmetric monoidal structure on Sp whose tensor product preserves colimits separately in each variable, and whose unit object is the sphere spectrum $\mathbb{S}$. After briefly recalling the definition of Sp and of the sphere spectrum $\mathbb{S}$, we then state Theorem 1.3.5, which states that the requirements above give rise to an essentially unique symmetric monoidal structure on Sp as desired, which we will call the *smash product*. We also note that this structure is "multiplicative", in the sense that it satisfies the usual distribution laws with respect to the direct sum of spectra. The latter is the tensor product in the (co)cartesian symmetric monoidal structure on Sp (these coincide as Sp is stable), and can be thought of as an "additive" structure.

We start with the few theoretical definitions needed.

Definition 1.3.1. Let $\mathcal{C}$ be an ∞-category which admits finite limits. A *spectrum object* of $\mathcal{C}$ is a reduced and excisive functor $X\colon \mathcal{S}_*^{\mathrm{fin}} \longrightarrow \mathcal{C}$. Denote by $\mathrm{Sp}(\mathcal{C})$ the full subcategory of $\mathrm{Fun}(\mathcal{S}_*^{\mathrm{fin}}, \mathcal{C})$ spanned by the spectrum objects.

It is a result of Lurie [Lur17, Corollary 1.4.2.17] that the ∞-category $\mathrm{Sp}(\mathcal{C})$ is stable. Moreover, it can be identified, by [Lur17, Remark 1.4.2.25], with the homotopy limit of the tower:

$$\cdots \to \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}.$$

Note that there is an obvious functor:

$$\Omega^\infty\colon\ \mathrm{Sp}(\mathcal{C}) \longrightarrow \mathcal{C}$$

which is given by evaluation at the 0-sphere $S^0 = \Delta^0_+$, seen as an object of $\mathcal{S}_*^{\mathrm{fin}}$.

Definition 1.3.2. The *∞-category of spectra* is defined as the ∞-category of spectrum objects of the ∞-category $\mathcal{S}$ of spaces:

$$\mathrm{Sp} := \mathrm{Sp}(\mathcal{S}).$$

An object of Sp is called a *spectrum*.

It can be shown that the functor $\Omega^\infty \colon \mathrm{Sp} := \mathrm{Sp}(\mathcal{S}) \to \mathcal{S}$ satisfies the hypotheses of the Adjoint Functor Theorem [Lur09, Corollary 5.5.2.9], so that it admits a left adjoint:

$$\Sigma^\infty_+ \colon \mathcal{S} \longrightarrow \mathrm{Sp}\,.$$

Definition 1.3.3. The *sphere spectrum* $\mathbb{S}$ is the image under Σ^∞_+ of the final object $* \in \mathcal{S}$.

Remark 1.3.4. Since $\mathcal{S}_*^{\mathrm{fin}}$ is a small pointed ∞-category and $\mathcal{S}$ is presentable, it follows by [Lur17, Remark 1.4.2.4] that Sp is a presentable ∞-category. In particular, it admits all small colimits. In fact, it is a consequence of [Lur17, Corollary 1.4.4.6] that Sp is the stable ∞-category that is freely generated under small colimits by a single object: the sphere spectrum.

We are now ready to state the main result of interest to us. It provides the desired symmetric monoidal structure on Sp, though in a very abstract way.

Theorem 1.3.5 ([Lur09, Corollary 4.8.2.19])**.** *There is an essentially unique symmetric monoidal ∞-category structure on* Sp *such that:*

(1) The sphere spectrum $\mathbb{S}$ is the unit object;

(2) The tensor product functor $\mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ *preserves colimits separately in each variable.*

Definition 1.3.6. We call the symmetric monoidal ∞-category structure on Sp of Theorem 1.3.5 the *smash product symmetric monoidal structure.*

This descends to the usual smash product of spectra on the homotopy category hSp, which coincides with the usual stable homotopy category. This was originally introduced by Boardman in [Boa70]; we will denote the smash product of two spectra $X, Y \in \mathrm{Sp}$ by $X \wedge Y$ in accordance with his original notation.

Remark 1.3.7. Before moving on to the next chapter, it should be noted that this symmetric monoidal structure is of course not the only one that we can put on Sp, though it is the one we will be mostly interested into. Recall that Example 1.2.13 provides a way to endow an ∞-category $\mathcal{C}$ with the (co)cartesian symmetric monoidal structure, where the tensor product is given by the (co)product in $\mathcal{C}$, provided that $\mathcal{C}$ admits all finite (co)products. Since Sp is stable, it follows in particular that it admits all finite products and coproducts, and moreover these coincide and are denoted by $\oplus$. We obtain an essentially unique symmetric monoidal structure on Sp, where the tensor product is given by direct sum of spectra.

As anticipated, this latter symmetric monoidal structure on Sp can be thought of as "additive", whereas the smash product can be thought of as "multiplicative". This is because they satisfy a suitable version of the distributive laws: given spectra $X, Y, Z \in \mathrm{Sp}$, we have that:

$$(X \oplus Y) \wedge Z \simeq (X \wedge Z) \oplus (Y \wedge Z)$$

and similarly when smashing on the other side.

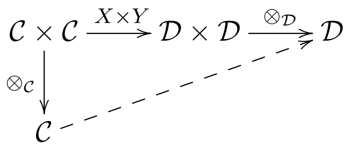
Chapter 2

Day convolution for ∞-categories

Let $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ be symmetric monoidal ∞-categories, where the cocartesian fibration to $\mathcal{F}_*$ is understood. In this chapter, we will describe a way to endow the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors between the underlying ∞-categories, under appropriate hypotheses, with a Day convolution symmetric monoidal structure, following work done by Glasman in [Gla16]. We will then make the construction of the Day convolution tensor product explicit in the example of filtered spectra.

Before going deep into theoretical details, in this introductory part we provide some intuition for how the construction works, looking at Day convolution in the ordinary context, and then we sketch how the ∞-categorical version works in the example of filtered spectra.

Example 2.0.1. Let us first recall how Day convolution works for 1-categories. Given symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$, we would like to endow $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with a symmetric monoidal structure. Of course, not any structure works; we would want it to descend from those of $\mathcal{C}$ and $\mathcal{D}$, as well as having some extra desirable features. A very natural construction is the following: given two functors $X, Y \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, we can take the left Kan extension in the following diagram:



Naturally, this does not always make sense. We need to require that $\mathcal{C}$ is small and that $\mathcal{D}$ has all small colimits to ensure that this left Kan extension always exists. This new functor, which we will denote $X \otimes_{\mathrm{Day}} Y$, is a good candidate for the tensor product of X and Y in a symmetric monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, and its value on an object C of $\mathcal{C}$ can be computed as:

$$X \otimes_{\mathrm{Day}} Y(C) = \operatorname*{colim}_{C_0 \otimes_{\mathcal{C}} C_1 \to C} X(C_0) \otimes_{\mathcal{D}} Y(C_1).$$

To make sure that this actually is the tensor product of a symmetric monoidal structure, we also need to require that $\otimes_{\mathcal{D}}$ preserves colimits in each variable. If this happens, then we obtain a symmetric monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ with tensor product $\otimes_{\mathrm{Day}}$ given as above, with the following features:

(1) $\otimes_{\mathrm{Day}}$ also preserves colimits in each variable;

(2) There is an equivalence of categories between the category of commutative algebra objects of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and the category of lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

This is the *Day convolution* symmetric monoidal structure on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Glasman's approach to Day convolution has the advantage of being very explicit in its constructions. This results in a tool that can be readily applied in practice: although everything is done with ∞-categories and so is inherently more complicated, things boil down to diagrams and formulas strongly resembling the 1-categorical case, as the same ideas are behind it. To highlight this, let us first sketch how Day convolution works for filtered spectra. This will be discussed in more detail later on in Section 2.4.

Example 2.0.2. The ∞-category of filtered spectra can be defined as the functor category $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$ (some authors define filtered objects to be functors from $\mathbb{Z}^{\mathrm{op}}$ instead of $\mathbb{Z}$; everything we say can be easily dualized). Suppose that we endow $\mathbb{Z}$ with the symmetric monoidal structure of Example 1.2.10, given by the sum, and Sp with the smash product defined in 1.3.6, obtaining symmetric monoidal ∞-categories $\mathbb{Z}^{\otimes} \to \mathcal{F}_*$ and $\mathrm{Sp}^{\otimes} \to \mathcal{F}_*$. Starting from these, Glasman's construction builds a symmetric monoidal ∞-category $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})^{\otimes} \to \mathcal{F}_*$ with in particular the following features:

(1) The underlying ∞-category is precisely the functor category $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$ of filtered spectra.

(2) 0-simplices over $\langle n \rangle \in \mathcal{F}_*$ correspond to products of n functors $\mathbb{Z} \to \mathrm{Sp}$.

Suppose we want to compute the tensor product in this symmetric monoidal structure of two filtered spectra $X, Y : \mathbb{Z} \to \mathrm{Sp}$. This will be another functor $\mathbb{Z} \to \mathrm{Sp}$, which we will denote $X \otimes Y$, which can be obtained by left Kan extending a diagram similar to the 1-categorical case. As it is also true for ∞-categories that we can compute left Kan extensions objectwise using colimits, we obtain the following formula:

$$(X \otimes Y)(n) = \operatorname*{colim}_{i+j \leq n} X(i) \wedge Y(j)$$

which has the clear advantage of being very explicit.

After this intuitive introduction, we now look in detail at the theory behind Glasman's construction.

2.1 Relative colimits and left Kan extensions along inclusions

Before looking at the actual construction of the Day convolution symmetric monoidal ∞-category, we recall useful definitions and facts about relative colimits as well as left Kan extensions along inclusions, relative to an inner fibration, following Lurie [Lur09, Sections 4.3.1 and 4.3.2]. This is because we will identify cocartesian edges of the Day convolution cocartesian fibration as products of such left Kan extensions, which are in turn defined via relative colimits. Ultimately, we want to recover a formula to compute the Day convolution tensor product at the level of objects, and this essentially comes from the computations for left Kan extensions. This section is highly theoretical and its purpose is to make the theory precise; it can be readily skipped if one is only interested in the ideas behind Day convolution for ∞-categories.

We first wish to extend the usual definition of colimit in the ∞-categorical setting, with which we assume the reader is familiar. We want to be able to take colimits not only with respect to an ∞-category $\mathcal{C}$, but also to an arbitrary inner fibration $f\colon \mathcal{C} \to \mathcal{D}$ of simplicial sets, that is, we want to have a relative version of colimits. We therefore have the following definition:

Definition 2.1.1. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration of simplicial sets, and let $\overline{p}\colon K^{\triangleright} \to \mathcal{C}$ be a diagram of simplicial sets. Let $p\colon K \to \mathcal{C}$ be the restriction of $\overline{p}$ to K, i.e. $p = \overline{p}|_K$. Then, we say that $\overline{p}$ is an *f-colimit of p*, or simply that $\overline{p}$ is a *colimit diagram*, if the induced functor:

$$\mathcal{C}_{\overline{p}/} \longrightarrow \mathcal{C}_{p/} \times_{\mathcal{D}_{fp/}} \mathcal{D}_{f\overline{p}/}$$

is a trivial fibration of simplicial sets. We usually refer to this generalization of colimits as *relative colimits*.

Remark 2.1.2. This is indeed a generalization of the usual ∞-categorical definition of colimit: if we let $\mathcal{D} = \Delta^0$ be a point and $\mathcal{C}$ be an ∞-category, so that $\mathcal{C} \to \Delta^0$ is an inner fibration, then the above definition reduces to requiring that the functor:

$$\mathcal{C}_{\overline{p}/} \longrightarrow \mathcal{C}_{p/}$$

is a trivial fibration. This means exactly that $\overline{p}$ is a colimit of p in the usual sense.

The main result about relative colimits that is of relevance to us is the following, which relates relative colimits to colimits in the fibers of a locally cocartesian fibration.

Proposition 2.1.3 ([Lur09, Lemma 4.3.1.10]). *Let $q\colon X \to S$ be a locally cocartesian fibration of ∞-categories, let $s \in S$ be an object of S, and let $\overline{p}\colon K^{\triangleright} \to X_s$ be a diagram onto the fiber X_s. The following are equivalent:*

(1) The composition:

$$K^{\triangleright} \xrightarrow{\overline{p}} X_s \to X$$

is a q-colimit diagram.

(2) For every edge $e\colon s \to s'$ of S, the composition:

$$K^{\triangleright} \xrightarrow{\overline{p}} X_s \xrightarrow{e_!} X_{s'}$$

is a colimit diagram in the ∞-category $X_{s'}$, where $e_!$ is the cocartesian push-forward of e.

We now introduce left Kan extensions along inclusions, which are defined by means of a relative colimit. We start with the following useful notation, employed throughout this chapter.

Notation 2.1.4. Let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory of an ∞-category $\mathcal{C}$, and let $q\colon K \to \mathcal{C}$ be any diagram. Denote by $\mathcal{C}^0_{/q}$ the pullback:

$$\mathcal{C}^0_{/q} := \mathcal{C}_{/q} \times_{\mathcal{C}} \mathcal{C}^0.$$

In particular, if C is an object of $\mathcal{C}$, corresponding to a map $\Delta^0 \xrightarrow{C} \mathcal{C}$, we denote the above by $\mathcal{C}^0_{/C}$. This is the full subcategory of $\mathcal{C}_{/C}$ spanned by all morphisms $C_0 \to C$ such that C_0 is an object of $\mathcal{C}^0$.

Suppose now we have a commutative diagram of ∞-categories:

$$\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \big\downarrow & \overset{F}{\nearrow} & \big\downarrow{\scriptstyle p} \\ \mathcal{C} & \longrightarrow & \mathcal{D}' \end{array}$$

where p is an inner fibration, and the map $\mathcal{C}^0 \hookrightarrow \mathcal{C}$ is the inclusion of a full subcategory, so that $F_0 = F|_{\mathcal{C}}$. We would like to define a notion of F being a left Kan extension of F_0. As noted by Lurie, intuitively we would want F to assume values as small as possible over $\mathcal{C}$, given the values of its restriction F_0. This can be formalized in the following way.

Definition 2.1.5. Consider a commutative diagram of ∞-categories:

$$\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \big\downarrow & \overset{F}{\nearrow} & \big\downarrow{\scriptstyle p} \\ \mathcal{C} & \longrightarrow & \mathcal{D}' \end{array}$$

with the same assumptions as above. We say that F is a *p-left Kan extension* of F_0 at an object C of $\mathcal{C}$ if the induced diagram:

$$\begin{array}{ccc} \mathcal{C}^0_{/C} & \xrightarrow{F_C} & \mathcal{D} \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ (\mathcal{C}^0_{/C})^{\triangleright} & \longrightarrow & \mathcal{D}' \end{array}$$

exhibits $F(C)$ as a p-colimit of F_C, where the cone point of $(\mathcal{C}^0_{/C})^{\triangleright}$ is mapped to $F(C)$. If this is true for every $C \in \mathcal{C}$, then we say that F is a *p-left Kan extension of* F_0.

Let us now recall some results from Lurie about left Kan extensions that will be useful throughout the rest of the chapter. First, we have two results regarding the behaviour of left Kan extensions under equivalences.

Lemma 2.1.6 ([Lur09, Lemma 4.3.2.5])**.** *Consider a commutative diagram of ∞-categories as in Definition 2.1.5. Suppose that C and C' are equivalent objects of $\mathcal{C}$. Then, F is a p-left Kan extension of F_0 at C if and only if it is so at C'.*

Lemma 2.1.7 ([Lur09, Lemma 4.3.2.6, (1)])**.** *Consider a commutative diagram of ∞-categories as in Definition 2.1.5. Let $F, F' \colon \mathcal{C} \to \mathcal{D}$ be equivalent functors in* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$*. Then, F is a p-left Kan extension of F_0 if and only if F' is.*

Finally, we have two results that will be useful in establishing existence of certain left Kan extensions. Note that for both, we require for p to be a categorical fibration, but we will only use them for cocartesian fibrations, which are in particular categorical fibrations by the dual statement of [Lur09, Proposition 3.3.1.7].

Lemma 2.1.8 ([Lur09, Lemma 4.3.2.12]). *Consider a diagram of ∞-categories:*

$$\begin{array}{ccc} & & \mathcal{D} \\ & & \downarrow{\scriptstyle p} \\ \mathcal{C} & \longrightarrow & \mathcal{D}' \end{array}$$

where p is a categorical fibration, and let $\mathcal{C}^0 \subseteq \mathcal{C}$ be a full subcategory. Let $n > 0$ and consider a commutative diagram:

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{f_0} & \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}, \mathcal{D}) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \mathrm{Fun}_{\mathcal{D}'}(\mathcal{C}^0, \mathcal{D}) \end{array}$$

such that the functor $F\colon \mathcal{C} \to \mathcal{D}$ given by evaluating f_0 on $\{0\} \subseteq \partial\Delta^n$ is a p-left Kan extension of $F|_{\mathcal{C}^0}$. Then, there exists a lift rendering the diagram commutative.

Lemma 2.1.9 ([Lur09, Lemma 4.3.2.13]). *Consider a diagram of ∞-categories:*

$$\begin{array}{ccc} \mathcal{C}^0 & \xrightarrow{F_0} & \mathcal{D} \\ \downarrow & \overset{F}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D}' \end{array}$$

where $\mathcal{C}^0 \subseteq \mathcal{C}$ is a full subcategory and p is a categorical fibration. Then, the following are equivalent:

(1) There exists a functor $F\colon \mathcal{C} \to \mathcal{D}$ making the above commutative and such that F is a p-left Kan extension of F_0.

(2) For every object $C \in \mathcal{C}$, the diagram:

$$\mathcal{C}^0_{/C} \to \mathcal{C}^0 \xrightarrow{F_0} \mathcal{D}$$

admits a p-colimit, compatible with the map $(\mathcal{C}^0_{/C})^{\triangleright} \to \mathcal{C} \xrightarrow{G} \mathcal{D}'$.

2.2 First steps towards Day convolution

We are now ready to develop a symmetric monoidal ∞-category structure for the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of functors between two symmetric monoidal ∞-categories $\mathcal{C}$ and $\mathcal{D}$: the Day convolution. We will follow the construction given by Glasman in [Gla16], expanding on many proofs and adding further context where needed.

The first goal is to define a simplicial set $\overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})}^{\otimes}$ and a fibration from this down to $\mathcal{F}_*$. This is not yet what the Day convolution will look like, but is a good first approximation, and it is easier to work with. We will then prove some results for this, in particular that it is a locally cocartesian fibration, and we will characterize the locally cocartesian edges in terms of relative left Kan extensions.

Let $\mathcal{C}^{\otimes} \to \mathcal{F}_*$ and $\mathcal{D}^{\otimes} \to \mathcal{F}_*$ be symmetric monoidal ∞-categories. In order to avoid size issues, suppose that $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ are small. Given a simplicial set K and a map

$k\colon K \to \mathcal{F}_*$ of simplicial sets, we denote by $\mathcal{C}_k^{\otimes}$ the pullback:

$$\begin{CD}
\mathcal{C}_k^{\otimes} @>>> \mathcal{C}^{\otimes} \\
@VVV @VVV \\
K @>k>> \mathcal{F}_*
\end{CD}$$

and analogously for $\mathcal{D}_k^{\otimes}$. In particular, for an object $S \in \mathcal{F}_*$ corresponding to a map $\Delta^0 \to \mathcal{F}_*$, $\mathcal{C}_S^{\otimes}$ denotes the pullback:

$$\begin{CD}
\mathcal{C}_S^{\otimes} @>>> \mathcal{C}^{\otimes} \\
@VVV @VVV \\
\Delta^0 @>S>> \mathcal{F}_*,
\end{CD}$$

and for a morphism f in $\mathcal{F}_*$ corresponding to a map $\Delta^1 \to \mathcal{F}_*$, $\mathcal{C}_f^{\otimes}$ denotes the pullback:

$$\begin{CD}
\mathcal{C}_f^{\otimes} @>>> \mathcal{C}^{\otimes} \\
@VVV @VVV \\
\Delta^1 @>f>> \mathcal{F}_*.
\end{CD}$$

The first step is to define a simplicial set $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$ with the idea that we would want it to roughly act as some sort of internal Hom over $\mathcal{F}_*$, adjoint to the pullback construction above. This will be made precise later on, once we refine $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$ to a more appropriate simplicial set, $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$. For the moment, the natural bijection required in the upcoming definition serves this purpose.

Definition 2.2.1. Let K be any simplicial set and $k\colon K \to \mathcal{F}_*$ be any map of simplicial sets. We define the simplicial set:

$$\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$$

by requiring that there is a bijection, natural in k:

$$\mathrm{Fun}_{\mathcal{F}_*}(K, \overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}) \simeq \mathrm{Fun}_{\mathcal{F}_*}(\mathcal{C}_k^{\otimes}, \mathcal{D}^{\otimes}).$$

More explicitly, we are requiring a bijection between the top arrows in the diagrams:

$$\begin{array}{ccccccc}
K & \longrightarrow & \overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}} & \qquad\simeq\qquad & \mathcal{C}_k^{\otimes} & \longrightarrow & \mathcal{D} \\
& {\scriptstyle k}\searrow \quad \swarrow & & & & \searrow \quad \swarrow & \\
& \mathcal{F}_* & & & & \mathcal{F}_* &
\end{array}$$

Remark 2.2.2. By choosing $K = \Delta^n$ for any $n \geq 0$ in the above definition, we recover the n-simplices of $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$. This also allows us to make explicit the arrow $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}} \to \mathcal{F}_*$ as we have commutativity of the left diagram for any such K. Moreover, naturality in k ensures that we indeed get a simplicial set, i.e. a functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$.

Remark 2.2.3. Any functor $\mathcal{C}_k^{\otimes} \to \mathcal{D}$ over $\mathcal{F}_*$ will factor through $\mathcal{D}_k^{\otimes}$ by the universal property of pullbacks. It follows that we can equivalently view the natural bijection of Definition 2.2.1 as:

$$\mathrm{Fun}_{\mathcal{F}_*}(K, \overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}) \simeq \mathrm{Fun}_{\mathcal{F}_*}(\mathcal{C}_k^{\otimes}, \mathcal{D}_k^{\otimes}).$$

In particular:

(1) The 0-simplices of $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$, which arise from functors $\Delta^0 \xrightarrow{S} \mathcal{F}_*$, can be thought of as a choice of a vertex S in $\mathcal{F}_*$ together with a functor $\mathcal{C}_S^{\otimes} \to \mathcal{D}_S^{\otimes}$ over $\mathcal{F}_*$.

(2) Similarly, the 1-simplices of $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$, which arise from functors $\Delta^1 \xrightarrow{f} \mathcal{F}_*$, can be thought of as a choice of an edge $f\colon S \to T$ in $\mathcal{F}_*$ together with a functor $\mathcal{C}_f^{\otimes} \to \mathcal{D}_f^{\otimes}$ over $\mathcal{F}_*$.

(3) A section of $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$, i.e. a map $\mathcal{F}_* \to \overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$ fitting into a commutative diagram:

$$\begin{array}{ccc} \mathcal{F}_* & \longrightarrow & \overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}} \\ & {\scriptstyle \mathrm{id}} \searrow \!\!\! = \quad \swarrow & \\ & \mathcal{F}_* & \end{array}$$

corresponds, by choosing $K = \mathcal{F}_*$ and $k = \mathrm{id}_{\mathcal{F}_*}$, to a functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $\mathcal{F}_*$, as we have $\mathcal{C}^{\otimes} \times_{\mathcal{F}_*} \mathcal{F}_* \simeq \mathcal{C}^{\otimes}$, and similarly for $\mathcal{D}^{\otimes}$.

We now prove some preliminary results on $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$. First, we want to show that it is an ∞-category.

Proposition 2.2.4. *The map $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}} \to \mathcal{F}_*$ is an inner fibration. In particular, $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$ is an ∞-category.*

Proof. Let $0 < i < n$ and consider a lifting problem:

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{k_0} & \overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}} \\ \big\downarrow & \nearrow & \big\downarrow \\ \Delta^n & \xrightarrow{k_1} & \mathcal{F}_*. \end{array}$$

The natural bijection of Definition 2.2.1 allows us to rewrite this as the lifting problem:

$$\begin{array}{ccc} \mathcal{C}_{k_0}^{\otimes} & \longrightarrow & \mathcal{D}^{\otimes} \\ {\scriptstyle f}\big\downarrow & \nearrow & \big\downarrow \\ \mathcal{C}_{k_1}^{\otimes} & \longrightarrow & \mathcal{F}_* \end{array}$$

where the horizontal arrows correspond to k_0 and k_1, respectively. We have a commutative diagram:

$$\begin{array}{ccccc} \mathcal{C}_{k_0}^{\otimes} & \xrightarrow{f} & \mathcal{C}_{k_1}^{\otimes} & \longrightarrow & \mathcal{C}^{\otimes} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \Lambda_i^n & \hookrightarrow & \Delta^n & \xrightarrow{k_1} & \mathcal{F}_* \end{array}$$

where the right square and the rectangle are pullbacks, and so also the left square is a pullback by the pasting law for pullbacks. Note that since $\Lambda^n_i \hookrightarrow \Delta^n$ is inner anodyne, it is also a categorical equivalence by [Lur18, Tag 01EJ]. Also, since $\mathcal{C}^\otimes \to \mathcal{F}_*$ is a cocartesian fibration, its pullback $\mathcal{C}^\otimes_{k_1} \to \Delta^n$ is also a cocartesian fibration. We can therefore apply [Lur09, Proposition 3.3.1.3] to get that $\mathcal{C}^\otimes_{k_0} \to \mathcal{C}^\otimes_{k_1}$ is a categorical equivalence. It is therefore inner anodyne, and $\mathcal{D} \to \mathcal{F}_*$ is an inner fibration by definition, thus the second lifting problem admits a solution. □

We now need to introduce the local version of Definitions 1.2.1 and 1.2.2, that is, we will introduce locally cocartesian edges and fibrations. This is useful as our next step will be to prove that the functor $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^\otimes} \to \mathcal{F}_*$ is a locally cocartesian fibration. It is not true however in general that it is also a cocartesian fibration, so we will have to refine it later after additional hypotheses to an actual cocartesian fibration, in order to obtain the Day convolution symmetric monoidal structure.

Definition 2.2.5. Let $p\colon X \to S$ be a functor between simplicial sets X and S, and let f be an edge of X; call its image $p(f) =: g$. Consider the pullback of simplicial sets:

$$\begin{array}{ccc} X_g & \longrightarrow & X \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ \Delta^1 & \xrightarrow{g} & S \end{array}$$

and let f' be the unique edge of X_g lying over f and such that $p'(f')$ is the unique non-degenerate edge of Δ^1. We say that f is *locally p-cocartesian* if f' is a p'-cocartesian edge.

Definition 2.2.6. A functor $p\colon X \to S$ of simplicial sets is a *locally cocartesian fibration* if

(1) p is an inner fibration;

(2) for every $x \in X_0$ and every edge $g\colon p(x) \to s$ in S_1, there exists a locally p-cocartesian lift $f\colon x \to \overline{s}$ of g.

We now wish to show that $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^\otimes} \to \mathcal{F}_*$ is a locally cocartesian fibration. This is not always true, but we need to add the extra hypothesis that $\mathcal{D}^\otimes_S$ has all small colimits, for every $S \in \mathcal{F}_*$ to ensure that certain relative left Kan extensions exist. Indeed, to prove that the above fibration is locally cocartesian, we spell out a way of obtaining locally cocartesian edges in $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^\otimes}$ from certain relative left Kan extensions. We will then check that these indeed exist, thanks to the extra hypothesis, rendering the above map a locally cocartesian fibration.

Therefore, suppose now for the rest of this chapter that $\mathcal{D}^\otimes$ is such that $\mathcal{D}^\otimes_S$ has all small colimits, for every $S \in \mathcal{F}_*$. Recall that we can think of edges of $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^\otimes}$ as pairs of an edge $f\colon S \to T$ in $\mathcal{F}_*$ and a functor $\mathcal{C}^\otimes_f \to \mathcal{D}^\otimes_f$ over $\mathcal{F}_*$, which we will call F_f. Restricting F_f to $\mathcal{C}^\otimes_S \subseteq \mathcal{C}^\otimes_f$ yields a functor $F_S\colon \mathcal{C}^\otimes_S \to \mathcal{D}^\otimes_S$, which together with S forms a vertex of $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^\otimes}$.

Lemma 2.2.7. *Let $(f\colon S \to T, F_f\colon \mathcal{C}^\otimes_f \to \mathcal{D}^\otimes_f)$ be an edge of $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^\otimes}$. Suppose* *that the diagram:*

$$\begin{array}{ccccc} \mathcal{C}^{\otimes}_S & \xrightarrow{F_S} & \mathcal{D}^{\otimes}_S & \hookrightarrow & \mathcal{D}^{\otimes}_f \\ \downarrow & & & \nearrow_{F_f} & \downarrow p \\ \mathcal{C}^{\otimes}_f & & \longrightarrow & & \Delta^1 \end{array} \qquad (F_0 : \mathcal{C}^{\otimes}_S \to \mathcal{D}^{\otimes}_f)$$

exhibits F_f as a p-left Kan extension of F_0. Then, (f, F_f) is a locally cocartesian edge of $\overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}}$.

Proof. Suppose first that F_f is a p-left Kan extension of F_0. We wish to show that the edge (f, F_f) of $\overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}}$ is locally cocartesian, that is, its corresponding edge in $\overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}}_f \to \Delta^1$ is cocartesian. To show this, consider a lifting problem:

$$\begin{array}{ccc} \Lambda^n_0 & \longrightarrow & \overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}}_f \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & \Delta^1 \end{array}$$

where the top arrow maps the first edge of Λ^n_0 to F_f, and therefore the vertex 0 to F_0. Therefore, the bottom arrow must map the vertex 0 of Δ^n to 0, and all other vertices to 1. Writing $\Lambda^n_0 = \Delta^0 * \partial\Delta^{n-1}$ and $\Delta^n = \Delta^0 * \Delta^{n-1}$ and using the join-slice adjunction, we get the equivalent lifting problem:

$$\begin{array}{ccc} \partial\Delta^{n-1} & \longrightarrow & \left(\overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}}_f\right)_{F_0/} \\ \downarrow & \nearrow & \\ \Delta^{n-1} & & \end{array}$$

which is in turn equivalent to the following:

$$\begin{array}{ccc} \partial\Delta^{n-1} & \longrightarrow & \mathrm{Fun}_{\Delta^1}(\mathcal{C}^{\otimes}_f, \mathcal{D}^{\otimes}_f) \\ \downarrow & \nearrow & \downarrow \\ \Delta^{n-1} & \longrightarrow & \mathrm{Fun}(\mathcal{C}^{\otimes}_S, \mathcal{D}^{\otimes}_S) \end{array}$$

where the right vertical arrow is restriction to $\mathcal{C}^{\otimes}_S$, and the bottom horizontal arrow is constant at F_0. This is because the n-simplices of $(\overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}}_f)_{F_0/}$ correspond to $(n+1)$-simplices of $\mathrm{Fun}_{\Delta^1}(\mathcal{C}^{\otimes}_f, \mathcal{D}^{\otimes}_f)$ with 0-th vertex F_0.

Then, by Lemma 2.1.8, this diagram admits a lift, and so (f, F_f) is locally cocartesian. □

Lemma 2.2.8. *Given $f : S \to T$, there exists a functor $F : \mathcal{C}^{\otimes}_f \to \mathcal{D}^{\otimes}_f$ which is a p-left Kan extension of F_0, as in Lemma 2.2.7.*

Proof. The proof of this is nearly identical to the proof of [Lur09, Corollary 4.3.1.11], via the equivalent characterization for p-left Kan extensions given in 2.1.9, so we will not repeat it here. For further context, see the respective result in the paper by Glasman, [Gla16, Lemma 2.5]. □

As a direct consequence we have:

Corollary 2.2.9. *The functor* $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}} \to \mathcal{F}_*$ *is a locally cocartesian fibration.*

Proof. Combine Lemma 2.2.8 and Lemma 2.2.7. □

We can also show that the converse of Lemma 2.2.7 holds, thereby furnishing a characterization of locally cocartesian edges in $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$. These correspond precisely with p-left Kan extensions of F_0.

Corollary 2.2.10. *If* (f, F_f) *is a locally cocartesian edge of* $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$*, then the diagram of Lemma 2.2.7 exhibits* F_f *as a* p*-left Kan extension of* F_0*.*

Proof. By Corollary 2.2.10, for any morphism f in $\mathcal{F}_*$, we can choose a locally cocartesian edge (f, F_f) in $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$, corresponding to a left Kan extension as in the diagram in Lemma 2.2.7. Given any other locally cocartesian edge (f, F'_f) with the same source as (f, F_f), as they are both locally cocartesian, they are equivalent as edges in $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$. Therefore, the corresponding functors $F_f, F'_f \colon \mathcal{C}^{\otimes}_f \to \mathcal{D}^{\otimes}_f$ are equivalent. As F_f is a left Kan extension of F_0, by 2.1.7, also F'_f must be a left Kan extension of F_0, so we conclude. □

We can now spell out the following important remark, which is a direct consequence of this correspondence and will be useful later for direct computations of cocartesian pushforwards of functors.

Remark 2.2.11. The proof of 2.2.8 shows that given an object $X \in \mathcal{C}^{\otimes}_T$, the functor:

$$(\mathcal{C}^{\otimes}_S)_{/X} \to \mathcal{C}^{\otimes}_S \xrightarrow{F_0} \mathcal{D}^{\otimes}_S \hookrightarrow \mathcal{D}^{\otimes}_f$$

admits a p-colimit. Let:

$$(\mathcal{C}^{\otimes}_S)^{\triangleright}_{/X} \xrightarrow{\overline{F}} \mathcal{D}^{\otimes}_S \hookrightarrow \mathcal{D}^{\otimes}_f$$

be a p-colimit diagram for the above. We wish to apply Proposition 2.1.3 to get an equivalent characterization, as the above diagram is precisely in the form required in the hypotheses of the Proposition: it factors through the inclusion of a fiber.

Note that $p \colon \mathcal{D}^{\otimes}_f \to \Delta^1$ is a cocartesian fibration by 1.2.4, as it is the pullback of the cocartesian fibration $\mathcal{D}^{\otimes} \to \mathcal{F}_*$. In particular, p is a locally cocartesian fibration of ∞-categories, so we can indeed apply 2.1.3. The only nontrivial edge of Δ^1 is $\{0\} \to \{1\}$, and a cocartesian lift is $f_! \colon \mathcal{D}^{\otimes}_S \to \mathcal{D}^{\otimes}_T$, cocartesian pushforward of f. Therefore, we get that the following are equivalent:

(1) $\overline{F}$ composed with the inclusion of the fiber $\mathcal{D}^{\otimes}_S \hookrightarrow \mathcal{D}^{\otimes}_f$ is a p-colimit diagram.

(2) $f_! \circ \overline{F}$ is a colimit diagram in the ∞-category $\mathcal{D}^{\otimes}_T$.

In particular, given an S°-tuple $\{F_s\}_{s \in S^{\circ}}$ of functors $F_s \colon \mathcal{C} \to \mathcal{D}$, we can compute the cone vertex in $\mathcal{D}^{\otimes}_T$ of the colimit diagram $f_! \circ \overline{F}$ as:

$$\operatorname*{colim}_{Y \in (\mathcal{C}^{\otimes}_S)_{/X}} f_!(F_s(Y_s))$$

where $Y = \{Y_s\}_{s \in S^{\circ}}$ via the Segal condition for $\mathcal{C}^{\otimes}$.

2.3 Construction of the Day convolution

We have now shown successfully that the simplicial set $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$ that we constructed gives rise to a locally cocartesian fibration. However, it is not true in general that this same fibration is also cocartesian. In order to have this, we will need to impose some further conditions on the ∞-category $\mathcal{D}$, and we will also need to reduce $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$ to a smaller simplicial set. This reflects the need in the ordinary case for $\otimes_{\mathcal{D}}$ to commute with colimits in each variable, as well as the Segal condition. We will also need to impose an analogous condition on edges.

Construction 2.3.1. Recall that given a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ and any $S \in \mathcal{F}_*$, by the Segal condition we have an equivalence:

$$\mathcal{C}^{\otimes}_S \simeq \prod_{i \in S^{\circ}} \mathcal{C} =: \mathcal{C}^S$$

which arises from the inert maps ρ_i. Note that the notation $\mathcal{C}^S$, which we will use from now on, means the product of copies of $\mathcal{C} \simeq \mathcal{C}^{\otimes}_{\langle 1 \rangle}$ indexed over S°, and not S.

Consider now an edge $f\colon S \to T$ of $\mathcal{F}_*$. We wish to obtain a similar decomposition for $\mathcal{C}^{\otimes}_f$, given informally by the various preimages under f of elements of T. In fact, given $f \in \mathcal{F}_*$, the set $f^{-1}(*)$ determines a map $f^{-1}(*) \to *$ in $\mathcal{F}_*$, and given any $t \in T^{\circ}$, we have a map $f^{-1}(t)_+ \to \{t\}_+$, where $(\cdot)_+$ is the addition of the basepoint $*$. Note that if $f^{-1}(t)$ is empty, this is simply the inclusion $\{*\} \hookrightarrow \{t\}_+$. Also note that all such maps $f^{-1}(t)_+ \to \{t\}_+$ are active. Call β_f and $\mu_{f,t}$ respectively the maps $f^{-1}(*) \to *$ and $f^{-1}(t)_+ \to \{t\}_+$ for every $t \in T^{\circ}$.

Such a decomposition of f in simpler maps gives rise to a decomposition of the slice category:

$$(\mathcal{F}_*)_{/T} \simeq \left(\prod_{t \in T^{\circ}} (\mathcal{F}_*)^{\mathrm{act}}_{/\{t\}+} \right) \times \mathcal{F}_*$$

where any $f\colon S \to T$ is decomposed into $\mu_{f,t}$, for any $t \in T^{\circ}$, and β_f; the inverse construction is obvious. It follows that we also get a decomposition for the pullback $\mathcal{C}^{\otimes} \times_{\mathcal{F}_*} (\mathcal{F}_*)_{/T}$:

$$\mathcal{C}^{\otimes} \times_{\mathcal{F}_*} (\mathcal{F}_*)_{/T} \simeq \left(\prod_{t \in T^{\circ}} \mathcal{C}^{\otimes} \times_{\mathcal{F}_*} (\mathcal{F}_*)^{\mathrm{act}}_{/\{t\}_+} \right) \times \mathcal{C}^{\otimes}.$$

Note now that we have $\mathcal{C}^{\otimes}_f \simeq (\mathcal{C}^{\otimes} \times_{\mathcal{F}_*} (\mathcal{F}_*)_{/T})_f$, as shown from the diagram:

$$\begin{array}{ccccc} (\mathcal{C}^{\otimes} \times_{\mathcal{F}_*} (\mathcal{F}_*)_{/T})_f & \longrightarrow & \mathcal{C}^{\otimes} \times_{\mathcal{F}_*} (\mathcal{F}_*)_{/T} & \longrightarrow & \mathcal{C}^{\otimes} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^1 & \xrightarrow{\quad f \quad} & (\mathcal{F}_*)_{/T} & \longrightarrow & \mathcal{F}_* \end{array}$$

where both squares are pullbacks, and so the rectangle is also a pullback. A similar fact holds for any edge in $\mathcal{F}_*$. It follows that in particular the above decomposition gives:

$$\mathcal{C}^{\otimes}_f \simeq \left(\prod_{\Delta^1, t \in T^{\circ}} \mathcal{C}^{\otimes}_{\mu_{f,t}} \right) \times \mathcal{C}^{\otimes}_{\beta_f}$$

where by the $\prod$ symbol with subscript Δ^1 we mean taking the pullback over Δ^1 along the canonical maps $\mathcal{C}^{\otimes}_{\mu_{f,t}} \to \Delta^1$; we will also employ later this notation. This is the desired decomposition for $\mathcal{C}^{\otimes}_{f}$.

We will soon cut down the simplicial set $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$ by using the above decomposition for vertices and edges, requiring them to be "diagonal" with respect to it. Before doing that, we recall what it means for the tensor product on $\mathcal{D}$ to preserve colimits in each variable, and give an equivalent characterization for it. This will be the second requirement in order to obtain a symmetric monoidal ∞-category.

Definition 2.3.2. Let $\mathcal{D}^{\otimes} \to \mathcal{F}_*$ be a symmetric monoidal ∞-category, and suppose that the underlying ∞-category $\mathcal{D}$ admits all (small) colimits. We say that the tensor product of $\mathcal{D}^{\otimes}$ *preserves colimits in each variable* if for each object $X \in \mathcal{D}$, the composite functors:

$$\mathcal{D} \xrightarrow{(X,-)} \mathcal{D} \times \mathcal{D} \xrightarrow{\mu} \mathcal{D}$$

and

$$\mathcal{D} \xrightarrow{(-,X)} \mathcal{D} \times \mathcal{D} \xrightarrow{\mu} \mathcal{D}$$

are colimit-preserving functors, where $\mu\colon \mathcal{D} \times \mathcal{D} \simeq \mathcal{D}^{\otimes}_{\langle 2\rangle} \to \mathcal{D}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{D}$ comes from the unique active morphism $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}_*$.

Remark 2.3.3. The condition just stated for a tensor product to preserve colimits in each variable is equivalent to an apparently stronger one, which requires an essentially analogous condition for every $f\colon S \to T$. We are about to state this in the upcoming Lemma 2.3.4.

Before doing that though, recall the colimit formula given in Remark 2.2.11. The second condition in Lemma 2.3.4 essentially tells us that in the colimit formula we can "take out" the cocartesian pushforward $f_!$ when looking at each individual component. That is, for any $t \in T^{\circ}$, we have an equivalence:

$$\left(\operatorname*{colim}_{Y \in (\mathcal{C}^{\otimes}_{S})_{/X}} f_!(F_s(Y_s))\right)_t \simeq f_! \left(\operatorname*{colim}_{Y \in (\mathcal{C}^{\otimes}_{S})_{/X}} F_s(Y_s)\right)_t$$

where the t at the subscript means that we are considering the component corresponding to t as an object of $\mathcal{D}^{\otimes}_{T} \simeq \mathcal{D}^{T}$.

Lemma 2.3.4. *Let $\mathcal{D}^{\otimes} \to \mathcal{F}_*$ be a symmetric monoidal ∞-category, and suppose that the underlying ∞-category $\mathcal{D}$ admits all (small) colimits. The following are equivalent:*

(1) The tensor product of $\mathcal{D}^{\otimes}$ preserves colimits in each variable.

(2) Let $f\colon S \to T$ be a morphism in nFin, let $\{K_s\}_{s\in S^{\circ}}$ be simplicial sets, and for every $s \in S^{\circ}$, let $\varphi_s\colon K_s \to \mathcal{D}$ be a functor. Call $K := \prod_{s\in S^{\circ}} K_s$, and let φ be the product of the maps φ_s:

$$\varphi\colon K \xrightarrow{\prod_s \varphi_s} \mathcal{D}^{S} \simeq \mathcal{D}^{\otimes}_{S}.$$

Suppose we have a map $\varphi^{\triangleright} \to \mathcal{D}^{\otimes}_{S}$, restricting to φ on K, and such that componentwise it gives rise to a colimit diagram, i.e. for every $s_0 \in S^{\circ}$ and every

$y \in \prod_{s \neq s_0} K_s$, the composite:

$$K_{s_0}^{\triangleright} \xrightarrow{(-,y)} K^{\triangleright} \xrightarrow{\varphi^{\triangleright}} \mathcal{D}_S^{\otimes} \xrightarrow{\pi_s} \mathcal{D}$$

is a colimit diagram. Then, the same holds for the cocartesian pushforward $f_!(\varphi^{\triangleright})$, given by composing $\varphi^{\triangleright}$ with the functor $\mathcal{D}_S^{\otimes} \to \mathcal{D}_T^{\otimes}$ associated to f. That is, for every $t \in T^{\circ}$ and every $z \in \prod_{f(s) \neq t} K_s$, the composite:

$$\Big(\prod_{f(s)=t} K_s\Big)^{\triangleright} \xrightarrow{(-,z)} K^{\triangleright} \xrightarrow{f_!(\varphi^{\triangleright})} \mathcal{D}_T^{\otimes} \xrightarrow{\pi_t} \mathcal{D}$$

is a colimit diagram.

Proof. It is clear that the second condition generalizes the first one: we can obtain (1) by considering $f\colon \langle 2\rangle \to \langle 1\rangle$ the unique active map, and setting $K_1 = *$ trivial.

For the reverse implication, one gets it by reducing first to considering only active morphisms in $\mathcal{F}_*$, and then by factoring these as compositions of active morphisms where each preimage has at most two elements. This allows to deduce (2) from (1); for further details, see the respecive statement [Gla16, Lemma 2.7]. □

We now finally give the main definition of this section: we define the appropriate simplicial set to serve as total space in the cocartesian fibration which will result in a symmetric monoidal ∞-category structure for $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The idea is to restrict $\overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}}$ by allowing only vertices and edges which are "diagonal" after decomposing the fibers $\mathcal{C}_S^{\otimes}, \mathcal{D}_S^{\otimes}$ and $\mathcal{C}_f^{\otimes}, \mathcal{D}_f^{\otimes}$ over sets S and morphisms f of $\mathcal{F}_*$, as in Construction 2.3.1. This is needed to ensure that the Segal condition holds, as motivated by the following counterexample.

Example 2.3.5. Suppose $\mathcal{C}^{\otimes} = \mathcal{D}^{\otimes} = \mathbb{Z}^{\otimes}$, the symmetric monoidal ∞-category coming from $\mathbb{Z}$ with the sum. The fibers over $\langle 1\rangle$ and $\langle 2\rangle$ are evidently $\mathbb{Z}$ and $\mathbb{Z}^2$ respectively. Then, consider $\overline{\mathrm{Fun}(\mathbb{Z}, \mathbb{Z})^{\otimes}}$ as defined above: the fiber over $\langle 1\rangle$ is given by functors $\mathbb{Z} \to \mathbb{Z}$, whereas the fiber over $\langle 2\rangle$ is given by functors $\mathbb{Z}^2 \to \mathbb{Z}^2$. The latter is evidently not equivalent to the product of two copies of the former. However, restricting the fiber over $\langle 2\rangle$ to only those functors $\mathbb{Z}^2 \to \mathbb{Z}^2$ which are diagonal, that is, are given by a diagonal matrix, works.

Suppose now for the rest of this section that $\mathcal{D}$ admits all small colimits and that $\mathcal{D}^{\otimes}$ satisfies the equivalent conditions of Lemma 2.3.4.

Definition 2.3.6. Define the simplicial set:

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes} \subseteq \overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}}$$

as the largest subsimplicial set of $\overline{\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}}$ such that:

(1) its vertices are those given by pairs of a vertex S of $\mathcal{F}_*$ and a functor $F\colon \mathcal{C}_S^{\otimes} \to \mathcal{D}_S^{\otimes}$ such that F lies in the essential image of the diagonal inclusion:

$$\prod_{s \in S^{\circ}} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}(\prod_{s \in S^{\circ}} \mathcal{C}, \prod_{s \in S^{\circ}} \mathcal{D}) \cong \mathrm{Fun}(\mathcal{C}_S^{\otimes}, \mathcal{D}_S^{\otimes}).$$

(2) its edges are those given by pairs of an edge $f\colon S \to T$ of $\mathcal{F}_*$ and a functor $F\colon \mathcal{C}^{\otimes}_f \to \mathcal{D}^{\otimes}_f$ such that F lies in the essential image of the diagonal inclusion:

$$\left(\prod_{t\in T^\circ} \mathrm{Fun}_{\Delta^1}(\mathcal{C}^{\otimes}_{\mu_{f,t}}, \mathcal{D}^{\otimes}_{\mu_{f,t}})\right) \times \mathrm{Fun}_{\Delta^1}(\mathcal{C}^{\otimes}_{\beta}, \mathcal{D}^{\otimes}_{\beta}) \longrightarrow \mathrm{Fun}_{\Delta^1}(\mathcal{C}^{\otimes}_{f}, \mathcal{D}^{\otimes}_{f})$$

where f is decomposed into maps β_f and $\mu_{f,t}$ as in Construction 2.3.1.

This is a reasonable definition: note that the fiber over a vertex S in $\mathcal{F}_*$ is evidently equivalent to $\prod_{s\in S^\circ} \mathrm{Fun}(\mathcal{C},\mathcal{D})$, which is precisely what we would expect from the Segal condition on the restricted functor $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes} \to \mathcal{F}_*$. We now proceed to show that this is indeed a cocartesian fibration, and that it satisfies the Segal condition (we also have to check that the above is induced by the inert morphisms ρ_i), thus obtaining a symmetric monoidal ∞-category structure on $\mathrm{Fun}(\mathcal{C},\mathcal{D})$, the *Day convolution symmetric monoidal ∞-category.*

Proposition 2.3.7. *The functor $p\colon \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes} \to \mathcal{F}_*$ is an inner fibration. In particular, $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$ is an ∞-subcategory of $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$.*

Proof. We need to check that p lifts against any inner horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$. As we have imposed conditions only on vertices and edges, and if $n > 2$ the inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ doesn't add any vertices or edges, we only need to check that p lifts against $\Lambda^2_1 \hookrightarrow \Delta^2$. Therefore, consider a lifting problem:

$$\begin{array}{ccc} \Lambda^2_1 & \xrightarrow{\nu} & \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes} \\ \big\downarrow & & \big\downarrow p \\ \Delta^2 & \xrightarrow{\rho} & \mathcal{F}_* \end{array}$$

and call $\overline{\nu}$ the composite $p\circ\nu\colon \Lambda^2_1 \to \mathcal{F}_*$. This corresponds to specifying two composable morphisms in $\mathcal{F}_*$; let these be $f\colon S\to T$ and $g\colon T\to U$. Then, $\overline{\nu}$ is the data of f and g together with a map

$$\kappa\colon \mathcal{C}^{\otimes}_{\nu} \to \mathcal{D}^{\otimes}_{\nu}$$

over the horn ν. Since κ is an edge in $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$, it can be decomposed up to equivalence as a product of maps κ_u coming from the decomposition of its edge κ_g over g. More explicitly, g decomposes as maps g_u defined as:

$$g_u := \mu_{g,u}\colon g^{-1}(u)_+ \longrightarrow \{u\}_+$$

if $u \neq *$, and as:

$$g_u := \beta_g\colon g^{-1}(*) \longrightarrow *$$

if $u = *$. Then, associated to this we get a decomposition for the map f in maps f_u defined as:

$$f_u\colon (gf)^{-1}(u)_+ \longrightarrow g^{-1}(u)_+$$

if $u \neq *$, and as:

$$f_u\colon (gf)^{-1}(*) \longrightarrow g^{-1}(*)$$

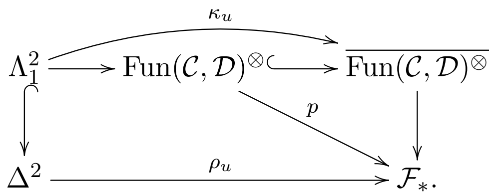
when $u = *$. The morphisms f_u and g_u specify a inner horn $\nu_u\colon \Lambda^2_1 \to \mathcal{F}_*$ for every $u\in U$. We therefore obtain functors:

$$\kappa_u\colon \mathcal{C}^{\otimes}_{\nu_u} \longrightarrow \mathcal{D}^{\otimes}_{\nu_u}$$

lying over ν_u, and a decomposition of κ:

$$\kappa = \prod_{\Lambda^2_1, u \in U} \kappa_u.$$

By the definition of edges in $\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$, each κ_u corresponds to a map $\Lambda^2_1 \to \overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$ that factors through $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$, so we get a diagram:



In Proposition 2.2.4, we proved that the rightmost map is an inner fibration, so we get a lift $\Delta^2 \to \overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}}$. Moreover, this lift trivially factors through $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$, since all edges of Δ^2 are sent to edges in $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$. This is clearly true for the images of the edges 01 and 12 as they come from κ, and also for the image of the edge 02, as this lies over the active morphism $(gf)^{-1}(u)_+ \to \{u\}_+$.

Therefore, we get a lift $\rho_u\colon \Delta^2 \to \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$ in the above diagram, which corresponds to a functor:

$$\lambda_u\colon \mathcal{C}^{\otimes}_{\rho_u} \longrightarrow \mathcal{D}^{\otimes}_{\rho_u},$$

that specifies a 2-simplex in $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$, together with the 2-simplex in $\mathcal{F}_*$ given by ρ_u. Taking the product of these maps, we obtain a functor:

$$\lambda = \prod_{\Delta^2, u \in U} \lambda_u$$

which specifies a 2-simplex in $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$, together with the 2-simplex in $\mathcal{F}_*$ given by ρ. This is a solution to the starting lifting problem, so we are done. □

Remark 2.3.8. Note that $p\colon \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes} \to \mathcal{F}_*$ is also a locally cocartesian fibration. Indeed, given $f\colon S \to T$ edge in $\mathcal{F}_*$, if $T = *$ or if f is active and $T^\circ = \{t\}$ has only one element, then the decomposition of f is trivial and we have an equality of fibers:

$$\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}_f = (\overline{\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}})_f$$

thus locally cocartesian lifts exist and correspond to p-left Kan extensions as in Remark 2.2.7. Otherwise, after decomposing f in such parts, and so $\mathcal{C}^{\otimes}_f$ and $\mathcal{D}^{\otimes}_f$, we can take products of p-left Kan extensions to obtain locally cocartesian lifts. This also gives a complete description of locally cocartesian edges in $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$.

We are now ready to prove that the map $p\colon \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes} \to \mathcal{F}_*$ gives rise to a symmetric monoidal ∞-category structure on $\mathrm{Fun}(\mathcal{C},\mathcal{D})$. We first show that it is a cocartesian fibration, and then that it satisfies the Segal condition. For the former, we make use of the following theoretical result, which establishes when a locally cocartesian fibration is also a cocartesian fibration.

Lemma 2.3.9. *Let $p\colon X \to S$ be a locally cocartesian fibration of simplicial sets. Then, the following are equivalent:*

(1) p is a cocartesian fibration.

(2) The composition of locally cocartesian edges is again a locally cocartesian edge. By this we mean that for any map $\Delta^2 \to X$ such that the edges 01 and 12 are mapped to locally cocartesian edges of X, then also 02 is mapped to a locally cocartesian edge.

(3) Every locally cocartesian edge is also cocartesian.

Proof. This is [Lur18, Tag 01V6]. □

Proposition 2.3.10. *The map $p\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes} \to \mathcal{F}_*$ is a cocartesian fibration.*

Proof. As p is a locally cocartesian fibration, thanks to Lemma 2.3.9, we can prove that locally cocartesian edges compose to another locally cocartesian edge in order to obtain the thesis.

View locally cocartesian edges as p-left Kan extensions, or products of such, as noted in Remark 2.3.8. Without loss of generality, we can then assume that we are dealing with locally cocartesian edges lying over f and g in $\mathcal{F}_*$ such that f is active with image $U = \langle 1 \rangle$. This is true because of the decomposition of edges over a Δ^2 in $\mathcal{F}_*$, in the same fashion as in the proof of the Proposition above.

Recall that we have p-left Kan extensions if and only if they realize certain colimit cones as in Lemma 2.2.8. Then, we have to show that taking consecutive colimits along the functors corresponding to the two inner edges lying over f and g yields a colimit along the other edge lying over gf. In other words, we want to show that we have an equivalence given by the canonical map:

$$\operatorname*{colim}_{Y \in (\mathcal{C}_S^{\otimes})_{/X}} (gf)_!(F_s(Y_s)) \xrightarrow{\simeq} \operatorname*{colim}_{Z \in (\mathcal{C}_T^{\otimes})_{/X}} g_! \left(\operatorname*{colim}_{W \in \left(\mathcal{C}^{\otimes}_{f^{-1}(t)_+}\right)_{/Z_t}} f_!(F_s(W_s)) \right)_t$$

where $Y = \{Y_s\}_{s \in S^\circ)}$, $Z = \{Z_t\}_{t \in T^\circ)}$ and $W = \{W_s\}_{s \in f^{-1}(t))}$ via the Segal condition. But this is a direct consequence of the assumption we made on $\mathcal{D}^{\otimes}$: it satisfies the equivalent conditions of Lemma 2.3.4, so the thesis follows from the observation done in the subsequent Remark 2.3.3. Indeed, we can rewrite the left colimit as:

$$\operatorname*{colim}_{Y \in (\mathcal{C}_S^{\otimes})_{/X}} (gf)_!(F_s(Y_s)) \simeq \operatorname*{colim}_{Z \in (\mathcal{C}_T^{\otimes})_{/X}} \left(\operatorname*{colim}_{W \in \left(\mathcal{C}^{\otimes}_{f^{-1}(t)_+}\right)_{/Z_t}} g_!(f_!(F_s(W_s))) \right)_t$$

and then take out $g_!$ to get the desired equivalence. □

Theorem 2.3.11. *The map $p\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes} \to \mathcal{F}_*$ is a symmetric monoidal ∞-category.*

Proof. The only thing we have left to check is the Segal condition. Thanks to the definition of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$, we already have at out disposal an equivalence of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}_S$ with $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^S$. Therefore, we just need to check that the maps:

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})^S \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}_S \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

are the projection onto the i-th factor, for every $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ inducing the map to the right. For this, simply note that the product decomposition of $\mathcal{C}^{\otimes}_{\rho_i}$ becomes:

$$\mathcal{C}^{\otimes}_{\rho_i} \simeq (\mathcal{C}\times\Delta^1)\times_{\Delta^1}\prod_{\Delta^1, j\neq i}\mathcal{C}^{\triangleright}$$

where the first factor comes from the pushforward of the identity in the i-th factor, and all others arise from maps $\langle 1\rangle \to \{*\}$. Then, locally cocartesian arrows over each ρ_i were described in Remark 2.3.8, and from that the conclusion follows easily. $\square$

We have therefore succeeded in endowing the functor category $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ with a symmetric monoidal ∞-category structure. This is particularly well behaved in the sense that it satisfies desirable properties, analogous to those in the ordinary case. These are summed up by the following two Propositions; these are [Gla16, Proposition 2.12 and Lemma 2.13], and we refer to Glasman's paper for the proofs.

Proposition 2.3.12. *A commutative monoid in* $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$ *is exactly a lax symmetric monoidal functor from* $\mathcal{C}^{\otimes}$ *to* $\mathcal{D}^{\otimes}$*. That is, we have an equivalence:*

$$\mathrm{CAlg}(\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}) \simeq \mathrm{Fun}^{\mathrm{Lax}}(\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}).$$

Proposition 2.3.13. *Under the hypotheses we had until now,* $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$ *also satisfies the equivalent conditions of Lemma 2.3.4, that is, it preserves colimits componentwise.*

2.4 Example: Day convolution for filtered spectra

In this final section, we expand on the structure of the Day convolution, obtaining a more explicit formula for computing cocartesian pushforwards of morphisms in $\mathcal{F}_*$. We will do this in particular in the case of the active morphism $\langle 2\rangle \to \langle 1\rangle$, since its cocartesian pushforward can be regarded as the actual tensor product functor. We then discuss this in the example of filtered spectra. The key point is the following remark.

Remark 2.4.1. By Lemma 2.3.9, the cocartesian edges of $p\colon \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes} \to \mathcal{F}_*$ are now precisely its locally cocartesian edges, as described in Remark 2.3.8. Therefore, they correspond to (products of) p-left Kan extensions as in Remark 2.2.7. In particular, we can now write down explicitly the behaviour of the cocartesian pushforward functor $f_!$ associated to an edge $f\colon S\to T$ of $\mathcal{F}_*$:

- Given a cocartesian edge (f, F_f) over f, its cocartesian pushforward sends, up to equivalence, a functor $F_S \in \mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}_S \simeq \mathrm{Fun}(\mathcal{C}^{\otimes}_S,\mathcal{D}^{\otimes}_S)$ to the restriction of F_f to $\mathcal{C}^{\otimes}_T$, i.e. to a functor $F_T\colon \mathcal{C}^{\otimes}_T \to \mathcal{D}^{\otimes}_T$. We can compute the value of F_T on objects of $\mathcal{C}^{\otimes}_T$ via the formula that was obtained in 2.2.11. Given $X\in\mathcal{C}^{\otimes}_T$, we have:
$$F_T(X) = \operatorname*{colim}_{Y\in(\mathcal{C}^{\otimes}_S)_{/X}} f_!(F_s(Y_s))$$
where $Y=\{Y_s\}_{s\in S^{\circ}}$ as $\mathcal{C}^{\otimes}_S \simeq \mathcal{C}^S$.

In particular, consider the active morphism $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}_*$. Its cocartesian pushforward is a map:

$$\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}_{\langle 2\rangle} \simeq \mathrm{Fun}(\mathcal{C},\mathcal{D})^2 \longrightarrow \mathrm{Fun}(\mathcal{C},\mathcal{D})$$

which essentially gives the Day convolution tensor product of two functors from $\mathcal{C}$ to $\mathcal{D}$. The above Remark 2.4.1 allows us to write down an explicit formula for its value on objects. Given two functors $F, G\colon \mathcal{C} \to \mathcal{D}$, their Day convolution tensor product is a functor $F \otimes_{\mathrm{Day}} G\colon \mathcal{C} \to \mathcal{D}$ whose value on an object $C \in \mathcal{C}$ is:

$$(F \otimes_{\mathrm{Day}} G)(C) = \operatorname*{colim}_{C_0 \otimes_{\mathcal{C}} C_1 \to C} F(C_0) \otimes_{\mathcal{D}} G(C_1)$$

where the subscript $C_0 \otimes_{\mathcal{C}} C_1 \to C$ of the colimit is shorthand for taking the colimit over the full subcategory of $\mathcal{C}_{/C}$ spanned by all such morphisms.

We can make this even more explicit for filtered spectra. First, we define these, and then we verify that the hypotheses needed in order to get a Day convolution symmetric monoidal structure are satisfied.

Definition 2.4.2. The *∞-category of filtered spectra* can be defined as the functor category $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$ of functors from $\mathbb{Z}$ to the ∞-category of spectra.

An object X of this ∞-category can in particular be viewed as a sequence of spectra $X(n)$ for every $n \in \mathbb{Z}$, together with morphisms of spectra $X(n) \to X(n+1)$ for every $n \in \mathbb{Z}$. Note that we are choosing an ascending filtration; if one wishes to work with a descending filtration instead, the first category should be $\mathbb{Z}^{\mathrm{op}}$ instead of $\mathbb{Z}$. However, this choice is not important at all as everything is easily dualized.

We wish to endow $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$ with a symmetric monoidal ∞-category structure via Day convolution, where on $\mathbb{Z}$ we take the symmetric monoidal structure induced by the sum, as in Example 1.2.10, and on Sp we take the smash product symmetric monoidal structure of Theorem 1.3.5.

Note that the ∞-category of spectra is such that:

(1) Sp admits all small colimits, by Remark 1.3.4;

(2) The smash product of spectra preserves colimits in each variable, by 1.3.5.

In particular, all hypotheses needed for the Day convolution construction are satisfied, and so we get a symmetric monoidal ∞-category:

$$\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp}) \longrightarrow \mathcal{F}_*$$

where the tensor product $X \otimes Y$ of two filtered spectra $X, Y \in \mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$ at the level of objects is given by:

$$(X \otimes Y)(n) = \operatorname*{colim}_{i+j \leq n} X(i) \wedge Y(j)$$

for any $n \in \mathbb{Z}$. This is just an application of the formula of Remark 2.4.1, as we have a morphism from the tensor product $i + j$ in $\mathbb{Z}$ to n if and only if $i + j \leq n$, and in that case such a morphism is unique. This gives a very explicit and nice way of computing the Day convolution tensor product on filtered spectra.

Chapter 3

∞-operads and Day convolution

Symmetric monoidal ∞-categories are just a special case of the more general notion of ∞-operad. Lurie, in his book [Lur17, Section 2.2.6], provides a detailed construction of a Day convolution for $\mathcal{O}$-monoidal ∞-categories, where $\mathcal{O}$ is an ∞-operad. Glasman's construction results from applying this in the particular case of $\mathcal{O} = \mathcal{F}_*$ is the trivial ∞-operad. However, in his construction Lurie uses more abstract tools to develop Day convolution and prove the main results. This approach then has the advantage of being very abstract and thus applicable in situations where this is needed, and where explicit computations are not the priority.

We start with an overview of the theory of colored operads and ∞-operads, as developed in [Lur17, Chapter 2.1], reporting the key definitions and results and giving some more intuition when needed. Then, we discuss the Day convolution construction, and discuss how it provides a robust, and more abstract, generalization of the construction studied in Chapter 2.

3.1 operads and ∞-operads

In this first section, we discuss briefly the definition of a colored operad and its generalization to ∞-categories. The goal is not to dive into details but rather to give an overview of how the previous chapters generalize, and so we will omit any technicalities.

A colored operad, in the classical sense, should be something that generalizes the idea of symmetric monoidal categories of encoding a binary operation. The goal is being able to encode also operations with any finite number of inputs.

To build intuition, start with an ordinary category: it consists of objects and morphisms. That is, for every pair of objects X and Y there is a set $\mathrm{Hom}(X, Y)$ of morphisms from X to Y. Every such morphism can be thought of as a "unary" operation with input space the single object X, and output space the single object Y.

Consider now a symmetric monoidal category. In addition to the previous, we also have a binary operation, the tensor product. Given objects X_1, X_2 and Y now, we can talk about bilinear maps from X_1 and X_2 to Y. Such maps are in bijection with morphisms in $\mathrm{Hom}(X_1 \otimes X_2, Y)$, and similarly to before they resemble a binary operation, with two input spaces X_1 and X_2, and an output space Y. Therefore, in a symmetric monoidal category we have a "space of bilinear morphisms" from every pair of objects X_1 and X_2 to a single object Y. Moreover, we can also use the tensor product to define n-linear operations, as being in correspondence with morphisms in $\mathrm{Hom}(X_1 \otimes \cdots \otimes X_n, Y)$. There is no ambiguity here thanks to the associativity of $\otimes$.

A colored operad generalizes this idea: for every n-uple of objects $\{X_1, \dots, X_n\}$ in some collection S of so-called *colors*, and for every other color Y in S, we require to have a *set of multimorphisms* $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{1\leq i\leq n}, Y)$ from $\{X_i\}_{1\leq i\leq n}$ to Y. The idea is that this set encodes n-ary operations from the n colors $X_1, \dots, X_n$ to Y. Of course, together with this data one has suitable composition laws, and these are all required to satisfy the expected associativity and unitary conditions. A more formal definition is the following; details on the composition maps and all conditions are omitted, for these we refer to [Lur17, Definition 2.1.1.1].

Definition 3.1.1. A *colored operad* $\mathcal{O}$ consists of:

(1) A collection S of *colors* of $\mathcal{O}$.

(2) For every finite set I, every collection $\{X_i\}_{i\in I}$ of colors in S, and any other color Y, a set:
$$\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$$
of *multimorphisms*, or simply *morphisms*, from $\{X_i\}_{i\in I}$ to Y.

(3) Suitable composition maps, required to satisfy appropriate associativity conditions.

(4) Identity elements $\mathrm{id}_X \in \mathrm{Mul}_{\mathcal{O}}(\{X\}, X)$ for every color X, satisfying appropriate unitary conditions.

Given a colored operad $\mathcal{O}$, its *underlying category*, still denoted $\mathcal{O}$, is the category with the same objects and with:

$$\mathrm{Hom}_{\mathcal{O}}(X, Y) = \mathrm{Mul}_{\mathcal{O}}(\{X\}, Y).$$

Remark 3.1.2. Colored operads can be seen as a generalization of ordinary categories: if $\mathcal{C}$ is any category, then we can view it as a colored operad with the same objects by setting the multimorphism sets to be equal to the hom sets of $\mathcal{C}$ whenever the input is a single color, and empty otherwise. That is, we have:

$$\mathrm{Mul}_{\mathcal{C}}(\{X_i\}_{i\in I}, Y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \text{if } \{X_i\}_{i\in I} = \{X\}; \\ \emptyset & \text{otherwise.} \end{cases}$$

The underlying category of this colored operad is then $\mathcal{C}$. In fact, if one defines a suitable category COp of colored operads, then this construction is left adjoint to the forgetful functor from COp to Cat giving the underlying category.

Remark 3.1.3. Colored operads also generalize symmetric monoidal categories. Given $\mathcal{C}$ a symmetric monoidal category, we can view it as a colored operad by setting:
$$\mathrm{Mul}_{\mathcal{C}}(\{X_i\}_{i\in I}, Y) = \mathrm{Hom}_{\mathcal{C}}(\bigotimes_{i\in I} X_i, Y)$$

and the original tensor product structure can be recovered up to canonical isomorphism thanks to Yoneda's Lemma: the object $X \otimes Y$, up to canonical isomorphism, corepresents the functor:

$$Z \longmapsto \mathrm{Mul}_{\mathcal{C}}(\{X, Y\}, Z).$$

Example 3.1.4. An nontrivial example of a colored operad is given by rooted trees: there is a colored operad with a single color, and with multimorphisms at level n given by trees with one root and n leaves, numbered from 1 to n. To compose a tree with n leaves with n other trees, with respectively $k_1, \dots, k_n$ leaves, one has to attach them to the leaves of the first one in the order given by the labeling. The result is a tree with $k_1 + \cdots + k_n$ leaves.

The key point now is that in analogy with Construction 1.1.12, starting from a colored operad $\mathcal{O}$ we can build a new category $\mathcal{O}^{\otimes}$, which comes equipped with a forgetful functor to $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}_*$. This satisfies the following:

(1) For every object $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ and any inert morphism $f\colon \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}_*$, there exists a p-cocartesian lift $\overline{f}\colon C \to C'$ of f, whose source is C. In particular, on the lines of Remark 1.1.8, this gives a functor $f_!\colon \mathcal{O}^{\otimes}_{\langle m\rangle} \to \mathcal{O}^{\otimes}_{\langle n\rangle}$ between fibers.

(2) The fiber $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ can be identified with the underlying category of $\mathcal{O}$. More generally, for every $n \geq 0$, denote by $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ the usual characteristic morphisms, which are inert. Then, the functors $\rho_{i,!}\colon \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$ induce an equivalence:

$$\mathcal{O}^{\otimes}_{\langle n\rangle} \xrightarrow{\simeq} \prod_{1\leq i\leq n} \mathcal{O}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{O}^n$$

so that objects in $\mathcal{O}^{\otimes}_{\langle n\rangle}$ can be identified with n-uples of objects of $\mathcal{O}$. When $n = 0$, we recover the identity elements for every $X \in \mathcal{O}$.

(3) Let $\{X_i\}_{i\in I}$ be an n-uple of objects of $\mathcal{O}$, and denote by $\overline{X}$ the corresponding object of $\mathcal{O}^{\otimes}_{\langle n\rangle}$. Then, for any other $Y \in \mathcal{O}$, the multimorphisms in $\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I}, Y)$ can be identified with the morphisms $\overline{X} \to Y$ in $\mathcal{O}^{\otimes}$ lying over the unique active morphism $\langle n\rangle \to \langle 1\rangle$ of $\mathcal{F}_*$.

(4) One can also recover the composition law in a more complicated way, which we will not discuss. However, we note the following: let $\overline{X} \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ and $\overline{Y} \in \mathcal{O}^{\otimes}_{\langle n\rangle}$, and for every $1 \leq i \leq n$, denote by Y_i the cocartesian pushforward of Y along $\rho_{i,!}$. Then, for every $f\colon \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}_*$, there is a decomposition:

$$\mathrm{Hom}^{f}_{\mathcal{O}^{\otimes}}(\overline{X}, \overline{Y}) \simeq \prod_{1\leq i\leq n} \mathrm{Hom}^{\rho_i\circ f}_{\mathcal{O}^{\otimes}}(\overline{Y}, Y_i)$$

where the superscripts f and $\rho_i \circ f$ denote those morphisms lying over f and $\rho_i \circ f$, respectively. The existence of this fact should be enough motivation for one of the requirements in the definition of ∞-operads, which will essentially be a higher categorical analogue.

The datum of a functor $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}_*$ satisfying the above conditions is in turn enough to obtain a colored operad. Thus, we can regard this as an equivalent definition for a colored operad. In particular, as was the case for symmetric monoidal categories, this is way better suited to be generalized to ∞-categories.

Definition 3.1.5. An ∞ operad consists of an ∞-category $\mathcal{O}^{\otimes}$ together with a functor $p\colon \mathcal{O}^{\otimes} \to \mathcal{F}_*$ which satisfies the following:

(1) For every object $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ and any inert morphism $f\colon \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}_*$, there exists a p-cocartesian lift $\overline{f}\colon C \to C'$ of f, with source C. In particular, we get a functor $f_!\colon \mathcal{O}^{\otimes}_{\langle m\rangle} \to \mathcal{O}^{\otimes}_{\langle n\rangle}$.

(2) Consider objects $C \in \mathcal{O}^{\otimes}_{\langle m\rangle}$ and $C' \in \mathcal{O}^{\otimes}_{\langle n\rangle}$; for every $1 \leq i \leq n$, denote by C'_i the cocartesian pushforward of C' along $(\rho_i)_!$, obtained from the choice of a p-cocartesian lift $C' \to C'_i$ of ρ_i. Also let $f\colon \langle m\rangle \to \langle n\rangle$ be any morphism in $\mathcal{F}_*$; denote by $\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C')$ the union of the connected components of $\mathrm{Map}_{\mathcal{O}^{\otimes}}(C, C')$ lying over f, and similarly for $\mathrm{Map}^{\rho_i\circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$. Then, the induced functor:

$$\mathrm{Map}^f_{\mathcal{O}^{\otimes}}(C, C') \xrightarrow{\simeq} \prod_{1\leq i\leq n} \mathrm{Map}^{\rho_i\circ f}_{\mathcal{O}^{\otimes}}(C, C'_i)$$

is a homotopy equivalence.

(3) For every finite collection of objects $C_1, \ldots, C_n \in \mathcal{O}^{\otimes}_{\langle 1\rangle}$, there exists an object $C \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ and p-cocartesian morphisms $C \to C_i$ covering ρ_i for every $i = 1, \ldots, n$.

The fiber $\mathcal{O}^{\otimes}_{\langle 1\rangle}$ will be denoted by $\mathcal{O}$, and referred to as the *underlying ∞-category* of $\mathcal{O}^{\otimes}$.

Remark 3.1.6. Note that we drop the adjective "colored" from the notation to make it simpler. Classically, an operad is a colored operad with a single object X.

Remark 3.1.7. Condition (3) of the previous Definition, assuming (1) and (2), is equivalent to the following, which looks more familiar:

(3') For every $n \geq 0$, there is an equivalence of ∞-categories:

$$\mathcal{O}^{\otimes}_{\langle n\rangle} \xrightarrow{\simeq} \prod_{1\leq i\leq n} \mathcal{O}^{\otimes}_{\langle 1\rangle}$$

induced by the functors $(\rho_i)_!\colon \mathcal{O}^{\otimes}_{\langle n\rangle} \to \mathcal{O}^{\otimes}_{\langle 1\rangle}$.

We will refer to this as the Segal condition. If $X \in \mathcal{O}^{\otimes}_{\langle n\rangle}$ is an object, we will sometimes write $X = X_1 \oplus \cdots \oplus X_n$ where each $X_i \in \mathcal{O}^{\otimes}_{\langle 1\rangle} \simeq \mathcal{O}$ to mean its decomposition according to the Segal condition.

Definition 3.1.5 looks very reasonable. For example, it includes colored operads in the expected way: if $\mathcal{O}^{\otimes} \to \mathcal{F}_*$ is a colored operad, then the nerve construction gives an ∞-operad $\mathrm{N}(\mathcal{O}^{\otimes}) \to \mathrm{N}(\mathcal{F}_*) =: \mathcal{F}_*$. It is also a weakening of the definition of symmetric monoidal ∞-categories: here we require cocartesian lifts not for every morphism in $\mathcal{F}_*$ but only for the inert ones, and whereas the functor to $\mathcal{F}_*$ for symmetric monoidal ∞-categories was thus a cocartesian fibration, here for ∞-operads it can be shown that it is just a categorical fibration in general. Moreover, condition (2) in the definition of ∞-operads can be seen to be always true for symmetric monoidal ∞-categories. The decomposition of slices that we studied in detail in Construction 2.3.1 in fact resembles this.

Example 3.1.8. The identity functor $\mathcal{F}_* \xrightarrow{\text{id}} \mathcal{F}_*$ makes $\mathcal{F}_*$ into an ∞-operad, also called the *commutative ∞-operad*. We will still denote this by $\mathcal{F}_*$.

In accordance to the situation for colored operads, we can also define spaces of multimorphisms in this context.

Notation 3.1.9. Let $\mathcal{O}^\otimes$ be an ∞-operad and consider objects $X_1, \ldots, X_n$ and Y in $\mathcal{O}^\otimes_{\langle 1 \rangle} \simeq \mathcal{O}$. Then, $\text{Mul}_{\mathcal{O}}(\{X_i\}_{1\leq i \leq n}, Y)$ will denote the union of the connected components of $\text{Map}_{\mathcal{O}^\otimes}(X_1 \oplus \cdots \oplus X_n, Y)$ lying over the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ of $\mathcal{F}_*$.

One can then define appropriate notions of maps between ∞-operads, fibrations and cocartesian fibrations of ∞-operads.

Definition 3.1.10. Let $\mathcal{O}^\otimes$ and $\mathcal{O}'^\otimes$ be two ∞-operads. A functor $f\colon \mathcal{O}^\otimes \to \mathcal{O}'^\otimes$ of simplicial sets is called a *map of ∞-operads* if:

(1) The following diagram commutes:

$$\begin{array}{ccc} \mathcal{O}^\otimes & \xrightarrow{\quad f \quad} & \mathcal{O}'^\otimes \\ & \searrow \quad \swarrow & \\ & \mathcal{F}_* & \end{array}$$

(2) f maps inert edges of $\mathcal{O}^\otimes$ to inert edges of $\mathcal{O}'^\otimes$.

The full subcategory of $\text{Fun}(\mathcal{O}^\otimes, \mathcal{O}'^\otimes)$ spanned by the maps of ∞-operads is denoted by $\text{Alg}_{\mathcal{O}}(\mathcal{O}')$.

Remark 3.1.11. There is an ∞-category of ∞-operads, denoted by Op_∞, with morphisms given by the maps of ∞-operads.

Definition 3.1.12. A map $f\colon \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ of ∞-operads is a *fibration of ∞-operads* if it is a categorical fibration.

We now look at cocartesian fibrations of ∞-operads. Their definition is motivated by the following result, where we note that $\mathcal{C}^\otimes$ is not a priori an ∞-operad.

Proposition 3.1.13 ([Lur17, Proposition 2.1.2.12])**.** *Let $\mathcal{O}^\otimes$ be an ∞-operad, and let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a cocartesian fibration of simplicial sets. Then, the following are equivalent:*

(1) For every object $T \in \mathcal{O}^\otimes$, decomposed as $T = T_1 \oplus \cdots \oplus T_n$, the cocartesian pushforwards of the (inert) morphisms $T \to T_i$ induce an equivalence of ∞-categories:

$$\mathcal{C}^\otimes_T \xrightarrow{\simeq} \prod_{1\leq i\leq n} \mathcal{C}^\otimes_{T_i}.$$

(2) The composite $\mathcal{C}^\otimes \to \mathcal{O}^\otimes \to \mathcal{F}_*$ exhibits $\mathcal{C}^\otimes$ as an ∞-operad.

The first condition strongly resembles the Segal condition of symmetric monoidal ∞-categories, whereas the second condition is something we would want to be true when considering cocartesian fibrations of ∞-operads. We therefore define them as follows:

Definition 3.1.14. A map $f\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ of ∞-operads is a *cocartesian fibration of ∞-operads* if it is a cocartesian fibration of simplicial sets and if satisfies the equivalent conditions of Proposition 3.1.13. We will also say that *p exhibits $\mathcal{C}^{\otimes}$ as an $\mathcal{O}$-monoidal ∞-category*, or simply that *$\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal ∞-category.*

Example 3.1.15. A symmetric monoidal ∞-category is then an ∞-category $\mathcal{C}^{\otimes}$ with a cocartesian fibration of ∞-operads $\mathcal{C}^{\otimes} \to \mathcal{F}_* = \mathcal{F}_*$.

Finally, we can define appropriate ∞-categories of algebra objects.

Definition 3.1.16. Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a fibration of ∞-operads, and let $\alpha\colon \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of ∞-operads. Then:

(1) The full subcategory of $\mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$, of functors over $\mathcal{O}^{\otimes}$, spanned by the maps of ∞-operads will be denoted by $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$. This is the ∞-category of *$\mathcal{O}'$-algebra objects of $\mathcal{C}$.*

(2) If $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes}$ and α is the identity, the ∞-category of $\mathcal{O}$-algebra objects of $\mathcal{C}$ will be denoted by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.

(3) If $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes} = \mathcal{F}_*$, the ∞-category of algebra objects of $\mathcal{C}$ will be denoted $\mathrm{CAlg}(\mathcal{C})$, and referred to as the *∞-category of commutative algebra objects of $\mathcal{C}$.*

3.2 Day convolution for ∞-operads

Let us now discuss Lurie's approach to Day convolution for ∞-operads, as in [Lur17, Section 2.2.6]; we will give a brief account of this, focusing on the key steps and avoiding proofs. Again, his construction is more abstract and does not resemble that of Glasman in [Gla16], besides the end result.

The situation is here generalized by considering $\mathcal{O}$-monoidal ∞-categories $\mathcal{C}$ and $\mathcal{D}$, and studying a Day convolution construction for these. We get back to the situation for symmetric monoidal ∞-categories when considering $\mathcal{O}^{\otimes} = \mathcal{F}_*$ the commutative ∞-operad.

We begin with the main definition, which at first impact may seem rather obscure. We give a hopefully more approachable interpretation right after this in Remark 3.2.3, looking only at the case that is of interest to us to obtain the Day convolution.

Definition 3.2.1. Let $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of ∞-operads, and consider two fibrations of ∞-operads:

$$\widetilde{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes} \qquad \text{and} \qquad \widetilde{\mathcal{O}}^{\otimes} \to \mathcal{O}^{\otimes}.$$

We say that a morphism:

$$\alpha\colon \widetilde{\mathcal{O}}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} \longrightarrow \widetilde{\mathcal{C}}^{\otimes}$$

of ∞-operads *exhibits $\widetilde{\mathcal{O}}^{\otimes}$ as a norm of $\widetilde{\mathcal{C}}^{\otimes}$ along p* if:

(1) The following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathcal{O}}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} & \xrightarrow{\alpha} & \widetilde{\mathcal{C}}^{\otimes} \\ & \searrow \qquad \swarrow & \\ & \mathcal{C} & \end{array}$$

(2) For any map $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ of ∞-operads, composition with α induces an equivalence:

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\widetilde{\mathcal{O}}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{C}}(\widetilde{\mathcal{C}}).$$

Remark 3.2.2. In particular, applying (2) to the case $\mathcal{O}'^{\otimes} = \mathcal{O}^{\otimes}$, we obtain an equivalence:

$$\mathrm{Alg}_{/\mathcal{O}}(\widetilde{\mathcal{O}}) \xrightarrow{\simeq} \mathrm{Alg}_{/\mathcal{C}}(\widetilde{\mathcal{C}}).$$

Remark 3.2.3. This definition seems rather obscure, so let us discuss now its relevance in the construction of Day convolution, and see what it specializes to in the case of symmetric monoidal ∞-categories. That is, let $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of ∞-operads, i.e. an $\mathcal{O}$-monoidal ∞-category, and let $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be any fibration of ∞-operads. Then, by taking its pullback along p we obtain a fibration of ∞-operads:

$$\mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} \longrightarrow \mathcal{C}^{\otimes}$$

which will play the role of the first fibration of Definition 3.2.1. Then, the Theorem that we are about to state will guarantee the existence of a fibration of ∞-operads, which we denote:

$$\mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$$

and a map:

$$\alpha\colon \mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes} \times \mathcal{C}^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$$

exhibiting $\mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes}$ as a norm of $\mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes}$ along p. The map α looks suspiciously like an evaluation map, and the notation used suggests that this turns out to be precisely the Day convolution monoidal structure, under some further hypotheses that we will see in a bit. Setting $\mathcal{O}^{\otimes}$ to be the commutative ∞-operad $\mathcal{F}_*$, we recover the Day convolution symmetric monoidal ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes} \to \mathcal{F}_*$.

Theorem 3.2.4 ([Lur17, Theorem 2.2.6.2])**.** *Let $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of ∞-operads, and let $\widetilde{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes}$ be a fibration of ∞-operads. Then, there exists another fibration*

$$\mathrm{Nm}_{\mathcal{C}/\mathcal{O}}(\widetilde{\mathcal{C}})^{\otimes} \to \mathcal{O}^{\otimes}$$

of ∞-operads, and a map α exhibiting $\mathrm{Nm}_{\mathcal{C}/\mathcal{O}}(\widetilde{\mathcal{C}})$ as a norm of $\widetilde{\mathcal{C}}$ along p.

This Theorem guarantees existence. It also follows rather smoothly from Definition 3.2.1 that we have uniqueness of $\widetilde{\mathcal{O}}^{\otimes}$ and α up to equivalence, and in fact up to a contractible space of choices, as noted by Lurie.

In the situation of Definition 3.2.1, we also have a way to characterize the fibers of $\widetilde{\mathcal{O}}^{\otimes}$ over an object $X \in \mathcal{O}$. This is the following:

Proposition 3.2.5 ([Lur17, Proposition 2.2.6.4])**.** *In the situation of Definition 3.2.1, for every object $X \in \mathcal{O}$ there is an equivalence of ∞-categories:*

$$\widetilde{\mathcal{O}}_X \xrightarrow{\simeq} \mathrm{Fun}_{\mathcal{C}_X}(\mathcal{C}_X, \widetilde{\mathcal{C}}_X).$$

Note that this is true only when $X \in \mathcal{O}$ is an object in the underlying ∞-category, and not for every $X \in \mathcal{O}^{\otimes}$; see [Lur17, Warning 2.2.6.5].

Remark 3.2.6. Suppose more generally that we have $X \in \mathcal{O}^{\otimes}_{\langle n \rangle}$, writing it as $X = X_1 \oplus \cdots \oplus X_n$ where each $X_i \in \mathcal{O}$. Then, as we have an equivalence $\widetilde{\mathcal{O}}^{\otimes}_X \simeq \prod_i \widetilde{\mathcal{O}}^{\otimes}_{X_i}$, we obtain an equivalence:

$$\widetilde{\mathcal{O}}^{\otimes}_X \simeq \prod_{1 \leq i \leq n} \mathrm{Fun}_{\mathcal{C}_{X_i}}(\mathcal{C}_{X_i}, \widetilde{\mathcal{C}}_{X_i}).$$

When $\widetilde{\mathcal{O}}^{\otimes} = \mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes}$ of Remark 3.2.3, this tells us that we can identify an object of the fiber $\mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes}_X$ with a collection of functors $\{F_i \colon \mathcal{C}_{X_i} \to \mathcal{D}_{X_i}\}_{1 \leq i \leq n}$.

Remark 3.2.7. Going back to the situation of Remark 3.2.3, we see that:

(1) Composition with α induces an equivalence:

$$\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}).$$

This is the generalization of Proposition 2.3.12.

(2) For each $X \in \mathcal{O}$, α induces an equivalence between fibers:

$$\mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})_X \xrightarrow{\simeq} \mathrm{Fun}(\mathcal{C}_X, \mathcal{D}_X).$$

This is reminiscent of how we defined simplices for the Day convolution of symmetric monoidal ∞-categories.

In the particular case where $\mathcal{O}^{\otimes} = \mathcal{F}_*$, we recover precisely the Day convolution discussed in Chapter 2. The underlying ∞-category is equivalent to $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, and its commutative algebra objects correspond to lax symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ (see again Proposition 2.3.12).

Of course, in order to ensure that this indeed resulted in a symmetric monoidal ∞-category we had to have additional hypotheses on $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$. A similar discussion must be made here for $\mathcal{O}$-monoidal ∞-categories. The first step is to ensure that we obtain a locally cocartesian fibration of ∞-operads:

Proposition 3.2.8 ([Lur17, Corollary 2.2.6.14])**.** *Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be cocartesian fibrations of ∞-operads, and let κ be an uncountable cardinal. Suppose that:*

(1) For each $X \in \mathcal{O}$, the fiber $\mathcal{C}_X$ is essentially κ-small. That is, it is equivalent to a κ-small ∞-category.

(2) For each $Y \in \mathcal{O}$, the fiber $\mathcal{D}_Y$ admits κ-small colimits.

Then, the canonical map $\theta \colon \mathrm{Fun}^{\mathcal{O}}(\mathcal{C}, \mathcal{D})^{\otimes} \to \mathcal{O}^{\otimes}$ is a locally cocartesian fibration of ∞-operads.

Remark 3.2.9. The proof of this result in [Lur17] is a generalization to the discussion we did leading to the proof of Corollary 2.2.9. θ being locally cocartesian is equivalent to the existence of certain left Kan extensions, which in fact do always exist, rendering θ locally cocartesian. Through this, we can give a concrete description of the cocartesian pushforward functors associated to any $\phi \in \mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i \in I}, Y)$. Recall that this notation means that ϕ is a morphism $\phi \colon X = X_1 \oplus \cdots \oplus X_n \to Y$ in $\mathcal{O}^{\otimes}$ lying over the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ of $\mathcal{F}_*$. Using the Segal condition and the equivalence of Remark 3.2.6, the cocartesian pushforwards look as follows:

$$\prod_{i\in I}\mathrm{Fun}(\mathcal{C}_{X_i},\mathcal{D}_{X_i})\simeq\prod_{i\in I}\mathrm{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{D})^{\otimes}_{X_i}\longrightarrow\mathrm{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{D})_Y\simeq\mathrm{Fun}(\mathcal{C}_Y,\mathcal{D}_Y).$$

Given a collection of functors $\{F_i\colon \mathcal{C}_{X_i}\to\mathcal{D}_{X_i}\}$, these are carried to a functor $G\colon \mathcal{C}_Y\to\mathcal{D}_Y$ which on objects $C\in\mathcal{C}_Y$ is given by the formula:

$$G(C)=\operatorname*{colim}_{\otimes_\phi\{C_i\}_i\to C}\bigotimes_{\phi}F_i(C_i),$$

where $\otimes_\phi$ denotes the pushforward functors associated to ϕ in $\mathcal{C}$ and $\mathcal{D}$. The colimit is taken over the slice category $(\prod\mathcal{C}_{X_i})\times_{\mathcal{C}_Y}(\mathcal{C}_Y)_{/C}$, whose objects are all possible morphisms $\otimes_\phi\{C_i\}_i\to C$. This formula strongly resembles the one we obtained earlier in 2.4.1.

Finally, we need to add one further hypothesis to make sure that $\theta\colon\mathrm{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{D})^{\otimes}\to\mathcal{O}^{\otimes}$ is moreover cocartesian, making $\mathrm{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{D})^{\otimes}$ into an $\mathcal{O}$-monoidal ∞-category. This is in analogy with Lemma 2.3.4.

Proposition 3.2.10. *Under the hypotheses of Proposition 3.2.8, suppose further that for each $\phi\in\mathrm{Mul}_{\mathcal{O}}(\{X_i\}_{i\in I},Y)$, the associated cocartesian pushforward functor in $\mathcal{D}^{\otimes}$:*

$$\otimes_\phi\colon\prod_{i\in I}\mathcal{D}_{X_i}\to\mathcal{D}_Y$$

preserves κ-small colimits in each variable separately. Then, $\theta\colon\mathrm{Fun}^{\mathcal{O}}(\mathcal{C},\mathcal{D})^{\otimes}\to\mathcal{O}^{\otimes}$ is a cocartesian fibration of ∞-operads, that is, an $\mathcal{O}$-monoidal ∞-category.

This whole discussion can be specialized to $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ being symmetric monoidal ∞-categories, resulting in the construction by Glasman that we studied in Chapter 2.

Example 3.2.11. Let $\mathcal{O}=\mathcal{F}_*$, so that we recover Day convolution for symmetric monoidal ∞-categories, and let us look back at the example of filtered spectra that was studied in section 2.4. Consider two filtered spectra $X,Y\colon\mathbb{Z}\to\mathrm{Sp}$; then, we can recover the formula that we had obtained there for their Day convolution tensor product:

$$(X\otimes Y)(n)=\operatorname*{colim}_{i+j\leq n}X(i)\wedge Y(j).$$

This is done by specializing Remark 3.2.9 to the choice of ϕ being precisely the unique active morphism $\langle 2\rangle\to\langle 1\rangle$. Indeed, the cocartesian pushforwards $\otimes_\phi$ associated to this ϕ are precisely the tensor products, so in $\mathbb{Z}$ we get the sum and in Sp we get the smash product.

Chapter 4

G-∞-categories and G-symmetric monoidal G-∞-categories

The goal of this chapter is to introduce the theory of G-∞-categories, that is, parametrized ∞-categories over the orbit category $\mathcal{O}rb_G$ of a finite group G. Such a structure arises from the need to encode a "weak action" ρ of G on an ∞-category $\mathcal{C}$: we wish to have a weak equivalence $\rho(g)\colon \mathcal{C} \xrightarrow{\simeq} \mathcal{C}$, such that $\rho(gh) = \rho(g) \circ \rho(h)$. It turns out that one should also remember more information, such as the fixed-point ∞-category $\mathcal{C}^H$ for every subgroup $H \leq G$, together with equivalences $\mathcal{C}^H \xrightarrow{\simeq} \mathcal{C}^{gHg^{-1}}$ for every $H \leq G$, coming from the conjugation action on subgroups.

A simple 1-categorical example of this is given by the complex representations of G, which are collected in the category $R(G)$. For every orbit G/H, we want to remember what the H-objects are, and in this case they are the complex representations of $H \leq G$. Moreover, we have that every map $G/H \to G/K$ between orbits gives contravariantly a functor $R(K) \to R(H)$; in case $H \leq K \leq G$ then this is the usual restriction Res^K_H.

The right structure to encode this type of information is a cocartesian fibration over $\mathcal{O}rb_G^{\mathrm{op}}$. By straightening-unstraightening, this corresponds to a functor $\mathcal{O}rb_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$, which encodes all the desired information: the values on an orbit G/H are the "H-objects", and the various maps in $\mathcal{O}rb_G$ give contravariantly functors between them as desired. This is in fact a special case of parametrized higher category theory, as developed by Barwick, Dotto, Glasman, Nardin and Shah; see for example [Bar+16a], [Bar+16b], [Sha22] and [Sha23].

4.1 G-∞-categories

The theory of parametrized ∞-categories can be developed in great generality by parametrizing over any ∞-category $\mathcal{T}$. In fact, in analogy with the discussion above, it is easy to come up with a satisfying definition:

Definition 4.1.1. Let $\mathcal{T}$ be any ∞-category. A *$\mathcal{T}$-∞-category*, or alternatively a *$\mathcal{T}$-parametrized ∞-category*, consists of an ∞-category $\underline{\mathcal{C}}$ together with a cocartesian fibration:

$$p\colon \underline{\mathcal{C}} \longrightarrow \mathcal{T}^{\mathrm{op}}.$$

Remark 4.1.2. Via straightening-unstraightening, this corresponds to a functor $\mathcal{T}^{\mathrm{op}} \to \mathrm{Cat}_\infty$. We can therefore think of G-∞-categories as presheaves of ∞-categories over $\mathcal{T}$, thus a generalization of the usual theory of ∞-categories.

For more general purposes, this is the right definition, as lots of theory can be developed in this generality. In this thesis though we are only interested in parametrizations given by a finite group G, and so we restrict our point of view to these. We will therefore often use the adjective "equivariant" in place of "parametrized". From now on, let G denote any finite group. We first recall the definition of its orbit category.

Definition 4.1.3. The *orbit category* $\mathcal{O}rb_G$ of G is the category whose objects are the quotients G/H, for every $H \leq G$ subgroup, and whose morphisms are G-equivariant maps between orbits.

In particular:

- if $H \leq K \leq G$ are subgroups, we get a map $G/H \to G/K \cong (G/H)/(K/H)$ in $\mathcal{O}rb_G$, which is just the projection;
- if $H \leq G$ and $g \in G$, we get a map $G/H \to G/(gHg^{-1})$ given by conjugation by g.

We can in particular specialize the above Definition 4.1.1 to this equivariant setting.

Definition 4.1.4. A *G-∞-category* consists of an ∞-category $\underline{\mathcal{C}}$ together with a cocartesian fibration:

$$p\colon \underline{\mathcal{C}} \to \mathcal{O}rb_G^{\mathrm{op}}.$$

We will denote the fiber over a G/H by $\underline{\mathcal{C}}_{G/H}$, and in particular the fiber over G/G will also be denoted by $\mathcal{C}$.

Notation 4.1.5. We will often omit writing the map to $\mathcal{O}rb_G^{\mathrm{op}}$ when it is understood from context, and instead just speak of a G-∞-category $\underline{\mathcal{C}}$. For this first part, we will generally try to make it clear that we are talking about G-∞-categories, and not just ∞-categories, by underlining them.

Equivalently, by straightening-unstraightening, a G-∞-category is a functor $\underline{\mathcal{C}}\colon \mathcal{O}rb_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$, that is, a Cat_∞-valued presheaf on $\mathcal{O}rb_G$. Given $H \leq G$, we can think of the fibers $\underline{\mathcal{C}}_{G/H}$ as "H-objects" of the G-∞-category; we have that $\underline{\mathcal{C}}(G/H) = \underline{\mathcal{C}}_{G/H}$. Using this point of view, it is clear that any morphism $G/H \to G/K$ in $\mathcal{O}rb_G$ gives contravariantly a functor $\underline{\mathcal{C}}_{G/K} \to \underline{\mathcal{C}}_{G/H}$. These are usually called *restriction functors.*

In the following, we will alternate between the two points of view of cocartesian fibrations over $\mathcal{O}rb_G^{\mathrm{op}}$ and Cat_∞-valued presheaves on $\mathcal{O}rb_G$. This is because the former is easier to work with, whereas the latter is better suited to build intuition on the behaviour of some G-∞-category.

Example 4.1.6. If $\mathcal{C}$ is any ∞-category, we have a projection functor:

$$p\colon \mathcal{C} \times \mathcal{O}rb_G^{\mathrm{op}} \longrightarrow \mathcal{O}rb_G^{\mathrm{op}}$$

which is clearly cocartesian, and so a G-∞-category. This is the *trivial G-∞-category* on $\mathcal{C}$. Viewing this as a presheaf, the value on every orbit G/H is constant at $\mathcal{C}$, and the restriction functors are all the identity of $\mathcal{C}$.

In particular, the category $\mathbb{Z}$, whose morphisms are given by the $\leq$ relation, can be turned trivially into a G-∞-category. This is then given as a presheaf by a functor:

$$\underline{\mathbb{Z}}\colon \mathcal{O}rb_G^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$$

which maps every orbit G/H to $\mathbb{Z}$, and every morphism $G/H \to G/K$ to the identity of $\mathbb{Z}$.

Example 4.1.7. If $\mathcal{C}$ is an ordinary category and we have a Grothendieck opfibration $p\colon \mathcal{C} \to \mathcal{O}rb_G^{\mathrm{op}}$, then the nerve construction makes $\mathrm{N}(\mathcal{C})$ into a G-∞-category via $\mathrm{N}(p)$.

Remark 4.1.8. Let $\underline{\mathcal{C}}$ be a G-∞-category, viewed as a Cat_∞-valued presheaf on $\mathcal{O}rb_G$, and let $H \leq G$. Note that we have an obvious inclusion $\mathcal{O}rb_H \hookrightarrow \mathcal{O}rb_G$. Then, $\underline{\mathcal{C}}$ can be restricted to an H-∞-category by taking the composite:

$$\mathcal{O}rb_H \hookrightarrow \mathcal{O}rb_G \xrightarrow{\underline{\mathcal{C}}} \mathrm{Cat}_\infty .$$

Viewing instead $\underline{\mathcal{C}}$ as a cocartesian fibration over $\mathcal{O}rb_G^{op}$, this corresponds to taking the pullback along $\mathcal{O}rb_H \hookrightarrow \mathcal{O}rb_G$. We will call the resulting H-∞-category the *H-restriction of* $\underline{\mathcal{C}}$.

If instead $H, K \leq G$ are conjugate, we have an equivalence $\mathcal{O}rb_H \simeq \mathcal{O}rb_K$. Then, to go from a K-∞-category to an H-∞-category, and viceversa, we can still just pre-compose by this equivalence, when viewing them as presheaves, or take the pullback if we view them as cocartesian fibrations.

We also need to define a suitable generalization of functors to this setting. This is quite straightforward when using the viewpoint of cocartesian fibrations over $\mathcal{O}rb_G^{\mathrm{op}}$, and is motivated by the fact that such a functor should "preserve" the cocartesian structure of the fibrations.

Definition 4.1.9. Let $\underline{\mathcal{C}} \xrightarrow{p} \mathcal{O}rb_G^{\mathrm{op}}$ and $\underline{\mathcal{D}} \xrightarrow{q} \mathcal{O}rb_G^{\mathrm{op}}$ be G-∞-categories. A *G-functor* from $\underline{\mathcal{C}}$ to $\underline{\mathcal{D}}$ is a functor $F\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ of ∞-categories over $\mathcal{O}rb_G^{\mathrm{op}}$:

$$\begin{array}{ccc} \underline{\mathcal{C}} & \xrightarrow{\quad F \quad} & \underline{\mathcal{D}} \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle q} & \\ & \mathcal{O}rb_G^{\mathrm{op}} & \end{array}$$

such that it sends p-cocartesian edges to q-cocartesian edges.

Remark 4.1.10. Given G-∞-categories $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ as above, the collection of G-functors from $\underline{\mathcal{C}}$ to $\underline{\mathcal{D}}$ can be organized into an ∞-category, which we will write $\mathrm{Fun}_G(\mathcal{C}, \mathcal{D})$. This can be further refined into a G-∞-category $\underline{\mathrm{Fun}}_G(\mathcal{C}, \mathcal{D})$. If we see this as a presheaf:

$$\underline{\mathrm{Fun}}_G(\mathcal{C}, \mathcal{D})\colon \mathcal{O}rb_G^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$$

then it will send an orbit G/H to the ∞-category $\mathrm{Fun}_H(\mathcal{C}, \mathcal{D})$ of functors between the H-restrictions of $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ (see Remark 4.1.8), and a morphism $G/H \to G/K$ to the obvious restriction $\mathrm{Fun}_K(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_H(\mathcal{C}, \mathcal{D})$. Note that for example if $H \leq K$ then it is clear that any K-functor is also an H-functor, as $\mathcal{O}rb_H \hookrightarrow \mathcal{O}rb_K$.

Remark 4.1.11. G-∞-categories and G-functors between them can be collected into an ∞-category $\mathrm{Cat}_{\infty,G}$, the *∞-category of G-∞-categories*. By straightening-unstraightening, we have:

$$\mathrm{Cat}_{\infty,G} \simeq \mathrm{Fun}(\mathcal{O}rb_G^{\mathrm{op}}, \mathrm{Cat}_\infty)$$

so that, once again, we can think of G-∞-categories as presheaves of ∞-categories over $\mathcal{O}rb_G^{\mathrm{op}}$. For example, a G-functor, between $\underline{\mathcal{C}} \to \mathcal{O}rb_G^{\mathrm{op}}$ and $\underline{\mathcal{D}} \to \mathcal{O}rb_G^{\mathrm{op}}$, viewed as cocartesian fibrations, corresponds to a natural transformation between functors $\underline{\mathcal{C}}, \underline{\mathcal{D}} \colon \mathcal{O}rb_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

We also have a concept of G-spaces, analogous to the usual spaces in Cat_∞, also called Kan complexes. Recall that a Kan complex is equivalently an ∞-groupoid, that is, an ∞-category where all edges are invertible. This can be generalized as follows:

Definition 4.1.12. A *G-space* is a G-∞-category $\underline{\mathcal{C}} \xrightarrow{p} \mathcal{O}rb_G^{\mathrm{op}}$ such that every edge of $\underline{\mathcal{C}}$ is p-cocartesian.

Remark 4.1.13. In particular in this case, the fibration p is not only cocartesian but also a left fibration, and viceversa every left fibration as above gives rise to a G-space. The fibers are thus not only ∞-categories but also Kan complexes, and the straightening-unstraightening correspondence yields a functor $\mathcal{O}rb_G^{\mathrm{op}} \to \mathcal{S}$ which lands in the ∞-category $\mathcal{S}$ of spaces. G-spaces are collected in an ∞-category $\mathcal{S}_G$, which can be defined as the full subcategory of $\mathrm{Cat}_{\infty,G}$ on the G-spaces. We have, again by straightening-unstraightening:

$$\mathcal{S}_G \simeq \mathrm{Fun}(\mathcal{O}rb_G^{\mathrm{op}}, \mathcal{S}).$$

This is usually called the *∞-category of genuine G-spaces*.

Remark 4.1.14. The ∞-categories just mentioned, $\mathrm{Cat}_{\infty,G}$ and $\mathcal{S}_G$, can be refined into G-∞-categories, which we will write as $\underline{\mathrm{Cat}}_{\infty,G}$ and $\underline{\mathcal{S}}_G$ respectively. These correspond to functors:

$$\underline{\mathrm{Cat}}_{\infty,G}, \underline{\mathcal{S}}_G \colon \mathcal{O}rb_G^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$$

for which in particular we have:

- The value of $\underline{\mathrm{Cat}}_{\infty,G}$ on an orbit G/H is $\mathrm{Cat}_{\infty,H}$, whereas the value on a morphism $G/H \to G/K$ is given by the restriction functors of Remark 4.1.8.
- The value of $\underline{\mathcal{S}}_G$ on an orbit G/H is $\mathcal{S}_H$, and the value on a morphism $G/H \to G/K$ is as above.

Alternatively, $\underline{\mathcal{S}}_G$ as cocartesian fibration over $\mathcal{O}rb_G^{\mathrm{op}}$ can also be obtained from $\underline{\mathrm{Cat}}_{\infty,G}$ by restricting each fiber to the H-spaces only, which are collected in a full subcategory of $\underline{\mathrm{Cat}}_{\infty,H}$.

Some more examples of G-∞-categories will be discussed later on, together with added monoidal structures.

4.2 G-symmetric monoidal G-∞-categories

Our next goal is to give the definition of G-symmetric monoidal G-∞-categories, that is, the equivariant generalization of symmetric monoidal ∞-categories. We will define them as certain kinds of cocartesian fibrations satisfying a Segal condition, in analogy to the non-parametrized setting in Definition 1.2.7. To do this, we first need to discuss in detail the construction of the category $\underline{\mathcal{F}}_{G,*}$, a variant of the category of pointed G-sets, which turns out to be the right choice as target of the above fibration, playing the same role that $\mathcal{F}_*$ was playing earlier. We mostly follow sections of the paper [NS22] by Nardin and Shah.

Start with G a finite group, and $\mathcal{O}rb_G$ its orbit category.

Definition 4.2.1. The (1-)category $\mathcal{F}_G$ of finite G-sets is defined as the finite coproduct completion of $\mathcal{O}rb_G$.

A way to realize $\mathcal{F}_G$ is by taking the full subcategory of the presheaf category $\mathrm{Fun}(\mathcal{O}rb_G^{\mathrm{op}}, \mathrm{Set})$ over finite coproducts of representable functors. We can think of its objects as finite coproducts of orbits G/H; we will often write these as:

$$\coprod_{i \in I} G/H_i$$

where I is a finite index set and each $H_i \leq G$ is a subgroup. Morphisms in $\mathcal{F}_G$ are equivariant maps of G-sets.

Of particular importance is the definition of the Span category of finite G-sets. We recall it here as a reminder (and first example) for the construction of Span categories, which can be done with more generality, and will shortly come into play.

Definition 4.2.2. The *Span category* $\mathrm{Span}(\mathcal{F}_G)$ is the $(2,1)$-category with:

- Objects: the same as $\mathcal{F}_G$.
- (1)-morphisms: spans of finite G-sets. That is, given finite G-sets S, T, a morphism from S to T in $\mathrm{Span}(\mathcal{F}_G)$ is a diagram:

$$S \xleftarrow{\;f\;} U \xrightarrow{\;g\;} T$$

  where f and g are morphisms in $\mathcal{F}_G$. This is called a *span diagram*. Composition is done by taking the pullback.
- 2-morphisms: isomorphisms between spans.

This is also called the *effective Burnside* $(2,1)$*-category of* G.

Remark 4.2.3. We can view $\mathrm{Span}(\mathcal{F}_G)$ as an ∞-category via the Duskin nerve construction. Moreover, in analogy with [NS22, Remark 2.5.2], it follows that $\mathrm{Span}(\mathcal{F}_G)$ is equivalent to the nerve of a 1-category, where morphisms are representatives of equivalence classes of spans, up to isomorphism of spans. In the following, we will sometimes abuse notation and still write $\mathrm{Span}(\mathcal{F}_G)$ to mean the latter, so the nerve of a 1-category.

Remark 4.2.4. Note that the product in $\mathrm{Span}(\mathcal{F}_G)$ is, somewhat unintuitively, given by the coproduct of G-sets. For example, the product in $\mathrm{Span}(\mathcal{F}_G)$ between two objects T and T' is the object $T \coprod T'$. This can be proven by showing that $T \coprod T'$ satisfies the universal property of the product.

Remark 4.2.5. Although the pointed version of $\mathcal{F}_G$ might seem like the right candidate to serve as the base category in this framework, it is not the right choice. This is because it is not a G-∞-category, which is something we want here.

We now therefore go back to the construction of $\underline{\mathcal{F}}_{G,*}$. This will be a G-∞-category, extending the notion of finite pointed G-sets, and therefore will be the right choice to work on.

Notation 4.2.6. First, we introduce the following useful notation.

(1) For any orbit G/H, let:

$$\mathcal{F}_G^{/H} := \mathrm{Ar}(\mathcal{F}_G) \times_{\mathrm{ev}_1, \mathcal{F}_G} \{G/H\}$$

denote the slice category over G/H. Its objects are morphisms $S \to G/H$ in $\mathcal{F}_G$, and its morphisms are commutative triangles:

$$\begin{array}{ccc} S & \longrightarrow & T \\ & \searrow \quad \swarrow & \\ & G/H & \end{array}$$

(2) Similarly, for any orbit G/H, let:

$$\mathcal{F}_{G,*}^{/H} := \left(\mathcal{F}_G^{/H}\right)^{\mathrm{id}_{G/H}/}$$

denote the slice category whose objects are commutative triangles in $\mathcal{F}_G$:

$$\begin{array}{ccc} G/H & \longrightarrow & S \\ & \parallel \quad \swarrow & \\ & G/H & \end{array}$$

and whose morphisms are the obvious ones.

Definition 4.2.7. Let:

$$\underline{\mathcal{F}} := \mathrm{Ar}(\mathcal{F}_G) \times_{\mathrm{ev}_1, \mathcal{F}_G} \mathcal{O}rb_G$$

be the category whose objects are morphisms $S \to G/H$ in $\mathcal{F}_G$ with target any orbit G/H, and whose morphisms are commutative squares:

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \downarrow & & \downarrow \\ G/H & \xrightarrow{g} & G/K \end{array}$$

We wish to define wide subcategories of $\underline{\mathcal{F}}$, by considering only certain kinds of morphisms.

Definition 4.2.8. Denote a morphism ψ in $\underline{\mathcal{F}}$ as in Definition 4.2.7. We define the following wide subcategories of $\underline{\mathcal{F}}$:

(1) $\underline{\mathcal{F}}^{\mathrm{tdeg}} \subset \underline{\mathcal{F}}$ contains only those morphisms ψ for which g is the identity of G/H.

(2) $\underline{F}^{\mathrm{cart}} \subset \underline{F}$ contains only those morphisms ψ for the induced map to the pullback:

$$S \longrightarrow T \times_{G/K} G/H$$

is a summand inclusion of G-sets.

We are now ready to define the appropriate parametrized version of finite pointed G-sets.

Definition 4.2.9. We define the $(2,1)$-category of *pointed G-sets* to be the Span category:

$$\underline{\mathcal{F}}_{G,*} := \mathrm{Span}(\underline{F}, \underline{F}^{\mathrm{cart}}, \underline{F}^{\mathrm{tdeg}})$$

where the notation $\mathrm{Span}(\underline{F}, \underline{F}^{\mathrm{cart}}, \underline{F}^{\mathrm{tdeg}})$ means that it has:

- Objects: the same as $\underline{F}$.
- 1-morphisms: spans of morphisms where the left one is in $\underline{F}^{\mathrm{cart}}$ and the right one is in $\underline{F}^{\mathrm{tdeg}}$. That is, a morphism in $\underline{\mathcal{F}}_{G,*}$ is a commutative diagram:

$$\begin{array}{ccccc} T & \longleftarrow & S' & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ G/K & \longleftarrow & G/H & = & G/H \end{array}$$

  where the left square induces a summand inclusion of S' in the pullback.
- 2-morphisms: morphisms of spans as above.

Remark 4.2.10. As claimed in [NS22, Remark 2.5.2], it follows that $\underline{\mathcal{F}}_{G,*}$ is equivalent to the nerve of a 1-category, by picking representatives for morphisms in each isomorphism class of spans.

We can again view $\underline{\mathcal{F}}_{G,*}$ as an ∞-category via the Duskin nerve. Moreover, $\underline{\mathcal{F}}_{G,*}$ be turned into a G-∞-category via the evaluation functor on the target:

$$\mathrm{ev}_1 \colon \underline{\mathcal{F}}_{G,*} \longrightarrow \mathcal{O}rb_G^{\mathrm{op}}$$

which maps an object $S \to G/H$ of $\underline{\mathcal{F}}_{G,*}$ to G/H, and a morphism ψ of $\underline{\mathcal{F}}_{G,*}$:

$$\begin{array}{ccccc} T & \longleftarrow & S' & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ G/K & \underset{g}{\longleftarrow} & G/H & = & G/H \end{array}$$

to $g^{\mathrm{op}} \colon G/K \to G/H$.

Remark 4.2.11. By [Nar16, Lemma 4.9], the ev_1-cocartesian edges are exactly those morphisms ψ, written as above, for which the maps $S' \to S$ and $S' \to T \times_{G/K} G/H$ are equivalences.

Lemma 4.2.12. *The functor* $\mathrm{ev}_1 \colon \underline{\mathcal{F}}_{G,*} \to \mathcal{O}rb_G^{\mathrm{op}}$ *is a cocartesian fibration, thus turning* $\underline{\mathcal{F}}_{G,*}$ *into a G-∞-category.*

Proof. Recall that $\underline{\mathcal{F}}_{G,*}$ is equivalent, by Remark 4.2.10, to the nerve of a 1-category. We can therefore check that an edge in $\underline{\mathcal{F}}_{G,*}$ is cocartesian by checking that the appropriate lift exists just for the inner horn inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$.

Consider a morphism $g^{\mathrm{op}}\colon G/K \to G/H$ in $\mathcal{O}rb_G^{\mathrm{op}}$, coming from a morphism $g\colon G/H \to G/K$, and let $T \to G/K$ be an object of $\underline{\mathcal{F}}_{G,*}$, which lies over G/K. We wish to find a cocartesian lift of g^{op} with source $T \to G/K$. The morphism:

$$\begin{array}{ccccc} T & \longleftarrow & T \times_{G/K} G/H & = & T \times_{G/K} G/H \\ \downarrow & & \downarrow & & \downarrow \\ G/K & \underset{g}{\longleftarrow} & G/H & = & G/H \end{array}$$

lies above g^{op}, and it is clearly cocartesian by Remark 4.2.11. □

Definition 4.2.13. We declare a morphism ψ of $\underline{\mathcal{F}}_{G,*}$ to be:

(1) *inert* if $S' \to S$ is an equivalence;

(2) *active* if $S' \to T \times_{G/K} G/H$ is an equivalence.

Example 4.2.14. For example, fix a morphism $g\colon G/H \to G/K$ in $\mathcal{O}rb_G$. The following two morphisms in $\underline{\mathcal{F}}_{G,*}$ are respectively inert and active:

$$\begin{array}{ccccc} G/K & \overset{g}{\longleftarrow} & G/H & = & G/H \\ \| & & \| & & \| \\ G/K & \overset{g}{\longleftarrow} & G/H & = & G/H \end{array} \qquad \begin{array}{ccccc} G/H & = & G/H & \overset{g}{\longrightarrow} & G/K \\ \downarrow g & & \downarrow g & & \| \\ G/K & = & G/K & = & G/K \end{array}$$

Remark 4.2.15. Note that a morphism ψ is both inert and active if and only if it is ev_1 cocartesian, by Remark 4.2.11.

Remark 4.2.16. The category $\underline{\mathcal{F}}_{G,*}$ is what we were looking for to replace $\mathcal{F}_*$ in this equivariant setting. Indeed, we can see $\underline{\mathcal{F}}_{G,*}$ as a generalization of $\mathcal{F}_*$ thanks to the existence of an equivalence of categories:

$$\mathcal{F}_* \simeq \mathrm{Span}(\mathcal{F}, \mathcal{F}^{\mathrm{inj}}, \mathcal{F})$$

where $\mathcal{F}^{\mathrm{inj}} \subset \mathcal{F}$ is the wide subcategory of $\mathcal{F}$ on injective maps. Here, the notation employed means that the left morphisms in spans must be in $\mathcal{F}^{\mathrm{inj}}$, whereas there is no restriction for the ones to the right. The idea is that the left map of the span encodes which portion of $\langle m \rangle$ is sent to the basepoint of $\langle n \rangle$, so the inert part of f, whereas the right map encodes the active part of f.

Let us describe how this equivalence works. The key point is the existence of a functor:

$$i\colon \mathcal{F}_* \longrightarrow \mathrm{Span}(\mathcal{F})$$

which lands into the slightly bigger category $\mathrm{Span}(\mathcal{F})$ and maps a finite set T to $T \setminus \{*\}$, removing the basepoint, and a morphism $S \xrightarrow{f} T$ to the span:

$$\begin{array}{ccccc} & & S \setminus f^{-1}(\{*\}) & & \\ & \swarrow & & \searrow^{f|_{S \setminus f^{-1}(\{*\})}} & \\ S \setminus \{*\} & & & & T \setminus \{*\} \end{array}$$

where the left map is the inclusion and the right map is the restriction of f. It turns out that i provides an equivalence:

$$\mathcal{F}_* \simeq \mathrm{Span}(\mathcal{F}, \mathcal{F}^{\mathrm{inj}}, \mathcal{F})$$

between $\mathcal{F}_*$ and the category $\mathrm{Span}(\mathcal{F}, \mathcal{F}^{\mathrm{inj}}, \mathcal{F})$, which is the wide subcategory of $\mathrm{Span}(\mathcal{F})$ where the left morphisms in the spans must be injective.

The inverse of this equivalence maps a finite set T to T_+, adjoining a basepoint, and it maps a span:

$$\begin{array}{ccccc} & & S & & \\ & \swarrow & & \searrow & \\ T & & & & U \end{array}$$

to the map $T_+ \to U_+$ defined by mapping all elements of $T \setminus S \subseteq T_+$ to the basepoint of U_+, and all other elements of T_+, which are then in $S \subseteq T$, by using the map $S \to U$ in the span.

The idea is that a span as above can encode the inert and active parts of any morphism f in $\mathcal{F}_*$: the left morphism $S \hookrightarrow T$ specifies which elements are sent to the basepoint, and the right morphism instead describes the action of f on all other elements, so its active part. One can then define notions of inert and active morphisms also in $\mathrm{Span}(\mathcal{F}, \mathcal{F}^{\mathrm{inj}}, \mathcal{F})$, simply by declaring a span as above to be active if $S \hookrightarrow T$ is an equality, and inert if $S \to U$ is an equality. Then, the functor i providing the above equivalence also preserves the notions of inert and active morphisms.

This should be enough motivation then for why $\underline{\mathcal{F}}_{G,*}$ is the right generalization of $\mathcal{F}_*$; for example, the requirement for the left part of a span to give a summand inclusion is analogous to requiring only morphisms in $\mathcal{F}^{\mathrm{inj}}$. The definitions of inert and active morphisms are also in analogy with the non-equivariant case.

Remark 4.2.17. There is a functor of G-∞-categories:

$$I\colon \mathcal{O}rb_G^{\mathrm{op}} \longrightarrow \underline{\mathcal{F}}_{G,*}$$

which maps an orbit G/H to $G/H \xrightarrow{\mathrm{id}} G/H$, and a morphism $g\colon G/H \to G/K$ to the inert morphism:

$$\begin{array}{ccccc} G/K & \xleftarrow{g} & G/H & = & G/H \\ \| & & \| & & \| \\ G/K & \xleftarrow{g} & G/H & = & G/H \end{array}$$

of Example 4.2.14. We refer to this functor as the *identity section*.

We also need a notion of characteristic morphisms analogous to the morphisms $\rho_i\colon \langle n \rangle \to \langle 1 \rangle$ of Definition 1.1.9. This is done by first decomposing a G-set T into orbits, and then, for every such orbit G/H_i, by mapping into it via an inert morphism. Informally, the characteristic morphism should send G/H_i to itself, and all other orbits in T should "disappear".

Definition 4.2.18. Let $T \xrightarrow{f} G/K$ be an object in $\underline{\mathcal{F}}_{G,*}$. Decompose T into orbits:

$$T = \coprod_{i \in I} G/H_i.$$

For any orbit G/H_i, the *characteristic morphism*:

$$\rho_i \colon f \longrightarrow I(G/H_i)$$

is defined as the span:

$$\begin{array}{ccccc} T & \hookleftarrow & G/H_i & = & G/H_i \\ \downarrow{\scriptstyle f} & & \| & & \| \\ G/K & \longleftarrow & G/H_i & = & G/H_i \end{array}$$

in $\underline{\mathcal{F}}_{G,*}$. Note that this is inert, as were the $\rho_i \colon \langle n\rangle \to \langle 1\rangle$.

We are now ready to define G-symmetric monoidal G-∞-categories. Similarly to the non-parametrized Definition 1.2.7, we require there to exist a cocartesian fibration over $\underline{\mathcal{F}}_{G,*}$, together with a parametrized Segal condition induced by the characteristic morphisms ρ_i.

Definition 4.2.19. A *G-symmetric monoidal G-∞-category* consists of a G-∞-category $\mathcal{C}^{\otimes} \to \mathcal{O}rb_G^{\mathrm{op}}$ together with a G-functor $p \colon \mathcal{C}^{\otimes} \to \underline{\mathcal{F}}_{G,*}$ over $\mathcal{O}rb_G^{\mathrm{op}}$ with the following requirements:

(1) p is a cocartesian fibration.

(2) Let $T \xrightarrow{f} G/K$ be an object in $\underline{\mathcal{F}}_{G,*}$, with $T = \coprod_{i\in I} G/H_i$. Since p is a cocartesian fibration, the characteristic morphisms ρ_i induce functors:

$$\mathcal{C}^{\otimes}_f \to \mathcal{C}^{\otimes}_{G/H_i}$$

between fibers over f and over $I(G/H_i)$. Then, we require that these functors together induce an equivalence:

$$\mathcal{C}^{\otimes}_f \xrightarrow{\simeq} \prod_{i\in I} \mathcal{C}^{\otimes}_{G/H_i}.$$

We will often omit the map to $\underline{\mathcal{F}}_{G,*}$ from the notation if it is clear from the context.

Definition 4.2.20. The *underlying G-∞-category* of $\mathcal{C}^{\otimes} \to \underline{\mathcal{F}}_{G,*}$ is the pullback:

$$\mathcal{C} := \mathcal{O}rb_G^{\mathrm{op}} \times_{\underline{\mathcal{F}}_{G,*}} \mathcal{C}^{\otimes}$$

along the identity section $I \colon \mathcal{O}rb_G^{\mathrm{op}} \to \underline{\mathcal{F}}_{G,*}$.

Notation 4.2.21. From now on, to avoid clumping up the notation we will not add an underline anymore to G-symmetric monoidal G-∞-categories, or to their underlying G-∞-categories.

Definition 4.2.22. A morphism of G-symmetric monoidal G-∞-categories from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ is a G-functor $F \colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ compatible with the maps to $\underline{\mathcal{F}}_{G,*}$.

Remark 4.2.23. G-symmetric monoidal G-∞-categories and morphisms between them can be collected into an ∞-category, which will be denoted by $\mathrm{Cat}_G^{\otimes}$.

We now highlight an important result from [NS22], which gives an equivalent way of talking about G-symmetric monoidal G-∞-categories. This is useful in practice when discussing examples.

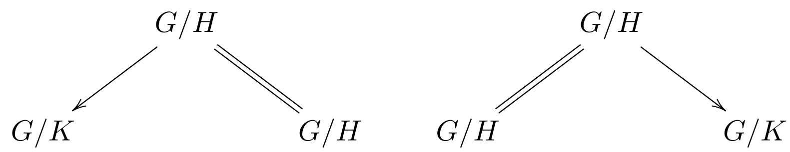
Theorem 4.2.24 ([NS22, Theorem 2.3.9])**.** *There is an equivalence of ∞-categories:*

$$\mathrm{Cat}_G^{\otimes} \simeq \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{F}_G), \mathrm{Cat}_\infty)$$

between $\mathrm{Cat}_G^{\otimes}$ *and the ∞-category of product-preserving functors from* $\mathrm{Span}(\mathcal{F}_G)$ *to* Cat_∞.

Remark 4.2.25. When we say that a functor is product-preserving, we really mean that it commutes with the coproduct of G-sets in the base G-sets of objects of $\mathrm{Span}(\mathcal{F}_G)$; see Remark 4.2.4.

Note that this recovers the ideas of the more classical definition of G-symmetric monoidal categories in the ordinary context; see for example [HH16] by Hill and Hopkins. Product-preserving functors from $\mathrm{Span}(\mathcal{F}_G)$ to Cat_∞ are the right generalization to the ∞-categorical setting of ordinary Mackey functors. In particular, at the level of objects it is enough to know what such a functor does on orbits G/H to recover its behaviour on all objects, since it is product-preserving. Moreover, if we consider $g\colon G/H \to G/K$ any map in $\mathcal{O}rb_G$, then in $\mathrm{Span}(\mathcal{F}_G)$ we have the spans:



and if we let $\mathcal{M} \in \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{F}_G), \mathrm{Cat}_\infty)$, then:

- the first span gives contravariantly a functor $\mathcal{M}(G/K) \to \mathcal{M}(G/H)$ of ∞-categories, which we can view as a *restriction* functor, in particular for the case $H \leq K \leq G$;
- the second span gives covariantly a functor $\mathcal{M}(G/H) \to \mathcal{M}(G/K)$ of ∞-categories, which we can view as a *transfer*, or *norm*, functor.

Finally, formulas analogous to those for ordinary Mackey functors hold. All this motivates the following definition:

Definition 4.2.26. Let $\mathcal{C}$ be a semiadditive ∞-category. The ∞-category of *G-Mackey functors* with values in $\mathcal{C}$ is defined as:

$$\mathrm{Mack}(\mathcal{C}) := \mathrm{Fun}^{\times}(\mathrm{Span}(\mathcal{F}_G), \mathcal{C}).$$

In particular, Theorem 4.2.24 gives an equivalence:

$$\mathrm{Cat}_G^{\otimes} \simeq \mathrm{Mack}(\mathrm{Cat}_\infty).$$

4.3 Examples of G-symmetric monoidal G-∞-categories

In this section, we introduce and discuss a good number of examples of G-symmetric monoidal G-∞-categories.

First off, we expect that the notion of G-symmetric monoidal G-∞-categories subsumes that of symmetric monoidal ∞-categories, given in Definition 1.2.7. This is indeed true, and requires only a small amount of work. The idea is that we want to forget about the group structure arising from the structure of $\mathcal{O}rb_G$, and instead treat every orbit as a point, so that every map between them becomes trivial.

Construction 4.3.1. Recall from Remark 4.2.16 that we have an equivalence of categories:

$$\mathcal{F}_* \simeq \mathrm{Span}(\mathcal{F}, \mathcal{F}^{\mathrm{inj}}, \mathcal{F}).$$

This means that, through the straightening-unstraightening equivalence, we can view a symmetric monoidal ∞-category as a functor $\mathrm{Span}(\mathcal{F}, \mathcal{F}^{\mathrm{inj}}, \mathcal{F}) \to \mathrm{Cat}_\infty$. This is moreover product-preserving, as a consequence of the Segal condition.

In order to make this into a G-symmetric monoidal G-∞-category, with trivial G-action, we wish to define a functor:

$$\Phi \colon \underline{\mathcal{F}}_{G,*} \longrightarrow \mathrm{Span}(\mathcal{F}, \mathcal{F}^{\mathrm{inj}}, \mathcal{F})$$

which "forgets" the differences between orbits. We define it as follows:

- Given an object $T \to G/H$, write $T = \coprod_{i \in I} G/H_i$, where without loss of generality we can assume $I = \{1, \dots, n\}$. Then, Φ maps $T \to G/H$ to I itself.
- Consider a morphism ψ of $\underline{\mathcal{F}}_{G,*}$:

$$\begin{array}{ccccc} T & \xleftarrow{\;g\;} & S' & \xrightarrow{\;f\;} & S \\ \downarrow & & \downarrow & & \downarrow \\ G/K & \longleftarrow & G/H & = & G/H \end{array}$$

  and decompose T, S and S' into orbits:

$$T = \coprod_{i \in I} G/H_i, \quad S' = \coprod_{j \in J} G/H_j, \quad S = \coprod_{k \in K} G/H_k$$

  where we can assume (up to relabeling and reordering) that $I = \{1, \dots, m\}$, $J = \{1, \dots, \overline{m}\} \subseteq I$ and $K = \{1, \dots, n\}$. Then, Φ sends ψ to a span in $\mathrm{Span}(\mathcal{F}, \mathcal{F}^{\mathrm{inj}}, \mathcal{F})$ from I to K:

$$I \xhookleftarrow{\;\overline{g}\;} J \xrightarrow{\;\overline{f}\;} K$$

  where $\overline{g}$ maps a $j \in J$ to the unique element of I such that g maps G/H_j to $G/H_{\overline{g}(j)}$, and similarly $\overline{f}$ maps a $j \in J$ to the unique element of K such that f maps G/H_j to $G/H_{\overline{f}(j)}$. Note that this is well-defined as f is equivariant, and g induces a summand inclusion on the pullback. Moreover, $\overline{g}$ is injective for the same reason.
- On 2-morphisms, Φ is defined in the obvious way.

Note that the above data is enough to define Φ, as the Duskin nerve construction is fully faithful (see [Lur18, Tag 00AU]).

We therefore obtain a functor Φ as desired, whose role is to "forget" the equivariant structure. Moreover, it is clearly product-preserving. We can now use Φ to view symmetric monoidal ∞-categories as trivially-parametrized G-symmetric monoidal G-∞-categories.

Example 4.3.2. Let $\mathcal{C}^{\otimes} \to \mathcal{F}_*$ be a symmetric monoidal ∞-category, which again we can view as a product-preserving functor

$$\mathcal{C}^{\otimes}\colon \operatorname{Span}(\mathcal{F}, \mathcal{F}^{\text{inj}}, \mathcal{F}) \longrightarrow \operatorname{Cat}_\infty .$$

Consider the composite functor:

$$p\colon \underline{\mathcal{F}}_{G,*} \xrightarrow{\Phi} \operatorname{Span}(\mathcal{F}, \mathcal{F}^{\text{inj}}, \mathcal{F}) \xrightarrow{\mathcal{C}^{\otimes}} \operatorname{Cat}_\infty .$$

which is product-preserving. Then, by straightening-unstraightening, this is a G-symmetric monoidal G-∞-category, as being product-preserving gives the Segal condition. The fiber of p over some T does not depend on the individual orbits G/H composing T, but rather in the number of them, and the restriction and norm maps similarly depend only on the tensor product functor of $\mathcal{C}^{\otimes}$.

Example 4.3.3. Recall that in Example 1.2.10, we have seen that we have a symmetric monoidal ∞-category which models the sum as the tensor product on $\mathbb{Z}$. Then, the construction that we have just seen makes this in particular into a G-symmetric monoidal G-∞-category, with trivial equivariant action. We get a product-preserving functor:

$$\mathbb{Z}^{\otimes}\colon \operatorname{Span}(\mathcal{F}_G) \longrightarrow \operatorname{Cat}_\infty$$

such that the value on an orbit G/H is always $\mathbb{Z}$, and so the value on a G-set T is $\mathbb{Z}^n$, where T is decomposed as coproduct of n orbits. This is because the functor is product-preserving, which is is analogous to the Segal condition. Moreover, all restriction and norm maps induced by a $g\colon G/H \to G/K$ are the identity of $\mathbb{Z}$, and all others are computed using the tensor product on $\mathbb{Z}$, i.e. the sum. For example, the norm map induced by any codiagonal $G/H \coprod G/H \to G/H$ is the sum $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

We now discuss the important example of G-spectra, or more specifically, of genuine G-spectra.

Definition 4.3.4. The ∞-category of *genuine G-spectra* is defined as the ∞-category of G-Mackey functors with values in Sp:

$$\operatorname{Sp}^G := \operatorname{Mack}(\operatorname{Sp}).$$

One can then endow Sp^G with a "multiplicative" symmetric monoidal structure with Day convolution, after giving $\operatorname{Span}(\mathcal{F}_G)$ an appropriate symmetric monoidal structure, and taking on Sp the symmetric monoidal structure with the smash product. This results in a symmetric monoidal structure on Sp^G which is the usual smash product of genuine G-spectra.

However, this is not quite enough. We wish to first make genuine G-spectra into a G-∞-category, and then endow it with a G-symmetric monoidal structure, therefore taking the equivariant part into consideration. This can be done in a rather abstract way, for the details of which we refer to [NS22, Example 2.4.2]; we just report the main result and the features of the G-symmetric monoidal structure on Sp^G. First, we need to extend Sp^G into a G-∞-category.

Definition 4.3.5. The *G-∞-category of genuine G-spectra* $\underline{\operatorname{Sp}^G}$ is defined as the cocartesian fibration over $\mathcal{O}rb_G^{\text{op}}$ classified by the functor:

$$\mathcal{O}rb_G^{\text{op}} \longrightarrow \operatorname{Cat}_\infty$$

which maps an orbit G/H to the ∞-category Sp^H, and a morphism $g\colon G/H \longrightarrow G/K$ to the usual restriction $\mathrm{Sp}^K \to \mathrm{Sp}^H$.

We now have the following result, endowing $\underline{\mathrm{Sp}^G}$ with a multiplicative G-symmetric monoidal structure.

Proposition 4.3.6. *There is a G-symmetric monoidal G-∞-category:*

$$(\underline{\mathrm{Sp}^G})^{\otimes} \longrightarrow \underline{\mathcal{F}}_{G,*}$$

such that:

- *its underlying G-∞-category is $\underline{\mathrm{Sp}^G}$;*
- *fiberwise the tensor product is given by the smash product of G-spectra;*
- *the norm maps are the Hill-Hopkins-Ravenel norm functors.*

To be more explicit, this G-symmetric monoidal G-∞-category is obtained via 4.2.24 after providing a suitable functor:

$$\underline{\mathrm{Sp}^G}\colon \mathrm{Span}(\mathcal{F}_G) \longrightarrow \mathrm{Cat}_\infty\,.$$

This functor is such that:

- Its value on an orbit G/H is Sp^H. Therefore, its value on a G-set $T = \coprod_{i\in I} G/H_i$ is $\prod i \in I\,\mathrm{Sp}^{H_i}$, as the functor preserves products. Again, this reflects the equivalence one gets from the Segal condition on fibers over objects.
- Given a $g\colon G/H \to G/K$, the restriction functors $\mathrm{Sp}^K \to \mathrm{Sp}^H$ are obtained precomposing with the map $\mathrm{Span}(\mathcal{F}_H) \to \mathrm{Span}(\mathcal{F}_K)$ induced by g. These are the usual restriction functors $\mathrm{Sp}^K \to \mathrm{Sp}^H$.
- Given a $g\colon G/H \to G/K$, the norm functors $\mathrm{Sp}^H \to \mathrm{Sp}^K$ are precisely the Hill-Hopkins-Ravenel norm functors, described in [HHR16, Section 2.3.3] (see also [BH21, Remark 9.10]).
- In the fiber over any orbit G/H, given the codiagonal $G/H \coprod G/H \to G/H$, the corresponding norm functor $\mathrm{Sp}^H \times \mathrm{Sp}^H \to \mathrm{Sp}^H$ is the smash product of H-spectra.

This G-symmetric monoidal structure is "multiplicative", as it extends the smash product of G-spectra, which is so. The "additive" counterpart is realized by taking the direct sum instead as tensor product, analogously to Remark 1.3.7 in the non-parametrized case. This is the result of a similar construction of cartesian and cocartesian G-symmetric monoidal structures on G-∞-categories admitting all limits and colimits. These are the same also here for G-spectra since coproducts and products of them coincide, and are called direct sums. We have the following:

Proposition 4.3.7. *There is a G-symmetric monoidal G-∞-category:*

$$(\underline{\mathrm{Sp}^G})^{\oplus} \longrightarrow \underline{\mathcal{F}}_{G,*}$$

such that:

- *its underlying G-∞-category is $\underline{\mathrm{Sp}^G}$;*

- *fiberwise the tensor product is given by the direct sum of G-spectra;*
- *the norm maps associated to a $g\colon G/H \to G/K$ are the usual induction maps, which coincide with the coinduction ones. We write them as:*

$$\mathrm{Ind}_H^K\colon\ \mathrm{Sp}^H \longrightarrow \mathrm{Sp}^K\,.$$

Let us now discuss, as a further example, filtered G-spectra. Of course, one can just define the ∞-category of *filtered G-spectra* as the functor category $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp}^G)$. However, also in this case we want to consider a parametrized version. This is not much more complicated.

Definition 4.3.8. The *G-∞-category of filtered G-spectra* $\mathrm{Fil}(\underline{\mathrm{Sp}}^G)$ is defined as the cocartesian fibration over $\mathcal{O}rb_G^{\mathrm{op}}$ classified by the functor:

$$\mathcal{O}rb_G^{\mathrm{op}} \longrightarrow \mathrm{Cat}_\infty$$

which maps an orbit G/H to the ∞-category $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp}^H)$ of filtered H-spectra, and a morphism $g^{\mathrm{op}}\colon G/K \longrightarrow G/K$ to postcomposition with the restriction $\mathrm{Sp}^K \to \mathrm{Sp}^H$.

Example 4.3.9. We can endow $\mathrm{Fil}(\underline{\mathrm{Sp}}^G)$ with an additive G-symmetric monoidal G-∞-category structure. The norm maps $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp}^H) \to \mathrm{Fun}(\mathbb{Z}, \mathrm{Sp}^K)$ coming from a $G/H \to G/K$ are just postcomposition with Ind_H^K, which are the norm maps in the (co)cartesian G-symmetric monoidal structure on G-spectra of Proposition 4.3.7.

Chapter 5

G-∞-operads and Day convolution

In this final chapter we give an overview of the construction of a Day convolution tensor product in the equivariant setting, for $\mathcal{O}$-monoidal G-∞-categories, where $\mathcal{O}$ is a G-∞-operad. Nardin and Shah in [NS22] develop parametrized higher algebra over a more general ∞-category $\mathcal{T}$, satisfying a handful of extra hypotheses. They then obtain a Day convolution product more generally for $\mathcal{O}$-promonoidal G-∞-categories, of which $\mathcal{O}$-monoidal G-∞-categories are a special case. We are not interested in this level of generality, and so we will follow their work by setting $\mathcal{T} = \mathcal{O}rb_G$, as well as restricting just to $\mathcal{O}$-monoidal G-∞-categories. This will simplify most of the definitions and constructions, without losing the scope of their work.

We also note that Nardin and Shah's paper is firstly a generalization of Lurie's treatment of Day convolution in [Lur17], which was discussed in Chapter 3. As the ideas are analogous, we try to follow the structure of Chapter 3, sometimes diverging from the order in which Nardin and Shah present their results. This is also motivated from the fact that they sometimes state and prove results using advanced tools which are outside the scope of this thesis.

5.1 G-∞-operads

In this section we discuss the definition of G-∞-operads, as well as various relevant notions of maps between them, and the algebra categories in which these are collected.

We start right away with the definition of G-∞-operads. Although seemingly complex, this is just a generalization to the equivariant context of Definition 3.1.5.

Definition 5.1.1. A G-∞-operad consists of a G-∞-category $\mathcal{O}^{\otimes}$ together with a G-functor $p\colon \mathcal{O}^{\otimes} \to \underline{\mathcal{F}}_{G,*}$ which satisfies the following:

(1) For every inert morphism $\psi\colon f \to g$ of $\underline{\mathcal{F}}_{G,*}$ and every object $x \in \mathcal{O}^{\otimes}_f$, there exists a p-cocartesian lift $x \to x'$ of ψ, with source x. In particular, we obtain a functor $\psi_!\colon \mathcal{O}^{\otimes}_f \to \mathcal{O}^{\otimes}_g$.

(2) For any object $f\colon T \to G/H$ of $\underline{\mathcal{F}}_{G,*}$, recall that that we have characteristic morphisms ρ_i for every orbit G/H_i in the decomposition of T. Then, these

induce an equivalence:

$$\mathcal{O}^{\otimes}_{f} \xrightarrow{\prod(\rho_i)_!} \prod_{i \in I} \mathcal{O}^{\otimes}_{G/H_i}.$$

(3) Let $\psi\colon f \to g$ be a morphism in $\underline{\mathcal{F}}_{G,*}$ between two objects $f\colon T \to G/H$ and $g\colon T' \to G/H'$, and let $x \in \mathcal{O}^{\otimes}_{f}$ and $x' \in \mathcal{O}^{\otimes}_{g}$. For every orbit G/H_i in the decomposition of T', denote by C'_i the cocartesian pushforward under $(\rho_i)_!$ of x'. Then, the induced map:

$$\mathrm{Map}^{\psi}_{\mathcal{O}^{\otimes}}(x, x') \xrightarrow{\simeq} \prod_{i \in I} \mathrm{Map}^{\rho_i \circ \psi}_{\mathcal{O}^{\otimes}}(x, x'_i)$$

is an equivalence.

The *underlying G-∞-category* of $\mathcal{O}^{\otimes}$ is given by the pullback $\mathcal{O} := \mathcal{O}rb_G^{\mathrm{op}} \times_{\underline{\mathcal{F}}_{G,*}} \mathcal{O}^{\otimes}$ along the identity section $\mathcal{O}rb_G^{\mathrm{op}} \hookrightarrow \underline{\mathcal{F}}_{G,*}$ of Remark 4.2.17.

As usual, we may omit to mention the G-functor p and simply refer to $\mathcal{O}^{\otimes}$ being a G-∞-operad.

Remark 5.1.2. Similarly to the situation in Chapter 3, this definition does not clearly show why it should generalize the more familiar notion of a colored operad, with colors and spaces of multimorphisms; the motivation there was given right before Definition 3.1.5. A similar discussion can be made in this context: from the data of a G-∞-operad one can recover the data of spaces of multimorphisms, and appropriate composition laws, satisfying the right axioms. This discussion is done in detail in [NS22, Remark 2.1.11].

Example 5.1.3. The identity of $\underline{\mathcal{F}}_{G,*}$ makes it into a G-∞-operad, which we will still denote by $\underline{\mathcal{F}}_{G,*}$.

Definition 5.1.4. Let $p\colon \mathcal{O}^{\otimes} \to \underline{\mathcal{F}}_{G,*}$ be a G-∞-operad. Then a morphism of $\mathcal{O}^{\otimes}$ is called:

(1) *inert* if it is p-cocartesian over an inert edge of $\underline{\mathcal{F}}_{G,*}$;

(2) *active* if it factors as a p-cocartesian edge followed by an edge lying over a fiberwise active edge of $\underline{\mathcal{F}}_{G,*}$.

We now discuss the notions of maps, fibrations and cocartesian fibrations of G-∞-operads. Again the definitions resemble those in the non-parametrized context.

Definition 5.1.5. Let $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$ be two G-∞-operads. A G-functor $q\colon \mathcal{O}^{\otimes} \to \mathcal{O}'^{\otimes}$ is called a *map of G-∞-operads* if:

(1) The following diagram of G-functors commutes:

$$\begin{array}{ccc} \mathcal{O}^{\otimes} & \xrightarrow{\quad q \quad} & \mathcal{O}'^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathcal{F}_{*} & \end{array}$$

(2) q maps inert edges of $\mathcal{O}^{\otimes}$ to inert edges of $\mathcal{O}'^{\otimes}$.

Definition 5.1.6. A map $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ of G-∞-operads is called a *fibration of G-∞-operads* if it is a categorical fibration.

We can then collect these into appropriate ∞-categories and G-∞-categories of algebra objects. These all come in pairs, an ∞-category and the corresponding G-∞-category. First, let us fix some notation.

Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be two fibrations of G-∞-operads. Denote by $\mathrm{Fun}_{/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ the ∞-category of functors from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ over $\mathcal{O}^{\otimes}$, and we denote by $\underline{\mathrm{Fun}}_{/\mathcal{O}^{\otimes},G}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ the G-∞-category of G-functors from $\mathcal{C}^{\otimes}$ to $\mathcal{D}^{\otimes}$ over $\mathcal{O}$.

Definition 5.1.7. With notation as above:

(1) We denote by $\mathrm{Alg}_{\mathcal{O},G}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathrm{Fun}_{/\mathcal{O}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the morphisms of G-∞-operads. This is an ∞-category.

(2) Similarly, $\underline{\mathrm{Alg}}_{\mathcal{O},G}(\mathcal{C}, \mathcal{D})$ denotes the full G-subcategory of $\underline{\mathrm{Fun}}_{/\mathcal{O}^{\otimes},G}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned again by the morphisms of G-∞-operads. This is now a G-∞-category.

The obvious simplifications of the notation, as in Definition 3.1.16, apply. In particular, if $\mathcal{C}^{\otimes} = \mathcal{O}^{\otimes} = \underline{\mathcal{F}}_{G,*}$, then we will denote $\mathrm{Alg}_{\mathcal{O},G}(\mathcal{C}, \mathcal{D})$ by $\mathrm{CAlg}_G(\mathcal{D})$, the ∞-category of *commutative algebra objects* in $\mathcal{D}^{\otimes}$. Also note that this notation differs slightly from that of 3.1.16, for the sake of clarity.

We now finally discuss cocartesian fibration of G-∞-operads.

Definition 5.1.8. A map $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ of G-∞-operads is called a *cocartesian fibration of G-∞-operads* if q is a cocartesian fibration. In this case, we will also say that $\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal G-∞-category.

Remark 5.1.9. This definition of cocartesian fibration of G-∞-categories is analogous to that of Lurie, in that it essentially requires the composites to exhibit $\mathcal{C}^{\otimes}$ as a G-∞-operad. Indeed, the following Proposition states that this condition is still equivalent to asking an equivariant version of the Segal condition.

Proposition 5.1.10 ([NS22, Proposition 2.2.6])**.** *Let $p\colon \mathcal{O}^{\otimes} \to \underline{\mathcal{F}}_{G,*}$ be a G-∞-operad, and let $q\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of G-∞-operads. Then, the following are equivalent:*

(1) The composite $p \circ q\colon \mathcal{C}^{\otimes} \to \underline{\mathcal{F}}_{G,*}$ exhibits $\mathcal{C}^{\otimes}$ as a G-∞-operad.

(2) For every object $f\colon T \to G/H$ in $\underline{\mathcal{F}}_{G,*}$ and every $x \in \mathcal{O}^{\otimes}_{f}$, denote by x_i the cocartesian pushforward of x via $(\rho_i)_!$ for any orbit G/H_i of T. Then, the cocartesian pushforwards of the inert morphisms $x \to x_i$ in $\mathcal{O}^{\otimes}$ induce an equivalence:

$$\mathcal{C}^{\otimes}_{x} \xrightarrow{\simeq} \prod_{i \in I} \mathcal{C}^{\otimes}_{x_i}.$$

Remark 5.1.11. If $\mathcal{O}^{\otimes} = \underline{\mathcal{F}}_{G,*}$, then we recover the notion of G-symmetric monoidal G-∞-category that was introduced in Definition 4.2.19.

Some of the cocartesian pushforwards are of particular importance, as they play the same role of the norm functors of G-symmetric monoidal G-∞-categories.

Definition 5.1.12. Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal G-∞-category. Given an active morphism $f\colon x \to y$ in $\mathcal{O}^{\otimes}$, its cocartesian pushforward is called the *norm functor* for f, and it is denoted by

$$f_{\otimes}\colon \mathcal{C}^{\otimes}_{x} \to \mathcal{C}^{\otimes}_{y}.$$

As just mentioned, the norm functors of G-symmetric monoidal G-∞-categories are a special case of this. Given $g\colon G/H \to G/K$, they can be obtained by setting $\mathcal{O}^{\otimes} = \underline{\mathcal{F}}_{G,*}$ and taking ψ to be the following fiberwise active morphism:

$$\begin{array}{ccccc} G/H & = & G/H & \longrightarrow & G/K \\ \downarrow & & \downarrow & & \| \\ G/K & = & G/K & = & G/K \end{array}$$

We have now introduced an equivariant version of ∞-operads. Note however that in the definition we are still making use of regular fibers and mapping spaces. In order to look at Day convolution here, we first need to extend these notions to the equivariant setting, fibers in particular. These will then appear in an equivariant generalization of conditions (2) and (3) of the Definition 5.1.1 of G-∞-operads.

Definition 5.1.13. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a G-functor between G-∞-categories, and let σ be an n-simplex of $\mathcal{D}$, that is, a functor $\sigma\colon \Delta^n \to \mathcal{D}$ (viewing $\mathcal{D}$ just as ∞-category). The *G-fiber of $\mathcal{C}$ over σ* is defined as the pullback:

$$\mathcal{C}_{\underline{\sigma}} := \Delta^n \times_{\mathcal{D}} \mathrm{Ar}^{\mathrm{cocart}}(\mathcal{D}) \times_{\mathcal{D}} \mathcal{C}$$

where the maps $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{D}) \to \mathcal{D}$ are ev_0 and ev_1, respectively.

If $n = 0$, given $d \in \mathcal{D}$, the objects of the G-fiber $\mathcal{C}_{\underline{d}}$ are given by pairs of an object $c \in \mathcal{C}$ and a cocartesian arrow $d \to F(c)$.

Remark 5.1.14. Note that the notion just introduced of G-fiber is indeed a generalization of the usual fiber. This is because identity morphisms are always cocartesian, and so restricting the above to just the identity in $\mathrm{Ar}^{\mathrm{cocart}}(\mathcal{D})$ recovers the usual fiber.

Remark 5.1.15. G-fibers are important for two reasons:

(1) We can extend condition (2) of Definition 5.1.1, which is the Segal condition in this context, to an analogue result on G-fibers. This is called the G-Segal condition. We will not state it here as it would require extra theory, and it goes beyond the scope of this thesis; the relevant reference is [NS22, Corollary 2.3.4]. However, we note that it is used in the proof of Theorem 4.2.24, as it allows to characterize G-symmetric monoidal G-∞-categories as G-commutative monoids in $\underline{\mathrm{Cat}}_{\infty,G}$.

(2) The norm functor $f_{\otimes}$ of Definition 5.1.12 can be canonically extended to a functor between parametrized fibers:

$$f_{\otimes}\colon \mathcal{C}^{\otimes}_{\underline{x}} \to \mathcal{C}^{\otimes}_{\underline{y}}.$$

which we will still denote by $f_{\otimes}$.

One can then also define G-mapping spaces, and provide a generalization of condition (3) of Definition 5.1.1. This is of little importance to us, so we just reference [NS22, Notation 2.3.6 and 2.3.8].

Remark 5.1.16. Nardin and Shah also define in [NS22, Section 2.6] appropriate model structures which allow for the definition of an ∞-category of (small) G-∞-operads, denoted $\mathrm{Op}_{\infty,G}$. The objects are G-∞-operads, and the morphisms are maps of ∞-operads in the sense of Definition 5.1.5. The ∞-category $\mathrm{Cat}_G^{\otimes}$ of G-symmetric monoidal G-∞-categories is then the full subcategory of $\mathrm{Op}_{\infty,G}$ on them.

5.2 Equivariant Day convolution

In [NS22, Chapter 3], Nardin and Shah construct an equivariant version of Day convolution, which we will refer to as *G-Day convolution.* This, when defined, is the internal Hom for G-∞-operads fibered over an arbitrary G-∞-operad $\mathcal{O}^{\otimes}$. Again, they work in a larger generality for the parametrizing category, but we restrict to the equivariant context.

In their setting, they work with *$\mathcal{O}$-promonoidal* G-∞-categories, which generalize the $\mathcal{O}$-monoidal G-∞-categories we introduced in Definition 5.1.8. Given an $\mathcal{O}$-promonoidal G-∞-category $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ and any fibration $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ of G-∞-operads, they construct the G-Day convolution:

$$\widetilde{\mathrm{Fun}}_{\mathcal{O},G}(\mathcal{C},\mathcal{D})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$$

and then add extra hypotheses on $\mathcal{D}^{\otimes}$ to make this $\mathcal{O}$-monoidal. Once more, we don't need the extra layer of generality of $\mathcal{O}$-promonoidal G-∞-categories, as understanding its relevance requires lots of extra theory and context. Also, any $\mathcal{O}$-monoidal G-∞-category is in particular $\mathcal{O}$-promonoidal, as noted in [NS22, Example 3.1.2], therefore we restrict to $\mathcal{O}$-monoidal G-∞-categories only. We will also present the results in a slightly different way compared to Nardin and Shah, to make the main ideas more clear, and still keeping similarities with Lurie's approach discussed in Chapter 3.

First off, we define norms, in analogy with Definition 3.2.1.

Definition 5.2.1. Let $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a cocartesian fibration of G-∞-operads, that is, $\mathcal{C}^{\otimes}$ is an $\mathcal{O}$-monoidal G-∞-categories. Consider two fibrations of G-∞-operads:

$$\widetilde{\mathcal{C}}^{\otimes} \to \mathcal{C}^{\otimes} \qquad \text{and} \qquad \widetilde{\mathcal{O}}^{\otimes} \to \mathcal{O}^{\otimes}.$$

We say that a morphism:

$$\alpha\colon \widetilde{\mathcal{O}}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} \longrightarrow \widetilde{\mathcal{C}}^{\otimes}$$

of G-∞-operads *exhibits $\widetilde{\mathcal{O}}^{\otimes}$ as a norm of $\widetilde{\mathcal{C}}^{\otimes}$ along p* if:

(1) The following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathcal{O}}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} & \xrightarrow{\alpha} & \widetilde{\mathcal{C}}^{\otimes} \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

(2) For any map $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ of G-∞-operads, composition with α induces an equivalence of G-∞-categories:

$$\underline{\mathrm{Alg}}_{\mathcal{O},G}(\mathcal{O}',\widetilde{\mathcal{O}}) \xrightarrow{\simeq} \underline{\mathrm{Alg}}_{\mathcal{C},G}(\mathcal{O}' \times_{\mathcal{O}} \mathcal{C},\widetilde{\mathcal{C}})$$

which in particular restricts to an equivalence of ∞-categories:

$$\mathrm{Alg}_{\mathcal{O},G}(\mathcal{O}',\widetilde{\mathcal{O}}) \xrightarrow{\simeq} \mathrm{Alg}_{\mathcal{C},G}(\mathcal{O}' \times_{\mathcal{O}} \mathcal{C},\widetilde{\mathcal{C}}).$$

Remark 5.2.2. We recognize the second condition to be one of the usual requirements for a Day convolution tensor product; see for example Proposition 2.3.12 in the case of symmetric monoidal ∞-categories, as well as Remark 3.2.7 and the previous discussion. Here the strategy is the same: we will now state an existence result for norms that will result in the existence in particular of a norm for the fibration $\mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} \to \mathcal{C}^{\otimes}$, where $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration of G-∞-operads, and $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ is any fibration of G-∞-operads. This, upon further hypotheses on $\mathcal{D}^{\otimes}$, will result in a Day convolution $\mathcal{O}$-monoidal structure on the functor category $\underline{\mathrm{Fun}}_G(\mathcal{C}, \mathcal{D})$ with the desired property.

Let us go back to the general situation of Definition 5.2.1 to state this existence result.

Theorem 5.2.3. *With the notation of Definition 5.2.1, if $\mathcal{C}^{\otimes}$ is $\mathcal{O}$-promonoidal, then there exists another fibration of G-∞-operads:*

$$\mathrm{Nm}_{\mathcal{C}/\mathcal{O}}(\widetilde{\mathcal{C}})^{\otimes} \to \mathcal{O}^{\otimes}$$

and a map α exhibiting $\mathrm{Nm}_{\mathcal{C}/\mathcal{O}}(\widetilde{\mathcal{C}})$ as a norm of $\widetilde{\mathcal{C}}$ along p.

Remark 5.2.4. This Theorem, together with the definition of norm, is not really stated in [NS22], but it is a consequence of [NS22, Corollary 3.1.8]. This in turn follows from two previous propositions which employ model categorical notations and tools, and so we decided to instead state the main consequence in more understandable terms. In fact, after they obtain the norm as a consequence of these, they then proceed to prove in [NS22, Proposition 3.1.7] that it satisfies condition (2) of Definition 5.2.1, which essentially amounts to proving existence.

Once again, note that if we take $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a cocartesian fibration of G-∞-operads, $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ any fibration of G-∞-operads, and we take their pullback, using the Theorem we obtain a suitable fibration over $\mathcal{O}^{\otimes}$ that behaves as a Day convolution, and so is a good candidate. We denote this fibration by:

$$\widetilde{\mathrm{Fun}}_{\mathcal{O},G}(\mathcal{C}, \mathcal{D})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$$

and if $\mathcal{O}^{\otimes} = \underline{\mathcal{F}}_{G,*}$, we more simply denote it by:

$$\underline{\mathrm{Fun}}_G(\mathcal{C}, \mathcal{D})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$$

Remark 5.2.5. The fact that this could be the right fibration is further strengthened by [NS22, Proposition 3.1.9], as the underlying G-∞-category is the right one. In particular, when $\mathcal{O}^{\otimes} = \underline{\mathcal{F}}_{G,*}$, the underlying G-∞-category of $\underline{\mathrm{Fun}}_G(\mathcal{C}, \mathcal{D})^{\otimes}$ is precisely $\underline{\mathrm{Fun}}_G(\mathcal{C}, \mathcal{D})$, the G-∞-category of G-functors from $\mathcal{C}$ to $\mathcal{D}$.

5.3 $\mathcal{O}$-monoidality of the G-Day convolution

We now look at the extra conditions needed to turn $\widetilde{\mathrm{Fun}}_{\mathcal{O},G}(\mathcal{C}, \mathcal{D})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ into an $\mathcal{O}$-monoidal G-∞-category.

Warning 5.3.1. For this last section, some knowledge of parametrized colimits and parametrized left Kan extensions is needed. This would be too much to add, and the theoretical behaviour is essentially analogous to the non-parametrized variants, although the situation is rendered much more complicated by the presence of many more "points". For example, G-colimits are taken with respect to $\mathcal{O}rb_G$, which has many possible "points", i.e. all orbits G/H, whereas an ordinary colimit can be seen as parametrized over the trivial category $\{*\}$, which has a single point and thus results in a simpler situation. For the interested reader, Shah in [Sha23] and [Sha22] develops parametrized higher category theory with the same notations and conventions as in [NS22].

First off, we have the following result which establishes when $\widetilde{\mathrm{Fun}}_{\mathcal{O},G}(\mathcal{C},\mathcal{D})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is a locally cocartesian fibration, and gives a characterization of cocartesian edges. As expected from the analogous results in the non-parametrized setting, we need some sort of cocompleteness on the target G-∞-operad.

Proposition 5.3.2 ([NS22, Proposition 3.2.2])**.** *Let $p\colon \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal G-∞-category, and let $q\colon \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ be an $\mathcal{O}$-monoidal G-∞-category such that for every orbit G/H and every object $x \in \mathcal{O}_{G/H}$, the G-fiber $\mathcal{D}^{\otimes}_{\underline{x}}$ is $\mathcal{O}rb_G^{/H} \simeq \mathcal{O}rb_H$-cocomplete. Then, the following hold:*

(1) The G-Day convolution $\widetilde{\mathrm{Fun}}_{\mathcal{O},G}(\mathcal{C},\mathcal{D})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is a locally cocartesian fibration, and for every $x \in \mathcal{O}_{G/H}$, its G-fiber $\widetilde{\mathrm{Fun}}_{\mathcal{O},G}(\mathcal{C},\mathcal{D})^{\otimes}_{\underline{x}}$ is also $\mathcal{O}rb_H$-cocomplete.

(2) Let $\alpha\colon x \to y$ be an edge in $\mathcal{O}^{\otimes}_{G/H}$ with $y \in \mathcal{O}_{G/H}$, lying over the morphism ψ:

$$\begin{array}{ccccc} T & = & T & \longrightarrow & G/H \\ \downarrow & & \downarrow & & \| \\ G/H & = & G/H & = & G/H \end{array}$$

of $\underline{\mathbb{F}}_{G,*}$. Then, a fiberwise active edge $\widetilde{\alpha}$ of $\widetilde{\mathrm{Fun}}_{\mathcal{O},G}(\mathcal{C},\mathcal{D})^{\otimes}$ is locally cocartesian if and only if the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{C}^{\otimes}_{\underline{x}} & \xrightarrow{F} & \mathcal{D}^{\otimes}_{\underline{x}} & \hookrightarrow & \mathcal{D}^{\otimes}_{\underline{\alpha}} & \xrightarrow{P_\alpha} & \mathcal{D}_{\underline{y}} \\ \big\downarrow & & & \nearrow_{H} & & & \big\downarrow \\ \mathcal{C}^{\otimes}_{\underline{\alpha}} & & & \longrightarrow & & & \mathcal{O}rb_H^{\mathrm{op}} \end{array}$$

exhibits H as an $\mathcal{O}rb_H$-left Kan extension. Here, H and F are determined by the datum of $\widetilde{\alpha}$, and P_α is obtained from a cocartesian pushforward.

Remark 5.3.3. As a consequence, we also have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{C}^{\otimes}_{\underline{x}} & \xrightarrow{F} & \mathcal{D}^{\otimes}_{\underline{x}} & \xrightarrow{\alpha_\otimes} & \mathcal{D}_{\underline{y}} \\ {\scriptstyle\alpha_\otimes}\big\downarrow & & \nearrow_{G} & & \\ \mathcal{C}_{\underline{y}} & & & & \end{array}$$

together with a natural transformation η which exhibits G as an $\mathcal{O}rb_H$-left Kan extension of $\alpha_\otimes \circ F$ along $\alpha_\otimes$, where G is the restriction of $P_\alpha \circ H$ to $\mathcal{C}_{\underline{y}}$ and $\alpha_\otimes$ are the norm functors of $\mathcal{C}^{\otimes}$ and $\mathcal{D}^{\otimes}$ given by α. This diagram is again resembling of the ordinary situation, in which a very similar diagram occurs.

This in theory should lead to a way of computing cocartesian pushforwards for the Day convolution. However, in practice these are hard to compute explicitly, as they involve parametrized colimits, which in general do not simplify into easier expressions. We are therefore content with this theoretical characterization.

Finally, in analogy with the conditions stated in Lemma 2.3.4 and Proposition 3.2.10, we have to add a further requirement on the cocartesian pushforward functors in $\mathcal{D}^{\otimes}$ associated to fiberwise active edges α of $\mathcal{O}^{\otimes}$ lifting a morphism ψ as in point (2) of Proposition 5.3.2, i.e. with image $[G/H = G/H] \in \underline{\mathcal{F}}_{G,*}$. Informally, we want all of these to commute with colimits in a specific sense: compared to the non-equivariant case, where we required commutativity componentwise, here we have an extra layer of complexity given by the many "points" of $\mathcal{O}rb_G$, that is, all the orbits G/H, as well as the morphisms between them. The right notion is that of a *distributive* functor, which is defined in [NS22, Definition 3.2.3]. An $\mathcal{O}$-monoidal G-∞-category $\mathcal{C}^{\otimes}$ is then called *distributive* if all the cocartesian pushforwards $\alpha_{\otimes}$ of morphisms α as above are distributive. We then have the following Theorem, which is the endpoint of our treaty on Day convolution.

Theorem 5.3.4 ([NS22, Theorem 3.2.6])**.** *In the situation of Proposition 5.3.2, suppose moreover that $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ is distributive. Then, $\widetilde{\mathrm{Fun}}_{\mathcal{O},G}(\mathcal{C},\mathcal{D})^{\otimes} \longrightarrow \mathcal{O}^{\otimes}$ is an $\mathcal{O}$-monoidal G-∞-category, and it is also distributive.*

We conclude with one final example which, although simple, highlights the usefulness of Day convolution in the equivariant context.

Example 5.3.5. Recall that $\underline{\mathcal{S}}_G$ is the G-∞-category of genuine G-spaces. We can view it as a G-symmetric monoidal G-∞-category with the cartesian structure, so that the norm map associated to $G/H \to G/K$ is the induction map Ind_H^K. Then, following [NS22, Example 3.2.8], we can describe the Day convolution structure for the G-∞-category of *pointed G-spaces*, denoted $\underline{\mathcal{S}}_{G,*}$. Given $g\colon G/H \to G/K$ associated with $H \leq K \leq G$, the norm map in the Day convolution amounts to just postcomposing with the norm map of $\underline{\mathcal{S}}_G$, that is, with Ind_H^K: for a real K-representation R, the representation sphere S^R is sent to $S^{\mathrm{Ind}_H^K R}$

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