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TOPOLOGICAL ASPECTS OF NON-ABELIAN CHERN-SIMONS THEORY

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Abstract

We introduce the non-abelian Chern-Simons functional in 3 dimensions and the corresponding partition function. Via the Faddeev-Popov gauge fixing procedure and the stationary phase approximation we evaluate the 1-loop contribution to the partition function.
In order to investigate the metric dependence of the resulting expression, we make use of some mathematical results by Albert Schwarz and the Atiyah-Patodi-Singer index theorem.
The final result shows that the partition function depends on the metric in a simple, controllable way. We also elaborate on the possibility of removing such dependence by adopting a different choice of regularization scheme.

Contents

Introduction

Background and motivation

In theoretical physics, gauge theories serve as the foundational framework for understanding the fundamental interactions in nature. The electromagnetic force, weak force, and strong force are all described by gauge theories, where the behaviour of fields is governed by symmetries and local invariance principles. Among these, the Chern-Simons theory holds a special place due to its rich topological structure and potential applications in various domains, ranging from quantum field theory to condensed matter physics and mathematical physics.

The Chern-Simons theory was first introduced by the mathematicians Shiing-Shen Chern and James Harris Simons in the 1970s. The theory might seem deceitfully simple, as it requires a relatively small number of ingredients in order to be defined. One usually picks a Lie group G, a principal bundle over a 3-dimensional manifold M with structure group G and a gauge field A (technically this is a connection form, that is a differential 1-form on M with values in the Lie algebra $\mathfrak{g}$). Then the Chern-Simons action is defined by

$$\mathcal{S}_{CS}[A] = \frac{k}{4\pi} \int_M \operatorname{tr} \left(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A \right) .$$

Here k is the *level* of the theory, a constant which determines the strength of interactions. The precise mathematical formulation of the theory will be carried out in Chapter 1, however one can already notice an important aspect: the action does not contain any reference to a choice of metric on the manifold M. For this reason, Chern-Simons theory falls under the class of the so-called *topological field theories* (of Schwarz type).
It is important to note that this property holds explicitly for a classical theory: when quantizing the theory, for example via the path integral formalism, it is possible for an *anomaly* to arise. Indeed, one of the main questions we tackle in this thesis is whether the partition function of the non-abelian Chern-Simons theory does depend on the metric.

In order to address this question we will study the partition function of the theory with a semi-classical approximation, by expanding around the solutions of the classical equations of motion and focusing on the 1-loop contribution. Let us give a preview of the result we obtain at the end of Chapter 1, after carrying out the gauge fixing procedure:

$$\begin{aligned} \mathcal{Z}(M) &\approx \frac{1}{\mathrm{Vol}(Z_G)} \sum_\alpha e^{ikI_{CS}[A^{(\alpha)}]} \frac{\det \Delta_0}{\sqrt{\det L_-}} \\ &= \frac{1}{\mathrm{Vol}(Z_G)} \sum_\alpha e^{ikI_{CS}[A^{(\alpha)}]} \frac{(\det \Delta_0)\, e^{-i\pi\eta(L_-)/4}}{\sqrt{|\det L_-|}} . \end{aligned}$$

For now let us overlook the details of this expression and focus instead on some specific terms, namely the determinants of the differential operators Δ_0 and L_-.
The appearance of determinants of some operators is common in quantum field theory when evaluating the partition function. In general, these quantities cannot simply be defined as products of eigenvalues, since they might not converge; moreover, they rely on the choice of a metric on M, thus apparently spoiling the topological property of the theory.
For these reasons, most of our work will be devoted to introducing some technical and quite diverse mathematical tools, which will be used to rigorously define and study the contributions to the partition function at 1-loop order.

Abelian and non-abelian gauge groups

The simplest version of Chern-Simons theory, known as the abelian Chern-Simons theory, is defined for a gauge field corresponding to an abelian group, such as the unitary group U(1). In this case the action takes a simpler form:

$$\mathcal{S}_{CS}[A] = \frac{k}{4\pi} \int_M \mathrm{tr}\,(A \wedge \mathrm{d}A)\ ,$$

where the disappearance of second term is due to the fact that the structure constants of an abelian Lie group are identically zero.

The introduction of a non-abelian gauge group, such as SU(N) for $N > 1$, fundamentally alters the structure of the theory. The non-abelian generalization of Chern-Simons theory has several important features that distinguish it from the abelian case:

- Self-interactions of the gauge field: In non-abelian Chern-Simons theory, the field strength tensor $F = A \wedge \mathrm{d}A + A \wedge A$ involves the gauge field interacting with itself, unlike the abelian case, where the field strength is simply the exterior derivative of the gauge field.
- Non-trivial field configurations: The non-abelian gauge fields allow for more complicated field configurations, such as instantons or monopoles, which do not appear in the abelian case. These are localized, non-perturbative solutions that play a central role in understanding the quantum behaviour of the theory, and their presence can lead to interesting phenomena such as quantum anomalies.
- Knot theory: The non-abelian Chern-Simons theory also has connections to knot theory, where the gauge field configurations can be interpreted in terms of links and knots. This provides a powerful mathematical tool for studying topological invariants in quantum field theory. In particular, in 1989 Edward Witten proved that the partition function of the theory with the addition of Wilson loops can reproduce some known knot polynomials, which are topological invariants used to distinguish knots from one another.

Some physical applications

Chern-Simons theory has obtained increasing interest in recent years for its applications to quantum gravity, topological quantum field theory, and string theory. It has also found important uses in condensed matter physics, where it can model fractional quantum Hall states, bosonization of fermionic systems, and quantum computation.

- Condensed matter physics: In condensed matter physics, Chern-Simons theory has been applied to the study of topological phases of matter. The fractional quantum Hall effect (FQHE), in which the Hall conductance is quantized in fractional units, can be described using a Chern-Simons theory.
  Furthermore, the Chern-Simons functional plays a role in the relation between fermionic and bosonic systems. We will elaborate on this aspect in the last Chapter.
- Topological quantum computing: the non-abelian Chern-Simons theory is also a key component in the study of topological quantum computation. In this framework, quantum computation is based on topologically protected quantum states, which are resistant to local noise. Non-abelian anyons, which arise in Chern-Simons theories, have been proposed as the building blocks for topologically protected qubits.
- Quantum gravity and string theory: in string theory, the non-abelian Chern-Simons theory plays a role in describing topologically twisted theories and in the study of the AdS/CFT correspondence. This is in turn related to the duality between Chern-Simons theory in 3 dimensions and the Wess-Zumino-Witten (WZW) model, a 2-dimensional conformal field theory.

Structure of the thesis

This work is structured as follows.

In Chapter 1 we provide a review of the mathematical foundations of non-abelian Chern-Simons theory, which is properly formulated in the language of fibre bundles. Then we begin the evaluation of the partition function in the weak coupling limit via the method of stationary phase.

Chapter 2 is dedicated to the rigorous definition of the partition function of a quadratic functional, and the study of its dependence on the metric of the underlying space. We apply these general results to the semi-classical approximation of Chapter 1 and explain the appearance of a topological invariant known as Ray-Singer torsion.

In Chapter 3 we explore some profound mathematical theorems concerning the so-called index problem: to compute the analytical index of an elliptic operator D using only topological data related to D and to the vector bundle on which it acts. Similarly to Chapter 2, these concepts are applied to the evaluation of the Chern-Simons partition function.

The last Chapter contains some comments on the topological nature of the non-abelian Chern-Simons theory, an overview of some generalizations of the results presented in our work and a mention to an interesting physical application of this gauge theory.

1. The non-abelian Chern-Simons theory

This first Chapter is dedicated to the introduction of the non-abelian Chern-Simons theory. We recall some geometrical notions regarding the theory of principal bundles. The Chern-Simons action is defined and the quantization of its "level" (a parameter related to the strength of interactions) is motivated. We introduce the the partition function of the theory and the main observables: correlation functions of Wilson loops. Finally we begin the evaluation of the Chern-Simons partition function in the weak coupling limit, employing the stationary phase approximation.

1.1 Geometrical preliminaries

Let us begin by describing the mathematical framework of our theory. We briefly recall some notions and results concerning connections on principal fibre bundles. We also fix our notation for this Chapter by giving coordinate-free and coordinate-dependent expressions, as both types will be useful in the following Sections.
For more information on the general theory of fibre bundles see [22]; for a rigorous exposition of the connection between gauge theories and the mathematical language of principal fibre bundles we refer to [9].

Consider an oriented manifold M without boundary and a compact simple Lie group G (the *gauge group* of our theory), whose Lie algebra will be denoted by $\mathfrak{g}$. Fix a principal G-bundle $P \xrightarrow{\pi} M$; the space of k-forms on P will be denoted by $\Lambda^k(P)$.
Equip P with a connection 1-form $\omega \in \Lambda^1(P) \otimes \mathfrak{g}$. In physics one often works with *local* connection 1-forms: let $U \subset M$ an open subset and $\psi_u : U \times G \to \pi^{-1}(U)$ a local trivialization; denote by $\sigma_u : U \to P$ the local section satisfying $\psi_u(b, g) = \sigma_u(b)g$. Then the corresponding local connection 1-form (usually called *gauge field* in the physics literature) is defined by $A_u = \sigma_u^* \omega \in \Lambda^1(U) \otimes \mathfrak{g}$.

In the following we will assume that G is a matrix Lie group; this assumptions is not strictly necessary, but it will simplify our notation, by allowing us to make sense of expressions such as "$g^{-1}\,\mathrm{d}g$" and "$\varphi \wedge \eta$".
If $\psi_v : V \times G \to \pi^{-1}(V)$ is another local trivialization and $g_{uv} : U \cap V \to G$ is the transition function from U to V, then the transformation law of A_u is

$$A_v = g_{uv}^{-1} A_u g_{uv} + g_{uv}^{-1}\,\mathrm{d}g_{uv} \ , \tag{1.1}$$

where $\mathrm{d}g_{uv}$ denotes the differential of the matrix-valued function g_{uv}.

Given a connection 1-form ω on a principal G-bundle $P \xrightarrow{\pi} M$, we can uniquely write any $X \in T_pP$ as $X = X^V + X^H$ where X^V is vertical (i.e. $\pi_*(X^V) = 0$) and X^H is horizontal (i.e. $\omega(X^H) = 0$). If $\varphi \in \Lambda^k(P) \otimes \mathfrak{g}$ then we define $\varphi^H \in \Lambda^k(P) \otimes \mathfrak{g}$ by $\varphi^H(X_1, \ldots, X_k) \equiv \varphi(X_1^H, \ldots, X_k^H)$.
We define the *covariant exterior derivative* of $\varphi \in \Lambda^k(P) \otimes \mathfrak{g}$ as $D\varphi = (\mathrm{d}\varphi)^H \in \Lambda^{k+1}(P) \otimes \mathfrak{g}$, where $\mathrm{d}\varphi$ is the usual exterior derivative of φ. Then the curvature of the connection ω is defined by $\Omega = D\omega \in \Lambda^2(P) \otimes \mathfrak{g}$; the local curvature 2-forms (*gauge field strength* in physics) are $F_u = \sigma_u^* \Omega \in \Lambda(U) \otimes \mathfrak{g}$. If $\Omega = 0$ the connection ω is said to be *flat*.
The transformation law of F_u is

$$F_v = g_{uv}^{-1} F_u g_{uv} \, . \tag{1.2}$$

Choose local coordinates $\{x^\mu\}$, $\mu = 1, \ldots, d = \dim M$ on U and let $\{T_a\}$, $a = 1, \ldots, n = \dim \mathfrak{g}$ be a basis of $\mathfrak{g}$; we have the following expressions (suppressing the subscript "u" corresponding to the open subset $U \subset M$)

$$A(x) = A_\mu(x)\,\mathrm{d}x^\mu = A^a_\mu(x)\,\mathrm{d}x^\mu \otimes T_a \tag{1.3}$$

$$F(x) = \frac{1}{2}F_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}F^a_{\mu\nu}(x)\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes T_a\,. \tag{1.4}$$

Let $\varphi = \varphi^a \otimes T_a \in \Lambda^p(U) \otimes \mathfrak{g}$, $\eta = \eta^a \otimes T_a \in \Lambda^q(U) \otimes \mathfrak{g}$. We adopt the notation

$$\begin{aligned}[\varphi, \eta] &\equiv \varphi^a \wedge \eta^b \otimes [T_a, T_b]_{\mathfrak{g}} \\ &= \varphi^a \wedge \eta^b \otimes f_{ab}{}^c\, T_c\,,\end{aligned} \tag{1.5}$$

where $[\cdot,\cdot]_{\mathfrak{g}}$ denotes the Lie bracket and $f_{ab}{}^c$ are the structure constants of $\mathfrak{g}$.
Using $\varphi^a \wedge \eta^b = (-1)^{pq}\,\eta^b \wedge \varphi^a$ and $[T_a, T_b] = T_aT_b - T_bT_a$ for a matrix group, we can also write

$$[\varphi, \eta] = \varphi \wedge \eta - (-1)^{pq}\,\eta \wedge \varphi\,. \tag{1.6}$$

Here φ and η are regarded as matrices of $\mathbb{R}$-valued forms and $\varphi \wedge \eta$ is matrix multiplication where the entries are multiplied via the usual wedge product, that is $\varphi \wedge \eta = \varphi^a \wedge \eta^b \otimes T_aT_b$.
In particular, for $\varphi = \eta$ and when p, q are odd we get

$$[\varphi, \varphi] = 2\varphi \wedge \varphi\,, \tag{1.7}$$

which is non-zero in general (however, $[\varphi, \varphi] = 0$ if at least one of p and q is even).

The local connection and curvature forms satisfy the relation

$$F = \mathrm{d}A + \frac{1}{2}[A, A] = \mathrm{d}A + A \wedge A\,, \tag{1.8}$$

which yields the familiar components expressions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_{\mathfrak{g}} \tag{1.9}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f_{bc}{}^a\, A^b_\mu A^c_\nu\,. \tag{1.10}$$

1.2 The Chern-Simons action

In this Section define the main object of our study: the (non-abelian) Chern-Simons action. We explicitly compute its variation under general gauge transformations; by requiring the theory to be gauge invariant we obtain a quantization law, which is actually related to the topology of the gauge group G.
We also observe that the set of stationary points of the action (in other words, the solutions to the classical equations of motion) coincides with the space of flat connections. This fact will be relevant in Section 1.4, where the partition function is evaluated in the weak coupling limit via stationary phase approximation.

In the rest of the present work we will always assume dim $M = 3$ (M can be interpreted as (2+1)-dimensional spacetime). We will also assume that the principal G-bundle is *trivial*, that is $P \cong M \times G$; the local connection 1-form will be denoted by $A = A_\mu\,\mathrm{d}x^\mu = A^a_\mu\,\mathrm{d}x^\mu \otimes T_a$.

Definition 1.1. The Chern-Simons action is defined by

$$\begin{aligned}\mathcal{S}_{CS}[A] &= \frac{k}{4\pi}\int_M \mathrm{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right) \\ &= \frac{k}{8\pi}\int_M \mathrm{d}^3x\,\varepsilon^{\mu\nu\rho}\,\mathrm{tr}\left(A_\mu(\partial_\nu A_\rho - \partial_\rho A_\nu) + \frac{2}{3}A_\mu[A_\nu, A_\rho]_{\mathfrak{g}}\right)\,,\end{aligned} \tag{1.11}$$

where tr() is simply the trace of a matrix (recall that we are assuming G is a matrix Lie group for simplicity) and k is a constant called *level* of the theory.

Remark. If G is not a matrix group, one can define an invariant symmetric bilinear form tr() as a multiple of the Killing form on $\mathfrak{g}$ (since $\mathfrak{g}$ is a simple Lie algebra); the normalization of tr() will be fixed shortly.

Our first goal is to compute the variation of the Chern-Simons action under a gauge transformation, i.e. a change of local trivialization. If we denote by g the transition function, then

$$A \to A' = g^{-1}Ag + g^{-1}\,\mathrm{d}g\ . \tag{1.12}$$

Starting from $0 = \mathrm{d}(gg^{-1}) = \mathrm{d}g\, g^{-1} + g\, \mathrm{d}(g^{-1})$, we obtain $\mathrm{d}(g^{-1}) = -g^{-1}\,\mathrm{d}g\, g^{-1}$. Then $\mathrm{d}A$ transforms as

$$\mathrm{d}A \longrightarrow \mathrm{d}A' = -g^{-1}\,\mathrm{d}g\, g^{-1} \wedge Ag + g^{-1}\,\mathrm{d}A\, g - g^{-1}A \wedge \mathrm{d}g - g^{-1}\,\mathrm{d}g\, g^{-1} \wedge \mathrm{d}g\ , \tag{1.13}$$

where we used $d^2 = 0$.
Let us set $\mathrm{tr}(A \wedge \mathrm{d}A) \equiv \text{(I)}$, $\frac{2}{3}\mathrm{tr}(A \wedge A \wedge A) \equiv \text{(II)}$ and compute their variation under (1.12). Using the cyclicity of tr() we obtain

$$\begin{aligned}
\text{(I)}' = \mathrm{tr}(&-A \wedge \mathrm{d}g\, g^{-1} \wedge A + A \wedge \mathrm{d}A - g^{-1}A \wedge A \wedge \mathrm{d}g - g^{-1}A \wedge \mathrm{d}g\, g^{-1} \wedge \mathrm{d}g \\
&- \mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\, g^{-1} \wedge A + \mathrm{d}g \wedge g^{-1}\,\mathrm{d}A - g^{-1}\,\mathrm{d}g \wedge g^{-1}A \wedge \mathrm{d}g - g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\, g^{-1} \wedge \mathrm{d}g) \\
= \mathrm{tr}(&A \wedge \mathrm{d}A - 2A \wedge A \wedge \mathrm{d}g\, g^{-1} - 3A \wedge \mathrm{d}g\, g^{-1} \wedge \mathrm{d}g\, g^{-1} \\
&+ \mathrm{d}g\, g^{-1} \wedge \mathrm{d}A - g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g)\,.
\end{aligned}$$

A similar computation gives

$$\begin{aligned}
\text{(II)}' = \frac{2}{3}\mathrm{tr}(&A \wedge A \wedge A + g^{-1}A \wedge A \wedge \mathrm{d}g + A \wedge \mathrm{d}g \wedge g^{-1}A + g^{-1}A \wedge \mathrm{d}g \wedge g^{-1}\,\mathrm{d}g + \mathrm{d}g \wedge g^{-1}A \wedge A \\
&+ g^{-1}\,\mathrm{d}g \wedge g^{-1}A \wedge \mathrm{d}g + \mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}A + g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g) \\
= \frac{2}{3}\mathrm{tr}(&A \wedge A \wedge A + 3A \wedge A \wedge \mathrm{d}g\, g^{-1} + 3A \wedge \mathrm{d}g\, g^{-1} \wedge \mathrm{d}g\, g^{-1} + g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g)
\end{aligned}$$

and therefore

$$\text{(I)}' + \text{(II)}' = \text{(I)} + \text{(II)} + \mathrm{tr}\left(-A \wedge \mathrm{d}g\, g^{-1} \wedge \mathrm{d}g\, g^{-1} + \mathrm{d}g\, g^{-1} \wedge \mathrm{d}A - \frac{1}{3}\, g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\right).$$

Remark. The cyclicity of tr() holds in this case because we are taking the trace of matrices of 3-forms. For example, let $A, B, C, D \in \Lambda^1(M) \otimes \mathfrak{g}$, then

$$\begin{aligned}
\mathrm{tr}(A \wedge B \wedge C) &= A^a \wedge B^b \wedge C^c \otimes \mathrm{tr}(T_aT_bT_c) \\
&= A^a \wedge B^b \wedge C^c \otimes \mathrm{tr}(T_cT_aT_b) \\
&= C^c \wedge A^a \wedge B^b \otimes \mathrm{tr}(T_cT_aT_b) \\
&= \mathrm{tr}(C \wedge A \wedge B)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}(A \wedge \mathrm{d}B) &= A^a \wedge (\mathrm{d}B)^b \otimes \mathrm{tr}(T_aT_b) \\
&= A^a \wedge (\mathrm{d}B)^b \otimes \mathrm{tr}(T_bT_a) \\
&= (\mathrm{d}B)^b \wedge A^a \otimes \mathrm{tr}(T_bT_a) \\
&= \mathrm{tr}(\mathrm{d}B \wedge A)\,,
\end{aligned}$$

however

$$\begin{aligned}
\mathrm{tr}(A \wedge B \wedge C \wedge D) &= A^a \wedge B^b \wedge C^c \wedge D^d \otimes \mathrm{tr}(T_aT_bT_cT_d) \\
&= A^a \wedge B^b \wedge C^c \wedge D^d \otimes \mathrm{tr}(T_dT_aT_bT_c) \\
&= -D^d \wedge A^a \wedge B^b \wedge C^c \otimes \mathrm{tr}(T_dT_aT_bT_c) \\
&= -\mathrm{tr}(D \wedge A \wedge B \wedge C)\,.
\end{aligned}$$

In general one can prove that the following graded cyclic property of the trace holds [7]: let $\varphi \in \Lambda^p(M) \otimes \mathfrak{g}$, $\eta \in \Lambda^q(M) \otimes \mathfrak{g}$, then

$$\operatorname{tr}(\varphi \wedge \eta) = (-1)^{pq} \operatorname{tr}(\eta \wedge \varphi)\,. \tag{1.14}$$

Using the properties of the exterior derivative we find

$$\mathrm{d}(\mathrm{d}g\, g^{-1} \wedge A) = \mathrm{d}g \wedge g^{-1}\, \mathrm{d}g\, g^{-1} \wedge A - \mathrm{d}g\, g^{-1} \wedge \mathrm{d}A\,. \tag{1.15}$$

We conclude that the variation of the Chern-Simons action is

$$\delta\mathcal{S}_{CS}[A] = -\frac{k}{4\pi} \int \left(\mathrm{d}\left(\operatorname{tr}(\mathrm{d}g\, g^{-1} \wedge A)\right) + \frac{1}{3}\operatorname{tr}\left(g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\right)\right). \tag{1.16}$$

The first term is a total derivative which vanishes if suitable boundary conditions are imposed. The second term is related to a quantity known as *winding number*, which depends on the topology of the group G.

Definition 1.2. The winding number of the gauge transformation g is defined by

$$w(g) = \frac{1}{24\pi^2} \int \operatorname{tr}\left(g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\right)\,. \tag{1.17}$$

One can prove that the winding number is quantized, that is $w(g) = (\text{const}) \cdot n,\ n \in \mathbb{Z}$. It follows that we can fix the normalization of tr() in such a way that $w(g)$ is precisely an integer. This result is related to a general property of the gauge group: if G is a compact simple Lie group, then $\pi_3(G) \cong \mathbb{Z}$, where $\pi_n(X)$ denotes the n-th homotopy group of the topological space X [11]; this in turn implies that each gauge transformation g is classified by an integer $w(g) \in \mathbb{Z}$.

Remark. See Chapter 7 of [13] for a proof of the fact that the winding number is quantized (in the context of instantons) and a discussion of the case $G = SU(2)$. The generalization to any compact simple Lie group follows from a result of Raoul Bott [11].

This observation implies that the Chern-Simons action changes non-trivially under a gauge transformation with $w(g) \neq 0$, i.e. transformations which are not homotopic to the identity map (also called *large gauge transformations*). However, the quantum theory remains well-defined as long as the *amplitude* $e^{i\mathcal{S}_{CS}}$ is gauge invariant.
By the above calculations we can write

$$\exp\left(i\delta\mathcal{S}_{CS}\right) = \exp\left(-2\pi i k w(g)\right) \tag{1.18}$$

Therefore one must impose the condition $kw(g) \in \mathbb{Z}$, which implies $k \in \mathbb{Z}$, that is the level of the Chern-Simons theory must be an integer.

To conclude this Section we consider the classical equation of motion of the theory with action $\mathcal{S}_{CS}$. It is immediate to check that they are equivalent to the condition

$$F = \mathrm{d}A + A \wedge A = 0\,. \tag{1.19}$$

In other words, the stationary points of the Chern-Simons action correspond to the space of flat connections of the principal G-bundle.

1.3 Observables: Wilson loops and knots

We briefly introduce the observables that one can compute in a topological field theory such as the Chern-Simons theory. These observables are constructed from objects known as Wilson loops, which are gauge invariant operators that encode all gauge information about the connection. Moreover, they play a fundamental role in lattice field theory and in 3-dimensional manifolds they are related to knot

theory.
Our main references for this Section are [34] and [7].

Let us begin by noting that to define (1.11) we did not make use of a metric on the manifold M. Following [34] we define a *topological field theory* as a theory in which the observables do not depend on the choice of metric on M. In particular, the Chern-Simons action defines a "Schwarz"-type topological field theory, that is a theory where both the action and the fields do not involve the metric at all.
The next question to address concerns what kind of observables one can compute in a topological field theory. By definition they must be independent of the metric, therefore the usual gauge invariant local operators would not be appropriate. Instead, we will consider observables arising from *Wilson loops*, which are constructed as follows.

Definition 1.3. Let γ be an oriented closed curve on M, called a *knot*. One can then integrate the local connection 1-form A over γ to yield an element of the Lie algebra $\mathfrak{g}$. The path-ordered exponential of this integral is an element of G, called the *holonomy* of the connection around γ:

$$h[\gamma] = \mathcal{P} \exp \oint_\gamma A \,. \tag{1.20}$$

Let R be an irreducible representation of the group G. The *Wilson loop* is defined as the trace, in the representation R, of the holonomy:

$$W_R[\gamma] = \mathrm{tr}_R \left(\mathcal{P} \exp \oint_\gamma A \right) \,. \tag{1.21}$$

Definition 1.4. Let $\gamma_1, \ldots, \gamma_r$ be r oriented and non-intersecting closed curves in M. Their disjoint union $L = \bigcup_{i=1}^{r} \gamma_i$ is called a *link*. Fix an irreducible representation R_i of G for each knot γ_i. The observables in the Chern-Simons theory are given by *correlation functions* of Wilson loops, that is vacuum expectation values of products of Wilson loops, which we formally define as

$$\langle W_{R_1}[\gamma_1], \ldots, W_{R_r}[\gamma_r] \rangle = \frac{1}{\mathcal{Z}(M)} \int \mathscr{D}A \, e^{i\mathcal{S}_{CS}[A]} \prod_{i=1}^{r} W_{R_i}[\gamma_i] \equiv \mathcal{Z}(M; L) \,, \tag{1.22}$$

where the *partition function* in absence of links is

$$\mathcal{Z}(M) = \int \mathscr{D}A \, e^{i\mathcal{S}_{CS}[A]} \,. \tag{1.23}$$

Here it is understood that Feynman's path integral must be performed over all equivalence classes of gauge transformations.

Observe that, just like for the Chern-Simons action $\mathcal{S}_{CS}$, the definition of Wilson loop never requires one to choose a metric on M. One could then be tempted to conclude that both the partition function and the correlation functions are topological invariants of the manifold M. However, there is one more detail we have to worry about: there may be an anomaly in the quantum theory. In the present context an anomaly could arise via a "hidden" dependence of the path integral measure on the Riemannian metric. We will introduce this important aspect in the next Section, when trying to evaluate the path integral in the weak coupling limit (i.e. for large k). The rest of this thesis will be mostly devoted to describing some of the mathematical techniques that can be employed to address this question.

Remark. Under a gauge transformation g the holonomy transforms as $h'[\gamma] = g\, h[\gamma]\, g^{-1}$. On the other hand, Wilson loops are gauge invariant: physically observable quantities in a gauge theory should not change under gauge transformations, hence we construct observables from $W_R[\gamma]$ and not $h[\gamma]$. We refer to [7] for more details.

We conclude this Section by mentioning the fundamental connection between the non-abelian Chern-Simons theory and the mathematical field of knot theory, formulated by Edward Witten in 1989 [35]. If M is the 3-sphere S^3 and $G = \mathrm{SU}(2)$, then $\mathcal{Z}(S^3; L)$ reproduces precisely a kind of knot polynomial (i.e. a knot invariant that is a polynomial in some variables) known as Jones polynomial. This can be generalized to other gauge groups to yield different kind of knot polynomials: for example, $G = \mathrm{SO}(N)$ and $G = \mathrm{SU}(N)$ are related to the Kauffman polynomial and the HOMFLY polynomial respectively. Moreover, Witten proposed a way to generalize the Jones polynomial from S^3 to arbitrary 3-manifolds, via a procedure called "surgery on links".

1.4 The path integral in the weak coupling limit

In this Section we apply the stationary phase approximation to the Chern-Simons partition function and obtain an expression for the 1-loop contribution. We briefly discuss the validity of this result in comparison to another method used to evaluate the partition function. Finally we anticipate the relation between the 1-loop contribution and some known topological invariants, which we will elaborate on in the following Chapters.

To proceed we make some simplifying assumptions: we will consider the Chern-Simons partition function in absence of Wilson loops and we will assume that the level k is large, which is equivalent to saying that the theory is weakly coupled (intuitively, the parameter k plays the same role as $1/\hbar$ in the path integral formalism). This essentially allows us to evaluate the path integral via the stationary phase approximation; in particular, we will focus on the 1-loop contribution to the partition function and try to express it in terms of topological invariants.
This 1-loop approximation has been studied extensively in [16], [19] and [28]. We will sketch the derivation of the final result with no claim to completeness; in Chapter 2 and 3 we will describe the nature and properties of some interesting topological invariants that appear in this asymptotic formula.

In the large k limit, $\mathcal{Z}(M)$ is dominated by contributions from the points of stationary phase, that is the flat connections $\{A^{(\alpha)}\}$ which satisfy the classical equations of motion (1.19). We will assume that the cohomology groups $H^1_A(M)$ and $H^0_A(M)$ are trivial. As hinted in [16], these conditions are equivalent to requiring that the flat connections are isolated (modulo gauge equivalence) and that there are no automorphisms on the space of flat connections with fixed points. By Poincarè duality, one can conclude that $H^2_A(M)$ and $H^3_A(M)$ are also trivial.
For clarity we note that said cohomology groups are constructed from the complex

$$0 \to \Lambda^0(M) \otimes \mathfrak{g} \xrightarrow{D} \Lambda^1(M) \otimes \mathfrak{g} \xrightarrow{D} \Lambda^2(M) \otimes \mathfrak{g} \xrightarrow{D} \Lambda^3(M) \otimes \mathfrak{g} \to 0\,, \tag{1.24}$$

where D is the covariant exterior derivative with respect to the connection $A^{(\alpha)}$, and $D^2 = 0$ holds because $A^{(\alpha)}$ is flat.
These requirements should become clearer after the construction carried out in Chapter 2, where they play a fundamental role in determining the dependence of the partition function on the metric that we equip M with.

Under these assumptions, the integrand is wildly oscillatory and the partition function can be expressed as a sum of terms associated to stationary points:

$$\mathcal{Z}(M) = \sum_\alpha \mathcal{Z}^{(\alpha)}(M)\,. \tag{1.25}$$

Let us fix a flat connection $A^{(\alpha)}$. In order to evaluate the contribution $\mathcal{Z}^{(\alpha)}(M)$ we expand

$$A = A^{(\alpha)} + B\,, \tag{1.26}$$

where $A^{(\alpha)}$ is regarded as a "background" connection and B as a "fluctuation" around $A^{(\alpha)}$.
Inserting this expression into (1.11) and keeping the first non-vanishing term in B one finds, for a

particular flat connection $A^{(\alpha)}$

$$\mathcal{S}_{CS}[A^{(\alpha)}+B] \approx \mathcal{S}_{CS}[A^{(\alpha)}] + \frac{k}{4\pi}\int_M \mathrm{tr}(B\wedge DB)\,, \tag{1.27}$$

where DB denotes the covariant exterior derivative of B with respect to $A^{(\alpha)}$.
Observe that the linear term in B vanishes (up to a total derivative) since the connection $A^{(\alpha)}$ is flat:

$$\int_M \mathrm{tr}(A^{(\alpha)}\wedge \mathrm{d}B + B\wedge \mathrm{d}A^{(\alpha)} + 2B\wedge A^{(\alpha)}\wedge A^{(\alpha)}) = \int_M \mathrm{tr}(\mathrm{d}(B\wedge A^{(\alpha)}) + 2B\wedge(\mathrm{d}A^{(\alpha)} + A^{(\alpha)}\wedge A^{(\alpha)}))\,.$$

This is expected, as we are expanding around a stationary point of the action.

Let us denote the *classical Chern-Simons invariant* of the connection $A^{(\alpha)}$ by

$$I_{CS}[A^{(\alpha)}] = \frac{1}{4\pi}\int_M \mathrm{tr}\left(A^{(\alpha)}\wedge \mathrm{d}A^{(\alpha)} + \frac{2}{3}A^{(\alpha)}\wedge A^{(\alpha)}\wedge A^{(\alpha)}\right)\,. \tag{1.28}$$

This quantity is a purely topological invariant and it has been widely analysed (see for example [14]); we will not discuss it further.
So far we have obtained

$$\mathcal{Z}(M) \approx \sum_\alpha e^{ikI_{CS}[A^{(\alpha)}]}\int \mathscr{D}B\,\exp\left(\frac{ik}{4\pi}\int_M \mathrm{tr}(B\wedge DB)\right)\,. \tag{1.29}$$

Now a problem arises: in order to perform the path integral in $\mathscr{D}B$ a gauge fixing condition must be imposed on the field B. This in turn requires one to fix a metric on M, apparently spoiling the topological nature of the Chern-Simons theory. Following Witten we choose a covariant condition (by "covariant" we mean with respect to the connection $A^{(\alpha)}$):

$$D_\mu B^\mu = 0\,, \tag{1.30}$$

where D_μ denotes the covariant exterior derivative in local coordinates, that is

$$D_\mu = \partial_\mu + [A_\mu^{(\alpha)},\cdot]\,. \tag{1.31}$$

Let us explicitly carry out the gauge fixing procedure in the BRST formalism. Following [8] we introduce the anticommuting ghost fields $c\in\Lambda^0(M)\otimes\mathfrak{g}, \bar{c}\in\Lambda^3(M)\otimes\mathfrak{g}$ (by anticommuting we mean $c^ac^b = -c^bc^a$ and similarly for $c\bar{c}, \bar{c}\bar{c}$) and the auxiliary field $\phi\in\Lambda^3(M)\otimes\mathfrak{g}$. The standard BRST transformation laws are

$$\delta A_\mu = iD_\mu c,\quad \delta c = \frac{1}{2}[c,c],\quad \delta\bar{c} = \phi,\quad \delta\phi = 0\,. \tag{1.32}$$

Observe that the transformation denoted by δ turns commuting fields into anticommuting ones and vice-versa (equivalently one can replace δ by $\varepsilon\delta$, where ε is an anticommuting arbitrary parameter, i.e. a Grassmann number).
The action after gauge fixing can be written as

$$\mathcal{S}'_{CS} = \mathcal{S}_{CS} + \delta\mathcal{V} \tag{1.33}$$

where $\mathcal{V}$ is a functional chosen in such a way that $\mathcal{S}'_{CS}$ is non-degenerate.
Let $\star$ denote the Hodge operator which maps k forms to $3-k$ forms; we define it so that $\star^2 = 1$ for a Riemannian 3-manifold. The induced positive-definite inner product on $\Lambda^k(M)\otimes\mathfrak{g}$ is denoted in this Section by $\langle\cdot,\cdot\rangle$.
We choose the following form of the gauge-fixing Lagrangian:

$$\mathcal{V} = \frac{k}{2\pi}\int_M \mathrm{tr}(\bar{c}\star D\star B) \tag{1.34}$$

so that

$$\delta\mathcal{V} = \frac{k}{2\pi}\int_M \mathrm{tr}(\phi\star D\star B + i\bar{c}\star D\star Dc)\,. \tag{1.35}$$

Now the path integral in (1.29) becomes

$$\int \mathscr{D}B\,\mathscr{D}\phi\,\mathscr{D}\bar{c}\,\mathscr{D}c\;\exp\left(\frac{k}{4\pi}\int_M \mathrm{tr}(iB\wedge DB + 2i\phi\star D\star B - 2\bar{c}\star D\star Dc)\right)\,. \tag{1.36}$$

After the field redefinition

$$B' = \sqrt{\frac{k}{2\pi}}B,\quad c' = \sqrt{\frac{k}{2\pi}}c,\quad \phi' = \sqrt{\frac{k}{2\pi}}\phi,\quad \bar{c}' = \sqrt{\frac{k}{2\pi}}\star\bar{c} \tag{1.37}$$

this is equivalent to

$$\int \mathscr{D}B'\,\mathscr{D}\phi'\,\mathscr{D}\bar{c}'\,\mathscr{D}c'\;\exp\left(\frac{1}{2}\int_M \mathrm{tr}(iB'\wedge DB' + 2i\phi'\star D\star B') - \int_M \mathrm{tr}(\bar{c}'D\star Dc')\right) \tag{1.38}$$

(observe that now $\bar{c}'$ is a $\mathfrak{g}$-valued zero form).
Let $\Lambda_+ = \Lambda^0\oplus\Lambda^2$ and $\Lambda_- = \Lambda^1\oplus\Lambda^3$. It is useful to combine B' and ϕ' into a field $H = (B',\phi')$, which is valued in $\Lambda_-(M)\otimes\mathfrak{g}$. The path integral can then be rewritten as

$$\int \mathscr{D}H\,\mathscr{D}\bar{c}'\,\mathscr{D}c'\;\exp\left(-\frac{i}{2}\langle H, L_-H\rangle + \langle\bar{c}',\Delta_0 c'\rangle\right)\,, \tag{1.39}$$

where $\Delta_0 = \star D\star D$ is the standard Laplacian on $\Lambda^0(M)\otimes\mathfrak{g}$ and $L_- = (\star D + D\star)J$ is a standard elliptic operator on $\Lambda_-(M)\otimes\mathfrak{g}$. Here J is defined as follows: $J\varphi = -\varphi$ if $\varphi\in\Lambda^3$ and $J\varphi = +\varphi$ if $\varphi\in\Lambda^1$.
Now introduce a set of orthonormal eigenfunctions of Δ_0 and L_-:

$$\Delta_0\psi_i = u_i\psi_i,\quad \langle\psi_i,\psi_j\rangle = \delta_{ij} \tag{1.40}$$

$$L_-\varphi_i = v_i\varphi_i,\quad \langle\varphi_i,\varphi_j\rangle = \delta_{ij}\,. \tag{1.41}$$

We also expand $c' = \sum_i c_i\psi_i$, $\bar{c}' = \sum_i \bar{c}_i\psi_i$, $H = \sum_i h_i\varphi_i$ and we interpret the path integral measures to be

$$\mathscr{D}\bar{c}'\,\mathscr{D}c' = \prod_i \mathrm{d}\bar{c}_i\,\mathrm{d}c_i\,,\quad \mathscr{D}H = \prod_i \frac{\mathrm{d}h_i}{\sqrt{2\pi}}\,. \tag{1.42}$$

Now we make use of the Berezin integral

$$\int \mathrm{d}\bar{c}\,\mathrm{d}c\,e^{u\bar{c}c} = u \tag{1.43}$$

and the Fresnel integral

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}h}{\sqrt{2\pi}}e^{-ivh^2/2} = \frac{e^{-i\pi\,\mathrm{sign}(v)/4}}{\sqrt{|v|}}\,. \tag{1.44}$$

Finally we get to the following expression of the 1-loop approximation of the partition function

$$\begin{aligned}\mathcal{Z}(M) &\approx \frac{1}{\mathrm{Vol}(Z_G)}\sum_\alpha e^{ikI_{CS}[A^{(\alpha)}]}\frac{\det\Delta_0}{\sqrt{\det L_-}}\\ &= \frac{1}{\mathrm{Vol}(Z_G)}\sum_\alpha e^{ikI_{CS}[A^{(\alpha)}]}\frac{(\det\Delta_0)\,e^{-i\pi\eta(L_-)/4}}{\sqrt{|\det L_-|}}\,.\end{aligned} \tag{1.45}$$

Here $\mathrm{Vol}(Z_G)$ denotes the number of elements in the center Z_G of the group G (see [28] for the reasoning behind its appearance). We formally defined $\det\Delta_0 = \prod_i u_i$, $|\det L_-| = \prod_i |v_i|$, while $\eta(L_-) = \sum_i \mathrm{sign}(v_i)$ is the signature of $\langle H, L_-H\rangle$.

We stress that (to our knowledge) (1.45) has not been proven rigorously yet. In particular, the path integral measures (1.42) are not guaranteed to be well defined; a rigorous discussion on the evaluation of the partition function is carried out in Chapter 2, although restricted to some simpler cases. That being said, the perturbative expansion is not the only way to evaluate the Chern-Simons partition function: Witten introduced another method, sometimes called "surgery calculus", which is based upon a construction of M as a surgery on a link in S^3. It has been proved in [27] that this surgery calculus applied to $\mathcal{Z}(M)$ yields a topological invariant of the manifold M. The comparison between both methods of computing Witten's invariant has been carried out in some specific cases, we cite for example [16], [19] and [28]. A complete agreement between the perturbative approximation and the surgery calculus method has been found in all these papers.

We will discuss in Chapter 2 how the determinants appearing in (1.45) should be properly defined. Towards the end of the same Chapter we also sketch how the absolute value of their ratio can be expressed in terms of a topological invariant known as Ray-Singer torsion [26, 25].
The phase of this ratio will be discussed in Chapter 3 and linked to another quantity called η-invariant, introduced by Atiyah, Patodi and Singer in [3, 4, 5].
Can we conclude then that the Chern-Simons partition function truly is a topological invariant? Unfortunately the phase arising from the η-invariant has an "anomalous" dependence on the metric. The good news is that one can exactly determine said dependence and therefore account for it.

2. The Ray-Singer torsion

In this Chapter we explore the relation between the absolute value of the ratio of the determinants appearing in (1.45) and the topological invariant known as Ray-Singer torsion [26]. The following analysis, which is largely based on [29] and [30], concerns more generally the partition function of a degenerate quadratic functional and its dependence on the geometry of the underlying space.
This Chapter is more mathematical in nature, however we might omit some technical steps in our proofs. When that is the case we will refer to the relevant literature.

2.1 Regular operators and regularized determinants

In this Section we introduce a technique known as zeta-function regularization [12]. In general, the determinant of an infinite-dimensional operator on a Hilbert space cannot be simply defined as the product of its eigenvalues, since it might not converge. Some operators, however, admit a generalized version of the determinant known as regularized determinant (sometimes also called functional determinant). We consider below a class of such operators, called regular operators; we will also define regular pairs of operators and families of regular operators, that is operators which smoothly depend on a real parameter.

The first step is to generalize the familiar notion of determinant of a matrix to the case of infinite-dimensional operators.

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space and $B : \mathcal{H} \to \mathcal{H}$ a non-negative, self-adjoint, trace class operator. Then we say that B is *regular* if there exist coefficients $\alpha_k(B)$ such that

$$\mathrm{Tr}(e^{-Bt} - \Pi_B) = \sum_{k\in K} \alpha_k(B) t^{-k} + O(t^{\varepsilon}) \quad \text{for} \quad t \to 0^+ \tag{2.1}$$

where Π_B is the projector on the kernel of B, $\varepsilon > 0$ and K is a finite set of non-negative numbers.

In the following, when taking the trace of an operator we will always implicitly assume that it is of the trace class.

Now recall the definition of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} \mathrm{d}t, \quad s \in \mathbb{C}, \, \mathrm{Re}(s) > 1\,,$$

where $\Gamma(s)$ is the gamma function.
Similarly, we can define a generalized zeta function associated to an operator B.

Definition 2.2. Given a regular operator B, its *associated zeta function* for large $\mathrm{Re}(s)$ is

$$\zeta_B(s) = \sum_j \lambda_j^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \mathrm{Tr}(e^{-Bt} - \Pi_B) t^{s-1} \mathrm{d}t \tag{2.2}$$

where λ_j are the non-zero eigenvalues of B.
For other values of $\mathrm{Re}(s)$ the function $\zeta_B(s)$ can be defined by means of analytic continuation. For a regular operator B, we choose the following form of the analytic continuation of $\zeta_B(s)$ in the half-plane

$\mathrm{Re}(s) > -\varepsilon$, $\varepsilon > 0$

$$\begin{aligned} \zeta_B(s) = \frac{1}{\Gamma(s)} \Bigg(\sum_{k\in K} \frac{\alpha_k(B)}{s-k} &+ \int_1^\infty \mathrm{Tr}(e^{-Bt} - \Pi_B) t^{s-1} \mathrm{d}t \\ &+ \int_0^1 \left(\mathrm{Tr}(e^{-Bt} - \Pi_B) - \sum_{k\in K} \alpha_k(B) t^{-k} \right) t^{s-1} \mathrm{d}t \Bigg) . \end{aligned} \tag{2.3}$$

Now observe that for a finite-dimensional operator B we have

$$\frac{\mathrm{d}}{\mathrm{d}s} \zeta_B(s) \bigg|_{s=0} = -\sum_j \log \lambda_j = -\log \prod_j \lambda_j \, .$$

This observation combined with the fact that $\zeta_B(s)$ is analytic at $s = 0$ motivates the following definition.

Definition 2.3. Given a regular operator B, its *regularized determinant* $\det'(B)$ is defined by the formula

$$\log \det{}'(B) = -\frac{\mathrm{d}}{\mathrm{d}s} \zeta_B(s) \bigg|_{s=0} . \tag{2.4}$$

Using expression (2.3) one can write

$$\begin{aligned} \log \det{}'(B) = &\sum_{k\in K,\, k\neq 0} k^{-1} \alpha_k(B) + \Gamma'(1) \alpha_0(B) \\ &- \int_1^\infty \mathrm{Tr}(e^{-Bt} - \Pi_B) t^{-1} \mathrm{d}t \\ &- \int_0^1 \left(\mathrm{Tr}(e^{-Bt} - \Pi_B) - \sum_{k\in K} \alpha_k(B) t^{-k} \right) t^{-1} \mathrm{d}t \, , \end{aligned} \tag{2.5}$$

where we used

$$\frac{1}{\Gamma(0)} = 0 \, , \quad \frac{\mathrm{d}}{\mathrm{d}s} \frac{1}{\Gamma(s)} \bigg|_{s=0} = 1 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s} \frac{1}{s\Gamma(s)} \bigg|_{s=0} = -\Gamma'(1)$$

(the first two relations can be derived, for example, from the infinite product expansion of the reciprocal gamma function

$$\frac{1}{\Gamma(s)} = s \prod_{n=1}^\infty \frac{1 + s/n}{(1 + 1/n)^s} \, ,$$

while the last one directly follows from the property $s\Gamma(s) = \Gamma(s+1)$).

Remark. A similar definition can be given if the operator B^*B is regular, where B is an operator between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and B^* is the adjoint of B:

$$\det{}'(B) = \exp \left(-\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} \zeta_{B^*B}(s) \bigg|_{s=0} \right) = \det{}'(B^*B)^{1/2} \, . \tag{2.6}$$

The two definitions are equivalent if B is a self-adjoint operator.

The procedure outlined above is called *zeta-function regularization.* In the context of perturbation theory, this technique has some advantages compared to other commonly used methods of regularization. For example, in curved space-times it is not always clear how to apply dimensional regularization, while zeta-function regularization can yield unambiguous results (for a deeper discussion, see Chapter 1 of [12]).

In the rest of this Chapter we will often consider operators which continuously depend on a parameter $0 \leq u \leq 1$ and their relation with pairs of other operators. We define these objects below.

Definition 2.4. A smooth *regular family of operators* is a family of regular operators $B(u)$, $0 \leq u \leq 1$, satisfying

$$\left| \frac{\mathrm{d}}{\mathrm{d}u} \left(\mathrm{Tr}(e^{-B(u)t} - \Pi_B) - \sum_{k \in K} \alpha_k(B(u)) t^{-k} \right) \right| \leq C t^{\varepsilon} \quad \text{for} \quad 0 \leq t \leq 1 \tag{2.7}$$

and

$$\left| \frac{\mathrm{d}}{\mathrm{d}u} \mathrm{Tr}(e^{-B(u)t} - \Pi_B) \right| \leq C_N t^{-N} \quad \text{for} \quad t > 1\,. \tag{2.8}$$

Here K is a finite set of non-negative numbers, N is an arbitrary number, the positive constants ε, C, C_N do not depend on u.

Definition 2.5. The operators A, B acting on a Hilbert space $\mathcal{H}$ form a *regular pair* (A, B) if the following conditions are satisfied:

- B is a non-negative self-adjoint operator;
- there exist coefficients $\beta_k(A|B)$ such that

$$\mathrm{Tr}\, A(e^{-Bt} - \Pi_B) = \sum_{k \in K} \beta_k(A|B) t^{-k} + O(t^{\varepsilon}) \quad \text{for} \quad t \to 0^+ \tag{2.9}$$

  and

$$\mathrm{Tr}\, A(e^{-Bt} - \Pi_B) = O(t^{-N}) \quad \text{for} \quad t \to \infty\,. \tag{2.10}$$

Here K denotes a finite set of non-negative numbers, $\varepsilon > 0$, N is an arbitrary number.

2.2 The partition function of a quadratic functional

The partition function is a fundamental object in quantum field theory, as it permits one to study expectation values and correlation functions, which are in turn related to physical observables. Its formulation in terms of Feynman's path integral, which was used in the previous Chapter, is sometimes difficult to work with, especially when trying to rigorously establish some of its key properties.
That being said, in some cases the formal computation of the partition function as a path integral can result in the simple product of (regularized) determinants of appropriate operators; in particular, this happens when we are dealing with quadratic functionals. One can then choose to *define* the partition function by such a product of determinants.
This Section is dedicated to defining the partition function of a quadratic functional, with particular emphasis to the case of a degenerate functional, that is a functional which is invariant under some kind of "symmetry" transformation. Degenerate functionals are ubiquitous in gauge theories, which include the non-abelian Chern-Simons theory.

Let Γ_0 be a pre-Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle$ and $S : \Gamma_0 \to \Gamma_0$ a self-adjoint operator. Then a quadratic functional $\mathcal{S}$ on Γ_0 is defined by

$$\mathcal{S}(f) = \langle Sf, f \rangle\,. \tag{2.11}$$

We say that $\mathcal{S}$ is non-degenerate if

$$Sf = 0 \quad \Longleftrightarrow \quad f = 0\,.$$

If $\mathcal{S}$ is non-degenerate and S^2 is a regular operator then we define the partition function of $\mathcal{S}$ as

$$\mathcal{Z} = \det{}'(S)^{-1/2} = \det{}'(S^2)^{-1/4}\,.$$

However, we will be more interested in the case of a *degenerate* quadratic functional, as is the Chern-Simons Lagrangian (more generally, the action in gauge theories is invariant under the infinite-dimensional group of local gauge transformations and therefore the corresponding Lagrangian is degenerate).

Let Γ_1 be another pre-Hilbert space and $T : \Gamma_1 \to \Gamma_0$ an operator which satisfies

$$\mathcal{S}(f + Th) = \mathcal{S}(f) \quad \forall h \in \Gamma_1 \,. \tag{2.12}$$

This means that every element $h \in \Gamma_1$ generates a symmetry transformation of $\mathcal{S}$ and if $T \neq 0$ the functional $\mathcal{S}$ is degenerate.

Lemma 2.1. *The symmetry condition (2.12) is equivalent to $ST = 0$.*

Proof. If $ST = 0$ then

$$\begin{aligned}
\mathcal{S}(f + Th) &= \langle S(f + Th), f + Th\rangle \\
&= \langle Sf, f\rangle + \langle STh, f\rangle + \langle Sf, Th\rangle + \langle STh, Th\rangle \\
&= \langle Sf, f\rangle + \langle f, STh\rangle \\
&= \mathcal{S}(f)\,.
\end{aligned}$$

On the other hand, if $\mathcal{S}(f + Th) = \mathcal{S}(f)$ then

$$\langle STh, f\rangle + \overline{\langle STh, f\rangle} + \langle STh, Th\rangle = 0 \quad \forall h \in \Gamma_1$$

which implies $ST = 0$. □

We further assume that the adjoint operator $T^* : \Gamma_0 \to \Gamma_1$ exists and that the operators S^2 and T^*T are regular.

Definition 2.6. Under these assumptions, we define the *partition function* of the degenerate functional $\mathcal{S}$ as

$$\mathcal{Z} = \det{}'(S)^{-1/2} \det{}'(T)\,. \tag{2.13}$$

Remark. This definition, as well as the more general expression (2.37), can be interpreted in terms of the Faddeev-Popov procedure applied to gauge theories in quantum field theory. For more details see the Appendix of [29].

Lemma 2.2. *The partition function of a degenerate functional $\mathcal{S}$ can be written as*

$$\mathcal{Z} = \det{}'(\Box_0)^{-1/4} \det{}'(\Box_1)^{3/4} \tag{2.14}$$

where

$$\begin{aligned}
\Box_0 &= S^2 + TT^* \\
\Box_1 &= T^*T\,.
\end{aligned} \tag{2.15}$$

We provide a sketch of the proof below.

Proof. By definition $\det'(S) = \det'(S^2)^{1/2}$, $\det'(T) = \det'(T^*T)^{1/2}$ and therefore

$$\mathcal{Z} = \det{}'(S^2)^{-1/4} \det{}'(T^*T)^{1/2}\,.$$

One can check that the operator $S^2 + TT^*$ is regular. Lemma 2.1 implies $S^2TT^* = 0$ and $TT^*S^2 = 0$, which in turn can be used to prove the relation

$$\det{}'(S^2 + TT^*) = \det{}'(S^2) \det{}'(TT^*)\,. \tag{2.16}$$

We give an intuitive argument to justify the validity of such an expression. Clearly S^2 and TT^* commute, and it is well known that in this case one can find a basis $\mathcal{B}$ of common eigenvectors which diagonalizes both S^2 and TT^*.

Let $f \in \mathcal{B} \setminus \{0\}$ and denote the corresponding eigenvalue of S^2 and TT^* by σ and τ respectively, i.e. $S^2 f = \sigma f,\ TT^* f = \tau f$. Assume that $\sigma \neq 0$, then

$$0 = TT^* S^2 f = TT^* \sigma f = \sigma \tau f$$

and therefore $\sigma \neq 0 \implies \tau = 0$. Similarly, we have $\tau \neq 0 \implies \sigma = 0$.
This implies that the operator $S^2 + TT^*$ has eigenvalues (σ_i, τ_j), where σ_i and τ_j denote the eigenvalues of S^2 and TT^*; hence the regularized determinant of $S^2 + TT^*$ can be factorized as $\det'(S^2)\det'(TT^*)$. It can be proven that the non-zero eigenvalues of TT^* and T^*T are the same, therefore their regularized determinants also coincide. Combining this observation with relation (2.16) concludes the proof. □

2.3 Variation of the partition function

Our goal now is to take the partition function written as in lemma (2.2) and study its variation with respect to the choice of scalar products on the relevant Hilbert spaces. We parametrize the change of these scalar products via some self-adjoint operators which smoothly depend on a real parameter $0 \leq u \leq 1$ (here the definition of regular family from Section 2.1 comes into play). We then proceed to formally compute an explicit expression for the variation of the partition function.
Afterwards we make this result rigorous by specifying our analysis to the case of differential operators acting on a compact manifold; we further require the operators (2.15) to be elliptic.
Finally we state a lemma which guarantees the independence of the partition function on the choice of scalar products under some specific assumptions.

We begin this Section by emphasizing that definition 2.6 explicitly makes use of the operator $T : \Gamma_1 \to \Gamma_0$ and the scalar products in the spaces Γ_0 and Γ_1. This observation leads to the following questions:

- How does the partition function $\mathcal{Z}$ depend on T, $\langle \cdot, \cdot \rangle_{\Gamma_0}$ and $\langle \cdot, \cdot \rangle_{\Gamma_1}$?
- In which cases, if any, is $\mathcal{Z}$ invariant under variation of said objects?

To answer these questions we will make use of a result which we state in the following form.

Lemma 2.3. *Let*

$$\sigma(u) = \sum_{0 \leq q \leq m} \lambda_q \log \det'(A_q(u)) \tag{2.17}$$

where $A_q(u)$ is a smooth regular family of operators for every q and the coefficients λ_q do not depend on u.
Let us suppose that

$$\frac{\mathrm{d}}{\mathrm{d}u} \sum_{0 \leq q \leq m} \lambda_q \mathrm{Tr}\left(e^{-tA_q(u)} - \Pi_{A_q}\right) = t\,\frac{\mathrm{d}}{\mathrm{d}t} \sum_{0 \leq r \leq n} \mathrm{Tr}\, R_r(u) \left(e^{-tT_r(u)} - \Pi_{T_r}\right) \tag{2.18}$$

where $(R_r(u), T_r(u))$ is a regular pair of operators for every u and r. Then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}u} = \sum_{0 \leq r \leq n} \beta_0(R_r(u)|T_r(u))\,. \tag{2.19}$$

Proof. If condition (2.18) holds, then using (2.7) and (2.9) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}u} \sum_q \lambda_q \alpha_k(A_q(u)) = -k \sum_r \beta_k(R_r(u)|T_r(u))\,. \tag{2.20}$$

Now we compute the derivative of $\sigma(u)$ using the explicit form of the regularized determinant (2.5),

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}u} =& \sum_{k\neq 0} k^{-1}\frac{\mathrm{d}}{\mathrm{d}u}\sum_q \lambda_q \alpha_k(A_q) + \Gamma'(1)\frac{\mathrm{d}}{\mathrm{d}u}\sum_q \lambda_q\alpha_0(A_q)\\ &- \int_1^\infty \frac{\mathrm{d}}{\mathrm{d}u}\sum_q \lambda_q \mathrm{Tr}\left(e^{-tA_q} - \Pi_{A_q}\right) t^{-1}\mathrm{d}t\\ &- \int_0^1 \frac{\mathrm{d}}{\mathrm{d}u}\sum_q \lambda_q\left(\mathrm{Tr}(e^{-tA_q} - \Pi_{A_q}) - \sum_k \alpha_k(A_q)t^{-k}\right)t^{-1}\mathrm{d}t\,.\end{aligned}$$

By (2.18) and (2.20) we get

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}u} =& -\sum_{k\neq 0}\sum_r \beta_k(R_r|T_r)\\ &- \int_1^\infty \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\left(\sum_r R_r\left(e^{-tT_r} - \Pi_{T_r}\right)\right)\mathrm{d}t\\ &- \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Tr}\left(\sum_r R_r\left(e^{-tT_r} - \Pi_{T_r}\right)\right)\mathrm{d}t + \sum_k\sum_r \beta_k(R_r|T_r)\,.\end{aligned}$$

Using the hypothesis of regularity of the pair (R_r, T_r), we obtain (2.19). □

We will study the variation of the partition function $\mathcal{Z}$ under variation of the scalar products in Γ_0 and Γ_1. We denote them by $\langle\cdot,\cdot\rangle_0^u$ and $\langle\cdot,\cdot\rangle_1^u$ respectively, with $0 \leq u \leq 1$ which parametrizes said variation. To be more concrete, we define self-adjoint operators $B_i^u : \Gamma_i \to \Gamma_i$ (where $i = 0, 1$) by

$$\frac{\mathrm{d}}{\mathrm{d}u}\langle f, g\rangle_i^u = \langle B_i^u f, g\rangle_i^u = \langle f, B_i^u g\rangle_i^u\,. \tag{2.21}$$

Remark. The operator B^u has a simple form in the case of finite-dimensional vector spaces. Fix a basis and denote by g_{ij}, g^{ij} the components of the metric and the inverse metric respectively. Then by explicit computation we have

$$(B^u)^i{}_j = g^{ik}\frac{\mathrm{d}g_{kj}}{\mathrm{d}u}\,.$$

We are not aware of an analogous interpretation in the case of general Hilbert spaces.

According to lemma 2.2 the partition function can be written as

$$\mathcal{Z}(u) = \det{}'(\Box_0(u))^{-1/4}\det{}'(\Box_1(u))^{3/4} \tag{2.22}$$

where

$$\begin{aligned}\Box_0(u) &= S^2(u) + TT^*(u)\\ \Box_1(u) &= T^*(u)T\end{aligned} \tag{2.23}$$

and the u-dependent operators $S(u)$, $T^*(u)$ are defined by

$$\begin{aligned}\mathcal{S}(f) &= \langle S(u)f, f\rangle_0^u = \langle f, S(u)f\rangle_0^u\\ \langle T^*(u)f, g\rangle_1^u &= \langle f, Tg\rangle_0^u\end{aligned} \tag{2.24}$$

(T does not depend on u since its definition does not involve any inner product).
Using these definitions and (2.21) one finds

$$\begin{aligned}\frac{\mathrm{d}S(u)}{\mathrm{d}u} &= -B_0^u S(u)\\ \frac{\mathrm{d}T^*(u)}{\mathrm{d}u} &= T^*(u)B_0^u - B_1^u T^*(u)\end{aligned} \tag{2.25}$$

and therefore

$$\begin{aligned} \frac{\mathrm{d}\Box_0(u)}{\mathrm{d}u} &= -B_0 S^2 - S B_0 S + T T^* B_0 - T B_1 T^* \\ \frac{\mathrm{d}\Box_1(u)}{\mathrm{d}u} &= T^* B_0 T - B_1 T^* T \,. \end{aligned} \tag{2.26}$$

By expanding $e^{-t\Box_0}$ and $e^{-t\Box_1}$ and recalling $ST = 0$ one can check that the following relations hold

$$\begin{aligned} e^{-t\Box_0} T &= T e^{-t\Box_1} \\ e^{-t\Box_1} T^* &= T^* e^{-t\Box_0} \\ e^{-t\Box_0} S &= S e^{-t\Box_0} \,. \end{aligned} \tag{2.27}$$

Now observe that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}u}\left(-\frac{1}{4}\mathrm{Tr}\, e^{-t\Box_0(u)} + \frac{3}{4}\mathrm{Tr}\, e^{-t\Box_1(u)}\right) \\ &= \frac{1}{4} t\,\mathrm{Tr}\left(\frac{\mathrm{d}\Box_0}{\mathrm{d}u} e^{-t\Box_0}\right) - \frac{3}{4} t\,\mathrm{Tr}\left(\frac{\mathrm{d}\Box_1}{\mathrm{d}u} e^{-t\Box_1}\right) \\ &= \frac{1}{4} t\,\mathrm{Tr}\left((-B_0 S^2 - S B_0 S + T T^* B_0 - T B_1 T^*) e^{-t\Box_0}\right) - \frac{3}{4} t\,\mathrm{Tr}\left((T^* B_0 T - B_1 T^* T) e^{-t\Box_1}\right) \\ &= \frac{1}{4} t\,\mathrm{Tr}\left(-2 B_0 S^2 e^{-t\Box_0} + B_0 T T^* e^{-t\Box_0} - B_1 T^* T e^{-t\Box_1}\right) - \frac{3}{4} t\,\mathrm{Tr}\left(B_0 T T^* e^{-t\Box_0} - B_1 T^* T e^{-t\Box_1}\right) \\ &= -\frac{1}{2} t\,\mathrm{Tr}\left(B_0 \Box_0 e^{-t\Box_0}\right) + \frac{1}{2} t\,\mathrm{Tr}\left(B_1 \Box_1 e^{-t\Box_1}\right) \\ &= t\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\mathrm{Tr}\, B_0 e^{-t\Box_0} - \frac{1}{2}\mathrm{Tr}\, B_1 e^{-t\Box_1}\right) . \end{aligned} \tag{2.28}$$

In order to make use of lemma 2.3 we identify

$$\sigma(u) = \sum_{0\le q\le m} \lambda_q \log \det{}'(A_q(u)) = \log \mathcal{Z}(u) \tag{2.29}$$

and make the assumption that $\mathrm{Tr}\,\Pi_{\Box_i} = \dim\ker \Box_i(u)$ do not depend on u. We do not have an intuitive explanation for this hypothesis, however when applying our results we will always consider the case $\Pi_{\Box_i} = 0$ (see definition 2.11 below). In the notation of lemma 2.3 we have

$$\lambda_0 = -\frac{1}{4}, \quad \lambda_1 = \frac{3}{4}, \quad A_0 = \Box_0 = T_0, \quad A_1 = \Box_1 = T_1, \quad R_0 = \frac{1}{2} B_0, \quad R_1 = \frac{1}{2} B_1 \,.$$

Finally we obtain that the variation of $\log \mathcal{Z}$ under variation of the scalar products in Γ_0 and Γ_1 can be written as

$$\frac{\mathrm{d}\log \mathcal{Z}(u)}{\mathrm{d}u} = \frac{1}{2}\beta_0(B_0^u|\Box_0(u)) - \frac{1}{2}\beta_0(B_1^u|\Box_1(u)) \,. \tag{2.30}$$

Remark. The above analysis is not entirely rigorous, however our formal manipulations can be fully justified using the lemmas found in Section 1 of [29], which apply specifically to the case of differential operators acting on a compact manifold. We will indeed restrict ourselves to such a case below.

Following [22], we define now differential operators and elliptic operators, which will also play a role in Chapter 3 when discussing the index theorem.

Definition 2.7. Let E and F be complex vector bundles over a smooth manifold M. A *differential operator* A is a linear map

$$A : \Gamma(M, E) \to \Gamma(M, F)$$

where $\Gamma(M, E)$, $\Gamma(M, F)$ denote the space of sections of E and F respectively.
Take a chart $U \subset M$ over which E and F are trivial; denote the local coordinates of U as x^μ. We introduce the following notation,

$$D_\lambda = \frac{\partial^{\lambda_1+\cdots+\lambda_n}}{\partial(x^1)^{\lambda_1}\ldots\partial(x^n)^{\lambda_n}} \qquad \lambda_j \in \mathbb{Z}, \lambda_j \ge 0 \,.$$

If $\dim E = k$ and $\dim F = k'$, the most general form of A is

$$(As(x))^a = \sum_{\substack{\lambda_1+\cdots+\lambda_n \leq N \\ 1\leq b\leq k}} \left(A^{\lambda_1\ldots\lambda_n}\right)^a{}_b (x)\, D_\lambda s^b(x) \qquad 1 \leq a \leq k' \tag{2.31}$$

where $s(x)$ is a section of E. The integer N is called the *order* of A.

Definition 2.8. The *symbol* of the differential operator A is a $k' \times k$ matrix

$$\sigma(A,\xi) = \sum_{\lambda_1+\cdots+\lambda_n = N} \left(A^{\lambda_1\ldots\lambda_n}\right)^a{}_b (x)\, (\xi_1)_{\lambda_1}\ldots(\xi_n)_{\lambda_n} \tag{2.32}$$

where $\xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n$. In the case $k = k'$, if $\sigma(A,\xi)$ is invertible for each $x \in M$ and for each $\xi \in \mathbb{R}^n \setminus \{0\}$, the operator A is said to be *elliptic*.

Consider now two complex vector bundles $E_0 \xrightarrow{\pi_0} M_0$, $E_1 \xrightarrow{\pi_1} M_1$, where M_0 and M_1 are (smooth) compact manifolds, and equip them with hermitian metrics (i.e. there is a hermitian scalar product on each fibre). Let Γ_0 and Γ_1 be the spaces of sections of E_0 and E_1 respectively; then the scalar products in Γ_0 and Γ_1 are naturally defined.
Following our previous notation, let $\mathcal{S}$ be a quadratic functional on Γ_0 defined by the differential operator $S : \Gamma_0 \to \Gamma_0$:

$$\mathcal{S}(f) = \langle Sf, f\rangle = \langle f, Sf\rangle\,.$$

Let $T : \Gamma_1 \to \Gamma_0$ be a differential operator and use the scalar products in Γ_0, Γ_1 to define

$$\Box_0 = S^2 + TT^*$$
$$\Box_1 = T^*T\,,$$

so that we can write the partition function of $\mathcal{S}$ as

$$\mathcal{Z} = \det{}'(\Box_0)^{-1/4}\, \det{}'(\Box_1)^{3/4}\,.$$

The functional $\mathcal{S}$ will be called elliptic with respect to T if the operators $\Box_0$ and $\Box_1$ are elliptic.
Assume now that we have a smooth family of hermitian metrics in E_i, with $i = 0,1$, parametrized by $0 \leq u \leq 1$. The variation of the corresponding scalar products $\langle\cdot,\cdot\rangle_i^u$ in Γ_i can be described by means of the differential operators B_i^u, which are defined by

$$\frac{\mathrm{d}}{\mathrm{d}u}\langle f,g\rangle_i^u = \langle B_i^u f, g\rangle_i^u = \langle f, B_i^u g\rangle_i^u\,.$$

After imposing these requirements we can more precisely state the result that we formally obtained in (2.30).

Theorem 2.1. *The variation of the partition function $\mathcal{Z}$ of an elliptic functional $\mathcal{S}$ with respect to the variation of hermitian metrics in E_0, E_1 can be written as*

$$\frac{\mathrm{d}\log\mathcal{Z}(u)}{\mathrm{d}u} = \frac{1}{2}\beta_0(B_0^u|\Box_0(u)) - \frac{1}{2}\beta_0(B_1^u|\Box_1(u))\,. \tag{2.33}$$

This theorem has significant implications: it can be used in some particular cases to determine whether the partition function of a quadratic functional is dependent on the metric or not.
In order to elaborate on this aspect we first need to provide a result connected to the asymptotic coefficients $\beta_0(B_i^u|\Box_i)$ that appear in (2.33). This result is stated in Section 1 of [29], where it is claimed that it can be deduced from the content of [31] and [18].

Lemma 2.4. *Let M be a compact manifold of dimension n. Let A be a differential operator on M and let B be a self-adjoint non-negative elliptic differential operator on M of order m.*
Then there exist coefficients $\psi_k(A|B)$ such that

$$\mathrm{Tr}\, Ae^{-Bt} = \sum_k \psi_k(A|B)t^{-k} + O(t^\varepsilon) \quad \text{for} \quad t \to 0^+\,, \tag{2.34}$$

where $\varepsilon > 0$ and $k = (n-2l)/m$ with $l \in \mathbb{Z}$. In particular, if M is odd-dimensional then $\psi_0(A|B) = 0$.

Therefore for a regular pair (A,B) we have the following relations between the asymptotic coefficients $\beta_k(A|B)$ and $\psi_k(A|B)$:

$$\beta_0(A|B) = \psi_0(A|B) - \operatorname{Tr} A\Pi_B, \qquad \beta_k(A|B) = \psi_k(A|B) \quad \text{for} \quad k > 0\,. \tag{2.35}$$

Let us apply this result to the pairs of operators $(B_i^u, \Box_i(u))$ which appear in Theorem 2.1. If the following conditions are both satisfied:

1. the operators $\Box_0(u), \Box_1(u)$ have trivial kernel, i.e. $\Pi_{\Box_0} = \Pi_{\Box_1} = 0$;
2. the base manifolds M_0, M_1 have odd dimension,

then the partition function of the quadratic functional $\mathcal{S}$ is independent of the choice of hermitian metrics in E_0, E_1.

2.4 Resolvents

The goal of this Section is to generalize theorem 2.1 to the case where we have N linear operators $T_i : \Gamma_i \to \Gamma_{i-1}$ which satisfy the specific condition $T_i T_{i+1} = 0$. In order to do so we define the notion of resolvent and partition function with respect to a resolvent. The latter is again related to the interpretation of the partition function as a functional integral and to the Faddeev-Popov gauge fixing procedure for gauge theories, as discussed in [29].
After giving these definitions, we explicitly carry out the formal calculations leading to the desired result (similarly to theorem 2.1, it technically only applies to the case of elliptic resolvent).

Definition 2.9. Let $\mathcal{S}$ be a quadratic functional on the pre-Hilbert space Γ_0. A sequence of pre-Hilbert spaces Γ_i and operators $T_i : \Gamma_i \to \Gamma_{i-1}$, $i = 1, \ldots, N$, is called a *resolvent* of the functional $\mathcal{S}$ if the following conditions are satisfied:

$$\begin{aligned} &\mathcal{S}(f + T_1 h) = \mathcal{S}(f) \quad \forall h \in \Gamma_1\,, \\ &T_i T_{i+1} = 0\,. \end{aligned} \tag{2.36}$$

The resolvent will be denoted by $\{\Gamma_i, T_i\}$.

Observe that if $\mathcal{S} = 0$ the notion of resolvent coincides with the one of *complex.*

Now let S be an operator on Γ_0 satisfying $\mathcal{S}(f) = \langle Sf, f\rangle = \langle f, Sf\rangle$ and let T_i^* be the adjoint operator of T_i. We assume that the operators S^2 and $T_i^* T_i$ are regular. Then we have the following generalization of definition 2.6.

Definition 2.10. The partition function of the quadratic functional $\mathcal{S}$ with respect to the resolvent $\{\Gamma_i, T_i\}$ is defined as

$$\mathcal{Z} = \det{}'(S)^{-1/2} \prod_{1 \le i \le N} \det{}'(T_i)^{(-1)^{i-1}}\,. \tag{2.37}$$

Mimicking the last Section we define the regular operators

$$\begin{aligned} \Box_0 &= S^2 + T_1 T_1^* \\ \Box_i &= T_i^* T_i + T_{i+1} T_{i+1}^* \quad i = 1, \ldots, N\,, \end{aligned} \tag{2.38}$$

and observe that

$$\det{}'(\Box_0) = \det{}'(S^2)\det{}'(T_1 T_1^*) = \det{}'(S)^2 \det{}'(T_1)^2$$

$$\det{}'(\Box_i) = \det{}'(T_i)^2 \det{}'(T_{i+1})^2.$$

By direct computation one can check that the partition function can be represented in the form

$$\mathcal{Z} = \prod_{0\le i\le N} \det{}'(\Box_i)^{\nu_i} \tag{2.39}$$

where $\nu_i = (-1)^{i+1}(2i+1)/4$.

Consider now $N+1$ complex vector bundles $E_0 \xrightarrow{\pi_0} M_0$, $E_i \xrightarrow{\pi_i} M_i$, $i = 1, \dots, N$, equip them with hermitian metrics and denote the spaces of their sections by Γ_i. We say that $\{\Gamma_i, T_i\}$ is an *elliptic resolvent* of $\mathcal{S}$ if $S : \Gamma_0 \to \Gamma_0$ and $T_i : \Gamma_i \to \Gamma_{i-1}$ are differential operators of order m (independent of i) and one can choose hermitian metrics in E_i in such a way that the operators $\Box_i$ are elliptic. Observe that if $N = 1$ then $\mathcal{S}$ is simply an elliptic functional with respect to T_1.

Remark. We note that to define the operators S and $\Box_i$ we had to fix some hermitian metrics on our vector bundles. However, one can prove that the ellipticity of $\Box_i$ does not depend on the choice of said metrics. This assertion is well known in the case of an elliptic complex, i.e. for $\mathcal{S} = 0$; it can be extended to the case $\mathcal{S} \neq 0$ as done in [29]. Clearly, this also applies to the operators $\Box_0, \Box_1$ defined in the previous Section.

Assume now that we have a smooth family of hermitian metrics in E_i depending on the parameter $0 \le u \le 1$; the variation of the corresponding family of scalar products $\langle\cdot,\cdot\rangle_i^u$ in Γ_i can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}u}\langle f, g\rangle_i^u = \langle B_i^u f, g\rangle_i^u = \langle f, B_i^u g\rangle_i^u\,.$$

The partition function of the functional $\mathcal{S}$ with respect to the resolvent $\{\Gamma_i, T_i\}$ and the scalar products $\langle\cdot,\cdot\rangle_i^u$ will be denoted by $\mathcal{Z}(u)$.

Theorem 2.2.

$$\frac{\mathrm{d}\log\mathcal{Z}(u)}{\mathrm{d}u} = \frac{1}{2}\sum_{0\le i\le N}(-1)^i\beta_0(B_i^u|\Box_i(u))\,. \tag{2.40}$$

Proof. The proof is similar to the one of theorem 2.1. We explicitly carry out the formal calculations, which can be rigorously justified, as noted earlier.
We first compute

$$\begin{aligned}\frac{\mathrm{d}S(u)}{\mathrm{d}u} &= -B_0^u S(u)\\ \frac{\mathrm{d}T_i^*(u)}{\mathrm{d}u} &= T_i^*(u)B_{i-1}^u - B_i^u T_i^*(u)\,,\end{aligned} \tag{2.41}$$

which imply

$$\begin{aligned}\frac{\mathrm{d}\Box_0(u)}{\mathrm{d}u} &= -B_0S^2 - SB_0S + T_1T_1^*B_0 - T_1B_1T_1^*\\ \frac{\mathrm{d}\Box_i(u)}{\mathrm{d}u} &= T_i^*B_{i-1}T_i - B_iT_i^*T_i + T_{i+1}T_{i+1}^*B_i - T_{i+1}B_{i+1}T_{i+1}^*\,.\end{aligned} \tag{2.42}$$

One can verify the relations

$$\begin{aligned}e^{-t\,\Box_{i-1}}T_i &= T_ie^{-t\,\Box_i}\\ e^{-t\,\Box_i}T_i^* &= T_i^*e^{-t\,\Box_{i-1}}\\ e^{-t\,\Box_0}S &= Se^{-t\,\Box_0}\,,\end{aligned} \tag{2.43}$$

by using not only the property $ST_1 = 0$ but also $T_iT_{i+1} = 0$.

Now we need to compute the quantity

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}u}\left(\sum_{0\le i\le N}\nu_i \mathrm{Tr}\, e^{-t\,\Box_i(u)}\right)\\
&\quad= -t\,\nu_0 \mathrm{Tr}\left(\frac{\mathrm{d}\Box_0}{\mathrm{d}u}e^{-t\,\Box_0}\right) - t\sum_{1\le i\le N}\nu_i \mathrm{Tr}\left(\frac{\mathrm{d}\Box_i}{\mathrm{d}u}e^{-t\,\Box_i}\right)\\
&\quad= -t\,\nu_0 \mathrm{Tr}\left((-B_0S^2 - SB_0S + T_1T_1^*B_0 - T_1B_1T_1^*)e^{-t\,\Box_0}\right)\\
&\qquad - t\sum_i \nu_i \mathrm{Tr}\left((T_i^*B_{i-1}T_i - B_iT_i^*T_i + T_{i+1}T_{i+1}^*B_i - T_{i+1}B_{i+1}T_{i+1}^*)e^{-t\,\Box_i}\right)\\
&\quad= -t\,\nu_0 \mathrm{Tr}\left((-2B_0S^2 + B_0T_1T_1^*)e^{-t\,\Box_0} - B_1T_1^*T_1e^{-t\,\Box_1}\right)\\
&\qquad - t\sum_i \nu_i \mathrm{Tr}\left(B_{i-1}T_iT_i^*e^{-t\,\Box_{i-1}} + (-B_iT_i^*T_i + B_iT_{i+1}T_{i+1}^*)e^{-t\,\Box_i} - B_{i+1}T_{i+1}^*T_{i+1}e^{-t\,\Box_{i+1}}\right)\\
&\quad= -t\,\mathrm{Tr}\left(B_0(-2\nu_0S^2 + \nu_0T_1T_1^* + \nu_1T_1T_1^*)e^{-t\,\Box_0}\right)\\
&\qquad - t\sum_i \mathrm{Tr}\left(B_i(-\nu_{i-1}T_i^*T_i - \nu_iT_i^*T_i + \nu_iT_{i+1}T_{i+1}^* + \nu_{i+1}T_{i+1}T_{i+1}^*)e^{-t\,\Box_i}\right)
\end{aligned} \tag{2.44}$$

Recalling that $\nu_i = (-1)^{i+1}(2i+1)/4$, we find

$$\nu_0 = -\frac{1}{4}, \quad \nu_i + \nu_{i+1} = \frac{(-1)^i}{2}$$

and therefore

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}u}\left(\sum_{0\le i\le N}\nu_i \mathrm{Tr}\, e^{-t\,\Box_i(u)}\right)\\
&\quad= -\frac{1}{2}t\left(\mathrm{Tr}\left(B_0(S^2 + T_1T_1^*)e^{-t\,\Box_0}\right) + \sum_{1\le i\le N}(-1)^i \mathrm{Tr}\left(B_i(T_i^*T_i + T_{i+1}T_{i+1}^*)e^{-t\,\Box_i}\right)\right)\\
&\quad= -\frac{1}{2}t\sum_{0\le i\le N}(-1)^i \mathrm{Tr}\left(B_i\Box_ie^{-t\,\Box_i}\right)\\
&\quad= t\,\frac{\mathrm{d}}{\mathrm{d}t}\sum_{0\le i\le N}\frac{(-1)^i}{2}\mathrm{Tr}\left(B_ie^{-t\,\Box_i}\right)
\end{aligned} \tag{2.45}$$

Assuming that $\mathrm{Tr}\,\Pi_{\Box_i} = \dim\ker\Box_i(u)$ do not depend on u we apply once again lemma 2.3 and conclude the proof. □

As done with theorem 2.1, we can apply lemma 2.4 to rewrite (2.40) as

$$\frac{\mathrm{d}\log\mathcal{Z}(u)}{\mathrm{d}u} = \frac{1}{2}\sum_{0\le i\le N}(-1)^i\left(\psi_0(B_i^u|\Box_i(u)) - \mathrm{Tr}\,B_i^u\Pi_{\Box_i}\right) \tag{2.46}$$

To conclude this Section we consider a specific class of resolvents called acyclic (this notion coincides with the one of acyclic complex in the case $\mathcal{S} = 0$).

Definition 2.11. A resolvent $\{\Gamma_i, T_i\}$, $i = 1, \dots, N$, is called *acyclic* if

$$\ker S = \mathrm{Im}T_1\,, \quad \ker T_i = \mathrm{Im}T_{i+1}\,. \tag{2.47}$$

If the resolvent $\{\Gamma_i, T_i\}$ is acyclic then

$$\Pi_{\Box_i} = 0\,, \quad i = 0, \dots, N\,. \tag{2.48}$$

We have thus reached the main theoretical result of this Chapter: *if the base manifold M is odd-dimensional and the elliptic resolvent $\{\Gamma_i, T_i\}$ of $\mathcal{S}$ is acyclic, then the partition function of the quadratic functional $\mathcal{S}$ is independent of the choice of hermitian structures.*

2.5 The Ray-Singer torsion as a partition function

In this Section the torsion of an elliptic complex and the Ray-Singer torsion are defined. The Ray-Singer torsion (or analytic torsion) is a topological invariant of Riemannian manifolds, whose properties were described by Daniel Ray and Isadore Singer in 1971 and 1973 [26], [25]. The Ray-Singer torsion is the analytic analogue of the Reidemeister torsion (or R-torsion), another invariant which was introduced in the 1930s in order to classify a type of 3-manifolds called lens spaces. Ray and Singer showed that the R-torsion and the analytic torsion share some properties, and their equivalence was proven independently by Jeff Cheeger and Werner Müller in 1978. For a detailed account of the historical background, the main Ray-Singer papers and some further developments see [20].
Afterwards we briefly discuss some steps which permit one to interpret the Ray-Singer torsion as partition function of a functional with respect to an appropriate elliptic resolvent. As observed below, the independence of the Ray-Singer torsion on the choice of metric can then be derived by directly applying the results obtained in the previous Section.
In the last part of this Section we switch back to the path integral formalism in order to explain the relation between the Ray-Singer torsion and the asymptotic expression for the Chern-Simons partition function (1.45).

We begin with the general definition of torsion of an elliptic complex (equivalently, an elliptic resolvent in the case $\mathcal{S} = 0$).

Definition 2.12. Let $\{\Gamma_i, T_i\}$, $i = 1, \dots, N,$ be an elliptic complex. Its *torsion* $\mathrm{Tor}(\Gamma_i, T_i)$ is defined by

$$\begin{aligned} \log \mathrm{Tor}(\Gamma_i, T_i) &= \frac{1}{4} \sum_{0 \le i \le N} (-1)^{i+1}(2i+1) \log \det{}'(\Box_i) \\ &= \frac{1}{2} \sum_{0 \le i \le N} (-1)^{i+1} i \log \det{}'(\Box_i) \end{aligned} \tag{2.49}$$

where $\Box_i = T_i^* T_i + T_{i+1} T_{i+1}^*$ (and we identify $\Box_0 = T_1 T_1^*$).

By comparing this definition with the general expression (2.39) we see that the partition function of the functional $\mathcal{S} = 0$ with respect to an elliptic complex coincides with the torsion of the same elliptic complex. Therefore one can study the dependence of $\mathrm{Tor}(\Gamma_i, T_i)$ on the metric by employing theorem 2.2.

With the above definition, one could simply state that the Ray-Singer torsion is the torsion of the de Rham complex. Given its importance in our discussion, however, we shall be more precise in formulating its definition.

Let M denote an oriented compact Riemannian manifold of dimension n. It is a well known fact (see for example [33]) that there is a one-to-one correspondence between flat principal G-bundles over M (up to isomorphisms) and representations of $\pi_1(M)$ into G (up to conjugacy). To every representation $\chi : \pi_1(M) \to O(m)$ we assign then a flat vector bundle ξ_χ (it is simply the vector bundle associated to the flat principal $O(m)$-bundle given by the above correspondence).
We denote the space of k-forms on M taking values in the fibres of ξ_χ by $\Lambda^k_\chi(M)$. The exterior derivative $d : \Lambda^k_\chi(M) \to \Lambda^{k+1}_\chi(M)$ has a formal adjoint $\delta : \Lambda^k_\chi(M) \to \Lambda^{k-1}_\chi(M)$. Given $\alpha_1 \in \Lambda^{k_1}_{\chi_1}(M)$ and $\alpha_2 \in \Lambda^{k_2}_{\chi_2}(M)$, the exterior product $\alpha_1 \wedge \alpha_2$ is considered as an element of $\Lambda^{k_1+k_1}_{\chi_1\otimes\chi_2}(M)$. The Riemannian metric on M, along with the standard inner product on $\mathbb{R}^m$, allows one to naturally define a map $\lambda : \Lambda^k_{\chi\otimes\chi} \to \Lambda^k(M)$, where $\Lambda^k(M)$ denotes the space of real valued k-forms on M.

Definition 2.13. The *Ray-Singer torsion* $T(M, \chi)$ is defined by

$$\log T(M, \chi) = \frac{1}{2} \sum_{0 \le i \le n} (-1)^{i+1} i \log \det{}'(\Delta_i^\chi)\,, \tag{2.50}$$

where $\Delta_k^\chi = \delta d + d\delta$ is the Laplacian on k-forms.

As shown by Ray and Singer, if the cohomology groups $H^k(M, \xi_\chi)$ are trivial, then $T(M, \chi)$ does not depend on the choice of Riemannian metric on M. It is possible to derive the same conclusion using the results presented earlier in this Chapter.

Let us first discuss how the Ray-Singer torsion can be interpreted as a partition function. Consider the representations $\chi_i : \pi_1(M) \to O(m_i)$, $i = 0, \ldots, n$, where $O(m_i)$ are the groups of orthogonal transformations of the m_i-dimensional Euclidean space $\mathbb{E}_i$; the corresponding flat vector bundles will be denoted by ξ_i. Fix invertible linear operators $A_i : \mathbb{E}_i \to \mathbb{E}_{n-i-1}$ satisfying $A_i \chi_i = \chi_{n-i-1}$; these operators induce linear maps $\hat{A}_i : \Lambda^k_{\chi_i}(M) \to \Lambda^k_{\chi_{n-i-1}}(M)$.
Our candidate functional is

$$\mathcal{S} = \sum_{0 \le r \le n} \int_M \lambda\left(\hat{A}_r \alpha^r \wedge d\beta^{n-r-1}\right) , \tag{2.51}$$

where $\alpha^r \in \Lambda^r_{\chi_r}(M)$.
Without loss of generality one can assume that $A_r^* = (-1)^{n(r+1)} A_{n-r-1}$. Let us construct an elliptic resolvent $\{\Gamma_i, T_i\}$ of the functional (2.51). We take

$$\Gamma_i = \bigoplus_{0 \le r \le n} \Lambda^{r-i}_{\chi_r}(M) \tag{2.52}$$

and the operators T_i can be defined as appropriate exterior derivatives. Now one can check that the partition function with respect to the elliptic resolvent $\{\Gamma_i, T_i\}$ can be expressed as $T(M, \chi_i)$.
Observe that for even-dimensional manifolds Hodge duality implies $T(M, \chi) = 1$. For odd-dimensional manifolds, if the cohomology groups of M with coefficients in ξ_χ are trivial, then theorem 2.2 and lemma 2.4 can be used to conclude that $T(M, \chi)$ does not depend on the choice of Riemannian metric on M.

We will consider a specific case of (2.51), as it is directly related to the argument proposed by Witten which we introduced in the previous Chapter. Let $n = 3$ and

$$\mathcal{S} = \int_M \lambda(\alpha \wedge \mathrm{d}\beta) , \tag{2.53}$$

where $\alpha, \beta \in \Lambda^1_{\chi_1}(M)$.
Recall that in the weak coupling limit (i.e. for large k) the leading contribution in the Chern-Simons action which depends on the quantum fluctuation B takes the form

$$\mathcal{S}_{(1)} = \frac{k}{4\pi} \int_M \mathrm{tr}(B \wedge DB) , \tag{2.54}$$

where D is the covariant exterior derivative with respect to the flat "background" connection $A^{(\alpha)}$ and tr() plays the same role as λ.
To relate these expressions it is convenient to work in the path integral formalism. We schematically set

$$\gamma_\pm = \frac{1}{2}(\alpha \pm \beta) , \tag{2.55}$$

so that $\gamma_+^2 - \gamma_-^2 = \alpha\beta$. The path integral corresponding to (2.53) can be written as

$$\begin{aligned} &\int \mathscr{D}\alpha \mathscr{D}\beta \, \exp\left(i \int \lambda(\alpha \wedge d\beta)\right) \\ &\quad = \int \mathscr{D}\gamma_+ \, \exp\left(i \int \lambda(\gamma_+ \wedge d\gamma_+)\right) \int \mathscr{D}\gamma_- \, \exp\left(-i \int \lambda(\gamma_- \wedge d\gamma_-)\right) \\ &\quad = \left| \int \mathscr{D}\gamma_+ \, \exp\left(i \int \lambda(\gamma_+ \wedge d\gamma_+)\right) \right|^2 \end{aligned} \tag{2.56}$$

In other words, the path integral (formally equivalent to the partition function) of (2.53) is the absolute value squared of the path integral of the 1-loop contribution to the Chern-Simons theory in the weak coupling limit. This leads to a factor

$$e^{i\theta(A^{(\alpha)})}\sqrt{T(M,A^{(\alpha)})} \tag{2.57}$$

in the Chern-Simons partition function, where $T(M,A^{(\alpha)})$ is the Ray-Singer torsion of the flat connection $A^{(\alpha)}$ and $e^{i\theta(A^{(\alpha)})}$ is a phase factor.
By comparing this result with expression (1.45) we can write

$$\mathcal{Z}(M) \approx \frac{1}{\mathrm{Vol}(Z_G)} \sum_{\alpha} e^{ikI_{CS}[A^{(\alpha)}]} e^{-i\pi\eta(L_-)/4} \sqrt{T(M,A^{(\alpha)})}\,. \tag{2.58}$$

As we discussed earlier, $T(M,A^{(\alpha)})$ is a topological invariant of the 3-manifold M (under some assumptions); the phase factor, however, requires further study. We will describe its meaning and properties in the next Chapter.

Remark. In this final discussion we have implicitly made use of a crucial fact: since the connection is flat, the covariant derivative satisfies $D^2 = 0$. This implies that D can be treated as a differential and the above construction is still valid.

3. The Atiyah-Patodi-Singer index theorem

In this Chapter we conclude the evaluation of the 1-loop approximation of the Chern-Simons partition function. In particular, we express the phase $\exp(-i\pi\eta(L_-)/4)$ appearing in (2.58) in terms of the classical Chern-Simons invariant and another quantity that does not depend on the flat connection $A^{(\alpha)}$. In order to do so, we first need to describe a result known as Atiyah-Patodi-Singer (APS) index theorem [3, 4, 5] and the relevant mathematical background. The APS theorem is the generalization to manifolds with boundary of the Atiyah-Singer (AS) index theorem, which establishes a deep connection between topology, geometry and analysis.
We will mostly follow the approach and notation of [22] and [32].

3.1 Characteristic classes

The goal of this Section is to introduce characteristic classes, which are a fundamental tool used in the classification of fibre bundles. We follow the Chern-Weil approach, which establishes a connection between invariant polynomials of the curvature form and de Rham cohomology classes. We begin by defining Weil polynomials and stating the Chern-Weil theorem. Then we define some common characteristic classes for complex vector bundles (Chern classes, Chern characters, Todd classes) and real vector bundles (Pontryagin classes, Euler classes, $\hat{A}$ genus). We briefly discuss how to express them in terms of the eigenvalues of the curvature, interpreted as a matrix of 2-forms.
The main references for this Section are [22] and [9].

Throughout this Chapter G will denote a matrix Lie group, and its Lie algebra will be denoted by $\mathfrak{g}$.

Definition 3.1. A *Weil polynomial* of degree m is a symmetric multilinear map $\tilde{P} : \mathfrak{g} \times \overset{m}{\cdots} \times \mathfrak{g} \to \mathbb{C}$ such that

$$\tilde{P}(\mathrm{Ad}_g A_1, \dots, \mathrm{Ad}_g A_m) = \tilde{P}(A_1, \dots, A_m) \tag{3.1}$$

for all $g \in G$ and $A_i \in \mathfrak{g}$, where the adjoint map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\mathrm{Ad}_g A = g^{-1} A g \,. \tag{3.2}$$

The space of Weil polynomials of degree m is denoted by $I^m(G)$.

Definition 3.2. Let $\tilde{P} \in I^i(G)$ and $\tilde{Q} \in I^j(G)$, then $\tilde{P}\tilde{Q} \in I^{i+j}(G)$ is defined by

$$(\tilde{P}\tilde{Q})(A_1, \dots, A_{i+j}) = \frac{1}{(i+j)!} \sum_{\sigma} \tilde{P}(A_{\sigma_1}, \dots, A_{\sigma_i})\, \tilde{Q}(A_{\sigma_{i+1}}, \dots, A_{\sigma_{i+j}}) \tag{3.3}$$

where σ ranges over all permutations of $\{1, \dots, i+j\}$.
With this product the sum $I^*(G) \equiv \bigoplus_m I^m(G)$ is an algebra, called the *Weil algebra.*

Let $\tilde{P} \in I^m(G)$. We use the following shorthand notation for the diagonal combination

$$P(A) \equiv \tilde{P}(A, \overset{m}{\dots}, A) \,. \tag{3.4}$$

We extend the domain of Weil polynomial from $\mathfrak{g}$ to $\mathfrak{g}$-valued differential forms on M by defining

$$\tilde{P}(A_1\eta_1, \dots, A_m\eta_m) \equiv \eta_1 \wedge \cdots \wedge \eta_m \tilde{P}(A_1, \dots, A_m) \,, \tag{3.5}$$

where $A_i \in \mathfrak{g}$, $\eta_i \in \Lambda^{k_i}(M)$. Observe that $\tilde{P}(A_1\eta_1, \dots, A_m\eta_m)$ is an element of Λ^k, where $k = \sum_i k_i$.
The diagonal combination is

$$P(A\eta) \equiv \eta \wedge \overset{m}{\cdots} \wedge \eta P(A) \,. \tag{3.6}$$

Consider now a principal G-bundle $P \xrightarrow{\pi} M$; equip it with a local connection A (in general, a set of local connections $\{A_i\}$). We will be interested in Weil polynomials of the form $P(F)$, where F denotes the field strength corresponding to the local connection A. The reason lies in the following theorem, also known as *Chern-Weil theorem*.

Theorem 3.1. *Let P be a Weil polynomial. Then $P(F)$ satisfies*

1. $\mathrm{d}P(F) = 0$.
2. *Let F and F' be local curvature 2-forms corresponding to different local connections A and A'. Then the difference $P(F) - P(F')$ is exact.*

The Chern-Weil theorem implies that any Weil polynomial P defines a cohomology class of M. Moreover, this class is independent of the connection chosen, by the second part of the theorem. The cohomology class such defined is called the *characteristic class*. The characteristic class defined by a Weil polynomial P is denoted by $\chi_E(P)$, where E is a fibre bundle on which connections and curvatures are defined.

Remark. Since a principal bundle and its associated bundles share the same gauge potentials and field strengths, the Chern-Weil theorem applies equally to both bundles. Accordingly, E can be either a principal bundle or a vector bundle.

It is simple to check that the map

$$\chi_E : I^*(G) \to H^*(M), \quad P \mapsto \chi_E(P) \tag{3.7}$$

is a homomorphism (*Chern-Weil homomorphism*). In other words, one can write

$$(PQ)(F) = P(F) \wedge Q(F)\,. \tag{3.8}$$

Let $F : N \to M$ be a smooth map. For the pullback bundle f^*E of E, we have the so-called *naturality*

$$\chi_{f^*E} = f^*\chi_E\,. \tag{3.9}$$

Characteristic classes can be used to verify whether two fibre bundles are different or not. In order to make this claim precise we give another definition and state a theorem whose proof can be found in [9].

Definition 3.3. Let $P \xrightarrow{\pi} M$ and $P' \xrightarrow{\pi'} M$ be two principal G-bundles. We say that they are *equivalent* if there exists a map $\lambda : P \to P'$ such that

$$\lambda(pg) = \lambda(p)g \quad \text{and} \quad \pi'(\lambda(p)) = \pi(p) \tag{3.10}$$

for all $p \in P$ and $g \in G$.

Theorem 3.2. *The Chern-Weil homomorphisms $I^*(G) \to H^*(M)$ for equivalent bundles are the same.*
Consequently, the characteristic classes for a trivial principal fibre bundle are all trivial.

Starting from a principal fibre bundle endowed with a connection, one can build many different characteristic class by choosing appropriate Weil polynomials. In the following we will define the most commonly used characteristic classes, some of which will play a role in the rest of this Chapter, in particular when describing the content of the index theorem.

Chern classes, Chern characters and Todd classes

We first focus on a *complex* vector bundle $E \xrightarrow{\pi} M$ with fibre $\mathbb{C}^k$; we denote $m = \dim M$. Here the structure group G is a subgroup of $\mathrm{GL}(k, \mathbb{C})$, and the gauge field A and field strength F are $\mathfrak{g}$-valued differential forms on M.

Definition 3.4. The *total Chern class* is defined by

$$c(F) \equiv \det\left(\mathbb{I} + \frac{i}{2\pi}F\right). \tag{3.11}$$

Observe that $c(F)$ can be expanded as

$$c(F) = 1 + c_1(F) + c_2(F) + \cdots + c_k(F) \tag{3.12}$$

where $c_j(F) \in \Lambda^{2j}(M)$ is called the j-th Chern class and $c_k(F) = \det(iF/2\pi)$. In particular, $c_j(F) = 0$ if $2j > m$ or $j > k$.
It is usually inconvenient to compute the Chern classes by expanding the determinant; we can instead diagonalize the curvature form F, interpreted as a matrix of 2-forms. Let $g \in \mathrm{GL}(k, \mathbb{C})$ such that

$$g^{-1}\left(\frac{iF}{2\pi}\right)g = \mathrm{diag}(x_1, \ldots, x_k) \equiv D\,, \tag{3.13}$$

where x_j is a 2-form. The total Chern class can be written as

$$\begin{aligned}
\det(\mathbb{I} + D) &= \det\left(\mathrm{diag}(1 + x_1, \ldots, 1 + x_k)\right) \\
&= \prod_{j=1}^{k}(1 + x_j) \\
&= 1 + (x_1 + \cdots + x_k) + (x_1x_2 + \cdots + x_{k-1}x_k) + \cdots + (x_1x_2\cdots x_k) \\
&= 1 + \mathrm{tr}D + \frac{1}{2}\left((\mathrm{tr}D)^2 - \mathrm{tr}D^2\right) + \cdots + \det D\,.
\end{aligned} \tag{3.14}$$

One can then read the expression for each Chern class:

$$\begin{aligned}
c_0(F) &= 1 \\
c_1(F) &= \mathrm{tr}D = \frac{i}{2\pi}\mathrm{tr}F \\
c_2(F) &= \frac{1}{2}\left((\mathrm{tr}D)^2 - \mathrm{tr}D^2\right) = \frac{1}{2}\left(\frac{i}{2\pi}\right)^2\left(\mathrm{tr}F \wedge \mathrm{tr}F - \mathrm{tr}(F \wedge F)\right) \\
&\;\;\vdots \\
c_k(F) &= \det D = \left(\frac{i}{2\pi}\right)^k \det F\,.
\end{aligned}$$

The next characteristic class we introduce is related to the Chern class and it explicitly appears in the statement of the APS theorem.

Definition 3.5. The *total Chern character* is defined by

$$\mathrm{ch}(F) \equiv \mathrm{tr}\exp\left(\frac{iF}{2\pi}\right) \tag{3.15}$$

By definition, $\mathrm{ch}(F)$ can be expanded as

$$\mathrm{ch}(F) = \sum_j \frac{1}{j!}\mathrm{tr}\left(\frac{iF}{2\pi}\right)^j \equiv \sum_j \mathrm{ch}_j(F)\,, \tag{3.16}$$

where $\mathrm{ch}_j(F) \in \Lambda^{2j}(M)$ is the j-th Chern character. Clearly, $\mathrm{ch}_j(F) = 0$ if $2j > m$.
Using the diagonalization procedure outlined above, one can also write

$$\mathrm{ch}(F) = \mathrm{tr}\exp(D) = \sum_{j=1}^{k}\exp(x_j) = \sum_{j=1}^{k}\left(1 + x_j + \frac{1}{2!}x_j^2 + \frac{1}{3!}x_j^3 + \cdots\right) \tag{3.17}$$

Each Chern character can therefore be expressed in terms of the Chern classes. The first few expressions are

$$\begin{aligned}\mathrm{ch}_0(F) &= k\\ \mathrm{ch}_1(F) &= c_1(F)\\ \mathrm{ch}_2(F) &= \frac{1}{2}\left(c_1(F)^2 - 2c_2(F)\right) .\end{aligned}$$

To conclude we briefly define another characteristic class associated to a complex vector bundle, which will make an appearance in the next Section. The *total Todd class* is formally defined by

$$\mathrm{Td}(F) = \prod_j \frac{x_j}{1 - e^{-x_j}} \tag{3.18}$$

(we note that this definition is not universal as different conventions may be used in the literature). Similarly to the Chern character, the Todd class can be expanded in terms of the Chern classes:

$$\mathrm{Td}(F) = 1 + \frac{1}{2}c_1(F) + \frac{1}{12}\left(c_1(F)^2 + c_2(F)\right) + \cdots \tag{3.19}$$

Pontryagin classes, Euler classes and $\hat{A}$ genus

Here we consider a *real* vector bundle $E \xrightarrow{\pi} M$ with fibre dimension k and $\dim M = m$. The structure group $\mathrm{GL}(k, \mathbb{R})$ can always be reduced to $G = \mathrm{O}(k)$. Since the generators of the Lie algebra $\mathfrak{g} = \mathfrak{o}(k)$ are skew-symmetric, the field strength F of E is also skew-symmetric.
It is known that a skew-symmetric matrix can be reduced to block diagonal form by an element of $\mathrm{GL}(k, \mathbb{R})$:

$$\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & \lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

It can instead be diagonalized by an element of $\mathrm{GL}(k, \mathbb{C})$; its eigenvalues are purely imaginary and they come in pairs:

$$\begin{pmatrix} i\lambda_1 & & & & \\ & -i\lambda_1 & & & \\ & & i\lambda_2 & & \\ & & & -i\lambda_2 & \\ & & & & \ddots \end{pmatrix}$$

(if k is odd then the last eigenvalue is set to zero).

Definition 3.6. The *total Pontryagin class* is defined by

$$p(F) \equiv \det\left(\mathbb{I} + \frac{F}{2\pi}\right) . \tag{3.20}$$

Observe that

$$\det\left(\mathbb{I} + \frac{F}{2\pi}\right) = \det\left(\mathbb{I} - \frac{F}{2\pi}\right) \tag{3.21}$$

and therefore $p(F)$ is an even function of F. The total Pontryagin class can be expanded as

$$p(F) = 1 + p_1(F) + p_2(F) + \cdots , \tag{3.22}$$

where the j-th Pontryagin class $p_j(F)$ is a polynomial of order $2j$ and therefore is an element of $\Lambda^{4j}(M)$. We have $p_j(F) = 0$ if $2j > k$ or $4j > m$.

As recalled above, $F/2\pi$ can be diagonalized by an element of $\mathrm{GL}(k,\mathbb{C})$ to $D = \mathrm{diag}(-ix_1, ix_1, \dots)$, where $x_i = \lambda_i/2\pi$, λ_i being the eigenvalues of F. The total Pontryagin class can be written as

$$p(F) = \det(\mathbb{I} + D) = \prod_{i=1}^{k/2}(1 + x_i^2)\,, \tag{3.23}$$

which implies

$$p_j(F) = \sum_{i_1<i_2<\cdots<i_j}^{k/2} x_{i_1}^2 x_{i_2}^2 \cdots x_{i_j}^2 \tag{3.24}$$

(here and in the following it is understood that $k/2$ must be replaced by $(k-1)/2$ if k is odd).
By observing that

$$\mathrm{tr}\left(\frac{F}{2\pi}\right)^{2j} = \mathrm{tr}D^{2j} = 2(-1)^j \sum_{i=1}^{k/2} x_i^{2j} \tag{3.25}$$

one can write each $p_j(F)$ in terms of the curvature.
It is also possible to express the Pontryagin classes in terms of Chern classes (in particular, in terms of even Chern classes, since $c_j(F)$ is a polynomial of order j in F). In order to do so, it is first necessary to complexify the fibre of the real vector bundle E; the resulting complex vector bundle will be denoted by $E^{\mathbb{C}}$. We have

$$\begin{aligned}\det(\mathbb{I} + iD) &= \det\begin{pmatrix} 1+x_1 & & & & \\ & 1-x_1 & & & \\ & & 1+x_2 & & \\ & & & 1-x_2 & \\ & & & & \ddots \end{pmatrix} \\ &= \prod_{i=1}^{k/2}(1 - x_i^2) = 1 - p_1(D) + p_2(D) - \cdots \end{aligned}$$

and therefore

$$p_j(E) = (-1)^j c_{2j}(E^{\mathbb{C}})\,. \tag{3.26}$$

We remark that although $p_j(F)$ vanishes as a differential form for $4j > m$, one can still formally compute $p_j(A)$ for a matrix A. Indeed this is the starting point for the definition of the next characteristic class.

In the following we require the manifold M to be even dimensional, i.e. $m = 2l$, and the real vector bundle we consider is the tangent bundle TM. It is always possible to reduce the structure group of TM down to $\mathrm{SO}(2l)$ by employing an orthonormal frame.
The *Euler class* e of M is formally defined as the square root of the l-th Pontryagin class,

$$e(M)e(M) = p_l(F)\,. \tag{3.27}$$

Here $p_l(F)$ vanishes as a differential form since it is a $4l$-form on a $2l$-manifold, however it can be formally computed for a $2l \times 2l$ matrix to yield the $2l$-form $e(M)$. If M is odd-dimensional we set $e(M) = 0$.
An equivalent definition is

Definition 3.7.

$$e(M) = \mathrm{Pf}\left(\frac{F}{2\pi}\right)\,, \tag{3.28}$$

where $\mathrm{Pf}(A)$ is the *Pfaffian* of a $2l \times 2l$ skew-symmetric matrix A, a polynomial defined by the relation

$$\mathrm{Pf}(A)^2 = \det A\,. \tag{3.29}$$

The Pfaffian can be written as

$$\mathrm{Pf}(A) = \frac{(-1)^l}{2^l l!} \sum_\sigma \mathrm{sign}(\sigma) A_{\sigma(1)\sigma(2)} \cdots A_{\sigma(2l-1)\sigma(2l)} \, . \tag{3.30}$$

By putting F in block diagonal form as above and taking $x_i = -\lambda_i / 2\pi$ one finds

$$e(A) = \prod_{i=1}^{l} x_i \, . \tag{3.31}$$

We conclude this section by defining another quantity which will be relevant for the APS theorem: the $\hat{A}$ *genus* (or Dirac genus). It is a formal power series

$$\hat{A}(F) = \prod_{j=1}^{k} \frac{x_j/2}{\sinh(x_j/2)} \, . \tag{3.32}$$

This is an even function of x_j and it can be written in terms of the Pontryagin classes

$$\hat{A}(F) = 1 - \frac{1}{24} p_1 + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots \, . \tag{3.33}$$

3.2 Analytical and topological index

In this Section we introduce Fredholm operators, a special class of elliptic differential operators. We define the analytical index of a Fredholm operator and more in general of an elliptic complex. We state the Atiyah-Singer index theorem, which provides a connection between analysis and topology for manifolds without boundary.
For an accessible discussion on the index theorem we refer to [32]; for an overview of the motivations, proof techniques and variations of the theorem see [15].

Let $D : \Gamma(M,E) \to \Gamma(M,F)$ be an elliptic operator, as defined in the previous Chapter. We assume that the vector bundles E and F are endowed with hermitian metrics, so that we can define the adjoint operator $D^* : \Gamma(M,F) \to \Gamma(M,E)$.
We recall that the *cokernel* of an operator D is the set

$$\mathrm{coker} D \equiv \Gamma(M,F) / \, \mathrm{Im}\, D \, . \tag{3.34}$$

Definition 3.8. An elliptic operator D is said to be a *Fredholm operator* if $\ker D$ and $\mathrm{coker} D$ are both finite-dimensional.
The *analytical index* of a Fredholm operator is the integer defined by

$$\mathrm{ind} D = \dim \ker D - \dim \mathrm{coker} D \, . \tag{3.35}$$

Remark. It is known (see for example [10]) that elliptic operators on a compact manifold are Fredholm operators; indeed, recall that the base space of the Chern-Simons theory under examination is assumed to be a compact 3-manifold.

One can show that $\mathrm{coker} D \cong \ker D^*$ for a Fredholm operator D, therefore the analytical index of D can also be expressed as

$$\mathrm{ind} D = \dim \ker D - \dim \ker D^* \, . \tag{3.36}$$

The definition can be generalized to the case of an elliptic complex, that is a sequence (E_i, D_i) of vector bundles E_i (over a compact manifold M) and Fredholm operators $D_i : \Gamma(M, E_i) \to \Gamma(M, E_{i+1})$ such that $D_i D_{i-1} = 0$ for any i.
Analogously to the de Rham cohomology groups, we define

$$H^i(E,D) \equiv \ker D_i / \, \mathrm{Im}\, D_{i-1} \, . \tag{3.37}$$

Definition 3.9. The analytical index of the elliptic complex (E_i, D_i) is defined by

$$\mathrm{ind}(E, D) = \sum_{i=0}^{m} (-1)^i \dim H^i(E, D) \,. \tag{3.38}$$

Given an elliptic complex (E_i, D_i), we can define the Laplacian operators $\Delta_i : \Gamma(M, E_i) \to \Gamma(M, E_i)$ as

$$\Delta_i \equiv D_{i-1} D_{i-1}^* + D_i^* D_i \,. \tag{3.39}$$

It can be shown that $H^i(E, D) \cong \ker \Delta_i$, and therefore

$$\mathrm{ind}(E, D) = \sum_{i=0}^{m} (-1)^i \dim \ker \Delta_i \,. \tag{3.40}$$

In order to understand how the two definitions of analytical index are related it is sufficient to consider the Fredholm operator $D : \Gamma(M, E) \to \Gamma(M, F)$ and formally add zero to both sides; we obtain the elliptic complex

$$0 \xrightarrow{i} \Gamma(M, E) \xrightarrow{D} \Gamma(M, F) \xrightarrow{\phi} 0 \,, \tag{3.41}$$

where i is the inclusion and ϕ is the zero map.
Then we identify $\Gamma(M, E) \equiv \Gamma(M, E_0)$, $\Gamma(M, F) \equiv \Gamma(M, E_1)$ and obtain

$$\begin{aligned} \mathrm{ind}(E, D) &= \dim H^0 - \dim H^1 \\ &= (\dim \ker D - \dim \mathrm{Im}\, i) - (\dim \ker \phi - \dim \mathrm{Im}\, D) \\ &= \dim \ker D - (\dim \Gamma(M, F) - \dim \mathrm{Im}\, D) \\ &= \dim \ker D - \dim \mathrm{coker} D = \mathrm{ind} D \,. \end{aligned} \tag{3.42}$$

The importance of the index defined above lies in the following fact: *the analytical index of a Fredholm operator is invariant under arbitrary small (in the operator norm sense) perturbations.* Intuitively, while the dimensions of $\ker D$ and $\ker D^*$ are not invariant under perturbations, the two jump by the same amount, so that their difference (the index) remains constant. For a precise formulation of these statements and their proof we refer to [10].

This observation implies that the analytical index has properties analogous to homotopy invariants in algebraic topology, such as the Euler characteristic $\chi(M)$ of a compact manifold M. Indeed, it is possible to show that the Euler characteristic $\chi(M)$ coincides with the index of the de Rham complex over a compact manifold M without boundary. In particular, by defining the Laplacian on k-forms

$$\Delta_k \equiv d_{k-1} \delta_{k-1} + \delta_k d_k \tag{3.43}$$

(where $d_k : \Lambda^k(M) \to \Lambda^{k+1}(M)$ is the differential and $\delta_k : \Lambda^{k+1}(M) \to \Lambda^k(M)$ is its adjoint) one can write

$$\chi(M) = \sum_{k=1}^{m} (-1)^k \dim \ker \Delta_k \,. \tag{3.44}$$

This expression signals a deep connection between different fields of mathematics: the left-hand side is a purely *topological* quantity (which can be computed, for example, by triangulating M), while the right-hand side is expressed in terms of the space of solutions of the *analytic* equation $\Delta_k \omega = 0$.
Moreover, a theorem known as the *Chern-Gauss-Bonnet theorem* asserts that the Euler characteristic of a compact orientable manifold M without boundary can be computed in terms of its Euler class:

$$\int_M e(M) = \chi(M) \,. \tag{3.45}$$

By combining the last two expressions we obtain a relation between the index of the de Rham complex and the Euler class of M

$$\mathrm{ind}_{\mathrm{dR}}(M) = \sum_k (-1)^k \dim \ker \Delta_k = \int_M e(M) \,, \tag{3.46}$$

which is a typical form of an index theorem.
A similar result holds for a holomorphic vector bundle V over a complex manifold X. The theorem, known as Hirzebruch-Riemann-Roch theorem, relates the index of an elliptic complex, the (twisted) Dolbeault complex, to the Todd class $\mathrm{Td}(TM)$ and the Chern character $\mathrm{ch}(X)$ (see [32] for the details). It turns out that these two results are special cases of the celebrated *Atiyah-Singer index theorem*, which we state below.

Theorem 3.3. *Let (E, D) be an elliptic complex over an m-dimensional compact manifold M without a boundary. The index of this complex is given by*

$$\mathrm{ind}(E, D) = (-1)^{m(m+1)/2} \int_M \mathrm{ch}\left(\bigoplus_r (-1)^r E_r\right) \left.\frac{\mathrm{Td}(TM^{\mathbb{C}})}{e(TM)}\right|_{\mathrm{vol}} \tag{3.47}$$

Here $TM^{\mathbb{C}}$ denotes the complexified tangent bundle of M and the subscript $|_{\mathrm{vol}}$ indicates that in the integrand only m-forms are picked up.

Remark. This formulation of the theorem is valid as long as the Euler class $e(TM)$ satisfies certain technical conditions. This is not always the case: for example, if M admits a non-vanishing vector field then $e(TM)$ vanishes. For more information on said conditions and a more general version of the theorem we refer to [32]; the original statement can be found in [6].

As hinted above, this theorem expresses the index of an elliptic complex, which is an analytical quantity, as the integral of certain characteristic classes, which are instead topological invariants. The term on the right-hand side is sometimes referred to as *topological index*; with this definition, the Atiyah-Singer index theorem asserts that analytical and topological index coincide. Moreover, while the left-hand side of the formula is an integer by definition, it is by no means evident a priori that the right-hand side should be an integer too.
As a final remark, we note that in general it is easier to compute the index via the integral of appropriate characteristic classes, as the kernel of a general elliptic operator and its adjoint can be notoriously difficult to study. Still, the AS index theorem can give some information about these spaces: for example, if the right-hand side of the formula is positive and the complex consists of a single operator D, then we can conclude that the partial differential equation $Du = 0$ has at least one solution.

3.3 The Atiyah-Patodi-Singer theorem

In the previous Section we presented an important result regarding the index of an elliptic operator acting on sections of a vector bundle over a manifold without boundary. The next logical question is the following: does the Atiyah-Singer index theorem hold for manifolds *with a boundary*? In general, the answer is negative. Nevertheless, in a series of three papers by M. F. Atiyah, V. K. Patodi and I. M. Singer [3, 4, 5] a generalization of the original AS index theorem to manifolds with a boundary was discussed and proved, along with several applications.
In this Section we begin with a simple motivating example related to the classical Gauss-Bonnet theorem for 2-manifolds. We recall the notion of Dirac operator and define its associated eta function. Then we state a simpler version of the Atiyah-Patodi-Singer (APS) index theorem. Finally we return to our original goal of evaluating the non-abelian Chern-Simons partition function, and discuss whether the phase appearing in (2.58) can be expressed in terms of topological invariants or not.

Let us recall the classical Gauss-Bonnet theorem for Riemannian surfaces. Let X be a compact Riemannian 2-manifold and $K : X \to \mathbb{R}$ its Gaussian curvature, then the Euler characteristic $\chi(X)$ is given by

$$2\pi\chi(X) = \int_X K \,\mathrm{d}\mu \tag{3.48}$$

where $\mathrm{d}\mu$ is the Riemannian measure.
If X has a boundary there is an additional contribution due to the geodesics curvature $\kappa : \partial X \to \mathbb{R}$,

$$2\pi\chi(X) = \int_X K \,\mathrm{d}\mu + \int_{\partial X} \kappa \,\mathrm{d}\tilde{\mu} \,, \tag{3.49}$$

where $\mathrm{d}\tilde{\mu}$ is the induced measure on the boundary. This new boundary-dependent term is similar in spirit to the boundary terms that appears in the APS index theorem. We also observe that if a neighbourhood of ∂X in X is isometric to the cylinder $[0, \varepsilon) \times \partial X$ with its product metric for some $\varepsilon > 0$, then the boundary term vanishes [15].

We now recall the definition of Dirac operator, which can be thought of as a first-order differential operator $\mathcal{D}$ on a vector bundle E such that $\mathcal{D}^2 = \Delta$, where Δ is the Laplacian operator on E.
Let $S(N)$ be a spinor bundle over an n-dimensional orientable manifold N. The sections of this bundle are denoted by $\Delta(N) \equiv \Gamma(N, S(N))$. We assume that $n = 2l$, i.e. N is even-dimensional. The spin group $\mathrm{Spin}(n)$ is generated by n Dirac matrices γ^α, which satisfy

$$(\gamma^\alpha)^* = \gamma^\alpha, \quad \{\gamma^\alpha, \gamma^\beta\} = 2\delta^{\alpha\beta} \,. \tag{3.50}$$

The *Dirac operator* on $\sigma \in \Delta(N)$ is given by

$$i\nabla\!\!\!/\,\sigma \equiv i\gamma^\mu(\partial_\mu + \omega_\mu)\sigma \,, \tag{3.51}$$

Here ω_μ is the spin connection and $\gamma^\mu \equiv e^\mu{}_\alpha \gamma^\alpha$, where $e^\mu{}_\alpha$ are the vielbein.

It is known from the theory of Clifford algebras that $\Delta(N)$ can be separated into eigenspaces of positive/negative chirality (by chirality we mean the eigenvalues of $\gamma^{n+1} \equiv (i)^l \gamma^1 \cdots \gamma^m$, which are ± 1),

$$\Delta(N) = \Delta^+(N) \oplus \Delta^-(N) \tag{3.52}$$

and the Dirac operator can be written as

$$i\nabla\!\!\!/ = \begin{pmatrix} 0 & \mathcal{D}^* \\ \mathcal{D} & 0 \end{pmatrix} . \tag{3.53}$$

One can show that $\mathcal{D} : \Delta^+(N) \to \Delta^-(N)$ and $\mathcal{D}^* : \Delta^-(N) \to \Delta^+(N)$ is indeed the adjoint of $\mathcal{D}$. It can also be useful to write $\mathcal{D} = i\nabla\!\!\!/ P^+$ and $\mathcal{D}^* = i\nabla\!\!\!/ P^-$, where $P^\pm$ are the projection operators on $\Delta^\pm$. Therefore we have a two-term complex called the *spin complex*.
We mention another application of the AS index theorem. Let us denote the number of positive/negative chirality zero-modes by $\nu_\pm$:

$$\nu_+ = \dim\ker\mathcal{D}, \quad \nu_- = \dim\ker\mathcal{D}^* \,. \tag{3.54}$$

Then after some calculations one can show that the index of the spin complex is given by

$$\mathrm{ind}\mathcal{D} = \nu_+ - \nu_- = \int_N \hat{A}(TN)\Big|_{vol} \,. \tag{3.55}$$

In particular, since the Dirac genus $\hat{A}(TN)$ contains only $4j$-forms, $\nu_+ - \nu_-$ vanishes unless n is divisible by 4.

If A is a gauge field on a vector bundle E, the *twisted Dirac operator* $i\nabla\!\!\!/(A)$ acts on sections of $S(N) \otimes E$. It is defined analogously to the Dirac operator,

$$i\nabla\!\!\!/(A) = \begin{pmatrix} 0 & \mathcal{D}_E^* \\ \mathcal{D}_E & 0 \end{pmatrix} , \tag{3.56}$$

where $\mathcal{D}_E : \Delta^+(N) \otimes E \to \Delta^-(N) \otimes E$ is given by

$$\mathcal{D}_E = i\gamma^\mu(\partial_\mu + \omega_\mu + A_\mu)P^+ \,. \tag{3.57}$$

Given an operator $\mathcal{D}$ it can be useful to determine the net difference between the number of its positive and negative eigenvalues, sometimes called the signature of $\mathcal{D}$. Formally we could define it as

$$n(\mathcal{D}) = \sum_{\lambda>0} 1 - \sum_{\lambda<0} 1\,, \tag{3.58}$$

where the eigenvalues λ are counted by taking into account their multiplicity.
This quantity is not well defined, however, for a general operator on a infinite-dimensional Hilbert space, and therefore we need a way to *regularize* it. This is very similar to what we have done in the previous Chapter by introducing a well defined notion of determinant of an operator via the procedure of zeta-function regularization. To regularize the signature we will employ another function, which is similar in form to the Dirichlet eta function (hence the name)

$$\eta_{Dirichlet}(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}\,, \tag{3.59}$$

which converges for $\mathrm{Re}(s) > 0$.

Definition 3.10. Given a Dirac operator $\mathcal{D}$, its *associated eta function* for large $\mathrm{Re}(s)$ is defined by

$$\eta(\mathcal{D}, s) = \sum_{\lambda\neq 0} (\mathrm{sign}\lambda)|\lambda|^{-s}\,, \tag{3.60}$$

where λ are the eigenvalues of $\mathcal{D}$. For other values of $\mathrm{Re}(s)$ the function $\eta(\mathcal{D}, s)$ can be defined by means of analytic continuation.
It can be shown that, under proper boundary conditions, $\eta(\mathcal{D}, s)$ has no pole at $s = 0$. Therefore we define the *eta invariant* (or signature) of $\mathcal{D}$ as

$$\eta(\mathcal{D}) \equiv \eta(\mathcal{D}, 0)\,. \tag{3.61}$$

We provide below the statement of the APS theorem as found in [22], since it uses the mathematical language we have adopted so far. We note, however, that in order to evaluate the 1-loop contribution to the Chern-Simons partition function, a different formulation of the theorem has been used in [35] (and in subsequent works such as [16]). This general version of the theorem can be found in the original papers by Atiyah-Patodi-Singer [3, 4, 5], along with a far more precise and advanced discussion, which unfortunately is outside the scope of the present work.

Let M be an odd-dimensional manifold, $\dim M = 2l + 1$. Let $i\nabla\!\!\!\!/\,(A_t)$ be a hermitian twisted Dirac operator on M interacting with a time-dependent gauge field A_t, $t \in I = [0, 1]$. Let us define a $(2l + 2)$-dimensional Dirac operator by

$$i\tilde{\mathcal{D}} = \begin{pmatrix} 0 & \mathcal{D} \\ \mathcal{D}^* & 0 \end{pmatrix}\,, \tag{3.62}$$

where

$$\mathcal{D} = i\partial_t - \nabla\!\!\!\!/\,(A_t), \quad \mathcal{D}^* = i\partial_t + \nabla\!\!\!\!/\,(A_t)\,. \tag{3.63}$$

Theorem 3.4. *With the above notation,*

$$\mathrm{ind}\mathcal{D} = \int_{M\times I} \hat{A}(F)\mathrm{ch}(F)\Big|_{\mathrm{vol}} - \frac{1}{2}\left[\eta(i\nabla\!\!\!\!/\,(A_1)) - \eta(i\nabla\!\!\!\!/\,(A_0))\right]\,, \tag{3.64}$$

where the subscript $|_{\mathrm{vol}}$ *indicates that in the integrand only top-forms are picked up.*

Remark. In the original work by Atiyah et al. the manifold M is identified as the boundary of the manifold on which the Dirac operator ($\tilde{\mathcal{D}}$ above) is defined. An important point that we eluded is that one has to impose suitable boundary conditions in order to rigorously state the theorem. We limit ourselves to mentioning that *local* boundary conditions are obstructed, in general, as shown by Atiyah and Bott in 1964 [2]. A decade later this obstacle was overcome by Atiyah-Patodi-Singer, who introduced *global* boundary conditions that exist for any generalized Dirac operator.

Let us recall the 1-loop approximation of the Chern-Simons partition function that we obtained at the end of the previous Chapter:

$$\mathcal{Z}(M) \approx \frac{1}{\mathrm{Vol}(Z_G)} \sum_{\alpha} e^{ikI_{CS}[A^{(\alpha)}]} e^{-i\pi\eta(L_-)/4} \sqrt{T(M, A^{(\alpha)})} \,. \tag{3.65}$$

Here $\eta(L_-)$ is regularized as we explained above,

$$\eta(L_-) \equiv \eta(L_-, s)|_{s=0} \,, \tag{3.66}$$

and the operator $L_- : \Lambda_-(M) \otimes \mathfrak{g} \to \Lambda_-(M) \otimes \mathfrak{g}$ is defined by (here and in the following D denotes the exterior covariant derivative)

$$L_- = (\star D + D\star)J \,, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \,. \tag{3.67}$$

In [8] it is claimed that by interpreting the 3-manifold M as the boundary of an appropriate 4-manifold and applying the APS index theorem one obtains the relation

$$\eta(A) = \eta(0) - \frac{4h}{\pi} I_{CS}[A] \pmod 2 \,. \tag{3.68}$$

Here $\eta(A)$ and $\eta(0)$ denote the η-invariant of L_- defined using two different connections: an arbitrary connection A (not necessarily flat) and the trivial connection (it is well defined since the principal bundle is trivial). h is the dual Coxeter number of $\mathfrak{g}$, a quantity which appears in the study of representations of a Lie algebra. The (mod 2) term is related to the fact that η jumps by 2 when an eigenvalue passes through 0; it only contributes a fourth root of unity in the partition function (see [16] for more details).

When we consider the trivial connection, the operator L_- takes a simpler form, namely it is the direct sum of $\dim G$ copies of the operator $D_- = (\star d + d\star)J : \Lambda_-(M) \to \Lambda_-(M)$. The η-invariant of D_- is denoted by η_{grav} since D_- only depends on the metric; therefore we can write

$$\eta(0) = (\dim G)\, \eta_{\mathrm{grav}} \,. \tag{3.69}$$

Combining these results we obtain

$$\mathcal{Z}(M) \approx \frac{e^{-i\pi(\dim G)\eta_{\mathrm{grav}}/4}}{\mathrm{Vol}(Z_G)} \sum_{\alpha} e^{i(k+h)I_{CS}[A^{(\alpha)}]} \sqrt{T(M, A^{(\alpha)})} \,. \tag{3.70}$$

We conclude this Chapter with some remarks.

- The 1-loop contributions shift the effective value of the level from k to $k + h$.
- The terms which depend on the flat connection $A^{(\alpha)}$ are all topological invariants of the manifold M.
- The pre-factor depends on the metric (and on the gauge group G) in a controllable manner, but not on the connection. In the next Chapter we will make some comments on the possibility of removing this dependence by employing a different regularization scheme.

4. Conclusions and outlook

Let us summarize the results we presented throughout this work.
We introduced the classical action $\mathcal{S}_{CS}$ of the non-abelian Chern-Simons theory and constructed its partition function via Feynman's path integral approach. By imposing that the amplitude $e^{i\mathcal{S}_{CS}}$ be gauge invariant we showed that the level of the quantum theory must be an integer.
We proceeded to evaluate the partition function via a semi-classical approximation in the weak coupling limit: we expanded the action around the solutions of the classical equations of motion (the gauge fields with vanishing curvature) and considered the first non-vanishing term in the perturbations. The gauge-fixing of the path integral was explicitly carried out via standard BRST transformations. The final result, expression (1.45), depends on quantities that are ill-defined: determinant and signature of infinite-dimensional operators. Some convenient regularization procedures were explained in Chapters 2 and 3. Moreover, expression (1.45) does not make very clear how the partition function depends on the metric, which directly enters the definition of the operators Δ_0 and L_-. Our main goal was then to present a more explicit formula for the 1-loop partition function and to discuss to what extent the quantum Chern-Simons theory can be regarded as a topological field theory.

In Chapter 2 we used a method known as zeta-function regularization to properly define the determinant of an operator. We gave a mathematically rigorous definition of partition function of a quadratic functional as a product of (regularized) determinants of appropriate operators. We explained the steps leading to an explicit formula for the variation of the partition function under a change of the metric; this result can be especially useful in order to determine in which cases said partition function does not depend on the metric at all.
The main application of this discussion was to confirm that the Ray-Singer torsion, interpreted as a partition function, is indeed a topological invariant of a manifold if certain conditions are satisfied. Finally, we used the path integral formalism to show how the Ray-Singer torsion enters the expression of the Chern-Simons partition function in the semi-classical approximation (2.58).

Chapter 3 was dedicated to the presentation of some important mathematical results which connect seemingly distant fields: topology, geometry and analysis. After an overview of the relevant background (characteristic classes in the Chern-Weil approach, the index of a Fredholm operator, Dirac operators on a vector bundle), we stated the Atiyah-Patodi-Singer and Atiyah-Singer index theorems, which apply to manifolds with and without boundary respectively. The APS theorem, in particular, was used to further simplify the 1-loop expression of the partition function by interpreting the base manifold M as the boundary of a 4-dimensional manifold.

We shall now make some remarks on the metric dependence of the non-abelian Chern-Simons theory. Recall what we obtained at the end of the last Chapter:

$$\mathcal{Z}(M) \approx \frac{e^{-i\pi(\dim G)\eta_{\text{grav}}/4}}{\mathrm{Vol}(Z_G)} \sum_{\alpha} e^{i(k+h)I_{CS}[A^{(\alpha)}]} \sqrt{T(M, A^{(\alpha)})}\,.$$

The only non-topologically invariant term in this expression is the phase $e^{-i\pi(\dim G)\eta_{\text{grav}}/4}$. According to [8, 35], one can compensate this metric-dependent term by changing regularization scheme, which is equivalent to adding a local counterterm to the original action. Since the problematic phase factor only depends on the metric (and not on the gauge field), the counterterm should be constructed from the metric alone as well.
Let us briefly comment on the possible form that such a counterterm could have. If ω is the Levi-Civita connection on the spinor bundle of the base manifold M, then the *gravitational* Chern-Simons

functional is defined by

$$I_{CS}[\omega] = \frac{1}{4\pi} \int_M \operatorname{tr}\left(\omega \wedge \mathrm{d}\omega + \frac{2}{3}\omega \wedge \omega \wedge \omega\right) .$$

It is claimed in [8, 35] that another consequence of the APS theorem is that the combination

$$-\frac{1}{4}\eta_{\text{grav}} + \frac{I_{CS}[\omega]}{24\pi}$$

is a topological invariant. Here it is necessary to at least mention a technical point: the gravitational Chern-Simons term $I_{CS}[\omega]$ is well-defined as a real-valued functional only on a *framed* 3-manifold M, that is a manifold with a choice of trivialization of the tangent bundle TM (such a trivialization always exists for an oriented 3-manifold, see [23]). Moreover, it has been proven [1, 16] that there exists a choice of framing (called "canonical" framing) such that the combination $(-\eta_{\text{grav}}/4 + I[\omega]/24\pi)$ vanishes identically.
It appears then that the topological nature of the non-abelian Chern-Simons theory can be interpreted in two ways. One can write the action in a manifestly covariant manner (i.e. without any reference to a background metric); then the 1-loop partition function of the quantum theory has an anomalous dependence on the metric, which takes the form of a simple phase factor independent of the gauge field. Alternatively, one can add by hand a gravitational counterterm to the action (thus spoiling the original covariance) together with a choice of framing; the result, somewhat counter-intuitively, is a partition function which can be expressed entirely in terms of topological invariants (of a framed manifold).

It is now appropriate to comment on some limits of the present work. In order to evaluate the Chern-Simons partition function we made some important simplifying assumptions.
First of all, we assumed that the de Rham cohomology $H^*_A(M)$ was trivial. A generalization of the 1-loop expression (3.70) to the case $H^0_A(M) \neq 0$ was conjectured in [16]. A more detailed analysis was carried out in [19, 28], along with the application to some specific 3-manifolds. In particular, our relatively simple discussion of the Ray-Singer torsion requires an in-depth analysis if the cohomology is not trivial [19].
We also required the gauge group G of the theory to be compact. The case of non-compact group was first studied in [8], where it was shown that the correct results cannot be obtained from those of the compact groups by simple analytic continuation.
Moreover, we avoided the inclusion of Wilson loops (equivalently, knots) in our theory. The perturbative calculations can be found in the original work by Witten [35] for the specific case $M = S^3$, $G = \mathrm{U}(1)$; the same paper contains a discussion on the necessity of giving a framing of each knot (this is a normal vector field along the knot, not to be confused with a framing of the 3-manifold).
Finally, we remark again that the perturbative treatment is not the only way to approach the study of non-abelian Chern-Simons theory. In fact, one of the most relevant results of [35] was showing that the theory can be exactly solved by using non-perturbative methods and the relation to the Wess-Zumino-Witten (WZW) model, a conformal field theory in two dimensions. For a discussion of this topic and the connection between Chern-Simons theory and topological string theory we refer to the review [21].

To conclude we mention an interesting physical application of Chern-Simons theory. It was observed in [24] that a Chern-Simons term in the action of a (2+1)-dimensional abelian gauge theory can reverse the statistics of charged particles. In particular, it was shown that some particles that are bosons at large momenta behave as fermions at small momenta. This was done by considering the partition function of some bosons coupled to a gauge field, with an additional abelian Chern-Simons term; by taking the small momentum limit (thus neglecting short range interactions among the particles), it was found that the propagator in momentum space has the form of a fermionic propagator.
A related, more detailed discussion can be found in [17], where a non-abelian *bosonization formula* was derived for a system of interacting, non-relativistic fermions confined to a plane. Let us briefly describe this result.

Consider a (2+1)-dimensional system of identical, non-relativistic fermions with charge e and half-integer spin S in an electromagnetic field, described by the vector potential A. Let $eA + B$ and V be U(1) and SU(2) gauge fields respectively, and construct a $\mathrm{U}(1) \times \mathrm{SU}(2)$ Chern-Simons theory with levels $(1/(2l+1), \pm 2S)$, $l = 0, 1, 2, \ldots$. Then the partition function of the *fermions* can be written in terms of the expectation value (computed with respect to the CS theory defined above) of the partition function of *bosons* of spin S.
One can recover another known bosonization formula for the abelian case by setting $V = 0$ and choosing again $k = 1/(2l+1)$ for the U(1) Chern-Simons level.
An important caveat is that the formula does not hold if there are interaction terms coupling the spin of the particles to an external electromagnetic field; this is due to the non-abelian character of the group SU(2). We also mention that this result can be extended to the case where the particles carry some internal symmetry groups other than SU(2), if certain conditions (called "fusion rules") are satisfied.

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