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Non-invertible continuous global symmetries and their symmetry theory from type II string theory

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Contents

Abstract

Generalized global symmetries are ubiquitous in quantum field theory (QFT) models, which describe the physics of particle interactions or low-energy matter phases. The generalization comes from extending the notion of charge conservation to the existence of topological (extended) operators in the theory. Generalized symmetries are important because they provide extra selection rules, which have been ignored until very recently. Non-invertible symmetries, which do not form a mathematical group, are one of such generalizations. For instance, they are present in Maxwell theory, Maxwell theory coupled to axions, and quantum electrodynamics (QED). They can be of discrete and finite type or of continuous type. The goal of this thesis is to systematically derive the non-invertible symmetries structure of 4d and 3d field theories constructed from type II string theories. The symmetry structure of a given QFT is encoded by another field theory (Symmetry Theory) defined on a space, which has an extra dimension. The physical QFT lives at the boundary of the symmetry theory. The study of the topological operators in the bulk symmetry theory and how they behave, depending on the choice of boundary condition, provides the full symmetry structure. The thesis consists of studying type II string theories backgrounds and implementing a dimensional reduction to derive the symmetry theory. The explicit backgrounds are provided by a Calabi-Yau geometry called the Conifold and the product manifold consisting of the Conifold times a circle.

Introduction

Symmetries have always been a very meaningful concept in Physics, fundamental to develop new theories and also to characterize and distinguish between various models. The description of the natural phenomena in modern times is encoded in the formalism of quantum field theories, where symmetries are one of the most important features. Moreover, also from the phenomenological point of view, symmetries are very relevant, since they constitute guiding principles to produce outcomes that can be compared with experiments and contribute to validate a theory. Given all these motivations, we can see that try to find better descriptions of symmetries in order to achieve a deeper understanding of this concept is very useful.

In the last decade, the notion of symmetries has been generalized with outstanding results, through the definition of generalized global symmetries. With the work [1], it has been developed a framework that broadens the horizons of symmetries. The usual global symmetries are now called 0-form symmetries, since they act on local operators with zero dimension in the Hilbert space of a theory. One of the generalizations is given by the notion of higher-form symmetries, in which the charged objects under the symmetries are higher-dimensional operators, with support on lines, surfaces or hypersurfaces. The starting point is the usual way of characterizing symmetries through Noether theorem and conserved currents, where the action of a symmetry is encoded in unitary operators. They form group representations and are invariant under the time evolution, since they are defined as the exponential of the conserved charges. With higher-form symmetries, the notion of conservation is extended to the one of topological invariance of the symmetry generators. In this way, we can see that all the ordinary symmetries are described by topological defects, but the inverse is not true. This leads to other generalizations: we can use topological operators that do not belong to a group representation to define new symmetries. If we consider topological operators that do not admit an inverse we get the notion of non-invertible symmetries. In this work we will explore this kind of symmetries considering in particular higher-form non-invertible symmetries.

Given the fundamental relevance of symmetries, we need to develop the better tools in order to study and present them. In recent years, it has been introduced the framework of Symmetry Topological Field Theory, SymTFT, which aims to describe all the symmetries of a d-dimensional QFT, by using a theory with an additional dimension. If we consider a QFT defined on M_d, its SymTFT is a $(d+1)$-dimensional topological field theory on M_{d+1} with the requirement $\partial M_{d+1} = M_d$. In this picture, we can build what is called sandwich construction, considering the additional dimension of the SymTFT as valued in an interval at which extremes we have the two boundaries of M_{d+1}. One is the physical boundary in which lies the d-dimensional QFT and the other one is the topological boundary on which we impose the boundary conditions on the bulk fields. In this way we can determine which topological operators can be projected on the physical theory and so can become the generators of the symmetries we want to describe. A possible generalization of this structure is given by the concept of Symmetry Theory, SymTh, where the purpose is the same but we lose the requirement of having a topological theory in the bulk. Now the kinetic terms for the bulk fields are also important and in this way we managed to fully describe not only discrete symmetries but also continuous non-finite symmetries.

The aim of this work is to derive all the generalized continuous symmetries of theories in four and three dimensions, that can be constructed from type II superstring theories. In order to do so we build the SymTh in five and four dimensions respectively and we characterize the global symmetries through the study of the topological operators of the SymTh. The last ingredient of this work so is given by string theory, since it constitutes the UV origin of the SymTh that will be derived in an top-down approach.

In chapter 1, starting from the presentation of ordinary symmetries, we develop the formalism to describe generalized symmetries. We focus first on higher-form symmetries, giving an overlook of both continuous and discrete global symmetries and then we pass to the description of non-invertible symmetries. We provide some relevant examples that show how generalized symmetries are widely present in all the known QFTs, in particular we discuss them in Maxwell theory, QED and chiral QCD. In chapter 2 we present first the construction of the SymTFT and then the generalization to the SymTh. With some examples we can see how these structures can represent symmetries and anomalies of the physical QFTs that we put on the boundary.

Chapter 3 is devoted to outline some relevant aspects of string theory, in particular to delineate the low-energy actions of IIA and IIB superstring theories. We need to discuss at the beginning some aspects of the bosonic string and to understand how to determine the spacetime dimensions in which the string lives. Since the superstring requires $D = 10$ for consistency, we explain how to get rid of the extra dimensions using Kaluza-Klein compactification. In order to get to the five and four dimensional SymTh that we are interested in, we consider for the internal space a particular Calabi-Yau geometry given by the conifold, whose main features are underlined in the last part of the discussion.

In chapter 4 finally we arrive to the dense core of the work. In the first part, we restrict the type II supergravity actions to the flux sector and we present the dimensional reduction obtained expanding the fluxes in terms of the internal geometry. For type IIB the background is given by the near horizon conifold geometry in the first case and by the conifold times a circle in the second case. In order to apply the same procedure to type IIA theory we present how to apply the T-duality to pass from IIB to IIA theory. In the second part of the chapter we analyze the topological operators of the resulting Symmetry Theories, studying which defects can be projected on the boundary by imposing suitable boundary conditions. The different sets of boundary conditions, that we are allowed to fix for the fields, give rise to a variety of different boundary QFTs. The two main examples that we discuss are the 4d axion-Maxwell model as boundary QFT for the 5d SymTh and the 3d Goldstone-Maxwell model for the 4d SymTh. These two models, besides being relevant per se to describe axions and Goldstone bosons, present interesting examples of non-invertible global symmetries that deserve to be studied.

Chapter 1

Generalized Symmetries

Symmetries are a fundamental notion in physics, relevant in an amount of different fields, from high energy physics to condensed matter and useful for various purposes, from characterizing a quantum field theory and studying its behavior under the renormalization group flow, to giving selection rules in scattering processes that can be matched with experiments.

First of all it is necessary to understand that the kind of symmetries we address in this context are the global ones and not the local gauge symmetries. In fact, even though the fundamental forces of nature are described with local gauge interactions, we can see that gauge invariance is not an observable symmetry, but it's just an unphysical redundancy introduced to have locality and Lorentz invariance in relativistic QFTs. It can be easily shown because if in a theory with a gauge symmetry we compute correlation functions involving gauge non invariant operators, we get always zero. The gauge symmetry in fact relates the correlators up to an arbitrary local phase

$$\langle O'_1(x_1)...O'_n(x_n)\rangle = e^{i\alpha(x_1)q_1+\ ...\ +i\alpha(x_n)q_n}\,\langle O_1(x_1)...O_n(x_n)\rangle = 0\ , \tag{1.1}$$

so the only possibility to satisfy this equality is with vanishing correlation functions. On the contrary, if we consider a global symmetry, that for the generic operator O_k gives the transformation $O_k(x_k) \to e^{iq_k\alpha}\,O_k(x_k)$, with the parameter α fixed and globally defined, then the correlation functions can be written as

$$\langle O'_1(x_1)...O'_n(x_n)\rangle = e^{i\alpha\sum_k q_k}\,\langle O_1(x_1)...O_n(x_n)\rangle = \begin{cases} 0 & \text{if} \sum\limits_k q_k \neq 0 \\ \neq 0 & \text{if} \sum\limits_k q_k = 0 \end{cases} \tag{1.2}$$

So, since the expression can be non vanishing if the total charge of the operator inside the correlator is zero, we conclude that only global symmetries are observable, in the sense that they impose selection rules on the correlation functions. Moreover, the global symmetries are the only ones that can be spontaneously broken, since in order to have SSB of a gauge symmetry we would need a local non-invariant operator with a non vanishing vacuum expectation value and this is not possible according to (1.1). Global symmetries can also give powerful tools to constrain QFT dynamics, giving useful parameters to match UV and IR theories, for example with the notion of 't Hooft anomalies, that involve only global currents and are constant under the Renormalization Group flow.

Since the history of symmetries goes back a long time, there have been many considerations that try to generalize the notion of symmetries avoiding the prescription of Coleman-Mandula theorem [30]. This allows only the direct product of the Poincaré group with the internal symmetry group as the most general symmetries of an analytic scattering matrix. The two famous exceptions that generalize the Poincarè group are the conformal group and the supersymmetry.

The more recent concept of generalized global symmetries instead is the one introduced by D. Gaiotto, A. Kapustin, N. Seiberg and B. Willet in [1], that has led to another large class of symmetries that avoid Coleman-Mandula theorem. Despite seeming exotic at first, these are symmetries present in a lot of familiar theories and they give a new perspective on known phenomena leading to powerful new results.

The wide notion of generalized global symmetries [1]-[10], includes four different classes:

- higher-form symmetries, where the action of the symmetry operators is not only restricted to local objects but can be generalized to higher-dimensional charged operators like lines, surfaces and hypersurfaces;
- higher-groups, that are constituted by higher-form symmetries of different form degrees that act together in the theory and also mix each other [6];
- non-invertible symmetries, in which the symmetry operator has no inverse in the fusion algebra, that is no more the one of a group but needs to be generalized to a fusion category;
- subsystem symmetries, that involve mostly condensed matter systems like fracton models. The charges of these symmetries do not depend only on the topology of the manifold in which they are realized, but also on their shape and location, [26]-[27].

In the following we will analyze the first and the third type, focusing on examples and applications in well known models as Maxwell theory, QED and chiral QCD.

1.1 Higher-form symmetries

In the attempt of addressing the first kind of generalization we need to focus first on how we describe the ordinary global symmetries of a QFT, that we will call 0-form symmetries since they act on local 0-dimensional operators. In order to do this, we need to distinguish between continuous and discrete symmetries and we will see that the generalization to higher-form symmetries, [3]-[4], that can be performed naturally in the continuous case translates also in the discrete case where, in general, we do not have a conserved current.

1.1.1 Continuous global symmetries

Let's consider a d-dimensional QFT with action $S[\phi]$, function of the set of all dynamical fields, which is invariant under ordinary continuous global symmetry transformations given by the group $G^{(0)}$. The existence of a continuous global symmetry implies the presence of a conserved current because of Noether theorem [29]

$$\partial_\mu j^\mu(x) = 0 \tag{1.3}$$

In order to use a framework in which the generalization will come easy, we want to adopt the language of differential form, [28], so the current can be expressed by a 1-form and the conservation equation translates into $j_1 = j_\mu \, dx^\mu$ being co-closed

$$d * j_1 = 0 \tag{1.4}$$

The conserved current $*j_1$ can be integrated over a $(d-1)$-dimensional surface in order to define the conserved charge Q associated to the symmetry. In the familiar four-dimensional

Minkowskian case we know that the charge is conserved in time $\dot{Q} = 0$, because the integral that defines the charge is performed over all the *space* directions of spacetime. If we consider instead the Euclidean framework we see that the direction that will correspond to time, namely the one that would be Wick rotated, can be chosen arbitrarily, so the charge is defined integrating the current on a generic surface Σ_{d-1}. The result indeed is that the charge is invariant under any deformation and this means that it is topological.

$$Q(\Sigma_{d-1}) = \int_{\Sigma_{d-1}} *j_1 \tag{1.5}$$

We can always describe symmetries in terms of Euclidean field theories since all the states of Lorentzian field theories can be defined using Euclidean path integral.

The next step is then to define the action of the symmetry $G^{(0)}$ on the local charged operators of the theory $O_R(x)$, that transform in the representation R of the group symmetry. In general, the action of a symmetry on the associated Hilbert space is encoded in the unitary operator given by the exponential of the conserved charge, according to Wigner theorem. This operator can be generalized in the Euclidean theory to

$$U_g(\Sigma_{d-1}) = exp\Big(i\lambda \int_{\Sigma_{d-1}} *j_1\Big) \tag{1.6}$$

which is topological since Σ_{d-1} can be any $(d-1)$-dimensional surface. The operator $U_g(\Sigma_{d-1})$ is called symmetry defect operator and it satisfies the group multiplication law given by

$$U_{g_1}(\Sigma) \cdot U_{g_2}(\Sigma) = U_{g_1 g_2}(\Sigma) \tag{1.7}$$

The topological nature of the symmetry defect can be shown using the conservation of the current, in fact choosing two homotopic surfaces Σ_{d-1} and Σ'_{d-1}, that can be deformed one into the other by smooth deformations, we get:

$$U_g(\Sigma_{d-1}) \cdot U_{g^{-1}}(\Sigma'_{d-1}) = exp\left(i\lambda \int_{\Sigma_{d-1}} *j_1 - i\lambda \int_{\Sigma'_{d-1}} *j_1\right) = exp\Big(i\lambda \int_{M_d} d * j_1\Big) = \mathbb{1} \tag{1.8}$$

where the manifold M_d is such that $\partial M_d = \Sigma_{d-1} \cup \overline{\Sigma'}_{d-1}$, with $\overline{\Sigma'}_{d-1}$ that is Σ'_{d-1} with opposite orientation. The result of the relation (1.8) is that

$$U_g(\Sigma'_{d-1}) \cong U_g(\Sigma_{d-1}) \tag{1.9}$$

and this implies that the $U_g(\Sigma_{d-1})$ operator is topological. In order to derive the action of the symmetry operator on the charged operators, we need to consider the Ward identities, which are the analog of the classical laws of conservation in the QFTs and which give relations between correlation functions of the theory. Considering the partition function Z of a theory with action $S[\Phi]$ we compute the generic correlation function of one field as

$$\langle \phi(y) \rangle \equiv \frac{1}{Z} \int \mathcal{D}\Phi \; \phi(y) \; e^{iS[\Phi]} \tag{1.10}$$

where Φ represents all the fields of the theory. In this formalism it is possible to derive the Ward identities for the current j^μ of a 0-form symmetry in absence of anomalies

$$\partial_\mu \langle j^\mu(x)\phi(y) \rangle = -i\delta^{(d)}(x-y) \langle \delta\phi(y) \rangle \tag{1.11}$$

If we express it with differential forms by integrating over a d-dimensional region M_d with boundary Σ_{d-1} we get for the left hand side

$$\int_{M_d} \langle d * j_1\, \phi(y)\rangle = \int_{\Sigma_{d-1}} \langle * j_1\, \phi(y)\rangle = \langle Q(\Sigma_{d-1})\, \phi(y)\rangle \tag{1.12}$$

so (1.11) becomes the relation that gives the action of the symmetry on the field ϕ

$$\langle Q(\Sigma_{d-1})\, \phi(y)\rangle = -i \int_{M_d} d^d x\, \delta^{(d)}(x-y)\, \langle \delta\phi(y)\rangle = -i\ \mathrm{Link}(\Sigma, y)\, \langle \delta\phi(y)\rangle \tag{1.13}$$

We have defined the linking number of Σ_{d-1} and y as

$$\mathrm{Link}(\Sigma_{d-1}, y) = \int_{M_d} d^d x\, \delta^{(d)}(x-y) \tag{1.14}$$

that is 0 if the point y is outside of the region M_d and is 1 if y is inside, since in d dimensions a point can always link with a $(d-1)$-dimensional manifold. The linking number is a topological quantity since it does not depend on the shape of Σ_{d-1}, that can be deformed as long as in the changing we do not cross the point y.

If we translate the Ward identities (1.11) for a generic operator[1]:

$$\partial_\mu j^\mu(x) O_R(y) = \delta^{(d)}(x-y) R(T^a) O_R(y) \tag{1.15}$$

we can derive the action of the symmetry operator as

$$U_g(\Sigma_{d-1}) O_R(y) = R(g)\ O_R(y) U_g(\Sigma'_{d-1}) \tag{1.16}$$

where Σ_{d-1} is the surface that links the point y, while Σ'_{d-1} does not, so it is obtained with a smooth deformation that crosses the point, see Fig. 1.1. $R(g)$ is the realization of the symmetry group element g in the same representation of O_R, if we consider an algebra given by the generators T^a we get that the representation of g is $e^{i\lambda^a T^a}$.

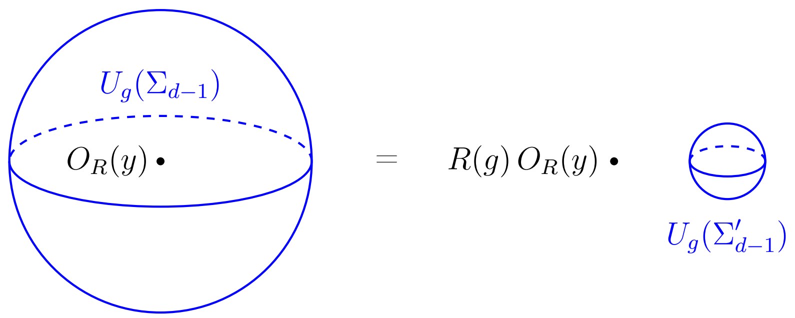

Figure 1.1: Example of linking of a local operator in d dimensions and of the action of the symmetry. After the deformation of Σ_{d-1} transforming O_R, the operator $U_g(\Sigma'_{d-1})$ can shrink to zero.

[1] We change also the bracket notation used before, omitting the vacuum expectation value.

The relation (1.16) can be computed explicitly as

$$\begin{aligned} U_g(\Sigma_{d-1})O_R(y)U_{g^{-1}}(\Sigma'_{d-1}) &= \exp\left(i\lambda^a \int_{M_d} d^dx\ \partial_\mu j^\mu(x)\right) O_R(y) \\ &= \sum_{n=0}^{\infty} \frac{(i\lambda^a)^n}{n!} \left(\int_{M_d} d^dx\ \partial_\mu j^\mu(x)\right)^n O_R(y) \\ &= \sum_{n=0}^{\infty} \frac{(i\lambda^a R(T^a))^n}{n!} \left(\int_{M_d} d^dx\ \delta^{(d)}(x-y)\right)^n O_R(y) \\ &= \sum_{n=0}^{\infty} \frac{(i\lambda^a R(T^a))^n}{n!}\ O_R(y) \\ &= R(g)\ O_R(y) \end{aligned} \tag{1.17}$$

in which we have expanded the exponential and inserted the Ward identities.

In general, the conserved current encodes the local features of a symmetry, such as Ward identities or 't Hooft anomalies, that are not naturally understandable at the level of the conserved charges. A way to keep these information accessible is to couple the current to a background gauge field by adding an appropriated term in the action. The adjective gauge to characterize the background field is just the usual name to address it, but should not be confusing, we are considering a field which is fixed, non-dynamical, so is not summed up in the path integral.

For an ordinary global symmetry, since the closed current $*j_1$ has codimension one, we need a 1-form background field, which is the connection of a $G^{(0)}$ bundle on spacetime. The action so contains a term where the field A_1 acts as a classical source for the current, so that the correlation functions with the current j_μ are given by $-\frac{\delta Z}{\delta A^\mu(x)}$, where $Z[A]$ is the partition function of the theory that depends on the background field.

$$S \supset i \int A_1 \wedge *j_1 \tag{1.18}$$

We can see that the action is invariant under small gauge transformations of the background $A_1 \to A_1 + d\lambda$ using the conservation of the current after the integration by parts. The introduction of the background field A_1 is useful because when we ask if a transformation is a symmetry, we can easily answer by checking if the partition function is invariant under the gauge transformations of A_1. In this way we get also a natural way to study anomalies.

Generalization to higher-forms

The natural generalization that gives rise to the notion of higher-form global symmetries is to consider a p-form symmetry given by a conserved current that is a $(p+1)$-form such that

$$d * J_{p+1} = 0 \tag{1.19}$$

This allows to define a conserved charge and the operator:

$$U_g(\Sigma_{d-p-1}) = e^{i\alpha \int_{\Sigma_{d-p-1}} *J_{p+1}} \tag{1.20}$$

which is topological and obeys a group multiplication law in the same way explained for ordinary symmetries. This operator encodes the action of the symmetry on p-dimensional objects $O_q(\Sigma_p)$ of charge q that can link with the $(d-p-1)$-dimensional surface

$$U_g(\Sigma_{d-p-1})O_q(\Sigma_p) = e^{i\alpha q}\ O_q(\Sigma_p)\, U_g(\Sigma'_{d-p-1}) \tag{1.21}$$

where Σ'_{d-p-1} is the deformed surface that does not link with Σ_p and α is the representation of the group element g in the Lie algebra.

An interesting property of higher-form symmetries is that they are always abelian and so only 0-form symmetries can be encoded by non-abelian groups. In fact to define a non-abelian symmetry we need a notion of ordering for the product of symmetry defect operators and this is clearly present for 0-form symmetries, for which the defects have codimension one. Since we have only one transverse direction, we cannot exchange the ordering with topological manipulation. Instead for higher form symmetries, the defects has always codimension bigger than one and as we see for example in Fig. (1.2) we can deform smoothly the defects and manage to exchange the ordering.

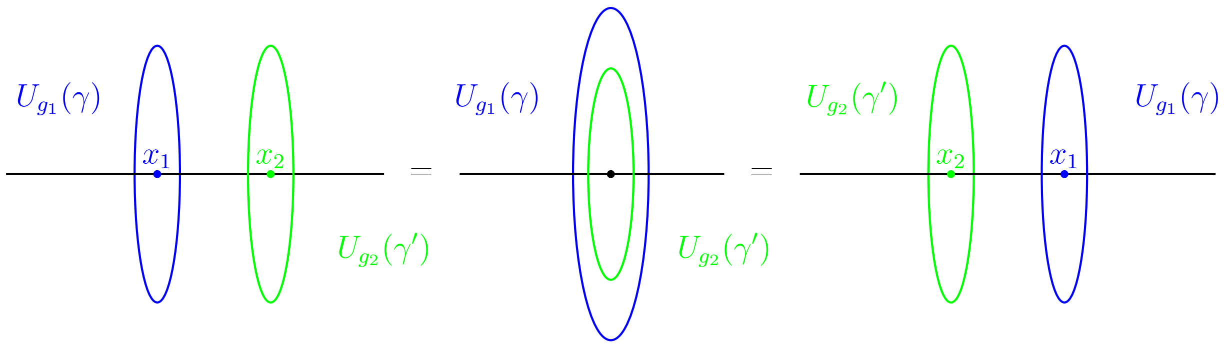

Figure 1.2: Higher-form symmetries can only be abelian: example of line defects in three dimensions, we can deform one line to exchange the order without crossing

Spontaneous symmetry breaking of higher-form symmetries

When a continuous global 0-form symmetry is spontaneously broken $G^{(0)} \to H^{(0)}$, this gives rise to an associated Goldstone boson that takes values in the Lie algebra of the coset space G/H. The symmetry is non-linearly realized for the Goldstone bosons as a shift symmetry along the broken directions. For a p-form global symmetry the Goldstone field associated with its SSB is a p-form since it must be shifted by the gauge parameter λ_p that comes from the background gauge field transformation $B_{p+1} \to B_{p+1} + d\lambda_p$.

The order parameter of the spontaneous symmetry breaking seen as a phase transition is given by the vacuum expectation value of the charged operator under the symmetry. Since the charge operator needs to act trivially on the vacuum state, then any charged operator should have vanishing vacuum expectation value if the symmetry is unbroken

$$0 = \langle 0|[Q, O(x)]|0\rangle = \langle 0|O(x)|0\rangle \tag{1.22}$$

Spontaneous symmetry breaking is also related to the notion of confinement, because if a charged operator O_p has a non trivial vacuum expectation value once inserted along a p-dimensional sphere of infinite radius

$$\lim_{R\to\infty} \langle O_p(S^p_R)\rangle \neq 0 \tag{1.23}$$

then the p-form symmetry is spontaneously broken and the operator survives in the IR, so it can be said that the operator deconfines in the IR. On the other hand if an operator has

$$\lim_{R\to\infty} \langle O_p(S^p_R)\rangle = 0 \tag{1.24}$$

then it does not exist in the IR and we can say that it confines. The terminology recall the more familiar one used in the context of quark confinement, the meaning is analogous since we speak about confinement of operators when they are not present once the theory flows in the IR. In the same way when the quarks cannot be observed as free particles but only inside hadrons at low-energy we get quark confinement. Summing up, the statement is that: if a charged p-dimensional operator deconfines, then the symmetry is spontaneously broken, while if all charged operators confine, then the symmetry is unbroken.

Restricting to the case of line operators, the distinction between these two situations is given by the perimeter or area law of the asymptotic vacuum expectation value. The area law for the vev implies that the operator confines, while instead the perimeter law implies deconfinement, since the operator can be modified by a counter term in order to explicitly satisfy the condition (1.23).

1.1.2 Anomalies

In order to enter in the discussion about anomalies, we need to distinguish between three types:

- ABJ anomalies, that occur when we have a transformation that is a symmetry of the classical lagrangian, but that is not a symmetry of the full quantum partition function since it does not leaves invariant for example the path integral measure. This anomaly depends on dynamical fields of the theory, so it is not constant under the RG flow and its presence implies that the transformation is *not* a global symmetry of the QFT.
- gauge anomalies, in which the path integral is not invariant under the gauge transformations of the dynamical gauge fields. This type of anomaly is not allowed in a consistent model since it makes the theory ill-defined.
- 't Hooft anomalies, that concern background fields of a theory and that are the ones on which we focus in this section

A p-form continuous global symmetry $G^{(p)}$ is said to have a pure 't Hooft anomaly when the transformations of the background field B_{p+1}, associated to the symmetry that acts also on the dynamical fields,

$$B_{p+1} \to B^g_{p+1} = B_{p+1} + d\lambda_p \tag{1.25}$$

do not leave the partition function invariant

$$Z[B_{p+1}] \neq Z[B^g_{p+1}] \tag{1.26}$$

and this inequality remains true even if we add to the partition function all the possible counter terms that depend on the background field B_{p+1}.

The 't Hooft anomalies constitute a powerful tool to study the renormalization group flow of the quantum field theories, since they are RG invariant quantities that constrain the ways in which symmetries are matched between the UV and the IR. For example the fact that a global symmetry is preserved tells us that terms that explicitly breaks the symmetry cannot flow to the IR.

In order to show the modern description of anomalies that uses the mechanism of anomaly inflow from a topological QFT, we begin introducing the concept of symmetry protected topological phase, SPT. A d-dimensional SPT phase of a set $\mathcal{S}$ of p-form global symmetries is an invertible d-dimensional topological QFT endowed with the symmetries $\mathcal{S}$, such that its partition function with all the background fields turned off is trivially equal to one. This means indeed that the SPT phase is like a trivial theory when symmetries are not considered.

Let's take a $(d+1)$-dimensional SPT phase in order to define an anomaly theory. Its partition function is invariant under all the gauge transformations of the background fields, so in the

SPT phase we do not have 't Hooft anomalies. However, if we consider the theory on an open manifold M_{d+1} that has boundary $M_d = \partial M_{d+1}$ we can see that the theory will present an anomaly. In fact, the partition function in which we perform the symmetry transformations is equal to the original one times a factor that is localized on the boundary M_d and that depends on the background fields and the parameters of their transformations.

$$Z[B_{p_i+1} + \delta\lambda_{p_i}] = Z[B_{p_i+1}] \times Z_\partial[B_{p_i+1}, \lambda_{p_i}] \tag{1.27}$$

This means that defining the SPT phase on an open manifold gives rise to a 't Hooft anomaly. This is convenient for our purposes, since if we take a d-dimensional QFT that lives on M_d and has the same symmetries of the SPT phase but presents also a 't Hooft anomaly given by

$$\mathcal{I}_d[B_{p_i+1}, \lambda_{p_i}] = Z_\partial^{-1}[B_{p_i+1}, \lambda_{p_i}] \tag{1.28}$$

we get that the the combined theory, given by the $(d+1)$-dimensional SPT phase and the d-dimensional boundary QFT, is anomaly free. So reasoning in reverse, given a d-dimensional QFT, its anomaly theory with partition function that we can call $\mathcal{I}_{d+1}[B_{p_i+1}]$ is given by the $(d+1)$-dimensional symmetry protected topological phase for which condition (1.28) is true. For a continuous symmetry the partition function of the anomaly theory can be seen as the exponential of an effective action

$$\mathcal{I}_{d+1}[B_{p_i+1}] = e^{2\pi i \mathcal{A}_{d+1}} \tag{1.29}$$

which is given by what is called the anomalous phase, that can be used to define the anomaly polynomial in $d+2$ dimensions

$$\mathcal{I}_{d+2} = d\mathcal{A}_{d+1} \tag{1.30}$$

The descent mechanism then can be used in order to come back to the 't Hooft anomaly in the d-dimensional theory. This is the use of the concept of anomaly inflow, [23], that exploit the decoupled $(d+1)$-dimensional theory to understand that the 't Hooft anomalies are invariant under the RG flow. It can be seen by the fact that the anomaly theory, as well as the anomaly polynomial, being topological are invariant flowing between the IR and the UV, though they can be used to match the anomaly of the d-dimensional QFT along the RG flow, [24].

1.1.3 Higher-form symmetries in Maxwell theory

The free Maxwell theory is the emblematic example of a model with higher-form symmetries. Let's start with the action of a compact $U(1)$ 1-form gauge field a in d dimensions

$$S[a] = \int -\frac{1}{2e^2} f \wedge *f \tag{1.31}$$

where $f = da$ is the field strength and the electric charge is quantized as a consequence of a being a compact $U(1)$ field. The extended operators of this theory are the Wilson lines

$$W_q(\gamma) = e^{iq\int_\gamma a} \tag{1.32}$$

in which q is the electric charge and γ must be infinitely long or closed in order to have an operator that is gauge invariant. This line operator represents the worldline of a probe charged particle not dynamical. From the equation of motion and the Bianchi identity we get two conservation equations for two currents that we can define as

$$\frac{1}{e^2}\, d * f = d * J_e = 0 \qquad\qquad df = \frac{1}{2\pi} d * (*f) = d * J_m = 0 \tag{1.33}$$

Therefore we see that the theory has two types of higher-form symmetries, a 1-form electric symmetry with the 2-form current J_e and a $(d-3)$-form magnetic symmetry with the $(d-2)$-form current J_m. The topological operators that realize these symmetries are given by the exponential of the conserved charges

$$\begin{aligned} U_e(\alpha, \Sigma_{d-2}) &= e^{i\alpha \int_{\Sigma_{d-2}} *J_e} = e^{i\alpha \int_{\Sigma_{d-2}} *f} \\ U_m(\beta, \Sigma_2) &= e^{i\beta \int_{\Sigma_2} *J_m} = e^{i\beta \int_{\Sigma_2} f} \end{aligned} \tag{1.34}$$

So in general the higher-form symmetries of a free Maxwell theory in d dimensions are

$$U(1)_e^{(1)} \times U(1)_m^{(d-3)} \tag{1.35}$$

and we can notice some special cases: in $d = 2$ there is no magnetic symmetry, while in $d = 4$ both electric and magnetic $U(1)$ are 1-form symmetries. In a generic dimension, the magnetic symmetry can be better characterized looking at the operators charged under it, let's define the dual gauge field $\tilde{a}$ as

$$*f = d\tilde{a} \tag{1.36}$$

where $\tilde{a}$ is a $(d-3)$-form that allows to construct the 't Hooft surface operators of charge m

$$T_m(\Gamma_{d-3}) = e^{i2\pi m \int_{\Gamma_{d-3}} \tilde{a}} \tag{1.37}$$

in which m is the magnetic charge and the support Γ_{d-3} is a $(d-3)$-dimensional surface. These gauge invariant extended objects are the charged objects under the magnetic symmetry, while the charged operators under the electric symmetry are the Wilson lines. The action of the topological symmetry defects on the charged operators is given by the non-trivial linking between their supports, Σ' is homotopic to Σ and does not link the support of the charged objects

$$\begin{aligned} \langle U_e(\alpha, \Sigma_{d-2})\ W_q(\gamma)\rangle &= e^{iq\alpha \ \mathrm{Link}(\Sigma_{d-2},\gamma)}\ \langle W_q(\gamma)\ U_e(\alpha, \Sigma'_{d-2})\rangle \\ \langle U_m(\beta, \Sigma_2)\ T_m(\Gamma_{d-3})\rangle &= e^{im\beta \ \mathrm{Link}(\Sigma_2,\Gamma_{d-3})}\ \langle T_m(\Gamma_{d-3})\ U_m(\beta, \Sigma'_2)\rangle \end{aligned} \tag{1.38}$$

Let's restrict now the study to the four-dimensional Maxwell theory, where we have that both the electrically and magnetically charged objects are of dimension one: Wilson lines and 't Hooft lines. In order to describe its global higher-form symmetries,

$$U(1)_e^{(1)} \times U(1)_m^{(1)} \tag{1.39}$$

the theory can be coupled to background gauge fields, which are 2-form fields, B_2^e and B_2^m, so the action becomes

$$S = \frac{1}{2e^2} \int (f - B_2^e) \wedge *(f - B_2^e) + \frac{i}{2\pi} \int B_2^m \wedge f \tag{1.40}$$

The coupling of B_2^m is the standard one with the current of the magnetic symmetry, instead in order to couple the other background field B_2^e to the electric current we need to introduce also a local counter term of the form $B_2^e \wedge *B_2^e$ that does not affect the dynamics of the theory since it depends only on the background field. This allows to have the kinetic term invariant under background gauge transformations, since f and B_2^e transform in the same way. The field strength in fact is not invariant since we are considering large gauge transformations $a \to a + \lambda_1^e$.

Coupling the currents to background gauge fields allows to naturally study the 't Hooft anomalies of the global symmetries of interests. The presence of a 't Hooft anomaly is encoded in the non-invariance of the partition function under background gauge transformations with all the background fluxes activated. In the case we are considering the action (1.40) is not invariant under the gauge transformations of the electric background field $B_2^e \to B_2^e + d\lambda_1^e$, because also the field a transforms non-linearly under it. The photon field a can indeed be seen as the Nambu-Goldstone boson of the $U(1)_e^{(1)}$, since a symmetry is spontaneously broken when is non-linearly realized for the Goldstone field, [20]. The order parameter of this SSB is given by the vacuum expectation value of the Wilson lines of a that becomes non trivial. If we apply the field transformations, the variation of the action is given by

$$\delta S = \frac{i}{2\pi} \int B_2^m \wedge d\lambda_1^e \tag{1.41}$$

which translates in the partition function as

$$Z[B_2^e + d\lambda_1^e, B_2^m] = e^{\frac{i}{2\pi} \int B_2^m \wedge d\lambda_1^e} \cdot Z[B_2^e, B_2^m] \tag{1.42}$$

This underlines that the gauging of the two 1-form symmetries of Maxwell theory cannot be done simultaneously. With the choice of counter terms in the action (1.40) we see that only the magnetic 1-form symmetry leaves the action invariant and so it can be gauged. It can be shown that there is not a possible choice of counter terms that makes the theory invariant under both symmetries.

In order to give a concrete example of the topic discussed in the paragraph of anomalies we can write the anomaly theory for the 4-dimensional Maxwell theory as

$$\mathcal{I}_{d+1} = e^{2\pi i \int B_2^e \wedge dB_2^m} \tag{1.43}$$

so the anomaly polynomial is given by

$$\mathcal{I}_{d+2} = \int dB_2^e \wedge dB_2^m \tag{1.44}$$

Matter fields and explicit breaking of higher-form symmetries

We consider now the Maxwell theory with a matter field ϕ of charge q under the gauge group $U(1)$, which can be in the scalar or spinor representation of the Lorentz group and can be massive or massless. The introduction of the matter field breaks the $U(1)$ 1-form symmetry of the pure Maxwell theory to its $\mathbb{Z}_q^{(1)}$ subgroup, since the presence in the theory of matter fields give rise to non genuine local operators that screens a charge q Wilson line, because the line it is allowed to end on the matter field, as shown in Fig. 1.3

Figure 1.3: A matter field gives a non genuine operator living at the end of a Wilson line

A non genuine operator is an operator which is attached to a collection of higher dimensional operators, [8]. We can see that the field ϕ need to be attached to something since it does not define a genuine operator, because it is not gauge invariant. Under a gauge transformation of the $U(1)$ gauge field

$$A(x) \to A(x) - \frac{d\theta(x)}{2\pi} \tag{1.45}$$

the matter field transform as

$$\phi(x) \to e^{iq\theta(x)}\phi(x) \tag{1.46}$$

The matter field changes with a phase in the same way, with opposite sign, as a Wilson line of charge q, that ends on the point x

$$W_q(L) = e^{2\pi i q \int_L A} \to e^{-iq\int_{\partial L}\theta}\ W_q(L) = e^{-iq\theta(x)} W_q(L) \tag{1.47}$$

since we have $\partial L = x$. Indeed combining together the matter field and the Wilson line we get a gauge invariant quantity that constitutes the non genuine operator

$$\phi(x) W_q(L) \tag{1.48}$$

This is a way to see the explicit breaking of an higher-form symmetry, in fact, if an operator O_p can be completely screened, namely there exists another operator O_{p-1} on which O_p can end, then O_p becomes invisible to the symmetry operators. For example, if we have a topological defect operator that links the Wilson line ending on a point, we can deform the defect in order to reach the ending of the line and unlink the operators without doing nothing to the Wilson line, as shown in Fig (1.4). In this sense the use of the term screening becomes clear.

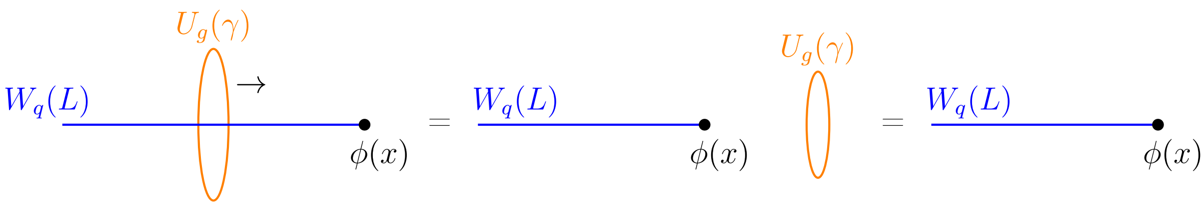

Figure 1.4: The Wilson lines that end on the local operator becomes invisible under the global $U(1)^{(1)}$ symmetry, since the defect can be smoothly moved and then shrink to zero.

1.1.4 Discrete global symmetries

Let's consider now the case of discrete symmetries that would be relevant in the introduction of the SymTFT in the next chapter. In general, for this type of symmetries we do not have a conserved current, but we can still use the tools developed so far, such as the background gauge fields associated to the symmetry and the topological symmetry defect operators.

A complete description of discrete global symmetries would require a presentation of discrete gauge theory, [11], where the better language encodes the discrete gauge fields as co-chains defined with respect to a triangulation on spacetime. A typical action for a discrete gauge theory in d-dimensions would be

$$2\pi \int a_p \cup \delta b_{d-p-1} \tag{1.49}$$

where $\cup$, known as the cup product, is the discrete version for cohomology classes of the wedge product. The field a_p is a $G^{(p)}$-valued p co-chain, which assigns an element of the group to every p-simplex of the triangulation and the field b_{d-p-1} is a $G'^{(p)}$-valued $(d-p-1)$ co-chain.

Using this framework it is possible to show how to build the symmetry operators and to show that they are topological using a discrete version of the Stokes theorem and the equations of motion coming from the action. However, we will not enter in the details of this language since it is something that goes beyond the aim of this work. What we do instead is to focus on a $\mathbb{Z}_N$ gauge theory, which is a special case since $\mathbb{Z}_N$ is a discrete subgroup of the continuous $U(1)$, so we can try to describe the $\mathbb{Z}_N$ gauge fields as the restriction of the $U(1)$ connections, requiring holonomies valued in $\mathbb{Z}_N$:

$$e^{i\oint_\gamma A} = e^{\frac{2\pi i n}{N}} \qquad n = 0, 1, ..., N-1 \tag{1.50}$$

This restriction on the holonomies implies that the connection is flat. In fact, since the values of the holonomies belong to a discrete group, one holonomy is invariant under a continuous deformation of γ, because we would need a discontinuous transformation to change the element of the discrete group. As a consequence if we calculate the holonomy along a trivial γ, that can be contracted to a point, the result should be trivial, but the holonomy can also be expressed as

$$e^{i \oint_\gamma A} = e^{\int_D dA} \tag{1.51}$$

considering the path as the boundary of a disk $\gamma = \partial D$. The field strength of A so must be trivial along cycles that are contractible to a point, namely we can say that the $\mathbb{Z}_N$ connection A must be locally flat, then the complete definition of the field strength of a discrete gauge theory would require the notion of Bockstein map in discrete cohomology.

As we said before, in order to describe the $\mathbb{Z}_N$ gauge field A we can use the associate $U(1)$ connection $\hat{A}$, denoted as integral lift, that should have the same holonomies as the original field and a period that matches the one of A $\mathrm{mod}_{\mathbb{Z}}$, so the different choices for the $U(1)$ gauge field are related by large gauge transformations. In general a $U(1)$ connection is not flat and we can use the correspondence between $U(1)$ and $\mathbb{Z}_N$ to get the same result one could get in cohomology

$$\frac{F}{2\pi} := \frac{d\hat{A}}{2\pi} \mod_N \tag{1.52}$$

BF theory

The most characteristic example of a discrete gauge theory is the BF theory, which describes $\mathbb{Z}_N$ p-form gauge fields by using the formalism of $U(1)$ gauge theories with suitable constraints. Let's write the action of a p-form $U(1)$ gauge field A_p and a $(d-p-1)$-form $U(1)$ gauge field B_{d-p-1}

$$S = \frac{iN}{2\pi} \int B_{d-p-1} \wedge dA_p \tag{1.53}$$

In the following we will see how this action is appropriated to describe a theory in which only $\mathbb{Z}_N$-valued gauge fields would contribute to the path integral, so the field strengths are flat and the Wilson lines have charges that take values in $\mathbb{Z}_N$.

First we notice that the theory is topological since the action does not contain the spacetime metric, namely the Hodge star operator does not appear in the expression (1.53). The equations of motion confirm the fact that there are no propagating degrees of freedom in the theory that is truly topological

$$N\frac{dA_p}{2\pi} = 0 \qquad\qquad N\frac{dB_{d-p-1}}{2\pi} = 0 \tag{1.54}$$

and describes flat gauge fields. Moreover, the action is invariant under the gauge transformations of the fields given by a $(p-1)$ and a $(d-p-2)$ gauge parameters respectively for A_p and B_{d-p-1}

$$A_p \to A_p + d\Lambda_{p-1} \qquad\qquad B_{d-p-1} \to B_{d-p-1} + d\Lambda_{d-p-2} \tag{1.55}$$

We can define gauge invariant Wilson operators with support on closed or infinitely extended manifolds:

$$W_n(\Gamma_p) = e^{in\int_{\Gamma_p} A_p} \qquad\qquad V_m(\tilde{\Gamma}_{d-p-1}) = e^{im\int_{\tilde{\Gamma}_{d-p-1}} B_{d-p-1}} \tag{1.56}$$

which are topological since the gauge fields are flat. The charges are $n, m = 1, ..., N-1$ because if we insert the Wilson operators in the partition function of the theory for example with a term like

$$in \int A_p \wedge \delta^{(d-p)}(x \in \Gamma_p) \tag{1.57}$$

we can see that this modifies the equation of motion of the field B_{d-p-1}, that becomes

$$N \frac{dB_{d-p-1}}{2\pi} = n\delta^{(d-p)}(x \in \Gamma_p) \tag{1.58}$$

So, for every Wilson operator with charge $n \geq N$ we can perform a gauge transformation of the field B_{d-p-1} that reduces the charge n dividing by multiples of N.

Another proof that the action (1.53) describes a $\mathbb{Z}_N$ gauge theory is given by the fact that the $p = 1$ BF action in 4 dimensions comes from the Abelian Higgs Model, with the Higgs field of charge N, in the low-energy limit in fact the $U(1)$ symmetry is broken down to its subgroup $\mathbb{Z}_N$.

In order to analyze the global symmetries of the theory we cannot use the Noether currents, since in general for discrete symmetries that do not come from the breaking of continuous symmetries we do not have a current associated to the transformation. For example the d-dimensional BF theory has $\mathbb{Z}_N^{(p)} \times \mathbb{Z}_N^{(d-p-1)}$ higher-form symmetries that act on the Wilson operators of the theory and are non-linearly realized on the fields

$$A_p \to A_p + \frac{1}{N}\epsilon_p \qquad\qquad B_{d-p-1} \to B_{d-p-1} + \frac{1}{N}\epsilon_{d-p-1} \tag{1.59}$$

with $\oint \epsilon_p = 2\pi\mathbb{Z}$ and $\oint \epsilon_{d-p-1} = 2\pi\mathbb{Z}$.

If we try to see the equations of motion of the theory (1.54) as conservation equations for two currents we can define two symmetry defect operators

$$\begin{aligned} U_{\frac{2\pi ik}{N}}(\Sigma_p) &= e^{\frac{2\pi ik}{N}\oint_{\Sigma_p}\frac{NA_p}{2\pi}} = e^{ik\oint_{\Sigma_p}A_p} \\ \tilde{U}_{\frac{2\pi ik}{N}}(\tilde{\Sigma}_{d-p-1}) &= e^{\frac{2\pi ik}{N}\oint_{\tilde{\Sigma}_{d-p-1}}\frac{NB_{d-p-1}}{2\pi}} = e^{ik\oint_{\tilde{\Sigma}_{d-p-1}}B_{d-p-1}} \end{aligned} \tag{1.60}$$

So, even if the currents are not well defined being not gauge invariant, the symmetry defect operators can be constructed and are topological and gauge invariant for $\mathbb{Z}_N$ fields. The result of this procedure leads us to show that the symmetry defects for the $\mathbb{Z}_N^{(p)}$ symmetry that shifts the field A_p is exactly the Wilson operator for the field B_{d-p-1}. On the contrary, the topological operator for the symmetry $\mathbb{Z}_N^{(d-p-1)}$ that shifts B_{d-p-1} is the Wilson operator constructed with the field A_p.

The action of the defects on the charged operators is always given by linking as it can be already seen by the fact that the two sets of Wilson operators for the two fields have non trivial linking

$$\langle W_n(\Gamma)V_m(\tilde{\Gamma})\rangle = e^{2\pi i\,\frac{mn}{N}\,\mathrm{Link}(\Gamma,\tilde{\Gamma})} \tag{1.61}$$

The correspondence between the charged operators and the topological defects underlined before is due to the fact that the Wilson operator of one field gives a source of fractional charge for the other gauge field, as discussed above once we insert the W_n in the path integral.

Background gauge fields

We have seen that in general to describe the symmetries of a theory we can couple it to a background gauge field. Moreover, considering discrete symmetries that have only flat fields, we can think about the background flat connections as a network of defects. A flat connection is defined in the principal bundle that realizes the gauge theory in some base manifold, it contains the information on how to compare the fibers define over different points, allowing to construct the notion of parallel transport. If we cover the base manifold with a collection of open sets $\mathcal{U}_\alpha$, we need the prescription that the transition functions $\psi_{\alpha\beta}$ defined in the intersection of two open sets satisfy, in triple intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$, the condition:

$$\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\delta} = \mathbb{1} \tag{1.62}$$

The relation between defects and background gauge fields is clear also for ordinary symmetries, since a generic symmetry defect operator $U_{g_0}(\Sigma_{d-1})$ gives a background gauge transformation along its world-volume

$$g(x) = e^{i\lambda_0 \Theta(x,\Sigma)} \tag{1.63}$$

where $\Theta(x, \Sigma)$ is the Heaviside step function that is 0 at one side of Σ_{d-1} and 1 on the other. This implies the presence of a background gauge field proportional to a delta function $\delta(x \in \Sigma_{d-1})$ and to the unit vector orthogonal to the $(d-1)$-dimensional surface.

So, the flat connections can be described in terms of the topological defects and their junctions and also other notions such as global 't Hooft anomalies can be encoded in this point of view, implying the absence of some appropriated junctions. Moreover, gauging a discrete symmetry, which means to sum over all flat connections, can be seen as summing over all possible insertions of the defect operators. For continuous global symmetries instead, we can always have more general background connections, which are not flat, so they are not encoded in the network of topological defects.

1.2 Non-invertible symmetries

After the discussion presented so far we can summarize the fundamental result of this new way of treating symmetries as

$$\text{Symmetries} \leftrightarrow \text{topological operators}$$

All ordinary symmetries are realized in terms of topological defects, but if we think in the opposite direction we can see that not all topological operators correspond to traditional symmetries. The fact that the topological defects are more abundant allows the generalization discussed in the previous sections of higher-form symmetries, if we consider defects with codimension higher than one. Another fundamental generalization arises if we consider topological operators with no inverse, they correspond to transformations that are not group-like symmetries and would be called non-invertible symmetries, [4],[9].

This kind of symmetries deviates a lot from the usual picture of ordinary symmetries in quantum mechanics, that according to Wigner theorem are implemented by unitary operators which have inverses. However, the non-invertible symmetries leads to interesting results such as new conservation laws, new selection rules and constraints on the renormalization group flow, [21].

The complete structure of the space of topological defects can be complicated in general space-time dimensions and can be really addressed in the framework of category theory, which contains the special case of groups, but also much more structures. We will not enter in the details of categories but before proceeding in the characterization of non-invertible symmetries we can focus on some comments about topological defects. As explained in the section about continuous higher-form symmetries, we consider topological operators in the Euclidean spacetime in order to enlarge the concept of conservation in time to the one of topological invariance. The result is that every global symmetry should be interpreted as an operator or a defect.

Ordinary symmetries are given by quantities conserved in time that act as operators in the Hilbert space of the theory and this is what we get when we construct the topological operator by exponentiating the integral over the *space* manifold of the conserved current of a continuous symmetry. When instead the M_{d-1} manifold is extended also in the time direction and localized in one of the spatial directions, the topological operator that we define in the usual way should be considered as a defect that modifies the quantization of the theory and gives rise to a twisted Hilbert space.

Let's for example consider the theory of a free complex scalar field Φ in 2 dimensions

$$\mathfrak{L} = \partial_\mu \Phi \partial^\mu \Phi^\dagger \tag{1.64}$$

with $U(1)$ global symmetry defined in a space given by the coordinate x that takes values in S^1. In canonical quantization the field has periodic boundary conditions:

$$\Phi(\tau, x + 2\pi) = \Phi(\tau, x) \tag{1.65}$$

If we insert the topological operator that realizes the $U(1)$ symmetry $\Phi \to e^{i\theta}\Phi$ along the time direction, it is clear that we change the boundary conditions for the field, since going around S^1 we pass through the defect that modifies the scalar field, see Fig. 1.5, so:

$$\Phi(\tau, x + 2\pi) = e^{i\theta}\Phi(\tau, x) \tag{1.66}$$

and we get a twisted Hilbert space that depends on the $U(1)$ group element $\theta \in [0, 2\pi)$.

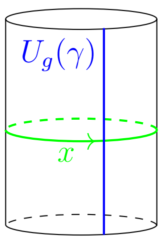

Figure 1.5: The insertion of the defect along the time direction gives the twisted Hilbert space

So depending on whether the support of the U_g is extended in time or not, we get a defect that modifies the space of states giving a twisted Hilbert space, or an operator that coincides with the classical notion of symmetry in the Hilbert space. This is what is called operator/defect principle and it is a fundamental notion that helps to establish what are the allowed symmetries of a QFT that give consistent euclidean correlation functions. Considering a linear combination of the defects $\mathcal{D}_1$ and $\mathcal{D}_2$ with fractional coefficients or negative signs, for example $\mathcal{D}_1 - 1/2\ \mathcal{D}_2$, we see that it does not define a consistent topological defect. In fact, there is no Hilbert space associated to it, so even if it is a conserved operator, since $\mathcal{D}_1$ and $\mathcal{D}_2$ are conserved, this combination cannot define a symmetry generator. In the end both features, operator and defect, are encoded in $U_g(M_{d-1})$ built in the euclidean framework, so as done before we will continue to refer to it as operator or defect without distinctions.

Let's examine now the specific case of non-invertible symmetries. Considering a q-form global symmetry, if we insert two $(d-q-1)$-dimensional topological operators $\mathcal{D}_a$ and $\mathcal{D}_b$, for a general group like symmetry they should satisfy the group multiplication law

$$\mathcal{D}_a \times \mathcal{D}_b = \mathcal{D}_{ab} \tag{1.67}$$

Once we have instead a non-invertible symmetry, we find more general fusion rules for the topological defects, for example:

$$\mathcal{D}_a \times \mathcal{D}_b = \sum_c \mathcal{D}_c \tag{1.68}$$

In general these defect do not need to have an inverse, namely there is no operator $\mathcal{D}_a^{-1}$ such that

$$\mathcal{D}_a \times \mathcal{D}_a^{-1} = \mathbb{1} \tag{1.69}$$

Non-invertible symmetries are widely present in theories in $1+1$ dimensions and can be studied in the framework of rational conformal field theories, where the topological operators are line defects that satisfy

$$\mathcal{L}_a \times \mathcal{L}_b = \sum_c N_{ab}^c \;\; \mathcal{L}_c \tag{1.70}$$

with $N_{ab}^c \in \mathbb{Z}_{\geq 0}$, that lies in a fusion category. For example in 2 dimensions such defects can be obtained if we take linear combinations of invertible symmetry operators with positive integer coefficients. The result is a non-invertible symmetry defect that is not a simple one, since it is built as a sum of defects.

Moreover, we can underline that the non-invertible nature is not a feature that concerns only 0-form symmetries, but is a notion that can mix with the other generalization leading to the definition of higher-form non-invertible symmetries.

1.2.1 Higher gauging

One possible way to insert a non-invertible defect in a theory is given by the construction of the condensation defect, which arises when in a d-dimensional theory we gauge a p-form global symmetry $G^{(p)}$ along a fixed q-dimensional surface. In absence of anomalies, the symmetry can be gauged in the full spacetime and this means, in the case of discrete symmetries, to sum over the mesh of the topological defects of dimension $(d-p-1)$ on the entire manifold. However, since this is not always the case, instead of the entire spacetime we can consider the gauging of the symmetry $G^{(p)}$ in a surface Σ_q, in which some 't Hooft anomalies that obstruct the total space gauging may be trivialized. It is realized by summing over all the insertions of the defect operators of dimension $(d-p-1)$ restricted to the q-dimensional surface and this can be done introducing an associated gauge field B_{p+1} which only lives on the surface.

This procedure is known as higher gauging, [18]-[19]: without changing the bulk of the quantum system, it generates the non-invertible condensation defects, which are a generalization of the projection operators and get the name from the context of anyon condensation, [25]. They are simple defects because they cannot be written using other operators of the same dimension, but they are constituted by a network of topological defects of lower dimensions, since this is exactly what gives the gauging of a symmetry on the surface Σ_q.

Let's consider as an example a topological QFT in $2+1$ dimensions given by the continuum $\mathbb{Z}_2$ gauge theory, which consists of a BF theory with two $U(1)$ 1-form gauge fields with lagrangian:

$$\mathfrak{L} = \frac{2i}{2\pi} \; a \, db \tag{1.71}$$

the theory has the following topological operators

$$W_e(\gamma) = e^{i \oint a} \qquad V_m(\gamma') = e^{i \oint b} \qquad U_f(\gamma'') = e^{i \oint a+b} \tag{1.72}$$

which give the 1-form global symmetries $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}$. We can gauge one $\mathbb{Z}_2^{(1)}$ subgroup generated by one operator and get as a result the condensation defects, [19], [9], for example:

$$\mathcal{C}(\Sigma) = \frac{1}{\sqrt{|H^1(\Sigma, \mathbb{Z}_2)|}} \sum_{\gamma \in H^1(\Sigma, \mathbb{Z}_2)} W_e(\gamma) \tag{1.73}$$

where we sum over the $\mathbb{Z}_2$ non-trivial cycles on Σ given by the cohomology class $H^1(\Sigma, \mathbb{Z}_2)$ and we need also an appropriated normalization. The condensation defect is a surface defect made of lines, so in some sense it gives something in the middle between a 0-form and a 1-form symmetry.

In general, in order to insert in the path integral a condensation defect we need to associate the background field of the p-form symmetry we are gauging to a topological $(d-q)$-dimensional phase given by BF actions η

$$\mathcal{C}_\eta(\Sigma_q) \sim \sum_{B_{p+1}} \exp\left(i \int \eta[B_{p+1}]\right) \tag{1.74}$$

where the gauging is given by the sum over all the gauge inequivalent background fields and we omitted a normalization factor. In the previous particular example we couple a background field to the theory, considering the action $\eta[B] = a\, dB$ and then we perform the condensation summing over the configurations of B as in (1.74).

If we gauge a discrete p-form symmetry $G^{(p)}$ on half of the spacetime we get what is called half gauging procedure, that gives as a result a topological interface between two theories, the initial one $\mathcal{T}$ and the one obtained after the gauging $\mathcal{T}/G^{(p)}$. If the system is invariant under the gauging, namely the theory $\mathcal{T}/G^{(p)}$ is equivalent to the original one, then the interface becomes a topological defect in a unique well defined theory.

1.2.2 Non invertible defect in QED

Given the discussion done so far about discrete non-invertible symmetries, we can now focus on what is the most famous example of a continuous non-invertible symmetry, that is present in the well known QED theory and that indeed shows that non-invertible symmetries exist in nature, [12]-[13].

Let's consider the 4-dimensional QED with a $U(1)$ gauge field a and a Dirac massless fermion ψ of charge one. Denoting $f = da$ the dynamical field strength of the theory, the action is given by

$$S = \int \frac{1}{2g^2} f \wedge *f + i\overline{\psi} \not{D} \psi \tag{1.75}$$

The fermion field is minimally coupled to the gauge field through the covariant derivative $\not{D} = (\partial_\mu - iA_\mu)\gamma^\mu$. The action is left invariant by the $U(1)_A$ chiral symmetry that acts on the fermion field as

$$\psi \to e^{i\alpha\gamma_5}\, \psi \tag{1.76}$$

The chiral transformation is a symmetry at the classical level, since the fields variations are $\delta\psi = i\alpha\gamma_5\psi$ and $\delta\overline{\psi} = \gamma^0(-i\alpha\gamma_5)\psi^\dagger = i\alpha\gamma_5\overline{\psi}$, so given the fact that γ_5 anticommutes with all the gamma matrices, we get that the kinetic terms for the fermions is invariant. The symmetry would be broken even at the classical level if we consider massive fermions, because the mass term would not be invariant. The current associated to the symmetry is given by

$$j_A^\mu(x) = \overline{\psi}\, \gamma^\mu \gamma_5\, \psi(x) \tag{1.77}$$

At the quantum level however this transformation has an ABJ anomaly, [14], so it's not a real symmetry of the theory. Since the path integral measure also transforms, we get that the partition function is not invariant under $U(1)_A$ chiral. We can adopt Fujikawa formalism to rewrite the anomaly and the result is

$$d * j_A = \frac{1}{8\pi^2}\ f \wedge f \tag{1.78}$$

where the classical current conservation equation is modified with a non zero contribution proportional to the field strength. The non-conservation equation means that the usual symmetry defect operator that we can try to built as

$$U_g(\Sigma_3) = e^{i\alpha \oint_{\Sigma_3} *j_A} \tag{1.79}$$

is not topological, since if we deform the 3-dimensional support of the defect we need to use the current equation and the result is no more invariant

$$U_g(\Sigma_3') = U_g(\Sigma_3) \cdot e^{i\alpha \int_{\xi_4} d*j_A} = U_g(\Sigma_3) \cdot e^{i\alpha \int_{\xi_4} \frac{1}{8\pi^2}\ f\wedge f} \tag{1.80}$$

Here ξ_4 is a four dimensional manifold whose boundary is $\partial\xi_4 = \Sigma_3 \cup \overline{\Sigma_3'}$, where the orientation of Σ_3' has been reversed. A first attempt to solve this issue is to define a conserved current doing a redefinition in the equation (1.78)

$$*j' = *j_A - \frac{1}{8\pi^2}\ a \wedge da \tag{1.81}$$

since $d * j' = 0$ we see that the following defect operator is topological

$$U_g'(\Sigma_3) = e^{i\alpha \oint_{\Sigma_3} *j'} = e^{i\alpha \oint_{\Sigma_3} \left(*j_A - \frac{1}{8\pi^2}\ a\wedge da\right)} \tag{1.82}$$

The operator however is still not well-defined, the problem now is that it is not gauge invariant. It can be seen that the $U(1)_A$ chiral is truly a symmetry only in flat spacetime, since it determines a selection rule that is confirmed by experiments, namely it gives the helicity conservation law for the scattering amplitudes of electrons. The idea is that in flat spacetime there are no $U(1)$ instantons because the homotopy group is trivial $\pi_3(U(1)) = 0$ and so the $U(1)_A$ symmetry is unbroken, since its amount of violation is proportional to $f \wedge f$ which is the instanton term.

The problem arises if the three dimensional space has non trivial topology, for example, if there are monopoles in the theory the symmetry is broken. Moreover, the defect operator in general is not gauge invariant since it contains the Chern-Simons partition function

$$e^{\frac{iN}{4\pi} \oint a\wedge da} \tag{1.83}$$

that is gauge invariant only if the level N is an integer. In this case, inside equation (1.82), the level is $\alpha/2\pi$. These considerations seem not very convincing, because global symmetries should be characteristic of the model that we are studying and independent from the structure of spacetime. We will see in the following another interpretation: the chiral symmetry is not completely destroyed by the ABJ anomaly, but instead it comes back as a non-invertible global symmetry. If we restric to the case of rational rotation angles

$$\alpha = \frac{2\pi}{N} \tag{1.84}$$

we see that the Chern-Simons term that appears in the would-be symmetry operator (1.82)

$$\frac{i}{4\pi N}\int a\wedge da \tag{1.85}$$

has the same expression of the action of the fractional quantum Hall state in $2+1$ dimensions with filling fraction $\nu = 1/N$. a is the background gauge field given by the magnetic field used in the Hall experiment. In that framework there is a known way to obtain a well-defined gauge invariant action

$$\int \frac{iN}{4\pi}A\wedge dA + \frac{i}{2\pi}A\wedge da \tag{1.86}$$

in which A is an additional dynamical $U(1)$ gauge field. Now the levels of the Chern-Simons terms are properly quantized so gauge invariance is restored. The two actions can not be naively compared since one is gauge invariant and the other is not, but if we think about integrating out the additional field A from (1.86) we see that morally the result is the original Chern-Simons term.

Finally, we understand that the suitable definition of the topological gauge invariant symmetry defect in 4-dimensional QED is given by

$$D_{\frac{1}{N}}(\Sigma_3) = \int \mathcal{D}A\, \exp\Big(\oint_{\Sigma_3}\frac{2\pi i}{N} * j_A + \frac{iN}{4\pi}A\wedge dA + \frac{i}{2\pi}A\wedge da\Big) \tag{1.87}$$

The topological dressing comes form the fractional quantum Hall theory, with the insertion of the auxiliary 1-form field A over which there is a path integration. This additional field lives only in the closed support Σ_3 of the defect and it does not introduce new information in the model, the bulk physics is unchanged away from Σ_3.

Let's show now explicitly that the symmetry defect is a non-invertible operator, first we can see that it is non-unitary since if we compute

$$\begin{aligned} D_{\frac{1}{N}}(\Sigma_3)\times D^{\dagger}_{\frac{1}{N}}(\Sigma_3) &= \int \mathcal{D}A\mathcal{D}B\;\exp\left(\oint_{\Sigma_3}\frac{iN}{4\pi}A\wedge dA - \frac{iN}{4\pi}B\wedge dB + \frac{i}{2\pi}(A-B)\wedge da\right)\\ &= \mathcal{C}_N(\Sigma_3) \end{aligned} \tag{1.88}$$

where $\mathcal{C}_N$ is the condensation defect that we get with the higher gauging procedure for the $\mathcal{Z}_N^{(1)}$ symmetry. We get that $D_{\frac{1}{N}}(\Sigma_3)$ is non unitary since $D_{\frac{1}{N}}(\Sigma_3)\times D^{\dagger}_{\frac{1}{N}}(\Sigma_3) \neq \mathbb{1}$.

Another additional consideration is that the traditional group structure of the symmetry is broken since for example $D_{\frac{1}{N}}(\Sigma_3)\times D_{\frac{1}{N}}(\Sigma_3) \neq D_{\frac{2}{N}}(\Sigma_3)$

$$D_{\frac{1}{N}}(\Sigma_3)\times D_{\frac{1}{N}}(\Sigma_3) = e^{\frac{4\pi i}{N}\int *j_5}\int \mathcal{D}A\mathcal{D}B\;\exp\left(\oint_{\Sigma_3}\frac{iN}{4\pi}(A\wedge dA + B\wedge dB) + \frac{i}{2\pi}(A+B)\wedge da\right) \tag{1.89}$$

The definitive proof that it is a non-invertible operator comes from the construction we have done, in fact the fractional quantum Hall state is a non-invertible topological theory in three dimensions, as in general all the Chern-Simons gauge theories. So it does not exist another topological QFT, such that its tensor product with the initial one gives a trivial TQFT. The defect operator can also be generalized from $D_{\frac{1}{N}}$ to $D_{\frac{p}{N}}$, with $p\in\mathbb{N}$.

Half-gauging

All the arguments presented so far are pretty convincing to understand that the defect is really topological, an additional rigorous proof of this comes from the half-gauging construction of this non-invertible symmetry operator. Since QED, due to the Bianchi identity, has a magnetic 1-form symmetry $U(1)^{(1)}_m$ that acts on the 't Hooft lines, we can couple the theory to a 2-form background gauge field B by adding

$$iB \wedge *j^{(1)}_m = \frac{i}{2\pi} B \wedge f \tag{1.90}$$

to the QED lagrangian. We can then study the gauging of the $\mathbb{Z}^{(1)}_N$ subgroup of this symmetry and see how it implies the presence of the non-invertible defect (1.87). In order to do this, we promote the field B to a dynamical field b and we introduce also a 1-form $U(1)$ dynamical gauge field c. It acts as a Lagrange multiplier imposing some constraints on b through the coupling

$$\frac{iN}{2\pi} b \wedge dc \tag{1.91}$$

which implies that b becomes a $\mathbb{Z}_N$ 2-form gauge field. In fact $db = 0$, so it is flat and with holonomies in $\mathbb{Z}_N$, then the path integration over b is what realizes the gauging of the $\mathbb{Z}^{(1)}_N$ symmetry. In addition, we are always free to add to the QED lagrangian a counter term that depends only on the field b.

$$\frac{i}{2\pi} b \wedge f + \frac{iN}{2\pi} b \wedge dc + \frac{iNk}{4\pi} b \wedge b \tag{1.92}$$

It can be seen, by integrating out the field c, that the additional terms that realize the gauging also imply a discrete shift of the θ-angle of QED, since k is defined as $p\,k = 1 \text{ mod}_N$.

$$\theta \to \theta - \frac{2\pi p}{N} \tag{1.93}$$

A shift of the θ-angle can be undone by an axial rotation of the fermions (1.76), choosing $\alpha = 2\pi p/N$. We can exploit this fact in order to gauge the $\mathbb{Z}^{(1)}_N$ symmetry only on half of the spacetime; we add all the terms that give the gauging (1.92) for example in the region $x > 0$ and then we insert in the lagrangian also the integral of the axial current, that comes from the shift of the action once we apply the fermionic axial transformation. Since the path integral measure also transforms, we need to add even the term proportional to $f \wedge f$ which gives the anomalous current conservation. Finally the action becomes

$$\int_{x>0} \mathfrak{L}_{QED} + \frac{2\pi i p}{2N} \oint_{x=0} *j_A + \int_{x\geq 0} \mathfrak{L}_{QED} + \frac{ip}{4\pi N} f \wedge f + \frac{i}{2\pi} b \wedge f + \frac{iN}{2\pi} b \wedge dc + \frac{iNk}{4\pi} b \wedge b \tag{1.94}$$

where the gauge fields b, c are defined only in the region $x \geq 0$ and fixed by Dirichlet boundary conditions at $x = 0$. Since in the region $x > 0$ we are implementing the discrete gauging and also the axial transformations and their effects balance each other, we get the original QED theory also in this region. The only thing that survives is the defect at $x = 0$, which is exactly $D_{\frac{p}{N}}$ and it is manifestly topological since the boundary conditions on the fields at $x = 0$ are topological.

Action of the non-invertible symmetry

This construction of the non-invertible defect from half-gauging, as a combination of the gauging of $\mathbb{Z}^{(1)}_N$ and the axial rotations, helps also to understand how the non-invertible symmetry acts on the objects of the theory. The fermions are not affected by the gauging so $D_{\frac{p}{N}}$ acts on them as an invertible chiral rotation with rational angle. Once we look instead at the action on 't Hooft lines, we can see the non-invertible nature of the defect. Considering a simple 't Hooft line $H(\gamma)$, when it is going through the defect operator it becomes a non-genuine operator, as can be seen in Fig. 1.6.

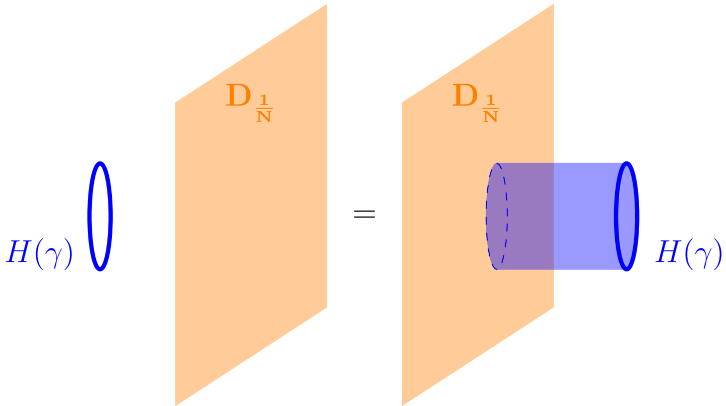

Figure 1.6: Action of the non-invertible defect on 't Hooft lines $H(\gamma)$

The explanation for this comes from the fact that after the gauging of the $\mathbb{Z}_N^{(1)}$ subgroup of the magnetic 1-form symmetry, the 't Hooft lines are no more gauge invariant, while the combination

$$H(\gamma)\ e^{i\frac{p}{N}\int_\Sigma f} \tag{1.95}$$

where $\partial\Sigma = \gamma$, is gauge invariant. So, going across the non-invertible defect the 't Hooft line becomes attached to this topological surface which is extended between the symmetry operator and the 't Hooft line. As described by the Witten effect [16], where one argues the existence of dyons, namely particles with both electric and magnetic charges, now the 't Hooft line gains a fractional electric charge p/N. In the case of [16], it is shown how the electric charge of a magnetic monopole in a $U(1)$ gauge theory can be connected to the θ-angle of the CP violating term of the theory. The one described here represents the typical action of a non-invertible symmetry on the charged objects, the same thing happens for example in the Ising CFT where local spin operators are mapped into non local ones by the non-invertible defect that implements the Kramers-Wannier duality, [22].

Finally, summing up, the result of this section is that the ABJ anomaly can be cured by the fractional quantum Hall state and survives as a non-invertible symmetry for every rational angle. However, it is still not a continuous symmetry even if the rational numbers are dense in the real line. A possible solution is given by changing the topological dressing of the defect, using non compact fields with values in $\mathbb{R}$, so that we can describe the axial rotations for any real angle, as it is done in [17].

1.2.3 Non-invertible symmetries and pions decay

The problem of pion decay was the motivation that started the study of the ABJ anomaly [15]. The dominant decay channel for the neutral pion π^0 is the one in two photons

$$\pi^0 \to \gamma\gamma \tag{1.96}$$

that is observed experimentally. However, this type of interaction comes from a coupling given by $\pi^0 F \wedge F$ in the chiral lagrangian of QCD, which unfortunately is not invariant under the shift symmetry of the pion.

It is interesting to show then how the results obtained in QED can be translated in the QCD framework and can solve this problem. Let's consider QCD below the electroweak scale, where the gauge group is $U(1)_{EM}$. The classical chiral symmetry in the massless quark limit acts for example on the up and down quarks as

$$\begin{pmatrix} u \\ d \end{pmatrix} \to e^{i\alpha\gamma_5\sigma_3} \begin{pmatrix} u \\ d \end{pmatrix} \tag{1.97}$$

with axial current given by

$$j^\mu_{A_3} = \frac{1}{2}\bar{u}\gamma_5\gamma^\mu u - \frac{1}{2}\bar{d}\gamma_5\gamma^\mu d \tag{1.98}$$

If this transformation is really a symmetry then once it is spontaneously broken the Goldstone boson that we would get is the neutral pion π^0, which transform non linearly as

$$\pi^0 \to \pi^0 - 2\alpha f_\pi \tag{1.99}$$

where the shift depends on the pion decay constant f_π. This transformation is the one that modifies the coupling $\pi^0 F \wedge F$, so the solution usually considered was to determine the coupling coefficient by matching with the anomalous equation for the current of the broken chiral symmetry

$$d * j_{A_3} = \frac{1}{8\pi^2}\, F \wedge F \tag{1.100}$$

where F is the electromagnetic field strength, with gauge field A. The aspect that is a bit strange of this reasoning is that in order to derive a result that agrees with the experiments we match the absence of a global symmetry, instead of trying to discover a new global symmetry that is what is usually done to get selection rules.

In order to find another interpretation, we can translate what we computed in QED and see that there is a non-invertible symmetry also in QCD and that it could play some role to explain the pion decay. We need just to substitute the current j_A of the chiral symmetry with the QCD current j_{A_3} and we get the non-invertible symmetry defect

$$D_{\frac{1}{N}}(\Sigma_3) = \int \mathcal{D}c \, \exp\Big(\oint_{\Sigma_3} \frac{2\pi i}{N} * j_{A_3} + \frac{iN}{4\pi} c \wedge dc + \frac{i}{2\pi} c \wedge dA \Big) \tag{1.101}$$

In the low-energy chiral theory the axial current can be written as $j^\mu_{A_3} = -f_\pi \partial^\mu \pi^0 + ...$ and the terms relevant for our purposes in the pion lagrangian are

$$\mathfrak{L}_{IR} = \frac{1}{2} d\pi^0 \wedge *d\pi^0 + ig\, \pi^0\, F \wedge F \tag{1.102}$$

In order to determine the coupling g that can be compared with the decay measurements we need to insert the defect in the theory, for example at $x = 0$, so the action becomes

$$\int_{x<0 \cup x>0} \Big(\frac{1}{2} d\pi^0 \wedge *d\pi^0 + ig\, \pi^0\, F \wedge F \Big) + \int_{x=0} \Big(\frac{2\pi i}{N} * j_{A_3} + \frac{iN}{4\pi} c \wedge dc + \frac{i}{2\pi} c \wedge dA \Big) \tag{1.103}$$

Because of the current j_{A_3} inside the defect operator, we see that the pion field is discontinuous across the defect computing its equations of motion

$$\pi^0\Big|_{x=0^+} - \pi^0\Big|_{x=0^-} = -\frac{2\pi}{N} f_\pi \tag{1.104}$$

We need to consider also the equations of motion for the auxiliary field c and the gauge field A coming from the action and evaluated at $x = 0$

$$N dc + F = 0 \qquad\qquad 2i\, g \left(\pi^0\Big|_{x=0^+} - \pi^0\Big|_{x=0^-} \right) F = \frac{i}{2\pi} dc \tag{1.105}$$

If we match these three equations we see that they are all simultaneously consistent if

$$g = \frac{1}{8\pi^2 f_\pi} \tag{1.106}$$

which gives exactly a result compatible with experiments. This important outcome gives one practical motivation for the study of non-invertible symmetries, whose effect can really be seen in nature and this shows how by revising the known models we can get new interesting results. In the following we will focus on the way to characterize and study all the generalized global symmetries of a QFT, introducing the tools of SymTFT and SymTh.

Chapter 2

A theory for symmetries and anomalies

2.1 Symmetry Topological Field Theory: SymTFT

Let's consider a d-dimensional quantum field theory $\mathcal{T}$ on a spacetime manifold M_d in which we have a global symmetry structure that is determined by a collection of topological operators. The idea is to build a $(d+1)$-dimensional topological QFT to describe these topological symmetry operators on a manifold M_{d+1} which has boundary M_d. The logic is the same of the picture of anomaly inflow that allows to study the anomalies of a QFT. With the anomaly polynomial in fact we can write an anomaly in a topological way in a space that is $(d+2)$-dimensional by using Chern-Simons terms, then with the descent mechanism we can come back to the anomaly in d-dimensions.

The topological $(d+1)$-dimensional theory that we can build, called in short SymTFT: Symmetry Topological Field Theory, [31]-[41], is an useful construction because it gives a uniform mechanism to extract the topological sector of a QFT. In addition, it encodes in terms of the different choices of boundary conditions the possible symmetries generators and charged objects of the boundary theory. The defect operators of the SymTFT would become the topological symmetry operators of the QFT on the boundary, but their braiding and fusion would be determined by the behavior of the bulk operators in the TQFT. Therefore the SymTFT is a general concept that has a role in studying generalized symmetries, charges and relation among symmetries via gaugings. As we will see, along with the SymTh, it constitutes a close interlink with holography and string theory constructions of QFTs.

2.1.1 Symmetry and charge operators in the sandwich construction

We start with a theory $\mathcal{T}$ in d dimensions with a symmetry $\mathcal{S}$, given in general by a collection of topological defects that form a higher-fusion category. The SymTFT is the (d+1) topological field theory with two boundaries: the physical one $\mathcal{B}^{phys}_{\mathcal{T}}$ and the topological one $\mathcal{B}^{sym}_{\mathcal{S}}$ where the symmetry is realized. In general, a non trivial SymTFT will have a dependence on the choice of the manifold M_{d+1}, for example if it has non trivial bulk cycles away from its boundary, namely a non trivial cohomology, then the partition function of the topological theory will sum over all possible topological operators wrapping these cycles.

The point is to choose the $(d+1)$-dimensional manifold in a clever way: $M_{d+1} = M_d \times [0,1]$ where the extremes of the interval correspond to the two boundaries, so the picture we can build is presented in Fig.(2.1) and goes under the name of sandwich construction. Since the bulk theory is topological there will be no dependence on the bulk physics and since we do not want to introduce additional degrees of freedom with this construction the symmetry boundary must be topological, namely a gapped boundary in which lives the symmetry $\mathcal{S}$. The theory does not depend on the size of the interval so it can be shrink to zero and the result of the compactification is the original theory with symmetry $\mathcal{S}$, because the product of the topological boundary and of the physical one gives the partition function of the QFT.

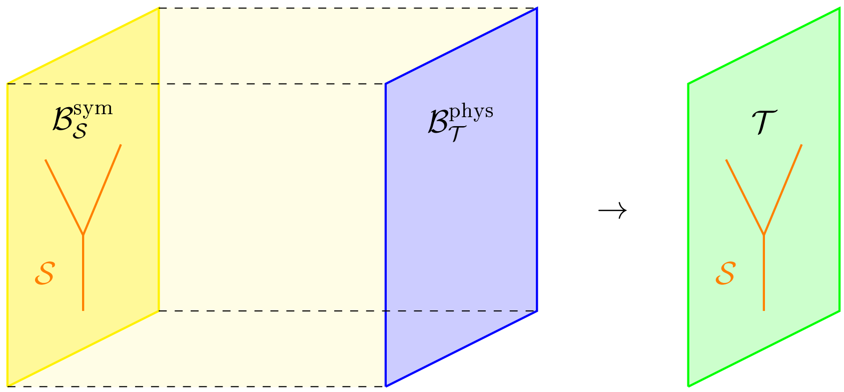

Figure 2.1: The SymTFT sandwich construction, after the compactification we get the physical theory with the symmetry that is realized in the symmetry boundary

If we consider a genuine q-dimensional operator $\mathcal{O}_q$ in the d-dimensional QFT, we can see how it should be realized in the sandwich construction, according to [33]. The simplest possibility is that it corresponds to a q-dimensional operator $\mathcal{O'}_q$ living on the physical boundary $\mathcal{B}^{phys}_{\mathcal{T}}$ of the SymTFT without being attached to any bulk operator. However, since the symmetries live on the symmetry boundary $\mathcal{B}^{sym}_{\mathcal{S}}$ the operator $\mathcal{O'}_q$ is left invariant by the action of every symmetry, and that means that the operator $\mathcal{O}_q$ must be uncharged. In fact, as shown in figure (2.2), in the physical theory the symmetry defect can be moved across $\mathcal{O}_q$ with no effect on the operator, since in the SymTFT the operator is on the physical boundary and the defects can be moved freely on the symmetry boundary.

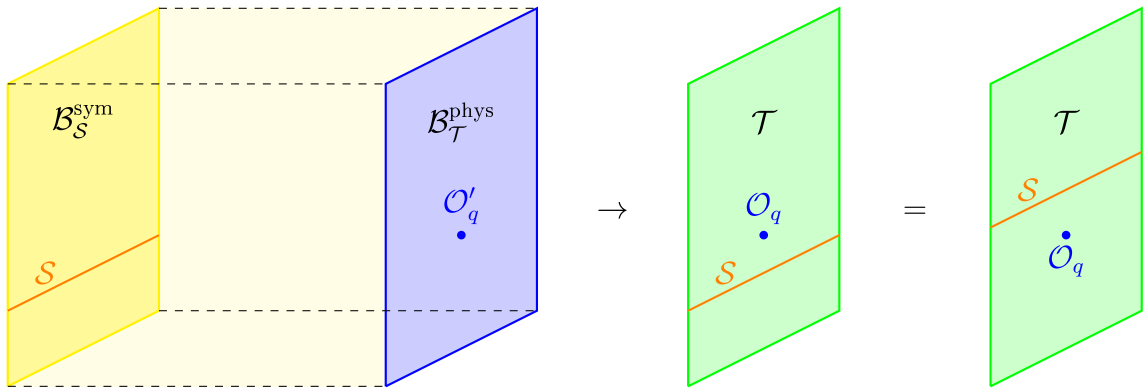

Figure 2.2: The realization of uncharged operators on the SymTFT

If the boundary operator $\mathcal{O}_q$ carries a charge under the symmetry, then it must be attached to a simple bulk topological operator $\mathcal{Q}_{q+1}$ that has to end on both boundaries. Let's call ξ_q the q-dimensional topological operator given by the intersection of $\mathcal{Q}_{q+1}$ with the symmetry boundary $\mathcal{B}^{sym}_{\mathcal{S}}$. Now the action of the symmetry $\mathcal{S}$ on the charged operator $\mathcal{O}_q$ is encoded in how the topological defect that generates the symmetry and lies on the symmetry boundary acts on the operator ξ_q. The action of the symmetry gives the map $\xi_q \to \xi'_q$, so performing the sandwich compactification, keeping the end $\mathcal{P}_q$ fixed, the result in the physical theory is the transformation $\mathcal{O}_q \to \mathcal{O}'_q$ under the symmetry $\mathcal{S}$, see figure (2.3).

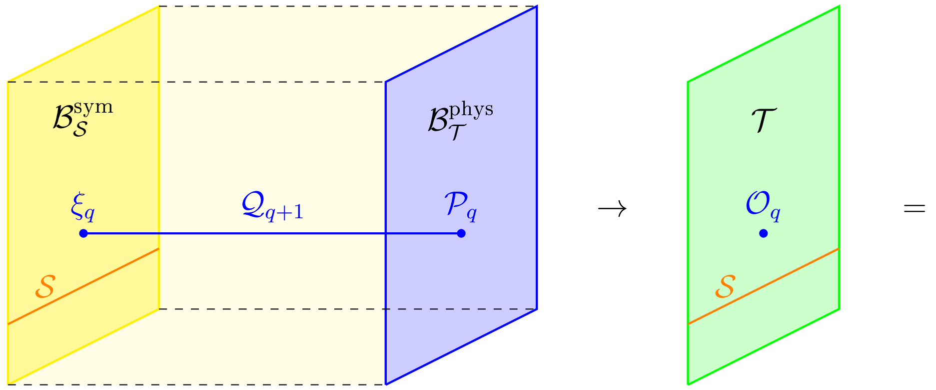

Figure 2.3: The realization of a charged operator on the SymTFT.

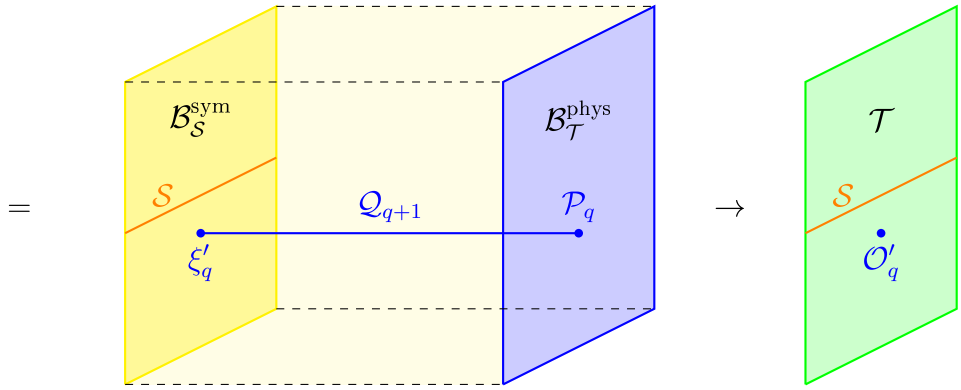

Figure 2.4: The action of the symmetry $\mathcal{S}$ on a charged operator in the SymTFT: moving the symmetry defect across the operator ξ_q has the effect to map it to ξ'_q and so as a result to map $\mathcal{O}_q$ into $\mathcal{O}'_q$ in the boundary QFT.

In this general framework we can show also the action of a non-invertible symmetry for which in general may happen that a q-dimensional operator is mapped into another operator by attaching a $(q+1)$-dimensional defect S_{q+1} to it. So if we start as before with a charged operator $\mathcal{O}_q$ that is realized in the SymTFT with a $(q+1)$-dimensional operator than ends on $\mathcal{P}_q$ on the physical boundary and on ξ_q on the symmetry boundary, we can see that the non-invertible symmetry maps ξ_q into ξ'_q attached to S_{q+1}. Accordingly the operator $\mathcal{O}'_q$ of the boundary QFT obtained after the interval compactification is also attached to a topological defect D_{q+1} as in figure (2.5). One usually says that the operator $\mathcal{O}'_q$ lives in the twisted sector of the symmetry $\mathcal{S}$, namely is a non-genuine operator living at the end of a $(q+1)$-dimensional defect.

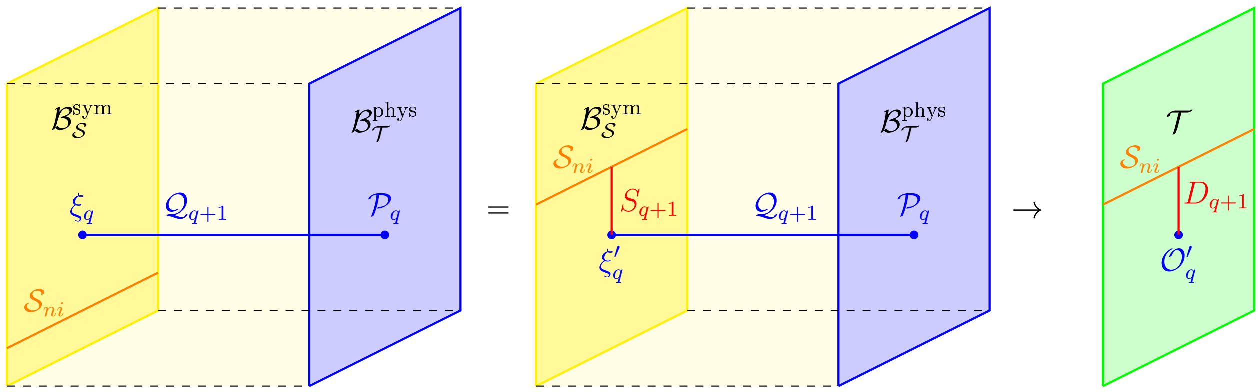

Figure 2.5: The action of a non-invertible symmetry $\mathcal{S}_{ni}$ on a q-dimensional operator represented in the SymTFT.

2.1.2 An example of SymTFT

When the symmetry is an invertible group-like symmetry the SymTFT is a BF theory and the building of the SymTFT can be thought as the $(d+1)$-dimensional gauging of the symmetry $\mathcal{S}$ in the abelian case. The topological defects of the SymTFT can have two different types of canonical boundary conditions:

- Dirichlet boundary conditions, so the defect ends on the two boundaries and gives rise to a charged operator in the physical QFT;
- Neumann boundary conditions, so the defect of the bulk theory can be projected and results in a topological generator of a symmetry of the QFT.

Let's consider an example: a 0-form $\mathbb{Z}_N$ symmetry in a 2d theory $\mathcal{T}$. The SymTFT can be written as a BF term using $U(1)$ valued fields

$$S_{BF} = \frac{N}{2\pi} \int_{M_3} b_1 \wedge dc_1 \tag{2.1}$$

This theory is clearly topological with no propagating degrees of freedom as follows from the equations of motion

$$\frac{N}{2\pi} db_1 = 0 \qquad\qquad \frac{N}{2\pi} dc_1 = 0 \tag{2.2}$$

The gauge fields need to be flat, but this does not mean that the theory is trivial in fact since the only configurations that matters are those for which the holonomies are $\mathbb{Z}_N$-valued, the theory has a set of non-trivial gauge invariant operators. In the partition function of the BF theory the sum over the field configurations cancels all contributions except the ones that satisfy

$$N \oint \frac{db_1}{2\pi} \in \mathbb{Z} \qquad\qquad N \oint \frac{dc_1}{2\pi} \in \mathbb{Z} \tag{2.3}$$

localizing the path integral over $\mathbb{Z}_N$ gauge fields. The result is that the theory has a collection of gauge invariant Wilson operators that are topological since the gauge fields are flat

$$W_n(\Gamma) = e^{in \int_\Gamma b_1} \qquad\qquad V_m(\tilde{\Gamma}) = e^{im \int_{\tilde{\Gamma}} c_1} \tag{2.4}$$

where the charges of the Wilson lines are $n, m = 1, ..., N-1$ since the charge N Wilson operators are trivial line operators as they are invisible in the theory due to the localization of the path integral on $\mathbb{Z}_N$ fields.

Summing up, a BF theory written in this way is the appropriate SymTFT for a QFT with a 0-form $\mathbb{Z}_N$ symmetry, so the next aspect to consider is given by the boundary conditions that can be imposed for the 1-form fields on the topological symmetry boundary. For example, Dirichlet b.c. for b_1 means that the field does not fluctuate at the boundary and so it becomes a background field after the interval compactification, this boundary condition corresponds to insert a delta function $\delta(b_1 - B)$ where B is a non-dynamical background field. Neumann b.c. instead indicates that the field remains dynamical in the physical theory.

Upon imposing the boundary conditions, the topological Wilson lines will determine the symmetry generators and the generalized charges in the physical QFT. With the choice:

$$b_1 \; \textit{Dirichlet}, \; c_1 \; \textit{Neumann}$$

the b_1 Wilson lines W_n can end and so become point-like operators in the boundary QFT, whereas the c_1 Wilson lines V_m cannot end and remain topological operators on the boundary, as we can see in Fig.(2.6).

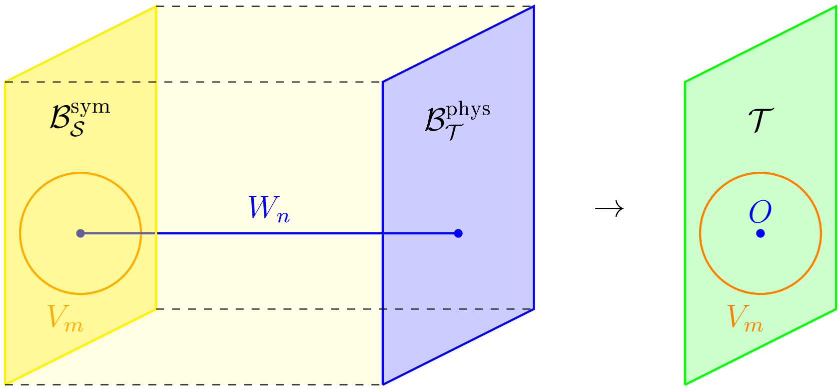

Figure 2.6: Wilson lines in the SymTFT with different boundary conditions on the fields.

To understand how the defect acts on the local operator in the resulting boundary QFT we need to consider the action of one topological operator on the other in the SymTFT, that is given by the fact that the Wilson operators have a non-trivial braiding relation:

$$\langle W_n(\Gamma) V_m(\tilde{\Gamma}) \rangle = e^{2\pi i \frac{mn}{N} \text{Link}(\Gamma,\tilde{\Gamma})} \tag{2.5}$$

where $\text{Link}(\Gamma, \tilde{\Gamma})$ is the linking number of the two curves. Since we are in three dimensions, two curves can link because the rule is: dimension of the space equal to the sum of the dimensions of the manifolds that link plus one and here we have $3 = 1 + 1 + 1$.

2.1.3 SymTFT and anomalies

One powerful feature of the SymTFT is that it provides a way to encode both the global symmetries of a QFT and their anomalies. In the SymTFT the anomalies are incorporated adding the suitable Chern-Simons terms, for example in a $4d$ QFT with a $\mathbb{Z}_N^{(0)}$ global symmetry, its anomaly is given by the $5d$ SPT phase

$$\mathcal{A} = \frac{i}{24\pi^2} \int A_1 \wedge dA_1 \wedge dA_1 \tag{2.6}$$

where the A_1 is the $U(1)$ representative of a $\mathbb{Z}_N$ gauge field. The SymTFT of this theory is given by:

$$S_{Sym} = \frac{iN}{2\pi} \int da_1 \wedge b_3 + \frac{i}{24\pi^2} \int a_1 \wedge da_1 \wedge da_1 \tag{2.7}$$

Recalling that anomalies prevent the gauging of the corresponding symmetries, we see that the Chern-Simons terms obstruct the possibility to impose Neumann boundary conditions. The additional topological terms in fact provide a shift in the equations of motion, giving to the corresponding Wilson operators $V_m(\Sigma)$ of b_3 a non trivial expectation value, that depends on the triple self-intersection linking number. As a result, the attempt to impose Neumann boundary conditions fails because the only way to vanish the variation of the action at the boundary $\partial M_4 = M_3$, given by:

$$\delta S_{Sym} = \frac{iN}{2\pi} \int_{M_3} \delta a_1 \wedge \left(b_3 + \frac{1}{6\pi N} a_1 \wedge da_1 \right) \tag{2.8}$$

is to impose the standard Dirichlet boundary condition $\delta a_1\big|_{M_3} = 0$ or the one that can be called the would-be Neumann boundary condition

$$\frac{N}{2\pi} b_3 + \frac{1}{12\pi^2} a_1 \wedge da_1 \,\Big|_{M_3} = 0 \tag{2.9}$$

However, the problem is that this relation is not gauge invariant and not compatible with the bulk equations of motion, so the only possibility is to impose $a_1 \wedge da_1 = 0$ and $b_3 = 0$ and these restrictions are not suitable boundary conditions.

In conclusion, anomalies obstruct the existence of a trivially gapped phase, in the sense that there are no pair of boundary conditions that we can impose on the interval so that the path integral describes a trivially gapped phase with a dynamical field. Considering the boundary QFT this is equivalent to say that an anomaly does not allow the theory to flow to a trivially gapped phase in the IR.

2.2 Symmetry theory: SymTh

The same discussion described so far for the Symmetry Topological Field Theory for discrete symmetries can be modified losing the prescription of having a topological field theory in the bulk. In this case we can build a Symmetry Theory, called SymTh [58], with the same purpose of the SymTFT, namely to encode all the symmetries and anomalies of a boundary physical QFT. The inspiration comes also from holography where the boundary theory is dual to the full bulk gravity, [42].

The advantage is that now we can fully describe in a simple way also continuous global symmetries as well as the finite symmetries, that have a natural realization also in the SymTFT. We can encode in the SymTh also continuous symmetries with non-flat connections that are not automatically included in the SymTFT approach. In the SymTh we can built the sandwich construction considering the total space as $M_{d+1} = M_d \times I$, where I is a finite interval, analogously to the SymTFT, that gives two boundaries corresponding to the extremes of the interval.

A difference with the SymTFT is that now the kinetic terms for the bulk fields play an important role in the discussion, so the SymTh is not invariant under the RG flow since it is not topological and it must be considered as an effective description. However, even if the whole theory is not topological, in order to describe symmetries we focus on its topological properties that are universal and preserved under the Renormalization Group flow.

2.2.1 An example of SymTh

The typical constructions for a SymTh are given by weakly coupled Maxwell terms plus interactions terms given by Chern-Simons couplings. The simplest example we can make is how to built the SymTh for a d-dimensional QFT with a $U(1)$ p-form global symmetry. This symmetry can be described by a Maxwell theory in $(d+1)$-dimension with a $(p+1)$-form gauge field a_{p+1}

$$S_{d+1} = -\frac{1}{2}\int_{M_{d+1}} da_{p+1} \wedge *da_{p+1} \tag{2.10}$$

where the Hodge star operator is $* = *_{d+1}$ and the coupling, assumed small, is absorbed with a redefinition of the gauge field.

The point is to study the topological operators of the theory and to understand how they describe the symmetries of the d-dimensional QFT depending on the boundary conditions that we impose on the a_{p+1} field. The two conserved currents are given by

$$d * J_{p+2} = d * f_{p+2} = 0 \qquad\qquad d * J_{d-p-1} = df_{p+2} = 0 \tag{2.11}$$

where $f_{p+2} = da_{a+1}$ and the conservation equations are respectively the equation of motion of the Maxwell theory and the Bianchi identity. We can write explicitly the topological operators defined by these conserved currents

$$\begin{aligned} U_\alpha(\Sigma_{d-p-1}) &= \exp\left(i\alpha \int_{\Sigma_{d-p-1}} *J_{p+2}\right) = \exp\left(i\alpha \int_{\Sigma_{d-p-1}} *f_{p+2}\right) \\ U_\beta(\Sigma_{p+2}) &= \exp\left(i\beta \int_{\Sigma_{p+2}} *J_{d-p-1}\right) = \exp\left(i\beta \int_{\Sigma_{p+2}} f_{p+2}\right) \end{aligned} \tag{2.12}$$

with the parameters of the transformations defined in $[0, 2\pi)$. This topological operators act on the Wilson surfaces defined as

$$W_q(M_{p+1}) = e^{i\,q \int_{M_{p+1}} a_{p+1}} \qquad\qquad V_m(M_{d-p-2}) = e^{i\,m \int_{M_{d-p-2}} b_{d-p-2}} \tag{2.13}$$

since the field a_{p+1} admit a magnetic dual field b_{d-p-2}, defined such that $*_{d+1} da_{p+1} = db_{d-p-2}$. We notice that the Wilson operators are not topological since for example $da_{p+1} \neq 0$ which is a different behavior respect to the discrete symmetries that are described by a BF theory, where we have flat configurations as a result of the equations of motion. The topological operators act by linking on the Wilson surfaces, in the following expressions the Σ surfaces are the ones that links the support of the Wilson operators, while the $\tilde{\Sigma}$ are the deformed version obtained after the unbraiding

$$\begin{aligned} \langle U_\alpha(\Sigma_{d-p-1}) W_q(M_{p+1})\rangle &= \exp(iq\alpha \ \text{Link}(\Sigma_{d-p-1}, M_{p+1})) \,\langle W_q(M_{p+1}) U_\alpha(\tilde{\Sigma}_{d-p-1})\rangle \\ \langle U_\beta(\Sigma_{p+2}) V_m(M_{d-p-2})\rangle &= \exp(im\beta \ \text{Link}(\Sigma_{p+2}, M_{d-p-2})) \,\langle V_m(M_{d-p-2}) U_\beta(\tilde{\Sigma}_{p+2})\rangle \end{aligned} \tag{2.14}$$

Imposing the boundary conditions we can determine which Wilson surfaces can end on the boundary and as a result we see that the topological operator that links them acts faithfully as a symmetry generator in the boundary QFT. In order to find the possible boundary conditions for this SymTh, we require the vanishing of the boundary variation of the action

$$\delta S\big|_{\partial M_{d+1}} = \int_{\partial M_{d+1}} \delta a_{p+1} \wedge *_{d+1} da_{p+1} \tag{2.15}$$

the first possibility is given by the Dirichlet boundary condition

$$D: \quad \delta a_{p+1}\big|_{\partial M_{d+1}} = 0 \tag{2.16}$$

This implies that the value of the field is fixed to a background value on the boundary so the gauge transformations $a_{p+1} \to a_{p+1} + d\lambda_p$ must vanish on ∂M_{d+1}. As a result the Wilson surfaces $W_q(M_{p+1})$ can end on the boundary and the topological operator that links them $U_\alpha(\Sigma_{d-p-1})$ can be projected giving rise to the $U(1)$ p-form symmetry operator on the boundary. Instead since the dual field b_{d-p-2} remains freely varying, the other Wilson surfaces $V_m(M_{d-p-2})$ cannot end on the boundary and so the projection of $U_\beta(\Sigma_{p+2})$ gives a non-faithful operator on the boundary QFT.

The second possibility to satisfy $\delta S\big|_{\partial M_{d+1}} = 0$ is given by the Neumann boundary condition

$$N: \quad *_{d+1} da_{p+1}\big|_{\partial M_{d+1}} = 0 \tag{2.17}$$

that says that the field a_{p+1} is allowed to vary freely at the boundary so its Wilson surfaces cannot end on the boundary. The condition for the dual field b_{d-p-2} can be understand introducing the dual field in the action from the beginning considering it as a Lagrange multiplier field that induces the Bianchi identity $df_{p+2} = 0$ once integrated out

$$S_{d+1}^{\text{dual}} = -\int_{M_{d+1}} \frac{1}{2} da_{p+1} \wedge *da_{p+1} - da_{p+1} \wedge db_{d-p-2} \tag{2.18}$$

and instead once we integrate out a_{p+1} we get the definition of the dual field $*da_{p+1} = db_{d-p-2}$. So that, substituting a_{p+1}, the action can be rewritten as

$$S_{d+1}^{\text{dual}} = \int_{M_{d+1}} -\frac{1}{2} db_{d-p-2} \wedge *db_{d-p-2} \tag{2.19}$$

Looking at its boundary variation we can conclude that

$$D(a_{p+1}) = N(b_{d-p-2}) \qquad N(a_{p+1}) = D(b_{d-p-2}) \tag{2.20}$$

since we have seen that when a_{p+1} is fixed at the boundary then the dual field is freely varying and viceversa. The Dirichlet and Neumann boundary conditions for the same field are mutually exclusive and they are all free boundary conditions. In fact, the bulk fields will induce a free theory living on the boundary given by the gauge field a_{p+1} if we impose $N(a_{p+1})$ boundary conditions or by the dual field b_{d-p-2} if we have $D(a_{p+1})$ b.c.

2.2.2 Sandwich construction for the SymTh

Despite the bulk theory not being topological we are always interested into isolating the symmetry sector of the physical boundary QFT, trying to factor out the dynamics of the gauge field in the bulk and this is exactly what is done implementing the sandwich construction. If we recall the sandwich picture of the SymTFT we remember that the shrinking of the interval to zero was possible only because the bulk theory was topological. Now instead the partition function of the Maxwell theory is not topological and will depend on the length of the interval, so the solution is to factorize the partition function, separating the boundary part due to the physical QFT from the quantum fluctuation of the configurations of the bulk theory.

If we consider, as usual, the SymTh as living in the manifold $M_{d+1} = M_d \times [0, L]$ the simple case is to impose Dirichlet boundary conditions for both $x = 0$ and $x = L$ boundaries. Such conditions can be written as states in the braket notation

$$\begin{aligned} \langle D(a_{p+1,0})|_{x=0} &= \int \mathcal{D}a_{p+1} \langle a_{p+1}| \, \delta(a_{p+1,0} - a_{p+1}) \\ |D(a_{p+1,L})\rangle_{x=L} &= \int \mathcal{D}a_{p+1} \delta(a_{p+1,L} - a_{p+1}) \, |a_{p+1}\rangle \end{aligned} \tag{2.21}$$

Since the braket of these boundary conditions gives the delta function $\delta(a_{p+1,0} - a_{p+1,L})$, the partition function of this theory corresponds to the euclidean propagator of a Maxwell theory with space manifold M_d: $G_d(a_{p+1,0}, a_{p+1,L}, L)$ that reduces to a delta function for $L \to 0$. The point is to factorize the partition function in order to separate the bulk physics and check that the result is the same.

We can decompose any field configuration in the $(d+1)$-dimensional theory considering the fluctuations around a classical solution $a_{p+1,cl}$ which is the configuration obtained fixing the gauge and making a choice of $a_{p+1,0}$ and $a_{p+1,L}$. The fluctuation field can be called $a_{p+1,\delta}$ and should be vanishing on the boundary, so the action can be written as

$$S = S_{cl}(a_{p+1,cl}) + S_{\delta}(a_{p+1,\delta}) \tag{2.22}$$

and consequently the partition function decomposes as the classical contribution times the path integral over all the fluctuations in the bulk.

$$Z = e^{-S_{cl}} \int_{a_{p+1,\delta}|_{\partial M_{d+1}}=0} \mathcal{D}a_{p+1,\delta} e^{-S_\delta} = e^{-S_{cl}} \cdot Z_{\text{bulk}} \tag{2.23}$$

With the classical partition function we can define a functional normalized to 1 in the space of classical configuration

$$\frac{e^{-S_{cl}(a_{p+1,cl})}}{\int \mathcal{D}a_{p+1,cl} \;\; e^{-S_{cl}(a_{p+1,cl})}} \tag{2.24}$$

so we see that it is a functional with support only on $a_{p+1,0} = a_{p+1,L}$, since as $L \to 0$ every classical solution with $a_{p+1,0} \neq a_{p+1,L}$ is exponentially suppressed by the action going to infinity in the exponential. As a result the functional (2.24) is a delta function as wanted, since it is also normalized to one

$$\langle D(a_{p+1,0})|D(a_{p+1,L})\rangle = Z_{\text{bulk}} \cdot e^{-S_{cl}} \sim_{L\to 0} \delta(a_{p+1,0} - a_{p+1,L}) \tag{2.25}$$

The same argument can be repeated and extended if we change the boundary conditions or if we add terms to the original action of the SymTh, the important result is that we can decouple the bulk physics from the boundary separating in the partition function the factor Z_{bulk}. Furthermore, the key point is that the action of the topological operators of the SymTh, that can be projected to realize the symmetries in the boundary theory, does not depend on the size L of the interval. So, understand how to take the limit $L \to 0$ gives information about the theory that we are describing, but it is not necessary in order to study the symmetries of the boundary QFT.

Chapter 3

Elements of string theory and compactification

In the first part of this chapter we will outline some relevant aspects of string theory, not in order to give a complete introduction, but only to present the notions that will be necessary for the understanding of the next chapter. The purpose is the characterization of the type II superstring theory, its flux sector and the T-duality that relates IIA and IIB theories. In order to do this we start with a brief presentation of the bosonic string and the symmetries of its action. The second part instead will address the issue of compactification of the extra dimensions in order to get to the four dimensional universe in which we live. We will consider a particular case of the Calabi-Yau manifolds given by the conifold geometry.

3.1 Basic notions of string theory

String theory was developed in the late 1960s, initially it was proposed to describe the strong nuclear force and then it became the most suitable candidate for a quantum theory of gravity unifying the forces of nature in a single quantum mechanical framework. At low energies string theory naturally gives rise to general relativity, gauge theories and fields. For this discussions we follow [47]-[50].

3.1.1 The bosonic string

The starting point of the theory is that at the fundamental level, matter does not consist of point-like particles but rather of strings, lines and loops. The strings sweep out a worldsheet, which is the analogue of the worldline for a point-like particle, but it's a surface in spacetime that can be parametrized by $X^\mu(\tau,\sigma)$, depending on the two coordinates of the worldsheet τ and σ. For closed strings we consider σ periodic requiring $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$.

In order to describe the dynamics of the strings, we need an action that should be proportional to the area of the worldsheet and independent from the parametrization (τ,σ). We can define then the Nambu-Goto action for the relativistic string as

$$S_{N-G} = -T \int d\sigma d\tau \sqrt{-\det\gamma} \tag{3.1}$$

where the induced metric on the worldsheet is $\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$, with α and β that can be σ or τ. T is the tension of the string, namely the mass per unit length, historically defined as

$$T = \frac{1}{2\pi\alpha'} \tag{3.2}$$

using α' which is called universal Regge slope. Since we are interested in performing the quantization of the string, it is better to use an equivalent formulation, given by the Polyakov action for the bosonic string:

$$S_P = \int -\frac{1}{4\pi\alpha'} d^2\sigma\sqrt{-g}\; g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} \tag{3.3}$$

where we introduce a new field $g_{\alpha\beta}$, which is the dynamical metric on the worldsheet and $g = \det(g_{\alpha\beta})$. It is an independent variable that can be determined by its own equation of motion and can be written as

$$g_{\alpha\beta} = 2\, f(\sigma, \tau)\; \partial_\alpha X \cdot \partial_\beta X \tag{3.4}$$

with $f(\sigma, \tau)$ that is a conformal factor, that simplifies when we compute the equation of motion for the field X^μ

$$\partial_\alpha(\sqrt{-g}\; g^{\alpha\beta}\partial_\beta X^\mu) = 0 \tag{3.5}$$

Symmetries of the theory

Let's consider now the symmetries of the Polyakov action. Firstly, we have the Poincarè invariance which can be seen as a global symmetry from the point of view of the worldsheet and acts as $X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu$. Another symmetry is the one given by reparametrization invariance, since we can redefine the worldsheet coordinates with a diffeomorphism as

$$(\sigma, \tau) \to (\tilde{\sigma}(\sigma, \tau), \tilde{\tau}(\sigma, \tau)) \tag{3.6}$$

Under this transformation the field X^μ changes as a scalar and the metric with two jacobian matrices as usually does a two tensor. Lastly, we have what is called Weyl invariance, that leaves invariant X^μ and acts on the metric as

$$g_{\alpha\beta}(\sigma, \tau) \to \Omega^2(\sigma, \tau) g_{\alpha\beta}(\sigma, \tau) \tag{3.7}$$

This is a gauge symmetry of the string, since the parameter of the transformation depends on the worldsheet coordinates. We can see that the action is invariant because the factors from the transformations of $\sqrt{-g}$ and $g^{\alpha\beta}$ cancel each other since we are in the two dimensional worldsheet. We can exploit the ambiguity given by the gauge symmetry to choose the metric in the conformal gauge with $\phi(\sigma, \tau)$ that is a function of the worldsheet.

$$g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta} \tag{3.8}$$

A useful thing to notice is that any 2-dimensional theory of gravity with both diffeomorphism and Weyl invariance, when the background metric is fixed corresponds to a conformally invariant theory, namely a theory that is independent on any length scales. When the metric is dynamical, as in string theory, the conformal gauge symmetry correspond to residual gauge transformations, namely diffeomorphisms that can be compensated by a Weyl transformation.

From the study of conformal field theories (CFT), [52]-[53], we can recall the useful notion of operator product expansion, called OPE, that allows to describe the behavior of two local operators inserted at nearby points. The product is approximated by an expansion of operators at one of these points

$$O_i(z, \bar{z}) O_j(w, \bar{w}) = \sum_k C^k_{ij}(z - w, \bar{z} - \bar{w}) O_k(w, \bar{w}) \tag{3.9}$$

Using this to expand the product of the stress-energy tensors $T(z)T(w)$ we can define the most important quantity that characterizes a conformal field theory, namely the central charge c, which is the coefficient of the term

$$\frac{c}{2\,(z - w)^4} \tag{3.10}$$

in the OPE of $T(z)T(w)$. The central charge gives information about the degrees of the freedom of the CFT.

Since gauge symmetries are just redundancies and not real symmetries, they are not allowed to have anomalies, because that would bring to inconsistencies. It can be seen that the only way to avoid an anomaly on Weyl symmetry is to impose that the central charge of the conformal field theory is vanishing. However, in general to have a non trivial CFT we always have $c > 0$.

The solution comes from the path integral quantization of Polyakov action, which should be done by an integration only over the physically distinct configurations not related by diffeomorphisms and Weyl transformations. Recalling the quantization of Yang-Mills theories, the trick is to introduce ghost fields, that allow to write the Faddeev-Popov determinant and to cancel the unphysical gauge degrees of freedom. The relevant result is that the central charge of the CFT that describes the ghost system is given by

$$c = -26 \tag{3.11}$$

As a consequence the theory of strings is forced to have exactly the right degrees of freedom to cancel the ghost contribution to the central charge. This means that we need a CFT for the strings with $c = 26$, one possibility is to consider a theory with D scalar fields X^μ, where each one has $c = 1$, so we need

$$D = 26 \tag{3.12}$$

that, in this special case of a CFT composed only of scalar fields, is the critical dimension of spacetime for the bosonic string theory.

Spectrum of the string and physical states

There are various methods to perform the quantization of the bosonic string, if we use the path integral formulation outlined above we get the BRST quantization. We can see that in the theory where we have introduced the ghost fields we have an additional invariance of the action under fermionic transformations called BRST symmetry, which gives the following variation of the scalar fields:

$$\delta_B X^\mu = i\epsilon \left(c\partial + \bar{c}\bar{\partial}\right) X^\mu \tag{3.13}$$

and also transforms the ghost fields. It is a fermionic symmetry because it mixes commuting and anticommuting objects, so the parameter ϵ must be anticommuting. From the current of this symmetry we can build the conserved charge $\mathcal{Q}_B$, which has the relevant property of being nilpotent in order to be conserved once we change the gauge fixing

$$\mathcal{Q}_B^2 = 0 \tag{3.14}$$

This operator allows to define the true physical states of the theory as a set of equivalence classes given by the cohomology of $\mathcal{Q}_B$, namely the states that are closed: annihilated by $\mathcal{Q}_B$, but not exact: they cannot be written as $|\psi\rangle = \mathcal{Q}_B |\chi\rangle$. The elements of this space

$$\mathcal{H} = \frac{\mathcal{H}_{\text{closed}}}{\mathcal{H}_{\text{exact}}} \tag{3.15}$$

are equivalence classes, so two closed forms belong to the same class if their difference is an exact form. This procedure defines the spectrum of the theory, solving the problem that arises from the mode expansions of the X^μ fields. The general oscillatory modes that solve the equations of motion (3.5) in fact includes states of negative norm, that need to be ruled out by imposing some constraints by hands. Now instead, we get a systematic way to define the physical states and with BRST quantization we have also an additional reason to require $c = 26$ for the CFT that describes the string degrees of freedom. In fact, only with this value of the central charge we can have a nilpotent BRST charge.

Open strings and interactions

The objects that we can have in the theory are closed or open strings, the dynamics of open strings is still described by Polyakov action, but we need also suitable boundary conditions on the end points. Considering the evolution between two configurations at τ_i and τ_f, the two possibilities to vanish the boundary variation of the action are:

- Neumann boundary condition, given by $\partial_\sigma X^\mu = 0$ at $\sigma = 0, \pi$, so this implies that the end points of the string δX^μ are freely varying;
- Dirichlet boundary condition, which sets $\delta X^\mu = 0$ at $\sigma = 0, \pi$, meaning that the end points of the string are fixed at some constant position in space.

An interesting combination is to impose Dirichlet b.c. for some coordinates and Neumann b.c. for the others, see Fig. 3.1, so that the end points of the string are constrained to lie on a $(p+1)$-dimensional hypersurface in spacetime, which is called Dp-brane. D stands for Dirichlet and p indicates the spatial dimensions of the brane. The branes should be considered as truly dynamical objects of the theory, that indeed contains not only strings but also higher-dimensional hypersurfaces.

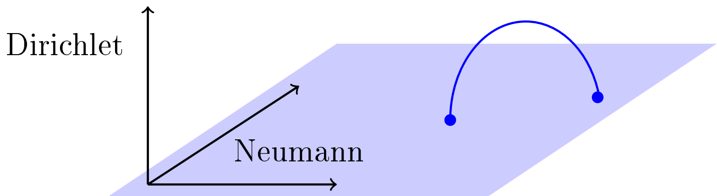

Figure 3.1: In the coordinates represented by the vertical axis we impose Dirichlet b.c. and on the others Neumann b.c so we fix the blue hypersurface in which the end points of the string live

All the information about interaction of strings are already contained in the free theory, because two interacting strings form a smooth worldsheet which locally looks like a free propagating string. Since there are no local off-shell gauge invariant observables in a theory of gravity, we cannot compute correlation functions as in QFT. The only thing that we have is the S-matrix, which is obtained by taking to infinity the points in the correlation function, where the gauge transformations vanish asymptotically. Perturbative expansions in string theory correspond to sum up worldsheets of different topology, namely to sum over Riemann surfaces of increasing genus, spheres at three level, torus at one loop and so on, with the insertion of operators that represent the initial and final states.

3.1.2 Adding fermions to the theory

The spectrum of the bosonic string does not contain fermions, that of course are fundamental in order to describe nature. If we add to the theory the fermionic modes, the resulting worldsheet theory is required to be supersymmetric. To take into account the supersymmetry that relates bosons and fermions we use what is called Ramond-Neveu-Schwarz (RNS) formalism, introducing along with the bosonic field X^μ a new fermionic field $\psi^\mu(\sigma, \tau)$ which is an anticommuting spinor on the worldsheet.

The action that describes both bosonic and fermionic degrees of freedom in conformal gauge is then

$$S = -\frac{1}{2\pi}\int d\sigma d\tau \left(\partial_\alpha X_\mu \partial^\alpha X^\mu + \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu\right) \tag{3.16}$$

in which ρ^α represents the two-dimensional Dirac matrices, satisfying the Clifford algebra $\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta}$ with $\alpha, \beta = 0, 1$. With the addition of supersymmetry the action is invariant under the superconformal symmetry that generalizes the notion of conformal field theories and ensures the elimination of negative-norm states in the spectrum. Then to determine the configurations of closed or open strings, we need to impose some specific boundary conditions, that now are called Ramond (R) or Neveu-Schwarz (NS), for example for closed strings they correspond to periodicity and antiperiodicity respectively.

Once we write down the modes in the spectrum of closed strings, we get two sets of fermionic modes, corresponding to left and right moving sectors. In order to project out the unphysical states we need to choose in the NS-sector only states with positive G-parity, which means that the operator

$$G = (-1)^{F+1} \tag{3.17}$$

defined in the NS sector is positive. F is the number of worldsheet fermionic excitations, so we are forced to have always an odd number of fermions in the NS states. In the R-sector instead, we can choose freely the sign of G and the result is that we can build two different types of superstring theories

- IIB theory, in which the left and right moving R-sector ground states have the same chirality and so positive G-parity. The gauge fields that are present only in type IIB are a scalar C, a 2-form $C_{\mu\nu}$ and a 4-form field $C_{\mu\nu\rho\sigma}$, with the restriction that the 4-form has a self-dual field strength $F_5 = *F_5$.
- IIA theory, where instead the two R-sectors have ground states with opposite chirality. The corresponding gauge fields are a 1-form C_μ and a 3-form $C_{\mu\nu\rho}$ and are called Ramond-Ramond fields.

Both theories have the same content in terms of fields in the NS-NS sector, which consists of the dilaton ϕ, the antisymmetric tensor $B_{\mu\nu}$ and the graviton $G_{\mu\nu}$, that are three massless bosonic fields.

Applying the quantization procedure, we find the critical dimension for the superstring, that is necessarily fixed in order to cancel Weyl anomaly. With the addition of supersymmetry the theory of ghosts that we need in the path integral quantization is different and requires the string degrees of freedom to give as central charge $c = 15$. Then, if we add bosons to the theory we need to add also the corresponding fermions because of supersymmetry. We should take into account that free bosons have $c = 1$ and free fermions have $c = 1/2$ in order to find that the critical dimension in this case is

$$D = 10 \tag{3.18}$$

Branes and fluxes in the superstring theories

The notion of Dp-branes introduced for the bosonic strings exists also in superstring theories. If in type IIA and IIB we insert Dp-branes as dynamical objects, we obtain a theory that has closed strings and also open strings that end on the branes. Some Dp-branes can carry a conserved charge that guarantees their stability and their presence usually breaks some of the symmetries of the superstring vacuum.

The superstring theories contain a great number of massless antisymmetric tensor gauge fields, that can be realized as differential forms and used to define the corresponding gauge invariant field strengths. If we think about Maxwell theory, the field strength F describes both electric and magnetic fields and can be used to define the electric and magnetic charges as

$$e = \int_{S^2} *F \qquad\qquad m = \int_{S^2} F \tag{3.19}$$

So we can use the analogy with Maxwell theory to understand how branes are electrically or magnetically charged under a given field strength. In general, p-branes couple to $(p+1)$-form gauge fields, since they carry a charge and so they can act as sources for the gauge fields. This can be understood looking at the case of a $D0$-brane which corresponds to a point-like particle and constitutes a source for a 1-form field. Its charge is given by the gaussian integral of $*F$ over a $(D-2)$-sphere, which is the surface that surrounds a point in D dimensions. Then the magnetic dual of this point-like particle carries the magnetic charge given by the integral of F on S^2.

So considering a p-brane, we say that it's coupled to a $(p+1)$-form gauge field A_{p+1}, in the sense that it's electrically charged by

$$\mu_e = \int_{S^{D-p-2}} *F_{p+2} \tag{3.20}$$

where $F_{p+2} = dA_{p+1}$ is the field strength and S^{D-p-2} is the sphere that we need to surround a p-brane. The magnetic dual charge is given by

$$\mu_m = \int_{S^{p+2}} F_{p+2} \tag{3.21}$$

and we notice that a $(p+2)$-dimensional sphere surrounds a $(D-p-4)$-brane. So in ten dimensions the magnetic dual of a p-brane is given by a $(6-p)$-brane.

Looking at the superstring theories of type II we have that IIA contains gauge fields that are 1-forms and 3-forms, so the stable branes in this case are the ones that carry the corresponding charges, they are Dp-branes with $p = 0, 2, 4, 6, 8$. This reflects the fact that the fluxes of the IIA theory are $F_0, F_2, F_4, F_6, F_8, F_{10}$. In type IIB theory instead the RR-sector contains gauge fields that are 0-form, 2-form and 4-form. The stable branes then are a $D1$-brane that is electrically charged under the 2-form gauge field and a $D5$-brane which is magnetically charged under it. Moreover, the 4-form field is coupled to a $D3$-brane both electrically and magnetically, so this brane carries a self-dual charge and this reflects the self-duality of the 5-dimensional field strength. We can introduce also a $D9$-brane, so the fluxes that are present in IIB superstring theory are F_1, F_3, F_5, F_7, F_9.

3.1.3 Low-energy effective actions

In order to define a consistent theory of gravity we need to consider how to describe strings moving in a curved background. In conformal gauge, the worldsheet theory for the bosonic string in a general non flat spacetime with metric $G_{\mu\nu}(X)$ is given by the interacting action

$$S = \frac{1}{4\pi\alpha'} \int d\sigma d\tau \; G_{\mu\nu}(X)\partial_\alpha X^\mu \partial^\alpha X^\nu \tag{3.22}$$

where the spacetime metric $G_{\mu\nu}(X)$ can be expanded considering small fluctuations around the flat metric. It can be shown that this metric does not need to be artificially added to the action, but instead it's built using the graviton states that are already contained in the string theory. Moreover we should underline that the perturbative description is suitable for this theory if it is weakly coupled and this is true when the string length scale $\sqrt{\alpha'}$ is much smaller than the radius of curvature of the geometry we are considering.

Let's proceed with the quantization of the interacting action presented above, in particular we need to check that with the renormalization procedure the Weyl invariance is still preserved, since it's a gauge symmetry that cannot be lost. We need to consider the β-functions of the theory, which describe how the fields and the couplings change with the renormalization energy scale μ. The requirement is that the β-functions are vanishing so that we find a theory independent from μ

$$\beta(G_{\mu\nu}) \sim \mu\frac{\partial}{\partial\mu}G_{\mu\nu} = 0 \tag{3.23}$$

The same reasoning must be repeated for the other couplings of the theory, the ones with the dilaton Φ and the antisymmetric tensor $B_{\mu\nu}$, from which we define the gauge invariant field strength $H = dB$. The dilaton coupling in particular doesn't respect Weyl invariance, but this problem can be solved because the non invariance is compensated by the one-loop contributions coming from the coupling to $G_{\mu\nu}$ and $B_{\mu\nu}$. Finally we can present the three β-functions, that at one-loop are

$$\begin{aligned} \beta_{\mu\nu}(G) &= \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\rho\sigma}H_\nu^{\ \rho\sigma} \\ \beta_{\mu\nu}(B) &= -\frac{\alpha'}{2}\nabla^\lambda H_{\lambda\mu\nu} + \alpha'\nabla^\lambda\Phi H_{\lambda\mu\nu} \\ \beta(\Phi) &= -\frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi - \frac{\alpha'}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} \end{aligned} \tag{3.24}$$

where $R_{\mu\nu}$ is the Ricci tensor. They need to satisfy

$$\beta_{\mu\nu}(G) = \beta_{\mu\nu}(B) = \beta(\Phi) = 0 \tag{3.25}$$

in order to define a string theory in a general curved background in which Weyl invariance is preserved. These three equations can be seen as equations of motion coming from a 26-dimensional spacetime action, which is the low-energy effective action for the bosonic string. We say that it's a low-energy description since we want to determine the β-function only up to one-loop.

$$S_b = \frac{1}{2\kappa_0^2}\int d^{26}X\ \sqrt{-G}\ e^{-2\Phi}\Big(R - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} + 4\,\partial_\mu\Phi\partial^\mu\Phi\Big) \tag{3.26}$$

where κ_0 is a constant that can be determined when we couple some sources to the theory. Taking the variation of the action respect to the three fields we can see that we get the equations for the β-functions

$$\delta S_b = \frac{1}{2\kappa_0^2\,\alpha'}\int d^{26}X\sqrt{-G}e^{-2\Phi}\Big(\delta G_{\mu\nu}\beta^{\mu\nu}(G) - \delta B_{\mu\nu}\beta^{\mu\nu}(B) - \Big(2\delta\Phi + \frac{1}{2}G^{\mu\nu}\delta G_{\mu\nu}\Big)(\beta_\rho^\rho(G) - 4\beta(\Phi))\Big) \tag{3.27}$$

The same procedure can be applied also to type II superstring theories in order to built the supergravity low-energy effective actions. The NS-NS sector is exactly given by the action S_b (3.26) and it's the same for IIA and IIB, then we have additional terms in the actions that describe the dynamics of the different fields of the two theories in the 10-dimensional spacetime. For type IIA we consider $H_3 = dB_2$ as usual and $F_2 = dC_1$, $F_4 = dC_3$, $\tilde{F}_4 = F_4 - C_1 \wedge H_3$ and in the following $|F_i|^2 = F_i \wedge *F_i$

$$S_{IIA} = S_b - \frac{1}{4\kappa_0^2}\int d^{10}X\Big(\sqrt{-G}\ (\,|F_2|^2 + |\tilde{F}_4|^2\,) + B_2 \wedge F_4 \wedge F_4\Big) \tag{3.28}$$

The type IIB instead contains H_3, $F_1 = dC_0$, $F_3 = dC_2$ and $F_5 = dC_4$, we can define also

$$\tilde{F}_3 = F_3 - C_0 \wedge H_3 \qquad\qquad \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \tag{3.29}$$

so the action is given by

$$S_{IIB} = S_b - \frac{1}{4\kappa_0^2} \int \left(\sqrt{-G} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right) + C_4 \wedge H_3 \wedge F_3 \right) \tag{3.30}$$

These low-energy actions for type II supergravity theories are the starting point for the next chapter, in which we will reformulate them in order to find suitable expressions to implement the dimensional reduction. About this we can say that the spacetime equations of motions given by the vanishing beta functions have various solutions and how to chose the right vacuum which gives the spacetime of our universe it's still an open problem. However one possible simple solution is the geometry given by the compactification of the extra dimensions, which is presented in the next sections.

3.1.4 T-duality

Let's present the notion of T-duality starting from the case of the bosonic string, in which one of the 25 spatial dimensions is compactified in a circle. The spacetime geometry so is given by

$$\mathbb{R}^{24,1} \times S^1 \tag{3.31}$$

where the coordinate of one direction takes values on a circle of radius R. The T-duality is a transformation that maps the radius of the circle as

$$R \to \tilde{R} = \frac{\alpha'}{R} \tag{3.32}$$

So this means that the compactification on a circle of radius R is physically equivalent to the compactification on a circle of radius $\tilde{R}$, which is an intuition based on the fact that the extended nature of the strings implies that they do not discriminate between large and small circles.

Let's look at the realization of T-duality from the Dp-branes point of view, their behavior depends on which direction we choose to compactify into a circle. If the direction is transverse with respect to the brane then we see that a p-brane transforms into a $(p+1)$-brane since with T-duality we exchange Neumann and Dirichlet boundary conditions. So analogously if we start with a p-brane wrapped around the circle direction, which means that the string has Neumann boundary conditions, then we get a $(p-1)$-brane localized in some point of the T-dual direction.

T-duality applied to type II superstring theories transforms type IIA theory on type IIB and viceversa, [51]. In fact, under this transformation only the right-moving sector of bosonic and fermionic fields changes sign and the difference between type IIA and IIB is exactly given by the relative chirality of the left and right moving sectors. The branes of IIA theory have p even, while the ones of IIB have p odd. This mapping is possible since, as we said before, the general rule is that if we perform the T-duality in one of the direction in which a p-brane is wrapped then it's mapped into a $(p-1)$-brane, localized in the dual circle of the other theory. So finally the branes that lie on the compactified direction and the ones that are unwrapped are interchanged.

Moreover this is what we will do also with the fluxes in the next chapter. The T-duality relates the fluxes of IIA and IIB theories in a way that mimic the correspondence between branes. So for example considering a flux F_i in IIB superstring theory, we take the coefficient of the part that is expanded on the compactified direction and that will contribute to the flux F_{i-1} of IIA theory. The part instead that is expanded only on the other coordinates of spacetime will enter in the flux F_{i+1} of IIA, after we add a dependence on the dual circle coordinate.

3.2 Compactification of the extra dimensions

We have seen that superstring theories live in a ten dimensional spacetime, so the point now is how we can connect a theory with all these extra dimensions to a universe described by a four dimensional spacetime. The idea is given by the Kaluza-Klein compactification. Since string theory is a theory of gravity, we are free to consider that the extra dimensions can be curled up. So we can find solutions to the vacuum Einstein equations $\mathcal{R}_{\mu\nu} = 0$, in which the metric is the direct product of two metrics and corresponds to a space geometry given by the product of an external and an internal manifold:

$$R^{1,3} \times X$$

where X is in general a compact 6-dimensional Ricci-flat manifold. If the characteristic length scale of the space X is small enough, then the presence of these extra dimensions would not be observed in experiments. Even if the internal manifolds are invisible, their topological properties determine the particle content and the structure of the four-dimensional theory.

Let's look at the simplest compactification, given by the background that we use above to explain T-duality:

$$\mathbb{R}^{1,24} \times S^1 \tag{3.33}$$

Considering a circle S^1 of radius r we have the periodicity condition on the coordinate X^{25}:

$$X^{25} \sim X^{25} + 2\pi r \tag{3.34}$$

We would like to describe the non-compact $\mathbb{R}^{1,24}$ Minkowski space at length scales greater than r so the motion on the compact coordinate is negligible. The metric can be parametrized by

$$ds^2 = \tilde{G}_{\mu\nu} dX^\mu dX^\nu + e^{2\tilde{\sigma}}(dX^{25} + A_\mu dX^\mu)^2 \tag{3.35}$$

where μ and ν indices take values $0, ..., 24$ over the non-compact directions and we use the metric $\tilde{G}_{\mu\nu}$ in which we reabsorbed the dilaton factor. We introduce also a scalar field $\tilde{\sigma}$ and a vector field A_μ which transforms as a gauge field, since the diffeomorphisms along the compact direction

$$X^{25} \to X^{25} + \Lambda(X^\mu) \tag{3.36}$$

can be seen as a gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda \tag{3.37}$$

The 26-dimensional action (3.26) can be rewritten using the Ricci scalar given by this parametrization of the metric

$$R^{(26)} = R^{(25)} - 2\, e^{-\tilde{\sigma}} \nabla^2 e^{\tilde{\sigma}} - \frac{1}{4} e^{2\tilde{\sigma}} F_{\mu\nu} F^{\mu\nu} \tag{3.38}$$

Considering the other fields in the low-energy effective action, it can be shown that the dilaton scalar field Φ reduces to a scalar field in 25 dimensions $\Phi^{(25)}$. So at low-energy we get a theory of one $U(1)$ gauge field A_μ and two massless scalars $\Phi^{(25)}$, $\tilde{\sigma}$ in a geometry with metric $G_{\mu\nu}$. If we consider all the fields independent from the coordinate X^{25}, we can write the action as

$$S = \frac{2\pi r}{2\kappa_0^2} \int d^{25}X \, \sqrt{-\tilde{G}} \, e^{-2\Phi^{(25)}} \Big(R^{(25)} - \frac{1}{4} e^{2\tilde{\sigma}} F_{\mu\nu} F^{\mu\nu} + 4\, \partial_\mu \Phi^{(25)} \partial^\mu \Phi^{(25)} - \partial_\mu \tilde{\sigma} \partial^\mu \tilde{\sigma} \Big) \tag{3.39}$$

It can be shown [49] also how to insert in the action the field $B_{\mu\nu}$, which after the compactification implies the presence of another vector field $\tilde{A}_\mu = B_{\mu 25}$. However, let's focus on the dilaton field and see what happens if it depends on the periodic direction $X^{(25)}$. We can use the Fourier expansion on the circle

$$\Phi(X^\mu, X^{25}) = \sum_{n=-\infty}^{\infty} \Phi_n(X^\mu)\, e^{\frac{inX^{25}}{r}} \tag{3.40}$$

so that the kinetic terms in the action (3.26) can be written as

$$\int d^{26}X\left(\partial_\mu\Phi\partial^\mu\Phi + (\partial_{25}\Phi)^2\right) = 2\pi r\int d^{25}X\sum_{n=-\infty}^{\infty}\left(\partial_\mu\Phi_n\partial^\mu\Phi_{-n} + \frac{n^2}{r^2}|\Phi_n|^2\right) \tag{3.41}$$

since the momentum in the periodic dimension is quantized as $p_{25} = n/r$. Through the Fourier decomposition we see that we have an infinite tower of scalar fields labeled by n which have mass

$$M_n^2 = -p^\mu p_\mu = \frac{n^2}{r^2} \tag{3.42}$$

If we consider the low-energy limit, or analogously length scales much larger than r, we see that all the $n \neq 0$ scalar modes are heavy when r is small and so can be neglected with respect to the massless zero mode $n = 0$. Only above the energy r^{-1} we can see the tower of Kaluza-Klein modes, but it is not what we are interested in to describe low-energy actions.

Considering the compactification from the point of view of the string we see that it is allowed a more general boundary condition for the closed string

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi m\, r \qquad m \in \mathbb{Z} \tag{3.43}$$

where the winding number m measures how many times the string winds around S^1. Looking at the mass spectrum it can be shown that the massless states that are present are the ones obtained with zero momentum and zero winding number and coincide with the fields present in the spacetime action discussed before.

3.2.1 Calabi-Yau manifolds

The most important kind of compact, complex manifolds that admit metrics with $\mathcal{R}_{\mu\nu} = 0$ are called Calabi-Yau manifolds. Compactification on these spaces leads to vacua with less supersymmetry in four dimensions, so they are phenomenologically interesting since supersymmetry is not observed in particle physics up to the energy scale of TeV.

In order to speak about Calabi-Yau manifolds we need to introduce the notion of hermitian manifolds, which are complex manifolds with transitions functions and their inverses that are holomorphic and such that we have a metric on the manifold that satisfies: $g_{ab} = g_{\bar{a}\bar{b}} = 0$. Then we can define a Kähler manifold, which is an hermitian manifold on which the Kähler form J is closed

$$J = ig_{a\bar{b}}\, dz^a \wedge d\bar{z}^{\bar{b}} \qquad\qquad dJ = 0 \tag{3.44}$$

This condition implies that the metric on these manifold can be written locally in the form of a derivative of some potential, called Kähler potential

$$g_{a\bar{b}} = \frac{\partial}{\partial z^a}\frac{\partial}{\partial \bar{z}^{\bar{b}}}\mathcal{K}(z, \bar{z}) \tag{3.45}$$

On a complex manifold we can define (p, q)-forms as having p holomorphic and q antiholomorphic indices. In terms of complex local coordinates, only the mixed components of the

Ricci tensor are non vanishing for an hermitian manifold, and we can define a $(1,1)$-form: $\mathcal{R} = iR_{a\bar{b}}dz^a \wedge d\bar{z}^{\bar{b}}$ called the Ricci form. For a Kähler manifold then the Ricci form is closed, so it is a representative of a cohomology class belonging to $H^{1,1}(M)$ that is called first Chern class

$$c_1 = \frac{1}{2\pi}[\mathcal{R}] \tag{3.46}$$

From the geometrical point of view, a defining property of Calabi-Yau manifolds is the presence of a covariantly constant spinor η, which is invariant under the parallel transport along any curve of the manifold. Considering the two chiralities $\eta_\pm$ of the spinor we can express the components of the Kähler form J as a bilinear

$$J_{mn} = \eta_+^\dagger \gamma_{mn}\, \eta_+ \tag{3.47}$$

and define the complex structure

$$\Omega = \frac{1}{6}\Omega_{mnp}\, dx^m \wedge dx^n \wedge dx^p \qquad\qquad \Omega_{mnp} = \eta_-^\dagger \gamma_{mnp}\, \eta_+ \tag{3.48}$$

using γ_{mn} and γ_{mnp} which are given by the antisymmetric product of two and three 6-dimensional gamma matrices respectively. Ω is a closed $(3,0)$ form that in a complex manifold can be globally parametrized by complex coordinates and in a Kähler manifold satisfies $J \wedge \Omega = 0$.

Finally we can give the definition of a Calabi-Yau n-fold as a Kähler manifold in n complex dimensions with $SU(n)$ holonomy and vanishing first Chern class. The only examples in two dimensions are the complex plane $\mathbb{C}$ and the two-torus T^2. The case of greatest interest is given by Calabi-Yau three-folds, which have six real dimensions (three complex) and so they allows to reduce the 10-dimensional space into four dimensions. The compactification on this type of manifolds of the type II superstring gives $\mathcal{N} = 2$ supersymmetric 4-dimensional theories.

An important characterization of the Calabi-Yau manifold is given by the Hodge numbers $h^{p,q}$, which count the number of harmonic (p,q)-forms on the manifold. These numbers are a decomposition of the more general topological numbers associated to a manifold which are the Betti numbers b_p:

$$b_k = \sum_{p=0}^{k} h^{p,k-p} \tag{3.49}$$

The Betti number b_p measures the dimension of the de Rham cohomology $H^p(M)$ of the manifold M, so it gives the number of independent harmonic p-forms on the manifold. Exploiting Poincarè duality we can see how the Betti numbers can be determined by counting the number of non-trivial cycles of the manifold, namely closed chain that are not boundaries. In our most relevant case, the cohomology of a Calabi-Yau 3-fold is completely characterized by the Hodge numbers $h^{1,1}$ and $h^{2,1}$. However, Calabi-Yau manifolds with some fixed Hodge numbers are not determined uniquely, since we have the possibility to make smooth deformations in the parameter space of their shapes and sizes, without changing the topology. This space, called moduli space is a continuously infinite family of manifolds. It can be seen also as the space of possible choices of the expectation values of the massless scalar fields of the theory in four dimensions, which are not fixed by the potential of the theory. Since the Hodge numbers does not completely fix the topology and the moduli fields parametrize the changes of size or shape of a manifold but not the changes of topology, another possibility is that the same numbers refer to topologically different Calabi-Yau manifolds and this is the case of conifold transitions.

Considering a fixed set of Hodge numbers, the spectrum of fluctuations of the given Calabi-Yau manifold comes from the deformations of the metric or of the antisymmetric tensor fields of the theory once we do the compactification. For example, in type II superstring we have the two-form field B_2 with equation of motion $d * dB_2 = 0$. Since the combination $d * d$ correspond to the Laplacian operator, once we consider the geometry of the 10-dimensional space as $M_4 \times M_6$, also the Laplacian decomposes and the number of massless 4-dimensional fields induced in the resulting theory is given by the number of zero modes of Δ_6, labeled by the Betti numbers. In the case of the 2-form antisymmetric tensor we get $b_2 = h^{1,1}$ scalar fields. The additional detail is that, for the Calabi-Yau three-fold, B_2 combines with the Kähler form J to give a complexified Käler form, so the result gives rise to $h^{1,1}$ massless *complex* scalar fields in four dimensions. From the expansion of the 10-dimensional fields in the cohomology of the internal manifold we get not only scalars but also all the p-dimensional forms that are allowed in the decomposition.

An important feature of the moduli space of a Calabi-Yau three-fold is that it decomposes into the product of two factors

$$\mathcal{M}(M) = \mathcal{M}^{2,1}(M) \times \mathcal{M}^{1,1}(M) \tag{3.50}$$

because it receives contribution both from the deformations of the Kähler form of the manifold M and the fluctuations of its complex structure. These second type of deformations corresponds to change the way we define the parametrization in complex coordinates of Ω and gives rise to the complex structure moduli z^i. In order to describe this sector of the moduli space we need to introduce a basis of the three-cycles A^I, B_J, with $I, J = 0, ..., h^{2,1}$, that satisfies the property $A^I \cap B_J = \delta^I_J$. Analogously we can introduce the dual cohomology basis (α_I, β^I) such that

$$\int_{A^J} \alpha_I = \int \alpha_I \wedge \beta^J = \sqrt{v}\ \delta^J_I \tag{3.51}$$

where v is the volume of the Calabi-Yau manifold. We will see in the last section of this chapter how to exploit these structures in order to implement the compactification of the IIB superstring theory on a Calabi-Yau three-fold.

3.2.2 Conifold geometry

An important role is played also by non-compact Calabi-Yau manifold, in particular because the moduli space, which is the space of all the parameters that define different compact Calabi-Yau manifolds, presents singular points. These are called conifold singularities and they are conveniently analyzed in terms of non-compact spaces obtained magnifying the region around a singularity of the three-fold. The non-compact Calabi-Yau space that we get is called conifold and its geometry is given by a cone, [43]-[46]. In fact it can be described as an hypersurface in $\mathbb{C}^4$ given by the equation:

$$\sum_{A=1}^{4} (w^A)^2 = 0 \qquad\qquad w \in \mathbb{C}^4 \tag{3.52}$$

which is a cone smooth everywhere except at $w^A = 0$. In fact if w^A is a solution then so it is λw^A with $\lambda \in \mathbb{C}$, so the surface is made up of complex lines through the origin and is therefore a cone. It is a manifold that can be considered as a simple non-compact Calabi-Yau 3-fold, a cone over the homogeneous space

$$T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)}$$

that is topologically equivalent to $S^2 \times S^3$ and is endowed with its Sasaki-Einstein metric. The singularity at the tip of the cone can be smoothed out in two ways, called deformation and resolution still preserving the Calabi-Yau structure. It's not hard to show that the base of the cone is $S^2 \times S^3$ because in general the base $\mathcal{N}$ is a manifold given by the intersection of the space of solutions of (3.52) with a sphere of radius r in $\mathbb{C}^4$ given by:

$$\sum_{A=1}^{4} |w^A|^2 = r^2 \tag{3.53}$$

If we consider the coordinates w^A as a four vector w on $\mathbb{C}^4$ and we separate the real and imaginary parts $w = x + iy$, then the equations that define the cone become:

$$x \cdot x = \frac{1}{2}r^2 \qquad y \cdot y = \frac{1}{2}r^2 \qquad x \cdot y = 0 \tag{3.54}$$

The first equation defines an S^3 with radius $r/\sqrt{2}$ and the other two define an S^2 fiber over S^3. Since all these bundles over S^3 are trivial we have that $\mathcal{N}$ has the topology of $S^2 \times S^3$. A metric on the cone can be written as:

$$ds^2 = dr^2 + r^2 d\Sigma^2 \qquad d\Sigma^2 = h_{ab}dx^a dx^b \tag{3.55}$$

where h_{ab} is a metric on the base $\mathcal{N}$. An n-dimensional cone admits a Ricci-flat metric if and only if its base manifold admits an Einstein metric with

$$R_{ab}(h) = (n-2)h_{ab} \tag{3.56}$$

Considering the general expression of the metric on manifolds $\mathcal{N}_{pq}$ that are fiber bundles over $S^2 \times S^2$ with $U(1)$ fibers, we can prove that $\mathcal{N}_{11}$ is $S^2 \times S^3$ and has the correct metric:

$$ds^2(\mathcal{N}_{11}) = \frac{4}{9}\Big(d\psi + \frac{1}{2}\cos\theta_1\, d\phi_1 + \frac{1}{2}\cos\theta_2\, d\phi_2\Big)^2 + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1\, d\phi_1) + \frac{1}{6}(d\theta_2^2 + \sin^2\theta_2\, d\phi_2) \tag{3.57}$$

In order to do that we rewrite equations (3.52) and (3.53) in terms of a matrix W defined by

$$W = \frac{1}{\sqrt{2}}w^A\sigma_A = \frac{1}{\sqrt{2}}\begin{pmatrix} w^3 + iw^4 & w^1 - iw^2 \\ w^1 + iw^2 & -w^3 + iw^4 \end{pmatrix} \tag{3.58}$$

with $\sigma_A = (\sigma_i, i\mathbb{1})$, where the σ_i are the Pauli matrices. Then the equations that define the cone become:

$$det\ W = 0 \qquad Tr(W^\dagger W) = r^2 \tag{3.59}$$

Defining a matrix normalized with the radius $Z = W/r$, the equations can be written in a simpler way

$$det\ Z = 0 \qquad Tr(Z^\dagger Z) = 1 \tag{3.60}$$

If we take a particular solution of these equations, for example

$$Z_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_1 + i\sigma_2) \tag{3.61}$$

the general solution can be written as $Z = LZ_0R^\dagger$, where L and R belong to $SU(2)$. We can check that some matrices, such that $(L, R) = (\theta, \theta^\dagger)$, leave Z_0 fixed

$$\theta = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \tag{3.62}$$

So the manifold $\mathcal{N}_{11}$ can be thought of as the set of matrices (L, R) with the identification $(L, R) = (L\theta, R\theta^\dagger)$, so it is clear that

$$\mathcal{N}_{11} = \frac{SU(2) \times SU(2)}{U(1)} = \frac{S^3 \times S^3}{U(1)} \tag{3.63}$$

where $U(1)$ is generated by θ and is embedded symmetrically in the direct product. For general p and q the space $\mathcal{N}_{pq}$ is the set of matrices with the identification $(L, R) = (L\theta^q, R\theta^{p\dagger})$.

Finally the Einstein metric can be obtained using the matrix Z that is the only $U(1)$ invariant quantity that can enter into the metric. We can check also that the Ricci-flat metric is compatible with the Kähler structure of the cone, meaning that the metric can be expressed as a derivative of some potential.

3.2.3 Type II theories dimensionally reduced on a Calabi-Yau 3-fold

In order to see the relevance of the conifold geometry for the dimensional reductions, let's consider first the compactification of the type II superstring theories on a Calabi-Yau three-fold. This leads to a four dimensional theory with $\mathcal{N} = 2$ supersymmetry, where the 4-dimensional fields come from the expansion of the 10-dimensional ones over the non-trivial cohomology of the manifold. The zero modes of the fluctuations of the 10-dimensional fields give rise to the moduli fields, which are organized in supermultiplets. Since they are massless states, they are labeled by helicity which is well defined and invariant under Lorentz transformations. The supermultiplets are characterized by the maximal helicity, so we get the supergravity multiplet if the greater helicity is 2, the vector multiplet if it is 1 and the hypermultiplet if it is 1/2.

If we focus on the type IIB theory, we can see that after the compactification we get $h^{2,1}$ abelian vector multiplets and $h^{1,1}+1$ hypermultiplets. The extra hypermultiplet is given by the dilaton, which gives rise to a complex scalar field. There is also the scalar a, usually called axion, coming from the dualization of the 2-form B_2: $*_4 dB_2 = da$. Looking only at the bosonic sector we can recognize the 10-dimensional origin of the massless fields in four dimensions, starting from the field content of the IIB theory: $G_{\mu\nu}$, B_2, Φ, C_0, C_2 and C_4:

- the $h^{2,1}$ vector supermultiplets contain contributions from C_4 and G_{ij};
- the $h^{1,1}$ hypermultiplets from C_4, G_{ij}, B_2 and C_2;
- the universal hypermultiplet gets contributions from Φ, C_0, B_2 and C_2;
- the gravity multiplet form $G_{\mu\nu}$ and C_4.

The result is that the total number of massless scalar fields is

$$2h^{2,1} + 4(h^{1,1} + 1) \tag{3.64}$$

and the number of massless vector fields is

$$h^{2,1} + 1 \tag{3.65}$$

To specify more details, if we consider the scalars we can divide the contributions of the moduli space from the one coming from the field content of the theory. We have $h^{2,1}$ complex massless scalars that parametrize the complex structure fluctuations and $h^{1,1}$ real massless scalars as moduli for the Kähler structure. Then we have the scalars coming from the fields B_2 and C_2, that can be expanded using harmonic 2-forms, from C_0 and the dilaton and from the expansion of C_4 using harmonic 4-forms. Considering that the multiplicity of these is given by the Betti numbers and adding also the axion we get the total number (3.64). These scalars correspond to the content of the $\mathcal{N} = 2$ supersymmetry multiplets given by $h^{2,1}$ vector multiplets and $h^{1,1}+1$ hypermultiplets. So, we see that we need the presence of $h^{2,1}$ vectors to complete the vector multiplets. They come from the expansion of C_4 in the basis of 3-forms (3.51) which leads also to another vector field that is the graviphoton, though we get to the number (3.65).

The case of the IIA theory is completely analogous, with the roles of the Kähler form and of the complex structure that need to be exchanged, so we can derive the correct total number of massless 4-dimensional fields with the substitution:

$$h^{2,1}(IIA) = h^{1,1}(IIB) \qquad\qquad h^{1,1}(IIA) = h^{2,1}(IIB) \tag{3.66}$$

This shows a realization of the T-duality, since it underlines the fact that IIB theory compactified on the Calabi-Yau three-fold M correspond to the IIA theory on the mirror manifold, with the Hodge number interchanged.

Considering now explicitly the low-energy action for the IIB theory (3.30), we can implement the compactification on the Calabi-Yau three-fold following [54]-[55]. In order to give a example, we focus in the case in which the fields H_3, C_0 and C_2 vanish and we expand the flux

$$F_5 = F^I \wedge \alpha_I - G_I \wedge \beta^I \tag{3.67}$$

in the cohomology basis (3.51) of the Calabi-Yau manifold, defining $F^I = dV^I$ and $G_I = dU_I$. V_I and U_I are one-forms which live in the four dimensional spacetime and correspond to abelian gauge fields. The self-duality condition on F_5 relates G_I to F^I with the condition

$$G_I = (Re\mathcal{M})_{IJ}F^J + (Im\mathcal{M})_{IJ} * F^J \tag{3.68}$$

defining the matrix $\mathcal{M}$ as in [54][1] with a function of the z^i fields. The relevant meaning of this relation is that G_I is the magnetic dual of F^I. The result of the reduction of both the matter sector and the sector of the Ricci scalar with the dilaton is given by

$$S_4 = \frac{1}{\kappa_{(4)}^2}\int \frac{1}{2}R \, - g_{i\bar{j}}\, dz^i \wedge * dz^{\bar{j}} - h_{ab}\, dq^a \wedge * dq^v + \frac{1}{2}F^I \wedge G_I \tag{3.69}$$

where $\kappa_{(4)}^2 = \kappa_0^2/v$ with v that is the volume of the internal manifold. The scalar fields z^i are the moduli of the complex structure with metric $g_{i\bar{j}}$ and the q^a are the scalars that belong to the hypermultiplets, moduli of the Kähler structure with metric h_{ab}.

We are ready to discuss now the particular case of the conifold geometry, which can be studied as a non-compact Calabi-Yau manifold, but by definition represents a singular point in the moduli space. For the values of the parameters at the conifold point the manifolds acquire some singularities. We can discuss then about conifold transition, which defines a Calabi-Yau manifold that connects the two regular configurations of the conifold: the deformed one obtained blowing up the S^3 factor and the resolved one given by expanding S^2. As explained in [56], some singularities appear in the effective action for a conifold transition because some of the heavy modes, that has been integrated out in the reduction, become massless at the singularity and so they need to be taken into account in the action.

[1]The definition of $\mathcal{M}$ goes beyond our purposes, the relevant information is the dependence on the scalars z^i and on a prepotential that can be defined in the complex structure moduli space, where the metric admit a Kähler potential.

In [55], it can be seen how the conifold transitions connects the homology cycles of the theory and the result of this translates in the way we write the four dimensional action of the IIB theory compactified on a conifold. Now, there are still $h^{2,1}$ vector multiplets and $h^{1,1}+1$ hypermultiplets, but we need to add also a set of P hypermultiplets coming from the D3-branes wrapped around the three-cycles of the conifold. Then, if we have Q homology relations these extra hypermultiplets became charged under the $(P-Q)$ vector fields, so we need an action that takes into account the coupling of the scalars charged under abelian gauge groups.

In addition, we can explain as in [62] how the gauge symmetries of the dimensionally reduced action arise from the reparametrization invariance and the gauge symmetries of the 10-dimensional fields. The scalars, that come from the expansion of the 10-dimensional fields in the non-trivial cohomology representatives, transform under these symmetries, for example the scalars that transform with a shift symmetry are recognized as axion fields. Since they are charged under the symmetries, in the 4-dimensional action the scalars need to be coupled to the vector fields. For example, through the definition of the covariant derivative we can realize the gauging of the symmetries and other coupling terms come from (3.68) since the matrix $\mathcal{M}$ depends on scalar fields.

Moreover, looking at the coupling of the gauge fields and of the scalars in the action obtained from the general compactification of IIB in the compact Calabi-Yau three-fold, we see that when we consider a non-compact manifold as the conifold the infinite volume implies that some couplings become negligible. The consequence is that the resulting theory has a bosonic sector that contains vectors coupled to the axion and some decoupled hypermultiplets. In the next chapter we will see a concrete example of the discussion presented so far since we will use the conifold geometry to derive the SymTh of various models. In particular we will build the theory that describes the symmetries of the axion-Maxwell model that contains a $U(1)$ vector field coupled to the axion field.

Chapter 4

SymTh from Type II Supergravity

The recent idea of separating the symmetries from the physical theory through the definition of what is called Symmetry Theory, SymTh, plays an important role in the study of generalized symmetries. This theory can be constructed directly starting from a specific QFT knowing its symmetries, or can be derived in a top-down perspective from 10-dimensional supergravity. The approach that we will follow in this chapter is the second one, in fact we will build a general SymTh starting from type IIA and IIB theories. Then, in a second moment we will impose some suitable boundary conditions on the fields of the SymTh in order to realize the symmetries of a specific physical QFT on the boundary.

We focus on the flux sector of the supergravity action, using a democratic formulation that includes both the supergravity fluxes and their Hodge duals in ten dimensions, the price to pay is to work in $10+1$ dimensions, see [57]. Let's delineate this formalism: if $F^{(a)}$ labels the set of fluxes of the theory, in order to avoid redundancies we can consider only magnetic sources, because an electric source for a given flux is a magnetic source for its Hodge dual. We denote $J^{(a)}$ the magnetic source for $F^{(a)}$ and we see that it modifies the Bianchi identity for the flux

$$dF^{(a)} = J^{(a)} = \delta(\mathcal{W}^{(a)}) \tag{4.1}$$

where the last equality is true if the source is localized, so $J^{(a)}$ is a delta function supported on a submanifold $\mathcal{W}^{(a)}$. These Bianchi identities can be derived from a topological action in $10+1$ dimensions:

$$S_{10+1} = \sum_{a,b} \int_{M_{10+1}} \frac{1}{2} k_{ab} F^{(a)} \wedge dF^{(b)} - k_{ab} F^{(a)} \wedge J^{(b)} \tag{4.2}$$

where k_{ab} is a constant non-degenerate matrix, that depends on the degree of the fluxes. The equations of motion derived with the variation of $F^{(a)}$ are given by

$$\sum_b k_{ab}(dF^{(b)} - J^{(b)}) = 0$$

which is equivalent to (4.1) because of the non-degeneracy of k_{ab}. These equations must be supplemented by the Hodge duality relations in 10 dimensions. In the (10+1)-dimensional topological action we can include also non trivial Chern-Simons terms, which are polynomials in the fluxes. In the case of the low-energy effective actions for type II superstring theories, we want to construct two topological actions in 10+1 dimensions that encode all the Bianchi identities and the equations of motion of the original theories. The result is the following, where

X_8 is a higher-derivative correction that depends on the curvature [66].

$$S^{IIA}_{10+1} = \int_{M_{10+1}} F_0 dF_{10} - F_2 dF_8 + F_4 dF_6 + H_3 dH_7 - H_3 \left(F_0 F_8 - F_2 F_6 + \frac{1}{2} F_4^2 + X_8 \right)$$
$$S^{IIB}_{10+1} = \int_{M_{10+1}} F_1 dF_9 - F_3 dF_7 + \frac{1}{2} F_5 dF_5 + H_3 dH_7 + H_3 \left(F_1 F_7 - F_3 F_5 \right)$$

together with the Hodge duality relations

$$H_7 = *_{10} H_3 \qquad\qquad F_p = (-1)^{\frac{p}{2}} *_{10} F_{10-p}$$

We underline that the IIB action should be considered as a pseudo-action since the self duality constraint $F_5 = *_{10} F_5$ needs to be imposed by hands. The following step is the dimensional reduction of the flux action on an appropriated geometrical background, which is a solution of the supergravity equations of motion. Then expanding the fluxes on the representative of the cohomologies of the internal manifold, we can perform the integration over the internal geometry and the extra auxiliary dimension, in order to obtain the Symmetry Theory.

4.1 II B theory reduced on M_5

Starting from the action of type IIB supergravity written in the democratic formulation, we can derive the 5-dimensional bulk action for a symmetry theory, as done in [58]. Considering a background given by the conifold geometry $C(T^{1,1})$, when we are close to the boundary at infinity we have:

$$M_{10} = M_5 \times T^{1,1} = M_5 \times S^2 \times S^3 \tag{4.3}$$

Recalling that the base of the conifold $T^{1,1}$ can be seen as a coset

$$T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)} \tag{4.4}$$

which topologically coincide with $S^2 \times S^3$ and can be considered as a U(1) fibration over $S^2 \times S^2$. In order to realize the dimensional reduction we expand the fluxes of IIB theory on the geometry, so we need the normalized volume forms of the two S^2, that are given by

$$v_i = \frac{1}{4\pi} \sin\theta_i \, d\theta_i \, d\phi_i \tag{4.5}$$

with $i = 1, 2$, so we can define

$$\omega_2 = -\frac{1}{\sqrt{2}}(v_1 - v_2) \qquad\qquad \omega_3 = \frac{1}{\sqrt{2}}(v_1 - v_2)\frac{D\psi}{2\pi} \tag{4.6}$$

Recalling

$$D\psi = d\psi + \frac{1}{2}\cos\theta_1 \, d\phi_1 + \frac{1}{2}\cos\theta_2 \, d\phi_2 \tag{4.7}$$

which comes from the metric (3.57). The normalization chosen is such that

$$\int_{T^{1,1}} \omega_2 \wedge \omega_3 = 1 \tag{4.8}$$

Using the explicit expressions of ω_2 and ω_3, we can easily show that:

$$d\omega_2 = 0 \qquad\qquad d\omega_3 = -\omega_2 \wedge da_1 \tag{4.9}$$

where the 1-form a_1 is introduced as the connection of the $U(1)$ fibration. a_1 is called U(1) isometry Reeb vector, [60]-[65], and it's inserted in the ansatz for the metric (3.57) and so in (4.7) with the replacement

$$\frac{d\psi}{2\pi} \rightarrow \frac{d\psi}{2\pi} + a_1 \tag{4.10}$$

since it is the Kaluza-Klein 1-form gauge field a_1 associated to the isometry ∂_ψ.

Finally, the ansatz for the fluxes of the IIB supergravity action is:

$$\begin{aligned}
F_3 &= dc_0 \wedge \omega_2 \\
H_3 &= db_2 \\
F_5 &= dc_1 \wedge \omega_3 - *_5 dc_1 \wedge \omega_2 \\
F_7 &= - *_{10} F_3 \\
H_7 &= *_{10} H_3 \\
F_1 &= F_9 = 0
\end{aligned} \tag{4.11}$$

In the following calculations we will use the decomposition of the Hodge star operator $*_{10}$ into $*_5 \wedge *_5$, paying attention to the sign that depends on the degrees of the forms at which the operator is applied, as explained in appendix A. The next thing to be noticed is that the $*_5$ acting on the internal space gives: $*_5\omega_2 = \omega_3$ and $*_5\omega_3 = \omega_2$, see appendix A for the proof. Inserting the fluxes in

$$S_{10+1} = \int_{M_{10+1}} F_1 dF_9 - F_3 dF_7 + \frac{1}{2} F_5 dF_5 + H_3 dH_7 + H_3\big(F_1 F_7 - F_3 F_5\big) \tag{4.12}$$

we can rewrite it term by term as:

$$\begin{aligned}
\int_{M_{10+1}} -F_3 dF_7 &= \int_{M_{10+1}} -\frac{1}{2} d\left(F_3 \wedge *_{10} F_3\right) = \int_{M_{10+1}} -\frac{1}{2} d\left(dc_0 \wedge \omega_2 \wedge *_5 dc_0 \wedge *_5 \omega_2\right) \\
&= \int_{M_{10}} -\frac{1}{2} dc_0 \wedge *_5 dc_0 \wedge \omega_2 \wedge \omega_3 = \int_{M_5} -\frac{1}{2} dc_0 \wedge *_5 dc_0
\end{aligned}$$

Noticing that the factor 1/2 of the kinetic terms comes from the fact that

$$F_3 \wedge dF_7 = -F_3 \wedge d *_{10} F_3 = \frac{1}{2} d(F_3 \wedge *_{10} F_3)$$

that applies also for the other terms. The action term with F_5 becomes:

$$\begin{aligned}
\int_{M_{10+1}} \frac{1}{2} F_5 dF_5 &= \frac{1}{2} \int_{M_{10+1}} (dc_1 \wedge \omega_3 - *_5 dc_1 \wedge \omega_2) \wedge d(dc_1 \wedge \omega_3 - *_5 dc_1 \wedge \omega_2) \\
&= \frac{1}{2} \int_{M_{10+1}} dc_1 \wedge \omega_3 \wedge d\left(dc_1 \wedge \omega_3\right) - *_5 dc_1 \wedge \omega_2 \wedge d\left(dc_1 \wedge \omega_3\right) \\
&\qquad - dc_1 \wedge \omega_3 \wedge d\left(*_5 dc_1 \wedge \omega_2\right) + *_5 dc_1 \wedge \omega_2 \wedge d\left(*_5 dc_1 \wedge \omega_2\right)
\end{aligned}$$

$$= \frac{1}{2}\int_{M_{10+1}} dc_1 \wedge \omega_3 \wedge dc_1 \wedge d\omega_3 - *_5 dc_1 \wedge \omega_2 \wedge dc_1 \wedge d\omega_3 - (-1)^{-5} d\left(dc_1 \wedge \omega_3 \wedge *_5 dc_1 \wedge \omega_2\right)$$
$$+ (-1)^{-5} dc_1 \wedge d\omega_3 \wedge *_5 dc_1 \wedge \omega_2 + *_5 dc_1 \wedge \omega_2 \wedge d *_5 dc_1 \wedge \omega_2$$
$$= \frac{1}{2}\int_{M_{10+1}} dc_1 \wedge dc_1 \wedge \omega_3 \wedge -\omega_2 \wedge da_1 - *_5 dc_1 \wedge \omega_2 \wedge dc_1 \wedge -\omega_2 \wedge da_1$$
$$- dc_1 \wedge -\omega_2 \wedge da_1 \wedge *_5 dc_1 \wedge \omega_2 + *_5 dc_1 \wedge \omega_2 \wedge d *_5 dc_1 \wedge \omega_2 + \frac{1}{2}\int_{M_{10}} dc_1 \wedge \omega_3 \wedge *_5 dc_1 \wedge \omega_2$$
$$= \int_{M_{5+1}} -\frac{1}{2} dc_1 \wedge dc_1 \wedge da_1 - \frac{1}{2}\int_{M_5} dc_1 \wedge *_5 dc_1$$
$$+ \int_{M_{10+1}} dc_1 \wedge *_5 dc_1 \wedge da_1 \wedge \omega_2 \wedge \omega_2 + \frac{1}{2} *_5 dc_1 \wedge d *_5 dc_1 \wedge \omega_2 \wedge \omega_2$$
$$= \int_{M_5} -\frac{1}{2} a_1 \wedge dc_1 \wedge dc_1 - \frac{1}{2} dc_1 \wedge *_5 dc_1$$

where the terms that do not have the combination $\omega_2 \wedge \omega_3$ give zero contribution to the integral since they have no support over $S^2 \times S^3$. The last two terms of the action can be rearranged as:

$$\int_{M_{10+1}} H_3 dH_7 - H_3 F_3 F_5 = \int_{M_{10+1}} -\frac{1}{2} d\left(H_3 \wedge *_{10} H_3\right) - db_2 \wedge dc_0 \wedge \omega_2 \wedge \left(dc_1 \wedge \omega_3 - *_5 dc_1 \wedge \omega_2\right)$$
$$= \int_{M_{10+1}} -\frac{1}{2} d\left(db_2 \wedge *_{10} db_2\right) - db_2 \wedge dc_0 \wedge \omega_2 \wedge dc_1 \wedge \omega_3 + db_2 \wedge dc_0 \wedge \omega_2 \wedge *_5 dc_1 \wedge \omega_2$$
$$= \int_{M_{10}} -\frac{1}{2} db_2 \wedge *_{10} db_2 - \int_{M_{5+1}} db_2 \wedge dc_0 \wedge dc_1 + \int_{M_{10+1}} *_5 dc_1 \wedge db_2 \wedge dc_0 \wedge \omega_2 \wedge \omega_2$$
$$= \int_{M_5} -\frac{1}{2} db_2 \wedge *_5 db_2 - b_2 \wedge dc_0 \wedge dc_1$$

Writing all the terms together and adding also the kinetic term for the U(1) Reeb vector, that comes from the metric and the Hilbert-Einstein term, the five dimensional reduced action becomes:

5-dimensional action from IIB

$$S_5^{IIB} = \int_{M_5} -\frac{1}{2} da_1 \wedge *_5 da_1 - \frac{1}{2} dc_0 \wedge *_5 dc_0 - \frac{1}{2} dc_1 \wedge *_5 dc_1 - \frac{1}{2} db_2 \wedge *_5 db_2$$
$$-\frac{1}{2} a_1 \wedge dc_1 \wedge dc_1 - b_2 \wedge dc_0 \wedge dc_1 \tag{4.13}$$

4.2 II A theory reduced on M_5

The same procedure can be repeated starting from the action of type IIA supergravity:

$$S_{10+1} = \int_{M_{10+1}} F_0 dF_{10} - F_2 dF_8 + F_4 dF_6 + H_3 dH_7 - H_3 \left(F_0 F_8 - F_2 F_6 + \frac{1}{2} F_4^2 + X_8\right) \tag{4.14}$$

The crucial point is that, in general, when we start from IIB supergravity theory in order to obtain the IIA theory we need to compactify one spacetime direction into a circle and apply

T-duality. In this case the near horizon geometry that we are considering is $T^{(1,1)}$, namely a S^1 bundle over $S^2 \times S^2$. When we perform the T-duality transformation over the S^1 we obtain the IIA theory on

$$M_{10} = M_5 \times S^2 \times S^2 \times S^1 \tag{4.15}$$

since the T-duality untwists the fibration and turns it into a direct product, [64]-[65]. As a result of the operation there is also a non trivial background flux induced in the IIA theory $H_3 = J_2 \wedge \omega_1$ where J_2 is the Kähler form of the space $S^2 \times S^2$, given by the sum of the volume forms of the two S^2 and ω_1 is the volume element of the circle. They are both closed forms: $dJ_2 = 0$ and $d\omega_1 = 0$. Now the $U(1)$ Kaluza-Klein vector a_1 can be introduced directly into the ansatz for the flux H_3, then we see that the kinetic term for a_1 follows naturally from the term $H_3 dH_7$ of the supergravity action. According to this, we can propose the following ansatz for the fluxes:

$$\begin{aligned}
H_3 &= db_2 + da_1 \wedge \omega_1 \\
F_4 &= dc_1 \wedge J_2 + dc_0 \wedge J_2 \wedge \omega_1 \\
F_6 &= - *_{10} F_4 = -(*_5 dc_1 \wedge J_2 \wedge \omega_1 + *_5 dc_0 \wedge J_2) \\
H_7 &= *_{10} H_3 \\
F_0 &= F_{10} = 0 \\
F_2 &= F_8 = 0 \\
X_8 &= 0
\end{aligned} \tag{4.16}$$

We consider the action term by term inserting the ansatz and keeping at the end only the terms with the right support over $S^2 \times S^2 \times S^1$, namely the ones with $J_2 \wedge J_2 \wedge \omega_1$ that can be correctly integrated.

$$\begin{aligned}
\int_{M_{10+1}} F_4 dF_6 &= \int_{M_{10+1}} \frac{1}{2} d\left(F_4 \wedge - *_{10} F_4\right) \\
&= \int_{M_{10+1}} -\frac{1}{2} d\big((dc_1 \wedge J_2 + dc_0 \wedge J_2 \wedge \omega_1) \wedge (*_5 dc_1 \wedge J_2 \wedge \omega_1 + *_5 dc_0 \wedge J_2)\big) \\
&= \int_{M_{10}} -\frac{1}{2} dc_1 \wedge J_2 \wedge *_5 dc_1 \wedge J_2 \wedge \omega_1 - \frac{1}{2} dc_0 \wedge J_2 \wedge \omega_1 \wedge *_5 dc_0 \wedge J_2 \\
&= \int_{M_{10}} -\frac{1}{2} dc_1 \wedge *_5 dc_1 \wedge J_2 \wedge J_2 \wedge \omega_1 - \frac{1}{2} dc_0 \wedge *_5 dc_0 \wedge J_2 \wedge J_2 \wedge \omega_1 \\
&= \int_{M_5} -\frac{1}{2} dc_1 \wedge *_5 dc_1 - \frac{1}{2} dc_0 \wedge *_5 dc_0
\end{aligned}$$

$$\begin{aligned}
\int_{M_{10+1}} H_3 dH_7 &= \int_{M_{10+1}} -\frac{1}{2} d\left(H_3 \wedge *_{10} H_7\right) \\
&= \int_{M_{10+1}} -\frac{1}{2} d\big((db_2 + da_1 \wedge \omega_1) \wedge (*_{10} db_2 - *_5 da_1 \wedge *_5 \omega_1)\big) \\
&= \int_{M_{10}} -\frac{1}{2} db_2 \wedge *_{10} db_2 + \frac{1}{2} da_1 \wedge \omega_1 \wedge *_5 da_1 \wedge J_2 \wedge J_2 \\
&= \int_{M_5} -\frac{1}{2} db_2 \wedge *_5 db_2 - \frac{1}{2} da_1 \wedge *_5 da_1
\end{aligned}$$

here the minus sign in front of the term with a_1 comes from the decomposition of the Hodge star operator $*_{10}$ as explained in (A.1).

$$
\begin{aligned}
\int_{M_{10+1}} -\frac{1}{2}H_3F_4^2 &= \int_{M_{10+1}} -\frac{1}{2}(db_2 + da_1 \wedge \omega_1) \wedge (dc_1 \wedge J_2 + dc_0 \wedge J_2 \wedge \omega_1)^2 \\
&= \int_{M_{10+1}} -\frac{1}{2}db_2 \wedge 2\, dc_1 \wedge J_2 \wedge dc_0 \wedge J_2 \wedge \omega_1 - \frac{1}{2}da_1 \wedge \omega_1 \wedge dc_1 \wedge J_2 \wedge dc_1 \wedge J_2 \\
&= \int_{M_{10+1}} -db_2 \wedge dc_0 \wedge dc_1 \wedge J_2 \wedge J_2 \wedge \omega_1 - \frac{1}{2}da_1 \wedge dc_1 \wedge dc_1 \wedge J_2 \wedge J_2 \wedge \omega_1 \\
&= \int_{M_5} -b_2 \wedge dc_0 \wedge dc_1 - \frac{1}{2}a_1 \wedge dc_1 \wedge dc_1
\end{aligned}
$$

The final result can be obtained collecting all the terms, we see that we get the same 5-dimensional action obtained before from IIB as expected.

5-dimensional action from IIA

$$
\begin{aligned}
S_5^{IIA} = \int_{M_5} & -\frac{1}{2}da_1 \wedge *_5 da_1 - \frac{1}{2}dc_1 \wedge *_5 dc_1 - \frac{1}{2}dc_0 \wedge *_5 dc_0 - \frac{1}{2}db_2 \wedge *_5 db_2 \\
& -\frac{1}{2}a_1 \wedge dc_1 \wedge dc_1 - b_2 \wedge dc_0 \wedge dc_1
\end{aligned}
\tag{4.17}
$$

4.3 II B theory reduced on M_4

The five dimensional action can be further reduced on a circle in order to obtain a theory in four dimensions, so now we have to decompose the total space as:

$$
M_{10} = M_5 \times S^2 \times S^3 = M_4 \times S^1 \times S^2 \times S^3 \tag{4.18}
$$

The ansatz for the fluxes of the IIB supergravity theory is the following:

$$
\begin{aligned}
F_3 &= dc_0 \wedge \omega_2 + dc_1 \wedge \omega_1 + dc_2 \\
H_3 &= db_2 + de_0 \wedge \omega_2 + de_1 \wedge \omega_1 \\
F_5 &= db_1 \wedge \omega_3 - *_4 db_1 \wedge \omega_1 \wedge \omega_2 + db_0 \wedge \omega_1 \wedge \omega_3 + *_4 db_0 \wedge \omega_2 \\
F_7 &= - *_{10} F_3 \\
H_7 &= *_{10} H_3 \\
F_1 &= F_9 = 0
\end{aligned}
\tag{4.19}
$$

With this ansatz we have F_5 self dual respect to $*_{10}$ as expected and we can consider as before $d\omega_2 = 0$, $d\omega_3 = -\omega_2 \wedge da_1$ and $d\omega_1 = 0$. Now we have that $*_{10}$ decomposes into $*_4 \wedge *_6$ and we get a minus from the epsilon tensors when we take the Hodge star of two forms with odd degree. Then it can be proven using polar coordinates that: $*_6\omega_1 = \omega_2 \wedge \omega_3$, $*_6\omega_2 = \omega_1 \wedge \omega_3$ and $*_6\omega_3 = -\omega_1 \wedge \omega_2$, see appendix A.

Inserting the fluxes in the action we get:

$$\int_{M_{10+1}} - F_3 dF_7 = \int_{M_{10+1}} -\frac{1}{2} d\left((dc_0 \wedge \omega_2 + dc_1 \wedge \omega_1 + dc_2) \wedge *_{10}(dc_0 \wedge \omega_2 + dc_1 \wedge \omega_1 + dc_2)\right)$$
$$= -\frac{1}{2}\int_{M_{10}} (dc_0 \wedge \omega_2 + dc_1 \wedge \omega_1 + dc_2) \wedge (*_4 dc_0 \wedge \omega_1 \wedge \omega_3 + *_4 dc_1 \wedge \omega_2 \wedge \omega_3 + *_{10} dc_2)$$
$$= -\frac{1}{2}\int_{M_{10}} dc_0 \wedge \omega_2 \wedge *_4 dc_0 \wedge \omega_1 \wedge \omega_3 + dc_1 \wedge \omega_1 \wedge *_4 dc_1 \wedge \omega_2 \wedge \omega_3 + dc_2 \wedge *_{10} dc_2$$
$$= \int_{M_4} -\frac{1}{2} dc_0 \wedge *_4 dc_0 - \frac{1}{2} dc_1 \wedge *_4 dc_1 - \frac{1}{2} dc_2 \wedge *_4 dc_2$$

$$\int_{M_{10+1}} \frac{1}{2} F_5 dF_5 = \frac{1}{2}\int_{M_{10+1}} (db_1 \wedge \omega_3 - *_4 db_1 \wedge \omega_1 \wedge \omega_2 + db_0 \wedge \omega_1 \wedge \omega_3 + *_4 db_0 \wedge \omega_2) \wedge$$
$$\wedge\, d\left(db_1 \wedge \omega_3 - *_4 db_1 \wedge \omega_1 \wedge \omega_2 + db_0 \wedge \omega_1 \wedge \omega_3 + *_4 db_0 \wedge \omega_2\right)$$
$$= \frac{1}{2}\int_{M_{10+1}} db_1 \wedge \omega_3 \wedge d\left(- *_4 db_1 \wedge \omega_1 \wedge \omega_2\right) + db_1 \wedge \omega_3 \wedge d\left(db_0 \wedge \omega_1 \wedge \omega_3\right)$$
$$+ db_0 \wedge \omega_1 \wedge \omega_3 \wedge d\left(db_1 \wedge \omega_3\right) + db_0 \wedge \omega_1 \wedge \omega_3 \wedge d\left(*_4 db_0 \wedge \omega_2\right)$$
$$= \frac{1}{2}\int_{M_{10+1}} -db_1 \wedge d *_4 db_1 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 + db_1 \wedge \omega_3 \wedge db_0 \wedge \omega_1 \wedge -\omega_2 \wedge da_1$$
$$+ db_0 \wedge \omega_1 \wedge \omega_3 \wedge db_1 \wedge -\omega_2 \wedge da_1 + db_0 \wedge d *_4 db_0 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3$$
$$= \int_{M_4} -\frac{1}{2} db_1 \wedge *_4 db_1 - \frac{1}{2} db_0 \wedge *_4 db_0 - a_1 \wedge db_1 \wedge db_0$$

$$\int_{M_{10+1}} H_3 dH_7 = \int_{M_{10+1}} -\frac{1}{2} d\left((db_2 + de_0 \wedge \omega_2 + de_1 \wedge \omega_1) \wedge *_{10}(db_2 + de_0 \wedge \omega_2 + de_1 \wedge \omega_1)\right)$$
$$= -\frac{1}{2}\int_{M_{10}} db_2 \wedge *_{10} db_2 + de_0 \wedge \omega_2 \wedge *_4 de_0 \wedge \omega_1 \wedge \omega_3 + de_1 \wedge \omega_1 \wedge *_4 de_1 \wedge \omega_2 \wedge \omega_3$$
$$= \int_{M_4} -\frac{1}{2} db_2 \wedge *_4 db_2 - \frac{1}{2} de_0 \wedge *_4 de_0 - \frac{1}{2} de_1 \wedge *_4 de_1$$

$$\int_{M_{10+1}} -H_3 F_3 F_5 = -\int_{M_{10+1}} (db_2 + de_0 \wedge \omega_2 + de_1 \wedge \omega_1) \wedge (dc_0 \wedge \omega_2 + dc_1 \wedge \omega_1 + dc_2) \wedge$$
$$\wedge\, (db_1 \wedge \omega_3 - *_4 db_1 \wedge \omega_1 \wedge \omega_2 + db_0 \wedge \omega_1 \wedge \omega_3 + *_4 db_0 \wedge \omega_2)$$
$$= -\int_{M_{10+1}} (db_2 \wedge dc_0 \wedge \omega_2 + db_2 \wedge dc_1 \wedge \omega_1 + db_2 \wedge dc_2 + de_0 \wedge \omega_2 \wedge dc_1 \wedge \omega_1$$
$$+ de_0 \wedge \omega_2 \wedge dc_2 + de_1 \wedge \omega_1 \wedge dc_0 \wedge \omega_2 + de_1 \wedge \omega_1 \wedge dc_2) \wedge$$
$$\wedge\, (db_1 \wedge \omega_3 - *_4 db_1 \wedge \omega_1 \wedge \omega_2 + db_0 \wedge \omega_1 \wedge \omega_3 + *_4 db_0 \wedge \omega_2)$$
$$= -\int_{M_{10+1}} db_2 \wedge dc_0 \wedge \omega_2 \wedge db_0 \wedge \omega_1 \wedge \omega_3 + de_0 \wedge \omega_2 \wedge dc_1 \wedge \omega_1 \wedge db_1 \wedge \omega_3$$
$$+ de_0 \wedge \omega_2 \wedge dc_2 \wedge db_0 \wedge \omega_1 \wedge \omega_3 + de_1 \wedge \omega_1 \wedge dc_0 \wedge \omega_2 \wedge db_1 \wedge \omega_3$$
$$= \int_{M_4} -b_2 \wedge dc_0 \wedge db_0 - e_0 \wedge dc_1 \wedge db_1 - e_0 \wedge dc_2 \wedge db_0 + e_1 \wedge db_1 \wedge dc_0$$

So the final result is:

4-dimensional action from IIB

$$S_4^{IIB} = \int_{M_4} -\frac{1}{2}dc_0 \wedge *_4 dc_0 - \frac{1}{2}dc_1 \wedge *_4 dc_1 - \frac{1}{2}db_1 \wedge *_4 db_1 - \frac{1}{2}dc_2 \wedge *_4 dc_2 - \frac{1}{2}db_0 \wedge *_4 db_0$$
$$-\frac{1}{2}db_2 \wedge *_4 db_2 - \frac{1}{2}de_0 \wedge *_4 de_0 - \frac{1}{2}da_1 \wedge *_4 da_1 - \frac{1}{2}de_1 \wedge *_4 de_1$$
$$-a_1 \wedge db_1 \wedge db_0 + b_2 \wedge db_0 \wedge dc_0 - e_0 \wedge dc_1 \wedge db_1 + e_0 \wedge db_0 \wedge dc_2 + e_1 \wedge db_1 \wedge dc_0$$

(4.20)

4.4 II A theory reduced on M_4

If we consider the total space as:

$$M_{10} = M_5 \times S^2 \times S^3 = M_4 \times S^1 \times S^2 \times S^3 \tag{4.21}$$

a possible ansatz for the fluxes of IIA supergravity can be written applying the T-duality on the additional S^1 starting from the ansatz of the IIB theory. For example in the flux F_3 we take the terms proportional to ω_1 to write a contribution in F_2 and the other terms go into F_4 adding the ω_1 factor, as explained in section 3.1.4. We obtain:

$$\begin{aligned}
F_2 &= dc_1 \\
H_3 &= -db_2 - de_0 \wedge \omega_2 \\
F_4 &= *_4 db_1 \wedge \omega_2 + db_0 \wedge \omega_3 + dc_2 \wedge \omega_1 + dc_0 \wedge \omega_1 \wedge \omega_2 \\
F_6 &= - *_{10} F_4 \\
F_8 &= *_{10} F_2 \\
H_7 &= *_{10} H_3 \\
F_0 &= F_{10} = 0 \\
X_8 &= 0
\end{aligned} \tag{4.22}$$

Considering as before $d\omega_2 = 0$, $d\omega_3 = -\omega_2 \wedge da_1$ and $d\omega_1 = 0$. Substituting in the terms of the action (4.14) considered one by one we get:

$$\int_{M_{10+1}} -F_2 dF_8 = \int_{M_{10+1}} -\frac{1}{2} d\left(dc_1 \wedge *_{10} dc_1\right) = \int_{M_4} -\frac{1}{2} dc_1 \wedge *_4 dc_1$$

$$\begin{aligned}
\int_{M_{10+1}} F_4 dF_6 = \int_{M_{10+1}} & (*_4 db_1 \wedge \omega_2 + db_0 \wedge \omega_3 + dc_2 \wedge \omega_1 + dc_0 \wedge \omega_1 \wedge \omega_2) \wedge \\
& \wedge d\ - *_{10}(*_4 db_1 \wedge \omega_2 + db_0 \wedge \omega_3 + dc_2 \wedge \omega_1 + dc_0 \wedge \omega_1 \wedge \omega_2) \\
= \int_{M_{10+1}} & - *_4 db_1 \wedge \omega_2 \wedge d *_{10} (*_4 db_1 \wedge \omega_2) - db_0 \wedge \omega_3 \wedge d *_{10} (db_0 \wedge \omega_3) \\
& - dc_2 \wedge \omega_1 \wedge d *_{10} (dc_2 \wedge \omega_1) - dc_0 \wedge \omega_1 \wedge \omega_2 \wedge d *_{10} (dc_0 \wedge \omega_1 \wedge \omega_2) \\
& - db_0 \wedge \omega_3 \wedge d *_{10} (*_4 db_1 \wedge \omega_2)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{M_{10+1}} \frac{1}{2} d\left(*_4 db_1 \wedge \omega_2 \wedge db_1 \wedge \omega_1 \wedge \omega_3\right) - \frac{1}{2} d\left(db_0 \wedge \omega_3 \wedge *_4 db_0 \wedge \omega_1 \wedge \omega_2\right) \\
&\qquad + \frac{1}{2} d\left(dc_2 \wedge \omega_1 \wedge *_4 dc_2 \wedge \omega_2 \wedge \omega_3\right) + \frac{1}{2} d\left(dc_0 \wedge \omega_1 \wedge \omega_2 \wedge *_4 dc_0 \wedge \omega_3\right) \\
&\qquad - db_0 \wedge \omega_3 \wedge db_1 \wedge \omega_1 \wedge d\omega_3 \\
&= \int_{M_{10}} \frac{1}{2} db_1 \wedge *_4 db_1 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 - \frac{1}{2} db_0 \wedge *_4 db_0 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 \\
&\qquad - \frac{1}{2} dc_2 \wedge *_4 dc_2 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 - \frac{1}{2} dc_0 \wedge *_4 dc_0 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 \\
&\qquad - \int_{M_{10+1}} da_1 \wedge db_1 \wedge db_0 \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 \\
&= \int_{M_4} \frac{1}{2} db_1 \wedge *_4 db_1 - \frac{1}{2} db_0 \wedge *_4 db_0 - \frac{1}{2} dc_2 \wedge *_4 dc_2 - \frac{1}{2} dc_0 \wedge *_4 dc_0 - a_1 \wedge db_1 \wedge db_0
\end{aligned}
$$

$$
\begin{aligned}
\int_{M_{10+1}} H_3 dH_7 &= \int_{M_{10+1}} -\frac{1}{2} d\big((db_2 + de_0 \wedge \omega_2) \wedge *_{10}(db_2 + de_0 \wedge \omega_2)\big) \\
&= \int_{M_{10}} -\frac{1}{2} db_2 \wedge *_{10} db_2 - \frac{1}{2} de_0 \wedge \omega_2 \wedge *_4 de_0 \wedge \omega_1 \wedge \omega_3 \\
&= \int_{M_4} -\frac{1}{2} db_2 \wedge *_4 db_2 - \frac{1}{2} de_0 \wedge *_4 de_0
\end{aligned}
$$

$$
\begin{aligned}
\int_{M_{10+1}} H_3 F_2 F_6 &= \int_{M_{10+1}} (db_2 + de_0 \wedge \omega_2) \wedge dc_1 \wedge *_{10}(*_4 db_1 \wedge \omega_2 + db_0 \wedge \omega_3 \\
&\qquad + dc_2 \wedge \omega_1 + dc_0 \wedge \omega_1 \wedge \omega_2) \\
&= \int_{M_{10+1}} (db_2 \wedge dc_1 + de_0 \wedge \omega_2 \wedge dc_1) \wedge (-db_1 \wedge \omega_1 \wedge \omega_3 - *_4 db_0 \wedge \omega_1 \wedge \omega_2 \\
&\qquad + *_4 dc_2 \wedge \omega_2 \wedge \omega_3 + *_4 dc_0 \wedge \omega_3) \\
&= \int_{M_{10+1}} -de_0 \wedge \omega_2 \wedge dc_1 \wedge db_1 \wedge \omega_1 \wedge \omega_3 \\
&= \int_{M_4} -e_0 \wedge dc_1 \wedge db_1
\end{aligned}
$$

$$
\begin{aligned}
\int_{M_{10+1}} -\frac{1}{2} H_3 F_4^2 &= \int_{M_{10+1}} -\frac{1}{2}(-db_2 - de_0 \wedge \omega_2) \wedge \\
&\qquad \wedge (*_4 db_1 \wedge \omega_2 + db_0 \wedge \omega_3 + dc_2 \wedge \omega_1 + dc_0 \wedge \omega_1 \wedge \omega_2)^2 \\
&= \int_{M_{10+1}} \frac{1}{2}(db_2 + de_0 \wedge \omega_2) \wedge 2(*_4 db_1 \wedge \omega_2 \wedge db_0 \wedge \omega_3 + *_4 db_1 \wedge \omega_2 \wedge dc_2 \wedge \omega_1 \\
&\qquad + *_4 db_1 \wedge \omega_2 \wedge dc_0 \wedge \omega_1 \wedge \omega_2 + db_0 \wedge \omega_3 \wedge dc_2 \wedge \omega_1 \\
&\qquad + db_0 \wedge \omega_3 \wedge dc_0 \wedge \omega_1 \wedge \omega_2 + dc_2 \wedge \omega_1 \wedge dc_0 \wedge \omega_1 \wedge \omega_2) \\
&= \int_{M_{10+1}} db_2 \wedge db_0 \wedge \omega_3 \wedge dc_0 \wedge \omega_1 \wedge \omega_2 + de_0 \wedge \omega_2 \wedge db_0 \wedge \omega_3 \wedge dc_2 \wedge \omega_1 \\
&= \int_{M_4} b_2 \wedge db_0 \wedge dc_0 + e_0 \wedge db_0 \wedge dc_2
\end{aligned}
$$

Finally the total four dimensional action is:

4-dimensional action from IIA

$$S_4^{IIA} = \int_{M_4} -\frac{1}{2}dc_1 \wedge *_4 dc_1 - \frac{1}{2}dc_0 \wedge *_4 dc_0 + \frac{1}{2}db_1 \wedge *_4 db_1 - \frac{1}{2}db_0 \wedge *_4 db_0 - \frac{1}{2}dc_2 \wedge *_4 dc_2$$
$$-\frac{1}{2}db_2 \wedge *_4 db_2 - \frac{1}{2}de_0 \wedge *_4 de_0 - \frac{1}{2}da_1 \wedge *_4 da_1$$
$$-a_1 \wedge db_1 \wedge db_0 + b_2 \wedge db_0 \wedge dc_0 - e_0 \wedge dc_1 \wedge db_1 + e_0 \wedge db_0 \wedge dc_2 \tag{4.23}$$

which is equivalent to the action from the reduction of the IIB theory (4.20). If we also activate e_1 in H_3 expanding the flux with the term $de_1 \wedge \omega_1$, we can have some ambiguity since the T-duality acting on the circle could give rise to some background component that interferes with the external field. In the following derivation instead we obtain the correct term with e_1 present in the IIB theory, since applying the T-duality on the fibration it's possible to maintain distinct the volume forms of the two S^1 factors.

4.4.1 Another way to apply the T-duality

The other possibility to built the ansatz for the fluxes of the IIA theory (4.14) is to apply the T-duality on the S^1 that constitutes the fiber bundle over $S^2 \times S^2$, giving the $S^2 \times S^3$ geometry of the IIB theory (4.3). The result as we already seen in the five dimensional case is that the T-duality trivialize the bundle giving a geometry that is the direct product:

$$M_4 \times S^1 \times S^2 \times S^1 \times S^2 \tag{4.24}$$

in the IIA theory with the addition of a component in the background flux H_3. The ansatz for the fluxes is the following

$$\begin{aligned}
H_3 &= db_2 + de_0 \wedge \omega_2 + de_1 \wedge \omega_1 + da_1 \wedge \sigma_1 \\
F_4 &= db_1 \wedge \omega_2 + db_0 \wedge \omega_2 \wedge \omega_1 + dc_2 \wedge \sigma_1 + dc_0 \wedge \omega_2 \wedge \sigma_1 + dc_1 \wedge \omega_1 \wedge \sigma_1 \\
F_6 &= - *_{10} F_4 \\
F_2 &= F_8 = 0 \\
H_7 &= *_{10} H_3 \\
F_0 &= F_{10} = 0 \\
X_8 &= 0
\end{aligned} \tag{4.25}$$

where we consider $\omega_2 = \dfrac{1}{\sqrt{2}}(v_1 + v_2)$ built with the volume forms of the two S^2. ω_1 is the volume form of the S^1 that is untouched by the T-duality and σ_1 is the volume form of the S^1 that originally was in the fiber bundle. They are normalized in such a way that

$$\int_{S^1 \times S^2 \times S^1 \times S^2} \omega_1 \wedge \omega_2 \wedge \sigma_1 \wedge \omega_2 = 1 \tag{4.26}$$

Taking into account the changes of sign when we apply the Hodge star operator we can derive:

$$
\begin{aligned}
F_6 &= -\big(*_4\, db_1 \wedge *_6\omega_2 - *_4 db_0 \wedge *_6(\omega_2 \wedge \omega_1) - *_4 dc_2 \wedge *_6\sigma_1 \\
&\quad - *_4 dc_0 \wedge *_6(\omega_2 \wedge \sigma_1) + *_4 dc_1 \wedge *_6(\omega_1 \wedge \sigma_1)\big) \\
&= -\big(*_4\, db_1 \wedge \omega_1 \wedge \sigma_1 \wedge \omega_2 - *_4 db_0 \wedge \sigma_1 \wedge \omega_2 + *_4 dc_2 \wedge \omega_1 \wedge \omega_2 \wedge \omega_2 \\
&\quad + *_4 dc_0 \wedge \omega_1 \wedge \omega_2 + *_4 dc_1 \wedge \omega_2 \wedge \omega_2\big)
\end{aligned}
$$

Inserting then the fluxes in the action term by term:

$$
\begin{aligned}
\int_{M_{10+1}} F_4 dF_6 &= \int_{M_{10+1}} -\frac{1}{2} d\Big((db_1 \wedge \omega_2 + db_0 \wedge \omega_2 \wedge \omega_1 + dc_2 \wedge \sigma_1 + dc_0 \wedge \omega_2 \wedge \sigma_1 + dc_1 \wedge \omega_1 \wedge \sigma_1) \\
&\quad \wedge (*_4 db_1 \wedge \omega_1 \wedge \sigma_1 \wedge \omega_2 - *_4 db_0 \wedge \sigma_1 \wedge \omega_2 + *_4 dc_2 \wedge \omega_1 \wedge \omega_2 \wedge \omega_2 \\
&\quad + *_4 dc_0 \wedge \omega_1 \wedge \omega_2 + *_4 dc_1 \wedge \omega_2 \wedge \omega_2)\Big) \\
&= -\frac{1}{2}\int_{M_{10}} db_1 \wedge \omega_2 \wedge *_4 db_1 \wedge \omega_1 \wedge \sigma_1 \wedge \omega_2 - db_0 \wedge \omega_2 \wedge \omega_1 \wedge *_4 db_0 \wedge \sigma_1 \wedge \omega_2 \\
&\quad + dc_2 \wedge \sigma_1 \wedge *_4 dc_2 \wedge \omega_1 \wedge \omega_2 \wedge \omega_2 + dc_0 \wedge \omega_2 \wedge \sigma_1 \wedge *_4 dc_0 \wedge \omega_1 \wedge \omega_2 \\
&\quad + dc_1 \wedge \omega_1 \wedge \sigma_1 \wedge *_4 dc_1 \wedge \omega_2 \wedge \omega_2 \\
&= \int_{M_4} -\frac{1}{2} db_1 \wedge *_4 db_1 - \frac{1}{2} db_0 \wedge *_4 db_0 - \frac{1}{2} dc_2 \wedge *_4 dc_2 - \frac{1}{2} dc_0 \wedge *_4 dc_0 - \frac{1}{2} dc_1 \wedge *_4 dc_1
\end{aligned}
$$

$$
\begin{aligned}
\int_{M_{10+1}} H_3 dH_7 &= \int_{M_{10+1}} -\frac{1}{2} d\Big((db_2 + de_0 \wedge \omega_2 + de_1 \wedge \omega_1 + da_1 \wedge \sigma_1) \wedge (*_{10} db_2 \\
&\quad + *_4 de_0 \wedge \omega_1 \wedge \omega_2 \wedge \sigma_1 + *_4 de_1 \wedge \omega_2 \wedge \sigma_1 \wedge \omega_2 - *_4 da_1 \wedge \omega_1 \wedge \omega_2 \wedge \omega_2)\Big) \\
&= \int_{M_{10+1}} -\frac{1}{2} db_2 \wedge *_{10} db_2 - \frac{1}{2} de_0 \wedge \omega_2 \wedge *_4 de_0 \wedge \omega_1 \wedge \omega_2 \wedge \sigma_1 \\
&\quad - \frac{1}{2} de_1 \wedge \omega_1 \wedge *_4 de_1 \wedge \omega_2 \wedge \sigma_1 \wedge \omega_2 + \frac{1}{2} da_1 \wedge \sigma_1 \wedge *_4 da_1 \wedge \omega_1 \wedge \omega_2 \wedge \omega_2 \\
&= \int_{M_4} -\frac{1}{2} db_2 \wedge *_4 db_2 - \frac{1}{2} de_0 \wedge *_4 de_0 - \frac{1}{2} de_1 \wedge *_4 de_1 - \frac{1}{2} da_1 \wedge *_4 da_1
\end{aligned}
$$

$$
\begin{aligned}
\int_{M_{10+1}} -\frac{1}{2} H_3 F_4^2 &= \int_{M_{10+1}} -\frac{1}{2}(db_2 + de_0 \wedge \omega_2 + de_1 \wedge \omega_1 + da_1 \wedge \sigma_1) \\
&\quad \wedge (db_1 \wedge \omega_2 + db_0 \wedge \omega_2 \wedge \omega_1 + dc_2 \wedge \sigma_1 + dc_0 \wedge \omega_2 \wedge \sigma_1 + dc_1 \wedge \omega_1 \wedge \sigma_1)^2 \\
&= \int_{M_{10+1}} -(db_2 + de_0 \wedge \omega_2 + de_1 \wedge \omega_1 + da_1 \wedge \sigma_1) \\
&\quad \wedge (db_1 \wedge \omega_2 \wedge db_0 \wedge \omega_2 \wedge \omega_1 + db_1 \wedge \omega_2 \wedge dc_2 \wedge \sigma_1 \\
&\quad + db_1 \wedge \omega_2 \wedge dc_0 \wedge \omega_2 \wedge \sigma_1 + db_1 \wedge \omega_2 \wedge dc_1 \wedge \omega_1 \wedge \sigma_1 \\
&\quad + db_0 \wedge \omega_2 \wedge \omega_1 \wedge dc_2 \wedge \sigma_1 + db_0 \wedge \omega_2 \wedge \omega_1 \wedge dc_0 \wedge \omega_2 \wedge \sigma_1) \\
&= -\int_{M_{10+1}} db_2 \wedge db_0 \wedge \omega_2 \wedge \omega_1 \wedge dc_0 \wedge \omega_2 \wedge \sigma_1 + de_0 \wedge \omega_2 \wedge db_1 \wedge \omega_2 \wedge dc_1 \wedge \omega_1 \wedge \sigma_1 \\
&\quad + de_0 \wedge \omega_2 \wedge db_0 \wedge \omega_2 \wedge \omega_1 \wedge dc_2 \wedge \sigma_1 + de_1 \wedge \omega_1 \wedge db_1 \wedge \omega_2 \wedge dc_0 \wedge \omega_2 \wedge \sigma_1 \\
&\quad + da_1 \wedge \sigma_1 \wedge db_1 \wedge \omega_2 \wedge db_0 \wedge \omega_2 \wedge \omega_1 \\
&= \int_{M_4} b_2 \wedge db_0 \wedge dc_0 - e_0 \wedge dc_1 \wedge db_1 + e_0 \wedge db_0 \wedge dc_2 + e_1 \wedge db_1 \wedge dc_0 - a_1 \wedge db_1 \wedge db_0
\end{aligned}
$$

So the final result is the following four dimensional action, which is exactly equal to the one obtained from IIB theory

4-dimensional action from IIA, T-duality on the fibration

$$
\begin{aligned}
S_4^{IIA} = \int_{M_4} & -\frac{1}{2}dc_1 \wedge *_4 dc_1 - \frac{1}{2}dc_0 \wedge *_4 dc_0 - \frac{1}{2}db_1 \wedge *_4 db_1 - \frac{1}{2}db_0 \wedge *_4 db_0 - \frac{1}{2}dc_2 \wedge *_4 dc_2 \\
& -\frac{1}{2}db_2 \wedge *_4 db_2 - \frac{1}{2}de_0 \wedge *_4 de_0 - \frac{1}{2}de_1 \wedge *_4 de_1 - \frac{1}{2}da_1 \wedge *_4 da_1 \\
& +b_2 \wedge db_0 \wedge dc_0 - e_0 \wedge dc_1 \wedge db_1 - e_0 \wedge dc_2 \wedge db_0 + e_1 \wedge db_1 \wedge dc_0 - a_1 \wedge db_1 \wedge db_0
\end{aligned}
\tag{4.27}
$$

4.5 Topological operators

4.5.1 Theories on M_5

From the Bianchi identities for the fields of the theory we can write the conservation laws that leads to topological operators.

$$
\begin{aligned}
d * j_3^a &= dda_1 = 0 \\
d * j_3^c &= ddc_1 = 0 \\
d * j_4^c &= ddc_0 = 0 \\
d * j_2^b &= ddb_2 = 0
\end{aligned}
\tag{4.28}
$$

where here the Hodge operator is $* = *_5$. The topological defect operators can be written as:

$$
\begin{aligned}
U_\alpha^a(M_2) &= e^{i\alpha \oint_{M_2} *j_3^a} = e^{i\alpha \oint_{M_2} da_1} \\
U_\beta^c(M_2) &= e^{i\beta \oint_{M_2} *j_3^c} = e^{i\beta \oint_{M_2} dc_1} \\
U_\gamma^c(M_1) &= e^{i\gamma \oint_{M_1} *j_4^c} = e^{i\gamma \oint_{M_1} dc_0} \\
U_\lambda^b(M_3) &= e^{i\lambda \oint_{M_3} *j_2^b} = e^{i\lambda \oint_{M_3} db_2}
\end{aligned}
\tag{4.29}
$$

The boundary conditions will determine which ones are faithful representations of symmetry generators and can be projected into the boundary theory. From the IIB dimensionally reduced action we get the following equations of motion

$$
\begin{aligned}
d * J_2^a &= d * da_1 = -\frac{1}{2}dc_1 \wedge dc_1 \\
d * J_2^c &= d * dc_1 = -da_1 \wedge dc_1 - db_2 \wedge dc_0 \\
d * J_1^c &= d * dc_0 = -db_2 \wedge dc_1 \\
d * J_3^b &= d * db_2 = dc_0 \wedge dc_1
\end{aligned}
\tag{4.30}
$$

that can be seen as non-conserved currents and can be used to build the topological operators that represent non-invertible symmetries, as explained in section 1.2.2

$$\begin{aligned}
D^{a_1}_{\frac{1}{N}}(M_3) &= \int \mathcal{D}W_1 \exp\left(2\pi i \oint_{M_2} \frac{1}{N} * J^a_2 - \frac{N}{2} W_1 \wedge dW_1 - W_1 \wedge dc_1\right) \\
D^{c_1}_{\frac{1}{N}}(M_3) &= \int \mathcal{D}W_1 \mathcal{D}X_1 \mathcal{D}Y_0 \mathcal{D}Z_2 \exp\Big(2\pi i \oint_{M_3} \frac{1}{N} * J^c_2 - N\, W_1 \wedge dX_1 - W_1 \wedge da_1 \\
&\qquad - X_1 \wedge dc_1 - N\, Y_0 \wedge dZ_2 - Y_0 \wedge db_2 - Z_2 \wedge dc_0\Big) \\
D^{c_0}_{\frac{1}{N}}(M_4) &= \int \mathcal{D}X_1 \mathcal{D}Z_2 \exp\left(2\pi i \oint_{M_4} \frac{1}{N} * J^c_1 - N\, X_1 \wedge dZ_2 - X_1 \wedge db_2 - Z_2 \wedge dc_1\right) \\
D^{b_2}_{\frac{1}{N}}(M_2) &= \int \mathcal{D}X_1 \mathcal{D}Y_0 \exp\left(2\pi i \oint_{M_2} \frac{1}{N} * J^b_3 + N\, Y_0 \wedge dX_1 + Y_0 \wedge dc_1 + X_1 \wedge dc_0\right)
\end{aligned} \tag{4.31}$$

For the five dimensional action obtained from IIA theory we have the same Bianchi identities and equations of motion since the action is the same.

4.5.2 Theories on M_4

Considering the field content of the theories dimensionally reduced on M_4 we get the Bianchi identities:

$$\begin{aligned}
d *_4 j^a_2 &= dda_1 = 0 \\
d *_4 j^c_3 &= ddc_0 = 0 \\
d *_4 j^c_2 &= ddc_1 = 0 \\
d *_4 j^c_1 &= ddc_2 = 0 \\
d *_4 j^b_3 &= ddb_0 = 0 \\
d *_4 j^b_2 &= ddb_1 = 0 \\
d *_4 j^b_1 &= ddb_2 = 0 \\
d *_4 j^e_3 &= dde_0 = 0 \\
d *_4 j^e_2 &= dde_1 = 0
\end{aligned} \tag{4.32}$$

As in the previous case exponentiating the integral of the conserved current we get the topological operators that define the defects of the bulk theory

$$\begin{aligned}
U^{a_1}_\alpha(M_2) &= e^{i\alpha \oint_{M_2} *_4 j^a_2} = e^{i\alpha \oint_{M_2} da_1} \\
U^{c_0}_\beta(M_1) &= e^{i\beta \oint_{M_1} *_4 j^c_3} = e^{i\beta \oint_{M_1} dc_0} \\
U^{c_1}_\gamma(M_2) &= e^{i\gamma \oint_{M_2} *_4 j^c_2} = e^{i\gamma \oint_{M_2} dc_1} \\
U^{c_2}_\lambda(M_3) &= e^{i\lambda \oint_{M_3} *_4 j^c_1} = e^{i\lambda \oint_{M_3} dc_2} \\
U^{b_0}_\eta(M_1) &= e^{i\eta \oint_{M_1} *_4 j^b_3} = e^{i\eta \oint_{M_1} db_0} \\
U^{b_1}_\theta(M_2) &= e^{i\theta \oint_{M_2} *_4 j^b_2} = e^{i\theta \oint_{M_2} db_1} \\
U^{b_2}_\sigma(M_3) &= e^{i\sigma \oint_{M_3} *_4 j^b_1} = e^{i\sigma \oint_{M_3} db_2} \\
U^{e_0}_\tau(M_1) &= e^{i\tau \oint_{M_1} *_4 j^e_3} = e^{i\tau \oint_{M_1} de_0} \\
U^{e_1}_\omega(M_2) &= e^{i\omega \oint_{M_2} *_4 j^e_2} = e^{i\omega \oint_{M_2} de_1}
\end{aligned} \tag{4.33}$$

From the type IIB action the equations of motion are:

$$\begin{aligned}
d *_4 J_2^a &= d *_4 da_1 = -db_1 \wedge db_0 \\
d *_4 J_1^c &= d *_4 dc_0 = -db_2 \wedge db_0 - de_1 \wedge db_1 \\
d *_4 J_2^c &= d *_4 dc_1 = -de_0 \wedge db_1 \\
d *_4 J_3^c &= d *_4 dc_2 = -de_0 \wedge db_0 \\
d *_4 J_1^b &= d *_4 db_0 = da_1 \wedge db_1 + db_2 \wedge dc_0 + de_0 \wedge dc_2 \\
d *_4 J_2^b &= d *_4 db_1 = -da_1 \wedge db_0 - de_0 \wedge dc_1 + de_1 \wedge dc_0 \\
d *_4 J_3^b &= d *_4 db_2 = -db_0 \wedge dc_0 \\
d *_4 J_1^e &= d *_4 de_0 = dc_1 \wedge db_1 - db_0 \wedge dc_2 \\
d *_4 J_2^e &= d *_4 de_1 = db_1 \wedge dc_0
\end{aligned} \tag{4.34}$$

and from these non-conserved currents we build the following topological operators:

$$\begin{aligned}
D^{a_1}_{\frac{1}{N}}(M_2) &= \int \mathcal{D}X_0 \mathcal{D}Y_1 \exp\Big(2\pi i \oint_{M_3} \frac{1}{N} * J_2^a - N\, Y_1 \wedge dX_0 - Y_1 \wedge db_0 - b_1 \wedge dX_0\Big) \\
D^{c_0}_{\frac{1}{N}}(M_3) &= \int \mathcal{D}X_0 \mathcal{D}Z_2 \mathcal{D}Y_1 \mathcal{D}V_1 \exp\Big(2\pi i \oint_{M_3} \frac{1}{N} * J_1^c - N\, Z_2 \wedge dX_0 - Z_2 \wedge db_0 - b_2 \wedge dX_0 \\
&\quad - N V_1 \wedge dY_1 - V_1 \wedge db_1 - e_1 \wedge dY_1\Big) \\
D^{c_1}_{\frac{1}{N}}(M_2) &= \int \mathcal{D}W_0 \mathcal{D}Y_1 \exp\Big(2\pi i \oint_{M_2} \frac{1}{N} * J_2^c - N\, W_0 \wedge dY_1 - W_0 \wedge db_1 - e_0 \wedge dY_1\Big) \\
D^{c_2}_{\frac{1}{N}}(M_1) &= \int \mathcal{D}W_0 \mathcal{D}X_0 \exp\Big(2\pi i \oint_{M_1} \frac{1}{N} * J_3^c - N\, W_0 \wedge dX_0 - W_0 \wedge db_0 - e_0 \wedge dX_0\Big) \\
D^{b_0}_{\frac{1}{N}}(M_3) &= \int \mathcal{D}V_1 \mathcal{D}Y_1 \mathcal{D}Z_2 \mathcal{D}X_0 \mathcal{D}W_0 \mathcal{D}P_2 \exp\Big(2\pi i \oint_{M_3} \frac{1}{N} * J_1^b + N\, V_1 \wedge dY_1 + V_1 \wedge db_1 \\
&\quad + a_1 \wedge dY_1 + N\, Z_2 \wedge dX_0 + Z_2 \wedge dc_0 + b_2 \wedge dX_0 + N\, W_0 \wedge dP_2 + W_0 \wedge dc_2 + e_0 \wedge dP_2\Big) \\
D^{b_1}_{\frac{1}{N}}(M_2) &= \int \mathcal{D}V_1 \mathcal{D}X_0 \mathcal{D}W_0 \mathcal{D}Y_1 \mathcal{D}R_1 \mathcal{D}Z_0 \exp\Big(2\pi i \oint_{M_2} \frac{1}{N} * J_2^b - N\, V_1 \wedge dX_0 - V_1 \wedge db_0 \\
&\quad - a_1 \wedge dX_0 - N\, W_0 \wedge dY_1 - W_0 \wedge dc_1 - e_0 \wedge dY_1 + N\, R_1 \wedge dZ_0 + R_1 \wedge dc_0 + e_1 \wedge dZ_0\Big) \\
D^{b_2}_{\frac{1}{N}}(M_1) &= \int \mathcal{D}W_0 \mathcal{D}X_0 \exp\Big(2\pi i \oint_{M_1} \frac{1}{N} * J_3^b - N\, W_0 \wedge dX_0 - W_0 \wedge dc_0 - b_0 \wedge dX_0\Big) \\
D^{e_0}_{\frac{1}{N}}(M_3) &= \int \mathcal{D}V_1 \mathcal{D}Y_1 \mathcal{D}X_0 \mathcal{D}Z_2 \exp\Big(2\pi i \oint_{M_3} \frac{1}{N} * J_1^e + N\, V_1 \wedge dY_1 + V_1 \wedge db_1 + c_1 \wedge dY_1 \\
&\quad - N\, X_0 \wedge dZ_2 - X_0 \wedge dc_2 - b_0 \wedge dZ_2\Big) \\
D^{e_1}_{\frac{1}{N}}(M_2) &= \int \mathcal{D}Y_1 \mathcal{D}X_0 \exp\Big(2\pi i \oint_{M_2} \frac{1}{N} * J_2^e + N\, Y_1 \wedge dX_0 + Y_1 \wedge dc_0 + b_1 \wedge dX_0\Big)
\end{aligned} \tag{4.35}$$

4.6 Boundary conditions

The set of boundary conditions is given by imposing the vanishing variation of the action at the boundary. Recalling that Dirichlet b.c. $\delta f\big|_{\partial M} = 0$ on a field f means that it does not fluctuate on the boundary theory, namely it becomes a non dynamical background field. Neumann b.c. instead reads as $*df\big|_{\partial M} = 0$ and means that the field f remains dynamical and freely varying.

4.6.1 SymTh on M_5

Looking at the variation of the IIB dimensionally reduced action on M_5 we get at the boundary:

$$\begin{aligned}\delta S_5^{IIB}\big|_{\partial M_5} = \int_{\partial M_5} & - \delta a_1 \wedge *_5 da_1 - \delta c_0 \wedge *_5 dc_0 - \delta c_1 \wedge *_5 dc_1 - \delta b_2 \wedge *_5 db_2 \\ & + 2\delta c_1 \wedge a_1 \wedge dc_1 - \delta c_0 \wedge b_2 \wedge dc_1 - \delta c_1 \wedge b_2 \wedge dc_0 \end{aligned} \tag{4.36}$$

We notice that in order to vanish the variation of the kinetic terms we can either impose Dirichlet or Neumann boundary conditions with no difference, for a Chern-Simons term instead it is sufficient to choose Dirichlet boundary conditions for one of the three fields that are in it. To see this we can consider the gauge transformations for the fields in $b_2 \wedge dc_1 \wedge dc_0$ for example. We see that this term is gauge invariant under the transformations of c_1, c_0 since it contains their field strength, while instead is not invariant under the gauge transformation of b_2. As a result we can only impose Dirichlet boundary conditions for the field b_2, while for the other two we can choose between Dirichlet and Neumann. In fact, c_1 and c_0 can be freely varying in the boundary since the theory remains gauge invariant, otherwise b_2 should become a fixed background field.

The same reasoning can be repeated exchanging the role of the fields in the discussion, since we can integrate by parts the original Chern-Simons term. By adding a boundary local counter term, it becomes for example $c_0 \wedge db_2 \wedge dc_1$ and now it is c_0 the field for which we can impose only Dirichlet boundary conditions. Finally, for this five dimensional SymTh a possible choice of boundary conditions is given by:

$$\delta a_1\big|_{\partial M_5} = 0 \qquad \delta b_2\big|_{\partial M_5} = 0 \qquad *_5\, dc_0\big|_{\partial M_5} = 0 \qquad *_5\, dc_1\big|_{\partial M_5} = 0 \tag{4.37}$$

The Dirichlet boundary conditions imply that the Wilson surfaces of the fields a_1 and b_2 can end on the four dimensional boundary and the operators that link them, $D^{a_1}_{\frac{1}{N}}(M_3)$ and $D^{b_2}_{\frac{1}{N}}(M_2)$ respectively, in (4.31), are projected on the boundary as symmetry generators. They generate a 0-form and a 1-form non-invertible global symmetry. It can be seen since they act on the Wilson surfaces of a_1 and b_2, that projected on the boundary become a point-like operator and a line operator. The Neumann boundary conditions for the fields c_0 and c_1 instead say that the Wilson surfaces with the dual fields can end on the boundary. So, the operators that can be faithful projected on the boundary are the ones that link them, $U^c_\gamma(M_1)$ and $U^c_\beta(M_2)$, in (4.29), that are generators of the invertible 1 and 2-form U(1) symmetries respectively.

The rest of the topological operators cannot be projected on the boundary since they link closed Wilson surfaces which do not end on the boundary. They are transparent in the interval compactification of the symmetry theory, since they do not act faithfully on the spectrum of operators of the boundary QFT.

These symmetries, invertible and non invertible, are the same ones present in the four dimensional axion-Maxwell model, [58], described by the following action

$$S_{a-M} = -\int_{M_4} \frac{1}{2} d\theta \wedge * d\theta + \frac{1}{2} f_2 \wedge * f_2 - \frac{k}{2}\theta\, f_2 \wedge f_2 \tag{4.38}$$

in which $k \in \mathbb{Z}$, $f_2 = da$ and θ is a periodic scalar $\theta \approx \theta + 2\pi$. Using the fields we can build four currents and their conservation equations that give rise to some topological operators of the same form of the ones coming from the symmetry theory.

4.6.2 SymTh on M_4

The same procedure leads to the boundary variation of the action on M_4

$$\begin{aligned}\delta S_4^{IIB}\big|_{\partial M_4} = \int_{\partial M_4} & - \delta a_1 \wedge *_4 da_1 - \delta c_0 \wedge *_4 dc_0 - \delta c_1 \wedge *_4 dc_1 - \delta c_2 \wedge *_4 dc_2 - \delta b_0 \wedge *_4 db_0 \\ & - \delta b_1 \wedge *_4 db_1 - \delta b_2 \wedge *_4 db_2 - \delta e_0 \wedge *_4 de_0 - \delta e_1 \wedge *_4 de_1 \\ & + \delta b_0 \wedge a_1 \wedge db_1 + \delta b_1 \wedge a_1 \wedge db_0 + b_0 \wedge b_2 \wedge dc_0 - \delta c_0 \wedge b_2 \wedge db_0 \\ & - \delta c_1 \wedge e_0 \wedge db_1 - \delta b_1 \wedge e_0 \wedge dc_1 + \delta b_0 \wedge e_0 \wedge dc_2 - \delta c_2 \wedge e_0 \wedge db_0 \\ & + \delta b_1 \wedge e_1 \wedge dc_0 - \delta c_0 \wedge e_1 \wedge db_1 \end{aligned} \tag{4.39}$$

Here imposing some specific boundary conditions we can obtain as boundary QFT the three-dimensional Goldstone-Maxwell model [59], which is given by a dynamical U(1) gauge field denoted by $c^{(1)}$ and a Goldstone boson χ coupled with the U(1) field with a θ-term.

$$S[c^{(1)}, \chi] = -\frac{1}{2g^2} \int f_c^{(2)} \wedge * f_c^{(2)} - \frac{1}{2} \int d\chi \wedge * d\chi + \frac{i\theta}{4\pi^2} \int d\chi \wedge f_c^{(2)} \tag{4.40}$$

The θ angle has 2π periodicity following from the flux quantization given by the conditions for any closed cycles Σ_1 and Σ_2:

$$\int_{\Sigma_1} \frac{d\chi}{2\pi} \in \mathbb{Z} \qquad \int_{\Sigma_2} \frac{f^{(2)}}{2\pi} \in \mathbb{Z} \tag{4.41}$$

From the terms of the action we see which global symmetries are present: the Maxwell kinetic term gives a magnetic 0-form symmetry acting on local monopoles operators and an electric 1-form symmetry acting on Wilson lines. The Goldstone boson transforms non-linearly under a $U(1)_A^{(0)}$ symmetry as $\chi \to \chi + \lambda_A^{(0)}$ and presents also a 1-form symmetry due to its dynamics. So the currents are given by

$$j_A^{(1)} = -i\, d\chi \qquad J_e^{(2)} = -\frac{i}{g^2} f_c^{(2)} \qquad j_m^{(1)} = \frac{1}{2\pi} * f_c^{(2)} \qquad J_B^{(2)} = \frac{1}{2\pi} * d\chi \tag{4.42}$$

From the Bianchi identities we get the conservation equations:

$$d * j_m^{(1)} = df_c^{(2)} = 0 \qquad\qquad d * J_B^{(2)} = dd\chi = 0 \tag{4.43}$$

and from the equations of motions for χ and $c^{(1)}$ we get the non-conservation equations for the other two currents, but first is necessary to notice that θ can be promoted to a spacetime dependent background field in the effective action and then it can be gauged. Namely we replace the θ angle by a compact dynamical field ϕ with periodicity 2π, obtaining the following action for the theory

$$S[c^{(1)}, \chi, \phi] = -\frac{1}{2g^2} \int f_c^{(2)} \wedge * f_c^{(2)} - \frac{1}{2} \int d\chi \wedge * d\chi - \frac{1}{2} \int d\phi \wedge * d\phi + \frac{ik}{4\pi^2} \int \phi \wedge d\chi \wedge f_c^{(2)} \tag{4.44}$$

The presence of ϕ allows two emergent global symmetries: a $U(1)_{\tilde{A}}^{(0)}$ which is the usual shift symmetry for the scalar field ϕ with current $J_{\tilde{A}}^{(1)} = -id\phi$ and a 1-form symmetry with conserved current given by

$$J_{\tilde{B}}^{(2)} = \frac{1}{2\pi} * d\phi \tag{4.45}$$

Finally from the equations of motion coming from the modified action (4.44) we get the non-conservation equations for the currents $j_A^{(1)}$ and $J_e^{(2)}$

$$d * j_A^{(1)} = -\frac{k}{4\pi^2} d\phi \wedge f_c^{(2)} \qquad\qquad d * J_e^{(2)} = -\frac{k}{4\pi^2} d\phi \wedge d\chi \tag{4.46}$$

and also for the emerging symmetry with current $J^{(1)}_{\tilde{A}}$ we get the non-conservation equation

$$d * J^{(1)}_{\tilde{A}} = -id * d\phi = \frac{k}{4\pi^2} d\chi \wedge f^{(2)}_c \tag{4.47}$$

So the full set of global symmetries of the three dimensional Goldstone-Maxwell model is given by

$$\mathbb{Z}^{(0)}_{k,A} \times \mathbb{Z}^{(0)}_{k,\tilde{A}} \times U(1)^{(0)}_m \times \mathbb{Z}^{(0)}_{k,e} \times U(1)^{(1)}_B \times U(1)^{(1)}_{\tilde{B}} \tag{4.48}$$

the $\mathbb{Z}^{(0)}_{k,A}$ and $\mathbb{Z}^{(0)}_{k,e}$ factors comes from the fact that the corresponding U(1) symmetries are explicitly broken. The second and the last symmetry groups instead are the emergent global symmetries induced by the gauging of θ.

Starting from the current equations (4.46) and (4.47), in the case $k = 1$, we can define three non-invertible symmetry generators. As usual we need to pay attention to build them in such a way that they are gauge invariant and topological.

$$D_{\frac{1}{N}}(\Sigma_2) = \int \mathcal{D}\xi \mathcal{D}v^{(1)} \exp\left(i \int_{\Sigma_2} \frac{2\pi}{N} * j^{(1)}_A + \frac{N}{2\pi}\xi dv^{(1)} + \frac{1}{2\pi}\xi f^{(2)}_c - \frac{1}{2\pi}\phi dv^{(1)}\right)$$

$$D_{\frac{1}{M}}(\Sigma_1) = \int \mathcal{D}\xi \mathcal{D}\tilde{\chi} \ \exp\left(i \int_{\Sigma_1} \frac{2\pi}{M} * J^{(2)}_e + \frac{M}{2\pi}\xi d + \frac{1}{2\pi}\xi d\chi - \frac{1}{2\pi}\phi d\tilde{\chi}\right) \tag{4.49}$$

$$D_{\frac{1}{N}}(\Sigma_2) = \int \mathcal{D}\xi \mathcal{D}v^{(1)} \exp\left(i \int_{\Sigma_2} \frac{2\pi}{N} * J^{(1)}_{\tilde{A}} + \frac{N}{2\pi}\xi dv^{(1)} + \frac{1}{2\pi}\xi f^{(2)}_c - \frac{1}{2\pi}\chi dv^{(1)}\right)$$

where the auxiliary ξ and $\tilde{\chi}$ are compact scalar fields and $v^{(1)}$ is a 1-form gauge field. We can check, as done in [59], that the fusion algebra of these operators does not follow the usual group-relation, since $\mathcal{D} \times \mathcal{D}^\dagger$ does not give the identity operator, but gives what is called condensation defect.

Coming back to the starting point of this discussion, we need to examine some boundary conditions that can be imposed to the four dimensional SymTh in order to obtain on the boundary theory the topological operators that characterize the Goldstone-Maxwell model. In the following possibilities we choose the minimal set of Dirichlet boundary conditions in order to vanish all the Chern-Simons terms in the boundary variation of the action given by (4.39).

Dirichlet b.c. for b_1, b_2, e_0 & Neumann b.c. for a_1, c_0, c_1, c_2, b_0, e_1;

Dirichlet b.c. for b_1, c_2, c_0 & Neumann b.c. for a_1, c_1, b_0, b_2, e_0, e_1

Dirichlet b.c. for a_1, b_2, e_0, b_1 & Neumann b.c. for c_0, c_1, c_2, b_0, e_1

In the first case the Wilson surfaces of b_1 and b_2 can end on the boundary so the topological operators that link them can be projected and become symmetry generators of the same dimension, but with co-dimension decreased by one, on the boundary QFT. The operators involved are $D^{b_1}_{\frac{1}{N}}(M_2)$ and $D^{b_2}_{\frac{1}{N}}(M_1)$ which are the topological defects of a 0-form and 1-form non-invertible global symmetry respectively. The field e_0 with Dirichlet boundary conditions is a local operator in the symmetry theory linked by the topological defect $D^{e_0}_{\frac{1}{N}}(M_3)$, but since the charge is already of dimension 0 it cannot be projected on the boundary. The first two defects match the non-invertible operators (4.49) obtained before from the three dimensional Goldstone-Maxwell model, but since $D^{e_0}_{\frac{1}{N}}(M_3)$ cannot act faithfully on the boundary theory then the non-invertible symmetry emerging from the gauging of the θ background field is missing. The second possibility for the boundary conditions leads to the same result exchanging the roles of b_2 and e_0 with c_2 and c_0 and the corresponding operators.

The Neumann boundary conditions on all the other fields of the SymTh imply that the dual surfaces of that fields can end on the boundary so all the topological operators built using them act faithfully on the boundary QFT. The only exception is the dual surface obtained from the field c_2 which has zero dimension since $*_4 dc_2 = d\tilde{c}_0$, so it cannot be projected on the boundary. With the first choice of b.c. indeed the operators $U_\beta^{c_0}(M_1)$, $U_\eta^{b_0}(M_1)$ and $U_\gamma^{c_1}(M_2)$ gives the three invertible global symmetries of the Goldstone-Maxwell model, a 0-form and two 1-forms symmetries, while the other topological defects defined from a_1 and e_1 give some additional 1-form symmetries.

All the others operators of the SymTh cannot be projected on the boundary, so they do not realize any symmetry. However since with these two choices of boundary conditions we cannot obtain all the symmetries of Goldstone-Maxwel model, we need to analyze the third one, given by:

$$\delta a_1\big|_{\partial M_4} = 0 \quad \delta b_2\big|_{\partial M_4} = 0 \quad \delta e_0\big|_{\partial M_4} = 0 \quad \delta b_1\big|_{\partial M_4} = 0 \quad \& \quad \text{Neumann b.c. for } \; c_0, c_1, c_2, b_0, e_1 \tag{4.50}$$

Now with these Dirichlet boundary conditions we have all the topological operators that realize the three non-invertible symmetries of Goldstone-Maxwell model. $D^{a_1}_{\frac{1}{N}}(M_2)$, $D^{b_1}_{\frac{1}{N}}(M_2)$ and $D^{b_2}_{\frac{1}{N}}(M_1)$, listed in (4.35), generate the two 0-forms and the 1-form non-invertible symmetry respectively. The invertible global symmetries instead are given by $U_\beta^{c_0}(M_1)$, $U_\eta^{b_0}(M_1)$ and $U_\omega^{e_1}(M_2)$ in (4.33), since c_0, b_0 and e_1 have Neumann boundary conditions. There is also one additional symmetry given by $U_\gamma^{c_1}(M_2)$ since also c_1 has its dual surface that can end on the boundary.

In order to explain this additional symmetry that seems to be absent in the Goldstone-Maxwell model, we can try to rewrite in a clever way the action (4.44), noticing that we have two scalar fields in the theory that seem to be interchangeable. We can express ϕ and χ in a doublet, writing the kinetic terms and the interaction term with some bilinear functions of the doublet variable. The kinetic terms are built using the identity matrix and the interaction term with the symplectic matrix S.

$$\begin{aligned} S[c^{(1)}, \chi, \phi] \supset & \int -\frac{1}{2} d \begin{pmatrix}\phi & \chi\end{pmatrix} \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} \wedge *d \begin{pmatrix}\phi \\ \chi\end{pmatrix} + \frac{ik}{4\pi^2}\frac{1}{2} \begin{pmatrix}\phi & \chi\end{pmatrix} \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} \wedge d \begin{pmatrix}\phi \\ \chi\end{pmatrix} \wedge f_c^{(2)} \\ = & \int -\frac{1}{2} d\phi \wedge *d\phi - \frac{1}{2} d\chi \wedge *d\chi + \frac{ik}{4\pi^2}\frac{1}{2}(-\chi \wedge d\phi + \phi \wedge d\chi) \wedge f_c^{(2)} \\ = & \int -\frac{1}{2} d\phi \wedge *d\phi - \frac{1}{2} d\chi \wedge *d\chi + \frac{ik}{4\pi^2} \phi \wedge d\chi \wedge f_c^{(2)} \end{aligned} \tag{4.51}$$

where we get to the last row integrating by parts the term $-\chi \wedge d\phi \wedge f_c^{(2)}$. The point is to look at the possible transformations given by a 2×2 matrix R that acts on the doublet $(\phi \;\; \chi)$ leaving invariant the action. The conditions that must be satisfied are

$$R\mathbb{1}R^T = RR^T = \mathbb{1} \qquad\qquad RSR^T = S \tag{4.52}$$

the first one says that the matrix needs to be orthogonal, so it can be parametrized by α that take values in $[0, 2\pi)$ and written as

$$R = \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix} \tag{4.53}$$

It is easy to check that this matrix leaves the symplectic matrix S invariant for every value of α. So given the isomorphism $SO(2) \cong U(1)$ between $R(\alpha)$ and $e^{i\alpha}$, we can see this as a $U(1)$ 0-form symmetry in the space of fields ϕ and χ. Luckily, it is exactly the additional symmetry that is given by the operator $U_\gamma^{c_1}(M_2)$ that acts on the dual surfaces for the c_1 field in the four dimensional SymTh.

We can compute the Noether conserved current looking at the infinitesimal transformation given by

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} \to \begin{pmatrix} \phi \\ \chi \end{pmatrix} + i\alpha \begin{pmatrix} -\chi \\ \phi \end{pmatrix} \tag{4.54}$$

We can check that the infinitesimal variation induced in the lagrangian by this transformation is zero and then compute the Noether current as

$$\begin{aligned} J^{(2)} &= \frac{d\mathcal{L}}{d\, d(\phi \quad \chi)} \begin{pmatrix} -\chi \\ \phi \end{pmatrix} \\ &= *d\phi \wedge \chi - *d\chi \wedge \phi + \frac{1}{2}\frac{ik}{4\pi^2}(\chi^2 + \phi^2) \wedge f_c^{(2)} \end{aligned} \tag{4.55}$$

where the conservation equation $dJ^{(2)} = 0$ is ensured by the Euler-Lagrange equations of motion for ϕ and χ. So our symmetry has a conserved one-form current $j^{(1)} = *J^{(2)}$ as expected and can be realized by an invertible topological operator.

Conclusions

Let's recall the relevant results of this work. In chapter 4, starting from the actions of type II supergravity we implemented the dimensional reduction by expanding the fluxes on the internal geometry and by integrating over the extra dimensions. We compared the type IIA and IIB theories, checking how the T-duality connects their fluxes and arriving to the same dimensionally reduced action for the two theories, as expected.

In the first part of the discussion the internal geometry was given by the conifold in the near horizon limit, so the reduction over the space

$$M_{10} = M_5 \times S^2 \times S^3$$

led to the five dimensional SymTh given by

$$S_5 = \int_{M_5} -\frac{1}{2}da_1 \wedge *_5 da_1 - \frac{1}{2}dc_0 \wedge *_5 dc_0 - \frac{1}{2}dc_1 \wedge *_5 dc_1 - \frac{1}{2}db_2 \wedge *_5 db_2$$
$$-\frac{1}{2}a_1 \wedge dc_1 \wedge dc_1 - b_2 \wedge dc_0 \wedge dc_1$$

The tricky part consisted on how to implement the T-duality between IIB and IIA in this geometry and the answer is carefully, since the S^1 fiber bundle structure over $S^2 \times S^2$ that was the background geometry for the IIB theory, once we applied the T-duality was trivialized. The bundle was untwisted and became for the IIA theory the direct product $S^2 \times S^2 \times S^1$, giving rise to an induced internal flux. Taking into account this in the IIA ansatz for the fluxes we managed to get exactly to the same 5-dimensional action that comes from IIB. Moreover, it was possible to study the topological operators of the SymTh, which realize both invertible and non-invertible global symmetries. In order to get as boundary theory the known axion-Maxwell model we presented a suitable set of boundary conditions on the SymTh fields. In this way we projected on the boundary all the topological defects necessary to generate the symmetries of the model.

In the second step we considered a different geometry

$$M_{10} = M_4 \times S^1 \times S^2 \times S^3$$

with an additional reduction on a circle, so we obtained the 4-dimensional SymTh that follows

$$S_4 = \int_{M_4} -\frac{1}{2}dc_0 \wedge *_4 dc_0 - \frac{1}{2}dc_1 \wedge *_4 dc_1 - \frac{1}{2}db_1 \wedge *_4 db_1 - \frac{1}{2}dc_2 \wedge *_4 dc_2 - \frac{1}{2}db_0 \wedge *_4 db_0$$
$$-\frac{1}{2}db_2 \wedge *_4 db_2 - \frac{1}{2}de_0 \wedge *_4 de_0 - \frac{1}{2}da_1 \wedge *_4 da_1 - \frac{1}{2}de_1 \wedge *_4 de_1$$
$$-a_1 \wedge db_1 \wedge db_0 + b_2 \wedge db_0 \wedge dc_0 - e_0 \wedge dc_1 \wedge db_1 + e_0 \wedge db_0 \wedge dc_2 + e_1 \wedge db_1 \wedge dc_0$$

Also in this case we built the topological operators and we managed to find the correspondence with the three dimensional Goldstone-Maxwell model, which presents three non-invertible and three invertible symmetries. Our four dimensional action was the symmetry theory for this boundary model, all its symmetries in fact could be obtained projecting the topological defects of the SymTh in the boundary after choosing an appropriated set of boundary conditions.

We saw that the SymTh allows to project also an additional invertible symmetry generator on the boundary theory, which corresponds to the symmetry that interchanges the two scalar fields of Goldstone-Maxwell theory. With this last remark we can understand the power of the symmetry theory approach derived from the supergravity actions: if we choose a sufficiently general ansatz, all the symmetries are already there in the complete set of topological defects that we can write. A general SymTh in fact can describe a variety of physical boundary theories, imposing the suitable boundary conditions that allow the right symmetry operators to act on the boundary.

Possible future outlook after this work are: to apply the formalism and the symmetry theories obtained in order to describe the symmetries of other physical models and also to extend the study of the non-invertible defects. In fact using non compact fields with values in $\mathbb{R}$ we could be able to define not only discrete but also continuous non-invertible symmetries. Additionally, it's possible to investigate the UV origin of the topological operators of the various theories in terms of branes in supergravity.

Appendix A

Hodge star operator

In the following section we collect various proofs of the results used in the derivation of the dimensionally reduced actions once we have the Hodge star operator applied on the fluxes. The decomposition of the Hodge star operator $*_{10}$ into $*_5 \wedge *_5$ can be written as

$$\begin{aligned} *_{10}(\alpha_1 \wedge \beta_2) &= \frac{1}{2\cdot 1\cdot 7!}\ \sqrt{-g_{10}}\ \epsilon^{i_1 i_2 i_3}{}_{i_4 \dots i_{10}}\ dx^{i_4\dots i_{10}}\ (\alpha_1)_{i_1}(\beta_2)_{i_2 i_3} \\ &= \frac{1}{2\cdot 1\cdot 7!}\cdot\frac{7!}{4!\cdot 3!}\ \sqrt{-g_5}\sqrt{-g_{T^{1,1}}}\ \epsilon^{i_1}{}_{i_4 i_5 i_6 i_7}\ \epsilon^{i_2 i_3}{}_{i_8 i_9 i_{10}}\ dx^{i_4\dots i_7} dx^{i_8\dots i_{10}}\ (\alpha_1)_{i_1}(\beta_2)_{i_2 i_3} \\ &= *_5\alpha_1 \wedge *_5\beta_2 \end{aligned} \tag{A.1}$$

paying attention to the sign that depend on the degrees of the forms at which the operator is applied. It's clear that the sign depends on the number of exchanges of the indices to get the two epsilon tensors with as first indices the ones of the two forms respectively. The only case in which we get a minus sign is the one in which the first form has even degree and the second has odd degree. With these considerations in mind we can also check that the ansatz for the flux F_5 is correctly Hodge self-dual.

Another useful relation is given by what happens once we apply $*_5$ on the internal volume forms ω_2 and ω_3. Using polar coordinates to express the volume forms $\omega_2 = \sin(\theta)\, d\theta\, d\beta$, $\omega_3 = \sin^2(\phi)\sin(\gamma)\, d\phi\, d\gamma\, d\alpha$ we can prove for example the relation:

$$\begin{aligned} *_5\,\omega_2 &= \frac{1}{2\cdot 3!}\sqrt{-g}\ \ \epsilon^{ab}{}_{cde}\ \ dx^c dx^d dx^e\ \ (\omega_2)_{ab} \\ &= \frac{1}{3!}\sqrt{-g}\ \ (g^{-1})^{\theta\theta}(g^{-1})^{\beta\beta}\ \ \epsilon_{\theta\beta cde}\ \ dx^c dx^d dx^e \sin\theta \\ &= \sin\theta\sin^2\phi\sin\gamma\frac{1}{\sin^2\theta}\ \ \epsilon_{\theta\beta\phi\gamma\alpha}\ \ dx^\phi dx^\gamma dx^\alpha \sin\theta \\ &= \sin^2\phi\sin\gamma\, d\phi\, d\gamma\, d\alpha = \omega_3 \end{aligned} \tag{A.2}$$

We omit the normalization of the volume forms but it's straightforward to insert them and check that all the relations we discuss stay the same. Analogously we can show that:

$$*_5\omega_3 = \omega_2 \tag{A.3}$$

In the previous calculation the metric we consider for the geometry $S^2 \times S^3$ is given by the matrix

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & & \\ & \sin^2\theta & & & \\ & & 1 & & \\ & & & \sin^2\phi & \\ & & & & \sin^2\phi\,\sin^2\gamma \end{pmatrix} \tag{A.4}$$

Considering instead the internal geometry for the IIA theory $S^2 \times S^2 \times S^1$ where we use as volume forms $J_2 = \frac{1}{2}(\omega_2^{(1)} + \omega_2^{(2)})$ and $\omega_1 = d\psi$ we have the metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & & \\ & \sin^2\theta_1 & & & \\ & & 1 & & \\ & & & \sin^2\theta_2 & \\ & & & & 1 \end{pmatrix} \tag{A.5}$$

so we find the relations:

$$\begin{aligned} *_5\ \omega_1 &= J_2 \wedge J_2 \\ *_5\ J_2 &= J_2 \wedge \omega_1 \\ *_5\ (J_2 \wedge \omega_1) &= \omega_2 \end{aligned} \tag{A.6}$$

We can present the proof of the first

$$\begin{aligned} *_5\omega_1 &= \frac{1}{4!}\sqrt{-g}\ \ \epsilon^{a}{}_{bcde}\ \ dx^b dx^c dx^d dx^e\ \ (\omega_1)_a \\ &= \sqrt{-g}\ \ (g^{-1})^{\psi\psi}\ \ \epsilon_{\psi bcde}\ \ dx^b dx^c dx^d dx^e \\ &= \sin\theta_1 \sin\theta_2\ \ \epsilon_{\psi\phi_1\theta_1\phi_2\theta_2}\ \ d\phi_1 d\theta_1 d\phi_2 d\theta_2 \\ &= \omega_1^{(1)}\ \omega_2^{(2)} = J_2 \wedge J_2 \end{aligned} \tag{A.7}$$

In the case of the dimensional reduction to M_4 we need to decompose the Hodge star operator $*_{10}$ into $*_4 \wedge *_6$ and this can be done as

$$\begin{aligned} *_{10}(\alpha_1 \wedge \beta_2) &= \frac{1}{2\cdot 1\cdot 7!}\ \sqrt{-g_{10}}\ \ \epsilon^{i_1 i_2 i_3}{}_{i_4\ldots i_{10}}\ dx^{i_4\ldots i_{10}}\ \ (\alpha_1)_{i_1}(\beta_2)_{i_2 i_3} \\ &= \frac{1}{2\cdot 1\cdot 7!}\cdot\frac{7!}{3!\cdot 4!}\ \sqrt{-g_5}\sqrt{-g_6}\ \ \epsilon^{i_1}{}_{i_4 i_5 i_6}\ \ \epsilon^{i_2 i_3}{}_{i_7 i_8 i_9 i_{10}}\ \ dx^{i_4\ldots i_6} dx^{i_7\ldots i_{10}}\ \ (\alpha_1)_{i_1}(\beta_2)_{i_2 i_3} \\ &= *_4\alpha_1 \wedge *_6\beta_2 \end{aligned} \tag{A.8}$$

Again the sign depends on the number of exchanges of the indices to get the two epsilon tensors with as first indices the ones of the two forms respectively. The only case in which we get a minus sign is the one in which $*_{10}$ acts on two forms of odd degree. Then looking at the action of $*_6$ on the volume forms of the internal geometry given by $S^1 \times S^2 \times S^3$ we can list:

$$\begin{aligned} *_6\ \omega_1 &= \omega_2 \wedge \omega_3 \\ *_6\ \omega_2 &= \omega_1 \wedge \omega_3 \\ *_6\ \omega_3 &= -\omega_1 \wedge \omega_2 \\ *_6\ (\omega_1 \wedge \omega_2) &= \omega_3 \\ *_6\ (\omega_1 \wedge \omega_3) &= \omega_2 \end{aligned} \tag{A.9}$$

considering ω_2, ω_3 as before in spherical coordinates and $\omega_1 = d\psi$. We can give the proof of the third relation for example

$$\begin{aligned} *_6\ \omega_3 &= \frac{1}{3!\cdot 3!}\sqrt{-g}\ \ \epsilon^{abc}{}_{def}\ \ dx^d dx^e dx^f\ \ (\omega_3)_{abc} \\ &= \frac{1}{3!}\sqrt{-g}\ \ (g^{-1})^{\phi\phi}(g^{-1})^{\gamma\gamma}\ \ (g^{-1})^{\alpha\alpha}\ \ \epsilon_{\phi\gamma\alpha def}\ \ dx^d dx^e dx^f\ \ (\omega_3)_{abc} \\ &= \sin\theta\sin^2\phi\sin\gamma\frac{1}{\sin^2\phi}\ \frac{1}{\sin^2\phi\sin^2\gamma}\ \ \epsilon_{\phi\gamma\alpha\psi\gamma\beta}\ \ d\psi d\theta d\beta\ \ \sin^2\phi\ \sin\gamma \\ &= \epsilon_{\phi\gamma\alpha\psi\theta\beta}\ d\psi \wedge \sin\theta\ d\theta \wedge d\beta \\ &= -\omega_1 \wedge \omega_2 \end{aligned} \tag{A.10}$$

The minus sign comes from the odd number of exchanges that we need to make in the epsilon tensor in order to obtain $\epsilon_{\psi\theta\beta\phi\gamma\alpha}$ which is equal to 1, since we choose the metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \sin^2\theta & & & \\ & & & 1 & & \\ & & & & \sin^2\phi & \\ & & & & & \sin^2\phi\,\sin^2\gamma \end{pmatrix} \tag{A.11}$$

If we consider instead as internal geometry $S^1 \times S^2 \times S^1 \times S^2$ as in the last case of chapter 4, the volume forms that we use are ω_2, ω_1 and σ_1. Always looking at the number of exchanges that we need to make in the epsilon tensor, we find:

$$\begin{aligned}
*_6\ \omega_2 &= \omega_1 \wedge \omega_2 \wedge \sigma_1 \\
*_6\ \omega_1 &= \omega_2 \wedge \sigma_1 \wedge \omega_2 \\
*_6\ \sigma_1 &= -\ \omega_1 \wedge \omega_2 \wedge \omega_2 \\
*_6\ (\omega_2 \wedge \omega_1) &= \omega_2 \wedge \sigma_1 \\
*_6\ (\omega_2 \wedge \sigma_1) &= -\ \omega_1 \wedge \omega_2 \\
*_6\ (\omega_1 \wedge \sigma_1) &= \omega_2 \wedge \omega_2
\end{aligned} \tag{A.12}$$

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