# UNIVERSITÀ DEGLI STUDI DI PADOVA

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**Master Degree in Physics**


**Final Dissertation**

# Stability of Euclidean Wormholes in the Axiverse


**Thesis supervisor**

**Prof. Luca Martucci**

**Candidate**

**Marco Arrighi**



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**Abstract**

In Axiverse models, namely models of pseudo-scalar fields coupled with gravity, it is common to encounter particular solutions known as *gravitational instantons*. These metric configurations are of interest across various theoretical fields of physics, such as cosmology and the phenomenological axion physics, but they are especially relevant to Quantum Gravity research. In particular, a specific class of solutions, referred to as *wormholes*, is expected to provide non-negligible contributions to low-energy Effective Field Theories derived from String Theory models. Wormholes consist in specific configurations of the Euclidean space-time metric that connect distant, non causally-related points of the universe. They also reveal interesting aspects of the theory, potentially offering insights into the structural features of Quantum Gravity.

Despite the fact that the known practical effects of wormholes have been studied for years, the issue of whether Euclidean wormholes should be considered in the low-energy limit remains unsolved. Indeed, it is necessary to investigate their stability in order to determine whether they could yield real physical contributions. This problem has been addressed multiple times over the last 40 years, but a definitive answer is still lacking.

Based on a recent work that confirms their stability in a simple model, this thesis aims to reproduce this result and further investigate the stability in a more general framework.

# Contents

# Introduction

## A walk through wormholes' history

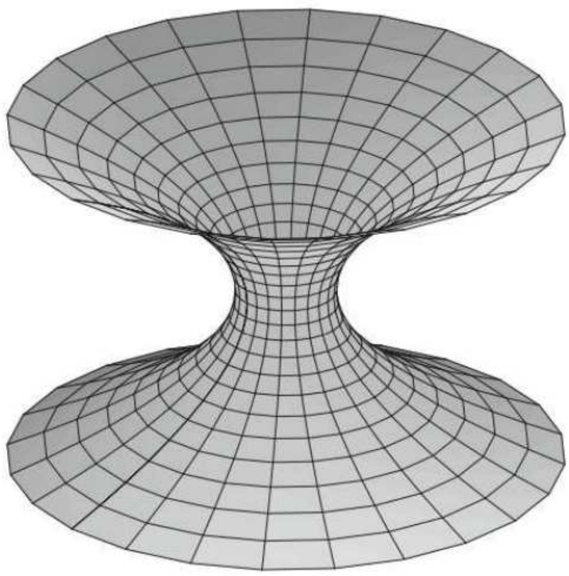
In the last century, a profound scientific revolution drastically changed our understanding of particle physics. One of the most remarkable achievements has been the development of the Standard Model (SM), which, through the framework of Quantum Mechanics (QM), successfully unified three of the four fundamental forces of nature: electromagnetism, the weak nuclear force, and the strong nuclear force.

However, a significant limitation of the SM is its inability to incorporate gravity, which is described by Albert Einstein's General Relativity (GR). Despite the importance of GR in cosmology, its validity is restricted to energy scales well below the Planck mass. At low energies, gravity is the weakest of the fundamental forces, but it becomes strongly coupled at high energy scales, rendering GR inadequate for the study of quantum systems. Consequently, GR is considered a plausible Effective Field Theory (EFT) derived from a more fundamental Quantum Field Theory (QFT).
Physicists hypothesize the existence of an ultraviolet (UV) complete theory at extremely high energies, capable of unifying the frameworks of QM and GR, the two pillars of modern physics. This ambitious goal is the primary objective of Quantum Gravity (QG) research: an ideal QG model would represent a consistent high-energy unification of these two low-energy limits.

Over the years, various proposals and approaches have been developed to construct a theory of QG. Unfortunately, confirming any of these proposals has proven extremely challenging due to the lack of experimental methods capable of testing such hypotheses. A recent theoretical strategy to address this challenge involves seeking indirect evidence. This effort follows two complementary directions, combining bottom-up and top-down approaches to effective theories.
The bottom-up approach explores "beyond the Standard Model" physics, investigating potential extensions of the SM while satisfying phenomenological constraints. Conversely, the top-down approach aims to identify the features of effective theories consistent with

a given UV-complete model. Such a high-energy theory must represent a QG framework, incorporating gravity into the quantum field formalism alongside the other fundamental interactions. Among the QG candidates, String Theory (ST) stands out as one of the most promising and is central to the motivation of this thesis. Briefly, ST replaces point-like particles with one-dimensional strings and extends the structure of spacetime to higher-dimensional manifolds.

Moreover, the hope of advancing QG research lies in studying extreme objects and scenarios, such as spacetime singularities (e.g., black holes) or the initial singularity of the universe hypothesized in the Big Bang model. These phenomena, although described within the context of GR, exhibit distinct quantum effects. A historically significant example is Hawking's discovery of black hole evaporation, wherein particle-antiparticle pairs are created due to vacuum dynamics, causing the black hole to lose mass.



Graphic representation of a wormhole.

In this context, one particularly intriguing phenomenon that may provide valuable insights into low-energy string models is that of Euclidean wormholes. A wormhole is a spacetime configuration resembling a tunnel connecting two causally disconnected regions of space-time.

The first examples of wormholes were studied in the Lorentzian framework of GR by Schwarzschild, Einstein, and Rosen. These solutions emerged as peculiar solutions to Einstein's equations under specific symmetry assumptions[1]. Subsequently, wormholes

[1]See [16] for a comprehensive review, including the different classes of wormhole solutions.

were examined for their geometric implications in the context of Euclidean QG. Notably, in the late 1980s, Giddings and Strominger discovered (in [1, 2]) a new class of wormhole solutions by studying Euclidean gravity coupled to scalar fields, known as axions. This framework gave rise to the so-called *Axiverse* models. These studies aimed to analyze processes involving topology change. More recently, such solutions have regained interest for their low-energy effects and their contributions to corresponding EFTs. Wormhole solutions appear as saddle points of the Euclidean action, making them essential in semi-classical analyses. However, verifying whether these solutions are true minima of the action is critical, as unstable configurations do not dominate the path integral. This question of stability remains a debated issue requiring further investigation.

These topics form the foundation of this thesis, which seeks to explore the key aspects of this general framework and highlight its connections to various branches of modern physics (e.g., cosmology, ST phenomenology, and beyond the SM physics). Thus, the primary aim of this work is to contribute to ongoing research on the stability of wormhole solutions.

## The thesis structure

This thesis is organized into two main parts. The first four chapters provide an overview of Euclidean wormholes, presenting key concepts and discussing relevant recent developments in the literature. The second part, contained in chapter 5, builds on recent findings to offer new insights into the stability of wormhole solutions.

Chapter 1 introduces the motivations behind studying Euclidean wormholes and their interesting features. Beginning with their discovery, it is addressed their early connection to inflation and topology change. It is then discussed the strong CP problem in the SM, highlighting the necessity of extending the SM. Finally, it is introduced the "Swampland" programme within the context of ST phenomenology.

Chapter 2 outlines the technical tools required to understand wormhole solutions. It emphasizes the significance of working in Euclidean spacetime and the semiclassical approximation, particularly in handling non-perturbative contributions. Additionally, it explains the formalism of duality, which allows scalar axions to be equivalently treated as three-forms.

Chapter 3 provides a detailed discussion of wormhole solutions, starting from simple models and extending to more general cases. Differences between the three classes of gravitational instantons are highlighted, with special attention given to the delicate choice of boundary conditions.

Chapter 4 examines the low-energy effects of wormhole solutions. Indeed, after integrating out wormholes from the path integral, it is left a bi-local operators in the EFT, which requires a peculiar physical interpretation. This chapter introduces the puzzle of $\alpha$ parameters and explores phenomenological implications for axion potentials and Swampland conjectures.

The core of the thesis is contained in the second part, i.e. chapter 5, addressing the stability of wormhole solutions. By introducing small field and metric perturbations, it is analyzed the issue regarding the possible presence of negative modes. It is calculated the second order action perturbation and the problem is simplified to a single-variable eigenvalue equation. The following analysis demonstrates wormhole stability, without requiring an explicit numerical analysis.

Finally, chapter 6 summarizes the significant contributions of wormhole physics and suggests potential directions for future research on this challenging topic.

# Chapter 1

# Motivations for studying wormholes

The first chapter of this thesis provides an introductory overview of the main areas of interest related to the physics of wormholes, covering both recent and older studies present in the extensive literature on the subject.
Before delving into the direct description of instanton solutions, it is useful to explore the underlying concepts and to provide an overview of all the aspects standing behind these kind of solutions. In particular, it is important to review different frameworks concerning models of pseudo-scalar fields coupled with gravity. In such scenarios, wormhole configurations are, in principle, allowed and are expected to contribute significantly to the low-energy limit of the theory. For each of these models, it is crucial to consider possible non-perturbative corrections to the action potential as well as other possible phenomenological effects.
In section 1.1, the early interest in the discovery of euclidean wormholes is presented, along with the original problems that motivated the birth of this field of research.
Other key areas related to the physics of gravitational instantons are briefly introduced in sections 1.2 and 1.3. These motivations and their related developments pertain not only to well-known problems but also to more recent progress achieved in the last decade.

## 1.1 The Euclidean wormholes discovery

In the end of the 1980's Giddings and Strominger achieved the important discovery of the first Euclidean wormhole solution, which is more carefully and thoroughly described in section 3.1 and 3.2.
This section focuses on the motivations that led them, along with several other physicists, to investigate these peculiar solutions. The framework where this research took place is essentially the same as the one where active investigations continue today, namely

ST. However, the earlier motivations were slightly different. In fact, the initial studies were primarily driven by formal aspects and cosmological motivations related to these models. The former research focused on the inflationary paradigm and the fundamental issue of quantum coherence. By contrast, today's primary area of investigation is string phenomenology (described in section 1.3), which examines the structure of the low-energy limits of string theory as a consequence of the UV features of the specific QG model under consideration.

### 1.1.1 Inflationary models with axions

Studies on the inflationary mechanism are of great interest in modern cosmology, despite being a longstanding topic. Indeed, cosmologists can coherently describe the history of the universe's evolution up to one second after its hypothetical beginning, thanks to the ΛCDM model. This theory makes significant predictions regarding, for example, the formation of the first light nuclei, as well as the birth of stars and macroscopic structures. However, the early phases of the universe's evolution remain unclear and may hide new mechanisms or phase transitions.

In the early universe, all fundamental particles were part of the so-called thermal bath—a hot and dense dynamical plasma where particles continuously interacted and collided violently. To address some non-trivial issues regarding the early universe's dynamics,[1] the existence of a prior phase of exponential expansion has been conjectured, commonly referred to as inflation. Following the logic of cosmological models, going back in time corresponds to increasing the temperature of the thermal bath distribution. It is important to note that the temperature parameter is proportional to the energy of the system. Therefore, during the early phase of expansion, when temperatures were extremely high, the effects of high-energy physics became highly significant. This provides a strong motivation for the study of QG.
Going into details, the dynamics of inflation are typically described using models with a few scalar fields (called inflatons) evolving within a Friedman-Robertson-Walker-Lemaître (FRWL) expanding universe.[2] These models differ primarily in the shape of the inflaton potential and the number of inflaton fields considered.

[1]It is widely accepted that, in the very early universe, there were no bound states of particles; without atoms, the universe was populated solely by the primordial thermal bath. The unresolved issues (which inflation aims to solve) include the cosmological horizon problem, the flatness problem, and the unwanted relics problem (related to grand-unification theories).

[2]The FRWL model describes an evolving homogeneous and isotropic spacetime, with a metric given by $ds^2 = a(t)^2 \left(-dt^2 + d\vec{x}^2\right)$. The parameter $a(t)$, known as the scale factor, depends only on time.

The relevance of inflation to the physics of wormholes stems from the fact that axions are considered good candidates for describing inflaton fields. However, axions exhibit a shift symmetry that can be broken at a certain energy scale.[3] The breaking of this global symmetry can be provided by the presence of non-perturbative effects that introduce corrections to the scalar potential. The potential plays a crucial role in inflationary models as it is one of the few factors guiding the slow-roll dynamics of inflaton fields.
Therefore, in studying axion models, it is natural to ask whether the effective contributions from gravitational instantons and wormhole configurations should be considered in the inflationary dynamics.
Without going into too much detail, and following the recent work of [12], the expected correction to the potential caused by a single wormhole is typically of the form:

$$\delta V = \mathcal{T} \cos(q\varphi)\, e^{-S} \,. \tag{1.1}$$

In this formula, $S$ represents the on-shell instanton action, which is approximately given by $S_{inst} \sim \frac{q}{f_{ax}}$.[4] Meanwhile, $\varphi$ is the axion field, and $\mathcal{T}$ is a numerical pre-factor. To explicitly estimate the correction $\delta V$, it is essential to compute both $S$ and $\mathcal{T}$, which depend on the specific instanton solution. In the simplest models, it has been shown that the correction $\delta V$ is typically small.[5] However, no general result has been found that allows wormhole contributions to be neglected a priori.
For this reason, their role remains unclear and requires further investigation, especially when dealing with more complex and structured models.

### 1.1.2 Quantum coherence problem

The first significant motivation that led many physicists to pursue studies on gravitational instantons was the issue of "quantum (in)coherence." While the problem is relatively simple to introduce, nowadays its possible resolution remains far from clear. In the framework of QG, certain hypothetical processes that lead to changes in spacetime topology are permitted because, at a fundamental level, there is no principle or motivation that inherently forbids such transformations. The classical understanding of spacetime as a fixed manifold might break down at a certain quantum scale, allowing for these kind of processes, potentially playing an important role in the UV theory. These topological transitions

[3] This feature is a common expectation in Quantum Gravity and will be discussed in more detail in section 1.3.

[4] Here, $q$ is the topological charge of the instanton, and $f_{ax}$ is the characteristic energy scale associated with the axion. Their roles are explained in more detail in the following chapters.

[5] An example of how to estimate the wormhole correction to the low-energy potential is provided in section 4.3. For a concrete estimation in inflationary models, it is again suggested the reference [12] or also the discussion of [13].

challenge the preservation of quantum coherence.

The universe is generally thought of as a four-dimensional manifold with a specific structure, described by its metric $g_{\mu\nu}$. However, some physical solutions, such as those involving gravitational instantons, suggest that 4d spacetime could evolve into a substantially different configuration. The result of this transition is a structure that mirrors the original spacetime but with the addition of a small extra region. This last portion of space can be physically interpreted as the emission of a spatial three-dimensional sphere evolving over time.[6]

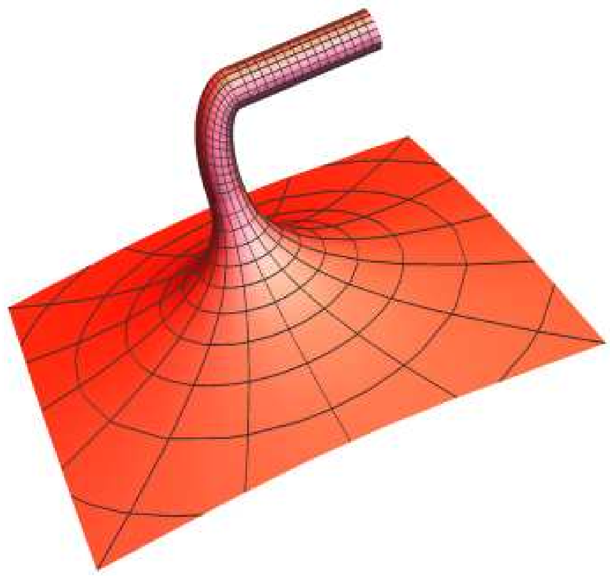


**Figure 1.1:** Pictorial representation of the emission of a "baby universe" from an asymptotically flat space-time.

The problem arises because the emission of the so-called "baby universe" has catastrophic consequences for the theory: it seems to lead to a loss of information, as the pure quantum mechanical state describing the universe would evolve into a mixed state, summing over all the possible couplings related to the newly created structure.[7]

[6]Two important remarks are needed for completeness. First, the nucleated structure could have a different nature; the three-sphere mentioned here is simply the most common case studied in the literature, but a priori, any topology could be considered. Moreover, these transitions are well-known in Euclidean spacetime models, while the situation in Minkowski spacetime is quite different.

[7]A toy model in [43] illustrates this problem more explicitly. Consider, for example, the creation of an electron-positron pair from a single electron state, a process well understood in any standard quantum field theory in a standard spacetime. However, in a spacetime with baby universe structures, the situation becomes more complex. Indeed, there are multiple possible emission sites for the pair, leading to a mixed state that is a superposition of different dynamics of the interaction. For example, one of the two particles created could fall into the baby universe, or either both.

The following provides a brief summary of the general argument, while the specific application to wormhole physics is outlined in sections 4.1 and 4.2, though it still lacks a complete explanation. Although the situation is not identical, it is worth noting that a similar problem is central to black hole physics. The presence of singularities at black hole cores is also related to the topology of spacetime, and the problem of black hole evaporation remains widely studied due to its implications for potential information loss, which contradicts the basic principles of quantum mechanics.

In the late 1980s, a group of theoretical physicists[8] realized that any physical process that modifies the spacetime topology could potentially violate the laws of quantum mechanics. A first solution to this issue was proposed by Coleman in [30], where he introduced key concepts such as baby universes and $\alpha$-parameters (further discussed in section 4.1 and 4.2). The same year, Giddings and Strominger published the papers [1] and [3], where they not only introduced the topic of Euclidean wormholes but also applied Coleman's ideas to these particular instanton solutions. They found that their results were in perfect agreement with Coleman's theoretical predictions, confirming the restoration of unitarity and thus, of quantum coherence.

The loss of coherence is caused by the creation of secondary structures from the space-time and their correlation with the parent universe. Any observable in the "initial" spacetime can be related to corresponding states in the "final" spacetime.[9] To formalize this idea, one could assume that the state of the universe is a tensor product of a fixed baby universe state and a second state related to the low energy effective field theory, as follows:

$$|\varphi\rangle = |\varphi_{BU}\rangle \otimes |\varphi_{EFT}\rangle \ . \tag{1.2}$$

In general, this state will evolve in time into an entangled state:

$$|\tilde{\varphi}\rangle = \sum_i c_i \, |\varphi\rangle^i_{BU} \otimes |\varphi\rangle^i_{EFT} \ . \tag{1.3}$$

From the EFT perspective, this leads to a clear loss of information. Taking the trace of the EFT state, and so erasing the baby universe one, it reults:

$$\langle\varphi|_{EFT} \, |\varphi\rangle_{EFT} \to \sum_i |c_i|^2 \, \langle\varphi|^i_{EFT} \, |\varphi\rangle^i_{EFT} \ , \tag{1.4}$$

[8] Mainly Hawking, Page, and Pope, as well as the authors of [48].

[9] Assuming that the nucleation of baby universes occurs at a specific Euclidean time, "initial" and "final" are used here to distinguish between the times before and after the nucleation of these branches.

which shows the transition from a pure to a mixed state.

The rigorous demonstration that restores coherence is non-trivial and requires introducing new concepts related to the low-energy relics of instanton solutions.[10] The key to resolve the apparent loss of quantum coherence lies in integrating out the wormholes from the theory, adopting the path integral formulation. The result is the emergence of new local operators coupled with new constants labeled as $\alpha^i$. While the precise values and origins of these parameters are unknown, a plausible interpretation is that they correspond to different kind of baby universes being nucleated.
In this view, the $\alpha$ parameters arise as superselection sectors in the full QG theory. Thus, the loss of information is not real; each baby universe corresponds to a different eigenstate of these parameters, leading to a well-defined unitary EFT for each sector.

A more general explanation has been proposed, considering the possibility that each baby universe state is given by a superposition of different $\alpha$ parameters. In this case, the issue is more intricate: rather than a loss of information, the process of nucleation highlights our ignorance of the specific values of the $\alpha$'s. Even if their values are fixed, there is no way for an observer in the parent universe to measure them. The superposition of $\alpha$'s would then represent a probability distribution, effectively describing a classical ensemble of different possible effective field theories.

While the solution outlined above seems to work, it is not universally accepted, and physicists continue to investigate this problem. Two recent papers that further explore this issue are [29] and [57].

## 1.2 Strong CP problem and axion particles

Axions are scalar particles that have garnered increasing interest in the particle physics community over the years. Today, they are studied across various theoretical frameworks: they are considered potential drivers of cosmic inflation, viable dark matter candidates, particles in the low-energy limits of string theory, and possible additions to the Standard Model in its extensions. Originally, axions were introduced in the 1970s to address the *strong CP problem*.[11]

[10] Described in chapter 4.
[11] This section is based on the work [72], but there are various references recasting this argument.

Despite the Standard Model's impressive success, it leaves some fundamental questions unanswered, so many physicists believe it is incomplete. One such puzzle is the strong CP problem, which remains a crucial aspect to resolve in particle physics.
The CPT theorem, developed by Schwinger, claims that every quantum field theory should remain invariant under the combination of three discrete symmetries:

- $C$: the charge conjugation, relating particles to their antiparticles by flipping the sign of their quantum charges;
- $P$: the parity transformation, which invertsthe spatial coordinates;
- $T$: time-reversal symmetry, which reverses the direction of time.

In the 1960s, the famous experiment by professor Wu demonstrated that CP violation occurs in the weak interaction sector of the SM. This violation is associated with a complex phase in the CKM matrix, which mixes the weak interactions of different quark families. While CP violation is a crucial ingredient for understanding the baryon asymmetry of the universe,[12] the amount of CP violation observed in the electroweak sector of the Standard Model is insufficient to explain the baryon asymmetry. This led physicists to search for other potential sources of CP violation, including in the strong interaction sector.

While CP is conserved at the perturbative level in quantum chromodynamics (QCD), non-perturbative effects could violate CP symmetry. An anomalous $U(1)$ symmetry leads to an additional non-perturbative term in the QCD Lagrangian[13]:

$$\mathcal{L}_{\not{CP}}^{QCD} = \frac{\theta_{QCD}}{32\pi^2} G_{\mu\nu} \tilde{G}^{\mu\nu} \ . \tag{1.5}$$

where $G_{\mu\nu}$ is the gluon field strength tensor, $\tilde{G}^{\mu\nu}$ is its dual, and $\theta_{QCD}$ is a parameter that governs the amount of CP violation. Similar terms are theoretically explained as contribution generated by instanton configuration, in this particular case by the Yang-Mills instanton.[14]
Experimental data on the neutron electric dipole moment constrain $\theta_{QCD}$ to be less than

[12] It is estimated that there is a significant mismatch between the number of baryons and antibaryons in the universe, which appears unnatural (the discrepancy in relative number densities is on the order of $10^{10}$). This prompted researchers to search for possible sources or mechanisms that could originate this asymmetry. Sakharov summarized the necessary conditions for such a mechanism: baryon number violation, C and CP symmetry violations, and processes occurring out of thermal equilibrium. Consequently, it is natural to ask whether such a mechanism might exist within the Standard Model itself.

[13] This term refers to a type of symmetry which is respected by the classical theory but is violated at the quantum level. In a perturbative approach, such violations only appear beyond the tree-level, meaning that become evident when quantum loop corrections are included in the analysis of the interaction.

[14] See section 2.3 for an overview of this kind of solution.

$10^{-10}$ , which is an unusually small value compared to other SM parameters, raising questions about its naturalness.

To resolve this unnaturalness, Peccei and Quinn proposed a mechanism involving a new anomalous U(1) symmetry. In their model, the $\theta_{QCD}$ parameter is replaced by a scalar field $\theta_{QCD} \equiv \frac{\phi}{f}$, whose dynamical evolution naturally explains the small value of $\theta_{QCD}$. This scalar field is the axion, a pseudo-scalar field that obeys a shift symmetry.[15] The energy scale $f$ indicates the characteristic value where the Peccei-Quinn symmetry breaks down, leading to a spontaneous symmetry breaking process.

Initially introduced to solve the strong CP problem, axions have since been generalized to a broad class of pseudo-scalar fields in various theoretical contexts. Today, they are studied not only in relation to the Standard Model but also in its phenomenological extensions, in string theory, in inflationary models and even as dark matter candidates.[16]

When gravity is included in these frameworks, particularly in the context of QG, new effects may arise. Non-perturbative objects, such as gravitational instantons and wormholes, could contribute with additional CP-violating terms. For example, a potential lagrangian with non-perturbative contributions might look like:

$$\mathcal{L}_{\not{CP}} = \frac{\theta_{EM}}{32\pi^2} G_{\mu\nu} \tilde{G^{\mu\nu}} + \frac{\theta_g}{32\pi^2} R_{\alpha\beta\mu\nu} \tilde{R}^{\alpha\beta\mu\nu} \tag{1.6}$$

where the first term represents non-perturbative electromagnetic contributions, and the second term involves gravitational effects through the Ricci tensor $R_{\alpha\beta\mu\nu}$. To be precise, this extra term becomes relevant once gravity is incorporated into the model but even in the absence of instantons. However, gravitational instantons can enhance this effect or potentially introduce additional corrections to the lagrangian.[17] These contributions from gravitational instantons or wormholes could play an important role in CP violation, though their exact nature remains unclear.

Thus, the historical motivation to solve the strong CP problem has significantly broadened interest in axions and their non-perturbative effects, highlighting their relevance not only in particles' phenomenology but also in cosmology, string theory, and quantum gravity research.

[15] So the relative action is invariant under the transformation $\phi \to \phi + a$ for any constant $a$. In the following this concept is taking into account again and explained with further details.

[16] See [68] and [69] for a complete overview about these fields.

[17] As shown in section 4.3, an explicit calculation demonstrates how the presence of wormholes in axion theories results in non-trivial corrections to the scalar field's potential.

## 1.3 The phenomenology of string theory

Discussions about axion fields and non-perturbative gravitational effects are key topics in string phenomenology. Indeed, considering any effective theory whose UV completion is based on String Theory, the presence of a large number of light axions typically follows. The number of axions can range from hundreds to even thousands, depending on the specific UV model under consideration.
To provide a complete description of what is meant by String Theory would require far more than a single chapter of this thesis. In this introduction, only the most fundamental aspects will be mentioned.

String Theory is mainly considered as an attempt to unify gravity with the other fundamental forces and to describe physics at very high energies. The original idea of String Theory is to replace the fundamental point particles of GR and QFT with tiny vibrating strings (or branes). These objects, which have a size comparable to the Planck characteristic length (about $10^{-35}$m), are primarily characterized by their tension (a sort of energy density). The theory is more fundamental in the sense that the same string, depending on how it vibrates, can in principle describe different particles with varying masses, making mass no longer an intrinsic property.[18]
To treat String Theory like a typical QFT and explicitly perform calculations, it is necessary to work within the low-energy approximation. This approach is usually called as String phenomenology.[19] In this approximation, only the massless states are considered, and the gravitational spacetime curvature must be weak. Furthermore, String Theory is fundamentally defined perturbatively, assuming weak interactions between strings. However, in the low-energy approximation, it is possible to encounter non-perturbative phenomena such as D-instantons or wormholes.

A notable aspect of String Theory is that it requires a new d-dimensional spacetime structure. In contrast, standard models in GR are based on the assumption that the universe has three spatial dimensions and one time dimension (with opposite metric sign), all unified in a four-dimensional spacetime structure. As a result, the geometry of the

[18]Conversely, it remains an important intrinsic feature the distinction between fermionic or bosonic strings, depending on the respective spin statistic.

[19]The low energy results are called *Supergravity* theories. In general, "low energy" refers to values below the string scale, but still above the Grand Unification scale, which is approximately $10^{15}$ GeV.

universe may vary[20], but it is constrained to be 4-dimensional. Physicists working on String Theory believe that when gravity is incorporated into the theory, it is essential to consider the effects of physics in "hidden" dimensions, which are compactified in a $(d-4)$-dimensional manifold that we do not directly experience.[21]
The number of dimensions is determined by the requirement for mathematical consistency (i.e., the theory must be rigorous and well-defined). For example, the old bosonic string model[22] requires 26 dimensions. When supersymmetry is included, generally the theory reduces to 10 dimensions. Without delving into the details of supersymmetry and its integration into QFT, it is important to note that five distinct supersymmetric perturbative string theories can be defined:

- Type I, which introduces an unoriented open string;
- Type IIA and Type IIB, which describe closed strings with different boundary conditions on the world-sheet;
- Heterotic theories, which feature non-abelian gauge groups $E_8 \times E_8$ and $SO(32)$.

Finally, in 1995, M-theory was developed, based on the assumption of an 11-dimensional spacetime. This model, the result of the so-called "second string revolution," unifies the five superstring theories listed above into a single framework. In this framework, the previous theories emerge as specific limits of the parameter space, and they are related non-perturbatively by various dualities.

In the following section, we present an overview of String EFTs and the idea of the compactification procedure necessary to achieve them. We will then illustrate a possible model of an EFT and the formalism commonly used in the literature to handle such theories. Without focusing on any particular ST, the following discussions are intended to be general and could, in principle, apply to any model that includes axions and gravity.

### 1.3.1 String EFT and compactification

One of the key concepts in particle physics is that of effective theories, which are characterized by a limited range of validity over a specific energy window.
The succession of discoveries over the last century has followed a step-by-step process,

[20] For example, it is completely different the case where the cosmological constant $\Lambda$ is less, equal or major than zero; Other possibilities considered are the cases of homogeneous or static universe, as well as the opposite options.
[21] All physical processes and experiments occur within our observable 4-dimensional spacetime. So, the new manifold, where the extra dimension are curled up, is not directly accessible.
[22] This is the fundamental toy model used to introduce String Theory, starting from the Nambu-Goto action. See references [85] and [86].

moving towards increasingly complex phenomena that require more precise and accurate explanations, and thus more fundamental theories. This approach allows scientists to use effective theories that provide strong predictive and computational power, but within a restricted domain. A low-energy theory is obtained by defining the so-called high energy cut-off scale[23], which can be adjusted by "integrating in or out" certain fields. The running of physical couplings also allows for the continuous variation of the cut-off if needed, although this is not always a trivial task.

A simple example of an EFT is classical physics, which breaks down when dealing with more fundamental quantum systems. A more standard example is Fermi's theory, which coincided with the discovery of the weak nuclear force. While it was sufficient to explain $\beta$-decay, it was inadequate for describing more complex processes where the physics of the Standard Model was required.[24]
This mechanism applies broadly across many areas of physics. Indeed, even the Standard Model of particle physics is unable to explain phenomena at energy scales higher than those probed by particle accelerators. This naturally raises the question: What lies beyond the SM, as we approach energies near the Grand Unification or Planck mass scales? In this context, particle phenomenology is focused on understanding how to extend the SM. Indeed, there is a broad class of possible extensions and UV-consistent theories. For example, an important feature that gives rise to a vast number of different possible extensions is the symmetry group of the theory. Each of the five main types of ST (cited in the previous section) is characterized by a different local symmetry group, and also the theories of "$grand\ unification$" present various new possibilities.[25]

Since the development of quantum mechanics and Quantum Field Theory, physicists have also pursued the goal of building a "theory of everything," attempting to explain every physical process and force. This ambitious task is inherently problematic because it requires experimental confirmation at energy scales that are currently unattainable. In practice, this effort led to the development of Quantum Gravity theories. Incorporating gravity into a QFT framework requires treating the spacetime metric as a standard field and developing a consistent procedure to quantize it. While this approach is possible, it does not avoid significant technical challenges. To understand why canonical quantization

[23]In other areas of physics, it may also be useful to consider a low-energy cut-off when the theories exhibit pathological behavior in the infrared (IR) limit. For the purposes of this thesis, it is unnecessary to address this issue.

[24]Fermi's theory involve only the low energy limit of interactions with "charge current", while the h.e. extension of the electroweak theory (contained in the SM) also includes the possibility of "neutral current" interactions.

[25]For the purposes of this thesis, the aspect of symmetry groups is not emphasized, while the concept of compactification is discussed in more details in the following sections.

is problematic, one must study black hole physics: black holes, which are well understood in the context of GR and low-energy regimes, reveal hidden features related to a large number of UV degrees of freedom when their thermodynamic aspects are considered. The problem arises because the simultaneous dependence on both infrared and ultraviolet scales causes a breakdown of the EFT approach, necessitating a coherent explanation that remains elusive.

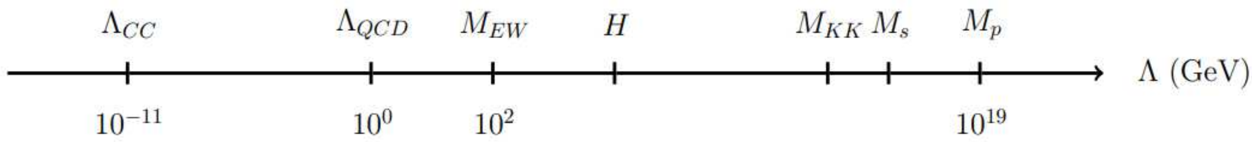



**Figure 1.2:** Key energy scales for particle physics, corresponding to theory cut-offs or phase transitions. The last three scales are the most relevant for this thesis: the Kaluza-Klein compactification scale, the string scale, and the Planck mass scale.

Despite these difficulties, in recent decades a new approach has gained traction in overcoming these challenges. The emerging idea is to base research on a hybrid strategy that combines bottom-up and top-down perspectives. This means that physicists believe that new physics can be better understood by combining beyond Standard Model physics with research on EFTs motivated by quantum gravity.

In particular, String Phenomenology is a field that seeks to identify the features and constraints that a low-energy effective field theory must satisfy in order to be consistent with, and directly related to, a high-energy quantum gravity theory. As the name suggests, the only self-consistent and widely accepted UV theory typically considered is String Theory.[26]

Starting from a ST model and integrating out some UV degrees of freedom, one obtains an EFT containing a renormalizable part as well as some non-renormalizable operators. The goal of String Phenomenology is to start from an EFT weakly coupled to gravity and investigate under which conditions it can be UV-completed into a quantum gravity model. Through this approach, one can derive constraints and criteria for ruling out inconsistent EFTs, known as the "Swampland Criteria".[27]

[26] String theory involves energy scales slightly below the Planck mass $M_{Pl} \sim 10^{19}$ GeV, with string contributions becoming significant around $M_s \sim 10^{17/18}$ GeV. See 1.1.

[27] Described in the subsection 1.3.2

**The idea of compactification**

With String Theory, it is possible to address several formal problems related to Quantum Gravity, but an obvious question arises from the framework: where are the extra dimensions required for a well-defined String Theory? Does the full space-time structure have any consequences for the standard four dimensional theory?

The basic idea to answer this question was first developed by Kaluza and Klein, who proposed that the invisible dimensions are curled up into a compact hidden manifold, leaving only four dimensions accessible at low energies. Thus, the new dimensions are somehow present in the structure of the 4-dimensional theory, even if their effects are often not explicitly observable. Technically, the way to handle this issue is a mechanism called "compactification."

The 4-dimensional (low-energy) theory is achieved by starting from a given String Theory model. Depending on the topology of the manifold in which the extra dimensions are wrapped, the final model contains a different number of fields and symmetries. For the purposes of this thesis, it is important to understand why there is a specific configuration of scalar fields (namely axions and saxions) in the EFT and how it depends on the compactified dimensions.

Another important aspect of compactification is that this procedure could also break the supersymmetry (SUSY) of the model, that is typically implied in the UV completion. The presence of SUSY (at least with its minimal structure $\mathcal{N} = 1$) in the compactified theory is important for two reasons: without supersymmetry, there would be tachyonic states in the spectrum[28]; Moreover, SUSY is necessary for hierarchy reasons.[29]

In order to retain $\mathcal{N} = 1$ residual supersymmetry, it is necessary to compactify the theory in a specific way[30]: in the following lines, the case of compactification of Type IIB string theory on a Calabi-Yau manifold will be described. However, the same mechanism is also directly related to models of heterotic string theory and F-theory. With some differences, it is even possible to extend the discussion to other perturbative regimes, such as Type I, Type IIA, and M-theory.

The following discussion aims to highlight some physical idea and the basics of the geometric framework behind this technique in the 10-dimensional case.

[28]These types of fields are non-physical. See, for example, [76] and [112] for a complete discussion.

[29]The problem of the energy scales' hierarchy is typically known in particle physics, for a detailed review it is suggested to read the references reviewing the main aspects of QFTs, like [83], [84] and [85].

[30]The first compactification mechanism was developed by Kaluza and Klein and consisted of an uplift of 4d space-time to a five-dimensional one, where the extra dimension was contained in a one-sphere $S^1$. An other interesting case for the development of this technique was the compactification on toroidal structures. See [85] for more details.

In formulae, the explanation can be translated by imposing that the metric of the full theory has the following shape:

$$ds^2_{10}(x;y) = e^{2h} g^4_{\mu\nu}(x) dx^\mu dx^\nu + g^6_{ab}(y) dy^a dy^b \tag{1.7}$$

where $ds^2_{10}$ is the global 10d space-time metric. Instead, the coordinates $x$ refers to the Minkoswki ones[31], while $ds^2_6$ is the metric of the interior six dimensional Calabi-Yau manifold "$X$", with coordinates $y$.
The line element of the 4d theory is paired with a conformal factor $e^{2h}$, that is crucial because it contains the relics of the compactified dimensions.
The Weyl rescaling factor reads:

$$e^{2h} = \frac{M^2_{Pl}\, l^2_d}{4\pi V_x} \tag{1.8}$$

where $V_X$ is the volume of the manifold $X$ (that should be a compact complex manifold, otherwise the description doesn't work).
In order to describe the perturbative regime of the theory and to find the origin of the scalars present in the EFT, it is necessary to introduce a few mathematical notions about the complex Riemann manifolds and their topological structure.
For our purposes, it is crucial to understand the geometrical origin of the pseudo-scalars. Axion-like fields generically arise in compactifications as Kaluza-Klein (KK) zero modes of antisymmetric tensors. Higher-order antisymmetric tensor fields typically give rise to a large number of zero modes, related to the Hodge numbers of the compact manifold. For example, the number of resulting massless scalar fields originating from a single tensor $B_{MN}$ is equal to the number of (homologically non-equivalent) closed two-cycles $C_i$[32] in the Calabi-Yau $X$. By taking the indices of the two-form along a cycle, one gets massless four-dimensional fields. Formally, the KK expansion for the two-form reads:

$$B_2 = \frac{1}{2\pi} \sum_i b_i(x) w^i(y) + \ldots\,, \tag{1.9}$$

where $w^i$ are closed non-exact two-forms dual to the cycles $C_i$.
This argument can be generalized for higher-form fields, and the number of cycles can become large, typically of order hundreds. A more general $(p+1)$ field can be expressed with a form strength $F_{p+1}$ and appears in the action as:

$$S = -\frac{1}{2}\int F_{p+1} \wedge \star F_{p+1} = -\frac{1}{2(p+1)!}\int d^dx \sqrt{-g} F_{\mu_1\ldots\mu_{p+1}} F^{\mu_1\ldots\mu_{p+1}}\,. \tag{1.10}$$

[31]Or to the 4 dimensional one considered, generally one could also work with AdS and dS spaces.

[32]Cycles are closed submanifolds useful in characterizing the homology class of a manifold, as each non-equivalent cycle can be associated with a "hole" in the manifold.

From the equation of motion $dF = 0$, one can write the tensor respect to a p-form potential $A$: $F_{p+1} = dA_p$.[33] A general solution can be found decomposing the potential $A$ into a basis of harmonic p-forms $w_{p,i}$ on the compact manifold:

$$A_p = \frac{1}{2\pi} \sum a_i(x) w_{p,i}(y) \ . \tag{1.11}$$

The $a_i$ fields resulting from the decomposition are pseudo scalars. From their definition it follows that they can be expressed as $a_i = \int_{C_{p,i}} A_p$ , where $C_{p,i}$ are p-cycles in the compact manifold. The sum over $i$ into the expression (1.11) counts the number of harmonic forms and expresses the topologically distinct ways $A_p$ can be integrated on the compact space. The number of the basis p-forms $w_{p,i}$ is determined by the number of homologically non-equivalent p-cycles, i.e. by the $p^{th}$ Betty number $b_p$.[34] Each axion is part of a complex field where an other modulus (the saxion) is controlling the size of the corresponding p-cycle.
Thus, the moduli (the saxions) arise as a consequence of the compactification, and their pairing with axions is a direct consequence of SUSY, which requires the existence of an appropriate form in supergravity.

Supersymmetry requires working with complex manifolds, making the concept of Kähler space important. This type of manifold is a Riemannian manifold with non-zero metric components $g^{i\bar{j}}$, which locally take the form:

$$g^{i\bar{j}} = \partial^i \partial^j K(z, \bar{z}) \tag{1.13}$$

where the key funcion K is called Kähler potential.
The Kähler form is a (1,1) closed form, $k^{i\bar{j}}$, which is the antisymmetric version of $g^{i\bar{j}}$, and it can be expressed as

$$J = \partial \bar{\partial} K \ . \tag{1.14}$$

[33] This is always possible as a consequence of the property of the exterior derivative $d^2 = 0$.

[34] Additional important geometrical tools are listed here to complete the discussion.
The de Rham cohomology $H_p$ of a manifold M is defined as the set of closed p-forms (i.e. such that $dA = 0$) modulo exact p-forms (i.e. such that $dA_p = B_{(p-1)}$), respect to the exterior derivative $d$. Their dimensions $b_p = H^p(M)$ are topological invariants known as Betti numbers, also related to the Euler characteristic of the manifold M.
For complex manifolds (like the Kähler and the Calabi-Yau), it is possible to define a generalization of this concept using the set of differential operators $\partial$ and $\bar{\partial}$ which map $(p, q)$ forms respectively to $(p+1, q)$ and $(p, q+1)$ forms. The operation must also satisfy the relation

$$d = \partial + \bar{\partial} \ . \tag{1.12}$$

So, the topologically invariant Dolbeault cohomology $H^{p,q}$ can be defined as the space of $\bar{\partial}$-closed (p, q) forms modulo $\bar{\partial}$-exact (p, q) forms. The dimensions of these space is known as Hodge number and it is crucial studying the CY manifolds.

The Kähler form can also be written as $J = ig^{i,\bar{j}} dx_i \wedge dx_{\bar{j}}$ using complex coordinates. It is part of the cohomology $H^{1,1}$ and its dimension (the Hodge number $h^{1,1}$) describes the number of possible (non-topologically equivalent) ways the metric can be deformed.
The resulting saxions emerging in the EFT can be labeled as $s^i = (s^0, s^a)$. The $s^a$ are the Kähler saxions, which arise from the decomposition of the Kähler form:

$$J = s^a \cdot \mathcal{D}_a \, , \tag{1.15}$$

where $\mathcal{D}_a$ is a basis defined to be dual to a set of divisors.[35] While $J$ lives in the Kähler cone, the "universal" saxion $s^0$ parameterize the volume of the complex compactified space.[36]
For instance, in the case of the heterotic string, $s^0$ is given by:

$$s^0 = \frac{1}{3!} e^{-2\phi} k_{abc} s^a \cdot s^b \cdot s^c = e^{-2\phi} V_X \, , \tag{1.18}$$

where $\phi$ is the 10d dilaton present in the full theory, and $k_{abc} \equiv D_a \cdot D_b \cdot D_c$ are the triple intersection numbers of $X$ in the chosen basis.
To see more details and some explicit examples about compactification, see the references from [83] to [100].
After this geometric introduction, the following section will outline the physical structure of a generic string EFT model obtained through a compactification mechanism.

### Effective models

Without assuming any specific String UV completion, let's summarize the key aspects of the resulting effective theory. This section is crucial in order to highlight the primary factors that need attention when dealing with the low-energy approximation of string theory.

[35] A divisor can be thought of as a linear combination of complex codimension-one cycles.
[36] To study the moduli space, it is important to introduce the notion of convex cones. First, the set of possible charges related to the discrete shift symmetry of the axion it is usually called $\mathcal{C}_I$. Additionally, the saxions are part of a vector space $V$, which also admits a dual space $V^*$.
The saxionic convex cone is defined as

$$\nabla \equiv \{ s \subset V | \, \langle s; q \rangle \; \forall q \subset C_I \} \tag{1.16}$$

The interior of the cone consists of vectors that satisfy this condition with a strictly positive product.
There is also a definition for the dual saxion cone: $\mathcal{P} = \{ \ell \in V^* \, | \, \ell_i = -\frac{1}{2} \frac{\partial K}{\partial s^i} \big|_{s \in \nabla} \}$. These notions are essential for various reasons, such as identifying specific regions like the perturbative regime of the EFT, which obeys the condition:

$$\langle q; s \rangle \; \gg 1 \, . \tag{1.17}$$

This regime is clearly parameterized by the saxionic moduli.
Conversely, the non-perturbative effects are expected to become relevant when the boundary of $\nabla$ is reached.

Effectively, each theory and its specific compactification into a four dimensional model is unique and can present non-trivial aspects. However, for the purpose of this thesis, it is important to understand why all the models with scalar massless fields coupled to gravity are physically meaningful.
Starting from the constrain of non-existence of global symmetries, the first requirement is that if the model contains standard axion particles (characterized by a shift symmetry), consequently one would expect such fields to acquire a large mass, which poses phenomenological issues.[37] Therefore, it becomes inevitable that this symmetry must be broken in some way, likely through gravitational effects.
Furthermore, it is also important to note that there is no reason forcing supersymmetry to break in the energy window considered, so it is commonly assumed that at least a $\mathcal{N} = 1$ SUSY survives the compactification process.
The axion in the system are typically normalized such that the shift symmetry takes the form

$$\varphi^i \to \varphi^i + 2\pi \tag{1.19}$$

where $\varphi^i$ are the axion scalar fields, with $i = 1, 2, ..N$. Supersymmetry provides the theory with "chiral supermultiplets", in which axions are contained in the lower component, always paired with other scalars, $s^i$, called *saxions*. The complex fields are combined together in

$$t^i = \varphi^i + is^i \; , \tag{1.20}$$

which preserves the invariance under the shift symmetry by $2\pi$.
Neglecting all the other possible matter fields, the bulk action involving scalars and gravity takes the following form:

$$\frac{M_{Pl}^2}{2} \int R * 1 - \frac{M_{Pl}^2}{2} \int \mathcal{G}_{ij}(s) \left( d\varphi^i \wedge *d\varphi^j + ds^i \wedge *ds^j \right) \tag{1.21}$$

where $\mathcal{G}_{ij}$ is a symmetric positive-definite matrix depending on the saxions. This ansatz for the action is general, but it is clear from the previous subsection that the number of scalars, as well as the matrix $\mathcal{G}ij$, strongly depends on the adopted compactification scheme.
The kinetic terms of the scalars are related to the so-called *Kähler Potential* $\mathcal{K}(s)$, which depends solely on the complex (saxionic) part of $t^i$. It is defined through the relation:

$$\mathcal{G}_{ij} \equiv \frac{1}{2} \frac{\partial^2 \mathcal{K}}{\partial s^i \partial s^j} \; . \tag{1.22}$$

[37] Otherwise, this kind of particles would have been observed or could be produced experimentally.

For a wide class of string models it has been proved that $\mathcal{K}$ is proportional to an homogeneous function of grade $n$: $P(\lambda s) = \lambda^n P(s)$.
Furthermore, in the perturbative regime of many string models, the Kähler potential exhibits a specific dependence:

$$\mathcal{K}(s) = -\log\left(P(s)\right) \tag{1.23}$$

which can be assumed as an underlying hypothesis throughout this dissertation.
The Hodge duality for the axion formulation will be discussed in detail in 2.4, but it can be anticipated here in the general supersymmetric context. Defining a two-form potential and its corresponding three-form field strength as $d\mathcal{B}_2 = \mathcal{H}_3 = -M_{Pl}^2 \mathcal{G}_{ij} * d\varphi^i$, the duality allows for the chiral multiplet to be exchanged with a linear multiplet whose lower component is the dual saxion $\ell_i$, defined as:

$$\ell_i \equiv -\frac{1}{2}\frac{\partial \mathcal{K}}{\partial s^i} \,. \tag{1.24}$$

The kinetic terms of the action include an extra additional contribution aside from the Kälher one:

$$\mathcal{F}(\ell_i) = \mathcal{K} + 2\,\ell_i\, s^i \,. \tag{1.25}$$

Thus, the action (1.21) can be expressed in its dual equivalent form:

$$\frac{M_{Pl}^2}{2}\int R*1 - \frac{M_{Pl}^2}{2}\int \mathcal{G}^{ij}\left(d\,\ell_i \wedge *d\,\ell_j\right) - \frac{1}{2M_{Pl}^2}\int \mathcal{G}^{ij}\left(\mathcal{H}_{3,i}\wedge *\mathcal{H}_{3,j}\right) \tag{1.26}$$

where $\mathcal{G}^{ij}$ is the inverse of (1.22), i.e.

$$\mathcal{G}^{ij} = -\frac{1}{2}\frac{\partial^2 \mathcal{F}}{\partial \ell^i \partial \ell^j} \,. \tag{1.27}$$

Moreover, it is possible to define the dual saxion fields as:

$$\ell_i = \frac{1}{2}\frac{\partial \mathcal{F}}{\partial s^i} \,. \tag{1.28}$$

Additionally, an extra boundary term arises, which must be included when considering the dual theory:

$$-i\int_{\partial M} a^i\, \mathcal{H}_{3,i} \,. \tag{1.29}$$

From the homogeneity of $P(s)$, it follows that $l_i s^i = \frac{n}{2}$, which, together with equation (1.25), leads to the relation:

$$\mathcal{F}(\ell) = \log\left(\tilde{P}(\ell)\right) \,, \tag{1.30}$$

valid up to a constant.
The function $\tilde{P}$ is the inverse of $P$, yet remains an homogeneous function of degree $n$. It is notable that, although $n$ could, in principle, take any integer value, in string-motivated models it has been shown that only values between 1 and 7 are accepted.[38] However, as highlighted in section 3.3, to obtain a consistent wormhole solution, $n > 3$ is required, and the other values are discarded.

In the following section, a brief summary of the conjectures regarding the phenomenological aspects of these string EFT models will be presented. The role of the wormhole solution will be highlighted in the last section of chapter 4, emphasizing its significance in this broad area of research.

### 1.3.2 The Swampland programme

To distinguish whether an EFT is viable, i.e. predictive and consistent with the low-energy limit of the SM, a new approach has been proposed, based on different "*swampland criteria*".
The term "*swampland*" refers to the set of anomaly-free and apparently consistent quantum EFTs that cannot be embedded in a UV-complete QG theory. On the other hand, the set of EFTs that are consistent with the QG completion assumed is typically called the "*landscape*". Consequently, any self-consistent EFT must either fall into the swampland or the landscape, as shown in figure 1.3.

Interestingly, the higher the cut-off of an EFT, the more challenging it becomes for the theory to fall within the landscape. It is particularly difficult to incorporate new lighter degrees of freedom, without spoiling the structure of the QFT.[39] As a result, the set of consistent theories narrows as energy increases.
The set of possibilities for establishing new constraints on low-energy theories is broad and heavily dependent on the assumed UV-completion and the specific analysis being performed.

A remarkable and surprising result from past research is that various criteria are interconnected, as illustrated in figure 1.4. In some cases, these criteria appear to be two sides of the same coin, while in other instances they are complementary, meaning that they

[38]For the full discussion about this result, see [15] and [11].
[39]This difficulty arises because, when the cut-off of the EFT is reached, the theory breaks down, and new degrees of freedom should be included.

require each other to make good phenomenological predictions.

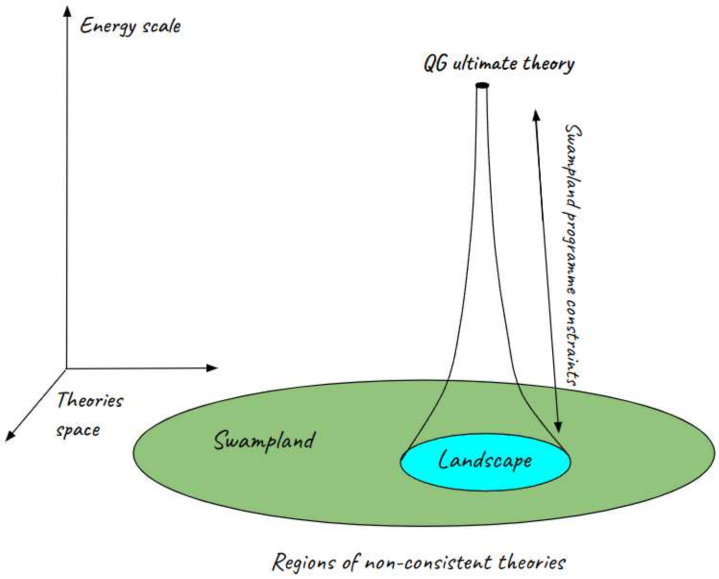



**Figure 1.3:** Graphic representation of the (effective) theories classification, using the constrains provided by the Swampland conjectures.

In situations where the criteria seem complementary, it remains unclear whether their relationship points to a more fundamental mechanism or if it simply confirm the common origin rooted in the UV theory.
String phenomenologists are actively investigating this complex area of research, aiming to validate the conjectures through various studies related to black hole physics, AdS/CFT duality, cosmological models, and even pure geometric approaches.
Indeed, numerous definitions and formulations of the swampland criteria exist, and it is challenging to determine if they can be universally verified within a general QG framework without relying on the powerful tools provided by ST.[40]

In the picture 1.4, a comprehensive summary of the main criteria studied in recent years is presented. For the purposes of this thesis, it is crucial to focus on the three criteria highlighted in black, as they form the core of the full swampland program. So, the three conjectures outlined below are the "No Global Symmetries Conjecture" (NGS), the "Weak Gravity Conjecture" (WGC), and the "Distance Conjecture" (DC); each of these is related to the topic of wormhole physics.[41] For further details, several complete and recent reviews discussing the current state of the art can be found in the bibliography,

[40] As previously emphasized, the swampland program consists of conjectures: none of them has been definitively proved, at best they hold only in specific frameworks or limits.
[41] The importance of wormhole solutions is presented in section 4.4.

see [91]-[100].

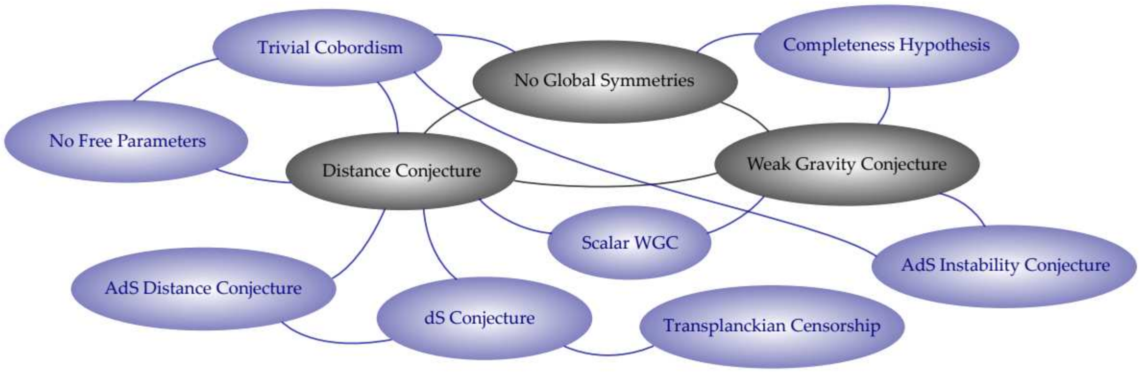



**Figure 1.4:** Recent map of the main swampland conjectures recently studied in literature. Taken from the lectures published in [91].

### The no global symmetries criterion

The first criterion considered is also the oldest one, generally referred to as the "No Global Symmetries Conjecture" (NGS). The name itself is quite explanatory, although various formulations exist in the literature. A general statement for this conjecture could be articulated as follows:

**(No Global Symmetries).** *In a Quantum Gravity theory there are no global symmetries; thus every symmetry of the theory must be either broken or gauged if it is global at lower energies.*

The notion of global symmetry refers to any map of local operators that is unitary, obeys a group law, and also commutes with the Hamiltonian to conserve the system's energy. There are various motivations for the NGS conjecture, most of which are connected to black hole physics or perturbative string arguments.
For instance, an older no-hair theorem proposed by Hawking posits that a black hole can be entirely described by its mass, angular momentum, and (gauge) charges. The existence of an additional global symmetry would lead to a divergent number of possible microstates, implying infinite entropy, which contradicts our understanding of black hole thermodynamics.
Moreover, in ST, it has been demonstrated that any global symmetry of a p-brane worldsheet corresponds to a gauge symmetry in spacetime. This indicates that there is no possibility of having a continuous global symmetry in this framework.
While the NGS is well-motivated and its connection to other conjectures has yielded significant results, it does not provide solid constraints in the low-energy limit. In practice,

it is sufficient to break a global symmetry at a certain energy scale or to gauge it by introducing extra degrees of freedom to bypass this limitation. Nevertheless, the conjecture serves as a valuable starting point for deriving further arguments and constraints.

**The distance conjecture**

Particles are typically classified according to their quantum numbers, with spin being one of the most important characteristics, as it determines the statistics of the fields. In 2012, the Higgs boson, a spin-0 particle, was experimentally observed. This particle behaves as a scalar field, like all particles with zero spin.
A key aspect of scalar fields is their tendency to adopt a vacuum expectation value (v.e.v.), which can be thought of as an homogeneous condensate that fills the entire universe. The v.e.v. is tied to a specific energy value, and in the case of the Higgs field, this value is crucial for explaining the masses of most SM particles through Yukawa interactions.
But where is the link to the Swampland?

In a generic QFT, the v.e.v. of a scalar field can be related to the cut-off scale, the energy boundary where the theory becomes ineffective. For all the EFTs in the landscape (those with a high-energy QG completion) the following inequality should hold:

$$\Lambda_0 < M_{Pl} \tag{1.31}$$

where $\Lambda_0$ indicates the h.e. cut-off. The "Distance Conjecture" (DC) imposes a stricter bound, which depends on all the vacuum expectation values of the effective theory. It can be shown that the more restrictive limit is given by:

$$\Lambda < \Lambda_0 \, e^{-\alpha\left(\frac{\langle|\phi|\rangle}{M_{Pl}}\right)} \tag{1.32}$$

where clearly $\langle| \, \phi \, |\rangle$ represents the absolute value of the canonical normalized scalar v.e.v..
An interesting consequence of the conjecture is that the lowering of the UV cut-off occurs exponentially, making the bound significantly more restrictive compared to the previous one.

Before defining this principle, it is important to explain its relevance within the context of ST. In order to work within a "standard" four-dimensional framework instead of the more the complex 10-dimensional string theory, some dimensions of space-time must be compactified. This process involves "compressing" a six-dimensional manifold into the boundary of space-time, which introduces additional massless scalar fields known as

*saxions*. The v.e.v. of these fields are referred to as moduli, and the set of all possible values for these parameters defines a geometrical manifold called the *moduli space*. Essentially, the moduli space can be viewed as a parameterization of all the possible EFTs resulting from this compactification process.
Since the moduli are not free parameters, it becomes particularly interesting to study the physical implications that arise from their dynamics within the moduli space. Varying them corresponds to changes in physical quantities such as couplings or the masses of the EFT's fields. The metric of the moduli space is determined by the kinetic terms of the scalar fields and can be used to define distances within this space.[42] This tool is particularly useful for exploring bounds associated with physical properties of effective theories (for instance, weak-strong duality limits, or the presence of gauged and non-gauged symmetries) as the moduli values shift.[43]
The premise leads directly to the definition:

**(Distance Conjecture).** *Considering a gravitational effective theory with a moduli space $\mathcal{M}$, parameterized by the expectation values of scalar fields $\phi^a$ which do not have any potential, given a point $\mathcal{Q}$ in $\mathcal{M}$ there always exists a point $\mathcal{P}$ in $\mathcal{M}$ at an infinite geodesic distance from it. Along the path connecting the two points, an infinite tower of states arises, with an energy scale $M(\mathcal{P}) \sim M(\mathcal{Q})e^{-\alpha\, d(\mathcal{P},\mathcal{Q})}$, which exponentially become asymptotically massless.*

The consequence of this conjecture is precisely the behavior of the cut-off scale anticipated by the equation (1.32). This implies that by varying the moduli values, one introduces lighter degrees of freedom, thereby lowering the threshold of validity of the effective theory.

One interesting aspect of the DC is its counter intuitive nature respect to the QFT expectation. Naively, one might assume that if a QG theory is valid at energies around $M_{Pl}$ (as in the bulk of $\mathcal{M}$), the physics of EFTs operating well below that threshold would be partially independent of the UV degrees of freedom (as in the bound of $\mathcal{M}$). In contrast, the DC posits that the separation of scales arises from the exponential decay of the tower of states, implying that UV physics still exerts a significant influence on the low-energy theory.[44]

[42] See the toy model in section 3.1 for a concrete example.

[43] There is also a formulation known as the "local distance conjecture," which studies local field variations within the same theory.

[44] This could be related to a fundamental principle that would also explain the UV-IR relation in objects such as black holes, as mentioned at the beginning of this chapter. Furthermore, this conjecture is reminiscent of the "naturalness problem", although it remains unclear whether it can formally resolve it.

An important application of the DC is linked to the weak gravity conjecture (discussed in the following subsection), as well as to the weak/strong coupling duality discovered by Witten. In this context, the cut-off is related to the size of the compactified dimensions and suggests that at energies exceeding the Planck mass $M_{Pl}$, the ten-dimensional ST should be replaced by an eleven-dimensional one.
Additionally, the DC has recently been studied in the context of inflationary physics and cosmology, with efforts aimed at elucidating the nature of the cosmological constant and the rate of the universe's expansion. All these potential applications and further details are outlined in the review of the swampland program cited in the bibliography.[45]

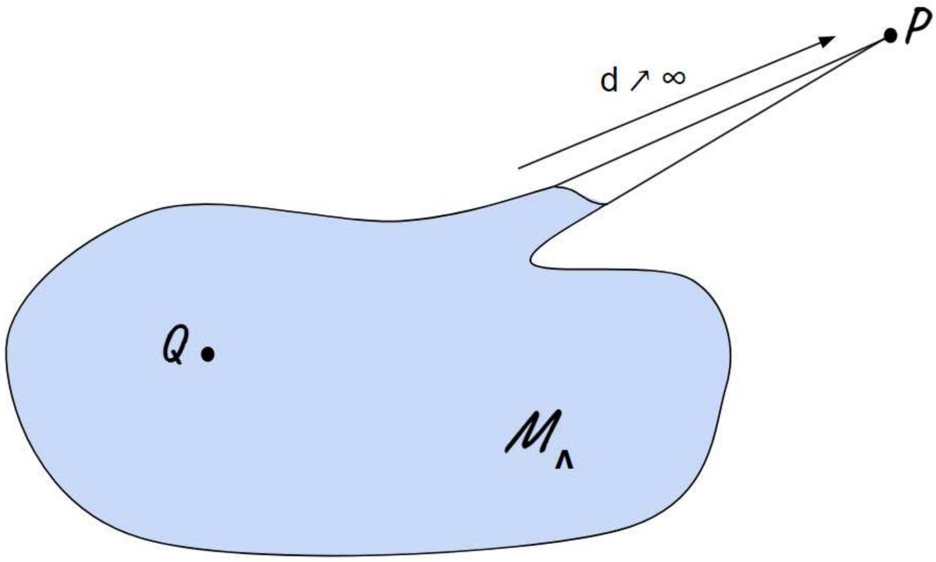



**Figure 1.5:** Moduli space of an EFT with UV cut-off Λ; the point P is at infinite distance respect to Q (in general it is possible that multiple point satisfy this condition) as stated in the definition of the conjecture.

### The weak gravity conjecture

One of the most studied swampland criteria is the "Weak Gravity Conjecture" (WGC). It is notable for having important feedback and applications in various models and contexts. For this reason, a clear definition does not exist; instead, the literature contains several formulations, each with slightly different nuances.
Starting from the general idea, the criterion is introduced with its generic formulation. The version related to axion physics and instantons will be briefly discussed later in 4.4.

[45]There are several discussions and application in literature, some of them contained in the articles [91]-[100].

In a certain sense, the WGC could be viewed as a continuation of the NGS conjecture. Effectively, it provides a bound that relates a particle's mass to its charge, thereby forbidding the possibility of sending the charge to zero and recovering a global symmetry. From another perspective, the "completeness spectrum" conjecture asserts that in a QG theory there should always exist charged states, although it does not provide any information about their masses. The WGC aims to fill this gap.
The main idea of the criterion is that if the effective theory contains a gauge symmetry, then some charged state should experience a repulsive force that is stronger than their gravitational interaction. This can be translated into a constraint for the mass-to-charge ratio as follows:

$$\frac{m}{q} \leq \sqrt{2} M_{Pl} \tag{1.33}$$

The inequality appears trivial from a classical point of view, because the gravitational interaction is more than 40 times weaker than the electromagnetic one. From the ST perspective, however, this aspect is less obvious because all fundamental interactions are unified, and there is no exchange of photons or gravitons. Instead, the interaction occurs between different strings. If one sends the string coupling $g_S$ to zero in order to weaken gravity, even the other forces automatically follows the same trend. Nevertheless, it has been firmly established that the WGC is satisfied in every ST context.[46]

Another important framework that strengthen the validity of the WGC comes from black hole physics, where the charge and mass of the black hole naturally appear in the description of the Reissner-Nordström solution. Considering the case of an extremal configuration, the following bound is proposed:

$$\left(\frac{Q_{bh}}{M_{bh}}\right)_{extremal} \leq \frac{q \cdot g}{m} \tag{1.34}$$

where $q$ and $m$ refer to a specific state that satisfies the WGC, and $g$ is the gauge coupling constant. This bound is important because it explains the possibility of a discharge process for extremal black holes, which would otherwise present theoretical issues.
Furthermore, considering the strong-weak duality, the conjecture can also been studied for magnetically charged objects (such as Polyakov monopoles or dyons). In this case, the WGC is once again verified, providing a natural bound for the theory's cut-off, similar to the one discussed for the DC in equations (1.32).

This last consideration leads to a simple version of the conjecture, which could be split into two parts, corresponding respectively to electric and magnetic charges:

[46]See the articles [95, 99] for a specific review of the WGC.

**(Elecectric Weak Gravity Conjecture).** *In a 4 dimensional gravitational theory with a $U(1)$ gauge symmetry and coupling constant $g$, the spectrum of the theory must include at least one particle with mass $m$ and charge $q$, such that*

$$m \leq \sqrt{2}\, g\, q\, M_{Pl} \tag{1.35}$$

and the complementary version for magnetic charges:

**(Magnetic Weak Gravity Conjecture).** *In a 4-dimensional gravitational theory with a $U(1)$ gauge symmetry and coupling constant $g$, the cut-off of the effective theory is bounded from above by:*

$$\Lambda \leq g\, M_{Pl} \tag{1.36}$$

It is interesting to note that the principle asserts the existence of certain states[47] that satisfy the given condition. However, it does not specify which states these should be. A stronger version of the conjecture has recently been introduced to address this ambiguity, aiming to force these relevant states to lie below the theory's cut-off, thereby offering clearer phenomenological constraints.[48]

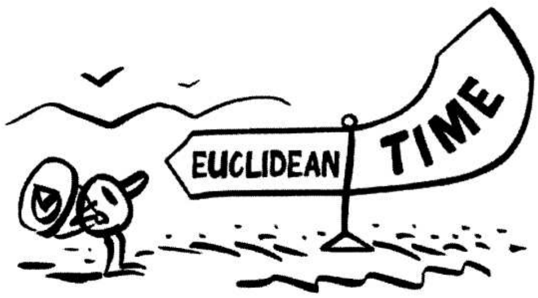
Despite the conjectures having already provided important constraints and insights into the low-energy limit of QG, many aspects still require deeper investigation and understanding. It would not be surprising if new results emerge from further studies of wormhole physics.

[47] To be precise, even a single state is sufficient, and the condition does not need to hold for every field in the theory.

[48] There are three different variants of this formulation, where the states satisfying the conjecture are respectively the one with the lowest charge, the one with the lowest mass and the one with the largest charge-to-mass ratio.

# Chapter 2

# Euclidean quantum gravity and non-perturbative contributions




Picture taken from [23].

Wormholes were first discovered in the 1950s by Schwarzschild, within the context of classical GR. The initial solution involves a structure of the space-time metric, which can be envisioned as a region of space connecting two black holes[1] via a bridge with a 2-sphere cross-section. An intriguing feature of this solution is that the two mouths of the wormhole open into causally disconnected regions of space-time. This characteristic was found in the radially symmetric Schwarzschild solution of classical GR in Minkowski signature. However, this class of mathematical solutions attracted less attention compared to black holes, as the wormhole itself shrinks as an observer approaches it, making it seem physically implausible.

Conversely, this thesis focuses on a different type of solution, characteristic of quantum gravity models. In this case, the wormholes are found by studying systems in Euclidean space-time. Although this assumption may initially seem less meaningful, it introduces a significant conceptual shift.

[1]To be precise, a couple of a black and a white holes.

Thus, the aim of chapter 2 is to explain why it is interesting to study Euclidean Quantum Gravity, and why it is convenient to adopt the semi-classical approximation procedure, which leads to the concept of a gravitational instanton.
In the latter part of the chapter, the thesis will also explore what a classical instanton solution looks like, along with the key duality in the formalism that will be essential for the further calculations of gravitational instantons in chapter 3.

## 2.1 Quantum gravity: Euclidean vs Minkowski signature

As previously mentioned, wormholes can be studied in both the Euclidean and Minkowski signatures, but these approaches are not always equivalent.
In QG, the most effective methods adopted for analyzing the models are either canonical quantization or the path integral approach. The former builds upon the classical framework of quantum mechanics by promoting dynamical quantities to operators, and then further promotes wave functions to quantum fields. The latter method involves integrating over all possible classical trajectories to achieve quantization.

While both approaches are often shown to be equivalent in particle physics, the choice of method can depend on the specific nature of the analysis being conducted. In this thesis, it is fundamental to assume the QFT version based on the path integral formulation.

The path integral of a classical particle traveling from an initial point $x$ to a final point y is defined as[2]:

$$\langle x|y\rangle = \int \mathcal{D}w\, e^{iS[w]/\hbar}\,, \tag{2.1}$$

where the idea behind this definition is that the transition amplitude accounts for all possible trajectories between the two points. The weight inside the path integral is proportional to the action $S$ of the system, divided by Planck's constant $\hbar$.
The path integral is widely used in QFT as a generalization of the previous definition. In the case of a simple theory with a quantum field $\varphi$ and an action $S[\varphi]$, the path integral is written as:

$$\mathcal{Z}(\varphi) = \int \mathcal{D}\varphi, e^{iS[\varphi]/\hbar}\,. \tag{2.2}$$

[2]The definition is general. Considering the Minkowski space-time, the two points refer to the 4d space: $w = (w_0; \vec{w})$.

Here, no explicit boundary conditions for the field are assumed, so the original concept of a "transition" between two points isn't directly applied. Nonetheless, the formalism proves useful for obtaining computational results.

In the case of a theory of gravity, a further generalization is required. QG research is based on the assumption that the metric $g$ should be treated as a quantum field in the UV limit. Thus, the path integral also includes the integration over the metric (or weights potential transitions between two boundary metrics, if such boundary conditions are imposed):

$$\mathcal{Z}(\varphi; g) = \int \mathcal{D}g\mathcal{D}\varphi \, e^{iS[\varphi;g]/\hbar} \, . \tag{2.3}$$

Regardless of the chosen metric signature, the previous discussion remains equally valid. So, what is the utility of switching to the Euclidean signature?

The main motivation is practical: the standard perturbative expansion[3] allows for the study of contributions arising from "small" deviations from the vacuum of the theory. However, the Euclidean formulation permits us to go further by including contributions from field configurations of a different nature, called *non-perturbative* contributions. This important aspect is further elaborated upon in the discussion of the semi-classical approximation in subsection 2.2.

In a four-dimensional space-time, the transition to Euclidean signature is achieved by performing a "Wick rotation", which involves transforming the time coordinate into its spatial Euclidean counterpart:

$$t \to -i\tau \, , \tag{2.4}$$

where $t$ denotes the Minkowski time coordinate and $\tau$ represents the "Euclidean time". Physically, the Wick rotation changes the signature $(- \, ; + \, ; + \, ; +)$, which distinguishes the "negative" time coordinate from the "positive" spatial ones, to $(+ \, ; + \, ; + \, ; +)$, where the time coordinate is positive-definite.
The immediate consequence of this rotation is that the path integral weight changes as follows:

$$e^{iS/\hbar} \to e^{-S_E/\hbar} \, . \tag{2.5}$$

This transformation is crucial because it significantly improves the convergence of the path integral. Indeed, the exponential term in Minkowski is an imaginary oscillating phase (due to the $i$ factor ahead the action), which can lead to weak or even undefined convergence of the integral. In some cases, the oscillation of the exponential is too strong

[3]Performed in Minkowski space-time notation.

and the path integral is ill-defined. By contrast, the Euclidean weight is real and often leads to a well-defined, convergent integral.
The advantages of the Euclidean signature extend further, but when gravity is included, the situation becomes more complex.
The pure Einstein-Hilbert action[4], including the appropriate boundary term (necessary to ensure a well-defined variational problem), can be written as:

$$S_E = -\frac{1}{16\pi G_N} \int_{\mathcal{M}} d^4x \sqrt{g}\, R - \frac{1}{8\pi G_N} \int_{\partial\mathcal{M}} d^3y \sqrt{h}\, K \tag{2.6}$$

where $\mathcal{M}$ denotes the Euclidean space-time, $\partial\mathcal{M}$ its boundary, $g$ is determinant of the Euclidean metric and $h$ is the determinant of the induced three-dimensional metric on the boundary, $R$ represents the scalar curvature and $K$ is the extrinsic curvature in the Gibbons-York term.[5]

A conceptual issue, known as the "$conformal\, factor\, problem$", arises because the theory should be invariant under "Weyl" conformal transformations, which re-scale the metric without altering its general structure:

$$g_{\mu\nu} \to e^{2w(x)} \tilde{g}_{\mu\nu} \, . \tag{2.7}$$

Applying this transformation to the Einstein-Hilbert action in Eq. (2.6), we get:

$$S_E = -\frac{1}{16\pi G_N} \int_{\mathcal{M}} d^4x \sqrt{\tilde{g}}\, e^{2w} \left( \tilde{R} + 6(\nabla w)^2 \right) - \frac{1}{8\pi G_N} \int_{\partial\mathcal{M}} d^3y \sqrt{\tilde{h}}\, e^{2w} K \, . \tag{2.8}$$

The problem emerges because the exponential factor $e^{2w}$ of the transformation is strictly positive and can be fixed arbitrarily large. Consequently, the Euclidean Einstein-Hilbert action is not bounded from below and can take on any negative value, making the associated path integral ill-defined.

This feature of gravitational theory sparked significant debate, but a solution was proposed by Page and later refined by Gibbons, Hawking, and Perry in [49]. Their solution involves rotating the contour of integration in the path integral, causing the conformal factor of the metric to become imaginary, thus ensuring the path integral converges. This is done by splitting the domain of integration in the path integral into conformally-equivalent classes of metric configurations. For each class, a specific metric configuration

[4]The cosmological constant is not considered here, as it does not affect the following discussion.
[5]Further details regarding the boundary term and the boundary conditions in gravitational models are discussed in section 3.4.

$\tilde{g}$ with $\tilde{R} = 0$ is chosen, because this condition ensures its regularity. The conformal factor is then integrated out around these configurations, and finally, the equivalence classes themselves are integrated out, yielding a well-defined one-loop approximation of the path integral.
More recently, an alternative technique has emerged studying wormholes in the context of quantum gravity, performing the analysis of gravitational models within the Minkowski framework. The key idea of this approach is to deform the integration contour of the path integral, working in a complexified field space. Applying this method, wormholes correspond to complex saddles of the Einstein-Hilbert action, though it remains unclear whether they always contribute. This approach, which relies on working with complex saddle points, requires the rigorous application of the Picard-Lefschetz theory. For more details on this parallel approach, see [8, 50].

Although not fully understood in all respects, the Euclidean path integral has been critical in several QG models, such as investigating black hole thermodynamics and the instability of Kaluza-Klein vacua.[6] Hence, it seems natural to adopt this tool for studying systems that involve topology change and wormhole solutions.

## The saddle-point approximation and Gaussian integrals

Another key reason for working in Euclidean space-time is that it allows for the use of mathematical and computational techniques often employed in fields like statistical mechanics.
To illustrate this, it is helpful to make a brief digression before proceeding, by focusing on an explicit example. The following toy model demonstrates the saddle-point approximation technique, which is widely used in QFT. For Euclidean instantons, this analysis is valuable because it introduces the concept of Gaussian integrals, which will be central in later chapters.

Even in the cases where the path integral is rigorously defined, it does not imply that it can always be computed explicitly. Nevertheless, in many cases, it is possible to apply the so-called saddle-point approximation. This approximation is performed around the extrema of the action, which correspond to the classical solutions of the equations of motion.

[6] See [21], for example, for details about this topic."

To construct the toy model, consider the following simple one-dimensional integral:

$$I = \int_{-\infty}^{+\infty} dx\, e^{-f(x)}\,, \tag{2.9}$$

evaluated along the real axis.
We assume that the function $f(x)$ can be analytically continued to the complex plane, with coordinates $z = x + iy$, and that there exists a stationary saddle point at $z_\star = iw$. Additionally, we require that the function $f(x)$ has a minimum at $z_\star$ while considering a contour parallel to the $x$-axis, such that $\partial_x^2 f(z)_{z_\star} > 0$.
At this point, we can Taylor expand the analytical continuation of the function $(f(z))$ around the saddle point, as follows:

$$f(z) = f(z_\star) + \frac{1}{2} f''(z)(z - z_\star)^2 + \ldots\,, \tag{2.10}$$

where the first derivative is omitted, as it corresponds to the variation of the action evaluated at the stationary point. Indeed, from a physical perspective, the saddle-point represents a classical solution to the equations of motion, for which the first variation of the action vanishes.
Substituting this expansion into the integral, we obtain the saddle-point approximation:

$$\begin{aligned} I &\approx \int_{-\infty+iw}^{+\infty+iw} dz\, e^{-f(z_\star)-\frac{1}{2}f''(z)(z-z_\star)^2} \\ &= \int_{-\infty}^{+\infty} dx\, e^{-f(iw)-\frac{1}{2}\partial_x^2 f(iw) x^2} \\ &= \left(\partial_x^2 f(iw)\right)^{-1/2} e^{-f(iw)}\,. \end{aligned} \tag{2.11}$$

The same result can be derived by choosing a different contour of integration. For example, one can achieve the same result by introducing a $\frac{\pi}{2}$-rotated variable $\tilde{z} = -iz$. In this case, the analogy with the computation of instanton amplitudes in the Euclidean metric becomes even clearer.

This argument can be generalized to various physical models where the function $f$ corresponds to the action and the saddle point represents non-trivial solutions to the equations of motion. The expansion then translates into a physical decomposition of the system into a background field (the classical solution) and its perturbations (quantum corrections). This decomposition is reflected in the result, where the exponential corresponds to the classical contribution to the integral, while the second derivative provides the quantum one-loop correction. Finally, the requirement of analytic continuation in this toy model generalizes to the Wick rotation necessary for transitioning to the Euclidean signature, where such computations become explicitly viable.

There is an important caveat to the previous discussion. The result of the saddle-point approximation is valid if and only if the Gaussian integral can be computed. In the toy model, this task is trivial as it involves a one-dimensional space. However, in physical situations, this is not always the case.
Let's now consider a Gaussian integral that depends on an arbitrary number of variables:

$$I = \int \prod_a dx^a e^{-\sum_a \lambda_{ab} x^a x^b} , \tag{2.12}$$

where $\lambda_{ab}$ is a real matrix. It is always possible to rotate the coordinates such that, in the new basis, the matrix $\lambda$ becomes diagonal. Thus, the integral becomes:

$$\begin{aligned} I &= \int \prod_a d\tilde{x}^a e^{-\sum_a \lambda_a \tilde{x}_a^2} \\ &= \prod_a \sqrt{\frac{\pi}{\lambda_a}} = \sqrt{\frac{\pi}{det\,\lambda}} . \end{aligned} \tag{2.13}$$

This result holds only if all the eigenvalues $\lambda_a$ are strictly positive. If some eigenvalues are negative, the result is no longer valid, and the situation becomes less clear. This is a crucial point for this thesis, as the stability analysis of wormhole solutions discussed in chapter 5 is based on this mathematical consideration.[7]

The example in this section demonstrates that transforming the complex equations of QG into a more manageable form is possible. The approach based on the Euclidean path integral provides deep insights and simplifies theoretical studies, making it an important tool in advancing the understanding of Quantum Gravity.

## 2.2 Non perturbative effects and semi-classical approximation

In the majority of short-distance physical systems, it is not possible to perform exact calculations and obtain precise solutions. For this reason, physicists often rely on approximation methods. There are two main approaches:

- *Perturbation theory*: this method is used when dealing with systems whose Hamil-

[7]Further comments on the consequences of having one or more negative eigenvalues in the spectrum are provided later, in chapter 5.

tonians can be expressed in the form $H = H_0 + \lambda H_p$, where $H_p$ is a perturbation of the classical Hamiltonian $H_0$ and $\lambda$ is a small parameter controlling the perturbative expansion; The energy eigenvalues are computed as an expansion series $E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + ...$, and similarly for the eigenvalues of all the other observables.

- *Semi − classical approximation*: this approach applies when a system deviates from the classical limit, where $\hbar \to 0$, such as in tunneling phenomena or other quantum mechanical transitions. So, quantum corrections are added to the classical contribution.

While perturbation theory has led to many important advances in particle physics, it comes with a caveat: it can fail by neglecting important contributions, often called non-perturbative effects. In many theories, addressing this limitation requires turning to the semi-classical approach.

Consider the path integral of the theory:

$$Z[\varphi] = \int_{\mathcal{C}} exp\left[\frac{-S_E(\varphi)}{\hbar}\right] , \tag{2.14}$$

its domain of integration $\mathcal{C}$ represents the space of all possible configurations of the field $\varphi$.

A natural question arises: if an exact computation of the path integral over all field configurations is not possible, are there specific configurations providing the leading physical contributions and dominating the integration?

The answer is yes. The configurations that dominate are close to the classical solutions of the equations of motion (the stationary points of the action). While considering only these classical configurations gives a rough approximation, it is not sufficient for an accurate description of the physics. The semi-classical approximation refines this by including the first-order quantum corrections.

The reasoning is as follows: the path integral weight $e^{-S_E}$ is dominated by the classical configurations, where $\delta S_E = 0$. However, classical solutions alone have zero measure in the integration and thus do not contribute to the path integral. The semi-classical approach addresses this by considering small quadratic fluctuations around these classical solutions, leading to more accurate predictions.[8]

[8] Indeed, while all solutions with a finite action have zero measure, the approximation considers a specific portion of the path integral domain which has non-zero measure.

In this approach, the field $\varphi$ is expanded around the classical solution $\varphi_0$:

$$\varphi = \varphi_0 + \phi\,, \tag{2.15}$$

where $\varphi_0$ corresponds to a stationary minimal point and $\phi$ to a small quantum fluctuation around it.
This expansion is based on the assumption that $|\phi| <<< |\varphi|$. In this way, the approximation allows for a systematic expansion of the path integral capturing both the classical contribution and the leading quantum corrections. This method even provides a more complete understanding of the system, especially in scenarios where non-perturbative effects play a significant role.

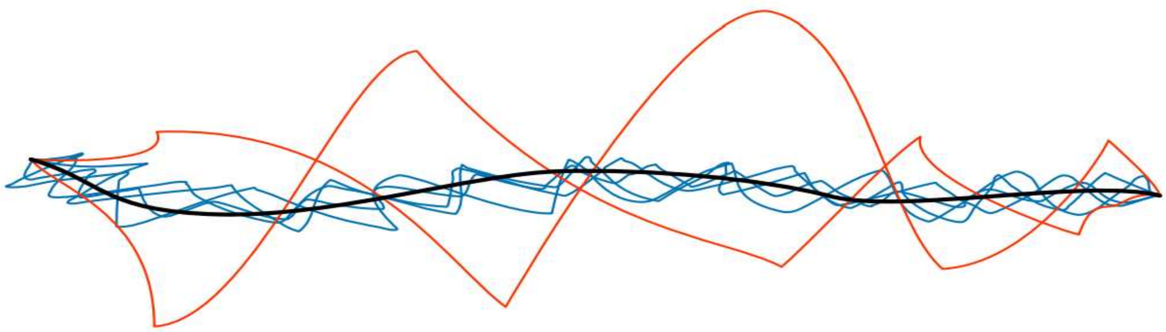


**Figure 2.1:** A sketchy representation of the possible field configurations in the path integral: the black line indicates a stationary configuration, the blue ones are configurations with small deviations with respect to the previous one, the red ones indicate paths that strongly deviate from the minima.

Plugging the expansion in the classical action leads to the following splitting of the action:

$$S(\varphi) = S(\varphi_0) + \mathcal{Q}(\phi) + \mathcal{R}(\phi)\,, \tag{2.16}$$

where the term $\mathcal{Q}$ includes the quadratic terms in the field fluctuation, while the "reminder" $\mathcal{R}$ represents all the higher-orders contributions.
Thus, the semi-classical approximation consists in neglecting all the higher-order terms of $\mathcal{R}$ due to their suppression relative to the quadratic ones. This allows to integrate out the classical solution and obtain a path integral that is quadratic (so Gaussian-shaped) and, therefore, more manageable.
If the stationary point considered in the expansion corresponds to a minimum of the action, the approximation may not yield more predictive power compared to the corresponding perturbation approach. For instance, in the action illustrated in figure 2.2, the perturbation approach involves selecting a minimum and expanding the field around it. In contrast, the semi-classical strategy split the integral into the main contributions from

configurations around each minimum:

$$I \simeq I_1 + I_2 + I_3 \, . \tag{2.17}$$

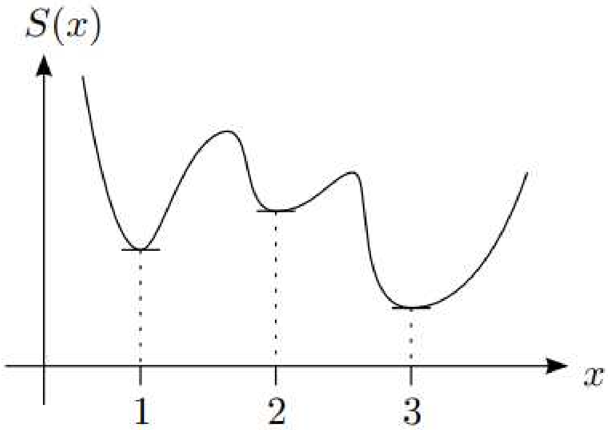



**Figure 2.2:** Example of a general action, with three minima.

While the outcomes of the two approaches may be similar, the second method can be enhanced by considering also non-trivial stationary configurations. Indeed, there exist pure non-perturbative saddle points that interpolate between different vacua. Consequently, the approximation in equation (2.17) can be refined by incorporating these additional contributions. Notably, instanton contributions become even more significant when dealing with potentials that do not admit true minima, as in various gravitational models.
This new class of solutions is referred to as "topological solitons". Depending on the specific model under consideration, different kinds of solutions have been identified, all linked to features of the space's topology.[9] The most common non-perturbative solutions studied include kinks, vortices, monopoles, and instantons. A defining characteristic of these objects is that they are topologically stable and represent finite-energy solutions to the classical equations of motion; moreover, instantons are typically discovered only within the Euclidean formalism. Indeed, the change in signature in this formalism permits these solutions to have a finite action. However, this does not imply that they lack a corresponding physical interpretation in the relative Minkowski QFT; for instance, such interpretations frequently arise in cases of quantum tunneling between vacua.[10]
A concrete example will be presented in the following section 2.3, introducing instanton solutions in two "non-gravitational" frameworks.

[9]For example, the dimension "$d$" of the space and the gauge group of the theory are both important, because they can lead to different kinds of topological solutions.
[10]See, for example, the Coleman lectures [101].

In summary, the Euclidean formulation is fundamental in this context because it facilitates the discovery of these field configurations. Wormholes are metric configurations of a similar non-perturbative nature, highlighting the necessity of understanding their potential contributions to the low-energy effective theories to which they belong.

## 2.3 Instantons in QFT

In the previous section, the concepts of non-perturbative contributions and topological objects were introduced. However, two explicit examples can further illustrate these ideas. The first is an instructive toy model where the soliton can be derived more easily. The second example, on the other hand, involves a more complex framework (i.e. Yang-Mills theories), that require additional mathematical concepts; thus, it will be described qualitatively without explicit calculations.
This section focuses on a specific type of topological solution known as the instanton. An instanton is a classical solution to the equations of motion with finite, non-zero action, derived in Euclidean space-time. These configurations are typically found in both "classical" quantum mechanics and QFT, where they often correspond to absolute minima of the action, providing dominant contributions to the path integral.

### A quantum mechanic toy model

The first model considered is a quantum-mechanical system with a double-well potential:

$$V(x) = \frac{1}{4}(x^2 - a^2)^2 \, . \tag{2.18}$$

In this case, the points of absolute minima, $x = \pm a$, are those typically used for a perturbative study. However, it can be shown that these are not the only critical points with physical relevance, both from a classical and quantum perspective.
When using the WKB approximation of the Schrödinger equation, a tunneling amplitude can be derived. Interestingly, the same result can be obtained through the path integral formalism, offering an alternative yet equivalent approach:

$$\langle x_f, t_f \,|\, x_i, t_i \rangle = \int \mathcal{D}x \, e^{-\frac{S_E}{\hbar}} \, , \tag{2.19}$$

where the action has already been Wick-rotated and is evaluated between the initial and final points, $x_i$ and $x_f$.

The initial action in Minkowski signature is given by

$$S_M = \int_{t_i}^{t_f} \left[ \frac{m^2}{2} \left( \frac{dx}{dt} \right)^2 - V(x) \right] dt^2 \,, \tag{2.20}$$

and can be transformed to its Euclidean counterpart as:

$$S_E = \int_{\tau_i}^{\tau_f} \left[ \frac{m^2}{2} \left( \frac{dx}{d\tau} \right)^2 + V(x) \right] d\tau^2 \,. \tag{2.21}$$

The explicit consequence it that the sign of the potential is flipped. This change is significant: in the Euclidean signature, the critical points $x = \pm a$ become maxima, as shown in figure 2.3.

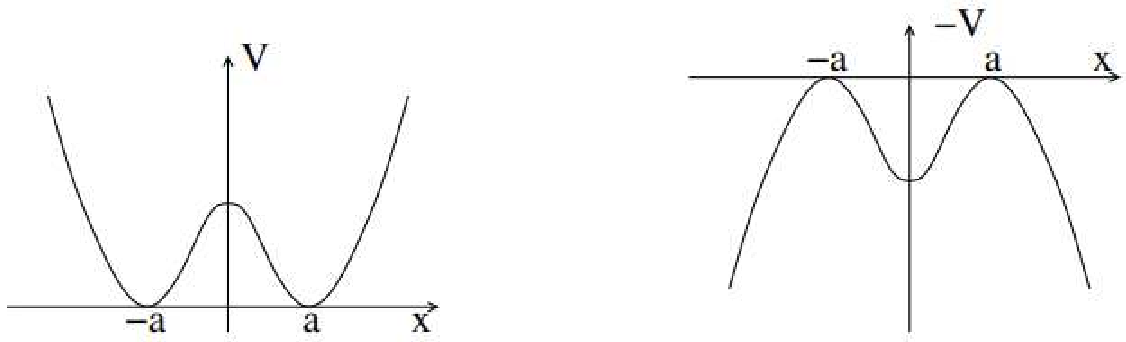



**Figure 2.3:** Double well potential considered in the model. The right figure corresponds to the Minkowski case, the left one to the Euclidean situation.

Consequently, a different solution to the equation of motion is expected to minimize the on-shell value of the action. This solution is what is sought in the path integral formalism, because it would dominate the semi-classical approximation.

Thus, setting $m = 1$ for simplicity, there is a useful trick that allows to rewrite the action in an equivalent form:

$$\begin{aligned} S_E &= \int_{\tau_i}^{\tau_f} \frac{1}{2} \left[ \left( \frac{dx}{d\tau} \right) - \sqrt{2V} \right]^2 d\tau + \sqrt{2} \int_{\tau_i}^{\tau_f} \frac{dx}{d\tau} \sqrt{V} d\tau \\ &= \int_{\tau_i}^{\tau_f} \frac{1}{2} \left[ \left( \frac{dx}{d\tau} \right) - \sqrt{2V} \right]^2 d\tau + \sqrt{2} \int_{x_i}^{x_f} \sqrt{V} dx \\ &= \int_{\tau_i}^{\tau_f} \frac{1}{2} \left[ \left( \frac{dx}{d\tau} \right) - \sqrt{2V} \right]^2 d\tau + \frac{1}{\sqrt{2}} \int_{x_i}^{x_f} dx (a^2 - x^2) \,, \end{aligned} \tag{2.22}$$

where in the final step it is substituted the explicit formula for the double well potential (2.18).

Fixing the extrema of integration at the two maximal points[11], and setting $a = 1$ for simplicity, the second integral yields a finite value:

$$S_E = \int_{\tau_i}^{\tau_f} \frac{1}{2} \left[ \left( \frac{dx}{d\tau} \right) - \sqrt{2V} \right]^2 d\tau + \frac{2\sqrt{2}}{3} \, . \tag{2.23}$$

The resulting action is strictly positive, with the lower bound $S_E \geq \frac{2\sqrt{2}}{3}$. This makes it straightforward to identify the configuration that minimizes the action, which is the one that saturates the inequality:

$$\left( \frac{dx}{d\tau} \right) = \sqrt{2V} = \frac{1 - x^2}{\sqrt{2}} \, . \tag{2.24}$$

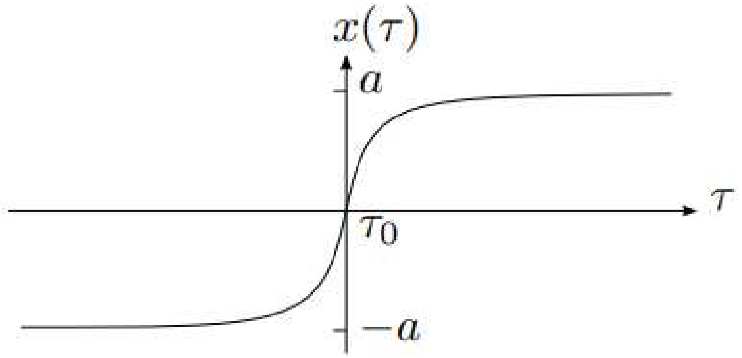



**Figure 2.4:** Profile of the instanton solution derived for the double well potential model.

Adopting the boundary conditions $x = \pm a$ for $\tau \to \pm\infty$, the previous equation leads to the following solution:

$$x(\tau) = tanh \left( \frac{\tau - \tau_0}{\sqrt{2}} \right) \, , \tag{2.25}$$

where $\tau_0$ is introduced as an integration constant.[12]

The configuration found in equation (2.25), depicted in figure 2.4, represents the instanton that interpolates between the two vacua $\pm 1$, also known as a "pseudo-particle". This solution is purely classical, and while its exact shape is not particularly significant, it is notable that the solution only deviates significantly from the asymptotic values (the minima of the potential in the Minkowski signature) within a small interval. Because of this, the instanton has zero measure in the path integral, which necessitates the use of the

[11] These boundary conditions lead to a "non-trivial" topological solution, which is why this configuration cannot be captured by the usual perturbative approach.

[12] This constant is commonly referred to as the "modulus" of the solution. Its physical meaning relates to the Euclidean time at which the solution experiences a rapid transition, significantly changing its slope.

semi-classical approximation, including Gaussian deviations, for a more precise analysis.

### The case of Yang-Mills instantons

The topic of this subsection is more complex than the previous model, so the discussion will be less technical. The goal here is not to provide too many details but to highlight that instantons are also important solutions in this class of models, which have been central to the development of particle physics. Indeed, research on non-abelian gauge theories has garnered significant interest in the particle physics community.
In the past, these types of theories were studied as extensions of Quantum Electrodynamics (QED) but with different gauge groups. Non-abelian gauge theories, also known as "Yang-Mills theories", are QFTs characterized by a non abelian gauge group, e.g. $SU(N)$.

The Yang-Mills action is typically written as follows:

$$S_{YM} = \frac{1}{2g^2}\int d^4x\, tr\big(F_{\mu\nu}F^{\mu\nu}\big)\,, \tag{2.26}$$

where $g$ is the coupling constant of the theory.
For simplicity, the discussion will focus on the simplest case, $SU(2)$, but the results can be generalized to any symmetry group $SU(N)$.
The gauge field of the theory, also known as the connection, is $A_\mu$, and it is encoded in the field strength tensor:

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]\,. \tag{2.27}$$

The connection $A_\mu$ is a vector field such that $A_\mu = g\,A^a_\mu\,T^a$, where $T^a$ are the generators of the gauge group. In the specific case of $SU(2)$, these generators are related to the Pauli matrices, up to a constant factor.
The action in equation (2.26) is invariant under gauge transformations, which transform the connection $A_\mu$ as follows:

$$A'_\mu = g\,A_\mu\,g^{-1} - g\,\partial_\mu\,g^{-1}\,, \tag{2.28}$$

for a certain $g(x) = e^{i\alpha^a T^a}$.
Studying the theory, a key task is identifying all the possible vacuum states. The classical vacuum is trivial, corresponding to the "pure static" gauge configuration $A_i(\vec{x}) = 0$, as well as its gauge-equivalent configurations $A_i(\vec{x}) = e^{-\alpha}A_i e^{+\alpha}$, obtained through gauge transformations generated by $\alpha(\vec{x}) = \alpha^a(\vec{x})\,T^a$.

In particular, it is enough to focus on the "small" gauge transformations, which satisfy $e^{\alpha} = 1$ at $|x| \to +\infty$. This condition is typically expressed with an explicit option for the function value, as the simplest choice $\alpha = 0$.
Mathematically, this problem is about finding maps that send the space of $SU(2)$ into itself, so the $\alpha : S^3 \to S^3$, where $S^3$ is trivially the three-dimensional space. These maps are known as "homotopic", meaning they can be continuously deformed into one another. Therefore, gauge-equivalent field configurations correspond to homotopic maps.
As a result, the vacua of the theory are classified based on their topological equivalence, with each homotopy class labeled by an index $N$, called Pontryagyn index.[13] The important conclusion is that there are infinitely many topologically distinct vacua, which are not physically equivalent because they cannot be transformed into one another by "small" gauge transformations. These vacua are known as $\theta$ vacua.
However, it is possible to have transition between different vacua using "large" gauge transformations.[14] This kind of gauge transformations is incorporated in the definition of specific field configurations, known as Yang-Mills instantons. Instantons are field configurations that interpolate between two topologically distinct (i.e. gauge-inequivalent) $\theta$ vacua[15] at the boundaries, while in the bulk correspond to "large" gauge transformations deviating from the pure static gauge configuration.
It can be explicitly shown that tunneling between two distinct vacua is non-zero, and that it is proportional to

$$e^{-S_E} \approx e^{\frac{-8\pi^2}{g^2}} , \tag{2.29}$$

indicating that these physical solutions must be included in the theory. The action used in the previous estimate corresponds to the case of a single instanton with $n = \pm 1$. This action demonstrates a typical feature of non-perturbative solutions: its dependence on the inverse of the coupling constant $\frac{1}{g}$. This feature also clearly justifies why a perturbative approach is unsuitable. A similar dependence will also appear in the case of gravitational instantons.

Among the non-abelian models, the case of the strong sector of the SM, with $SU(3)$ gauge group, is particularly important. In this case, Yang-Mills instantons generate additional contributions proportional to the gluon kinetic term, as mentioned in equation (1.5). Once these non-perturbative contributions are accounted for, the strong CP problem introduced in section 1.2 arises.

[13] For $SU(2)$, the homotopy group is $\Pi(S^3) = \mathbb{Z}$, meaning the vacua can be indexed by integers; In other gauge groups, similar indices exist, though the situation may slightly differ.
[14] These transformations do not satisfy the request of $\alpha = 0$, at the spatial boundary.
[15] Different $\theta$ vacua are characterized by different winding number and corresponds to distinct theories.

For a more deeper and detailed discussion on instantons, further reading is recommended, such as the reference in [41].

## 2.4 The Hodge duality

Before analyzing the models that include gravitational instantons, it is important to introduce a key concept in axion physics known as "Hodge duality".
This duality implies that axions can be described using two physically equivalent frameworks: either as classical pseudo-scalar fields or by replacing them with three-forms.[16]
To derive the duality, it is convenient to work within the path integral formulation, where the equivalence between the two formalisms becomes clearer. For simplicity, we can consider a basic model of a single axion field coupled to gravity, as any generalization from this is straightforward.
The total action for the model is split into two contributions:

$$S[g, H, \theta] = S_{\text{grav}}[g] + S_{\text{matter}}[g, H]\,, \tag{2.30}$$

where

$$S_{\text{grav}}[g] = \frac{1}{2}\int d^4x\sqrt{-g}R + S_{\text{boundary}} \tag{2.31}$$

is the classical Einstein-Hilbert action, plus a boundary term that it is not important to explicitly define for the moment.[17] In this context, the axion is expressed through the 3-form $H_{\mu\nu\rho}$, or simply $H$. Its corresponding action term reads:

$$S_{\text{matter}}[g, H] = \int d^4x \left(-\frac{1}{2}H \wedge *H\right)\,. \tag{2.32}$$

For simplicity, it is adopted the shortest notation with the wedge product $\wedge$.[18]
Furthermore, a (scalar) Lagrange multiplier $\varphi$ can be added in the path integral without any issue, as it does not produce any extra physical contribution. Therefore, within the path integral, the action (2.32) can be replaced by:

$$S^{\varphi}_{\text{matter}}[g, H, \varphi] = \int d^4x \left(-\frac{1}{2}H \wedge *H + \varphi\, dH\right)\,. \tag{2.33}$$

[16] This thesis focuses only on four-dimensional descriptions of the space-time. However, in a more general context, considering a $d$-dimensional space-time, axions are dual to $(d-1)$ strength fields, or equivalently to $(d-2)$ potentials. In this formulation, scalar fields can be rewritten as $H_{\mu\ldots\nu} = \frac{1}{2}\epsilon_{\mu\ldots\nu\rho}\epsilon^{\rho\alpha\beta\ldots\gamma}\partial_\alpha B_{\beta\ldots\gamma}$.
[17] For the models in chapter 3, it is essential to choose the boundary term carefully. See section 3.4 for further details on this.
[18] The explicit notation is straightforward: $\int w \wedge *w = \frac{1}{s!}\int\sqrt{|g|}w_{\alpha_1,\ldots\alpha_s}w^{\alpha_1,\ldots\alpha_s}$, for every s-form $w$.

The general form of the path integral partition function, following from its definition, is given by

$$Z[h,J] = \int_{g|_\partial = h, H|_\partial = J} DgDHD\varphi\, e^{iS[g,H,\varphi]}, \tag{2.34}$$

where the constraints $h$ and $J$ indicate that the boundary conditions for $g$ and $H$ are fixed, respectively.

It is important to note that the choice of boundary conditions for both the metric and the 3-form is critical for having a well defined problem. The most convenient choice is to assume Dirichlet boundary conditions for the three-form and Neumann conditions for the axion.[19]

With these assumptions, it is possible to integrate out the multiplier by applying the boundary condition for the three-form. The outcome of this procedure is the application of the closure property to $H$, i.e. $dH = 0$, which follows from the possibility of (locally) writing $H = dB$, combined with the important property of the exterior derivative $d^2 = 0$.[20]

On the other hand, the 3-form flux can be integrated out by completing the square in $H$ and rewriting the action as follows:

$$S_{\text{matter}}[g,H,\varphi] = \int \left( -\frac{1}{2}(H - *d\varphi) \wedge *(H - *d\varphi) - \frac{1}{2} d\varphi \wedge *d\varphi \right) + \int_\partial \varphi H\,. \tag{2.35}$$

In the last step it has been applied the relation $*^2 = (-1)^{s+1}$, which is valid in a 4d Lorentzian manifold while acting on a general $s$-form. Thus, the integral over the bulk is Gaussian, and the boundary condition sets $H|_\partial = J$ . So, the partition function can be rewrite as

$$Z[h,J] = \int_{g|_\partial = h} DgD\varphi\, e^{iS[g,\varphi]}\,, \tag{2.36}$$

where the action is given by

$$S[g,\varphi] = S_{\text{grav}}[g] + \int \left( -\frac{1}{2} d\varphi \wedge *d\varphi \right) + \int_\partial \varphi J. \tag{2.37}$$

The path integral can be further split into two components, one corresponding to the bulk contributions and the other to the boundary terms. The first part can be seen as a kind

[19] Further details regarding this choice are provided in section 3.4.

[20] In a four-dimensional framework, the explicit expression is $H_{\alpha\beta\gamma} = B_{(\alpha\beta\gamma)} = \partial_\alpha B_{\beta\gamma} + \partial_\beta B_{\gamma\alpha} + \partial_\gamma B_{\alpha\beta}$.

of Fourier transform:

$$\begin{aligned} Z[h,J] &= \int D\varphi_{\text{bound}}\, e^{i\int_\partial \varphi_{\text{bound}} J} \int_{g|_\partial = h\,, \varphi|_\partial=\varphi_{\text{bound}}} DgD\varphi\, e^{iS_{\text{grav}}[g]+i\int\left(-\frac{1}{2}d\varphi\wedge * d\varphi\right)} \\ &= \int D\varphi_{\text{bound}}\, e^{i\int_\partial \varphi_{\text{bound}} J} Z[h,\varphi_{\text{bound}}]. \end{aligned} \tag{2.38}$$

Thus, the path integral $Z[h,J]$ can be computed in two different ways.
Following the 3-form picture, it is given by

$$Z[h,J] = \int_{g|_\partial = h, H|_\partial = J} DgDH\delta[dH]\, e^{iS_{\text{grav}}[g]+i\int\left(-\frac{1}{2}H\wedge * H\right)}\,, \tag{2.39}$$

while adopting the scalar-axion formalism it results

$$Z[h,J] = \int D\varphi_{\text{bound}}\, e^{i\int_\partial \varphi_{\text{bound}} J} Z[h,\varphi_{\text{bound}}]\,. \tag{2.40}$$

In this formulation, it is possible to define the "Fourier transform":

$$Z[h,\varphi_{\text{bound}}] = \int_{g|_\partial = h, \varphi_\partial = \varphi_{\text{bound}}} DgD\varphi\, e^{iS_{\text{grav}}[g]+i\int\left(-\frac{1}{2}d\varphi\wedge * d\varphi\right)}\,. \tag{2.41}$$

The key conclusion from the previous derivation indicates that it is always possible to switch the dual formulations, by enforcing the following relation[21]:

$$\epsilon_{\mu\nu\alpha\beta}\partial^\beta\varphi = H_{\mu\nu\alpha}\,, \tag{2.42}$$

which is obtained by comparing the two partition function formulations (2.39) and (2.40). Equivalently, this condition can be expressed as:

$$H = -\star d\varphi\,. \tag{2.43}$$

The importance of Hodge duality lies in the ability to treat the axion as a three-form, which is perfectly equivalent to considering it as a standard scalar field. It is essential to note that when the signature is switched to the Euclidean one, the axion acquires a negative sign in the kinetic term.[22]

[21] In literature, the action is sometimes defined with an additional constant $f_{\text{axion}}$ to explicitly indicate the dependence on the characteristic energy of the axion. In this case, the constant would multiply the right side of relation (2.42). However, it is still possible to recast the situation presented here simply by scaling the fields.

[22] Indeed, the Levi-Civita tensor $\epsilon$ must be redefined when changing the signature, resulting in an additional i-factor. It follows that the three-form kinetic term has an extra negative sign (coming from the new $i^2$ factor of the $\epsilon$).

It is also crucial to emphasize that the duality works effectively only when the correct boundary conditions are imposed, according to the specific framework being used. In fact, in some earlier articles investigating the stability of gravitational instantons, erroneous theoretical results were obtained due to mistakes in the application of duality and the associated choice of boundary conditions.

# Chapter 3

# Gravitational instantons and wormholes physics

After having outlined the motivations and technical aspects that recommend using Euclidean signature to study these solutions within a semi-classical framework, it is possible to delve into gravitational instantons.

The instantons discussed earlier can be generalized to gravitational theories, and indeed, a broad set of related solutions has been explored in ST.[1] This thesis, however, will focus on a specific class of solutions that includes wormhole configurations.

Section 3.1 introduces a general sigma-model of scalar fields coupled with gravity. In this model, it is straightforward to derive the wormhole metric structure, highlighting its essential features.
Section 3.2 presents an extension of this model by including a dilaton field, motivated by string theory compactifications. Here, three different types of gravitational instantons are examined, each demonstrating a unique profile for the solutions.
Section 3.3 further generalizes the discussion, deriving gravitational instantons within general models of compactified string theory, regardless of the number or configuration of axion-saxion fields. Importantly, it is shown that the calculations can always be reduced to a form similar to the simplified model in section 3.2.
Lastly, section 3.4 provides valuable comments about the theme of boundary conditions imposition, required for precise treatment of these models.

[1]For an in-depth discussion of the so-called D-instantons, typical of type IIB supergravity, refer to [24, 41].

## 3.1 Simple example of gravitational instanton in a sigma model

To begin, it is examined a simple model of spacetime that includes a set of scalar axion fields, minimally coupled to gravity. Typically, it is incorporated a potential $V$ that depends on the scalars (allowing for interactions, self-interactions among the axions or mass terms). However, it is possible to consider the simple case without a potential to streamline the analysis, as the main features of instantonic solutions don't change under this assumption.

The action of the model, in Euclidean signature, is given by the following expression:

$$S_E = \int d^4x\sqrt{g}\left[-\frac{M_{Pl}^2}{2}R + \frac{1}{2}G_{IJ}g^{\mu\nu}\partial_\mu\varphi_I\partial^\nu\varphi_J\right] , \tag{3.1}$$

while its dual form is

$$S_E = \int d^4x\sqrt{g}\left[-\frac{M_{Pl}^2}{2}R + \frac{\mathcal{F}_{IJ}}{2}H^I_{\mu\nu\rho}H^{J,\mu\nu\rho}\right] . \tag{3.2}$$

The two actions are trivially equivalent, interchangeable applying the Hodge duality.

A boundary term could be included, but it is disregarded since it does not influence the system's dynamics.[2]

The notation here uses $R$ to denote the Ricci scalar curvature[3], $\varphi_I$ indicates scalar fields (axions, other scalars, or moduli of the theory) and $G_{IJ}$ is a functional term potentially dependent on other fields, introducing new interactions or higher-order operators for the scalars.

In the dual formulation $G_{IJ}$ is replaced by $\mathcal{F}_{IJ}$, and in this specific model, both are set to be proportional only to the scalars $\varphi_I$, excluding any additional fields.

Throughout this section, natural units are set. Thus, it is fixed $M_{Pl} = 1$.[4]

To impose rotational invariance, with symmetry group $O(4)$, it is adopted the following metric ansatz:

$$ds^2 = a^2(r)dr^2 + r^2 d\Omega_3^2 , \tag{3.3}$$

[2] However, in principle it is not correct to neglect it. A discussion about this term is given later, in section 3.4.

[3] For reference: $R = g^{\mu\nu}R_{\mu\nu}$, where $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ represents the Ricci tensor, $R^\alpha_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha_{\beta\nu} - \partial_\nu\Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}$ is the Riemann curvature tensor, and $\Gamma$ denotes the Levi-Civita connection derived from the metric: $\Gamma^\alpha_{\mu\nu} = \frac{g^{\alpha\beta}}{2}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right)$.

[4] The Planck mass is the constant $M_{Pl} = \frac{1}{\sqrt{G_N}} = 1.22\cdot 10^{19}GeV$, it is put equal 1 to simplify notation, rescaling all variables' mass dimension accordingly.

using polar coordinates, where the time-dependent parameter $a$ resembles the scale factor in the "RWFL" metric.[5] All fields in the theory are forced to respect this symmetry, so the scalars depend only on the radius $r$.

Applying the principle of stationary action and varying $S_E$ with respect to the axion $\varphi_K$, it is obtained the equation of motion for each scalar:

$$\left(\sqrt{g}g^{rr}G_{KJ}\varphi'^J\right)' = \frac{\sqrt{g}}{2}g^{rr}\partial_K G_{IJ}\varphi'^I\varphi'^J \ , \tag{3.4}$$

where the prime denotes a derivative with respect to $r$.

The equations of motion can also be viewed as geodesic equations for the fields, where $G$ represents a scalar metric. This becomes explicit with a change of coordinates, defining a $\tau$ time (Euclidean) variable such that $\frac{d\tau}{dr} = \sqrt{g}g^{rr}$; then the equations of motion become[6]

$$\partial_\tau^2\varphi^I + \Gamma^I_{JK}\partial_\tau\varphi^J\partial_\tau\varphi^K = 0 \tag{3.5}$$

or, equivalently

$$\partial_\tau(G_{IJ}\partial_\tau\varphi^I\partial_\tau\varphi^J) = 0 \ . \tag{3.6}$$

Using the ansatz (3.3), it is straightforward to obtain the relation

$$G_{IJ}\varphi'_I\varphi'_J = \frac{6C}{(g^{rr}\sqrt{g})^2} = \frac{6Ca^2}{r^6} \ , \tag{3.7}$$

where $6C$ is an integration constant, whose physical significance will become clear shortly. Due to radial symmetry, the "r-r" Einstein equation decouples. Here, the "r-r" component of the stress-energy tensor is $T_{rr} = \frac{1}{2}G_{IJ}\varphi'_I\varphi'_J$, while the corresponding Einstein tensor component is $G_{rr} = \frac{3}{r^2}(1-a^2)$.

By applying the Einstein equation with the expression in (3.7), it is derived the instantonic configuration. The Einstein equation, given the ansatz, becomes

$$\frac{3}{r^2}(1-a^2) = \frac{6Ca^2}{2r^6} \ , \tag{3.8}$$

which has a simple solution for the radial metric component:

$$a^2 = \left(1 + \frac{C}{r^4}\right)^{-1} \ . \tag{3.9}$$

[5] The Robertson-Walker-Friedman-Lemaitre (RWFL) metric is commonly used in cosmology to describe a homogeneous, isotropic expanding universe and is equivalent to equation (3.3) with Euclidean signature.

[6] Assuming $\Gamma$ as the relative Chistoffel symbol.

Depending on the sign of $C$ (whose field dependence is implicitly represented in equation (3.7)), the metric's topological structure changes significantly. Specifically, there are three types of gravitational instantons, which will be explored further in Section 3.2.
For now, it is interesting to consider the regime $C < 0$, so that the metric takes the form

$$ds^2 = \left(\frac{1}{1-|C|/r^4}\right) dr^2 + r^2 d\Omega_3^2 \,. \tag{3.10}$$

The resulting configuration is the Euclidean Giddings-Strominger *wormhole* solution.

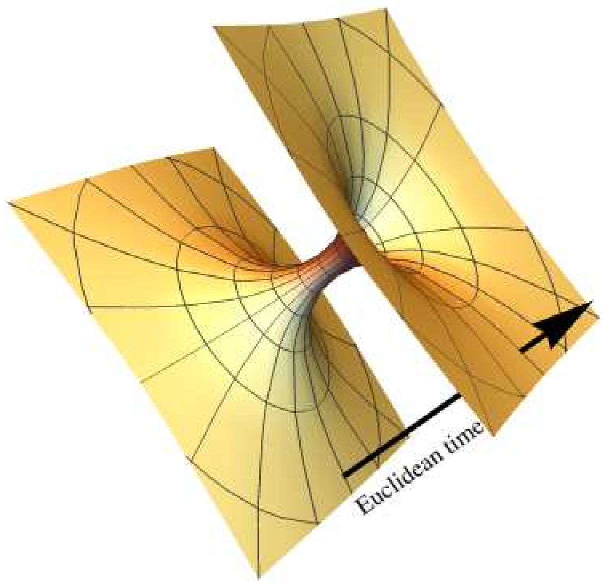



**Figure 3.1:** Graphic representation of a wormhole structure taken from [6].

Through a conformal transformation, it is possible to show that the wormhole metric is topologically equivalent to $S^3$ x $\mathbb{R}$. This implies that for each Euclidean "time" value $r$, the cross-section of the configuration is a 3-sphere $S^3$ with precisely a radius $r$.
The metric becomes asymptotically flat as $r \to +\infty$, recovering Minkowski spacetime (only asymptotically).
Conversely, as $r$ decreases, the metric approaches a singularity at $r_\star = |C|^{1/4}$. However, this singularity is merely a coordinate defect rather than a physical one: the parameter $r_\star$ represents the minimum radius of the wormhole's cross-section.
Specifically, the Ricci scalar is given by $R = \frac{6C}{r^6}$, and it remains positive for any $r > r_\star$; it vanishes precisely at $r = r_\star$, marking the endpoint of the wormhole throat.
Consequently, to construct the full wormhole configuration as depicted in figure 3.1, a

second copy of this instanton configuration must be "glued" at the singular point.[7].

To demonstrate that the singularity is not physical, we can redefine the variable as follows:

$$\left(\frac{d\tau}{dr}\right)^2 = \left(\frac{r^2}{r^4 + C}\right) . \tag{3.11}$$

With this substitution, the metric assumes a Robertson-Walker-like form

$$ds^2 = a(\tau)^2 \left(d\tau^2 + d\Omega_3\right) , \tag{3.12}$$

and the singularity is removed.
Using this variable redefinition, it is found that

$$a(\tau)^2 = \sqrt{|C|} \cdot cosh(2\tau) . \tag{3.13}$$

By substituting this explicit scale factor back into the RW metric and the equations of motion, we can recover the standard instanton profile. In this section, we do not perform the complete derivation, as a more comprehensive version is provided in the following model.

In the original work by Giddings and Strominger[8], the variable $|C|$ was calculated for a single axion coupled to gravity, yielding:

$$C = \frac{-q^2}{24\,\pi^4\, f^2} , \tag{3.14}$$

where the constant $f$ originates from a different initial action in their analysis, $S_H = \int d^4x\sqrt{g}\frac{1}{12f^2}H_{\mu\nu\delta}H^{\mu\nu\delta}$, and $q$ is the axion charge.
However, it is noteworthy that similar geometries arise in more general models, although with different values of $|C|$. By inserting the equations of motion into the action[9], the on-shell action can be computed as:

$$S_W = \frac{1}{f^2}\int H \wedge \star H = 2\frac{n^2}{2\pi^2 f^2}\int_{r_0}^{+\infty}\frac{dr}{r\sqrt{C + r^4}} = \frac{\pi\sqrt{6}}{4}\frac{|q|}{f} . \tag{3.15}$$

In general, the two mouths of a wormhole could either exist in causally disconnected re-

[7]In the next section, the axion's topological charge is introduced. The two copies of the metric (3.10) must have opposite sign charges, in order to conserve the theory's total charge.
[8]See the original papers [1] and [2].
[9]Note that a factor 2 is needed, in order to take into account the contributions from the two halves of the wormhole.

gions within the same universe, or in distinct universes.[10] The two situations are depicted in the diagram of figure 3.3.

## 3.2 Gravitational instantons in axion-dilaton models

Axiverse models have different physical applications, but the principle one is related to the phenomenology of ST. From the compactification procedure within the EFT framework, the low-energy ST model naturally suggests the presence of a large number of additional scalar fields.[11]
However, the following discussion considers a simplified model that can be adapted to any context involving scalars (or three-form fields) coupled with gravity, potentially alongside other matter fields. In the subsequent section 3.3, it will be demonstrated that similar solutions can be derived within models containing multiple scalar fields.

Using a Euclidean signature, a basic model featuring a single pair of scalar fields-a single axion and a dilaton-is described by the action:

$$S_E = \int d^4x\sqrt{g}\left[-\frac{M_{Pl}^2}{2}R + \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi + \frac{\mathcal{F}(\phi)}{2}H_{\mu\nu\rho}H^{\mu\nu\rho}\right] . \tag{3.16}$$

In this expression, the axion is represented through the three-form field $H$, while $\phi$ represents the dilaton.[12]
This action specifies a general setup, requiring an explicit form for $\mathcal{F}$ to define the interaction between the dilaton and the axion.
The simplest options, setting $\mathcal{F} = 0$ or $\mathcal{F} = \frac{1}{3! f_{ax}}$, would yield models with scalar fields minimally coupled to gravity, already discussed in the previous section 3.1.
It is more interesting to consider the following model, which involves an interaction term between the dilaton and the axion, obtained by setting

$$\mathcal{F} = \frac{exp(\frac{-\beta\phi}{M_{Pl}})}{3! f^2} . \tag{3.17}$$

Here, $\beta$ represents a coupling constant.

[10] This last possibility can be considered only in a multiverse framework. For further discussion on these scenarios, see the first two sections of chapter 4.

[11] The number and nature of these fields depend non-trivially on the specific string model and the topology of the compactified dimensions; in general, the number could reach the order of $10^{2/3}$.

[12] In a string theory context, the dilaton is often termed the "saxion," though the two are largely equivalent. More precisely, the saxion is the imaginary component of the lowest scalar field of a superfield, while the axion serves as its real component.

Expanding the exponential in the high-$M_{Pl}$ limit yelds

$$e^{-\frac{\beta\phi}{M_{Pl}}} \approx \left(1 - \frac{\beta\phi}{M_{Pl}}\right) , \tag{3.18}$$

where the leading term 1 serves as the kinetic term for the three-form field, and the second term $\frac{\beta\phi}{M_{Pl}}$ introduces the interaction.[13]

In the framework of string theory and considering a wormhole configuration, the coupling parameter $\beta$ must satisfy the constraint

$$0 \leq \beta \leq 2\sqrt{\frac{2}{3}} . \tag{3.19}$$

indicating a specific range where wormhole solutions are supported. This constraint on $\beta$ emerges from ensuring consistency within the compactified string effective theory, in which the dilaton and axion fields are coupled.

For now, the focus remains solely on the bulk contribution to the action, with the boundary term deferred to a later analysis in section 3.4.
Although the boundary term doesn't impact the geometry of the system, it is essential for obtaining the correct value of the on-shell action.
Moreover, the model could include a potential $V(\phi)$, whose presence would introduce modifications to the equations of motion without fundamentally altering the wormhole's structural properties neither affecting its stability, and so it is omitted here to keep the analysis concise.[14]

The next step is to derive the non-perturbative instantonic solutions by minimizing the action. The e.o.m. for the metric, or Einstein-Hilbert equations, can be obtained by evaluating

$$(\delta S_E)_{\delta g} = \int d^4x\sqrt{g}\ \delta g_{\mu\nu}\left[-\frac{M_{Pl}^2}{2}(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}) + \frac{1}{2}\nabla^\mu\phi\nabla^\nu\phi + T_H^{\mu\nu}\right] \tag{3.20}$$

where $T_H^{\mu\nu}$ is the stress-energy tensor of $H$. The Einstein tensor is defined as $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{M_{Pl}^2}T_{\mu\nu}^H$, while the Riemann curvature is defined as $R = g^{\mu\nu}R_{\mu\nu}$. The last

[13] A comprehensive analysis on the potential values of $\beta$, and the stability of instanton solutions varying this parameter, can be found in [14]. This study generalizes the original Giddings-Strominger solution to include a broader class of wormhole configurations with a massive dilaton.

[14] There are several articles studying models including a potential, as [9, 14, 16, 27]

term in the variation above provides the three-form field stress-energy tensor[15]:

$$T^H_{\mu\nu} = e^{-\beta\phi\kappa}\frac{M_{Pl}^2}{3!f^2}(3H_{\mu\alpha\beta}H_\nu^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}H^2)\,. \tag{3.21}$$

Following a similar procedure yields the equations of motion for both the dilaton and the three-form fields. Setting $M_{Pl}^2 = \frac{1}{k}$ in the subsequent calculations simplifies the physical constants.

Thus, the resulting equations of motion are as follows:

$$\begin{cases} R_{\mu\nu} = k\partial_\mu\phi\partial_\nu\phi + \frac{k}{2f^2}e^{-\beta\phi\sqrt{k}}H_{\mu\alpha\beta}H_\nu^{\alpha\beta} - \frac{k}{6f^2}e^{-\beta\phi\sqrt{k}}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}g_{\mu\nu} \\ \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) = \frac{-\beta\sqrt{k}}{12f^2}e^{-\beta\phi\sqrt{k}}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma} \\ \partial_\mu\left(\sqrt{g}e^{-\beta\phi\sqrt{k}}H^{\mu\alpha\beta}\right) = 0 \end{cases} \tag{3.22}$$

To obtain the instanton structure, an appropriate ansatz is required, as in previous models. The symmetry group of four-dimensional rotations, $O(4)$, is assumed to hold for this system, leading to a metric of the form:

$$ds^2 = a^2(r)dr^2 + b^2(r)d\Omega_3^2 \tag{3.23}$$

where $a$ and $b$ are functions of the Euclidean time $r$.[16]

Under this symmetry assumption, the fields are postulated to follow a similar structure:

$$\begin{cases} \phi = \phi(r) \\ H_{ijk} = q\cdot\epsilon^N_{ijk}\ ;\ H_{0ij} = 0 \end{cases} \tag{3.24}$$

where $q$ is an integer related to the axion's shift symmetry charge, which is quantized in string theory according to the Dirac quantization condition.[17]

The symbol $\epsilon^N$ represents a "normal" tensor density, which can also be expressed in terms of a "transversal" tensor $\epsilon^T$ as:

$$\epsilon^N_{ijk} = \frac{\epsilon^T_{ijk}}{\sqrt{\tilde{g}^{(3)}}}\,. \tag{3.25}$$

The unit volume element is thus $\int d^3x\ \epsilon^T = 2\pi^2b^3$, leading to the axion charge conservation

[15]It is necessary to consider the implicit definition of the three-form $H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} - \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} \equiv \partial_{[\mu}B_{\nu\rho]}$, where the square brakets indicates the anti-symmetrization of the indexes.

[16]For this general metric, the Ricci scalar curvature is given by: $R = \frac{a'bb' - a\left((b')^2 + bb''\right)}{a^3b^2}$. Additionally, the determinant of the metric can be expressed as: $\sqrt{g} = ab^3$. The components of the Einstein tensor are as follows: $G_{rr} = 3\frac{(b')^2}{b^2}$ and $G_{ij} = \delta_{ij}\left[\frac{(b')^2}{a^2} + 2\frac{bb''}{a^2} - 2\frac{a'bb'}{a^3}\right]$.

[17]See the articles [11] and [12] for more details about this feature.

relation:

$$\int dx^3 H = q \,. \tag{3.26}$$

The ansatz satisfies the equation of motion for $H$ automatically, while the Bianchi identity[18] imposes the constraint:

$$H(r) = \frac{q\epsilon^T}{2\pi^2 b^3} \,. \tag{3.27}$$

By substituting the ansatz and performing integration by parts with boundary conditions incorporated into $S_{bound}$, the $O(4)$-symmetric action becomes:

$$S_E = 2\pi^2 \int d\tau \left[ \frac{-3b\dot{b}^2}{a\kappa} + \frac{b^3\dot{\phi}^2}{2a} - \frac{3ab}{\kappa} + \frac{N^2 a}{b^3} e^{-\beta\phi\sqrt{\kappa}} \right] + S_{bound} \tag{3.28}$$

where the parameter $N \equiv \frac{q^2}{2f^2}$ simplifies the notation.
The ansatz transforms the equations of motion into:

$$\begin{cases} \frac{2b\ddot{b}}{a} + \frac{\dot{b}^2}{a} - a - \frac{2b\dot{b}\dot{a}}{a^2} + \kappa\frac{b^2\dot{\phi}^2}{2a} - \frac{\kappa N^2 a}{b^4} e^{-\beta\phi\sqrt{\kappa}} = 0 \,, \\ \frac{\dot{b}^2}{a^2} - 1 = \frac{\kappa}{3}\frac{\dot{\phi}^2}{2a^2} - \frac{\kappa N^2}{3b^4} e^{-\beta\phi\sqrt{\kappa}} \,, \\ \ddot{\phi} + \left(3\frac{\dot{b}}{b} - \frac{\dot{a}}{a}\right)\dot{\phi} = -\frac{\beta N^2 a^2\sqrt{\kappa}}{b^6} e^{-\beta\phi\sqrt{\kappa}} \,. \end{cases} \tag{3.29}$$

One may employ gauge freedom to fix either $a$ or $b$, as was done in section 3.1 by setting $b = r$ for simplicity. Using the $r - r$ metric equation of motion yields the explicit gravitational instanton form, as the one of eq.(3.10).
Solving these equations requires determining the dilaton profile, while the three-form solution remains fixed by symmetry.
To proceed, the sign of the constant $C$ must be set, as each possible option leads to a distinct metric configuration.

Substituting the analogous metric shape of (3.10), derived considering also the dilaton:

$$\begin{cases} a(r) = \frac{1}{\sqrt{1+\frac{C}{r^4}}} \,; \\ b(r) = r \end{cases} \tag{3.30}$$

and simplifying, the dilaton equation of motion becomes:

$$k\frac{r^2}{2}(\dot{\phi})^2\left(1 + \frac{C}{r^4}\right) - 3\frac{C}{r^4} = \frac{kN^2}{r^4} e^{-\beta\sqrt{k}\phi} \,. \tag{3.31}$$

[18] Given the possibility of writing $H = dB$ for a certain two form potential $B_{\mu\nu}$, as the exterior derivative satisfies $d^2 = 0$, the Bianchi identity automatically follows: $dH = 0$ or equally $\nabla_{[\mu} H_{\nu\sigma\delta]} = 0$

This equation can be solved by separating variables:

$$\int d\phi \left(\sqrt{N^2 e^{-\beta\sqrt{k}\phi} + 3\frac{C}{k}}\right)^{-1} = \pm \int dr \frac{\sqrt{2}}{r^3\sqrt{1+\frac{C}{r^4}}} \,. \tag{3.32}$$

The solution to the integral depends on the sign of $C$, while for the three-form the formula (3.27), derived imposing the Bianchi identity, still holds.

While solutions for $C \geq 0$ are known but less physically relevant, the case of $C < 0$ leads to the wormhole solution, which has significant implications.

### 3.2.1 Extremal and cored instantons

Let's begin with the simplest configuration, where $C = 0$.
In this case, the metric remains strictly positive, ensuring that there are no singularities. It turns out that the shape of the metric is simply flat; however, unlike Minkowski space, the origin must be excluded from the spacetime, as the fields and the three-form diverge at $r = 0$. Therefore, both fields exhibit non-trivial profiles.
To determine the dilaton function, the following integral equation need to be solved:

$$\int d\phi \left(\sqrt{N^2 e^{-\beta\sqrt{k}\phi}}\right)^{-1} = \pm\frac{\sqrt{2}}{2}\left(\frac{-1}{r^2}\right) \,. \tag{3.33}$$

This equation can be easily computed to yield:

$$e^{\beta\sqrt{k}\phi} = +\frac{N^2\beta^2 k}{8r^4} + f\,, \tag{3.34}$$

where the solution with a negative sign is discarded, as the only valid solution in the domain $r > 0$ would be the trivial one: $e^{\beta\sqrt{k}\phi} = 0$.
The integration constant $f$ must be determined by imposing the boundary condition for the dilaton, i.e. that $\phi \xrightarrow{r\to+\infty} 0$[19]. This condition leads us to the final form of the dilaton:

$$e^{\beta\sqrt{k}\phi} = +\frac{N^2\beta^2 k}{8r^4} + 1 \tag{3.35}$$

The plot of this specific spacetime configuration is shown in figure 3.2-b.

[19]The physical significance of this limit is tied to spacetime compactification: as the radius approaches infinity, the structure of spacetime asymptotically resembles flat space, since compactification primarily influences the bulk properties of the spacetime composition.

Next, the second case where $C > 0$ can be analyzed. This scenario is typically referred to as the "sub-extremal" case or "cored gravitational instantons" (the terminology draws an analogy to a specific types of black holes).
In this instance, the metric remains positive, but it diverges as it approaches the origin. The shape of the cored instanton configuration exhibits a naked singularity at $r = 0$, which raises questions regarding its physical significance.
Nevertheless, this type of instanton features a dilaton field profile that depends on a free parameter. For this reason, this solution is studied by embedding the model in a higher-dimensional space, where this family of solutions can be physically motivated, for instance, as Reissner-Nordström black holes (in a 5d space-time).[20]

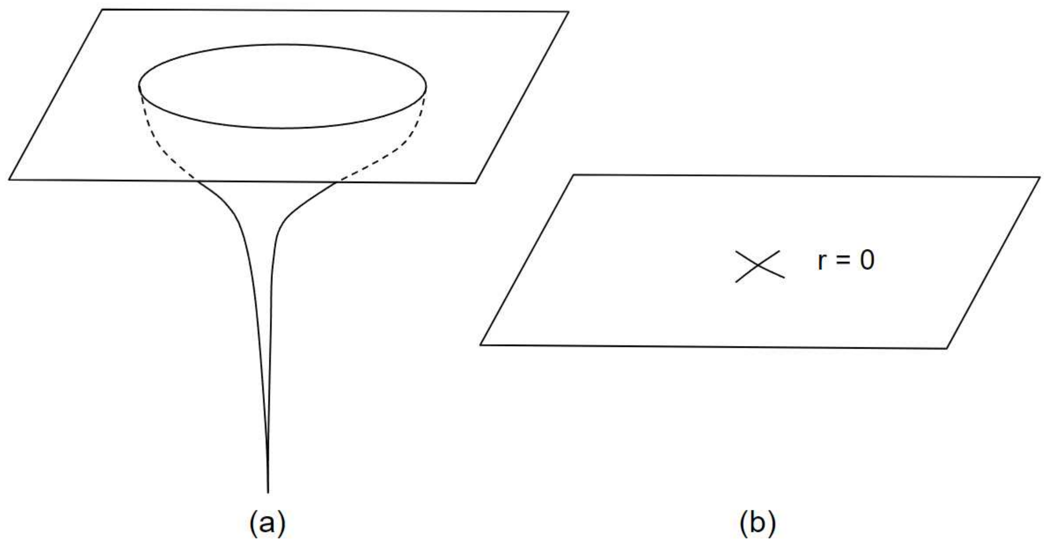



**Figure 3.2:** A cored gravitational Instanton and an extremal Instanton metric configurations.

A convenient approach to solve the equation in the $C > 0$ case involves a change of variables[21]

$$s = \frac{\sqrt{|C|}}{r^2}\,. \tag{3.36}$$

This substitution transforms the right side of equation (3.32) into:

$$\frac{-\sqrt{2}}{2\sqrt{C}} \int ds \frac{1}{\sqrt{1+s^2}} = \frac{-1}{\sqrt{2C}} \operatorname{arcsinh}\left(\frac{\sqrt{C}}{r^2}\right) + g\,, \tag{3.37}$$

where the integration constant $g$ will be fixed by imposing boundary conditions.

[20]This peculiar case shows also an important consistency with the weak gravity conjecture, introduced in section 1.3.2.

[21]Here, the absolute value is included to allow for a change of variable that will also be applicable in the wormhole case.

Instead, the left side can be rewritten as:

$$\sqrt{\frac{k}{3C}}\int d\phi\left(\sqrt{\frac{kN^2}{3C}e^{-\beta\sqrt{k}\phi}+1}\right)^{-1}=\frac{2}{\beta\sqrt{3C}}\int dx\frac{\cosh(x)}{\sqrt{1+\sinh^2(x)}}\,. \tag{3.38}$$

Here, we introduce a new variable $x$, such that:

$$\sinh(x)\equiv\frac{\sqrt{3C}}{N\sqrt{k}}e^{+\beta\sqrt{k}\frac{\phi}{2}}\,. \tag{3.39}$$

Using the fundamental relations for hyperbolic functions, the integral simplifies significantly. Equating the two sides of the equation results in:

$$\frac{+1}{\sqrt{2|C|}}\operatorname{arcsinh}\left(\frac{\sqrt{C}}{r^2}\right)+g=\frac{2\cdot x}{\beta\sqrt{3C}}\,. \tag{3.40}$$

Now, substituting back the inverse of (3.3.9) to eliminate the variable $x$ gives:

$$\frac{2}{\beta\sqrt{3C}}\operatorname{arcsinh}\left(\frac{\sqrt{3C}}{N\sqrt{k}}e^{+\beta\sqrt{k}\frac{\phi}{2}}\right)=\frac{+1}{\sqrt{2|C|}}\operatorname{arcsinh}\left(\frac{\sqrt{C}}{r^2}\right)+g\,. \tag{3.41}$$

So, this leads to the result:

$$e^{+\beta\sqrt{k}\phi}=\frac{kN^2}{3C}\sinh^2\left[\frac{\beta}{2}\sqrt{\frac{3}{2}}\operatorname{arcsinh}\left(\frac{\sqrt{C}}{r^2}\right)+g\right]\,. \tag{3.42}$$

As anticipated, the constant $g$ cannot be entirely removed and remains a free parameter, with only specific values allowed for it.
A detailed discussion on this matter is complex, as it requires an analysis of the microscopic origin of the solution in each specific model. For example, when embedding the solution in a five-dimensional space, it is found that the instanton is promoted to a charged black hole, where $g$ is constrained by its mass and charge.
Generally, it is shown that several values of this free parameter are forbidden, as they lead to unstable or physically inconsistent solutions.

However, applying the condition of vanishing dilaton in the limit of asymptotic radius yields a relation between $C$ and $g$:

$$\sinh^2(g)=\frac{3C}{kN^2}\,. \tag{3.43}$$

Using this relation, the dilaton profile can be expressed with the following resulting expression:

$$e^{+\beta\sqrt{k}\phi} = \frac{1}{\sinh^2(g)} \sinh^2 \left[ \frac{\beta}{2}\sqrt{\frac{3}{2}} \operatorname{arcsinh}\left(\frac{\sqrt{C}}{r^2}\right) + g \right] . \tag{3.44}$$

### 3.2.2 The wormhole configuration

Lastly, the most interesting instanton configuration is the "super-extremal" one, commonly referred to as the *wormhole*.

Assuming as hypothesis that $C < 0$, there is an immediate difference from the previous situations: the metric exhibits a singularity for $r^4 = L^4 \equiv -C$.
As previously noted, this is not a physical singularity but a coordinate one, which can be eliminated traightforwardly.[22]
To achieve a smooth metric, it suffices to change coordinates using $\frac{r^4}{L^4} = cosh(2u)$, resulting in the smooth metric:

$$ds^2 = r^2 \left(du^2 + d\Omega^2\right) , \tag{3.45}$$

defined for $u > 0$ (corresponding to $r > L$).
The new coordinate's symmetry under $u \to -u$ reflects that this metric describes both sides of the wormhole, where the minimum value $u = 0$ marks the wormhole's smallest radius.
Thus, the analysis continues as in the previous section, assuming $r > L$ .

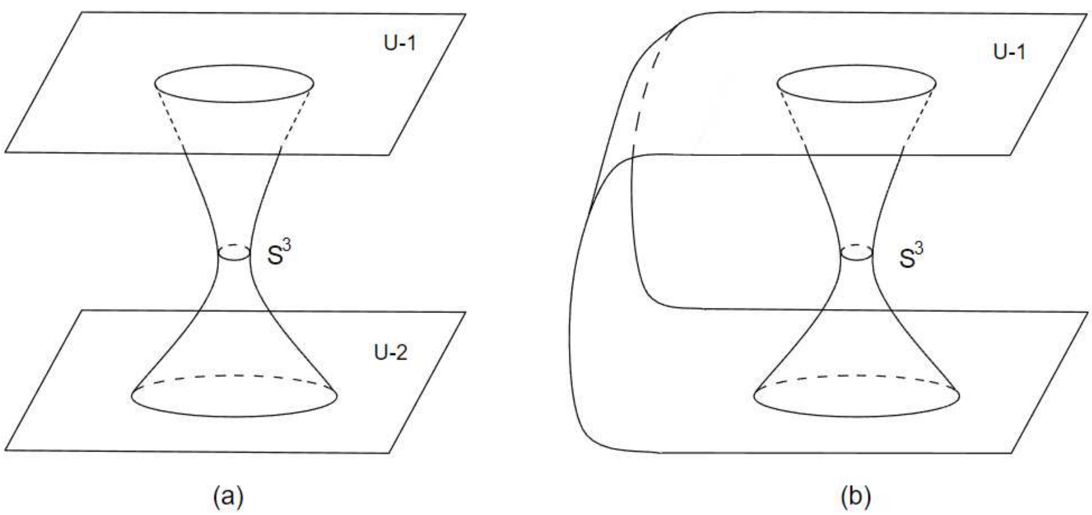



**Figure 3.3:** Two euclidean wormhole configurations respectively connecting two separate universes and two regions of the same space-time.

[22] This solution is widely accepted based on a similar argument in black hole physics, as in the Schwarzchild solution, where a coordinate singularity corresponds to the black hole horizon.

Figure 3.3 illustrates two possible wormhole configurations. Although structurally similar, the two situations are physically distinct: configuration ($a$) connects two separate universes, while ($b$) connects different regions within the same spacetime, akin to the setup first explored by Einstein and Rosen,[23]

For $C < 0$, equation (3.32) can be solved similarly to the cored configuration by defining the variable as in (3.36).
For the right side of the equation, a minus sign is instroduced for every $C$ term (as a consequence of the absolute value):

$$\frac{-\sqrt{2}}{2\sqrt{-C}} \int ds \frac{1}{\sqrt{1-s^2}} = \frac{-1}{\sqrt{-2C}} \arcsin\left(\frac{\sqrt{-C}}{r^2}\right) + h\,. \tag{3.46}$$

The left side, meanwhile, becomes

$$\sqrt{\frac{k}{-3C}} \int d\phi \left(\sqrt{\frac{kN^2}{3C} e^{-\beta\sqrt{k}\phi} + 1}\right)^{-1} = \frac{2}{\beta\sqrt{-3C}} \int dx \frac{\cos(x)}{\sqrt{1-\sin^2(x)}}\,, \tag{3.47}$$

where it is used that $\sin x = \sqrt{\frac{-3C}{N^2 k}} e^{+\beta\sqrt{k}\frac{\phi}{2}}$.
The dilaton profile can be found, after few algebric steps, giving:

$$e^{+\beta\sqrt{k}\phi} = \left(\frac{-N^2 k}{3C}\right) sin^2\left[\frac{\beta}{2}\sqrt{\frac{3}{2}} \arcsin\left(\frac{\sqrt{-C}}{r^2}\right) + h\right]\,. \tag{3.48}$$

The integration constant $h$ an be set by applying boundary conditions:

$$1 = \left(\frac{-N^2 k}{3C}\right) \sin^2(h)\,. \tag{3.49}$$

However, this is not sufficient alone; indeed, also the limit of $r = |C|^{\frac{1}{4}}$ should be considered:

$$\sin^2 h = e^{-\beta\sqrt{k}\phi_0}\,, \tag{3.50}$$

where $\phi_0$ represents the field value at the wormhole's minimum radius.
By combining the above conditions and plugging it into (3.48), it results:

$$h = \frac{\pi}{2}\left(1 - \frac{\beta}{2}\sqrt{\frac{3}{2}}\right)\,, \tag{3.51}$$

[23]In that case, the wormhole was not a Euclidean instanton but rather a Minkowski space-time structure linking two black holes at infinite spatial separation.

and the dilaton profile, free of the integration constant, reads:

$$e^{+\beta\sqrt{k}\phi} = \frac{1}{\cos^2\left(\frac{\beta\pi}{4}\sqrt{\frac{3}{2}}\right)}\cos^2\left[\frac{\beta}{2}\sqrt{\frac{3}{2}}\arccos\left(\frac{\sqrt{-C}}{r^2}\right)\right] \tag{3.52}$$

The resulting wormhole configuration is well-defined for $r \in \left[|C|^{\frac{1}{4}}; +\infty\right]$ if the couplings satisfy the constraint of (3.19).
However, this metric only describes one half of the wormhole structure shown in figure 3.1.
The full structure is given by metric (3.45), where the two $u$-coordinate intervals $[-\infty; 0]$ and $[0; +\infty]$ correspond to the two wormhole halves.

Lastly, it is essential that the two wormhole halves can be glued together only if their respective three-form charges are $q$ and $-q$.
Attempting to join two identical instantons would result in an un-physical axion-charge violation, which must be avoided in any consistent QFT framework.
In other words, the two halves of the wormhole structure must consist of an instanton and an anti-instanton pair.

The appeal of wormhole solutions lies in their peculiar property of enabling topological transitions in the spacetime metric.
These types of instantons can induce oscillations in both the fields within the model and the scale factor, oscillations that, in some cases, persist even under an analytic continuation to Minkowski space.
Such effects could carry profound physical implications[24], although the full extent of their relevance remains an open question.
Notably, these spacetime configurations commonly appear in axiverse models, underscoring the importance of further investigation into their properties and implications. However, before their relevance can be fully appreciated, it is essential to thoroughly examine the stability of these configurations. Any form of instability would undermine their physical significance, potentially negating their role in theoretical models.

[24]Some of them, explored in chapter 4.

## 3.3 Generalization to string-EFT models

The discussion of Euclidean instantons developed in previous sections can be extended to a broader context; in fact, it can be shown that wormhole solutions exist for a class of string EFTs.[25]

The following calculations will focus on the models introduced in section 1.3.1.
Considering the action of eq.(1.26), its Wick-rotated version is given by:

$$S = -\frac{M_{Pl}^2}{2}\int \sqrt{g}\,R + \frac{M_{Pl}^2}{2}\int \sqrt{g}\,\mathcal{G}^{ab} d\ell_a \wedge \star d\ell_b + \frac{1}{2M_{Pl}^2}\int \sqrt{g}\,\mathcal{G}^{ab}\left(\mathcal{H}_{3,a}\wedge \star\mathcal{H}_{3,b}\right)\,, \tag{3.53}$$

where $\ell_a$ represents the saxions and $\mathcal{H}_{3,a}$ denotes the three-forms.
Following the same strategy applied in previous models, it is firstly imposed an ansatz respecting rotational symmetry.
The three-form can once again be expressed in terms of the axion charge as:

$$\mathcal{H}_{3,a} = q_a vol_{S^3}\,, \tag{3.54}$$

consistent with eq.(3.24).
Substituting this ansatz into the equation (3.53) yields an action containing the following term:

$$2\pi\int d\tau\left[\frac{1}{2}\mathcal{G}^{ab}(\ell)\dot{\ell}_a\dot{\ell}_b - V_q(\ell)\right]\,, \tag{3.55}$$

along with the standard gravitational term (the Einstein Hilbert action).
Here, it is introduced a coordinate transformation involving a variable $\tau$, which also depends on the metric components.
Furthermore, the following potential can be defined:

$$V_q(\ell) \equiv -\frac{1}{2}\mathcal{G}^{ab}q_a q_b \equiv -\frac{1}{2}||q||^2\,, \tag{3.56}$$

where $||q||$ is defined as a norm, with $\mathcal{G}$ serving as the metric so that $||q||^2 \equiv \mathcal{G}^{ab}q_a q_b$.
This notation proves useful as the action (3.55) is reminiscent of a classical particle moving in a non-trivial metric $\mathcal{G}$ and subject to a potential $V_q$.
This analogy simplifies the calculations by allowing the definition of a conserved, dimensionless energy:

$$E(\ell) \equiv \frac{1}{2}\mathcal{G}^{ab}(\ell)\dot{\ell}_a\dot{\ell}_b + V_q(\ell)\,. \tag{3.57}$$

[25] For a detailed exposition, see [11].

The equation of motion for each dual saxion is then given by

$$\mathcal{G}^{ab}\frac{D\dot{\ell}_a}{d\tau} = -\frac{\partial V_q}{\partial \ell^b} \,, \tag{3.58}$$

or equivalently,

$$\mathcal{F}^{ab}\ddot{\ell}_a = \frac{1}{2}\mathcal{F}^{abc}\left(q_a q_b - \dot{\ell}_a\dot{\ell}_b\right) \,, \tag{3.59}$$

where $\mathcal{F}^{ab}$ and $\mathcal{F}^{abc}$ denote derivatives of the dual Kähler potential with respect to the dual saxions.

So, the analysis is reduced to the case where $V_q < 0$, hence $E < 0$, which is precisely the characteristic regime for wormhole solutions.[26]

The gauge can once again be fixed by selecting the metric:

$$ds^2 = a(r)^2 dr^2 + r^2 d\Omega^2 \,. \tag{3.60}$$

Minimizing the action (3.53) with respect to $a$ and using the definition of the energy $E$, the wormhole solution results:

$$a^2 = \frac{1}{1-\frac{|C|}{r^4}} = \frac{1}{1-\frac{|E|}{3\pi^2 r^4 M_{Pl}^4}} \,, \tag{3.61}$$

where the neck radius $|C|^{\frac{1}{4}}$ can be explicitly calculated as

$$|C|^{\frac{1}{4}} \equiv \frac{||q||_\star^2}{6\pi^2 M_{Pl}^4} \,. \tag{3.62}$$

The charge $||q||_\star$ takes its value at the dual saxion's neck point, $\ell_\star$.

The on-shell action then has the following expression:

$$S|_{on-shell} = 2\pi \int ||q||^2 \,. \tag{3.63}$$

With a specific Kähler potential, it is thus possible to calculate the associated wormhole configuration.

In particular, in various ST models, the Kähler potential often satisfies the condition of being proportional to a homogeneous polynomial of degree $n$, as discussed in 1.3.1.

This leads to the class of homogeneous wormhole configurations, which can be explored by considering the ansatz:

$$\ell(\tau) = \tilde{\ell}(\tau)\,\mathbf{q} \,, \tag{3.64}$$

for some $\tilde{\ell} > 0$.

[26] As will soon be clarified, the sign of $E$ corresponds to the one of the constant $C$ discussed previously.

With this assumption, it is straightforward to find that:

$$\mathcal{G}^{ab}q_a q_b = \frac{n}{2}\,, \tag{3.65}$$

where $n$ is the degree of the homogeneous polynomial.
In this way, the model with multiple saxions reduces to a simpler case with a single saxion $\tilde{\ell}$, having a kinetic potential $\tilde{\mathcal{F}} = n\, log(\tilde{\ell})$ and an effective potential $\tilde{V}(\tilde{\ell}) = \frac{-n}{4\ell^2}$.
Hence, it follows that:

$$|C| = \frac{n}{12\pi^2\tilde{\ell}_\star^2}\,, \tag{3.66}$$

and, asymptotically,

$$\tilde{\ell}_\infty(\tau) = \tilde{\ell}_\star cos\left(\frac{\pi}{2}\sqrt{\frac{3}{n}}\right)\,, \tag{3.67}$$

Combining this result with the requirement that $\ell > 0$, this implies the condition for smooth wormhole configurations:

$$n > 3\,. \tag{3.68}$$

This result demonstrates the generalization of wormhole solutions to a broad class of string-inspired models.

## 3.4 Boundary conditions

In the previous sections, the question of which boundary conditions to impose on the model was postponed.
However, this is a crucial aspect of any QG model because boundary terms can contribute to the on-shell action of the system, eventually also causing potential divergences.

Furthermore, certain calculations may require integration by parts, where it becomes necessary to consider the physical implications at the boundary.
In this brief section, we summarize the typical boundary conditions chosen to ensure a well-defined variational problem.
The following discussion will also be relevant for chapter 5, as stability calculations, in principle, require consideration of boundary contributions. Even though it turns out that these contributions do not affect the stability of the wormhole, an incorrect choice could lead to problems, especially when considering the duality in the axion formulation.

The standard approach for implementing the boundary in gravitational models, such as the one in (3.1), is to add the "Gibbons-Hawking-York" (GHY) term:

$$S_{GHY} = \int_{\partial U} d^3x \sqrt{\tilde{g}} \left( K - K_0 \right) \ , \tag{3.69}$$

since the action itself would otherwise produce terms involving second derivatives of the metric components, which complicate the variational principle.
Here, $\tilde{g}$ represents the determinant of the induced 3-dimensional metric on the boundary $\partial M$ of the space-time $M$.
Finally, $K$ denotes the trace of the extrinsic curvature of the boundary on $M$, while $K_0$ is the trace of the extrinsic curvature of the same boundary embedded in a flat space-time.

The extrinsic curvature is defined by

$$K \equiv \nabla_\mu n^\mu = \partial_\mu n^\mu + \Gamma^\mu_{\mu\nu} n^\nu \ , \tag{3.70}$$

where $n$ is a unit vector orthogonal to the boundary.
With the metric ansatz (3.12) ($ds^2 = a^2 \left( dr^2 + r^2 d\Omega^2 \right)$), it is possible to calculate:

$$K = \frac{3a'}{a^2} \tag{3.71}$$

and

$$K_0 = \frac{3}{a} \ . \tag{3.72}$$

Considering the model of (3.1), it is possible to obtain an explicit evaluation of the GHY term:

$$S_{GHY} = \frac{-3}{8\pi k} \int d\Omega_3 \left( a\dot{a} - a^2 \right) = \left( \frac{-2}{3\pi} \right) \frac{\sqrt{3\pi} q}{8kf} \tag{3.73}$$

where in the last step it is substituted the explicit solution of (3.13).

An alternative approach, adopted in recent studies, is to set boundary conditions on the fields directly through explicit variational computation.
This approach can be illustrated by considering the action (3.16).
When the metric ansatz of (3.23) is applied, i.e. $ds^2 = a^2 dr^2 + b^2 d\Omega_3^2$, the exact computation yields an additional boundary term in the action:

$$S_{bound} = 2\pi^2 \int dr \frac{d}{dr} \left( \frac{3b^2\dot{b}}{ka} \right) \ . \tag{3.74}$$

To ensure a well-defined variational problem, Dirichlet boundary conditions are imposed, requiring that the following conditions[27] are satisfied:

$$\left[\frac{3b^2}{ak}\delta b\right]_{r_n}^{r_a} = 0 \tag{3.75}$$

and

$$\left[\frac{b^3\dot{\phi}}{a}\delta\phi\right]_{r_n}^{r_a} = 0\,. \tag{3.76}$$

These constraints imply that the values of the dilaton $\phi$ and the metric component $b$ must be fixed at both the mouth of the wormhole, now positioned at $r = r_m$, and at the asymptotic limit, given by $r = r_a$.

Specifically, the Dirichlet conditions enforce:

$$\begin{cases} (\dot{b})_{r_m} = 0 \\ (\dot{b})_{r_a} = 1 \\ (\dot{\phi})_{r_m} = 0 \\ (\phi)_{r_a} = 0\,. \end{cases} \tag{3.77}$$

With these conditions imposed, there is no need to add the GHY term to the action.

It is important to note that the metric component $a$ remains unrestricted by these boundary conditions; its initial value may instead be determined by the Friedmann equation of motion of (3.29).

Additionally, the axion field value at the wormhole mouth remains as a free parameter.

Finally, the boundary conditions differ when using the axion-sclar formulation rather than the three-form.

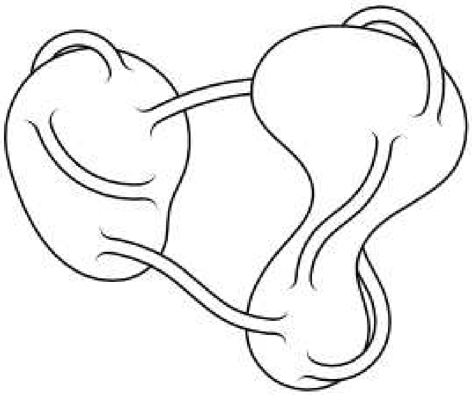
Hodge duality naturally links the boundary conditions in both frames: Dirichlet conditions for the three-form correspond to Neumann boundary conditions for the axion (i.e., their Fourier transformation).

This dual approach with the corresponding boundary conditions is explicitly employed in [10] to analyze wormhole stability.

[27] The second condition can be obtained considering the integration by part of the Dilaton kinetic term of eq.(3.28).

# Chapter 4

# Low energy contributions



Pictorial multi-universe system with multiple baby universe emissions. Taken from [13].

As previously emphasized, in QG it is natural to expect the existence of topology-changing processes. These types of transitions occur through phenomena similar to quantum tunneling.
In section 1.1.2, the concept underlying the "Loss of Quantum Coherence" debates from the 1980s was introduced. However, after years of study, physicists have generally concluded that gravitational instantons do not lead to transitions that violate quantum coherence. Nevertheless, an important question remains: what is the physical meaning of the topological transitions caused by wormholes?[1] Are there any possible interpretations or practical consequences?

This is a challenging topic, but a fruitful line of reasoning can be developed by exploring the low-energy effects arising from the presence of wormhole configurations in the theory. According to the EFT approach, it is reasonable to expect that, after integrating out the wormholes from the path integral, some remnants of these peculiar solutions would

[1]In this section, the distinction between wormholes and gravitational instantons is not emphasized, so the terms are used interchangeably unless otherwise specified.

survive in the low energy limit of the UV theory.
The results of this process are presented in section 4.1, which reveals an interesting puzzle in fundamental particle physics.
In section 4.2, a brief overview is provided of the reasoning that led physicists to the complex concept of "third quantization".
Then, section 4.3 offers an intuitive derivation of the explicit low-energy effects resulting from integrating out wormholes.
Finally, section 4.4 revisits the discussion on the swampland conjectures (introduced in section 1.3.2) and briefly explores their applications in the context of wormholes.

The core idea of this chapter is that wormhole configurations may provide insights pointing toward new physics, as well as serve as a potential tool to explore phenomenological aspects of ST.

## 4.1 The Bi-local operator and the $\alpha$ parameters

In the following discussion, it will be considered the simple case of a single gravitational instanton, although systems with multiple wormholes have been well studied and analyzed in the literature. The so-called "dilute gas of instantons" approximation is commonly adopted, where interactions between the wormholes are neglected. This assumption allows for the factorization of the contributions from different instantons, making the generalization from the simple case considered here to the most general case straightforward. When including instanton interactions, it turns out that, even though the dilute gas approximation is not strongly justified a priori, it still provides a good description of the model.[2]

In order to successfully integrate out the wormholes, it is implicitly required that the wormholes have a specific thickness, denoted by $\mathcal{L}$, and that the effective theory applies to energies below the scale $\frac{1}{\mathcal{L}}$. Otherwise, the entire argument becomes ill-defined from the outset due to the breakdown of the semi-classical approach. Thus, the foundational assumption is that there is a clear separation between three energy scales: the Planck mass (where QG effects become significant), the wormhole "characteristic energy" $\frac{1}{\mathcal{L}}$, and the low-energy effective theory regime.

[2]For example, the partition function of a dilute gas of wormholes can be expressed as a product of terms similar to the one in equation (4.3). This approximation was first introduced in the seminal work by Giddings and Strominger [1]. For a further comprehensive discussion, see also [64].

However, this assumption alone is not sufficient. It is also crucial that the two mouths[3] of the wormhole are far apart. In this way, the wormhole structure interacts with the "matter" (scalar) fields at two points that are completely independent of each other (in other words, that are causally disconnected). This lack of correlation can also be achieved by ensuring that the wormhole has no characteristic length; in general, it is sufficient that the separation between the two ends is much larger than the thickness $\mathcal{L}$.

Considering the simple case of a single generic instanton coupled to a background scalar field (the axion), the partition function in the path integral formulation is given by:

$$\mathcal{Z} = \int \mathcal{D}\phi \, e^{-S(\phi)} \left( \int d^4 x e^{-S_{Inst}} \right) . \tag{4.1}$$

In the low-energy limit, one can integrate out the instanton, obtaining an infinite set of new local operators for the effective theory:

$$S_{Inst}(x, \phi) = S^{\star}_{Inst} + a_1 \, \phi(x) + a_2 \, \phi^2(x) + a_3 \, (\partial\phi(x))^2 + \ldots , \tag{4.2}$$

where $S^{\star}_{Inst}$ denotes the unperturbed action evaluated at $\phi = 0$.
The generalization to the wormhole case is straightforward. However, in the gravitational case, a key feature is that there are two points where the scalar fields are located. So, the partition function is written as

$$\mathcal{Z} = \int \mathcal{D}g\mathcal{D}\phi \, e^{-S(\phi,g)} \left( \int d^4 x \sqrt{g(x)} \int d^4 y \sqrt{g(y)} e^{-S_{wh}(\phi,\, g;\, x,\, y)} \right) , \tag{4.3}$$

where $\int \mathcal{D}g$ represents the integration over all possible background metrics, with an implicit cutoff determined by the wormhole thickness $\mathcal{L}$.[4]
Following the previous case, integrating out the wormholes yields an infinite sum of new operators:

$$S_{wh}(\phi,\, g;\, x,\, y) = S^{\star}_{wh} + b_1 \, \phi(x) + b_2 \, \phi(y) + b_3 \, \phi(x)\phi(y) + b_4 \, (\partial\phi(x))^2 (\partial\phi(y))^2 + \ldots , \tag{4.4}$$

where again $S^{\star}_{wh}$ is evaluated on the unperturbed wormhole metric (with the scalar field $\phi$ vanishing).
The resulting effective action is composed of a sum of non-local operators. This outcome is inevitable when performing a double Taylor expansion, and moreover, there is no sym-

[3] In the earliest articles discussing the concept of wormholes, their endpoints were considered to be two black holes, often referred to as "mouths."

[4] It is notable that wormhole configurations exceeding the bounds of the EFT are naturally suppressed, since the weight becomes very small, making their contribution negligible.

metry to prevent the appearance of these new bi-local terms.
Using a more compact notation, the resulting EFT action is given by

$$S_{wh}(\phi,\, g;\, x,\, y) = S^{\star}_{wh} + \sum_{a,b} \left(\Delta_{ab}\, \mathcal{O}^a(x)\, \mathcal{O}^b(y)\right) \tag{4.5}$$

where $\Delta$ is a real symmetric matrix, independent from the space-time points $x$ and $y$.
At this point, it is possible to rewrite the partition function by expanding the sums over$a$ and $b$ and substituting into equation (4.3), resulting in

$$\mathcal{Z} = \int \mathcal{D}g\mathcal{D}\phi\; exp \left[-S(\phi, g) + \frac{1}{2}\int d^4x\sqrt{g(x)} \int d^4y\sqrt{g(y)} \left(\sum_{a,b} \Delta_{ab}\, \mathcal{O}^a(x)\, \mathcal{O}^b(y)\right)\right] . \tag{4.6}$$

Unfortunately, the newly generated operators are problematic because they lead to non-local interactions, which lack a clear physical interpretation.
However, it is possible to rewrite the effective action in terms of standard local operators. This issue can be addressed by using a Gaussian integral and introducing some new parameters, labeled by $\alpha_I$.
Leaving the sums over $I$ and $J$ implicit, the necessary Gaussian integral takes the form

$$\left(\prod_I \int d\alpha^I\right) exp\left(-\frac{1}{2}\alpha^I \Delta_{IJ}\, \alpha^J - \alpha^K A_K\right) = \sqrt{\frac{2\pi}{\det \Delta}} exp\left(-\frac{1}{2}A^I \Delta^{-1}_{IJ} A^J\right) \tag{4.7}$$

with the precaution of replacing the operators $A^I$ with the $\mathcal{O}^a$.
Thus, the final form of the partition function becomes:

$$\mathcal{Z} = \int \mathcal{D}g\mathcal{D}\phi e^{-S(\phi,g)} \left(\prod_I \int d\alpha^I\right) exp\left(-\frac{1}{2}\alpha^I \Delta_{IJ}\, \alpha^J\right) exp\left(\sum_I \alpha^I \int d^4x\sqrt{g}\, \mathcal{O}_I(x)\right) . \tag{4.8}$$

In a generic QFT, one can express the original action $S$ in terms of the same operator basis:

$$S(g, z) = \sum_I \left(z^I \int d^4x\sqrt{g}\; \mathcal{O}_I(x)\right) , \tag{4.9}$$

where the $z_I$ are the coupling constants of the original theory, which couple to the effective local operators.
By comparing the corrections from equation (4.8) with the expression in (4.9), it becomes evident that integrating out the wormholes results in a shift in each coupling constant of the theory, by a constant value $\alpha_I$.

Since the $\alpha$ parameters are independent of spacetime points, they do not violate any

gauge symmetry of the original action $S$, nor do they break Poincaré symmetry.
This resolves the issue of considering non-local operators, but at the cost of introducing new parameters whose role is not immediately clear.
In the following, it is reported the earlier interpretation of these parameters, as discussed in the article by Preskill [64].

At first, Coleman suggested that these new parameters could simply be determined experimentally and fixed, meaning that they have a precise value, even if initially unknown. However, it is not possible to practically do it. Moreover, it is expected an explanation for their appearance in the path integral, as well as for the Gaussian weight. The main difficulty is that it doesn't seem possible to predict their value from the QG theory.

A different interpretation was later proposed, again by Coleman, considering the $\alpha$'s as statistical objects which follow a Gaussian distribution. In this view, the integration over all possible values of each $\alpha_I$ represents an ensemble average, giving rise to a physical picture reminiscent of a "multiverse" model.[5]
In this framework, the couplings of each universe depend on the specific $\alpha$-value of the universe. Even though each possible $\alpha$-value (which could differ in a spacetime distinct from our own) has a Gaussian weighting, in each universe it assumes a specific value that influences all observable phenomena.
Coleman's reasoning was initially supported by an explicit estimate suggesting the vanishing of the cosmological constant, based on a calculation that relied on the existence of the $\alpha$ parameters. However, this aspect is no longer valid, as it has been understood that the cosmological constant, even if assume a small value, is non-zero.
Moreover, Preskill argued that the same reasoning could be generalized to determine all the constants of nature. Unfortunately, this approach did not yield the expected results. Preskill also offered a similar interpretation, aiming to clarify the physical meaning behind the $\alpha$-parameters. Coleman's argument was based on the assumption of a dilute gas of wormholes, valid only within a semi-classical approximation, while Preskill extended it to include possible interactions among instantons.

In this view, the multi-universe proposed by Coleman could be seen as a superposition of independent universes, each characterized by different values for the parameters and

[5]The original discussion was focused on the role of Euclidean wormholes in a flat background, but the similar cases in de Sitter (dS) and Anti-de Sitter (AdS) universes have also been extensively studied by the scientific community. There have been considered other interesting sides of this topic, like the possible relation of the $\alpha$'s to the cosmological constant (as well as other constants of nature) and the related problem of the "FKS catastrophe", predicting an over-density of wormholes in the multi-universe frameworks. This aspects are analyzed, for example, in [13, 20].

for the coupling constants. From this perspective, the loss of quantum coherence, as discussed by Hawking, Giddings, Strominger, and Coleman, can be interpreted as a loss of information regarding the $\alpha$-universes other than the one we inhabit. The existence of wormholes would then imply a fundamental indeterminacy in the constants of nature, although this indeterminacy would be inaccessible to any experiments conducted within a single universe.

This multiverse interpretation remains controversial and far from universally accepted. However, no alternative solution has yet emerged that satisfactorily resolves these issues.

## 4.2 Baby universes and third quantization

To gain a deeper understanding of the low-energy physics, an alternative perspective can be explored by slightly shifting the starting point.
Rather than considering the entire wormhole structure, formed by gluing together an instanton and an anti-instanton with opposite sign charges, it is possible to focus on the two individual halves. The Giddings-Strominger instanton (or anti-instanton) can be viewed as the nucleation of a three-sphere from Minkowski space-time, which then evolves over (Euclidean) time. This scenario is depicted in figure 4.1.

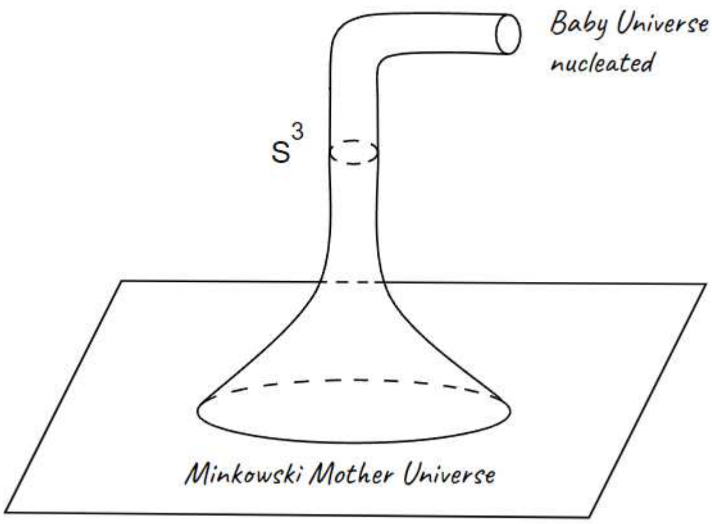



**Figure 4.1:** The single instanton depicted in the picture illustrates the idea of the process generating a new $S^3$ structure of space-time, which was not present in the original universe.

As previously mentioned, the instanton can either connect with its counterpart at a dis-

tant point within the same universe or potentially flow into a different universe (as shown in figure 3.3). In both cases, the topology change can be interpreted as a transition from Minkowski space to the same space-time, but now with an additional three-dimensional structure known as a "*baby universe*", which corresponds to the instanton tunnel.
In the context of a four-dimensional metric, the unitarity problem[6] is resolved by conjecturing that the newly nucleated 3-sphere is associated with a specific set of $\alpha$-eigenvalue. The main issue with this approach is that it treats the $\alpha$ parameters as free variables of the theory, with no connection to the fields or moduli of the theory.[7]

This puzzle is complex and remains a subject of ongoing debate in the literature. Therefore, the goal of this chapter is to outline the current state of the art and emphasize its key challenges.

Building on the conclusion of the previous section, the $\alpha$ parameters can be understood as eigenstates linked to specific baby universes, which do not disrupt the system's quantum coherence. Physically, these parameters act as the coupling constants in the effective theory derived from a model that includes gravitational instantons.
Coleman postulated that, by allowing for different kinds of baby universes (e.g., differing in size, charge, or topology), one can define "classical" creation and annihilation operators for them. Denoting the different types of baby universes by the index $i$[8], the corresponding ladder operators, called $a_i$ and $a_i^\dagger$, must satisfy the (bosonic) canonical commutation relation $[a_i; a_j^\dagger] = \delta_{ij}$.
Thus, the action can be expressed as:

$$S = S_0 + \sum_i (a_i + a_i^\dagger) \int d^4x \, O_i(x) \, , \tag{4.10}$$

where the $O_i$ are local operators associated with the respective parameters.
Defining the Hermitian operator $A_i \equiv \left(a_i + a_i^\dagger\right)$, it is possible to define an eigenvector $|\alpha\rangle$, such that its corresponding eigenvalue is the $\alpha_i$ parameter introduced in equation (4.8), formally:

$$A_i \, |\alpha\rangle = \alpha_i \, |\alpha\rangle \tag{4.11}$$

With these assumptions, the different eigenstates define the superselection sectors, labeled by $\alpha_i$, as discussed previously.

[6] An other equivalent way to call the quantum coherence issue, discussed in section 1.1.2

[7] This contradicts a key principle of the swampland program, which asserts that a consistent theory of QG should not have any free parameters.

[8] For example, one value of $i$ could label the Giddings-Strominger configuration with charge $n = 1$, and so on.

Furthermore, the action can be rewritten as:

$$S = S_0 + \sum_i \alpha_i \int d^4x\, O_i(x) \tag{4.12}$$

which reproduces the resulting action of equation (4.8), but extrapolated from the path integral weight.
In this framework, if the initial flat universe exists in a superposition of $\alpha$-eigenstates, the nucleation of a baby universe does not pose a unitarity problem. Instead, it merely represents the collapse of the universe's wave function.[9]

Furthermore, a different perspective soon gained attention, rooted in the idea that topology changes can be examined by using the path integral formulation over Euclidean space-time, leading to the formulation of a "*third quantization*" theory.[10]
This approach aims to provide a coherent physical meaning to the gravitational path integral. The integration over space-time geometries with fixed boundaries can be viewed as a type of Green function, which essentially weighs all the possible bulk structures. Unlike the standard Green function, related to the particles' interactions, in this case, the integral acts as a single universe propagator connecting fixed boundaries.

In this third-quantized framework, the field operator acts to create or annihilate second-quantized states of single universes (without any baby universe and no possibility of topology change), whose coupling constants are the eigenvalues of the dynamical field operators in this third-quantized theory (which can change topology and allow for baby universes).
The outcome is a formalism that is one-dimensional and provides an optimal framework for studying processes involving multiple interacting universes. It even introduces Feynman rules for such systems: vertices correspond to "mother" universes, propagators to wormholes, and the conservation of momentum to the conservation of global charges (such as the axion one).[11]
An interesting point of view of this approach has been proposed with an other swampland conjecture:

**(Baby universe hypothesis).** *In a unitary QG theory with a $d \geq 4$ space-time, the Hilbert space of a baby universe must be one-dimensional.*

This hypothesis might seem counterintuitive, especially compared to the naive expecta-

[9]See ref. [43] for more details. The idea of universe wave function is intuitive, but a rigorous definition and analysis can be found in various references, as in[4, 19, 50, 53, 60].
[10]Discussed, for example, in [4, 13, 64]
[11]This formalism was introduce by Giddings and Strominger in [4].

tions of effective theory approaches, which predict a large number of degrees of freedom in the infrared regime.
A possible solution to this puzzle has been proposed recently[12], suggesting that a type of gauge redundancy could exist, identifying multiple space-time topologies within a single universe wave function. However, the full implications of this idea remain unclear.
Notably, the "Baby Universe Hypothesis" aligns with the NGS conjecture (described in section 1.3.2), as the existence of a global symmetry in QG would imply that the Fock space of the baby universe wave function must have more than one dimension.

There are several studies exploring the applications of this formalism in addressing practical problems. For example, early arguments suggesting that the cosmological constant $\Lambda$ should vanish were derived within this context. More recent efforts, including attempts to solve the naturalness problem of the SM [13], have also seen interesting developments along these lines.

In summary, a deeper understanding of wormhole physics could prove crucial in the pursuit of a comprehensive theory of QG, along with resolving various related aspects of fundamental physics.

## 4.3 The Axion potential

This section follows the discussion in 1.2 and presents a concrete calculation to estimate the effects of a wormhole configuration on the axion potential. To quantify the effects of the wormhole, we first need to start from the process of integrating out the instanton discussed in the previous sections. The wormhole correction to the action of a parent universe (in the presence of multiple instantons) reads:

$$e^I = \langle \Psi_i | \, exp \left[ \sum_n e^{\frac{-S_w(n)}{2}} K_n \left( a_n + a^\dagger_{-n} \right) \int d^4x \sqrt{g(x)} \, \mathcal{O}(x) \right] | \Psi_f \rangle \ . \tag{4.13}$$

In this expression, the bra and ket $\Psi$ are introduced to define the initial and final states for the spacetime perturbed with a plasma of baby universes, while the term $K_n$ represents functions of the fields evaluated in the bulk.
Additionally, the "ladder" operators refer again to the creation and destruction of baby universes. It is interesting to note that the same topological structure can be obtained

[12] Two articles discussing this idea are [32, 57].
[13] See [37] for the more details.

from the annihilation of an instanton charged with $n$, as well as from the creation of an anti-instanton charged with $(-n)$. Indeed, the two cases both yield the same axionic charge violation.

Applying this to the case of G.S. wormholes (where $n$ takes the simple role of the axion charge), we can obtain an explicit evaluation for this contribution. The on-shell wormhole action can be easily computed[14], as previously done for equation (3.15), with the following result:

$$S_w = \frac{\pi\sqrt{6}}{4}\frac{n}{f} \,. \tag{4.14}$$

However, in the partition function term (4.13), only the half associated with the instanton (corresponding to a semi-wormhole structure) is considered.

By fixing both the initial and final eigenstates as simple baby universe states[15]

$$|\Psi_i\rangle = |\Psi_f\rangle = |\alpha\rangle \ , \tag{4.15}$$

the eigenvalue equation (4.11) remains valid. Consequently, it can be re-expressed as:

$$\left(a_n + a^\dagger_{-n}\right)|\alpha\rangle = \alpha_n\,|\alpha\rangle \tag{4.16}$$

Furthermore, we can rewrite:

$$\alpha_n = |\alpha_n|\,e^{i\delta_n} \ , \tag{4.17}$$

introducing a generic constant phase $\delta_n$.

The operator $\mathcal{O}_n$ associated with a specific $\alpha_n$ eigenvalue is $n$-charged; thus, under the shift transformation $\phi \to \phi + \epsilon f$, it transforms as $\mathcal{O}_n \to e^{in\epsilon}\mathcal{O}_n$.

From this argument, its form can be deduced, resulting:

$$\mathcal{O}_n(x) = exp\left(i\frac{n\phi}{f}\right)\mathcal{O}_s(x)\,, \tag{4.18}$$

where the new $\mathcal{O}_s$ is defined as a singlet operator.

With these previous assumptions, we find the following contribution to the action:

$$\begin{aligned} I &= \sum_m e^{\frac{-S_w(n)}{2}} K_n \int d^4x\sqrt{g(x)}\,\mathcal{O}_s(x)\ |\alpha_n|\ exp\left(\frac{in\phi}{f} + i\delta_n\right) \\ &= \sum_m e^{\frac{-S_w(n)}{2}} K_n \int d^4x\sqrt{g(x)}\,\mathcal{O}_s(x)\ |\alpha_n|\ cos\left(\frac{n\phi}{f} + \delta_n\right) \,. \end{aligned} \tag{4.19}$$

[14]It is obtained by taking the trace of the Einstein equation and multiplying by 2 to account for both the two halves of the wormhole, the instanton and the anti-instanton.

[15]If instead one fixes $|\Psi_i\rangle = |\Psi_f\rangle = |0\rangle$, then it would recover the Gaussian weight of (4.8).

Lastly, the singlet operator can be generally written as a series of terms that are $U(1)$ singlets, depending by numerical parameters:

$$\mathcal{O}_s(x) = 1 + a\,L^2\,\mathcal{R} + bL^4\partial^\mu\phi\partial_\mu\phi + \dots\,, \tag{4.20}$$

where $L$ is the minimal radius size of the wormhole throat, while $a$ and $b$ are order one parameters that, in principle, could be experimentally estimated.
By comparing the wormhole-originated potential to the QCD axion potential, we obtain an expression like:

$$V(\phi) = \Lambda_{qcd}^4\,cos\left(\frac{\phi}{f}\right) + \frac{e^{-S_w}}{L^4}\,cos\left(\frac{\phi}{f} + \delta\right)\,. \tag{4.21}$$

Here, we only consider the effect of a $n = \pm 1$ wormhole, as it is the dominant one. The outcome indicates that the instanton effects are significantly smaller compared to the QCD term, as the exponential factor suppresses them. However, it is not a good approximation to completely neglect it, as it could still provide un-expected effects.
For example, in QCD, the low-energy contribution is minimized at $\phi = 0$, while the gravitational term is proportional to an arbitrary phase $\delta$, which could be related to strong CP violation (quantifying the mismatch relative to the QCD term).
A deeper analysis could also lead to extra gravitational corrections to the axion masses and adjustments to the $\theta$ CP-term. Additionally, other studies have evaluated the implications related to different physical aspects, such as inflationary mechanisms or dark matter models proposing axion-like candidates.

## 4.4 Wormholes in the Swampland programme

Having examined the gravitational instanton solutions explicitly and their associated low-energy effects, we can return to the discussion in section 1.3.2 and analyze the role of wormholes in the swampland conjectures introduced there.

### No Global Symmetries

In the context of wormhole physics, it is possible to examine the NGS conjecture in relation to the $U(1)$ shift symmetry of the axion. This symmetry is a global one that, to satisfy the conjecture, should not be implemented in the QG completion.[16]

[16] In this section, we briefly discuss the case of the axion's symmetry, which can be easily generalized to the case of p-forms, as done in [91].

The invariance under the shift transformation is given by the following equation:

$$\varphi \to \varphi + a \, . \tag{4.22}$$

This implies that the density current $\mathcal{J} = d\varphi$ is conserved.
The transformation is rigorously defined by the following local operator:

$$U(x) = exp\left[i\lambda \int \star\mathcal{J}\right] \, . \tag{4.23}$$

As a consequence, there are local operators charged under this symmetry:

$$\mathcal{O}(x) = exp\left(iq\varphi\right) \, , \tag{4.24}$$

which leads to a violation of the conjecture.
In ST, the shift symmetry is restricted to a subgroup of discrete transformations:

$$\varphi \to \varphi + 2\pi f k \, , \tag{4.25}$$

where $k$ is an integer.
This result, well explained in [15], improve the situation, as this residual symmetry is gauged so does not violate the NGS criterion.

Thus, it is crucial that the global symmetry must either be gauged or broken. Physically, the key indication that global symmetries are violated in QG arises from non-perturbative contributions. A notable example discussed in the literature is that of black holes, which can "absorb" certain charges and subsequently evaporate, thereby violating conservation laws.
Similarly, wormholes can exhibit a related effect through their non-perturbative low energy corrections, "transporting" global charges to other universes through the nucleation of baby universes.[17]

## Distance Conjecture

The application of the distance conjecture to the case of gravitational instantons is quite technical. For a comprehensive overview of this topic, see [89, 93].

An illustrative example of applying the conjecture is found in models containing mul-

[17] This effect is discussed in [2]. For further details on the NGS conjecture, it is also recommended to consult [90].

tiple scalars, such as the sigma model discussed in section 3.1. Considering the action (3.1), the functional $G_{IJ}$ can be viewed as a metric on the moduli space of the theory. Indeed, the equations of motion (3.5) can be expressed in the form of a geodesic equation, where the Levi-Civita connection is defined with respect to the metric $G_{IJ}$.
The conjecture can be tested by evaluating the maximal distance that saxions can travel in moduli space, given by the following definition:

$$d(\phi_\star;\phi_\infty) \equiv \int_0^\infty d\tau \left( G_{IJ}(\phi) \frac{d\phi^I}{d\tau} \frac{d\phi^J}{d\tau} \right)^{\frac{1}{2}} . \tag{4.26}$$

This distance is calculated (for some specific models) with respect to the Euclidean time $\tau$, considering the maximum time interval along a semi-wormhole configuration.
The explicit calculation performed in the referenced article reveals that the distance is super-Planckian.[18]
In the opposite case, a breakdown of the EFT would ensue, making the low energy model fall into the swampland.
This result is significant, as it provides further confirmation of the distance conjecture.

## Weak Gravity Conjecture

It is interesting to explore the application of the weak gravity conjecture within the context of axion physics, where the wormhole solution offers further validation.
Following the review by Palti [97], we can define similar formulations of the conjecture for various models, including those with scalars, dilatons, p-forms, and even instantons. In this discussion, we will focus on the last option and its application to the wormhole solution.

In general, the states of the theory that must satisfy the WGC are not limited to standard fields corresponding to particles; they can also include extended objects, such as p-branes. In this context, axions can be considered as 0-forms with a coupling constant $\frac{1}{f}$. This leads to the following definition based on the general case of p-forms. The statement of the conjecture is as follows:

**(Axion Weak Gravity Conjecture).** *An Axion with decay constant $f$ should be coupled to one or more instantons, with constant action $S$, and satisfying the condition:*

$$S \leq \frac{M_{Pl}}{f} . \tag{4.27}$$

[18] Quantitatively, this means that $d > M_{Pl}$.

This formulation is applicable to both instantons that are solutions of gauge theories and those that have a gravitational nature.

In models where wormhole solutions are permitted, an explicit calculation of the on-shell action would provide direct confirmation of the conjecture, as the bound stated in (4.27) would be satisfied. According to the discussion in [12], this constraint is satisfied in models including wormhole solutions, and the conjecture appears to be valid, especially in the cases of microscopic wormholes. Even the estimate provided in equation (3.15) clearly satisfies the bound of the axion-WGC.

However, this area of research has become increasingly active in recent years. The verification of the WGC for macroscopic wormholes could lead to significant implications, such as strict constraints on inflationary models or the violation of slow-roll conditions. Consequently, physicists are actively seeking potential loopholes within the string theory framework.

# Chapter 5

# Stability of wormhole solutions

In the first four chapters, the general framework behind Euclidean wormholes and the most important related aspects have been discussed. Now, it is finally time to reach the core of the work. The focus of the thesis is the stability feature of wormhole solutions as found in the context of effective theories motivated by Quantum Gravity research.
In the following section, starting with a brief recap of the concepts introduced in chapter 2, the problem of negative modes in the stability analysis of solutions in GQ models is presented.

## 5.1 The problem of negative modes

In chapter 2, the concept of the gravitational path integral was introduced. Starting from the definition of the partition function in curved space QFT settings, the gravitational case is defined to allow metric fluctuations as well. Thus, the dynamical nature of space-time indicates the need to sum over different topologies of the universe, keeping the boundary conditions at infinity fixed. This is essentially what leads to the study of objects like wormholes.

Even though the path integral is formally defined in this way, it does not follow that it is always possible to compute it. For this reason, the saddle-point approximation around the extremes of the action is well motivated. However, it has been shown that it is insufficient to consider only the minima, and so (especially when working with the Euclidean signature) it is necessary to also consider the non-perturbative contributions given by saddle-points.

In this approach, fields are expanded around the background solution in the following way:

$$\varphi = \varphi_0 + \delta\varphi \tag{5.1}$$

where $\varphi_0$ indicates the background configuration and $\delta\varphi$ small perturbation around it. Thus, the action can be Taylor-expanded as a perturbative series:

$$S\left(\varphi_0 + \delta\varphi\right) \approx\; S\left(\varphi_0\right) + \frac{1}{2}\left(\delta\varphi\right)^2 \cdot \left(\frac{\delta^2 S}{\delta\,\varphi^2}\right)_{\varphi=\varphi_0} + \,\ldots \tag{5.2}$$

Obviously, the first-order term is not of interest because the first derivative of the action evaluated at the stationary configuration trivially vanishes.[1]
Conversely, physically relevant information about the validity of the approximation is encoded in the second-order perturbation. So it is important to examine it explicitly. Inserting the expansion into the path integral, it results:

$$\mathcal{Z} \approx e^{-S(\varphi_0)} \int \mathcal{D}\varphi\; \exp\left(-\delta^2 S\left(\varphi_0, \delta\varphi\right) + \mathcal{O}(\delta\varphi^3)\right)\;. \tag{5.3}$$

The exponential outside the integral represents the dominant classical contribution.[2] Instead, the quadratic term gives the one-loop contributions and can be computed as a Gaussian integral, as described in section 2.1.

However, a peculiar non-perturbative solution to the equation of motion does not necessarily correspond to a real physical phenomenon. For this reason, it is important to evaluate its stability, which means investigating whether it truly provides a physical contribution to the path integral of the theory.
Thus, the action fluctuation can be rewritten in the following schematic form:

$$\delta^2 S = \frac{1}{2}\int \delta\varphi\,\hat{\mathcal{M}}\,\delta\varphi^\dagger \tag{5.4}$$

and consequently, the path integral evaluation depends on the matrix $\hat{\mathcal{M}}$.
Indeed, the quadratic integration is well defined if and only if the action fluctuation is semi-positive definite:

$$\delta^2 S \geq 0\,, \tag{5.5}$$

otherwise, it is not possible to perform the integration obtaining a real valued contribution.
It follows that the stability of the solution is related to the signs of the eigenvalues of the

[1]Indeed, the background solution satisfies the equations of motion, giving $\left(\frac{\delta S}{\delta\Psi}\right)_{\Psi=\Psi_0} = 0$.
[2]To explicitly see that it is the dominant term, one should avoid assuming natural units and retain the parameter $\hbar$.

matrix $\hat{\mathcal{M}}$: the existence of negative modes around the critical point of the Euclidean path integral would imply the need for a different physical interpretation.
In fact, if $\delta^2 S$ exhibits negative eigenvalues, this would imply that the saddle point configuration is not a true minimum dominating the semi-classical expansion of the path integral, but something else.

In QFT models without gravity, the minima of the Euclidean action provide non-perturbative contributions to the vacuum energy. Moreover, in some cases, it has been shown that tunneling processes via instantons exhibit a negative mode in their perturbation spectrum.[3] In these physical cases, the negative mode is needed as it provides an imaginary energy contribution[4], and thus an instability. Consequently, these field configurations are interpreted as mediators of decay from "meta-stable" vacua.

When gravity is involved in the model, the situation is less clear, but the negative mode still exists.
Considering the "conformal factor problem" described in section 2.1, some studies have found that an infinite tower of negative modes would be present, as a consequence of the negative kinetic term. Interpreting this result has proven difficult, but in [117] it was shown that these negative modes could be removed through specific canonical transformations on the variables. Indeed, the gravitational action is redundant because it is invariant under diffeomorphisms. To select the "real" negative modes, it is necessary to carefully fix the gauge and remove the non-physical degrees of freedom. Thus, the appearance of multiple negative modes was a technical issue arising from the formalism used.

However, the problem of providing a physical interpretation for solutions with multiple negative modes remains an open debate. Indeed, the presence of negative modes would suggest that the saddle point is not a true minimum, and so it is expected the existence of a lower action stationary point.
Thus, it is crucial to precisely analyze the stability of peculiar solutions to determine their nature. Any instability with multiple negative eigenvalues would lack a clear physical interpretation and, in such cases, it would be natural to discard the solution.[5]

[3]This feature has been considered in various paper, for example a complete analysis is contained in [101].

[4]This is straightforward to see by considering the case of the Gaussian integral in (2.12) where the exponential factor has one negative eigenvalue.

[5]This conclusion was originally proposed by Coleman in [111], but more recent studies have adopted the same reasoning. For example, see also [102, 117].

## Wormhole stability in the past

Wormhole stability is a non-trivial yet crucial aspect that has divided the scientific community for years. Before considering an explicit stability analysis, it is useful to highlight the previous discussions on this topic present in the literature.
To quickly recap the evolution of studies on this subject, it is necessary to go back to 1996. In the first work [5], Rubakov and Shvedov showed that there is a negative mode in the fluctuation spectrum, undermining wormhole stability. Their result was interpreted as an instability of the universe due to its decay via the emission of baby universes. Following this, some papers seemed consistent with their findings.
However, their conclusion was not universally accepted, as it was not ruled out that the negative eigenvalue could result from the gauge redundancy of the conformally-invariant metric. Thus, in 2017, Alfonso and Urbano published an article [6] discussing various theoretical and phenomenological aspects of gravitational instantons in Euclidean QG. They argued that there are no negative modes in the wormhole perturbation spectrum, explaining that an incorrect assumption was made in the gauge fixing in previous papers. The following year, a new computation [7] presented an even more unfavorable scenario: a set of negative inhomogeneous modes was found in both flat and AdS spaces, challenging the wormhole's status as a Euclidean stationary point. The situation did not improve even when models with additional scalar fields or a cosmological constant were considered, as previously shown in [45].

Additionally, [22], published in 2021, revealed that the situation in various AdS models is far from clear, as negative modes were present in some cases but absent in others.
More recently, papers published over the last two years have found that wormhole solutions appear to be stable under specific configurations.
First, [8] shows that the choice of boundary conditions is crucial, and adopting Dirichlet conditions removes negative modes in the perturbation spectrum. Subsequently, the arguments in [9] further improved the situation by examining the more "physical" case of a massive dilaton field. Here, the existence of stable wormhole classes was confirmed, even when varying the dilaton mass and the coupling constant value.
Finally, this year it was shown in [10] that stability calculations lead to a fully positive set of eigenvalues when considering both Hodge-dual formulations. Notably, the results of this paper are in perfect agreement with the following analysis, despite differences in notation and the initial metric ansatz.

In the following, a stability analysis is presented, parallel to the recent studies cited above. But given the controversial history of this topic, further developments in the future are expected. At present, both the claim of the absence of negative modes and their possible physical interpretation remain unclear, thus constituting an open problem.

## 5.2 An introduction to the stability analysis

The aim of this chapter is to demonstrate the stability of wormhole solutions, with the expectation of obtaining results consistent with recent works [9, 10]. However, the procedure followed will be slightly different in order to make easier and more clear some steps. The cosmological perturbation theory used here is primarily based on the works of [103, 109, 112, 115, 116].

Wormhole stability is analyzed in a model consisting of a single pair of axion-dilaton scalars minimally coupled to gravity, as the one introduced in section 3.2.
It is assumed that the coupling $\beta$ satisfies the constraint (3.19). Additionally, only the case of a massless dilaton is considered. Indeed, as shown in [14] and later in [9], when the dilaton is massive, a bifurcation of wormhole solutions occurs. The splitting of solutions and their relative stability depend strongly on the value of the coupling. Consequently, the situation becomes more complex, as it has been proved that not all these solutions are stable.

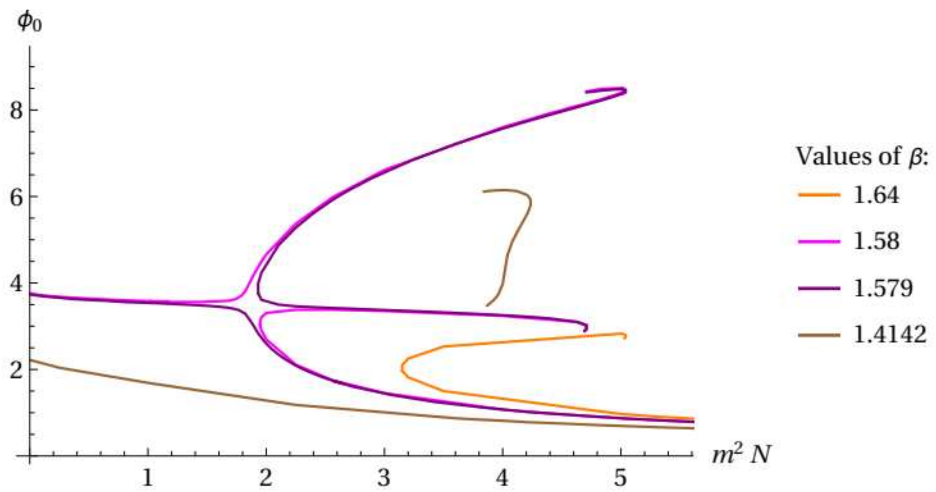



**Figure 5.1:** Figure taken from [9]. Displayed here, for four different coupling values, are the possible configurations that emerge as the initial dilaton condition $\phi_0$ varies with respect to the mass-charge combination $m^2N$. The two middle cases (colored with light and dark purple lines) show the bifurcation, although not all these configurations are stable.

In [10], it was demonstrated that, by carefully choosing the boundary conditions and applying integration by parts, stability computations using both the Hodge-dual formalisms yield the same result. This finding is significant, as previous research had assumed otherwise and mainly employed the three-form formalism.
Thus, it suffices to carry out the computation using one of the two options. Here, we use the three-form formulation, which was historically employed in both key studies asserting wormhole stability and instability. Moreover, small perturbation of the three form should be normalizable and possess a finite energy $\int \delta H \wedge \star \delta H < \infty$. The duality of section 2.4 ensures the correspondence to axion perturbation with finite energy $\int d\delta\varphi \wedge \star d\delta\varphi < \infty$, despite this fluctuation could not be normalizable, i.e. $\int \star(\delta\varphi)^2 \to \infty$, as a consequence of the axion shift symmetry. Thus, it is motivated the choice of the three-form formalism in order to work with normalizable variables and only requiring Dirichlet conditions without the need of calculating extra boundary terms.

The general approach to conducting the stability analysis is summarized in the following steps. First, starting from the wormhole solution derived in chapter 3 and defining an appropriate ansatz for metric and field fluctuations, it is possible to compute the second-order action perturbation (i.e., the second variation of the action).
The result can then be elaborated and simplified, for instance, using gauge transformations and changing variables. Consequently, the problem reduces to a one-dimensional study of a Schrödinger-like eigenvalues' equation of the form:

$$-\frac{d^2 w}{d\tau^2} + U \cdot w = \lambda \cdot z \cdot w \,, \tag{5.6}$$

where $w$ denotes the function parameterazing fluctuating modes, $U$ is the effective potential derived with the previous analysis, $\lambda$ represents the eigenvalues obtained respect to $w$ and $z$ is a (time dependent) measure which is necessary to solve the equation. With a numerical analysis, it is then possible to explicitly determine the $\lambda$ values and, most importantly, to establish whether all eigenvalues are positive or if any negative mode appears.

## 5.3 Background perturbations and derivation of the eigenvalue equation

The action for the model under consideration is given by (3.16), with the specific functional $\mathcal{F}$ fixed by the choice of (3.17).
Imposing rotational invariance symmetry under the group $O(4)$ and substituting the equa-

tions of motion (3.29), the Euclidean action becomes:

$$S = 2\pi^2 \int d\tau \left( -\frac{3}{\kappa} \left( \dot{a}^2 + a^2 \right) + \frac{a^2}{2} \dot{\phi}^2 + \frac{e^{-\beta\phi\sqrt{\kappa}} N^2}{a^2} \right) , \tag{5.7}$$

where it is also adopted the conformal gauge choice $a(\tau) = b(\tau)$ for the metric components. Under this same assumption, the equations of motion (3.29) become

$$\begin{cases} 0 = \frac{\dot{\mathcal{H}}}{\kappa} + \frac{1}{2}\dot{\phi}^2 + \frac{1}{\kappa a^2} - \frac{N^2}{a^6} e^{-\beta\phi\sqrt{\kappa}} \\ 0 = \frac{3\mathcal{H}^2}{\kappa} - \frac{3}{\kappa a^2} - \frac{1}{2}\dot{\phi}^2 + \frac{N^2}{a^6} e^{-\beta\phi\sqrt{\kappa}} \\ 0 = \ddot{\phi} + 3\mathcal{H}\dot{\phi} + \beta\sqrt{\kappa}\frac{N^2}{a^6} e^{-\beta\phi\sqrt{\kappa}} \end{cases} \tag{5.8}$$

where $\mathcal{H} \equiv \frac{\dot{a}}{a}$ represents the Hubble parameter.
For completeness, it is straightforwardly compute the on-shell action:

$$S_{\text{on-shell}} = 2\pi^2 \int d\tau \left( + \frac{2N^2 e^{-\beta\phi\sqrt{\kappa}}}{a^2} \right) . \tag{5.9}$$

At this point, the metric perturbation can be introduced, generally expressed in terms of scalar, vector, and tensor components. Thus, the perturbed Friedmann-Robertson-Walker metric reads

$$\begin{aligned} ds^2 &= \left( g_{\mu\nu} + \delta g_{\mu\nu} \right) dx^\mu dx^\nu = \\ &= a^2(\tau) \left[ (1 + 2A(\tau, x^i)) d\tau^2 + 2B_i dx^i d\tau + \left( \gamma_{ij} + h_{ij} \right) dx^i dx^j \right] , \end{aligned} \tag{5.10}$$

where the perturbations are $A$, $B_i$, and $h_{ij}$, with $\gamma_{ij}$ denoting the induced three-dimensional metric.
In this metric, the decomposition into scalar, vector, and tensor components is expressed as:

$$\begin{cases} B_i = \partial_i B + \tilde{B}_i \\ h_{ij} = -2\psi\gamma_{ij} + 2\partial_i\partial_j E + 2\partial_i \tilde{E}_j + \tilde{h}_{ij} \end{cases} \tag{5.11}$$

yielding four scalar perturbations ($A$, $B$, $\psi$, and $E$), two vector perturbations ($\tilde{B}_i$ and $\tilde{E}_j$), and one tensor perturbation ($\tilde{h}_{ij}$).

However, these three types of perturbations are independent at the first order. Moreover, vector and tensor perturbations cannot trigger instabilities as they are not sourced by the scalars of the theory. In fact, these two types of perturbations generally do not compromise stability; rather, they can contribute positively.[6]

[6] In [103], stability under tensor fluctuations around a wormhole solution was explicitly computed,

Therefore, the analysis can be limited to scalar perturbations, leading to the metric ansatz:

$$ds^2 = a^2(\tau)\left[(1+2A(\tau,x^i))d\tau^2 + (1-2\Psi(\eta,x^i))d\Omega_3^2\right] \ , \tag{5.12}$$

commonly referred to as the conformal Newton gauge.[7]

The fields are also expanded around the background wormhole solution:

$$\begin{cases} \phi = \phi(\tau) + \Phi(\tau, x^i) \\ H_{0ij} = 0 + \epsilon^N_{ijk}\delta^{kl}\partial_l W(\eta, x^i) = \frac{q}{a}\epsilon^T_{ijk}g^{kl}\partial_l W(\eta, x^i) \\ H_{ijk} = q\epsilon^N_{ijk}(1+T(\eta,x^i)) = \frac{q}{\sqrt{g(3)}}\epsilon^T_{ijk}(1+T(\eta,x^i)) \end{cases} \tag{5.13}$$

which introduces two additional scalar perturbations for the axion and a third one for the saxion.

The previous perturbation ansatz can now be plugged into the action to obtain an expansion like: $S(\varphi+\delta\varphi) = S(\varphi) + \delta^2\varphi\frac{\delta^2 S}{\delta\varphi^2} + \textit{higher orders}$. By integrating by parts[8] and using the equations of motion, the following second-order correction is obtained:

$$\begin{aligned} \delta S^{(2)} \equiv \sum_{i,j}\delta\varphi_i\frac{\delta^2 S}{\delta\varphi_i\delta\varphi_j}\delta\varphi_j = \int d^4x\sqrt{\gamma}(a)\Bigg[ & A^2\left(\frac{3}{\kappa} - \frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right) \\ & - 3\Psi^2\left(\frac{1}{\kappa} + \frac{3N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right) + 6A\Psi\left(\frac{1}{\kappa} - \frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right) + \frac{1}{\kappa}(2A-\Psi)\nabla^2\Psi \\ & + \left(\frac{3a^2}{\kappa}\right)\dot{\Psi}^2 + \left(\frac{6a\dot{a}}{\kappa}\right)A\dot{\Psi} + \Phi^2\left(+\frac{\beta^2\kappa N^2}{2a^4}e^{-\beta\phi\sqrt{\kappa}}\right) + A\Phi\left(+\beta\sqrt{\kappa}\frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right) \\ & - \left(\frac{a^2}{2}\right)\dot{\Phi}^2 + \frac{1}{2}\Phi\nabla^2\Phi + \left(a^2\right)\dot{\phi}A\dot{\Phi} + \left(\frac{6\beta\sqrt{\kappa}N^2}{a^4}\right)e^{-\beta\phi\sqrt{\kappa}}\Psi\Phi - \left(3a^2\right)\dot{\phi}\dot{\Psi}\Phi \\ & - \left(\frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right)(T^2 + W\nabla^2 W) - \left(\frac{2N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right)T(A+3\Psi-\beta\sqrt{\kappa}\Phi)\Bigg] \ , \end{aligned} \tag{5.14}$$

where $\gamma$ denotes the three-dimensional induced metric, and $\nabla$ its covariant derivative.

However, it is possible to rewrite equation (5.14) to simplify the problem.
Following the standard procedure[9], it is first useful to define the canonical conjugate momenta, $\Pi_\chi \equiv \frac{\delta S}{\delta\dot{\chi}}$, for each fluctuation variable. This will allow to integrate them out in

yielding a positive spectrum. The same result was confirmed in [10] for the dilaton-saxion model.

[7]Compared to expression (5.10), here it is switched to polar coordinates, using $\gamma_{ij}dx^i dx^j = d\Omega_3^2$.

[8]Adopting the Dirichlet boundary conditions discussed in section 3.4, it is allowed to perform this integration without adding extra terms.

[9]Described in details in [103].

the next step and thereby reduce the degrees of freedom in $S^{(2)}$.
With this procedure, it is possible to write the action fluctuations in the following Hamiltonian form:

$$S^{(2)} = \int d^4x\sqrt{\gamma}\left(\sum_i \Pi_{\varphi_i}\dot{\varphi}_i - h(\varphi, \Pi_\varphi) + \theta_i C_i\right) , \tag{5.15}$$

where the last term contains Lagrange multipliers $\theta_i$ that can be integrated out to yield constraints $C_i = 0$, thereby reducing the number of variables. This technique eliminates some terms from $S^{(2)}$.

The conjugate fields are given by:

$$\begin{cases} \Pi_\Phi = \sqrt{\gamma}a^3\left(A\dot{\phi} - \dot{\Phi}\right) \\ \Pi_\Psi = \sqrt{\gamma}a^3\left(6\frac{\dot{\Psi}}{k} - \frac{6\mathcal{H}}{k}A + 3\dot{\phi}\Phi\right) \\ \Pi_{T,A,W} = 0 \end{cases} \tag{5.16}$$

such that it is possible to invert them and substitute into (5.14) to obtain an equation analogous to (5.15). It is important to note that non-dynamical fields (which don't have a kinetic term in the action) do not give any constraint, and so their degrees of freedom are not physical and will then be removed. Thus, the initial five fluctuation variables contained in equation (5.14) can be reduced to only two dynamical ones, i.e., $\Phi$ and $\Psi$. Furthermore, these two remaining variables can be combined into a single gauge-invariant variable.

However, the previous procedure is not convenient as it requires some non-trivial calculations. Indeed, it is more convenient to first apply gauge transformations to change variables and thus work with (a smaller number of) gauge-invariant variables. This approach is commonly used in cosmology and particle physics, as it eliminates gauge redundancy, which could otherwise complicate the analysis.
Under the following gauge transformation[10]

$$\tau \to \tau + \lambda(\tau) , \tag{5.17}$$

the variable transformations are easily computed by applying the definition of Lie deriva-

[10]The gauge transformation is defined only with respect to the time variable. The motivation is explained later in the first comment of 5.4.

tives[11] as follows:

$$\begin{cases} A \to A + a\dot{\lambda} + a\mathcal{H}\lambda \\ \Psi \to \Psi - a\mathcal{H}\lambda \\ \Phi \to \Phi + a\dot{\phi}\lambda \end{cases} \tag{5.18}$$

At this point, an important consideration is needed before continuing with the calculation. The action perturbation contains terms with spatial derivatives (specifically, covariant derivatives, $\nabla$). However, a significant result in cosmological perturbation theory[12] allows neglecting inhomogeneous fluctuations. Indeed, if the system is stable when considering the homogeneous sector of the fluctuations, then inhomogeneous fluctuations can only increase the system's energy, enhancing stability. For this reason, analyzing the homogeneous sector is sufficient if it yields a stable solution; otherwise, inhomogeneous perturbations would need to be included in the analysis.

Thus, the action fluctuations is restricted to the homogeneous case by assuming $\nabla\varphi = 0$ for $\varphi = \Psi, \Phi, W$.

Additionally, applying the gauge transformation (5.17) shows that the combination $\left(\dot{T} - \nabla W\right)$ is gauge-invariant, and the Bianchi identity[13] of the three-form sets this combination to zero. It follows that, when restricting to homogeneous perturbations, the Euclidean-time derivative $\dot{T}$ vanishes, making the perturbation $T$ a constant term, which is negligible in the analysis.

Therefore, the significant terms of the second derivative are as follows:

$$\begin{aligned} S^{(2)} = \int d^4x\sqrt{\gamma}(a)\Bigg[ & A^2\left(\frac{3}{\kappa} - \frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right) - 3\Psi^2\left(\frac{1}{\kappa} + \frac{3N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right) \\ & + 6A\Psi\left(\frac{1}{\kappa} - \frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right) + \left(\frac{3a^2}{\kappa}\right)\dot{\Psi}^2 + \left(\frac{6a^2\mathcal{H}}{\kappa}\right)A\dot{\Psi} \\ & + \Phi^2\left(\frac{\beta^2\kappa N^2}{2a^4}e^{-\beta\phi\sqrt{\kappa}}\right) + A\Phi\left(\beta\sqrt{\kappa}\frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right) - \left(\frac{a^2}{2}\right)\dot{\Phi}^2 \\ & + \left(a^2\right)\dot{\phi}A\dot{\Phi} + \left(\frac{6\beta\sqrt{\kappa}N^2}{a^4}\right)e^{-\beta\phi\sqrt{\kappa}}\Psi\Phi - \left(3a^2\right)\dot{\phi}\dot{\Psi}\Phi\Bigg] \end{aligned} \tag{5.19}$$

Now, a new variable, which will later be interpreted as a kinetic factor, can be introduced:

$$\mathcal{Q} \equiv \frac{1}{a^2} - \frac{kN^2}{3a^6}e^{-\beta\phi\sqrt{k}} = \mathcal{H}^2 - \frac{k}{6}\dot{\phi}^2\,, \tag{5.20}$$

[11]The Lie derivative for the metric tensor, respect to an infinitesimal vector $v$, is defined as: $\mathcal{L}_v g_{\mu\nu} \equiv v^\sigma\partial_\sigma g_{\mu\nu} + g_{\mu\sigma}\partial_\nu v^\sigma + g_{\nu\sigma}\partial_\mu v^\sigma$.

[12]See the articles [103, 105, 106, 108, 109, 112, 115, 116] for detailed information on this topic and other applications of perturbation theory within the GR framework.

[13]See the footnote "18" before equation (3.27) for the definition of the Bianchi identity.

where in the last step it is substituted the equation of motion[14].
Now, the two degrees of freedom of the system, corresponding to the two dynamical fields $\Phi$ and $\Psi$, can be translated in the following two gauge-invariant[15] variables (typically used in cosmology):

$$\alpha = A + \frac{\dot{\Psi}}{H} + \left(\frac{\dot{H}}{H\dot{\phi}}\right)\Phi \tag{5.21}$$

and

$$\mathcal{R} = \Psi + \Phi\frac{H}{\dot{\phi}} :, \tag{5.22}$$

where $\alpha$ is called the "lapse" function, and $\mathcal{R}$ is a curvature perturbation variable.

By changing variables and performing the required algebraic manipulations[16], the fluctuation action can be rewritten as:

$$\delta S^{(2)} = \int d^4x\sqrt{\gamma}a^3\Bigg[\frac{3\mathcal{Q}}{k}\alpha^2 - \frac{\dot{\phi}^2}{2H^2}\dot{\mathcal{R}}^2 + \frac{\dot{\phi}^2}{H}\dot{\mathcal{R}}\alpha - 6\left(\frac{1}{k} - \frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right)\mathcal{R}\alpha + \\ + \left(3\frac{\beta\sqrt{k}\dot{\phi}N^2}{Ha^4}e^{-\beta\phi\sqrt{\kappa}} + 3(3 - \frac{2}{H^2})(\frac{1}{k} - \frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}})\right)\mathcal{R}^2\Bigg]. \tag{5.23}$$

The newly introduced lapse variable $\alpha$ is a non-dynamical spectator field. As a result, it is unnecessary to employ a Hamiltonian formalism for its treatment; instead, the variable can be directly eliminated by substituting its equation of motion, $\frac{\delta S}{\delta\alpha} = 0$, which gives:

$$\alpha = \left(1 - \frac{kN^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right)\frac{\mathcal{R}}{\mathcal{Q}} - \frac{k\dot{\phi}^2}{6H}\frac{\dot{\mathcal{R}}}{\mathcal{Q}} \tag{5.24}$$

Substituting this expression into the fluctuations in equation (5.23) and performing further algebraic simplifications leads to a compact result for the action perturbation, expressed entirely in terms of the curvature perturbation $\mathcal{R}$:

$$\delta S^{(2)} = \frac{1}{2}\int d^4x\sqrt{\gamma}a^3\left[\frac{\dot{\phi}^2}{\mathcal{Q}}\dot{\mathcal{R}}^2 + \mathcal{R}^2\left(-\frac{8\kappa}{3}\frac{\dot{\phi}^2}{\mathcal{Q}^2a^4} - 4\sqrt{\kappa}H\beta\frac{\dot{\phi}}{\mathcal{Q}^2}\frac{N^2}{a^8}e^{-\beta\phi\sqrt{\kappa}}\right)\right] \tag{5.25}$$

The eigenvalue equation, similar to the form in (5.6), is obtained by defining the effective potential as:

$$U \equiv \left(\frac{2}{a^4Q^2}\right)\left(-\frac{4\kappa}{3}\dot{\phi}^2 - 2\sqrt{\kappa}H\beta\dot{\phi}\frac{N^2}{a^4}e^{-\beta\phi\sqrt{\kappa}}\right) \tag{5.26}$$

[14]To be precise, the second one of (5.8).
[15]The gauge-invariance property is straightforwardly proved applying the transformations of (5.18).
[16]This step also involves using the equations of motion from (5.8) to simplify certain terms.

Thus, the second-order perturbation can be written in its final compact form:

$$\delta S^{(2)} = \frac{1}{2}\int d^4x\sqrt{\gamma}a^3\left[\frac{\dot{\phi}^2}{\mathcal{Q}}\dot{\mathcal{R}}^2 + U\mathcal{R}^2\right] . \tag{5.27}$$

## 5.4 The stability analysis

Before addressing the homogeneous Schrödinger-like equation derived previously, two fundamental points require clarification.

First, a precise treatment of second-order action perturbations requires decomposing each field as a sum of hyper-spherical harmonic functions, as follows:

$$\varphi(\tau, \vec{x}) = \sum_{k\geq 0}\varphi_k(\tau)Y_k(\vec{x}) , \tag{5.28}$$

where the spherical harmonic functions satisfy the eigenvalue equation $\nabla Y_j = -\lambda_j Y_j$ (with the eigenvalue degeneracy given by $\lambda_j = j(j+1)$). This approach reduces the problem to a one-dimensional eigenvalue equation for each mode, leading to the suppression of the label $j$ for notational simplicity.
As previously discussed in the calculation of perturbations, the modes of interest are the homogeneous ones (corresponding to $j = 0$). Non-homogeneous terms have been neglected, as their contributions do not affect the stability as shown in [105, 108]. Moreover, all the negative modes found studying wormhole stability in the past, were related to the homogeneous sector. Physically, this simplification reflects the $O(4)$ rotational symmetry imposed on the system.

A second essential aspect to address concerns the boundary conditions.
The calculations of the previous section followed the approach used in the model discussed in chapter 3, employing the Dirichlet boundary conditions outlined in section 3.4. The condition for the curvature variable translates into $\dot{\mathcal{R}}(\tau_{\text{inf}}) = 0$ and $\dot{\mathcal{R}}(\tau_m) = 0$, which straightforwardly follow from the boundary conditions on the dilaton and the scale factor. The approach adopted here improves upon the method used in [9], where the same result was obtained only by directly solving the eigenvalue equation.
Moreover, this choice is particularly advantageous, as it eliminates the need for additional terms in the second-order action fluctuations and facilitates integration by parts without

boundary complications. In contrast, adopting the scalar-axion formalism would require modifying the boundary conditions through the (Neumann) Fourier transformation of Dirichlet ones. Such an alternative approach introduces boundary fluctuations that must be included in the analysis, as shown in [10].

Finally, the result from the previous section can be analyzed. The eigenvalue equation is derived by integrating by parts equation (5.27) and introducing a test function $z(\tau)$, which is necessary to obtain a proper eigenvalue equation:

$$a^3\left(\frac{\dot{\phi}^2}{\mathcal{Q}}\dot{\mathcal{R}}^2+U\mathcal{R}^2\right)=-\frac{d}{d\tau}\left(\frac{a^3\dot{\phi}^2}{\mathcal{Q}}\dot{\mathcal{R}}\right)\mathcal{R}+a^3U\mathcal{R}^2=\lambda a^3 z\mathcal{R}^2\,. \tag{5.29}$$

The equation can be rewritten in a simplified form as

$$-\frac{1}{a^3}\frac{d}{d\tau}\left(\frac{a^3\dot{\phi}^2}{\mathcal{Q}}\dot{\mathcal{R}}\right)+U\mathcal{R}=\lambda z\mathcal{R}\,. \tag{5.30}$$

By expanding the derivative, it follows:

$$\ddot{\mathcal{R}}\left(\frac{-\dot{\phi}^2}{\mathcal{Q}}\right)+\dot{\mathcal{R}}\left(\frac{-2\dot{\phi}\ddot{\phi}}{\mathcal{Q}}-\frac{3H\dot{\phi}^2}{\mathcal{Q}}+\frac{\dot{\phi}^2\dot{\mathcal{Q}}}{\mathcal{Q}^2}\right)+\left(U-\lambda z\right)\mathcal{R}=0\,, \tag{5.31}$$

which can be solved numerically to explicitly determine all eigenvalues $\lambda$.

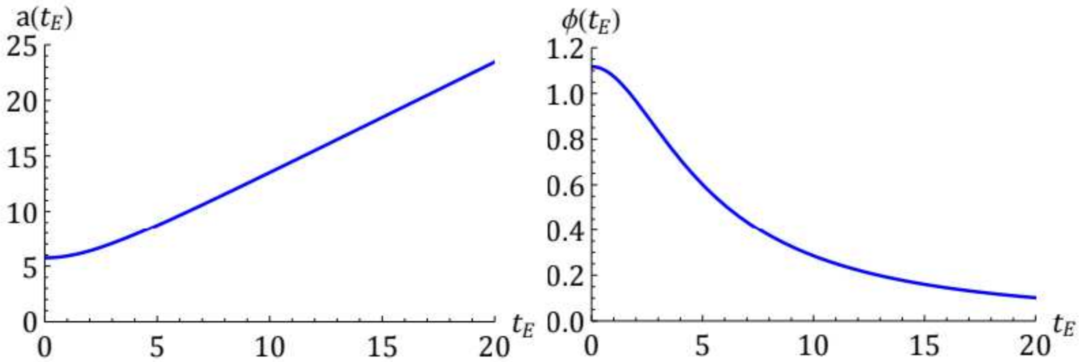



**Figure 5.2:** The left panel shows the profile of the universe's scale factor $a$ as a function of (Euclidean) time, while the right panel presents the dilaton field as a function of the same parameter. Both functions correspond to the wormhole solution analyzed in this study, with $\beta = 1$ for convenience.

A more efficient method to evaluate the stability of the wormhole can be employed. Using the explicit results from section 3.2 for the wormhole solution and the associated dilaton profile:

$$\begin{cases} a^2(\tau) = a_0^2 \cdot \cosh(2\tau) \\ \phi = \frac{1}{\sqrt{\kappa\beta}} \log\left[\frac{N^2}{3a_0^4}\cos^2\left(\sqrt{\frac{3}{2}}\frac{\beta}{2}\arccos\left(\frac{1}{\cosh(2\tau)}\right)\right)\right] \end{cases} \tag{5.32}$$

where the dilaton is obtained by inverting equation (3.52); instead, the scale factor constant is determined using equations (3.49) and (3.50), yielding $a_0 = \frac{N\sqrt{\kappa}}{\sqrt{3}}\cos\left(\frac{\pi}{4}\sqrt{\frac{3}{2}}\beta\right)$. These functions are shown in Figure 5.2, setting for concreteness $\beta = 1$.

It can thus be demonstrated that the two operators in equation (5.27) are well-defined along the wormhole and are strictly positive.
For a wormhole whose mouth is located at $r_m = 0$ and which asymptotically reaches infinity in $\tau$, the wormhole solution can be substituted into the potential $U$ and the kinetic factor $\mathcal{Q}$ to plot their respective functions, as shown in Figure 5.3.

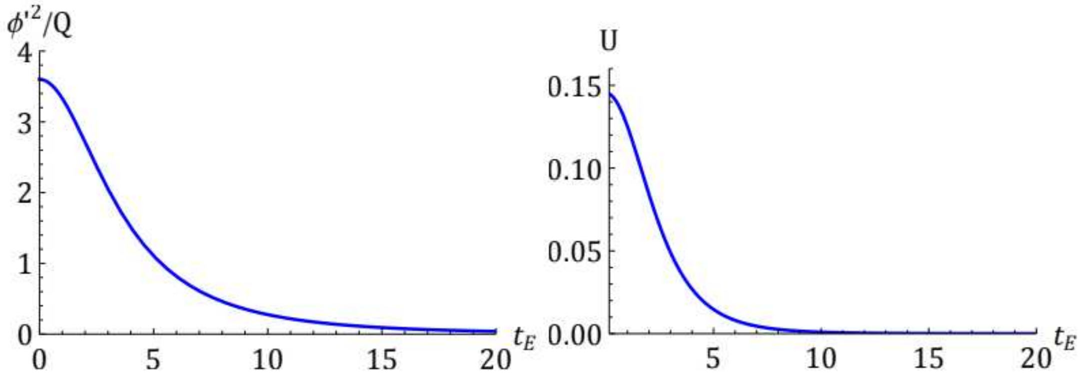



**Figure 5.3:** The two plots show the functions of the kinetic and effective potential factors of equation (5.27), respect to Euclidean time.

The result is clear: both solutions are positive, indicating that the eigenvalue equation cannot admit any negative value. Nevertheless, it is essential to carefully verify whether the two functionals are well-defined in both the asymptotic and wormhole mouth limits, as the eigenvalue equation might otherwise be ill-defined in these regions.
Considering the limit $\tau \to 0$, the Taylor expansion of equation (5.52) provides the behavior of the scale factor near the wormhole mouth:

$$a(\tau) = a_0 + \frac{\tau^2}{a_0} + \mathcal{O}(\tau^4)\,. \tag{5.33}$$

The Friedman equation of motion[17] also determines the dilaton behavior near zero, as related to $a_0$ through equation (3.49)[18].

[17]The second equation of (5.8).

[18]Recall that $a_0$ and $C$ are constants associated with the scale factor $a$, expressed in different coordi-

Consequently, the dilaton behaves as:

$$\phi(\tau) = \phi_0 - \left( \frac{\beta\sqrt{\kappa}}{2} \frac{N^2}{a_0^6} e^{-\beta\sqrt{\kappa}\phi_0} \right) \tau^2 + \mathcal{O}(\tau^4) \,. \tag{5.34}$$

Using these expansions, the behavior of $\frac{\dot{\phi}^2}{\mathcal{Q}}$ can also be calculated, demonstrating that it is well-defined in this limit. Substituting the previous expansions into equation (5.20) leads to an additional expansion in terms of $\tau^2$, yielding a regular result.

Conversely, the asymptotic limit $\tau \to \infty$ can be qualitatively analyzed. By expanding equation (5.32) in this regime, the approximate behavior is:

$$\begin{cases} a(\tau) \approx \tau \\ \phi(\tau) \approx 0 \end{cases} \tag{5.35}$$

Additionally, from their definitions, the behaviors of the potential and kinetic factors can be deduced:

$$\begin{cases} \mathcal{Q}(\tau) \approx \frac{1}{a^2} \approx \frac{1}{\tau^2} \\ U(\tau) \approx \frac{\dot{\phi}}{\mathcal{Q}} \cdot const \end{cases} \tag{5.36}$$

These approximations indicate that the ratio $\frac{\dot{\phi}^2}{\mathcal{Q}}$ tends to zero, potentially invalidating the eigenvalue equation.
However, a subtle adjustment resolves this issue. By considering the extended version of equation (5.31) and isolating the term $\frac{\dot{\phi}^2}{\mathcal{Q}}$, the equation becomes:

$$-\ddot{\mathcal{R}} + \dot{\mathcal{R}} \left( \frac{-2\ddot{\phi}}{\dot{\phi}} - 3H \, + \frac{\dot{\mathcal{Q}}}{\mathcal{Q}} \right) + (U - \lambda z) \left( \frac{\mathcal{Q}}{\dot{\phi}^2} \right) \mathcal{R} = 0 \,, \tag{5.37}$$

where the only one problematic term involves the weight function $z$. This function is arbitrary, and setting $z \equiv \frac{\dot{\phi}^2}{\mathcal{Q}}$ modifies the eigenvalue equation, ensuring it is now well-defined asymptotically.

This final verification of the wormhole mouth and asymptotic Euclidean-time limits concludes the stability analysis, demonstrating that the eigenvalue equation is well-defined and positive-definite. It follows that no perturbative mode can lower the action-energy of the wormhole solution, confirming its status as a stable absolute minimum of the model. For this reason, its physical contribution must be considered in the partition function within the semi-classical approximation.

nates. These constants are equivalent and can be easily transformed into one another.

# Chapter 6

# Conclusion and future prospects

This thesis focused on the topic of Euclidean wormholes, which are non-perturbative solutions to the equations of motion in the context of Axiverse models, containing scalar fields minimally coupled to gravity.

The thesis had two main goals. The first was to introduce the central topic and highlight the most significant frameworks where such solutions arise. To this end, digressions on string theory compactifications and the Swampland program were provided, along with comments on particle physics and associated phenomenological issues. Furthermore, the practical effects induced by the presence of wormhole solutions in the theory were examined, particularly the puzzle surrounding the interpretation of the $\alpha$ parameters.

The second and most important goal was to explore the stability of wormhole solutions. Indeed, the formal structure of Quantum Field Theory requires certain approximations to extract quantitative predictions from a given model. For this reason, it is crucial to determine whether, in addition to standard solutions to the equations of motion studied with perturbative approaches, non-perturbative ones should also be considered. Mathematically, this involves analyzing the stability of these solutions to establish whether they contribute physically meaningful effects.

Specifically, stability was studied within an Axiverse model including asymptotically flat gravity, a scalar field called the saxion motivated by UV string theory compactifications, and a three-form field equivalent to an axion. The stability analysis was performed by calculating the second-order action perturbation, achieved by expanding both the fields and the metric, after defining small fluctuations around the wormhole background configurations. By removing non-physical degrees of freedom and exploiting the gauge freedom of the model, the problem was reduced to a one-dimensional Schrödinger-like equation involving a single gauge-invariant curvature variable $\mathcal{R}$.

It was then demonstrated that no negative fluctuation modes are present in the model under consideration, showing that the wormhole solution is a stable minimum of the action, providing a dominant real contribution to the corresponding path integral. This analysis aligns perfectly with the results of [9] and [10], which were published during the development of this work.
However, as emphasized in the final chapter, it was also shown that the same results could be achieved using only the Euclidean formalism, where wormhole solutions naturally arise, and without requiring explicit numerical calculations of the modes' eigenvalues. This proves that the stability of wormhole solutions can be established through a simple yet rigorous analysis.

Despite the significant results achieved, it is important to note that the investigation was limited to a specific case. Therefore, further generalizations should be explored in the future. There are several ways to extend the results of this work by generalizing to different models. For instance, it would be interesting to analyze scenarios involving multiple saxions, massive saxions, or the inclusion of a potential term in the action. Additionally, one could consider models with a cosmological constant in a de Sitter (dS) or anti-de Sitter (AdS) curved spacetime.
While wormhole stability is expected to hold in these alternative setups, explicit calculations are necessary to confirm this feature in generalized frameworks.

In conclusion, wormhole saddle points are argued to be physically relevant due to their stability. This implies that understanding their low-energy implications is crucial, especially regarding the $\alpha$ parameter puzzle and its associated consequences. A deeper understanding of wormhole physics would also represent a concrete step forward in the pursuit of Quantum Gravity, offering new insights into the fundamental mechanisms of nature.

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