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Light in a prism
Love refracted
Show your colors
Flesh is a prison
I'm still a dreamcatcher

(Dreamcatcher, Pt. II – State Faults)

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Contents

Introduction

The principle moving this thesis work is the quest to understand the extent to which finitely generated groups are determined by their finite quotients. This fruitful question has often emerged throughout the history of group theory, frequently linked to problems of geometry and low-dimensional topology. In particular, we will be interested in the interplay between the finite quotients of a group and the property of being amenable.

The finite quotients of a group G are encoded in its *profinite completion* $\widehat{G}$, a compact topological group defined as the inverse limit of the inverse system of the finite quotients of G. It is natural to restrict our attention to the class of *residually finite groups*: indeed, if the group G contains elements that do not survive in any finite quotient, we certainly cannot hope to recover G from its profinite completion alone. Our fundamental challenge then becomes the following: which finitely generated residually finite groups G are *profinitely rigid*, i.e, they have the property that, for every finitely generated residually finite group H, then $\widehat{G} \cong \widehat{H}$ implies $G \cong H$?

In Chapter 1, we give an introduction on profinite groups, which are compact topological groups that can be built starting from a system of finite groups. After explaining their basic properties, in Section 1.3 we direct our attention to profinite completions of groups. The main goal is highlighting the fact that, given a finitely generated residually finite group G, its profinite completion $\widehat{G}$ is entirely characterized by the finite quotients of G (Theorem 1.3.22).

Chapter 2 focuses entirely on profinite rigidity: the goal is to develop a complete understanding of the circumstances in which finitely generated residually finite groups have isomorphic profinite completions. It is not difficult to show that finitely generated abelian groups are profinitely rigid (Proposition 2.1.1), but immediately beyond that we start to struggle. For instance, there are examples of virtually cyclic groups that are not profinitely rigid [Bau74], and we highlight in Section 2.1.2 that the corresponding question on free groups is still open. In Section 2.2, we turn our attention to *relative profinite rigidity*, meaning that we ask if $\widehat{G}$ distinguishes the group G from all other finitely generated residually finite groups in a certain class of groups. In this context, we discuss Grothendieck's question on finitely presented groups (Question 2.2.1) and the negative answer given by Bridson and Grunewald [BG04]. Furthermore, in Section 2.2.2 we briefly investigate the rich interplay between profinite completions and low-dimensional topology. We then conclude the second chapter in Section 2.3 with a focus on *profinite invariants*, meaning group properties which take the same value on finitely generated residually finite groups with isomorphic profinite completions.

In Chapter 3, we introduce the class of *amenable groups*. We start in Section 3.1 by defining amenable groups via the existence of invariant means and we then study the stability properties of amenability (Proposition 3.1.8). We continue in Sections 3.2 and 3.3 by presenting equivalent characterizations of amenability through the existence of almost invariant finite subsets (called *Følner sets*) and asymptotic properties of random walks on groups. We conclude in Section 3.4 by turning our attention to the class of *uniformly amenable groups*, obtained by adding a uniform condition to the definition of amenability.

Finally, Chapter 4 is devoted to the interplay between amenability and profinite completions. In this context, it is interesting to explore the following variation of Grothendieck's question:

Question. *Let A and G be two finitely generated residually finite groups, where A is amenable. Suppose that a morphism $u : A \to G$ induces an isomorphism $\widehat{u} : \widehat{A} \to \widehat{G}$ between the profinite completions. Is the group G amenable?*

In [KS23], S. Kionke and E. Schesler gave a negative answer to this question, proving that amenability is not a profinite invariant of finitely generated residually finite groups. The proof of the following theorem will take up most of Chapter 4.

Theorem (Kionke and Schesler [KS23])**.** *There exist an uncountable family of pairwise non-isomorphic, residually finite 18-generator groups $(G_j)_{j \in J}$ and a residually finite 6-generator group A with embeddings $u_j : A \to G_j$ such that:*
(i) $\widehat{u_j} : \widehat{A} \to \widehat{G_j}$ is an isomorphism for every $j \in J$,
(ii) A is amenable,
(iii) G_j is non-amenable for every $j \in J$.

The construction of the amenable group A and the non-amenable groups $(G_j)_{j \in J}$ of the statement exploits tools related to automorphisms of rooted trees, which we introduce in Section 4.1. The main ingredients of the proof are the following. First of all, we explain that we can regard the automorphism group $\mathrm{Aut}(T_X)$ of a rooted tree T_X as a profinite group. Then, in Section 4.2 we present in detail a technical construction, called *Ω-construction*, which provides a family of subgroups of $\mathrm{Aut}(T_X)$ with isomorphic profinite completions and such that the inclusions between these subgroups induce isomorphisms of the profinite completions. We proceed in Section 4.3

by showing how the Ω-construction can be implemented in order to build the groups A and $(G_j)_{j\in J}$ of the theorem. We conclude in Section 4.4 by showing that, in contrast to amenability, uniform amenability can be detected just by looking at the finite quotients of a group. Indeed, we have the following result.

Theorem (Kionke and Schesler [KS23]). *Let G_1 and G_2 be residually finite groups with $\widehat{G_1} \cong \widehat{G_2}$. Then, G_1 is uniformly amenable if and only if G_2 is uniformly amenable.*

Therefore, we obtain that uniform amenability is a profinite invariant.

Chapter 1

Profinite groups

The aim of this first chapter is to give an introduction to profinite groups. Profinite groups arise naturally in many diverse areas of mathematics, for instance as Galois groups of infinite Galois field extensions. The key idea behind their construction is that they are compact topological groups that can be assembled from a system of finite groups.

We start by introducing the definition of profinite groups through inverse limits and then focus on their topological properties, which reflect group-theoretic properties of the original system of finite groups. In particular, we show that profinite groups are compact, Hausdorff and totally disconnected topological groups, and we will notice that this can actually be given as an equivalent characterization of them. We then turn our attention to profinite completions of groups, which provide the mean for organizing efficiently the finite quotients of a group. In our discussion, we will mainly restrict to the class of residually finite groups.

The first extensive exposition on the theory of profinite groups appeared in the book ”Cohomologie Galoisienne” by Serre in 1964 [Ser79]. The main references for this chapter are the book by Wilson [Wil98] and the one by Ribes and Zalesskii [RZ00]. Furthermore, an accessible introduction to profinite groups can be found in Wilkes' lectures notes from Cambridge University [Gar20]

Before delving into the theory of profinite groups, let us start by briefly recalling the definition of topological groups.

Definition 1.1. A *topological group* is a group G endowed with a topology for which the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ are continuous. Here, $G \times G$ is endowed with the product topology. Furthermore, a *homomorphism of topological groups* $f : G \to H$ is a group homomorphism which is continuous with respect to the topologies of G and H.

We denote by TopGrps the category of topological groups and continuous group homomorphisms. Given two topological groups G and H, we say that *topologically isomorphic* or *isomorphic as topological groups* if there is a bijective function $f : G \to H$ which is both an isomorphism of groups and a homeomorphism. Moreover, all finite groups will be considered topological groups with the discrete topology. For a brief introduction to topological groups, we refer to [Wil98], §0.3.

1.1 Inverse limits and profinite groups

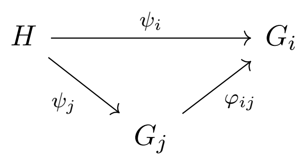
In order to give the formal definition of profinite groups, we start this section by recalling some basic notions regarding inverse systems and inverse limits of topological groups.

Definition 1.2. A *directed set* is a poset $(J, \preceq)$ such that, for every $i, j \in J$, there exists $k \in J$ satisfying $i \preceq k$ and $j \preceq k$.

An *inverse system* of topological groups $(G_j, \varphi_{ij})_{i,j\in J}$ over a directed set J consists of a family of topological groups $(G_j)_{j\in J}$ and a family of continuous group homomorphisms $\varphi_{ij} : G_j \to G_i$, defined whenever $i \preceq j$, such that $\varphi_{ii} = \mathrm{id}_{G_i}$ and $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ for $i \preceq j \preceq k$. The maps φ_{ij} are called *transition maps* of the inverse system.

We will usually omit the transition maps and directly write $(G_j)_{j\in J}$ to denote the inverse system of topological groups $(G_j, \varphi_{ij})_{i,j\in J}$.

Now, let H be a topological group, $(G_j)_{j\in J}$ an inverse system of topological groups and let $(\psi_j : H \to G_j)_{j\in J}$ be a family of continuous group homomorphisms. We say that the maps ψ_j are *compatible* if $\varphi_{ij} \circ \psi_j = \psi_i$ for all $i \preceq j$, that is, if the following diagram commutes:

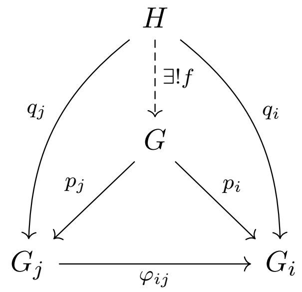


In the language of category theory, an inverse system consists of a direct set $(J, \preceq)$ and a functor from the corresponding poset category J to the category TopGrps. An *inverse limit* of an inverse system of topological groups is the limit of such functor J $\to$ TopGrps. Nevertheless, we will provide the definition assuming that the reader has no prior knowledge of category theory.

Definition 1.3. An *inverse limit* $(G, p_i)_{i\in J}$ of an inverse system of topological groups $(G_j, \varphi_{ij})_{i,j\in J}$ is a topological group G together with compatible continuous group homomorphisms

$$p_i : G \to G_i$$

for every $i \in J$, called *projections*, satisfying the following universal property: for every topological group H with compatible continuous group homomorphisms $q_j : H \to G_j$ for $j \in J$, there exists a unique continuous group homomorphism $f : H \to G$ such that $p_i \circ f = q_i$ for every $i \in J$.



We are interested in giving a more explicit description of such objects. Recall that the direct product of topological groups is again a topological group, endowed with the product topology, where the group operation is defined coordinate-wise.

Proposition 1.1.1. *Let $(G_j)_{j\in J}$ be an inverse system of topological groups. Then, the inverse limit of $(G_j)_{j\in J}$ exists and is unique. More specifically, it is given by*

$$G = \left\{ (g_j)_{j\in J} \in \prod_{j\in J} G_j \mid \varphi_{ij}(g_j) = g_i \text{ for all } i \preceq j \right\}.$$

Proof. Uniqueness of the inverse limit immediately follows from its universal property. Let us focus on the existence part: denote by $\widetilde{G}$ the set on the right hand side above. This is indeed a subgroup of the direct product $\prod_{j\in J} G_j$, because the maps φ_{ij} are group homomorphisms. For every $i \in J$, it is easily seen that the natural projection map $\prod_{j\in J} G_j \to G_i$ restricts to a continuous group homomorphism $p_i : \widetilde{G} \to G_i$ satisfying $\varphi_{ij} \circ p_j = p_i$ whenever $i \preceq j$. We now want to show that $(\widetilde{G}, p_i)_{i\in J}$ satisfies the universal property characterizing the inverse limit.

Let H be a topological group with compatible continuous group homomorphisms $q_i : H \to G_i$ for every $i \in J$. Define $f : H \to \prod_{j\in J} G_j$ by $f(h) = (q_j(h))_{j\in J}$: its image is contained in $\widetilde{G}$, thus we can regard it as a continuous group homomorphism $f : H \to \widetilde{G}$. Now, $p_i \circ f = q_i$ for every $i \in J$, and f is the unique map with this property from the definition of the direct product $\prod_{j\in J} G_j$. Therefore, $(\widetilde{G}, p_i)_{i\in J}$ is the inverse limit of the inverse system $(G_j)_{j\in J}$. □

Given an inverse system of topological groups $(G_j)_{j\in J}$, we will denote its inverse limit by $\varprojlim_{j\in J} G_j$, or directly $\varprojlim G_j$. We are finally ready to give the definition of profinite groups.

Definition 1.4. A *profinite group* is the inverse limit of an inverse system of finite groups.

1.2 Topological properties

We are now interested in understanding the topology of profinite groups: the explicit description in Proposition 1.1.1 helps us in this direction.

Let us consider an inverse system of finite groups $(G_j)_{j\in J}$ and its inverse limit $\varprojlim G_j$. Recall that we endow every finite group G_j with the discrete topology and the direct product $\prod_{j\in J} G_j$ with the product topology: then, the topology on the profinite group $\varprojlim G_j \leq \prod_{j\in J} G_j$ is simply the subspace topology.

Definition 1.5. A topological space is *totally disconnected* if every connected subspace has at most one element.

In other words, a topological space is totally disconnected if every point is its own connected component. For example, discrete spaces are totally disconnected, as well as the rational line.

Proposition 1.2.1. *A profinite group is a compact, Hausdorff and totally disconnected topological group.*

Proof. Let G be a profinite group: then, $G = \varprojlim G_j \leq \prod_{j\in J} G_j$, where $(G_j)_{j\in J}$ is an inverse system of finite groups with transition maps φ_{ij}. It is straightforward to check that, since each discrete group G_j is Hausdorff and totally disconnected, then the product $\prod_{j\in J} G_j$ is Hausdorff and totally disconnected as well. Furthermore, the discrete finite groups of the inverse system are compact, hence Tychonov's theorem ([Bou89], Chapter 1, Theorem 3) implies that their product is compact. As a subset of $\prod_{j\in J} G_j$ endowed with the subspace topology, the group G is Hausdorff and totally disconnected. To conclude that it is also compact, it suffices to show that it is a closed subset of the product. This follows from the fact that G can be written as the intersection $G = \bigcap_{i\preceq j} H_{ij}$, where

$$H_{ij} = \left\{ (g_j)_{j\in J} \in \prod_{j\in J} G_j \mid \varphi_{ij}(g_j) = g_i \right\} \text{ for } i \preceq j.$$

Each H_{ij} is closed in $\prod_{j\in J} G_j$, thus the profinite group G is closed in the product and this finally implies that it is a compact topological group. □

Remark 1.1. A standard result of general topology states that a continuous bijection from a compact space to a Hausdorff space is automatically a homeomorphism ([Wil98], Lemma 0.1.2). Thanks to Proposition 1.2.1, this gives us an easier way of showing that two profinite groups are topologically isomorphic: indeed, this now reduces to exhibiting a continuous group isomorphism between them. We will often use this fact without reference.

Proposition 1.2.2. *Let $(G_j)_{j\in J}$ be an inverse system of non-empty finite groups. Then, the profinite group $G = \varprojlim G_j$ is non-empty.*

Proof. As in the proof of Proposition 1.2.1, let us write the inverse limit G as the intersection $\bigcap_{i\preceq j} H_{ij}$. Assume, for a contradiction, that G is empty: then, we have an empty intersection of closed subsets in a compact space. We can therefore use the finite intersection property of compact spaces, which implies that

$$\bigcap_{s=1}^{n} H_{i_s j_s} = \varnothing$$

for some integer n and elements $i_s, j_s \in J$. Now, J is a directed set, hence there exists $k \in J$ such that $j_s \preceq k$ for every $s = 1, \dots, n$. Let us choose some $g_k \in G_k$, set $g_m = \varphi_{mk}(g_k)$ for $m \preceq k$ and define x_m arbitrarily for all other elements of J. This defines a new element $(g_j)_{j\in J} \in \prod_{j\in J} G_j$ which lies in $\bigcap_{s=1}^{n} H_{i_s j_s}$, since

$$\varphi_{i_s j_s}(g_{j_s}) = \varphi_{i_s j_s}(\varphi_{j_s k}(g_k)) = \varphi_{i_s k}(g_k) = g_{i_s}$$

for every $s = 1, \dots, n$. Therefore, this contradicts the finite intersection property of compact spaces and implies that the profinite group G is non-empty. □

Continuous homomorphisms between profinite groups often arise from families of continuous group homomorphisms between the finite groups of the inverse systems. We briefly see such construction.

Definition 1.6. Let $(G_j, \varphi^G_{ij})_{j\in J}$ and $(H_j, \varphi^H_{ij})_{j\in J}$ be two inverse systems of finite groups over the same directed poset J. A *morphism of inverse systems*

$$\alpha : (G_j)_{j\in J} \to (H_j)_{j\in J}$$

consists of a collection of continuous group homomorphisms $\alpha_j : G_j \to H_j$ for $j \in J$ such that, if $i \preceq j$, the following diagram commutes:

$$\begin{array}{ccc} G_j & \xrightarrow{\varphi^G_{ij}} & G_i \\ {\scriptstyle \alpha_j}\downarrow & & \downarrow{\scriptstyle \alpha_i} \\ H_j & \xrightarrow[\varphi^H_{ij}]{} & H.i \end{array}$$

The maps α_j are said to be the *components* of the morphism α.

Now, let $\alpha : (G_j)_{j\in J} \to (H_j)_{j\in J}$ be a morphism of inverse systems of finite groups with components $\alpha_j : G_j \to H_j$, and consider the inverse limits $G = \varprojlim G_j$ and $H = \varprojlim H_j$. Let us denote by $p_j^G : G \to G_j$ and $p_j^H : H \to H_j$ the projection maps. Then, the collection of compatible continuous group homomorphisms $\alpha_j \circ p_j^G : G \to H_j$ induces a continuous group homomorphism

$$\varprojlim \alpha = \varprojlim_{j\in J} \alpha_j : \varprojlim_{j\in J} G_j \to \varprojlim_{j\in J} H_j$$

such that $p_j^H \circ \varprojlim \alpha = \alpha_j \circ p_j^G$. Furthermore, if all the components of the morphism α are isomorphisms, then $\varprojlim \alpha$ is an isomorphism as well ([RZ00], Lemma 1.1.5).

We now present some useful results on the topological properties of profinite groups. We shall always consider an inverse system of finite groups $(G_j)_{j\in J}$ and its inverse limit $G = \varprojlim G_j$ with projections $p_j : G \to G_j$ for $j \in J$. Notice that the sets $p_j^{-1}(\{g_j\})$ with $j \in J$ and $g_j \in G_j$ form a basis for the topology on the profinite group G.

Lemma 1.2.3. *Let $G = \varprojlim G_j$ be a profinite group and let H be a topological group. A group homomorphism $f : H \to G$ is continuous if and only if every map $p_j \circ f : H \to G_j$ is continuous.*

Proof. This result follows from the definition of product topology, since the group homomorphism f can be regarded as a map $f : H \to G \leq \prod_{j\in J} G_j$ and the maps $p_j : G \to G_j$ are the restrictions of the natural projections. □

Lemma 1.2.4. *Let $G = \varprojlim G_j$ be a profinite group and let $X \subseteq G$ be a subset. If $p_j(X) = G_j$ for every $j \in J$, then X is dense in G.*

Proof. In order to show that X is a dense subset of G, it suffices to show that X has a non-empty intersection with every non-empty element of a basis of open subsets for the topology on G. We can therefore consider a basic open set of the form $U = p_j^{-1}(\{g_j\})$ for some $j \in J$ and $g_j \in G_j$: then, $X \cap U \neq \varnothing$ since $g_j \in p_j(X) = G_j$. □

The following crucial characterization of open subsets holds for general compact topological groups.

Proposition 1.2.5. *Let G be a compact topological group. A subgroup of G is open if and only if it is closed and it has finite index in G.*

Proof. Let H be a open subgroup of G. Then, the group G can be written as

$$G = \bigcup_{g\in G} gH,$$

where each gH is again open since G is a topological group. Therefore, $\{gH\}_{g\in G}$ is an open covering of G and, by compactness, we can extract a finite covering $\{g_1H, \dots, g_nH\}$ with $g_1, \dots, g_n \in G$. The fact that H has finite index in G follows from the fact that the elements $g_1, \dots, g_n$ form a finite set of coset representatives for H. In order to show that H is closed, it suffices to notice that $G \setminus H = \bigcup_{g\notin H} gH$ is open.

Conversely, assume that H is a closed subgroup of G with finite index, and let $\{g_1, \dots, g_n\}$ be a finite set of coset representatives for H in G: then, we can write

$$H = G \setminus \bigcup_{g_i \notin H} g_iH,$$

where each g_iH is closed since G is a topological group. Therefore, H is a open subset of G. □

Lemma 1.2.6. *Let $G = \varprojlim G_j$ be a profinite group. Then, the open normal subgroups $N_j = \ker(p_j)$ form a basis of open neighbourhoods of the identity in G.*

Proof. [RZ00], Lemma 2.1.1. □

Since we are working with topological groups, from a neighbourhood basis of the identity we can obtain neighbourhood basis of all the other points of G. Hence, the following corollary holds.

Corollary 1.2.7. *Let $G = \varprojlim G_j$ be a profinite group and let $g = (g_j)_{j\in J} \in G$. Then, the open cosets $gN_j = p_j^{-1}(\{g_j\})$ form a basis of open neighbourhoods of g in G.*

Given an open normal subgroup N of a group G, we shall always denote it as $N \trianglelefteq_o G$. We now present a useful way of describing the closure of a subset of a profinite group.

Proposition 1.2.8. *Let G be a profinite groups and let $X \subseteq G$ be a subset of G. Then,*

$$\overline{X} = \bigcap_{N \trianglelefteq_o G} XN.$$

Proof. Let N be an open normal subgroup of G. Then, we can write

$$XN = \bigcup_{x\in X} xN.$$

By Proposition 1.2.5, this is a finite union of closed cosets, hence XN is closed as well. This implies that $\overline{X} \subseteq \bigcap_{N \trianglelefteq_o G} XN$. Now, let us consider an element $g \in G$ such that $g \notin \overline{X}$: then, there exists an open neighbourhood U of g such that $X \cap U = \varnothing$. From Corollary 1.2.7, there thus exists an open normal subgroup $N \trianglelefteq_o G$ such that $g \in gN \subseteq U$. This implies that $g \notin XN$.

Indeed, if we could write $g = xn$ for some $x \in X$ and $n \in N$, then we would have $x = gn^{-1} \in gN \subseteq U$, which gives a contradiction. Therefore, $\bigcap_{N \trianglelefteq_o G} XN \subseteq \overline{X}$ and this completes the proof. □

Proposition 1.2.9. *Every closed subgroup of a profinite group is a profinite group.*

Proof. [RZ00], Corollary 1.1.8. □

In the following theorem, we finally encounter some fundamental equivalent characterizations of profinite groups. In particular, we get that the topological properties seen in Proposition 1.2.1 actually characterize profinite groups.

Theorem 1.2.10. *Let G be a topological group. Then, the following conditions are equivalent.*
(i) G is a profinite group;
(ii) G is topologically isomorphic to a closed subgroup of the Cartesian product of finite groups;
(iii) G is compact and

$$\bigcap_{N \trianglelefteq_o G} N = \{1_G\};$$

(iv) G is compact, Hausdorff and totally disconnected.

Proof. [Wil98], Corollary 1.2.4. □

Furthermore, given a profinite group, we can represent it explicitly as an inverse limit using its quotient groups.

Corollary 1.2.11. *Let G be a profinite group.*
(i) Let $\mathcal{N}$ be a collection of open normal subgroups of G forming a basis of neighbourhoods of the identity in G. Then,

$$G \cong \varprojlim_{N \in \mathcal{N}} G/N.$$

(ii) If, moreover, H is a closed subgroup of G, then

$$H \cong \varprojlim_{N \in \mathcal{N}} H/(H \cap N).$$

Proof. [Wil98], Theorem 1.2.5. □

Let us recall that, from Lemma 1.2.6, the kernels of the projection maps form a basis of open neighbourhoods of the identity in a profinite group. Therefore, Corollary 1.2.11 yields the isomorphism

$$G \cong \varprojlim_{j \in J} G/\ker(p_j).$$

We conclude by citing a result which implies that an infinite profinite group cannot be countable. This highlights the fact that profinite groups can be troublesome when simply considered as abstract groups, while they are much better behaved as topological groups.

Proposition 1.2.12. *Let G be a profinite group. Then, the cardinality of G is either finite or uncountable.*

Proof. [RZ00], Proposition 2.3.1. □

1.2.1 Examples

Let us now turn our attention to some fundamental examples of profinite groups.

Example 1.1. Every finite group endowed with the discrete topology is a profinite group. Indeed, every discrete finite group is compact, Hausdorff and totally disconnected, hence profinite by Theorem 1.2.10.

Example 1.2. An important example of profinite group is given by the group of *p-adic integers*. Let p be a prime number and consider the inverse system of finite groups

$$\cdots \to \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to \cdots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z},$$

where, for $n \leq m$, the transition maps $\varphi_{nm} : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ are given by the natural projection maps. We denote by

$$\mathbb{Z}_p = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$$

the inverse limit of such inverse system. Then, $\mathbb{Z}_p$ is an abelian group that inherits a natural ring structure from the finite rings $\mathbb{Z}/p^n\mathbb{Z}$. It is called the group, or ring, of *p-adic integers*, and it was first introduced by Hensel in 1908 [Hen08]. Following the description given in Proposition 1.1.1, we obtain that $\mathbb{Z}_p$ is given by

$$\mathbb{Z}_p = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \mid x_n \equiv x_m \bmod p^n \text{ for all } n \leq m \right\}.$$

It is also useful to identify $\mathbb{Z}_p$ with the set of power series

$$\mathbb{Z}_p \cong \left\{ \sum_{n=0}^{\infty} a_n p^n \mid a_n \in \mathbb{N},\ 0 \leq a_n < p \right\}.$$

It is not difficult to show that the ring $\mathbb{Z}_p$ is an integral domain. More details on p-adic integers can be found in [Wil98], §1.5.

The p-adic integers $\mathbb{Z}_p$ are a standard examples of a special kind of profinite groups, called *pro-p groups*.

Definition 1.7. Let p be a prime number. A *pro-p group* is the inverse limit of an inverse system of finite groups whose orders are a power of p.

Example 1.3. Consider the inverse system given by the cyclic groups $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}$, ordered by setting $n \preceq m$ whenever $n \mid m$. The transition maps $\varphi_{nm} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ are the natural projection maps for $n \mid m$. We denote by

$$\widehat{\mathbb{Z}} = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$$

the inverse limit of such inverse system, and we call it the *profinite completion* of $\mathbb{Z}$. We will see the general construction for such completions in Section 1.3. Similarly to the ring of p-adic integers, the profinite completion $\widehat{\mathbb{Z}}$ inherits a ring structure from that of the finite rings $\mathbb{Z}/n\mathbb{Z}$. Using the Chinese Remainder Theorem, one can prove that

$$\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$$

as topological rings. As a consequence, many questions about $\widehat{\mathbb{Z}}$ can be reduced to question about $\mathbb{Z}_p$. Nevertheless, in contrast to p-adics integers, the ring $\widehat{\mathbb{Z}}$ is not an integral domain.

Example 1.4. Another important class of profinite groups is given by matrix groups over the rings $\mathbb{Z}_p$ and $\widehat{\mathbb{Z}}$. Given a ring R, let us denote by $\mathrm{Mat}_{n\times m}(R)$ the group of $n \times m$ matrices of elements in R. Then, we can consider the profinite group

$$\mathrm{Mat}_{n\times m}(\mathbb{Z}_p) = \varprojlim_{k\in\mathbb{N}} \mathrm{Mat}_{n\times m}\,\mathbb{Z}/p^k\mathbb{Z}\,.$$

The determinant map in $\mathbb{Z}_p$ is continuous, since the ring operations in $\mathbb{Z}_p$ are continuous. This allows us to further consider the following profinite groups:

$$\mathrm{GL}_n(\mathbb{Z}_p) = \varprojlim_{k\in\mathbb{N}} \mathrm{GL}_n(\mathbb{Z}/p^k\mathbb{Z}) \cong \{A \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p) \mid \det(A) \in \mathbb{Z}_p^\times\},$$

$$\mathrm{SL}_n(\mathbb{Z}_p) = \varprojlim_{k\in\mathbb{N}} \mathrm{SL}_n(\mathbb{Z}/p^k\mathbb{Z}) \cong \{A \in \mathrm{Mat}_{n\times n}(\mathbb{Z}_p) \mid \det(A) = 1\}.$$

Analogously for the profinite completion $\widehat{\mathbb{Z}}$, we have the profinite group

$$\mathrm{Mat}_{n\times m}(\widehat{\mathbb{Z}}) = \varprojlim_{k\in\mathbb{N}} \mathrm{Mat}_{n\times n}\,\mathbb{Z}/n\mathbb{Z}\,.$$

We can use the fact that $\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$ in order to get that

$$\mathrm{Mat}_{n\times m}(\widehat{\mathbb{Z}}) \cong \prod_{p \text{ prime}} \mathrm{Mat}_{n\times m}(\mathbb{Z}_p).$$

Furthermore, we can again consider the general and special linear groups, which splits as $\mathrm{Mat}_{n\times m}(\widehat{\mathbb{Z}})$ does:

$$\mathrm{GL}_n(\widehat{\mathbb{Z}}) = \varprojlim_{k\in\mathbb{N}} \mathrm{GL}_n(\mathbb{Z}/k\mathbb{Z}) \cong \prod_{p \text{ prime}} \mathrm{GL}_n(\mathbb{Z}_p),$$

$$\mathrm{SL}_n(\widehat{\mathbb{Z}}) = \varprojlim_{k\in\mathbb{N}} \mathrm{SL}_n(\mathbb{Z}/k\mathbb{Z}) \cong \prod_{p \text{ prime}} \mathrm{SL}_n(\mathbb{Z}_p).$$

Example 1.5. Historically, interest in profinite groups first sparked among number theorists: the original motivation for studying such topological groups is the fact that they appear as Galois groups of (finite or infinite) Galois extensions of fields, equipped with the so-called *Krull topology*. Such topology was introduced by Krull in [Kru28] with the goal of generalizing the fundamental theory of Galois theory to extensions of infinite degrees. We write down the main results and refer to [RZ00], §2.11 for a complete survey on the matter.

Theorem 1.2.13. *Let K/F be a Galois extension and let $\mathcal{K} = \{K_i \mid i \in I\}$ be the collection of all intermediate subfields $F \subseteq K_i \subseteq K$ such that each K_i/F is a finite Galois extension. Then, the Galois group*

$$\mathrm{Gal}(K/F) \cong \varprojlim_{i\in I} \mathrm{Gal}(K_i/F)$$

is a profinite group.

For instance, we can easily see what happens for the *absolute Galois extensions* of finite fields. For a prime number p, let us consider the field $\mathbb{F}_p$ and its algebraic closure $\overline{\mathbb{F}_p}$: then, $\overline{\mathbb{F}_p}/\mathbb{F}_p$ is said to be the *absolute Galois extension* of $\mathbb{F}_p$. For every $n \in \mathbb{N}$, there exists a unique Galois extension $K_n/\mathbb{F}_p$ of degree n, whose Galois group is given by $\mathrm{Gal}(K_n/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$. Therefore, Theorem 1.2.13 implies that the Galois group of the extension $\overline{\mathbb{F}_p}/\mathbb{F}_p$ is given by

$$\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}.$$

Not only every Galois group can be interpreted as a profinite group: the next result, first proved by [Lep55], tells that the converse holds as well.

Theorem 1.2.14. *Let G be a profinite group. Then, there exists a Galois extension of fields K/L such that $G = Gal(K/L)$.*

Example 1.6. Profinite groups also appear in algebraic geometry as *étale fundamental groups* of schemes. A complete treatment of the subject was done by Grothendieck and can be found in [GR71].

1.2.2 Finitely generated profinite groups

Most of the profinite groups that we will encounter are finitely generated in the following sense.

Definition 1.8. Let G be a topological group and let $X \subseteq G$ be a subset. We say that X is a *(topological) generating set* for G if the abstract subgroup $\langle X \rangle$ generated by X is dense in G, and we write $G = \overline{\langle X \rangle}$. The group G is *(topologically) finitely generated* if it has a finite topological generating set.

Remark 1.2. Given a topological group G, a subset $X \subseteq G$ generates a dense subgroup in G if and only if $G = \langle X \rangle N$ for every $N \trianglelefteq_o G$. This follows immediately from Proposition 1.2.8, as

$$\overline{\langle X \rangle} = \bigcap_{N \trianglelefteq_o G} \langle X \rangle N.$$

Definition 1.9. Let G be a topological group. The *rank* $d(G)$ of G is the smallest cardinality of a topological generating set of G.

We shall generally omit the word "topologically" and directly speak about generating sets and finitely generated topological groups. Furthermore, given a positive integer d, a topological group is called a *d-generator group* if it admits a generating set containing at most d elements.

The next result tells us that we can determine whether a profinite group is a d-generator group just by examining its quotients by open normal subgroups.

Proposition 1.2.15. *Let d be a positive integer and let G be a profinite group. Then, G is a d-generator group if and only if G/N is a d-generator group for all open normal subgroups $N \trianglelefteq_o G$.*

Proof. [Wil98], Theorem 4.2.1. □

Proposition 1.2.16. *Let G be a finitely generated profinite group and let U be an open subgroup of G. Then, U is finitely generated.*

Proof. [Wil98], Proposition 4.3.1. □

Lemma 1.2.17. *Let G be a finitely generated profinite group. For every $n \in \mathbb{N}$, the number of open subgroups of G of index n is finite.*

Proof. [RZ00], Proposition 2.5.1. □

Example 1.7. An example of finitely generated profinite group is given by the Galois group $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$ of the absolute Galois extension $\overline{\mathbb{F}_p}/\mathbb{F}_p$ of Example 1.5. Indeed, the Galois group is topologically generated by the p-power Frobenius automorphism $\varphi : \overline{\mathbb{F}_p} \to \overline{\mathbb{F}_p}, x \mapsto x^p$.

In Proposition 1.2.5, we have seen that a finite index subgroup of a profinite group is open if and only if it is closed. Furthermore, open subgroups always have finite index: it is natural to ask under what circumstances the converse implication holds.

Question. Which are the profinite groups where all subgroups of finite index are open?

Following [RZ00], profinite groups with this property are called *strongly complete*. The topology of strongly complete groups is entirely determined by its algebraic structure. Indeed, it follows from Lemma 1.2.6 that the collection of all open subgroups of a profinite group forms a basis of open neighbourhoods of the identity, and each of these subgroups has finite index. Hence, if all subgroups of finite index are open, it is possible to reconstruct the topology of the profinite group just by taking all finite index subgroups as a basis of open neighbourhoods of the identity. We remark that there exist profinite groups that fail to be strongly complete: interesting examples are constructed in [RZ00], §4.2.

Serre proved in 1979 that finitely generated pro-p groups are strongly complete [Ser79]. In 2007, Nikolov and Segal were finally able to prove the result for all finitely generated profinite groups.

Theorem 1.2.18 (Nikolov and Segal [NS07])**.** *If G is a finitely generated profinite group, then every subgroup of finite index is open in G.*

We highlight that the proof of this result relies heavily on the classification of finite simple groups.

Remark 1.3. Notice that Theorem 1.2.18 is equivalent to the saying that, given a finitely generated profinite group G, then every group homomorphism from G to every other profinite group is continuous. We will often use this result without specific reference.

1.3 Profinite completions

This section is devoted to the study of *profinite completions* of groups, which are essentially profinite groups with the same finite quotients as the original group. We begin by presenting their construction and then delve into their properties.

Let G be a group, and let us denote by $\mathcal{N}$ the set of all its finite index normal subgroups. We write $N \trianglelefteq_f G$ for such subgroups. The set $\mathcal{N}$ can be made into a direct set by declaring that $M \preceq N$ if $N \leq M$ for $N, M \in \mathcal{N}$. If $M \preceq N$, let $\varphi_{MN} : G/N \to G/M$ be the natural projection homomorphism. Then,

$$\{G/N, \varphi_{NM}\}_{N,M \in \mathcal{N}}$$

is an inverse system of finite groups. The inverse limit

$$\widehat{G} = \varprojlim_{N \trianglelefteq_f G} G/N$$

of such inverse system is the *profinite completion* of the group G. It directly follows from the definition that profinite completions are profinite groups.

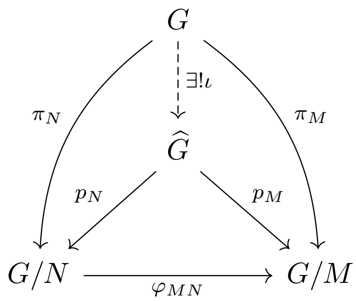
Given $N \trianglelefteq_f G$, let us denote by $\pi_N : G \to G/N$ the natural projection map. From the universal property of the inverse limit, we get a canonical homomorphism

$$\iota : G \to \widehat{G} \leq \prod_{N \trianglelefteq_f G} G/N$$
$$g \mapsto (gN)_{N \trianglelefteq_f G}$$

Such map $\iota : G \to \widehat{G}$ will be fundamental in our study of profinite completions. We immediately see some of its basic properties.

Proposition 1.3.1. *Let G be a group. Then, the image $\iota(G)$ of G in $\widehat{G}$ is a dense subgroup of $\widehat{G}$.*

Proof. This follows from Lemma 1.2.4, since from the diagram below we clearly have that $p_N(\iota(G)) = G/N$ for every $N \trianglelefteq_f G$.



□

Remark 1.4. Let G be a group. Proposition 1.3.1 implies that, if X is an abstract generating set of G, then its image $\iota(X)$ is a topological generating set of $\widehat{G}$. As a consequence, the profinite completion of a finitely generated group is a finitely generated profinite group.

The canonical map $\iota : G \to \widehat{G}$ satisfies the following useful universal property.

Proposition 1.3.2. *Let G be group, H a profinite group and $\varphi : G \to H$ a continuous group homomorphism. Then, there exists a unique continuous group homomorphism $\overline{\varphi} : \widehat{G} \to H$ such that $\overline{\varphi} \circ \iota = \varphi$.*

Proof. [Wil98], Proposition 1.4.2. □

We now wish to show that the profinite completion construction is functorial: given a group homomorphism $f : G \to H$, we want to define canonically an induced continuous homomorphism $\widehat{G} \to \widehat{H}$ between the completions. This can be done as in the following proposition.

Proposition 1.3.3. *Let $f : G \to H$ be a group homomorphism. Then, there exists a unique continuous group homomorphism $\widehat{f} : \widehat{G} \to \widehat{H}$ such that $\widehat{f} \circ \iota_G = \iota_H \circ f$.*

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \iota_G}\downarrow & & \downarrow{\scriptstyle \iota_H} \\ \widehat{G} & \xrightarrow[\widehat{f}]{} & \widehat{H} \end{array}$$

Proof. Let us consider the inverse system of finite quotients of the group H: since all the transition maps are surjective, one can show that the projections $p_N : \widehat{H} \to H/N$ for $N \trianglelefteq_f H$ are surjective as well (see [RZ00], Proposition 1.1.10). Consider the map $p_N \circ \iota_H \circ f : G \to H/N$: from the universal property stated in Proposition 1.3.2, there exists a unique continuous group homomorphism $q_N : \widehat{G} \to H/N$ such that $p_N \circ \iota_H \circ f = q_N \circ \iota_G$. By uniqueness, such maps q_N for $N \trianglelefteq_f H$ are compatible with the transition maps of the inverse system $\{H/N\}_{N \trianglelefteq_f H}$, therefore the universal property of the inverse limit yields a unique continuous group homomorphism $\widehat{f} : \widehat{G} \to \widehat{H}$ such that $p_N \circ \widehat{f} = q_N$. Now, for all $N \trianglelefteq_f H$ we have

$$p_N \circ \iota_H \circ f = p_N \circ \widehat{f} \circ \iota_G,$$

thus $\iota_H \circ f = \widehat{f} \circ \iota_G$ since the projections p_N are surjective. □

We say that the map $\widehat{f}$ is *induced* by f. From the uniqueness part of Proposition 1.3.3, we get that the defining functorial conditions indeed hold: $\widehat{f_1 \circ f_2} = \widehat{f_1} \circ \widehat{f_2}$ and $\widehat{\mathrm{id}_G} = \mathrm{id}_{\widehat{G}}$. We can therefore sum up the construction in the following lemma.

Lemma 1.3.4. *The profinite completion $\widehat{\cdot}$ is a functor from the category of abstract group to the category of profinite groups with morphisms being continuous group homomorphisms.*

Proof. [RZ00], Lemma 3.2.3. □

Proposition 1.3.5. *The profinite completion functor $\widehat{\cdot}$ is right exact, that is, if*

$$1 \to K \xrightarrow{f} G \xrightarrow{g} H \to 1$$

is an exact sequence of groups, then

$$\widehat{K} \xrightarrow{\widehat{f}} \widehat{G} \xrightarrow{\widehat{g}} \widehat{H} \to 1$$

is an exact sequence of profinite groups.

Proof. [RZ00], Proposition 3.2.5. □

Example 1.8. (i) Let G be a finite group. Its profinite completion $\widehat{G}$ is isomorphic to G endowed with the discrete topology.
(ii) Let G be a strongly complete group as defined in Section 1.2.2. Then, G is isomorphic to its own profinite completion. This is a consequence of Corollary 1.2.11, since

$$\widehat{G} = \varprojlim_{N \trianglelefteq_f G} G/N \cong \varprojlim_{N \trianglelefteq_o G} G/N \cong G.$$

In particular, if G is a finitely generated group, then its profinite completion is finitely generated (see Remark 1.4) and Theorem 1.2.18 implies that

$$\widehat{\widehat{G}} \cong \widehat{G}.$$

(iii) We have already introduced the profinite completion $\widehat{\mathbb{Z}}$ of the integers in Example 1.3. Recall that $\widehat{\mathbb{Z}}$ is the inverse limit of the inverse system $\{\mathbb{Z}/n\mathbb{Z}\}_{n\in\mathbb{N}}$, with transition maps given by the natural projection homomorphisms $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ for $n \mid m$.
(iv) Given a smooth complex projective variety, its étale fundamental group is isomorphic to the profinite completion of its topological fundamental group.
(v) The first example of an infinite finitely presented group with no non-trivial finite quotients was given by Higman [Hig51]. This so-called *Higman group* can be defined as the group H with the following presentation:

$$H = \langle (x_i)_{i\in\mathbb{Z}/4\mathbb{Z}} \mid x_i x_{i+1} x_i^{-1} = x_{i+1}^2 \text{ for all } i \in \mathbb{Z}/4\mathbb{Z}\rangle.$$

A detailed proof that this group has no non-trivial finite quotients can be found in [Ser77], §1.4. As a consequence, the profinite completion of the Higman group H is trivial. This immediately tells us that it is possible to construct non-isomorphic groups with isomorphic profinite completions, such as the trivial group and Higman group H.

It is interesting to know that, as was shown by Bridson and Wilton in [BW15], there is no algorithm that can determine whether or not a finitely presented group has a non-trivial finite quotient.

1.3.1 Residual finiteness

The name "completion" is justified by the fact that, if a given group G is *residually finite*, then it embeds as a dense subgroup in its profinite completion $\widehat{G}$. In this section, we explore such condition of residual finiteness and we give some important examples. A survey on the topic has been written by Magnus [Mag69].

Definition 1.10. A group G is said to be *residually finite* if for every non-trivial element $g \in G$ there exists a finite index normal subgroup $N \trianglelefteq_f G$ such that $g \notin N$.

Equivalently, a group is residually finite if for every non-trivial element $g \in G$ there exists a group homomorphism $\varphi : G \to Q$, where Q is finite, such that $\varphi(g) \neq 1_Q$.

Example 1.9. Finite groups are residually finite. Indeed, let G be a finite group and consider a non-trivial element $g \in G$: then, it suffices to consider the identity map $\varphi = \mathrm{id} : G \to G$ in order to have $\varphi(g) = g \neq 1_G$.

We immediately see the connection between residual finiteness and profinite completions.

Proposition 1.3.6. *A group G is residually finite if and only if the canonical map $\iota : G \to \widehat{G}$ is injective.*

Proof. This directly follows from the fact that the map $\iota : G \to \widehat{G}$ maps an element $g \in G$ to $(gN)_{N \trianglelefteq_f G} \in \widehat{G}$. □

Hence, if G is a residually finite group, we can use the canonical map ι to identify G with a subgroup of $\widehat{G}$: from now on, we will directly write $G \leq \widehat{G}$. Explicitly, we are identifying an element $g \in G$ with the element

$$(gN)_{N \trianglelefteq_f G} \in \widehat{G} \leq \prod_{N \trianglelefteq_f G} G/N.$$

Furthermore, Proposition 1.3.1 yields that the group G is a dense subgroup of $\widehat{G}$.

We now proceed with some standard examples and properties of residually finite groups.

Proposition 1.3.7. *The additive group $\mathbb{Z}$ is residually finite.*

Proof. Let us consider $k \in \mathbb{Z}$ such that $k \neq 0$. We can choose $n \in \mathbb{Z}$ such that $|k| < n$, hence the canonical homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ satisfies $\varphi(k) \neq 0$ and $\mathbb{Z}$ is residually finite. □

Proposition 1.3.8. *Every subgroup of a residually finite group is residually finite.*

Proof. Let G be a residually finite group and let H be a subgroup of G. Let h be a non-trivial element of H: by residual finiteness of G, there exists a group homomorphism $\varphi : G \to Q$, where Q is finite, such that $\varphi(h) \neq 1_Q$. Let $\varphi|_H : H \to Q$ be the restriction of φ to the subgroup H. Then, we get that $\varphi|_H(h) = \varphi(h) \neq 1_Q$, which implies that H is residually finite. □

Proposition 1.3.9. *Let $(G_j)_{j\in J}$ be a family of residually finite groups. Then, their direct product $G = \prod_{j\in J} G_j$ is residually finite.*

Proof. [CC10], Proposition 2.2.2. □

From the standard structure theorem of finitely generated abelian groups, we immediately get the following result.

Corollary 1.3.10. *Every finitely generated abelian group is residually finite.*

Proof. Let G be a finitely generated abelian group. Then, there exist $r \in \mathbb{N}$ and a finite abelian group A such that

$$G \cong \mathbb{Z}^r \times A.$$

The statement then follows from Example 1.9, Proposition 1.3.7 and Proposition 1.3.9. □

However, we will see later that there are abelian groups that fail to be residually finite.

Proposition 1.3.11. *Let G be an inverse limit of an inverse system of residually finite groups. Then, G is residually finite.*

Proof. Let $(G_j)_{j\in J}$ be an inverse system of residually finite groups such that $G = \varprojlim G_j$. Following the same idea of the proof of Proposition 1.1.1, the group G can be seen as a subgroup of the product $\prod_{j\in J} G_j$, hence we can conclude from Proposition 1.3.8 and Proposition 1.3.9 that G is residually finite. □

Corollary 1.3.12. *Every profinite group is residually finite.*

Proof. This directly follows from Example 1.9 and Proposition 1.3.11. □

Let us recall that, given a property of groups $\mathcal{P}$, we say that a group G is *virtually* $\mathcal{P}$ if G contains a subgroup of finite index which satisfies $\mathcal{P}$.

Proposition 1.3.13. *Every virtually residually finite group is residually finite.*

Proof. [CC10], Proposition 2.2.12. □

The following result for free group was first established by Levi [Lev33]

Theorem 1.3.14. *Every free group is residually finite.*

Proof. [CC10], Theorem 2.3.1. □

Recall that we say that a group G is *linear* if there exist an integer $n \geq 1$ and a field K such that G is isomorphic to a subgroup of $\mathrm{GL}_n(K)$.

Theorem 1.3.15 (Malcev, [Mal40])**.** *Every finitely generated linear group is residually finite.*

The converse implication does not hold: there exist residually finite groups which are not linear. An example was given by Druțu and Sapir in [DS05]. They were able to prove that the group with presentation

$$\langle a, t \mid t^2 a t^{-2} = a^2 \rangle$$

is a non-linear residually finite group.

Following [CC10], we now briefly present an example of groups that fail to be residually finite.

Definition 1.11. An abelian group G is *divisible* if, for every integer $n \geq 1$ and $g \in G$, there exists an element $h \in G$ such that $h^n = g$.

Example 1.10. The additive groups $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are divisible. More generally, the underlying additive group of every $\mathbb{Q}$-vector space is divisible.

Proposition 1.3.16. *Every non-trivial divisible group is not residually finite.*

Proof. Let G be a non-trivial divisible group. In order to show that it cannot be residually finite, we prove that every group homomorphism $\varphi : G \to Q$, where Q is a finite group, must be trivial. Indeed, let us take $g \in G$ and set $n = |Q|$: since G is divisible, there exists $h \in G$ such that $g = h^n$. If $\varphi : G \to Q$ is a homomorphism, then

$$\varphi(g) = \varphi(h^n) = \varphi(h)^n = 1_Q.$$

Therefore, the group G is not residually finite. □

We conclude this part by presenting a property that, under certain circumstances, imply residual finiteness.

Definition 1.12. A group G is *Hopfian*, or satisfies the *Hopf property*, if every surjective homomorphism $\varphi : G \to G$ is an isomorphism.

Example 1.11. (i) Every finite group is Hopfian.
(ii) Every simple group is Hopfian. Recall that a non-trivial group G is simple if its only normal subgroups are G and $\{1_G\}$.
(iii) There exist finitely generated groups that are not Hopfian: an example is given by the Baumslag-Solitar group

$$BS(2,3) = \langle a, b \mid ba^2 = a^3 b \rangle,$$

which is 2-generated but not Hopfian (see [BS62]). The following surjective homomorphism is not injective:

$$\varphi : BS(2,3) \to BS(2,3) : \begin{cases} a \mapsto a^2 \\ b \mapsto b \end{cases}$$

Indeed, $[a^b, a]$ is a non-trivial element in the kernel of φ.

Malcev discovered that, if a group is finitely generated, then being Hopfian implies being residually finite [Mal40] . In order to approach the proof, we first need an easy lemma.

Lemma 1.3.17. *Let G be a finitely generated group and let Q be a finite group. Then the set Hom(G, Q) is finite.*

Proof. Let X be a finite generated set of the group G. Every homomorphism $f : G \to Q$ is entirely determined by the images of the generators $f(x)$ for $x \in X$. Therefore, $\mathrm{Hom}(G, Q)$ contains at most $|Q|^{|X|}$ elements. □

Theorem 1.3.18. *Every finitely generated residually finite group is Hopfian.*

Proof. Let G be a finitely generated residually finite group and let $\varphi : G \to G$ be a surjective homomorphism of G. Let us consider a normal subgroup of finite index $N \trianglelefteq_f G$ and denote by $\pi_N : G \to G/N$ the canonical projection homomorphism. Define the map

$$\begin{aligned} \Phi : \mathrm{Hom}(G, G/N) &\to \mathrm{Hom}(G, G/N) \\ u &\mapsto u \circ \varphi. \end{aligned}$$

By surjectivity of φ, the map Φ is injective. Moreover, the set $\mathrm{Hom}(G, G/N)$ is finite from Lemma 1.3.17, hence Φ is bijective. In particular, there exists $\overline{u} \in \mathrm{Hom}(G, G/N)$ such that $\pi_N = \overline{u} \circ \varphi$, implying that

$$\ker(\varphi) \subseteq \ker(\pi_N) = N.$$

Then, $\ker(\varphi)$ is contained in the intersection of all normal subgroups of finite index of G, which is trivial since G is residually finite. Finally, φ is injective and the group G is Hopfian. □

Corollary 1.3.19. *Every finitely generated profinite group is Hopfian.*

Proof. Since profinite groups are residually finite by Corollary 1.3.12, this is an immediate consequence of Theorem 1.3.18. □

The converse implication of Theorem 1.3.18 does not hold.

Example 1.12. The Baumslag-Solitar group

$$B(2,4) = \langle a, b \mid ba^2b^{-1} = a^4 \rangle$$

is an example of finitely presented Hopfian group which is not residually finite (see [Mes72]).

1.3.2 Profinite completion and finite quotients

Defining the profinite completion of a group is a way of organizing its finite quotients in a new object, which will naturally be a profinite group. We now see that, for a finitely generated residually finite group G, its profinite completion $\widehat{G}$ is entirely characterized by the finite quotients of G. This means that two finitely generated residually finite groups have isomorphic profinite completions if and only if they have the same sets of isomorphism types of finite quotients (Theorem 1.3.22).

We start with a fundamental theorem that relates the finite index subgroups of a finitely generated residually finite group with the finite index subgroups of its profinite completion.

Theorem 1.3.20. *Let G be a finitely generated residually finite group. Identify G with its image in its profinite completion under the canonical map $\iota : G \hookrightarrow \widehat{G}$. We denote by $\overline{X}$ the closure in $\widehat{G}$ of a subset X of G.*
(i) Let us consider the map

$$\Phi : \{N \mid N \leq_f G\} \to \{U \mid U \leq_f \widehat{G}\}$$
$$N \mapsto \overline{N}.$$

Then, Φ is a one-to-one correspondence between the set of all finite index subgroups of G and those of $\widehat{G}$, whose inverse is given by

$$U \mapsto U \cap G.$$

In particular, if $U \leq_f \widehat{G}$, then $\overline{U \cap G} = U$.
(ii) The map Φ sends normal subgroups to normal subgroups.
(iii) Let $H, K \in \{N \mid N \leq_f G\}$ with $H \leq K$. Then, $[K : H] = [\overline{K} : \overline{H}]$. If, in addition, $H \trianglelefteq K$, then $K/H \cong \overline{K}/\overline{H}$.
(iv) If $H \leq_f G$, then $\widehat{H} \cong \overline{H}$.
(v) The map Φ is an isomorphism of lattices, i.e., if $H, K \in \{N \mid N \leq_f G\}$, then $\overline{H \cap K} = \overline{H} \cap \overline{K}$ and $\overline{\langle H, K\rangle} = \langle \overline{H}, \overline{K} \rangle$.

Proof. [RZ00], Proposition 3.2.2. Since we assume that the group G is finitely generated, we use Theorem 1.2.18 to replace "open" with "finite index" subgroup in the profinite case. □

Notice that, if two finitely generated residually finite groups G and H have isomorphic profinite completions, we do not automatically get a map between the abstract groups. However, in this situation, Theorem 1.3.20 implies that G and H have isomorphic lattices of finite index subgroups. Indeed, we have the following corollary.

Corollary 1.3.21. *Let G and H be finitely generated residually finite groups such that $\widehat{G} \cong \widehat{H}$. Then, there is a one-to-one correspondence*

$$\varphi : \{U \mid U \leq_f G\} \to \{V \mid V \leq_f H\}$$

between the set of finite index subgroups of G and the set of finite index subgroups of H. Let $H, K \in \{U \mid U \leq_f G\}$ with $H \leq K$. Then,
(i) $[K : H] = [\varphi(K) : \varphi(H)]$.
(ii) $H \trianglelefteq K$ *if and only if* $\varphi(H) \trianglelefteq \varphi(K)$.
(iii) If $H \trianglelefteq K$*, then* $K/H \cong \varphi(K)/\varphi(H)$.
(iv) $\widehat{H} \cong \widehat{\varphi(H)}$.
(v) The map φ is an isomorphism of lattices.

Proof. By assumption, we have an isomorphism $f : \widehat{G} \to \widehat{H}$ between the profinite completions. Let $U \leq_f G$: then, the correspondence φ is given by $U \longleftrightarrow f(\overline{U}) \cap H$. □

Now, Corollary 1.3.21 tells us that two finitely generated residually finite groups with isomorphic profinite completions have the same sets of isomorphism types of finite quotients. In the following theorem, we see that the converse also holds, implying therefore that the profinite completion of a group G encodes all the information about the finite quotients of G. This result first appeared in [Dix+82].

Theorem 1.3.22. *Let G and H be finitely generated residually finite groups. Then, $\widehat{G} \cong \widehat{H}$ if and only if G and H have the same sets of isomorphism types of finite quotients.*

Proof. As noticed, we have already obtained the "only if" implication. We now sketch the "if" implication. Let G and H be finitely generated residually finite groups with the same sets of isomorphism types of finite quotients: we need to show that G and H have isomorphic profinite completions. For all $n \in \mathbb{N}$, we define

$$G_n = \bigcap \{U \mid U \trianglelefteq_o G, [G : U] \leq n\} \text{ and } H_n = \bigcap \{U \mid U \trianglelefteq_o H, [H : U] \leq n\}.$$

Let us recall that, from Lemma 1.2.17, the number of open normal subgroups of index n is finite in both G and H, hence the two intersections above are finite. This implies that $G_n \trianglelefteq_o G$ and $H_n \trianglelefteq_o H$. One can then easily check that $\widehat{G} \cong \varprojlim G/G_n$ and $\widehat{H} \cong \varprojlim H/H_n$ (see [RZ00], Lemma 1.1.9). Since H/H_n is a finite quotient of H, by hypothesis there exists $K \trianglelefteq_o G$ such that $H/H_n \cong G/K$. The subgroup K is the intersection of some of the open normal subgroups of G with index at most n, therefore $G_n \subseteq K$ and

$$|G/G_n| \geq |G/K| = |H/H_n|.$$

By symmetry, $|H/H_n| \leq |G/G_n|$. This implies that, for every $n \in \mathbb{N}$, we have

$$G/G_n \cong H/H_n.$$

To conclude that $\widehat{G}$ and $\widehat{H}$ are isomorphic, we show that there is a morphism of inverse systems $\alpha : (G/G_n)_{n\in\mathbb{N}} \to (H/H_n)_{n\in\mathbb{N}}$ whose components are group isomorphisms. This implies that $\varprojlim \alpha : \widehat{G} \to \widehat{H}$ is an isomorphism (see [RZ00], Lemma 1.1.5). Let X_n be the set of all isomorphisms from G/G_n to H/H_n, and take an element $\alpha_{n+1} \in X_{n+1}$. Since $\alpha_{n+1}(G_n/G_{n+1}) = H_n/H_{n+1}$, the isomorphism α_{n+1} induces an isomorphism

$$\alpha_n : G/G_n \to H/H_n.$$

Let us define the map

$$\begin{aligned} \varphi_{n,n+1} : X_{n+1} &\to X_n \\ \alpha_{n+1} &\mapsto \alpha_n. \end{aligned}$$

If we denote by $\pi_n^G : G/G_{n+1} \to G/G_n$ and $\pi_n^H : H/H_{n+1} \to H/H_n$ the natural projection homomorphisms, then the following diagram commute:

$$\begin{array}{ccc} G/G_{n+1} & \xrightarrow{\alpha_{n+1}} & H/H_{n+1} \\ {\scriptstyle \pi_n^G}\downarrow & & \downarrow{\scriptstyle \pi_n^H} \\ G/G_n & \xrightarrow[\alpha_n]{} & H/H_n. \end{array}$$

We therefore have an inverse system $\{X_n, \varphi_{n,n+1}\}$ of non-empty finite sets. Its inverse limit is non-empty (this follows from the fact that the proof of Proposition 1.2.2 can be easily adapted to inverse systems of non-empty finite sets), thus there exists some $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \varprojlim X_n$. Such element is an isomorphism of the inverse systems

$$\alpha : (G/G_n)_{n\in\mathbb{N}} \to (H/H_n)_{n\in\mathbb{N}}.$$

Hence, α induces an isomorphism

$$\varprojlim \alpha : \varprojlim G/G_n \cong \widehat{G} \to \varprojlim H/H_n \cong \widehat{H}$$

between the profinite completions of G and H, and this concludes the proof.

□

Chapter 2

Profinite rigidity

It is an old idea to try to understand to what extent a finitely generated group is determined by its finite quotients. Indeed, a natural approach in group theory is to study how a group acts on sets, and in particular on finite ones. The action on finite sets provides information only on the finite images of the group, hence this method brings us to question how to distinguish groups via their finite quotients.

As we have seen in Chapter 1, finite quotients of a finitely generated group are encoded in its profinite completion. We can therefore rephrase the above problem in terms of the latter, and investigate under which assumptions two finitely generated groups have isomorphic profinite completions. It is natural to further restrict our attention to residually finite groups: if there are elements that do not survive in any finite quotient, we certainly cannot hope to recover the group from its profinite completion alone. We recall that, under this assumption, the canonical map $\iota : G \hookrightarrow \widehat{G}$ is an embedding, thus we can regard G as a subgroup of $\widehat{G}$. Interesting surveys on profinite rigidity were recently done by Reid ([Rei13; Rei18]), Rémy (Séminaire Bourbaki n.1221, [Rém24]) and Bridson ([MT], Chapter V, *Profinite Rigidity and Free Groups*.)

2.1 Absolute profinite rigidity

Let us start by giving the main definition of this chapter.

Definition 2.1. A finitely generated residually finite group G is *profinitely rigid (in the absolute sense)* if, for every finitely generated residually finite group H, then $\widehat{G} \cong \widehat{H}$ implies $G \cong H$.

We recall that the profinite completion of a finite group is isomorphic to the finite group itself (Example 1.8). Hence, it is immediately clear that finite groups are profinitely rigid. The first non-trivial example of profinitely rigid groups is given by abelian groups. Recall that finitely generated abelian groups are automatically residually finite by Corollary 1.3.10.

Proposition 2.1.1. *Let G be a finitely generated abelian group. Then, the group G is profinitely rigid.*

Proof. Let H be a finitely generated residually finite group such that $\widehat{H} \cong \widehat{G}$. First of all, we show that H is abelian. Assume, for a contradiction, that H is non-abelian: we can thus find a non-trivial commutator $[x, y]$ in H. Since H is residually finite, this non-trivial element survives in some finite quotient of H, i.e., there is a group homomorphism $\varphi : H \twoheadrightarrow Q$ with Q finite and $\varphi([x, y]) \neq 1_Q$. However, the fact that $\widehat{H} \cong \widehat{G}$ means that H and G have the same sets of isomorphism types of finite quotients (Theorem 1.3.22), and clearly all quotients of G are abelian. This implies that Q must be abelian as well, therefore $\varphi([x, y]) = [\varphi(x), \varphi(y)] = 1_Q$, which gives a contradiction.

From the structure theorem of finitely generated abelian groups, we can now write

$$G \cong \mathbb{Z}^r \times A, \quad H \cong \mathbb{Z}^s \times B$$

for some $r, s \in \mathbb{N}$ and A, B finite abelian groups. Let us consider the group G: both the rank r of the infinite cyclic component and the finite abelian group A of the above decomposition can be derived from set of finite quotients of G. Indeed, the rank r can be described as the greatest $n \in \mathbb{N}$ such that, for every prime p, there exists an epimorphism

$$G \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^n.$$

The group A is the largest (by size) finite group T such that, for every prime p, there exists an epimorphism $G \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^r \times T$. Since $\widehat{G} \cong \widehat{H}$, from the two descriptions above it immediately follows that $r = s$ and $A \cong B$, implying that the groups G and H are isomorphic. □

Given a group G, we recall that its *abelianization* G_{ab} is the quotient of G by its commutator subgroup $[G, G]$. Furthermore, we denote by $b_n(G)$ the n^{th} *Betti number* of a group G, which represents the rank of the n^{th} homology group of G.

Corollary 2.1.2. *Let G and H be finitely generated groups such that $\widehat{G} \cong \widehat{H}$. Then, $G_{ab} \cong H_{ab}$ and, in particular, $b_1(G) = b_1(H)$.*

Proof. The groups G and H have the same sets of isomorphism types of finite quotients (Theorem 1.3.22), thus in particular of abelian finite quotients. These are precisely the finite quotients of the abelianizations, hence Theorem 1.3.22 implies that $G_{\text{ab}} \cong H_{\text{ab}}$. Since the first homology group of a group G is precisely the abelianization of G, the result on the first Betti number immediately follows. □

However, as soon as we try to move slightly beyond the class of abelian groups, we start to struggle: profinite rigidity does not even hold for virtually cyclic groups.

Theorem 2.1.3 (Baumslag [Bau74])**.** *There exist two non-isomorphic, virtually cyclic groups G_1 and G_2 such that $\widehat{G_1} \cong \widehat{G_2}$.*

Proof. Baumslag actually proves a more precise result [Bau74]:

($*$) Let F be a finite cyclic group with an automorphism of order n, where n is different from $1, 2, 3, 4$ and 6. Then, there are at least two non-isomorphic cyclic extensions of F, say G_1 and G_2, with $\widehat{G_1} \cong \widehat{G_2}$.

Recall that the automorphism group of a cyclic group of order k is an abelian group of order $\phi(k)$, where ϕ is the Euler function. We can thus take F to be cyclic of order 11, so that it has an automorphism of order 5.

We give a sketch of the proof of ($*$). Let us consider a cyclic group F of order m generated by an element a, and assume that F admits an automorphism α of order n, with n different from $1, 2, 3, 4$ and 6. Let $r \in \mathbb{Z}$ be such that $\alpha(a) = a^r$. Using the fact that $\phi(n) > 2$, one can find $\ell \in \mathbb{Z}$ with $(\ell, n) = 1$ such that

$$\alpha^\ell \neq \alpha \quad \text{and} \quad \alpha^\ell \neq \alpha^{-1}.$$

Let G_1 be the split extension of F by an infinite cyclic group which induces the automorphism α on F, and let G_2 be the split extension of F by an infinite cyclic group which induces α^ℓ on F. Explicitly, we can present the groups G_1 and G_2 as

$$G_1 = \langle a, b \mid a^m = 1, b^{-1}ab = a^r\rangle, \;\; G_2 = \langle a, c \mid a^m = 1, c^{-1}ac = a^{r^\ell}\rangle.$$

Assume, for a contradiction, that there exists an isomorphism $\theta : G_1 \to G_2$. Since, by construction, F is the set of elements of finite order in both G_1 and G_2, we get that θ induces an automorphism of F. Hence, we can write $\alpha(a) = a^s$ with $(s, m) = 1$. The quotients G_1/F and G_2/F are both isomorphic to $\mathbb{Z}$, therefore we can also write $\theta(b) = c^\varepsilon a^t$ with $\varepsilon = \pm 1$ and $t \in \mathbb{Z}$. If $\varepsilon = 1$, meaning that $\theta(b) = ca^t$, then

$$\alpha(a^s) = a^{rs} = \theta(a^r) = \theta(bab^{-1}) = (ca^t)a^s(ca^t)^{-1} = ca^sc^{-1} = \alpha^\ell(a^s).$$

Since $(s, m) = 1$, this implies that $\alpha = \alpha^\ell$. Similarly, if $\varepsilon = -1$, we get that $\alpha^{-1} = \alpha^\ell$. These two equalities both contradict the choice of ℓ, proving that G_1 and G_2 are not isomorphic.

In order to show that G_1 and G_2 have isomorphic profinite completions, Bamuslag first proves that $G_1 \times \mathbb{Z} \cong G_2 \times \mathbb{Z}$. He finally concludes by using the following result of Hirshon [Hir69]: if A and B are two groups such that $A \times \mathbb{Z} \cong B \times \mathbb{Z}$, then A and B have the same sets of isomorphism types of finite quotients. □

Following Grunewald and Zalesskii [GZ11], we introduce the definition of *genus* of a group.

Definition 2.2. Let G be a finitely generated residually finite group. The *genus* of G is

$$\mathfrak{g}(G) := \text{IsoClasses}(\{H \mid \widehat{G} \cong \widehat{H}\}),$$

which is the set of isomorphism classes of groups having isomorphic profinite completion as the fixed group G.

This definition is given in analogy with the theory of integral quadratic forms in number theory: in that context, the genus of a quadratic form over $\mathbb{Z}$ is the set of integral equivalence classes of integral quadratic forms which are $\widehat{\mathbb{Z}}$-equivalent to the given form.

Remark 2.1. Using the definition of genus, we can give an equivalent characterization of profinitely rigid groups: a finitely generated residually finite group G is profinitely rigid if and only if its genus is given by the isomorphism class of G itself.

We conclude this section by presenting some examples of how the genus of a group can behave. Let us recall that a group G is said to be *metabelian* if its commutator subgroup $[G, G]$ is abelian. Pickel was able to construct finitely presented metabelian groups with infinite genus [Pic74]. Furthermore, Pickel studied the case of nilpotent groups: he proved that, given a finitely generated nilpotent group G, then its genus $\mathfrak{g}(G)$ consists of a finite number of isomorphism classes of nilpotent groups [Pic71]. Examples of non-isomorphic nilpotent groups of class 2 with isomorphic profinite completions were constructed by Grunewald and Scharkau [GS79].

Grunewald and Zalesskii gave examples of finitely generated virtually free groups of every finite rank which fail to be profinitely rigid [GZ11]. Nevertheless, they were able to prove that the genus of every finitely generated virtually free groups is finite.

A further interesting family of groups is given by *polycyclic-by-finite* groups. Recall that a group G is *polycyclic* if it admits a finite sequence of subgroups

$$\{1_G\} = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \ldots \trianglelefteq H_n = G$$

where H_{i+1}/H_i is a cyclic group for every $0 \leq i \leq n-1$. A group is said to be *polycyclic-by-finite* if it has a polycyclic normal subgroup of finite index. Grunewald, Pickel and Segal proved in [GS78] that the genus of every polycyclic-by-finite group is finite.

Examples of groups of uncountable genus were constructed by Pyber [Pyb04], Nekrashevych [Nek13] and Kionke and Schesler [KS23]. The latter will be the fundamental construction of Chapter 4. Moreover, examples of finitely presented residually finite groups with countably infinite genus were recently exhibited by Bridson, Grunewald and Reid [BRS23]. We will discuss this later in Theorem 2.2.7.

2.1.1 Profinite rigidity and free groups

A central challenge in the context of profinite rigidity is understanding what happens for the family of free groups. This intriguing problem was first posed by Remeslennikov and it can be found in the *Kourovka notebook* ([MK95], Question 5.48).

Open Question 2.1. For $n \geq 2$, let F_n be the free group of rank n. Is F_n profinitely rigid?

In other words, Remeslennikov asked whether every finitely generated residually finite group with the same finite quotients as the free group F_n must be free as well. This challenge is particularly fruitful, as it encourages us to understand how we can detected information about free groups from their finite quotients alone.

Recently, some important advances on Remeslennikov's question have been made. In 2023, Jaikin-Zapirain [Jai23] showed that a finitely generated residually finite group with the same profinite completion as a free group must be *parafree* in the sense of Baumslag [Bau67]. Let us recall that a group G is said to be *residually nilpotent* if, given a non-trivial element $g \in G$, there exists a normal subgroup $N \trianglelefteq G$ such that $g \notin N$ and G/N is a nilpotent group. Then, we say that a group is *parafree* if it is residually nilpotent and its quotients by the terms of its lower central series are the same as those of some free group. The recent result by Jaikin-Zapirain therefore allows us to shift our focus from free groups to a class of groups for which, similarly to finitely generated abelian groups, we can hope to prove a structure theorem which could allow us to finally give an answer to Remeslennikov's question.

Following [Rei13], we now investigate the behaviour of profinite completions of free groups. Analogously to Definition 1.9, given a group G we define its *rank* $d(G)$ as the minimal cardinality of a generating set of G.

Proposition 2.1.4. *Let G be a finitely generated group and, for $n \in \mathbb{N}$, let F_n be the free group of rank n. Assume that there exists a finite quotient Q of G such that $d(G) = d(Q)$. If $\widehat{G} \cong \widehat{F_n}$, then $G \cong F_n$.*

Proof. Since $\widehat{G} \cong \widehat{F_n}$, Theorem 1.3.22 implies that we can regard the group Q a quotient of F_n, implying that $n = d(F_n) \geq d(Q) = d(G)$. Hence, there exists an epimorphism $f : F_n \to G$ which a continuous group homomorphism $\widehat{f} : \widehat{F_n} \to \widehat{G}$ between the profinite completions. The map $\widehat{f}$ is again an epimorphism, since its image contains the dense subgroup G. As $\widehat{G} \cong \widehat{F_n}$, the Hopf property for finitely generated profinite groups (Corollary 1.3.19) implies that $\hat{f}$ is an isomorphism and, consequently, that f is an isomorphism as well. We can therefore conclude that $G \cong F_n$. □

Corollary 2.1.5. *Let G be a finitely generated group and, for $n \in \mathbb{N}$, let F_n be the free group of rank n. Assume that $d(G) = d(G_{ab})$. If $\widehat{G} \cong \widehat{F_n}$, then $G \cong F_n$.*

Proof. The finitely generated abelian group G_{ab} has a finite quotient of rank $d(G_{ab}) = d(G)$. We thus conclude by Proposition 2.1.4 that $G \cong F_n$. □

We can therefore immediately see what happens for finitely generated free groups with isomorphic profinite completions.

Proposition 2.1.6. *Let F and F' be finitely generated free groups. If $\widehat{F} \cong \widehat{F'}$, then $F \cong F'$.*

Proof. If F is a free group of rank n, its abelianization $F_{ab} \cong \mathbb{Z}^n$ has again rank n. Hence, Corollary 2.1.5 implies that $F \cong F'$. □

Let us now consider surface groups: for $g \geq 1$, the group

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1], \dots, [a_g, b_g] \rangle$$

is the fundamental group of a closed, connected orientable surface Σ_g of genus g.

Proposition 2.1.7. *For $g \geq 1$, there exists no free group F such that $\widehat{F} \cong \widehat{\pi_1(\Sigma_g)}$.*

Proof. A standard computation shows that the abelianization of $\pi_1(\Sigma_g)$ is isomorphic to $\mathbb{Z}^{2g}$. Then, both $\pi_1(\Sigma_g)$ and its abelianization have rank $2g$. Since $\pi_1(\Sigma_g)$ is not isomorphic to a free group, Corollary 2.1.5 immediately implies that $\pi_1(\Sigma_g)$ cannot have the same profinite completion as a free group. □

A similar result holds for the class of *right-angled Artin groups*, which are again closely related to free groups. Let Γ be a finite simplicial graph, and let $V = \{v_1, \dots, v_n\}$ and $E \subseteq V \times V$ be the sets of vertices and edges of Γ respectively. The *right-angled Artin group* associated with the graph Γ is given by the presentation

$$A = \langle v_1, \dots, v_n \mid [v_i, v_j] = 1 \text{ for all } i, j \text{ such that } \{v_i, v_j\} \in E \rangle.$$

This class of groups has been receiving growing attention in recent years. We notice that it includes the free groups of finite rank, corresponding to a graph Γ with no edges, and finitely generated free abelian groups $\mathbb{Z}^r$, corresponding to a complete graph Γ on r vertices.

Proposition 2.1.8. *Let A be a right-angled Artin group that is not free. Then, there exists no free group F such that $\widehat{F} \cong \widehat{A}$.*

Proof. Let A be the right-angled Artin group associated with a simplicial graph Γ having $|V| = n$. Then, the abelianization of A is isomorphic to the free abelian group $\mathbb{Z}^n$. Since both A and A_{ab} have the same rank n, the result holds again by Corollary 2.1.5. □

2.1.2 Full-sized profinitely rigid groups

Not only is it difficult to establish profinite rigidity of free groups, but also of finitely generated residually finite groups containing non-abelian free subgroups. Following [Bri+20], we will call these groups *full-sized*.

Let us have a look again at the proof of Proposition 2.1.1 on profinite rigidity of finitely generated abelian groups. The key idea used to show that H has to be abelian was to consider commutators and to work with the group homomorphism $\varphi : H \twoheadrightarrow Q$ given by the hypothesis on residual finiteness. In other words, we use the fact that an abelian group satisfies a *group law*.

Definition 2.3. Let G be a group. We say that G satisfies a *group law* if there exists a word $w(x_1, \ldots, x_n)$ in finitely many free variables such that $w(g_1, \ldots, g_n) = 1_G$ for every choice of elements $g_1, \ldots, g_n \in G$.

The approach seen in Proposition 2.1.1 can be generalized to groups satisfying other kinds of group laws, for instance nilpotent groups (we will see this in more details in Proposition 2.3.1).

However, it is straightforward to see that a full-sized group cannot satisfy a law. Under this assumption, it is thus not possible to use the same technique, and it has been difficult to establish the existence of profinitely rigid groups that do not satisfy a group law. The first example of full-sized profinitely rigid groups was given by Bridson, McReynolds, Reid and Spitler in 2020.

Theorem 2.1.9 (Bridson, McReynolds, Reid and Spitler [Bri+20], [Bri+21])**.** *There exist arithmetic lattices in* $\mathrm{PSL}(2, \mathbb{C})$ *and in* $\mathrm{PSL}(2, \mathbb{R})$ *that are profinitely rigid in the absolute sense.*

Let us recall that the group of orientation-preserving isometries of the real hyperbolic space is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$ in dimension 2 and to $\mathrm{PSL}(2, \mathbb{C})$ in dimension 3. A *lattice* in $\mathrm{PSL}(2, \mathbb{R})$, or in $\mathrm{PSL}(2, \mathbb{C})$, is a discrete subgroup of finite covolume. Given a lattice Γ of $\mathrm{PSL}(2, \mathbb{C})$, then the quotient $\mathbb{H}^3_{\mathbb{R}}/\Gamma$ is a hyperbolic 3-dimensional orbifold, which is a manifold if and only if Γ is torsion-free. In this case, Γ is precisely the fundamental group of $\mathbb{H}^3_{\mathbb{R}}/\Gamma$. Analogous results hold for lattices in $\mathrm{PSL}(2, \mathbb{R})$. The groups which are shown to be profinitely rigid in the absolute sense include:

- Bianchi's group $\mathrm{PSL}(\mathbb{Z}[\omega])$, where ω denotes a primitive cubic root of unity.
- The fundamental group $\pi_1(M_w)$ of the Weeks manifold M_w, which is the unique closed orientable hyperbolic 3-manifold of minimal volume among such 3-manifolds. This is an example of *uniform* lattice, meaning that the quotient $\mathbb{H}^3_{\mathbb{R}}/\Gamma$ is compact.

- A finite collection of triangle groups $\Delta(p, q, r)$ in $\mathrm{PSL}(2, \mathbb{R})$, which are the index-2 orientation-preserving subgroups of the reflection group associated to a hyperbolic triangle with interior angles $\pi/p, \pi/q$ and π/r for $p, q, r \in \mathbb{N}$.

We present a sketch of the proof regarding lattices in $\mathrm{PSL}(2, \mathbb{C})$. The key idea is to work with the so-called *representation rigidity*: the first two lattices above have, up to conjugacy, few irreducible representations into $\mathrm{PSL}(2, \mathbb{C})$, and these representations derive from the arithmetic structure of the lattice itself (see the introduction of [Bri+20]).

The first step of the proof consists in studying the *Zariski-dense representations* $\Gamma \to \mathrm{PSL}(2, \mathbb{C})$. If we denote by $q : \mathrm{SL}(2, \mathbb{C}) \twoheadrightarrow \mathrm{PSL}(2, \mathbb{C})$ the canonical quotient map, we say that a subgroup $H \leq \mathrm{PSL}(2, \mathbb{C})$ is Zariski-dense in $\mathrm{PSL}(2, \mathbb{C})$ if its preimage $q^{-1}(H)$ is Zariski-dense in $\mathrm{SL}(2, \mathbb{C})$. Informally, we say that a lattice Γ is *Galois-rigid* if the number of Zariski-dense representations $\Gamma \to \mathrm{PSL}(2, \mathbb{C})$ is the fewest possible ([Bri+20], Definition 4.4). Let Λ be a finitely generated, residually finite group with $\widehat{\Lambda} \cong \widehat{\Gamma}$: thanks to the aforementioned Galois-rigidity of the lattices considered, using arguments coming from the arithmetic structure of the lattices one can always construct a Zariski-dense representation $\rho : \Lambda \to \mathrm{PSL}(2, \mathbb{C})$ whose image is contained in Γ. To prove profinite rigidity, it then suffices to prove that ρ is injective and that its image in $\mathrm{PSL}(2, \mathbb{C})$ is exactly equal to the lattice Γ. The proof of the latter relies heavily on the topological properties of certain hyperbolic 3-manifolds, and it cannot be easily extended to other cases. Once we have that the image of the representation is given by Γ, it is not difficult to conclude: indeed, ρ induces an epimorphism $\hat{\rho} : \widehat{\Gamma} \to \widehat{\Gamma}$, and the Hopf property for profinite groups (Corollary 1.3.19) implies that $\hat{\rho}$, and thus ρ, are isomorphisms.

For the moment, these results on absolute profinite rigidity are known only for finitely many lattices in $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$. Nevertheless, Bridson highlights that it seems reasonable to pose the following conjecture (see [MT], p.236).

Conjecture 2.1. Every lattice in $\mathrm{PSL}(2, \mathbb{R})$ and $\mathrm{PSL}(2, \mathbb{C})$ is profinitely rigid in the absolute sense.

2.2 Relative profinite rigidity

In many cases, we are interested in establishing profinite rigidity within a certain class of groups, thus speaking of *relative profinite rigidity*. In other words, we impose additional conditions on the group H whose profinite completion is compared to that of the fixed group G. In this situation, we will speak about the genus of the group G *among* the chosen class of groups.

The most common setting is given by finitely presented groups, following Grothendieck's original motivation for his studies on profinite completions. In this context, we ask if it is possible that a finitely presented, residually finite group admits a finitely presented proper subgroup with the same profinite completion. Another interesting direction is given by restricting to groups of geometric origin, for to example fundamental groups of certain 3-dimensional manifolds, relying thus heavily on structures of 3-dimensional topology.

2.2.1 Grothendieck rigidity

Grothendieck proved a striking connection between the representation theory of a finitely generated group and its profinite completion [Gro70]. For a commutative ring A and a group G, we denote by $\mathrm{Rep}_G(A)$ the category of finitely presented A-modules with a G-action. This will be called the *representation category* of G over A: Grothendieck's goal was to recover the group from $\mathrm{Rep}_G(A)$. Notice that a group homomorphism $u : G_1 \to G_2$ induces a functor

$$u^* : \mathrm{Rep}_{G_2}(A) \to \mathrm{Rep}_{G_1}(A).$$

Theorem 2.2.1 (Grothendieck [Gro70])**.** *A morphism $u : G_1 \to G_2$ of finitely generated groups induces an equivalence of categories $u^* : Rep_{G_2}(A) \to Rep_{G_1}(A)$ if and only if $\widehat{u} : \widehat{G_1} \to \widehat{G_2}$ is an isomorphism.*

Having this result, Grothendieck wanted then to investigate what we can say about the morphism u knowing that it induces an isomorphism $\widehat{u} : \widehat{G_1} \to \widehat{G_2}$ between the profinite completions. This led him to pose the following problem.

Question (Grothendieck [Gro70])**.** Is a morphism of finitely presented, residually finite groups $u : G_1 \to G_2$ an isomorphism, if it induces an isomorphism $\widehat{u} : \widehat{G_1} \to \widehat{G_2}$ of the profinite completions?

The restriction on finite presentability goes back to Grothendieck's original motivation for studying profinite completions (see [BG04], Introduction). Indeed, Grothendieck was interested in understanding the connection between the topological and the algebraic fundamental group of a complex projective variety. Let X be a connected, smooth projective scheme over $\mathbb{C}$ and let X^{an} be the associated complex variety. Then, the topological fundamental group of X^{an} can be described as the étale fundamental group of X. As the complex variety X^{an} is compact and locally simply-connected, its topological fundamental group is always finitely presented.

Definition 2.4. Let G be a finitely generated residually finite group and let $u : H \hookrightarrow G$ be the inclusion of a subgroup H in G. We say that (G, H)

is a *Grothendieck pair* if u is not an isomorphism but it induces an isomorphism $\widehat{u} : \widehat{H} \to \widehat{G}$ of the profinite completions. Moreover, we say that the group G is *Grothendieck rigid* if no proper finitely generated subgroup gives a Grothendieck pair.

Finitely generated Grothendieck pairs were exhibited by Platonov and Tavgen [PT90]. Their idea to work with a fibre product associated with a well-chosen group quotient was subsequently employed by Bass and Lubotzky [BL00], who were able to construct a family of pairs of finitely generated residually finite groups where $\widehat{u} : \widehat{G_1} \to \widehat{G_2}$ is an isomorphism while $u : G_1 \to G_2$ is not. Again, these groups were not finitely presented.

A negative answer to Grothendieck's question was finally given by Bridson and Grunewald [BG04], who were the first to provide Grothendieck pairs of finitely presented groups. In the following statement, we mean *hyperbolic* in the sense of Gromov [Gro87], and the *dimension* of a group is its geometric dimension (hence, the group H has a compact, 2-dimensional Eilenberg-Maclane space $K(H, 1)$).

Theorem 2.2.2 (Bridson and Grunewald [BG04])**.** *There exist residually finite, 2-dimensional, hyperbolic groups H and finitely presented subgroups*

$$P \hookrightarrow \Gamma \coloneqq H \times H$$

of infinite index, such that:
(i) P is not abstractly isomorphic to Γ;
(ii) the inclusion $u : P \hookrightarrow \Gamma$ induces an isomorphism $\widehat{u} : \widehat{P} \to \widehat{\Gamma}$ of the profinite completions.

We now focus on giving a sketch of the proof of Theorem 2.2.2. A first, naive approach to the problem would be to consider a short exact sequence of finitely generated groups

$$1 \to N \to G \to Q \to 1$$

with $\widehat{Q} = \{1\}$: if the profinite completion functor were exact, we would immediately get $\widehat{N} \cong \widehat{G}$, allowing us to easily construct Grothendieck pairs. Unfortunately, this is false, but we will see that working with short exact sequences as above is a good starting point.

Let us a consider a finitely presented group Q satisfying $\widehat{Q} = \{1\}$ and $H_2(Q, \mathbb{Z}) = 0$. Since Q is finitely generated, we can certainly construct a short exact sequence

$$1 \to N \to F \to Q \to 1,$$

where F is a finitely generated free group. However, we have no control over the group N, which will in general be an infinitely generated free group. In order to overcome this problem, we employ the so-called *Rips* *construction*: Rips presented an algorithm that, for an arbitrary finitely presented group Q, gives a short exact sequence of groups

$$1 \to N \to H \to Q \to 1,$$

where H is a 2-dimensional hyperbolic group and N is a finitely generated group [Rip82]. A refinement of this construction due to Wise ensures in addition that the group H is residually finite, as needed in our setting [Wis03].

Following the idea of Platonov and Tavgen, it is now useful to introduce the *fibre product* associated to a quotient of groups. Let

$$1 \to N \to H \xrightarrow{\pi} Q \to 1$$

be a short exact sequence of groups. The associated *fibre product* $P \subseteq H \times H$ is defined as

$$P := \{(h_1, h_2) \in H \times H \mid \pi(h_1) = \pi(h_2)\}.$$

Lemma 2.2.3 (0-1-2 Lemma). *If H is finitely generated and Q is finitely presented, then P is finitely generated.*

Proof. [BG04], Lemma 2.1. □

Working with fibre products provides an effective way of constructing Grothendieck pairs. The following theorem is proved by Bass and Lubotzky [BL00] from arguments found in [PT90]. We notice that it is now necessary to impose the condition $H_2(Q, \mathbb{Z}) = 0$ on the group Q.

Theorem 2.2.4 (Platonov and Tavgen). *Let $1 \to N \to H \to Q \to 1$ be a short exact sequence of groups and let $P \subseteq H \times H$ be the associated fibre product. If H is finitely generated, $\widehat{Q} = \{1\}$ and $H_2(Q, \mathbb{Z}) = 0$, then the inclusion $u : P \to H \times H$ induces an isomorphism $\widehat{u} : \widehat{P} \to \widehat{H \times H}$ of the profinite completions.*

Proof. [BL00], Theorem 6.3. □

In order to conclude, it remains to establish when the fibre product constructed is finitely presented. This problem is discussed in details in [Bau+00] by Baumslag, Bridson, Miller and Short. Recall that a discrete group G is said to be of *type F_n* if there exists an Eilenberg-Maclane space $K(G, 1)$ with finite n-skeleton. Being of type F_1 is equivalent to being finitely generated, while being of type F_2 is equivalent to being finitely presented.

Theorem 2.2.5 (1-2-3 Theorem). *Let $1 \to N \to H \to Q \to 1$ be a short exact sequence of groups and let $P \subseteq H \times H$ be the associated fibre product. If N is finitely generated, H is finitely presented and Q is of type F_3, then the group P is finitely presented.*

Proof. [Bau+00], Sections 1 and 2. □

The name of Theorem 2.2.5 comes from the fact that the groups N, H and Q of the statement are assumed to be of type F_1, F_2 and F_3 respectively. Bridson and Grunewald gave explicit examples of groups Q satisfying the necessary properties of the construction. This finally allowed them to exhibit Grothendieck pairs of finitely presented groups.

More recently, Bridson constructed examples of finitely presented residually finite groups Γ containing an infinite sequence of non-isomorphic, finitely presented subgroups $P_n \hookrightarrow \Gamma$ such that the inclusion maps induces isomorphisms of profinite completions [Bri16].

Other results on Grothendieck rigidity

By applying Theorem 2.2.4 to epimorphisms $F \to Q$ from a free group F, Platonov and Tavgen were able to prove that the direct product $F \times F$ of two finitely generated, non-abelian free groups is not Grothendieck rigid [PT90]. Using a similar idea on suitable sequences of quotients $F \to Q_1 \to Q_2 \to \ldots$, it is also possible to construct infinitely many finitely generated groups P such that $P \hookrightarrow F \times F$ induces an isomorphism $\widehat{P} \cong \widehat{F \times F}$. However, such P cannot be finitely presented. This follows from a result of Baumslag and Roseblade [BR84], saying that a finitely presented subgroup of $F \times F$ that maps onto both factors and intersects each non-trivially must have finite index in $F \times F$. We can summarize these results in the following theorem.

Theorem 2.2.6. *Let F be a finitely generated, non-abelian free group.*
(i) There exist infinitely many non-isomorphic, finitely generated groups H such that the inclusion $u : H \hookrightarrow F \times F$ induces an isomorphism $\widehat{u} : \widehat{H} \to \widehat{F \times F}$.
(ii) There does not exist a finitely presented proper subgroup $H \subseteq F \times F$ such that the inclusion $u : H \hookrightarrow F \times F$ induces an isomorphism $\widehat{u} : \widehat{H} \to \widehat{F \times F}$.

Therefore, the direct product $F \times F$ of two finitely generated, non-abelian free groups is not Grothendieck rigid among finitely generated groups, but it is Grothendieck rigid among finitely presented groups.

We conclude this section by further highlighting the importance of finiteness properties in the context of profinite rigidity. Recently, Bridson, Reid and Spitler provided the first examples of finitely presented groups which are profinitely rigid among finitely presented groups but not among finitely generated groups [BRS23].

Theorem 2.2.7 (Bridson, Reid and Spitler [BRS23])**.** *There exist finitely presented residually finite groups Γ with the following properties:*
(i) $\Gamma \times \Gamma$ is profinitely rigid among all finitely presented residually finite groups.
(ii) There exist infinitely many non-isomorphic, finitely generated groups Λ such

that $\widehat{\Lambda} \cong \widehat{\Gamma \times \Gamma}$.
(iii) If Γ *is as in (ii), then there is an embedding* $\Lambda \hookrightarrow \Gamma \times \Gamma$ *that induces the isomorphism* $\widehat{\Lambda} \cong \widehat{\Gamma \times \Gamma}$.

Notice that, by parts (ii) and (iii) of Theorem 2.2.7, the groups $\Gamma \times \Gamma$ are the first example of groups whose genus $\mathfrak{g}(\Gamma \times \Gamma)$ is countably infinite. Furthermore, the embeddings $\Lambda \hookrightarrow \Gamma \times \Gamma$ as in (iii) provide Grothendieck pairs of finitely generated groups.

The groups Γ in the statement of Theorem 2.2.7 have a geometric origin, similarly to what happens in Theorem 2.1.9. Indeed, they are the fundamental groups of certain type of Seifert fibred space, and they have again particular arithmetic properties.

2.2.2 Profinite rigidity in topology

In this section, we restrict our attention to the class of fundamental groups of 3-dimensional manifolds, called 3-*manifold groups*. We refer to [AFW15] for a complete survey on such groups.

We notice that finitely generated 3-manifold groups are automatically residually finite (by [Hem87] and the Perelman's Geometrization Theorem) and finitely presented (by [Sco73]).

We start with a construction by Hempel [Hem14], who provided examples of compact 3-manifolds which cannot be distinguished by the finite quotients of their fundamental groups. In other words, Hempel was able to exhibit 3-manifold groups which are not profinitely rigid among the class of 3-manifold groups.

Example 2.1. There exist two closed Seifert fibred spaces M_1 and M_2 that are not homeomorphic but satisfy $\widehat{\pi_1(M_1)} \cong \widehat{\pi_1(M_2)}$. Let us quickly have a look at the construction of such spaces by Hempel [Hem14].

Let S be a closed orientable surface of genus $g \geq 2$, let $\varphi : S \to S$ be a periodic, orientation-preserving homeomorphism of S and let k be a relatively prime to the order of φ. We denote by M_φ and M_{φ^k} the mapping torus of φ and φ_k respectively, and define the groups $\Gamma_\varphi = \pi_1(M_\varphi)$ and $\Gamma_{\varphi^k} = \pi_1(M_{\varphi^k})$. Hempel proved that, in general, these groups will not be isomorphic. However, inspired by Baumslag (Theorem 2.1.3), he showed that that they have isomorphic profinite completions by proving that $\Gamma_\varphi \times \mathbb{Z} \cong \Gamma_{\varphi^k} \times \mathbb{Z}$.

Recently, Wilkes was able to prove that Hempel's examples are the only failures of profinite rigidity among all closed orientable Seifert fibred 3-manifolds [Wil17].

Theorem 2.2.8 (Wilkes [Wil17]). *Let* M *be a closed Seifert fibred space with infinite fundamental group. Then,* $\pi_1(M)$ *is profinitely rigid among all finitely* *generated 3-manifold groups unless M is as in Example 2.1, and the failure is precisely given by the construction in Example 2.1. In this case, $\pi_1(M)$ has infinite genus among all finitely generated 3-manifold groups.*

Nevertheless, there are other examples of 3-manifold groups that cannot be distinguished by their finite quotients alone. For instance, building on Stebe's construction of pairs of non-isomorphic groups with the same profinite completions [Ste72], Funar exhibited infinite families of Sol manifolds with the same finite quotients [Fun13].

If we restrict to hyperbolic 3-dimensional manifolds, we have the following recent result.

Theorem 2.2.9 (Liu [Liu23])**.** *Let M be a finite-volume hyperbolic 3-manifold group. Then, $\pi_1(M)$ has finite genus among all finitely generated 3-manifold groups.*

We may now ask if there are known examples of 3-manifold groups which are profinitely rigid among all finitely generated 3-manifold groups. Positive answers were given in some specific cases.

Theorem 2.2.10 (Bridson, Reid and Wilton [BRW17])**.** *The fundamental group of every once-punctured torus bundle over the circle is profinitely rigid among all finitely generated 3-manifold groups.*

We recall that the exterior $\mathbb{S}^3 \setminus \mathcal{K}$ of a knot $\mathcal{K} \subseteq \mathbb{S}^3$ is a compact, orientable 3-manifold whose fundamental group is called the *knot group* of $\mathcal{K}$.

Theorem 2.2.11 (Bridson and Reid [BR20])**.** *The figure-eight knot group is profinitely rigid among all finitely generated 3-manifold groups.*

Furthermore, regarding Grothendieck rigidity, we finally have a complete result concerning 3-manifold groups.

Theorem 2.2.12 (Sun [Sun23])**.** *All finitely generated 3-manifold groups are Grothendieck rigid.*

Theorem 2.2.12 generalizes previous partial results. Long and Reid proved that all fundamental groups of closed geometric 3-manifolds and all fundamental groups of finite volume hyperbolic 3-manifold groups are Grothendieck rigid [LR11]. Subsequently, Boileau and Friedl proved that Grothendieck rigidity also holds for all fundamental groups of compact, connected, orientable, irreducible 3-manifolds with empty or tori boundary [BF19].

Sun's proof of Theorem 2.2.12 is independent from the proofs of the above partial results. Indeed, the key idea employed by Sun is to give a characterization of *separability* of subgroups of 3-manifold groups.

Definition 2.5. Let G be a group and let H be a subgroup of G. We say that H is *separable* in G if, for every $g \in G \setminus H$, there exists a group homomorphism $\varphi : G \to Q$, with Q finite, such that $\varphi(g) \notin \varphi(H)$. Furthermore, a group G is *LERF (locally extended residually finite)* if every finitely generated subgroup of G is separable in G.

The following lemma, first stated in [LR11] and then in a slightly weaker version in [Sun23], highlights the fundamental role that separability plays in the context of Grothendieck rigidity. We follow the proof of [Sun23].

Lemma 2.2.13. *Let G be a group and let H be a proper subgroup of G that is separable in G. Then, (G, H) is not a Grothendieck pair.*

Proof. Since H is a proper subgroup of G, there exists $g \in G \setminus H$. Moreover, H is separable in G, hence there exists a finite index normal subgroup $N \trianglelefteq_f G$ such that the quotient map $\varphi : G \to G/N$ satisfies $\varphi(g) \notin \varphi(H)$. Let us denote by $i : H \hookrightarrow G$ the inclusion map, by $\bar{i} : H/N \cap H \to G/N$ the induced homomorphism on the quotients and by $\widehat{i} : \widehat{H} \to \widehat{G}$ the induced homomorphism between the profinite completions. Furthermore, let $q_M : \widehat{H} \to H/M$ for $M \trianglelefteq_f H$ and $p_N : \widehat{G} \to G/N$ for $N \trianglelefteq_f G$ be the projections. Recall that we can assume such projections to be surjective (see [RZ00], Proposition 1.1.10). We can now write the following commutative diagram:

$$
\begin{array}{ccc}
\widehat{H} & \xrightarrow{\ \hat{i}\ } & \widehat{G} \\
{\scriptstyle q_{N\cap H}}\big\downarrow & & \big\downarrow{\scriptstyle p_N} \\
H/N \cap H & \xrightarrow[\ \bar{i}\ ]{} & G/N
\end{array}
$$

Suppose that $\widehat{i} : \widehat{H} \to \widehat{G}$ is an isomorphism. From the surjectivity of the projection $p_N : \widehat{G} \to G/N$, we get that the composition map $p_N \circ \widehat{i} : \widehat{H} \to G/N$ is surjective as well. However, $\bar{i} \circ \pi_{N \cap H}$ is not surjective by the hypothesis on separability, as gN does not lie in the image of $\bar{i}$. We thus get a contradiction, proving that (G, H) is not a Grothendieck pair. □

Corollary 2.2.14. *Let G be a LERF group. Then, G is Grothendieck rigid.*

Proof. Recall that a group is Grothendieck rigid if no proper finitely generated subgroup gives a Grothendieck pair. Since being LERF also only concerns finitely generated subgroups, the result follows immediately from Lemma 2.2.13. □

Corollary 2.2.14 therefore implies that all LERF 3-manifold groups are Grothendieck rigid. In particular, given Agol's result on the fact that all hyperbolic 3-manifold groups are LERF [Ago13], we obtain an alternative proof of the Grothendieck rigidity of such groups, which was the main result of [LR11].

Nevertheless, as Sun points out, there are many 3-manifold groups that fail to be LERF. Hence, in order to complete the proof of Theorem 2.2.12, it remains to prove that every non-separable subgroup of a 3-manifold group does not give a Grothendieck pair. We briefly summarize his line of argument.

Let us consider a 3-manifold M with finitely generated fundamental group $G = \pi_1(M)$, and let $i : H \hookrightarrow G$ be a non-separable proper subgroup of G. Then, Sun proved in [Sun20] that there exists a subgroup $H_0 \leq H \leq G$ such that the normalizer $N_H(H_0)$ of H_0 in H satisfies $[N_H(H_0) : H_0] < \infty$, while the normalizer $N_G(H_0)$ of H_0 in G satisfies $[N_G(H_0) : H_0] = \infty$. Through tools of profinite groups and profinite Bass-Serre trees, Sun was able to show that the above behaviour of the normalizers passes to the profinite completions of the groups. Using this, he concluded that the induced map $\widehat{i} : \widehat{H} \to \widehat{G}$ is an isomorphism.

2.3 Profinite invariants

Up to now, we have been interested in understanding which groups can be completely recovered by their profinite completions alone, hence by their sets of isomorphisms types of finite quotients. Another interesting challenge concerns properties which can, or cannot, be detected from the profinite completion of a group.

Definition 2.6. A property $\mathcal{P}$ of groups is a *profinite invariant* if, for every two finitely generated, residually finite groups G and H with $\widehat{G} \cong \widehat{H}$, then G satisfies $\mathcal{P}$ if and only if H satisfies $\mathcal{P}$.

A first example of profinite invariant is given by the property of being abelian. This immediately follows from the first part of the proof of Proposition 2.1.1. Furthermore, the abelianization of a group and its first Betti number are profinite invariants (Corollary 2.1.2). As mentioned in Section 2.1.2, the key idea when working with abelian groups is to use the fact that they satisfy a *group law* (which, in this case, is simply the commutator law). We can use the same approach in order to prove the following result.

Proposition 2.3.1. *Being virtually nilpotent is a profinite invariant.*

Proof. Let G and H be two finitely generated residually finite groups satisfying $\widehat{G} \cong \widehat{H}$, and assume that G is virtually nilpotent. Then, there exists a finite index subgroup $K \leq_f G$ which is nilpotent. By Corollary 1.3.21, there is a one-to-one correspondence φ between the set of finite index subgroups of G and the set of finite index subgroups of H. Let us set $U = \varphi(K)$:

then, U is a finite index subgroup of H which satisfies $\widehat{K} \cong \widehat{U}$. In order to conclude, it suffices to show that U is a nilpotent group.

Since the K is a nilpotent subgroup of G, there exists some $c \in \mathbb{N}$ such that all the $(c+1)$-fold commutators in K are trivial, that is,

$$[x_1, \dots, x_{c+1}] = [[\dots[x_1, x_2], x_3] \dots, x_{c+1}] = 1_K$$

for all $x_1, \dots, x_{c+1} \in K$. The smallest c for which the above condition holds is the nilpotency class of the nilpotent group. Assume, for a contradiction, that there exists a non-trivial $(c+1)$-fold commutator $[x_1, \dots, x_{c+1}]$ in U. Since U is residually finite, there exists a group homomorphism $\pi : U \twoheadrightarrow Q$, with Q finite, such that $\pi([x_1, \dots, x_{c+1}]) \neq 1_Q$. Furthermore, the fact that $\widehat{K} \cong \widehat{U}$ implies, from Theorem 1.3.20, that K and U have the same sets of isomorphism types of finite quotients. Let us now recall that every group quotient of a nilpotent group of class c is nilpotent of class at most c. Indeed, if K is nilpotent of class c and $\{N_i\}_{i=1}^{c}$ is a central series of K, given $N \trianglelefteq K$ we have that $\{N_i N/N\}_{i=1}^{c}$ is a central series for G/N. Therefore, the group Q is nilpotent of class at most c and

$$\pi([x_1, \dots, x_{c+1}]) = [\pi(x_1), \dots, \pi(x_{c+1})] = 1_Q,$$

which gives a contradiction. Hence, U is a finite index nilpotent subgroup of H. We can therefore conclude that H is a virtually nilpotent group and that being virtually nilpotent is a profinite invariant. □

A celebrated theorem by Gromov tells us that a finitely generated group is virtually nilpotent if and only if it has polynomial growth [Gro81]. We refer to Section 3.2.1 for a brief introduction to growth types of groups.

Corollary 2.3.2. *Having polynomial growth is a profinite invariant.*

Proof. It immediately follows from Proposition 2.3.1 and Gromov theorem on polynomial growth [Gro81]. □

Another example of profinite invariant is given by the rate of *subgroup growth* of a group. Given a finitely generated group G and $n \in \mathbb{N}$, we denote by $a_n(G)$ the number of subgroups of index n in G. The *subgroup growth* of the group G is the asymptotic behaviour of the sequence $\{a_n(G)\}_{n \in \mathbb{N}}$. An extensive study on this topic was done by Lubotzky and Segal in [LS12]. The fact that the rate of subgroup growth is a profinite invariant is a direct consequence of Corollary 1.3.21.

A more surprising profinite invariant is given by *largeness*. The following definition was introduced by Gromov in [Gro82].

Definition 2.7. A group is *large* if it admits a finite index subgroup that has a free non-abelian quotient.

In order to understand the behaviour of largeness with respect to profinite completions, one needs to employ the useful characterization of large groups given by Lackenby.

Theorem 2.3.3 (Lackenby [Lac05]). *Let G be a finitely presented group. Then, the following are equivalent:*

1. *G is large;*
2. *there exists a nested sequence $G_1 \geq G_2 \geq \dots$ of finite index subgroups of G, each normal in G_1, such that:*
   (i) G_i/G_{i+1} *is abelian for all* $i \geq 0$*;*
   (ii) $\lim_{i\to\infty} \log([G_i : G_{i+1}])/[G : G_i] = \infty$*;*
   (iii) $\limsup_i d(G_i/G_{i+1})/[G : G_i] > 0$*, where* $d(\cdot)$ *denotes the rank of the group.*

As a consequence, we get that largeness is a profinite invariant among finitely presented groups.

Theorem 2.3.4 (Lackenby [Lac10]). *Let G and H be finitely presented groups with $\widehat{G} \cong \widehat{H}$. Then, G is large if and only if H is large.*

Proof. Let us assume that the group G is large. From Theorem 2.3.3, we get that G contains a nested sequence $\{G_i\}_{i\in I}$ of finite index subgroups, each satisfying the conditions above. Let us fix an index i and let $\widetilde{G}_i$ be the intersection of all the conjugates of G_i in G. We denote by φ the one-to-one correspondence between the set of finite index subgroups of G and the set of finite index subgroups of H given by Corollary 1.3.21. Set $K_i = \varphi(G_i)$ and $\widetilde{K}_i = \varphi(\widetilde{G}_i)$: then, $\widetilde{K}_i$ is a normal subgroup of K which is contained in K_i. Furthermore, we get an isomorphism

$$G/\widetilde{G}_i \to K/\widetilde{K}_i$$

sending $G_j/\widetilde{G}_i$ to $K_j/\widetilde{K}_i$ for every $j \leq i$. We have that $G_i \trianglelefteq G_1$ if and only if $G_i/\widetilde{G}_i \trianglelefteq G_1/\widetilde{G}_i$, and this implies that $K_i \trianglelefteq K_1$. The isomorphism

$$G/\widetilde{G}_{i+1} \to K/\widetilde{K}_{i+1}$$

sends $G_i/\widetilde{G}_{i+1}$ to $K_i/\widetilde{K}_{i+1}$ and $G_{i+1}/\widetilde{G}_{i+1}$ to $K_{i+1}/\widetilde{K}_{i+1}$. Therefore, the quotient K_i/K_{i+1} is isomorphic to G_i/G_{i+1} and we can conclude that the group K is large by Theorem 2.3.3. □

In Chapter 4, we will see that another interesting example of profinite invariant is given by *uniform amenability* (Theorem 4.4.4). This result was recently shown by Kionke and Schesler [KS23].

2.3.1 Non-profinite invariants

We now focus on properties that fail to be profinite invariant. The list of such properties has been expanding recently, and it includes several fundamental notions of geometric group theory.

- *Kazhdan's property (T)* was introduced by Kazhdan in the 1960's and it is characterized by properties of the unitary representations of locally compact groups [Kaz67]. It is a fundamental concept which is widely studied in various areas of mathematics, such as group theory, ergodic theory, differential geometry, operator algebras, combinatorics and computer science. For instance, it can be equivalently formulated in terms of fixed point properties of geometric actions. We refer to [BLV08] for a complete study on such notion. The fact that Kazhdan's property (T) is not a profinite invariant was proved by Aka [Aka12b], who was able to construct examples of non-isomorphic arithmetic groups with isomorphic profinite completions. In order to do so, he employed quadratic forms theory, their Clifford algebras and their spin groups. Furthermore, Aka proved that there exist lattices in semi-simple Lie groups which have the same completion but where the ambient Lie groups do not have the same rank [Aka12a]. As a consequence, we get that the rank of the ambient Lie group is not a profinite invariant as well.

- A consequence of Kazhdan's property (T) (*cf.* [Wat81]) is given by *property (FA)*, which is the property of having global fixed points for all actions by automorphisms on trees. This notion was introduced by Serre [Ser73]. In 2022, Cheetham-West, Lubotzky, Reid and Spitler showed that property (FA) is not a profinite invariant [Che+22].

- A group G is said to be *conjugacy separable* if every conjugacy class is closed in the profinite topology on G. Cotton-Barratt and Wilton proved that there exist two finitely presented residually finite groups with isomorphic profinite completions such that one is conjugacy separable and the other is not [CW12]. Hence, being conjugacy separable is not a profinite invariant.

- The classical *finiteness properties* are not profinite invariants (Lubotzky [Lub14]). Let us recall that a group G is of type F_n if there exists an Eilenberg-MacLane space $K(G,1)$ with finite n-skeleton. Being of type F_1 is equivalent to being finitely generated, while being of type F_2 is equivalent to being finitely presented. More specifically, Lubotzky proved that, for every $r, s \in \mathbb{N}$, there exist two finitely generated residually finite groups Γ_r and Γ_s such that Γ_r is of type F_r (and not F_{r+1}), Γ_s is of type F_s (and not F_{s+1}) and $\widehat{\Gamma}_r \cong \widehat{\Gamma}_s$. His proof relies on properties of arithmetic groups over positive characteristic function fields. Using similar ideas, he was able to show that being *residually solvable, residually nilpotent, residually p* and *torsion-free* are not profinite invariants. Furthermore, he proved hat *cohomological dimension* of a group is not a profinite property.

- *Bounded cohomology* is not a profinite invariant (Echtler and Kammeyer [EK24]). For a given discrete group Γ, let us denote by $H_b^*(\Gamma, \mathbb{R})$ its bounded cohomology with real coefficients. We refer to [Fri17] for a complete exposition on the topic. Echtler and Kammeyer were able to exhibit two spinor groups Γ and Λ with isomorphic profinite completions such that $H_b^2(\Gamma, \mathbb{R}) \cong \mathbb{R}$ and $H_b^2(\Lambda, \mathbb{R}) \cong 0$.

- Kammer and Sauer proved that *higher ℓ^2-Betti numbers* are not profinite invariants [KS20]. Their counterexamples build on the idea of Aka [Aka12b]: they constructed S-arithmetic groups Γ_1 and Γ_2 with isomorphic profinite completions such that $b_k^{(2)}(\Gamma_j) > 0$ if and only if $k = n_j$ for $n_1, n_2 \in \mathbb{N}$ with $n_1 \neq n_2$. Hence, for every $n \geq 2$, not even the n^{th} ℓ^2-Betti numbers being non-zero is a profinite invariant. However, in all examples constructed by Kammer and Sauer, they had that $n_1 \equiv n_2 \bmod 2$. This led them to investigate another possible profinite invariant: let us recall that the *Euler characteristic* of a group Γ can be computed as the alternating sum of the ℓ^2-Betti numbers [Lüc02], that is,
$$\chi(\Gamma) = \sum_{k \geq 0} (-1)^k b_k^{(2)}(\Gamma).$$
For arithmetic groups, ℓ^2-Betti numbers are known to be non-zero in at most one degree, and such non-zero ℓ^2-Betti number appears only when working with semisimple groups of fundamental rank zero. Inspired by this fact, Kammeyer, Kionke, Raimbault and Sauer were able to show that the *sign of the Euler characteristic* of arithmetic groups with the *congruence subgroup property* is a profinite invariant [Kam+20]. We refer to [Sur03] for an introduction to such congruence subgroup property. However, the profinite invariance does not extend to the Euler characteristic itself. Furthermore, if an arithmetic group Γ has vanishing Euler characteristic, we can introduce another invariant called *ℓ^2-torsion* $\rho^{(2)}(\Gamma)$ (see [Lüc02], Chapter 3). From the point of view of profinite rigidity, such ℓ^2-torsion behaves similarly to the Euler characteristic, meaning that its sign is a profinite invariant among arithmetic groups with the congruence subgroup property. In contrast to what we have said up to now, the first ℓ^2-Betti number is a profinite invariant among finitely presented groups. This was shown by Bridson, Conder and Reid ([BCR16], Corollary 3.3) as a consequence of Lück approximation theorem [Lüc94].

- *Amenability* is not a profinite invariant (Kionke and Schesler [KS23]). We introduce this property in Chapter 3, highlighting several different characterizations. We will then focus on the proof that amenability is not a profinite invariant in Chapter 4.

2.3.2 Quasi-morphisms

We conclude our discussion on profinite invariants with an open question regarding *quasi-morphisms*, which are closely related to bounded cohomology. An introduction to the topic can be found in [Fri17], Chapter 2.

Definition 2.8. Let G be a discrete group. A map $\varphi : G \to \mathbb{R}$ is a *quasi-morphism* on G if there exists a constant $D \geq 0$ such that

$$|\varphi(gh) - \varphi(g) - \varphi(h)| \leq D$$

for all $g, h \in G$. The least $D \geq 0$ for which the above inequality holds is called the *defect* of φ and it is denoted by $D(\varphi)$. The $\mathbb{R}$-vector space of quasi-morphisms on G is denoted by $\mathrm{QM}(G)$.

Definition 2.9. Let G be a discrete group. A quasi-morphism $\varphi : G \to \mathbb{R}$ is called *trivial* if there exists a group homomorphism $\varphi' : G \to \mathbb{R}$ such that

$$\sup_{g \in G} |\varphi'(g) - \varphi(g)| < \infty.$$

The subspace of $\mathrm{QM}(G)$ of all trivial quasi-morphisms on the group G is denoted by $\mathrm{QM}_0(G)$.

In other words, a quasi-morphism is non-trivial it is not at bounded distance from a group homomorphism. It is interesting to investigate the space $\mathrm{QM}(G)/\mathrm{QM}_0(G)$ of non-trivial quasi-morphisms. More specifically, in our context of profinite rigidity, the following open question was recently pointed out by Echtler and Kammeyer [EK24].

Open Question 2.2. Is the existence of non-trivial quasi-morphisms a profinite invariant?

In order to approach this problem, it is convenient to recall the strict connection between quasi-morphisms on a group G and the bounded cohomology of G with real coefficients. Let us consider the *comparison map* in degree two

$$c^2 : H_b^2(G, \mathbb{R}) \to H^2(G, \mathbb{R})$$

and denote by $EH_b^2(G, \mathbb{R})$ its kernel. The following result tells us that such kernel detects the non-trivial quasi-morphisms of a group.

Theorem 2.3.5. *Let G be a discrete group. Then, there is a canonical homomorphism*

$$\mathrm{QM}(G)/\mathrm{QM}_0(G) \cong EH_b^2(G,\mathbb{R}).$$

Proof. [Löh20], Theorem 2.5.15. □

Therefore, a way to tackle Question 2.2 consists in investigating how bounded cohomology behaves in the several different constructions of non-isomorphic groups with isomorphic profinite completions. For instance, in the construction of the finitely presented Grothendieck pairs by Bridson and Grunewald (Theorem 2.2.2), we work with a hyperbolic group H: in this case, it was shown by Epstein and Fujiwara the dimension of $H_b^2(H,\mathbb{R})$ as a vector space over $\mathbb{R}$ is the cardinal of the continuum [EF97].

For lattices in higher rank linear Lie groups, Burger and Monod showed that the comparison map in degree two is injective ([BMI01], Theorem 21). Thus, in this case we have no non-trivial quasi-morphisms and constructions with such higher rank Lie groups do not help us in answering Question 2.2. However, the situation is different for lattices in rank one Lie groups. One of the key ingredient to build pairs of non-isomorphic lattices in Lie groups with isomorphic profinite completions is given by the aforementioned *congruence subgroup property*. An old, but still unresolved, conjecture by Serre states that lattices in rank one Lie groups do not satisfy such property [Ser70]. Nevertheless, Lubotzky pointed out that lattices in the rank one groups $\mathrm{Sp}(n,1)$ and $F_{4(-20)}$ behave in many ways like higher rank lattices, and this suggest that they do indeed satisfy the congruence subgroup property [Lub05]. If this were the case, Echtler and Kammeyer showed that it would be possible to exhibit two arithmetic lattices Γ, Λ in $F_{4(-20)}$ with $\widehat{\Gamma} \cong \widehat{\Lambda}$, but $Eh_b^2(\Gamma,\mathbb{R}) \neq 0$ while $EH_b^2(\Lambda,\mathbb{R}) = 0$. This would imply that the existence of non-trivial quasi-morphisms is not a profinite invariant.

Chapter 3

Amenability

This chapter focuses entirely on the notion of amenability. The first definition of amenable groups was given by von Neumann in 1929, who coined the German name "meßbar" (translated in English as "measurable") during his studies on the Banach-Tarski Paradox [Neu29]. The term "amenable" was later introduced by Day [Day49].

Amenability has been playing a central role in many different areas of mathematics: this lies on the fact that amenable groups admit several different characterizations, e.g., via invariant means, Følner sets, decomposition properties, fixed point properties or random walks on groups. A detailed treatment of this class of groups can be found in [CC10] and [Löh17].

3.1 Amenability via means

We start by introducing amenable groups via the existence of invariant means. First of all, given a set X, we denote by $\ell^\infty(X)$ the set of all bounded functions from X to $\mathbb{R}$. Recall that $\ell^\infty(X)$ with pointwise addition and scalar multiplication is a real vector space. Moreover, it becomes a Banach space for the norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{x \in X} |f(x)|.$$

Definition 3.1. Let X be a set. A *mean* on X is a linear map $m : \ell^\infty(X) \to \mathbb{R}$ such that:
(i) $m(1) = 1$;
(ii) for every $f \in \ell^\infty(X)$ with $f \geq 0$ pointwise, then $m(f) \geq 0$.
We denote by $\mathcal{M}(X)$ the set of all means on X.

Let G be a group. Then, G acts on the space $\ell^\infty(G)$ as follows:

$$\begin{aligned} G \times \ell^\infty(G) &\longrightarrow \ell^\infty(G) \\ (g, f) &\longmapsto (h \mapsto f(g^{-1}h)). \end{aligned}$$

Definition 3.2. A group G is *amenable* if there exists a G-invariant mean on $\ell^\infty(G)$, i.e., a mean m on G such that $m(g \cdot f) = m(f)$ for all $f \in \ell^\infty(G)$ and all $g \in G$.

Before proceeding with some basic examples of amenable groups, we notice that an equivalent definition of amenability can be given using probability measures. We briefly present the construction and refer to [CC10], §4.1 for details.

Definition 3.3. Let X be a set. A *finitely additive probability measure* on X is a map $\mu : \mathcal{P}(X) \to [0,1]$ such that:
(i) $\mu(X) = 1$;
(ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ for all $A, B \in \mathcal{P}(X)$ with $A \cap B = \varnothing$.

Let us denote by $\mathcal{M}(X)$ the set of all means on X and by $\mathcal{PM}(X)$ the set of all finitely additive probability measure on X. Then, there is a natural bijection between $\mathcal{M}(X)$ and $\mathcal{PM}(X)$:

$$\begin{aligned} \mathcal{M}(X) &\longrightarrow \mathcal{PM}(X) \\ m &\longmapsto (A \mapsto m(\chi_A)), \end{aligned}$$

where we denote by χ_A the characteristic function of the subset $A \subseteq X$. Furthermore, given a group G, this naturally acts on $\mathcal{PM}(G)$ by

$$\begin{aligned} G \times \mathcal{PM}(G) &\to \mathcal{PM}(G) \\ (g, \mu) &\mapsto (A \mapsto \mu(g^{-1}A)). \end{aligned}$$

In the above bijection between $\mathcal{M}(X)$ and $\mathcal{PM}(X)$, we have that a mean m is G-invariant if and only if the associated probability measure is. As a consequence, the following proposition holds.

Proposition 3.1.1. *A group G is amenable if and only if there exists a G-invariant finitely additive probability measure* $\mu : \mathcal{P}(G) \to [0,1]$ *on G.*

Proof. [CC10], Proposition 4.4.4. □

3.1.1 First examples

We now present some basic important examples of (non-)amenable groups.

Proposition 3.1.2. *Every finite group is amenable.*

Proof. Let G be a finite group. Then, it is easily checked that the averaging operator

$$\begin{aligned} \ell^\infty(G) &\longrightarrow \mathbb{R} \\ f &\longmapsto \frac{1}{|G|} \sum_{g \in G} f(g) \end{aligned}$$

is a G-invariant mean on $\ell^\infty(G)$. □

Another important class of groups that satisfies the property of being amenable is given by abelian groups. The proof of the fact that every abelian groups is amenable lies on classical results of functional analysis. Recall that, given a set X, the dual space $(\ell^\infty(X))^*$ is a Banach space for the operator norm $\|\cdot\|$ defined by

$$\|u\| = \sup_{\|x\|_\infty \leq 1} |u(x)|$$

for all $u \in (\ell^\infty(X))^*$. Moreover, the space $(\ell^\infty(X))^*$ can be equipped with the *weak-$*$ topology*, which is the smallest topology for which the evaluation map

$$\begin{aligned} \Phi_x : (\ell^\infty(X))^* &\to \mathbb{R} \\ u &\mapsto u(x) \end{aligned}$$

is continuous for every $x \in X$.

Theorem 3.1.3. *Let X be a set. The set $\mathcal{M}(X)$ of all means on X is a convex compact subset of $(\ell^\infty(X))^*$ with respect to the weak-$*$ topology.*

Proof. [CC10], Theorem 4.2.1. □

Let us now consider a group G. The action of G on $\ell^\infty(G)$ is isometric (see [CC10], §4.3) and it induces a G-action by continuous linear maps on the dual $(\ell^\infty(G))^*$. More precisely, we have

$$\begin{aligned} G \times (\ell^\infty(G))^* &\to (\ell^\infty(G))^* \\ (g, u) &\mapsto (x \mapsto u(g^{-1}x). \end{aligned}$$

Furthermore, the set $\mathcal{M}(G) \subseteq (\ell^\infty(G))^*$ is invariant under such G-action.

Proposition 3.1.4. *Let G be a group. The action of G on $\mathcal{M}(G)$ is affine and continuous with respect to the weak-$*$ topology on $\mathcal{M}(G)$.*

Proof. [CC10], Proposition 4.3.1. □

We recall a fundamental result of functional analysis that we will need when working with abelian groups.

Theorem 3.1.5 (Markov-Kakutani fixed point theorem)**.** *Let V be a locally convex $\mathbb{R}$-vector space (e.g., a normed $\mathbb{R}$-vector space), let $A \times V \to V$ be an action of an abelian group A by continuous linear maps on V and let $M \subseteq V$ be a non-empty convex compact subset of V that is invariant under such action. Then, there exists $m \in M$ such that $g \cdot m = m$ for every $g \in A$.*

Proof. [Pat88], Proposition 0.14. □

Thanks to Theorem 3.1.5, we can finally prove the following.

Theorem 3.1.6. *Every abelian group is amenable.*

Proof. Let G be an abelian group and consider the space $(\ell^\infty(G))^*$ equipped with the weak-$*$ topology. From Theorem 3.1.3, the set $\mathcal{M}(G)$ is a non-empty convex and compact subset of $(\ell^\infty(G))^*$. Moreover, the action of G on $\mathcal{M}(G)$ is affine and continuous by Proposition 3.1.4. By the Markov-Kakutani fixed point theorem (Theorem 3.1.5), the set $\mathcal{M}(G)$ contains a G-fixed point m. Such fixed point m is clearly a G-invariant mean on G. Therefore, we can deduce that the group G is amenable. □

We conclude with a crucial example of a group that fails to be amenable.

Proposition 3.1.7. *The free group F_2 of rank 2 is not amenable.*

Proof. Assume, for a contradiction, that the group F_2 is amenable, that is, it admits an invariant mean m. Let $\{a, b\}$ be a free generating set of F_2, and let $A \subseteq F_2$ be the set of elements of F_2 whose reduced form starts with a non-trivial power of a. Since $F_2 = A \cup a^{-1} \cdot A$, we get

$$\begin{aligned} 1 = m(1) = m(\chi_{F_2}) &\leq m(\chi_A) + m(a^{-1} \cdot \chi_A) \\ &= m(\chi_A) + m(a^{-1} \cdot \chi_A) \\ &= 2m(\chi_A), \end{aligned}$$

which implies that $m(\chi_A) \geq 1/2$. On the other hand, we notice that the sets $A, b \cdot A$ and $b^2 \cdot A$ are disjoint, therefore

$$\begin{aligned} 1 = m(\chi_{F_2}) \geq m(\chi_{A \cup b \cdot A \cup b^2 \cdot A}) &= m(\chi_A) + m(\chi_{b \cdot A}) + m(\chi_{b^2 \cdot A}) \\ &= m(\chi_A) + m(b \cdot \chi_A) + m(b^2 \cdot \chi_A) \\ &= 3m(\chi_A) \\ &\geq \frac{3}{2}, \end{aligned}$$

which gives a contradiction. Thus, the free group F_2 is not amenable.

□

3.1.2 Stability properties of amenable groups

The class of amenable groups satisfies many useful properties: we start by showing that it is closed under the operations of taking subgroups, quotients, extensions and directed ascending unions. Among the consequences, we obtain that amenable groups cannot contain non-abelian free subgroups and that all virtually solvable groups are amenable.

Proposition 3.1.8. *(i) Every subgroup of an amenable group is amenable.*
(ii) Every homomorphic image of an amenable group is amenable.
(iii) Let

$$1 \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to 1$$

be an extension of groups. Then, G is amenable if and only if N and Q are amenable.

Proof. (i) Let G be an amenable group with a G-invariant mean m on $\ell^\infty(G)$ and let $H \leq G$ be a subgroup. Choose a set of representatives $R \subseteq G$ of right cosets of $H \setminus G$ and consider the map $\gamma : G \to H$ with

$$g \in \gamma(g)R$$

for every $g \in G$. Then,

$$\begin{aligned} m' : \ell^\infty(H) &\to \mathbb{R} \\ f &\mapsto m(f \circ \gamma) \end{aligned}$$

is a H-invariant mean on H. Indeed, it is straightforward to see that m' is still a mean, and the invariance comes from the fact that $m'(h \cdot f) = m((h \cdot f) \circ \gamma) = m(h \cdot (f \circ \gamma)) = m(f \circ \gamma) = m'(f)$ for every $h \in H$ and $f \in \ell^\infty(H)$.
(ii) Let G be an amenable group with a G-invariant mean m on $\ell^\infty(G)$ and let $\pi : G \to Q$ be a surjective group homomorphism. Then,

$$\begin{aligned} \overline{m} : \ell^\infty(Q) &\to \mathbb{R} \\ f &\mapsto m(f \circ \pi) \end{aligned}$$

is a Q-invariant mean on Q, where the invariance follows from the fact that π is surjective.
(iii) If G is amenable, then both N and Q are amenable by (i) and (ii). Conversely, assume that N and Q are amenable via the two invariant means m_N and m_Q. We may assume that $N \subseteq G$ and $Q = G/N$. Let $f \in \ell^\infty(G)$ and consider the map

$$\begin{aligned} \widetilde{f} : G/N &\to \mathbb{R} \\ gN &\mapsto m_N(n \mapsto f(gn)). \end{aligned}$$

Then, $\widetilde{f}$ is well defined since m_N is a N-invariant mean, and $\widetilde{f} \in \ell^\infty(G/N)$. Now, let us consider

$$\begin{aligned} m : \ell^\infty(G) &\to \mathbb{R} \\ f &\mapsto m_Q(\widetilde{f}). \end{aligned}$$

Clearly, m is a mean on G. We therefore have to show that m is G-invariant. Let $g \in G$ and $f \in \ell^\infty(G)$: for every $g' \in G$, we have

$$\widetilde{g \cdot f}(g'N) = \widetilde{f}(g^{-1}g'N) = gN \cdot \widetilde{f}(g'N).$$

Hence, $\widetilde{g \cdot f} = gN \cdot \widetilde{f}$. Since m_Q is a Q-invariant mean, we obtain

$$\begin{aligned} m(g \cdot f) &= m_Q(\widetilde{g \cdot f}) = m_Q(gN \cdot \widetilde{f}) \\ &= m_Q(\widetilde{f}) = m(f). \end{aligned}$$

Finally, we obtain that m is a G-invariant mean on G and this implies that the group G is amenable. □

Corollary 3.1.9. *Every group that contains a free subgroup of rank* 2 *is not amenable.*

Proof. This immediately follows from the fact that the free group F_2 of rank 2 is not amenable (Proposition 3.1.7) and that every subgroup of an amenable group is amenable (Proposition 3.1.8). □

Example 3.1. (i) Every non-abelian free group is not amenable.
(ii) The group $\mathrm{SL}_n(\mathbb{Z})$ is not amenable for $n \geq 2$. This follows from the fact that the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the matrices

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

is isomorphic to the free group F_2 of rank 2 ([CC10], Lemma 2.3.2.).

Corollary 3.1.10. *Suppose that G_1 and G_2 are amenable groups. Then, the group $G = G_1 \times G_2$ is amenable.*

Proof. Consider $N = \{(g, 1_{G_2}) \mid g \in G_1\}$: this is a normal subgroup of G isomorphic to G_1. Moreover, the quotient G/N is isomorphic to G_2. We can therefore conclude by Proposition 3.1.8 that the group G is amenable. □

It follows from Corollary 3.1.10 that every direct product of a finite number of amenable groups is amenable. However, this does not hold in general for infinite products (see [CC10], Remark 4.5.7).

Corollary 3.1.11. *Every solvable group is amenable.*

Proof. Let G be a solvable group. Recall that abelian groups are amenable by Proposition 3.1.6: then, by induction on the length of the derived series of G, we can conclude that G is amenable thanks to Proposition 3.1.8. □

Notice that the converse implication does not hold: there are many examples of finite (thus, amenable) groups that are not solvable, such as the symmetric group $\mathrm{Sym}(5)$.

Corollary 3.1.12. *Every nilpotent group is amenable.*

Proof. As every nilpotent group is solvable, from Corollary 3.1.11 we immediately obtain the nilpotent groups are amenable. □

Corollary 3.1.13. *Every virtually amenable group is amenable.*

Proof. Let G be a virtually amenable group: then, G contains an amenable subgroup H of finite index. Let us consider the subgroup of G defined as

$$K := \bigcap_{g \in G} gHg^{-1}.$$

The subgroup K can be seen as the stabilizer of the G-action on G/H given by left multiplication. As a consequence, K is a normal subgroup of finite index of G that is contained in H. Since H is amenable, K is amenable by Proposition 3.1.8. Furthermore, G/K is a finite group, and thus amenable by Proposition 3.1.2. We can therefore conclude by Proposition 3.1.8 that the group G is amenable. □

We are now interested in understanding the behaviour of directed ascending unions of amenable groups.

Lemma 3.1.14. *Let G be a group. Suppose that there is a net $(m_i)_{i \in I}$ in $\mathcal{M}(G)$ such that, for every $g \in G$, the net $(g \cdot m_i - m_i)_{i \in I}$ converges to 0 in $(\ell^\infty(G))^*$ for the weak-$*$ topology. Then, the group G is amenable.*

Proof. [CC10], Lemma 4.5.9. □

Proposition 3.1.15. *Let G be a group and let $(G_i)_{i \in I}$ be a directed ascending system of amenable subgroups of G such that $G = \bigcup_{i \in I} G_i$. Then, the group G is amenable.*

Proof. For every $i \in I$, let m_i be a G_i-invariant mean on the group G_i. Consider the family $\widetilde{m}_i : \ell^\infty(G) \to \mathbb{R}$ of means on G defined by

$$\begin{aligned} \widetilde{m}_i : \ell^\infty(G) &\to \mathbb{R} \\ f &\mapsto m_i(f|_{G_i}). \end{aligned}$$

Let us consider an element $g \in G$. Since $G = \bigcup_{i \in I} G_i$ and the subgroups $\{G_i\}_{i \in I}$ form a directed ascending system, there exists $i_0(g) \in I$ such that $g \in G_i$ for every $i \geq i_0(g)$. Then, given $f \in \ell^\infty(G)$ and $i \geq i_0(g)$, we have

$$(g \cdot \widetilde{m_i} - \widetilde{m_i})(f) = g \cdot \widetilde{m_i}(f) - \widetilde{m_i}(f) = g \cdot m_i(f|_{G_i}) - m_i(f|_{G_i}) = 0$$

from the fact that m_i is a G_i-invariant mean. Therefore, $g \cdot \widetilde{m_i} - \widetilde{m_i} = 0$ for all $i \geq i_0(g)$. By applying Lemma 3.1.14, we finally get that the group G is amenable. □

As a consequence of Proposition 3.1.15, we obtain that amenability is a local property of groups in the following sense.

Corollary 3.1.16. *Let G be a group. Then, G is amenable if and only if all its finitely generated subgroups are amenable.*

Proof. If the group G is amenable, then all of its subgroups are amenable by Proposition 3.1.8. Conversely, assume that all finitely generated subgroups of G are amenable: since they form a directed ascending system of subgroups covering all G, we can apply Proposition 3.1.8 to conclude that the group G is amenable. □

Corollary 3.1.17. *Let $(G_i)_{i\in I}$ be a family of amenable groups. Then, their direct sum $G = \bigoplus G_i$ is amenable.*

Proof. Let us consider a finitely generated subgroup of the direct sum G: this is a subgroup of some finite product of the groups G_i, thus it is amenable by Proposition 3.1.8 and Corollary 3.1.10. Therefore, Corollary 3.1.16 implies that the group G is amenable. □

We say that a group is *locally finite* if all its finitely generated subgroups are finite. As finite groups are amenable (Proposition 3.1.2), then Corollary 3.1.16 immediately implies the following.

Corollary 3.1.18. *Every locally finite group is amenable.*

Example 3.2. Given a set X, let us consider the group $\mathrm{Sym}_f(X)$ of finitely supported permutations on X, i.e., the bijections from X to X that coincide with the identity map outside of a finite set. Then, $\mathrm{Sym}_f(X)$ is locally finite ([CC10], Example 3.2.4). Indeed, let us consider a finite subset $\Sigma \subseteq \mathrm{Sym}_f(X)$: then, the subgroup of $\mathrm{Sym}_f(X)$ generated by Σ is finite, as it is isomorphic to a subgroup of $\mathrm{Sym}(Y)$, where Y is the union of the supports of the elements in Σ. As a consequence, we obtain from Corollary 3.1.18 that the group $\mathrm{Sym}_f(X)$ is amenable.

We conclude this section with a result which links amenability and hyperbolic groups.

Proposition 3.1.19. *Let G be a hyperbolic group. Then, either G is virtually cyclic or G is not amenable.*

Proof. Given a hyperbolic group G, then either G is virtually cyclic or it contains a free group of rank 2 ([Löh17], Corollary 8.3.17). Hence, we conclude by Corollary 3.1.9. □

A virtually cyclic hyperbolic groups is said to be *elementary hyperbolic*. Proposition 3.1.19 therefore yields that non-elementary hyperbolic groups are not amenable.

3.2 Følner condition

Another important characterization of amenability is given by the existence of *almost invariant* subsets, called *Følner sets*.

Definition 3.4. A group G satisfies the *Følner condition* if, for every $\varepsilon > 0$ and every finite subset $S \subseteq G$, there exists a non-empty finite subset $F \subseteq G$ such that

$$\frac{|sF\Delta F|}{|F|} \leq \varepsilon \text{ for every } s \in S.$$

Such finite subset F is called *(ε, S)-Følner set*, or directly *Følner set*.

Here, Δ denotes the symmetric difference of sets, i.e., $A\Delta B = (A \cup B) \setminus (A \cap B)$.

Definition 3.5. Let G be a countable group. A *Følner sequence* for G is a sequence $\{F_n\}_{n\in\mathbb{N}}$ of non-empty finite subsets of G such that

$$\lim_{n\to\infty} \frac{|gF_n\Delta F_n|}{|F_n|} = 0 \text{ for every } g \in G.$$

If G is an uncountable group, we can define a *Følner net* in the analogous way.

Proposition 3.2.1. *A countable group satisfies the Følner condition if and only if it admits a Følner sequence.*

Proof. Let G be a countable group and assume that it satisfies the Følner condition. Since G is countable, we can write it as the increasing union of finite sets $\{S_n\}_{n\in\mathbb{N}}$. For every n, set $\varepsilon_n = \frac{1}{n}$: then, the Følner condition yields a finite subsets $F_n \subseteq G$ such that $|sF_n\Delta F_n| \leq \frac{1}{n}|F_n|$ for every $s \in S_n$. For every $g \in G$, there exists $n_0 \in \mathbb{N}$ such that $g \in S_n$ for every $n \geq n_0$. Hence,

$$\lim_{n\to\infty} \frac{|gF_n\Delta F_n|}{|F_n|} = 0$$

for every $g \in G$ and this implies that $\{F_n\}_{n\in\mathbb{N}}$ is a Følner sequence for G.

Conversely, assume that the group G admits a Følner sequence. Let us fix $\varepsilon > 0$ and a finite subset $S \subseteq G$. For every $s \in S$, there exists $n(s) \in \mathbb{N}$ such that

$$\frac{|sF_n\Delta F_n|}{|F_n|} \leq \varepsilon \text{ for all } n \geq n(s).$$

Set $\overline{n} = \max_{s\in S} n(s)$: then, $F = F_{\overline{n}}$ is a (ε, S)-Følner set. Therefore, the group G satisfies the Følner condition. □

Example 3.3. Let G be a finite group. Then, the sequence $\{F_n\}_{n\in\mathbb{N}}$ with $F_n = G$ for every $n \in \mathbb{N}$ is a Følner sequence for G, since $gF_n\Delta F_n = \varnothing$ for every $g \in G$.

Example 3.4. The group $\mathbb{Z}$ admits the Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ given by $F_n = \{-n, \dots, n\}$. Indeed, for $z \in \mathbb{Z}$, we have that $z+F_n = \{z-n, \dots, z+n\}$. If $n \geq |z|$, then $|(z + F_n)\Delta F_n)| = 2z$, while $|F_n| = 2n + 1$. Hence,

$$\lim_{n\to\infty} \frac{|(z + F_n)\Delta F_n|}{|F_n|} = 0$$

for every $z \in \mathbb{Z}$. More generally, the group $\mathbb{Z}^r$ for $r \geq 1$ admits a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ given by

$$F_n = \{-n, \dots, n\}^r.$$

We now want to show that the Følner condition gives an equivalent characterization of amenability. In order to do so, we first need to introduce the so-called *Reiter condition*. Let us denote by $\mathrm{Prob}(G)$ the space of all probability measures on G:

$$\mathrm{Prob}(G) := \{\mu \in \ell^1(G) \mid \mu \geq 0 \text{ and } \|\mu\|_{\ell^1(G)} = \sum_{g\in G} \mu(g) = 1\}.$$

We recall that, given a probability measure $\mu \in \mathrm{Prob}(G)$, its *support* is given by

$$\mathrm{supp}(\mu) = \{g \in G \mid \mu(g) > 0\}.$$

Furthermore, the group G acts on the space $\mathrm{Prob}(G)$ via

$$\begin{aligned} G \times \mathrm{Prob}(G) &\to \mathrm{Prob}(G) \\ (g, \mu) &\mapsto (h \mapsto \mu(g^{-1}h)). \end{aligned}$$

Definition 3.6. A group G satisfies the *Reiter condition* if, for every $\varepsilon > 0$ and every finite subset $S \subseteq G$, there exists a finitely-supported probability measure $\mu \in \mathrm{Prob}(G)$ such that

$$\|s \cdot \mu - \mu\|_{\ell^1(G)} \leq \varepsilon$$

for all $s \in S$.

Theorem 3.2.2. *Let G be a group. Then, the following conditions are equivalent.*
(i) The group G is amenable.
(ii) The group G satisfies Reiter condition.
(iii) The group G satisfies Følner condition.

Proof. (i) $\Rightarrow$ (ii): [Pat88], Proposition 0.8.
(ii) $\Rightarrow$ (iii): Let us fix $\varepsilon' > 0$ and a finite subset $S \subseteq G$. Set $\varepsilon = \varepsilon' / |S|$. Then, Reiter condition implies that there exists a finitely-supported probability measure μ on G such that $\|s \cdot \mu - \mu\|_{\ell^1(G)} \leq \varepsilon$ for all $s \in S$. For $t \geq 0$, let

$E_\mu(t) = \{g \in G \mid \mu(g) > t\}$ be an upper level subset for μ. For every $g \in G$ and $s \in S$, we have

$$|s \cdot \mu(g) - \mu(g)| = \int_0^1 |\chi_{sE_\mu(t)}(g) - \chi_{E_\mu(t)}(g)| dt.$$

Summing over all $g \in G$, we obtain

$$\|s \cdot \mu - \mu\|_{\ell^1(G)} = \sum_{g \in G} |s \cdot \mu(g) - \mu(g)| = \int_0^1 |sE_\mu(t) \Delta E_\mu(t)| dt.$$

Furthermore,

$$\int_0^1 |E_\mu(t)| dt = \int_0^1 \sum_{g \in G} \chi_{E_\mu(t)}(g) dt = \sum_{g \in G} \mu(g) = 1.$$

Therefore, using the hypothesis on the probability measure μ, we have

$$\begin{aligned} \varepsilon' \int_0^1 |E_\mu(t)| dt = \varepsilon' = \varepsilon\,|S| &\geq \sum_{s \in S} \|s \cdot \mu - \mu\|_{\ell^1(G)} \\ &= \int_0^1 \sum_{s \in S} |sE_\mu(t) \Delta E_\mu(t)| dt. \end{aligned}$$

This implies that there exists $\bar{t} > 0$ such that

$$\sum_{s \in S} |sE_\mu(\bar{t}) \Delta E_\mu(\bar{t})| \leq \varepsilon'\,|E_\mu(\bar{t})|.$$

As a consequence, it suffices to take as (ε', S)-Følner set $F = E_\mu(\bar{t})$ to see that Følner condition is satisfied.
(iii) $\Rightarrow$ (i): [CC10], Theorem 4.9.2.

□

3.2.1 Amenability and growth

Amenability has an interesting connection with the *growth type* of a group, which can be proved using the characterization of amenable groups via the existence of Følner sets. We start by recalling the definition of growth types.

Let G be finitely generated group. A subset $S \subseteq G$ is called *symmetric* if $S = S^{-1}$. Let $S \subseteq G$ be a symmetric finite generating set. The *length function* $l_S : G \to \mathbb{N}$ on G with respect to S is defined by taking $l_S(g)$ to be the minimal integer $n \geq 0$ such that g can be expressed as a product of n elements in S, that is,

$$l_S(g) = \min\{n \geq 0 \mid g = s_1 \cdot s_n,\ s_i \in S \text{ for } 1 \leq i \leq n\}.$$

The *growth function* of G with respect to S is the function $\gamma_S^G : \mathbb{N} \to \mathbb{N}$ defined by

$$\gamma_S^G(n) = |B_S^G(n)| = |\{g \in G \mid l_S(g) \leq n\}|$$

for every $n \in \mathbb{N}$. Given two non-decreasing functions $\gamma, \gamma' : \mathbb{N} \to [0; +\infty)$, we say that γ' *dominates* γ, and we write $\gamma \preceq \gamma'$, if there exists an integer $c \geq 1$ such that $\gamma(n) \leq c\gamma'(n)$ for every $n \geq 1$. If $\gamma \preceq \gamma'$ and $\gamma' \preceq \gamma$, we say that γ and γ' are *equivalent* and we write $\gamma \sim \gamma'$. Notice that $\sim$ is an equivalence relation. Growth functions with respect to finite symmetric generating sets of a given finitely generated group are all equivalent ([CC10], Corollary 6.4.5). We can thus omit the subscript S of the generating set in γ_S^G and write $\gamma(G)$ to denote the equivalence class of the growth function of G. The latter is called the *growth type* of the group G.

Definition 3.7. Let G be a finitely generated group.

1. The group G has *polynomial growth* if $\gamma(G) \preceq n^a$ for some $a \in \mathbb{R}_{\geq 0}$.
2. The group G has *exponential growth* if $\gamma(G) \sim e^n$.
3. The group G has *subexpontential growth* if $\gamma(G) \nsim e^n$.

Let us now consider the limit

$$\omega = \lim_{n \to \infty} \gamma_G(n)^{1/n}.$$

This limit exists because the growth function is subadditive ([CC10], Proposition 6.5.2). The group G has exponential growth if and only if $\omega > 1$, while it has subexponential growth if and only if $\omega = 1$ ([CC10], Proposition 6.5.4 and Corollary 6.5.5). In particular, this implies that every finitely generated group has either exponential or subexponential growth. We can now prove the following result.

Theorem 3.2.3. *Every finitely generated group of subexponential growth is amenable.*

Proof. Let G be a finitely generated group of subexponential growth and let S be a finite symmetric generating set of G. In order to show that G is amenable, it suffices to check that the Følner condition holds for the generating set S. Let us write $B_S^G(n) = B(n)$: by hypothesis, we have that

$$\lim_{n \to \infty} \gamma_G(n)^{1/n} = \lim_{n \to \infty} |B(n)|^{1/n} = 1.$$

This implies that, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\frac{B(k+1)}{B(k)} < 1 + \varepsilon\,.$$

Indeed, if we assume that, for a contradiction, there exists $\varepsilon > 0$ such that the inequality $|B(k+1)| > (1+\varepsilon)|B(k)|$ holds for every $k \in \mathbb{N}$, then $|B(k+1)| > (1+\varepsilon)|B(1)|$. Hence, $\limsup_{k\in\mathbb{N}} |B(k)|^{1/k} \geq 1+\varepsilon$, which contradicts the hypothesis of subexponential growth. Therefore, for every $s \in S$, we have

$$\frac{|sB(k)\Delta B(k)|}{|B(k)|} \leq \frac{2(|B(k+1)\setminus B(k)|)}{|B(k)|} \leq 2\,\varepsilon\,.$$

Finally, by taking $F = B(k)$ as Følner set, Theorem 3.2.2 implies that the group G is amenable. □

Corollary 3.2.4. *Every finitely generated group that contains a free subgroup of rank* 2 *has exponential growth.*

Proof. Let G be a finitely generated group containing a free subgroup of rank 2. Then, G is non-amenable by Corollary 3.1.9. It therefore follows from Theorem 3.2.3 that the group G has exponential growth. □

Let us remark again that, from Corollary 3.1.9, we know that an immediate way of showing that a group is not amenable consists in exhibiting a free subgroup of rank 2: it is natural to ask if this is a characterization of amenability.

Question (von Neumann)**.** Does every non-amenable group contain a free subgroup of rank 2?

Historically, Thompson's group F was a first potential candidate for giving a negative answer to von Neumann's question, as it was already known that F does not contain free subgroups of rank 2. We recall that Thompson's group F can be described by the following finite presentation:

$$F = \langle x, y \mid [xy^{-1}, x^{-1}yx], [xy^{-1}, x^{-2}yx^2]\rangle.$$

However, its amenability remains to this day an open problem. Von Neumann's question was answered negatively in 1980 by Ol'shanskii [Ols80]: he proved that the Tarski monster group, an infinite group in which every non-trivial proper subgroup is cyclic of order a fixed prime p, is not amenable. Later examples of non-amenable groups that do not contain free subgroups of rank 2 were given by Adyan [Ady83]. He proved that the *free Burnside group* $B(m,n)$ is non-amenable for $m \geq 2$ and n odd with $n \geq 665$. The free Burnside group $B(m,n)$ is defined as the quotient of the free group F_m of rank m by the subgroup F_m^n of F_m generated by all n^{th} powers of elements of F_m. Furthermore, examples of finitely presented non-amenable groups with no proper subgroups isomorphic to F_2 were given by Ol'shanskii and Sapir [OS03].

We conclude this section by presenting an interesting class of groups introduced by Day in 1957 [Day57].

Definition 3.8. The class of *elementary amenable* groups is the smallest class of groups containing all abelian groups and all finite groups and that is closed under taking subgroups, quotients, extensions and directed ascending unions.

It follows from Proposition 3.1.2, Theorem 3.1.6 and Proposition 3.1.8 that elementary amenable groups are amenable. However, at the time of their introduction, there were no examples of amenable groups that fail to be elementary amenable.

Question (Day)**.** Does there exist an amenable group which is not elementary amenable?

Day's problem was resolved in 1984 by Grigorchuk [Gri80; Gri85], who provided the first example of finitely generated infinite group with intermediate growth (the so-called *first Grigorchuk group*). In particular, this implies that it is amenable by Theorem 3.2.3. Furthermore, Chou showed that an elementary amenable group must have either polynomial or exponential growth [Cho80]. This implies that Grigorchuk group indeed gives a negative answer to Day's problem.

3.3 Amenability via random walks on groups

This section is devoted to equivalent characterizations of amenability via random walks on groups. The term *random walk* was first introduced in a short note by Pearson [Pea05], who asked about the distribution of a random walk on a plane. Later, Poincaré discussed in [Poi12] how the shuffling of a card deck can be modeled by repeated multiplications of random independent elements in a finite permutation group. In 1921, Pólya was the first to study simple random walks on infinite abelian groups [Pól21]. The discussion on non-abelian groups was introduced by Kesten in his PhD thesis [Kes59b]: he was concerned on the relation between the structure of a group G and the properties of the spectrum of the operator on $\ell^2(G)$ associated to the symmetric random walk on G. He was then the first one to exhibit a connection between amenability and random walks, showing that a group is non-amenable if and only if, in a simple random walk, the probability of returning to the identity in $2n$ steps decays exponentially fast ([Kes59a], Theorem 3.3.1). A detailed historical introduction on the study of random walks on groups can be found in [SZ21], §1.1. In this section, we give the fundamental definitions and present the relevant results on amenability. We refer to [KV83] for an extensive study on the matter.

3.3.1 Basic definitions

Definition 3.9. Let G be a countable group and μ a probability measure on G. The *μ-random walk* (G, μ) is the Markov chain with state space G and transition probabilities

$$p(g, gh) = \mu(h),\ g, h \in G,$$

which are equivariant with respect to the left action of the group G on itself.

Namely, the position of the random walk (G, μ) can be obtained from the preceding one by right multiplication with the independent random group element chosen according to the distribution μ. Such element is called *increment*. Hence, if we assume that the random walk starts at time 0 from a point g_0, then its position at time n is given by

$$g_n = g_0 h_1 h_2 \dots h_n,$$

where $\{h_i\}_{i=1}^n$ is a sequence of independent μ-distributed increments. We will usually assume that the starting point is the trivial element $g_0 = 1_G$ of G. We use the notation

$$g \underset{h \sim \mu}{\longmapsto} gh$$

to depict the transition probabilities of the random walk (G, μ).

When working with finitely generated groups, we can consider the family of probability measures that are distributed uniformly on a symmetric finite generating set. The corresponding random walks are called *simple random walks*.

Example 3.5. The simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$ for $d \geq 1$ is the random walk (G, μ) where the probability measure μ is uniform on the canonical set of generators $\{e_i\}_{i=1}^d$, that is,

$$\mu(e_i) = \frac{1}{2d}$$

for $i = 1, \dots, d$.

Definition 3.10. Let μ_1 and μ_2 be two probability measure on G. Their *convolution* $\mu_1 * \mu_2$ is the image of the product measure $\mu_1 \times \mu_2$ via the map

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh. \end{aligned}$$

We denote by μ^{*n} the n-fold convolution power of the probability measure μ. Notice that the distribution of the position of the μ-random walk at time n precisely given by μ^{*n}.

Let us recall that the *support* of a probability measure μ on G is given by

$$\mathrm{supp}(\mu) = \{g \in G \mid \mu(g) > 0\}.$$

We say that the measure μ is *non-degenerate* if $\mathrm{supp}(\mu)$ generates G as a semigroup, that is, if

$$G = \bigcup_{n \geq 0} \mathrm{supp}(\mu)^n.$$

Furthermore, the probability measure μ is called *symmetric* if it is invariant by inversion, that is, if $\mu(g) = \mu(g^{-1})$ for every $g \in G$. We will often extend these adjectives from the probability measure μ to the μ-random walk on a group G.

We conclude this part by briefly introducing a special kind of random walks which will be useful in the next chapter.

Definition 3.11. A *random walks with internal degrees of freedom* (RWIDF) on a group G is a Markov chain whose state space is the product of G by a certain space X (the *space of degrees of freedom*) and whose transition probabilities are equivariant with the respect to the action of G on itself.

Thus, the transition probabilities of such random walks are given by

$$p((g,x),(gh,y)) = \mu_{xy}(h)$$

where $M = \{\mu_{xy} \mid x, y \in X\}$ is an order $|X|$ matrix of subprobability measures on G such that

$$\sum_{y \in X} \|\mu_{xy}\| = 1 \text{ for all } x \in X.$$

Here, $\|\cdot\|$ denotes the mass of a probability measure. We denote by (G, M) the RWIDF whose transition probabilities are encoded in the matrix M.

3.3.2 Characterizations of amenability

The first criterion for amenability based on asymptotic properties of random walks on groups relies on the notion of *spectral radius*.

Definition 3.12. Let G be a countable group and μ a probability measure on G. The *spectral radius* of the μ-random walk on G is defined as

$$\rho(G,\mu) := \limsup_{n\to\infty} \mu^{*2n}(1_G)^{1/2n}.$$

Theorem 3.3.1 (Kesten's criterion [Kes59a]). *Let G be a finitely generated group and let μ be a finitely supported symmetric probability measure on G. Then, the group G is amenable if and only if $\rho(G,\mu) = 1$.*

Recall that $\mu^{*2n}(1_G)$ is the probability of the random walk (G, μ) of returning to the identity after $2n$ steps. Hence, Kesten's criterion tells us that a group is non-amenable if and only if the probability of returning to the identity in $2n$ steps decays exponentially fast.

Theorem 3.3.2. *Let G be a countable group. Then, G is amenable if and only if there exists a non-degenerate probability measure μ on G such that*

$$\lim_{n\to\infty} \|g\mu^{*n} - \mu^{*n}\| = 0 \text{ for all } g \in G.$$

Proof. [KV83], Theorem 4.3. We note that sufficiency immediately follows from Reiter condition (Definition 3.6 and Theorem 3.2.2). □

Another fundamental concept in the context of random walks on groups is given by *entropy*, which was introduced by Shannon.

Definition 3.13. Let G be a countable group and μ a probability measure on G. The *entropy* of μ is defined as

$$H(\mu) := -\sum_{g\in G} \mu(g)\log(\mu(g)),$$

where we use the convention that $0 \cdot \log(0) = 0$. We say that the probability measure μ has *finite entropy* if $H(\mu) \leq \infty$.

Proposition 3.3.3. *Let G be a countable group and let μ_1 and μ_2 be two probability measures on G with finite entropies. Then, the entropy of their convolution is also finite and*

$$H(\mu_1 * \mu_2) \leq H(\mu_1) + H(\mu_2).$$

Proof. [KV83], Proposition 1.1. □

It follows from Proposition 3.3.3 that, given a probability measure μ on G, the sequence of entropies

$$\{H(\mu^{*n})\}_{n\geq 1}$$

of the n-fold convolutions of μ is subadditive. As a consequence, we get that $\lim_{n\to\infty} \frac{H(\mu^{*n})}{n}$ exists and it is equal $\inf_{n\geq 1} \frac{H(\mu^{*n})}{n}$. This allows us to give the following definition, which was first introduced by Avez [Ave72].

Definition 3.14. Let G be a countable group and μ a probability measure on G with finite entropy. The *entropy of the μ-random walk* (G, μ) is defined as

$$h(G, \mu) := \lim_{n\to\infty} \frac{H(\mu^{*n})}{n}.$$

The entropy plays a pivotal role in the understanding of asymptotic properties of the random walk (G, μ). It can be interpreted as the asymptotic mean quantity of information on one factor of the product $g_1 g_2 \cdots g_n$ of n independent random variables $g_1, \ldots, g_n$ in G distributed according to the measure μ ([KV83], Proposition 1.2).

In order to present a characterization of amenability through entropy of random walks, one needs to pass through the notion of *Poisson boundary*. We will not give the formal definition and refer the reader to [KV83], §0.3. Informally, given a countable group G with a probability measure μ, its Poisson boundary is a probability space that describes the asymptotic behaviour of the μ-random walk on G.

Theorem 3.3.4. *Let G be a countable group. Then, G is amenable if and only if it admits a non-degenerate random walk with a trivial Poisson boundary.*

Proof. [Aze06], Corollary after Proposition II.1 and [KV83], Theorem 4.4. □

Avez proved that every μ-random walk on a group G with μ finitely supported probability measure satisfying $h(G, \mu) = 0$ has a trivial Poisson boundary [Ave72]. It was later shown that the following more general result holds.

Theorem 3.3.5. *Let G be a countable group and μ a probability measure on G with finite entropy. Then, the Poisson boundary of the random walk (G, μ) is trivial if and only if $h(G, \mu) = 0$.*

Proof. [KV83], Theorem 1.1. □

Remark 3.1. The above criterion does not hold for a measure μ with infinite entropy. For instance, given an abelian group G with a measure μ such that $H(\mu) = \infty$, we have that $h(G, \mu) = 0$, but the Poisson boundary of G is always trivial (Choquet-Deny, [Cho60]).

We can finally extract the following result from Theorem 3.3.4 and Theorem 3.3.5.

Corollary 3.3.6. *Let G be a countable group. If there is a non-degenerate probability measure μ on G with $h(G, \mu) = 0$, then the group G is amenable.*

3.4 Uniform amenability

We now focus on the notion of *uniform amenability* of class of groups. Uniform amenability was first introduced by Keller in 1972 using methods of non-standard analysis [Kel72], and then independently defined by Bożejko in 1980 [Boż80]. As the name suggests, the key idea is to add a uniform condition to the definition of amenability: following Kionke and Schesler [KS23], we present the definition for classes of groups using a *uniform Følner condition*.

Definition 3.15. A class of groups $\mathfrak{K}$ is *uniformly amenable* if there is a function $m : \mathbb{R}_{>0} \times \mathbb{N} \to \mathbb{N}$ such that, for every $\varepsilon > 0$, every $G \in \mathfrak{K}$ and every finite subset $S \subseteq G$, there exists a non-empty finite subset $F \subseteq G$ satisfying
(i) $|F| \leq m(\varepsilon, |S|)$;
(ii) $|SF| \leq (1+\varepsilon)|F|$.
Such finite subset F is called *uniform Følner set*. In this case, we say that the class $\mathfrak{K}$ is m-uniformly amenable. Furthermore, we say that a group G is uniformly amenable if the class consisting only of G is uniformly amenable.

Remark 3.2. By replacing the function m in Definition 3.15 with $m'(\varepsilon, N) = \max_{k \leq N} m(\varepsilon, k)$, we will always assume that m is non-decreasing in the second argument.

We immediately notice that uniformly amenable groups are amenable by Theorem 3.2.2. This follows from the fact that a uniform Følner set as in Definition 3.15 is simply an (ε, S)-Følner set whose size is uniformly bounded in terms of ε and $|S|$. In order to see this, we first need an easy lemma.

Lemma 3.4.1. *Let G be a group. Consider a finite subset $F \subseteq G$ and two non-empty subsets $S \subseteq T \subseteq G$, where S is finite and $1_G \in T$. Then,*

$$|SF\Delta F| \leq 2|TF\Delta F|.$$

Proof. Let us consider three subsets F, S, T as in the statement. Then,

$$\begin{aligned} |SF\Delta F| &= |SF \setminus F| + |F \setminus SF| = |SF \setminus F| + |F| - |F \cap SF| \\ &\leq |SF \setminus F| + |SF| - |F \cap SF| = 2|SF \setminus F| \\ &\leq 2|TF \setminus F| = 2|TF\Delta F|. \end{aligned}$$

This concludes the proof. □

Now, assume that the uniform Følner condition holds for a group G and let us fix a finite subset $S \subseteq G$ and $\varepsilon > 0$. Consider the non-empty finite subset $T = S \cup \{1_G\}$: then, there is a uniform Følner set $F \subseteq G$ such that $|TF| \leq (1+\varepsilon)|F|$. For every $s \in S$, Lemma 3.4.1 yields

$$\begin{aligned} |sF\Delta F| &\leq 2|TF\Delta F| = 2|TF \setminus F| = 2|TF| - 2|F| \\ &\leq 2(1+\varepsilon)|F| - 2|F| = 2\varepsilon|F|. \end{aligned}$$

This finally implies that F is a (ε, S)-Følner set.

Example 3.6. (i) Let us fix $d \in \mathbb{N}$. The class $\mathfrak{Fin}_d$ of all finite groups of order at most d is uniformly amenable. Indeed, the map m is simply given by $m(\varepsilon, N) = d$ and the finite group G itself is a suitable uniform Følner set.
(ii) The class of abelian groups is uniformly amenable (see [BP78]).

Proposition 3.4.2. *Homomorphic images and subgroups of uniformly amenable groups are uniformly amenable. In addition, extensions of uniformly amenable groups are uniformly amenable.*

Proof. [Kel72], Theorem 4.5 and [Boż80], Theorems 3 and 4. □

Corollary 3.4.3. *Every solvable group is uniformly amenable.*

Proof. As in the proof of Corollary 3.1.11, we use Proposition 3.4.2 and the fact the abelian groups are uniformly amenable (Example 3.6), and proceed by induction on the length of the derived series of a solvable group. □

Proposition 3.4.4. *Let G be a group and let $(G_i)_{i\in I}$ be a directed ascending system of m-uniformly amenable subgroups of G such that $G = \bigcup_{i\in I} G_i$. Then, the group G is m-uniformly amenable.*

Proof. Let us fix $\varepsilon > 0$ and a finite subset $S \subseteq G$: then, $S \subseteq G_i$ for some $i \in I$. Hence, the result immediately follows from the fact that each G_i is m-uniformly amenable. □

Example 3.7 (Example 2.3 [KS23])**.** Let us consider $d \in \mathbb{N}$, a finite group G with $|G| = d$ and a set I. We show that the direct power G^I is uniformly amenable.

Let $S \subseteq G^I$ be a finite subset, say $S = \{s_1, \dots, s_n\}$ for some $n \in \mathbb{N}$: the goal is proving that the subgroup generated by S is a suitable uniform Følner set in G^I. Let us consider the set G^n of n-tuples in G: this is a finite set, hence we can enumerate its elements as $G^n = \{x^{(1)}, \dots, x^{(k)}\}$, where $k = d^n$ and $x^{(i)} = (x_1^{(i)}, \dots, x_n^{(i)})$ for $i = 1, \dots, k$. Furthermore, we consider the *universal subset* $U \subseteq G^k$ defined as $U = \{u_1, \dots, u_n\}$, with $u_j = (x^{(1)}, \dots, x_j^{(k)})$. Now, for every $i \in I$, we get an n-tuple

$$S(i) \coloneqq (s_1(i), \dots, s_n(i)) \in G^n$$

and a function $\phi : I \to \{1, \dots, k\}$ such that $S(i) = x^{(\phi(i))}$. Let us define

$$\alpha : G^k \to G^I,\ \alpha(g^{(1)}, \dots, g^{(k)})(i) = g^{\phi(i)}.$$

This homomorphism maps the universal subset $U \subseteq G^k$ to S: indeed, given $u_j \in U$, we have $\alpha(u_j^{(1)}, \dots, u_j^{(k)})(i) = u_j^{\phi(i)} = x_j^{\phi(i)} = s_j(i)$. As a consequence, the subgroup generated by S is isomorphic to a subfactor of the finite group G^k and it is therefore is a suitable uniform Følner set, with the function m being $m(\varepsilon, N) = |G^k| = d^{d^n}$. Finally, this implies that the direct power G^I is a uniformly amenable group.

The following proposition highlights a crucial property of uniformly amenable groups.

Proposition 3.4.5. *Let G be a uniformly amenable group. Then, G satisfies a group law.*

Proof. [Kel72], Corollary 5.9. □

Let $\mathfrak{K}$ be a class of groups. We say that a group G is *residually* $\mathfrak{K}$ if, for every non-trivial element $g \in G$, there exists a normal subgroup $N \trianglelefteq G$ such that $g \neq N$ and $G/N \in \mathfrak{K}$. The following result is an immediate consequence of Proposition 3.4.5.

Corollary 3.4.6. *Let $\mathfrak{K}$ be a class group such that the free group F_2 of rank 2 is residually $\mathfrak{K}$. Then, $\mathfrak{K}$ is not uniformly amenable.*

After Proposition 3.4.5, Keller asked the following questions [Kel72].

Question. Is every group that satisfies a group law amenable? Is every amenable group that satisfies a group law uniformly amenable?

The answer to the first question is negative: an example of non-amenable group satisfying a group law is given by free-Burnside groups of large exponent (see [Ady83]). However, we still have no such examples among finitely generated residually finite groups, as pointed out by de Cornulier and Mann [DM07].

A first example of an amenable group which is not uniformly amenable was given by Wyzoczanski [Wys88], who gave a further characterization of uniform amenability based on random walks on groups (Proposition 3.4.9).

Example 3.8 (Example IV [Wys88])**.** For a prime p, we denote by G_p be the group of 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{F}_p$. The group $G = \bigoplus_{p \text{ prime}} G_p$ is amenable but not uniformly amenable.

Furthermore, it is interesting to investigate the relation between elementary and uniformly amenable groups. Thanks to Proposition 3.4.5, we can give an example of finitely generated (elementary) amenable group that fails to be uniformly amenable.

Example 3.9. Let G be a group and let $\mathrm{Sym}_f(G)$ denote the group of finitely supported permutations on G introduced in Example 3.2. The group G acts on $\mathrm{Sym}_f(G)$ by

$$\begin{aligned} G \times \mathrm{Sym}_f(G) &\to \mathrm{Sym}_f(G) \\ (g, f) &\mapsto (x \mapsto gf(g^{-1}x)). \end{aligned}$$

Their semidirect product $\mathrm{Sym}_f(G) \rtimes G$ is sometimes called *Lampshuffler group*. This terminology was first introduced in [GM22], and we refer to [Sil23] for more details on such groups. Even if $\mathrm{Sym}_f(G)$ is not finitely generated, we have that if G is finitely generated then $\mathrm{Sym}_f(G) \rtimes G$ is finitely generated as well. Since $\mathrm{Sym}_f(G)$ is locally finite, it is (elementary) amenable (Corollary 3.1.18): then, $\mathrm{Sym}_f(G) \rtimes G$ is (elementary) amenable whenever G is. Let us consider $G = \mathbb{Z}$: the group $\mathrm{Sym}_f(\mathbb{Z}) \rtimes \mathbb{Z}$ contains every finite group as a subgroup, therefore it cannot satisfy a group law. The latter is consequence of residual finiteness of free groups. Indeed, recall that a group law is a non-trivial word in a free group: by residual finiteness, for every group law there exists a finite group where this word is again non-trivial, meaning that there exists a finite group that does not satisfy such law. Therefore, Proposition 3.4.5 implies that $\mathrm{Sym}_f(\mathbb{Z}) \rtimes \mathbb{Z}$ is a finitely generated (elementary) amenable group that is not uniformly amenable.

However, we still do not have any examples in the converse direction.

Open Question 3.1. Are there uniformly amenable groups which are not elementary amenable?

3.4.1 Characterizations of uniform amenability

We now present two useful characterizations of uniform amenability given by Kionke and Schesler [KS23].

Definition 3.16. A class of groups $\mathfrak{K}$ satisfies a *uniform isoperimetric inequality* if there is a function $\widetilde{m} : \mathbb{R}_{>0} \times \mathbb{N} \to \mathbb{N}$ such that, for every $\varepsilon > 0$, every $G \in \mathfrak{K}$ and every finite symmetric subset $S \subseteq G$, there exists a finite subset $E \subseteq G$ satisfying $|E| \leq \widetilde{m}(\varepsilon, |S|)$ and

$$\frac{|\partial_S E|}{|E|} \leq \varepsilon,$$

where $\partial_S E = SE \setminus E$ denotes the S-boundary of E.

Proposition 3.4.7. *A class of groups $\mathfrak{K}$ is uniformly amenable if and only if it satisfies a uniform isoperimetric inequality.*

Proof. Let us assume that the class $\mathfrak{K}$ is uniformly amenable for a function m, and set $\widetilde{m}(\varepsilon, N) := m(\varepsilon, N+1)$. Fix $\varepsilon > 0$, a group $G \in \mathfrak{K}$ and a finite symmetric subset $S \subseteq G$. Set $T = S \cup \{1_G\}$: since the class $\mathfrak{K}$ is uniformly amenable, we can find a finite subset $E \subseteq G$ satisfying $|E| \leq m(\varepsilon, |T|) = \widetilde{m}(\varepsilon, |S|)$ and

$$|TE| \leq (1+\varepsilon)|E|.$$

Now, $TE = (S \cup \{1_G\})E = E \cup \partial_S E$. Hence, $|TE| = |E| + |\partial_S E| \leq (1+\varepsilon)|E|$, implying that

$$\frac{|\partial_S E|}{|E|} \leq \varepsilon\,.$$

As a consequence, the class $\mathfrak{K}$ satisfies a uniform isoperimetric inequality.

Conversely, assume that the class $\mathfrak{K}$ satisfies a uniform isoperimetric inequality for a function $\widetilde{m}$, and define $m(\varepsilon, N) := \max_{k \leq 2N} \widetilde{m}(\varepsilon, k)$. Fix $\varepsilon > 0$, a group $G \in \mathfrak{K}$ and a finite subset $S \subseteq G$. Set $T = S \cup S^{-1}$: this is a symmetric finite subset of G, hence there exists a finite subset $E \subseteq G$ satisfying $|E| \leq \widetilde{m}(\varepsilon, |T|) \leq m(\varepsilon, |S|)$ and $|\partial_T E| \leq \varepsilon\,|E|$. Therefore,

$$|SE| \leq |TE| \leq |E| + |\partial_T E| \leq (1+\varepsilon)|E|,$$

which finally implies that the class $\mathfrak{K}$ is uniformly amenable. □

Let G be a group and μ a probability measure on G. Let us recall that, given an element $g \in G$, then $g \cdot \mu(h) = \mu(g^{-1}h)$.

Definition 3.17. A class of groups $\mathfrak{K}$ satisfies the *uniform Reiter condition* if there is a function $r : \mathbb{R}_{>0} \times \mathbb{N} \to \mathbb{N}$ such that, for every $\varepsilon > 0$, every $G \in \mathfrak{K}$ and every finite subset $S \subseteq G$, there exists a finitely supported probability measure μ on G satisfying $|\mathrm{supp}(\mu)| \leq r(\varepsilon, |S|)$ and

$$\|s \cdot \mu - \mu\|_{\ell^1(G)} \leq \varepsilon$$

for every $s \in S$.

Proposition 3.4.8. *A class of groups $\mathfrak{K}$ is uniformly amenable if and only if it satisfies the uniform Reiter condition.*

Proof. Let us assume that the class $\mathfrak{K}$ is uniformly amenable: from Proposition 3.4.7, we know that it satisfies a uniform isoperimetric inequality for a function $\widetilde{m}$. Fix $\varepsilon > 0$, a group $G \in \mathfrak{K}$ and a finite subset $S \subseteq G$. Set $T = S \cup S^{-1}$: by assumption, there exists a finite subset $E \subseteq G$ such that $|E| \leq \widetilde{m}(\varepsilon, |T|) \leq \widetilde{m}(\varepsilon, 2|S|)$ and

$$\frac{|\partial_T E|}{|E|} \leq \varepsilon,$$

where $\partial_T E = TE \setminus E$. Let μ be the uniform probability measure on E, i.e., $\mu(A) = |A|/|E|$ for every $A \subseteq E$. As $|\partial_T E| \leq \varepsilon\,|E|$, we have $|sE\Delta E| \leq \varepsilon\,|E|$ for every $s \in S$. Therefore,

$$\|s \cdot \mu - \mu\|_{\ell^1(G)} = \sum_{g\in G} |s \cdot \mu(g) - \mu(g)| = \sum_{g \in G} \frac{1}{|E|} \chi_{sE\Delta E}(g) = \frac{|sE\Delta E|}{|E|} \leq \varepsilon\,.$$

This implies that the class $\mathfrak{K}$ satisfies the uniform Reiter condition.

Conversely, let us assume that the class $\mathfrak{K}$ satisfies the uniform Reiter condition. Fix $\varepsilon' > 0$, a group $G \in \mathfrak{K}$ and a finite symmetric subset $S \subseteq G$. Let $\varepsilon = \varepsilon'/|S|$: by assumption, there exists a finitely supported probability measure μ on G satisfying $|\operatorname{supp}(\mu)| \leq r(\varepsilon, |S|)$ and

$$\|s \cdot \mu - \mu\|_{\ell^1(G)} \leq \varepsilon\,.$$

For all $t \in [0, 1]$, consider the level set $E_\mu(t) = \{g \in G \mid \mu(g) > t\}$. Notice that $|E_\mu(t)| \leq |\operatorname{supp}(\mu)| \leq r(\varepsilon, |S|)$. Then, as in the proof of Theorem 3.2.2, for every $g \in G$ we have

$$\|s \cdot \mu - \mu\|_{\ell^1(G)} = \int_0^1 |sE_\mu(t)\Delta E_\mu(t)|dt$$

Furthermore,

$$\begin{aligned}\varepsilon' = |S|\,\varepsilon &\geq \sum_{s\in S} \|\lambda_s^*(\mu) - \mu\|_{\ell^1(G)} \\ &= \int_0^1 \sum_{s\in S} |sE_\mu(t)\Delta E_\mu(t)|dt \geq \int_0^1 |\partial_S E_\mu(t)|dt.\end{aligned}$$

We now see that some level set satisfies a uniform isoperimetric inequality. Indeed, if we assume for a contradiction that $|\partial_S E_\mu(t)| > \varepsilon'\,|E_\mu(t)|$ for every $t \in [0, 1]$, then we would get

$$\int_0^1 |\partial_S E_\mu(t)|dt > \varepsilon' \int_0^1 |E_\mu(t)|dt = \varepsilon',$$

which gives a contradiction. □

Finally, we note that uniform amenability can be characterized by a uniform version of Kesten's condition on random walks seen in Theorem 3.3.1.

Definition 3.18. A class of groups $\mathfrak{K}$ satisfies a *uniform Kesten condition* if there is a function $k : \mathbb{R}_{>0} \times \mathbb{N} \to \mathbb{N}$ such that, for every $\varepsilon > 0$, every $G \in \mathfrak{K}$, every finitely supported probability measure μ on G and every $n \geq k(\varepsilon, |\operatorname{supp}(\mu)|)$, we have

$$\mu^{*2n}(1_G)^{1/2n} > 1 - \varepsilon\,.$$

Proposition 3.4.9. *A class of groups $\mathfrak{K}$ is uniformly amenable if and only if it satisfies the uniform Kesten condition.*

Proof. See [Wys88]. □

Chapter 4

Amenability is not a profinite invariant

In the context of profinite rigidity and amenability, it is interesting to investigate the following variation of Grothendieck's question 2.2.1:

Question. Let A and G be two finitely generated residually finite groups, where A is amenable. Suppose that a morphism $u : A \to G$ induces an isomorphism $\widehat{u} : \widehat{A} \to \widehat{G}$ between the profinite completions. Is the group G amenable?

In their recent work [KS23], S. Kionke and E. Schesler gave a negative answer to this question, proving therefore that amenability is not a profinite invariant of finitely generated residually finite groups. In other words, amenability is not a property that can be detected just by looking at the set of isomorphism types of finite quotients of such groups. The goal of this chapter is to present a detailed proof of the following result.

Theorem 4.0.1 (Kionke and Schesler [KS23])**.** *There exist an uncountable family of pairwise non-isomorphic, residually finite 18-generator groups* $(G_j)_{j\in J}$ *and a residually finite 6-generator group* A *with embeddings* $u_j : A \to G_j$ *such that:*
(i) $\widehat{u_j} : \widehat{A} \to \widehat{G_j}$ *is an isomorphism for every* $j \in J$*,*
(ii) A *is amenable,*
(iii) G_j *is non-amenable for every* $j \in J$*.*

Firstly, we note that we cannot use again the same construction that was employed in order to give a negative answer to the original Grothendieck's question. Indeed, let us recall that Bridson and Grunewald worked with the fibre product $P \leq H \times H$ associated to a group quotient of a non-elementary hyperbolic group H. The fibre product P projects onto H, and we have seen that every homomorphic image of an amenable group is amenable as well. Since the non-elementary hyperbolic group H constructed in [BG04] is non-amenable (Proposition 3.1.19), we cannot hope to use again the idea in order to approach this new question.

Kionke and Schesler proceeded along a completely different path, and decided to work with *automorphisms of rooted trees*. We give a brief summary of their proof of Theorem 4.0.1, which is inspired by methods of Segal [Seg00] and Nekrashevych [Nek13].

We start in Section 4.1 by introducing the basic definitions and notations regarding automorphisms of rooted trees constructed starting from a finite alphabet X. In particular, we focus on the fact the group $\mathrm{Aut}(T_X)$ can be made into a profinite group: this allows us to define the *tree completion* of a subgroup $G \leq \mathrm{Aut}(T_X)$ as its closure in $\mathrm{Aut}(T_X)$ with respect to the profinite topology. If G satisfies the *congruence subgroup property*, then its profinite and tree completion are isomorphic.

In Section 4.2, we present in detail the so-called Ω*-construction*: starting with a subgroup $G \leq \mathrm{Aut}(T_X)$, this construction gives rise to a new subgroup $\Gamma_{\mathrm{G}}^{\Omega} \leq \mathrm{Aut}(T_X)$ depending on a certain subset Ω of the boundary of the tree. If $G \leq \mathrm{Aut}(T_X)$ is perfect, self-similar and satisfies a certain property introduced by Segal [Seg00] (called *property H*), then the resulting groups $\Gamma_{\mathrm{G}}^{\Omega}$ are just infinite branch groups that satisfy the congruence subgroup property. After obtaining a clear description of their tree completion, we get that the profinite completion of $\Gamma_{\mathrm{G}}^{\Omega}$ is a permutational iterated wreath product that depends only on the action of G on the first level of the tree. As a consequence, the subgroups $\Gamma_{\mathrm{G}}^{\Omega}$ form a family of groups with isomorphic profinite completions, and such that the inclusions between them induce isomorphisms of the profinite completions. Furthermore, thanks to a rigidity result from Lavreniuk and Nekrashevych [LN02], we prove that the Ω-construction produces an uncountable family of pairwise non-isomorphic groups.

We proceed in Section 4.3 by showing how this construction can be implemented to achieve the result stated in Theorem 4.0.1. In order to obtain the amenable group A, we fix a prime number p and $n \in \mathbb{N}$ and we apply the construction to the special affine group $\mathrm{SAff}_n(\mathbb{F}_p)$ acting on the p^n-regular rooted tree by rooted automorphisms. Amenability of the resulting group follows from a result by Bartholdi, Kaimanovich and Nekrashevych [BKN10], whose proof relies on the characterization of amenability via asymptotic properties of random walks. Finally, we apply the construction to $\mathrm{SAff}_n(\mathbb{Z})$ acting self-similarly on the p^n-regular rooted tree. Merging the resulting groups with the amenable group A, we obtain the family of non-amenable groups $(G_j)_{j \in J}$ of Theorem 4.0.1.

We conclude this chapter with another interesting result due to Kionke and Schesler: they were able to prove that it suffices to pass from amenability to uniform amenability in order to get a property that is profinite invariant. Indeed, we have the following theorem.

Theorem 4.0.2 (Kionke and Schesler [KS23])**.** *Let G_1 and G_2 be residually finite groups with $\widehat{G_1} \cong \widehat{G_2}$. Then, G_1 is uniformly amenable if and only if G_2 is* *uniformly amenable.*

The proof of such result, discussed in Section 4.4, relies on the characterization of uniform amenability of a group via uniform amenability of a special family of its quotients. Notice that, as a consequence of Theorem 4.0.1 and Theorem 4.4.4, the Ω-construction of Section 4.2 gives us a new example of finitely generated amenable groups that fail to be uniformly amenable.

4.1 Groups acting on rooted trees

A *rooted tree* is a tree T with a distinguished vertex $\varnothing$, called the *root* of T. We are interested in rooted trees arising as Cayley graphs of free monoids, hence we start by giving the necessary notation and basic results. More details can be found in [Sid98], [GNS00], [BGŠ03] and [Nek05].

Let X be a non-empty finite set, which we call *alphabet*. Let X^* be the free monoid generated by X: its elements are finite words over X, i.e.,

$$X^* = \{x_1 x_2 \dots x_n \mid n \in \mathbb{N} \text{ and } x_i \in X \text{ for } i = 1, \dots, n\}.$$

The composition of words is given by concatenation, and we denote by $\varnothing$ the empty word. The *length* of a word $w = x_1 x_2 \dots x_n$ is simply the number of letters of w, and it is denoted by $|w|$.

The set X^* can be naturally considered as vertex set of a rooted tree. Indeed, let T_X be the Cayley graph associated to X^* with respect to X: the vertices are given by words over X, the root is the empty word $\varnothing$ and two words are connected by an edge if one can be obtained from the other by right-adjunction of a letter in the alphabet X. We say that two vertices are *adjacent* if they are connected by an edge.

The set $X^n \subseteq X^*$ of words of length n is called n^{th} *level* of the tree. The *boundary* ∂T_X of the tree is given by the set of infinite sequences over the alphabet X.

From now on, when we consider rooted trees we will always mean those T_X arising from the above construction.

Definition 4.1. Let T_X be a rooted tree. An *automorphism* of T_X is a bijection of the vertices $\alpha : X^* \to X^*$ that preserves the root and the adjacency of the vertices.

The set of all automorphisms of the rooted tree T_X is a group with respect to composition, and we will denote it by $\mathrm{Aut}(T_X)$. Since an automorphism $\alpha \in \mathrm{Aut}(T_X)$ fixes the root, one can easily obtain by induction that $\alpha(X^n) \subseteq X^n$. In other words, α preserves the level sets of the tree. This implies that we have a natural homomorphism

$$\pi_n : \mathrm{Aut}(T_X) \to \mathrm{Sym}(X^n).$$

We now define some important subgroups of $\mathrm{Aut}(T_X)$.

Definition 4.2. Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup and let v be a vertex of T_X.
(i) The *stabilizer* of v in G is the subgroup $\mathrm{St}_G(v) = \{\alpha \in G \mid \alpha(v) = v\}$.
(ii) For $n \in \mathbb{N}$, the *level n stabilizer* of G is defined as

$$\mathrm{St}_G(n) = \bigcap_{v \in X^n} \mathrm{St}_G(v).$$

Namely, the level n stabilizer of $G \leq \mathrm{Aut}(T_X)$ is the subgroup of G that consists of all automorphisms that fix all the vertices on the level n (and up).

Lemma 4.1.1. *Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup. For every $n \in \mathbb{N}$, the level n stabilizer $\mathrm{St}_G(n)$ is a finite index normal subgroup of G.*

Proof. This immediately follows from the fact that $\mathrm{St}_G(n)$ is the kernel of the restriction to G of the natural map $\pi_n : \mathrm{Aut}(T_X) \to \mathrm{Sym}(X^n)$. □

Furthermore, since

$$\bigcap_{n \in \mathbb{N}} \mathrm{St}_{\mathrm{Aut}(T_X)}(n) = \{1_{\mathrm{Aut}(T_X)}\},$$

we have that Lemma 4.1.1 implies the following corollary.

Corollary 4.1.2. *The group $\mathrm{Aut}(T_X)$ is residually finite.*

Another important family of subgroups of $\mathrm{Aut}(T_X)$ is given by *rigid stabilizers*. Let v be a vertex of T_X, and denote by $(T_X)_v$ the subtree of T_X whose vertex set is given by vX^*, i.e., the tree whose vertices are words starting with v. Then, $(T_X)_v$ is naturally isomorphic to T_X via the map

$$\begin{aligned} T_X &\to (T_X)_v \\ u &\mapsto vu. \end{aligned}$$

Definition 4.3. Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup and let v be a vertex of T_X.
(i) The *rigid stabilizer* of v in G, denote by $\mathrm{RiSt}_G(v)$, is the subgroup of $\mathrm{St}_G(v)$ consisting of all automorphisms of G that fix every vertex outside from the subtree $(T_X)_v$.
(ii) For $n \in \mathbb{N}$, the *rigid level n stabilizer* is defined as

$$\mathrm{RiSt}_G(n) = \left\langle \bigcup_{v \in X^n} \mathrm{RiSt}_G(v) \right\rangle.$$

It follows from the definition that automorphisms in different rigid stabilizers of vertices at the same level commute. Moreover,

$$\mathrm{RiSt}_G(n) = \prod_{v \in X^n} \mathrm{RiSt}_G(v).$$

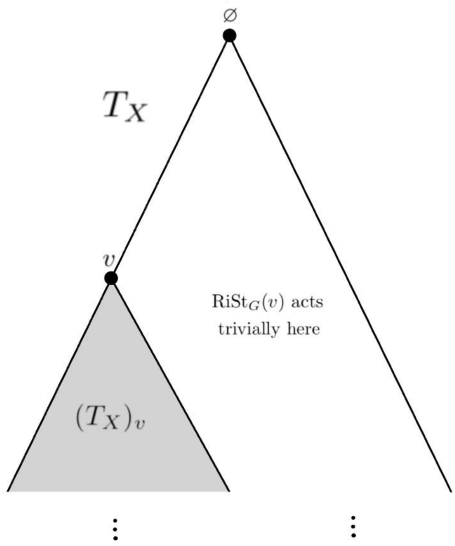


Figure 4.1: The rigid stabilizer of a vertex v in G.

If $G = \mathrm{Aut}(T_X)$, then the rigid level n stabilizers $\mathrm{RiSt}_{\mathrm{Aut}(T_X)}(n)$ coincide with the n level stabilizers. However, this is not true for general proper subgroups of $\mathrm{Aut}(T_X)$: rigid level n stabilizers may have infinite index, and they may even be trivial.

Definition 4.4. Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup. We say that G is a *branch group* if the index of the rigid stabilizer $\mathrm{RiSt}_G(n)$ in G is finite for every $n \in \mathbb{N}$.

Furthermore, it is useful to consider the following embedding of a subgroup in $\mathrm{Aut}(T_X)$.

Definition 4.5. Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup and $v \in X^n$. We define the embedding $\iota_v : G \to \mathrm{Aut}(T_X)$ as

$$\iota_v(\alpha)(uw) = \begin{cases} u\alpha(w) & \text{if } u = v \\ uw & \text{if } u \neq v \end{cases}$$

for every $\alpha \in G$, $w \in X^*$ and $u \in X^n$.

Therefore, $\iota_v(\alpha)$ acts as α on the subtree $(T_X)_v$ rooted at the vertex $v \in X^n$, and as the identity outside from such subtree. We notice that

$$\mathrm{RiSt}_G(v) = \iota_v(G) \cap G.$$

We are now interested in giving a convenient description of the automorphisms of T_X. We start with an easy example.

Example 4.1. A *rooted automorphism* α is an automorphism of T_X that permutes rigidly the main subtrees $(T_X)_x$ for $x \in X$ according to a permutation σ of $\mathrm{Sym}(X)$:

$$\alpha(xw) = \sigma(x)w$$

for every $x \in X$ and $w \in X^*$.

Recall that, since $\mathrm{Aut}(T_X)$ preserves the first level of T_X, we have a natural homomorphism

$$\begin{aligned} G &\longrightarrow \mathrm{Sym}(X) \\ \alpha &\longmapsto \sigma_\alpha. \end{aligned}$$

In this way, every automorphism $\alpha \in \mathrm{Aut}(T_X)$ determines a permutation $\sigma_\alpha \in \mathrm{Sym}(X)$. Given $x \in X$, the subtree $(T_X)_x$ is sent by α into $(T_X)_{\sigma_\alpha(x)}$. Since both $(T_X)_x$ and $(T_X)_{\alpha(x)}$ are naturally isomorphic to T_X, we can give the following definition.

Definition 4.6. Let $\alpha \in \mathrm{Aut}(T_X)$ and $x \in X$. The *state* of α at x is the unique automorphism $\alpha_x \in \mathrm{Aut}(T_X)$ such that

$$\alpha(xw) = \sigma_\alpha(x)\alpha_x(w)$$

for every $w \in X^*$.

Conversely, given a permutation $\sigma \in \mathrm{Sym}(X)$ and a collection of automorphisms $(\alpha_x)_{x\in X} \in \mathrm{Aut}(T_X)$, we can define as above an automorphism α of T_X. This gives a one-to-one correspondence

$$\alpha \mapsto (\sigma_\alpha, (\alpha_x)_{x\in X})$$

between $\mathrm{Aut}(T_X)$ and $\mathrm{Sym}(X) \times \mathrm{Aut}(T_X)^X$, where $\mathrm{Sym}(X) \times \mathrm{Aut}(T_X)^X$ acts on T_X by

$$((\sigma, (\alpha_x)), yw) = \sigma(y)\alpha_y(w)$$

for evert $y \in X$ and $w \in X^*$. We will directly write (α_x) instead of $(\alpha_x)_{x\in X}$, and we call $(\sigma_\alpha, (\alpha_x))$ the *wreath decomposition* of the automorphism α.

Using such decompositions, the multiplication rule in $\mathrm{Aut}(T_X)$ takes the following form:

$$(\sigma_\alpha, (\alpha_x)) \cdot (\sigma_\beta, (\beta_x)) = (\sigma_\alpha\sigma_\beta, (\alpha_{\sigma_\beta(x)}\beta_x)). \qquad (*)$$

Let us now consider the *permutational wreath product* $\mathrm{Sym}(X) \wr \mathrm{Aut}(T_X)$: this is defined as the semidirect product $\mathrm{Sym}(X) \ltimes \mathrm{Aut}(T_X)^X$, where $\mathrm{Sym}(X)$ acts on $\mathrm{Aut}(T_X)^X$ by permuting the factors. The multiplication rule in $(*)$ tells us that the wreath decomposition gives an isomorphism between $\mathrm{Aut}(T_X)$ and $\mathrm{Sym}(X) \wr \mathrm{Aut}(T_X)$:

$$\begin{aligned} \mathrm{Aut}(T_X) &\to \mathrm{Sym}(X) \wr \mathrm{Aut}(T_X) \\ \alpha &\mapsto (\sigma_\alpha, (\alpha_x)_{x\in X}). \end{aligned}$$

Notice that the subgroup $\mathrm{Sym}(X) \leq \mathrm{Sym}(X) \wr \mathrm{Aut}(T_X)$ is identified with the subgroup of rooted automorphisms $\alpha = (\sigma_\alpha, (\mathrm{id}_x))$ of Example 4.1. On the other hand, the subgroup $\mathrm{Aut}(T_X)^X \leq \mathrm{Sym}(X) \wr \mathrm{Aut}(T_X)$ is identified with the first level stabilizer $\mathrm{St}_{\mathrm{Aut}(T_X)}(1)$.

Definition 4.7. Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup. We say that G is *self-similar* if, for every $\alpha \in G$, the state α_x of α at x lies in G for every $x \in X$.

We conclude by highlighting that we can easily extend the notion of *state* of an automorphism α at $x \in X$ to the state at a vertex v of T_X. Since $\alpha((T_X)_v) = (T_X)_{\alpha(v)}$, by restricting α to the subtree $(T_X)_v$ and identifying its image $(T_X)_{\alpha(v)}$ with T_X, we can generalize Definition 4.6 to arbitrary words in X^*.

Definition 4.8. Let $\alpha \in \mathrm{Aut}(T_X)$ and $v \in X^*$. The *state* of α at v is the unique automorphism α_v of T_X such that

$$\alpha(vw) = \alpha(v)\alpha_v(w)$$

for every $w \in X^*$.

4.1.1 $\mathrm{Aut}(T_X)$ as a profinite group

Now, our goal is to show that, given a rooted tree T_X, then $\mathrm{Aut}(T_X)$ can be regarded as a profinite group. In order to do so, we start by considering the set of vertices $X^{\leq n}$ of length at most n for $n \in \mathbb{N}$. The resulting subtree will be the finite truncated tree denoted by $(T_X)_n$. The set of all automorphisms of $(T_X)_n$ is the finite group $\mathrm{Aut}((T_X)_n)$.

Remark 4.1. It immediately follows from the definition of automorphisms of rooted trees that every $\alpha \in \mathrm{Aut}((T_X)_n)$ is completely determined by its action on X^n, that is, by the images of the words of length n.

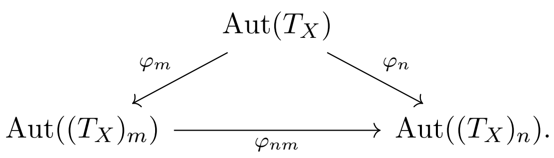
Let us denote by $\varphi_n : \mathrm{Aut}(T_X) \to \mathrm{Aut}((T_X)_n)$ the natural restriction homomorphism. We promptly obtain the following result.

Lemma 4.1.3. *Let $n \in \mathbb{N}$. Then,*

$$\mathrm{Aut}(T_X)/\,\mathrm{St}_{\mathrm{Aut}(T_X)}(n) \cong \mathrm{Aut}((T_X)_n).$$

Proof. The restriction homomorphism $\varphi_n : \mathrm{Aut}(T_X) \to \mathrm{Aut}((T_X)_n)$ is surjective and its kernel is given by $\mathrm{St}_{\mathrm{Aut}(T_X)}(n)$. □

Furthermore, whenever $n \leq m$, the restriction homomorphisms induce a homomorphism $\varphi_{nm} : \mathrm{Aut}((T_X)_m) \to \mathrm{Aut}((T_X)_n)$ which makes the following diagram commute:



Together with these maps φ_{nm}, the automorphism groups of the truncated trees $(T_X)_n$ form an inverse system of finite groups:

$$\cdots \to \mathrm{Aut}((T_X)_{n+1}) \to \mathrm{Aut}((T_X)_n) \to \cdots \to \mathrm{Aut}((T_X)_1) \cong \mathrm{Sym}(X).$$

The inverse limit of this inverse system is precisely given by the full automorphism group of T_X:

$$\mathrm{Aut}(T_X) = \varprojlim_{n\in\mathbb{N}} \mathrm{Aut}((T_X)_n)$$

together with the natural projection maps $\pi_n : \mathrm{Aut}(T_X) \to \mathrm{Aut}((T_X)_n)$. This can be easily seen by considering the injective group homomorphism

$$\varphi : \mathrm{Aut}(T_X) \to \prod_{n\in\mathbb{N}} \mathrm{Aut}((T_X)_n)$$
$$\alpha \mapsto (\varphi_n(\alpha))_{n\in\mathbb{N}}$$

and noticing that

$$\mathrm{im}(\varphi) \cong \{(\alpha_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathrm{Aut}(T_X) \mid \varphi_{nm}(\alpha_m) = \alpha_n \text{ for all } n \leq m\}.$$

Hence, we can conclude by Proposition 1.1.1 that $\mathrm{Aut}(T_X)$ is indeed the inverse limit of above the inverse system. Furthermore, thanks to Lemma 4.1.3, we get

$$\mathrm{Aut}(T_X) \cong \varprojlim_{n\in\mathbb{N}} \mathrm{Aut}(T_X)/\,\mathrm{St}_{\mathrm{Aut}(T_X)}(n).$$

We can finally write the following proposition.

Proposition 4.1.4. *The automorphism group of a rooted tree* $\mathrm{Aut}(T_X)$ *can be regarded as a profinite group where* $\{\mathrm{St}_{\mathrm{Aut}(T_X)}(n)\}_{n\in\mathbb{N}}$ *is a basis of neighbourhoods of the identity.*

Proof. The kernel of the projection $\pi_n : \mathrm{Aut}(T_X) \to \mathrm{Aut}((T_X)_n)$ is given by $\mathrm{St}_{\mathrm{Aut}(T_X)}(n)$. Therefore, Lemma 1.2.6 implies that the level n stabilizers form a basis of open neighbourhoods of the identity in $\mathrm{Aut}(T_X)$. □

4.1.2 The congruence subgroup property

From now on, we will always regard $\mathrm{Aut}(T_X)$ as a profinite group, that is, as a compact, Hausdorff and totally disconnected topological group. We call *profinite topology* the topology that makes $\mathrm{Aut}(T_X)$ into a profinite group. This allows us to give the following definition.

Definition 4.9. Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup. Its *tree completion* $\overline{G}$ is the closure of G in $\mathrm{Aut}(T_X)$ with respect to the profinite topology.

Proposition 4.1.5. *Let $G \leq \operatorname{Aut}(T_X)$ be a subgroup. Then,*

$$\overline{G} \cong \varprojlim_{n \in \mathbb{N}} G / \operatorname{St}_G(n).$$

Proof. The tree completion $\overline{G}$ is a closed subgroup of the profinite group $\operatorname{Aut}(T_X)$, hence it is a profinite group by Proposition 1.2.9. Moreover, the level n stabilizers $\{\operatorname{St}_{\operatorname{Aut}(T_X)}(n)\}_{n \in \mathbb{N}}$ form a basis of neighbourhoods of the identity in $\operatorname{Aut}(T_X)$ from Proposition 4.1.4 and $\operatorname{St}_{\operatorname{Aut}(T_X)}(n) \cap G = \operatorname{St}_G(n)$. We can therefore conclude by Corollary 1.2.11 that $\overline{G} \cong \varprojlim G / \operatorname{St}_G(n)$. □

Now, given a subgroup $G \leq \operatorname{Aut}(T_X)$, we have two *completions* of G: the profinite completion $\widehat{G}$ and the tree completion $\overline{G}$. Let us recall that we have a canonical homomorphism $\iota : G \to \widehat{G}$, which is an embedding if and only if G is residually finite. This is precisely the case when working in $\operatorname{Aut}(T_X)$, thanks to Corollary 4.1.2 and Proposition 1.3.8. From the universal property of ι (Proposition 1.3.2), we obtain a canonical homomorphism

$$\operatorname{res}_G : \widehat{G} \to \overline{G}$$

which makes the following diagram commute:

$$\begin{array}{ccc} G & \overset{\iota}{\hookrightarrow} & \widehat{G} \\ & \searrow & \big\downarrow \operatorname{res}_G \\ & & \overline{G}. \end{array}$$

The map res_G allows to extend the action of G on the rooted tree T_X to an action of $\widehat{G}$ on T_X. Furthermore, since G is dense both in $\overline{G}$ and $\widehat{G}$ (Proposition 1.3.1), we have that res_G is always surjective. A condition for res_G to be an isomorphism was introduced by Grigorchuk in [Gri00], §10.

Definition 4.10. Let $G \leq \operatorname{Aut}(T_X)$ be a subgroup. We say that G satisfies the *congruence subgroup property* (CSP) if, for every normal subgroup of finite index $N \trianglelefteq_f G$, there exists $n \in \mathbb{N}$ such that $\operatorname{St}_G(n) \leq N$.

Proposition 4.1.6. *Let $G \leq \operatorname{Aut}(T_X)$ be a subgroup. If G satisfies the congruence subgroup property, then*

$$\operatorname{res}_G : \widehat{G} \xrightarrow{\cong} \overline{G}$$

is an isomorphism.

Proof. This immediately follows from the descriptions of $\widehat{G}$ and $\overline{G}$, and the fact that the level n stabilizers $\{\operatorname{St}_{\operatorname{Aut}(T_X)}(n)\}_{n \in \mathbb{N}}$ form a basis of neighbourhoods of the identity in $\operatorname{Aut}(T_X)$ from Proposition 4.1.4. □

The congruence subgroup property will be a key ingredient in the proof of Theorem 4.0.1. Indeed, knowing that a group satisfies such property tells us that, in order to understand its profinite completion -the object that interests us in the first place-, it suffices to give a description of its tree completion. We will see that, in our situation, the latter will be easier.

We conclude this section with a useful result which provides a way of showing that a group satisfies the congruence subgroup property. Recall that we write $G' = [G, G]$ to denote the commutator subgroup of a group G.

Lemma 4.1.7. *Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup that acts level-transitively on the rooted tree T_X. Then, for every non-trivial normal subgroup $N \trianglelefteq G$, there is some $n \in \mathbb{N}$ with $\mathrm{RiSt}_G(n)' \leq N$.*

Proof. This was extracted by Segal ([Seg00], Lemma 4) from the proof of [Gri00], Theorem 4. □

Definition 4.11. An infinite group G is said to be *just infinite* if every proper quotient of G is finite.

Proposition 4.1.8. *Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup that acts level-transitively on T_X. Suppose that $\mathrm{St}_G(n) = \mathrm{RiSt}_G(n)$ for every $n \in \mathbb{N}$ and that, for every $v \in X^*$, the rigid stabilizer $\mathrm{RiSt}_G(v)$ is perfect. Then, G is just infinite and satisfies the congruence subgroup property.*

Proof. Let $N \trianglelefteq G$ be a non-trivial normal subgroup of G: we want to find a level n stabilizer $\mathrm{St}_G(n)$ for some $n \in \mathbb{N}$ such that $\mathrm{St}_G(n) \leq N$. From Lemma 4.1.7, there exists $n \in \mathbb{N}$ such that $\mathrm{RiSt}_G(n)' \leq N$. Since we are assuming that all the rigid stabilizers are perfect, we have

$$\mathrm{RiSt}_G(v) = \mathrm{RiSt}_G(v)' \leq \mathrm{RiSt}_G(n)'$$

for every $v \in X^n$. Furthermore, the level n stabilizer

$$\mathrm{St}_G(n) = \mathrm{RiSt}_G(n) = \left\langle \bigcup_{v \in X^n} \mathrm{RiSt}_G(v) \right\rangle$$

is generated by all the rigid stabilizers of vertices of level n, implying that

$$\mathrm{St}_G(n) = \mathrm{RiSt}_G(n)' \leq N.$$

This yields that G satisfies the congruence subgroup property. Moreover, G is just infinite since $\mathrm{St}_G(n)$ has finite index in G from Lemma 4.1.1. □

4.2 Ω-construction

This section is entirely focused on the *Ω-construction*, a technical construction which, given a perfect, self-similar subgroup $G \leq \mathrm{Aut}(T_X)$ satisfying *property H*, produces a new subgroup $\Gamma_G^\Omega \leq \mathrm{Aut}(T_X)$ depending on a certain subset Ω of the boundary of the tree T_X. After introducing the necessary definitions, we prove that the groups Γ_G^Ω are just-infinite branch groups satisfying the congruence subgroup property. Passing through the tree completion, we finally show that we obtain a family of groups with isomorphic profinite completions and such that the inclusions between them induce isomorphisms of the profinite completions.

We start by fixing a non-empty finite alphabet X and an element $o \in X$. We denote by $\mathcal{S}$ the space of infinite sequences $(\omega_n)_{n\in\mathbb{N}}$ over the set $X \setminus \{o\}$. Furthermore, we consider on $\mathcal{S}$ the *left-shift operator*

$$\begin{aligned} L : \mathcal{S} &\to \mathcal{S} \\ (\omega_1, \omega_2, \omega_3, \dots) &\mapsto (\omega_2, \omega_3, \dots). \end{aligned}$$

For every non-empty subset $\Omega \subseteq \mathcal{S}$, we denote by

$$L(\Omega) = \{L(\omega) \mid \omega \in \Omega\} \subseteq \mathcal{S}$$

its image under the shift operator.

Definition 4.12. Let us fix a sequence $\omega = (\omega_n)_{n\in\mathbb{N}} \in \mathcal{S}$. We define the homomorphism

$$\begin{aligned} \widetilde{\cdot}^{\omega} : \mathrm{Aut}(T_X) &\to \mathrm{Aut}(T_X) \cong \mathrm{Sym}(X) \wr \mathrm{Aut}(T_X) \\ \alpha &\mapsto \widetilde{\alpha}^{\omega} = (\mathrm{id}, (\widetilde{\alpha}_x)_{x\in X}) \end{aligned}$$

where

$$\tilde{\alpha}_x = \begin{cases} \widetilde{\alpha}^{L(\omega)} & \text{if } x = o \\ \alpha & \text{if } x = \omega_1 \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup. We define Γ_G^ω as the subgroup of $\mathrm{Aut}(T_X)$ generated by G and $\widetilde{G}^\omega$. For every non-empty subset $\Omega \subseteq \mathcal{S}$, we define Γ_G^Ω as the subgroup of $\mathrm{Aut}(T_X)$ generated by all groups Γ_G^ω with $\omega \in \Omega$.

Remark 4.2. (i) The groups G and $\widetilde{G}^\omega$ are isomorphic, since the homomorphism $\widetilde{\cdot}^{\omega}$ has trivial kernel.
(ii) The group Γ_G^ω is generate by automorphisms of the form $\alpha = (\sigma_\alpha, (\alpha_x))$ with either $\alpha_x \in G$ or $\alpha_x \in \widetilde{G}^{L(\omega)}$.

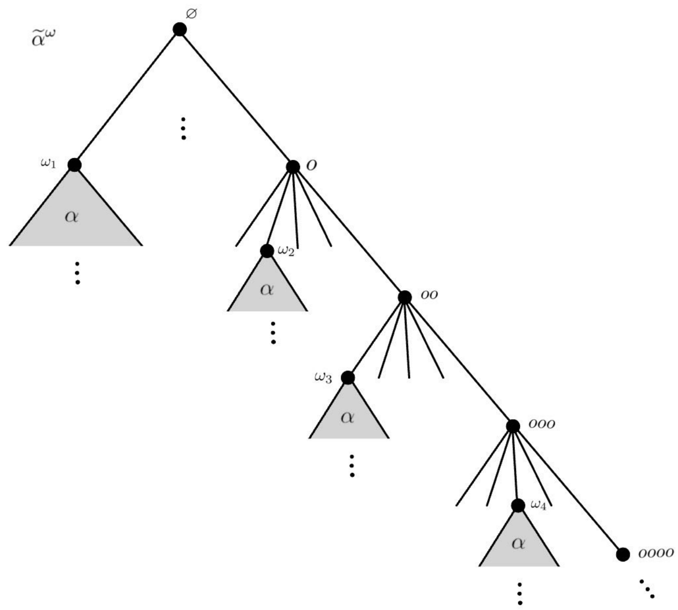


Figure 4.2: Visual representation of the automorphism $\widetilde{\alpha}^{\omega}$.

We now want to investigate how the groups $\Gamma_{\mathrm{G}}^{\Omega}$ behave. The first step is showing that they satisfy the congruence subgroup property: to this end, we wish to apply Proposition 4.1.8. We will therefore focus on understanding the rigid stabilizers $\mathrm{RiSt}_{\Gamma_{\mathrm{G}}^{\Omega}}(v)$ for $v \in X^*$. Before starting, we need to assume a further hypothesis on the initial subgroup $G \leq \mathrm{Aut}(T_X)$.

Definition 4.13. Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup. We say that G satisfies *property H* if the following hold:
(i) G acts transitively on X;
(ii) for all $x \neq y$ in X, there exists $\alpha \in \mathrm{St}_G(x)$ such that $\alpha \notin \mathrm{St}_G(y)$.

Let us recall that, given a subgroup $G \leq \mathrm{Aut}(T_X)$ and a vertex $v \in X^n$, in Definition 4.5 we have defined an embedding $\iota_v : G \to \mathrm{Aut}(T_X)$ behaving in the following way. If $\alpha \in G$, then $\iota_v(\alpha)$ acts as α on the subtree $(T_X)_v$ rooted at the vertex $v \in X^n$, and it acts as the identity outside from such subtree.

Lemma 4.2.1. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. For every $\omega = (\omega_n)_{n\in\mathbb{N}} \in \mathcal{S}$, we have*

$$\iota_{\omega_1}(G) \subseteq \mathrm{RiSt}_{\Gamma_G^{\omega}}(\omega_1).$$

Proof. Let us consider an arbitrary automorphism $\alpha \in G$ and its image $\widetilde{\alpha}^{\omega} = (\mathrm{id}, (\widetilde{\alpha}_x))$ in the group $\widetilde{G}^{\omega}$. Since G satisfies property H and $\omega_1 \neq o$, there exists $\beta \in G$ such that $\beta(\omega_1) = \omega_1$ and $\beta(o) \neq o$. We write $\beta = (\sigma, (\beta_x))$ for the wreath decomposition of β. We now conjugate $\widetilde{\alpha}^\omega$ by β:

$$\begin{aligned}\beta\widetilde{\alpha}^\omega\beta^{-1} &= (\sigma, (\beta_x)) \cdot (\mathrm{id}, (\widetilde{\alpha}_x)) \cdot (\sigma, (\beta_x))^{-1} \\ &= (\sigma, (\beta_x)) \cdot (\mathrm{id}, (\widetilde{\alpha}_x)) \cdot (\sigma^{-1}, (\beta^{-1}_{\sigma^{-1}(x)})) \\ &= (\mathrm{id}, (\beta_{\sigma^{-1}(x)}\widetilde{\alpha}_{\sigma^{-1}(x)}\beta^{-1}_{\sigma^{-1}(x)})).\end{aligned}$$

The state of $\beta\widetilde{\alpha}^\omega\beta^{-1}$ at $x = \omega_1$ is given by

$$\beta_{\omega_1}\widetilde{\alpha}_{\omega_1}\beta^{-1}_{\omega_1} = \beta_{\omega_1}\alpha\beta^{-1}_{\omega_1}.$$

As the group G is self-similar, we get that $\beta_{\omega_1}\alpha\beta^{-1}_{\omega_1} \in G$. Moreover, the state of $\beta\widetilde{\alpha}^\omega\beta^{-1}$ is the identity at every $x \neq \{\omega_1, \beta(o)\}$.

Now, let $\gamma \in G$ be another arbitrary automorphism, and consider the commutator

$$[\widetilde{\gamma}^\omega, \beta\widetilde{\alpha}^\omega\beta^{-1}] = (\mathrm{id}, ([\widetilde{\gamma}_x, \beta_{\sigma^{-1}(x)}\widetilde{\alpha}_{\sigma^{-1}(x)}\beta^{-1}_{\sigma^{-1}(x)}])).$$

Notice that $\widetilde{\gamma}$ acts only below ω_1 and o, while $\beta\widetilde{\alpha}^\omega\beta^{-1}$ acts only below ω_1 and $\beta(o)$. This implies that the state of such commutator is the identity at $x \neq \omega_1$, and it is given by $[\gamma, \beta_{\omega_1}\alpha\beta^{-1}_{\omega_1}]$ at $x = \omega_1$. As $[\widetilde{\gamma}^\omega, \beta\widetilde{\alpha}^\omega\beta^{-1}]$ acts only below ω_1, it lies in $\mathrm{RiSt}_{\Gamma^\omega_G}(\omega_1)$. More specifically, we get that

$$\iota_{\omega_1}([\gamma, \beta_{\omega_1}\alpha\beta^{-1}_{\omega_1}]) \subseteq \mathrm{RiSt}_{\Gamma^\omega_G}.$$

Now, since the group G is perfect and the two automorphisms $\alpha, \gamma \in G$ were chosen arbitrarily, we can conclude that $\iota_{\omega_1}(G) \subseteq \mathrm{RiSt}_{\Gamma^\omega_G}(\omega_1)$. □

Lemma 4.2.2. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. For every non-empty $\Omega \subseteq \mathcal{S}$ and every $x \in X$, we have*

$$\mathrm{RiSt}_{\Gamma^\Omega_G}(x) = \iota_x(\Gamma^{L(\Omega)}_G).$$

Proof. Let us consider an arbitrary $\omega = (\omega_n)_{n\in\mathbb{N}} \in \Omega$. Let $x \in X$ and $\alpha \in \mathrm{RiSt}_{\Gamma^\omega_G}(x)$: the automorphism α acts trivially outside from $(T_X)_x$ and it can be written as $\alpha = (\sigma_\alpha, (\alpha_x))$ with either $\alpha_x \in G$ or $\alpha_x \in \widetilde{G}^{L(\omega)}$. Therefore, $\mathrm{RiSt}_{\Gamma^\omega_G}(x) \subseteq \iota_x(\Gamma^{L(\omega)}_G)$. By arbitrariness, we get that

$$\mathrm{RiSt}_{\Gamma^\Omega_G}(x) \subseteq \iota_x(\Gamma^{L(\Omega)}_G)$$

holds for every $x \in X$. In order to prove the converse inclusion, let us start by recalling that, from Lemma 4.2.1, we have $\iota_{\omega_1}(G) \subseteq \mathrm{RiSt}_{\Gamma^\omega_G}(\omega_1)$. Since the group G acts transitively on X, there exists $\beta \in G$ such that $\beta(\omega_1) = o$. We can write $\beta = (\sigma_\beta, (\beta_x))$, where each β_x lies in G by self-similarity of G. Using the wreath decomposition, it is readily checked that

$$\beta\iota_{\omega_1}(G)\beta^{-1} = \iota_o(G).$$

Hence,

$$\iota_o(G) \subseteq \beta \,\mathrm{RiSt}_{\Gamma_G^\omega}(\omega_1)\beta^{-1} = \mathrm{RiSt}_{\Gamma_G^\omega}(o).$$

Given $\alpha \in G$, notice that $\widetilde{\alpha}^\omega \iota_{\omega_1}(\alpha)^{-1} = \iota_o(\widetilde{\alpha}^{L(\omega)})$. Therefore, a further application of Lemma 4.2.1 implies that

$$\iota_o(\widetilde{G}^{L(\omega)}) \subseteq \mathrm{RiSt}_{\Gamma_G^\omega}(o).$$

Since the group $\Gamma_G^{L(\Omega)}$ is generated by G and all $\widetilde{G}^{L(\Omega)}$ for $\omega \in \Omega$, we obtain

$$\iota_o(\Gamma_G^{L(\Omega)}) \subseteq \mathrm{RiSt}_{\Gamma_G^\Omega}(o).$$

Using again the fact that the group G is self-similar and acts transitively on X, we finally get that

$$\iota_x(\Gamma_G^{L(\Omega)}) \subseteq \mathrm{RiSt}_{\Gamma_G^\Omega}(x)$$

holds for every $x \in X$. □

Corollary 4.2.3. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. For every non-empty $\Omega \subseteq \mathcal{S}$ and every word $v \in X^n$ of length n, the rigid stabilizer of v in Γ_G^Ω is given by*

$$\mathrm{RiSt}_{\Gamma_G^\Omega}(v) = \iota_v(\Gamma_G^{L^n(\Omega)}).$$

Moreover, for every $n \in \mathbb{N}$, we have

$$\mathrm{RiSt}_{\Gamma_G^\Omega}(n) = \mathrm{St}_{\Gamma_G^\Omega}(n).$$

Proof. In order to prove the first part of the statement, we proceed by induction on the length n of the word v. If v is the empty word, then we have nothing to show. Assume that $\mathrm{RiSt}_{\Gamma_G^\Omega}(v) = \iota_v(\Gamma^{L^n(\Omega)})$ holds for some $n \in \mathbb{N}$, and let us consider a word $w \in X^{n+1}$ of the form $w = vx$, with $v \in X^n$ and $x \in X$. By induction,

$$\mathrm{RiSt}_{\Gamma_G^\Omega}(w) = \mathrm{RiSt}_{\mathrm{RiSt}_{\Gamma_G^\Omega}(v)}(vx) = \mathrm{RiSt}_{\iota_v(\Gamma_G^{L^n(\Omega)})}(vx).$$

Using the definition of ι_v, one can check that

$$\mathrm{RiSt}_{\iota_v(\Gamma_G^{L^n(\Omega)})}(vx) = \iota_v(\mathrm{RiSt}_{\Gamma_G^{L^n(\Omega)}}(x)).$$

Now, Lemma 4.2.2 implies that

$$\iota_v(\mathrm{RiSt}_{\Gamma_G^{L^n(\Omega)}}(x)) = \iota_v(\iota_x(\Gamma_G^{L^{n+1}(\Omega)})) = \iota_w(\Gamma_G^{L^{n+1}(\Omega)}).$$

Therefore, by putting all the above equalities together, we finally obtain

$$\mathrm{RiSt}_{\Gamma_G^\Omega}(w) = \iota_w(\Gamma_G^{L^{n+1}(\Omega)}).$$

Let us now consider $\omega \in \Omega$ and $\alpha \in \mathrm{St}_{\Gamma_G^\omega}(n)$. Using the wreath decomposition, we can write $\alpha = (\sigma_\alpha, (\alpha_x))$ with either $\alpha_x \in G$ or $\alpha_x \in \widetilde{G}^{L(\omega)}$. Hence, the action of α on a word of the form vw, with $v \in X^n$ and $w \in X^*$, is given by

$$\alpha(vw) = v\alpha'(w)$$

for some α' either in G or in $\widetilde{G}^{L^n(\omega)}$. This implies that $\mathrm{St}_{\Gamma_G^\Omega}(n)$ is contained in the group generated by all $\iota_v(\Gamma_G^{L^n(\Omega)})$ with $v \in X^n$. As we have shown that $\iota_v(\Gamma_G^{L^n(\Omega)}) = \mathrm{RiSt}_{\Gamma_G^\Omega}(v)$, the above inclusion yields

$$\mathrm{RiSt}_{\Gamma_G^\Omega}(n) = \mathrm{St}_{\Gamma_G^\Omega}(n)$$

and this concludes the proof. □

Corollary 4.2.4. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar group that satisfies property H. For every non-empty $\Omega \subseteq \mathcal{S}$, we have that Γ_G^Ω is a branch group. Moreover, for every $v \in X^*$, the rigid stabilizer $\mathrm{RiSt}_{\Gamma_G^\Omega}(v)$ acts level-transitively on $(T_X)_v$.*

Proof. From Corollary 4.2.3, we have that $\mathrm{St}_{\Gamma_G^\Omega}(n) = \mathrm{RiSt}_{\Gamma_G^\Omega}$ for every $n \in \mathbb{N}$. Since the level n stabilizers have finite index in Γ_G^Ω by Lemma 4.1.1, we immediately obtain that Γ_G^Ω is a branch group. In order to show that the rigid stabilizers act level-transitively, let us proceed by induction. Fix a word $v \in X^n$ and consider two vertices in the first level of $(T_X)_v$: they are of the form vx, vy for $x, y \in X$. Since the group G satisfies property H, it acts transitively on X, therefore there exists $\alpha \in G$ such that $\alpha(x) = y$. Now, the automorphism $\iota_v(\alpha)$ sends vx to vy. As $\iota_v(G) \subseteq \mathrm{RiSt}_{\Gamma_G^\Omega}(v)$, we get that $\mathrm{RiSt}_{\Gamma_G^\Omega}(v)$ acts transitively on the first level of $(T_X)_v$. Let us now assume that $\mathrm{RiSt}_{\Gamma_G^\Omega}(v)$ acts transitively on the n^{th} level of $(T_X)_v$ and consider two vertices at level $n+1$. We write them as $wx, w'y$ with $w, w' \in X^n$ and $x, y \in X$. By induction, there exists $\alpha \in \mathrm{RiSt}_{\Gamma_G^\Omega}(v)$ such that $\alpha(w) = w'$. Assume that $\alpha(wx) = w'x'$ for some $x' \in X$. Using again the fact that the group G acts transitively on X, we can find $\beta \in G$ such that $\beta(x') = y$. The automorphism $\iota_{w'}(\beta)$ sends $w'x'$ to $w'y$, and we can therefore conclude by the fact that $\iota_{w'}(G) \subseteq \mathrm{RiSt}_{\Gamma_G^\Omega}(w') \leq \mathrm{RiSt}_{\Gamma_G^\Omega}(v)$. □

We can finally apply Proposition 4.1.8 to the groups Γ_G^Ω.

Theorem 4.2.5. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. Then, for every non-empty subset $\Omega \subseteq \mathcal{S}$, the group Γ_G^Ω is just infinite and satisfies the congruence subgroup property.*

Proof. We know from Corollary 4.2.3 that $\mathrm{RiSt}_{\Gamma_G^\Omega}(n) = \mathrm{St}_{\Gamma_G^\Omega}(n)$ for every $n \in \mathbb{N}$, while Corollary 4.2.4 tells us that Γ_G^Ω acts level-transitively on T_X.

Furthermore, given a word $v \in X^*$, the group $\mathrm{RiSt}_{\Gamma_{\mathrm{G}}^{\Omega}}(v) = \iota_v(\Gamma_G^{L^n(\Omega)})$ is generated by isomorphic copies of the perfect group G, therefore it is perfect. We can thus conclude by Proposition 4.1.8 that $\Gamma_{\mathrm{G}}^{\Omega}$ is just infinite and satisfies the congruence subgroup property. □

Now that we know that the groups $\Gamma_{\mathrm{G}}^{\Omega}$ satisfy the congruence subgroup property, the next step is describing their tree completion

$$\overline{\Gamma_{\mathrm{G}}^{\Omega}} \cong \varprojlim_{n \in \mathbb{N}} \Gamma_{\mathrm{G}}^{\Omega} \,/\, \mathrm{St}_{\Gamma_{\mathrm{G}}^{\Omega}}(n).$$

Understanding the quotients $\Gamma_{\mathrm{G}}^{\Omega} \,/\, \mathrm{St}_{\Gamma_{\mathrm{G}}^{\Omega}}(n)$ means understanding the action of $\Gamma_{\mathrm{G}}^{\Omega}$ on each level of the tree T_X. Our goal is now to show that the groups $\Gamma_{\mathrm{G}}^{\Omega}$ act on every level as iterated permutational wreath products which depend only on the action of G on X, and not even on the starting subset $\Omega \subseteq \mathcal{S}$.

Let us recall that, given two groups G and H with actions on sets X and Y, their permutational wreath product $G \wr_X H$ is the semidirect product $G \ltimes H^X$, where G acts on H^X by permuting the coordinates. Then, the group $G \wr_X H$ acts naturally on $X \times Y$ by

$$((g, (h_x)_{x \in X}), (x, y)) = (g(x), h_x(y)).$$

Let $Q \leq \mathrm{Sym}(X)$ and $n \in \mathbb{N}$. We consider the n-iterated permutational wreath product of Q:

$$\wr_X^n Q = Q \wr_X (Q \wr_X (\cdots (Q \wr_X Q)) \cdots).$$

The group $\wr_X^n Q$ naturally acts on the set X^n. Hence, we get an induced action on the rooted tree T_X as follows: given $\alpha \in \wr_X^n Q$, we set $\alpha(vw) = \alpha(v)w$ for all $v \in X^n$ and $w \in X^*$. In this way, we can regard $\wr_X^n Q$ as a subgroup of $\mathrm{Aut}(T_X)$.

Proposition 4.2.6. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. Let $Q \leq \mathrm{Sym}(X)$ denote the image of G under the canonical action on X. Then, for every non-empty subset $\Omega \subseteq \mathcal{S}$ and every $n \in \mathbb{N}$, the image of $\Gamma_{\mathrm{G}}^{\Omega}$ in $\mathrm{Aut}(T_X)/\,\mathrm{St}_{\mathrm{Aut}(T_X)}(n)$ is given by $\wr_X^n Q$.*

Proof. Let us consider an automorphism $\alpha \in \Gamma_{\mathrm{G}}^{\Omega}$: then, α has wreath decomposition

$$\alpha = (\sigma_\alpha, (\alpha_x)), \tag{$*$}$$

with $\sigma_\alpha \in Q$ and $\alpha \in \Gamma_G^{L(\Omega)}$. The action of α on the truncated tree $(T_X)_n$ is uniquely determined by its action on X^n, i.e., on words of the form $v = x_1 \ldots x_n$ of length n over X. Now, the image of the word v under α is given by

$$\alpha(v) = \sigma_1(x_1) \ldots \sigma_n(x_n) \tag{$**$}$$

for some appropriate permutations $\sigma_i \in Q$, $i = 1, \ldots, n$. This implies that the image of $\Gamma_{\mathrm{G}}^{\Omega}$ in the quotient $\mathrm{Aut}(T_X)/\,\mathrm{St}_{\mathrm{Aut}(T_X)}(n)$ is contained in $\wr_X^n Q$.

Conversely, recall that Lemma 4.2.2 yields that $\mathrm{RiSt}_{\Gamma_{\mathrm{G}}^{\Omega}}(x) = \iota_x(\Gamma_G^{L(\Omega)})$ for every $x \in X$. Then, using $(*)$, we obtain that, for every permutation $\sigma \in Q$, its corresponding rooted automorphism, which we still denote by σ, is contained in $\Gamma_{\mathrm{G}}^{\Omega}$. As from Corollary 4.2.3 we know that $\mathrm{RiSt}_{\Gamma_{\mathrm{G}}^{\Omega}}(v) = \iota_v(\Gamma_G^{L^n(\Omega)})$ for every $v \in X^*$, we obtain that

$$\iota_v(\sigma) \in \mathrm{RiSt}_{\Gamma_{\mathrm{G}}^{\Omega}}(v)$$

for every $v \in X^*$ and every $\sigma \in Q$. On account of $(**)$, we can therefore conclude that the image of $\Gamma_{\mathrm{G}}^{\Omega}$ in $\mathrm{Aut}(T_X)/\,\mathrm{St}_{\mathrm{Aut}(T_X)}(n)$ is precisely given by $\wr_X^n Q$. □

The iterated permutational wreath products $(\wr_X^n Q)_{n\in\mathbb{N}}$ form an inverse system of finite groups, where the natural maps $\varphi_{nm} : \wr_X^m Q \to \wr_X^n Q$ for $n \leq m$ are given by restricting the natural action of $\wr_X^m Q$ on X^m to the first n coordinates. We can therefore consider the inverse limit

$$\wr_X^\infty Q \coloneqq \varprojlim_{n\in\mathbb{N}} \wr_X^n Q.$$

The group $\wr_X^\infty Q$ naturally acts on T_X, hence it can be regarded as a subgroup of $\mathrm{Aut}(T_X)$. We notice that $\wr_X^\infty Q$ is closed in $\mathrm{Aut}(T_X)$, and as such it is uniquely determined by its actions on all finite levels of T_X. Passing through the tree completion, the following result finally gives us a description of the profinite completion of the groups $\Gamma_{\mathrm{G}}^{\Omega}$.

Proposition 4.2.7. *Let $G \leq \mathrm{Aut}(T_X)$ be a perfect, self-similar subgroup that satisfies property H. Let $Q \leq \mathrm{Sym}(X)$ denote the image of G under the canonical action on X. For every non-empty subset $\Omega \subseteq \mathcal{S}$, the canonical map*

$$\mathrm{res}_{\Gamma_{\mathrm{G}}^{\Omega}} : \widehat{\Gamma_{\mathrm{G}}^{\Omega}} \to \overline{\Gamma_{\mathrm{G}}^{\Omega}}$$

defines an isomorphism from $\widehat{\Gamma_{\mathrm{G}}^{\Omega}}$ onto $\wr_X^\infty Q \leq \mathrm{Aut}(T_X)$.

Proof. From Proposition 4.2.6, we know that

$$\overline{\Gamma_{\mathrm{G}}^{\Omega}} \cong \varprojlim_{n\in\mathbb{N}} \Gamma_{\mathrm{G}}^{\Omega} \,/\, \mathrm{St}_{\Gamma_{\mathrm{G}}^{\Omega}}(n) \cong \varprojlim_{n\in\mathbb{N}} \wr_X^n Q = \wr_X^\infty Q.$$

Hence, Theorem 4.2.5 implies that $\mathrm{res}_{\Gamma_{\mathrm{G}}^{\Omega}}$ gives an isomorphism between $\widehat{\Gamma_{\mathrm{G}}^{\Omega}}$ and $\wr_X^\infty Q$. □

At last, we obtain a family groups with embeddings that induce isomorphisms of the profinite completions.

Theorem 4.2.8. *Let $G, H \leq \mathrm{Aut}(T_X)$ be perfect, self-similar subgroups that satisfy property H. If the images of G and H in $\mathrm{Sym}(X)$ coincide, then*

$$\widehat{\Gamma^{\Omega}_{G}} \cong \widehat{\Gamma^{\Omega'}_{H}}$$

for all non-empty subsets $\Omega, \Omega' \subseteq \mathcal{S}$. If, moreover, $G \leq H$ and $\Omega \subseteq \Omega'$, then $\Gamma^{\Omega}_{G} \leq \Gamma^{\Omega'}_{H}$ and the inclusion map

$$u : \Gamma^{\Omega}_{G} \hookrightarrow \Gamma^{\Omega'}_{H}$$

induces an isomorphism

$$\widehat{u} : \widehat{\Gamma^{\Omega}_{G}} \to \widehat{\Gamma^{\Omega'}_{H}}.$$

Proof. The first part of the statement is a direct consequence of Proposition 4.2.7. Let $G \leq H$ and $\Omega \subseteq \Omega'$: the inclusion $\Gamma^{\Omega}_{G} \subseteq \Gamma^{\Omega'}_{H}$ simply follows from the definition. Let us denote by Q the image of G and H in $\mathrm{Sym}(X)$. Using the definitions of the maps involved and the fact that $\overline{\Gamma^{\Omega}_{G}} \cong \wr^{\infty}_{X} Q \cong \overline{\Gamma^{\Omega'}_{H}}$, we obtain that the following diagram commutes:

$$\begin{array}{ccccc}
\Gamma^{\Omega}_{G} & \xrightarrow{u} & \Gamma^{\Omega'}_{H} & & \\
\downarrow & & \downarrow & \searrow & \\
\widehat{\Gamma^{\Omega}_{G}} & \xrightarrow{\widehat{u}} & \widehat{\Gamma^{\Omega'}_{H}} & \xrightarrow{\mathrm{res}_{\Gamma^{\Omega'}_{H}}} & \wr^{\infty}_{X} Q. \\
& \multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{\mathrm{res}_{\Gamma^{\Omega}_{G}}}} &
\end{array}$$

As a consequence,

$$\mathrm{res}_{\Gamma^{\Omega}_{G}} = \mathrm{res}_{\Gamma^{\Omega'}_{H}} \circ \widehat{u}.$$

From Proposition 4.2.7, we finally conclude that the induced map $\widehat{u}$ is an isomorphism between the profinite completions of Γ^{Ω}_{G} and $\Gamma^{\Omega'}_{H}$. □

4.2.1 An uncountable family of groups

In this section, we show that, by adding the assumption that the starting subgroup G is countable, we obtain that the family of groups Γ^{Ω}_{G}, where $\Omega \subseteq \mathcal{S}$ is a non-empty finite subset, contains uncountably many isomorphism types of groups. The key idea is to combine a rigidity result from Lavreniuk and Nekrashevych [LN02] with the notion of *support volume* of an automorphism $\alpha \in \mathrm{Aut}(T_X)$, which measures the set of elements in the boundary of the tree T_X that are moved by α.

Definition 4.14. Let $G \leq \mathrm{Aut}(T_X)$ be a subgroup. We say that G is *rigid* in $\mathrm{Aut}(T_X)$ if every automorphism φ of G is induced by a conjugation of T_X, that is, if there exists $\gamma \in \mathrm{Aut}(T_X)$ such that $\varphi(\alpha) = \gamma\alpha\gamma^{-1}$ for every $\alpha \in G$.

Proposition 4.2.9. *Let $G \leq \mathrm{Aut}(T_X)$ be a branch group. Suppose that, for every vertex $v \in X^*$, the rigid stabilizer $\mathrm{RiSt}_G(v)$ acts level-transitively on the subtree $(T_X)_v$. Then, G is rigid in $\mathrm{Aut}(T_X)$.*

Proof. This is a special case of [LN02], Proposition 8.1. □

Lemma 4.2.10. *Let $G_1, G_2 \leq \mathrm{Aut}(T_X)$ be two branch groups such that, for every $v \in X^*$, the rigid stabilizers $\mathrm{RiSt}_{G_1}(v), \mathrm{RiSt}_{G_2}(v)$ act level-transitively on the subtree $(T_X)_v$. Suppose that G_1 and G_2 satisfy the congruence subgroup property and that $\overline{G_1} = \overline{G_2}$. Then, every isomorphism between G_1 and G_2 is induced by a conjugation in $\mathrm{Aut}(T_X)$.*

Proof. Suppose that $\varphi : G_1 \to G_2$ is an isomorphism. Since the groups G_1 and G_2 satisfy the congruence subgroup property, we know that the maps $\mathrm{res}_{G_i} : \widehat{G_i} \to \overline{G_i}$ are isomorphisms for $i = 1, 2$. Let us set $\overline{G} \coloneqq \overline{G_1} = \overline{G_2}$ and

$$\overline{\varphi} \coloneqq \mathrm{res}_{G_2} \circ \widehat{\varphi} \circ \mathrm{res}_{G_1}^{-1},$$

so that $\widehat{\varphi}$ is an automorphism of $\overline{G}$ making the following diagram commute:

$$\begin{array}{ccc} \widehat{G_1} & \xrightarrow{\widehat{\varphi}} & \widehat{G_2} \\ {\scriptstyle \mathrm{res}_{G_1}} \downarrow & & \downarrow {\scriptstyle \mathrm{res}_{G_2}} \\ \overline{G} & \xrightarrow[\overline{\varphi}]{} & \overline{G}. \end{array}$$

Notice that, as $G_1 \subseteq \overline{G}$, then $\varphi(\alpha) = \overline{\varphi}(\alpha)$ for every $\alpha \in G_1$. Let $v \in X^*$. Since $\mathrm{RiSt}_{\overline{G}}(v)$ contains $\mathrm{RiSt}_{G_i}(v)$ for $i = 1, 2$, we can apply Proposition 4.2.9: this yields an automorphism $\gamma \in \mathrm{Aut}(T_X)$ such that $\overline{\varphi}(\alpha) = \gamma\alpha\gamma^{-1}$ for every $\alpha \in \overline{G}$. Hence, $\varphi(\alpha) = \overline{\varphi}(\alpha) = \gamma\alpha\gamma^{-1}$ holds for every $\alpha \in G_1$, meaning that the isomorphism φ is induced by a conjugation in $\mathrm{Aut}(T_X)$. □

Definition 4.15. Let $\alpha \in \mathrm{Aut}(T_X)$ be an automorphism and let $n \in \mathbb{N}$. We denote by

$$\mathrm{Fix}_n(\alpha) = \{v \in X^n \mid \alpha(v) = v\}$$

the set of words in X of length n which are fixed by α. Moreover, we define the *support volume* of α as

$$\mathrm{vol}(\alpha) \coloneqq \lim_{n\to\infty} \frac{|X^n \setminus \mathrm{Fix}_n(\alpha)|}{|X^n|}.$$

First of all, we need to check that such support volume is well-defined. This follows from the fact that, if a vertex v is not fixed by α, then no descendant of v is. Hence, $\frac{|X^n \setminus \mathrm{Fix}_n(\alpha)|}{|X^n|}$ is a sequence of numbers that is non-decreasing and bounded by 1. Notice that the support volume measures the set of elements of the boundary of the tree which are not fixed by α.

Remark 4.3. The support volume is invariant under conjugation. Indeed, given $\gamma \in \mathrm{Aut}(T_X)$, one can easily check that $\mathrm{Fix}_n(\gamma\alpha\gamma^{-1}) = \gamma(\mathrm{Fix}_n(\alpha))$, and this yields

$$\mathrm{vol}(\alpha) = \mathrm{vol}(\gamma\alpha\gamma^{-1}).$$

Let us recall that, given a non-empty finite alphabet X, we fix an element $o \in X$ and denote by $\mathcal{S}$ the space of infinite sequences $(\omega_n)_{n\in\mathbb{N}}$ over the set $X \setminus \{o\}$.

Theorem 4.2.11. *Let X be a finite set with $|X| \geq 3$, and let $G \leq \mathrm{Aut}(T_X)$ be a non-trivial subgroup. For every $\omega \in \mathcal{S}$, the set of support volumes*

$$\{\mathrm{vol}(\alpha) \mid \alpha \in \Gamma_G^{\{\omega,\omega'\}}, \omega' \in \mathcal{S}\} \subseteq [0,1]$$

is uncountable.

Proof. Let $A \subseteq \mathbb{N}$. Since $|X| \geq 3$, for every $n \in \mathbb{N}$ we can pick an element $z_n \in X \setminus \{o, \omega_n\}$. We define a new infinite sequence $\omega' = \omega'(A)$ over $\mathcal{S}$ by

$$\omega'_n = \begin{cases} \omega_n & \text{if } n \notin A \\ z_n & \text{if } n \in A. \end{cases}$$

Let $\alpha \in G$ be a non-trivial automorphism: in particular, $\mathrm{vol}(\alpha) > 0$. Let us define

$$g := (\widetilde{\alpha}^{\omega})^{-1}\widetilde{\alpha}^{\omega'} \in \Gamma_G^{\{\omega,\omega'\}}.$$

It is readily checked that the automorphism g acts like α on the subtree $(T_X)_{o^{n-1}z_n}$, like α^{-1} on $(T_X)_{o^{n-1}\omega_n}$ and it acts trivially elsewhere. Recall that, when computing the support volume, we only focus on those vertices where the automorphism acts non-trivially. Hence, using the fact that $\mathrm{vol}(\alpha) = \mathrm{vol}(\alpha^{-1})$, we immediately get that

$$\mathrm{vol}(g) = \sum_{n\in A} \frac{2\mathrm{vol}(\alpha)}{|X|^n} = 2\mathrm{vol}(\alpha) \sum_{n\in A} |X|^{-n}.$$

As $|X| \geq 3$, we notice that $\mathrm{vol}(g)$ uniquely determines the subset A, because the first non-trivial term dominates the series above. Since the number of subsets $A \subseteq \mathbb{N}$ is uncountable, we obtain that the set of volumes in the statement is uncountable. □

Corollary 4.2.12. *Let $G \leq \mathrm{Aut}(T_X)$ be a countable, perfect, self-similar subgroup that satisfies property H. For every $\omega \in \mathcal{S}$, the family of groups*

$$(\Gamma_G^{\{\omega,\omega'\}})_{\omega'\in\mathcal{S}}$$

contains uncountably many isomorphism types.

Proof. First of all, since the group G is countable, the groups $\Gamma_G^{\{\omega,\omega'\}}$ are countably generated and thus countable as well. From Corollary 4.2.4, we know that $\Gamma^{\{\omega,\omega'\}}$ are branch groups whose rigid stabilizers act level-transitively. Theorem 4.2.5 tells us that they satisfy the congruence subgroup property and, from Proposition 4.2.7, we get that their closure in $\mathrm{Aut}(T_X)$ does not depend on ω and ω'. We can therefore apply Lemma 4.2.10 to say that every isomorphism between these groups $\Gamma_G^{\{\omega,\omega'\}}$ is induced by a conjugation in $\mathrm{Aut}(T_X)$. Now, we have noticed that the support volume is invariant under conjugation, hence every isomorphism between these groups preserve the support volume of elements. Furthermore, the group G is perfect and acts non-trivially on X. This implies that $|X| \geq 5$: indeed, if we assume for a contradiction that $|X| \leq 4$, from the solvability of $\mathrm{Sym}(n)$ for $n \leq 4$ we would get a non-trivial abelian quotient of G, contradicting the fact that it is perfect. We can thus apply Theorem 4.2.11, which yields that the set of support volumes of elements in the family of groups $(\Gamma_G^{\{\omega,\omega'\}})_{\omega' \in \mathcal{S}}$ is uncountable. Finally, using the fact that $\Gamma_G^{\{\omega,\omega'\}}$ is countable and that isomorphisms preserve the support volumes, we can conclude that the family of groups $(\Gamma_G^{\{\omega,\omega'\}})_{\omega' \in \mathcal{S}}$ contains uncountably many isomorphism types. □

4.3 Applying the Ω-construction

Let us fix a prime p and $n \in \mathbb{N}$. The goal of this section is applying the Ω-construction to p^n-regular rooted trees and special affine groups acting on them, in order to construct the groups A and $(G_j)_{j \in J}$ of Theorem 4.0.1. We choose as alphabet the finite set

$$X = \{0, \dots, p-1\}^n.$$

We denote by $T_{p,n}$ the Cayley graph of X^* with respect to X: as always, this will be a rooted tree whose vertex set is given by words over X. Let $\mathcal{A}_{p,n}$ be set of infinite sequences over X. We take as distinguished element in X the tuple $o = (0, \dots, 0)$. We write $\mathcal{S}_{p,n} = \mathcal{A}_{p,n} \setminus \{o\}$.

Recall from Example 1.2 that the ring of *p-adic integers* $\mathbb{Z}_p$ can be described as set of series of the form

$$\sum_{i=0}^{\infty} x_i p^i, \; x_i \in \{0, \dots, p-1\}.$$

We notice that we can identify the set of infinite sequences $\mathcal{A}_{p,n}$ with the ring $\mathbb{Z}_p^n$. Let $\omega = (\omega_k)_{k \in \mathbb{N}} \in \mathcal{A}_{p,n}$: every ω_k is a word over $\{0, \dots, p-1\}^n$, thus it can be written as $\omega_k = (x_{k_1}, \dots, x_{k_n})$ with $x_{k_i} \in \{0, \dots, p-1\}$. The identification is therefore the following:

$$\begin{aligned} \mathcal{A}_{p,n} &\longrightarrow \mathbb{Z}_p^n \\ (\omega_k)_{k\in\mathbb{N}} &\longmapsto \sum_{k=0}^{\infty} \omega_k p^k, \end{aligned}$$

where we write $\sum_{k=0}^{\infty} \omega_k p^k$ to mean $(\sum_{k=0}^{\infty} x_{k_1} p^k, \dots, \sum_{k=0}^{\infty} x_{k_n} p^k)$. Similarly, we can identify the ℓ^{th} level of the tree $T_{p,n}$ with $(\mathbb{Z}/p^\ell\,\mathbb{Z})^n$.

Definition 4.16. Given a commutative ring R with unity and $n \in \mathbb{N}$, we denote by $\mathrm{SAff}_n(R) \cong R^n \rtimes \mathrm{SL}_n(R)$ the group of affine transformations of R^n such that their linear part lies in $\mathrm{SL}_n(R)$.

Let us consider $\mathrm{SAff}_n(\mathbb{Z}_p)$: from the identification above, its natural action on $\mathbb{Z}_p^n$ induces an action on the rooted tree $T_{p,n}$, and the action on the ℓ^{th} level of $T_{p,n}$ factors through $\mathrm{SAff}_n(\mathbb{Z}/p^\ell\,\mathbb{Z})$. This allows us to regard $\mathrm{SAff}_n(\mathbb{Z}_p)$ as a subgroup of $\mathrm{Aut}(T_{p,n})$. Furthermore, we can consider the subgroups $\mathrm{SAff}_n(\mathbb{Z}) \leq \mathrm{Aut}(T_{p,n})$ and $\mathrm{SAff}_n(\mathbb{F}_p) \leq \mathrm{Aut}(T_{p,n})$, where $\mathrm{SAff}_n(\mathbb{F}_p)$ acts on the tree via rooted automorphisms.

We finally apply the Ω-construction introduced in the previous section. Given a sequence $\omega = (\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}_{p,n}$, we define

$$\Gamma_{p,n}^{\omega} \coloneqq \Gamma_{\mathrm{SAff}_n(\mathbb{Z})}^{\omega}.$$

As before, for a non-empty subset $\Omega \subseteq \mathcal{S}_{p,n}$, we denote by $\Gamma_{p,n}^{\Omega}$ the subgroup of $\mathrm{Aut}(T_{p,n})$ generated by the groups $\Gamma_{p,n}^{\omega}$ for $\omega \in \Omega$. Moreover, we define

$$A_{p,n}^{\omega} \coloneqq \Gamma_{\mathrm{SAff}_n(\mathbb{F}_p)}^{\omega}$$

and analogously for the groups $A_{p,n}^{\Omega}$ for $\Omega \subseteq \mathcal{S}_{p,n}$.

Our first goal is to show that we can apply in this context the results obtained in the previous section for the groups of the form $\Gamma_{\mathrm{G}}^{\Omega}$. In order to do so, we need to investigate the properties of $\mathrm{SAff}_n(\mathbb{Z})$ and $\mathrm{SAff}_n(\mathbb{F}_p)$. We start by proving that they are perfect for $n \geq 3$: this result is not surprising, as it is well known that the special groups $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{F}_p)$ both behave in this way ([HO13], 1.2.15 and [Wil09], §3.3.2).

Lemma 4.3.1. *The groups* $\mathrm{SAff}_n(\mathbb{Z})$ *and* $\mathrm{SAff}_n(\mathbb{F}_p)$ *are perfect for* $n \geq 3$.

Proof. Let us consider $\mathrm{SAff}_n(\mathbb{Z}) = \mathbb{Z}^n \rtimes \mathrm{SL}_n(\mathbb{Z})$: since $\mathrm{SL}_n(\mathbb{Z})$ is perfect, it remains to show that every translation T_{e_i} by a standard unit vector $e_i \in \mathbb{Z}^n$ can be expressed as a commutator of elements in $\mathrm{SAff}_n(\mathbb{Z})$. For $1 \leq i < j \leq n$, consider the elementary matrices $E_{i,j} \in \mathrm{SAff}_n(\mathbb{Z})$ such that $E_{i,j} \cdot e_j = e_i + e_j$ and $E_{i,j} \cdot e_k = e_k$ for $k \neq j$. Then,

$$[T_{-e_j}, E_{i,j}] = T_{-e_j} E_{i,j} T_{-e_j}^{-1} E_{i,j}^{-1} = T_{-e_j} + T_{e_i+e_j} = T_{e_i}.$$

Hence, $\mathrm{SAff}_n(\mathbb{Z})$ is perfect. The proof for $\mathrm{SAff}_n(\mathbb{F}_p)$ is analogous. □

Lemma 4.3.2. *The subgroups* $\mathrm{SAff}_n(\mathbb{Z}), \mathrm{SAff}_n(\mathbb{F}_p) \leq \mathrm{Aut}(T_{p,n})$ *are self-similar and satisfy property* H*.*

Proof. Let $A \in \mathrm{SL}_n(\mathbb{Z})$ and $b \in \mathbb{Z}^n$. We denote by $\alpha \in \mathrm{SAff}_n(\mathbb{Z})$ the element defined by $\alpha(v) = Av + b$ for every vertex v of $T_{p,n}$. Let us write $v = x + pw$ with $x \in X$ and $w \in \mathbb{Z}_p^n$. Let $x' \in X$ and $b' \in \mathbb{Z}_p^n$ be such that $Ax + b = x' + pb'$. Then, we can write

$$\alpha(v) = A(x + pw) + b = Ax + b + pAw = x' + p(Aw + b').$$

This tells us that the state of α at x is given by $\alpha_x(w) = Aw + b'$ with $b' \in \mathbb{Z}$ as $Ax + b = x' + pb'$ with $b \in \mathbb{Z}^n$. Hence, $\alpha_x \in \mathrm{SAff}_n(\mathbb{Z})$ and $\mathrm{SAff}_n(\mathbb{Z})$ is self-similar. The proof that $\mathrm{SAff}_n(\mathbb{F}_p)$ is self-similar is analogous.

Now, the action of $\mathrm{SAff}_n(\mathbb{F}_p)$ on $\mathbb{F}_p^n$ is 2-transitive: this can be shown by proving that $\mathrm{SAff}_n(\mathbb{F}_p)$ acts transitively on $\mathbb{F}_p^n$ and that $\mathrm{St}_{\mathrm{SAff}_n(\mathbb{F}_p)}(0) = \mathrm{SL}_n(\mathbb{F}_p)$ acts transitively on $\mathbb{F}_p^n \setminus \{0\}$. As a consequence, $\mathrm{SAff}_n(\mathbb{F}_p)$ satisfies property H. The action of $\mathrm{SAff}_n(\mathbb{Z})$ on the first level of $T_{p,n}$ factors through the action of $\mathrm{SAff}_n(\mathbb{F}_p)$ on $\mathbb{F}_p^n$, thus we can conclude that $\mathrm{SAff}_n(\mathbb{Z})$ satisfies property H as well. □

We can therefore obtain the following result for the groups $\Gamma_{p,n}^{\Omega}$.

Corollary 4.3.3. *Let* $n \geq 3$ *and let* $\Omega, \Omega' \subseteq \mathcal{S}_{p,n}$ *be non-empty subsets.*
(i) The group $\Gamma_{p,n}^{\Omega}$ *is a just infinite, branch group that satisfies the congruence subgroup property and contains a non-abelian free subgroup. Furthermore, its profinite completion* $\widehat{\Gamma_{p,n}^{\Omega}}$ *does not depend on* Ω*.*
(ii) If $\Omega \subseteq \Omega'$*, the inclusion* $u : \Gamma_{p,n}^{\Omega} \to \Gamma_{p,n}^{\Omega'}$ *induces an isomorphism*

$$\widehat{u} : \widehat{\Gamma_{p,n}^{\Omega}} \to \widehat{\Gamma_{p,n}^{\Omega'}}$$

between the profinite completions.
(iii) If $\Gamma_{p,n}^{\Omega}$ *and* $\Gamma_{p,n}^{\Omega'}$ *are isomorphic groups, then they are conjugated in* $\mathrm{Aut}(T_{p,n})$*.*

Proof. (i) Recall that $\mathrm{SL}_n(\mathbb{Z})$ contains a non-abelian free subgroup ([CC10], Lemma 2.3.2.). Since $\Gamma_{p,n}^{\Omega}$ contains a copy of $\mathrm{SL}_n(\mathbb{Z})$, then it contains a non-abelian free subgroup as well. By Corollary 4.2.4, which can be applied thanks to Lemma 4.3.1 and Lemma 4.3.2, we get that $\Gamma_{p,n}^{\Omega}$ is a branch group and the rigid stabilizers act level-transitively. Furthermore, by Theorem 4.2.5, the group $\Gamma_{p,n}^{\Omega}$ is just infinite and satisfies the congruence subgroup property. By Proposition 4.2.7, its closure in $\mathrm{Aut}(T_{p,n})$ does not depend on the subset Ω. Hence, its profinite completion does not depend on Ω.
(ii) It follows from Theorem 4.2.8.
(iii) We can apply Lemma 4.2.10, therefore every isomorphism between $\Gamma_{p,n}^{\Omega}$ and $\Gamma_{p,n}^{\Omega'}$ is induced by a conjugation in $\mathrm{Aut}(T_{p,n})$.

□

4.3.1 Amenable groups acting on trees

The aim of this section is proving that, under some mild assumption on the infinite sequence $\omega = (\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}_{p,n}$, the group $A^{\omega}_{p,n} := \Gamma^{\omega}_{\mathrm{SAff}_n(\mathbb{F}_p)}$ is amenable. In order to do so, we employ a result proved in [BKN10] by Bartholdi, Kaimanovich and Nekrashevych, stating that *the group of all bounded automatic automorphisms of T_X is amenable*. The proof of this theorem relies on the characterization of amenability via asymptotic properties of random walks on groups. We start by introducing the necessary notation.

Recall that, given an automorphism $\alpha \in \mathrm{Aut}(T_X)$, we defined the *state* of α at $v \in X^*$ as the unique automorphism α_v of T_X such that

$$\alpha(vw) = \alpha(v)\alpha_v(w)$$

for every $w \in X^*$. We denote the *set of states* of α by

$$S(\alpha) = \{\alpha_w \mid w \in X^*\}.$$

Definition 4.17. An automorphism $\alpha \in \mathrm{Aut}(T_X)$ is *automatic* if the set $S(\alpha)$ is finite.

Example 4.2. Let α be a rooted automorphism of T_X, as in Example 4.1. Let $\omega = (\omega_k)_{k\in\mathbb{N}} \in \mathcal{S}$ and consider the automorphism $\widetilde{\alpha}^{\omega}$ of Definition 4.12. For $v \in X^*$, we want to describe the state of $\widetilde{\alpha}^{\omega}$ at v. A straightforward computation yields

$$\widetilde{\alpha}^{\omega}_v = \begin{cases} \widetilde{\alpha}^{L^{\ell}(\omega)} & \text{if } v = o^{\ell} \text{ for some } \ell \in \mathbb{N} \\ \alpha & \text{if } v = o^{\ell}\omega_{\ell} \text{ for some } \ell \in \mathbb{N} \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Therefore, $S(\widetilde{\alpha}^{\omega})$ is finite if and only if $\{L^{\ell}(\omega) \mid \ell \in \mathbb{N}\} \subseteq \mathcal{S}$ is finite, that is, if and only if there exists some $N \in \mathbb{N}$ such that $L^N(\omega)$ is periodic. In this case, we say that the sequence $\omega \in \Omega$ is *eventually periodic.*

Definition 4.18. An automorphism $\alpha \in \mathrm{Aut}(T_X)$ is *bounded* if there exists some constant $C \geq 0$ such that

$$|\{w \in X^n \mid \alpha_w \neq \mathrm{id}\}| \leq C$$

for every $n \in \mathbb{N}$.

Example 4.3. Let α be a rooted automorphism of T_X. It immediately follows from the definition that $\widetilde{\alpha}^{\omega}$ is bounded for every $\omega \in \mathcal{S}$.

Theorem 4.3.4 (Bartholdi, Kaimanovich and Nekrashevych [BKN10])**.** *The group of all bounded automatic automorphism of the rooted tree T_X is amenable.*

We immediately see how we can apply this theorem to our groups $A_{p,n}^{\Omega}$.

Proposition 4.3.5. *Let $G \leq \mathrm{Aut}(T_X)$ be a group of rooted automorphisms and let $\Omega \subseteq \mathcal{S}$ be a non-empty subset. Assume that every $\omega \in \Omega$ is eventually periodic. Then, the group $\Gamma_{\mathrm{G}}^{\Omega}$ is amenable.*

Proof. Thanks to Examples 4.2 and 4.3, we know that the group G is generated by bounded automatic automorphisms. Since every subgroup of an amenable group is amenable, the result directly follows from Theorem 4.3.4. □

Therefore, if we assume that the sequence $\omega \in \mathcal{S}_{p,n}$ is eventually periodic, we obtain that the group $A_{p,n}^{\omega}$ is amenable. We are now interested in giving a sketch of the proof of Theorem 4.3.4. Let us denote by $\mathfrak{BU}(X)$ the self-similar group of all bounded automatic automorphism of T_X. The two essential steps of the proof are the following:

- Reduce the question about amenability of $\mathfrak{BU}(X)$ to that about amenability of a special family of groups, called *Mother groups*.
- Prove the amenability of such Mother groups using asymptotic properties of random walks on them.

In order to approach the proof, we first present a useful way of visualizing the structure of $\mathrm{Aut}(T_X) = \mathrm{Sym}(X) \wr \mathrm{Aut}(T_X)$ through *generalized permutation matrices*. More details on this can be found in [Kai09]. Given a permutation $\sigma \in \mathrm{Sym}(X)$, we can always associate to it a *permutation matrix* M^{σ}, which is the order $|X|$ matrix whose entries are given by

$$M_{xy}^{\sigma} = \begin{cases} 1 & \text{if } y = \sigma(x) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that the map $\sigma \mapsto M^{\sigma}$ is a group isomorphism. In a similar manner, we can associate to the wreath decomposition of an automorphism $\alpha = (\sigma_\alpha, (\alpha_x)_{x\in X})$ the *generalized permutation matrix* M^{α}: its entries are given by

$$M_{xy}^{\alpha} = \begin{cases} \alpha_x & \text{if } y = \sigma_\alpha(x) \\ 0 & \text{otherwise.} \end{cases}$$

This allows us to identify the group $\mathrm{Sym}(X) \wr \mathrm{Aut}(T_X)$ with a subgroup of the matrix algebra $\mathrm{Mat}_{|X|}(\mathbb{C}[\mathrm{Aut}(T_X)])$ over the group ring of $\mathrm{Aut}(T_X)$. More generally, given a subgroup $G \leq \mathrm{Aut}(T_X)$, we denote by

$$\mathrm{Sym}(X; G) = G \wr \mathrm{Sym}(X)$$

the *group of generalized permutation matrices* of order $|X|$ with non-zero entries from the group G. We therefore have a natural isomorphism of the automorphism group $\mathrm{Aut}(T_X)$ and the group of generalized permutation matrices $\mathrm{Sym}(X;\mathrm{Aut}(T_X))$.

The above isomorphism allows us to give an equivalent definition of self-similarity: we say that a subgroup $G \leq \mathrm{Aut}(T_X)$ is *self-similar* if, for every $g \in G$, all entries of the matrices M^g belong G. In other words, the restriction of the isomorphism $\mathrm{Aut}(T_X) \to \mathrm{Sym}(X;\mathrm{Aut}(T_X))$ to G induces an embedding

$$\begin{aligned} G &\hookrightarrow \mathrm{Sym}(X;G) \\ g &\mapsto M^g. \end{aligned}$$

This embedding extends by linearity to an algebra homomorphism

$$\begin{aligned} \ell^1(G) &\hookrightarrow \mathrm{Mat}_{|X|}(\ell^1(G)) \\ \mu &\mapsto M^\mu = \sum_{g\in G} \mu(g) M^g. \end{aligned}$$

Let us now go back to the group of $\mathfrak{BU}(X)$ of all bounded automatic automorphisms of the rooted tree T_X. First of all, we show that $\mathfrak{BU}(X)$ contains a special family of groups.

Definition 4.19. Let us fix a distinguish element $o \in X$ in the finite alphabet X and set $\overline{X} = X \setminus \{o\}$. Let $A := \mathrm{Sym}(X)$ and $B := \mathrm{Sym}(\overline{X}; A)$, and recursively embed the groups A and B into $\mathrm{Sym}(X)$ as

$$A \ni a \mapsto (a, (1, \ldots, 1)), \quad B \ni b = (\sigma, (b_2, \ldots, b_d)) \mapsto (\sigma, (b, b_2, \ldots, b_d)),$$

assuming that $X = \{o = 1, \ldots, d\}$. The matrix presentations of a and b are given by

$$M^a = \phi_A(a), \quad M^b = \begin{pmatrix} b & 0 \\ 0 & \phi_B(b) \end{pmatrix},$$

where $\phi_A(a)$ and $\phi_B(b)$ are the permutation and the generalized permutation matrices corresponding to $a \in A$ and $b \in B$, respectively. Then, we define the *Mother group* $\mathfrak{M}(X)$ as the subgroup of $\mathrm{Aut}(T_X)$ generated by the finite groups A and B.

From the definition, it is straightforward to verify that both A and B are contained in $\mathfrak{BU}(X)$. Hence, the Mother group $\mathfrak{M}(X)$ is a subgroup of $\mathfrak{BU}(X)$. On the other hand, we have the following result.

Proposition 4.3.6. *Every finitely generated subgroup of $\mathfrak{BU}(X)$ can be embedded as a subgroup into the wreath product $\mathfrak{M}(X^N) \wr \mathrm{Sym}(X^N)$ for some $N \in \mathbb{N}$.*

Proof. [BKN10], Theorem 3.3. □

Theorem 4.3.7. *For any finite alphabet X, the associated Mother group $\mathfrak{M}(X)$ is amenable.*

Corollary 4.3.8 (=Theorem 4.3.4)**.** *The group $\mathfrak{BU}(X)$ is amenable.*

Proof. From Corollary 3.1.16, in order to show that $\mathfrak{BU}(X)$ is amenable it suffices to show that all its finitely generated subgroup are amenable. Let G be a finitely generated subgroup of $\mathfrak{BU}(X)$: from Proposition 4.3.6, there is some integer $N \in \mathbb{N}$ such that G embeds in $\mathfrak{M}(X^N) \wr \mathrm{Sym}(X^N)$. Since $\mathrm{Sym}(X^N)$ is finite (thus, amenable) and $\mathfrak{M}(X^N)$ is amenable from Theorem 4.3.7, from the fact that amenability is preserved under group extensions (Proposition 3.1.8) we get that $\mathfrak{M}(X^N) \wr \mathrm{Sym}(X^N)$ is amenable. We can conclude again by Proposition 3.1.8 that G is amenable. □

In order to prove amenability of the Mother groups $\mathfrak{M}(X)$ in Theorem 4.3.7, we will work with random walks on them. We start by noticing that there is a natural connection between random walks on self-similar groups and random walks with internal degrees of freedom introduced in Definition 3.11. Let $G \leq \mathrm{Aut}(T_X)$ be a self-similar group and let μ be a probability measure on G: the associated random walk (G, μ) has transition probabilities

$$g \underset{h\sim\mu}{\longmapsto} gh.$$

Then, by applying the self-similarity embedding $G \to \mathrm{Sym}(X; G),\ g \mapsto M^g$, we get a random walk on the generalized permutation group $\mathrm{Sym}(X; G)$ with transition probabilities

$$M^g \underset{h\sim\mu}{\longmapsto} M^g M^h.$$

The multiplication of the matrix M^g by the increment M^h is done row by row. Hence, we get a Markov chain on the space of rows of matrices in $\mathrm{Sym}(X; G)$, whose transition probabilities are given by

$$R \underset{h\sim\mu}{\longmapsto} RM^h.$$

Now, the rows of such matrices have only one non-zero entry, thus they can be described by the position and the value of this entry. This tells us that they can be identified with points in $G \times X$. Therefore, the above Markov chain can be interpreted as a Markov chain on $G \times X$ whose transition probabilities are invariant with respect to the left action of G on $G \times X$. This implies that such Markov chains are *random walks with internal degrees of freedom* parameterized by the alphabet X. Recall that such random walks can be described by matrices $M = (M_{xy})_{x,y\in X}$ of order $|X|$ whose entries are subprobability measures on G such that $\sum_{y\in X} ||M_{xy}|| = 1$ for every $x \in X$. In our situation, the matrix describing the RWIDF given by each row of matrices in $\mathrm{Sym}(X; G)$ is precisely the matrix

$$M^\mu = \sum_{g\in G} \mu(g) M^g,$$

which we introduced as image of the linear extension of the self-similarity embedding of G into $\mathrm{Sym}(X;G)$ to an embedding of the Banach algebra $\ell^1(G)$ into $\mathrm{Mat}_{|X|}(\ell^1(G))$.

Let us now consider the self-similar Mother group $\mathfrak{M}(X)$, and define on it the measure μ given as the convolution product of the uniform measures μ_A and μ_B on the finite subgroups A and B:

$$\mu = \mu_A * \mu_B.$$

Then, the matrix M^μ has a very special form:

$$M^\mu = M^{\mu_A} M^{\mu_B} = E_d \begin{pmatrix} \mu_B & 0 \\ 0 & \mu_A E_{d-1} \end{pmatrix},$$

where $d = |X|$ and E_d is the order d matrix with entries $1/d$. Hence, M^μ has entries

$$M^\mu_{xy} = \begin{cases} \mu_B/d & \text{if } y = o \\ \mu_A/d & \text{otherwise.} \end{cases}$$

The transition probabilities of the associated RWIDF $(\mathfrak{M}(X) \times X, M^\mu)$ do not depend on x. This implies that the projection of $(\mathfrak{M}(X) \times X, M^\mu)$ to the group $\mathfrak{M}(X)$ is the random walk $(\mathfrak{M}(X), \widetilde{\mu})$ determined by the measure

$$\widetilde{\mu} = \sum_{y \in X} M^\mu_{xy} = \frac{d-1}{d}\mu_A + \frac{1}{d}\mu_B.$$

The idea is then to compare the asymptotic entropies of μ and $\widetilde{\mu}$ in order to obtain that the asymptotic entropy $h(\mathfrak{M}(X), \mu)$ vanishes. Then, we can conclude that the Mother group $\mathfrak{M}(X)$ is amenable by Theorem 3.3.6. First of all, one can employ the *Münchhausen trick* introduced by Kaimanovich in [Kai05] in order to get the following inequality:

$$h(\mathfrak{M}(X), \mu) \leq d h(\mathfrak{M}(X), \widetilde{\mu}).$$

Furthermore, we notice that the measures μ_A and μ_B defined on the finite groups A and B are idempotent. Since $\widetilde{\mu}$ is a convex combination of them, its convolution powers $\widetilde{\mu}^{*n}$ are essentially convex combinations of the convolution powers of μ. From this, one gets that

$$h(\mathfrak{M}(X), \widetilde{\mu}) \leq \frac{d-1}{d^2} h(\mathfrak{M}(X), \mu).$$

Therefore, we can conclude that $h(\mathfrak{M}(X), \mu) = 0$ and that the Mother group $\mathfrak{M}(X)$ is amenable.

4.3.2 Conclusion

In order to conclude the proof of Theorem 4.0.1, we need to construct the family of non-amenable groups in which the amenable group $A^{\omega}_{p,n}$ embeds. The idea is to simply merge the groups $\Gamma^{\Omega}_{p,n}$ with $A^{\omega}_{p,n}$. Let $\Omega \subseteq \mathcal{S}_{p,n}$ be a non-empty subset: we denote by $G_{p,n}$ the subgroup of $\mathrm{Aut}(T_{p,n})$ generated by $\mathrm{SAff}_n(\mathbb{Z})$ and $\mathrm{SAff}_n(\mathbb{F}_p)$ acting as usual on $T_{p,n}$, and define

$$M^{\Omega}_{p,n} := \Gamma^{\Omega}_{G_{p,n}}.$$

Equivalently, $M^{\Omega}_{p,n}$ can be defined as the subgroup of $\mathrm{Aut}(T_{p,n})$ generated by $A^{\Omega}_{p,n}$ and $\Gamma^{\Omega}_{p,n}$.

Theorem 4.3.9. *Let us fix a prime number p and $n \geq 3$. Let $\omega \in \mathcal{S}_{p,n}$ be eventually periodic. Then, for every $\omega' \in \mathcal{S}_{p,n}$, the following hold:*
(i) $A^{\omega}_{p,n}$ is a 6-generated amenable group.
(ii) $M^{\{\omega,\omega'\}}_{p,n}$ is a 18-generated non-amenable group.
(iii) The inclusion $\iota : A^{\omega}_{p,n} \to M^{\{\omega,\omega'\}}_{p,n}$ induces an isomorphism

$$\widehat{\iota} : \widehat{A^{\omega}_{p,n}} \to \widehat{M^{\{\omega,\omega'\}}_{p,n}}$$

between the profinite completions.
(iv) The family $(M^{\omega,\omega'}_{p,n})_{\omega' \in \mathcal{S}_{p,n}}$ contains uncountably many pairwise non-isomorphic groups.

Proof. (i) The group $A^{\omega}_{p,n}$ is amenable by Proposition 4.3.5. Since $\mathrm{SL}_n(\mathbb{F}_p)$ is 2-generated (see [HR49]), then $\mathrm{SAff}_n(\mathbb{F}_p)$ is 3-generated. The group $A^{\omega}_{p,n}$ is generated by two isomorphic copies of $\mathrm{SAff}_n(\mathbb{F}_p)$, hence it is 6-generated.
(ii) Since $M^{\{\omega,\omega'\}}_{p,n}$ contains $\Gamma^{\{\omega,\omega'\}}_{p,n}$, it contains a non-abelian free subgroup by Corollary 4.3.3. Hence, it is non-amenable by Proposition 3.1.9. The group $\mathrm{SL}_n(\mathbb{Z})$ is 2-generated (see [HR49]), so $\mathrm{SAff}_n(\mathbb{Z})$ is 3-generated and $G_{p,n}$ is 6-generated. Now, $M^{\{\omega,\omega'\}}_{p,n}$ is generated by three copies of $G_{p,n}$, therefore it is 18-generated.
(iii) We need to check that we can apply Theorem 4.2.8. The groups $\mathrm{SAff}_n(\mathbb{F}_p)$ and $G_{p,n}$ are perfect, self-similar and satisfy property H by Lemma 4.3.1 and Lemma 4.3.2. Since the action of $\mathrm{SAff}_n(\mathbb{Z})$ on the first level of the tree is precisely the natural action of $\mathrm{SAff}_n(\mathbb{F}_p)$ on $\mathbb{F}_p^n$, the two groups $\mathrm{SAff}_n(\mathbb{Z})$ and $G_{p,n}$ have the same image in $\mathrm{Sym}(X)$. Finally, Theorem 4.2.8 implies that the inclusion $A^{\omega}_{p,n} \to M^{\{\omega,\omega'\}}_{p,n}$ induces an isomorphism of the profinite completions.
(iv) This immediately follows from Corollary 4.2.12.

□

4.4 Uniform amenability is a profinite invariant

The fact that amenability is not a profinite invariant may not be surprising. Indeed, when looking for profinite invariants, we restrict our attention only to the set of isomorphism types of finite quotients of a group, and we know that finite groups are always amenable. What is more interesting is that there is actually not much ”missing” from amenability in order to obtain a property that can be detected on the set of finite quotients: it suffices to add a uniform condition, and pass from amenability to uniform amenability. Let us recall that an amenable group G is uniformly amenable if, for every $\varepsilon > 0$ and every finite subset $S \subseteq G$, the size of an (ε, S)-Følner set can be uniformly bounded in terms of ε and $|S|$.

In this last section, we focus on the recent proof by Kionke and Schesler that uniform amenability is a profinite invariant [KS23]. The key idea is to give a characterization of such property using a special class of finite quotients of the group.

Proposition 4.4.1. *Let $\mathfrak{K}$ be a uniformly amenable class of groups. Then, the class of all quotients of groups in $\mathfrak{K}$ is uniformly amenable.*

Proof. Let us assume that the class $\mathfrak{K}$ is amenable by assuming that it satisfies the uniform Reiter condition for a function r (Definition 3.17 and Proposition 3.4.8). For a group $G \in \mathfrak{K}$ and a normal subgroup $N \trianglelefteq G$, we denote by $\pi_N : G \to G/N$ the canonical projection. Let us fix $\varepsilon > 0$ and a finite subset $S \subseteq G/N$: we can lift such S to a finite subset $S' \subseteq G$ such that $\pi_N(S') = S$ and $|S'| = |S|$. Using the uniform Reiter condition in G, there is a finitely supported probability measure μ' on G such that $|\mathrm{supp}(\mu')| \leq r(\varepsilon, |S|)$ and

$$\left\|s \cdot \mu' - \mu'\right\|_{\ell^1(G)} \leq \varepsilon$$

for every $s \in S$. Let

$$\mu = \pi_*(\mu')$$

be the pushforward measure of μ' on the quotient G/N. Clearly, we have $|\operatorname{supp}(\mu)| \leq |\operatorname{supp}(\mu')| \leq r(\varepsilon, |S|) = r(\varepsilon, |S'|)$. Furthermore,

$$\begin{aligned}
\|sN \cdot \mu - \mu\|_{\ell^1(G)} &= \sum_{x \in G/N} \left|\mu(s^{-1}x) - \mu(x)\right| \\
&= \sum_{x \in G/N} \left|\sum_{y \in x} \mu'(s^{-1}y) - \mu'(y)\right| \\
&\leq \sum_{h \in G} |\mu'(s^{-1}h) - \mu'(h)| = \left\|s \cdot \mu' - \mu'\right\|_{\ell^1(G)} \leq \varepsilon
\end{aligned}$$

for every $s \in S'$. This finally implies that the class of all quotients of groups in $\mathfrak{K}$ satisfies the uniform Reiter condition for the same function r as $\mathfrak{K}$, hence it is uniformly amenable. □

Let $\mathfrak{F}$ be a non-empty set of subgroups of a group. We say that $\mathfrak{F}$ is a *filter base* of subgroups if, for all $N, M \in \mathfrak{F}$, the intersection $N \cap M$ contains an element of $\mathfrak{F}$.

Theorem 4.4.2. *Let G be a group and let $\mathfrak{F}$ be a filter base of normal subgroups of G such that $\bigcap_{N\in\mathfrak{F}} N = \{1_G\}$. Then, the group G is uniformly amenable if and only if the class $\{G/N \mid N \in \mathfrak{F}\}$ is uniformly amenable.*

Proof. Let us assume that G is uniformly amenable. Then, Proposition 4.4.1 immediately tells us that the class of quotients $\{G/N \mid N \in \mathfrak{K}\}$ is uniformly amenable.

Suppose now that the class $\{G/N \mid N \in \mathfrak{F}\}$ is uniformly amenable for a function $m : \mathbb{R}_{>0} \times \mathbb{N} \to \mathbb{N}$. Let us fix $\varepsilon > 0$ and a finite subset $S \subseteq G$, and denote again by $\pi_N : G \to G/N$ the natural projection for $N \in \mathfrak{F}$. Let C_N denote the set of finite subsets $F \subseteq G$ with $1_G \in F$ that satisfy $|F| \leq m(\varepsilon, |S|)$ and

$$|\pi_N(SF)| \leq (1+\varepsilon)|F|.$$

The sets C_N are non-empty by assumption, as we can construct such F by lifting Følner sets of G/N. Furthermore, for $N \subseteq M$, we have $C_N \subseteq C_M$. Every set in C_N has bounded cardinality, therefore the C_N are closed, and thus compact, in the Fell topology on the power set $\mathcal{P}(G)$. We refer to [Bee93] for an introduction to the Fell topology. Now, compactness implies that there exists a finite set

$$F \in \bigcap_{N\in\mathfrak{F}} C_N,$$

which is non-empty and contains 1_G. Let $N \in \mathfrak{F}$ be sufficiently small such that distinct elements in SF represent distinct cosets in G/N. Then,

$$|SF| = |\pi_N(SF)| \leq (1+\varepsilon)|F|,$$

which implies that F is a Følner set and the group G satisfies the uniform Følner condition. We conclude that G is uniformly amenable. □

Corollary 4.4.3. *Let G be a profinite group. If some dense subgroup $H \leq G$ is uniformly amenable, then G is uniformly amenable.*

Proof. Let $\mathfrak{F}$ be a filter of open normal subgroups of G. Since the subgroup H is uniformly amenable, from Proposition 4.4.1 we immediately know that the class of quotients $\{H/(H \cap N) \mid N \in \mathfrak{F}\}$ is uniformly amenable. Now, H is dense in G, hence Proposition 1.2.8 tells us that $HN = G$ for every open normal subgroup $N \trianglelefteq_o G$. Therefore,

$$H/(H \cap N) \cong G/N$$

for every $N \trianglelefteq_o G$. Finally, Theorem 4.4.2 implies that the profinite group G is uniformly amenable. □

Theorem 4.4.4 (Kionke and Schesler [KS23]). *Let G_1 and G_2 be residually finite groups with $\widehat{G_1} \cong \widehat{G_2}$. Then, G_1 is uniformly amenable if and only if G_2 is uniformly amenable.*

Proof. Since both G_1 and G_2 are residually finite, they are dense subgroups of $\widehat{G_1}$ and $\widehat{G_2}$, respectively. Assume that the group G_1 is uniformly amenable: from Corollary 4.4.3, we get that $\widehat{G_1} \cong \widehat{G_2}$ is uniformly amenable. Finally, Proposition 3.4.2 implies that G_2 is uniformly amenable. □

We can therefore conclude from Theorem 4.4.4 that uniform amenability is a profinite invariant.

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Ich habe die Arbeit selbständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in § 26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.

Regensburg, 22 July 2024.

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