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Fourier analysis and sub-Laplacian on a 3-step Carnot group

Nicolò Tedesco

Supervisors:
Prof. Davide Barilari†, Prof. Dario Prandi‡

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† Università degli studi di Padova - Dipartimento di Matematica "Tullio Levi-Civita"
‡ Université Paris-Saclay - CentraleSupélec, Laboratoire des signaux et systèmes

Abstract

The aim of this internship project is to extend recent results on harmonic analysis for nilpotent Lie groups ([3], [4]) to a 3-step stratified group carrying a sub-Riemannian structure. We will consider the Lie group N_4, the unique connected and simply connected nilpotent Lie group whose Lie algebra is $\mathfrak{n}_4$, the Lie algebra of stricly upper triangular 4×4 matrices. In particular, $\mathfrak{n}_4$ is stratified with grading $(3, 5, 6)$. We will highlight some spectral properties of the sub-Laplacian and we will explicitly determine the *Plancherel measure* associated to the group.

The choice of studying this group is motivated by a two-fold purpose. First of all, the group will serve as a model for the study of basic results on nilpotent Lie groups (choices of coordinates, adjoint and coadjoint action, Haar measure,...), together with the celebrated Kirillov theory of Lie groups representation. Moreover, we will investigate general questions regarding Fourier theory on nilpotent Lie groups, giving the proof of a *Fourier inversion* formula. We will also compare our results with the well understood cases of the Heisenberg and Engel group, with a particular attention to this last one, which shares with the group N_4 the property of being 3-step stratified.

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Introduction

The aim of this internship project is to extend recent results on harmonic analysis for nilpotent Lie groups ([3], [4]) to a 3-step stratified group carrying a sub-Riemannian structure. The object of this study is the Lie group N_4, the unique connected and simply connected nilpotent Lie group whose Lie algebra is $\mathfrak{n}_4$, the Lie algebra of strictly upper triangular 4×4 real matrices. Once fixed a basis $\{X_1, X_2, X_3, X_{12}, X_{23}, X_0\}$ of the Lie algebra, the only nontrivial brackets are:

$$[X_1, X_2] = X_{12}, \quad [X_2, X_3] = X_{23}$$
$$[X_1, X_{23}] = X_0, \quad [X_{12}, X_3] = X_0.$$

The group will serve as a model for the study of general questions regarding nilpotent Lie groups, such as the classification of their unitary and irreducible representations and the computation of the Plancherel measure. The thesis is divided in three chapters.

The purpose of the first chapter is to fix the notation and to collect some useful results on nilpotent Lie groups. Most of the material is taken from [1], [7], [8].
After recalling some definitions about Lie algebras and Lie groups we define the *exponential coordinates* for nilpotent Lie groups, making use of the *Baker-Campbell-Hausdorff* formula. After this, we define the so-called *Malcev coordinates*, whose advantage is to give a simpler expression of the group operation, and we explicit these systems of coordinates on the group N_4. We also describe a connection between the theory of nilpotent Lie groups and sub-Riemannian geometry, following [2]: namely, we show how to equip a nilpotent and stratified Lie group G with a sub-Riemannian structure and define the sub-Laplacian Δ_G.
In the second part of the chapter we recall the definition of *Haar measure* for locally compact groups and we highlight three important facts about nilpotent Lie groups: that they are *unimodular*, that a Haar measure can be *transferred* from the Lebesgue measure on $\mathbb{R}^n$ by the coordinate maps and that it is always possible to endow the space of cosets $H\backslash G$ with a G-invariant measure.
Finally, we recall some basic properties of symplectic and Poisson manifolds. In particular, we describe the Poisson structure of the dual of the Lie algebra $\mathfrak{g}^*$, showing that the symplectic leaves coincide with the orbits of the coadjoint action of G on $\mathfrak{g}^*$. We conclude describing the foliation of $\mathfrak{n}_4^*$.

The second chapter is devoted to the study of the representation theory for nilpotent Lie groups, with a focus on the *orbit method* developed by A.A. Kirillov (see [13], [12], [11]). The main sources for this chapter are [7], [12] and [4], Appendix A.
We start giving the definition of unitary representation of a group G, that is a map

$\mathcal{R}: G \to \mathcal{U}(\mathcal{H})$, where $\mathcal{H}$ is an Hilbert space, such that:

$$\mathcal{R}(gg') = \mathcal{R}(g) \circ \mathcal{R}(g') \quad \forall g, g' \in G.$$

In paticular, we classify the *unirreps* (i.e. unitary, irreducible and strongly continuous representations) of nilpotent Lie groups. The classification is related with the foliation of $\mathfrak{g}^*$, therefore we start showing an alternative method for determining the foliation based on the identification of Casimir functions, following [4]. After this, we describe the construction of the *induced representations* of G; the basic idea is the following. Fixing $\eta \in \mathfrak{g}^*$ and a *polarizing subalgebra* $\mathfrak{h}$ for η, we can easily obtain a representation $\chi_{\eta,\mathfrak{h}} : H \to \mathbb{S}^1$ of the subgroup $H = \exp_G(\mathfrak{h}) \subseteq G$; starting from $\chi_{\eta,\mathfrak{h}}$, we can *induce* a representation

$$\mathcal{R}_{\eta,\mathfrak{h}} : G \to L^2(H \backslash G, d\mu),$$

where μ is a right invariant measure on $H\backslash G$. Kirillov proved in [13] that whenever G is nilpotent, such $\mathcal{R}_{\eta,\mathfrak{h}}$ are, up to equivalence, all the possible unirreps of G, and if η, η' lie on the same coadjoint orbit then the corresponding unirreps are equivalent. Another important fact that we show is that such representations can be realized in the more familiar spaces $L^2(\mathbb{R}^m; \mathbb{C})$, where m depends on $\dim(\mathfrak{h})$. The application of this method on N_4 determines five classes of representations. We will be mostly interested in the two-dimensional ones:

$$\mathcal{R}_{\alpha,\gamma} : N_4 \to L^2(\mathbb{R}^2; \mathbb{C})$$
$$[\mathcal{R}_{\alpha,\gamma}(\psi(x))f](\theta_1, \theta_3) = e^{i\left(\alpha(x_0+\theta_1 x_{23}-\theta_3 x_{12}-\theta_1\theta_3 x_2)+\frac{\gamma}{\alpha}x_2\right)} f(x_1+\theta_1, x_3+\theta_3),$$

where $(\alpha, \gamma) \in \mathbb{R}^* \times \mathbb{R}$.
We also discuss a different method to obtain a representation, that works if the group carries a structure $G = H \rtimes K$ when a representation of H is known. We finaly apply this alternative method to $N_4 = \mathbb{H}^2 \rtimes \mathbb{R}$, obtaining some more representations.
In the second part of the chapter, we explain how to extend a unirrep $(\mathcal{R}, \mathcal{H})$ of a Lie group G to a representation of its Lie algebra $\mathfrak{g}$ on $\mathcal{H}$. The construction is then generalized to obtain a representation of the *universal eveloping algebra* $\mathfrak{u}(\mathfrak{g})$ and of left-invariant differential operators on G. An important role is played by the operators $\mathcal{R}(\phi)$, defined as:

$$\mathcal{R}(\phi) := \int_G \phi(g)\mathcal{R}(g)dg, \quad \phi \in \mathcal{C}_c^\infty(G).$$

We conclude focusing on the sub-Laplacian on stratified groups. The theory of sub-Laplacians is one of the most investigated fields in sub-Riemannian geometry (see, for instance, [2], [6]) and it has been studied also in the context of nilpotent Lie groups, for instance on the Heisenberg and Engel group (see [4], [3]). For the group N_4 we find:

$$\mathcal{R}_{\alpha,\gamma}(-\Delta_{N_4}) = -\partial^2_{\theta_1} - \partial^2_{\theta_3} + \left(\frac{\gamma}{\alpha} - \alpha\theta_1\theta_3\right)^2$$

and we show that this operator, as in the cases of Heisenberg and Engel groups, has purely discrete spectrum.

The third chapter is based on the observation that operators $\mathcal{R}(\phi)$ can interpreted as Fourier transforms. The main goal is to prove a *Fourier inversion* equality for nilpotent

Lie groups, i.e. the existence of a *Plancherel measure* $|\operatorname{Pf}(\eta)|d\eta$ and a subspace $Q \subseteq \mathfrak{g}^*$ such that:

$$\phi(1_G) = \frac{1}{(2\pi)^k} \int_Q \operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) |\operatorname{Pf}(\eta)| d\eta \quad \forall \phi \in \mathcal{S}(G),$$

where k is some power that will be explicited. The main source is [9].

The construction is splitted in two parts. In the first part we prove a *Trace Theorem*, which states the existence of a measure ν_η on the coadjoint orbit of any $\eta \in \mathfrak{g}^*$ that permits to rewrite the trace of the operators $\mathcal{R}_{\eta,\mathfrak{h}}(\phi)$ with a simple integral expression. In the second part we define an invariant Zariski open $U \subseteq \mathfrak{g}^*$, together with a subspace $Q \subseteq \mathfrak{g}^*$ that can be used to parametrize *simultaneously* the orbis in U, where we will prove the Fourier inversion. We give some practical rules to identify U and Q. Our equality is finally obtained after expliciting the measure ν_η of the Trace Theorem and an analog of the Plancherel theorem follows directly from this result.
The chapter concludes with the application of these theoretical results to the group N_4 and the Heisenberg and Engel groups, giving an explicit expressions for the Plancherel measure and emphasizing some analogies and differences.

Let us conclude with some general observations.

- In Section 2.4.2 we derive two classes of representations inherited by the structure of semidirect product. We did not find whether the representations belonging to the second class are reducible or not.

- As for the Heisenberg and Engel group, the open Zariski U is precisely the collection of the top-dimensional orbits that are not contained in the hyperplane $\{h_0 = 0\}$, where h_0 denotes the dual variable of the center of $\mathfrak{g}$. Another interesting analogy is that in each of these cases the center is one-dimensional and the density of the Plancherel measure depends only on the variable corresponding to the center. One possible extension of this work could be the application of this theory to nilpotent groups with higher-dimensional center.

- Again, as for the Heisenberg and Engel group, the representation $\mathcal{R}_{\alpha,\lambda}(-\Delta_{N_4})$ is a Schrödinger operator $-\Delta + P(x)^2$, where P is a polynomial, which has purely discrete spectrum.

Chapter 1

Basic results on nilpotent Lie groups

1.1 The group and its operation

1.1.1 Lie algebras

Definition 1.1.1. A Lie algebra is a vector space $\mathfrak{g}$ equipped with a Lie bracket, i.e. a bilinear map

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

verifying:

(i) $[X,Y] = -[Y,X] \quad \forall X,Y \in \mathfrak{g}$ (*antisymmetry*)

(ii) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \forall X,Y,Z \in \mathfrak{g}$ (*Jacobi identity*).

We call $\zeta(\mathfrak{g}) := \{X \in \mathfrak{g} \mid [X,Y] = 0 \ \forall\ Y \in \mathfrak{g}\}$ the *center* of $\mathfrak{g}$.

Remark 1.1.2. Let M be a smooth manifold. The collection of smooth vector fields $\mathrm{Vec}(M)$, equipped with the standard Lie bracket:

$$[X,Y] = XY - YX \quad \forall X,Y \in \mathrm{Vec}(M)$$

is a Lie algebra.

Definition 1.1.3. Let $\mathfrak{g}$ be a Lie algebra. A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a *Lie subalgebra* of $\mathfrak{g}$ if $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$.

Definition 1.1.4. Let $(\mathfrak{g}_1, [\cdot,\cdot]_1), (\mathfrak{g}_2, [\cdot,\cdot]_2)$ be Lie algebras. A *Lie algebra homomorphism* is a linear homomorphism $\psi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

$$\psi([X,Y]_1) = [\psi(X), \psi(Y)]_2 \quad \forall X,Y \in \mathfrak{g}_1.$$

If ψ is also a linear isomorphism, it is called a *Lie algebra isomorphism.*

Definition 1.1.5. The *descending central series* of a Lie algebra $\mathfrak{g}$ is defined inductively:

$$\mathfrak{g}^1 := \mathfrak{g}, \quad \mathfrak{g}^{n+1} := [\mathfrak{g}, \mathfrak{g}^n]$$

We say that $\mathfrak{g}$ is *nilpotent* if there exists $n \in \mathbb{N}$ such that $\mathfrak{g}^{n+1} = \{0\}$; in particular, it is *n-step* nilpotent if n is minimal for such property.

For the next definition, we follow [14].

Definition 1.1.6. A Lie algebra is called *stratified* if it can be decomposed in a direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_r \text{ with } [\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1},$$

where $\mathfrak{g}_r \neq \{0\}$ and $\mathfrak{g}_{r+1} = \{0\}$.

For all $n \in \mathbb{N}$, an example of nilpotent Lie algebra is the vector space:

$$\mathfrak{n}_n \subset M_n(\mathbb{R}),$$

defined as the set of $n \times n$ strictly upper triangular real matrices, equipped with the usual matrix commutator ($[A, B] = AB - BA$). It is an $(n-1)$-step nilpotent Lie algebra of dimension $n(n-1)/2$ and it is stratified with decomposition:

$$\mathfrak{n}_n = (\mathfrak{n}_n)_1 \oplus ... \oplus (\mathfrak{n}_n)_{n-1}$$
$$(\mathfrak{n}_n)_k := \operatorname{span}\{E_{i,i+k} \mid i = 1, ..., n-k\},$$

where $E_{i,j}$ denotes the $n \times n$ matrix with the $(i,j)-th$ entry equal to 1 and all the others equal to zero. We will study the Lie algebra $\mathfrak{n}_4$, together with the associated Lie group N_4.

Remark 1.1.7. The Lie algebra $\mathfrak{n}_4$ corresponds to the one denoted with the symbol $\mathfrak{g}_{6,12}$ in [5].

1.1.2 Lie groups and Lie algebras, left-invariant vector fields

Definition 1.1.8. A *Lie group G* is a topological group equipped with a smooth manifold structure, such that the multiplication:

$$\mu : G \times G \to G, \quad \mu(g, h) = gh$$

and the inversion

$$\iota : G \to G, \quad \iota(g) = g^{-1}$$

are smooth maps.

Notation 1. For any $g \in G$ we denote the *left translation*, *right translation* and *conjugation by g* respectively:

$$L_g : G \to G, \quad h \mapsto gh$$
$$R_g : G \to G, \quad h \mapsto hg$$
$$C_g : G \to G, \quad h \mapsto ghg^{-1}.$$

Corollary 1.1.9. By definition of Lie group the maps L_g, R_g, C_g are smooth diffeomorphisms of G into itself.

Definition 1.1.10. Let G, H be Lie groups. A *Lie group homomorphism* from G to H is a group homomorphism that is also smooth. If the map is also a group isomorphism, we call it a *Lie group isomorphism*.

Let G be a Lie group. The group operation on G induces a Lie algebra structure on $\mathfrak{g} := T_{1_G}G$, as we now explain.

Definition 1.1.11. Let G be a Lie group and $\tilde{X} \in \mathrm{Vec}(G)$ a smooth vector field. $\tilde{X}$ is called *left-invariant* if

$$(L_g)_* \tilde{X} = \tilde{X} \quad \forall g \in G,$$

where $(L_g)_* \tilde{X}$ denotes the push-forward of $\tilde{X}$ via L_g. The collection of left-invariant vector fields on G is denoted as $\mathrm{Vec}_L(G)$.

In particular, a left-invariant $\tilde{X}$ must verify:

$$\tilde{X}(g) = (L_g)_{*,1_G} \tilde{X}(1_G) \quad \forall g \in G. \tag{1.1}$$

Using the standard identity $(L_g)_*[\tilde{X}, \tilde{Y}] = [(L_g)_* \tilde{X}, (L_g)_* \tilde{Y}]$ we deduce that $\mathrm{Vec}_L(G)$ is a Lie subalgebra of $\mathrm{Vec}(G)$.

The equation (1.1) shows that a left-invariant vector field is *completely determined* by its value at identity 1_G.

Notation 2. Fixed $X \in \mathfrak{g}$, we denote with the symbol $\tilde{X} \in \mathrm{Vec_L}(\mathfrak{g})$ the unique left-invariant vector field such that:

$$\tilde{X}(1_G) = X.$$

Proposition 1.1.12. *Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$. The correspondence:*

$$\lambda : \mathfrak{g} \to \mathrm{Vec}_L(G), \quad \lambda(X) = \tilde{X}$$

is a linear isomorphism.

Finally, we can use the above correpondence to define a Lie algebra structure on $\mathfrak{g}$ letting:

$$[X, Y] := [\tilde{X}, \tilde{Y}](1_G) \quad \forall X, Y \in \mathfrak{g}. \tag{1.2}$$

With this definition, the map (1.2) automatically becomes a Lie algebra isomorphism.

Definition 1.1.13. If G is a Lie group, the tangent space $T_{1_G}G$ is called the *Lie algebra of G*. We denote it with the symbol $\mathfrak{g}$.

Definition 1.1.14. A Lie group is called *nilpotent* if its Lie algebra is nilpotent.

1.1.3 Exponential map

Notation 3. Let G be a Lie group, $\tilde{X} \in \mathrm{Vec}(G)$. We denote with the symbol $\Phi_t^{\tilde{X}}(g)$ or $e^{t\tilde{X}}(g)$ the value at $t \in \mathbb{R}$ of the flow of $\tilde{X}$ starting in $g \in G$ at $t_0 = 0$.

Definition 1.1.15. Let G be a Lie group with Lie algebra $\mathfrak{g}$. We define the *exponential map*:

$$\exp_G : \mathfrak{g} \to G, \quad X \mapsto \exp_G(X) := \Phi_1^{\tilde{X}}(1_G).$$

Proposition 1.1.16. *The exponential map verifies:*

(i) $\exp_G$ *is a smooth map;*

(ii) $\exp_G(0) = 1_G$*;*

(iii) $\exp_G(tX) = \Phi_t^{\tilde{X}}(1_G)$ *for all* $X \in \mathfrak{g},\ t \in \mathbb{R}$*;*

(iv) $\exp_G((t+s)X) = \exp_G(tX)\ \exp_G(sX)$ *for all* $X \in \mathfrak{g},\ t, s \in \mathbb{R}$*;*

(v) $\exp_G$ *is a local diffeomorphism at* $0 \in \mathfrak{g}$.

Remark 1.1.17. In general, $\exp_G$ is neither surjective nor injective, and it may fail to be a local diffeomorphism away from zero.

Proposition 1.1.18 (Fundamental property of exponential map)**.** *Let* G, H *be Lie groups,* $\Psi : G \to H$ *a Lie group homomorphism and denote* $\psi = \Psi_{*,1_G}$ *the differential at identity. We have:*

- ψ *is a Lie algebra homomorphism;*
- $\Psi \circ \exp_G = \exp_H \circ \psi$.

The last statement precisely means that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\Psi} & H \\ {\scriptstyle \exp_G}\big\uparrow & & \big\uparrow{\scriptstyle \exp_H} \\ \mathfrak{g} & \xrightarrow{\psi} & \mathfrak{h} \end{array}$$

Notation 4. In the following, we will sometimes write e^X in place of $\exp_G(X)$, when no confusion with other notations is possible.

Definition 1.1.19. Let G be a Lie group. If $g \in G$ belongs to the neighborhood of 1_G where $\exp_G^{-1}$ is defined, we set:

$$\log_G(g) := (\exp_G)^{-1}(g) \in \mathfrak{g}.$$

1.1.4 Lie group actions, adjoint action

Definition 1.1.20. Let G be a Lie group and M a smooth manifold. A *(left) action* of G on M is a smooth map:

$$\Psi : G \times M \to M, \quad (g, m) \mapsto \Psi_g(m)$$

such that:

(i) $\Psi_{1_G} = \mathrm{id}_M$;

(ii) $\Psi_{gh} = \Psi_g \circ \Psi_h \quad \forall g, h \in G$.

If (ii) is replaced by:

(ii)' $\Psi_{gh} = \Psi_h \circ \Psi_g \quad \forall g, h \in G$

then we call Ψ a *right action*.

Definition 1.1.21. Let Ψ be a Lie group action of G on a smooth manifold M. Fixed $m \in M$, we define the *orbit* $\mathcal{O}_m$ as:

$$\mathcal{O}_m := \{\Psi_g(m) \mid g \in G\}.$$

Moreover, we call *stabilizer* of m in G the Lie subgroup:

$$\mathrm{stab}_G(m) := \{g \in G \mid \Psi_g(m) = m\}.$$

Let's now focus on a particular action of a Lie group G on its Lie algebra $\mathfrak{g}$.

Let again C_g denote the conjugation by $g \in G$ and denote $\mathrm{Ad}_g := (C_g)_{*,1_G}$ the tangent map computed at 1_G. We define the map

$$\mathrm{Ad} : G \to GL(\mathfrak{g}), \;\; g \mapsto \mathrm{Ad}_g,$$

denoting $\mathrm{ad} := (\mathrm{Ad})_{*,1_G} : \mathfrak{g} \to L(\mathfrak{g})$ (here $L(\mathfrak{g})$ denotes the vector space of linear endomorphisms of $\mathfrak{g}$). This costruction permits to associate any $X \in \mathfrak{g}$ with the linear map

$$[\mathrm{ad}(X)](\cdot) : \mathfrak{g} \to \mathfrak{g}.$$

Proposition 1.1.22. *For any* $X, Y \in \mathfrak{g}$ *we have* $[\mathrm{ad}(X)](Y) = [X, Y]$.

Equipping $GL(\mathfrak{g})$ with its natural Lie group structure, we have the following:

Proposition 1.1.23. *The map* Ad *defined above is a Lie group homomorphism.*

Corollary 1.1.24. From Proposition 1.1.23 and Proposition 1.1.18 we deduce the commutativity the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}} & GL(\mathfrak{g}) \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_{L(\mathfrak{g})}} \\ \mathfrak{g} & \xrightarrow{\mathrm{ad}} & L(\mathfrak{g}) \end{array}$$

Here $\exp_{L(\mathfrak{g})}$ indicates the usual matrix exponential: $\exp_{L(\mathfrak{g})}(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

With a slight abuse of notation, we are now ready to introduce the *adjoint action.*

Definition 1.1.25. The *adjoint action* of a Lie group G on its Lie algebra $\mathfrak{g}$ is the Lie group action:

$$\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}, \quad (g, X) \mapsto \mathrm{Ad}_g(X).$$

1.1.5 Stratified Lie groups, sub-Riemannian structure

In this section we introduce the notion of sub-Riemannian manifold, following [2], with the purpose of defining the *sub-Laplacian* on nilpotent and stratified Lie groups. The details of this construction are out of the scope of this work, for a detailed exposition see [2].

Notation 5. To avoid confusion, we will denote with the symbol X, Y smooth vector fields on a general smooth manifold M. We keep the usual notation $\tilde{X}, \tilde{Y}$ for left-invariant vector fields on a Lie group G.

Definition 1.1.26. A (n, m) *sub-Riemannian manifold* is a triple $(M, \mathcal{D}, \sigma)$, where:

- M is a connected smooth manifold of dimension n;

- $\mathcal{D}$ is a smooth distribution of rank $m \leq n$ satisfying the condition:

$$\text{span}\{[\mathsf{X}_1, [\dots, [\mathsf{X}_{k-1}, \mathsf{X}_k] \dots] \mid \mathsf{X}_i \in \text{Vec}_H(M), k \in \mathbb{N}\} = T_m M,$$

where $\text{Vec}_H(M)$ is the set of *horizontal smooth vector fields*, i.e.

$$\text{Vec}_H(M) := \{\mathsf{X} \in \text{Vec}(M) \mid \mathsf{X}(m) \in \mathcal{D}(m) \quad \forall m \in M\}.$$

- σ_m is a Riemannian metric on $\mathcal{D}(m)$ that is smooth as a function of m.

Remark 1.1.27. Of course, if $n = m$, the above definition coincides with the one of *Riemannian manifold* and no distribution is needed.

Let G be a nilpotent stratified Lie group, with stratification

$$\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_r$$

as in Definition 1.1.6. We can equip G with a *left-invariant* sub-Riemannian structure fixing a positive definite quadratic form $\sigma(1_G)$ on $\mathfrak{g}_1$ and defining:

- the *left-invariant distribution*

$$\mathcal{D}(g) := (L_g)_{*,1_G}\mathfrak{g}_1;$$

- the quadratic form:

$$\sigma(g)(v, w) := \sigma(1_G)((L_{g^{-1}})_{*,g}(v), (L_{g^{-1}})_{*,g}(w)) \quad \forall g \in G, \ \forall v, w \in \mathcal{D}(g).$$

In particular, we can fix a basis $\{X_1, \dots, X_k\}$ of $\mathfrak{g}_1$ and define $\sigma(1_G)$ so that the basis is orthonormal with respect to this metric. With this choice:

$$\mathcal{D}(g) = \text{span}\{\tilde{X}_1(g), \dots, \tilde{X}_k(g)\},$$
$$\sigma(g)(\tilde{X}_i(g), \tilde{X}_j(g)) = \delta_{ij}.$$

Definition 1.1.28. Let $(M, \mathcal{D}, \sigma)$ be a sub-Riemannian manifold. The *horizontal gradient* if the unique operator

$$\nabla_{sr} : \mathcal{C}^\infty(M) \to \text{Vec}_H(M)$$

such that:

$$\sigma(m)(\nabla_{sr}\phi(m), v) = \langle d\phi(m), v\rangle \quad \forall m \in M, v \in \mathcal{D}(m), \phi \in \mathcal{C}^\infty(M).$$

If G is a nilpotent stratified Lie group we denote the sub-Riemennian gradient with the symbol ∇_G. With the above notation, one can verify that:

$$\nabla_G \phi = \sum_{i=1}^{k} (\tilde{X}_i \phi)\tilde{X}_i \quad \forall \phi \in \mathcal{C}^\infty(G),$$

where the $\tilde{X}_i$'s are the left-invariant vector fields, orthonormal for σ, associated to the basis of $\mathfrak{g}_1$ that we fixed.

We just mention that on sub-Riemannian manifolds it is possible to define an *intrinsic volume* μ_{sr}, which permits also to define a sub-Riemannian divergence, setting as usual:

$$\mathcal{L}_{\mathsf{X}}\mu_{sr} = (\text{div}_{sr} \mathsf{X})\mu_{sr}, \quad \forall \, \mathsf{X} \in \text{Vec}(M)$$

and consequently a sub-Riemannian Laplacian (or *sub-Laplacian)*:

$$\Delta_{sr}\phi := \text{div}_{sr}(\nabla_{sr}(\phi)), \quad \forall \phi \in \mathcal{C}^\infty(M).$$

Notation 6. If G is a stratified group as above, we denote the sub-Laplacian with the symbol $\widetilde{\Delta_G}$.

We conclude giving a simple expression for the sub-Laplacian on stratified, unimodular[1] Lie groups.

Proposition 1.1.29. *Let G be a unimodular, stratified Lie group with stratification $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_r$ as above. Fix a basis $\{X_1, \ldots, X_k\}$ of $\mathfrak{g}_1$ and equip G with a sub-Riemannian structure $(G, \mathcal{D}, \sigma)$ such that vector fields $\tilde{X}_1, \ldots, \tilde{X}_k$ are orthonormal for σ. Then:*

$$\widetilde{\Delta_G} = \sum_{i=1}^{k} \tilde{X}_i^{\,2}.$$

1.1.6 From the Lie algebra to exponential coordinates

In this section we follow [7].

Definition 1.1.30. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$ we define:

$$X * Y := \log_G(e^X e^Y)$$

Remark 1.1.31. The map $* : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is well defined on a small neighborhood of 0. Fixing a system of coordinates on $\mathfrak{g}$, this map permits to reconstruct G with its operation, up to Lie group isomorphism, at least locally around the identity 1_G.

Proposition 1.1.32 (BCH formula)**.** *In the above setting it holds:*

$$X * Y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[\sum_{\substack{p_i+q_i>0 \\ 1\le i\le n}} \frac{\left(\sum_{i=1}^n p_i + q_i\right)^{-1}}{p_1!q_1!...p_n!q_n!} (\mathrm{ad}(X))^{p_1}(\mathrm{ad}(Y))^{q_1}...(\mathrm{ad}(X))^{p_n}(\mathrm{ad}(Y))^{q_n-1}Y \right],$$

where we recall that $[\mathrm{ad}(X)]Y = [X, Y]$. If $q_n = 0$ the last factor is $(\mathrm{ad}(X))^{p_n-1}X$. If $q_n > 1$ or $q_n = 0$ and $p_n > 1$, then the summand is automatically zero.

Expliciting the first terms, we find:

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\Big([X, [X, Y]] - [Y, [X, Y]]\Big) + ... \tag{1.3}$$

Notice that the sum on the right hand side is finite if G is nilpotent.

The BCH formula is useful to express the group operation on a neighborhood of 1_G via the exponential map. We may ask if, under some more assumptions, is possible to extract *global* informations with the same procedure. The answer is positive if the Lie group is nilpotent, connected and simply connected, which we assume from now on.

Theorem 1.1.33. *Let G be a connected and simply connected nilpotent Lie group and let $\mathfrak{g}$ be its Lie algebra. Then:*

[1]See Section 1.2.1.

(i) $\exp_G$ *is a smooth diffeomorphism;*

(ii) The BCH formula holds for all $X, Y \in \mathfrak{g}$.

The above proposition permits to identify an n-dimensional nilpotent Lie group with a corresponding Lie group $(\mathbb{R}^n, \star)$ as we now describe. Choose a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$, and define the coordinate map:

$$\tilde{\psi} : \mathbb{R}^n \to \mathfrak{g}, \quad \tilde{\psi}(x_1, \ldots, x_n) = x_1 X_1 + \cdots + x_n X_n.$$

Assuming that G is connected and simply connected, $\exp_G$ is a global diffeomorphism, hence we can identify $\exp_G(X) \in G$ with the corresponding $X \in \mathfrak{g}$. Finally, we obtain a system of coordinates coordinates (i.e a global parametrization) for G letting:

$$\psi : \mathbb{R}^n \to G, \quad \psi = \exp_G \circ \tilde{\psi}.$$

Since the coordinate map ψ is **global**, we can identify $(G, \cdot)$ with $(\mathbb{R}^n, \star)$, where, letting $x, y \in \mathbb{R}^n$ we set:

$$x \star y := \psi^{-1}(\psi(x)\psi(y)). \tag{1.4}$$

Remark 1.1.34. Under the above hypotheses, G can be covered with a single chart (G, ψ^{-1}), so it is in particular an *analytic* manifold. In such coordinates, the multiplication is an analytic map (thanks to BCH formula), hence G is also an analytic Lie group and $\exp_G$ is an analytic diffeomorphism.

Definition 1.1.35. The above defined coordinates are called *exponential coordinates* or *canonical coordinates of the first kind.*

Notation 7. From now on, with the expression *nilpotent Lie group* we will indicate a connected, simply connected nilpotent Lie group.

Together with the group operation, it is useful to express also the conjugation via $\exp_G$. From Proposition 1.1.18 we have:

$$C_g(e^Y) = e^{\mathrm{Ad}_g(Y)} \quad \forall g \in G, \forall Y \in \mathfrak{g},$$

so in particular, taking $g = e^X$ we have : $e^X e^Y e^{-X} = e^{(\mathrm{Ad}_{e^X}(Y))}$ for all $Y \in \mathfrak{g}$. Moreover, from Corollary 1.1.24:

$$\mathrm{Ad}_{e^X}(Y) = e^{\mathrm{ad}(X)}Y = \sum_{k=0}^{\infty} \frac{1}{k!}[\mathrm{ad}(X)]^k Y \quad \forall X, Y \in \mathfrak{g}.$$

In conclusion:

$$e^X e^Y e^{-X} = e^{C(X,Y)} \quad \forall X, Y \in \mathfrak{g}, \text{ where } C(X,Y) = \sum_{k=0}^{\infty} \frac{1}{k!}[\underbrace{X, ..., [X}_{k \text{ times}}, Y]].$$

1.1.7 Exponential coordinates on N_4

Let's realize the above construction when $G = N_4$. We consider a 6-dimensional Lie algebra, stratified as described in Section 1.1.1 , therefore we will identify $N_4 \cong (\mathbb{R}^6, \star)$ with operation $\star$ that we now determine. With the notation introduced in Section 1.1.1 consider the basis $\{X_1, X_2, X_3, X_{12}, X_{23}, X_0\}$, where:

$$\begin{gathered} X_1 = E_{1,2},\ X_2 = E_{2,3},\ X_3 = E_{3,4} \\ X_{12} = E_{1,3},\ X_{23} = E_{2,4} \\ X_0 = E_{1,4} \end{gathered}$$

so that the only nontrivial Lie brackets are:

$$\begin{aligned} [X_1, X_2] &= X_{12}, \quad [X_2, X_3] = X_{23} \\ [X_1, X_{23}] &= X_0, \quad [X_{12}, X_3] = X_0. \end{aligned}$$

As explained above, we let $X = x_1 X_1 + \cdots + x_0 X_0,\ X' = x_1' X_1 + \cdots + x_0' X_0$ and we compute:

$$\begin{aligned} &[X, X'] = (x_1 x_2' - x_2 x_1') X_{12} + (x_2 x_3' - x_3 x_2') X_{23} + (x_1 x_{23}' - x_{23} x_1' + x_{12} x_3' - x_3 x_{12}') X_0 \\ &[X, [X, X']] - [X', [X, X']] = \alpha(X, X') X_0, \end{aligned}$$

where $\alpha(X, X') := (x_1 - x_1')(x_2 x_3' - x_3 x_2') - (x_3 - x_3')(x_1 x_2' - x_2 x_1')$. According to (1.3), we find:

$$\begin{aligned} X * X' = &(x_1 + x_1') X_1 \ + (x_2 + x_2') X_2 \ + (x_3 + x_3') X_3 \\ &+ \left(x_{12} + x_{12}' + \frac{1}{2}(x_1 x_2' - x_2 x_1') \right) X_{12} + \left(x_{23} + x_{23}' + \frac{1}{2}(x_2 x_3' - x_3 x_2') \right) X_{23} \\ &+ \left(x_0 + x_0' + \frac{1}{2}(x_1 x_{23}' - x_{23} x_1' + x_{12} x_3' - x_3 x_{12}') + \frac{1}{12}\alpha(X, X') \right) X_0. \end{aligned}$$

As in (1.4), we identify our group with $(\mathbb{R}^6, \star)$, where:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_{12} \\ x_{23} \\ x_0 \end{pmatrix} \star \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_{12}' \\ x_{23}' \\ x_0' \end{pmatrix} := \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' \\ x_{12} + x_{12}' + \frac{1}{2}(x_1 x_2' - x_2 x_1') \\ x_{23} + x_{23}' + \frac{1}{2}(x_2 x_3' - x_3 x_2') \\ x_0 + x_0' + \frac{1}{2}(x_1 x_{23}' - x_{23} x_1' + x_{12} x_3' - x_3 x_{12}') + \frac{1}{12}\alpha(X, X') \end{pmatrix}$$

With this operation we have $1_{N_4} = 0$ and $(x_1, x_2, x_3, x_{12}, x_{23}, x_0)^{-1} = -(x_1, x_2, x_3, x_{12}, x_{23}, x_0)$.

Denoting $\partial_i = \partial/\partial_{x_i}$, a basis of left-invariant vector fields (that we denote X_i in place of

$\tilde{X}_i$) in these coordinates is:

$$\begin{aligned} X_1 &= \partial_1 - \frac{1}{2}x_2\partial_{12} + \left(-\frac{1}{2}x_{23} + \frac{1}{12}x_2x_3\right)\partial_0, \\ X_2 &= \partial_2 + \frac{1}{2}x_1\partial_{12} - \frac{1}{2}x_3\partial_{23} - \frac{1}{6}x_1x_3\partial_0, \\ X_3 &= \partial_3 + \frac{1}{2}x_2\partial_{23} + \left(\frac{1}{2}x_{12} + \frac{1}{12}x_1x_2\right)\partial_0, \\ X_{12} &= \partial_{12} - \frac{1}{2}x_3\partial_0, \\ X_{23} &= \partial_{23} + \frac{1}{2}x_1\partial_0 \\ X_0 &= \partial_0. \end{aligned}$$

We can write the *horizontal* distribution $\mathcal{D} := \operatorname{span}\{X_1, X_2, X_3\}$ in terms of differential forms as:

$$\mathcal{D} = \bigcap_{i=1}^{3} \ker(\omega_i),$$

where

$$\begin{aligned} \omega_1 &:= x_2dx_1 - x_1dx_2 + 2dx_{12}, \\ \omega_2 &:= x_3dx_2 - x_2dx_3 + 2dx_{23}, \\ \omega_3 &:= 3x_{23}dx_1 - 3x_{12}dx_3 + x_3dx_{12} - x_1dx_{23} + 6dx_0. \end{aligned}$$

1.1.8 Malcev coordinates

We now describe a different coordinate systems, that we call *Malcev coordinates*, again following [7]. The advantage of working in Malcev coordinates is that group operations have, in general, simpler expressions. As before, we will identify an n-dimensional nilpotent group G with $(\mathbb{R}^n, \bullet)$ via a coordinate map $\psi : \mathbb{R}^n \to G$, where:

$$x \bullet y = \psi^{-1}(\psi(x)\psi(y)) \quad x, y \in \mathbb{R}^n.$$

First of all, we introduce the notion of *Malcev basis.*

Definition 1.1.36. Let $\mathfrak{g}$ be a nilpotent n-dimensional Lie algebra and $\mathfrak{g}_1 \subseteq ... \subseteq \mathfrak{g}_k \subseteq \mathfrak{g}$ be subalgebras with $\dim \mathfrak{g}_j = m_j$. Let $\{X_1, ..., X_n\}$ be a basis for $\mathfrak{g}$; we call it a *weak Malcev basis for* $\mathfrak{g}$ *through* $\mathfrak{g}_1, ..., \mathfrak{g}_k$ if:

(i) For each m, $\mathfrak{h}_m := \operatorname{span}\{X_1, ..., X_m\}$ is a subalgebra of $\mathfrak{g}$,

(ii) $\mathfrak{h}_{m_j} = \mathfrak{g}_j$ for $j = 1, ..., k$.

If (i) is replaced with:

(i)' For each m, $\mathfrak{h}_m := \operatorname{span}\{X_1, ...X_m\}$ is an *ideal* of $\mathfrak{g}$, i.e. $[\mathfrak{g}_j, \mathfrak{g}] \subseteq \mathfrak{g}_j$,

such basis is called a *strong Malcev basis for* $\mathfrak{g}$ *through* $\mathfrak{g}_1, ..., \mathfrak{g}_k$.

Definition 1.1.37. If $k = 0$ we call $\{X_1, \ldots, X_n\}$ simply *weak (resp. strong) Malcev basis* for $\mathfrak{g}$.

Theorem 1.1.38. *Let $\mathfrak{g}$, $\mathfrak{g}_1 \subseteq ... \subseteq \mathfrak{g}_k \subseteq \mathfrak{g}$ as above. There exists a weak Malcev basis for $\mathfrak{g}$ through $\mathfrak{g}_1, ..., \mathfrak{g}_k$. If each $\mathfrak{g}_j$ is an ideal of $\mathfrak{g}$, there exists also a strong Malcev basis through $\mathfrak{g}_1, ..., \mathfrak{g}_k$.*

Before showing how to obtain Malcev coordinates with the use of Malcev bases, we introduce the notion of *polynomial map.*

Definition 1.1.39. Let V, W be vector spaces. A map $f : V \to W$ is *polynomial* if its components are polynomial in one (hence any) pair of bases. We call f a *polynomial diffeomorphism* if it is invertible and f, f^{-1} are polynomial.

Definition 1.1.40. Let G be a nilpotent Lie group. A map $\phi : G \to G$ is called *polynomial diffeomorphism* if $\log_G \circ \phi \circ \exp_G : \mathfrak{g} \to \mathfrak{g}$ is a polynomial diffeomorphism. Moreover, a *polynomial coordinate map* for G is a polynomial diffeomorphism $\psi : \mathbb{R}^n \to G$, where we identify G and $\mathfrak{g}$ by $\exp_G$.

Theorem 1.1.41. *Let $\{X_1, ..., X_n\}$ be a strong Malcev basis for a nilpotent Lie algebra $\mathfrak{g}$ and G be the associated Lie group. Define the map:*

$$\psi : \mathbb{R}^n \to G, \quad s \mapsto \psi(s) = e^{s_1 X_1} \dots e^{s_n X_n} = e^{s_1 X_1 * \dots * s_n X_s}.$$

We have:

(i) $\psi(s) = \exp_G(\sum_{j=1}^n P_j(s) X_j)$, *where the P_j's are polynomial maps;*

(ii) *For each $j = 1, ..., k$, we have $P_j(s) = s_j +$ (polynomial in $s_{j+1}, ..., s_n$) ;*

(iii) *The map $\log_G \circ \psi$ is a polynomial diffeomorphism of $\mathbb{R}^n$ with polynomial inverse;*

(iv) *If $\mathfrak{g}_k = \mathrm{span}\{X_1, ..., X_k\}$ and $G_k = \exp_G(\mathfrak{g}_k)$, then $G_k = \exp_G(\mathbb{R}X_1) ... \exp_G(\mathbb{R}X_k)$.*

Definition 1.1.42. Malcev coordinates are also called *canonical coordinates of the second kind.*

Properties (iii) and (iv) remain true also if $\{X_1, \dots, X_n\}$ is a weak Malcev basis.

We have the following relation between coordinates of first and second kind:

Proposition 1.1.43. *Let $\mathcal{X} = \{X_1, \dots, X_n\}$ be basis for a nilpotent Lie algebra $\mathfrak{g}$ and let G be the associated nilpotent Lie group, which we equip with exponential coordinates associated to $\mathcal{X}$. Assume that $\mathcal{X}$ is either a weak or strong Malcev basis.*

(i) *There exists a polynomial diffeomorphism that permits to pass from Malcev to exponential coordinates. Its Jacobian determinant is identically 1. The same holds if we consider any of the two kind of coordinates associated to a different basis of $\mathfrak{g}$, but the constant may differ from 1.*

(ii) *In either exponential or Malcev coordinates, the multiplications L_g, R_g have determinant of Jacobian identically equal to 1 for all $g \in G$.*

(iii) *Let H be a normal subgroup compatible with the basis, i.e. $H = \exp_G(\mathfrak{h})$, $\mathfrak{h} = \mathrm{span}\{X_1, \dots, X_k\}$ for some $k \leq n$. Equipping H with the restriction of the coordinates on G, the conjugation C_g, $g \in G$, has Jacobian determinant identically equal to 1.*

1.1.9 Malcev coordinates in N_4

Notation 8. In the following, we use BCH formula to write e^{X+Y} in place of $e^X e^Y$ whenever $X, Y \in \mathfrak{g}$ commute.

With the notation of the previous section, we let:

$$\begin{gathered}\mathfrak{g}_1 := (\mathfrak{n}_4)_3 = \{X_0\}\\ \mathfrak{g}_2 := \operatorname{span}\{\mathfrak{g}_1, (\mathfrak{n}_4)_2\} = \operatorname{span}\{X_0, X_{12}, X_{23}\}\\ \mathfrak{g}_3 := \operatorname{span}\{X_0, X_{12}, X_{23}, X_2\}\\ \mathfrak{g}_4 := \mathfrak{n}_4.\end{gathered}$$

A strong Malcev basis through $\mathfrak{g}_1, \dots, \mathfrak{g}_4$ for the Lie algebra $\mathfrak{n}_4$ is

$$\mathcal{X} = \{X_0, X_{12}, X_{23}, X_2, X_1, X_3\}.$$

The coordinate map of the theorem reads

$$\psi(x_0, x_{12}, x_{23}, x_2, x_1, x_3) = e^{x_0 X_0 + x_{12} X_{12} + x_{23} X_{23} + x_2 X_2} e^{x_1 X_1 + x_3 X_3}. \tag{1.5}$$

Letting $x = (x_1, x_2, x_3, x_{12}, x_{23}, x_0)$ indicate the generic element of $\mathbb{R}^6$ we define:

$$x \bullet x' = \psi^{-1}(\psi(x)\psi(x')), \tag{1.6}$$

and we identify our group with $(\mathbb{R}^6, \bullet)$, where:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_{12} \\ x_{23} \\ x_0 \end{pmatrix} \bullet \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_{12}' \\ x_{23}' \\ x_0' \end{pmatrix} := \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' \\ x_{12} + x_{12}' + x_1 x_2' \\ x_{23} + x_{23}' - x_3 x_2' \\ x_0 + x_0' + x_1 x_{23}' - x_3 x_{12}' - x_1 x_3 x_2' \end{pmatrix}.$$

Again we have $1_{N_4} = 0$ and

$$(x_1, x_2, x_3, x_{12}, x_{23}, x_0)^{-1} = -(x_1, x_2, x_3, x_{12} - x_1 x_2, x_{23} + x_2 x_3, x_0 - x_1 x_{23} + x_3 x_{12} - x_1 x_2 x_3).$$

Again with a small abuse of notation, a basis of left-invariant vector fields with this choice of coordinates is:

$$\begin{gathered} X_1 = \partial_1, \quad X_2 = \partial_2 + x_1 \partial_{12} - x_3 \partial_{23} - x_1 x_3 \partial_0, \quad X_3 = \partial_3 \\ X_{12} = \partial_{12} - x_3 \partial_0, \quad X_{23} = \partial_{23} + x_1 \partial_0 \\ X_0 = \partial_0. \end{gathered}$$

We can write $\mathcal{D} := \operatorname{span}\{X_1, X_2, X_3\} = \bigcap_{i=1}^3 \ker(\tilde{\omega}_i)$, where:

$$\begin{aligned} \tilde{\omega}_1 &:= -x_1 dx_{23} + dx_0, \\ \tilde{\omega}_2 &:= x_3 dx_{12} + dx_0, \\ \tilde{\omega}_3 &:= x_1 x_3 dx_2 + dx_0. \end{aligned}$$

Let us compute explicitly the change of variable between exponential and Malcev coordinates. Using the BCH we have:

$$\exp(x_0X_0 + \cdots + x_2X_2)\exp(x_1X_1 + x_3X_3) =$$
$$= \exp\left(\sum_{i=1}^{3} x_iX_i + \left(x_{12} - \frac{x_2x_1}{2}\right)X_{12} + \left(x_{23} + \frac{x_2x_3}{2}\right)X_{23} + \left(x_0 + \frac{x_{12}x_3 - x_{23}x_1}{2} - \frac{x_1x_2x_3}{6}\right)X_0\right). \tag{1.7}$$

Inverting the above equality we deduce how to pass from exponential to Malcev coordinates. Denoting $x := (x_1, .., x_0)$ and letting $\Psi : \mathbb{R}^6 \to \mathbb{R}^6$,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_{12} \\ x_{23} \\ x_0 \end{pmatrix} = x \mapsto \Psi(x) := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_{12} + \frac{x_1x_2}{2} \\ x_{23} - \frac{x_2x_3}{2} \\ x_0 + \frac{x_1x_{23}}{2} - \frac{x_3x_{12}}{2} - \frac{x_1x_2x_3}{3} \end{pmatrix},$$

we find

$$\Psi(x \star x') = \Psi(x) \bullet \Psi(x') \quad \forall x, x' \in \mathbb{R}^6.$$

1.2 Nilpotent Lie groups and integration

In this section, we follow [7] and [8]. With the expression *locally compact group* we will indicate a topological, Hausdorff group that is also locally compact. With the term *Radon measure* we will indicate a Borel measure that is finite on compact sets, outer regular on Borel sets and inner regular on open sets. The symbol $\mathcal{C}_c^+(G)$ will denote continuous functions from G to $[0, \infty)$, compactly supported in G.

1.2.1 Haar measure

Definition 1.2.1. Let G be a locally compact group. A *left Haar measure* on G is a Radon measure verifying:

$$\mu(gE) = \mu(E) \quad \text{for all } E \text{ borelian subset of } G,\ \forall g \in G.$$

Proposition 1.2.2. *Let G be a locally compact group and μ a Radon measure on G. Such μ is a left Haar measure if and only if*

$$\int_G f(gt)d\mu(t) = \int_G f(t)d\mu(t) \quad \forall f \in \mathcal{C}_c^+(G),\ \forall g \in G.$$

Theorem 1.2.3. *Let G be a locally compact group. There exists a left Haar measure on G; such measure is unique up to multiplication for a positive constant.*

The construction of the Haar measure for a general locally compact group G is complicated. However, in the case of a Lie group a left Haar measure can be recovered easily: it is enough to fix an $n-$form α on the Lie algebra (where $n = \dim(G)$) and define a *left-invariant* volume form ω on G letting:

$$\omega(g) := (L_{g^{-1}})^*\alpha \quad \forall g \in G.$$

This permits to obtain a functional acting on $C_c(G)$ by $f \mapsto \int_G f d\omega$, which corresponds to a Radon measure on G by the Riest representation theorem.

Another possibility to recover the Haar measure is to fix an inner product on the Lie algebra $\sigma(1_G) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ and use it to define a left-invariant 2-form on G, letting again:

$$\sigma(g)(v, w) := \sigma(1_G)((L_{g^{-1}})_{*,g}(v), (L_{g^{-1}})_{*,g}(w)) \quad \forall g \in G, \ \forall v, w \in T_g G.$$

A Haar measure is obtained considering the associated Riemannian volume V_σ and proceding as in the previous case.

On a nilpotent Lie group the Haar measure can be obtained explicitly using the coordinate maps that we discussed in the previous sections. Namely, it can be *transported* by the coordinate maps if we fix Lebesgue measure on $\mathbb{R}^n$, as we now describe (see [7] for the proof).

Theorem 1.2.4. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension n and G be the associated nilpotent Lie group.*

(i) The push forward through the map $\exp_G : \mathfrak{g} \to G$ of the Lebesgue measure on $\mathfrak{g}$ is a left Haar measure on G.

(ii) The push forward through any polynomial coordinate map $\psi : \mathbb{R}^n \to G$ of the Lebesgue measure on $\mathbb{R}^n$ is a left Haar measure on G.

(iii) Fix a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ and let $\tilde{\psi} : \mathbb{R}^n \to \mathfrak{g}$ define coordinates on $\mathfrak{g}$ associated to such basis (i.e. $\tilde{\psi}(t) = \sum_i t_i X_i$). Let:

- *m be the Lebesgue measure on $\mathbb{R}^n$*
- *$\mu = \tilde{\psi}_*(m)$, measure on $\mathfrak{g}$*
- *$\mu_1 = (\exp_G)_*(\mu)$ as in (i)*
- *$\nu = \psi_*(m)$ as in (ii).*

We have:

$$\nu = |\det J(\psi^{-1} \circ \exp_G \circ \tilde{\psi})(0)| \mu_1.$$

Of course, all the results presented above can be adapted to define a *right Haar measure*, which shares the same properties as the left one. We now discuss some conditions under which a left Haar measure can be also a right Haar measure.

Consider again a locally compact group G with a left Haar measure μ. Fixing $g \in G$, we let

$$\mu_g(E) := \mu(Eg),$$

where E is any Borel subset of G. Such μ_g is again a left Haar measure, hence by Theorem 1.2.3 there must exist $\Delta(g) \in (0, \infty)$ such that

$$\mu_g = \Delta(g)\mu.$$

In particular, again by Theorem 1.2.3, $\Delta(g)$ does not depend on μ, hence we have a well defined map $\Delta : G \to (0, \infty)$, that we call *modular function of G*.

Proposition 1.2.5. *The map Δ is a continuous homomorphism from G to $(0,\infty)$, i.e. $\Delta(gg') = \Delta(g)\Delta(g')$. For all $f \in L^1(G, d\mu)$ it holds:*

$$\int_G f(tg)d\mu(t) = \Delta(g^{-1}) \int_G f(t)d\mu(t).$$

Definition 1.2.6. A locally compact group G is called *unimodular* if $\Delta \equiv 1$ on G, i.e. if any left Haar measure on G is also a right Haar measure.

Proposition 1.2.7. *Let G be a connected Lie group, we have:*

$$\Delta(g) = \det(\mathrm{Ad}_{g^{-1}}) \quad \forall g \in G.$$

Corollary 1.2.8. From Proposition 1.1.43-(*ii*) and Proposition 1.2.7 we deduce that any nilpotent Lie group is unimodular.

We recall an useful formula related with the modular function.

Proposition 1.2.9. *Let G be a locally compact group, μ a left-invariant measure on G. We have:*

$$d\mu(g^{-1}) = \Delta(g^{-1})d\mu(g).$$

We deduce that, if G is unimodular, the change of variable of integration $g \mapsto g^{-1}$ reads:

$$\mu(g^{-1}) = \mu(g).$$

Finally, we give the definition of *group convolution.*

Definition 1.2.10. Let G be a locally compact group and μ a left Haar measure on G. For $\phi, \psi \in L^1(G; d\mu)$ we define:

$$\phi * \psi(g) := \int_G \phi(t)\psi(t^{-1}g)d\mu(t).$$

Using Fubini's theorem (see [8]) it is possible to prove that:

$$\|\phi * \psi\|_{L^1(G;d\mu)} \leq \|\phi\|_{L^1(G;d\mu)}\|\psi\|_{L^1(G;d\mu)}.$$

1.2.2 Invariant measure on cosets

At some point we will be interested in endowing the space of cosets $H\backslash G$, where H is a closed subgroup of G, with a measure that is invariant under the action of G. This will be always possible for nilpotent Lie groups and for unimodular groups in general.

Let G be a locally compact group, $H \subseteq G$ a closed subgroup and indicate with dg, dh a right Haar measure on G, H respectively. Let $p : G \to H\backslash G$ the projection $g \mapsto Hg$. Equipping $H\backslash G$ with the quotient topolgy, define the map:

$$P : \mathcal{C}_c(G) \to \mathcal{C}_c(H\backslash G), \quad P\phi(Hg) := \int_H \phi(hg)dh.$$

Proposition 1.2.11. *For all $f \in \mathcal{C}_c(H\backslash G)$ there exists $\phi \in \mathcal{C}_c(G)$ such that $P\phi = f$, supp$(f) = p($supp$(\phi))$, and $\phi \geq 0$ if $f \geq 0$.*

Theorem 1.2.12. *Let G be a locally compact Lie group, $H \subseteq G$ a closed subgroup. There exists a G-invariant measure μ on $H\backslash G$ if and only if the restriction of the modular function Δ_G of G to H is the modular function of H, i.e:*

$$\Delta_G|_H = \Delta_H. \tag{1.8}$$

Such measure is unique up to a positive multiplicative constant and chosing properly this factor:

$$\int_G \phi(g)dg = \int_{H\backslash G} P\phi(Hg)d\mu(Hg) = \int_{H\backslash G}\left(\int_H \phi(hg)dh\right)d\mu(Hg) \quad \forall \phi \in \mathcal{C}_c(G). \tag{1.9}$$

Corollary 1.2.13. If G is a nilpotent Lie group there exists a G-invariant measure on $H\backslash G$. Indeed, (1.8) is verified since G is unimodular.

As the following Proposition shows, the formula (1.9) has a *transitive* property (see [9], Chapter 3 for more details).

Proposition 1.2.14. *Let K, H be closed subgroups of G such that $K \subseteq H \subseteq G$. Indicating by $\mu_{H,G}$ a G-invariant measure on H and by $\mu_{K,H}$ an H-invariant measure on K, the definition:*

$$\int_{K\backslash G} f(Kg)d\mu_{K,G} := \int_{H\backslash G}\left(\int_{K\backslash H} f(Khg)d\mu_{K,H}\right)d\mu_{H,G} \quad \forall f \in \mathcal{C}_c(G/K).$$

determines a G-invariant measure $\mu_{K,G}$ on $K\backslash G$.

We now focus on the case when G is a nilpotent Lie group, following [7]. In this case, making use of a Malcev basis, the presented construction can be explicited in coordinates.

Proposition 1.2.15. *Let G be an n-dimensional nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a k-dimensional subalgebra and consider a weak Malcev basis $\{X_1, \ldots, X_n\}$ for $\mathfrak{g}$ through $\mathfrak{h}$. Let $H := \exp_G(\mathfrak{h})$. Denoting $m = n - k$, the map $\Psi : \mathbb{R}^m \to H\backslash G$ defined by:*

$$\Psi(t_1, \ldots, t_m) := H(\exp_G(t_1 X_{k+1}) \ldots \exp_G(t_m X_n))$$

is an analytic diffeomorphism. The push-forward of the Lebesgue measure on $\mathbb{R}^m$ through Ψ defines a G-invariant measure on $H\backslash G$.

We can also compute explicitly integrals with respect to the invariant measure on $H\backslash G$ thanks to the following proposition.

Proposition 1.2.16. *In the setting of Theorem 1.2.12, let again $p : G \to H\backslash G$ be the projection on cosets and fix a function $\beta \in \mathcal{C}(G)$ such that:*

(i) $\beta \geq 0$

(ii) For any compact subset $K \subseteq H\backslash G$ also $supp(\beta) \cap p^{-1}(K)$ is compact

(iii) $\int_H \beta(hg)dh = 1 \quad \forall g \in G.$

Then we have

$$\int_{H\backslash G} \phi(Hg)d\mu(Hg) = \int_G \beta(g)(\phi \circ p)(g)dg \quad \forall \phi \in \mathcal{C}_c(H\backslash G).$$

1.3 Symplectic manifolds and coadjoint orbits

1.3.1 Symplectic and Poisson manifolds

In this section we follow [1]. We refer to it for the omitted proofs.

Definition 1.3.1. A *symplectic manifold* is a pair (M, ω), where M is a smooth manifold and ω is a 2- form, called *symplectic form*, verifying:

(i) ω is nondegenerate;

(ii) ω is a *closed*, i.e. $\mathrm{d}\omega = 0$.

Definition 1.3.2. Let $(M, \omega_M), (N, \omega_N)$ be symplectic manifolds. A smooth map $\psi : M \to N$ is called *symplectic* if:

$$\psi^* \omega_N = \omega_M.$$

Remark 1.3.3. By the assumption of nondegeneracy of ω, every symplectic manifold is necessarily even-dimensional.

The symplectic structure permits to associate to any function $h \in \mathcal{C}^\infty(M)$ a vector field $\vec{h}$ letting $i_{\vec{h}}\omega = -dh$, i.e:

$$\omega(\cdot, \vec{h}) = dh.$$

Definition 1.3.4. The above defined vector field $\vec{h}$ is called the *Hamiltonian vector field* associated to h.

Definition 1.3.5. A *Poisson manifold* is a pair $(M, \{\cdot,\cdot\})$, where the *Poisson brackets*

$$\{\cdot,\cdot\} : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M),$$

are such that for all $h, f, g \in \mathcal{C}^\infty(M)$:

(I) $\{h, g\} = -\{g, h\}$ (*skew-symmetry*);

(II) $\{\cdot,\cdot\}$ is bilinear;

(III) $\{h, fg\} = f\{h, g\} + g\{h, f\}$ (*Leibnitz rule*);

(IV) $\{h, \{f, g\}\} + \{f, \{g, h\}\} + \{g, \{h, f\}\} = 0$ (*Jacobi identity*).

Definition 1.3.6. Let $(M, \{\cdot,\cdot\}_M), (N, \{\cdot,\cdot\}_N)$ be Poisson manifolds. A smooth map $\psi : M \to N$ is called *Poisson* if:

$$\{h, g\}_N \circ \psi = \{h \circ \psi, g \circ \psi\}_M \quad \forall h, g \in \mathcal{C}^\infty(N).$$

Remark 1.3.7. Every symplectic manifold is also a Poisson manifold. Indeed, with the above notation, we can define

$$\{h, g\} := \omega(\vec{h}, \vec{g}).$$

Using the properties of the symplectic form and the definition of Hamiltonian vector field it is easy to verify that such $\{\cdot,\cdot\}$ is a Poisson bracket.

The following Proposition establishes a correspondence between Lie brackets on $\mathrm{Vec}(M)$ and Poisson brackets on $\mathcal{C}^\infty(M)$ on symplectic manifolds.

Proposition 1.3.8. *Let M be a* ***symplectic*** *manifold, $h, g \in \mathcal{C}^\infty(M)$. The vector field $[\vec{h}, \vec{g}] \in \mathrm{Vec}(M)$ is Hamiltonian, and :*

$$[\vec{h}, \vec{g}] = \overrightarrow{\{h, g\}}.$$

On Poisson manifolds, the notion of Hamiltonian vector field can be recovered using the previous formulas. Indeed, fixed $h \in \mathcal{C}^\infty(M)$, we can define

$$D_h : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M), \quad D_h(g) := \{h, g\}.$$

Thanks to property (II) of Definition 1.3.5, such D_h is a derivation on $\mathcal{C}^\infty(M)$, hence there exists a unique vector field, that we denote again by $\vec{h}$, such that

$$\vec{h}(g) = \{h, g\} \quad \forall h, g \in \mathcal{C}^\infty(M).$$

Definition 1.3.9. The above defined vector field $\vec{h}$ is called *Poisson vector field* associated to h.

Proposition 1.3.10. *Let M be a* ***Poisson*** *manifold, $h, g \in \mathcal{C}^\infty(M)$. The vector field $[\vec{h}, \vec{g}] \in \mathrm{Vec}(M)$ is Poisson, and :*

$$[\vec{h}, \vec{g}] = \overrightarrow{\{h, g\}}.$$

Poisson manifolds are not symplectic in general. However, they can always be *foliated* with symplectic manifolds, as we now explain.

Definition 1.3.11. Let M be a Poisson manifold, the *characteristic distribution of Poisson structure $\mathscr{P}$* is defined as:

$$\mathscr{P}_m := \{\vec{h}(m) \mid h \in \mathcal{C}^\infty(M)\}.$$

The dimension of $\mathscr{P}$ at one point $m \in M$ is called *rank of Poisson structure at m.*

Remark 1.3.12. In general, the distribution $\mathscr{P}$ has not constant rank.

Definition 1.3.13. Fixing $m \in M$, we define the *symplectic leaf through m* as the set of points $n \in M$ such that

$$n = e^{t_d \vec{h}_d} \circ \cdots \circ e^{t_1 \vec{h}_1}(m)$$

for some $d \in \mathbb{N}, t_1 \ldots, t_d \in \mathbb{R}, \vec{h}_1, \ldots, \vec{h}_d \in \mathscr{P}$. We denote it with the symbol Σ_m.

Theorem 1.3.14. *Every Poisson manifold M is the disjoint union of its symplectic leaves. Every symplectic leaf $\Sigma = \Sigma_m$ is an immersed submanifold of M and*

$$T_m \Sigma_m = \mathscr{P}_m;$$

moreover, it is equipped with a symplectic form ω_Σ, such that the inclusion $\iota : \Sigma \to M$ is a Poisson map.

Remark 1.3.15. On each symplectic leaf Σ, the symplectic form ω_Σ is defined by:

$$(\omega_\Sigma)_m(\vec{h}(m), \vec{g}(m)) := \{h, g\}(m) \quad \forall h, g \in \mathcal{C}^\infty(M).$$

Definition 1.3.16. Let M be a Poisson manifold. A *Casimir* function on M is $c \in \mathcal{C}^\infty(M)$ such that $\{c, h\} = 0$ for all $h \in \mathcal{C}^\infty(M)$.

Remark 1.3.17. If c is a Casimir function, symplectic leaves are contained in the level sets of c.

1.3.2 Coadjoint action, Poisson structure on $\mathfrak{g}^*$

Following what we have done in Section 1.1.4 for the adjoint action, we define

$$\mathrm{Ad}^* : G \to GL(\mathfrak{g}^*), \quad g \mapsto (\mathrm{Ad}_{g^{-1}})^*.$$

We denote the differential at identity $\mathrm{ad}^* := (\mathrm{Ad}^*)_{*,1_G} : \mathfrak{g} \to L(\mathfrak{g}^*)$.

Proposition 1.3.18. *The map* Ad^* *defined above is a Lie group homomorphism. Moreover, it holds:*

$$\langle (\mathrm{ad}^*(X))(\eta), Y\rangle = \langle \eta, [Y, X]\rangle \ \forall \eta \in \mathfrak{g}^*, \ \forall X, Y \in \mathfrak{g}.$$

Again with a small abuse of notation:

Definition 1.3.19. The *coadjoint action* of G on the dual of its Lie algebra $\mathfrak{g}^*$ is the (*left*) Lie group action:

$$\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad (\mathrm{Ad}^*)_g(\eta) = (\mathrm{Ad}_{g^{-1}})^*(\eta), \quad \forall \eta \in \mathfrak{g}^*$$

where by definition

$$\langle (\mathrm{Ad}_{g^{-1}})^*(\eta), X\rangle = \langle \eta, \mathrm{Ad}_{g^{-1}}(X)\rangle \quad \forall \eta \in \mathfrak{g}^*, \forall X \in \mathfrak{g}.$$

We now show that on a Lie group G the dual of the Lie algebra $\mathfrak{g}^*$ can be equipped with a structure of Poisson manifold.

Considering $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$, at any $\eta \in \mathfrak{g}^*$ we have $df(\eta) \in T^*_\eta \mathfrak{g}^*$. Thanks to the identifications:

$$T^*_\eta \mathfrak{g}^* = (T_\eta \mathfrak{g}^*)^* = \mathfrak{g}^{**} = \mathfrak{g}.$$

we can think at $df(\eta)$ as an element of $\mathfrak{g}$. More precisely, we interpret $df(\eta)$ as the unique element of $\mathfrak{g}$ such that:

$$\langle \lambda, df(\eta)\rangle = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\eta + \varepsilon\lambda) \quad \forall \lambda \in \mathfrak{g}^*.$$

Definition 1.3.20. For $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$ we define the *Lie-Poisson brackets* as :

$$\{f, g\}(\eta) := \langle \eta, [df(\eta), dg(\eta)]\rangle,$$

where we interpreted df, dg as described above.

In view of what we proved in Section 1.3.1, we can *foliate* $\mathfrak{g}^*$ with symplectic leaves. It is possible to prove that these leaves coincide with the orbits of the coadjoint action that we have just defined.

Theorem 1.3.21. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. With the above defined* Lie-Poisson brackets*, the dual of the Lie algebra $\mathfrak{g}^*$ has the structure of a Poisson manifold; the symplectic leaves coincide with the orbits of the coadjoint action* Ad^*_g *of G on $\mathfrak{g}^*$.*

Therefore, with the notation introduced in the previous sections, for all $\eta \in \mathfrak{g}^*$ we have:

$$\mathcal{O}_\eta = \Sigma_\eta = \{\lambda \in \mathfrak{g}^* \mid \lambda = (\mathrm{Ad}^*)_g(\eta), \text{ for some } g \in G\}.$$

1.3.3 Radicals, polarizing subalgebras

Definition 1.3.22. Let $\eta \in \mathfrak{g}^*$ and denote $B_\eta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ the bilinear form defined by:

$$B_\eta(X, Y) := \langle \eta, [X, Y] \rangle \quad \forall X, Y \in \mathfrak{g}.$$

We define the *radical* of B_η as:

$$\mathfrak{r}_\eta := \{Y \in \mathfrak{g} \mid B_\eta(X, Y) = 0 \quad \forall X \in \mathfrak{g}\}.$$

Notice that B_η defines a skew-symmetric and nondegenerate bilinear form on $\mathfrak{g}/\mathfrak{r}_\eta$, so in particular this space is even dimensional.

Maximal isotropic subspaces for B_η have dimension $k = n - \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{r}_\eta)$. Fixed $\eta \in \mathfrak{g}^*$, one may ask if there exists any Lie subalgebra of $\mathfrak{g}$ that is also a maximal isotropic subspace for B_η.

Definition 1.3.23. Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\eta \in \mathfrak{g}^*$. A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called *polarizing* or *maximal subordinate* for η if it is also a maximal isotropic subspace for B_η.

Proposition 1.3.24. *Let $\mathfrak{g}$ be a nilpotent Lie algebra, $\eta \in \mathfrak{g}^*$. There always exists a polarizing subalgebra for η.*

For all $\eta \in \mathfrak{g}^*$ there is, in general, more the one polarizing subalgebra. However, one desirable property of $\mathfrak{h}$ is that $\mathfrak{r}_\eta \subseteq \mathfrak{h}$. A standard way to choose such $\mathfrak{h}$ is the following.

Proposition 1.3.25. *Let $\{X_1, \dots, X_n\}$ be a strong Malcev basis for $\mathfrak{g}$ and let $\mathfrak{g}_j = \operatorname{span}\{X_1, \dots, X_j\}, \quad \forall j = 1, \dots, n$. Fixed $\eta \in \mathfrak{g}^*$, let*

$$\eta_j := \eta|_{\mathfrak{g}_j}.$$

Then, $\mathfrak{h} = \mathfrak{r}_{\eta_1} + \cdots + \mathfrak{r}_{\eta_n}$ is a polarizing subalgebra for η.

The following lemma gives a property of the coadjoint action that will be useful later.

Lemma 1.3.26. *Let $\eta \in \mathfrak{g}^*$ and $\mathfrak{h} \subseteq \mathfrak{g}$ be a polarizing subalgebra for η, $H = \exp_G(\mathfrak{h})$. Letting $\mathfrak{h}^\perp$ indicate the orthogonal of $\mathfrak{h}$ in $\mathfrak{g}^*$, we have:*

$$(\mathrm{Ad}^*(H))(\eta) = \eta + \mathfrak{h}^\perp.$$

Before concluding, let us give another useful interpretation of the radical $\mathfrak{r}_\eta$. Fix $\eta \in \mathfrak{g}^*$ and notice that:

$$\begin{aligned}\mathfrak{r}_\eta &= \{Y \in \mathfrak{g} \mid B_\eta(X, Y) = 0 \ \forall X \in \mathfrak{g}\} = \{Y \in \mathfrak{g} \mid \langle \eta, [X, Y] \rangle = 0 \ \forall X \in \mathfrak{g}\} \\ &= \{Y \in \mathfrak{g} \mid (\mathrm{ad}^*(Y)(\eta))(X) = 0 \ \forall X \in \mathfrak{g}\} = \{Y \in \mathfrak{g} \mid \mathrm{ad}^*(Y)(\eta) = 0\}.\end{aligned}$$

Remembering that Ad^* is a Lie group homomorphism and Proposition 1.1.18, we deduce that

$$\mathrm{stab}_G(\eta) = \{g \in G \mid (\mathrm{Ad}^*)_g(\eta) = \eta\} = \exp_G(\mathfrak{r}_\eta) \quad \forall \eta \in \mathfrak{g}^*,$$

where we recall that $\mathrm{stab}_G(\eta)$ is the stabilizer of η under the coadjoint action of G on $\mathfrak{g}^*$.

Notation 9. We will denote also $R_\eta = \exp_G(\mathfrak{r}_\eta)$.

1.3.4 Coadjoint orbits on $\mathfrak{n}_4^*$: first method

In this section we give a first classification of coadjoint orbits for the group N_4. In the next Chapter we will give an alternative method to determine them, which will require less computations.

We develop this first computation in Malcev coordinates associated to the strong Malcev basis:

$$\mathcal{X} = \{X_0, X_{12}, X_{23}, X_2, X_1, X_3\}.$$

Again we denote $\psi : \mathbb{R}^6 \to N_4$ the coordinate map described in (1.5).

Fixing $x \in \mathbb{R}^6$, $Y = y_1 Y_1 + \cdots + y_0 Y_0 \in \mathfrak{n}_4$, the adjoint action reads:

$$\mathrm{Ad}_{\psi(x)}(Y) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ -x_2 y_1 + x_1 y_2 + y_{12} \\ -x_3 y_2 + x_2 y_3 + y_{23} \\ -x_{23} y_1 - x_1 x_3 y_2 + x_{12} y_3 - x_3 y_{12} + x_1 y_{23} + y_0 \end{pmatrix}.$$

Let now:

$$\mathcal{X}^* = \{X_0^*, X_{12}^*, X_{23}^*, X_2^*, X_1^*, X_3^*\}$$

be the dual basis of $\mathcal{X}$, and write $\eta = \eta_1 X_1^* + \cdots + \eta_0 X_0^*$ the generic element of $\mathfrak{n}_4^*$. We have:

$$(Ad^*)_{\psi(x)}(\eta) = \begin{pmatrix} \eta_1 + x_2\eta_{12} + (x_{23} + x_2 x_3)\eta_0 \\ \eta_2 - x_1\eta_{12} + x_3\eta_{23} - x_1 x_3 \eta_0 \\ \eta_3 - x_2\eta_{23} - (x_{12} - x_1 x_2)\eta_0 \\ \eta_{12} + x_3\eta_0 \\ \eta_{23} - x_1\eta_0 \\ \eta_0 \end{pmatrix}.$$

If $\eta_0 = 0$ we have different cases.

• If also $\eta_{12} = \eta_{23} = 0$ we find

$$\mathcal{O}_\eta = \{\eta\}$$

and the orbits of this type are all the singletons $\{(\eta_1, \eta_2, \eta_3, 0, 0, 0)\}$ with $\eta_1, \eta_2, \eta_3 \in \mathbb{R}$.

• Otherwise, if $\eta_{12} \neq 0$ or $\eta_{23} \neq 0$ we have planes:

$$\begin{aligned} \mathcal{O}_\eta &= \left\{(\eta_1 + x_2\eta_{12}, \eta_2 - x_1\eta_{12} + x_3\eta_{23}, \eta_3 - x_2\eta_{23}, \eta_{12}, \eta_{23}, 0) \mid x \in \mathbb{R}^6\right\} \\ &= \left\{(\eta_1 + t_1\eta_{12}, \eta_2 + t_2, \eta_3 - t_1\eta_{23}, \eta_{12}, \eta_{23}, 0) \mid (t_1, t_2) \in \mathbb{R}^2\right\}. \end{aligned} \tag{1.10}$$

More specifically, three possible cases occur:

- If $\eta_{12} \neq 0 = \eta_{23}$ the orbit is:

$$\mathcal{O}_\eta = \left\{(\eta_1 + t_1\eta_{12}, \eta_2 + t_2, \eta_3, \eta_{12}, 0, 0) \mid (t_1, t_2) \in \mathbb{R}^2\right\},$$

  and we can fix the representative $(0, 0, \eta_3, \eta_{12}, 0, 0)$.

- If $\eta_{23} \neq 0 = \eta_{12}$ the orbit is:

$$\mathcal{O}_\eta = \left\{(\eta_1, \eta_2 + t_2, \eta_3 - t_1\eta_{23}, 0, \eta_{23}, 0) \mid (t_1, t_2) \in \mathbb{R}^2\right\},$$

  and we can fix the representative $(\eta_1, 0, 0, 0, \eta_{23}, 0)$.

- If $\eta_{23} \neq 0 \neq \eta_{12}$ the orbit is described by (1.10) and we can fix the representative $(\eta_1, 0, \eta_3, \eta_{12}, \eta_{23}, 0)$.

If $\eta_0 \neq 0$, we can choose

$$x_1 = \frac{\eta_{23}}{\eta_0},\ x_3 = -\frac{\eta_{12}}{\eta_0},\ x_{12} = \frac{\eta_3}{\eta_0},\ x_{23} = -\frac{\eta_1}{\eta_0},$$

obtaining $\tilde{\eta} := (0, \gamma, 0, 0, \eta_0) \in \mathcal{O}_\eta$, where $\gamma = \eta_2 - \eta_{12}\eta_{23}/\eta_0$. Therefore we find:

$$\mathcal{O}_\eta = \mathcal{O}_{\tilde{\eta}} = \left\{\left(s_1, \gamma + \frac{t_1 t_2}{\eta_0}, s_2, t_1, t_2, \eta_0\right) \mid (t_1, t_2, s_1, s_2) \in \mathbb{R}^4\right\},$$

which is the product of a plane and an hyperboloid. We can fix the representative $(0, \gamma, 0, 0, 0, \eta_0)$.

Remark 1.3.27. To sum up, we have three types of orbits: points, planes, products of planes and hyperboloids.

Chapter 2

Representation theory

2.1 Introduction

Definition 2.1.1. A *unitary representation* of a Lie group G is a pair $(\mathcal{R}, \mathcal{H})$ where $\mathcal{H}$ is an Hilbert space and $\mathcal{R} : G \to \mathcal{U}(\mathcal{H})$ is a group homomorphism, i.e:

$$\mathcal{R}(g) \circ \mathcal{R}(g') = \mathcal{R}(gg') \quad \forall g, g' \in G.$$

The representation is called *one dimensional* if $\dim(\mathcal{H}) = 1$.

Remark 2.1.2. If $\mathcal{H} = \mathbb{C}$, the representation is one dimensional and takes value on $\mathcal{U}(\mathbb{C}) = \mathbb{S}^1$.

Notation 10. We will use the term *representation* to indicate a unitary representation.

We list here some more definitions.

Definition 2.1.3. Let $(\mathcal{R}, \mathcal{H})$ be a representation of a Lie group G. A subspace $V \subseteq \mathcal{H}$ is called *invariant* if $\mathcal{R}(g)V \subseteq V$ for all $g \in G$.

Definition 2.1.4. A representation $(\mathcal{R}, \mathcal{H})$ is called:

- *strongly continuous* if for any $v \in \mathcal{H}$ the function $G \ni g \mapsto \mathcal{R}(g)v \in \mathcal{H}$ is continuous;
- *irreducible* if the only invariant closed subspaces are trivial.

Definition 2.1.5. Let G be a Lie group, two representations $(\mathcal{R}_1, \mathcal{H}_1), (\mathcal{R}_2, \mathcal{H}_2)$ of G are called *equivalent* if there exists a unitary map $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that:

$$T \circ \mathcal{R}_1(g) = \mathcal{R}_2(g) \circ T \quad \forall g \in G.$$

2.2 The orbit method

The orbit method developed by A.A. Kirillov[1] permits to determine all the possible unitary, irreducible and continuous representations of a nilpotent Lie group G once the symplectic foliation of the dual of the Lie algebra $\mathfrak{g}^*$ is understood.

We start describing an alternative method for determinig the foliation of $\mathfrak{g}^*$, following the approach presented in [4], Appendix A.

[1]See [12], [13] for a detailed exposition of this theory.

2.2.1 Foliation of $\mathfrak{g}^*$

The method is based on the choice of a convenient system of coordinates on $\mathfrak{g}^*$ and it works independently from the choice of coordinates on G. The nilpotency of G is not required for the moment. As we will see, some of the results presented below will hold automatically if we choose exponential or Malcev coordinates on a nilpotent Lie group G.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Fix a basis of left-invariant vector fields $\tilde{X}_1, ..., \tilde{X}_n$ on G, and define $c_{ij}^k \in \mathbb{R}$, $i, j, k = 1, ..., n$ letting:

$$[\tilde{X}_i, \tilde{X}_j] = \sum_{k=1}^n c_{ij}^k \tilde{X}_k.$$

Notice that, due to left-invariance, the c_{ij}^k's are constant. On the cotangent bundle T^*G fix bundle coordinates $T^*G \ni (g, \lambda) \mapsto (x, p) \in \mathbb{R}^{2n}$ and define the functions

$$h_i(g, \lambda) := p \cdot \tilde{X}_i(x), \quad i = 1, \ldots, n,$$

where we have identified the vector fields $\tilde{X}_i$ with their expression in coordinates. Such functions h_i satisfy the following relation:

$$\{h_i, h_j\}(g, \lambda) = p \cdot [\tilde{X}_i, \tilde{X}_j](x) = \sum_{k=1}^n c_{ij}^k p \cdot \tilde{X}_k(x) = \sum_{k=1}^n c_{ij}^k h_k(g, \lambda) \quad \forall i, j = 1, \ldots, n.$$

Here we have used the fact that, again by left invariance, $h_i(g, \lambda) = \mathsf{h}_i((L_g)^*_{1_G}\lambda)$, where

$$\mathsf{h}_\mathsf{i} : \mathfrak{g}^* \to \mathbb{R}, \quad \mathsf{h}_\mathsf{i}(\eta) = \langle \eta, X_i \rangle,$$

together with the following[2]:

Proposition 2.2.1. *Let h_1, h_2 be two left-invariant functions on T^*G, i.e:*

$$h_i(g, \lambda) = \mathsf{h}_\mathsf{i}((L_g)^*_{1_G}\lambda),$$

where $\mathsf{h}_i : \mathfrak{g}^* \to \mathbb{R}$ is smooth. Then also $\{h_1, h_2\}$ is left-invariant , and:

$$\{h_1, h_2\}(g, \lambda) = \{\mathsf{h}_1, \mathsf{h}_2\}((L_g)^*_{1_G}\lambda),$$

where the last bracket is the Lie-Poisson bracket in $\mathfrak{g}^*$.

In particular, such h_i's are smooth functions T^*G that can be used as coordinates on $\mathfrak{g}^*$.

Remark 2.2.2. If we start from a basis $\mathcal{X} = \{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ and choose exponential or Malcev coordinates on G, then such h_i's are the coordinates associated to the dual basis $\mathcal{X}^*$ and the condition $\{h_i, h_j\} = c_{ij}^k h_k$ automatically holds.

In order to determine the coadjoint orbits in $\mathfrak{g}^*$, we want to find Casimir functions. We rewrite functions $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ as $f = f(h_1, \ldots, h_n)$ and the condition for being Casimir reduces to:

$$0 = \{f, h_j\} = \sum_{i=1}^n \frac{\partial f}{\partial h_i}\{h_i, h_j\} = \sum_{i,k=1}^n \frac{\partial f}{\partial h_i} c_{ij}^k h_k \quad \text{for } j = 1, \ldots, n. \tag{2.1}$$

[2] See [1], Chapter 18.

Indeed if (2.1) holds, for all $a \in \mathcal{C}^\infty(\mathfrak{g}^*)$ we have:

$$\{f, a\} = \sum_{j=1}^{n} \frac{\partial a}{\partial h_j}\{f, h_j\} = 0.$$

Recall now[3] that the Poisson vector field associated to a generic smooth function f, which we indicate again by $\vec{f}$, is defined by $\vec{f}(a) = \{f, a\}$ for all $a \in \mathcal{C}^\infty(\mathfrak{g}^*)$. But we have:

$$\{f, a\} = \sum_{i,j=1}^{n} \frac{\partial f}{\partial h_i}\frac{\partial a}{\partial h_j}\{h_i, h_j\} = \sum_{i,j,k=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k h_k \frac{\partial a}{\partial h_j},$$

hence we deduce

$$\vec{f} = \sum_{i,j,k=1}^{n} \frac{\partial f}{\partial h_i} c_{ij}^k h_k \partial_{h_j}.$$

In particular,

$$\vec{h_i} = \sum_{j,k=1}^{n} c_{ij}^k h_k \partial_{h_j}. \tag{2.2}$$

Remark 2.2.3. Fixed $\eta \in \mathfrak{g}^*$, the coadjoint orbit is $\mathcal{O}_\eta = \{e^{t_1\vec{h_1}} \circ \cdots \circ e^{t_n\vec{h_n}}(\eta) \mid t_i \in \mathbb{R}\}$.

2.2.2 One-dimensional representations, induced representatons

In this section we expose the orbit method, following [7] and [4], Appendix A.

Let G be a nilpotent Lie group. We first obtain a one dimensional representation of a particular Lie subgroup $H \subseteq G$ in the following way:

- fix an element $\eta \in \mathfrak{g}^*$;
- choose a *polarizing* subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Denote by $H = \exp_G(\mathfrak{h})$ the corresponding Lie subgroup;
- define the unitary representation $\chi_{\eta,\mathfrak{h}} : H \to \mathbb{S}^1$ on $\mathbb{C}$ letting:

$$\chi_{\eta,\mathfrak{h}}(\exp_G(X)) := e^{i\langle \eta, X\rangle} \quad \forall X \in \mathfrak{h}.$$

The above defined $\chi_{\eta,\mathfrak{h}}$ is actually a representation. Indeed given $X, Y \in \mathfrak{h}$, using the BCH formula we find:

$$\begin{aligned}\chi_{\eta,\mathfrak{h}}(\exp_G(X)\exp_G(Y)) &= \chi_{\eta,\mathfrak{h}}(\exp_G(X * Y)) = e^{i\langle \eta, X*Y\rangle} = \\ &= e^{i\langle \eta, X+Y\rangle} = \chi_{\eta,\mathfrak{h}}(\exp_G(X))\chi_{\eta,\mathfrak{h}}(\exp_G(Y)),\end{aligned}$$

where we use that $\mathfrak{h}$ is polarizing for η.

In order to *induce* a representation of G starting from the above described representation of H, we follow a general method.

Let $\chi : H \to \mathcal{U}(V)$ be a a unitary representation of a Lie subgroup of G on the Hilbert space V. We build a new Hilbert space W and a, so called, *induced* representation

[3] see Section 1.3.1.

$\mathcal{R} : G \to \mathcal{U}(W)$ as follows. We let the representation $\mathcal{R}$ act on measurable functions $\phi : G \to V$ such that:

$$\phi(hg) = \chi(h)(\phi(g)) \quad \forall h \in H, g \in G. \tag{2.3}$$

In particular, since χ is unitary, the quantity $\|\phi(g)\|_V$ is constant along the H-cosets Hg, $g \in G$, therefore we can define

$$\|\phi(Hg)\|_V := \|\phi(hg)\|_V \text{ for some (hence, any) } h \in H.$$

Since G is nilpotent, we know from Section 1.2.2 that there exists a right-invariant measure μ on $H\backslash G$, so we can restrict to the functions:

$$\phi : G \to V \text{ such that } \int_{H\backslash G} \|\phi(Hg)\|_V^2 d\mu(Hg) \ < \infty. \tag{2.4}$$

We finally let

$$W := \{\phi : G \to V \mid (2.3) \text{ and } (2.4) \text{ hold}\}.$$

Given $\phi_1, \phi_2 \in W$, also $\langle\phi_1(g), \phi_2(g)\rangle_V$ is constant along the H-cosets, so it is well defined the quantity $\langle\phi_1(Hg), \phi_2(Hg)\rangle_V$. The space W is Hilbert[4] for the scalar product:

$$\langle\phi_1, \phi_2\rangle_W := \int_{H\backslash G} \langle\phi_1(Hg), \phi_2(Hg)\rangle_V d\mu(Hg).$$

Finally, we define the *induced representation* $\mathcal{R} : G \to \mathcal{U}(W)$ with the right action:

$$[\mathcal{R}(g)\phi](g') := \phi(g'g) \quad \forall \phi \in W, \ \forall g, g' \in G. \tag{2.5}$$

Notation 11. We will denote with the symbol $\mathcal{R}_{\eta,\mathfrak{h}}$ the representation induced by $\chi_{\eta,\mathfrak{h}}$ defined above.

Such $\mathcal{R}_{\eta,\mathfrak{h}}$ are unitary and continuous. The above procedure can be replicated with the weaker assumption that the subalgebra $\mathfrak{h}$ is only an isotropic subspace for B_η, i.e. $\langle\eta, [\mathfrak{h}, \mathfrak{h}]\rangle = 0$. However, *maximal isotropy* of $\mathfrak{h}$ ensures irreducibility of $\mathcal{R}_{\eta,\mathfrak{h}}$. See [7], [13] for a more detailed discussion of this point.

This construction permits to classify, up to equivalence, all the unitary, irreducible and continuous representations of a nilpotent Lie group G. A proof of the following theorem can be found in [7], [13].

Theorem 2.2.4 (Kirillov)**.** *Let G be a nilpotent Lie group. The following two statements hold true:*

(i) Every unitary, irreducible and continuous representation of G is equivalent to one of the $\mathcal{R}_{\eta,\mathfrak{h}}$'s, for some $\eta \in \mathfrak{g}^*$, $\mathfrak{h}$ polarizing subalgebra for η.

(ii) Two representations $\mathcal{R}_{\eta,\mathfrak{h}}, \mathcal{R}_{\eta',\mathfrak{h}'}$ are equivalent if and only if η, η' belong to the same coadjoint orbit in $\mathfrak{g}^*$.

Notation 12. To simplify the notation, we will use the term *unirrep* to indicate a unitary, continuous and irreducible representation.

[4]See Remark 2.2.5 below.

2.2.3 A more concrete realization of the representation

The construction of unirreps described in the previous Section can be realized in a more *concrete* setting, as we now describe. We divide the exposition in three steps. The first two steps do not strictly require the nilpotency of G (assuming the existence of the G-invariant measure on cosets) and may work in more general settings.

First step: we replace the space W with $L^2(H\backslash G; V)$.

Assume again that $\chi : H \to \mathcal{U}(V)$ is a representation of a subgroup $H \subseteq G$ and denote with $\mathcal{R}$ the induced unitary representation on W.

Let's consider the projection $p : G \to H\backslash G$ and choose a cross-section $s : H\backslash G \to G$ (i.e: $p \circ s = id_{H\backslash G}$). We let $K := s(H\backslash G)$ and we write $G = HK$, i.e:

$$\forall g \in G, \quad g = hs(x) \quad \text{for some } h \in H, x \in H\backslash G.$$

By construction of W, any function $\phi \in W$ is completely determined by its values on K. Moreover, the map $\phi \mapsto f := \phi \circ s$ is an isometry from W to $L^2(H\backslash G; V)$, hence we can get a representation of G in $L^2(H\backslash G; V)$ letting:

$$[\mathcal{R}(g)f](x) := [\mathcal{R}(g)\phi](s(x)) \quad \forall g \in G, \forall x \in H\backslash G, \text{ if } f = \phi \circ s.$$

Remark 2.2.5. Given $f \in L^2(H\backslash G; V)$ we can always associate $\phi \in W$ such that $f = \phi \circ s$, defining:

$$\phi(hs(x)) := \chi(h)f(x) \quad \forall h \in H, \ \forall x \in H\backslash G.$$

This proves that W is isomorphic to $L^2(H\backslash G; V)$ and, in particular, complete.

Observe that if G is nilpotent and $\chi = \chi_{\eta,\mathfrak{h}}$, the above mentioned section s always exists. This is as an easy consequence of Proposition 1.2.15; we will describe s explicitly in the third step.

Second step: we explicit the action of G on $L^2(H\backslash G; V)$.

In view of the previous step and the assuption (2.3) on the functions in W, we can reduce to explicit the terms:

$$[\mathcal{R}(g)\phi](s(x)), \quad \phi \in W, x \in H\backslash G.$$

By definition of induced representation (2.5) we have $[\mathcal{R}(g)\phi](s(x)) = \phi(s(x)g)$. At this point, **assuming** that we are able to solve the so-called *Master equation*

$$s(x)g = h(x, g)s(y), \quad \text{for some } h(x, g) \in H, y \in H\backslash G, \tag{2.6}$$

and using (2.3), we can explicit the action of G on $f \in L^2(H\backslash G; V)$ as follows:

$$\begin{aligned}[\mathcal{R}(g)f](x) = [\mathcal{R}(g)\phi](s(x)) = \phi(s(x)g) = \phi(h(x, g)s(y)) = \\ = \chi(h(x, g))\phi(s(y)) = \chi(h(x, g))f(y).\end{aligned}$$

Third step: realization on $L^2(\mathbb{R}^m; \mathbb{C})$.

We now concentrate on the case when G is nilpotent, $H = \exp_G(\mathfrak{h})$ and $\chi = \chi_{\eta,\mathfrak{h}}$. We let $k = \dim(\mathfrak{h})$ and $m = n - k$.

Let $\mathcal{X} = \{X_1, \dots, X_n\}$ be a weak Malcev basis for $\mathfrak{g}$ through $\mathfrak{h}$ (existence is guaranteed by Theorem 1.1.38). By Proposition 1.2.15 we can choose coordinates on $H\backslash G$:

$$\Psi(t_1, \dots, t_m) = H \exp_G(t_1 X_{k+1}) \dots \exp_G(t_m X_n),$$

which transport the Lebesgue measure on $\mathbb{R}^m$ to a G-invariant measure on $H\backslash G$. We can also choose the section $s : H\backslash G \to G$ defined as:

$$s(He^{t_1 X_{k+1}} \dots e^{t_m X_n}) := e^{t_1 X_{k+1}} \dots e^{t_m X_n}.$$

Being $\mathcal{X}$ a Malcev basis, the Master equation (2.6) always has a solution. Namely, for each $(x_1, \dots, x_n) \in \mathbb{R}^n$ and $(t_1, \dots, t_m) \in \mathbb{R}^m$ there exists $(y_1, \dots, y_k, t'_1, \dots, t'_m) \in \mathbb{R}^n$ such that:

$$e^{x_1 X_1} \dots e^{x_n X_n} \cdot e^{t_1 X_{k+1}} \dots e^{t_m X_n} = e^{y_1 X_1} \dots e^{y_k X_k} \cdot e^{t'_1 X_{k+1}} \dots e^{t'_m X_n}.$$

Again thanks to Proposition 1.2.15, we can identify

$$L^2(H\backslash G, d\mu; \mathbb{C}) \ni f \approx \tilde{f} \in L^2(\mathbb{R}^m, dt; \mathbb{C}),$$

where $d\mu$ is the invariant measure on $H\backslash G$ and dt is the Lebesgue measure on $\mathbb{R}^m$, letting

$$f(He^{t_1 X_{k+1}} \dots e^{t_m X_n}) = \tilde{f}(t_1, \dots, t_m).$$

Finally, choosing the Malcev coordinates $\psi : \mathbb{R}^n \to G$ associated to the basis $\mathcal{X}$, we find a realization on $L^2(\mathbb{R}^m, dt; \mathbb{C})$:

$$\begin{aligned}
[\mathcal{R}(\psi(x))\tilde{f}](t_1, \dots, t_m) &= [\mathcal{R}(\psi(x))\phi](e^{t_1 X_{k+1}} \dots e^{t_m X_n}) \\
&= \phi(e^{x_1 X_1} \dots e^{x_n X_n} \cdot e^{t_1 X_{k+1}} \dots e^{t_m X_n}) \\
&= \phi(e^{y_1 X_1} \dots e^{y_k X_k} \cdot e^{t'_1 X_{k+1}} \dots e^{t'_m X_n}) = \\
&= \chi(e^{y_1 X_1} \dots e^{y_k X_k})\tilde{f}(t'_1, \dots, t'_m).
\end{aligned}$$

Definition 2.2.6. The above described realization in $L^2(\mathbb{R}^m; \mathbb{C})$ is called *basis realization* of $\mathcal{R} = \mathcal{R}_{\eta,\mathfrak{h}}$.

Remark 2.2.7. We stress an obious fact that may be useful for computations. Writing $\eta = \eta_1 X_1^* + \dots + \eta_n X_n^*$, it is **not true**, in general, that:

$$\chi_{\eta,\mathfrak{h}}(e^{y_1 X_1} \dots e^{y_k X_k}) = e^{i(y_1\eta_1 + \dots + y_k \eta_k)},$$

since we are using Malcev coordinates on H. This is true only when $\mathfrak{h}$ is commutative.

2.3 Representations of N_4

In this section we apply the theoretical results described in the previous sections to deduce, up to equivalence, all the unirreps of the group N_4.

Foliation of $\mathfrak{n}_4^*$.
The only nontrivial brackets are:

$$\begin{aligned}
[X_1, X_2] &= X_{12}, \quad [X_2, X_3] = X_{23} \\
[X_1, X_{23}] &= X_0, \quad [X_{12}, X_3] = X_0,
\end{aligned}$$

therefore the only nonzero coefficients are

$$c_{1,2}^{12} = c_{2,3}^{23} = c_{12,3}^{0} = c_{1,23}^{0} = 1$$
$$c_{2,1}^{12} = c_{3,2}^{23} = c_{3,12}^{0} = c_{23,1}^{0} = -1.$$

Using (2.2) we find the associated Poisson vector fields:

$$\vec{h_1} = h_{12}\partial_{h_2} + h_0\partial_{h_{23}}, \quad \vec{h_2} = h_{23}\partial_{h_3} - h_{12}\partial_{h_1}, \quad \vec{h_3} = -h_{23}\partial_{h_2} - h_0\partial_{h_{12}}$$
$$\vec{h_{12}} = h_0\partial_{h_3}, \quad \vec{h_{23}} = -h_0\partial_{h_1}, \quad \vec{h_0} = 0.$$

We immediately deduce that h_0 is a Casimir. We look for another Casimir: we expect at least another one, since $\dim(N_4) = 6$ and symplectic leaves are even dimensional (however, we already know the existence of four-dimensional orbits from Section 1.3.4, so we look for exacly one more Casimir).

The condition (2.1) reads:

$$\begin{aligned}
0 = \{f, h_1\} &= -\frac{\partial f}{\partial h_2}h_{12} - \frac{\partial f}{\partial h_{23}}h_0 \\
0 = \{f, h_2\} &= \frac{\partial f}{\partial h_1}h_{12} - \frac{\partial f}{\partial h_3}h_{23} \\
0 = \{f, h_3\} &= \frac{\partial f}{\partial h_2}h_{23} + \frac{\partial f}{\partial h_{12}}h_0 \\
0 = \{f, h_{12}\} &= -\frac{\partial f}{\partial h_3}h_0 \\
0 = \{f, h_{23}\} &= \frac{\partial f}{\partial h_1}h_0.
\end{aligned}$$

while we already know $\{f, h_0\} = 0$ by definition of Poisson vector field. From the above conditions we find the second Casimir function:

$$f := h_2h_0 - h_{12}h_{23}.$$

This permits to conclude that coadjoint orbits are contained in the level sets:

$$\begin{cases} h_0 = \alpha \\ h_2h_0 - h_{12}h_{23} = \gamma \end{cases} \quad \text{for } \alpha, \gamma \in \mathbb{R}. \tag{2.7}$$

Let's now fix $\eta = (h_1, \dots, h_0) \in \mathfrak{g}^*$ contained in one of these level sets and describe $\mathcal{O}_\eta$ depending on the values of α and γ.

If $\alpha = \gamma = 0$, the orbit is contained in

$$\begin{cases} h_0 = 0 \\ h_{12}h_{23} = 0 \end{cases}$$

and remembering Remark 2.2.3 we find three possibilities:

- If $h_{12} = h_{23} = 0$ the orbit reduces to a singleton $\mathcal{O}_\eta = \{\eta\} = \{(h_1, h_2, h_3, 0, 0, 0)\}$.

- If $h_{12} \neq 0 = h_{23}$ we have the plane:

$$\mathcal{O}_\eta = \{e^{t_1 h_{12}\partial_{h_2}} \circ e^{t_2 h_{12}\partial_{h_1}}(\eta) \mid t_i \in \mathbb{R}\}.$$

  We fix the element $\boxed{(0, 0, h_3, h_{12}, 0, 0)}$ on the orbit.

- If $h_{23} \neq 0 = h_{12}$ we have the plane:

$$\mathcal{O}_\eta = \{e^{t_1 h_{23}\partial_{h_3}} \circ e^{t_2 h_{23}\partial_{h_2}}(\eta) \mid t_i \in \mathbb{R}\}$$

  We fix the element $\boxed{(h_1, 0, 0, 0, h_{23}, 0)}$ on the orbit.

If $\alpha = 0, \gamma \neq 0$, the orbit is the plane:

$$\mathcal{O}_\eta = \left\{ e^{t_1 h_{12}\partial_{h_2}} \circ e^{t_2\left(-\frac{\gamma}{h_{12}}\partial_{h_3} - h_{12}\partial_{h_1}\right)} \circ e^{t_3 \frac{\gamma}{h_{12}}\partial_{h_2}}(\eta) \;\middle|\; t_i \in \mathbb{R} \right\}.$$

We fix the element $\boxed{(h_1, 0, h_3, h_{12}, -\gamma/h_{12}, 0)}$ on the orbit (notice that $h_{12} \neq 0$ because the second condition in (2.7) is $h_{12}h_{23} = -\gamma \neq 0$).

If $\alpha \neq 0$ the orbit is four dimensional (the distribution $\mathscr{P}(\lambda) = \mathrm{span}\{\vec{h_1}(\lambda), \dots, \vec{h_0}(\lambda)\}$ has constant rank equal to 4 on $\{h_0 = \alpha\}$) and it is completely described by the condition (2.7). We fix the element $\boxed{(0, \gamma/\alpha, 0, 0, 0, \alpha)}$ on the orbit.

At this point, thanks to Theorem 2.2.4 we are able to describe, up to equivalence, all the unitary irreducible representations of N_4.

Unirreps of N_4.

To simplify the notation, we write $\chi, \mathcal{R}$ respectively in place of $\chi_{\eta,\mathfrak{h}}, \mathcal{R}_{\eta,\mathfrak{h}}$. Again, we will indicate with $\psi : \mathbb{R}^6 \to G$ the Malcev coordinate map (1.5).

Remeber that for $\eta \in \mathfrak{g}^*$ the dimension of a polarizing subalgebra is $n - \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{r}_\eta)$. But $\dim(\mathfrak{g}/\mathfrak{r}_\eta)$ is the dimension of the coadjoint orbit of η[5], hence from the previous descripton we deduce the dimension of polarizing subalgebras in each case.

If $\alpha = \gamma = 0$ we follow the three cases presented above.

- **If** $h_{12} = h_{23} = 0$ a polarizing subalgebra for $\eta = (h_1, h_2, h_3, 0, 0, 0)$ is

$$\boxed{\mathfrak{h} = \mathfrak{g}}$$

  and the representation is one-dimensional. Following (1.7), we have:

$$e^{x_0 X_0 + \cdots + x_2 X_2} e^{x_1 X_1 + x_3 X_3} =$$
$$= e^{\left(x_0 - \frac{x_1 x_{23}}{2} + \frac{x_3 x_{12}}{2} - \frac{x_1 x_2 x_3}{6}\right)X_0 + \left(x_{12} - \frac{x_1 x_2}{2}\right)X_{12} + \left(x_{23} + \frac{x_2 x_3}{2}\right)X_{23} + x_2 X_2 + x_1 X_1 + x_3 X_3}.$$

  Letting ψ as above, the representation reads:

$$\boxed{\chi : N_4 \to \mathbb{S}^1, \quad \chi(\psi(x)) = e^{i(h_1 x_1 + h_2 x_2 + h_3 x_3)}}$$

[5]Indeed $\mathcal{O}_\eta \approx \mathfrak{g}^*/\,\mathrm{stab}(\eta)$, hence $\dim(\mathcal{O}_\eta) = \dim(\mathfrak{g}^*) - \dim(\mathrm{stab}(\eta)) = \dim(\mathfrak{g}) - \dim(\mathfrak{r}_\eta)$. We will come back on this in the last chapter.

- **If** $h_{12} \neq 0 = h_{23}$ a polarizing subalgebra for $\eta = (0, 0, h_3, h_{12}, 0, 0)$ is

$$\boxed{\mathfrak{h} = \{X_0, X_{12}, X_{23}, X_2, X_3\}.}$$

  We have:

$$e^{x_0X_0+x_{12}X_{12}+x_{23}X_{23}+x_2X_2}e^{x_3X_3} = e^{\left(x_0+\frac{x_{12}x_3}{2}\right)X_0+x_{12}X_{12}+\left(x_{23}+\frac{x_2x_3}{2}X_{23}\right)+x_2X_2+x_3X_3}$$

  and the unidimensional representation of $H = \exp_{N_4}(\mathfrak{h})$ is

$$\chi : H \to \mathbb{S}^1, \quad \chi(e^{x_0X_0+x_{12}X_{12}+x_{23}X_{23}+x_2X_2}e^{x_3X_3}) = e^{i(h_3x_3+h_{12}x_{12})}.$$

  The Master equation reads:

$$e^{\theta X_1}\left(e^{x_0X_0+x_{12}X_{12}+x_{23}X_{23}+x_2X_2}e^{x_3X_3}e^{x_1X_1}\right) = \\ = \left(e^{(x_0+\theta x_{23})X_0+(x_{12}+\theta x_2)X_{12}+x_{23}X_{23}+x_2X_2}e^{x_3X_3}\right)e^{(x_1+\theta)X_1}.$$

  We conclude that, letting ψ as above, the induced representation is:

$$\boxed{\mathcal{R} : N_4 \to L^2(\mathbb{R}; \mathbb{C}), \quad [\mathcal{R}(\psi(x))f](\theta) = e^{i(h_3x_3+h_2(x_{12}+\theta x_2))}f(x_1 + \theta).}$$

- **If** $h_{23} \neq 0 = h_{12}$, the computations are analogous to the previous case. A polarizing subalgebra for $\eta = (h_1, 0, 0, 0, h_{23}, 0)$ is

$$\boxed{\mathfrak{h} = \{X_0, X_{12}, X_{23}, X_2, X_1\}}$$

  and the representation is:

$$\boxed{\mathcal{R} : N_4 \to L^2(\mathbb{R}; \mathbb{C}), \quad [\mathcal{R}(\psi(x))f](\theta) = e^{i(h_1x_1+h_2(x_{23}-\theta x_2))}f(x_3 + \theta).}$$

  Here again ψ is the usual coordinate map.

If $\alpha = 0, \gamma \neq 0$ a polarizing subalgebra for $\eta = (h_1, 0, h_3, h_{12}, -\gamma/h_{12}, 0)$ is

$$\boxed{\mathfrak{h} = \{X_0, X_{12}, X_{23}, X_1, X_3\}.}$$

We have:

$$e^{x_0X_0+x_{12}X_{12}+x_{23}X_{23}}e^{x_1X_1+x_3X_3} = e^{\left(x_0+\frac{1}{2}(x_{12}x_3-x_{23}x_1)\right)X_0+x_{12}X_{12}+x_{23}X_{23}+x_1X_1+x_3X_3}$$

and the unidimensional representation of $H = \exp_{N_4}(\mathfrak{h})$ is

$$\chi : H \to \mathbb{S}^1, \quad \chi(e^{x_0X_0+x_{12}X_{12}+x_{23}X_{23}}e^{x_1X_1+x_3X_3}) = e^{i(h_1x_1+h_3x_3+h_{12}x_{12}-\frac{\gamma}{h_{12}}x_{23})}.$$

The Master equation reads:

$$e^{\theta X_2}\left(e^{x_0X_0+x_{12}X_{12}+x_{23}}e^{x_1X_1+x_3X_3}e^{x_2X_2}\right) = \\ = \left(e^{(x_0+\theta x_1x_3)X_0+(x_{12}-\theta x_1)X_{12}+(x_{23}+\theta x_3)X_{23}}e^{x_1X_1+x_3X_3}\right)e^{(x_2+\theta)X_2}. \quad (2.8)$$

In place of the usual coordinate map, we choose the more convenient:

$$\phi : \mathbb{R}^6 \to N_4, \quad \phi(x) = e^{x_0 X_0 + x_{12} X_{12} + x_{23}} e^{x_1 X_1 + x_3 X_3} e^{x_2 X_2}.$$

Finally, the induced representation reads:

$$\mathcal{R} : N_4 \to L^2(\mathbb{R}; \mathbb{C}), \tag{2.9}$$
$$[\mathcal{R}(\phi(x))f](\theta) = e^{i(h_1 x_1 + h_3 x_3 + h_{12}(x_{12} - \theta x_1) - (x_{23} + \theta x_3)\frac{\gamma}{h_{12}})} f(x_2 + \theta).$$

If $\alpha \neq 0$, a polarizing subalgebra for $\eta = (0, \frac{\gamma}{\alpha}, 0, 0, 0, \alpha)$ is

$$\boxed{\mathfrak{h} = \{X_0, X_{12}, X_{23}, X_2\}}$$

(notice that it is commutative) and the unidimensional representation of $H = \exp_{N_4}(\mathfrak{h})$ is:

$$\chi : H \to \mathbb{S}^1, \quad \chi(e^{x_0 X_0 + x_{12} X_{12} + x_{23} X_{23} + x_2 X_2}) = e^{i\left(\alpha x_0 + \frac{\gamma}{\alpha} x_2\right)}.$$

The Master equation reads:

$$(e^{\theta_1 X_1 + \theta_3 X_3})(e^{x_0 X_0 + x_{12} X_{12} + x_{23} X_{23} + x_2 X_2} e^{x_1 X_1 + x_3 X_3}) =$$
$$(e^{(x_0 + \theta_1 x_{23} - \theta_3 x_{12} - \theta_1 \theta_3 x_2) X_0 + (x_{12} + \theta_1 x_2) X_{12} + (x_{23} - \theta_3 x_2) X_{23} + x_2 X_2})(e^{(x_1 + \theta_1) X_1 + (x_3 + \theta_3) X_3}).$$

With the usual choice of coordinates, the induced representation is:

$$\mathcal{R} : N_4 \to L^2(\mathbb{R}^2; \mathbb{C}) \tag{2.10}$$
$$[\mathcal{R}(\psi(x))f](\theta_1, \theta_3) = e^{i\left(\alpha(x_0 + \theta_1 x_{23} - \theta_3 x_{12} - \theta_1 \theta_3 x_2) + \frac{\gamma}{\alpha} x_2\right)} f(x_1 + \theta_1, x_3 + \theta_3).$$

2.4 Semidirect product and representation

2.4.1 Abstract construction

In Section 2.2 we have seen how to obtain a representation of a nilpotent Lie group G once representations of a Lie subgroup $H \subseteq G$ is known. This method works, in particular, if G carries a structure of semidirect product:

$$G = H \rtimes K,$$

where H, K are subgroups of G and H is normal. We start giving some general properties of semidirect product of Lie groups.

Definition 2.4.1. Let H, K be groups and $\varphi : K \to \mathrm{Aut}(H)$ be a homomorphism. The *outer semidirect product* $H \rtimes_\varphi K$ is the group $(H \times K, \cdot)$ with operation:

$$(h_1, k_1) \cdot (h_2, k_2) := (h_1 \varphi_{k_1}(h_2), k_1 k_2).$$

One can check that $H \rtimes_\varphi K$ is actually a group with identity $(1_H, 1_K)$ and

$$(h,k)^{-1} = (\varphi_{k^{-1}}(h^{-1}), k^{-1}) \quad \forall h \in H, \forall k \in K.$$

If H and K are Lie groups, the product is intended also in the sense of smooth manifolds. In this case, $H \rtimes_\varphi K$ is also a Lie group if the map φ is smooth and acts by Lie group homomorphisms (See [17], Section 2.2.4).

The semidirect product contains the *copies* of H, K defined respectively:

$$H' = H \times \{1_K\}, \quad K' = \{1_H\} \times K.$$

Writing explicitly the conjugation:

$$(h_1,k_1)(h_2,k_2)(h_1,k_1)^{-1} = (h_1\varphi_{k_1}(h_2)\varphi_{k_1k_2k_1^{-1}}(h_1^{-1}), k_1k_2k_1^{-1}),$$

we deduce that H', K' are closed subgroups of $H \rtimes_\varphi K$ and H' is normal.

We recall a criterion to establish when a group is (isomorphic to) a semidirect product.

Proposition 2.4.2. *Le G be a group and H, K be subgroups such that:*

- *$G = HK$, i.e. for all $g \in G$ we can find $h \in H$ and $k \in K$ such that $g = hk$;*
- *$H \cap K = \{1_G\}$;*
- *H is normal in G.*

Then, the map

$$\mu : H \rtimes_\varphi K \to G, \quad (h,k) \mapsto hk$$

is an isomorphism, where $\varphi : K \to \mathrm{Aut}(H)$ is the conjugation $\varphi_k(h) = khk^{-1}$.

This criterion works also for Lie groups, but we need to assume that H, K are closed subgroups and that G is simply connected. In this case, μ is also a Lie group isomorphism.[6]

Notation 13. In the above situation, we simply write $G = H \rtimes K$ and we call it *inner* semidirect product.

In particular, if $G = H \rtimes K$ it holds:

$$H\backslash G = H\backslash HK \cong (H \cap K)\backslash K = K; \tag{2.11}$$

and if $G = H \rtimes K$ is a simply connected Lie group as above, such isomorphism is also a Lie group isomorphism (see [10], Section 11).

Together with the notion of semidirect product of Lie groups, it is useful to introduce a notion of semidirect product of Lie algebras. We follow [17].

Definition 2.4.3. Let $\mathfrak{g}$ be a Lie algebra, its *derivation Lie algebra* $\mathrm{Der}(\mathfrak{g})$ is the Lie algebra of derivations on $\mathfrak{g}$, endowed with the commutator Lie bracket.

[6] Actually, these assumptions are more than what we need. See also [10], Section 11.

Definition 2.4.4. Let $\mathfrak{h}$ and $\mathfrak{k}$ be Lie algebras and let $\varphi_* : \mathfrak{k} \to \mathrm{Der}(\mathfrak{h})$ be a Lie algebra homomorphism. The *semidirect product Lie algebra* $\mathfrak{h} \rtimes \mathfrak{k}$ is the vector space $\mathfrak{h} \times \mathfrak{k}$ equipped with the Lie bracket:

$$[(X,T),(X',T')] := [[X,X'] + \varphi_*(T)X' - \varphi_*(T')X, [T,T']], \tag{2.12}$$

for $X, X' \in \mathfrak{h}$, $T, T' \in \mathfrak{k}$.

As we expect, there is a strong relation between the notion of semidirect product of Lie groups and Lie algebras.

Proposition 2.4.5. *Let $G = H \rtimes_\varphi K$ as above and denote by $\mathfrak{g}$ its Lie algebra. For any $X \in \mathfrak{h}, T \in \mathfrak{k}$, the adjoint action on $\mathfrak{g}$ reads:*

$$\mathrm{Ad}^G_{(h,k)}(X,T) = (\mathrm{Ad}^H_h \circ (\varphi_k)_{*,1_H}(X) + (L_h)_{*,h^{-1}} \circ (\varphi_{(\cdot)}(h^{-1}))_{*,1_K} \circ \mathrm{Ad}^K_k(T), \mathrm{Ad}^K_k(T)).$$

Taking the derivative at identity, we find:

$$[\mathrm{ad}^G(X,T)](X',T') = \Big([\mathrm{ad}^H(X)](X') + \varphi_{*,(1_H,1_K)}(X',T) - \varphi_{*,(1_H,1_K)}(X,T'), [\mathrm{ad}^K(T)](T')\Big), \tag{2.13}$$

where $\varphi_{*,(1_H,1_K)}$ indicates the differential of φ seen as a map defined on the product $H \times K$.

Setting $\varphi_* := \varphi_{*,(1_H,1_K)}$, which is a derivation of $\mathfrak{h}$, from (2.13) we deduce that

$$\mathrm{Lie}(H \rtimes_\varphi K) = \mathfrak{h} \rtimes_{\varphi_*} \mathfrak{k}.$$

Let's now turn our attention to nilpotent Lie groups, starting with an observation. Assume that G is a nilpotent Lie group and $H, K \subseteq G$ are closed subgroups such that $G = H \rtimes K$. Choosing $\{X_1, \dots, X_k\}$, $\{Y_1, \dots, Y_m\}$ weak Malcev bases for $\mathfrak{h}$ and $\mathfrak{k}$ respectively, using (2.12) we see that $\{X_1, \dots, X_k, Y_1, \dots, Y_m\}$ is also a weak Malcev basis for $\mathfrak{g}$. We can use this observation, together with the identity (2.11), to revisit the second and third steps of Section 2.2.3 as follows.

Under the same assumptions, let $\chi : H \to \mathcal{U}(V)$ be a unirrep of H, where is V an Hilbert space. Remembering (2.11), since we have chosen a Malcev basis through $\mathfrak{h}$, we can put coordinates on $H\backslash G$ using 1.2.15, defining

$$\Psi : \mathbb{R}^m \to H\backslash G, \quad \Psi(t_1, \dots, t_m) := He^{t_1 X_{k+1}} \dots e^{t_m X_n},$$

and since $H\backslash G \cong K$, if we identify

$$He^{t_1 X_{k+1}} \dots e^{t_m X_n} \approx e^{t_1 X_{k+1}} \dots e^{t_m X_n}$$

we can realize the representation on $L^2(K; V)$. Finally, since $\{X_{k+1}, \dots, X_n\}$ is a weak Malcev basis for K, we can realize the action in $L^2(\mathbb{R}^m; V)$ following Step 3 of Section 2.2.3.

2.4.2 Representation of $N_4 = \mathbb{H}^2 \rtimes \mathbb{R}$

We recall that the Heisenberg group $\mathbb{H}^n$ is the unique (connected and simply connected) nilpotent Lie group with Lie algebra

$$\mathfrak{h}_n := \text{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\},$$

where the only nonzero brackets are $[X_i, Y_i] = Z, i = 1, \dots, n$. With the strong Malcev basis $\{Z, Y_1, \dots, Y_n, X_1, \dots, X_n\}$, we can identify

$$\mathbb{H}^n \cong (\mathbb{R}^{2n+1}, \cdot_{\mathbb{H}^n}),$$

where indicating with $x \cdot y$ the Euclidean scalar product between $x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n$, the operation is:

$$(x, y, z) \cdot_{\mathbb{H}^n} (x', y', z') = (x + x', y + y', z + z' + x \cdot y').$$

The orbit method (see [7]) gives two families of unitary and irreducible representations:

- The family one dimensional representations $\chi_{\mu,\nu} : \mathbb{H}^n \to \mathbb{S}^1$,

$$\chi_{\mu,\nu}(x, y, z) = e^{i(\mu \cdot x + \nu \cdot y)}, \quad \mu, \nu \in \mathbb{R}^n.$$

- The family of representations $\mathcal{R}_\lambda : \mathbb{H}^n \to L^2(\mathbb{R}^n; \mathbb{C})$, $\lambda \neq 0$,

$$[\mathcal{R}_\lambda(x, y, z) f](\theta) = e^{i\lambda(z + \theta \cdot y)} f(x + \theta), \quad \theta \in \mathbb{R}^n.$$

Consider now the group N_4, identified with $\mathbb{R}^6$ with the usual Malcev coordinates. We prove that $N_4 = H \rtimes K$ with $H \cong \mathbb{H}^2, K \cong \mathbb{R}$ and deduce some representations of G as explained in the previous section.

Define:

$$\mathfrak{h} := \text{span}\{X_0, X_{12}, X_{23}, X_1, X_3\}, \quad H := \exp_{N_4}(\mathfrak{h})$$
$$\mathfrak{k} = \mathbb{R}X_2, \quad K := \exp_{N_4}(\mathfrak{k}).$$

Such H and K are closed Lie subgroups of N_4; notice that the coordinates inherited by H are exactly the Malcev coordinates associated with the basis $\{X_0, X_{12}, X_{23}, X_1, X_3\}$ of $\mathfrak{h}$ via $\exp_H = \exp_G |_{\mathfrak{h}}$, and the same trivially holds for K. In particular, H coincides with the subset $\{x_2 = 0\}$ of N_4.

Clearly $H \cap K = \{1_{N_4}\}$ and a straightforward calculation shows that $H \trianglelefteq N_4$ (observe that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$). Notice also that, with the notation introduced in (1.6), we have:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_{12} \\ x_{23} \\ x_0 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ x_3 \\ x_{12} - x_1 x_2 \\ x_{23} + x_2 x_3 \\ x_0 + x_1 x_2 x_3 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ x_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

hence $N_4 = HK$, and we conclude that $N_4 \cong H \rtimes K$ thanks to Proposition 2.4.2.

Observe now that the map $\Psi : H \to \mathbb{H}^2$ defined in coordinates by

$$\Psi(x_1, 0, x_3, x_{12}, x_{23}, x_0) = (x_1, x_3, x_{23}, -x_{12}, x_0)$$

is a Lie group isomorphism. Consistently with the notation adopted in (2.9), we let:

$$\phi : \mathbb{R}^6 \to N_4, \quad \phi(x) = e^{x_0 X_0 + x_{12} X_{12} + x_{23}} e^{x_1 X_1 + x_3 X_3} e^{x_2 X_2}$$

and, rembembering the representations we already have for $\mathbb{H}^2$, composing with Ψ we find the following representations of H.

- The family one dimensional representations $\chi^H_{\mu,\nu} : H \to \mathbb{S}^1$,
$$\chi_{\mu,\nu}(\phi(x_1, 0, x_3, x_{12}, x_{23}, x_0)) = e^{i(\mu\cdot(x_1,x_3)+\nu\cdot(x_{23},-x_{12}))}, \quad \mu, \nu \in \mathbb{R}^2.$$
- The family of representations $\mathcal{R}^H_\lambda : H \to L^2(\mathbb{R}^2; \mathbb{C})$, $\lambda \neq 0$,
$$[\mathcal{R}^H_\lambda(\phi(x_1, 0, x_3, x_{12}, x_{23}, x_0))f](\theta) = e^{i\lambda(x_0+\theta_1 x_{23}-\theta_3 x_{12})} f(\theta_1+x_1, \theta_3+x_3), \quad (\theta_1, \theta_3) \in \mathbb{R}^2.$$

Let's compute the corresponding induced representations.

• First we consider the representations of N_4 induced by the $\chi^H_{\mu,\nu}$, where $\mu = (\mu_1, \mu_3), \nu = (\nu_1, \nu_3) \in \mathbb{R}^2$.

By our previous discussion we can realize the representation in $L^2(\mathbb{R}; \mathbb{C})$. We let $e^{\theta X_2}$ indicate the generic element of K. The Master equation is already solved in (2.8), and we get:

$$\mathcal{R}_{\mu,\nu} : N_4 \to \mathcal{U}(L^2(\mathbb{R}; \mathbb{C}))$$
$$[\mathcal{R}_{\mu,\nu}(\phi(x))f](\theta) = e^{i(\mu_1 x_1+\mu_3 x_3+\nu_1(x_{23}+\theta x_3)-\nu_3(x_{12}-\theta x_1))} f(\theta + x_2).$$

Remark 2.4.6. We have recovered the representations corresponding to the case $\alpha = 0, \gamma \neq 0$ described by (2.9).

• We now detail the representations of G induced by the $\mathcal{R}^H_\lambda$, $\lambda \neq 0$.

We can realize the representation on $L^2(\mathbb{R}; L^2(\mathbb{R}^2; \mathbb{C}))$. Letting $e^{\zeta X_2}$ indicate the generic element of K, following the above described steps, we obtain:

$$\mathcal{R}_\lambda : N_4 \to \mathcal{U}(L^2(\mathbb{R}; L^2(\mathbb{R}^2; \mathbb{C})))$$
$$[\mathcal{R}_\lambda(\phi(x))f](\zeta)|_{(\theta_1,\theta_3)} = e^{i\lambda(x_0+\zeta x_1 x_3+\theta_1(x_{23}+\zeta x_3)-\theta_3(x_{12}-\zeta x_1))} \left(f(\zeta + x_2)|_{(\theta_1+x_1,\theta_3+x_3)} \right).$$

Identifying $L^2(\mathbb{R}; L^2(\mathbb{R}^2; \mathbb{C}))$ with $L^2(\mathbb{R}^3; \mathbb{C})$ and renaming $\zeta = \theta_2$ we find:

$$\mathcal{R}_\lambda : N_4 \to \mathcal{U}(L^2(\mathbb{R}^3; \mathbb{C}))$$
$$[\mathcal{R}_\lambda(\phi(x))f]_{(\theta_1,\theta_2,\theta_3)} = e^{i\lambda(x_0+\theta_2 x_1 x_3+\theta_1(x_{23}+\theta_2 x_3)-\theta_3(x_{12}-\theta_2 x_1))} f(\theta_1 + x_1, \theta_2 + x_2, \theta_3 + x_3).$$

Remark 2.4.7. We did not establish whether such $\mathcal{R}_\lambda$ are irreducible, hence equivalent to one of the representations described in Section 2.3, or not.

2.5 Representation of $\mathfrak{g}$ and $\mathfrak{u}(\mathfrak{g})$

Let G be a Lie group (not necessarily nilpotent) with Lie algebra $\mathfrak{g}$ and $(\mathcal{R}, \mathcal{H})$ be a unirrep of G. We describe how to represent the elements of $\mathfrak{g}$ and left-invariant differential operators on G using $\mathcal{R}$, following [7] and [12].

2.5.1 Representation of $\mathfrak{g}$

The basic idea for representing an element of the Lie algebra $X \in \mathfrak{g}$ through $\mathcal{R}$ is to differentiate the representation along a curve that is tangent to X at identity, i.e, fixed $X \in \mathfrak{g}$, let:

$$\mathcal{R}(X) := \frac{d}{dt}\Big|_{t=0} \mathcal{R}(e^{tX}). \tag{2.14}$$

However, in general, the $\mathcal{R}(X)$'s cannot be bounded operators on $\mathcal{H}$: some attention to their domain of definition must be paid. For this reason, we introduce the class of *smooth vectors* in $\mathcal{H}$.

To any $v \in \mathcal{H}$ we associate a function $r_v : G \to \mathcal{H}$ defined as:

$$r_v(g) := \mathcal{R}(g)v \quad \forall g \in G.$$

Definition 2.5.1. We say that $v \in \mathcal{H}$ is a $\mathcal{C}^k$ *vector*, writing $v \in \mathcal{H}^k$, if

$$r_v \in \mathcal{C}^k(G; \mathcal{H}).$$

Moreover, we say that v is a $\mathcal{C}^\infty$ *vector*, or *smooth vector*, if $v \in \mathcal{H}^\infty := \bigcap_{k\geq 1} \mathcal{H}^k$.

The above defined $\mathcal{H}^k, \mathcal{H}^\infty$ are G-stable. Let's prove that $\mathcal{H}^\infty$ is also dense in $\mathcal{H}$. To this aim, we introduce the class of *smoothing operators*, i.e. the family of operators:

$$\mathcal{R}(\phi) := \int_G \phi(g)\mathcal{R}(g)dg, \quad \phi \in \mathcal{C}_c^\infty(G),$$

where dg indicates a left Haar measure on G. The following Lemma shows an important property of such operators.

Lemma 2.5.2. *For all $X \in \mathfrak{g}$ and $\phi \in \mathcal{C}_c^\infty(G)$ we have:*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{R}(e^{tX})\mathcal{R}(\phi)v = \mathcal{R}(\tilde{X}_R\phi)v \quad \forall v \in \mathcal{H},$$

where $\tilde{X}_R$ indicates the right-invariant *vector field associated to X.*

The proof is based on the definition of right-invariant vector field and change of variable (See [12]). We will explicit the details of a completely analogous computation in the proof of Proposition 2.5.12.

From Lemma 2.5.2 we deduce that for all $v \in \mathcal{H}$ and $\phi \in \mathcal{C}_c^\infty(G)$ it holds $\mathcal{R}(\phi)v \in \mathcal{H}^\infty$.

Proposition 2.5.3. *The subspace $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.*

Proof. Thanks to the previous observation, it is enough to prove that $\{\mathcal{R}(\phi)v \mid \phi \in \mathcal{C}_c^\infty(G),\ v \in \mathcal{H}\}$ is dense in $\mathcal{H}$. Let u be any vector in $\mathcal{H}$; being $\mathcal{R}$ strongly continuous, for all $\varepsilon > 0$ we can find a neighborhood U of 1_G such that

$$\|\mathcal{R}(g)u - \mathcal{R}(1_G)u\|_{\mathcal{H}} = \|\mathcal{R}(g)u - u\|_{\mathcal{H}} < \varepsilon \quad \forall g \in U.$$

Choose now $\phi \in \mathcal{C}_c^\infty(G)$ supported in U, $\phi \geq 0$, such that $\int_U \phi(g)dg = 1$. We have:

$$\|\mathcal{R}(\phi)u - u\|_{\mathcal{H}} = \left\| \int_U \phi(g)(\mathcal{R}(g)u - u)dg \right\|_{\mathcal{H}} \leq \varepsilon,$$

hence the density of $\mathcal{H}^\infty$. □

We are now ready to describe the representation of $\mathfrak{g}$ through $\mathcal{R}$. From our discussion, we know that there exists a dense subspace where we are allowed to compute the differentiation (2.14). Therefore, we start considering the operators:

$$((\mathcal{R}(X))_0, \mathcal{H}^\infty), \quad (\mathcal{R}(X))_0 v := \left.\frac{d}{dt}\right|_{t=0} \mathcal{R}(\exp_G(tX))v \quad \forall v \in \mathcal{H}^\infty.$$

Proposition 2.5.4. *Let $G, \mathcal{R}, \mathcal{H}$ as above. For any $X \in \mathfrak{g}$ the operator*

$$-i(\mathcal{R}(X))_0 := -i \left.\frac{d}{dt}\right|_{t=0} \mathcal{R}(\exp_G(tX)),$$

is essentially self-adjoint on $\mathcal{H}^\infty$.

Proof. We let $A := -i\mathcal{R}(X)_0$. First we notice it is a symmetric operator. Indeed for all $v, w \in \mathcal{H}^\infty$ we have:

$$\begin{aligned}\langle Av, w \rangle &= \left.\frac{d}{dt}\right|_{t=0} \langle -i\mathcal{R}(\exp_G(tX))v, w \rangle = \left.\frac{d}{dt}\right|_{t=0} \langle v, i\mathcal{R}(\exp_G(tX))^* w \rangle \\ &= \left.\frac{d}{dt}\right|_{t=0} \langle v, i\mathcal{R}(\exp_G(-tX))w \rangle = \langle v, Aw \rangle.\end{aligned}$$

To prove essential self-adjointness, we use this general fact: an unbounded operator $(A, \mathrm{D}(A))$ on a Hilbert space $\mathcal{H}$ is essentially self-adjoint if and only if $\ker(A^* \pm i) = 0$. Let $v \in \mathcal{H}^\infty$ and $w \in \ker(A^* \pm i)$ and consider the function:

$$f(t) = \langle \mathcal{R}(\exp_G(tX)v, w \rangle.$$

We prove that $w = 0$. We have:

$$\begin{aligned}\frac{d}{dt} f(t) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \langle \mathcal{R}(\exp_G(\varepsilon X))\mathcal{R}(\exp_G(tX))v, w \rangle = \langle iA\mathcal{R}(\exp_G(tX))v, w \rangle \\ &= \langle \mathcal{R}(\exp_G(tX))v, -iA^* w \rangle = \pm \langle \mathcal{R}(\exp_G(tX))v, w \rangle = \pm f(t),\end{aligned}$$

hence $f(t) = ce^{\pm t}$ for some $c \in \mathbb{R}$. But since $\mathcal{R}$ is unitary, f is bounded, indeed:

$$|f(t)| \leq \|v\|_{\mathcal{H}} \|w\|_{\mathcal{H}},$$

hence $c = 0$. In particular, fixing $t = 0$ we deduce that w is orthogonal to $\mathcal{H}^\infty$, and being $\mathcal{H}^\infty$ dense in $\mathcal{H}$ we conclude that $w = 0$, as we wanted. □

We will let $(\mathcal{R}(X), \mathrm{D}(\mathcal{R}(X))$ be the closure of the operator $((\mathcal{R}(X))_0, \mathcal{H}^\infty)$, which is skew-adjoint on its domain. Let us be more precise on the definition of $\mathrm{D}(\mathcal{R}(X))$, using the following results.

Proposition 2.5.5. *In the above setting, letting $A = -i\mathcal{R}(X)$ we have*

$$\mathcal{R}(\exp_G(tX)) = e^{itA},$$

where the right-hand side is defined by functional calculus for self-adjoint operators.

Proof. By the very definition of e^{itA} we deduce that equality must hold on $\mathrm{D}(A)$. But $\mathrm{D}(A)$ is dense in $\mathcal{H}$ and both $\mathcal{R}(\exp_G(tX))$ and e^{itA} are bounded operators, hence they must coincide on the whole $\mathcal{H}$. □

We conclude stating a general result, whose proof can be found in [18].

Theorem 2.5.6. *Let $(A, \mathrm{D}(A))$ be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and define e^{itA} by functional calculus for self-adjoint operators. We have:*

(i) For $v \in \mathrm{D}(A)$,

$$\lim_{t \to 0} \frac{e^{itA}v - v}{t} = iAt$$

(ii) Let $v \in \mathcal{H}$. If the limit

$$\lim_{t \to 0} \frac{e^{itA}v - v}{t}$$

exists, then $v \in \mathrm{D}(A)$.

We can now give a more precise definition of $(\mathcal{R}(X), \mathrm{D}(\mathcal{R}(X)))$.

Definition 2.5.7. Let $X \in \mathfrak{g}$ we define the operator $(\mathcal{R}(X), \mathrm{D}(\mathcal{R}(X)))$ letting:

$$\mathrm{D}(\mathcal{R}(X)) := \left\{ v \in \mathcal{H} \;\middle|\; \lim_{t \to 0} \frac{1}{t} \left(\mathcal{R}(\exp_G(tX))v - v \right) \in \mathcal{H} \right\}$$
$$\mathcal{R}(X)v := \lim_{t \to 0} \frac{1}{t} \left(\mathcal{R}(\exp_G(tX))v - v \right) \quad \forall v \in \mathrm{D}(\mathcal{R}(X)).$$

2.5.2 Representation of $\mathfrak{u}(\mathfrak{g})$, the sub-Laplacian

Abstract construction. As in the first chapter, we denote by X, Y the elements of $\mathfrak{g}$ and by $\tilde{X}, \tilde{Y}$ the associated left-invariant vector fields.

Definition 2.5.8. Let $\mathfrak{g}$ be a Lie algebra, the *universal enveloping algebra* associated to $\mathfrak{g}$ is the associative algebra $\mathfrak{u}(\mathfrak{g})$ satisfying:

(i) $\mathfrak{u}(\mathfrak{g})$ has a unit, contains $\mathfrak{g}$ as vector subspace and is generated by $\mathfrak{g}$ as an associative algebra.

(ii) $[X, Y] = XY - YX$ in $\mathfrak{u}(\mathfrak{g})$ for any $X, Y \in \mathfrak{g}$.

(iii) If $\mathfrak{b}$ is any associative algebra with unit and $\lambda : \mathfrak{g} \to \mathfrak{b}$ is any linear map verifying:

$$\lambda([X, Y]) = \lambda(X)\lambda(Y) - \lambda(Y)\lambda(X), \quad \forall X, Y \in \mathfrak{g}$$

then there exists a unique homomorphism $\lambda' : \mathfrak{u}(\mathfrak{g}) \to \mathfrak{b}$ such that $\lambda'(1) = 1$ and $\lambda' |_{\mathfrak{g}} = \lambda$.

A concrete way to think at $\mathfrak{u}(\mathfrak{g})$ in Lie groups is the following. We know that there is an identification $\lambda : \mathfrak{g} \to \mathrm{Vec}_L(G)$ which associates an element $X \in \mathfrak{g}$ with $\tilde{X} \in \mathrm{Vec}_L(G)$. Such λ can be extended to a homomorphism into the algebra $\mathfrak{u}(\mathfrak{g})_L$ of left-invariant differential operators on G, which is a realization of $\mathfrak{u}(\mathfrak{g})$, fixing a basis of $\mathfrak{g}$. Indeed, once a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ is fixed, a basis of $\mathfrak{u}(\mathfrak{g})$ is given by the elements:

$$X^\alpha := X_1^{\alpha_1} \dots X_n^{\alpha_n}, \quad \alpha \in \mathbb{N}^n \text{ multi-index.} \tag{2.15}$$

The extension of λ can be obtaining letting $X^\alpha \in \mathfrak{u}(\mathfrak{g})$ corresponds to the left-invariant operator $\tilde{X}^\alpha := \tilde{X}_1^{\alpha_1} \dots \tilde{X}_n^{\alpha_n}$ and by linearity. As for the elements of $\mathfrak{g}$, we can represent the elements of $\mathfrak{u}(\mathfrak{g})$ through $\mathcal{R}$. Again, in principle we are able to differentiate only *smooth* vectors, therefore we adopt the following convention.

Let $A \in \mathfrak{u}(\mathfrak{g})$ and write it as a combination of elements X^α as in (2.15). Each element $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ is represented by the operator:

$$\mathcal{R}(X^\alpha)_0 = \mathcal{R}(X_1)_0^{\alpha_1} \dots \mathcal{R}(X_n)_0^{\alpha_n}.$$

A representation of A is then obtained by linearity.

The sub-Laplacian

Let us focus our attention on a particular left-invariant differential operator that plays an important role when the group G is nilpotent and its Lie algebra $\mathfrak{g}$ is stratified. We recall that a Lie algebra is *stratified* it can be decomposed in a direct sum:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r, \quad [\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}.$$

As we explained in Section 1.1.5, choosing a basis $\{X_1, \dots, X_k\}$ of $\mathfrak{g}_1$, we can equip G with a sub-Riemannian structure such that the sub-Laplacian is the left-invariant differential operator:

$$\widetilde{\Delta_G} := \sum_{i=1}^{k} \tilde{X}_i^2.$$

Such operator corresponds to the element:

$$\Delta_G = X_1^2 + \dots + X_k^2 \in \mathfrak{u}(\mathfrak{g}).$$

According to the previous section, if $(\mathcal{R}, \mathcal{H})$ is a unirrep of G, we can define:

$$\mathcal{R}(-\Delta_G)_0 = -\mathcal{R}(X_1)_0^2 \dots - \mathcal{R}(X_k)_0^2.$$

We know that each $\mathcal{R}(X_i)_0$ is skew-symmetric on $\mathcal{H}^\infty$, hence $\mathcal{R}(-\Delta_G)_0$ is symmetric and positive on the same domain. Therefore, it admits a unique self-adjoint extension, the *Friedrichs extension*[7], that we denote:

$$(\mathcal{R}(-\Delta_G), \mathrm{D}(\mathcal{R}(-\Delta_G))).$$

In particular, letting Q be the domain of the closure of the quadratic form associated to $\mathcal{R}(-\Delta_G)$, we have:

$$\mathrm{D}(\mathcal{R}(-\Delta_G)) = \{v \in Q \mid \exists w \in \mathcal{H} \text{ s.t. } \langle v, u \rangle_Q = \langle w, u \rangle \ \forall u \in Q\}$$
$$\mathcal{R}(-\Delta_G)v = w.$$

[7]See, [15], Chapter 3.

2.5.3 Fourier transform and representation

We turn our attention back to operators of the type:

$$\mathcal{R}(\phi) := \int_G \phi(g)\mathcal{R}(g)dg, \quad \phi \in \mathcal{C}_c^\infty(G),$$

where dg indicates a Haar measure on a nilpotent Lie group G. Such operators can be interpreted as a *Fourier transform*, and we will discuss this intepretation on the last chapter. Here, we are interested in rewriting $\mathcal{R}(\tilde{A}\phi)$, for $A \in \mathfrak{u}(\mathfrak{g})$, in terms of $\mathcal{R}(\phi)$ and representation of elements of $\mathfrak{u}(\mathfrak{g})$.

We start recalling a basic result that will be useful.

Lemma 2.5.9. *Let $X \in \mathfrak{g}$. The flow of the associated left-invariant vector field $\tilde{X}$ is a right translation, namely:*

$$e^{t\tilde{X}}(g) = g\exp_G(tX) \quad \forall g \in G,\ \forall t \in \mathbb{R}.$$

Proof. See [1], Lemma 7.22-7.23. □

Let's now consider the bilinear form on $\mathcal{C}_c^\infty(G)$: $(\phi, \psi) := \int_G \phi(g)\psi(g)dg$.
Defining the *formal transpose* of an element of the basis of the universal enveloping algebra

$$X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n} \in \mathfrak{u}(\mathfrak{g})$$

as:

$$(X^\alpha)^T := (-1)^{|\alpha|} X_n^{\alpha_n} \cdots X_1^{\alpha_1},$$

we extend the above definition to any $A \in \mathfrak{u}(\mathfrak{g})$. Then, we can consider the transpose of any left-invariant operator $\tilde{A}$ on G, corresponding to $A \in \mathfrak{u}(\mathfrak{g})$, letting $(\tilde{A})^T := \widetilde{(A^T)}$. We prove that such transpose actually behaves as we expect with respect to the bilinear form we fixed on $\mathcal{C}_c^\infty(G)$.

Proposition 2.5.10. *Let G be a nilpotent Lie group. For all $\phi, \psi \in \mathcal{C}_c^\infty(G)$ it holds:*

$$(\tilde{A}^T\phi, \psi) = (\phi, \tilde{A}\psi).$$

Before the proof we need to prove a technical lemma. Recall that, according to Section 1.2.1, up to choosing an appropriate rescaling of the Haar measure dg, we have the equality:

$$\int_G \phi \, dg = \int_G \phi \, dV_\sigma \quad \forall \phi \in \mathcal{C}_c^\infty(G),$$

where dV_σ is the Riemannian volume associated to a left-invariant 2-form σ.

Lemma 2.5.11. *Let G be a unimodular Lie group and $\tilde{X} \in \mathrm{Vec}_L(G)$. Then we have* $\mathrm{div}_{V_\sigma}(\tilde{X}) = 0$.

Proof (Lemma). By definition we have

$$(\mathrm{div}_{V_\sigma}(\tilde{X}))V_\sigma = \mathcal{L}_{\tilde{X}} V_\sigma = \left.\frac{d}{dt}\right|_{t=0} (e^{t\tilde{X}})^* V_\sigma,$$

where $\mathcal{L}$ indicates the Lie derivative. Remembering Lemma 2.5.9 and the hypothesis of unimodularity:

$$(\operatorname{div}_{V_\sigma}(\tilde{X}))V_\sigma = \frac{d}{dt}\bigg|_{t=0} (R_{\exp_G(tX)})^* V_\sigma = \frac{d}{dt}\bigg|_{t=0} V_\sigma = 0.$$

We conclude that $\operatorname{div}_{V_\sigma}(\tilde{X}) = 0$ since V_σ is a volume form on G. □

Proof (Proposition 2.5.10). It is sufficient to prove it only for $A = X \in \mathfrak{g}$.
Start from the equality:

$$\operatorname{div}_{V_\sigma}(\phi\psi\tilde{X}) = \phi(\tilde{X}\psi) + \psi(\tilde{X}\phi) + \phi\psi \operatorname{div}_{V_\sigma}(\tilde{X}),$$

and notice that the last summand vanishes thanks to Lemma 2.5.11. Integrate over G, obtaining:

$$\int_G \operatorname{div}_{V_\sigma}(\phi\psi\tilde{X}) dV_\sigma = \int_G \operatorname{div}_{V_\sigma}(\phi\psi\tilde{X}) dg = \int_G \phi(\tilde{X}\psi) + \psi(\tilde{X}\phi) dg.$$

The left hand side is zero thanks to the Riemannian divergence formula, remembering that $\partial G = \varnothing$ since G is diffeomorphic to $\mathbb{R}^n$. □

Finally, we show how to rewrite the operators $\mathcal{R}(\tilde{A}\phi)$ with $\phi \in \mathcal{C}_c^\infty(G), A \in \mathfrak{u}(\mathfrak{g})$, as a composition of simpler terms.

Proposition 2.5.12. *Denoting by $\tilde{A}$ the left-invariant differential operator associated to $A \in \mathfrak{u}(\mathfrak{g})$, we have:*

$$\mathcal{R}(\tilde{A}\phi) = \mathcal{R}(\phi) \circ \mathcal{R}(A^T) \quad \forall \phi \in \mathcal{C}_c^\infty(G), \forall A \in \mathfrak{u}(\mathfrak{g}),$$

where the equality shall be intended in the sense of operators acting on $\mathcal{H}^\infty$.

Proof. It is sufficient to prove it for $X \in \mathfrak{g}$, whose formal transpose is $X^T = -X$.
We have:

$$\begin{aligned}
\mathcal{R}(\tilde{X}\phi) &= \int_G \tilde{X}\phi(g)\mathcal{R}(g)dg = \frac{d}{dt}\bigg|_{t=0} \int_G \phi(e^{t\tilde{X}}(g))\mathcal{R}(g)dg \\
&= \frac{d}{dt}\bigg|_{t=0} \int_G \phi(g \exp_G(tX))\mathcal{R}(g)dg = \frac{d}{dt}\bigg|_{t=0} \int_G \phi(g)\mathcal{R}(g \exp_G(-tX))dg \\
&= \int_G \phi(g)\mathcal{R}(g)\mathcal{R}(-X)dg = \mathcal{R}(\phi) \circ \mathcal{R}(-X),
\end{aligned}$$

where we used the fact that we are acting in $\mathcal{H}^\infty \subseteq D(\mathcal{R}(-X))$, together with the Lemma 2.5.9. □

We conclude this section proving some properties of the operators $\mathcal{R}(\phi)$ that justify their interpretation as Fourier transforms.

Proposition 2.5.13. *Let G be a unimodular Lie group and $(\mathcal{R}, \mathcal{H})$ a unirrep of G. For all $\phi, \psi \in \mathcal{C}_c^\infty(G)$ the following statements hold true.*

(i) $\mathcal{R}(\phi \circ R_g) = \mathcal{R}(\phi) \circ \mathcal{R}(g)^*$, *for all $g \in G$.*

(ii) *Setting* $\phi^*(g) := \overline{\phi(g^{-1})}$ *we have* $\mathcal{R}(\phi^*) = \mathcal{R}(\phi)^*$.

(iii) $\mathcal{R}(\phi * \psi) = \mathcal{R}(\phi) \circ \mathcal{R}(\psi)$.

Proof. (i) With a change of variable, we have:

$$\mathcal{R}(\phi \circ R_g) = \int_G \phi(g'g)\mathcal{R}(g')dg'$$
$$= \int_G \phi(g')\mathcal{R}(g') \circ \mathcal{R}(g^{-1})dg' = \mathcal{R}(\phi) \circ \mathcal{R}(g)^*.$$

(ii) Let $v, w \in \mathcal{H}$. Denoting with $\langle \cdot, \cdot \rangle$ the scalar product of $\mathcal{H}$, we have:

$$\langle v, \mathcal{R}(\phi^*)w \rangle = \int_G \langle v, \overline{\phi(g^{-1})}\mathcal{R}(g)w \rangle dg = \int_G \phi(g^{-1})\langle \mathcal{R}(g^{-1})v, w \rangle dg.$$

And we conclude applying the change of variable $g^{-1} \mapsto g$, using Proposition 1.2.9.

(iii) By definition, for $v \in \mathcal{H}$ we have:

$$\mathcal{R}(\phi) \circ \mathcal{R}(\psi)v = \int_G \phi(g)\mathcal{R}(g) \left(\int_G \psi(g')\mathcal{R}(g')vdg' \right) dg.$$

Using the definition of representation, together with the invariance of dg' and changing the order of integration we have:

$$\mathcal{R}(\phi) \circ \mathcal{R}(\psi)v = \int_G \phi(g) \left(\int_G \psi(g')\mathcal{R}(gg')vdg' \right) dg$$
$$= \int_G \phi(g) \left(\int_G \psi(g^{-1}g')\mathcal{R}(g')vdg' \right) dg$$
$$= \int_G \left(\int_G \phi(g)\psi(g^{-1}g')dg \right) \mathcal{R}(g')dg' = \mathcal{R}(\phi * \psi)(g).$$

□

2.5.4 The case of basis realizations

In this section, we consider basis realizations of the representations, i.e. realizations in spaces $L^2(\mathbb{R}^m; \mathbb{C})$, obtained as described in Section 2.2.3.

We start with a computation that shows how elements $X \in \mathfrak{g}$ can be represented as polynomial differential operators on $L^2(\mathbb{R}^m; \mathbb{C})$. Let G be a nilpotent Lie group with Lie algebra $\mathfrak{g}$, and consider a unirrep $\mathcal{R} = \mathcal{R}_{\eta,\mathfrak{h}}$, realized in $L^2(\mathbb{R}^m; \mathbb{C})$ as in Section 2.2.3; again we let:

$$\dim(\mathfrak{h}) = k, \quad m = n - k.$$

Let $\{X_1, \ldots, X_n\}$ be the weak Malcev basis for $\mathfrak{g}$ through $\mathfrak{h}$ chosen in the construction of the representation, and define the corresponding polynomial coordinate map $\psi : \mathbb{R}^n \to G$, which has also polynomial inverse.

Define the polynomials $P_i(t, s)$ for $i = 1, \ldots, n$, by:

$$\psi^{-1}(\psi(t)\psi(s)) = (P_1(t, s), \ldots, P_n(t, s)), \quad t, s \in \mathbb{R}^n.$$

For any $1 \leq q \leq n$, fixing $f \in \mathrm{D}(\mathcal{R}(X_q)) \subseteq L^2(\mathbb{R}^d)$, we have by definition:

$$[\mathcal{R}(X_q)f](\theta) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} [\mathcal{R}(\exp_G(\varepsilon X_q))f](\theta).$$

Identifying as usual $f(e^{\theta_1 X_{k+1}} \dots e^{\theta_m X_n}) \approx f(\theta_1, \dots, \theta_m)$, by definition of P_i it holds :

$$[\mathcal{R}(\exp_G(\varepsilon X_q))f](\theta) = \\ e^{i\langle \eta, P_1(t(\theta),s(\varepsilon))X_1 + \dots + P_k(t(\theta),s(\varepsilon))X_k\rangle} f(P_{k+1}(t(\theta), s(\varepsilon)), \dots, P_n(t(\theta), s(\varepsilon))),$$

where $t(\theta) = (0, \dots, 0, \theta_1, \dots, \theta_m)$ and $s(\varepsilon) = (0, \dots, 0, \varepsilon, 0, \dots, 0)$, where only the q-th entry is nonvanishing .

Finally, taking the derivative at $\varepsilon = 0$ we find:

$$[\mathcal{R}(X_q)f](\theta) = \sum_{j=1}^{k} i\langle \eta, X_j\rangle \frac{\partial P_j}{\partial s_q}(t(\theta), 0) f(\theta) + \sum_{j=1}^{m} \frac{\partial P_{k+j}}{\partial s_q}(t(\theta), 0) \frac{\partial f}{\partial \theta_j}(\theta). \quad (2.16)$$

Remark 2.5.14. For computations it is useful to notice that (2.16) is coincides with the derivative at zero of $[\mathcal{R}(\psi(x))f](\theta)$ computed with respect to the variable x_q.
The formula (2.16) shows how to represent an element X_q of the basis as a polynomial differential operator on $L^2(\mathbb{R}^m; \mathbb{C})$. It is possible to show that the represetations of left-invariant differential operators on G *cover* all the polynomial differential operators on $L^2(\mathbb{R}^m; \mathbb{C})$. Namely, denoted by $\mathcal{P}(\mathbb{R}^m)$ the algebra of such operators, it is possible to obtain a surjective map (with a small abuse of notation):

$$\mathcal{R} : \mathfrak{u}(\mathfrak{g}) \to \mathcal{P}(\mathbb{R}^m).$$

We state here the main result, see [7] for a detailed exposition.

Theorem 2.5.15. *In the above setting, taking* any *Malcev basis through* $\mathfrak{h}$*, we have:*

(i) $\mathcal{R} : \mathfrak{u}(\mathfrak{g}) \to \mathcal{P}(\mathbb{R}^m)$ *is a surjective map.*

(ii) For any $X \in \mathfrak{g}$*, the corresponding* $\mathcal{R}(X)$ *is a differential operator of degree either* 0 *or* 1*.*

(iii) The polynomial coefficients of $\mathcal{R}(X)$ *depend polynomially on* X*.*

Moreover, the set of smooth vectors $\mathcal{H}^\infty$ *coincides with* $\mathcal{S}(\mathbb{R}^m)$*.*

2.5.5 Application to N_4, spectral properties of the sub-Laplacian

We apply the theoretical results obtained in this section to the group N_4, with a particular interest in studying the representations of the sub-Laplacian.

We consider only the representations corresponding to 4-dimensional orbits in $\mathfrak{n}_4^*$, described in (2.10). For each value of $\alpha \neq 0, \gamma \in \mathbb{R}$, we denote the corresponding representation with the symbol $\mathcal{R}_{\alpha,\gamma}$.

With the choice of Malcev coordinates made in Section 2.3 (here P_i corresponds to the coordinate x_i), rewriting the formula (2.16), we find:

$$[\mathcal{R}_{\alpha,\gamma}(X_q)f](\theta) = i\alpha \frac{\partial P_0}{\partial s_q}(t(\theta), 0) f(\theta) + i\frac{\gamma}{\alpha}\frac{\partial P_2}{\partial s_q}(t(\theta), 0) f(\theta) + \\ + \frac{\partial P_1}{\partial s_q}(t(\theta), 0)\frac{\partial f}{\partial \theta_1}(\theta) + \frac{\partial P_3}{\partial s_q}(t(\theta), 0)\frac{\partial f}{\partial \theta_3}(\theta),$$

where $\theta = (\theta_1, \theta_3)$.
Hence:

$$\mathcal{R}_{\alpha,\gamma}(X_1) = \partial_{\theta_1} \quad \mathcal{R}_{\alpha,\gamma}(X_2) = i\left(\frac{\gamma}{\alpha} - \alpha\theta_1\theta_3\right) \quad \mathcal{R}_{\alpha,\gamma}(X_3) = \partial_{\theta_3},$$

and the sub-Laplacian reads:

$$\mathcal{R}_{\alpha,\gamma}(-\Delta_{N_4}) = -\partial_{\theta_1}^2 - \partial_{\theta_3}^2 + \left(\frac{\gamma}{\alpha} - \alpha\theta_1\theta_3\right)^2.$$

Let's prove that the representation of the sub-Laplacian has purely discrete spectrum. We adapt the first argument presented in [19], which corresponds to the case $\gamma = 0, \alpha = 1$.

For simplifying the notation, we study the class of operators acting on $L^2(\mathbb{R}^2)$:

$$H = -\Delta + (a - bxy)^2, \quad (a, b) \in \mathbb{R} \times \mathbb{R}^*.$$

Following our convention, we should define H on the domain $\mathcal{S}(\mathbb{R}^2)$, but being it symmetric and positive we can work on its unique self-adjoint (Friedrichs) extension. The domain of the associated quadratic form is:

$$Q(H) = \{u \in H^1(\mathbb{R}^2) \mid (a - bxy)|u(x, y)| \in L^2(\mathbb{R}^2)\},$$

and the quadratic form reads:

$$q_H(u) = \int_{\mathbb{R}^2} |\nabla u(x, y)|^2 + (a - bxy)^2 |u(x, y)|^2 dxdy, \quad u \in Q(H).$$

Notice that, in the sense of quadratic forms, we have:

$$\begin{aligned} -\Delta + (a - bxy)^2 &= \frac{1}{2}\left(-\partial_x^2 + (a - bxy)^2\right) + \frac{1}{2}\left(-\partial_y^2 + (a - bxy)^2\right) - \frac{1}{2}\Delta \quad (2.17) \\ &\geq \frac{1}{2}(-\Delta + |b|(|y| + |x|)) =: H_1, \end{aligned}$$

where we have used the property of the harmonic oscillator:

$$-\frac{d^2}{dx^2} + (c - \omega x)^2 \geq |\omega|, \quad c \in \mathbb{R}, \omega \neq 0.$$

The operator H_1 has purely discrete spectrum, since it is a Schrödinger operator with potential $V = |b|(|x| + |y|)$ that tends to $+\infty$ at infinity. The inequality (2.17) implies that the form domain of H is cointained in the form domain of H_1. To deduce that also H has purely discrete spectrum, we recall the following characterization[8] of the minimum of the essential spectrum σ_{ess}.

Lemma 2.5.16. *Let $(A, D(A))$ be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and indicate with $Q(A)$ the domain of the associated quadratic form q_A. Assuming that A is bounded below, we have:*

$$\min \sigma_{ess}(A) = \min\left\{\liminf_{n\to\infty} q_A(v_n) \;\middle|\; (v_n)_n \in Q(A)^{\mathbb{N}},\ \|v_n\|_{\mathcal{H}} = 1,\ v_n \rightharpoonup 0\right\},$$

with the convention that both the sides of the equation take value $+\infty$ if $\sigma_{ess}(A) = \emptyset$.

[8]See [15], Chapter 5.

In particular, the previous result implies that:

$$\liminf_{n\to\infty} q_{H_1}(v_n) = +\infty \quad \forall (v_n)_n \in Q(H_1)^{\mathbb{N}},\ \|v_n\|_{\mathcal{H}} = 1,\ v_n \rightharpoonup 0$$

and since $Q(H) \subseteq Q(H_1)$ we deduce that the same must hold for H, hence $\sigma_{ess}(H) = \emptyset$, i.e. H has purely discrete spectrum.

Remark 2.5.17. A more direct proof could be also performed applying the result denoted as Theorem 3.5 in [16].

Chapter 3

Plancherel measure

Let G be a nilpotent Lie group and fix $\mathcal{R}_{\eta,\mathfrak{h}}$ a unirrep of G. In the previous chapter we discussed the operators:

$$\mathcal{R}_{\eta,\mathfrak{h}}(\phi) = \int_G \phi(g)\mathcal{R}_{\eta,\mathfrak{h}}(g)dg,$$

that can be interpreted as Fourier transforms of ϕ, and one may ask if usual properties of the Fourier transform (Plancherel theorem, Fourier inversion, ...) hold. The answer is positive, and this chapter is devoted to the exposition this theory, following [9]. As we will see, a central role is played by the classification of the representations given by the orbit method.

3.1 The Trace Theorem

Let G be a nilpotent Lie group and indicate with $\mathcal{R}_{\eta,\mathfrak{h}}$ its unitary, continuous and irreducible representations. The goal of this section is to prove that for all $\eta \in \mathfrak{g}^*$ there exists an invariant measure ν_η on the coadjoint orbit $\mathcal{O}_\eta$ satisfying the following equality:

$$\mathrm{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) = \int_{\mathcal{O}_\eta} \mathscr{F}(\phi \circ \exp_G)(\lambda)d\nu_\eta(\lambda) \quad \forall \phi \in \mathcal{S}(G),$$

where $\mathscr{F}$ denotes the Euclidean Fourier transform. We will give the proof only of the most important result: for the missing ones, see [9].

Notation 14. In the following, for $f \in \mathcal{S}(\mathbb{R}^n)$ the Euclidean Fourier transform is defined:

$$\mathscr{F}f(\xi) = \int_{\mathbb{R}^n} f(x)e^{i\langle \xi,x\rangle}dx.$$

3.1.1 Parametrizations and invariant measures on orbits

We first discuss different ways to define invariant measures on the coadjoint orbits.

In Section 1.3.2 we introduced the coadjoint action of G on $\mathfrak{g}^*$, which is a *left* action (i.e. $(Ad^*)_{gh} = (Ad^*)_g \circ (Ad^*)_h$). For a reason that will be clear later (see Remark 3.1.6) we will need to consider the *right* action:

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, \quad (g, \eta) \mapsto (Ad^*)_{g^{-1}}(\eta).$$

Of course, the left and right actions give the same orbits in $\mathfrak{g}^*$. Fixed $\eta \in \mathfrak{g}^*$, we can identify its coadjoint orbit with the quotient $R_\eta \backslash G$ (whose elements are the *right* cosets $R_\eta g,\ g \in G$):

$$\mathcal{O}_\eta \approx R_\eta \backslash G,$$

where $R_\eta = \mathrm{stab}_G(\eta) = \exp_G(\mathfrak{r}_\eta)$. The identification can be realized via the map:

$$R_\eta \backslash G \to \mathcal{O}_\eta, \quad R_\eta g \mapsto (Ad^*)_{g^{-1}}(\eta), \tag{3.1}$$

which is a bijection (see [7], pag. 26-27). In particular, the relation (3.1) determines a bijection between invariant measures on $R_\eta \backslash G$ and on $\mathcal{O}_\eta$.

Since R_η is a closed subgroup, we can put coordinates on $R_\eta \backslash G$ and define on it an invariant measure as explained in Section 1.2.2. Such measure is unique up to a positive factor. Using (3.1), we deduce also the uniqueness up to a positive factor of invariant measure on $\mathcal{O}_\eta$. However, once a measure on $R_\eta \backslash G$ is fixed, it is not easy to obtain explicitly the corresponding measure on $\mathcal{O}_\eta$, since the bijection (3.1) is not easily expressed in coordinates.

We now discuss an alterntive way to define an invariant measure on $\mathcal{O}_\eta$.

Proposition 3.1.1. *Let G be a nilpotent Lie group, $\eta \in \mathfrak{g}^*$ with $\dim(\mathcal{O}_\eta) = d$ and fix $\mathcal{X} = \{X_1, \ldots, X_n\}$ a strong Malcev basis for $\mathfrak{g}$. There exist n polynomial maps $P_1, \ldots, P_n$ and d indices $\{j_1, \ldots, j_d\}$ such that:*

- $\mathcal{O}_\eta = \{\sum_{j=1}^n P_j(x) X_j^* \mid x \in \mathbb{R}^d\}$
- $P_{j_k}(x) = x_{j_k}$ *for $k = 1, \ldots, d$*
- *P_j depends only on the x_k's such that $j_k \leq j$.*

This polynomial parametrization of the orbit permits to define another measure on $\mathcal{O}_\eta$, which turns out to be G-invariant.

Proposition 3.1.2. *Let $G, \eta, \mathcal{X}, P_j$ as above. The equality:*

$$\int_{\mathcal{O}_\eta} \varphi(\lambda) d\mu_\eta(\lambda) = \int_{\mathbb{R}^d} \varphi\left(\sum_{j=1}^n P_j(x) X_j^*\right) dx, \quad \forall \varphi \in \mathcal{C}_c(\mathfrak{g}^*)$$

defines a G-invariant measure μ_η on $\mathcal{O}_\eta$.

In view of the previous discussion, we deduce the following: once we fix an invariant measure on $\mathcal{O}_\eta$, inherited from the identification (3.1), the measure μ_η must be one of its multiples. Later we will need to determine explicitly this constant.

3.1.2 Trace Theorem

We start pointing out that the operators $\mathcal{R}_{\eta,\mathfrak{h}}(\phi)$ are trace-class whenever $\phi \in \mathcal{S}(G)$, and their kernel is a Schwartz function.

Definition 3.1.3. Let G be a nilpotent Lie group, the *Schwartz space* $\mathcal{S}(G)$ is the space of functions ϕ defined on G such that $\phi \circ \psi \in \mathcal{S}(\mathbb{R}^n)$, where $\psi : \mathbb{R}^n \to G$ is any polynomial coordinate map.

Proposition 3.1.4. *Let $\mathcal{R}_{\eta,\mathfrak{h}}$ be a unirrep of the nilpotent Lie group G, realized in $L^2(\mathbb{R}^m;\mathbb{C})$ for a particular $m \in \mathbb{N}$. Letting $\phi \in \mathcal{S}(G)$, the operator $\mathcal{R}_{\eta,\mathfrak{h}}(\phi)$ is trace-class on $L^2(\mathbb{R}^m;\mathbb{C})$ and the kernel belongs to the space $\mathcal{S}(\mathbb{R}^m \times \mathbb{R}^m)$.*

Let us accept this result and show a computation that will be useful in the proof of the Trace Theorem.

Fix one unirrep $\mathcal{R}_{\eta,\mathfrak{h}}$, letting as usual $H = \exp_G(\mathfrak{h})$, and assume from now on that $\mathfrak{h}$ is an *ideal* of $\mathfrak{g}$ (we will need to choose strong Malcev bases through $\mathfrak{h}$). Let $\phi \in \mathcal{S}(G)$ and consider the realization of $\mathcal{R}_{\eta,\mathfrak{h}}$ in the space W discussed in Section 2.2.3. Fixing a function $f \in W$, letting dg, dh denote a Haar measure on G, H and $d\mu$ an invariant measure on $H\backslash G$ (such that Theorem 1.2.12 holds), we have:

$$\begin{aligned}
[\mathcal{R}_{\eta,\mathfrak{h}}(\phi)f](t) &= \int_G \phi(g)[\mathcal{R}_{\eta,\mathfrak{h}}(g)f](t)dg = \int_G \phi(g)f(tg)dg \\
&= \int_G \phi(t^{-1}g)f(g)dg = \int_{H\backslash G}\left(\int_H \phi(t^{-1}hg)f(hg)dh\right)d\mu(Hg) \\
&= \int_{H\backslash G}\left(\int_H \phi(t^{-1}hg)\chi_{\eta,\mathfrak{h}}(h)dh\right)f(g)d\mu(Hg),
\end{aligned}$$

where we used the definition of induced representation, together with Theorem 1.2.12. Recall now that we can identify $L^2(H\backslash G; d\mu)$ with $L^2(\mathbb{R}^m;\mathbb{C})$, where usual properties of the trace hold. Therefore, letting $\phi_{\eta,\mathfrak{h}}(t,g) := \int_H \phi(t^{-1}hg)\chi_{\eta,\mathfrak{h}}(h)dh$ we find the explicit expression for the trace:

$$\operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) = \int_{H\backslash G} \phi_{\eta,\mathfrak{h}}(g,g)d\mu(Hg).$$

We are now ready to prove the Trace Theorem. The proof will not be constructive, but the measure ν_η will be explicited later.

Theorem 3.1.5 (Trace)**.** *Let G be a nilpotent Lie group and $\mathcal{R}_{\eta,\mathfrak{h}}$ a unirrep of G. There exists a unique G-invariant measure ν_η on $\mathcal{O}_\eta$ such that:*

$$\operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) = \int_{\mathcal{O}_\eta} \mathscr{F}(\phi \circ \exp_G)(\lambda)d\nu_\eta(\lambda), \quad \phi \in \mathcal{S}(G).$$

Proof. We let $\dim(\mathfrak{h}) = k$, $n - k = m$.
By the previous discussion, denoting $\mu_{H,G}$ the invariant measure on $H\backslash G$ we have:

$$\begin{aligned}
\operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) &= \int_{H\backslash G} \phi_{\eta,\mathfrak{h}}(g,g)d\mu_{H,G}(Hg) \\
&= \int_{H\backslash G}\left(\int_H \phi(g^{-1}hg)\chi_{\eta,\mathfrak{h}}(h)dh\right)d\mu_{H,G}(Hg).
\end{aligned}$$

Choosing exponential coordinates on H and using that conjugations have Jacobian determinant equal to 1 on H, we can rewrite:

$$\begin{aligned}
\operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) &= \int_{H\backslash G}\left(\int_H \phi(h)\chi_\eta(ghg^{-1})dh\right)d\mu_{H,G} \qquad (3.2)\\
&= \int_{H\backslash G}\left(\int_{\mathfrak{h}} \phi \circ \exp_G(X)e^{i\langle \eta, Ad_g(X)\rangle}dX\right)d\mu_{H,G} \\
&= \frac{1}{(2\pi)^{n-k}}\int_{H\backslash G}\left(\int_{\mathfrak{h}^\perp} \mathscr{F}(\phi \circ \exp_G)((Ad^*)_{g^{-1}}(\eta+\lambda))d\lambda\right)d\mu_{H,G},
\end{aligned}$$

where in the last line we used a consequence of the Fourier inversion formula (see Lemma 3.1.7 below).

Let's now fix a strong Malcev basis through $\mathfrak{h}$; thanks to the equality $Ad^*(H)(\eta) = \eta + \mathfrak{h}^\perp$ (Lemma 1.3.26), we deduce that the only elements of the dual basis whose action is nontrivial on η are spanned by the last m elements $X^*_{k+1}, \dots, X^*_n$. Hence we can use Proposition 3.1.2 to find:

$$\int_{\mathfrak{h}^\perp} \varphi(\eta + \lambda) d\lambda = \int_{\mathbb{R}^m} \varphi\left(\eta + \sum_{j=k+1}^{n} x_j X_j^*\right) dx = \int_{R_\eta \backslash H} \varphi((Ad^*)_{h^{-1}}(\eta)) d\mu_{R_\eta, H}(R_\eta h),$$

for all $\varphi \in \mathcal{S}(\mathfrak{g}^*)$, where $\mu_{R_\eta,H}$ denotes an H-invariant measure on $R_\eta \backslash H$. We can finally rewrite:

$$\operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) = \frac{1}{(2\pi)^{n-k}} \int_{H\backslash G} \left(\int_{R_\eta \backslash H} \mathscr{F}(\phi \circ \exp_G)((Ad^*)_{g^{-1}}(Ad^*)_{h^{-1}}(\eta)) d\mu_{R_\eta,H} \right) d\mu_{H,G}$$

and use Theorem 1.2.14 to deduce the existence of a measure ν_η on $\mathcal{O}_\eta$ such that:

$$\operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) = \int_{\mathcal{O}_\eta} \mathscr{F}(\phi \circ \exp_G)(\lambda) d\nu_\eta(\lambda) \quad \forall \phi \in \mathcal{S}(G),$$

as we wanted. □

Remark 3.1.6. The presence of the term $(Ad^*)_{g^{-1}}(\eta + \lambda)$ in the last line of (3.2) is the reason why we are forced to consider the *right* action of G on $\mathfrak{g}^*$ in place of the usual one.

Lemma 3.1.7. *Let $\mathfrak{g}$ be a vector space and fix a subspace $V \subseteq \mathfrak{g}$ with* $\dim(V) = k$. *Let v' indicate the generic element of $V^\perp \subseteq \mathfrak{g}^*$. For any $\eta \in \mathfrak{g}^*$ it holds:*

$$\int_V f(v) e^{i\langle \eta, v\rangle} dv = \frac{1}{(2\pi)^{n-k}} \int_{V^\perp} \mathscr{F} f(\eta + v') dv' \quad \forall f \in \mathcal{S}(\mathfrak{g}).$$

Proof. Fix a decomposition $\mathfrak{g} = V \oplus W$. By Fourier inversion, writing $\mathfrak{g}^* \ni \lambda = v' + w'$ we have:

$$\begin{aligned} f(v) &= \frac{1}{(2\pi)^n} \int_{\mathfrak{g}^*} \mathscr{F} f(\lambda) e^{-i\langle \lambda, v\rangle} d\lambda \\ &= \frac{1}{(2\pi)^n} \int_{W^\perp} \int_{V^\perp} \mathscr{F} f(v' + w') e^{-i\langle w', v\rangle} dv' dw'. \end{aligned}$$

Multiplying by $e^{i\langle \eta, v\rangle}$, integrating in dv and using the Fourier inversion on $V, W^\perp$:

$$\int_V f(v) e^{i\langle \eta, v\rangle} dv = \frac{1}{(2\pi)^{n-k}} \int_{V^\perp} \mathscr{F} f(\eta + v') dv'.$$

□

3.2 Fourier inversion formula and Plancherel Theorem

The goal of this Section is to prove the following (Fourier inversion) equality:

$$\phi(1_G) = \frac{1}{(2\pi)^{n-\frac{d}{2}}} \int_Q \mathrm{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi))|\,\mathrm{Pf}(\eta)|d\eta, \quad \phi \in \mathcal{S}(G),$$

where d is the maximal dimension of a coadjoint orbit, Q is a subspace of $\mathfrak{g}^*$ and $|\,\mathrm{Pf}(\eta)|d\eta$ is a *Plancherel* measure on Q to be specified. To obtain it, we will select a Zariski open $U \subseteq \mathfrak{g}^*$, which is the union of the so-called *generic* orbits, such that each orbit in U intersects Q in exactly one point. We will finally give a *simultaneous* parametrization of the orbits in U and the result will follow applying the Trace Theorem on each of these orbits.

3.2.1 Generic orbits, simultaneous parametrization

This section is devoted to the definition of Q, U and the *simultaneous* parametrization.

Fix $\mathcal{X} = \{X_1, \dots, X_n\}$ a strong Malcev basis for $\mathfrak{g}$ and let $\mathfrak{g}_j := \mathrm{span}\{X_1, \dots, X_j\}$. For any $\eta \in \mathfrak{g}^*$ define the set of indices:

$$I^\eta := \{1 \leq j \leq n \mid \mathfrak{r}_\eta + \mathfrak{g}_{j-1} \neq \mathfrak{r}_\eta + \mathfrak{g}_j\}.$$

For any subset $I \subseteq \{1, \dots, n\}$ define:

$$\mathfrak{g}^*_I := \{\eta \in \mathfrak{g}^* \mid I^\eta = I\}$$

and let $\mathcal{I}^{\mathcal{X}} := \{I \subseteq \{1, \dots, n\} \mid \mathfrak{g}^*_I \neq \emptyset\}$. On $\mathcal{I}^{\mathcal{X}}$ we define an ordering as follows. Given $I = \{i_1, \dots, i_k\}, J = \{j_1, \dots, j_h\} \in \mathcal{I}^{\mathcal{X}}$ we say that $I > J$ if

$$\begin{cases} k > h \quad \textit{or} \\ k = h \text{ and there exists } 1 \leq r \leq k \text{ s.t. } i_1 = j_1, \dots, i_{r-1} = j_{r-1}, i_r < j_r. \end{cases}$$

From $\mathcal{I}^{\mathcal{X}}$, we select the maximal element $I^{max} = \{j_1, \dots, j_d\}$.

Definition 3.2.1. Fix $J \in \mathcal{I}^{\mathcal{X}}$. For $\eta \in \mathfrak{g}^*$ we define the *Pfaffian*:

$$\mathrm{Pf}_J(\eta) = (\det(\langle \eta, [X_i, X_j] \rangle_{i,j \in J}))^{1/2}.$$

Proposition 3.2.2. *Let $J \in \mathcal{I}^{\mathcal{X}}$. We have:*

$$\mathfrak{g}^*_J = \{\eta \in \mathfrak{g}^* \mid \mathrm{Pf}_J(\eta) \neq 0,\ \mathrm{Pf}_I(\eta) = 0 \quad \forall I > J\}.$$

The proof is based on the following result, together with the fact that for $\eta \in \mathfrak{g}^*$ the bilinear and skew-symmetric form $B_\eta(X, Y) = \langle \eta, [X, Y] \rangle$ is nondegenerate on $\mathfrak{g}/\mathfrak{r}_\eta$ (see Section 1.3.3).

Lemma 3.2.3. *Let V be an n-dimensional vector space and ω a nondegenerate skew-symmetric bilinear form on V. A family of vectors $\{Z_1, \dots, Z_n\} \subseteq V$ is linear independent if and only if:*

$$\det(\omega(Z_i, Z_j)_{i,j=1,\dots,n}) \neq 0.$$

Proof of Proposition 3.2.2. First we prove "$\subseteq$". Let $J = \{j_1, \dots, j_h\} \in \mathcal{I}^{\mathcal{X}}$ and fix $\eta \in \mathfrak{g}^*_J$. By Lemma 3.2.3 we have $\mathrm{Pf}_J(\eta) \neq 0$, since by definition $\{X_j \mid j \in J\}$ is linearly independent modulo $\mathfrak{r}_\eta$. Choose now $I = \{i_1, \dots, i_k\} > J$; there are two possibilites:

- If $|I| > |J|$ then $\{X_i \mid i \in I\}$ is not linearly independent modulo $\mathfrak{r}_\eta$, hence by Lemma 3.2.3 we deduce $\mathrm{Pf}_I(\eta) = 0$.
- If $|I| = |J|$ then there exists r such that $i_1 = j_1, \dots, i_{r-1} = j_{r-1}, i_r < j_r$. This means, by definition of $\mathfrak{g}^*_J$, that $\{X_{i_1}, \dots, X_{i_r}\}$ is not linear independent modulo $\mathfrak{r}_\eta$, hence $\mathrm{Pf}_I(\eta) = 0$.

Let's prove "$\supseteq$". Assume $\mathrm{Pf}_J(\eta) \neq 0$ and $\mathrm{Pf}_I(\eta) = 0$ for all $I > J$. Then:

- $J \subseteq I^\eta$: if not, the smallest index $j \in J \setminus I^\eta$ would give $X_j \in \mathfrak{r}_\eta \mod \mathfrak{g}_{j-1}$, from which $\mathrm{Pf}_J(\eta) = 0$.
- $J = I^\eta$: assume by contradiction $|J| < |I^\eta|$; then $J < I^\eta$ and $\mathrm{Pf}_{I^\eta}(\eta) = 0$, which is a contradiction.

$\square$

Corollary 3.2.4. We have: $\mathfrak{g}^*_{I^{max}} = \{\eta \in \mathfrak{g}^* \mid \mathrm{Pf}_{I^{max}}(\eta)^2 \neq 0\}$. In particular, $\mathfrak{g}^*_{I^{max}}$ is a Zariski open set, whose complement has zero Lebesgue measure.

Notation 15. From now on, we will denote

$$U := \mathfrak{g}^*_{I^{max}}, \quad \mathrm{Pf}(\eta) := \mathrm{Pf}_{I^{max}}(\eta), \quad |I^{max}| = d.$$

The following proposition describes the *simultaneous parametrization* of the orbits contained in U. We state it for I^{max}, but the same holds true for any $I \in \mathcal{I}^{\mathcal{X}}$.

Proposition 3.2.5. *There exist n rational nonsingular functions, that we denote by $R_j = R^{I^{max}}_{j,\mathcal{X}} : \mathbb{R}^d \times \mathfrak{g}^* \to \mathbb{R}$, satisfying the following properties.*

(i) For $\eta \in \mathfrak{g}^*$ fixed, the function $R_j(\cdot, \eta)$ is polynomial.

(ii) For $\eta \in \mathfrak{g}^*$ fixed, $R_{j_i}(\cdot, \eta) = x_i$ for all $j_i \in I^{max}$.

(iii) If $j_i < j < j_{i+1}$, the map $R_j(\cdot, \eta)$ depends only on $x_1, \dots, x_i$.

(iv) For all $\eta \in U$, the maps $R_j(\cdot, \eta)$ depend only on the coadjoint orbit of η and send $\mathbb{R}^d$ on $\mathcal{O}_\eta$ diffeomorphically, namely:

$$\mathcal{O}_\eta = \left\{ \sum_{j=1}^n R_j(x, \eta) X_j^* \;\middle|\; x \in \mathbb{R}^d \right\}.$$

In particular, for $j \notin I^{max}$ it holds $R_j(x, \lambda) = \lambda_j + R'_j(\lambda_1, \dots, \lambda_{j-1})$, where again R'_j is a rational nonsingular map.

We now define the vector subspace $Q \subseteq \mathfrak{g}^*$ letting:

$$Q := \left\{ \mathfrak{g}^* \ni \eta = \sum_{j=1}^n \eta_j X_j^* \;\middle|\; \eta_j = 0 \text{ for } j \in I^{max} \right\}.$$

With this definition, thanks to Proposition 3.2.5-*(ii)* we deduce that each orbit in U intersects Q in **exactly one point** η_0, that is:

$$\eta_0 = \sum_{j=1}^n R_j(0,\eta) X_j^*.$$

Remark 3.2.6. Notice that for $\eta \in U$ fixed, the properties *(i),(ii),(iii),(iv)* of Proposition 3.2.5 coincide with those of the P_j's in Proposition 3.1.1, with $\{j_1, \dots, j_d\} = I^{max}$. In particular, we can use the $R_j(\cdot, \eta)$'s to define an invariant measure on $\mathcal{O}_\eta$ as in Proposition 3.1.2.

Remark 3.2.7. The intersection $U \cap Q$ is used to parametrize the set U. We also have:

$$U \cap Q = \{\eta \in Q \mid \mathrm{Pf}(\eta) \neq 0\}.$$

Practical rules to identify I^{max}

We collect here some rules that may be useful to identify the set I^{max}. We know that $|I^{max}| = d$, where d denotes the maximal dimension of a coadjoint orbit in $\mathfrak{g}^*$. We can determine I^{max} in following way.

- Having classified the coadjoint orbits, consider only the d-dimensional ones.
- For each of these orbit, fix one element η and write its radical $\mathfrak{r}_\eta$ in the basis $\mathcal{X}$. Notice that we always have $\zeta(\mathfrak{g}) \subseteq \mathfrak{r}_\eta$.
- For each η, compute the set of indices I^η and select the maximal one I^{max}.
- As described above, U is the set of orbits whose representative verifies $I^\eta = I^{max}$, $Q = \mathrm{span}\{X_i^* \mid i \notin I^{max}\}$.

3.2.2 Fourier inversion, Plancherel Theorem

From now on, we adopt the following notation. As usual, we let G be a nilpotent Lie group, $\dim(G) = n$, and we fix (for the moment) $\mathcal{R}_{\eta,\mathfrak{h}}$ a unirrep of G. If $\eta \in U$ we let

$$\dim(\mathcal{O}_\eta) = |I^{max}| = d.$$

If $\mathfrak{h}$ $(\supseteq \mathfrak{r}_\eta)$ is a polarizing subalgebra for η, we let

$$\dim(\mathfrak{h}) = k, \quad m = n - k.$$

Again we need to assume that $\mathfrak{h}$ is also an ideal. In particular, by definition of polarizing subalgebra $m = \frac{1}{2}(\dim(\mathfrak{g}) - \dim(\mathfrak{r}_\eta)) = \frac{d}{2}$.

Fix a strong Malcev basis $\mathcal{X}$ as above and consider the corresponding functions $R_j = R_{j,\mathcal{X}}^{I^{max}}$. For each $\eta \in \mathfrak{g}^*$ we can choose two possible invariant measures on the coadjoint orbit $\mathcal{O}_\eta$:

- The measure ν_η, given by the Trace Theorem.
- Following Remark 3.2.6, the invariant measure μ_η described by:

$$\int_{\mathcal{O}_\eta} \varphi(\lambda) d\mu_\eta(\lambda) = \int_{\mathbb{R}^d} \varphi\left(\sum_{j=1}^{n} R_j(t,\eta)\right) dt \quad \forall \varphi \in \mathcal{C}_c(\mathcal{O}_\eta).$$

As we will see, to prove the final theorem we need to determine the factor relating μ_η and ν_η. This is the goal of the next proposition.

Proposition 3.2.8. *Let $\eta \in U$; with the above defined notation, we have the equality:*

$$\mu_\eta = (2\pi)^{\frac{d}{2}} |\operatorname{Pf}(\eta)| \ \nu_\eta.$$

Proof. The proof consists in expliciting two changes of variable. For $\eta \in U$ fix a polarizing subalgebra $\mathfrak{h}$ ($\supseteq \mathfrak{r}_\eta$). Consider the basis:

$$\mathcal{Y} := \{Y_1, \dots, Y_n\} = \{\underbrace{\overbrace{E_1, \dots, E_r}^{}}_{\text{basis of } \mathfrak{r}_\eta}\!\!\overbrace{, C_1, \dots, C_m}^{\text{basis of } \mathfrak{h}}, D_1, \dots, D_m\}$$

so that $\{E_1, \dots, E_r\}$ is a basis of $\mathfrak{r}_\eta$, $\{E_1, \dots, E_r, C_1, \dots, C_m\}$ is a basis of $\mathfrak{h}$. The proof is divided in two steps.

Step 1. We start from the equality:

$$\operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) = \frac{1}{(2\pi)^{n-k}} \int_{H\backslash G} \left(\int_{\mathfrak{h}^\perp} \mathscr{F}(\phi \circ \exp_G)((Ad^*)_{g^{-1}}(\eta + \lambda)) d\lambda \right) d\mu_{H,G} \ ,$$

which is obtained in the proof of the Trace Theorem, fixing on H and $H\backslash G$ Malcev coordinates associated to $\mathcal{Y}$. Notice that the Lebesgue measure $d\lambda$ on $\mathfrak{h}^\perp$ can be transferred to an invariant measure on $R_\eta\backslash H$ via the correspondence:

$$R_\eta\backslash H \ni \ R_\eta h \mapsto (Ad^*)_{h^{-1}}(\eta) = \eta + \lambda \ \in \eta + \mathfrak{h}^\perp,$$

where we recall that $(Ad^*)(H) = \eta + \mathfrak{h}^\perp$. Such measure must be a multiple of the invariant measure $\mu_{R_\eta,H}$ induced by $\{C_1, \dots, C_m\}$, i.e. we can write:

$$\operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) = \frac{\alpha}{(2\pi)^{n-k}} \int_{H\backslash G} \left(\int_{R_\eta\backslash H} \mathscr{F}(\phi \circ \exp_G)((Ad^*)_{g^{-1}}(Ad^*)_{h^{-1}}(\eta)) d\mu_{R_\eta,H} \right) d\mu_{H,G} \tag{3.3}$$

for some $\alpha \in \mathbb{R}$. To compute α explicitly we consider

$$\Psi : \mathbb{R}^m \to R_\eta\backslash H, \quad t \mapsto R_\eta e^{t_1 C_1} \dots e^{t_m C_m},$$

and we write:

$$(Ad^*)_{\Psi(t)^{-1}}(\eta) = \eta + \sum_{j=1}^{m} Q_j(t) D_j^*.$$

The constant α is given by:

$$\alpha = \left| \det \left(\left(\frac{\partial}{\partial t_i}\bigg|_{t=0} Q_j(t) \right)_{i,j=1,\dots,m} \right) \right|.$$

We have:

$$\frac{\partial}{\partial t_i}\bigg|_{t=0} Q_j(t) = \frac{\partial}{\partial t_i}\bigg|_{t=0} \langle (Ad^*)_{\Psi(t)^{-1}}(\eta), D_j\rangle = \frac{\partial}{\partial t_i}\bigg|_{t=0} \langle \eta, Ad_{e^{t_1C_1}\dots e^{t_mC_m}}(D_j)\rangle$$
$$= \frac{\partial}{\partial t_i}\bigg|_{t=0} \langle \eta, Ad_{e^{t_iC_i}}(D_j)\rangle = \langle \eta, [C_i, D_j]\rangle,$$

hence we conclude that

$$\alpha = |\det(\langle \eta, [C_i, D_j]\rangle_{i,j=1,\dots,m})|.$$

Step 2. We use the definitions of $\mu_{H,G}$ and $\mu_{R_\eta,H}$ to rewrite (3.3):

$$\mathrm{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) = \frac{\alpha}{(2\pi)^{n-k}} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \mathscr{F}(\phi\circ \exp_G)\left(\prod_{i=0}^{m-1}(Ad^*)_{e^{-(t_{m-i}D_{m-i})}} \prod_{h=0}^{m-1}(Ad^*)_{e^{-(s_{m-h}C_{m-h})}}(\eta)\right) dsdt.$$

We now compute the change of variable from the basis $\mathcal{Y}$ to the basis $\mathcal{X}$. We write:

$$\prod_{i=0}^{m-1}(Ad^*)_{e^{-(t_{m-i}D_{m-i})}} \prod_{h=0}^{m-1}(Ad^*)_{e^{-(s_{m-h}C_{m-h})}}(\eta) = \sum_{j=1}^{n} Q_j(t,s)X_j^*,$$

and by Proposition 3.2.5-*(ii)* the change of variable reads:

$$x_i = Q_{j_i}(t,s) \quad i = 1,\dots,d,$$

where $I^{max} = \{j_1, \dots, j_d\}$. This gives rise to a constant $\beta \in \mathbb{R}$ so that for all $\varphi \in \mathcal{C}_c(\mathfrak{g}^*)$:

$$\int_{\mathcal{O}_\eta} \varphi(\lambda)d\mu_\eta(\lambda) = \int_{\mathbb{R}^d} \varphi\left(\sum_{j=1}^{n} R_j(x,\eta)X_j^*\right) dx = \beta \int_{R_\eta\backslash G} \varphi\left((Ad^*)_{g^{-1}}(\eta)\right) d\mu_{R_\eta,G}.$$

We have:

$$\beta = \left|\det\left(\begin{pmatrix} \frac{\partial}{\partial t_p}\Big|_{t=s=0} Q_{j_i}(t,s) & , & \frac{\partial}{\partial s_q}\Big|_{t=s=0} Q_{j_i}(t,s) \end{pmatrix}_{i=1,\dots,d;\ q,p=1,\dots,m}\right)\right|,$$

and we compute

$$\frac{\partial}{\partial t_p}\bigg|_{t=s=0} Q_{j_i}(t,s) = \frac{\partial}{\partial t_p}\bigg|_{t=s=0} \langle (Ad^*)_{e^{-(t_pD_p)}}(\eta), X_{j_i}\rangle = \langle \eta, [D_p, X_{j_i}]\rangle,$$
$$\frac{\partial}{\partial s_q}\bigg|_{t=s=0} Q_{j_i}(t,s) = (\dots) = \langle \eta, [C_q, X_{j_i}]\rangle.$$

Hence:

$$\beta = \left|\det\begin{pmatrix} \langle \eta, [D_p, X_{j_i}]\rangle \\ \langle \eta, [C_q, X_{j_i}]\rangle \end{pmatrix}\right|.$$

We rewrite C_q, D_p in the basis X_j, modulo $\mathfrak{r}_\eta$, letting

$$D_p = \sum_{i=1}^{d} d_{pi}X_{j_i} \mod \mathfrak{r}_\eta, \quad C_q = \sum_{i=1}^{d} c_{qi}X_{j_i} \mod \mathfrak{r}_\eta.$$

Hence considering the $d \times d$ matrix

$$A = \begin{pmatrix} d_{pi} \\ \\ c_{qi} \end{pmatrix}$$

we have:

$$\beta = |\det(A)||\operatorname{Pf}(\eta)|^2.$$

To conclude notice that, being $\mathfrak{h}$ polarizing for η, it holds:

$$\alpha = |\det(\langle \eta, [C_q, D_p] \rangle)| = \left| \det \left(\begin{pmatrix} \langle \eta, [D_q, D_{p'}] \rangle & \langle \eta, [D_p, C_q] \rangle \\ \langle \eta, [C_q, D_p] \rangle & \langle \eta, [C_q, C_{q'}] \rangle \end{pmatrix} \right) \right|^{1/2}$$

and

$$\det \left(\begin{pmatrix} \langle \eta, [D_q, D_{p'}] \rangle & \langle \eta, [D_p, C_q] \rangle \\ \langle \eta, [C_q, D_p] \rangle & \langle \eta, [C_q, C_{q'}] \rangle \end{pmatrix} \right) = \det \left(\begin{pmatrix} d_{ph} \\ \\ c_{qh} \end{pmatrix} \begin{pmatrix} \langle \eta, [X_{j_h}, X_{j_i}] \rangle \end{pmatrix} \begin{pmatrix} d_{ip} & c_{iq} \end{pmatrix} \right),$$

so we finally get:

$$\alpha = |\det(A)||\operatorname{Pf}(\eta)|.$$

Conclusion. Putting everything together, on the one hand we find:

$$\begin{aligned} \operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) &= \frac{\alpha}{(2\pi)^{n-k}} \int_{H\backslash G} \left(\int_{R_\eta \backslash H} \mathscr{F}(\phi \circ \exp_G)((Ad^*)_{g^{-1}}(Ad^*)_{h^{-1}}(\eta)) d\mu_{R_\eta \backslash H} \right) d\mu_{H\backslash G} \\ &= \frac{1}{(2\pi)^{n-k}} \frac{\alpha}{\beta} \int_{\mathcal{O}_\eta} \mathscr{F}(\phi \circ \exp_G)(\lambda) d\mu_\eta(\lambda) \\ &= \frac{1}{(2\pi)^{n-k} |\operatorname{Pf}(\eta)|} \int_{\mathcal{O}_\eta} \mathscr{F}(\phi \circ \exp_G)(\lambda) d\mu_\eta(\lambda), \end{aligned}$$

while, by definition of ν_η,

$$\operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) = \int_{\mathcal{O}_\eta} \mathscr{F}(\phi \circ \exp_G)(\lambda) d\nu_\eta(\lambda),$$

therefore we conclude, by arbitrariness of $\phi \in \mathcal{S}(G)$, that

$$\mu_\eta = (2\pi)^{n-k} |\operatorname{Pf}(\eta)| \, \nu_\eta.$$

□

We are now ready to prove the main theorem of this chapter.

Theorem 3.2.9 (Fourier inversion)**.** *In the above setting, for all $\phi \in \mathcal{S}(G)$ we have:*

$$\phi(1_G) = \frac{1}{(2\pi)^{n-\frac{d}{2}}} \int_Q \operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) |\operatorname{Pf}(\eta)| d\eta,$$

where again d denotes the maximal dimension of a coadjoint orbit in $\mathfrak{g}^*$.

Proof. Choose a strong Malcev basis $\mathcal{X} = \{X_1, \dots, X_n\}$ and functions $R_j = R_{j,\mathcal{X}}^{I^{max}} : \mathbb{R}^d \times \mathfrak{g}^* \to \mathbb{R}$ as above. Using the porperties of exponential map and a classical Fourier identity we can rewrite:

$$\phi(1_G) = \frac{1}{(2\pi)^n} \int_{\mathfrak{g}^*} \mathscr{F}(\phi \circ \exp_G)(\lambda) d\lambda = \frac{1}{(2\pi)^n} \int_{\mathfrak{g}^*} \mathscr{F}(\phi \circ \exp_G) \left(\sum_{j=1}^n \lambda_j X_j^* \right) d\lambda.$$

Since $\mathfrak{g}^* \setminus U$ has zero Lebesgue measure we can operate the change of variable:

$$\phi(1_G) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d \times Q} \mathscr{F}(\phi \circ \exp_G) \left(\sum_{j=1}^n R_j(x, \eta) X_j^* \right) dx d\eta,$$

where no constant appears by construction of the R_j's. Thanks to Fubini Theorem and Proposition 3.2.8 we find:

$$\begin{aligned}
\phi(1_G) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d \times Q} \mathscr{F}(\phi \circ \exp_G) \left(\sum_{j=1}^n R_j(x, \eta) X_j^* \right) dx d\eta \\
&= \frac{1}{(2\pi)^n} \int_Q \int_{\mathcal{O}_\eta} \mathscr{F}(\phi \circ \exp_G)(\lambda) d\mu_\eta(\lambda) d\eta \\
&= \frac{(2\pi)^{\frac{d}{2}}}{(2\pi)^n} \int_Q |\operatorname{Pf}(\eta)| \left(\int_{\mathcal{O}_\eta} \mathscr{F}(\phi \circ \exp_G)(\lambda) d\nu_\eta(\lambda) \right) d\eta \\
&= \frac{1}{(2\pi)^{n-\frac{d}{2}}} \int_Q \operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) |\operatorname{Pf}(\eta)| d\eta.
\end{aligned}$$

□

Corollary 3.2.10. For all $\phi \in \mathcal{S}(G),\ g \in G$ we have:

$$\phi(g) = \frac{1}{(2\pi)^{n-\frac{d}{2}}} \int_Q \operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi) \circ \mathcal{R}_{\eta,\mathfrak{h}}(g)^*) |\operatorname{Pf}(\eta)| d\eta,$$

Proof. It is enough to apply the previus result to $\phi \circ R_g$ in place of ϕ and use property *(i)* of Proposition 2.5.13. □

We conclude proving a result that corresponds to the *Plancherel theorem* for the Euclidean Fourier transform.

Theorem 3.2.11 (Plancherel)**.** *In the above setting we have:*

$$\|\phi\|_{L^2(G)}^2 = \frac{1}{(2\pi)^{n-\frac{d}{2}}} \int_Q \|\mathcal{R}_{\eta,\mathfrak{h}}(\phi)\|_{HS}^2 |\operatorname{Pf}(\eta)| d\eta, \quad \forall \phi \in \mathcal{S}(G).$$

Proof. Define $\phi^*(g) = \overline{\phi(g^{-1})}$, so that $\mathcal{R}_{\eta,\mathfrak{h}}(\phi^*) = \mathcal{R}_{\eta,\mathfrak{h}}(\phi)^*$ as in Proposion 2.5.13. Notice that:

$$\|\phi\|_{L^2}^2 = \int_G \phi(g) \overline{\phi(g)} dg = \phi * \phi^*(1_G).$$

Using the Fourier inversion and the property

$$\mathcal{R}_{\eta,\mathfrak{h}}(\phi * \psi) = \mathcal{R}_{\eta,\mathfrak{h}}(\phi) \circ \mathcal{R}_{\eta,\mathfrak{h}}(\psi),$$

we have:

$$\begin{aligned}\|\phi\|_{L^2}^2 &= \frac{1}{(2\pi)^{n-\frac{d}{2}}}\int_Q \mathrm{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi*\phi^*))|\,\mathrm{Pf}(\eta)|d\eta \\ &= \frac{1}{(2\pi)^{n-\frac{d}{2}}}\int_Q \mathrm{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)\circ\mathcal{R}_{\eta,\mathfrak{h}}(\phi)^*)|\,\mathrm{Pf}(\eta)|d\eta \\ &= \frac{1}{(2\pi)^{n-\frac{d}{2}}}\int_Q \|\mathcal{R}_{\eta,\mathfrak{h}}(\phi)\|_{HS}^2|\,\mathrm{Pf}(\eta)|d\eta.\end{aligned}$$

□

3.3 Plancherel measure on N_4

To be consistent with the notation of the previous section, we rename the usual Malcev basis:

$$\mathcal{Y} = \{Y_1,\dots,Y_6\} = \{X_0, X_{12}, X_{23}, X_2, X_1, X_3\}.$$

We also let $\mathcal{Y}^* = \{Y_1^*,\dots,Y_6^*\}$ the dual basis, and we indicate $(\lambda_1,\dots,\lambda_6)$ the generic element $\lambda = \lambda_1 Y_1^* + \dots + \lambda_6 Y_6^* \in \mathfrak{n}_4^*$.

We follow the rules presented in Section 3.2.1 to determine I^{max}. First notice that $|I^{max}| = 4$ and, rewriting the system (2.7), the four-dimensional orbits are the ones described by:

$$\begin{cases}\lambda_1 = \alpha \\ \lambda_4\lambda_1 - \lambda_2\lambda_3 = \gamma\end{cases} \qquad \text{for } (\alpha,\gamma)\in\mathbb{R}^*\times\mathbb{R}.$$

On each orbit we fix the element $\eta = (\alpha, 0, 0, \gamma/\alpha, 0, 0)$. One easily verifies that for such η it holds:

$$\mathfrak{r}_\eta = \mathrm{span}\{Y_1, Y_4\} = \mathrm{span}\{X_0, X_2\},$$

hence $I^\eta = \{2,3,5,6\}$. We conclude that $I^{max} = \{2,3,5,6\}$.

Consequently, we have

$$\begin{gathered} U = \{\text{four-dimensional orbits}\} \\ Q = \mathrm{span}\{Y_1^*, Y_4^*\} \\ U\cap Q = \{(\alpha,0,0,\gamma/\alpha,0,0) \mid (\alpha,\gamma)\in\mathbb{R}^*\times\mathbb{R}\} \simeq \mathbb{R}^*\times\mathbb{R}.\end{gathered}$$

Remark 3.3.1. Alternatively, we could have directly verified that for any $\lambda\in U$ it holds:

$$\mathfrak{r}_\lambda = \mathrm{span}\{Y_1, \lambda_3Y_2 + \lambda_2Y_3 - \lambda_1Y_4\} = \mathrm{span}\{X_0, h_{23}X_{12} + h_{12}X_{23} - h_0X_2\},$$

which, as we expect, gives $I^\lambda = \{2,3,5,6\}$.

Let now $\eta = (\eta_1,\dots,\eta_6)\in U$, we have:

$$|\,\mathrm{Pf}(\eta)|^2 = \det((\langle\eta,[Y_i,Y_j]\rangle)_{i,j=2,3,5,6}) = \det\left(\begin{pmatrix} 0 & 0 & 0 & \eta_1 \\ 0 & 0 & -\eta_1 & 0 \\ 0 & \eta_1 & 0 & 0 \\ -\eta_1 & 0 & 0 & 0\end{pmatrix}\right) = \eta_1^4,$$

hence $\mathrm{Pf}(\eta) = \eta_1^2$ and:

$$\phi(1_{N_4}) = \frac{1}{(2\pi)^4} \int_Q \mathrm{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi))\ \eta_1^2 d\eta_1 d\eta_4 \quad \forall \phi \in \mathcal{S}(N_4).$$

Finally, writing $\eta = (\alpha, 0, 0, \gamma/\alpha, 0, 0) \in U \cap Q$ and indicating with $\mathcal{R}_{\alpha,\gamma}$ in place of $\mathcal{R}_{\eta,\mathfrak{h}}$ the corresponding unirrep, with the change of variable

$$\begin{cases} \eta_1 = \alpha \\ \eta_4 = \gamma/\alpha \end{cases}$$

we find:

$$\phi(1_{N_4}) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^2} \mathrm{Tr}(\mathcal{R}_{\alpha,\gamma}(\phi))\ |\alpha| d\alpha d\gamma \quad \forall \phi \in \mathcal{S}(N_4).$$

3.4 Further examples: Heisenberg and Engel groups

The Heisenberg group

We introduced the Heisenberg group $\mathbb{H}^n$ in Section 2.4.2. We consider the case $n = 1$, the computations are completely analogous for bigger n. We work with the coordinates associated to the Malcev basis:

$$\mathcal{Y} = \{Y_1, Y_2, Y_3\} = \{Z, Y, X\}.$$

Again we let $(\lambda_1, \lambda_2, \lambda_3)$ denote the element $\lambda = \lambda_1 Y_1^* + \lambda_2 Y_2^* + \lambda_3 Y_3^* \in \mathfrak{h}^*$. With the techiniques presented in Section 2.2.1 we realize that the coadjoint orbits are contained in the level sets: $\{\lambda_1 = \alpha\}$, for $\alpha \in \mathbb{R}$. There are two kind of orbits:

(i) If $\alpha = 0$ the orbit is the singleton $\{(0, \lambda_2, \lambda_3)\}$;

(ii) If $\alpha \neq 0$ the orbit is the plane $\{\lambda_1 = \alpha\}$.

For $\eta = (\alpha, \lambda_2, \lambda_3)$, $\alpha \neq 0$, we have $\mathfrak{r}_\eta = \mathrm{span}\{Y_1\}$, hence $I^{max} = \{2, 3\}$ and

$$U = \{\lambda_1 \neq 0\}, \quad Q = \mathrm{span}\{Y_1^*\}.$$

For $\eta = (\eta_1, \eta_2, \eta_3) \in U$ the Pfaffian is:

$$|\mathrm{Pf}(\eta)|^2 = \det(\langle \eta, [Y_i, Y_j] \rangle_{i,j=2,3}) = \det\left(\begin{pmatrix} 0 & -\eta_1 \\ \eta_1 & 0 \end{pmatrix} \right) = \eta_1^2,$$

hence

$$\phi(1_{\mathbb{H}}) = \frac{1}{(2\pi)^2} \int_Q \mathrm{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) |\eta_1| d\eta_1 \quad \forall \phi \in \mathcal{S}(\mathbb{H}).$$

Finally, writing $\eta = (\alpha, 0, 0) \in U \cap Q$ and indicating with $\mathcal{R}_\alpha$ the corresponding unirrep we find:

$$\phi(1_{\mathbb{H}}) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \mathrm{Tr}(\mathcal{R}_\alpha(\phi))\ |\alpha| d\alpha \quad \forall \phi \in \mathcal{S}(\mathbb{H}).$$

Remark 3.4.1. On $\mathbb{H}^n$ the same procedure gives the Plancherel measure $|\alpha|^n d\alpha$.

The Engel group

We conclude with another example: the Engel group. A detailed presentation of this group can be found in [4], [7]. The interesting property in this case is that the Zariski open $U \subseteq \mathfrak{g}^*$ does not contain all the top-dimensional orbits.

The Engel group $\mathbb{E}$ is the unique nilpotent Lie group with lie algebra $\mathfrak{e} = \operatorname{span}\{X_1, \dots, X_4\}$ with the only nontrivial Lie brackets:

$$[X_1, X_2] = X_3$$
$$[X_1, X_3] = X_4.$$

A strong Malcev basis for $\mathfrak{e}$ is $\mathcal{Y} = \{Y_1, \dots, Y_4\} = \{X_4, X_3, X_2, X_1\}$. Denoting with $\psi : \mathbb{R}^4 \to \mathbb{E}$ the associated Malcev coordinate map we find $\mathbb{E} \cong (\mathbb{R}^4, \bullet)$, with

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \bullet \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \begin{pmatrix} x_1 + x_1' \\ x_2 + x_2' \\ x_3 + x_3' + x_1 x_2' \\ x_4 + x_4' + x_1 x_3' + \frac{1}{2} x_1^2 x_2' \end{pmatrix}.$$

Using the techiniques presented in Section 2.2.1 (see also [4], Appendix A) and the notations adopted in the previous sections for the group N_4, we deduce that the coadjoint orbits are described by the system of equations:

$$\begin{cases} h_4 = \mu \\ \frac{1}{2} h_3^2 - h_2 h_4 = \nu \end{cases} \qquad \text{for } (\mu, \nu) \in \mathbb{R}^2. \tag{3.4}$$

There are three types of orbits.

(i) If $\mu = \nu = 0$, the orbits reduce to singletons $\{(h_1, h_2, 0, 0)\}$.

(ii) If $\mu = 0, \nu \neq 0$, the orbits are planes $\{h_4 = 0, h_3 = \pm\sqrt{2\nu}\}$. We fix the element $\boxed{(0, 0, \pm\sqrt{2\nu}, 0)}$ on each of them.

(iii) If $\mu \neq 0$, the orbits are 2-dimensional and are described by the system (3.4). On each of them we fix the element $\boxed{(0, -\nu/\mu, 0, \mu)}$.

Rewriting in the basis $\mathcal{Y}$, we rephrase (3.4) into:

$$\begin{cases} \lambda_1 = \mu \\ \frac{1}{2} \lambda_2^2 - \lambda_3 \lambda_1 = \nu \end{cases} \qquad \text{for } (\mu, \nu) \in \mathbb{R}^2. \tag{3.5}$$

To identify I^{max} we consider only the two-dimensional orbits, whose representatives are:

(ii) $\eta = (0, \pm\sqrt{2\nu}, 0, 0)$ if $\mu = 0, \nu \neq 0$.

(iii) $\eta = (\mu, 0, -\nu/\mu, 0)$ if $\mu \neq 0$.

In case (ii), we have $\mathfrak{r}_\eta = \operatorname{span}\{Y_1, Y_2\}$, hence $I^\eta = \{3,4\}$. In case (iii), $\mathfrak{r}_\eta = \operatorname{span}\{Y_1, Y_3\}$, hence

$$I^\eta = \{2,4\} = I^{max}.$$

Consequently:

$$U = \{\text{orbits corresponding to } \ \mu \neq 0 \text{ in } (3.5)\},$$
$$Q = \operatorname{span}\{Y_1^*, Y_3^*\},$$
$$U \cap Q = \{(\lambda, 0, -\nu/\mu, 0) \mid \mu \in \mathbb{R}^*, \nu \in \mathbb{R}\} \simeq \mathbb{R}^* \times \mathbb{R}.$$

Remark 3.4.2. Notice that in this case the set U does not contain *all* the top-dimensional orbits: the planes described in (ii) are excluded.

Let now $\eta = (\eta_1, \ldots, \eta_4) \in U$, we have:

$$|\operatorname{Pf}(\eta)|^2 = \det((\langle \eta, [Y_i, Y_j]\rangle)_{i,j=2,4}) = \det\left(\begin{pmatrix} 0 & -\eta_1 \\ \eta_1 & 0 \end{pmatrix}\right) = \eta_1^2,$$

hence $\operatorname{Pf}(\eta) = |\eta_1|$ and

$$\phi(1_{\mathbb{E}}) = \frac{1}{(2\pi)^3} \int_Q \operatorname{Tr}(\mathcal{R}_{\eta,\mathfrak{h}}(\phi)) \ |\eta_1| d\eta_1 d\eta_3 \quad \forall \phi \in \mathcal{S}(\mathbb{E}).$$

Finally, writing $\eta = (\mu, 0, -\nu/\mu, 0) \in U \cap Q$, and indicating with $\mathcal{R}_{\mu,\nu}$ the corresponding unirrep, with the change of variable

$$\begin{cases} \eta_1 = \mu \\ \eta_3 = -\nu/\mu \end{cases}$$

we find:

$$\phi(1_{\mathbb{E}}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^2} \operatorname{Tr}(\mathcal{R}_{\mu,\nu}(\phi)) \ d\mu d\nu \quad \forall \phi \in \mathcal{S}(\mathbb{E}).$$

Remark 3.4.3. Notice that in all the three examples the center of the group is one-dimensional and the Plancherel measure depends only on the variable associated to the center.

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