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On Stark's Conjecture and Elliptic Units for Complex Cubic Fields

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Introduction

In this thesis we will follow and expand the recent article [BCG23] by Nicolas Bergeron, Pierre Charollois and Luis E. Garcia.

For abelian extensions of quadratic imaginary fields[1], it is widely known the construction of special units[2], called elliptic units, obtained as special values of an elliptic modular function called Theta function (see e.g. [Sha87] Chapter III, Section 2).
Their importance is the similarity that they have to the cyclotomic units in the case of abelian number fields, i.e. abelian extensions of $\mathbb{Q}$. First of all, recall that we say that an extension K/k is "abelian" if it is a Galois extension, and its Galois group $Gal(K/k)$ is an abelian group. All abelian extensions of $\mathbb{Q}$ can be obtained by adjoining to $\mathbb{Q}$ some roots of unity. In the same way, all abelian extensions of K quadratic imaginary field can be obtained by adjoining to K certain values of elliptic functions.

The aim of this thesis is to expand this concept to an higher dimension, working with abelian extensions of complex cubic fields instead of imaginary quadratic. Recall that a "complex cubic field" is the splitting field of an irreducible polynomial of degree 3 over $\mathbb{Q}$ with two non-real roots.
The first difference that we will encounter is the new transcendental function that will determine the new units: in the complex cubic case it will be constructed starting from the classical Elliptic Gamma function and not from the Theta function, which was used for quadratic imaginary fields. The definitions of both these functions, as well as the construction of the Smoothed Elliptic Gamma function $\Gamma_{\mathfrak{a},h}(w,z;L)$, will be described in detail in the first Chapter.
This Chapter will then present all the notations that will be needed throughout this thesis, and will end with the statement of the Main Conjecture, that relates special values of the Smoothed Elliptic Gamma function to units in $K(\mathfrak{f})$, where K is the complex cubic field we start from, and $K(\mathfrak{f})$ is the Narrow Ray Class field of K of conductor a non zero ideal $\mathfrak{f}$ of $\mathscr{O}_K$ (which is a finite abelian extension of K ramified only at the ideal $\mathfrak{f}$). We now present a simplified statement of the Main Conjecture: define complex numbers $\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})$ depending on $\mathfrak{f}$ as well as on other data (ideals $\mathfrak{a},\mathfrak{b}$ and a complex number h).

Conjecture 1. *i) The numbers $\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1}) \in \mathbb{C}^\times$ belong to $K(\mathfrak{f})$ and are actually units in that number field, i.e. $\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1}) \in \mathscr{O}^\times_{K(\mathfrak{f})}$;*

ii) For an element $\sigma_\mathfrak{b} \in Gal(K(\mathfrak{f})/K)$ the following reciprocity law holds:

$$\sigma_\mathfrak{b}(\Gamma_{\mathfrak{a},h}(\mathfrak{f})) = \Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})$$

[1] A number field K is called quadratic imaginary if it is of the shape $K := \mathbb{Q}(\sqrt{-d})$, for d positive square-free.
[2] For a field K, we refer as a "unit in K" to a unit in its ring of integers, i.e. to an element in $\mathscr{O}_K^\times$.

In Chapter 2 we will step back from our main objective to review an interesting proof of the two classical Kronecker Limit Formulas (KLF) following as model the paper [BFK15] by K. Bannai, H. Furusho and S. Kobayashi; this will help us to get acclimatized with the statement of a New Kronecker Limit Formula that will be presented in Chapter 3, which is crucial to relate the Main Conjecture to the more known Stark's Conjecture.
Chapter 3 will develop the language of partial zeta functions and Hecke L-functions, which will be needed in the statement of the New KLF. We will use the results from the previous Chapter to prove a KLF for imaginary quadratic fields. We will finish this Chapter by pointing out similarities between this latter formula and the New KLF from the main article.

In Chapter 4 we will talk about the already mentioned Stark's Conjecture. This is one of the most important unsolved problems in Algebraic Number Theory, and it predicts the existence of special units which are related to Hecke L-functions (or zeta functions, depending on the version). We give now a simplified version; for the precise statement see Conjecture 5.

Conjecture 2 (Stark's Conjecture)**.** *For an abelian extension K/k, fix a place v of k that splits completely in K and let w be one of its lifts in K. Then there exists a unit $\varepsilon \in K$ such that its valuation is equal to 1 for all places of K that do not lie above v and such that its valuation for the place w is given explicitly by*

$$\log|\varepsilon^{\sigma}|_w = -e\zeta'(0,\sigma), \ \ \forall\sigma \in Gal(K/k)$$

Moreover, ε is such that the extension $K(\varepsilon^{1/e})/k$ is abelian[3]*, where e is the number of roots of unity contained in K.*

This Conjecture was believed by Tate [Tat84] to be a "vague form" of Hilbert's 12-th problem; he also wrote that maybe the key to solving the Conjecture is the use of an "unknown transcendental function", whose existence was predicted by Hilbert.
In this thesis we will first of all present Stark's Conjecture in its more general form, to later dive in the abelian case (i.e. regarding abelian extensions K/k) which is a more precise version of the one just stated.
We will then relate this abelian formulation to the Main Conjecture using the New KLF. What will come out is that, if both Conjectures (Stark's and the one of [BCG23]) hold, then Stark's units (ε from Conjecture 2) and the ones predicted by the authors of [BCG23] ($\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})$ from Conjecture 1) are related by an explicit formula. The authors of the article [BCG23] believe that in the case of complex cubic fields, the "unknown transcendental function" mentioned by Tate is precisely the Elliptic Gamma function.

Finally we will devote Chapter 5 to the long proof of the New KLF.
In the Appendix we will present the proof of Stark's Conjecture in a known case: that of quadratic imaginary fields, which will use the KLF for quadratic imaginary fields proved in Chapter 3.

The theory proposed in the main article [BCG23] is developing in the very months in which this thesis is being written. It is even more recent an article by a student of Charollois, P. L. L. Morain [Mor24], in which he expands the conjecture and the construction we will see in this thesis from the case of a complex cubic field (which has one real embedding and two complex conjugate embeddings) to the case of "almost totally real field" (ATR) (i.e. a field with exactly one couple of complex conjugate embeddings). The one proposed by Morain is the most natural generalization one can do of the subject, given that the two most important cases so far, imaginary quadratic fields and complex cubic fields, are both ATR.

[3] Note that this condition resembles the one of cyclotomic units and elliptic units mentioned before: also this unit ε, if existed, would be useful to obtain new abelian extensions of the base field k.

Contents

Chapter 1

Values of the Elliptic Gamma function

In this Chapter we will fix the setting that will be carried throughout all the thesis. In Section 1.1 we will construct the Elliptic Gamma function depending on a chosen lattice L. In Section 1.2 we will consider a specific lattice L on the complex cubic field K; we will then define the three ideals that will appear in both the Main Conjecture and the New KLF (called $\mathfrak{f}, \mathfrak{b}, \mathfrak{a}$) and a particular complex number h. Next, the Smoothed Elliptic Gamma function will be defined, as well as the element $\Gamma_{\mathfrak{a},h}(L) \in \mathbb{C}^{\times}$, crucial for the Main Conjecture. Section 1.3 will present the Main Conjecture and some first consequences based on properties previously mentioned.
We begin this Chapter by recalling two classical functions.

1.0.1 Theta function

Elliptic units are, as said in the Introduction, special units in abelian extensions of imaginary quadratic number fields. One key object in this theory is the Theta function defined for $z \in \mathbb{C}$, $\tau \in \mathscr{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ as the absolutely convergent infinite product

$$\theta_0(z,\tau) := \prod_{n\geq 0} \left(1 - e^{2\pi i(n\tau+z)}\right)\left(1 - e^{2\pi i((n+1)\tau-z)}\right)$$

This defines an holomorphic function on $\mathbb{C} \times \mathscr{H}$. θ_0 has some well known periodicity properties

$$\theta_0(z+1,\tau) = \theta_0(z,\tau); \qquad \theta_0(z+\tau,\tau) = -e^{-2\pi i z}\,\theta_0(z,\tau)$$

in the variable z as well as transformation properties in the variable $\tau \in \mathscr{H}$ under the usual action of $SL_2(\mathbb{Z})$. Elliptic units are obtained as quotients of values of θ_0 at special points (v,τ) with v rational and τ a quadratic irrationality in $\mathscr{H}$. In the following we want to extend this concept to the setting of complex cubic fields and to do so we will need at first a proper generalization of the Theta function: the Elliptic Gamma function. Theta functions will be used in the following to investigate in the quadratic imaginary case the motivation that lead the authors of [BCG23] to formulate certain results in the complex cubic case.

1.0.2 Classical Elliptic Gamma function

For $(z,\tau,\sigma) \in \mathbb{C} \times \mathscr{H} \times \mathscr{H}$ we define $\Gamma(z,\tau,\sigma)$ by the absolutely convergent infinite product:

$$\Gamma(z,\tau,\sigma) := \prod_{j,\,k=0}^{\infty} \frac{1-e^{2\pi i((j+1)\tau+(k+1)\sigma-z)}}{1-e^{2\pi i(j\tau+k\sigma+z)}}$$

Its domain can be extended to $(z,\tau,\sigma) \in \mathbb{C} \times (\mathbb{C}-\mathbb{R})^2$ by the rules

$$\Gamma(z,\tau,\sigma) = \frac{1}{\Gamma(z-\tau,-\tau,\sigma)} = \frac{1}{\Gamma(z-\sigma,\tau,-\sigma)}$$

Γ has $SL_3(\mathbb{Z})$-modular properties similar to the ones θ_0 has for $SL_2(\mathbb{Z})$ (the three periods involved are $1,\tau,\sigma$); the proof of this fact relies on the following properties, that came directly from the definition:

$$\Gamma(z+1,\tau,\sigma) = \Gamma(z,\tau,\sigma); \qquad \Gamma(z+\tau,\tau,\sigma) = \theta_0(z,\sigma)\,\Gamma(z,\tau,\sigma)$$

$$\Gamma(z+\sigma,\tau,\sigma) = \theta_0(z,\tau)\,\Gamma(z,\tau,\sigma)$$

These show the existence of three periods 1, τ, σ with respect to the complex variable z.
As previously said, the classical Elliptic Gamma function will be the cornerstone of our theory as we will create a new type of units for complex cubic number fields starting from "special" quotients of this function. To define these quotients, we'll first need to define a new type of Elliptic Gamma function, depending on a lattice L and on two elements of the dual of such lattice. This will be treated in the next Section, in which we will also see how this new Elliptic Gamma function relates to the one just defined.

1.1 The Elliptic Gamma function

Before defining the Elliptic Gamma function $\Gamma_{a,b}(w,z;L)$, we need to fix some notation, that will be carried throughout the rest of the thesis, unless otherwise stated.
Let's denote L a free abelian group of rank 3 and $\Lambda := \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$ its dual. In the following we'll often exchange the notion of duality between the two as we are in finite dimension, so L can be identified as Λ's dual too. It will be useful to see Λ as a vector space over $\mathbb{C}$ and to do so we extend it by scalars: call $V := \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, $V_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, $V_{\mathbb{C}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$. Fix an orientation of L, call it $\det : \Lambda^3_{\mathbb{Z}} L \to \mathbb{Z}$; if a triple of elements of L are sent to a positive integer, they are said to be positive oriented, while if they are sent to a negative integer, they are said to be negative oriented; a triple of elements of L is sent to 0 by the map det if and only if they are linearly dependent over $\mathbb{Z}$. The orientation on L naturally extends to an orientation on the vector spaces V, $V_{\mathbb{R}}$, $V_{\mathbb{C}}$.
We say that an element $a \in \Lambda$ is **primitive** iff $\frac{a}{n} \notin \Lambda\ \forall n \in \mathbb{Z}_{>1}$. We then easily see that there is a one-to-one correspondence between primitive elements in Λ and oriented planes in $L \otimes \mathbb{Q}$, obtained by sending a primitive element $a \in \Lambda$ to $H(a) := \ker(a) = \{\alpha \in L\colon a(\alpha) = 0\}$. It is indeed trivial to see that if an element b of Λ isn't primitive, then it is an integer multiple of a primitive, call it c, and then b and c define the same oriented plane since these are seen in $L \otimes \mathbb{Q}$, hence we can clear out denominators.
$H(a)$ divides L into two subsets:

$$L_+(a) := \{\alpha \in L\colon a(\alpha) > 0\} \quad L_-(a) := \{\alpha \in L\colon a(\alpha) \leq 0\}$$

Note that the latter contains $H(a)$.
We say that an ordered basis λ,μ of $H(a)$ is (**positively**) **oriented** if $\forall \delta \in L$ such that $\delta(a) > 0$ then

$\det(\lambda,\mu,\delta) > 0$.
Given $a \in \Lambda$ primitive, we define $U_a := \{z \in V_{\mathbb{C}}\colon \operatorname{Im}(\lambda(z)\overline{\mu(z)}) > 0\} \subset V_{\mathbb{C}}$ for any oriented basis (λ,μ) of $H(a)$ (some different characterization may be found in [Fel+08] and will be needed later). One easily sees that U_a is an open subset of $V_{\mathbb{C}}$.

We start now with the construction of the Elliptic Gamma function: fix (a,b) an ordered couple of linearly independent primitive elements of Λ. They determine oriented planes $H(a)$, $H(b) \subset L\otimes\mathbb{Q}$ that we claim intersect in a line spanned by the element $\det(a,b,\cdot) \in \Lambda^* = L$.

Proof. First of all note that $\det(a,b,\cdot) \in \Lambda^*$ as it takes an element of Λ and gives back an integer. Since an element in the line generated by $\det(a,b,\cdot)$ can be written as $\alpha := q\cdot\det(a,b,\cdot)$, $q \in \mathbb{Q}$, it then follows that $a(\alpha) = a(q\cdot\det(a,b,\cdot)) = q\cdot\det(a,b,a) = q\cdot 0 = 0$ so $\alpha \in H(a)$. Similarly we get $\alpha \in H(b)$, so that the line spanned by $\det(a,b,\cdot)$ is contained in the intersection between $H(a)$ and $H(b)$. The claim follows by linear independence of a,b that yields that such intersection cannot be more than 1-dimensional. $\square$

In general the element $\det(a,b,\cdot)$ is not primitive, but we can write it as

$$\det(a,b,\cdot) = s\cdot\gamma,\ \exists!\ s \in \mathbb{N}_{>0},\ \exists!\ \gamma \in L \text{ primitive}$$

Definition 1. *To the ordered pair (a,b) we also associate*

$$C_{+-}(a,b) := \{\delta \in L\colon\ \delta(a) > 0,\ \delta(b) \le 0\} \subset L$$

$$C_{-+}(a,b) := \{\delta \in L\colon\ \delta(a) \le 0,\ \delta(b) > 0\} \subset L$$

We are then ready to define for L,a,b as above the **Elliptic Gamma function**:

$$\Gamma_{a,b}(w,z;L) := \frac{\prod_{\delta\in C_{+-}(a,b)/\mathbb{Z}\gamma}\left(1-e^{-2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}{\prod_{\delta\in C_{-+}(a,b)/\mathbb{Z}\gamma}\left(1-e^{2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)} \tag{1.1}$$

This quotient of infinite products converges to a meromorphic function on $\mathbb{C}\times(U_a\cap U_b)$ as we'll prove in the next Proposition.

Remark 1.
- *Just by looking at the definition and recalling the periodicity of the complex exponential, we see that the Elliptic Gamma function has simple zeroes for $w = \delta(z)+n\gamma(z)$ for $\delta \in C_{+-}(a,b)$ and $n \in \mathbb{Z}$, and simple poles for $w = \delta(z)+n\gamma(z)$ for $\delta \in C_{-+}(a,b)$, $n \in \mathbb{Z}$.*

- *We gave this definition in the case of a,b linearly independent; when $a = b$ we conventionally set $\Gamma_{a,a}(w,z;L) := 1$ while if $a = -b$ we don't need to define anything as the domain of definition is empty as $U_a\cap U_{-a} = \varnothing$, condition easily checked by the above definition.*

- *In the definition of $\Gamma_{a,b}$ appears a quotient $C_{+-}(a,b)/\mathbb{Z}\gamma$ (respectively $C_{-+}(a,b)/\mathbb{Z}\gamma$), so we shall explain what the action of $\mathbb{Z}\gamma$ on $C_{+-}(a,b)$ (respectively $C_{-+}(a,b)$) is. This is simply an action by translation: first note that $C_{+-}(a,b) = (\mathbb{Q}_{>0}\alpha+\mathbb{Q}_{\le 0}\beta+\mathbb{Q}\gamma)\cap L$ where we choose $\alpha \in H(b)$, $\beta \in H(a)$ such that $\alpha(a) > 0$, $\beta(b) > 0$, this equality will be explained in the next proof. An element $\delta \in C_{+-}(a,b)$ can then be written as $\delta = q^+\alpha+q^-\beta+q\gamma$, so that the action of $\mathbb{Z}\gamma$ becomes clear: $\delta+\mathbb{Z}\gamma = q^+\alpha+q^-\beta+(q+\mathbb{Z})\gamma \subset C_{+-}(a,b)$. Similarly one can see the action of $\mathbb{Z}\gamma$ on $C_{-+}(a,b)$.*

Proposition 1. *The ratio of infinite products defining $\Gamma_{a,b}(w,z;L)$ converges to a meromorphic function on $\mathbb{C}\times(U_a\cap U_b)$*

Proof. We already claimed in the previous remark that we can write $C_{+-}(a,b) = (\mathbb{Q}_{>0}\alpha + \mathbb{Q}_{\leq 0}\beta + \mathbb{Q}\gamma) \cap L$ and $C_{-+}(a,b) = (\mathbb{Q}_{\leq 0}\alpha + \mathbb{Q}_{>0}\beta + \mathbb{Q}\gamma) \cap L$, where we choose $\alpha \in H(b)$, $\beta \in H(a)$ primitive elements such that $\alpha(a) > 0$, $\beta(b) > 0$. Let's prove e.g. the second equality: if $\delta \in$ RHS, we write it $\delta = q^-\alpha + q^+\beta + q\gamma$ with $q^-, q^+, q \in \mathbb{Z}$ (as they are respectively in $\mathbb{Q}^- \cap L, \mathbb{Q}^+ \cap L, \mathbb{Q} \cap L$ and by the fact that α, β, γ are primitive). By definition of γ we have that $\gamma(a) = \frac{1}{s}\det(a,b,a) = 0$ and similarly $\gamma(b) = 0$ so that $\delta(a) = q^-\alpha(a) + q^+\beta(a) + q\gamma(a) = q^-\alpha(a) + 0 + 0 \leq 0$ as $q^- \leq 0$ and $\alpha(a) > 0$. Similarly we see $\delta(b) > 0$ so that $\delta \in$ LHS. The other inclusion comes easily from definition, so we have our claim.
We want to deduce the condition for convergence of the infinite product at denominator: by what written above we get $C_{-+}(a,b)/\mathbb{Z}\gamma \approx (\mathbb{Z}_{\leq 0}\alpha + \mathbb{Z}_{>0}\beta)$ and we need that the real part of the exponent of e is negative to converge, i.e. need $\operatorname{Im}\left(\frac{\alpha(z)}{\gamma(z)}\right) < 0$ and $\operatorname{Im}\left(\frac{\beta(z)}{\gamma(z)}\right) > 0$ thanks to the known signs of the integer coefficients. Note that w is ininfluent when making considerations about convergence: indeed $\operatorname{Im}\left(\frac{\delta(z)-w}{\gamma(z)}\right) = \operatorname{Im}\left(\frac{\delta(z)}{\gamma(z)}\right) - \operatorname{Im}\left(\frac{w}{\gamma(z)}\right)$ and the second term is fixed (wrt δ). So the sign of the imaginary part of the exponent of e, $\operatorname{Im}\left(\frac{\delta(z)-w}{\gamma(z)}\right)$, is the same as the sign of $\operatorname{Im}\left(\frac{\delta(z)}{\gamma(z)}\right)$ for all but finitely many δ's. We can then forget about w as finitely many terms don't influence convergence.
Note that by construction (β, γ) is a basis of $H(a)$ since $\beta(a) = 0 = \gamma(a)$ and they are linearly independent, as seen when computed against b.
Claim: (β, γ) is an oriented basis of $H(a)$.
Proof: First note that (α, β, γ) is an oriented basis for $L \otimes \mathbb{Q}$, thanks to the definition of γ as $\frac{1}{s}\det(a,b,\cdot)$ with $\frac{1}{s} > 0$. We can then apply the definition: choose ε such that $\varepsilon(a) > 0$, since $\beta(b) > 0, \beta(a) = 0$ we can wlog assume that $\varepsilon(b) = 0$ up to adding a multiple of β, so that $\varepsilon = t \cdot \alpha$ for some $t \in \mathbb{Z}_{>0}$. So $\det(\beta, \gamma, \varepsilon) = \det(\varepsilon, \beta, \gamma) = t\det(\alpha, \beta, \gamma) > 0$ so that (β, γ) is an oriented basis of $H(a)$. ■
Similarly we see (γ, α) is an oriented basis of $H(b)$.
The convergence conditions are

$$\operatorname{Im}\left(\frac{\beta(z)}{\gamma(z)}\right) > 0, (\beta, \gamma) \text{ oriented basis for } H(a) \Leftrightarrow z \in U_a$$

$$\operatorname{Im}\left(\frac{\alpha(z)}{\gamma(z)}\right) < 0 \Leftrightarrow \operatorname{Im}\left(\frac{\gamma(z)}{\alpha(z)}\right) > 0, (\gamma, \alpha) \text{ oriented basis for } H(b) \Leftrightarrow z \in U_b$$

where we used for the last implications the equivalent conditions in the definitions of U_a, U_b from [Fel+08]. In the same way, we see that also the numerator converges iff $z \in U_a \cap U_b$, thus ending the proof. □

To conclude this Section, we want to express $\Gamma_{a,b}$ in terms of the classical Elliptic Gamma functions of last Subsection. Recall we chose $\alpha, \beta \in L$ such that $\alpha(b) = \beta(a) = 0$, $\alpha(a) > 0$, $\beta(b) > 0$ and recall the equivalent definition of $U_a \cap U_b$ from [Fel+08], which is

$$U_a \cap U_b = \left\{ z \in V_{\mathbb{C}} \colon \operatorname{Im}\left(\frac{\alpha(z)}{\gamma(z)}\right) < 0, \operatorname{Im}\left(\frac{\beta(z)}{\gamma(z)}\right) > 0 \right\}$$

Call

$$F := \left\{ \delta \in L \colon 0 \leq \delta(a) < \alpha(a),\ 0 \leq \delta(b) < \beta(b) \right\}$$

Proposition 2.

$$\Gamma_{a,b}(w,z;L) = \prod_{\bar{\delta} \in F/\mathbb{Z}\gamma} \Gamma\left(\frac{w + \bar{\delta}(z)}{\gamma(z)}, \frac{\alpha(z)}{\gamma(z)}, \frac{\beta(z)}{\gamma(z)}\right)$$

Remark 2. *This product is finite.*

Proof. One intuitively sees that, modulo $\mathbb{Z}\gamma$, each $\delta \in C_{+-}(a,b)$ can be uniquely written as $\delta = -\bar{\delta} + (j+1)\alpha - k\beta$ with $\bar{\delta} \in F,\ j,k \in \mathbb{Z}_{\geq 0}$. The idea to see it is that $(j+1)\alpha$ quantifies how much $\delta(a)$ is greater than 0 with an error strictly smaller than $\alpha(a)$ and similarly $-k\beta$ quantifies how much $\delta(b) < 0$ with an error of at most $\beta(b)$ and then $-\bar{\delta} \in F$ is used to remove such error.
In the same way we see that $\delta \in C_{-+}(a,b)$ can be uniquely written as $\delta = -\bar{\delta} - j\alpha + (k+1)\beta$ with $\bar{\delta} \in F,\ j,k \in \mathbb{Z}_{\geq 0}$, modulo $\mathbb{Z}\gamma$.
Now, recall the definitions and a property of the classical Elliptic Gamma function of previous Subsection:

$$\Gamma_{a,b}(w,z;L) = \frac{\prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma} \left(1 - e^{-2\pi i \left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}{\prod_{\delta \in C_{-+}(a,b)/\mathbb{Z}\gamma} \left(1 - e^{2\pi i \left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}$$

$$\Gamma(z,\tau,\sigma) = \prod_{j,\,k=0}^{\infty} \frac{1 - e^{2\pi i((j+1)\tau+(k+1)\sigma-z)}}{1 - e^{2\pi i(j\tau+k\sigma+z)}}, \qquad \Gamma(z,\tau,\sigma) = \frac{1}{\Gamma(z-\tau,-\tau,\sigma)}$$

so that

$$\prod_{\bar{\delta}\in F/\mathbb{Z}\gamma} \Gamma\left(\frac{w+\bar{\delta}(z)}{\gamma(z)}, \frac{\alpha(z)}{\gamma(z)}, \frac{\beta(z)}{\gamma(z)}\right) = \prod_{\bar{\delta}\in F/\mathbb{Z}\gamma} \frac{1}{\Gamma\left(\frac{w+\bar{\delta}(z)}{\gamma(z)} - \frac{\alpha(z)}{\gamma(z)}, -\frac{\alpha(z)}{\gamma(z)}, \frac{\beta(z)}{\gamma(z)}\right)} =$$

$$= \prod_{\bar{\delta}\in F/\mathbb{Z}\gamma} \prod_{j,\,k=0}^{\infty} \frac{1 - e^{2\pi i\left(-j\frac{\alpha(z)}{\gamma(z)} + k\frac{\beta(z)}{\gamma(z)} + \frac{w+\bar{\delta}(z)}{\gamma(z)} - \frac{\alpha(z)}{\gamma(z)}\right)}}{1 - e^{2\pi i\left(-(j+1)\frac{\alpha(z)}{\gamma(z)} + (k+1)\frac{\beta(z)}{\gamma(z)} - \frac{w+\bar{\delta}(z)}{\gamma(z)} + \frac{\alpha(z)}{\gamma(z)}\right)}} =$$

$$= \prod_{\bar{\delta}\in F/\mathbb{Z}\gamma} \prod_{j,\,k=0}^{\infty} \frac{1 - \exp\left(2\pi i\left(\frac{-(j+1)\alpha(z)+k\beta(z)+w+\bar{\delta}(z)}{\gamma(z)}\right)\right)}{1 - \exp\left(2\pi i\left(\frac{-j\alpha(z)+(k+1)\beta(z)-w-\bar{\delta}(z)}{\gamma(z)}\right)\right)} =$$

Now, thanks to the correspondence previously discussed, since in the numerator we want to get $\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma$ we have that $\delta = -\bar{\delta} + (j+1)\alpha - k\beta$ so that we can rewrite the exponent $-(j+1)\alpha(z) + k\beta(z) + w + \bar{\delta}(z) = w - \delta(z)$. Analogously for the denominator get $-j\alpha(z) + (k+1)\beta(z) - w - \bar{\delta}(z) = \delta(z) - w$ (for $\delta \in C_{-+}(a,b)/\mathbb{Z}\gamma$). We can then conclude

$$= \frac{\prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma} \left(1 - e^{2\pi i \left(\frac{-\delta(z)+w}{\gamma(z)}\right)}\right)}{\prod_{\delta \in C_{-+}(a,b)/\mathbb{Z}\gamma} \left(1 - e^{2\pi i \left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)} = \Gamma_{a,b}(w,z;L)$$

□

We conclude with a quick unproven fact: $SL_3(\mathbb{Z}) \cong SL(\Lambda)$ acts on Λ, V, $V_{\mathbb{C}}$ as explained in [Fel+08]. Moreover this action is transitive on primitive elements of Λ, i.e. if $a \in \Lambda$ is primitive, then also ga is primitive for $g \in SL_3(\mathbb{Z})$; one can show that $U_{ga} = gU_a$ and that $\Gamma_{ga,gb}(w,gz;L) = \Gamma_{a,b}(w,z;L)$ for $z \in U_a \cap U_b$ ($\Rightarrow gz \in U_{ga} \cap U_{gb}$ by above, so the LHS is well defined).

1.2 Complex cubic fields and values of the Elliptic Gamma function

We begin this Section with some heavy notation, that will be needed for the rest of the thesis.

Let K be a complex cubic number field over $\mathbb{Q}$, $\mathscr{O}_K$ its ring of integers and $\sigma_{\mathbb{R}} \colon K \hookrightarrow \mathbb{R}$ the unique real embedding of K[1]. Fix a complex embedding $\sigma_{\mathbb{C}} \colon K \hookrightarrow \mathbb{C}$ of K that will remain fixed for the rest of the thesis[2]. The map $x \mapsto \big(\sigma_{\mathbb{R}}(x), \sigma_{\mathbb{C}}(x)\big)$ induces an isomorphism $K \otimes \mathbb{R} \cong \mathbb{R} \times \mathbb{C}$, obtained extending by scalars. This allows us to give K the orientation obtained by pulling back the usual orientation of $\mathbb{R} \times \mathbb{C}$, after taking basis $(1_{\mathbb{R}}, 1_{\mathbb{C}}, i_{\mathbb{C}})$ for the latter.
Let $\mathfrak{f}$ and $\mathfrak{b}$ be non zero ideals of $\mathscr{O}_K$ with $(\mathfrak{f}, \mathfrak{b}) = 1$. Write $L := \mathfrak{f}\mathfrak{b}^{-1}$; it's a fractional ideal of K (clear by definition) and a free $\mathbb{Z}$-module of rank three (as all fractional ideals are, see e.g. [Neu99] Chapter I [2.10]); moreover the unit $1_K \in K$ is a primitive $\mathfrak{f}$-division point of L. Here primitive has the same meaning as in the previous Section, while we say that $y \in K = \mathfrak{f}\mathfrak{b}^{-1}\mathbb{Q}$ is an **$\mathfrak{f}$-division point** iff $\forall x \in \mathfrak{f}$, $xy = 0$ in $\mathfrak{f}\mathfrak{b}^{-1}\mathbb{Q}/\mathfrak{f}\mathfrak{b}^{-1}$; this is indeed true for 1_K as $\forall x \in \mathfrak{f}$, $x \cdot 1_K = x \in \mathfrak{f} \subset \mathfrak{f}\mathfrak{b}^{-1}$ so it is 0 in $\mathfrak{f}\mathfrak{b}^{-1}\mathbb{Q}/\mathfrak{f}\mathfrak{b}^{-1}$.
Call q the smallest positive integer in $L \cap \mathbb{Z}$; since $\mathbb{Z}$ is a PID we can find $f \in \mathfrak{f}$, $b \in \mathfrak{b}$ such that $\left(\frac{f}{b}\right) = L \cap \mathbb{Z} \subset \mathbb{Q}$, so $q = f$ (note that since $\mathfrak{b}^{-1}$ contains fractional elements, q is equivalently the smallest positive integer in $\mathfrak{f} \cap \mathbb{Z}$). Notice that q is the order of $L \otimes \mathbb{Q}/L$, i.e. $q \cdot 1_K \in L$ and no number $0 < r < q$ belongs to L by definition of q, so that $r \cdot 1_K \notin L$.
The group of units

$$\mathscr{O}_{K,\mathfrak{f}}^{+,\times} := \left\{ u \in \mathscr{O}_K^{\times} \mid u - 1_K \in \mathfrak{f} \text{ and } \sigma_{\mathbb{R}}(u) > 0 \right\} \leq \mathscr{O}_K^{\times} \tag{1.2}$$

has rank 1, i.e. is of finite index in $\mathscr{O}_K^{\times}$. This equivalence comes from the fact that $\mathscr{O}_K^{\times}$ has rank 1 by the Dirichlet's units theorem; indeed this famous result tells us that $\mathscr{O}_K^{\times} \cong \mu_K \times \mathbb{Z}$ (in particular has rank 1), where μ_K is the set of roots of unity contained in K. Now to see that $\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$ has finite index in $\mathscr{O}_K^{\times}$, consider the map

$$\varphi \colon \mathscr{O}_K^{\times} \to \big(\mathscr{O}_K/\mathfrak{f}\big)^{\times} \quad \varepsilon \mapsto \varepsilon \mod \mathfrak{f}$$

By definition it's clear that $\mathscr{O}_{K,\mathfrak{f}}^{+,\times} \subset \ker \varphi$, so that $\big(\mathscr{O}_K^{\times}/\mathscr{O}_{K,\mathfrak{f}}^{+,\times}\big) \subset \big(\mathscr{O}_K/\mathfrak{f}\big)^{\times}$ and the latter is finite (as $\mathscr{O}_K/\mathfrak{f}$ itself is already finite). So we get that $\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$ has finite index in $\mathscr{O}_K^{\times}$ and consequently that $\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$ has too rank 1.
$\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$ acts by multiplication on K (seen as a 3-dimensional $\mathbb{Q}$-vector space) and preserves the lattice L (multiplication by any unit preserves a lattice; in particular this is true for elements of $\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$).
We will assume throughout that $\mathscr{O}_{K,\mathfrak{f}}^{+,\times} = \mathscr{O}_K^{\times} \cap (1 + \mathfrak{f})$, i.e. that there are no units in $\mathscr{O}_K^{\times} \cap (1 + \mathfrak{f})$ of negative norm. This in particular implies $-1 \notin 1 + \mathfrak{f} \Rightarrow -2 \notin \mathfrak{f} \Rightarrow 2 \notin \mathfrak{f}$ so that $q \geq 3$, and also that $\mathfrak{f} \neq \mathscr{O}_K$.

Claim: We can always find a generator $\varepsilon \in \mathscr{O}_{K,\mathfrak{f}}^{+,\times}$ such that $\varepsilon_{\mathbb{R}} := \sigma_{\mathbb{R}}(\varepsilon) \in (0,1)$.
Proof: As previously said $\mathscr{O}_{K,\mathfrak{f}}^{+,\times} \subset \mathscr{O}_K^{\times}$ is of finite index and by Dirichlet's units theorem we can write the latter as $\mathscr{O}_K^{\times} \cong \mu_K \times \mathbb{Z}$, so that we can also write $\mathscr{O}_{K,\mathfrak{f}}^{+,\times} \cong \{1\} \times n\mathbb{Z}$ (the first term is $\{1\}$ by assumption on the norm of units in $\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$). We then choose an ε generating $n\mathbb{Z}$. If $\sigma_{\mathbb{R}}(\varepsilon) < 1$ (as it is by assumption > 0) we're done, while if $\sigma_{\mathbb{R}}(\varepsilon) > 1$, we just choose ε^{-1} which is still a generator and has now real part < 1. The

[1] If we write $K = \mathbb{Q}(\alpha)$, this embedding sends α to the real root of its minimal polynomial.
[2] This sends α to one of the two complex conjugate roots of its minimal polynomial.

case $\sigma_{\mathbb{R}}(\varepsilon) = 1$ cannot hold, since $\varepsilon \in \mathscr{O}_K^\times \Rightarrow 1 = N(\varepsilon) = \sigma_{\mathbb{R}}(\varepsilon)|\sigma_{\mathbb{C}}(\varepsilon)|^2 \Rightarrow \sigma_{\mathbb{C}}(\varepsilon) = 1 \Rightarrow \varepsilon \in \mu_K$ so that $\varepsilon = 1$ and it cannot be a generator of $\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$. ■

Definition 2. *We say that $\lambda \in L$ is* **admissible** *if:*

- $\frac{\lambda}{q}$ *is congruent to* 1_K *modulo L;*
- $\frac{\lambda}{N(\mathfrak{a})} \in \mathfrak{a}^{-1}L$ *and generates the quotient* $\mathfrak{a}^{-1}L/L$ *for an ideal* $\mathfrak{a} \subset \mathscr{O}_K$ *relatively prime to* $N(\mathfrak{f})$ *and such that* $N(\mathfrak{a})$ *is an odd prime.*

Remark 3. 1. *The conditions in the definition can be restated with others equivalent (the second and the third conditions must be taken together):*
$\cdot\ \frac{\lambda}{q} \equiv_L 1_K \leftrightarrow \lambda \equiv_{qL} q;$
$\cdot\ \frac{\lambda}{N(\mathfrak{a})} \in \mathfrak{a}^{-1}L \leftrightarrow \lambda \in N(\mathfrak{a})\mathfrak{a}^{-1}L;$
$\cdot$ *Asking that* $\frac{\lambda}{N(\mathfrak{a})}$ *generates the quotient* $\mathfrak{a}^{-1}L/L$ *is equivalent to asking that* $\lambda \not\equiv 0 \mod N(\mathfrak{a})L$. *Indeed, since* $N(\mathfrak{a})$ *is an odd prime, every non zero element is automatically a generator; so* $\lambda \not\equiv 0 \mod N(\mathfrak{a})L \leftrightarrow \frac{\lambda}{N(\mathfrak{a})} \not\equiv_L 0$, *united with the fact that* $\frac{\lambda}{N(\mathfrak{a})} \in \mathfrak{a}^{-1}L$, *says that* $\frac{\lambda}{N(\mathfrak{a})}$ *generates* $\mathfrak{a}^{-1}L/L$.

2. *Existence of such admissible elements is guaranteed by the Chinese Reminder Theorem, which says, in our setting, that*
$$N(\mathfrak{a})\mathfrak{a}^{-1}\mathfrak{f}/\big(qN(\mathfrak{a})\big)\mathfrak{f} \cong \mathfrak{f}/q\mathfrak{f} \times N(\mathfrak{a})\mathfrak{a}^{-1}/\big(N(\mathfrak{a})\big)$$
This later yields (after a couple of simple algebraic considerations) that
$$N(\mathfrak{a})\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}/\big(qN(\mathfrak{a})\big)\mathfrak{f}\mathfrak{b}^{-1} \cong \mathfrak{f}\mathfrak{b}^{-1}/q\mathfrak{f}\mathfrak{b}^{-1} \times N(\mathfrak{a})\mathfrak{a}^{-1}/\big(N(\mathfrak{a})\big)$$
Now, coprimality between $\mathfrak{f}$ *and* $\mathfrak{b}$ *gives us that* $N(\mathfrak{a})\mathfrak{a}^{-1}/\big(N(\mathfrak{a})\big) \cong N(\mathfrak{a})\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}/\big(N(\mathfrak{a})\big)\mathfrak{f}\mathfrak{b}^{-1}$, *so that we have the isomorphism (recall* $L = \mathfrak{f}\mathfrak{b}^{-1}$*)*
$$N(\mathfrak{a})\mathfrak{a}^{-1}L/\big(qN(\mathfrak{a})\big)L \cong L/qL \times N(\mathfrak{a})\mathfrak{a}^{-1}L/\big(N(\mathfrak{a})\big)L \tag{1.3}$$
An element λ *admissible for L is now simply the preimage of* $\big(q \mod qL, \zeta \neq 0\big)$ *via the isomorphism Eq. 1.3, which satisfies the conditions of point 1 by construction.*

3. *If* λ *is admissible for L, and if* $u \in \mathscr{O}_{K,\mathfrak{f}}^{+,\times}$ *then* $u\lambda$ *is admissible for L.*
Indeed $\frac{u\lambda}{q} \equiv_L u1_K = u \equiv_L 1_K$, *as* $u \equiv_{\mathfrak{f}} 1_K$ *and consequently also modulo* $L = \mathfrak{f}\mathfrak{b}^{-1}$. $u \in \mathscr{O}_K^\times$ *so that* $\lambda \in N(\mathfrak{a})\mathfrak{a}^{-1}L \Rightarrow u\lambda \in N(\mathfrak{a})\mathfrak{a}^{-1}L$ *and* $\lambda \not\equiv 0 \mod N(\mathfrak{a})L \Rightarrow u\lambda \not\equiv 0 \mod N(\mathfrak{a})L$.

4. *If* λ *is admissible for* $L = \mathfrak{f}\mathfrak{b}^{-1}$, *and if* $\alpha \in 1+\mathfrak{f}$ *then* $\alpha^{-1}\lambda$ *is admissible for* $\mathfrak{f}((\alpha)\mathfrak{b})^{-1}$.
Indeed $\frac{\alpha^{-1}\lambda}{q} \equiv_{\mathfrak{f}((\alpha)\mathfrak{b})^{-1}} 1_K$ *for* $\frac{\lambda}{q} \equiv_{\mathfrak{f}\mathfrak{b}^{-1}} 1_K$. *Moreover* $\lambda \in N(\mathfrak{a})\mathfrak{a}^{-1}L \Rightarrow \alpha^{-1}\lambda \in (\alpha^{-1})N(\mathfrak{a})\mathfrak{a}^{-1}L = N(\mathfrak{a})\mathfrak{a}^{-1}\mathfrak{f}((\alpha)\mathfrak{b})^{-1}$ *and* $\lambda \not\equiv 0 \mod N(\mathfrak{a})L \Rightarrow \alpha^{-1}\lambda \not\equiv 0 \mod (\alpha^{-1})N(\mathfrak{a})L = N(\mathfrak{a})\mathfrak{f}((\alpha)\mathfrak{b})^{-1}$.

Fix from now on an ideal $\mathfrak{a}$ as above ($\mathfrak{a} \subset \mathscr{O}_K$ relatively prime to $N(\mathfrak{f})$ and such that $N(\mathfrak{a})$ is an odd prime) and $h := \frac{\lambda}{q}$ with λ admissible for L.
Restricting $\sigma_{\mathbb{C}}\colon K \hookrightarrow \mathbb{C}$ to L defines a non zero element $\mathbf{x} \in V_{\mathbb{C}} = \mathrm{Hom}(L, \mathbb{C})$ and a point $\langle \mathbf{x} \rangle \in \mathbf{X} := \frac{V_{\mathbb{C}} - \mathbb{C} \cdot V_{\mathbb{R}}}{\mathbb{C}^\times}$ (this last notation will only be used in Chapter 5). Write $\det\colon \Lambda_{\mathbb{Z}}^3 L \to \mathbb{Z}$ for the orientation on L induced by the one on K already seen.

The unit ε, as all elements of $\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$, acts by multiplication on K and preserves L, but we see that it preserves also its orientation; one can indeed show with a long but simple computation of linear algebra that $\det(\varepsilon\gamma_1 \wedge \varepsilon\gamma_2 \wedge \varepsilon\gamma_3) = N(\varepsilon)\det(\gamma_1 \wedge \gamma_2 \wedge \gamma_3) = \det(\gamma_1 \wedge \gamma_2 \wedge \gamma_3)$, so that the sign is preserved, hence the orientation.
So ε defines a linear map in $SL(L)$ and, by duality, a linear map in $SL(\Lambda)$, which will be noted again ε.

Remark 4. $\mathbf{x} \in V_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ *is an eigenvector for the action of* ε.

Proof. To see what $\varepsilon \circ \mathbf{x}$ is, we compute it against the general element $\frac{f}{b} \in L$: first of all by definition of $\mathbf{x}$ we have $\mathbf{x}\left(\frac{f}{b}\right) = \sigma_{\mathbb{C}}\left(\frac{f}{b}\right)$, while by definition the action of ε on L makes $\varepsilon\left(\frac{f}{b}\right) = \varepsilon \cdot \frac{f}{b}$. So that:

$$(\varepsilon \circ \mathbf{x})\left(\frac{f}{b}\right) = \mathbf{x}\left(\varepsilon^{-1}\left(\frac{f}{b}\right)\right) = \mathbf{x}\left(\varepsilon^{-1} \cdot \frac{f}{b}\right) = \sigma_{\mathbb{C}}\left(\varepsilon^{-1} \cdot \frac{f}{b}\right) = \sigma_{\mathbb{C}}(\varepsilon^{-1}) \cdot \sigma_{\mathbb{C}}\left(\frac{f}{b}\right) = \sigma_{\mathbb{C}}(\varepsilon^{-1}) \cdot \mathbf{x}\left(\frac{f}{b}\right)$$

So that $\varepsilon \circ \mathbf{x} = \sigma_{\mathbb{C}}(\varepsilon^{-1}) \cdot \mathbf{x}$, i.e. $\mathbf{x}$ is an eigenvector for the action of ε with eigenvalue $\sigma_{\mathbb{C}}(\varepsilon^{-1})$. □

Lemma 1. *Let* $\lambda \in L$ *be admissible. Then there exists only one primitive element* $a = a_\lambda \in \Lambda$ *such that*

i) $\det(a, \varepsilon a, \cdot) \in \Lambda^* = L$ *is a non zero multiple of* λ *(in particular non zero integer multiple);*

ii) $\mathbf{x} \in U_a = \{z \in V_{\mathbb{C}} \mid \operatorname{Im}(\mu(z)\overline{\nu(z)}) > 0,$ *for any oriented basis* (μ, ν) *of* $H(a)\}$

Proof. Claim: [1] ε has no rational eigenvalues.
Proof: Indeed, since ε is a generator of $\mathscr{O}_K$ over $\mathbb{Z}$, hence of K over $\mathbb{Q}$, we have $K = \mathbb{Q}(\varepsilon) \cong \mathbb{Q}[x]/\big(f(x)\big)$ where $f(x) \in \mathbb{Q}[x]$ is the minimal polynomial of ε. The usual embedding of K in $\mathbb{C}$ makes us see that $\sigma_{\mathbb{R}}(\varepsilon) \in \mathbb{R} \setminus \mathbb{Q}$, because if it was in $\mathbb{Q}$ then f would have a rational root against its irreducibility. ■
So λ and $\varepsilon^{-1}\lambda$ are linearly independent in $L \otimes \mathbb{Q}$, because if λ was a rational multiple of $\varepsilon^{-1}\lambda$ it would imply that λ is an eigenvector for ε with rational eigenvalue against what just seen.
Claim: [2] Finding a such that $\det(a, \varepsilon a, \cdot) = n\lambda$ for some $n \in \mathbb{Z}$ is equivalent to finding $a \in \ker(\lambda) \cap \ker(\varepsilon^{-1}\lambda)$.
Proof: ($\Leftarrow$): if $a \in \ker(\lambda) \cap \ker(\varepsilon^{-1}\lambda) \Rightarrow a, \varepsilon a \in \ker(\lambda)$. Consider $\det(a, \varepsilon a, \cdot) \in L = \Lambda^*$. If $\mu \in \Lambda$ is in $\ker(\lambda)$, then all $a, \varepsilon a, \mu \in \ker(\lambda)$, so they are linearly dependent, in particular $\det(a, \varepsilon a, \mu) = 0 \Rightarrow \ker(\lambda) \subset \ker(\det(a, \varepsilon a, \cdot))$ and since they are both 1-dimensional, λ and $\det(a, \varepsilon a, \cdot)$ must differ by a non zero integer factor.
($\Rightarrow$): assume $\det(a, \varepsilon a, \mu) = n\lambda(\mu)$, $\forall \mu \in \Lambda$. If $\mu \in \ker(\lambda) \Rightarrow \det(a, \varepsilon a, \mu) = 0$ so that $a, \varepsilon a, \mu$ must be linearly dependent. Since this must work $\forall \mu \in \ker(\lambda)$, we must have $a, \varepsilon a \in \ker(\lambda) \Rightarrow a \in \ker(\lambda) \cap \ker(\varepsilon^{-1}\lambda)$. ■
Now, since we have that $\lambda, \varepsilon^{-1}\lambda$ are linearly independent, then $\ker(\lambda) \cap \ker(\varepsilon^{-1}\lambda)$ is a line passing through 0, or better a discrete set of points along a line for we are in Λ. Of those points, only the two closer to zero (call them $\pm a$) are primitive, while the others are multiples. Those $\pm a$ correspond to the two possible orientations of the same plane $H(\pm a) \subset L \otimes \mathbb{Q}$, and only one of those satisfies condition *ii*). □

Fix $\mathfrak{a}, h$ as above; write a for the element of Λ given by previous lemma (with the λ that defines h) and define $b := \varepsilon a$. Condition *i*) of the lemma tells us that a, b both vanish at λ (e.g. for a see that $\lambda(a) = \frac{1}{n}\det(a, b, a) = 0$). Hence, since $\mathfrak{a}^{-1}L = L + \frac{\lambda}{N(\mathfrak{a})}\mathbb{Z}$ (for recall $\frac{\lambda}{N(\mathfrak{a})}$ generates the quotient $\mathfrak{a}^{-1}L/L$), a and b take integral values on $\mathfrak{a}^{-1}L$, as the fractional part would depend on λ that is 0 against both.
The linear forms $a, b \in \Lambda$ define then elements in $\operatorname{Hom}_{\mathbb{Z}}(\mathfrak{a}^{-1}L, \mathbb{Z})$, that are moreover primitive for a, b are (b is primitive for what said at the end of last Section). Therefore, $\Gamma_{a,b}(w, z; \mathfrak{a}^{-1}L)$ is well defined in the sense of Eq. 1.1 (recall a and $b = \varepsilon a$ are also linearly independent).
Write $\gamma = \gamma(a, b) \in L$ for the primitive element determined uniquely by $\det(a, b, \cdot) = s\gamma$, $\exists s \geq 1$.

Lemma 2.

$$\Gamma_{a,b}(w,z;\mathfrak{a}^{-1}L)=\prod_{k=0}^{N(\mathfrak{a})-1}\Gamma_{a,b}\left(w+k\frac{\gamma(z)}{N(\mathfrak{a})},z;L\right)$$

Proof. We have that $\lambda,\gamma\in L$ both belong to the line $\ker(a)\cap\ker(\varepsilon a)$ and γ is primitive, so $\exists n\in\mathbb{N}$ such that $\lambda=n\gamma$. Moreover $\frac{\lambda}{N(\mathfrak{a})}\notin L$ (as $\lambda\not\equiv 0 \mod N(\mathfrak{a})L$), and since $\gamma=\frac{1}{n}\lambda\in L$ we need that $\frac{n}{N(\mathfrak{a})}\notin\mathbb{Z}$ (or else $\gamma=\frac{1}{n}\lambda\in L\Rightarrow\frac{\lambda}{N(\mathfrak{a})}=\frac{n}{N(\mathfrak{a})}\cdot\frac{1}{n}\lambda\in L$ ↯). This means that $N(\mathfrak{a})\nmid n$, i.e. $\gcd(N(\mathfrak{a}),n)=1$ as $N(\mathfrak{a})$ is prime.
Now Bezout tells us that $\exists x,y\in\mathbb{Z}$ such that $nx+N(\mathfrak{a})y=1$ so that

$$\frac{x\lambda}{N(\mathfrak{a})}=\frac{xn\gamma}{N(\mathfrak{a})}=\left(\frac{1}{N(\mathfrak{a})}-y\right)\gamma=\frac{\gamma}{N(\mathfrak{a})}-y\gamma$$

and hence

$$\mathfrak{a}^{-1}L=L+\frac{\lambda}{N(\mathfrak{a})}\mathbb{Z}=L+\frac{\gamma}{N(\mathfrak{a})}\mathbb{Z}$$

for $y\in\mathbb{Z},\ \gamma\in L\Rightarrow y\gamma\in L$. Now complete $\gamma\neq 0$ to a basis of L: $\{\lambda_1,\lambda_2,\gamma\}$. Then $\left\{\lambda_1,\lambda_2,\frac{\gamma}{N(\mathfrak{a})}\right\}$ is a basis of $\mathfrak{a}^{-1}L$ (for it has both the L part and the $\frac{\gamma}{N(\mathfrak{a})}\mathbb{Z}$ part).
The map sending a vector

$$L\ni\alpha\lambda_1+\beta\lambda_2+c\gamma\mapsto\alpha\lambda_1+\beta\lambda_2+c\frac{\gamma}{N(\mathfrak{a})}\in\mathfrak{a}^{-1}L$$

induces a bijection between $C_{+-}(a,b)$ and $C'_{+-}(a,b):=\{\delta'\in\mathfrak{a}^{-1}L\mid\delta'(a)>0,\delta'(b)\leq 0\}$. Moreover classes of the first modulo $\mathbb{Z}\gamma$ correspond to classes of the second modulo $\frac{\mathbb{Z}\gamma}{N(\mathfrak{a})}$. An equivalent result holds for $C_{-+}(a,b)$ and $C'_{-+}(a,b)$ defined similarly. To get the claim, start by definition (adapted from Eq. 1.1)

$$\Gamma_{a,b}(w,z;\mathfrak{a}^{-1}L)=\frac{\prod_{\delta'\in C'_{+-}(a,b)/\mathbb{Z}\frac{\gamma}{N(\mathfrak{a})}}\left(1-exp\left(-2\pi i\left(\frac{\delta'(z)-w}{\frac{\gamma(z)}{N(\mathfrak{a})}}\right)\right)\right)}{\prod_{\delta'\in C'_{-+}(a,b)/\mathbb{Z}\frac{\gamma}{N(\mathfrak{a})}}\left(1-exp\left(2\pi i\left(\frac{\delta'(z)-w}{\frac{\gamma(z)}{N(\mathfrak{a})}}\right)\right)\right)}=$$

Now, thanks to the correspondence $C_{+-}(a,b)\leftrightarrow C'_{+-}(a,b)$ just discussed and by regrouping carefully, i.e. wrt the fractional part left when passing from quotienting modulo $\mathbb{Z}\frac{\gamma}{N(\mathfrak{a})}$ to quotienting modulo $\mathbb{Z}\gamma$ (the only terms that can remain are of the shape $\frac{k}{N(\mathfrak{a})}\gamma$ with $0\leq k\lneq N(\mathfrak{a})$), we get:

$$=\prod_{k=0}^{N(\mathfrak{a})-1}\frac{\prod_{\delta\in C_{+-}(a,b)/\mathbb{Z}\gamma}\left(1-exp\left(-2\pi i\left(\frac{\delta(z)-k\frac{\gamma(z)}{N(\mathfrak{a})}-w}{\gamma(z)}\right)\right)\right)}{\prod_{\delta\in C_{-+}(a,b)/\mathbb{Z}\gamma}\left(1-exp\left(2\pi i\left(\frac{\delta(z)-k\frac{\gamma(z)}{N(\mathfrak{a})}-w}{\gamma(z)}\right)\right)\right)}=$$

$$=\prod_{k=0}^{N(\mathfrak{a})-1}\frac{\prod_{\delta\in C_{+-}(a,b)/\mathbb{Z}\gamma}\left(1-exp\left(-2\pi i\left(\frac{\delta(z)-\left(w+k\frac{\gamma(z)}{N(\mathfrak{a})}\right)}{\gamma(z)}\right)\right)\right)}{\prod_{\delta\in C_{-+}(a,b)/\mathbb{Z}\gamma}\left(1-exp\left(2\pi i\left(\frac{\delta(z)-\left(w+k\frac{\gamma(z)}{N(\mathfrak{a})}\right)}{\gamma(z)}\right)\right)\right)}=\prod_{k=0}^{N(\mathfrak{a})-1}\Gamma_{a,b}\left(w+k\frac{\gamma(z)}{N(\mathfrak{a})},z;L\right)$$

□

This last result tells us that there exists a relation between the Elliptic Gamma function of lattice L and the one of lattice $\mathfrak{a}^{-1}L$, as a product where the first entry (the one of w) changes. We want now to "average" this result in the following, crucial definition:

Definition 3. *Given a lattice L, an ideal $\mathfrak{a}$ and an $h = \frac{\lambda}{q}$ as before, and $a, b \in \Lambda$ as in Lemma 1, we define*

$$\Gamma_{\mathfrak{a},h}(w,z;L) := \frac{\Gamma_{a,b}(w,z;\mathfrak{a}^{-1}L)}{\Gamma_{a,b}(w,z;L)^{N(\mathfrak{a})}}$$

called the **Smoothed Elliptic Gamma function***.*

Note that this quotient converges to a meromorphic function on $\mathbb{C}\times(U_a\cap U_b)$ for the two Elliptic Gamma functions at numerator and denominator do (cf. Proposition 1). Moreover since the Elliptic Gamma function $\Gamma_{a,b}$ satisfies $\Gamma_{b,a} = \Gamma_{a,b}^{-1}$, as can be seen directly from Eq. 1.1 together with Definition 1, we get:

$$\Gamma_{\mathfrak{a},h}(w,z;L) := \frac{\Gamma_{b,\varepsilon^{-1}b}(w,z;L)^{N(\mathfrak{a})}}{\Gamma_{b,\varepsilon^{-1}b}(w,z;\mathfrak{a}^{-1}L)}$$

Remark 5. *The notation for $\Gamma_{\mathfrak{a},h}(w,z;L)$ helps us see that it depends only on the choice of $\mathfrak{a}$, h and oriented lattice L. We will see in following remarks and in the statement of the Main Conjecture that in some cases the dependence from h can be omitted.*

We want now to evaluate the Smoothed Elliptic Gamma function at particular values $w,z\in\mathbb{C}\times(U_a\cap U_b)$. For how we chose a in Lemma 1, we have that $\mathbf{x}\in U_a$. Moreover, after choosing (μ,ν) an oriented basis for $H(a)$, it's easy to see that $(\varepsilon\mu,\varepsilon\nu)$ is an oriented basis for $H(b) = H(\varepsilon a)$, for ε preserves orientation. Now $\mathbf{x}\in U_a \Rightarrow \mathrm{Im}(\mu(\mathbf{x})\overline{\nu(\mathbf{x})}) > 0$ and since $\varepsilon(\mathbf{x}) = \sigma_{\mathbb{C}}(\varepsilon)^{-1}\mathbf{x}$ we get that

$$0 < \mathrm{Im}(\mu(\mathbf{x})\overline{\nu(\mathbf{x})}) = \mathrm{Im}\left(\sigma_{\mathbb{C}}(\varepsilon)\mu(\varepsilon(\mathbf{x}))\overline{\sigma_{\mathbb{C}}(\varepsilon)}\;\overline{\nu(\varepsilon(\mathbf{x}))}\right) = \|\,\sigma_{\mathbb{C}}(\varepsilon)\|^2\,\mathrm{Im}(\varepsilon\mu(\mathbf{x})\overline{\varepsilon\nu(\mathbf{x})})$$

and since $\|\,\sigma_{\mathbb{C}}(\varepsilon)\|^2 > 0$ we get $\mathrm{Im}(\varepsilon\mu(\mathbf{x})\overline{\varepsilon\nu(\mathbf{x})}) > 0$ that proves $\mathbf{x}\in U_b$ too.
Thus we can evaluate $\Gamma_{\mathfrak{a},h}(w,z;L)$ at $z = \mathbf{x}\in U_a\cap U_b$.
Since $\mathfrak{a}$ is chosen to be relatively prime to $N(\mathfrak{f})$ and for how we took q, we get that $(N(\mathfrak{a}),q) = 1$. Indeed we chose q such that $(q) = \mathfrak{f}\cap\mathbb{Z}$ so that $q|N(\mathfrak{f})$. Moreover call $l\in\mathbb{Z}$ such that $\mathfrak{a}\cap\mathbb{Z} = (l)$ (such an l exists for $\mathbb{Z}$ is a PID) then $l|N(\mathfrak{a})$ and the latter is prime so they must be equal; $N(\mathfrak{f})$ relatively prime to $\mathfrak{a}$ gives $(l,N(\mathfrak{f})) = 1 \Rightarrow (N(\mathfrak{a}),N(\mathfrak{f})) = 1 \Rightarrow (N(\mathfrak{a}),q) = 1$.
This tells us that $h = \frac{\lambda}{q} \notin \frac{\mathbb{Z}\lambda}{N(\mathfrak{a})}$. Moreover $h = \frac{\lambda}{q}\notin L$, as by definition we need that $\frac{\lambda}{q}\equiv 1_K \mod L$, so that

$$h = \frac{\lambda}{q} \notin L + \frac{\mathbb{Z}\lambda}{N(\mathfrak{a})} = \mathfrak{a}^{-1}L$$

Remembering what the divisor of $\Gamma_{a,b}$ was (cf. Remark 1), we want to see that the complex number $w = h(\mathbf{x})$ lies away from the zeroes and the poles of $\Gamma_{\mathfrak{a},h}(w,\mathbf{x};L)$; since zeroes and poles form a discrete set, it's enough to prove that it's not one of them. Zeroes/poles of $\Gamma_{a,b}(w,\mathbf{x};L)$ are $w = \delta(\mathbf{x}) + n\gamma(\mathbf{x})$ for some $\delta\in L$ (more precisely in $C_{+-}(a,b), C_{-+}(a,b)$ but L is enough for this case), $n\in\mathbb{Z}$. If $h(\mathbf{x})$ was a zero/pole of $\Gamma_{a,b}(w,\mathbf{x};L)$ then $h(\mathbf{x}) = \delta(\mathbf{x}) + n\gamma(\mathbf{x})$, which would mean $h = \delta + n\gamma \in L + \mathbb{Z}\gamma \subset \mathfrak{a}^{-1}L$ ↯. Analogously zeros/poles of $\Gamma_{a,b}(w,\mathbf{x};\mathfrak{a}^{-1}L) = \prod_k \Gamma_{a,b}(w + k\frac{\gamma(\mathbf{x})}{N(\mathfrak{a})},\mathbf{x};L)$, cf. Lemma 2, are $w = \left(\delta + n\gamma - \frac{k\gamma}{N(\mathfrak{a})}\right)(\mathbf{x})$ so if $h(\mathbf{x})$ was one of those then we'd have

$$h = \delta + \frac{(nN(\mathfrak{a})-k)\gamma}{N(\mathfrak{a})} \in L + \frac{\mathbb{Z}\gamma}{N(\mathfrak{a})} \text{ ↯}$$

We can then define the main object of our theory:

Definition 4. *For an oriented lattice L, ideal $\mathfrak{a}$, $h = \frac{\lambda}{q}$ with λ admissible and a, b as in Lemma 1, define the unit in $\mathbb{C}$ (for we've just proven it's neither a zero nor a pole)*

$$\Gamma_{\mathfrak{a},h}(L) := \Gamma_{\mathfrak{a},h}(h(\mathbf{x}), \mathbf{x}; L) \in \mathbb{C}^\times$$

We will see in the next section how these values don't depend on h under certain hypothesis. We now give a few remarks that will help us to see some easy consequences of the Main Conjecture.

Remark 6. *1. If $u \in \mathscr{O}_{K,\mathfrak{f}}^{+,\times}$, we have*

$$\Gamma_{\mathfrak{a},uh}(L) = \Gamma_{\mathfrak{a},h}(L) \tag{1.4}$$

Proof. We saw in Remark 3 that for $u \in \mathscr{O}_{K,\mathfrak{f}}^{+,\times}$, if λ is admissible for L, then so is also $u\lambda$. We claim that the element $a = a_\lambda \in \Lambda$ defined in Lemma 1 satisfies $ua_\lambda = a_{u\lambda}$.
Indeed, by recalling the proof of the Lemma, we need $a_{u\lambda} \in \ker(u\lambda) \cap \ker(\varepsilon^{-1}u\lambda)$. If $a_\lambda \in \ker(\lambda) \Rightarrow \lambda(a_\lambda) = 0 \Rightarrow u\lambda(ua_\lambda) = \lambda(a_\lambda) = 0$ too by definition of dual action, so that $ua_\lambda \in \ker(u\lambda)$ and similarly $ua_\lambda \in \ker(\varepsilon^{-1}u\lambda)$. Moreover ua_λ is primitive for we're just multiplying for a unit, and $u \in \mathscr{O}_{K,\mathfrak{f}}^{+,\times}$ ensures that the orientation remains the same so that also the condition *ii*) from the Lemma is satisfied.
We can see, working as we did in the proof of Remark 4, that since u has positive norm, $\mathbf{x}$ is an eigenvector for u. We can now prove the claim:

$$\Gamma_{\mathfrak{a},uh}(L) = \Gamma_{\mathfrak{a},uh}(uh(\mathbf{x}), \mathbf{x}; L) = \frac{\Gamma_{ua,ub}(uh(\mathbf{x}), \mathbf{x}; \mathfrak{a}^{-1}L)}{\Gamma_{ua,ub}(uh(\mathbf{x}), \mathbf{x}; L)^{N(\mathfrak{a})}}$$

By definition and for what we just said about $a_{u\lambda}$ and consequently for b. Now look at the numerator of $\Gamma_{ua,ub}(uh(\mathbf{x}), \mathbf{x}; L)$ using Eq. 1.1. Recall that $C_{+-}(a,b) = \{\delta \in L \mid \delta(a) > 0,\ \delta(b) \leq 0\}$ and similarly

$$C_{+-}(ua,ub) = \{\delta \in L \mid (u^{-1}\delta)(a) = \delta(ua) > 0,\ (u^{-1}\delta)(b) = \delta(ub) \leq 0\}$$

We can then easily see that there is a correspondence between the two cones given by

$$C_{+-}(a,b) \overset{1:1}{\longleftrightarrow} C_{+-}(ua,ub); \quad \delta \mapsto u\delta$$

Note also that $s\gamma_{ua,ub} = \det(ua,ub,\cdot) = nu\lambda(\cdot)$. For Lemma 1.i) we have that $nu\lambda(\cdot) = un\lambda(\cdot) = u\det(a,b,\cdot) = us\gamma_{a,b}$ so that $s\gamma_{ua,ub} = us\gamma_{a,b} \Rightarrow \gamma_{ua,ub} = u\gamma_{a,b}$. So

$$\text{Numerator of } \Gamma_{ua,ub}(uh(\mathbf{x}), \mathbf{x}; L) = \prod_{\delta \in C_{+-}(ua,ub)/\mathbb{Z}\gamma_{ua,ub}} \left(1 - e^{-2\pi i\left(\frac{\delta(\mathbf{x}) - uh(\mathbf{x})}{\gamma_{ua,ub}(\mathbf{x})}\right)}\right) =$$

$$= \prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma_{a,b}} \left(1 - e^{-2\pi i\left(\frac{u\delta(\mathbf{x}) - uh(\mathbf{x})}{u\gamma_{a,b}(\mathbf{x})}\right)}\right) = \prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma_{a,b}} \left(1 - e^{-2\pi i\left(\frac{\delta(\mathbf{x}) - h(\mathbf{x})}{\gamma_{a,b}(\mathbf{x})}\right)}\right)$$

Which is the numerator of $\Gamma_{a,b}(h(\mathbf{x}), \mathbf{x}; L)$. Analogously we see that the denominators coincide and also the numerators and denominators of $\Gamma_{ua,ub}(uh(\mathbf{x}), \mathbf{x}; \mathfrak{a}^{-1}L)$ with those of $\Gamma_{a,b}(h(\mathbf{x}), \mathbf{x}; \mathfrak{a}^{-1}L)$ (by using that it is a product of Elliptic Gamma functions wrt lattice L by Lemma 2), so that $\Gamma_{\mathfrak{a},uh}(L) = \Gamma_{\mathfrak{a},h}(L)$. □

2. *If* $\alpha \in 1+\mathfrak{f}$ *with* $\sigma_{\mathbb{R}}(\alpha) > 0$*, then:*

$$\Gamma_{\mathfrak{a},\alpha^{-1}h}((\alpha^{-1})L) = \Gamma_{\mathfrak{a},h}(L) \tag{1.5}$$

Proof. $N(\alpha) = \sigma_{\mathbb{R}}(\alpha)\|\sigma_{\mathbb{C}}(\alpha)\|^2 > 0$ so multiplication by α preserves orientation on K and $\mathbf{x}$ is an eigenvector for α (as we proved for ε and also for u in the previous point). Recall we proved in Remark 3.4 that if λ is admissible for L then $\alpha^{-1}\lambda$ is admissible for $(\alpha)^{-1}L$. The element $a_{\alpha^{-1}\lambda} \in \mathrm{Hom}((\alpha)^{-1}L,\mathbb{Z})$ given by Lemma 1 satisfies $a_{\alpha^{-1}\lambda} = \alpha^{-1}a_{\lambda}$. Indeed $a_{\lambda} \in \ker(\lambda) \Rightarrow \lambda(a_{\lambda}) = 0 \Rightarrow \alpha^{-1}\lambda(\alpha^{-1}a_{\lambda}) = \lambda(a_{\lambda}) = 0$ too (always by definition of dual action), so that $\alpha^{-1}a_{\lambda} \in \ker(\alpha^{-1}\lambda)$ and similarly $\alpha^{-1}a_{\lambda} \in \ker(\varepsilon^{-1}\alpha^{-1}\lambda)$. Moreover $\alpha^{-1}a_{\lambda}$ is primitive as an element of $\mathrm{Hom}((\alpha)^{-1}L,\mathbb{Z})$ for a_{λ} is primitive in $\mathrm{Hom}(L,\mathbb{Z}) = \Lambda$; also α preserves orientation by what said above, so that also condition *ii*) of the Lemma is satisfied by $\alpha^{-1}a_{\lambda}$. We can now prove the claim; by definitions plus what we just said about $\alpha^{-1}a$ and consequently $\alpha^{-1}b$:

$$\Gamma_{\mathfrak{a},\alpha^{-1}h}((\alpha^{-1})L) = \Gamma_{\mathfrak{a},\alpha^{-1}h}(\alpha^{-1}h(\mathbf{x}),\mathbf{x};(\alpha^{-1})L) = \frac{\Gamma_{\alpha^{-1}a,\alpha^{-1}b}(\alpha^{-1}h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}(\alpha^{-1})L)}{\Gamma_{\alpha^{-1}a,\alpha^{-1}b}(\alpha^{-1}h(\mathbf{x}),\mathbf{x};(\alpha^{-1})L)^{N(\mathfrak{a})}}$$

Look at the numerator of $\Gamma_{\alpha^{-1}a,\alpha^{-1}b}(\alpha^{-1}h(\mathbf{x}),\mathbf{x};(\alpha^{-1})L)$. As in the previous proof we have

$$C_{+-}(\alpha^{-1}a,\alpha^{-1}b) = \{\delta \in (\alpha^{-1})L \mid (\alpha\delta)(a) = \delta(\alpha^{-1}a) > 0,\ (\alpha\delta)(b) = \delta(\alpha^{-1}b) \leq 0\}$$

We can then see that there is a correspondence given by

$$C_{+-}(a,b) \overset{1:1}{\longleftrightarrow} C_{+-}(\alpha^{-1}a,\alpha^{-1}b); \quad \delta \mapsto \alpha^{-1}\delta$$

Also like in previous proof we get $\gamma_{\alpha^{-1}a,\alpha^{-1}b} = \alpha^{-1}\gamma_{a,b}$. The numerator of $\Gamma_{\alpha^{-1}a,\alpha^{-1}b}(\alpha^{-1}h(\mathbf{x}),\mathbf{x};(\alpha^{-1})L)$ is then equal to

$$\prod_{\delta \in C_{+-}(\alpha^{-1}a,\alpha^{-1}b)/\mathbb{Z}\gamma_{\alpha^{-1}a,\alpha^{-1}b}} \left(1 - e^{-2\pi i\left(\frac{\delta(\mathbf{x}) - \alpha^{-1}h(\mathbf{x})}{\gamma_{\alpha^{-1}a,\alpha^{-1}b}(\mathbf{x})}\right)}\right) =$$

$$= \prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma_{a,b}} \left(1 - e^{-2\pi i\left(\frac{\alpha^{-1}\delta(\mathbf{x}) - \alpha^{-1}h(\mathbf{x})}{\alpha^{-1}\gamma_{a,b}(\mathbf{x})}\right)}\right) = \prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma_{a,b}} \left(1 - e^{-2\pi i\left(\frac{\delta(\mathbf{x}) - h(\mathbf{x})}{\gamma_{a,b}(\mathbf{x})}\right)}\right)$$

which is the numerator of $\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};L)$. Analogously the denominators coincide, and also numerators and denominators of $\Gamma_{\alpha^{-1}a,\alpha^{-1}b}(\alpha^{-1}h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}(\alpha^{-1})L)$ and $\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}L)$ (thanks again to Lemma 2) so that $\Gamma_{\mathfrak{a},\alpha^{-1}h}((\alpha^{-1})L) = \Gamma_{\mathfrak{a},h}(L)$. □

3. *If* $\alpha \in -1+\mathfrak{f}$ *with* $\sigma_{\mathbb{R}}(\alpha) > 0$*, then:*

$$\Gamma_{\mathfrak{a},-\alpha^{-1}h}((\alpha^{-1})L) = \Gamma_{\mathfrak{a},h}(-h(\mathbf{x}),\mathbf{x};L) \tag{1.6}$$

Proof. First of all notice that if λ is admissible for L, then $-\alpha^{-1}\lambda$ is admissible for $(\alpha^{-1})L$. Indeed $\frac{-\alpha^{-1}\lambda}{q} \equiv_{(\alpha)^{-1}L} 1_K$ for $\frac{\lambda}{q} \equiv_L 1_K$ and $-\alpha^{-1} \equiv_{\mathfrak{f}} 1_K$. Moreover $\lambda \in N(\mathfrak{a})\mathfrak{a}^{-1}L \Rightarrow -\alpha^{-1}\lambda \in (\alpha^{-1})N(\mathfrak{a})\mathfrak{a}^{-1}L = N(\mathfrak{a})\mathfrak{a}^{-1}(\alpha)^{-1}L$, and $\lambda \not\equiv 0 \mod N(\mathfrak{a})L \Rightarrow -\alpha^{-1}\lambda \not\equiv 0 \mod (\alpha^{-1})N(\mathfrak{a})L = N(\mathfrak{a})(\alpha^{-1})L$.
Notice that the primitive element associated to $-\alpha^{-1}\lambda$ from Lemma 1 is the same as in previous remark, i.e. $a_{-\alpha^{-1}\lambda} = \alpha^{-1}a_{\lambda}$. Indeed the sign doesn't influence the element $a_{-\alpha^{-1}\lambda}$ since the condition

ii) of the Lemma, $\mathbf{x} \in U_{a_{-\alpha^{-1}\lambda}}$, depends only on the lattice for which $-\alpha^{-1}\lambda$ is admissible, which is $(\alpha^{-1})L$ as it was in last remark.
We get as in previous cases that $\mathbf{x}$ is an eigenvector for α that has $\sigma_{\mathbb{R}}(\alpha) > 0$ and $\gamma_{\alpha^{-1}a,\alpha^{-1}b} = \alpha^{-1}\gamma_{a,b}$. Now since by definition:

$$\Gamma_{\mathfrak{a},-\alpha^{-1}h}((\alpha^{-1})L) = \Gamma_{\mathfrak{a},-\alpha^{-1}h}(-\alpha^{-1}h(\mathbf{x}),\mathbf{x};(\alpha^{-1})L) =$$

$$= \frac{\Gamma_{\alpha^{-1}a,\alpha^{-1}b}(-\alpha^{-1}h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}(\alpha^{-1})L)}{\Gamma_{\alpha^{-1}a,\alpha^{-1}b}(-\alpha^{-1}h(\mathbf{x}),\mathbf{x};(\alpha^{-1})L)^{N(\mathfrak{a})}}$$

and by the same correspondence

$$C_{+-}(a,b) \overset{1:1}{\longleftrightarrow} C_{+-}(\alpha^{-1}a,\alpha^{-1}b); \quad \delta \mapsto \alpha^{-1}\delta$$

of previous remark we get that the numerator of $\Gamma_{\alpha^{-1}a,\alpha^{-1}b}(-\alpha^{-1}h(\mathbf{x}),\mathbf{x};(\alpha^{-1})L)$ is

$$\prod_{\delta \in C_{+-}(\alpha^{-1}a,\alpha^{-1}b)/\mathbb{Z}\gamma_{\alpha^{-1}a,\alpha^{-1}b}} \left(1 - e^{-2\pi i\left(\frac{\delta(\mathbf{x})+\alpha^{-1}h(\mathbf{x})}{\gamma_{\alpha^{-1}a,\alpha^{-1}b}(\mathbf{x})}\right)}\right) =$$

$$= \prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma_{a,b}} \left(1 - e^{-2\pi i\left(\frac{\alpha^{-1}\delta(\mathbf{x})+\alpha^{-1}h(\mathbf{x})}{\alpha^{-1}\gamma_{a,b}(\mathbf{x})}\right)}\right) = \prod_{\delta \in C_{+-}(a,b)/\mathbb{Z}\gamma_{a,b}} \left(1 - e^{-2\pi i\left(\frac{\delta(\mathbf{x})+h(\mathbf{x})}{\gamma_{a,b}(\mathbf{x})}\right)}\right)$$

is the numerator of $\Gamma_{a,b}(-h(\mathbf{x}),\mathbf{x};L)$. Getting similar results for the other factors, we obtain the wanted result. $\square$

Notice that in this case we're unable to state a clean expression between units obtained from the Smoothed Elliptic Gamma functions as we did in the previous two remarks. A better result in this sense will be obtained in the last Chapter (Remark 24, Eq. 5.36) when we'll prove that, up to a third root of unity, trivial in the case $\mathfrak{a}$ *coprime with 6,* $\Gamma_{\mathfrak{a},h}(-h(\mathbf{x}),\mathbf{x};L) = \Gamma_{\mathfrak{a},h}(L)^{-1}$, *so that*

$$\Gamma_{\mathfrak{a},-\alpha^{-1}h}((\alpha^{-1})L) = \Gamma_{\mathfrak{a},h}(L)^{-1} \tag{1.7}$$

Remark 7. *We may wonder what happens if we chose the other complex embedding at the beginning of this Section. Write* $\Gamma_{\mathfrak{a},h}(L,\sigma_{\mathbb{C}}) = \Gamma_{\mathfrak{a},h}(L)$ *for the unit we just constructed. If we switched* $\sigma_{\mathbb{C}}$ *with* $\overline{\sigma_{\mathbb{C}}}$, *the orientation of K would change as we would pull back the usual orientation of* $\mathbb{R}\times\mathbb{C}$ *wrt* $x \mapsto \big(\sigma_{\mathbb{R}}(x),\overline{\sigma_{\mathbb{C}}}(x)\big)$ *instead of* $x \mapsto \big(\sigma_{\mathbb{R}}(x),\sigma_{\mathbb{C}}(x)\big)$.
Also $\mathbf{x}$ *would now be the restriction of* $\overline{\sigma_{\mathbb{C}}}$, *so that the element* a_λ *associated to* λ *admissible for L as in Lemma 1 stays the same: the condition i) is precisely the same, while for condition ii) both the orientation of* U_{a_λ} *and* $\mathbf{x}$ *add complex conjugate factors that cancel each other.*
So a_λ, b_λ *remain the same and consequently also the* α,β *from Proposition 1; in the proof of that result, we saw that* γ *completed* α,β *to an oriented basis of K, so that the change of orientation of K makes* γ *change sign. Recall from Proposition 2 that*

$$\Gamma_{a,b}(w,z;L) = \prod_{\bar\delta \in F/\mathbb{Z}\gamma} \Gamma\left(\frac{w+\bar\delta(z)}{\gamma(z)},\frac{\alpha(z)}{\gamma(z)},\frac{\beta(z)}{\gamma(z)}\right)$$

with α, β, γ *as above; so, computing it with* $z = \mathbf{x}$, $w = h(\mathbf{x})$, *we get*

$$\Gamma_{\mathfrak{a},h}(L, \sigma_{\mathbb{C}}) = \Gamma_{a,b}(h(\mathbf{x}), \mathbf{x}; L) = \prod_{\bar{\delta} \in F/\mathbb{Z}\gamma} \Gamma\left(\frac{h(\mathbf{x}) + \bar{\delta}(\mathbf{x})}{\gamma(\mathbf{x})}, \frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})}, \frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})} \right)$$

Denote $\tau := \frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})}$ *and* $\sigma := \frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})}$. *If we consider* $\overline{\sigma_{\mathbb{C}}}$ *instead of* $\sigma_{\mathbb{C}}$, τ *and* σ *change into* $-\overline{\tau}$ *and* $-\overline{\sigma}$, *as* $\mathbf{x}$ *become its complex conjugate and* γ *adds the minus sign. The same holds for* $\frac{h(\mathbf{x})+\bar{\delta}(\mathbf{x})}{\gamma(\mathbf{x})}$. *Between the properties of the classical Elliptic Gamma function from Subsection 1.0.2, we saw that* $\Gamma(-\overline{z}, -\overline{\tau}, -\overline{\sigma}) = \overline{\Gamma(z, \tau, \sigma)}$, *so that*

$$\Gamma_{\mathfrak{a},h}(L, \overline{\sigma_{\mathbb{C}}}) = \prod_{\bar{\delta} \in F/\mathbb{Z}\gamma} \Gamma\left(-\overline{\frac{h(\mathbf{x}) + \bar{\delta}(\mathbf{x})}{\gamma(\mathbf{x})}}, -\overline{\frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})}}, -\overline{\frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})}} \right) =$$

$$= \overline{\prod_{\bar{\delta} \in F/\mathbb{Z}\gamma} \Gamma\left(\frac{h(\mathbf{x}) + \bar{\delta}(\mathbf{x})}{\gamma(\mathbf{x})}, \frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})}, \frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})} \right)} = \overline{\Gamma_{\mathfrak{a},h}(L, \sigma_{\mathbb{C}})}$$

So if we exchange the complex embedding with its complex conjugate, also the units we constructed in this Section change into their complex conjugates.

1.3 The main Conjecture

Let $\mathfrak{f}$ be a non zero ideal in $\mathscr{O}_K$ different from $\mathscr{O}_K$, as in previous Section. Let $K(\mathfrak{f})$ be the narrow ray class field of K of conductor $\mathfrak{f}$, that is a particular finite abelian extension of K which has the property of being ramified only at $\mathfrak{f}$[3].

Notation 1.

- $I(\mathfrak{f})$*:= the group of fractional ideals of K generated by ideals coprime to* $\mathfrak{f}$*;*
- $P(\mathfrak{f})$*:= set of principal fractional ideals* (α) *generated by* $\alpha \in K$ *such that* $v_{\mathfrak{p}}(\alpha - 1) \geq v_{\mathfrak{p}}(\mathfrak{f})$, $\forall \mathfrak{p}$ *prime dividing* $\mathfrak{f}$*;*
- $P(\mathfrak{f})^{+}$*:=*$P(\mathfrak{f}) \cap \{\sigma_{\mathbb{R}}(\alpha) > 0\}$*;*
- $\mathscr{C}l_K^{+}(\mathfrak{f}) := I(\mathfrak{f})/P(\mathfrak{f})^{+}$

Recall a fact from class field theory: the group $\mathscr{C}l_K^{+}(\mathfrak{f})$ is finite and naturally isomorphic to $Gal(K(\mathfrak{f})/K)$ via the Artin map.
Briefly recall that for L/K a Galois extension of number fields, if $\mathfrak{B} \subset \mathscr{O}_L$ is a prime ideal lying over $\mathfrak{b} \subset \mathscr{O}_K$ (i.e. $\mathfrak{B} \cap \mathscr{O}_K = \mathfrak{b}$), and $e(\mathfrak{B}/\mathfrak{b}) = 1$, i.e. is an unramified extension at $\mathfrak{b}$, then $\exists!$[4] $Frob(\mathfrak{B}/\mathfrak{b}) \in Gal(L/K)$ corresponding to the usual Frobenius morphism of $Gal(\kappa(\mathfrak{B})/\kappa(\mathfrak{b}))$ (this is a finite extension of the finite residue field $\kappa(\mathfrak{b})$, so the Frobenius is the usual morphism that raises an element of $\kappa(\mathfrak{B})$ to the $|\kappa(\mathfrak{b})|$-th power).
In our case we have moreover that the extension $K(\mathfrak{f})/K$ is abelian ($K(\mathfrak{f})$ is L in our particular case), so that

[3] Its full definition comes from Global Class Field theory and would require too many preliminary objects to be given in full detail. The only property of $K(\mathfrak{f})$ that we'll need is the one recalled.

[4] The uniqueness comes from the extension being unramified at $\mathfrak{b}$.

$Frob(\mathfrak{B}/\mathfrak{b})$ only depends on $\mathfrak{b}$: indeed if we choose another $\mathfrak{B}'$ lying over $\mathfrak{b}$, this will be the conjugate of $\mathfrak{B}$ via an element of $Gal(K(\mathfrak{f})/K)$, say σ. Now Galois theory gives that

$$Frob(\sigma(\mathfrak{B})/\mathfrak{b}) = \sigma Frob(\mathfrak{B}/\mathfrak{b})\sigma^{-1} = Frob(\mathfrak{B}/\mathfrak{b})$$

where the last equality holds for abelianity. This allows us to express the most important direction of the Artin map; for $\mathfrak{b}$ prime ideal coprime with $\mathfrak{f}$ (that gives also $e(\mathfrak{B}/\mathfrak{b}) = 1$ by definition of narrow ray class field):

$$\begin{aligned} \mathscr{C}l_K^+(\mathfrak{f}) &\xrightarrow{\sim} Gal(K(\mathfrak{f})/K) \\ [\mathfrak{b}] &\longmapsto Frob(\mathfrak{b}) = Frob(\mathfrak{B}/\mathfrak{b}) \text{ for any } \mathfrak{B} \subset \mathscr{O}_L \text{ prime ideal lying over } \mathfrak{b} \end{aligned}$$

We will denote this morphism lifted from $\mathscr{C}l_K^+(\mathfrak{f})$ to $I(\mathfrak{f})$ as σ:

$$\mathfrak{c} \in I(\mathfrak{f}) \mapsto \sigma_{\mathfrak{c}} \in Gal(K(\mathfrak{f})/K)$$

Conjecture 3. *Let $\mathfrak{f}$ be an ideal of $\mathscr{O}_K$ such that $K(\mathfrak{f})$ is totally complex, fix a complex embedding $K(\mathfrak{f}) \hookrightarrow \mathbb{C}$ that extends the fixed embedding $\sigma_{\mathbb{C}}\colon K \hookrightarrow \mathbb{C}$ of last Section. Let $\mathfrak{b}$ be an ideal of $\mathscr{O}_K$ such that $(\mathfrak{f},\mathfrak{b}) = 1$; define the lattice $L := \mathfrak{f}\mathfrak{b}^{-1}$.*
Take $\mathfrak{a} \subset \mathscr{O}_K$ coprime with $6N(\mathfrak{f})$ and such that $N(\mathfrak{a})$ is an odd prime.
Let $h = \frac{\lambda}{q}$ with $\lambda \in L$ admissible and q smallest positive integer in $\mathfrak{f} \cap \mathbb{Z}$, i.e. $(q) = \mathfrak{f} \cap \mathbb{Z}$.

i) $\Gamma_{\mathfrak{a},h}(L) \in \mathbb{C}^\times$ only depends on L and $\mathfrak{a}$, and is the image in $\mathbb{C}$ via the fixed embedding $K(\mathfrak{f}) \hookrightarrow \mathbb{C}$ of a unit $u_{L,\mathfrak{a}} \in \mathscr{O}_{K(\mathfrak{f})}^\times$;

ii) If w is an archimedean place of $K(\mathfrak{f})$ lying above the real embedding $\sigma_{\mathbb{R}}$ of K, then $|u_{L,\mathfrak{a}}|_w = 1$;

iii) $\forall \mathfrak{c} \in I(\mathfrak{f})$ the following reciprocity law holds:

$$\sigma_{\mathfrak{c}}(u_{L,\mathfrak{a}}) = u_{\mathfrak{c}^{-1}L,\mathfrak{a}}$$

Recall that we say that an archimedean place w **lies above** $\sigma_{\mathbb{R}}\colon K \hookrightarrow \mathbb{R}$ iff

$$|\cdot|_w\big|_K \sim |\sigma_{\mathbb{R}}(\cdot)|$$

i.e. iff the restriction of w to K is equivalent to the norm over $\mathbb{R}$ of the valuation $\sigma_{\mathbb{R}}$. The concept of lying above will be often used in Chapters 3,4.

Remark 8. *1. When $\mathfrak{a}$ isn't coprime with 6, we still Conjecture that $\Gamma_{\mathfrak{a},h}(L)$ is algebraic, but in general we need to raise it to a certain power to get a number that is the image in $\mathbb{C}^\times$ of a unit in $K(\mathfrak{f})$ (it seems by numerical examples that the 12-th power is enough). Also the resulting unit only depends on L and $\mathfrak{a}$ up to a root of unity in $K(\mathfrak{f})$.*

2. Part i) of the Conjecture (non dependence from h) implies that $\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})$ only depends on $\mathfrak{b}$ through its class in the narrow ray class group $\mathscr{C}l_K^+(\mathfrak{f})$. Indeed the ideal $\mathfrak{b}(\alpha)$ for $\alpha \in 1+\mathfrak{f}$ ($\Rightarrow v_{\mathfrak{p}}(\alpha-1) \geq v_{\mathfrak{p}}(\mathfrak{f})$, $\forall \mathfrak{p}|\mathfrak{f}$ prime), $\sigma_{\mathbb{R}}(\alpha) > 0$, i.e. another element in the same class of $\mathfrak{b}$ modulo $P(\mathfrak{f})^+$, gives the same $\Gamma_{\mathfrak{a},h}$ by Remark 6.2.
This is consistent with part iii) of the Conjecture: as the Artin map is from $\mathscr{C}l_K^+(\mathfrak{f})$, then, if seen from $I(\mathfrak{f})$, it factors through $P(\mathfrak{f})^+$. Indeed, by above we get that two ideals $\mathfrak{c},\mathfrak{c}'$ congruent modulo $P(\mathfrak{f})^+$, to which correspond the same morphism $\sigma_{\mathfrak{c}} = \sigma_{\mathfrak{c}'}$, give the same unit via the reciprocity law: $u_{\mathfrak{c}^{-1}L,\mathfrak{a}} = u_{\mathfrak{c}'^{-1}L,\mathfrak{a}}$.

3. *Let $\tau \in Gal(K(\mathfrak{f})/K)$ correspond to the unique non-trivial element of the order two subgroup $P(\mathfrak{f})/P(\mathfrak{f})^+$ of $\mathscr{C}l_K^+(\mathfrak{f})$ via the Artin map.*

$$\tau \leftrightarrow \alpha \in P(\mathfrak{f}) \setminus P(\mathfrak{f})^+ \text{ i.e. } \alpha \in 1+\mathfrak{f},\ \sigma_{\mathbb{R}}(\alpha) < 0 \Leftrightarrow -\alpha \in -1+\mathfrak{f},\ \sigma_{\mathbb{R}}(-\alpha) > 0$$

Now Remark 6.3 (working with $-\alpha$) tells us

$$\Gamma_{\mathfrak{a},\alpha^{-1}h}((-\alpha^{-1})L) = \Gamma_{\mathfrak{a},h}(-h(\mathbf{x}),\mathbf{x};L) = \Gamma_{\mathfrak{a},h}(h(\mathbf{x}),\mathbf{x};L) = \Gamma_{\mathfrak{a},h}(L)$$

where the second equality holds because part $i)$ of the Conjecture tells us that $\Gamma_{\mathfrak{a},h}(L)$ doesn't depend on h. Now part $iii)$ of the Conjecture gives

$$\tau(u_{L,\mathfrak{a}}) = u_{(-\alpha^{-1})L,\mathfrak{a}} = u_{L,\mathfrak{a}}^{-1}$$

where the last equality holds for Eq. 1.7 together with the correspondence of point $i)$ of the Conjecture.

Chapter 2

Classical Kronecker Limit Formulas

In Chapter 3 we will present a new type of Kronecker Limit Formula created by the authors of [BCG23] and regarding the Main Conjecture. It will be convenient to review now the two classical Limit Formulas, together with their proofs. We will follow throughout the article [BFK15] by Kenichi Bannai, Hidekazu Furusho and Shinichi Kobayashi.

Let $\Gamma \subset \mathbb{C}$ be a lattice and call A the area of its fundamental domain divided by π.

Definition 5. *For $a \in \mathbb{Z}$, $z_0, w_0 \in \mathbb{C}$, $s \in \mathbb{C}$ define:*

$$K_a^*(z_0, w_0, s) = K_a^*(z_0, w_0, s; \Gamma) := \sum_{\gamma \in \Gamma \setminus \{-z_0\}} \frac{(\overline{z_0} + \overline{\gamma})^a}{|z_0 + \gamma|^{2s}} \chi_{w_0}(\gamma)$$

where

$$\chi_w(z) = \exp\left[\frac{(z\overline{w} - w\overline{z})}{A}\right], \quad \forall z, w \in \mathbb{C}$$

The series defining $K_a^*(z_0, w_0, s)$ converges for $\mathrm{Re}(s) > \frac{a}{2} + 1$, but one may give it meaning for general $s \in \mathbb{C}$ by analytic continuation. Precisely we have the following

Proposition 3. *i) The function $K_a^*(z_0, w_0, s)$ for s continues meromorphically to a function on the whole $\mathbb{C}$, with a simple pole only at $s = 1$, in the case $a = 0$ and $w_0 \in \Gamma$;*

ii) The function $K_a^*(z_0, w_0, s)$ satisfies the functional equation

$$\Gamma(s) K_a^*(z_0, w_0, s) = A^{a+1-2s} \Gamma(a + 1 - s) K_a^*(w_0, z_0, a + 1 - s) \chi_{z_0}(w_0)$$

A proof of this proposition will be given only for the case $a = 0$ in the Section about the first Limit Formula.

We will use throughout $\theta(z)$, the reduced theta function associated to the divisor $[0]$ of $\mathbb{C}/\Gamma$, normalized so that $\theta'(0) = 1$. The existence of this function can be derived by working with the usual σ functions of elliptic curves (see e.g. [Ste13] Chapter 2) but we'll just take it as a fact. This function can be proven to satisfy the transformation formula for $\gamma \in \Gamma$

$$\theta(z + \gamma) = \alpha(\gamma) \exp\left(\frac{z\gamma}{A} + \frac{|\gamma|^2}{2A}\right) \theta(z) \tag{2.1}$$

where $\alpha(\gamma) = -1$ if $\gamma \notin 2\Gamma$ and $\alpha(\gamma) = 1$ otherwise.

Definition 6. *The quotient*

$$\Theta(z,w) := \frac{\theta(z+w)}{\theta(z)\theta(w)}, \quad z,w \in \mathbb{C}$$

is called the ***Kronecker Theta function****.*

Observe that $K_a^*(z,w,s)$ is defined for all $z,w \in \mathbb{C}$ but, as a function, it's not continuous for $z,w \in \Gamma$. To solve this, we just define $K_a(z,w,s)$ as the $\mathscr{C}^\infty$-function defined as $K_a(z,w,s) := K_a^*(z,w,s)$ for $z,w \in \mathbb{C}\setminus\Gamma$ and simply undefined for $z,w \in \Gamma$.
Since $K_a^*(z,w,s)$ for $z \in \Gamma$ is defined by removing the summand that would have a pole, we have

$$\lim_{z\to 0}\left[K_1(z,w,1) - \frac{1}{z}\right] = K_1^*(0,w,1) \tag{2.2}$$

where notice that $K_1(z,w,1)$ is always defined in a neighbourhood of $z=0$ for Γ is discrete. This actually works more generally, e.g. for $a=s$: this would give $\frac{(\overline{z_0}+\overline{\gamma})^a}{|z_0+\gamma|^{2a}} = \frac{1}{(z_0+\gamma)^a}$ which is just $\frac{1}{\gamma^a}$ when $z_0 = 0$, which is indeed the term that would be removed in the case $0 \in \Gamma$. Anyway for the use we'll have for this formula, it's enough to consider the case $a=s=1$ of Eq. 2.2.

Theorem 1.

$$\Theta(z,w) = \exp\left(\frac{z\overline{w}}{A}\right)K_1(z,w,1)$$

A proof of this theorem is not useful for our scopes and is then omitted. The reader can take a look at [Wei76] Chapter VIII, Section 4(7) or [BK10] Theorem 2.10.

Theorem 2 (Kronecker Limit Formulas)**.** *Call* $c := \lim_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)$ *the Euler constant,* Δ *the discriminant of* Γ*:* $\Delta := g_2^3 - 27g_3^2$ *where* $g_k := \sum_{\gamma\in\Gamma\setminus\{0\}}\gamma^{-2k}$*. Then:*

i)

$$\lim_{s\to 1}\left(AK_0^*(0,0,s) - \frac{1}{s-1}\right) = -\frac{1}{12}\log|\Delta|^2 - 2\log A + 2c$$

ii) For $z \notin \Gamma$

$$AK_0^*(0,z,1) = -\log|\theta(z)|^2 + \frac{|z|^2}{A} - \frac{1}{12}\log|\Delta|^2$$

The aim of this Chapter is to prove the two formulas, with some ideas and techniques that will reveal useful when working with the New Kronecker Limit Formula. Even if these proofs are not the originals by Kronecker, we will still maintain the main idea: to prove at first the second Limit Formula and then to use it to prove the first.

2.1 Proof of the second Limit Formula

We want to deduce Theorem 2.*ii*) from Theorem 1. In particular, it will reveal itself a key instrument in the proof the following

Proposition 4. $\exists C \in \mathbb{C}$ *constant such that* $-AK_0^*(0,z,1) + C = \log|\theta(z)|^2 - \frac{|z|^2}{A}$ *for all* $z \notin \Gamma$.

Proof. By Theorem 1 we have

$$\Theta(z,w)-\frac{1}{z}=\exp\left(\frac{z\overline{w}}{A}\right)K_1(z,w,1)-\frac{1}{z}=$$

$$=\exp\left(\frac{z\overline{w}}{A}\right)\left(K_1(z,w,1)-\frac{1}{z}\right)+\frac{1}{z}\left(\exp\left(\frac{z\overline{w}}{A}\right)-1\right)$$

Computing the limits for $z\to 0$, we get

$$\lim_{z\to 0}\left(\Theta(z,w)-\frac{1}{z}\right)=\lim_{z\to 0}\left[e^{\frac{z\overline{w}}{A}}\left(K_1(z,w,1)-\frac{1}{z}\right)+\frac{1}{z}\left(e^{\frac{z\overline{w}}{A}}-1\right)\right]=$$

$$=\lim_{z\to 0}e^{\frac{z\overline{w}}{A}}\left(K_1(z,w,1)-\frac{1}{z}\right)+\lim_{z\to 0}\frac{\overline{w}}{A}\cdot\frac{1}{\frac{z\overline{w}}{A}}\left(e^{\frac{z\overline{w}}{A}}-1\right)=K_1^*(0,w,1)+\frac{\overline{w}}{A}$$

where we used Eq. 2.2 and two common facts: that $\lim_{z\to 0}e^{\frac{z\overline{w}}{A}}=1$ and that $\lim_{z\to 0}\frac{1}{\frac{z\overline{w}}{A}}\left(e^{\frac{z\overline{w}}{A}}-1\right)=1$.
On the other hand, by definition $\Theta(z,w)=\frac{\theta(z+w)}{\theta(z)\theta(w)}$ so that

$$\lim_{z\to 0}\left(\Theta(z,w)-\frac{1}{z}\right)=\lim_{z\to 0}\left(\frac{\theta(z+w)}{\theta(z)\theta(w)}-\frac{1}{z}\right)=\frac{1}{\theta(w)}\lim_{z\to 0}\frac{z\theta(z+w)-\theta(z)\theta(w)}{\theta(z)z}=$$

Now, since $\theta'(0)=1$, we have $\theta(z)\sim z$ around 0 so we can reduce

$$=\frac{1}{\theta(w)}\lim_{z\to 0}\frac{\theta(z+w)-\theta(w)}{z}=\frac{1}{\theta(w)}\theta'(w)$$

by definition. By putting all together we get

$$K_1^*(0,w,1)+\frac{\overline{w}}{A}=\frac{\theta'(w)}{\theta(w)}=\frac{\partial}{\partial w}\log\theta(w)$$

Replacing w with z and since $\frac{\partial}{\partial z}\frac{z\overline{z}}{A}=\frac{\overline{z}}{A}$ we get

$$K_1^*(0,z,1)=\frac{\partial}{\partial z}\left(\log\theta(z)-\frac{z\overline{z}}{A}\right)\tag{2.3}$$

Now define

$$\Xi(z):=\log|\theta(z)|^2-\frac{|z|^2}{A}$$

In order to compute $\frac{\partial\Xi(z)}{\partial z}$ and $\frac{\partial\Xi(z)}{\partial\overline{z}}$ let's recall a few facts about holomorphic functions: $f\colon U\to\mathbb{C}$ for $U\subset\mathbb{C}$ is holomorphic iff $\forall z_0\in U$ we can write $f(z)=\sum_{n\geq 0}a_n(z-z_0)^n$ (i.e. f is locally analytic). If that is the case, $\overline{f}\colon U\xrightarrow{f}\mathbb{C}\xrightarrow{\overline{(\cdot)}}\mathbb{C}$ is locally around each $z_0\in U$ of the form $\overline{f(z)}=\sum_{n\geq 0}\overline{a_n}\,\overline{(z-z_0)^n}$ so that we see that it only depends on $\overline{z}$ and not on z; moreover by looking at the coefficients we easily see that $\frac{\partial}{\partial\overline{z}}\overline{f(z)}=\overline{\frac{\partial}{\partial z}f(z)}$. By applying this to the functions defining Ξ, which are all holomorphic, we see that $\frac{\partial}{\partial z}\log\overline{\theta(z)}=0$ (we can say it without worrying about branches of logarithm, just because the numerator of that partial derivative is $\frac{\partial}{\partial z}\overline{\theta(z)}$ which is already 0) and in the same way get $\frac{\partial}{\partial \bar{z}}\log\theta(z)=0$. From Eq. 2.3 and $\frac{\partial}{\partial z}\log\overline{\theta(z)}=0$ we get

$$\frac{\partial \Xi(z)}{\partial z}=K_1^*(0,z,1)$$

On the other hand

$$\frac{\partial \Xi(z)}{\partial \bar{z}}=\frac{\partial \log\overline{\theta(z)}}{\partial \bar{z}}+\frac{\partial \log\theta(z)}{\partial \bar{z}}-\frac{\partial}{\partial \bar{z}}\frac{z\bar{z}}{A}=\overline{\frac{\partial}{\partial z}\log\theta(z)}+0-\frac{z}{A}=$$

where for the second equality we used $\frac{\partial}{\partial \bar{z}}\overline{f(z)}=\overline{\frac{\partial}{\partial z}f(z)}$ with $f(z)=\log\theta(z)$ and $\frac{\partial}{\partial \bar{z}}\log\theta(z)=0$

$$=\overline{\frac{\partial}{\partial z}\log\theta(z)}-\frac{\overline{\bar{z}}}{A}=\overline{\frac{\partial}{\partial z}\log\theta(z)-\frac{\bar{z}}{A}}=\overline{K_1^*(0,z,1)}$$

again thanks to Eq. 2.3.
We now plan to use the formulas from Lemma A.1 in [BKT10] (recall A is the area of the fundamental domain of Γ divided by π):

$$\partial_w K_a(z,w,s)=-\frac{1}{A}\big(K_{a+1}(z,w,s)-\bar{z}K_a(z,w,s)\big);\quad \partial_{\overline{w}}K_a(z,w,s)=\frac{1}{A}\big(K_{a-1}(z,w,s-1)-zK_a(z,w,s)\big)$$

The first easily implies that

$$A\frac{\partial}{\partial z}K_0^*(0,z,1)=-K_1^*(0,z,1)-0=-K_1^*(0,z,1)$$

While from the second we get

$$A\frac{\partial}{\partial \bar{z}}K_0^*(0,z,1)=K_{-1}^*(0,z,0)-0\overset{!}{=}-\overline{K_1^*(0,z,1)}$$

To prove $K_{-1}^*(0,z,0)=-\overline{K_1^*(0,z,1)}$ we cannot directly use the formula defining $K_a^*(z,w,s)$, for we are not in the domain for s in which that explicit formula holds. Take an s such that $\mathrm{Re}(s)>\frac{-1}{2}+1$ ($\Rightarrow \mathrm{Re}(1+s)>\frac{1}{2}+1$), so that by Definition 5

$$K_{-1}^*(0,z,s)=\sum_{\gamma\in\Gamma\setminus\{0\}}\frac{\overline{\gamma}^{-1}}{|\gamma|^{2s}}\chi_z(\gamma)=\sum_{\gamma\in\Gamma\setminus\{0\}}\frac{\exp\left[\frac{\gamma\bar{z}-z\overline{\gamma}}{A}\right]}{\overline{\gamma}|\gamma|^{2s}}$$

$$K_1^*(0,z,1+s)=\sum_{\gamma\in\Gamma\setminus\{0\}}\frac{\overline{\gamma}^{1}}{|\gamma|^{2+2s}}\chi_z(\gamma)=\sum_{\gamma\in\Gamma\setminus\{0\}}\frac{\exp\left[\frac{\gamma\bar{z}-z\overline{\gamma}}{A}\right]}{\gamma|\gamma|^{2s}}$$

Now, A is real so that $\frac{\gamma\bar{z}-z\overline{\gamma}}{A}$ is purely imaginary $\Rightarrow$ its complex conjugate has simply the opposite sign in the exponent, so we get

$$-\overline{K_1^*(0,z,1+s)}=\sum_{\gamma\in\Gamma\setminus\{0\}}\frac{\exp\left[\frac{-\gamma\bar{z}+z\overline{\gamma}}{A}\right]}{-\overline{\gamma}|\gamma|^{2s}}=$$

by re-indexing $\gamma':=-\gamma$

$$=\sum_{\gamma'\in\Gamma\setminus\{0\}}\frac{\exp\left[\frac{\gamma'\bar{z}-z\overline{\gamma'}}{A}\right]}{\overline{\gamma'}|\gamma'|^{2s}}=K_{-1}^*(0,z,s)$$

Now, since the equality works for $\mathrm{Re}(s)$ large enough and since we know that analytic continuation of $K_a^*(z,w,s)$ is holomorphic in the case we need (away from $a=0$), we get that two holomorphic functions on a connected domain $\mathbb{C}\setminus\Gamma$ that agree for $\mathrm{Re}(s)$ large enough must agree everywhere. In particular $K_{-1}^*(0,z,0) = -\overline{K_1^*(0,z,1)}$.
So from

$$A\frac{\partial}{\partial z}K_0^*(0,z,1) = -K_1^*(0,z,1), \quad A\frac{\partial}{\partial \bar{z}}K_0^*(0,z,1) = -\overline{K_1^*(0,z,1)}$$
$$\frac{\partial \Xi(z)}{\partial z} = K_1^*(0,z,1), \quad \frac{\partial \Xi(z)}{\partial \bar{z}} = \overline{K_1^*(0,z,1)}$$

we get that

$$\frac{\partial}{\partial z}\Big[\Xi(z) + AK_0^*(0,z,1)\Big] = 0 = \frac{\partial}{\partial \bar{z}}\Big[\Xi(z) + AK_0^*(0,z,1)\Big]$$

so that $\Xi(z) + AK_0^*(0,z,1) = \log|\theta(z)|^2 - \frac{|z|^2}{A} + AK_0^*(0,z,1)$ is constant equal to C.

□

Now we want to determine C.

Notation 2. *Denote $z_n \neq 0$ for $n \geq 1$ to indicate $z_n \in \frac{1}{n}\Gamma/\Gamma \setminus \{0\}$.*

Lemma 3 (Distribution relation)**.**

$$\sum_{z_n \neq 0} K_0^*(0,z_n,1) = -\frac{2\log n}{A}$$

where the sum runs over all n- torsion points z_n of $\mathbb{C}/\Gamma$ except zero, i.e. points P that aren't in Γ but such that $nP \in \Gamma$ (precisely $P \in \frac{1}{n}\Gamma/\Gamma \setminus \{0\}$).

Proof. First of all we claim that the following holds

$$\sum_{z_n \in \frac{1}{n}\Gamma/\Gamma} \chi_{z_n}(\gamma) = \begin{cases} n^2, & \text{if } \gamma \in n\Gamma; \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

To prove this, call $G := \Gamma/n\Gamma$ and $\hat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ its group of characters. Since $|\Gamma/n\Gamma| = n^2$, it's enough to show that
Claim: $\frac{1}{n}\Gamma/\Gamma \xrightarrow{\sim} \hat{G}$; $z_n \mapsto \chi_{z_n}$.
Indeed we know that for characters

$$\sum_{\chi_{z_n} \in \hat{G}} \chi_{z_n}(\gamma) = \begin{cases} |G| = |\Gamma/n\Gamma| = n^2, & \text{if } \gamma = e_G, \text{ i.e. } \gamma \in n\Gamma; \\ 0, & \text{otherwise.} \end{cases}$$

Proving the claim would then yield that $\sum_{z_n \in \frac{1}{n}\Gamma/\Gamma} \chi_{z_n}(\gamma) = \sum_{\chi_{z_n} \in \hat{G}} \chi_{z_n}(\gamma)$, hence Eq. 2.4.
Proof: Consider the map

$$\langle \cdot, \cdot \rangle : \Gamma/n\Gamma \times \frac{1}{n}\Gamma/\Gamma \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}; \quad (z,w) \mapsto \frac{z\overline{w} - w\overline{z}}{2\pi i A}$$

we use it to prove an additive version of the claim, precisely:

$$\psi\colon \frac{1}{n}\Gamma/\Gamma \xrightarrow{\sim} \mathrm{Hom}_+\Big(\Gamma/n\Gamma, \frac{1}{n}\mathbb{Z}/\mathbb{Z}\Big); \quad z_n \mapsto \langle \cdot, z_n \rangle \tag{2.5}$$

This would get $\frac{1}{n}\Gamma/\Gamma \xrightarrow{\sim} \hat{G} = \mathrm{Hom}_\cdot(G,\mathbb{C}^\times)$; $z_n \mapsto \chi_{z_n}$ simply by passing from additive to multiplicative homomorphisms, by applying the exponential map.
Injectivity of Eq 2.5: let z_n be an element mapped to 0, i.e. $\forall w \in \Gamma/n\Gamma$, $(\psi(z_n))(w) = \langle w, z_n\rangle = 0$; by looking at its definition, this implies that $z_n\overline{w} = \overline{z_n\overline{w}} \Rightarrow z_n\overline{w} \in \mathbb{R}\ \forall w$, which can happen iff $z_n = 0$.
Given that ψ is a map between finite groups, injectivity would allow us to conclude that Eq. 2.5 is an isomorphism if we prove that the cardinalities of domain and codomain are the same (this will give bijectivity; we also need that ψ is an homomorphism, but that is clearly seen). Since $|\frac{1}{n}\Gamma/\Gamma| = n^2$, we need to prove that $|\mathrm{Hom}_+(\Gamma/n\Gamma, \frac{1}{n}\mathbb{Z}/\mathbb{Z})| = n^2$ as well. Let ω_1, ω_2 be generators of Γ, i.e. such that $\Gamma = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$; being an homomorphism, we have that the image of ψ is determined by the images of the two generators, so it suffices to count the possible choices for those. Call $z_1 := \psi(\omega_1)$; since we work in $\Gamma/n\Gamma$, we have that $n\omega_1 = 0$, so that, since ψ is an homomorphism, we need to impose that $0 = \psi(n\omega_1) = n\psi(\omega_1) = nz_1$, i.e. that $z_1 \in \frac{1}{n}\mathbb{Z}/\mathbb{Z}$. The same is true for the image z_2 of ω_2. So we have $n = |\frac{1}{n}\mathbb{Z}/\mathbb{Z}|$ choices for both ω_1 and ω_2 for a total of n^2 which is then the cardinality of the group $\mathrm{Hom}_+(\Gamma/n\Gamma, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$.
This concludes the proof that ψ is an isomorphism, which then gives Eq. 2.4. ■

Therefore we have

$$\frac{1}{n^2}\sum_{z_n\in\frac{1}{n}\Gamma/\Gamma} K_0^*(0,z_n,s) = \frac{1}{n^2}\sum_{z_n}\sum_{\gamma\in\Gamma\setminus\{0\}} \frac{\cancel{\overline{\gamma}^0}}{|\gamma|^{2s}}\chi_{z_n}(\gamma) = \frac{1}{n^2}\sum_{\gamma\in\Gamma\setminus\{0\}}\sum_{z_n}\frac{1}{|\gamma|^{2s}}\chi_{z_n}(\gamma) = \frac{1}{n^2}\sum_{\gamma\in\Gamma\setminus\{0\}}\frac{1}{|\gamma|^{2s}}\sum_{z_n}\chi_{z_n}(\gamma) =$$

by definition of $K_0^*(0,z_n,s)$, exchanging the sums and since $0 \in \Gamma$. Now, since Eq. 2.4 tells us that $\sum_{z_n}\chi_{z_n}(\gamma)$ kills all γ's but those in $n\Gamma$,

$$= \frac{1}{n^2}\sum_{\gamma\in n\Gamma\setminus\{0\}}\frac{1}{|\gamma|^{2s}}\sum_{z_n}\chi_{z_n}(\gamma) = \frac{1}{n^2}\sum_{\gamma\in n\Gamma\setminus\{0\}}\frac{1}{|\gamma|^{2s}}n^2 = \sum_{\gamma\in n\Gamma\setminus\{0\}}\frac{1}{|\gamma|^{2s}} =$$

by changing variable and calling $\gamma' = \frac{1}{n}\gamma$, and since $\chi_0(w) = 1\forall w$,

$$= \sum_{\gamma'\in\Gamma\setminus\{0\}}\frac{1}{|\gamma'|^{2s}n^{2s}} = \frac{1}{n^{2s}}\sum_{\gamma'\in\Gamma\setminus\{0\}}\frac{1}{|\gamma'|^{2s}}\chi_0(\gamma) = \frac{1}{n^{2s}}K_0^*(0,0,s)$$

Where the last equality holds by definition for $\mathrm{Re}(s) >> 1$, hence $\forall s \in \mathbb{C}$ by analytic continuation as saw in previous proof. Thus we obtained

$$\frac{1}{n^2}\sum_{z_n\in\frac{1}{n}\Gamma/\Gamma} K_0^*(0,z_n,s) = \frac{1}{n^{2s}}K_0^*(0,0,s) \Rightarrow \frac{1}{n^2}\sum_{z_n\neq 0} K_0^*(0,z_n,s) = \left(\frac{1}{n^{2s}} - \frac{1}{n^2}\right)K_0^*(0,0,s)$$

Now we need to use Equation (31) from [Wei76] Chapter VIII Section 13:

$$\begin{aligned}\Gamma(s)K_a(x,x_0,s) =& I_\infty(T,x,x_0,a,s) + A^{a+1-2s}I_\infty(A^{-2}T^{-1},x_0,x,a,a+1-s)\chi(-x)-\\ &-\varepsilon\frac{T^s}{s}\chi(-x) + \varepsilon' A^{-a-1}\frac{T^{s-a-1}}{s-a-1}\end{aligned} \tag{2.6}$$

where $\varepsilon = 1$ if $a = 0, x \in \Gamma$ and 0 otherwise; $\varepsilon' = 1$ if $a = 0, x_0 \in \Gamma$ and 0 otherwise; $I_\infty(T,\dots) := \int_T^\infty \theta_a^*(t,x,x_0)\,dt$ (this resembles a similar one that we'll encounter in the next Section before the proof of the first Limit Formula); A is the same value as before and χ is the character we called χ_{x_0} (even if Weil's notation doesn't write it explicitly, it does depend on x_0). [Wei76] also tells us that I_∞ is entire, so it doesn't have poles. We want to compute the residue of $K_0^*(0,0,s)$ at $s=1$ by using Eq. 2.6: notice that $\Gamma(1)=1$ and in our case $a=0=x=x_0 \Rightarrow \varepsilon=\varepsilon'=1$. The last two summands in the RHS of 2.6 (the only two that can have poles, since I_∞ is entire) become

$$-\varepsilon\frac{T^s}{s}\chi(-x)+\varepsilon' A^{-a-1}\frac{T^{s-a-1}}{s-a-1} \Rightarrow -\frac{T^s}{s}\chi(0)+A^{-1}\frac{T^{s-1}}{s-1}$$

The first term has no pole at $s=1$, while the second does with residue $A^{-1}T^0=\frac{1}{A}$. So, locally around $s=1$, we can write

$$K_0^*(0,0,s)=\frac{1}{A(s-1)}+O(1)$$

so that we have by above

$$\frac{1}{n^2}\sum_{z_n\neq 0}K_0^*(0,z_n,s)=\left(\frac{1}{n^{2s}}-\frac{1}{n^2}\right)K_0^*(0,0,s)=\left(\frac{1}{n^{2s}}-\frac{1}{n^2}\right)\frac{1}{A(s-1)}+\left(\frac{1}{n^{2s}}-\frac{1}{n^2}\right)O(1)$$

in a neighbourhood of $s=1$, where $\left(\frac{1}{n^{2s}}-\frac{1}{n^2}\right)O(1)$ tends to 0 for $s\to 1$. Then, by taking on both sides limits for $s\to 1$ and using de l'Hôpital rule (in the second equality of the second line):

$$\frac{1}{n^2}\sum_{z_n\neq 0}K_0^*(0,z_n,1)=\lim_{s\to 1}\frac{1}{n^2}\sum_{z_n\neq 0}K_0^*(0,z_n,s)=\lim_{s\to 1}\left(\frac{1}{n^{2s}}-\frac{1}{n^2}\right)K_0^*(0,0,s)=$$

$$=\frac{1}{A}\lim_{s\to 1}\frac{\frac{1}{n^{2s}}-\frac{1}{n^2}}{s-1}+0=\frac{1}{A}\frac{\partial}{\partial s}\left(\frac{1}{n^{2s}}\right)\Bigg|_{s=1}=\frac{1}{A}\frac{\partial}{\partial s}\left(\frac{1}{(n^2)^s}\right)\Bigg|_{s=1}=$$

$$=\frac{1}{A}\frac{\partial e^{-s\log n^2}}{\partial s}\Bigg|_{s=1}=\frac{1}{A}\left(-\frac{2\log n}{n^2}\right)=-\frac{2\log n}{An^2}$$

so that, multiplying by n^2 both sides, we get the result of the lemma. $\square$

Choose $n\geq 1$; $\forall z_n\in\frac{1}{n}\Gamma/\Gamma\setminus\{0\}$, we have that $z_n\notin\Gamma$ (or else it would be 0 in the quotient); we're then in the conditions of Proposition 4. Since the number of $z_n\in\frac{1}{n}\Gamma/\Gamma\setminus\{0\}$ is n^2-1, we get

$$\begin{aligned}(n^2-1)C&=\sum_{z_n\neq 0}\left(\log|\theta(z_n)|^2-\frac{|z_n|^2}{A}+AK_0^*(0,z_n,1)\right)=\\&=\sum_{z_n\neq 0}\left(\log|\theta(z_n)|^2-\frac{|z_n|^2}{A}\right)-2\log n\end{aligned}\tag{2.7}$$

where we used the Distribution Relation, Lemma 3, for the second equality.
We now want to explicitly compute the sum left in Eq. 2.7 in terms of Δ.

Proposition 5.

$$\frac{1}{4}\log|\Delta'|^2=-\sum_{z_2\neq 0}\left(\log|\theta(z_2)|^2-\frac{|z_2|^2}{A}\right)$$

where z_2 runs over non trivial 2-torsion points of $\mathbb{C}/\Gamma$. Δ' is defined by writing the equation of the elliptic curve $\mathbb{C}/\Gamma$ as

$$y^2=4x^3-g_2x-g_3=4(x-e_1)(x-e_2)(x-e_3)\Rightarrow\Delta':=(e_1-e_2)^2(e_1-e_3)^2(e_3-e_2)^2$$

Proof. Note

$$(x-e_1)(x-e_2)(x-e_3) = \prod_{z_2 \neq 0} (x - \wp(z_2)) \tag{2.8}$$

where $\wp$ is the Weierstrass-$\wp$ function. To see this, recall (see e.g. [Ste13] Theorem 2.9) that the Weierstrass-$\wp$ function and its derivative satisfy the equation $\wp'(x) = 4\wp^3(x) - g_2\wp(x) - g_3$ with the same g_2, g_3 as ours (defined in Theorem 2). Moreover, recall that $\wp'$ is an odd elliptic function, so it must be zero when computed against 2-torsion points. This tells us that $(\wp(z_2), \wp'(z_2)) = (\wp(z_2), 0)$, with z_2 non-trivial torsion points are solutions of the Weierstrass equation of the elliptic curve $\mathbb{C}/\Gamma$, so that Eq. 2.8 holds.[1]

If we write $\Gamma = \mathbb{Z}\,\omega_1 + \mathbb{Z}\,\omega_2$, we can wlog assume thanks to Eq. 2.8 that $e_1 = \wp\big(\frac{\omega_1}{2}\big)$, $e_2 = \wp\big(\frac{\omega_2}{2}\big)$, $e_3 = \wp\big(\frac{\omega_1+\omega_2}{2}\big)$. From Ex 12, page 20 of [Ste13] we have[2]

$$\theta(z+w)\theta(z-w)\theta(z)^{-2}\theta(w)^{-2} = \wp(w) - \wp(z), \qquad w \notin \Gamma$$

where θ is the reduced theta function we defined in last Section. We will use this formula thrice, taking as w the three non trivial 2-torsion points, which satisfy condition $w \notin \Gamma$. This yields

$$\theta\Big(\frac{\omega_1+\omega_2}{2}\Big)\theta\Big(\frac{\omega_1-\omega_2}{2}\Big)\theta\Big(\frac{\omega_1}{2}\Big)^{-2}\theta\Big(\frac{\omega_2}{2}\Big)^{-2} = \wp\Big(\frac{\omega_2}{2}\Big) - \wp\Big(\frac{\omega_1}{2}\Big) = e_2 - e_1$$

$$\theta\Big(\omega_1 + \frac{\omega_2}{2}\Big)\theta\Big(\frac{\omega_2}{2}\Big)\theta\Big(\frac{\omega_1+\omega_2}{2}\Big)^{-2}\theta\Big(\frac{\omega_1}{2}\Big)^{-2} = \wp\Big(\frac{\omega_1}{2}\Big) - \wp\Big(\frac{\omega_1+\omega_2}{2}\Big) = e_1 - e_3$$

$$\theta\Big(\omega_2 + \frac{\omega_1}{2}\Big)\theta\Big(\frac{\omega_1}{2}\Big)\theta\Big(\frac{\omega_1+\omega_2}{2}\Big)^{-2}\theta\Big(\frac{\omega_2}{2}\Big)^{-2} = \wp\Big(\frac{\omega_2}{2}\Big) - \wp\Big(\frac{\omega_1+\omega_2}{2}\Big) = e_2 - e_3$$

Remember that we had Eq. 2.1: $\theta(z+\gamma) = \alpha(\gamma)\exp\big(\frac{z\gamma}{A} + \frac{|\gamma|^2}{2A}\big)\theta(z)$ where $\gamma \in \Gamma$ and $\alpha(\gamma) = -1$ if $\gamma \notin 2\Gamma$ and $\alpha(\gamma) = 1$ otherwise.

$$\Delta' = (e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2 = \theta\Big(\frac{\omega_1+\omega_2}{2}\Big)^{-6}\theta\Big(\frac{\omega_1}{2}\Big)^{-6}\theta\Big(\frac{\omega_2}{2}\Big)^{-6}\cdot$$

$$\cdot\theta\Big(\frac{\omega_1-\omega_2}{2}\Big)^2\theta\Big(\omega_1+\frac{\omega_2}{2}\Big)^2\theta\Big(\omega_2+\frac{\omega_1}{2}\Big)^2 = \circledast$$

Now using Eq. 2.1 (the square for the exponentials simply cancels the 2 in the denominator):

$$\theta\Big(\frac{\omega_1-\omega_2}{2}\Big)^2 = \theta\Big(\frac{\omega_1+\omega_2}{2} - \omega_2\Big)^2 = (\pm 1)^2\exp\Big(-\frac{(\omega_1+\omega_2)\overline{\omega_2}}{A} + \frac{\omega_2\overline{\omega_2}}{A}\Big)\theta\Big(\frac{\omega_1+\omega_2}{2}\Big)^2 =$$

$$= \exp\Big(-\frac{\omega_1\overline{\omega_2}}{A}\Big)\theta\Big(\frac{\omega_1+\omega_2}{2}\Big)^2$$

And similarly

$$\theta\Big(\omega_1+\frac{\omega_2}{2}\Big)^2 = \exp\Big(\frac{\omega_2\overline{\omega_1}}{A} + \frac{\omega_1\overline{\omega_1}}{A}\Big)\theta\Big(\frac{\omega_2}{2}\Big)^2$$

[1] The equality can be proven also in the completely opposite direction: by recalling the definition of sum on the elliptic curve (the operation that gives it structure as a commutative group), one can see by drawing a picture that the zeros of the Weierstrass equation $(e_i, 0)$ are 0 when summed with themselves, i.e. they're the non trivial 2-torsion points.

[2] The reference presents this formula in the form "$\sigma(z+w)\sigma(z-w)\sigma(z)^{-2}\sigma(w)^{-2} = \wp(w) - \wp(z)$"; we can rephrase it with the reduced theta function instead of σ thanks to [BK10], which constructs explicitly θ and affirms that "The function $\theta(z)$ is the Weierstrass-σ function $\sigma(z)$ up to a simple exponential factor"; this factor is made explicit at page 11 of the cited paper.

$$\theta\Big(\omega_2+\frac{\omega_1}{2}\Big)^2=\exp\Big(\frac{\omega_1\overline{\omega_2}}{A}+\frac{\omega_2\overline{\omega_2}}{A}\Big)\theta\Big(\frac{\omega_1}{2}\Big)^2$$

so that

$$\circledast=\theta\Big(\frac{\omega_1+\omega_2}{2}\Big)^{-4}\theta\Big(\frac{\omega_1}{2}\Big)^{-4}\theta\Big(\frac{\omega_2}{2}\Big)^{-4}\cdot\exp\Big(\frac{-\omega_1\overline{\omega_2}+\omega_2\overline{\omega_1}+\omega_1\overline{\omega_1}+\omega_1\overline{\omega_2}+\omega_2\overline{\omega_2}}{A}\Big)=$$

$$=\theta\Big(\frac{\omega_1+\omega_2}{2}\Big)^{-4}\theta\Big(\frac{\omega_1}{2}\Big)^{-4}\theta\Big(\frac{\omega_2}{2}\Big)^{-4}\cdot\exp\Big(\frac{\omega_2\overline{\omega_1}+\omega_1\overline{\omega_1}+\omega_2\overline{\omega_2}}{A}\Big)$$

A formula for $\overline{\Delta'}$ is obtained by conjugating:

$$\overline{\Delta'}=\overline{\theta\Big(\frac{\omega_1+\omega_2}{2}\Big)^{-4}}\,\overline{\theta\Big(\frac{\omega_1}{2}\Big)^{-4}}\,\overline{\theta\Big(\frac{\omega_2}{2}\Big)^{-4}}\,\overline{\cdot\exp\Big(\frac{\omega_2\overline{\omega_1}+\omega_1\overline{\omega_1}+\omega_2\overline{\omega_2}}{A}\Big)}=$$

$$=\overline{\theta\Big(\frac{\omega_1+\omega_2}{2}\Big)}^{-4}\,\overline{\theta\Big(\frac{\omega_1}{2}\Big)}^{-4}\,\overline{\theta\Big(\frac{\omega_2}{2}\Big)}^{-4}\cdot\exp\Big(\frac{\mathrm{Re}(\omega_2\overline{\omega_1})-\mathrm{Im}(\omega_2\overline{\omega_1})+\omega_1\overline{\omega_1}+\omega_2\overline{\omega_2}}{A}\Big)$$

as $\omega_1\overline{\omega_1}$ and $\omega_2\overline{\omega_2}$ are real numbers. We now need to multiply $\overline{\Delta'}$ and Δ' to get $|\Delta'|^2$ from the statement. Clearly the terms $\theta(z_2)^{-4}$ and their conjugates make together $\big(|\theta(z_2)|^2\big)^{-4}$; we need to be more careful with the exp term:

$$\exp\Big(\frac{\mathrm{Re}(\omega_2\overline{\omega_1})+\mathrm{Im}(\omega_2\overline{\omega_1})+|\omega_1|^2+|\omega_2|^2}{A}\Big)\cdot\exp\Big(\frac{\mathrm{Re}(\omega_2\overline{\omega_1})-\mathrm{Im}(\omega_2\overline{\omega_1})+|\omega_1|^2+|\omega_2|^2}{A}\Big)=$$

$$=\exp\Big(\frac{(2\,\mathrm{Re}(\omega_2\overline{\omega_1})+|\omega_1|^2+|\omega_2|^2)+|\omega_1|^2+|\omega_2|^2}{A}\Big)=\exp\Big(\frac{|\omega_1+\omega_2|^2+|\omega_1|^2+|\omega_2|^2}{A}\Big)=$$

$$=\exp\Big(-4\cdot\big(-\frac{|\frac{\omega_1+\omega_2}{2}|^2+|\frac{\omega_1}{2}|^2+|\frac{\omega_2}{2}|^2}{A}\big)\Big)=\Big(\prod_{z_2\neq0}\exp\big(\frac{-|z_2|^2}{A}\big)\Big)^{-4}$$

So that, putting all together, we get a formula for $|\Delta'|^2$:

$$|\Delta'|^2=\prod_{z_2\neq0}\Big(|\theta(z_2)|^2\Big)^{-4}\cdot\exp\Big(-\frac{|z_2|^2}{A}\Big)^{-4}$$

Now, taking logarithm and dividing by 4, get

$$\frac{1}{4}\log|\Delta'|^2=-\Big(\sum_{z_2\neq0}\log|\theta(z_2)|^2-\frac{|z_2|^2}{A}\Big)$$

which is the wanted formula. Note that this doesn't depend on the choice of the branch of logarithm as we're working with positive real numbers.

$\square$

We can finally prove Theorem 2.*ii*)

Proof of Second Limit Formula. By direct computation, we see from proof of Theorem 2.9 in [Ste13], that $\Delta=2^4\Delta'$. Using Eq. 2.7 with $n=2$ and previous proposition, get

$$C=\frac{1}{2^2-1}\Big[\sum_{z_2\neq0}\Big(\log|\theta(z_2)|^2-\frac{|z_2|^2}{A}\Big)-2\log2\Big]=\frac{1}{3}\Big(-\frac{1}{4}\log|\Delta'|^2-2\log2\Big)=$$

$$-\frac{1}{12}\Big(\log|\Delta'|^2+8\log 2\Big)=-\frac{1}{12}\log 2^8|\Delta'|^2=-\frac{1}{12}\log|\Delta|^2$$

Remembering how C was taken in Proposition 4, we get

$$\log|\theta(z)|^2-\frac{|z|^2}{A}=-AK_0^*(0,z,1)-\frac{1}{12}\log|\Delta|^2$$

That concludes the proof of the second Limit Formula. □

2.2 Proof of the first Limit Formula

We pass now to the proof of the first Limit Formula, which will use the second one that was proved in last Section. We will also prove here a special case of Proposition 3, i.e. when $a=0$, and to do so we will first define a couple of new objects, that we already encountered in a similar form during the proof of the Distribution relation, Lemma 3.

Definition 7. • *For fixed $z_0,w_0\in\mathbb{C}$, Γ a lattice and A like always, we define for $t>0$:*

$$\theta^*(t,z_0,w_0):={\sum_{\gamma}}^{*}\exp\left(-\frac{t|z_0+\gamma|^2}{A}\right)\chi_{w_0}(\gamma),$$

where $\sum^*$ denotes the sum over $\gamma\in\Gamma\setminus\{-z_0\}$ if $z_0\in\Gamma$.

•

$$I(z_0,w_0,s):=\int_1^\infty\theta^*(t,z_0,w_0)t^{s-1}\,dt$$

Note that $I(z_0,w_0,s)$ resembles a Mellin transform and it is in fact a part of the integral defining it. Moreover, it's easy to see that the integral defining it converges $\forall s\in\mathbb{C}$, as θ^* is of exponential decrease while t^{s-1} of polynomial growth for $|s|\to\infty$.
We already saw in the proof of Lemma 3 the following formula (now adapted to our current notation):

$$A^s\Gamma(s)K_0^*(z_0,w_0,s)=I(z_0,w_0,s)-\frac{\delta_{z_0}}{s}\chi_{z_0}(w_0)+I(w_0,z_0,1-s)\chi_{z_0}(w_0)+\frac{\delta_{w_0}}{s-1}\tag{2.9}$$

where $\delta_x=1$ if $x\in\Gamma$ and 0 otherwise.
A proof of this can be found in the already cited [Wei76] Chapter VIII Section 13; the idea is to write $\Gamma(s)$ as the Mellin transform of e^{-t}, split the integral into two parts: $\int_0^1$, $\int_1^\infty$ and use the functional equation for θ^* (also in [Wei76], equation (30), Chapter VIII Section 13).

Remark 9. *Equation 2.9 gives the meromorphic continuation of $K_0^*(z_0,w_0,s)$ and its functional equation from Proposition 3:*

$$\Gamma(s)K_0^*(z_0,w_0,s)=A^{1-2s}\Gamma(1-s)K_0^*(w_0,z_0,1-s)\chi_{z_0}(w_0)$$

Proof. Since all the functions appearing in Eq. 2.9 are meromorphic ($\Gamma(s)$, $\frac{1}{s}$, $\frac{1}{s-1}$) or even entire ($I(\cdot,\cdot,s)$, A^s), and since zeros of $\Gamma(s)$ are polynomials, the expression that we get after dividing both sides of Eq. 2.9 by $\Gamma(s)A^s$ makes $K_0^*(z_0,w_0,s)$ into a meromorphic function on $\mathbb{C}$.
For the functional equation, from Eq. 2.9 we get, switching z_0 and w_0 and with $1-s$ instead of s, that

$$A^{1-s}\Gamma(1-s)K_0^*(w_0,z_0,1-s)=I(w_0,z_0,1-s)-\frac{\delta_{w_0}}{1-s}\chi_{w_0}(z_0)+I(z_0,w_0,s)\chi_{w_0}(z_0)+\frac{\delta_{z_0}}{-s}$$

Multiply now both sides by $\chi_{z_0}(w_0)$ and notice that

$$\chi_{w_0}(z_0)\chi_{z_0}(w_0) = \exp\Big(\frac{z_0\overline{w_0} - w_0\overline{z_0}}{A}\Big)\exp\Big(\frac{w_0\overline{z_0} - z_0\overline{w_0}}{A}\Big) = \exp(0) = 1$$

Get

$$A^{1-s}\Gamma(1-s)K_0^*(w_0,z_0,1-s)\chi_{z_0}(w_0) = I(w_0,z_0,1-s)\chi_{z_0}(w_0) - \frac{\delta_{w_0}}{1-s}\chi_{w_0}(z_0)\chi_{z_0}(w_0) +$$

$$+I(z_0,w_0,s)\chi_{w_0}(z_0)\chi_{z_0}(w_0) - \frac{\delta_{z_0}}{s}\chi_{z_0}(w_0) = I(w_0,z_0,1-s)\chi_{z_0}(w_0) - \frac{\delta_{w_0}}{1-s} +$$

$$+I(z_0,w_0,s) - \frac{\delta_{z_0}}{s}\chi_{z_0}(w_0) = A^s\Gamma(s)K_0^*(z_0,w_0,s)$$

again for Eq. 2.9. Dividing both sides for A^s we get the functional equation. □

We finally give the proof of Theorem 2.*i*).

Proof of First Limit Formula. By Eq. 2.9, since $\chi_0(0) = 1$ and $0 \in \Gamma \Rightarrow \delta_0 = 1$ (so we'll omit these terms)

$$A^{s-1}\Gamma(s)\Big(AK_0^*(0,0,s) - \frac{1}{s-1}\Big) = I(0,0,s) - \frac{1}{s} + I(0,0,1-s) + \frac{1}{s-1} - \frac{A^{s-1}\Gamma(s)}{s-1} =$$

$$= I(0,0,s) - \frac{1}{s} + I(0,0,1-s) - \frac{A^{s-1}\Gamma(s) - 1}{s-1}$$

Taking $\lim_{s\to 1}$ get (since $A^0 = 1$, $\Gamma(1) = 0! = 1$)

$$\lim_{s\to 1}\Big(AK_0^*(0,0,s) - \frac{1}{s-1}\Big) = I(0,0,1) - 1 + I(0,0,0) - \log A + c \tag{2.10}$$

where we used de l'Hôpital rule to see that

$$\lim_{s\to 1}\frac{A^{s-1}\Gamma(s)-1}{s-1} = \lim_{s\to 1}\frac{\Big(\frac{d}{ds}A^{s-1}\Big)\Gamma(s) + A^{s-1}\Gamma'(s)}{1} = \lim_{s\to 1}\log A\, A^{s-1}\Gamma(s) + A^{s-1}\Gamma'(s) =$$

$$= \log A \cdot 1 + 1 \cdot \Gamma'(1) = \log A - c$$

for $\Gamma'(1) = -c$ where c is the Euler constant (defined in the statement of Theorem 2).
On the other hand, taking $z_0 = 0 \in \Gamma$, $w_0 = z \notin \Gamma$, $s = 1$, always for Eq. 2.9

$$\begin{aligned} AK_0^*(0,z,1) = A^1\Gamma(1)K_0^*(0,z,1) = I(0,z,1) - \frac{1}{1}\chi_0(z) + I(z,0,0)\chi_0(z) + 0 = \\ = I(0,z,1) - 1 + I(z,0,0) \end{aligned} \tag{2.11}$$

as $\chi_0(z) = 1$. Note $\lim_{z\to 0} I(0,z,1) = I(0,0,1)$ as it is continuous.
It will reveal useful to define for $z \neq 0$

$$I^*(z,0,s) := I(z,0,s) - \int_1^\infty \exp\Big(-\frac{t|z|^2}{A}\Big)t^{s-1}\,dt$$

which satisfies $\lim_{z\to 0} I^*(z,0,0) = I(0,0,0)$. Indeed by inserting in the definition just written those of I and θ^*, remembering the definition of $\sum^*$ and noticing that, since $z \to 0$ and $0 \in \Gamma$, we can safely assume that $z \notin \Gamma$ in a neighbourhood of 0, so that we don't have anything to exclude, we have that

$$I^*(z,0,s) = \int_1^\infty t^{s-1}\left[\sum_{\gamma\in\Gamma} \exp\Big(-\frac{t|z+\gamma|^2}{A}\Big)\cdot 1 - \exp\Big(-\frac{t|z|^2}{A}\Big)\right] dt =$$

$$= \int_1^\infty t^{s-1}\left[\sum_{0\neq\gamma\in\Gamma} \exp\Big(-\frac{t|z+\gamma|^2}{A}\Big)\right] dt$$

Now if we take the limit for $z \to 0$ the summand in the last integral corresponds to $\theta^*(t,0,0)$ so that $\lim_{z\to 0} I^*(z,0,s) = I(0,0,s)$, $\forall s$ and in particular for $s=0$, which was what we needed.
Rewrite $\Gamma(s) - \frac{1}{s}$ in the following way:

$$\begin{aligned}\Gamma(s) - \frac{1}{s} &= \int_0^\infty e^{-t}t^{s-1}\,dt - \frac{1}{s} = \int_{\frac{|z|^2}{A}}^\infty e^{-t}t^{s-1}\,dt + \int_0^{\frac{|z|^2}{A}} e^{-t}t^{s-1}\,dt - \frac{1}{s} = \\ &= \int_{\frac{|z|^2}{A}}^\infty e^{-t}t^{s-1}\,dt + \int_0^{\frac{|z|^2}{A}} (e^{-t}-1)t^{s-1}\,dt + \frac{1}{s}\left[\Big(\frac{|z|^2}{A}\Big)^s - 1\right]\end{aligned} \tag{2.12}$$

as $\int_0^{\frac{|z|^2}{A}} t^{s-1}\,dt = \frac{1}{s}\Big(\frac{|z|^2}{A}\Big)^s$. It is a known property of the Euler's constant that $-c = \lim_{s\to 0}\big(\Gamma(s) - \frac{1}{s}\big)$. So, taking $s \to 0$ on both sides of Eq. 2.12, we get that LHS $\to -c$, while for RHS we just have to notice that $\lim_{s\to 0}\frac{1}{s}\left[\Big(\frac{|z|^2}{A}\Big)^s - 1\right] = \log\Big(\frac{|z|^2}{A}\Big)$ to get the equality

$$-c = \int_{\frac{|z|^2}{A}}^\infty e^{-t}t^{-1}\,dt + \int_0^{\frac{|z|^2}{A}} (e^{-t}-1)t^{-1}\,dt + \log\Big(\frac{|z|^2}{A}\Big) \tag{2.13}$$

So, starting from Eq. 2.11,

$$AK_0^*(0,z,1) = I(0,z,1) - 1 + I(z,0,0) = I(0,z,1) - 1 + I^*(z,0,0) + \int_1^\infty \exp\Big(-\frac{t|z|^2}{A}\Big)t^{-1}\,dt = ⊛$$

by using Eq. 2.13 after the change of variable $r := \frac{A}{|z|^2}t \Rightarrow dt = \frac{|z|^2}{A}dr$ that makes

$$\int_1^\infty \exp\Big(-\frac{r|z|^2}{A}\Big)r^{-1}\,dr = \int_{\frac{|z|^2}{A}}^\infty e^{-t}t^{-1}\,dt$$

$$⊛ = I(0,z,1) - 1 + I^*(z,0,0) - c - \int_0^{\frac{|z|^2}{A}} (e^{-t}-1)t^{-1}\,dt - \log\Big(\frac{|z|^2}{A}\Big)$$

Therefore, using $\lim_{z\to 0} I^*(z,0,0) = I(0,0,0)$:

$$\lim_{z\to 0}\Big(AK_0^*(0,z,1) + \log|z|^2\Big) = \lim_{z\to 0}\Big(I(0,z,1) - 1 + I^*(z,0,0) - c - \int_0^{\frac{|z|^2}{A}} (e^{-t}-1)t^{-1}\,dt +$$

$$+\log\left(|z|^2\cdot\frac{A}{|z|^2}\right)\Bigg) = I(0,0,1)-1+I(0,0,0)-c-0+\log A$$

We can conclude: start from Eq. 2.10

$$\lim_{s\to 1}\left(AK_0^*(0,0,s)-\frac{1}{s-1}\right) = I(0,0,1)-1+I(0,0,0)-\log A+c=$$

$$=I(0,0,1)-1+I(0,0,0)-c+\log A-2\log A+2c=\lim_{z\to 0}\left(AK_0^*(0,z,1)+\log|z|^2\right)-2\log A+2c=$$

since by second Limit Formula we have $AK_0^*(0,z,1) = -\log|\theta(z)|^2+\frac{|z|^2}{A}-\frac{1}{12}\log|\Delta|^2$

$$=\lim_{z\to 0}\left(\log\frac{|z|^2}{|\theta(z)|^2}+\frac{|z|^2}{A}-\frac{1}{12}\log|\Delta|^2\right)-2\log A+2c=\log 1+0-\frac{1}{12}\log|\Delta|^2-2\log A+2c=$$

$$-\frac{1}{12}\log|\Delta|^2-2\log A+2c$$

as desired, where we used that $z\sim\theta(z)$ to get $\lim_{z\to 0}\frac{|z|^2}{|\theta(z)|^2}=1$. $\square$

Chapter 3

A Limit Formula for Hecke L-functions of complex cubic fields

We will soon present the New Kronecker's Limit Formula. Before doing that, we will review the theory of partial zeta functions and Hecke L-functions, by defining them and proving some important characterizations. The New KLF

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0) = \log|\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})|^2 \tag{3.1}$$

will be presented in Section 3.3, along with the definition of the Smoothed partial zeta function $\zeta_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},s)$. One may object that it's not so easy to find at first glance the similarities between this formula and the classical KLFs, Theorem 2 of the previous Chapter. To that concern, in Section 3.4 we will derive from the classical Limit Formulas, a version of KLF that holds for imaginary quadratic fields, which resembles the New KLF Eq. 3.1 and was probably what motivated the authors of [BCG23] to formulate the latter.

3.1 Partial Zeta functions

Partial zeta functions are part of the language we need to state the New Kronecker Limit Formula for complex cubic fields.
Fix a class C in $\mathscr{C}l_K^+(\mathfrak{f})$ consisting of elements $\mathfrak{b}(\alpha)$, where $\mathfrak{b}$ is a fixed fractional ideal generated by ideals coprime with $\mathfrak{f}$, while (α) varies over the $\alpha \in K$ such that $v_{\mathfrak{p}}(\alpha - 1) \geq v_{\mathfrak{p}}(\mathfrak{f})$, $\forall \mathfrak{p}|\mathfrak{f}$ primes and $\sigma_{\mathbb{R}}(\alpha) > 0$. Call σ the element in $Gal(K(\mathfrak{f})/K)$ that corresponds to C via the Artin map.

Definition 8.

$$\zeta_{\mathfrak{f}}(C,s) = \zeta_{\mathfrak{f}}(\sigma,s) := \sum_{\mathfrak{c}\in C} N(\mathfrak{c})^{-s}, \ \ \mathrm{Re}(s) > 1$$

*where the sum runs over all integral ideals $\mathfrak{c}$ in the ideal class C, is the **partial zeta function** of C.*

Notice that this definition resembles the one of the Dedekind zeta function, with the particularity that here we sum only on ideals that satisfy some condition depending on $\mathfrak{f}$. One can see that, if we sum $\zeta_{\mathfrak{f}}$'s for all possible such $\mathfrak{f}$'s, we get indeed the Dedekind zeta function. This is the reason behind the name of **partial** zeta function.

Remark 10. *If we write $C = [\mathfrak{b}]$ for an integral ideal $\mathfrak{b}$ of $\mathscr{O}_K$ and call $L := \mathfrak{f}\mathfrak{b}^{-1}$, we want a more explicit way to describe the integral ideals in the class C, i.e. those which appear in the sum defining $\zeta_{\mathfrak{f}}(C,s)$:*

$$\{\mathfrak{a} \subset \mathscr{O}_K \mid \mathfrak{a} \in C\} = \{\mathfrak{b}\mu \mid \ \mu \in (1+L)^+/\mathscr{O}_{K,\mathfrak{f}}^{+,\times}\}$$

where $(1+L)^+$ are the elements of $(1+L)$ with real norm > 0.

Proof. Consider the map

$$\begin{aligned}\Phi\colon\ (1+L)^+ &\to \{\mathfrak{a} \subset \mathscr{O}_K \mid \mathfrak{a} \in C\} \\ \mu &\mapsto \mathfrak{b}\mu\end{aligned}$$

We want to prove that this map is bijective (after some quotient).
The map is surjective. Let $\mathfrak{a} \in$ RHS; $\mathfrak{a} \in C$ means that $\mathfrak{a} = \mathfrak{b}\mu$, $\exists \mu \in K$ such that $v_{\mathfrak{p}}(\mu - 1) \geq v_{\mathfrak{p}}(\mathfrak{f})$, $\forall \mathfrak{p}$ primes dividing $\mathfrak{f}$, and $\sigma_{\mathbb{R}}(\mu) > 0$. These conditions tells us first that $\mu \in 1+L$ and then, since $\sigma_{\mathbb{R}}(\mu) > 0$, that actually $\mu \in (1+L)^+$. So $\mathfrak{a}$ comes from an element $\mu \in (1+L)^+$ via Φ, proving surjectivity.
We claim that $\ker\Phi = \mathscr{O}_{K,\mathfrak{f}}^{+,\times}$, so that the map becomes injective when we quotient by it. The claim is equivalent to proving that $\mathfrak{b}\mu = \mathfrak{b}\mu'$ iff $\mu,\mu' \in (1+L)^+$ differ by a $u \in \mathscr{O}_{K,\mathfrak{f}}^{+,\times}$. If $\mu' = \mu u$, $u \in \mathscr{O}_{K,\mathfrak{f}}^{+,\times} \Rightarrow$
$\mathfrak{b}\mu = \mathfrak{b}\mu u = \mathfrak{b}\mu'$ because two ideals that differ by a unit are the same ideal. Conversely, if $\mathfrak{b}\mu = \mathfrak{b}\mu'$ then the elements μ,μ' must differ by a unit $u \in \mathscr{O}_K^{\times}$. Moreover, since $\mu,\mu' \in (1+L)^+$, then $u \in \mathscr{O}_K^{\times} \cap (1+L)^+ = \mathscr{O}_{K,\mathfrak{f}}^{+,\times}$ and we're done. ◻

We can then see the partial zeta function as associated to an integral ideal $\mathfrak{b}$ coprime with $\mathfrak{f}$ as

$$\zeta_{\mathfrak{f}}(C,s) = \zeta_{\mathfrak{f}}(\mathfrak{b},s) = N(\mathfrak{b})^{-s} \sum_{\mu \in (1+L)^+/\mathscr{O}_{K,\mathfrak{f}}^{+,\times}} N(\mu)^{-s}$$

always for $\mathrm{Re}(s) > 1$, where we pulled out the norm of $\mathfrak{b}$ since the norm is multiplicative. We denote

$$\zeta(1_K,L,s) := \sum_{\mu \in (1+L)^+/\mathscr{O}_{K,\mathfrak{f}}^{+,\times}} N(\mu)^{-s}$$

Remark 11.

$$\zeta(1_K,L,s) = \frac{1}{2}\left(\sum_{\mu \in (1+L)/\mathscr{O}_{K,\mathfrak{f}}^{+,\times}} \frac{1}{|N(\mu)|^s} + \sum_{\mu \in (1+L)/\mathscr{O}_{K,\mathfrak{f}}^{+,\times}} \frac{\mathrm{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s}\right)$$

Where notice that this sum is on classes of $(1+L)$ instead of $(1+L)^+$ like before. Indeed:

$$\begin{aligned}\text{RHS} = \frac{1}{2}\Bigg(&\sum_{\mu:\ \mathrm{sgn}(\mu_{\mathbb{R}})>0} \frac{1}{|N(\mu)|^s} + \sum_{\mu:\ \mathrm{sgn}(\mu_{\mathbb{R}})<0} \frac{1}{|N(\mu)|^s} + \sum_{\mu:\ \mathrm{sgn}(\mu_{\mathbb{R}})>0} \frac{+1}{|N(\mu)|^s} + \\ &+ \sum_{\mu:\ \mathrm{sgn}(\mu_{\mathbb{R}})<0} \frac{-1}{|N(\mu)|^s}\Bigg) = \sum_{\substack{\mu \in (1+L)/\mathscr{O}_{K,\mathfrak{f}}^{+,\times} \\ \mathrm{sgn}(\mu_{\mathbb{R}})>0}} \frac{1}{|N(\mu)|^s} = \sum_{\mu \in (1+L)^+/\mathscr{O}_{K,\mathfrak{f}}^{+,\times}} \frac{1}{N(\mu)^s} = \text{LHS}\end{aligned}$$

We aim to express this sum as linear combination of Hecke L-functions, which will be defined in the next Section.

3.2 Hecke L-functions

Let χ be a character of $\mathscr{C}l_K^+(\mathfrak{f})$, i.e. a group homomorphism $\chi\colon \mathscr{C}l_K^+(\mathfrak{f}) \to \mathbb{C}^\times$. Since $\mathscr{C}l_K^+(\mathfrak{f}) = I(\mathfrak{f})/P(\mathfrak{f})^+$, it makes sense to consider the restriction of χ to $P(\mathfrak{f})/P(\mathfrak{f})^+$; since the zero class must be mapped to $1_{\mathbb{C}}$, we have only two possibility for $\chi\big([(\alpha)]\big)$ for $(\alpha) \in P(\mathfrak{f}) \setminus P(\mathfrak{f})^+$:

$$\begin{cases} \chi\big([(\alpha)]\big) = -1 & \Rightarrow \chi \text{ restricted to } P(\mathfrak{f})/P(\mathfrak{f})^+ \text{ has order } 2; \\ \chi\big([(\alpha)]\big) = 1 & \Rightarrow \chi \text{ restricted to } P(\mathfrak{f})/P(\mathfrak{f})^+ \text{ has order } 1. \end{cases}$$

We say χ is **odd** in the first case, and **even** in the second.
For χ a character of $\mathscr{C}l_K^+(\mathfrak{f})$ and $\mathrm{Re}(s) > 1$ define an L-function[1]:

$$L_{\mathfrak{f}}(\chi,s) := \sum_{\substack{\mathfrak{a} \subset \mathscr{O}_K \\ (\mathfrak{a},\mathfrak{f})=1}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \prod_{\substack{(\mathfrak{p},\mathfrak{f})=1 \\ \mathfrak{p} \text{ prime}}} \Big(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\Big)^{-1} \tag{3.2}$$

where the expression as Euler product appears similar to the more famous one for the Dedekind zeta function

$$\sum_{\mathfrak{a} \subset \mathscr{O}_K} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \frac{1}{1 - N(\mathfrak{p})^{-s}}$$

and is proved similarly, see e.g. [Neu99] pag 515.
This $L_{\mathfrak{f}}(\chi,s)$ satisfies a functional equation with Γ factors:

$$\Gamma(\chi,s) = \begin{cases} \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s) & \text{if } \chi \text{ even;} \\ \Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s) & \text{if } \chi \text{ odd.} \end{cases}$$

where $\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\big(\frac{s}{2}\big)$ and $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s)$. See [Sch88] for a more specific take on the subject.
Thanks to this functional equation, it can be proven that $\Gamma(\chi,s)L_{\mathfrak{f}}(\chi,s)$ is always entire, apart from the case $\chi = \chi_0$ the trivial character, in which it has a pole at $s = 1$ and is holomorphic elsewhere. Note that $L_{\mathfrak{f}}(\chi_0,s)$ is the Dedekind zeta function with the extra condition for ideals to be coprime with $\mathfrak{f}$, denote it $\zeta_{K,\mathfrak{f}}$

$$\zeta_{K,\mathfrak{f}} = \sum_{\mathfrak{a} \subset \mathscr{O}_K,\, (\mathfrak{a},\mathfrak{f})=1} \frac{1}{N(\mathfrak{a})^s}$$

Assume $\chi \neq \chi_0$ even. Since the completed L-function $\Gamma(\chi,s)L_{\mathfrak{f}}(\chi,s)$ is entire, then $L_{\mathfrak{f}}(\chi,s)$ has at least a double zero at $s = 0$, because $\Gamma_{\mathbb{R}}(s), \Gamma_{\mathbb{C}}(s)$ both have a simple pole at $s = 0$ (for the Γ function has simple poles at non negative integers).
Take now $\chi = \chi_0$; then $L_{\mathfrak{f}}(\chi_0,s) = \zeta_{K,\mathfrak{f}}(s) = \zeta_K(s)\prod_{\mathfrak{p}|\mathfrak{f}}(1 - N(\mathfrak{p})^{-s})$ where ζ_K is the usual Dedekind zeta function and the last equality is obtained from the Euler product of the Dedekind zeta function written above.
Now, $L_{\mathfrak{f}}(\chi_0,s)$ has analytic continuation to all of $\mathbb{C}$ with exactly one simple pole at $s = 1$ (it's a particular property of Hecke L-functions, see [Neu99] page 524). Since $\mathfrak{f} \neq \mathscr{O}_K$, also in this case $L_{\mathfrak{f}}(\chi_0,s)$ must have a zero of at least multiplicity 2 at $s = 0$ for $L_{\mathfrak{f}}(\chi_0,s)\Gamma(\chi,s)$ is holomorphic at 0 and $\Gamma(\chi,s)$ has again a double pole at 0 (as χ_0 is even).
Summarizing, for all χ even (both equal and not equal to the trivial character) we have that $L_{\mathfrak{f}}(\chi,s)$ has a zero of multiplicity ≥ 2 at $s = 0$, so that $L'_{\mathfrak{f}}(\chi,0) = 0$ for all χ even.

[1] Historically L-functions were defined as Euler product and later the writing as Dirichlet series was proven, thus following a similar construction to the one of Dedekind zeta function, see for example Tate's book [Tat84].

3.2.1 Meromorphic continuation of partial L-functions

The vanishing of $L'_{\mathfrak{f}}(\chi,0)$ for χ even gives the following expression for partial zeta functions:

Lemma 4.

$$\zeta'(1_K,L,0) = \frac{1}{2}\frac{d}{ds}\left(\sum_{\mu\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}} \frac{\operatorname{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s}\right)\Bigg|_{s=0}$$

Remark 12. *This expression may appear strange, as we defined $\zeta(1_K,L,s)$ only for $\mathrm{Re}(s)>1$, and not just by choice, but because its definition is well-posed (i.e. the series absolute converges) only in that region. The expression of Lemma 4 is to be intended for analytic continuations. In particular: the series on the RHS of Lemma 4 cannot be differentiated and evaluated at $s=0$; one needs to consider the series for $\mathrm{Re}(s)>1$, differentiate it there, extend its derivative by analytic continuation and evaluate this new function at $s=0$. This remark will be useful for some proofs inside the proof of the New Kronecker's Limit Formula (Chapter 5).*

Proof. We want to use the formula proved in Remark 11. In particular, we need to show that the derivative of the first sum in that formula is 0 when evaluated at $s=0$.
First rewrite that sum as

$$\sum_{\mu\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}} \frac{1}{|N(\mu)|^s} = \sum_{\mu\in(1+L)/\mathscr{O}^{\times}_{K,\mathfrak{f}}} \frac{1}{|N(\mu)|^s}$$

where notice that in the second sum we only quotient by $\mathscr{O}^{\times}_{K,\mathfrak{f}}$. This holds for the existence of the short exact sequence

$$0\to\{1\}\to(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}\twoheadrightarrow(1+L)/\mathscr{O}^{\times}_{K,\mathfrak{f}}\to 0$$

The crucial condition here is that the second term of the sequence is $\{1\}$ and not $\{\pm1\}$. This holds since in Section 1.2 we assumed that there are no units of negative norm, so that $-1\notin 1+\mathfrak{f}$, in particular $-1\notin 1+L$.
We claim that this sum can be expressed as linear combination of L-functions $L_{\mathfrak{f}}(\chi,s)$ with χ even: call $\mathfrak{a}_1,\dots,\mathfrak{a}_t$ some integral ideals representing the classes in $\mathscr{C}l_K(\mathfrak{f}) = I(\mathfrak{f})/P(\mathfrak{f})$ (which are known to be a finite number), then
<u>Claim:</u> [1]

$$L_{\mathfrak{f}}(\chi,s) = \sum_{j=1}^{t}\frac{\chi(\mathfrak{a}_j)}{N(\mathfrak{a}_j)^s}\cdot\sum_{\mu\in(1+\mathfrak{f}\mathfrak{a}_j^{-1})/\mathscr{O}^{\times}_{K,\mathfrak{f}}}\frac{1}{|N(\mu)|^s} \tag{3.3}$$

<u>Proof:</u> By definition we have $L_{\mathfrak{f}}(\chi,s) = \sum_{(\mathfrak{a},\mathfrak{f})=1}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}$ and each $\mathfrak{a}$ in this sum is in the same class as one of the $\mathfrak{a}_j$, $1\le j\le t$ in $\mathscr{C}l_K(\mathfrak{f})$; call $[\mathfrak{a}_j]$ such class. So

$$L_{\mathfrak{f}}(\chi,s) = \sum_{(\mathfrak{a},\mathfrak{f})=1}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{j=1}^{t}\sum_{\mathfrak{a}\in[\mathfrak{a}_j]}\frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}$$

Now $\mathfrak{a}\in[\mathfrak{a}_j]\Rightarrow\mathfrak{a}=\mathfrak{a}_j(\mu)$, some $(\mu)\in P(\mathfrak{f})$. That χ is even means by definition that $\chi\big((\mu)\big)=1$ and by multiplicativity of the character, this yields $\chi(\mathfrak{a})=\chi(\mathfrak{a}_j)$. So

$$L_{\mathfrak{f}}(\chi,s) = \sum_{j=1}^{t}\sum_{\mathfrak{a}\in[\mathfrak{a}_j]}\frac{\chi(\mathfrak{a}_j)}{N(\mathfrak{a})^s} = \sum_{j=1}^{t}\chi(\mathfrak{a}_j)\sum_{\mathfrak{a}\in[\mathfrak{a}_j]}N(\mathfrak{a})^{-s} = \sum_{j=1}^{t}\chi(\mathfrak{a}_j)\zeta_{\mathfrak{f}}([\mathfrak{a}_j],s) \tag{3.4}$$

by the first definition we gave in this Chapter. Later on, in Section 3.1, we saw partial zeta functions associated to an integral ideal as

$$\zeta_{\mathfrak{f}}([\mathfrak{a}_j],s) = \zeta_{\mathfrak{f}}(\mathfrak{a}_j,s) = N(\mathfrak{a}_j)^{-s} \sum_{\mu \in (1+L)^+/\mathscr{O}_{K,\mathfrak{f}}^{+,\times}} N(\mu)^{-s} =$$

as in this case $L = \mathfrak{f}\mathfrak{a}_j^{-1}$

$$= N(\mathfrak{a}_j)^{-s} \sum_{\mu \in (1+\mathfrak{f}\mathfrak{a}_j^{-1})^+/\mathscr{O}_{K,\mathfrak{f}}^{+,\times}} N(\mu)^{-s} = N(\mathfrak{a}_j)^{-s} \sum_{\mu \in (1+\mathfrak{f}\mathfrak{a}_j^{-1})/\mathscr{O}_{K,\mathfrak{f}}^{\times}} |N(\mu)|^{-s}$$

that, together with Eq. 3.4, gives the claim. ■

<u>Claim:</u> [2] $\forall 1 \leq j \leq t$, we have:

$$\zeta_{\mathfrak{f}}([\mathfrak{a}_j],s) = N(\mathfrak{a}_j)^{-s} \sum_{\mu \in (1+\mathfrak{f}\mathfrak{a}_j^{-1})/\mathscr{O}_{K,\mathfrak{f}}^{\times}} |N(\mu)|^{-s} = \frac{1}{t}\sum_{\chi} \chi(\mathfrak{a}_j)^{-1} L_{\mathfrak{f}}(\chi,s)$$

where we're summing over the characters $\chi : \mathscr{C}l_K(\mathfrak{f}) \to \mathbb{C}^\times$ and $t := |\mathscr{C}l_K(\mathfrak{f})|$.
<u>Proof:</u> Thanks to Claim 1, we have that

$$\sum_{\chi} \chi(\mathfrak{a}_j)^{-1} L_{\mathfrak{f}}(\chi,s) = \sum_{\chi} \chi(\mathfrak{a}_j)^{-1} \sum_{i=1}^{t} \chi(\mathfrak{a}_i)\zeta_{\mathfrak{f}}([\mathfrak{a}_i],s) = \sum_{i=1}^{t}\sum_{\chi} \chi(\mathfrak{a}_i\mathfrak{a}_j^{-1})\zeta_{\mathfrak{f}}([\mathfrak{a}_i],s) = \sum_{i=1}^{t} \zeta_{\mathfrak{f}}([\mathfrak{a}_i],s)\sum_{\chi}\chi(\mathfrak{a}_i\mathfrak{a}_j^{-1}) \tag{3.5}$$

where we can exchange sums because the one over i is finite and we use multiplicativity of the characters. Now, since for G finite abelian group and $\hat{G} = \mathrm{Hom}(G,\mathbb{C}^\times)$, we have for $g \in G$ that

$$\sum_{\chi \in \hat{G}} \chi(g) = \begin{cases} |G| & \text{if } g = e_G; \\ 0 & \text{otherwise.} \end{cases}$$

with $G = \mathscr{C}l_K(\mathfrak{f})$, since $\mathfrak{a}_i\mathfrak{a}_j^{-1} \in \mathscr{C}l_K(\mathfrak{f})$, we get that

$$\sum_{\chi \in \widehat{\mathscr{C}l_K(\mathfrak{f})}} \chi(\mathfrak{a}_i\mathfrak{a}_j^{-1}) = \begin{cases} |\mathscr{C}l_K(\mathfrak{f})| = t & \text{if } i = j; \\ 0 & \text{otherwise.} \end{cases} \tag{3.6}$$

Note that LHS of Eq. 3.6 is precisely the inner sum in the RHS of Eq.3.5, so that the outer sum on the RHS of Eq. 3.5 has only one non-zero summand, for $i = j$, giving us

$$\sum_{\chi} \chi(\mathfrak{a}_j)^{-1} L_{\mathfrak{f}}(\chi,s) = t\zeta_{\mathfrak{f}}([\mathfrak{a}_j],s)$$

and the claim holds. ■

Note that the characters of $\mathscr{C}l_K(\mathfrak{f})$ correspond to the even characters of $\mathscr{C}l_K^+(\mathfrak{f})$, so that $\forall \chi : \mathscr{C}l_K(\mathfrak{f}) \longrightarrow \mathbb{C}^\times$ character, we have $L_{\mathfrak{f}}'(\chi,0) = 0$ by this correspondence, together with considerations done at the beginning

of the Section.
Differentiating both sides of Claim 2 and evaluating at $s=0$ gives

$$\frac{d}{ds}\left(\frac{1}{t}\sum_{\chi}\chi(\mathfrak{a}_j)^{-1}L_{\mathfrak{f}}(\chi,s)\right)\Big|_{s=0}=0=\frac{d}{ds}\left(N(\mathfrak{a}_j)^{-s}\sum_{\mu\in(1+\mathfrak{f}\mathfrak{a}_j^{-1})/\mathscr{O}_{K,\mathfrak{f}}^{\times}}|N(\mu)|^{-s}\right)\Big|_{s=0}=$$

$$=\left(-sN(\mathfrak{a}_j)^{-s-1}\sum_{\mu\in(1+\mathfrak{f}\mathfrak{a}_j^{-1})/\mathscr{O}_{K,\mathfrak{f}}^{\times}}|N(\mu)|^{-s}\right)\Big|_{s=0}+\left(N(\mathfrak{a}_j)^{-s}\frac{d}{ds}\sum_{\mu\in(1+\mathfrak{f}\mathfrak{a}_j^{-1})/\mathscr{O}_{K,\mathfrak{f}}^{\times}}|N(\mu)|^{-s}\right)\Big|_{s=0}=$$

$$=0+1\cdot\frac{d}{ds}\left(\sum_{\mu\in(1+\mathfrak{f}\mathfrak{a}_j^{-1})/\mathscr{O}_{K,\mathfrak{f}}^{\times}}|N(\mu)|^{-s}\right)\Big|_{s=0}$$

so that actually the first sum in the formula of Remark 11 doesn't influence the derivative in $s=0$. The result of this lemma is then immediate. $\square$

This result will be useful in the proof of the New Kronecker Limit Formula that we will state in the next Section (but will be proved only in Chapter 5).

3.3 A New Kronecker Limit Formula

Consider $\mathfrak{a},\mathfrak{b},\mathfrak{f}$ ideals of $\mathscr{O}_K$ as they were taken in Chapter 1, that is: $\mathfrak{f},\mathfrak{b}$ non zero ideals such that $(\mathfrak{f},\mathfrak{b})=1$ and moreover $\mathfrak{f}\neq\mathscr{O}_K$; $\mathfrak{a}$ relatively prime to $N(\mathfrak{f})$ and such that $N(\mathfrak{a})$ is an odd prime. Let C be a class in $\mathscr{C}l_K^+(\mathfrak{f})$; we've seen that the partial zeta function associated to C is $\zeta_{\mathfrak{f}}(C,s)=\sum_{\mathfrak{c}\in C}N(\mathfrak{c})^{-s}$, $\mathrm{Re}(s)>1$ for $\mathfrak{c}$ integral ideals in the class C; we've also seen how to associate the partial zeta function to an integral ideal $\mathfrak{b}$ of $\mathscr{O}_K$, by considering the one for the class $C=[\mathfrak{b}]$. This allows us to give the following

Definition 9. *For $\mathfrak{a},\mathfrak{b},\mathfrak{f}$ as above [2], we define the **smoothed partial zeta function** as:*

$$\zeta_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},s):=\zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{b},s)-N(\mathfrak{a})\zeta_{\mathfrak{f}}(\mathfrak{b},s),\ \ \mathrm{Re}(s)>1$$

The next theorem is strongly reminiscent of the Limit Formulas expressing the derivative at $s=0$ of Hecke L-functions of imaginary quadratic fields in terms of logarithms and absolute values of elliptic units.

Theorem 3 (New Kronecker's Limit Formula)**.** *The modulus of $\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})$ defined in Chapter 1 doesn't depend on h and*

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0)=\log|\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})|^2$$

Note that dependence of $\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})$ on $\mathfrak{a}$ and L is seen by the presence of $\mathfrak{a}$ and $\mathfrak{f},\mathfrak{b}$ respectively on the left hand side of the formula.
A complete proof of this theorem can be found in Chapter 4 of [BCG23]; we will prove most of it in Chapter 5.

[2] It actually suffices that they're ideals of $\mathscr{O}_K$ such that $\mathfrak{a}$ and $\mathfrak{b}$ are coprime with $\mathfrak{f}$, to be coherent with the notations of Section 3.1.

Remark 13. *Assume the Main Conjecture 3 holds. Denote v the complex embedding of $K(\mathfrak{f})$ extending $\sigma_{\mathbb{C}}\colon K \hookrightarrow \mathbb{C}$ fixed in its statement; we have*

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0) = \log|u_{L,\mathfrak{a}}|_v$$

Indeed the Conjecture tells us that $\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1}) \in \mathbb{C}^\times$ corresponds via the embedding $v\colon K(\mathfrak{f}) \hookrightarrow \mathbb{C}$ to the unit $u_{L,\mathfrak{a}} \in \mathscr{O}^\times_{K(\mathfrak{f})}$. Moreover since v is complex, the archimedean place $|\cdot|_v$ correspond via v to the archimedean norm on $\mathbb{C}$, i.e. the complex module.

It is probably far from obvious how Theorem 3 resembles the classical two Limit Formulas as presented in Chapter 2. In the next Section we aim to present more in detail this resemblance, by looking at a Corollary that express the Second Limit Formula in the case of K imaginary quadratic field. As often in the study we're doing, the new results are motivated by the existence of more known ones for the imaginary quadratic case, that we try to expand for the complex cubic case we work on.

Corollary 1. *For K imaginary quadratic field and $\mathfrak{f}$ a non zero integral ideal different from $\mathscr{O}_K$, we have*

$$\zeta'_{\mathfrak{f}}(\sigma_{\mathfrak{b}},0) = \zeta'_{\mathfrak{f}}(\mathfrak{b},0) = -\frac{1}{6qw_{\mathfrak{f}}}\log|g_{\mathfrak{f}}(\mathfrak{b})|$$

for $[\mathfrak{b}] \in \mathscr{C}l_K(\mathfrak{f})$ and $\sigma_{\mathfrak{b}}$ the corresponding element in $Gal(K(\mathfrak{f})/K)$ via the Artin map, while q is again the smallest positive integer in $\mathfrak{f}\cap\mathbb{Z}$. (The elements $w_{\mathfrak{f}}$ and $g_{\mathfrak{f}}(\mathfrak{b})$ will be defined in the next Section).

This corollary presents some similarities with the New Kronecker Limit formula (Theorem 3). We will now derive this formula from the second Kronecker Limit Formula, Theorem 2.*ii*) proved in Section 2.1.

3.4 A Kronecker Limit Formula for imaginary quadratic fields

This Section simply aims to the proof of Corollary 1; its notation will be consistent with the one developed for partial zeta functions in the first Section of this Chapter, as well as with the notations of Chapter 2 about the proof of classical KLF's.
Let $K/\mathbb{Q}$ be an imaginary quadratic field (this notation for K is used only throughout this Section; later K will return the complex cubic field of the rest of the thesis).
Fix $\mathfrak{f}$ an integral ideal and choose $[\mathfrak{b}] \in \mathscr{C}l_K(\mathfrak{f}) := I(\mathfrak{f})/P(\mathfrak{f})$.
Call $w_{\mathfrak{f}} := |\mathscr{O}^\times_{K,\mathfrak{f}}| = |\{\alpha \in \mathscr{O}^\times_K : \alpha \equiv 1 \mod \mathfrak{f}\}|$.

Remark 14. *Note that we used in the statement of the Corollary 1 and in this Section so far only classes in $\mathscr{C}l_K(\mathfrak{f})$ instead of in $\mathscr{C}l^+_K(\mathfrak{f})$. For the imaginary quadratic case the two notations are equivalent. Indeed recall that we defined in Section 1.3: $P(\mathfrak{f})^+$:=$P(\mathfrak{f}) \cap \{\sigma_{\mathbb{R}}(\alpha) > 0\}$ where $\sigma_{\mathbb{R}}$ was the real embedding of that complex cubic field. We're now in a case in which there is no such real embedding, so that $P(\mathfrak{f})^+ = P(\mathfrak{f})$ and consequently $\mathscr{C}l_K(\mathfrak{f}) = \mathscr{C}l^+_K(\mathfrak{f})$. This will create some small differences with the previously established theory that will be pointed out each time.*

Recall the definition of the partial zeta function associated to a class $[\mathfrak{b}] \in \mathscr{C}l_K(\mathfrak{f})$, cf. previous remark, is (we write $\mathfrak{a} \subset \mathscr{O}_K$ to indicate that we sum only over integral ideals):

$$\zeta_{\mathfrak{f}}([\mathfrak{b}],s) = \sum_{\substack{\mathfrak{a}\subset\mathscr{O}_K \\ \mathfrak{a}\in[\mathfrak{b}]}} N(\mathfrak{a})^{-s},\ \ \mathrm{Re}(s) > 1$$

Let's see a property that relates partial zeta functions to the Eisenstein-Kronecker series of Chapter 2.

Proposition 6.

$$\zeta_{\mathfrak{f}}([\mathfrak{b}],s) = \frac{1}{w_{\mathfrak{f}} N(\mathfrak{b})^s} K_0^*(1,0,s;\mathfrak{f}\mathfrak{b}^{-1}) \tag{3.7}$$

where $K_0^*(z_0,w_0,s;\mathfrak{f}\mathfrak{b}^{-1})$ *was defined for* $\mathrm{Re}(s) > 1+\frac{a}{2}$ *at the beginning of Chapter 2 as*

$$K_a^*(z_0,w_0,s;\Gamma) := \sum_{\gamma\in\Gamma\setminus\{-z_0\}} \frac{(\overline{z_0}+\overline{\gamma})^a}{|z_0+\gamma|^{2s}} \chi_{w_0}(\gamma)$$

Proof. We prove Eq. 3.7 for $\mathrm{Re}(s) >> 1$ where both functions converge absolutely; then the claim would follow by analytic continuation as done multiple times in Chapter 2.
Similarly to what we did in Remark 10

$$\zeta_{\mathfrak{f}}([\mathfrak{b}],s) = \frac{1}{N(\mathfrak{b})^s} \sum_{\lambda\in(1+\mathfrak{f}\mathfrak{b}^{-1})/\mathscr{O}_{K,\mathfrak{f}}^{\times}} \frac{1}{|\lambda|^{2s}} =$$

where for the first equality we used a bijection $(1+\mathfrak{f}\mathfrak{b}^{-1})/\mathscr{O}_{K,\mathfrak{f}}^{\times} \cong \{\mathfrak{a}\in[\mathfrak{b}] \mid \mathfrak{a}\subset\mathscr{O}_K\}$; $\lambda\mapsto\lambda\mathfrak{b}$ which is the same as the bijection proved in Remark 10 just adapted to our case $\mathscr{C}l_K(\mathfrak{f}) = \mathscr{C}l_K^+(\mathfrak{f})$

$$= \frac{1}{w_{\mathfrak{f}} N(\mathfrak{b})^s} \sum_{\lambda\in(1+\mathfrak{f}\mathfrak{b}^{-1})} \frac{1}{|\lambda|^{2s}} = \frac{1}{w_{\mathfrak{f}} N(\mathfrak{b})^s} K_0^*(1,0,s;\mathfrak{f}\mathfrak{b}^{-1})$$

just by looking at the definitions of $w_{\mathfrak{f}}$ and $K_0^*(z_0,w_0,s;\mathfrak{f}\mathfrak{b}^{-1})$ above. □

Notation 3. *From now on, for the rest of this Section let's simply write* $\zeta_{\mathfrak{f}}(\mathfrak{b},s) := \zeta_{\mathfrak{f}}([\mathfrak{b}],s)$ *for a class* $[\mathfrak{b}]\in\mathscr{C}l_K(\mathfrak{f})$.

We want to use the functional equation of $K_a^*(z_0,w_0,s;\Gamma)$, proved for the case $a=0$ in Remark 9. To do so, denote $\Lambda(z_0,w_0,s;\mathfrak{f}\mathfrak{b}^{-1}) := \Gamma(s)A^sK_0^*(z_0,w_0,s;\mathfrak{f}\mathfrak{b}^{-1})$ (where recall that A is the area of the fundamental domain of $\mathfrak{f}\mathfrak{b}^{-1}$ divided by π). The functional equation from Remark 9 then becomes

$$\Lambda(z_0,w_0,s;\mathfrak{f}\mathfrak{b}^{-1}) = \Lambda(w_0,z_0,1-s;\mathfrak{f}\mathfrak{b}^{-1})\chi_{z_0}(w_0)$$

where $\chi_w(z) = \exp\left[\frac{(z\overline{w}-w\overline{z})}{A}\right], \forall z,w\in\mathbb{C}$. Second Kronecker Limit Formula, Theorem 2.*ii*) tells us that for a $z\notin\mathfrak{f}\mathfrak{b}^{-1}$ (this will be the case as we will consider $z=1$ in the following, and $\mathfrak{b}$ is taken coprime with $\mathfrak{f}$):

$$\Lambda(0,z,s;\mathfrak{f}\mathfrak{b}^{-1})\Big|_{s=1} = AK_0^*(0,z,s;\mathfrak{f}\mathfrak{b}^{-1})\Big|_{s=1} = -\log|\theta(z)|^2 + \frac{|z|^2}{A} - \frac{1}{12}\log|\Delta|^2$$

where the first equality holds since $\Gamma(s)A^s\big|_{s=1} = A$. Now, the functional equation gives

$$\Lambda(0,z,s;\mathfrak{f}\mathfrak{b}^{-1})\Big|_{s=1} = \Lambda(z,0,1-s;\mathfrak{f}\mathfrak{b}^{-1})\Big|_{s=1}\cdot 1 = \Lambda(z,0,s;\mathfrak{f}\mathfrak{b}^{-1})\Big|_{s=0}$$

as $\chi_0(z)=1$ and where the second equality holds by changing variable; combining the last two chains of equalities we obtain

$$\Lambda(z,0,s;\mathfrak{f}\mathfrak{b}^{-1})\Big|_{s=0} = -\log|\theta(z)|^2 + \frac{|z|^2}{A} - \frac{1}{12}\log|\Delta|^2 \tag{3.8}$$

Putting together the definition of Λ and Eq. 3.7 we get

$$\zeta_{\mathfrak{f}}(\mathfrak{b},s) = \frac{1}{w_{\mathfrak{f}} N(\mathfrak{b})^s} K_0^*(1,0,s;\mathfrak{f}\mathfrak{b}^{-1}) = \frac{1}{w_{\mathfrak{f}} N(\mathfrak{b})^s} \frac{\Lambda(1,0,s;\mathfrak{f}\mathfrak{b}^{-1})}{\Gamma(s) A^s}$$

Recall that $\Gamma(s)$ has a simple pole at $s=0$ with residue 1, so that $\frac{1}{\Gamma(s)} = s + O(s^2)$ near $s=0$. This implies

$$\zeta_{\mathfrak{f}}(\mathfrak{b},s)\Big|_{s=0} = \left[\frac{1}{w_{\mathfrak{f}}(N(\mathfrak{b})A)^s}\Lambda(1,0,s;\mathfrak{f}\mathfrak{b}^{-1})s + \frac{1}{w_{\mathfrak{f}}(N(\mathfrak{b})A)^s}\Lambda(1,0,s;\mathfrak{f}\mathfrak{b}^{-1})O(s^2)\right]_{s=0} \Rightarrow$$

$$\zeta'_{\mathfrak{f}}(\mathfrak{b},s)\Big|_{s=0} = \frac{1}{w_{\mathfrak{f}}(N(\mathfrak{b})A)^s}\Lambda(1,0,s;\mathfrak{f}\mathfrak{b}^{-1})\Big|_{s=0} = \frac{1}{w_{\mathfrak{f}}\cdot 1}\left(-\log|\theta(1)|^2 + \frac{|1|^2}{A} - \frac{1}{12}\log|\Delta|^2\right)$$

where we used Eq. 3.8 for the last equality.
Recall what we aim to prove is $\zeta'_{\mathfrak{f}}(\mathfrak{b},0) = -\frac{1}{6qw_{\mathfrak{f}}}\log|g_{\mathfrak{f}}(\mathfrak{b})|$ and what we currently have is $\zeta'_{\mathfrak{f}}(\mathfrak{b},0) = \frac{1}{w_{\mathfrak{f}}}\big(-\log|\theta(1)|^2 + \frac{1}{A} - \frac{1}{12}\log|\Delta|^2\big)$, so that we're left to prove

$$-\frac{1}{6q}\log|g_{\mathfrak{f}}(\mathfrak{b})| = -\log|\theta(1)|^2 + \frac{1}{A} - \frac{1}{12}\log|\Delta|^2 \tag{3.9}$$

Remark 15. *Dependence from $\mathfrak{f},\mathfrak{b}$, even if implicit, is present on the right hand side of Eq. 3.9: indeed A depends on the area of the fundamental domain of $\mathfrak{f}\mathfrak{b}^{-1}$, Δ is the discriminant of the lattice $\mathfrak{f}\mathfrak{b}^{-1}$ (defined in Theorem 2) and the theta function θ as defined in Chapter 2, is associated to the divisor $[0]$ of $\mathbb{C}/\mathfrak{f}\mathfrak{b}^{-1}$. All those dependencies are omitted from the notation to ease the reading, but one should keep in mind that they exist.*

It's now convenient to finally define the function $g_{\mathfrak{f}}(\mathfrak{b})$ that appears in the left hand side of Eq. 3.9 as well as in Corollary 1. We need first to recall classical elliptic functions associated to a lattice L.

Definition 10.
- *For a lattice L, $z \in \mathbb{C}$, the Weierstrass-σ function is defined as:*

$$\sigma(z) = \sigma(z;L) := z\prod_{w\in L\setminus\{0\}}\left(1-\frac{z}{w}\right)e^{\frac{z}{w}+\frac{1}{2}\left(\frac{z}{w}\right)^2}$$

- *For L,z as above, the Weierstrass-ζ function is defined*

$$\zeta(z) = \zeta(z;L) = \frac{d\log\sigma(z)}{dz} = \frac{\sigma'(z)}{\sigma(z)} = \frac{1}{z} + \sum_{w\in L\setminus\{0\}}\left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right)$$

- *Given a lattice L, it's a classical exercise in elliptic function theory to prove that exists an $\mathbb{R}$-linear function $\eta(\cdot) = \eta(\cdot;L)\colon L\otimes\mathbb{R}\to\mathbb{C}$, such that $\zeta(z+w;L) = \zeta(z;L) + \eta(w;L)$ for $z\in\mathbb{C}$ and $w\in L$ (note that the definition is given directly only from L and then extended to $L\otimes\mathbb{R}$ by $\mathbb{R}$-linearity).*

Remark 16. *For the following two definitions, we will follow the notation of [Tat84]. To be completely consistent with this notation, it's best to say that Tate defines $w_{\mathfrak{f}}$ (that for us is the cardinality of $\mathcal{O}^{\times}_{K,\mathfrak{f}}$) as the cardinality of the set $\{\zeta\in\mu(K) : \zeta\equiv 1 \mod \mathfrak{f}\}$. The two definitions are equivalent in the case of K quadratic imaginary field as Dirichlet's unit Theorem tells us that $\mathcal{O}_K^{\times}\cong\mu(K)\times\mathbb{Z}^{0+1-1}\cong\mu(K)$ for we have an extension of degree 2, completely imaginary.*

Recall that we defined q to be the smallest positive integer in $\mathfrak{f}\cap\mathbb{Z}$.

Definition 11.
- *We define for $z\in\mathbb{C}$ and L a lattice:*
$$G(z;L) := e^{-6z\eta(z;L)}\sigma(z;L)^{12}\Delta(L)$$
where $\Delta(L)$ is the usual discriminant associated to a lattice as defined in Theorem 2;
- *For each class $[\mathfrak{b}]\in\mathscr{C}l_K(\mathfrak{f})$ we define $g_{\mathfrak{f}}([\mathfrak{b}]) = g_{\mathfrak{f}}(\mathfrak{b}) := G(1,\mathfrak{f}\mathfrak{b}^{-1})^q$ where we can use the second notation for one can prove that the definition doesn't depend on the element of the class (see Appendix A for a proof). The element $g_{\mathfrak{f}}([\mathfrak{b}])$ is called Siegel-Ramachandra invariant.*

With these definition we can reformulate the LHS of Eq. 3.9 as $-\frac{1}{6q}\log|g_{\mathfrak{f}}(\mathfrak{b})| = -\frac{1}{6}\log|G(1,\mathfrak{f}\mathfrak{b}^{-1})|$. Let's analyze it:

$$-\frac{1}{6}\log|G(1,\mathfrak{f}\mathfrak{b}^{-1})| = -\frac{1}{6}\log|e^{-6\cdot 1\cdot\eta(1;\mathfrak{f}\mathfrak{b}^{-1})}\sigma(1;\mathfrak{f}\mathfrak{b}^{-1})^{12}\Delta(\mathfrak{f}\mathfrak{b}^{-1})| =$$
$$= -\frac{1}{6}\Big(\log|e^{-6\eta(1;\mathfrak{f}\mathfrak{b}^{-1})}| + \log|\sigma(1;\mathfrak{f}\mathfrak{b}^{-1})^{12}| + \log|\Delta(\mathfrak{f}\mathfrak{b}^{-1})|\Big) =$$

now we omit to write the lattice as it is always $\mathfrak{f}\mathfrak{b}^{-1}$ both here and in the RHS of Eq. 3.9 (cf. Remark 15)

$$= \frac{-6}{-6}\log|e^{\eta(1)}| + \frac{12}{-6}\log|\sigma(1)| + \frac{1}{-6}\log|\Delta| = \mathrm{Re}(\eta(1)) - 2\log|\sigma(1)| - \frac{1}{6}\log|\Delta|$$

So we currently have

$$-\frac{1}{6q}\log|g_{\mathfrak{f}}(\mathfrak{b})| = \mathrm{Re}(\eta(1)) - 2\log|\sigma(1)| - \frac{1}{6}\log|\Delta| \tag{3.10}$$

We want to investigate further the properties of the θ function appearing in Eq. 3.9 (and in all Chapter 2). The next results can be found in [BK10]. At page 11 of the cited paper, the function θ is explicitly constructed to satisfy the conditions of Chapter 2, and it can be explicitly expressed as $\theta(z) = \exp\left[-\frac{e_2^*}{2}z^2\right]\sigma(z)$. The value e_2^* is explicitly defined in the paper, but we only care for a relation that it satisfies: $\eta(z) = \frac{\overline{z}}{A} + e_2^* z$. One can check that this η function is the same as the one from Definition 10 by confronting the transformation formula for the σ-function at the beginning of page 11 of [BK10] and the claim of exercise 18, Chapter 2 of [Ste13].

These two formulas, adapted to our case $z=1$, yield $e_2^* = \eta(1) - \frac{1}{A} \Rightarrow$

$$\theta(1) = \exp\left[-\frac{\eta(1)}{2} + \frac{1}{2A}\right]\sigma(1)$$

This allows us to write the term $\log|\theta(1)|^2$ in the right hand side of Eq. 3.9 as

$$\log|\theta(1)|^2 = \log\exp\left(-\frac{\eta(1)}{2} + \frac{1}{2A} - \frac{\overline{\eta(1)}}{2} + \frac{1}{2A}\right) + \log|\sigma(1)|^2 = -\mathrm{Re}(\eta(1)) + \frac{1}{A} + \log|\sigma(1)|^2$$

Finally, by looking again at the whole RHS of Eq. 3.9 we have

$$-\log|\theta(1)|^2 + \frac{1}{A} - \frac{1}{12}\log|\Delta|^2 = \mathrm{Re}(\eta(1)) - \frac{1}{A} - \log|\sigma(1)|^2 + \frac{1}{A} - \frac{1}{12}\log|\Delta|^2 =$$
$$= \mathrm{Re}(\eta(1)) - 2\log|\sigma(1)| - 2\cdot\frac{1}{12}\log|\Delta| = -\frac{1}{6q}\log|g_{\mathfrak{f}}(\mathfrak{b})|$$

thanks to Eq. 3.10. We have then proved Eq. 3.9, hence Corollary 1.

Remark 17. *The KLF for quadratic imaginary field we just proved, appears in [Tat84] (Ch. IV, proof of Prop. 3.9) as (adapted to our notation)*

$$\zeta_{\mathfrak{f}}'([\mathfrak{b}],0) = -\frac{1}{12qw_{\mathfrak{f}}}\log|g_{\mathfrak{f}}(\mathfrak{b})| \tag{3.11}$$

We see that it differs from our formula by a factor 2: our formula has a 6 at denominator instead of Tate's 12. Tate doesn't include a full proof of his formula, but he states it relying on other references. I believe that both formulas are correct, and the reason for the discrepancy comes from an inner notation used implicitly by Tate when citing the paper [Sta80] by Stark. In Section 3, Eq. (11) of Stark's article appears a 2 factor more in the definition of G respect to ours; that, I think, explains the difference in the result. To further support our formula, I want to cite a paper by Ramachandra [Ram64], cited by Tate himself when giving the formula Eq. 3.11, which presents in computations in Section 3 (around pages 111-112) a factor 6 compatible with ours.
We will use our formula (with the factor 6) also in the Appendix, when proving Stark's conjecture for imaginary quadratic fields.

Chapter 4

Stark's Conjecture

The Main Conjecture 3 we gave in Chapter 1 is particularly relevant for its relation with the famous Stark's Conjecture. The first two Sections of this Chapter aim to give to the reader who isn't familiar with this result, a brief overview of the aspects more relevant for us, in particular the case of the Conjecture regarding Abelian extension, treated in Section 4.2. All references in this Chapter are to Tate's book [Tat84], unless otherwise mentioned. The first two Sections adopt the notation of Tate's book, to help the reader who intend on study the reference and confront it with the next pages. We'll switch back to our main notation in Section 4.3; the reader will be guided in this change by Remark 20.
In Section 4.3 we will explain in detail the similarities between the Main Conjecture and Stark's Conjecture; in particular we will derive thanks to the New KLF an explicit formula that relates the units of the two Conjectures:

$$u_{\text{Stark}}^{N(\mathfrak{a})-\sigma_{\mathfrak{a}}} = u_{\mathfrak{f},\mathfrak{a}}^{e_{\mathfrak{f}}}$$

4.1 General case

Let k be a number field of finite degree over $\mathbb{Q}$ and let K be a finite Galois extension of k with Galois group denoted G. It's a good idea for the reader to keep track of what this elements are in our case (that of Chapter 1) due to the change of notation: k is our complex cubic field K, while the Galois extension K just mentioned is the narrow ray class field $K(\mathfrak{f})$. Our case (always from Chapter 1) however has the stronger hypothesis of the extension being abelian, that we don't assume now; that's why in this Section we will only sketch the results, that will be expressed more in detail in the next two Sections that concentrate on the abelian case.
Let $\chi\colon G \to \mathbb{C}$ be a character and choose a representation that realizes that character, say $\rho\colon G \to GL(V)$[1].
Choose S a finite set of places of k, that contains all the archimedean places (those are in finite number since we chose k to be an extension of $\mathbb{Q}$ of finite degree) and denote S_K the set of places of K that lie over a place of k that belongs to S. We define the Artin L-function associated to S, χ for $\text{Re}(s) > 1$ as:

$$L_S(s,\chi) := \prod_{\mathfrak{p}\notin S} \det\left(1 - \frac{\sigma_{\mathfrak{P}}}{N(\mathfrak{p})^{-s}}\Big| V^{I_{\mathfrak{P}}}\right)^{-1}$$

[1]The realization of a character is basically the inverse process to the one that associates to a representation $\rho\colon G \to GL(V)$ a character χ_ρ as $\chi_\rho(g) := Tr(\rho(g)) \in \mathbb{C}$. We will refer at the realization as either V or ρ.

where $\mathfrak{P}$ is an arbitrary place of K that lies over $\mathfrak{p}$, and $\sigma_{\mathfrak{P}}$ is an element in $G_{\mathfrak{P}}/I_{\mathfrak{P}}$ where $G_{\mathfrak{P}}$ and $I_{\mathfrak{P}}$ are respectively the decomposition group and the inertia subgroup of $\mathfrak{P}$ over k. (Recall that $G_{\mathfrak{P}} = \{\sigma \in G \mid \sigma(\mathfrak{P}) = \mathfrak{P}\}$; $I_{\mathfrak{P}} = \{\sigma \in G \mid \sigma(a) \equiv a \mod \mathfrak{P}\, \forall a \in \mathscr{O}_K\}$). Tate proved in his book that the Artin L-function doesn't depend on the choice of the realization V but only on the character that make the realization, so the notation is consistent. $L_S(s,\chi)$ can be analytically extended to a meromorphic function on the whole complex plane[2]. This extension allows us to write in a neighbourhood of $s = 0$:

$$L_S(s,\chi) = c(\chi)s^{r(\chi)} + O(s^{r(\chi)+1})$$

with $r(\chi) \geq 0$[3].
We now need to define Stark's regulator, the last element needed in the statement of the Conjecture. Define:

$$X = \left\{ \sum_{w\in S_K} n_w w \mid n_w \in \mathbb{Z}, \sum_{w\in S_K} n_w = 0 \right\}$$

(Note that the sum is finite as both S and the extension K/k are finite, hence so is also S_K.)

$$U = \{x \in K^\times \mid |x|_w = 1\ \forall w \notin S_K\} \text{ are the } S_K\text{-unities of } K$$

Then the map

$$\lambda : U \to \mathbb{R} \otimes_{\mathbb{Z}} X;\ \ u \mapsto \sum_{w\in S_K} \log|u|_w w$$

is easily seen to be well defined, as the classical formula from [Neu99] (Chapter III, Theorem 1.3) tells us that $\prod_{\tilde{w}} |u|_{\tilde{w}} = 1$ where the product runs over all places of K; $u \in U$ imposes that also $\prod_{w\in S_K} |u|_w = 1$ as the valuation is 1 for all other places, in particular $0 = \log\left(\prod_{w\in S_K} |u|_w\right) = \sum_{w\in S_K} \log|u|_w$, so the map is indeed well defined. Moreover, it can be proved that λ induces isomorphisms $\mathbb{R}\otimes_{\mathbb{Z}}U \cong \mathbb{R}\otimes_{\mathbb{Z}}X$, $\mathbb{C}\otimes_{\mathbb{Z}}U \cong \mathbb{C}\otimes_{\mathbb{Z}}X$ compatibles with the natural actions of G on U and X (which come from the action of G on S_K that permutes places w dividing v, for all $v \in S$). So $\mathbb{Q}\otimes_{\mathbb{Z}}U$ and $\mathbb{Q}\otimes_{\mathbb{Z}}X$ are $\mathbb{Q}$-isomorphic as G-representation (or equivalently as $\mathbb{Q}[G]$-modules); call $f\colon\ \mathbb{Q}\otimes_{\mathbb{Z}}X \xrightarrow{\sim} \mathbb{Q}\otimes_{\mathbb{Z}}U$ such an isomorphism and, by abuse of notation, denote by f also its complexified, $f\colon\ \mathbb{C}\otimes_{\mathbb{Z}}X \xrightarrow{\sim} \mathbb{C}\otimes_{\mathbb{Z}}U$. The composition $\lambda \circ f$ is then an automorphism of $\mathbb{C}\otimes_{\mathbb{Z}}X$ and by functoriality we get

$$\mathrm{Hom}_G(V^*, \mathbb{C}\otimes_{\mathbb{Z}}X) \xrightarrow{(\lambda\circ f)_V} \mathrm{Hom}_G(V^*, \mathbb{C}\otimes_{\mathbb{Z}}X);\ \ \varphi \mapsto \lambda \circ f \circ \varphi$$

where V^* is the contragredient of V (in this case this is simply the dual representation, as G is finite). Finally we can define **Stark's regulator** as

$$R(\chi, f) := \det\left((\lambda \circ f)_V\right)$$

This doesn't depend on the realization V of χ, but does depend on f. We can now state

Conjecture 4 (Stark's Conjecture)**.** *Call* $A(\chi, f) := \frac{R(\chi,f)}{c(\chi)} \in \mathbb{C}$. *Then* $\forall \alpha \in Aut(\mathbb{C})$, *we have the relation:*

$$A(\chi, f)^\alpha = A(\chi^\alpha, f)$$

where the character χ^α *is defined as* $\chi^\alpha := \alpha \circ \chi\colon G \to \mathbb{C} \to \mathbb{C}$ *and* $A(\chi, f)^\alpha$ *simply denotes the action of* $\alpha \in Aut(\mathbb{C})$ *on* $A(\chi, f) \in \mathbb{C}$.

[2]The analytic extension was conjectured by Artin to be entire if χ doesn't contain the trivial character in its writing as combination of monomial irreducible characters, in the sense of Brauer's theorem, see Chapter 0.5.

[3]Its value can be written more explicitly in different ways; e.g. on Tate's Chapter 1, proposition 3.4 is proven that $r(\chi) = \dim_{\mathbb{C}} \mathrm{Hom}_G(V^*, \mathbb{C}\otimes_{\mathbb{Z}}X)$, where X is defined in the following lines.

We conclude this Section with a list of facts, which proofs can be found in Tate's book.

Remark 18.
- *We said that Stark's regulator depends on f and one may ask if the truthfulness of Stark's Conjecture may also depend on this choice. To that concern, it is proved that, if the Conjecture holds for one particular choice of isomorphism $f\colon\ \mathbb{C}\otimes_{\mathbb{Z}}X \xrightarrow{\sim} \mathbb{C}\otimes_{\mathbb{Z}}U$, then it holds for all possible choices (Ch. 1, Prop. 6.2).*
- *Truthfulness of Stark's Conjecture doesn't depend on the choice of S, i.e. if it holds for a suitable S (finite and containing all archimedean places), then it holds for all other such S's (Ch. 1, Prop. 7.3).*
- *To prove truthfulness of Stark's Conjecture, one may reduce to the case $K/\mathbb{Q}$ finite Galois extension; indeed if the Conjecture holds for all such extensions, then it holds for all finite Galois extensions K/k with k number field of finite degree over $\mathbb{Q}$ (Ch.1, Prop 7.2). Moreover it's enough to prove it for all irreducible characters of dimension 1, for if it holds in this case, then it holds for all characters of G. (This second part comes from the already cited Brauer's Theorem.)*
- *If $r(\chi)=0$ (the minimal exponent in the expansion of $L_S(s,\chi)$ near $s=0$), the Conjecture is true (Ch. 3, Thm. 1.3).*
- *If $r(\chi)=1$, the Conjecture is equivalent to the existence of some units in K (Ch. 3, Prop. 2.1).*

4.2 Abelian case

We consider now the case in which the extension K/k is abelian, which means that its Galois group $G := Gal(K/k)$ is abelian (the hypothesis of K/k being Galois, necessary for this definition, must hold here like in the general case). Call $\mu(K)$ the group of roots of unity in K and $e := |\mu(K)|$. Assume the set of places on k, always called S satisfies:

a) S contains all archimedean places of k and the places that are ramified in K;

b) S contains at least a place that splits completely in K;

c) $|S|\geq 2$.

Recall that, if we call S_K the set of places of K that divide some place of k contained in S, we've defined: $U=\{x\in K^\times \mid |x|_w = 1\ \forall w\notin S_K\}$, the S_K- unities of K. Moreover we define:

$$U^{ab}_{K/k} := \left\{u\in U \mid K\left(u^{\frac{1}{e}}\right) \text{ is an abelian extension of } k\right\}$$

Choose v a place of S which splits completely in K (that exists since S satisfies assumption b)) and call w one of its lifts to K. Define, if $|S|\geq 3$

$$U^{(v)} := \left\{u\in U \mid |u|_{w'} = 1\ \forall w'\nmid v\right\}$$

and, if $S=\{v,v'\}$,

$$U^{(v)} := \left\{u\in U \mid |u|_{\sigma w'} = |u|_{w'}\ \forall\sigma\in G,\ \forall w' \text{ lift of } v' \text{ to } K\right\}$$

We can now state a version of Stark's Conjecture for abelian extensions:

Conjecture 5. *Let K/k be an abelian extension. If S satisfies the hypothesis $a) - c)$ above and if v is a place of S that splits completely in K, and w a lift of v to K, then there exist a unit $\varepsilon \in U_{K/k}^{ab} \cap U^{(v)}$ such that*

$$\log|\varepsilon^{\sigma}|_w = -e\zeta'(0,\sigma), \ \forall \sigma \in G \tag{4.1}$$

or equivalently such that

$$L'(0,\chi) = -\frac{1}{e}\sum_{\sigma\in G}\chi(\sigma)\log|\varepsilon^{\sigma}|_w, \ \forall \chi \in \widehat{G}$$

Recall the notation ε^{σ} indicates the action of G on K which is on the left. Recall the function L here is the Artin L-function defined in last Section (dependence on S is present even if omitted from the notation). Similarly the zeta function $\zeta(s,\sigma) = \zeta_{S,k}(s,\sigma)$ is defined as

$$\zeta_{S,k}(s,\sigma) = \sum_{\mathfrak{a}\notin S,\mathfrak{a}\in[\sigma]} N(\mathfrak{a})^{-s}$$

where $[\sigma]$ is the class of ideals that correspond to σ under the Artin map. The equivalence between the two expressions in Conjecture 5 is clear.

Remark 19. *Every other ε that satisfies the given conditions differs from the initial one only by a root of unity in K.*

Proof. In the case $|S| \geq 3$, the condition $\varepsilon \in U^{(v)}$ already gives us the valuation for all places of K that lie above a place different from v (those valuations are all $= 1$, just by looking at the definition of $U^{(v)}$ when $|S| \geq 3$); moreover Eq. 4.1 gives us the valuation for a certain place w above v, that automatically fixes valuation for all other places above v (the action of $G = Gal(K/k)$ is transitive on the places: a $\sigma \in G$ sends w to $\sigma w := |\sigma(\cdot)|_w$). The valuation of ε is then fixed for all places of K, so that another element that satisfy the conditions of the Conjecture must differ from the initial one only by a root of unity in K.
In the case $S = \{v, v'\}$ we must pay more attention as the definition of $U^{(v)}$ changes. The condition $\varepsilon \in U$ already fixes its valuation ($= 1$) for all places apart from those above v and v'; Eq. 4.1 gives us like before the valuation of ε for all places above v (again by transitivity of the action of G on the places above v). We're left to wonder about the valuation of ε for the places above v'; we know, by condition $\varepsilon \in U^{(v)}$ with its definition in this case, together with transitivity of G on the set of places above v', that the norm of ε is the same for all the places above v'. To conclude we use again the formula from [Neu99] (Chapter III, Theorem 1.3)

$$\forall x \in K^{\times} \quad \prod_{\widetilde{w} \text{ place of } K} |x|_{\widetilde{w}} = 1$$

In our case we get for ε that

$$\prod_{w'|v'}|\varepsilon|_{w'} = \left(\prod_{w|v}|\varepsilon|_w \cdot \prod_{\widetilde{w}\nmid v,v'}|\varepsilon|_{\widetilde{w}}\right)^{-1} = \left(\prod_{w|v}|\varepsilon|_w \cdot 1\right)^{-1}$$

Now since $|\varepsilon|_{w'}$ is the same for all places above v', let's say there are ℓ of them, we get explicitly a valuation also for places $w'|v'$:

$$|\varepsilon|_{w'} = \left(\prod_{w|v}|\varepsilon|_w^{-1}\right)^{\frac{1}{\ell}}$$

Having a fixed valuation of ε for all places of K, we have that ε is determined up to root of unity also in this case. $\square$

A proof of Stark's Conjecture for abelian extensions of imaginary quadratic number fields is presented in the appendix A.

4.3 Relation with the Main Conjecture

We now want to see how the Main Conjecture 3 we stated in Section 1.3 is related to Stark's Conjecture. To see it, we need first of all to adapt the setting of last Section to the one of Chapter 1; our extension was indeed a particular case of abelian extension: a narrow ray class extension. Recall that $\mathfrak{f}$, as defined in Section 1.2, is a prime integral ideal different from the whole $\mathcal{O}_K$. Let $e_{\mathfrak{f}}$ be the number of roots of unity in $K(\mathfrak{f})$ and fix a complex place v of $K(\mathfrak{f})$ above the fixed complex place of K, $\sigma_{\mathbb{C}}$, chosen at the beginning of Chapter 1. We first state Stark's Conjecture as it is stated in the article [BCG23] and see how this is equivalent to the version from [Tat84] we gave in last Section (in the particular case of the narrow ray class extension $K(\mathfrak{f})/K$).

Conjecture 6 (Stark's Conjecture). *$\exists u_{Stark} \in K(\mathfrak{f})^{\times}$ unit such that:*

- *If w is a place of $K(\mathfrak{f})$ that does not divide $\sigma_{\mathbb{C}}$, then $|u_{Stark}|_w = 1$;*
- *$\forall \rho \in Gal(K(\mathfrak{f})/K)$,*
$$\zeta_{\mathfrak{f}}'(\rho,0) = -\frac{1}{e_{\mathfrak{f}}}\log|u_{Stark}^{\rho}|_v \tag{4.2}$$
- *$K(\mathfrak{f})(u_{Stark}^{1/e_{\mathfrak{f}}})/K$ is an abelian extension.*

Remark 20. *This statement is equivalent to Conjecture 5 of previous Section for a particular choice of S. First of all note that $e_{\mathfrak{f}} = e$ as they both represent $|\mu(K(\mathfrak{f}))|$ in this case.*
Choose $S = \{\sigma_{\mathbb{R}}, \sigma_{\mathbb{C}}, \overline{\sigma_{\mathbb{C}}}, \mathfrak{f}\}$; we claim that this choice satisfies the conditions of last Section. Indeed the first three are the archimedean places of K, that must belong to S; also $\mathfrak{f}$, the only ideal that is ramified in $K(\mathfrak{f})$ by properties of narrow ray class field, must belong to S for it to respect condition a). Note that also condition b) is respected since complex places always split completely. Clearly $|S| \geq 2$.
We have $u_{Stark} \in U$ because if w is a place $w \notin S_K$, then it cannot divide $\sigma_{\mathbb{C}}$ which belongs to S, so by the first condition of Conjecture 6 we get $|u_{Stark}|_w = 1$.

- *First condition is equivalent to "$u_{Stark} \in U^{(\sigma_{\mathbb{C}})}$" (see definition in previous Section) for we're in the case $|S| \geq 3$;*
- *Let's prove that second condition is equivalent to Tate's: "$\log|\varepsilon^{\sigma}|_w = -e\zeta'(0,\sigma)$, $\forall \sigma \in G$", which in the current notation becomes "$\log|u_{Stark}^{\rho}|_v = -e_{\mathfrak{f}}\zeta'(0,\rho)$, $\forall \rho \in Gal(K(\mathfrak{f})/K)$". Proving the equivalence simply boils down to proving the equality between Tate's zeta function $\zeta(s,\rho)$ and the $\zeta_{\mathfrak{f}}(\rho,s)$ that we defined in Chapter 3. This is just a matter of notation, indeed*
$$\text{Tate's } \zeta(s,\rho) = \sum_{\substack{(\mathfrak{a},S)=1 \\ (\mathfrak{a},K(\mathfrak{f})/K)=\rho}} N(\mathfrak{a})^{-s} \overset{!}{=} \zeta_{\mathfrak{f}}(\rho,s) = \sum_{\mathfrak{c}\in C} N(\mathfrak{c})^{-s}$$
The equality $\overset{!}{=}$ holds as the condition $(\mathfrak{a},S) = 1$ is equivalent to asking that $\mathfrak{a}$ is an integral ideal coprime to $\mathfrak{f}$, since $\mathfrak{f}$ is an element of S, and the other elements of S don't impose further condition being archimedean places; so this imposes $\mathfrak{a}$ as an integral element of $I(\mathfrak{f})$. $(\mathfrak{a},K(\mathfrak{f})/K) = \rho$ means that $\mathfrak{a}$ belongs to the class in $\mathscr{C}l_K^+(\mathfrak{f})$ corresponding to ρ via the Artin map, that we call C. Then the two sums are really the same.

- *The final condition is equivalent by definition to "$u_{Stark} \in U^{ab}_{K(\mathfrak{f})/K}$".*

So we have the wanted equivalence.

Note that the Conjecture specifies the valuation of u_{Stark} for every place of $K(\mathfrak{f})$; indeed it gives valuation =1 for places that don't divide $\sigma_{\mathbb{C}}$ and then give the explicit value for v, a chosen place above $\sigma_{\mathbb{C}}$. To conclude, it's enough to remember that the Galois group $G = Gal(K(\mathfrak{f})/K)$ acts transitively on the places above $\sigma_{\mathbb{C}}$, so that Eq. 4.2 implicitly tells us the valuation for all places above $\sigma_{\mathbb{C}}$. This gives us that u_{Stark} is uniquely determined up to root of unity. This ambiguity can be removed using the ideal $\mathfrak{a}$.

Proposition 7. *If $\mathfrak{a}$ is coprime to $6\mathfrak{f}$ (not only to $\mathfrak{f}$), then the element $\sigma_{\mathfrak{a}} - N(\mathfrak{a}) \in \mathbb{Z}[Gal(K(\mathfrak{f})/K)]$, the free module over the group $Gal(K(\mathfrak{f})/K)$, kills roots of unity in $K(\mathfrak{f})$, so that $u_{Stark}^{\sigma_{\mathfrak{a}}-N(\mathfrak{a})}$ becomes a well defined unit in $K(\mathfrak{f})$.*

Proof. First recall that $\sigma_{\mathfrak{a}}$ is the element of the Galois group associated to the class of the ideal $\mathfrak{a}$ via the Artin map. That $\sigma_{\mathfrak{a}} - N(\mathfrak{a}) \in \mathbb{Z}[Gal(K(\mathfrak{f})/K)]$ is then clear since $\sigma_{\mathfrak{a}} \in Gal(K(\mathfrak{f})/K)$ and the norm is an integer times the identity morphism. We plan to use a result from [Tat84] (Chapter IV, Lemma 1.1). It tells us that if S is a set of places of K, that contains archimedean places, those which ramify in $K(\mathfrak{f})$ and those which divide $e = |\mu(K(\mathfrak{f}))|$, then the annihilator of the $\mathbb{Z}[G]$-module $\mu(K(\mathfrak{f}))$ is generated over $\mathbb{Z}$ by the elements $\sigma_{\mathfrak{p}} - N(\mathfrak{p})$ for $\mathfrak{p}$ running over the prime ideals not contained in S. It is convenient to start by analyzing the number e.
If $K(\mathfrak{f})$ contains a p-th root of unity with p a prime coprime to $\mathfrak{f}$, then the ramification degree over $\mathbb{Q}$ of a prime above p is at least $p-1$. Indeed, call ε the p-th root of unity contained in $K(\mathfrak{f})$; consider the chain of extensions of fields $\mathbb{Q} \subset \mathbb{Q}(\varepsilon) \subset K(\mathfrak{f})$. It's known that, when adjoining a p-th root of unity, the ideal (p) becomes totally ramified, in particular its ramification degree is the same as the degree of the extension $\mathbb{Q}(\varepsilon)/\mathbb{Q}$, i.e. $p-1$. Consider now an ideal of $K(\mathfrak{f})$ lying over the only ideal of $\mathbb{Q}(\varepsilon)$ over (p): by multiplicativity, its ramification degree over (p) is at least $p-1$.
Call $\mathfrak{p}$ a prime ideal of K that lies over $(p) \subset \mathbb{Q}$; moreover call $\mathfrak{P}$ a prime ideal of $K(\mathfrak{f})$ lying over $\mathfrak{p}$. What we proved so far is that $e(\mathfrak{P}/(p)) \geq p-1$. Given the definition of $K(\mathfrak{f})$ and the fact that $\mathfrak{f}$ is coprime with p (and so also with $\mathfrak{p}$), we get that $e(\mathfrak{P}/\mathfrak{p}) = 1$, so that, again by multiplicativity of the ramification degree, $e(\mathfrak{p}/(p)) \geq p-1$. Now, since $[K:\mathbb{Q}] = 3$, and since $\mathfrak{p}$ is a prime of K, we have that $p-1 \leq e(\mathfrak{p}/(p)) \leq [K:\mathbb{Q}] = 3$, so that p must be less or equal to 3 (as $p=4$ isn't a prime), i.e. $K(\mathfrak{f})$ can only contain 2,3-th roots of unity. This tells us that the only prime numbers coprime with $\mathfrak{f}$ that can divide e are 2 and 3.
A good candidate for S for the extension $K(\mathfrak{f})/K$ is $S = \{\sigma_{\mathbb{R}}, \sigma_{\mathbb{C}}, \overline{\sigma}_{\mathbb{C}}, \mathfrak{f}\} \cup \{\mathfrak{p}|2\} \cup \{\mathfrak{q}|3\}$[4] as it contains archimedeans, the only place that ramifies and those who may divide e.
Since the ideal $\mathfrak{a}$ of the claim is coprime to $6\mathfrak{f}$, then it's neither $\mathfrak{f}$ nor a place above 2 or 3, and clearly it cannot be one of the archimedean places; then $\mathfrak{a}$ is a prime ideal of K not in S, so that the result from Tate tells us that $\sigma_{\mathfrak{a}} - N(\mathfrak{a})$ is among the generators of the annihilator of $\mu(K(\mathfrak{f}))$ and in particular belongs to it. So the element $\sigma_{\mathfrak{a}} - N(\mathfrak{a}) \in \mathbb{Z}[Gal(K(\mathfrak{f})/K)]$ kills roots of unity in $K(\mathfrak{f})$. $\square$

We are ready to write explicitly a relation between the unit u_{Stark} of Stark's Conjecture and $u_{L,\mathfrak{a}}$ of our Main Conjecture. Recall that the unit $u_{L,\mathfrak{a}}$ is explicitly determined by the condition of being the preimage via $v\colon K(\mathfrak{f}) \hookrightarrow \mathbb{C}$ of $\Gamma_{\mathfrak{a},h}(L) \in \mathbb{C}^{\times}$, and is then well defined.

[4] 2 and 3 are places of $\mathbb{Q}$, but here we need to consider places of K lying over them.

The well defined unit $u_{\text{Stark}}^{\sigma_{\mathfrak{a}}-N(\mathfrak{a})}$ of $K(\mathfrak{f})$ also satisfies

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0) = -\frac{1}{e_{\mathfrak{f}}}\log|u_{\text{Stark}}^{(\sigma_{\mathfrak{a}}-N(\mathfrak{a}))\sigma_{\mathfrak{b}}}|_v \tag{4.3}$$

Indeed

$$\zeta_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},s) = \zeta_{\mathfrak{f}}(\mathfrak{ab},s) - N(\mathfrak{a})\zeta_{\mathfrak{f}}(\mathfrak{b},s) \Rightarrow \zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0) = \zeta'_{\mathfrak{f}}(\mathfrak{ab},0) - N(\mathfrak{a})\zeta'_{\mathfrak{f}}(\mathfrak{b},0)$$

and since u_{Stark} satisfies Eq. 4.2 for all $\rho \in Gal(K(\mathfrak{f})/K)$, in particular for $\sigma_{\mathfrak{ab}}$ and $\sigma_{\mathfrak{b}}$,

$$\zeta'_{\mathfrak{f}}(\mathfrak{ab},0) = \zeta'_{\mathfrak{f}}(\sigma_{\mathfrak{ab}},0) = -\frac{1}{e_{\mathfrak{f}}}\log|u_{\text{Stark}}^{\sigma_{\mathfrak{ab}}}|_v = -\frac{1}{e_{\mathfrak{f}}}\log|u_{\text{Stark}}^{\sigma_{\mathfrak{a}}\sigma_{\mathfrak{b}}}|_v$$

$$N(\mathfrak{a})\zeta'_{\mathfrak{f}}(\mathfrak{b},0) = N(\mathfrak{a})\zeta'_{\mathfrak{f}}(\sigma_{\mathfrak{b}},0) = -\frac{N(\mathfrak{a})}{e_{\mathfrak{f}}}\log|u_{\text{Stark}}^{\sigma_{\mathfrak{b}}}|_v = -\frac{1}{e_{\mathfrak{f}}}\log|u_{\text{Stark}}^{N(\mathfrak{a})\sigma_{\mathfrak{b}}}|_v$$

which gives us the wanted formula:

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0) = \zeta'_{\mathfrak{f}}(\mathfrak{ab},0) - N(\mathfrak{a})\zeta'_{\mathfrak{f}}(\mathfrak{b},0) = -\frac{1}{e_{\mathfrak{f}}}\log|u_{\text{Stark}}^{\sigma_{\mathfrak{a}}\sigma_{\mathfrak{b}}-N(\mathfrak{a})\sigma_{\mathfrak{b}}}|_v$$

The relation between Stark's Conjecture 6 and the Main Conjecture 3 comes now thanks to the "New Kronecker's Limit Formula", Theorem 3, seen at the end of last Chapter (or better from Remark 13): this tells us that, if the Main Conjecture holds, the LHS of Eq. 4.3 is equal to $\log|u_{L,\mathfrak{a}}|_v$ so that, if both Conjectures hold, we obtain a relation between our unit and Stark's:

$$\log|u_{L,\mathfrak{a}}|_v = \zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0) = -\frac{1}{e_{\mathfrak{f}}}\log\left|u_{\text{Stark}}^{(\sigma_{\mathfrak{a}}-N(\mathfrak{a}))\sigma_{\mathfrak{b}}}\right|_v \Leftrightarrow |u_{L,\mathfrak{a}}|_v = \left|u_{\text{Stark}}^{-(\sigma_{\mathfrak{a}}-N(\mathfrak{a}))\sigma_{\mathfrak{b}}e_{\mathfrak{f}}^{-1}}\right|_v^{5}$$

for a chosen place v above $\sigma_{\mathbb{C}}$, hence for all places above $\sigma_{\mathbb{C}}$ for considerations like before. Moreover u_{Stark}, and consequently $u_{\text{Stark}}^{-(\sigma_{\mathfrak{a}}-N(\mathfrak{a}))\sigma_{\mathfrak{b}}e_{\mathfrak{f}}^{-1}}$, have norm equal to 1 for all places of $K(\mathfrak{f})$ not above $\sigma_{\mathbb{C}}$ by the first condition of Conjecture 6; the same holds for $u_{L,\mathfrak{a}}$ too: indeed condition "$u_{L,\mathfrak{a}} \in \mathscr{O}^{\times}_{K(\mathfrak{f})}$" gives that the norm of $u_{L,\mathfrak{a}}$ is 1 for all finite places (the non archimedeans), while property *ii*) of the Main Conjecture fixes the valuation of $u_{L,\mathfrak{a}}$ equal to 1 for the places over $\sigma_{\mathbb{R}}$. We then have that $u_{L,\mathfrak{a}}$ and $u_{\text{Stark}}^{-(\sigma_{\mathfrak{a}}-N(\mathfrak{a}))\sigma_{\mathfrak{b}}e_{\mathfrak{f}}^{-1}}$ differ by a root of unity, that is killed by raising everything to the $e_{\mathfrak{f}}$-th power, giving

$$u_{\text{Stark}}^{N(\mathfrak{a})-\sigma_{\mathfrak{a}}} = u_{L,\mathfrak{a}}^{e_{\mathfrak{f}}\sigma_{\mathfrak{b}}^{-1}} = \left(\sigma_{\mathfrak{b}^{-1}}u_{L,\mathfrak{a}}\right)^{e_{\mathfrak{f}}} = \left(u_{\mathfrak{bfb}^{-1},\mathfrak{a}}\right)^{e_{\mathfrak{f}}} = u_{\mathfrak{f},\mathfrak{a}}^{e_{\mathfrak{f}}}$$

where we used property *iii*) of the Main Conjecture for the third equality.
The relation

$$u_{\text{Stark}}^{N(\mathfrak{a})-\sigma_{\mathfrak{a}}} = u_{\mathfrak{f},\mathfrak{a}}^{e_{\mathfrak{f}}}$$

deeply connects the two units (in the case that both Conjectures are true), and becomes a consistent discovery that supports Stark's prediction of $u_{\text{Stark}}^{N(\mathfrak{a})-\sigma_{\mathfrak{a}}}$ being a $e_{\mathfrak{f}}$-power in $K(\mathfrak{f})$.

[5] Taking $e_{\mathfrak{f}}$-th root is well defined up to $e_{\mathfrak{f}}$-th root of unity, but the norm $|\cdot|_v$ doesn't depend on this choice.

The Main Conjecture proposes itself as a candidate to be a key step in the proof of Stark's Conjecture for abelian extensions of complex cubic fields. The article [BCG23] presents many numerical examples that support the truthfulness of the Main Conjecture.
The last thing that we aim to do, is to prove the New Kronecker's Limit Formula, which main purpose we've just seen. This will be the topic of the next and final chapter.

Chapter 5

Proof of the New Kronecker's Limit Formula

In this Chapter we will prove the New KLF, Theorem 3 from Section 3.3. To do so, we will follow the main article [BCG23] in which this formula first appeared. The purpose of this Chapter is to explain how ,starting from the derivative of the smoothed partial zeta function $\zeta_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},s)$, we can arrive to the logarithm of the Smoothed Elliptic Gamma function $\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})$ from Section 1.2. These two elements are really different as was clear from their definitions: the first more analytic, while the second more algebraic.
The following proof will be really long and tedious in some passages; some other parts will be skipped as they just provide technical results that don't have use in our scopes. Some parts will present long computation, that I believe are important to see why the functions of Chapter 1 are defined in a certain way.
We will begin with a plan of the proof to help the reader to concentrate on the parts that they're more curious about.
We can schematize the proof as a chain of equalities:

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0) = \frac{3}{\sqrt{\pi}}\int_{\langle\varepsilon\rangle} E_{\psi,\mathfrak{a}}(h,\mathfrak{f}\mathfrak{b}^{-1},0) = \frac{3}{\sqrt{\pi}}\int_0^\infty\int_0^\infty F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b})\frac{dt_1}{t_1}\frac{dt}{t} =$$

$$= \frac{3}{\sqrt{\pi}}\Big(I_{+-}(h)+I_{-+}(h)\Big) = \log|\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})|^2$$

In Section 5.1 we will briefly define the Eisenstein series E_ψ and the smoothed Eisenstein series $E_{\psi,\mathfrak{a}}$, as well as the closed oriented geodesic $\langle\varepsilon\rangle$, where ε is the same unit from Chapter 1. We'll then prove a result that, together with Lemma 4 of Subsection 3.2.1, will yield the first equality.
In Section 5.2 we will define the function $F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b})$ and relate it to the smoothed Eisenstein series.
In Section 5.3 we'll start to go towards the Elliptic Gamma function by rewriting $F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b})$ as a series and regroup the terms of that series to define $I_{+-}(h)$ and $I_{-+}(h)$. Also in this Section we will compute all integrals so they won't appear in the last one.
In Section 5.4, we'll basically conclude. This Section will present the most direct computations. Here most of the results from Chapter 1 will be directly used or mentioned.
Finally in Section 5.5 we will do some final considerations, and prove a result that we used without proof in Remarks 6.3 and 8.

5.1 First equality

Recall some notations from Chapter 1. We fix a lattice $L = \mathfrak{f}\mathfrak{b}^{-1}$ and write $\Lambda = \mathrm{Hom}(L, \mathbb{Z})$; for $\mathbf{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ define $V_{\mathbf{K}} = \Lambda \otimes \mathbf{K}$. By fixing a basis of L, we get an isomorphism $L \cong \mathbb{Z}^3$. We fixed a complex embedding $\sigma_{\mathbb{C}} \colon K \hookrightarrow \mathbb{C}$ and an orientation of K compatible with $\sigma_{\mathbb{C}}$. Finally, we fixed an ideal $\mathfrak{a}$ and an element $h = \lambda/q \in L \otimes \mathbb{Q}$. Write a for the element provided by Lemma 1 and $b = \varepsilon a$, with ε a generator of $\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$ (in particular $\varepsilon_{\mathbb{R}} |\varepsilon_{\mathbb{C}}|^2 = 1$) with $\varepsilon_{\mathbb{R}} \in (0,1)$.
The next definitions will be given very briefly as the specifics would only be needed in technical proofs that are of no use for this thesis. A curious reader should read more in detail Section 4.2 of [BCG23].
Recall we called $\mathbf{X} := \frac{V_{\mathbb{C}} - \mathbb{C} \cdot V_{\mathbb{R}}}{\mathbb{C}^\times}$ the space in which $\mathbf{x}$, the restriction of $\sigma_{\mathbb{C}}$ to L lived. The group $G := GL(V_{\mathbb{R}})^+ \cong GL_3(\mathbb{R})^+$ acts naturally on $\mathbf{X}$. Denote Q the quadratic form on $V_{\mathbb{R}}^\vee$ defined by $Q(x) := x_1^2 + x_2^2 + x_3^2$; its isometry group $H = SO(Q) \cong SO(3)$ is a maximal compact subgroup of G. Denote then $Z^0 \cong \mathbb{R}_{\geq 0}$ the connected component of the center of G. Write

$$\mathbf{S} := G/H \quad \text{and} \quad \overline{\mathbf{S}} := \mathbf{S}/Z^0$$

Moreover define $\Gamma_1(h) := \{g \in GL(L) \mid gh \equiv h \mod L\}$.
Now, we want to define the closed geodesic $\langle \varepsilon \rangle$. Since $h = \lambda/q \in \mathfrak{b}^{-1}$ (as $\lambda \in L$ and q is the minimum integer in $\mathfrak{f}$), we have that $(\varepsilon - 1)h \in \mathfrak{f}\mathfrak{b}^{-1} = L$, so $\varepsilon \in \Gamma_1(h)$ (ε can be seen as an element of $GL(L)$ as discussed in Section 1.2). For any $t_1 > 0$, define a metric $H_{t_1} \in \mathbf{S}$ by

$$H_{t_1}(v) := t_1^{-2}\, \sigma_{\mathbb{R}}(v)^2 + t_1 |\sigma_{\mathbb{C}}(v)|^2, \;\; v \in V_{\mathbb{R}}^\vee$$

and call $\overline{H_{t_1}}$ the corresponding element in $\overline{\mathbf{S}}$. Then the assignment

$$\iota \colon \mathbb{R}_{>0} \to \mathbf{S} \times \mathbf{X};\;\; t_1 \mapsto (H_{t_1}, \langle \mathbf{x} \rangle)$$

induces a map (recall $\varepsilon_{\mathbb{R}} = \sigma_{\mathbb{R}}(\varepsilon) > 0$)

$$\iota \colon \mathbb{R}_{>0} / \langle \varepsilon_{\mathbb{R}}^n \mid n \in \mathbb{Z} \rangle \to (\overline{\mathbf{S}} \times \mathbf{X})/\Gamma_1(h)$$

Its image is a closed oriented geodesic in $(\overline{\mathbf{S}} \times \mathbf{X})/\Gamma_1(h)$ that we denote $\langle \varepsilon \rangle$.
For $v \in V_{\mathbb{R}}^\vee$, we define a differential form $\psi(v) \in \Omega^1(\mathbf{S} \times \mathbf{X})$; its explicit definition isn't important for our purposes and would require many additional construction; however it can be proven an explicit formula for its pull-back via ι that we will take as a definition:

$$\iota^* \psi(v) = e^{-t_1^{-2} \sigma_{\mathbb{R}}(v)^2 - t_1 |\sigma_{\mathbb{C}}(v)|^2} t_1^{-1}\, \sigma_{\mathbb{R}}(v) \frac{dt_1}{t_1} \tag{5.1}$$

It can be proven that $\psi(v)$ satisfies $g^* \psi(gv) = \psi(v)$ for all $g \in G$; we will take this as a fact.

Definition 12.
- *Define for $t > 0$:*

$$\Theta_{\psi_t}(h, L) := \sum_{\lambda \in h+L} \psi(t\lambda) \;\in \Omega^1(\mathbf{S} \times \mathbf{X})$$

- *Then set*

$$E_\psi(h, L, s) := \int_0^\infty \Theta_{\psi_t}(h, L) t^{3s} \frac{dt}{t} \in \Omega^1(\overline{\mathbf{S}} \times \mathbf{X})$$

Note that the integral defining $E_\psi(h,L,s)$ converges for any $s\in\mathbb{C}$; this follows by splitting the integration domains into intervals $(0,1)$ and $(1,\infty)$ and applying Poisson summation formula. Moreover by the property of $\psi(v)$ stated above, the differential forms $E_\psi(h,L,s)$ and $\Theta_{\psi_t}(h,L)$ are invariant under $\Gamma_1(h)$, so that

$$\Theta_{\psi_t}(h,L)\in\Omega^1(\mathbf{S}\times\mathbf{X})^{\Gamma_1(h)}\cong\Omega^1(\mathbf{S}\times\mathbf{X}/\Gamma_1(h))$$

$$E_\psi(h,L,s)\in\Omega^1(\overline{\mathbf{S}}\times\mathbf{X})^{\Gamma_1(h)}\cong\Omega^1(\overline{\mathbf{S}}\times\mathbf{X}/\Gamma_1(h))$$

We use the Eisenstein series and the closed geodesic to prove a key result for the proof of the first equality.

Lemma 5.

$$\int_{\langle\varepsilon\rangle}E_\psi(h,L,s)=\frac{1}{6}\Gamma(s)\Gamma\left(\frac{s+1}{2}\right)\sum_{v\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\frac{\operatorname{sgn}(\sigma_\mathbb{R}(v))}{|N(v)|^s}$$

Proof. Eq. 5.1 evidently remains invariant if we replace (v,t_1) with $(\varepsilon v,\varepsilon_\mathbb{R}t_1)$, as $\varepsilon_\mathbb{R}|\varepsilon_\mathbb{C}|^2=1$. Moreover, as $h\equiv 1 \mod L$, when we write explicitly the definition of E_ψ and then of Θ_{ψ_t}, we can write $v\in 1+L$ instead of $v\in h+L$ as index of the sum. Moreover, by pulling back via ι, the integration on the closed geodesic $\langle\varepsilon\rangle\subset(\overline{\mathbf{S}}\times\mathbf{X})/\Gamma_1(h)$ becomes the integration on the interval $(t_0\varepsilon_\mathbb{R},t_0)\subset\mathbb{R}_{>0}$ for any $t_0\in\mathbb{R}_{>0}$, explicitly

$$\int_{\langle\varepsilon\rangle}E_\psi(h,L,s)=\int_{t_0\varepsilon_\mathbb{R}}^{t_0}\iota^*E_\psi(h,L,s)$$

By plugging in definitions (the pull-back allows us to use Eq. 5.1)

$$\int_{\langle\varepsilon\rangle}E_\psi(h,L,s)=\int_{t_0\varepsilon_\mathbb{R}}^{t_0}\int_0^\infty\sum_{v\in 1+L}e^{-t_1^{-2}t^2\,\sigma_\mathbb{R}(v)^2-t_1t^2|\,\sigma_\mathbb{C}(v)|^2}tt_1^{-1}\,\sigma_\mathbb{R}(v)\frac{dt_1}{t_1}t^{3s}\frac{dt}{t}=$$

$$=\sum_{v\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\sigma_\mathbb{R}(v)\int_0^\infty\int_0^\infty e^{-t_1^{-2}t^2\,\sigma_\mathbb{R}(v)^2-t_1t^2|\,\sigma_\mathbb{C}(v)|^2}t_1^{-1}t^{3s+1}\frac{dt_1}{t_1}\frac{dt}{t}$$

We obtain the second equality by rescaling the external integral (the one in t_1). Since the expression for $\iota^*\psi(v)$ remains invariant by replacing (v,t_1) with $(\varepsilon v,\varepsilon_\mathbb{R}t_1)$, and since ε is a generator of $\mathscr{O}^{+,\times}_{K,\mathfrak{f}}$, by regrouping expressions that correspond to the same $v\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}$ (i.e. the $\varepsilon^n v$ for all $n\in\mathbb{Z}$) we can extend the integral from $\int_{t_0\varepsilon_\mathbb{R}}^{t_0}$ to the whole $\mathbb{R}_{>0}$ (by uniting all integrals of the shape $\int_{t_0\varepsilon_\mathbb{R}^{n+1}}^{t_0\varepsilon_\mathbb{R}^n}$ for all $n\in\mathbb{Z}$).
Now, to compute the integral, operate the change of variables $u:=t_1^{-1}t,\ w:=t_1t^2$ which gives

$$\int_0^\infty\int_0^\infty e^{-u^2\,\sigma_\mathbb{R}(v)^2-w|\,\sigma_\mathbb{C}(v)|^2}(u^2w^{-1})^{1/3}(uw)^{s+1/3}\frac{1}{3}\frac{dudw}{uw}=$$

$$=\frac{1}{3}\left(\int_0^\infty e^{-u^2\,\sigma_\mathbb{R}(v)^2}u^{1+s}\frac{du}{u}\right)\left(\int_0^\infty e^{-w|\,\sigma_\mathbb{C}(v)|^2}w^s\frac{dw}{w}\right)=\frac{1}{3}\cdot 1\cdot 2$$

Direct computations with change of variables $\tilde{u}=u^2\,\sigma_\mathbb{R}(v)^2$ and $\tilde{w}=w|\,\sigma_\mathbb{C}(v)|^2$ give

$$1=\int_0^\infty e^{-\tilde{u}}\frac{\tilde{u}^{s/2-1/2}}{2|\,\sigma_\mathbb{R}(v)|^{1+s}}d\tilde{u}=\frac{1}{2|\,\sigma_\mathbb{R}(v)|^{1+s}}\int_0^\infty e^{-\tilde{u}}\tilde{u}^{\left(\frac{s+1}{2}-1\right)}d\tilde{u}=\frac{1}{2|\,\sigma_\mathbb{R}(v)|^{1+s}}\Gamma\left(\frac{s+1}{2}\right)$$

$$2 = \int_0^\infty e^{-\tilde{w}} \frac{\tilde{w}^{s-1}}{|\sigma_{\mathbb{C}}(v)|^{2s-2}} \frac{d\tilde{w}}{|\sigma_{\mathbb{C}}(v)|^2} = \frac{1}{|\sigma_{\mathbb{C}}(v)|^{2s}}\Gamma(s)$$

by definition of Γ-function as Mellin transform. Putting all together

$$\int_{\langle\varepsilon\rangle} E_\psi(h,L,s) = \sum_{v\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}} \sigma_{\mathbb{R}}(v)\frac{1}{6}\frac{1}{|\sigma_{\mathbb{R}}(v)|^{1+s}}\Gamma\left(\frac{s+1}{2}\right)\frac{1}{|\sigma_{\mathbb{C}}(v)|^{2s}}\Gamma(s) =$$

$$= \frac{1}{6}\Gamma(s)\Gamma\left(\frac{s+1}{2}\right)\sum_{v\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}} \frac{\sigma_{\mathbb{R}}(v)}{|\sigma_{\mathbb{R}}(v)|}\frac{1}{\left(\sigma_{\mathbb{R}}(v)\cdot|\sigma_{\mathbb{C}}(v)|^2\right)^s}$$

which is the claim of the proof. □

Definition 13. *For an auxiliary integral ideal $\mathfrak{a}$ as in Section 1.2 define the smoothed-$\mathfrak{a}$ versions of the previous functions:*

$$\Theta_{\psi_t,\mathfrak{a}}(h,L) := \Theta_{\psi_t}(h,\mathfrak{a}^{-1}L) - N(\mathfrak{a})\Theta_{\psi_t}(h,L)$$

$$E_{\psi,\mathfrak{a}}(h,L,s) := E_\psi(h,\mathfrak{a}^{-1}L,s) - N(\mathfrak{a})E_\psi(h,L,s)$$

We can now prove the first equality by combining the results from Lemmas 4 and 5.

Proposition 8.

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0) = \frac{3}{\sqrt{\pi}}\int_{\langle\varepsilon\rangle} E_{\psi,\mathfrak{a}}(h,\mathfrak{f}\mathfrak{b}^{-1},0)$$

Proof. First recall that

$$\zeta_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},s) := \zeta_{\mathfrak{f}}(\mathfrak{a}\mathfrak{b},s) - N(\mathfrak{a})\zeta_{\mathfrak{f}}(\mathfrak{b},s)$$

This definition is given for $\mathrm{Re}(s) > 1$, the function is then extended by analytic continuation. Note that throughout this proof, many equalities are to be understood as equalities of functions extended outside their regions of definition by analytic continuation. Recall Remark 12 for a better insight.
Using Lemma 5, we get that the Right Hand Side of Proposition 8 can be rewritten as: (use throughout $L := \mathfrak{f}\mathfrak{b}^{-1}$)

$$\frac{3}{\sqrt{\pi}}\int_{\langle\varepsilon\rangle} E_{\psi,\mathfrak{a}}(h,L,0) = \frac{3}{\sqrt{\pi}}\left[\int_{<\varepsilon>} E_\psi(h,\mathfrak{a}^{-1}L,0) - N(\mathfrak{a})\int_{<\varepsilon>} E_\psi(h,L,0)\right] =$$

$$= \frac{3}{\sqrt{\pi}}\frac{1}{6}\Gamma\left(\frac{1}{2}\right)\left[\Gamma(s)\left(\sum_{\mu\in(1+\mathfrak{a}^{-1}L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\frac{\mathrm{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s} - N(\mathfrak{a})\sum_{\mu\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\frac{\mathrm{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s}\right)\right]_{s=0} =$$

$$= \left[\frac{1}{2}\Gamma(s)\sum_{\mu\in(1+\mathfrak{a}^{-1}L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\frac{\mathrm{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s} - \frac{N(\mathfrak{a})}{2}\Gamma(s)\sum_{\mu\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\frac{\mathrm{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s}\right]_{s=0} = *$$

as $\Gamma(1/2) = \sqrt{\pi}$. We now plan to use Lemma 4, which said that

$$\zeta'(1_K,L,0) = \frac{1}{2}\frac{d}{ds}\left(\sum_{\mu\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\frac{\mathrm{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s}\right)\Bigg|_{s=0} \tag{5.2}$$

where recall that we have $\zeta_{\mathfrak{f}}(\mathfrak{b},s) = N(\mathfrak{b})^{-s}\zeta(1_K,\mathfrak{f}\mathfrak{b}^{-1},s)$. Recall that partial zeta functions have (like Dedekind zeta functions) zeroes of order $\geq (r+s'-1)$ (where r is the number of real embeddings of the field K and s' the number of pairs of complex conjugate embeddings) at $s=0$, so that $\zeta(1_K,L,s)\big|_{s=0} = 0$. Thanks to that, by deriving the product and evaluating at $s=0$, we get that $\zeta'_{\mathfrak{f}}(\mathfrak{b},0) = \zeta'(1_K,\mathfrak{f}\mathfrak{b}^{-1},0)$ so that $\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0) = \zeta'(1_K,\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1},0) - N(\mathfrak{a})\zeta'(1_K,\mathfrak{f}\mathfrak{b}^{-1},0)$

Claim: For both lattices $L = \mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}$ and $L = \mathfrak{f}\mathfrak{b}^{-1}$, we have

$$\left(\Gamma(s)\cdot\sum_{\mu\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\frac{\operatorname{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s}\right)\Bigg|_{s=0} = \frac{d}{ds}\left(\sum_{\mu\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\frac{\operatorname{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s}\right)\Bigg|_{s=0}$$

This claim, together with Eq. 5.2, gives us that the chain of equalities above continue as

$$* = \zeta'(1_K,\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1},0) - N(\mathfrak{a})\zeta'(1_K,\mathfrak{f}\mathfrak{b}^{-1},0) = \zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0)$$

thus concluding the proof. It remains to prove the claim.

Proof: Call $\phi(s)$ the analytic continuation of the function $\sum_{\mu\in(1+L)/\mathscr{O}^{+,\times}_{K,\mathfrak{f}}}\frac{\operatorname{sgn}(\mu_{\mathbb{R}})}{|N(\mu)|^s}$ and write its Taylor series as $a_0 + a_1 s + \ldots$. We know that $a_0 = \phi(0) = 0$ for considerations done in Chapter 3, and $a_1 = \left(\frac{d}{ds}\phi(s)\right)\big|_{s=0}$. On the other hand, we know that $\Gamma(s) = s^{-1} + O(1)$ so that $\Gamma(s)\phi(s) = 0\cdot s^{-1} + a_1 + O(s)$; evaluating at zero we obtain

$$\Big(\Gamma(s)\phi(s)\Big)\Big|_{s=0} = a_1 = \left(\frac{d}{ds}\phi(s)\right)\Bigg|_{s=0}$$

which is exactly what we needed (as recall that also for the equality in the claim we were implicitly working with analytic continuations). ■ □

5.2 Second equality

Before introducing the function $F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b})$, we want to make some considerations on how to work when integrating on the closed geodesic $\langle\varepsilon\rangle$. We will state some results from [BCG23] which proofs need the bundle structure that we didn't define. However these results give fairly explicit formulas that we'll need in the following.

For $t_1 > 0$ define metrics $H_{a,t_1}, H_{\varepsilon_{\mathbb{R}}b,t_1} \in \mathbf{S}$ (recall a was defined in Lemma 1 and $b = \varepsilon a$):

$$H_{a,t_1}(v) = t_1^{-2}a(v)^2 + t_1|\sigma_{\mathbb{C}}(v)|^2, \quad H_{\varepsilon_{\mathbb{R}}b,t_1}(v) = t_1^{-2}\varepsilon_{\mathbb{R}}^2 b(v)^2 + t_1|\sigma_{\mathbb{C}}(v)|^2$$

for $v \in V_{\mathbb{R}}^{\vee}$. The first one yields the map

$$\iota_{\langle a\rangle}\colon\ \mathbb{R}_{>0} \to \overline{\mathbf{S}}\times\mathbf{X};\ \ t_1 \mapsto (\overline{H_{a,t_1}},\langle\mathbf{x}\rangle)$$

where $\overline{H_{a,t_1}}$ corresponds to $H_{a,t_1} \in \mathbf{S}$ in the quotient $\overline{\mathbf{S}} = \mathbf{S}/Z^0$. One can see that, contrary to the map ι of previous Section, this map doesn't have compact image in $(\overline{\mathbf{S}}\times\mathbf{X})/\Gamma_1(h)$. However, as $t_1\to\infty$, the image of $\iota_{\langle a\rangle}$ approaches the closed geodesic defined by ι. More precisely:

Lemma 6 ([BCG23], Lemma 4.3)**.** *Let $\alpha \in \Omega^1((\overline{\mathbf{S}}\times\mathbf{X})/\Gamma_1(h))$. Then*

$$\int_{\langle\varepsilon\rangle}\alpha = \int_{t_1\varepsilon_{\mathbb{R}}}^{t_1}\iota^*\alpha = \lim_{t_1\to\infty}\int_{t_1\varepsilon_{\mathbb{R}}}^{t_1}\iota^*_{\langle a\rangle}\alpha$$

Note that the first equality we already knew (and we used it in the proof of Lemma 5). We will be interested in using this result with $\alpha = E_\psi(h,L,s)$.
We pull-back via $\iota_{\langle a\rangle}$ to get another couple of important technical functions:

Definition 14. • *For positive real numbers t, t_1, define $\Theta_a(t_1,t,h,L)$ by*

$$\iota^*_{\langle a\rangle}\Theta_{\psi_t}(h,L) = \Theta_a(t_1,t,h,L)\frac{dt_1}{t_1}$$

Definitions of $\Theta_{\psi_t}(h,L)$ and $\iota_{\langle a\rangle}$ allow us to write explicitly

$$\Theta_a(t_1,t,h,L) = \sum_{v\in h+L}\psi^0(tv)_{a,t_1}, \quad \psi^0(v)_{a,t_1} := -t_1^{-1}a(v)e^{-t_1^{-2}a(v)^2 - t_1|\sigma_{\mathbb{C}}(v)|^2}$$

• *Similarly, define $\Theta_{a,\mathfrak{a}}(t_1,t,h,L)$ by*

$$\iota^*_{\langle a\rangle}\Theta_{\psi_t,\mathfrak{a}}(h,L) = \Theta_{a,\mathfrak{a}}(t_1,t,h,L)\frac{dt_1}{t_1}$$

we have explicitly $\Theta_{a,\mathfrak{a}}(t_1,t,h,L) = \Theta_a(t_1,t,h,\mathfrak{a}^{-1}L) - N(\mathfrak{a})\Theta_a(t_1,t,h,L)$.

• *The exact same definitions can be given for $\varepsilon_{\mathbb{R}}b$ instead of a.*

We give without proof another technical result that will allow us to exchange series and integrals without worrying about convergence.

Lemma 7 ([BCG23], Lemma 4.4)**.** *For any $s \in \mathbb{C}$, the Mellin transform*

$$\int_0^\infty |\Theta_{a,\mathfrak{a}}(t_1,t,h,L)|t^s\frac{dt}{t}$$

decreases rapidly as $t_1 \to 0$. In particular the form $\iota^*_{\langle a\rangle}E_{\psi,\mathfrak{a}}(h,L,s)$ is rapidly decreasing as $t_1 \to 0$.

Definition 15. $F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b}) := \Theta_{\varepsilon_{\mathbb{R}}b,\mathfrak{a}}(t_1,t,h,\mathfrak{f}\mathfrak{b}^{-1}) - \Theta_{a,\mathfrak{a}}(t_1,t,h,\mathfrak{f}\mathfrak{b}^{-1})$

Proposition 9. *The function $F_{\mathfrak{f},\mathfrak{a}}(h,\mathfrak{b})$ belongs to $L^1\left(\mathbb{R}^2_{>0}, \frac{dt_1dt}{t_1t}\right)$ and:*

$$\int_{\langle\varepsilon\rangle}E_{\psi,\mathfrak{a}}(h,\mathfrak{f}\mathfrak{b}^{-1},0) = \int_0^\infty\int_0^\infty F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b})\frac{dt_1}{t_1}\frac{dt}{t}$$

Proof. The integrability claim deeply relies on Lemma 7 as well as on Lemma 4.6 of [BCG23]. We will concentrate on proving explicitly the equality.
Lemma 6 tells us with $\alpha = E_{\psi,\mathfrak{a}}(h,\mathfrak{f}\mathfrak{b}^{-1},0) \in \Omega^1((\overline{\mathbf{S}}\times\mathbf{X})/\Gamma_1(h))$ that, after operating the change of variables $T_1 := t_1\varepsilon_{\mathbb{R}}$, where t_1 is the variable from Lemma 6,

$$\int_{\langle\varepsilon\rangle}E_{\psi,\mathfrak{a}}(h,\mathfrak{f}\mathfrak{b}^{-1},0) = \lim_{T_1\to\infty}\int_{T_1}^{T_1\varepsilon_{\mathbb{R}}^{-1}}\iota^*_{\langle a\rangle}E_{\psi,\mathfrak{a}}(h,\mathfrak{f}\mathfrak{b}^{-1},0) =$$

$$= \lim_{T_1\to\infty}\left[\int_{T_0}^{T_1\varepsilon_{\mathbb{R}}^{-1}}\iota^*_{\langle a\rangle}E_{\psi,\mathfrak{a}}(h,\mathfrak{f}\mathfrak{b}^{-1},0) - \int_{T_0}^{T_1}\iota^*_{\langle a\rangle}E_{\psi,\mathfrak{a}}(h,\mathfrak{f}\mathfrak{b}^{-1},0)\right] =$$

for any $T_0 > 0$. Now, thanks to Lemma 7, we may let $T_0 \to 0$ so that we can continue as

$$= \lim_{T_1 \to \infty} \left[\int_0^{T_1 \varepsilon_{\mathbb{R}}^{-1}} \iota_{\langle a \rangle}^* E_{\psi,\mathfrak{a}}(h, \mathfrak{f}\mathfrak{b}^{-1}, 0) - \int_0^{T_1} \iota_{\langle a \rangle}^* E_{\psi,\mathfrak{a}}(h, \mathfrak{f}\mathfrak{b}^{-1}, 0) \right] = \circledast$$

Consider now just what's between square brackets: plugging in definition of $E_{\psi,\mathfrak{a}}$, the first integral can be rewritten as

$$\int_0^{T_1 \varepsilon_{\mathbb{R}}^{-1}} \iota_{\langle a \rangle}^* E_{\psi,\mathfrak{a}}(h, \mathfrak{f}\mathfrak{b}^{-1}, 0) = \int_0^{T_1 \varepsilon_{\mathbb{R}}^{-1}} \iota_{\langle a \rangle}^* \left[\int_0^\infty \Theta_{\psi_t,\mathfrak{a}}(h, \mathfrak{f}\mathfrak{b}^{-1}) \frac{dt}{t} \right] =$$

by recalling Definition 14

$$= \int_0^{T_1 \varepsilon_{\mathbb{R}}^{-1}} \int_0^\infty \Theta_{a,\mathfrak{a}}(t_1, t, h, \mathfrak{f}\mathfrak{b}^{-1}) \frac{dt_1}{t_1} \frac{dt}{t} = \int_0^{T_1} \int_0^\infty \Theta_{a,\mathfrak{a}}(t_2 \varepsilon_{\mathbb{R}}^{-1}, t, h, \mathfrak{f}\mathfrak{b}^{-1}) \frac{dt_2}{t_2} \frac{dt}{t}$$

after operating a change of variable $t_2 := t_1 \varepsilon_{\mathbb{R}}$. By obtaining a similar expression for the second integral (this time we're already integrating on $\int_0^{T_1}$ so we don't need changes of variables) and adding the two, we get

$$\circledast = \lim_{T_1 \to \infty} \left[\int_0^{T_1} \int_0^\infty \Big(\Theta_{a,\mathfrak{a}}(t_1 \varepsilon_{\mathbb{R}}^{-1}, t, h, \mathfrak{f}\mathfrak{b}^{-1}) - \Theta_{a,\mathfrak{a}}(t_1, t, h, \mathfrak{f}\mathfrak{b}^{-1}) \Big) \frac{dt_1}{t_1} \frac{dt}{t} \right] =$$

$$= \int_0^\infty \int_0^\infty \Big(\Theta_{a,\mathfrak{a}}(t_1 \varepsilon_{\mathbb{R}}^{-1}, t, h, \mathfrak{f}\mathfrak{b}^{-1}) - \Theta_{a,\mathfrak{a}}(t_1, t, h, \mathfrak{f}\mathfrak{b}^{-1}) \Big) \frac{dt_1}{t_1} \frac{dt}{t}$$

<u>Claim:</u> $\Theta_{a,\mathfrak{a}}(t_1 \varepsilon_{\mathbb{R}}^{-1}, t, h, \mathfrak{f}\mathfrak{b}^{-1}) = \Theta_{\varepsilon_{\mathbb{R}} b,\mathfrak{a}}(t_1, t, h, \mathfrak{f}\mathfrak{b}^{-1})$
Proving this claim would mean that the term inside the above integral is $F_{\mathfrak{f},\mathfrak{a}}(t_1, t, h, \mathfrak{b})$, thus concluding the proof.
<u>Proof:</u> Using $N(\varepsilon) = 1$ and $b = \varepsilon a$ we see that $\psi^0(\varepsilon^{-1} v)_{a, t_1 \varepsilon_{\mathbb{R}}^{-1}} = \psi^0(v)_{\varepsilon_{\mathbb{R}} b, t_1}$ with ψ^0 defined in Def. 14. Indeed (recall the action of ε is by multiplication, cf. Chapter 1):

$$\psi^0(\varepsilon^{-1} v)_{a, t_1 \varepsilon_{\mathbb{R}}^{-1}} = -\varepsilon_{\mathbb{R}} t_1^{-1} a(\varepsilon^{-1} v) e^{-\varepsilon_{\mathbb{R}}^2 t_1^{-2} a(\varepsilon^{-1} v)^2 - t_1 \varepsilon_{\mathbb{R}}^{-1} |\sigma_{\mathbb{C}}(\varepsilon^{-1} v)|^2} =$$

$$= -t_1^{-1} \varepsilon_{\mathbb{R}} \varepsilon a(v) e^{-t_1^{-2} [\varepsilon_{\mathbb{R}} \varepsilon a(v)]^2 - t_1 [\varepsilon_{\mathbb{R}} |\sigma_{\mathbb{C}}(\varepsilon)|^2]^{-1} |\sigma_{\mathbb{C}}(v)|^2} =$$

$$= -t_1^{-1} \varepsilon_{\mathbb{R}} b(v) e^{-t_1^{-2} [\varepsilon_{\mathbb{R}} b(v)]^2 - t_1 \cdot 1 \cdot |\sigma_{\mathbb{C}}(v)|^2} = \psi^0(v)_{\varepsilon_{\mathbb{R}} b, t_1}$$

Note that ε stabilizes the coset $h + L$. Indeed, as said in last Section, $h \in \mathfrak{b}^{-1}$, so that we can write $v \in h + L$ as $v = \frac{1}{b'} + \frac{f}{b}$ with $f \in \mathfrak{f}$, $b, b' \in \mathfrak{b}$. So, since $\varepsilon - 1 \in \mathfrak{f}$, call it f', then $\varepsilon v - v = f'(\frac{1}{b'} + \frac{f}{b}) = \frac{f'b + f'fb'}{bb'} \in L = \mathfrak{f}\mathfrak{b}^{-1}$ by properties of ideals ($f'b, f'fb' \in \mathfrak{f}$ as e.g. $f' \in \mathfrak{f}$ and $b, fb' \in \mathscr{O}_K$; for the same reason $bb' \in \mathfrak{b}$; moreover $f'b + f'fb' \in \mathfrak{f}$ as both summands are in $\mathfrak{f}$). So $h + L$ is fixed by ε.
We can finally prove the claim

$$\Theta_{a,\mathfrak{a}}(t_1 \varepsilon_{\mathbb{R}}^{-1}, t, h, \mathfrak{f}\mathfrak{b}^{-1}) = \Theta_a(t_1 \varepsilon_{\mathbb{R}}^{-1}, t, h, \mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}) - N(\mathfrak{a}) \Theta_a(t_1 \varepsilon_{\mathbb{R}}^{-1}, t, h, \mathfrak{f}\mathfrak{b}^{-1}) =$$

$$= \sum_{v \in h + \mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}} \psi^0(tv)_{a, t_1 \varepsilon_{\mathbb{R}}^{-1}} - N(\mathfrak{a}) \sum_{v \in h + \mathfrak{f}\mathfrak{b}^{-1}} \psi^0(tv)_{a, t_1 \varepsilon_{\mathbb{R}}^{-1}} =$$

Now since ε fixes $h+\mathfrak{f}\mathfrak{b}^{-1}$ (and one can show also $h+\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}$) we can multiply v by ε^{-1} and the sum remains the same, only re-indexed

$$= \sum_{v\in h+\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}} \psi^0(t\varepsilon^{-1}v)_{a,t_1\varepsilon_{\mathbb{R}}^{-1}} - N(\mathfrak{a}) \sum_{v\in h+\mathfrak{f}\mathfrak{b}^{-1}} \psi^0(t\varepsilon^{-1}v)_{a,t_1\varepsilon_{\mathbb{R}}^{-1}} =$$

since $\psi^0(\varepsilon^{-1}v)_{a,t_1\varepsilon_{\mathbb{R}}^{-1}} = \psi^0(v)_{\varepsilon_{\mathbb{R}}b,t_1}$

$$= \sum_{v\in h+\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}} \psi^0(tv)_{\varepsilon_{\mathbb{R}}b,t_1} - N(\mathfrak{a}) \sum_{v\in h+\mathfrak{f}\mathfrak{b}^{-1}} \psi^0(tv)_{\varepsilon_{\mathbb{R}}b,t_1} =$$

$$= \Theta_{\varepsilon_{\mathbb{R}}b}(t_1,t,h,\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}) - N(\mathfrak{a})\Theta_{\varepsilon_{\mathbb{R}}b}(t_1,t,h,\mathfrak{f}\mathfrak{b}^{-1}) = \Theta_{\varepsilon_{\mathbb{R}}b,\mathfrak{a}}(t_1,t,h,\mathfrak{f}\mathfrak{b}^{-1})$$

proving the claim and ending the proof. ■ □

5.3 Third equality

In this Section we start to relate $F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b})$ to the Elliptic Gamma functions. First of all we want to rewrite it in a more convenient way. Define

$$f_{a,\mathfrak{a}}(t_1,t,v) := \sum_{j=0}^{N(\mathfrak{a})-1} \left(\psi^0\Big(t(v+j\frac{\gamma}{N(\mathfrak{a})})\Big)_{\varepsilon_{\mathbb{R}}b,t_1} - \psi^0\Big(t(v+j\frac{\gamma}{N(\mathfrak{a})})\Big)_{a,t_1} \right) -$$

$$-N(\mathfrak{a})\Big(\psi^0(tv)_{\varepsilon_{\mathbb{R}}b,t_1} - \psi^0(tv)_{a,t_1}\Big)$$

Proposition 10. *Always with $L=\mathfrak{f}\mathfrak{b}^{-1}$, we have*

$$F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b}) = \sum_{v\in h+L} f_{a,\mathfrak{a}}(t_1,t,v) = \sum_{w\in L/\mathbb{Z}\gamma}\sum_{k\in\mathbb{Z}} f_{a,\mathfrak{a}}(t_1,t,h+w+k\gamma)$$

Proof. The second equality is just a matter of re-indexing and regrouping carefully. The first one is less trivial and requires a detailed explanation. Recall definitions 15, 14.

$$F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b}) = \Theta_{\varepsilon_{\mathbb{R}}b,\mathfrak{a}}(t_1,t,h,\mathfrak{f}\mathfrak{b}^{-1}) - \Theta_{a,\mathfrak{a}}(t_1,t,h,\mathfrak{f}\mathfrak{b}^{-1}) = \Theta_{\varepsilon_{\mathbb{R}}b}(t_1,t,h,\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}) -$$

$$-N(\mathfrak{a})\Theta_{\varepsilon_{\mathbb{R}}b}(t_1,t,h,\mathfrak{f}\mathfrak{b}^{-1}) - \Theta_a(t_1,t,h,\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}) + N(\mathfrak{a})\Theta_a(t_1,t,h,\mathfrak{f}\mathfrak{b}^{-1}) =$$

$$= \sum_{w\in h+\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1}} \Big(\psi^0(tw)_{\varepsilon_{\mathbb{R}}b,t_1} - \psi^0(tw)_{a,t_1}\Big) - N(\mathfrak{a})\cdot \sum_{v\in h+\mathfrak{f}\mathfrak{b}^{-1}} \Big(\psi^0(tv)_{\varepsilon_{\mathbb{R}}b,t_1} - \psi^0(tv)_{a,t_1}\Big) = \circledast$$

Now recall we proved in the proof of Lemma 2 in Section 1.2 that $\mathfrak{a}^{-1}L = L + \frac{\gamma}{N(\mathfrak{a})}\mathbb{Z}$ with γ the third element of the oriented basis we chose for $L=\mathfrak{f}\mathfrak{b}^{-1}$. This allows us to write $w\in h+\mathfrak{a}^{-1}L$ as $w = v + j\frac{\gamma}{N(\mathfrak{a})}$ for $v\in h+L$ and some $j\in\mathbb{Z}$. More precisely we need to choose j an integer in $[0,N(\mathfrak{a})-1]$ to avoid repetitions. We can then rewrite

$$\circledast = \sum_{v\in h+L}\Bigg[\sum_{j=0}^{N(\mathfrak{a})} \Big(\psi^0\big(t(v+j\frac{\gamma}{N(\mathfrak{a})})\big)_{\varepsilon_{\mathbb{R}}b,t_1} - \psi^0\big(t(v+j\frac{\gamma}{N(\mathfrak{a})})\big)_{a,t_1}\Big) -$$

$$-N(\mathfrak{a})\cdot\left(\psi^0(tv)_{\varepsilon_{\mathbb{R}}b,t_1}-\psi^0(tv)_{a,t_1}\right)\Bigg]=\sum_{v\in h+L}f_{a,\mathfrak{a}}(t_1,t,v)$$

by its definition, proving the proposition. □

Lemma 8 ([BCG23], Lemma 4.7)**.**

$$\int_0^\infty\int_0^\infty F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b})\frac{dt_1}{t_1}\frac{dt}{t}=\sum_{w\in L/\mathbb{Z}\gamma}\int_0^\infty\int_0^\infty\left(\sum_{k\in\mathbb{Z}}f_{a,\mathfrak{a}}(t_1,t,h+w+k\gamma)\right)\frac{dt_1}{t_1}\frac{dt}{t}$$

Basically this lemma simply tells us that we can exchange the outer sum in Proposition 10 with the two integrals. To do so, one needs to make some considerations on integrability. We skip this proof that can be found in [BCG23].

We're now directly headed towards the proof of the New Kronecker's Limit Formula 3. We start by changing the variables on the RHS of Lemma 8: call

$$u:=\pi|\gamma(\mathbf{x})|^{-2}t_1^{-1}t^{-2},\ \ v:=t_1^{-1}t\ \Rightarrow\ \frac{dt_1dt}{t_1t}=\frac{dvdu}{3uv}$$

Note that u,v are real positive, since t,t_1 were. So for all $w\in L/\mathbb{Z}\gamma$, we have

$$\int_0^\infty\int_0^\infty\left(\sum_{k\in\mathbb{Z}}f_{a,\mathfrak{a}}(t_1,t,h+w+k\gamma)\right)\frac{dt_1}{t_1}\frac{dt}{t}=\frac{1}{3}\int_0^\infty\int_0^\infty\left(\sum_{k\in\mathbb{Z}}f^{\#}_{a,\mathfrak{a}}(u,v,h+w+k\gamma)\right)\frac{dudv}{uv}\tag{5.3}$$

simply by substituting $f^{\#}_{a,\mathfrak{a}}(u,v,h+w+k\gamma)$ defined as

$$f^{\#}_{a,\mathfrak{a}}(u,v,w):=\sum_{j=0}^{N(\mathfrak{a})-1}\left(\psi^{\#}\Big(u,v,w+j\frac{\gamma}{N(\mathfrak{a})}\Big)_{\varepsilon_{\mathbb{R}}b,t_1}-\psi^{\#}\Big(u,v,w+j\frac{\gamma}{N(\mathfrak{a})}\Big)_{a,t_1}\right)-$$

$$-N(\mathfrak{a})\Big(\psi^{\#}(u,v,w)_{\varepsilon_{\mathbb{R}}b,t_1}-\psi^{\#}(u,v,w)_{a,t_1}\Big)\tag{5.4}$$

with $\psi^{\#}(u,v,w)_{a,t_1}$ and $\psi^{\#}(u,v,w)_{\varepsilon_{\mathbb{R}}b,t_1}$ defined as:

$$\begin{aligned}\psi^{\#}(u,v,w)_{a,t_1}&:=-va(w)e^{-v^2a(w)^2}e^{-\pi u^{-1}\left|\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right|^2}\\ \psi^{\#}(u,v,w)_{\varepsilon_{\mathbb{R}}b,t_1}&:=-v\varepsilon_{\mathbb{R}}b(w)e^{-v^2\varepsilon_{\mathbb{R}}^2b(w)^2}e^{-\pi u^{-1}\left|\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right|^2}\end{aligned}\tag{5.5}$$

Let's see that this substitution actually works; before we had (recalling $\sigma_{\mathbb{C}}(w)=w(\mathbf{x})$ by definition of $\mathbf{x}$):

$$\psi^0(w)_{a,t_1}=-t_1^{-1}a(w)e^{-t_1^{-2}a(w)^2}e^{-t_1|w(\mathbf{x})|^2}$$

Since $t\in\mathbb{R}_{>0}$, we have that $a(tw)=ta(w)$, $\mathbf{x}(tw)=t\mathbf{x}(w)$, so that

$$\psi^0(tw)_{a,t_1}=-t_1^{-1}ta(w)e^{-t_1^{-2}t^2a(w)^2}e^{-t_1t^2|w(\mathbf{x})|^2}=$$

$$=-va(w)e^{-v^2a(w)^2}e^{-u^{-1}\cdot\frac{\pi}{|\gamma(\mathbf{x})|^2}|w(\mathbf{x})|^2}=\psi^{\#}(u,v,w)_{a,t_1}$$

And a similar result is obtained for $\psi^{\#}(u,v,w)_{\varepsilon_{\mathbb{R}}b,t_1}$ and for $w+j\frac{\gamma}{N(\mathfrak{a})}$ instead of w. We can then conclude that

$$f_{a,\mathfrak{a}}(t_1,t,h+w+k\gamma)=f^{\#}_{a,\mathfrak{a}}(u,v,h+w+k\gamma),\ \forall w\in L/\mathbb{Z}\gamma,\ \forall k\in\mathbb{Z}$$

so that Eq. 5.3 holds $\forall w\in L/\mathbb{Z}\gamma$.
Recall the formula for Poisson summation, which gives for $\alpha\in\mathbb{C}$, $c\in\mathbb{R}_{>0}$:

$$\sum_{k\in\mathbb{Z}}e^{-\pi c^{-1}(k+\alpha)^2}=\sqrt{c}\sum_{k\in\mathbb{Z}}e^{-\pi k^2c+2\pi ik\alpha} \tag{5.6}$$

So that we get (note first that since $k\in\mathbb{Z}\subset\mathbb{R}$, $\left|k+\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right|^2=\left(k+\operatorname{Re}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)\right)^2+\operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2$ and the latter doesn't depend on k):

$$\sum_{k\in\mathbb{Z}}e^{-\pi u^{-1}\left|k+\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right|^2}=e^{-\pi u^{-1}\operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2}\sum_{k\in\mathbb{Z}}e^{-\pi u^{-1}\left(k+\operatorname{Re}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)\right)^2}=$$

using Eq. 5.6 with $c=u$, $\alpha=\operatorname{Re}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)$

$$=\sqrt{u}e^{-\pi u^{-1}\operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2}\sum_{k\in\mathbb{Z}}e^{-\pi k^2u+2\pi ik\operatorname{Re}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)} \tag{5.7}$$

Now we want to work with $\sum_{k\in\mathbb{Z}}f^{\#}_{a,\mathfrak{a}}(u,v,h+w+k\gamma)$ and rearrange it thanks to Eq. 5.7. Let's first of all do some considerations based on facts from Section 1.2. By definition of γ, straight before Lemma 2, we recall that $\gamma(a)=\gamma(b)=0$; moreover in the proof of Lemma 2 we saw that $\lambda=n\gamma$ for some positive integer n, so that $h=\lambda/q$ is a rational multiple of γ. In particular $h(a)=h(b)=0$.
When plugging in the definition of $f^{\#}_{a,\mathfrak{a}}$ in the sum we want to compute, we obtain inside the formula for $\psi^{\#}(u,v,h+w+k\gamma)_{a,t_1}$ a term $a(h+w+k\gamma)$ (see Eq. 5.5); by linearity, together with the previous considerations, this simply equals $a(w)$. In the same way also $a(h+w+(k+j/N(\mathfrak{a}))\gamma)=a(w)$ and the same replacing a with $\varepsilon_{\mathbb{R}}b$. We then have by plugging in definitions of $\psi^{\#}$ (Eq. 5.5) in Eq. 5.4 and thanks to considerations above (denote $N:=N(\mathfrak{a})$):

$$\begin{aligned}\sum_{k\in\mathbb{Z}}f^{\#}_{a,\mathfrak{a}}(u,v,h+w+k\gamma)&=\sum_{k\in\mathbb{Z}}\Bigg\{\sum_{j=0}^{N-1}\Bigg[\Bigg(-v\varepsilon_{\mathbb{R}}b(w)e^{-v^2\varepsilon_{\mathbb{R}}^2b(w)^2}e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k+\frac{j}{N}\right|^2}\Bigg)-\\
&-\Bigg(-va(w)e^{-v^2a(w)^2}e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k+\frac{j}{N}\right|^2}\Bigg)\Bigg]-N\cdot\Bigg[\Bigg(-v\varepsilon_{\mathbb{R}}b(w)e^{-v^2\varepsilon_{\mathbb{R}}^2b(w)^2}e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k\right|^2}\Bigg)-\\
&-\Bigg(-va(w)e^{-v^2a(w)^2}e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k\right|^2}\Bigg)\Bigg]\Bigg\}=\circledast\end{aligned}$$

Denote

$$A:=-va(w)e^{-v^2a(w)^2},\quad B:=-v\varepsilon_{\mathbb{R}}b(w)e^{-v^2\varepsilon_{\mathbb{R}}^2b(w)^2}$$

So that

$$\begin{aligned}\circledast&=\sum_{k\in\mathbb{Z}}\Bigg\{\sum_{j=0}^{N-1}\Bigg[(B-A)e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k+\frac{j}{N}\right|^2}\Bigg]-N\cdot(B-A)e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k\right|^2}\Bigg\}=\\
&=-N\cdot(B-A)\cdot\sum_{k\in\mathbb{Z}}\Bigg\{e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k\right|^2}-\frac{1}{N}\sum_{j=0}^{N-1}e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k+\frac{j}{N}\right|^2}\Bigg\}=\star\end{aligned}$$

We now want to apply Eq. 5.7: recall that h is a rational multiple of γ, so that $\mathrm{Im}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right) = \mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)$. The second sum needs to be analyzed individually, note that $\sum_{k\in\mathbb{Z}}\sum_{j=0}^{N-1}\left(k+\frac{j}{N}\right) = \sum_{k\in\mathbb{Z}}\frac{k}{N}$, so we can write

$$\sum_{k\in\mathbb{Z}}\frac{1}{N}\sum_{j=0}^{N-1} e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k+\frac{j}{N}\right|^2} = \frac{1}{N}\sum_{k\in\mathbb{Z}} e^{-\pi u^{-1}\left|\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+\frac{k}{N}\right|^2} = \frac{1}{N}\sum_{k\in\mathbb{Z}} e^{-\pi\frac{u^{-1}}{N^2}\left|\frac{N(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}+k\right|^2} =$$

By calling $u' := N^2 u \Rightarrow u'^{-1} = u^{-1}/N^2$ and using Eq. 5.7:

$$= \frac{1}{N}\sqrt{N^2 u}\, e^{-\pi u^{-1}N^{-2}\,\mathrm{Im}\left(\frac{N(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2}\sum_{k\in\mathbb{Z}} e^{-\pi k^2 u N^2 + 2\pi i k\,\mathrm{Re}\left(\frac{N(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)} =$$

$$= \sqrt{u}\, e^{-\pi u^{-1}\,\mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2}\sum_{k\in\mathbb{Z}} e^{-\pi(Nk)^2 u + 2\pi i N k\,\mathrm{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)}$$

So, by putting this expression in the formula above (and using Eq. 5.7 for the first sum):

$$\star = -N(B-A)u^{1/2} e^{-\pi u^{-1}\,\mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2}\left[\sum_{k\in\mathbb{Z}} e^{-\pi k^2 u + 2\pi i k\,\mathrm{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)} - \sum_{k\in\mathbb{Z}} e^{-\pi(Nk)^2 u + 2\pi i N k\,\mathrm{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right] =$$

$$= -N\left(va(w)e^{-v^2 a(w)^2} - v\varepsilon_{\mathbb{R}} b(w) e^{-v^2\varepsilon_{\mathbb{R}}^2 b(w)^2}\right)u^{1/2}\sum_{\substack{k\in\mathbb{Z}\\ N\nmid k}} e^{-\pi\left(k^2 u + \mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2 u^{-1}\right)} e^{2\pi i k\,\mathrm{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)} \tag{5.8}$$

We want to simplify as soon as possible this horrible expression, but we can appreciate how it contains two clearly separated parts wrt the variables u, v, which will reveal really useful in computing the integral. Recall that Eq. 5.8 is just a rewriting of the integrand in Eq. 5.3. Note that by changing variable $x := yv$

$$\int_0^\infty y e^{-y^2 v^2} dv = \mathrm{sgn}(y)\int_0^\infty e^{-x^2} dx = \mathrm{sgn}(y)\frac{\sqrt{\pi}}{2}$$

This easy computation helps us evaluate the integral in v. Explicitly

$$\int_0^\infty \left(va(w)e^{-v^2 a(w)^2} - v\varepsilon_{\mathbb{R}} b(w) e^{-v^2\varepsilon_{\mathbb{R}}^2 b(w)^2}\right)\frac{dv}{v} = \int_0^\infty a(w)e^{-v^2 a(w)^2}dv - \int_0^\infty \varepsilon_{\mathbb{R}} b(w) e^{-v^2\varepsilon_{\mathbb{R}}^2 b(w)^2}dv =$$

$$= \mathrm{sgn}(a(w))\frac{\sqrt{\pi}}{2} - \mathrm{sgn}(b(w))\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}\Big(\mathrm{sgn}(a(w)) - \mathrm{sgn}(b(w))\Big)$$

We didn't consider the sign of $\varepsilon_{\mathbb{R}}$, which is always positive by construction so $\mathrm{sgn}(\varepsilon_{\mathbb{R}} b(w)) = \mathrm{sgn}(b(w))$. Then the only non-zero terms in Eq. 5.3 are those with $\mathrm{sgn}(a(w)) \neq \mathrm{sgn}(b(w))$. To ease up notation, we give the following

Definition 16.
- *$\overline{C}_{+-} := \overline{C_{+-}(a,b)} = \{\delta \in L \mid \delta(a) \geq 0,\ \delta(b) \leq 0\}$; $\overline{C}_{-+} := \overline{C_{-+}(a,b)} = \{\delta \in L \mid \delta(a) \leq 0,\ \delta(b) \geq 0\}$ are the closures of the cones $C_{+-} := C_{+-}(a,b)$ and $C_{-+} := C_{-+}(a,b)$ of Chapter 1.*

- *For all $w \in L/\mathbb{Z}\gamma$, define $\chi(w) := (\mathrm{sgn}(a(w)) - \mathrm{sgn}(b(w)))/2$ with the convention $\mathrm{sgn}(0) = 0$.*

Note that χ has support in $\overline{C}_{+-} \cup \overline{C}_{-+}$; in particular, the only terms from Eq. 5.3 which are not zero come from $w \in (L \cap (\overline{C}_{+-} \cup \overline{C}_{-+}))/\mathbb{Z}\gamma$.
We can then rewrite the sum from Lemma 8 as $I_{+-}(h) + I_{-+}(h)$ where

$$I_{+-}(h) = -\frac{\sqrt{\pi}}{3} N \sum_{w \in (L \cap \overline{C}_{+-})/\mathbb{Z}\gamma} \chi(w) \sum_{\substack{k \in \mathbb{Z} \\ N \nmid k}} e^{2\pi i k \operatorname{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)} \int_0^\infty e^{-\pi\left(k^2 u + \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2 u^{-1}\right)} u^{1/2} \frac{du}{u} \tag{5.9}$$

$$I_{-+}(h) = -\frac{\sqrt{\pi}}{3} N \sum_{w \in (L \cap \overline{C}_{-+})/\mathbb{Z}\gamma} \chi(w) \sum_{\substack{k \in \mathbb{Z} \\ N \nmid k}} e^{2\pi i k \operatorname{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)} \int_0^\infty e^{-\pi\left(k^2 u + \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2 u^{-1}\right)} \frac{du}{u^{1/2}} \tag{5.10}$$

obtained simply by putting together all we said so far and pulling out from the integral all the terms that don't depend from u.[1] From now on we lose the notation $L \cap \overline{C}_{+-}$, resp. $L \cap \overline{C}_{-+}$ in all sums to write simply $\overline{C}_{+-}$, resp. $\overline{C}_{-+}$ as all the cones are already contained in L.
We conclude the Section by computing the above integrals (which are equal).

Notation 4. *Denote* $e(z) := e^{2\pi i z}$

Definition 17. *For a, b positive real numbers, define the McDonald-Bessel function*

$$K_s(a,b) := \int_0^\infty e^{-(a^2 t + b^2/t)} t^s \frac{dt}{t}$$

Note that

$$\int_0^\infty e^{-\pi(k^2 u + y^2 u^{-1})} \frac{du}{u^{1/2}} = K_{1/2}(\pi^{1/2}|k|, \pi^{1/2}|y|)$$

We will need some of its properties (see [Lan87], Chapter 20.3): Property "K2" tells us $K_s(a,b) = \left(\frac{b}{a}\right)^s K_s(ab)$; Property "K5" gives $K_{1/2}(c) = \sqrt{\frac{\pi}{c}} e^{-2c}$. Uniting the two, we get

$$K_{1/2}(\pi^{1/2}|k|, \pi^{1/2}|y|) = \left|\frac{y}{k}\right|^{1/2} K_{1/2}(\pi|ky|) = \left|\frac{y}{k}\right|^{1/2} \sqrt{\frac{\pi}{\pi|ky|}} e^{-2\pi|ky|} = \frac{1}{|k|} e^{-2\pi|ky|}$$

With $y = \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)$ we get

$$e^{2\pi i k \operatorname{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)} \int_0^\infty e^{-\pi\left(k^2 u + \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)^2 u^{-1}\right)} \frac{du}{u^{1/2}} = e^{2\pi i k \operatorname{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)} \frac{1}{|k|} e^{-2\pi\left|k \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)\right|} =$$

Dividing in cases regarding the sign of $k \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) \in \mathbb{R}$

$$= \begin{cases} \frac{1}{|k|} e^{2\pi i k\left[\operatorname{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right) + i \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)\right]} & \text{if } k \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) > 0 \\ \frac{1}{|k|} e^{2\pi i k\left[\operatorname{Re}\left(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right) - i \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right)\right]} & \text{if } k \operatorname{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) < 0 \end{cases} =$$

[1] The infinite sums indexed by k converge absolutely since the condition $\mathbf{x} \in U_a \cap U_b$ from Chapter 1 guarantees the function $w \mapsto |\operatorname{Im}(w(\mathbf{x})/\gamma(\mathbf{x})|$ grows at least linearly in the cones $\overline{C}_{+-}$, $\overline{C}_{-+}$. The negative exponent gives rapid decrease, hence absolute convergence. It is then legit to exchange series and integral as we did.

recalling that $\mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) = \mathrm{Im}\left(\frac{(w+h)(\mathbf{x})}{\gamma(\mathbf{x})}\right)$ and Notation 4

$$= \begin{cases} \frac{1}{|k|} e\left(k \cdot \overline{\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}}\right) \text{ if } k\,\mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) > 0 \\ \frac{1}{|k|} e\left(k \cdot \frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right) \text{ if } k\,\mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) < 0 \end{cases}$$

Recall that $\mathbf{x} \in U_a \cap U_b$ so that $\mathrm{Im}\left(\frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})}\right) < 0$ and $\mathrm{Im}\left(\frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})}\right) > 0$ (Notations as in Chapter 1). So if $w \in C_{+-} \Rightarrow w(a) > 0, w(b) \leq 0 \Rightarrow w = j\alpha + l\beta$ with $j > 0, l \leq 0$ so that $\mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) = \mathrm{Im}\left(\frac{j\alpha(\mathbf{x})+l\beta(\mathbf{x})}{\gamma(\mathbf{x})}\right) = j\,\mathrm{Im}\left(\frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})}\right) + l\,\mathrm{Im}\left(\frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})}\right) < 0$. Analogously, if $w \in C_{-+}$ then $\mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) > 0$.
We can then rewrite I_{+-}, I_{-+} dividing the cases k positive and negative. Since N divides 0, we can consider the sum only on $k \geq 1$, $N \nmid k$ and put in the summand both k and $-k$; this way we get explicitly that: for $w \in C_{+-}$, "$k\,\mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) > 0 \Leftrightarrow k < 0$" and for $w \in C_{-+}$, "$k\,\mathrm{Im}\left(\frac{w(\mathbf{x})}{\gamma(\mathbf{x})}\right) > 0 \Leftrightarrow k > 0$".
Thus substituting in Equations 5.9, 5.10:

$$I_{+-}(h) = \frac{\sqrt{\pi}}{3} \sum_{w \in \overline{C}_{+-}/\mathbb{Z}\gamma} \chi(w) \sum_{\substack{k\geq 1 \\ N\nmid k}} -\frac{N}{k}\left(e\left(k \cdot \overline{\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}}\right) + e\left(-k \cdot \frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)\right) \tag{5.11}$$

$$I_{-+}(h) = \frac{\sqrt{\pi}}{3} \sum_{w \in \overline{C}_{-+}/\mathbb{Z}\gamma} \chi(w) \sum_{\substack{k\geq 1 \\ N\nmid k}} -\frac{N}{k}\left(e\left(-k \cdot \overline{\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}}\right) + e\left(k \cdot \frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}\right)\right) \tag{5.12}$$

In next Section we will further develop these expressions, going towards a formula that uses Elliptic Gamma Functions.

5.4 Fourth equality

We pick up straight from Equations 5.11, 5.12.
Note that $\overline{e(z)} = e(-\overline{z})$. Moreover, in the inverse process to the one we did in Eq. 5.8, we go back to writing $\sum_{k\geq 1, N\nmid k} f(k) = \sum_{k \geq 1}(f(k) - f(Nk))$, obtaining for $I_{+-}(h)$ (denote $\Phi := \frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}$):

$$I_{+-}(h) = \frac{\sqrt{\pi}}{3} \sum_{w \in \overline{C}_{+-}/\mathbb{Z}\gamma} \chi(w) \sum_{k\geq 1} \left[-\frac{N}{k}\left(e(k\overline{\Phi}) + e(-k\Phi)\right) + \frac{N}{Nk}\left(e(Nk\overline{\Phi}) + e(-Nk\Phi)\right)\right] =$$

$$= \frac{\sqrt{\pi}}{3} \sum_{w \in \overline{C}_{+-}/\mathbb{Z}\gamma} \chi(w) \sum_{k\geq 1} \left[-\frac{N}{k}\left(\overline{e(-k\Phi)} + e(-k\Phi)\right) + \frac{1}{k}\left(\overline{e(-Nk\Phi)} + e(-Nk\Phi)\right)\right]$$

Now thanks to Taylor series, we know

$$\sum_{k\geq 1} \frac{\overline{e(-k\Phi)}}{k} = \sum_{k\geq 1} \frac{(\overline{e(-\Phi)})^k}{k} = \log \frac{1}{1 - \overline{e(-\Phi)}}$$

And analogously for the other terms, this way we get

$$\sum_{k\geq 1} \left[-\frac{N}{k}\left(\overline{e(-k\Phi)} + e(-k\Phi)\right) + \frac{1}{k}\left(\overline{e(-Nk\Phi)} + e(-Nk\Phi)\right)\right] =$$

$$= -N\log\frac{1}{1-\overline{e(-\Phi)}} - N\log\frac{1}{1-e(-\Phi)} + \log\frac{1}{1-\overline{e(-N\Phi)}} + \log\frac{1}{1-e(-N\Phi)} =$$

$$= \log(1-\overline{e(-\Phi)})^N + \log(1-e(-\Phi))^N - \log(1-\overline{e(-N\Phi)}) - \log(1-e(-N\Phi)) =$$

$$= \log\left[\frac{(1-\overline{e(-\Phi)})^N(1-e(-\Phi))^N}{(1-\overline{e(-N\Phi)})(1-e(-N\Phi))}\right] = \log\left|\frac{(1-e(-\Phi))^N}{(1-e(-N\Phi))}\right|^2$$

so that

$$I_{+-}(h) = \frac{\sqrt{\pi}}{3}\sum_{w\in\overline{C}_{+-}/\mathbb{Z}\gamma}\chi(w)\log\left|\frac{(1-e(-\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(-N\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}))}\right|^2 \tag{5.13}$$

and analogously

$$I_{-+}(h) = \frac{\sqrt{\pi}}{3}\sum_{w\in\overline{C}_{-+}/\mathbb{Z}\gamma}\chi(w)\log\left|\frac{(1-e(\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(N\frac{(h+w)(\mathbf{x})}{\gamma(\mathbf{x})}))}\right|^2 \tag{5.14}$$

Now, use the fact that χ is odd (clear from Def. 16) and replace w with $-w$ (note that if $w\in\overline{C}_{+-}$ then $-w\in\overline{C}_{-+}$ and vice versa, by computing against a and b). So we sum in I_{+-} over $w\in C_{-+}/\mathbb{Z}\gamma$ and then use $-w$ instead of w (and vice versa for I_{-+}):

$$I_{+-}(h) = -\frac{\sqrt{\pi}}{3}\sum_{w\in\overline{C}_{-+}/\mathbb{Z}\gamma}\chi(w)\log\left|\frac{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}\right|^2 \tag{5.15}$$

$$I_{-+}(h) = -\frac{\sqrt{\pi}}{3}\sum_{w\in\overline{C}_{+-}/\mathbb{Z}\gamma}\chi(w)\log\left|\frac{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}\right|^2 \tag{5.16}$$

Remark 21. *Let $\mathbb{1}_X$ denote the indicator function of a set X. Then*

$$\chi|_{\overline{C}_{+-}} = \mathbb{1}_{C_{+-}} - \frac{1}{2}\mathbb{1}_{\overline{C}_{+-}\cap H(b)} + \frac{1}{2}\mathbb{1}_{\overline{C}_{+-}\cap H(a)};\ \chi|_{\overline{C}_{-+}} = -\mathbb{1}_{C_{-+}} - \frac{1}{2}\mathbb{1}_{\overline{C}_{-+}\cap H(b)} + \frac{1}{2}\mathbb{1}_{\overline{C}_{-+}\cap H(a)} \tag{5.17}$$

Proof. Recall $H(a)$, $H(b)$ are the kernel of $a\in\Lambda$, resp. b, inside L, i.e. the elements of L that are 0 when computed against a, resp. b. The result is then just a direct computation, see briefly the first equality: if $w\in\overline{C}_{+-}$, the cases are:

- $w(a) = w(b) = 0 \Rightarrow w\in\overline{C}_{+-}\cap H(a)\cap H(b) \Rightarrow \chi(w) = 0 = 0-\frac{1}{2}+\frac{1}{2} = RHS(w)$;
- $w(a) > 0, w(b) = 0 \Rightarrow w\in\overline{C}_{+-}\cap H(b)\setminus H(a) \Rightarrow \chi(w) = \frac{1-0}{2} = 1-\frac{1}{2}+\frac{1}{2}\cdot 0 = RHS(w)$;
- $w(a) = 0, w(b) < 0 \Rightarrow w\in\overline{C}_{+-}\cap H(a)\setminus H(b) \Rightarrow \chi(w) = \frac{0-(-1)}{2} = 0-\frac{1}{2}\cdot 0+\frac{1}{2} = RHS(w)$;
- $w(a) > 0, w(b) < 0 \Rightarrow w\in\overline{C}_{+-}\setminus(H(a)\cup H(b)) \Rightarrow \chi(w) = \frac{1-(-1)}{2} = 1 = 1-\frac{1}{2}\cdot 0+\frac{1}{2}\cdot 0 = RHS(w)$.

The equality for $w\in\overline{C}_{-+}$ is done in the same way. $\square$

Eq. 5.17 will help us to write explicitly $\chi(w)$ when summing $I_{+-}(h)$ and $I_{-+}(h)$. We will divide the sum in two part. First we'll sum on the vectors w lying on the boundary of the cone $\overline{C}_{+-}\cup\overline{C}_{-+}$ (i.e. considering the contribution of the second and third term in both equalities in 5.17) and then on $C_{+-}\cup C_{-+}$ (i.e. considering the first term). The result will be:

Proposition 11.

$$\frac{3}{\sqrt{\pi}}(I_{+-}(h)+I_{-+}(h)) = \log\left|\frac{\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}L)}{\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};L)^N}\right|^2 - \log\left|\frac{\theta_0^{(N)}(z_0,\sigma)}{\theta_0^{(N)}(z_0,-\tau)}\right| \tag{5.18}$$

where $\theta_0^{(N)},\sigma,\tau,z_0$ will be defined in the proof.

Proof. By substituting Equations 5.15, 5.16 we get

$$\frac{3}{\sqrt{\pi}}(I_{+-}(h)+I_{-+}(h)) = -\Bigg[\sum_{w\in\overline{C}_{-+}/\mathbb{Z}\gamma}\chi(w)\log\left|\frac{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}\right|^2+$$

$$+\sum_{w\in\overline{C}_{+-}/\mathbb{Z}\gamma}\chi(w)\log\left|\frac{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}\right|^2\Bigg] =$$

Indicate

$$\Psi_{-+} := \left|\frac{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}\right|; \quad \Psi_{+-} := \left|\frac{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}\right|$$

Thanks to Eq. 5.17:

$$= -\Bigg[-\sum_{w\in C_{-+}/\mathbb{Z}\gamma}\log\Psi_{-+}^2 - \frac{1}{2}\sum_{w\in(\overline{C}_{-+}\cap H(b))/\mathbb{Z}\gamma}\log\Psi_{-+}^2 + \frac{1}{2}\sum_{w\in(\overline{C}_{-+}\cap H(a))/\mathbb{Z}\gamma}\log\Psi_{-+}^2 -$$

$$+\sum_{w\in C_{+-}/\mathbb{Z}\gamma}\log\Psi_{+-}^2 - \frac{1}{2}\sum_{w\in(\overline{C}_{+-}\cap H(b))/\mathbb{Z}\gamma}\log\Psi_{+-}^2 + \frac{1}{2}\sum_{w\in(\overline{C}_{+-}\cap H(a))/\mathbb{Z}\gamma}\log\Psi_{+-}^2\Bigg] =$$

simplifying the denominators with the 2-powers in the argument of the logarithm

$$\begin{aligned} = 1 + 2 := &\Bigg[\sum_{w\in C_{-+}/\mathbb{Z}\gamma}\log\Psi_{-+}^2 - \sum_{w\in C_{+-}/\mathbb{Z}\gamma}\log\Psi_{+-}^2\Bigg] + \Bigg[\sum_{w\in(\overline{C}_{-+}\cap H(b))/\mathbb{Z}\gamma}\log\Psi_{-+} + \\ &+\sum_{w\in(\overline{C}_{+-}\cap H(b))/\mathbb{Z}\gamma}\log\Psi_{+-} - \sum_{w\in(\overline{C}_{-+}\cap H(a))/\mathbb{Z}\gamma}\log\Psi_{-+} - \sum_{w\in(\overline{C}_{+-}\cap H(a))/\mathbb{Z}\gamma}\log\Psi_{+-}\Bigg] \end{aligned} \tag{5.19}$$

<u>Claim:</u> [1]

$$2 = -\log\left|\frac{\theta_0^{(N)}(z_0,\sigma)}{\theta_0^{(N)}(z_0,-\tau)}\right|$$

<u>Proof:</u> We will define in this part the missing elements that appear in the statement.
Pick $\alpha\in H(b)$ such that $\alpha(a)>0$ and $L\cap H(b)=\mathbb{Z}\alpha+\mathbb{Z}\gamma$. Then (γ,α) is an oriented basis of $H(b)$ and so $\tau := \frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})}$ has negative imaginary part. Similarly choose $\beta\in H(a)$ such that $\beta(b)>0$ and $L\cap H(a)=\mathbb{Z}\beta+\mathbb{Z}\gamma$; $\sigma := \frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})}$ has positive imaginary part since (β,γ) is an oriented basis of $H(a)$. (We make all these considerations without proofs for we actually already proved them in the construction done in the proof of Proposition 1.)

Notation 5. *Denote $q_z := e^{2\pi i z}$; $z_0 := \frac{h(\mathbf{x})}{\gamma(\mathbf{x})} = \frac{n}{q}$ where n was the positive integer such that $\lambda = n\gamma$ whose existence was proven in the proof of Lemma 2. Finally define*

$$\theta_0(z,-\tau) := \prod_{m\geq 0}(1-q_z q_{-\tau}^m)(1-q_z^{-1}q_{-\tau}^{m+1})$$

and similarly $\theta_0(z,\sigma)$.

The contribution coming to the boundary term from $\overline{C}_{-+}\cap H(b)$ is:

$$\sum_{\substack{w\in(\overline{C}_{-+}\cap H(b))/\mathbb{Z}\gamma \\ w\neq 0}} \log\left|\frac{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}\right| \overset{!}{=} \sum_{m\geq 1}\log\left|\frac{(1-q_{z_0}^{-1}q_{-\tau}^m)^N}{1-q_{Nz_0}^{-1}q_{-N\tau}^m}\right| \tag{5.20}$$

where we omitted the term $w=0$ from the sum, for it would cancel out with the corresponding term $w=0$ in $H(a)$ (in the last sum of Eq. 5.19) that has opposite sign. We now explain the equality in Eq. 5.20: $w\in\overline{C}_{-+}\cap H(b)$ can be written as $w=m'\alpha+l\gamma$ with $m'<0$ as $w(a)<0$ (we removed the case $w=0$) and $\alpha(a)>0$; since we quotient modulo $\mathbb{Z}\gamma$, we can assume $w=m'\alpha$. Then we have

$$e\Big(\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}\Big) = e\Big(\frac{-h(\mathbf{x})}{\gamma(\mathbf{x})}\Big)\cdot e\Big(\frac{m'\alpha(\mathbf{x})}{\gamma(\mathbf{x})}\Big) = q_{z_0}^{-1}\cdot e\Big(\frac{-m'(-\alpha)(\mathbf{x})}{\gamma(\mathbf{x})}\Big) = q_{z_0}^{-1}\cdot q_{-\tau}^{-m'}$$

$$e\Big(\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}\Big) = e\Big(\frac{-Nh(\mathbf{x})}{\gamma(\mathbf{x})}\Big)\cdot e\Big(\frac{Nm'\alpha(\mathbf{x})}{\gamma(\mathbf{x})}\Big) = q_{Nz_0}^{-1}\cdot e\Big(\frac{-m'(-N\alpha)(\mathbf{x})}{\gamma(\mathbf{x})}\Big) = q_{Nz_0}^{-1}\cdot q_{-N\tau}^{-m'}$$

Consider instead of m', $m := -m' > 0$. Summing over all $w\in\overline{C}_{-+}\cap H(b)$ is then equivalent to summing over all $m>0$, which gets Eq. 5.20.
In the same way, the contribute given by $\overline{C}_{+-}\cap H(b)$ is:

$$\sum_{w\in(\overline{C}_{+-}\cap H(b))/\mathbb{Z}\gamma} \log\left|\frac{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}\right| = \sum_{m\geq 0}\log\left|\frac{(1-q_{z_0}q_{-\tau}^m)^N}{1-q_{Nz_0}q_{-N\tau}^m}\right| \tag{5.21}$$

This time the condition $w(a)\geq 0 \Rightarrow w=m\alpha+l\gamma$, $m\geq 0$; we don't care about γ part thanks to the quotient, so we get $\frac{-w(\mathbf{x})}{\gamma(\mathbf{x})} = m\cdot(-\tau)$ so the sum is indexed by $m\geq 0$.
Summing Equations 5.20 and 5.21 we get the total contribution from vectors $w\in(\overline{C}_{+-}\cup\overline{C}_{-+})\cap H(b)\setminus\{w=0\}$ which is (re-index Eq. 5.20 as $m\geq 0$ writing inside $m+1$ instead of m)

$$\sum_{m\geq 0}\left(\log\left|\frac{(1-q_{z_0}^{-1}q_{-\tau}^{m+1})^N}{1-q_{Nz_0}^{-1}q_{-N\tau}^{m+1}}\right| + \log\left|\frac{(1-q_{z_0}q_{-\tau}^m)^N}{1-q_{Nz_0}q_{-N\tau}^m}\right|\right) = \log\prod_{m\geq 0}\left|\frac{(1-q_{z_0}^{-1}q_{-\tau}^{m+1})^N\cdot(1-q_{z_0}q_{-\tau}^m)^N}{(1-q_{Nz_0}^{-1}q_{-N\tau}^{m+1})(1-q_{Nz_0}q_{-N\tau}^m)}\right| =$$

$$= \log\left|\frac{\theta_0(z_0,-\tau)^N}{\theta_0(Nz_0,-N\tau)}\right| = \log|\theta_0^{(N)}(z_0,-\tau)|$$

where $\theta_0^{(N)}(z,-\tau) := \frac{\theta_0(z,-\tau)^N}{\theta_0(Nz,-N\tau)}$ and $\theta_0^{(N)}(z,\sigma)$ is defined similarly.
As intersecting with $H(b)$ we used $-\tau$, we'll use σ when intersecting with $H(a)$. Completely similar computations give that the contribution given to the boundary term by vectors in $(\overline{C}_{+-}\cup\overline{C}_{-+})\cap H(a)\setminus\{w=0\}$ (recall $w=0$ cancels out with the one in $H(b)$) give $-\log|\theta_0^{(N)}(z,\sigma)|$, since the coefficient is negative.

By adding both contributions to the boundary term $H(a)$ and $H(b)$, we get the second summand from Eq. 5.18:

$$2 = -\log\left|\frac{\theta_0^{(N)}(z_0,\sigma)}{\theta_0^{(N)}(z_0,-\tau)}\right|$$

■

Claim: [2]

$$1 = \log\left|\frac{\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}L)}{\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};L)^N}\right|^2$$

Proof: We use the $-$ sign before the sum $w\in C_{+-}/\mathbb{Z}\gamma$ in Eq. 5.19 to switch numerator and denominator in the argument of the logarithm:

$$1 = \sum_{w\in C_{-+}/\mathbb{Z}\gamma}\log\left|\frac{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}{(1-e(\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}\right|^2 + \sum_{w\in C_{+-}/\mathbb{Z}\gamma}\log\left|\frac{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})}))}{(1-e(-\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})}))^N}\right|^2 =$$

$$= \log\left|\prod_{w\in C_{-+}/\mathbb{Z}\gamma}\frac{(1-e^{2\pi i(\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})})})^N}{(1-e^{2\pi i(\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})})})}\cdot\prod_{w\in C_{+-}/\mathbb{Z}\gamma}\frac{(1-e^{-2\pi i(\frac{-h(\mathbf{x})+w(\mathbf{x})}{(\gamma/N)(\mathbf{x})})})}{(1-e^{-2\pi i(\frac{-h(\mathbf{x})+w(\mathbf{x})}{\gamma(\mathbf{x})})})^N}\right|^2 =$$

$$= \log\left|\left(\frac{\prod_{w\in C_{+-}/\mathbb{Z}\gamma}(1-e^{-2\pi i(\frac{w(\mathbf{x})-h(\mathbf{x})}{\gamma(\mathbf{x})})})}{\prod_{w\in C_{-+}/\mathbb{Z}\gamma}(1-e^{2\pi i(\frac{w(\mathbf{x})-h(\mathbf{x})}{\gamma(\mathbf{x})})})}\right)^{-N}\cdot\frac{\prod_{w\in C_{+-}/\mathbb{Z}\gamma}(1-e^{-2\pi i(\frac{w(\mathbf{x})-h(\mathbf{x})}{(\gamma/N)(\mathbf{x})})})}{\prod_{w\in C_{-+}/\mathbb{Z}\gamma}(1-e^{2\pi i(\frac{w(\mathbf{x})-h(\mathbf{x})}{(\gamma/N)(\mathbf{x})})})}\right|^2 =$$

Now, recall we saw in the proof of Lemma 2 that $\mathfrak{a}^{-1}L = L+\frac{\gamma}{N}\mathbb{Z}$, so that the second element in the above product is $\Gamma_{a,b}(h(\mathbf{x}),(\mathbf{x});\mathfrak{a}^{-1}L)$, we then conclude thanks to the definitions from Chapter 1 (Eq. 1.1)

$$= \log\left|(\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};L))^{-N}\cdot\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}L)\right|^2 = \log\left|\frac{\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}L)}{(\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};L))^N}\right|^2$$

This gives Claim 2 and finishes the proof. ■

□

Remark 22. *The proof of this Proposition in the main article [BCG23], page 34, presents a "+" sign in front of the boundary term, i.e. the one we called 2, while we had a "−" sign. The discrepancy is irrelevant, as next lemma tells us that the boundary term is 0, so its sign doesn't affect the proof in any way. I believe though that our signs are correct.*

Lemma 9.

$$\log\left|\frac{\theta_0^{(N)}(z_0,\sigma)}{\theta_0^{(N)}(z_0,-\tau)}\right| = 0 \quad \textit{with } N := N(\mathfrak{a})$$

Proof. Let's begin with some considerations about the relation between σ and $-\tau$.
Since $b=\varepsilon a$, multiplication by ε induces an isomorphism $L\cap H(a)\cong L\cap H(b)$; we can then write the image of the basis of $L\cap H(a)$, (β,γ), with respect to the basis of $L\cap H(b)$, (α,γ).

$$\varepsilon\beta = -a'\alpha+b'\gamma;\quad \varepsilon\gamma = -c\alpha+d\gamma.\quad a',b',c,d\in\mathbb{Z} \text{ s.t. } a'd-b'c=\pm 1 \tag{5.22}$$

We evaluate these expressions at $\mathbf{x}$, denote $\varepsilon_{\mathbb{C}} := \varepsilon(\mathbf{x}) \in \mathbb{C}^{\times}$

$$\varepsilon_{\mathbb{C}}\beta(\mathbf{x}) = (\varepsilon\beta)(\mathbf{x}) = -a'\alpha(\mathbf{x}) + b'\gamma(\mathbf{x}); \quad \varepsilon_{\mathbb{C}}\gamma(\mathbf{x}) = (\varepsilon\gamma)(\mathbf{x}) = -c\alpha(\mathbf{x}) + d\gamma(\mathbf{x})$$

Divide everything by $\gamma(\mathbf{x})$, get

$$\varepsilon_{\mathbb{C}}\frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})} = -a'\frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})} + b'; \quad \varepsilon_{\mathbb{C}} = -c\frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})} + d$$

Now recalling $-\tau = -\frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})}$ and $\sigma = \frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})}$, by dividing the first equality by the second we get

$$\sigma = \varepsilon_{\mathbb{C}}\frac{\beta(\mathbf{x})}{\gamma(\mathbf{x})}\varepsilon_{\mathbb{C}}^{-1} = \left(-a'\frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})} + b'\right)\left(-c\frac{\alpha(\mathbf{x})}{\gamma(\mathbf{x})} + d\right)^{-1} = \frac{a'(-\tau) + b'}{c(-\tau) + d} \tag{5.23}$$

Since σ and $-\tau$ have both positive imaginary part (recall $(-\alpha, \gamma)$, resp. (β, γ) are positively oriented basis of $H(b)$, resp. $H(a)$), the matrix of the action that sends $-\tau$ to σ must have positive determinant, i.e. $a'd - b'c = +1$ (it is actually the usual action of $SL_2(\mathbb{Z})$ on $\mathscr{H}$).
Now, recall once again that in the proof of Lemma 2, we proved that $\mathfrak{a}^{-1}L = L + \mathbb{Z}\frac{\gamma}{N}$. This yields, intersecting with $H(a), H(b)$, that

$$\mathfrak{a}^{-1}L \cap H(a) = \mathbb{Z}\beta + \mathbb{Z}\frac{\gamma}{N}; \qquad \mathfrak{a}^{-1}L \cap H(b) = \mathbb{Z}\alpha + \mathbb{Z}\frac{\gamma}{N} \tag{5.24}$$

Since multiplication by ε preserves $\mathfrak{a}^{-1}L$ (as it is just a unit), then $\varepsilon\frac{\gamma}{N} \in \mathfrak{a}^{-1}L$. From Eq. 5.22, we get $\varepsilon\frac{\gamma}{N} = -\frac{c}{N}\alpha + \frac{d}{N}\gamma$. It then belongs to $H(b)$ as well, hence to $\mathfrak{a}^{-1}L \cap H(b) = \mathbb{Z}\alpha + \mathbb{Z}\frac{\gamma}{N}$. This implies that $\frac{c}{N} \in \mathbb{Z}$, hence that N divides c.
Moreover $\frac{\lambda}{q} \equiv 1 \mod L$, for how λ was taken, and $\varepsilon \equiv 1 \mod \mathfrak{f}$ as it belongs to $\mathscr{O}_{K,\mathfrak{f}}^{+,\times}$, hence $\varepsilon\frac{\lambda}{q} = \varepsilon + \varepsilon\left(\frac{\lambda}{q} - 1\right) \in \varepsilon + L$, for $\varepsilon\left(\frac{\lambda}{q} - 1\right) \in L$ as ε preserves L. Moreover $\varepsilon + L = 1 + L = \frac{\lambda}{q} + L$ for above, so that $\varepsilon\frac{\lambda}{q} \in \frac{\lambda}{q} + L$ or equivalently $\varepsilon\frac{\lambda}{q} - \frac{\lambda}{q} \in L$.
Thus, recalling that $\lambda = n\gamma$, $\exists n \in \mathbb{Z}_{>0}$ and that $\gamma \in H(b)$, also using Eq. 5.22:

$$\frac{-nc}{q}\alpha + \frac{(nd-n)}{q}\gamma = \frac{n(-c\alpha + d\gamma - \gamma)}{q} = \frac{n(\varepsilon\gamma - \gamma)}{q} = \frac{\varepsilon\lambda - \lambda}{q} \in L \cap H(b) = \mathbb{Z}\alpha + \mathbb{Z}\gamma$$

give that $\frac{-nc}{q}, \frac{(nd-n)}{q} \in \mathbb{Z}$ so that

$$nc \equiv 0 \mod q; \qquad n(d-1) \equiv 0 \mod q \tag{5.25}$$

After these pure computational considerations, let's start to properly analyze the function $\theta_0(z, -\tau)$; recall its definition:

$$\theta_0(z, -\tau) := \prod_{m \geq 0}(1 - q_z q_{-\tau}^m)(1 - q_z^{-1} q_{-\tau}^{m+1}) \tag{5.26}$$

Directly from that, one can check the three following properties:

$$\begin{gathered} \theta_0(z+1, -\tau) = \theta_0(z, -\tau); \qquad \theta_0(z-\tau, -\tau) = -e^{-2\pi i z}\theta_0(z, -\tau); \\ \theta_0(Nz, -N\tau) = \prod_{j=0}^{N-1}\theta_0\left(z + \frac{j}{N}, -\tau\right) \end{gathered} \tag{5.27}$$

Claim: [1] These transformation properties give that $\theta_0^{(N)}(z,-\tau) = \frac{\theta_0(z,-\tau)^N}{\theta_0(Nz,-N\tau)}$ is periodic wrt the lattice $\mathbb{Z}+\mathbb{Z}\tau$.
Proof:

$$\theta_0^{(N)}(z+1,-\tau) = \frac{\theta_0(z+1,-\tau)^N}{\theta_0(N(z+1),-N\tau)} = \frac{\theta_0(z,-\tau)^N}{\prod_{j=0}^{N-1}\theta_0(z+1+\frac{j}{N},-\tau)} =$$

$$= \frac{\theta_0(z,-\tau)^N}{\prod_{j=0}^{N-1}\theta_0(z+\frac{j}{N},-\tau)} = \frac{\theta_0(z,-\tau)^N}{\theta_0(Nz,-N\tau)} = \theta_0^{(N)}(z,-\tau)$$

where we used the first and third properties of Eq. 5.27. Similarly, using also the second property:

$$\theta_0^{(N)}(z-\tau,-\tau) = \frac{\theta_0(z-\tau,-\tau)^N}{\theta_0(N(z-\tau),-N\tau)} = \frac{(-e^{-2\pi iz})^N\theta_0(z,-\tau)^N}{\prod_{j=0}^{N-1}\theta_0(z-\tau+\frac{j}{N},-\tau)} =$$

$$= \frac{(-e^{-2\pi iz})^N\theta_0(z,-\tau)^N}{\prod_{j=0}^{N-1}-e^{-2\pi i(z+\frac{j}{N})}\theta_0(z+\frac{j}{N},-\tau)} = \frac{(-e^{-2\pi iz})^N}{\prod_{j=0}^{N-1}-e^{-2\pi i(z+\frac{j}{N})}}\frac{\theta_0(z,-\tau)^N}{\prod_{j=0}^{N-1}\theta_0(z+\frac{j}{N},-\tau)} =$$

$$= \frac{(-1)^N}{(-1)^N}e^{-2\pi i\left(Nz-\sum_{j=0}^{N-1}(z+\frac{j}{N})\right)}\frac{\theta_0(z,-\tau)^N}{\theta_0(Nz,-N\tau)} = 1\cdot\theta_0^{(N)}(z,-\tau)$$

as we have that

$$e^{-2\pi i\left(Nz-\sum_{j=0}^{N-1}(z+\frac{j}{N})\right)} = e^{-2\pi i\left(-\frac{1}{N}\cdot\frac{N(N-1)}{2}\right)} = e^{(N-1)\pi i} = 1$$

where the last equality holds since the exponential is $2\pi i$- periodic and N is odd, so that $N-1$ is even.
From $\theta_0^{(N)}(z+1,-\tau) = \theta_0^{(N)}(z,-\tau)$ and $\theta_0^{(N)}(z-\tau,-\tau) = \theta_0^{(N)}(z,-\tau)$ we get that $\theta_0^{(N)}(\cdot,-\tau)$ is $1,\tau$- periodic, hence the claim. ■
The claim tells us that $\theta_0^{(N)}(z,-\tau)$ is a function on the elliptic curve $E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. Note that $\theta_0(\cdot,-\tau)$ is entire (clear from definition Eq. 5.26) and has simple zeroes at $\mathbb{Z}+\mathbb{Z}\tau$. Indeed the first two properties of Eq. 5.27 tell us that $\theta_0(\cdot,-\tau)$ is periodic in the $\mathbb{Z}$-direction and "quasi-periodic" in the $\mathbb{Z}\tau$-direction. It suffices to prove explicitly that $\theta_0(1,-\tau)=0$ and $\theta_0(\tau,-\tau)=0$; the first yields that $\theta_0(\cdot,-\tau)$ is zero at all integers, while the second that $\theta_0(\cdot,-\tau)$ is zero at all points $l\tau$, $l\in\mathbb{Z}$ by induction on l (divide in cases l positive and negative), hence at all points of $\mathbb{Z}+\mathbb{Z}\tau$. For $m=0$ we have $(1-q_1q_{-\tau}^0) = (1-e^{2\pi i}) = 0$, and that annihilates all the product, so that $\theta_0(1,-\tau)=0$. Similarly, for $m=1$, we have $(1-q_\tau q_{-\tau}) = (1-1) = 0$ and that kills all the product, giving $\theta_0(\tau,-\tau)=0$.
So $\theta_0^{(N)}(z,-\tau) = (\theta_0(z,-\tau)^N)/(\prod_{j=0}^{N-1}\theta_0(z+\frac{j}{N},-\tau))$ is a meromorphic function on E_τ with a zero $z=0$ of multiplicity N and simple poles $z=\frac{j}{N}$ for j integer between 0 and $N-1$ (it should be $z=-\frac{j}{N}$, but we can forget the sign by periodicity). Hence its divisor is $N[0]-\sum_{j=0}^{N-1}\left[\frac{j}{N}\right]$.
Define now the function $\tilde{\theta}_0^{(N)}(z,-\tau) := \theta_0^{(N)}\left(\frac{z}{c(-\tau)+d},\frac{a'(-\tau)+b'}{c(-\tau)+d}\right)$. Thanks to Claim 1, it's easy to prove that $\tilde{\theta}_0^{(N)}(z,-\tau)$ is periodic wrt the lattice $\Lambda := \langle c(-\tau)+d, a'(-\tau)+b'\rangle$. Indeed

$$\tilde{\theta}_0^{(N)}(z+(c(-\tau)+d),-\tau) = \theta_0^{(N)}\left(\frac{z+(c(-\tau)+d)}{c(-\tau)+d},\frac{a'(-\tau)+b'}{c(-\tau)+d}\right) =$$

$$= \theta_0^{(N)}\left(\frac{z}{c(-\tau)+d}+1,\frac{a'(-\tau)+b'}{c(-\tau)+d}\right) = \theta_0^{(N)}\left(\frac{z}{c(-\tau)+d},\frac{a'(-\tau)+b'}{c(-\tau)+d}\right) = \tilde{\theta}_0^{(N)}(z,-\tau)$$

and similarly $\tilde{\theta}_0^{(N)}(z+(a'(-\tau)+b'),-\tau)=\tilde{\theta}_0^{(N)}(z,-\tau)$. Note moreover that since $a'd-b'c=+1$, $\Lambda=\mathbb{Z}+\mathbb{Z}\tau$.
Claim: [2] The function $\tilde{\theta}_0^{(N)}(z,-\tau)$ is meromorphic on $E_\tau=\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ with divisor also $N[0]-\sum_{j=0}^{N-1}\left[\frac{j}{N}\right]$.
Proof: Call $\tilde{c}:=c(-\tau)+d$ and $\tilde{a}:=a'(-\tau)+b'$. We use it to reduce the computation to θ_0 that we know is entire with only zero $z=0$ in E_τ.

$$\tilde{\theta}_0^{(N)}(z,-\tau)=\theta_0^{(N)}\left(\frac{z}{\tilde{c}},\frac{\tilde{a}}{\tilde{c}}\right)=\frac{\theta_0(\frac{z}{\tilde{c}},\frac{\tilde{a}}{\tilde{c}})^N}{\theta_0(\frac{Nz}{\tilde{c}},\frac{N\tilde{a}}{\tilde{c}})}=\frac{\theta_0(\frac{z}{\tilde{c}},\frac{\tilde{a}}{\tilde{c}})^N}{\prod_{j=0}^{N}\theta_0(\frac{z}{\tilde{c}}+\frac{j}{N},\frac{\tilde{a}}{\tilde{c}})}$$

by using the third property of Eq. 5.27 as usual. Now, the numerator gives us the zeroes, which are $\frac{z}{\tilde{c}}=0\Rightarrow z=0$ with multiplicity N. In the same way the denominator gives us the poles which are $\frac{z}{\tilde{c}}=\frac{j}{N}$ (again there should be a $-$, but it's fine by periodicity), which is equivalent to

$$z=\frac{\tilde{c}}{N}j=\frac{c(-\tau)+d}{N}j=\left(\frac{c}{N}j\right)(-\tau)+\frac{dj}{N}\equiv\frac{dj}{N}\quad\mod\mathbb{Z}+\mathbb{Z}\tau$$

where we used $\frac{c}{N}\in\mathbb{Z}$, proved before. Moreover, since $N\mid c$, the equality $a'd-b'c=1$ tells us that $(N,d)=1$ or else $a'd-b'c$ would be a multiple of the non trivial gcd between c and d. This allows us to conclude that for j that runs on the integers between 0 and $N-1$, so does dj. In particular the simple poles of $\tilde{\theta}_0^{(N)}(z,-\tau)$ are given by $z=\frac{j}{N},\ j=0,\dots,N-1$.
Then $\tilde{\theta}_0^{(N)}(z,-\tau)$ has the wanted divisor.

■

Claim 2 tells us that $\tilde{\theta}_0^{(N)}(z,-\tau)$ and $\theta_0^{(N)}(z,-\tau)$ have the same divisors as meromorphic functions on E_τ. Uniqueness principle then tells us that

$$\tilde{\theta}_0^{(N)}(z,-\tau)=C\theta_0^{(N)}(z,-\tau),\qquad\exists C\in\mathbb{C}^\times\tag{5.28}$$

Claim: [3] $C^3=1$ and moreover, if $(6,N)=1$, then $C=1$.
Proof: The proof relies on two relations that can be directly checked from Eq. 5.26. If $(l,N)=1$

$$\prod_{s,t\in\mathbb{Z}/l\mathbb{Z}}\theta_0^{(N)}\left(\frac{z}{l}+\frac{s+t(-\tau)}{l},-\tau\right)=\theta_0^{(N)}(z,-\tau)\tag{5.29}$$

$$\prod_{s,t\in\mathbb{Z}/l\mathbb{Z}}\tilde{\theta}_0^{(N)}\left(\frac{z}{l}+\frac{s+t(-\tau)}{l},-\tau\right)=\tilde{\theta}_0^{(N)}(z,-\tau)\tag{5.30}$$

One can check that $z\in E_\tau\Rightarrow\frac{z}{l}+\frac{s+t(-\tau)}{l}\in E_\tau$, so that $\theta_0^{(N)}\left(\frac{z}{l}+\frac{s+t(-\tau)}{l},-\tau\right),\tilde{\theta}_0^{(N)}\left(\frac{z}{l}+\frac{s+t(-\tau)}{l},-\tau\right)$ are functions on E_τ since $\theta_0^{(N)}(z,-\tau)$ and $\tilde{\theta}_0^{(N)}(z,-\tau)$ are. So we still have

$$\tilde{\theta}_0^{(N)}\left(\frac{z}{l}+\frac{s+t(-\tau)}{l},-\tau\right)=C\theta_0^{(N)}\left(\frac{z}{l}+\frac{s+t(-\tau)}{l},-\tau\right)\quad\forall s,t\in\mathbb{Z}/l\mathbb{Z}$$

with the same constant C as before. By multiplying over all $s,t\in\mathbb{Z}/l\mathbb{Z}$ (for a total of l^2 terms), we get, thanks to Eq. 5.29 and 5.30, that $\tilde{\theta}_0^{(N)}(z,-\tau)=C^{l^2}\theta_0^{(N)}(z,-\tau)$, so that $C=C^{l^2}$ for any positive integer l coprime with N.
Thus since N is odd, taking $l=2$, we get $C^4=C\Rightarrow C^3=1$.

Moreover, if $(6,N)=1$, taking also $l=3$, $C^9=C=C^4 \Rightarrow C=1$. ■

Given that C is a root of unity, let's call it $\zeta := C$. Recalling Eq. 5.23:

$$\theta_0^{(N)}\left(\frac{z}{c(-\tau)+d},\sigma\right)=\tilde{\theta}_0^{(N)}(z,-\tau)=\zeta\theta_0^{(N)}(z,-\tau) \tag{5.31}$$

so that $\theta_0^{(N)}(z',\sigma)=\zeta\theta_0^{(N)}(z'(-c\tau+d),-\tau)$ calling $z' := (-c\tau+d)^{-1}z$.

Now, $z_0=\frac{n}{q}$ and thanks to Eq. 5.25, we get

$$z_0(-c\tau+d)=-\frac{nc}{q}\tau+\frac{n(d-1)}{q}+\frac{n}{q}\equiv\frac{n}{q}=z_0 \mod \mathbb{Z}+\mathbb{Z}\tau$$

as $-\frac{nc}{q},\frac{n(d-1)}{q}\in\mathbb{Z}$. Then, since $\theta_0^{(N)}$ is $\mathbb{Z}+\mathbb{Z}\tau$-periodic, we get:

$$\theta_0^{(N)}(z_0,\sigma)=\zeta\theta_0^{(N)}(z_0(-c\tau+d),-\tau)=\zeta\theta_0^{(N)}(z_0,-\tau) \tag{5.32}$$

with ζ a third root of unity, trivial when $(6,N)=1$. In any case

$$\left|\frac{\theta_0^{(N)}(z_0,\sigma)}{\theta_0^{(N)}(z_0,-\tau)}\right|=|\zeta|=1\Rightarrow\log\left|\frac{\theta_0^{(N)}(z_0,\sigma)}{\theta_0^{(N)}(z_0,-\tau)}\right|=0$$

□

5.5 Final considerations

Thanks to Lemma 9, Proposition 11 gives us (together with all previous partial results):

$$\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0)=\frac{3}{\sqrt{\pi}}\int_{\langle\varepsilon\rangle}E_{\psi,\mathfrak{a}}(h,\mathfrak{f}\mathfrak{b}^{-1},0)=\frac{3}{\sqrt{\pi}}\int_0^\infty\int_0^\infty F_{\mathfrak{f},\mathfrak{a}}(t_1,t,h,\mathfrak{b})\frac{dt_1}{t_1}\frac{dt}{t}=$$

$$=\frac{3}{\sqrt{\pi}}\Big(I_{+-}(h)+I_{-+}(h)\Big)=\log\left|\frac{\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}\mathfrak{f}\mathfrak{b}^{-1})}{\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};\mathfrak{f}\mathfrak{b}^{-1})^N}\right|^2-\log\left|\frac{\theta_0^{(N)}(z_0,\sigma)}{\theta_0^{(N)}(z_0,-\tau)}\right|=$$

$$=\log\left|\Gamma_{\mathfrak{a},h}(h(\mathbf{x}),\mathbf{x};\mathfrak{f}\mathfrak{b}^{-1})\right|^2-0=\log|\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})|^2$$

by Definitions 3, 4.

We've then proved the New Kronecker's Limit Formula, Theorem 3. We conclude with some final considerations.

Remark 23. *We proved the first part of point i) of the Main Conjecture 3. Indeed we have that* $\log|\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})|^2=\zeta'_{\mathfrak{f},\mathfrak{a}}(\mathfrak{b},0)$*, and, since the Right Hand Side doesn't depend on h, neither does the Left Hand Side; in particular* $\Gamma_{\mathfrak{a},h}(\mathfrak{f}\mathfrak{b}^{-1})$ *only depends on L and* $\mathfrak{a}$*.*

Remark 24. *Let's begin by noticing that*

$$\Gamma_{a,b}(-w,z;L)\Gamma_{a,b}(w,z;L)=\frac{\prod_{\delta\in C_{+-}/\mathbb{Z}\gamma}\left(1-e^{-2\pi i\left(\frac{\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{-2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}{\prod_{\delta\in C_{-+}/\mathbb{Z}\gamma}\left(1-e^{2\pi i\left(\frac{\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)} \tag{5.33}$$

simply by definition, Eq. 1.1. Notice that $\delta \in C^{o}_{-+} \Leftrightarrow -\delta \in C^{o}_{+-}$, as the interiors of the two cones are symmetric as clear from their definition, Def. 1. Looking at Eq. 5.33 we can then rewrite the terms of the interiors as:

$$\frac{\prod_{\delta\in C^{o}_{+-}/\mathbb{Z}\gamma}\left(1-e^{-2\pi i\left(\frac{\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{-2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}{\prod_{\delta\in C^{o}_{-+}/\mathbb{Z}\gamma}\left(1-e^{2\pi i\left(\frac{\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}=$$

$$=\prod_{\delta\in C^{o}_{+-}/\mathbb{Z}\gamma}\frac{\left(1-e^{-2\pi i\left(\frac{\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{-2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}{\left(1-e^{2\pi i\left(\frac{-\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{2\pi i\left(\frac{-\delta(z)-w}{\gamma(z)}\right)}\right)}=$$

$$=\prod_{\delta\in C^{o}_{+-}/\mathbb{Z}\gamma}\frac{\left(1-e^{-2\pi i\left(\frac{\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{-2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}{\left(1-e^{-2\pi i\left(\frac{\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{-2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}=1$$

The only terms worth considering in Eq. 5.33 are those which lie on the boundary, i.e.:

$$\Gamma_{a,b}(-w,z;L)\Gamma_{a,b}(w,z;L)=\frac{\prod_{\delta\in(C_{+-}\cap H(b))/\mathbb{Z}\gamma}\left(1-e^{-2\pi i\left(\frac{\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{-2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)}{\prod_{\delta\in(C_{-+}\cap H(a))/\mathbb{Z}\gamma}\left(1-e^{2\pi i\left(\frac{\delta(z)+w}{\gamma(z)}\right)}\right)\left(1-e^{2\pi i\left(\frac{\delta(z)-w}{\gamma(z)}\right)}\right)} \tag{5.34}$$

We now plan to use this formula to compute $\Gamma_{\mathfrak{a},h}(h(\mathbf{x}),\mathbf{x};L)\Gamma_{\mathfrak{a},h}(-h(\mathbf{x}),\mathbf{x};L)$, which was something we were left to do from Chapter 1. By Def. 3:

$$\Gamma_{\mathfrak{a},h}(h(\mathbf{x}),\mathbf{x};L)\Gamma_{\mathfrak{a},h}(-h(\mathbf{x}),\mathbf{x};L)=\frac{\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}L)\Gamma_{a,b}(-h(\mathbf{x}),\mathbf{x};\mathfrak{a}^{-1}L)}{[\Gamma_{a,b}(h(\mathbf{x}),\mathbf{x};L)\Gamma_{a,b}(-h(\mathbf{x}),\mathbf{x};L)]^{N(\mathfrak{a})}}=\frac{1}{2^{N(\mathfrak{a})}}$$

we get thanks to Eq. 5.34

$$2=\frac{\prod_{\delta\in(C_{+-}\cap H(b))/\mathbb{Z}\gamma}\left(1-e^{-2\pi i\left(\frac{\delta(\mathbf{x})+h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)\left(1-e^{-2\pi i\left(\frac{\delta(\mathbf{x})-h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)}{\prod_{\delta\in(C_{-+}\cap H(a))/\mathbb{Z}\gamma}\left(1-e^{2\pi i\left(\frac{\delta(\mathbf{x})+h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)\left(1-e^{2\pi i\left(\frac{\delta(\mathbf{x})-h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)}=$$

Now, by classical considerations (see Claim 1 in proof of Proposition 11) we can write, since we're quotienting by $\mathbb{Z}\gamma$, $\delta=n\alpha$ in the numerator and $\delta=m\beta$ in the denominator, some $n,m>0$ (as we need $\delta(a)>0$ in the numerator and $\delta(b)>0$ in the denominator), so that

$$=\frac{\prod_{n>0}\left(1-e^{-2\pi i\left(\frac{n\alpha(\mathbf{x})+h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)\left(1-e^{-2\pi i\left(\frac{n\alpha(\mathbf{x})-h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)}{\prod_{m>0}\left(1-e^{2\pi i\left(\frac{m\beta(\mathbf{x})+h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)\left(1-e^{2\pi i\left(\frac{m\beta(\mathbf{x})-h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)}\cdot\frac{\left(1-e^{-2\pi i\left(\frac{0-h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)}{\left(1-e^{2\pi i\left(\frac{0+h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)}=$$

where we also multiplied by a fraction equal to 1. This allows us to regroup the terms in the following way: in the numerator the exponent with $-h$ runs over $n\geq 0$ (the case $n=0$ is the one we just added) so we can consider in the numerator a product over $n\geq 0$ and re-index the exponent with $+h$ (which starts from $n=1$)

as $n+1$ instead of n. In the same way for the denominator, we multiply over $m \geq 0$ but we re-index the exponent with $-h$ (which starts from $m=1$) as $m+1$. This way we obtain

$$= \frac{\prod_{n\geq 0}\left(1-e^{-2\pi i\left(\frac{(n+1)\alpha(\mathbf{x})+h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)\left(1-e^{-2\pi i\left(\frac{n\alpha(\mathbf{x})-h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)}{\prod_{m\geq 0}\left(1-e^{2\pi i\left(\frac{m\beta(\mathbf{x})+h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)\left(1-e^{2\pi i\left(\frac{(m+1)\beta(\mathbf{x})-h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)} =$$

$$= \frac{\prod_{n\geq 0}\left(1-e^{2\pi i\left(\frac{-(n+1)\alpha(\mathbf{x})}{\gamma(\mathbf{x})}+\frac{-h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)\left(1-e^{2\pi i\left(\frac{-n\alpha(\mathbf{x})}{\gamma(\mathbf{x})}+\frac{h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)}{\prod_{m\geq 0}\left(1-e^{2\pi i\left(\frac{m\beta(\mathbf{x})}{\gamma(\mathbf{x})}+\frac{h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)\left(1-e^{2\pi i\left(\frac{(m+1)\beta(\mathbf{x})}{\gamma(\mathbf{x})}+\frac{-h(\mathbf{x})}{\gamma(\mathbf{x})}\right)}\right)} =$$

$$= \frac{\prod_{n\geq 0}(1-q_{z_0}^{-1}q_{-\tau}^{n+1})(1-q_{z_0}q_{-\tau}^{n})}{\prod_{m\geq 0}(1-q_{z_0}q_{\sigma}^{m})(1-q_{z_0}^{-1}q_{\sigma}^{m+1})} = \frac{\theta_0(z_0,-\tau)}{\theta_0(z_0,\sigma)}$$

Analogously for 1, using additionally that, thanks to Lemma 2,

$$1 = \prod_{k=0}^{N(\mathfrak{a})-1}\Gamma_{a,b}\left(h(\mathbf{x})+k\frac{\gamma(\mathbf{x})}{N(\mathfrak{a})},\mathbf{x};L\right)\Gamma_{a,b}\left(-h(\mathbf{x})+k\frac{\gamma(\mathbf{x})}{N(\mathfrak{a})},\mathbf{x};L\right)$$

and third property of Eq. 5.27: $\theta_0(Nz,-N\tau) = \prod_{j=0}^{N}\theta_0(z+\frac{j}{N},-\tau)$ that holds also for σ instead of $-\tau$, we get that

$$1 = \frac{\theta_0(Nz_0,-N\tau)}{\theta_0(Nz_0,-N\sigma)}$$

We conclude that

$$\Gamma_{\mathfrak{a},h}(h(\mathbf{x}),\mathbf{x};L)\Gamma_{\mathfrak{a},h}(-h(\mathbf{x}),\mathbf{x};L) = \frac{\theta_0(Nz_0,-N\tau)}{\theta_0(Nz_0,-N\sigma)}\cdot\left(\frac{\theta_0(z_0,\sigma)}{\theta_0(z_0,-\tau)}\right)^{N(\mathfrak{a})} = \frac{\theta_0(z_0,\sigma)^{(N(\mathfrak{a}))}}{\theta_0(z_0,-\tau)^{(N(\mathfrak{a}))}} \tag{5.35}$$

Eq. 5.32 in the proof of Lemma 9 gives that the last quotient of Eq. 5.35 equals to a third root of unity, say ζ, trivial when $(6,N(\mathfrak{a}))=1$.
We've obtained that, up to a third root of unity, trivial when the ideal $\mathfrak{a}$ is coprime with 6, we have

$$\Gamma_{\mathfrak{a},h}(-h(\mathbf{x}),\mathbf{x};L) = \Gamma_{\mathfrak{a},h}(L)^{-1} \tag{5.36}$$

This result was used in Remark 6.3, and later in Remark 8.3. We can consider now those results fully proved.

Appendix A

Proof of Stark's Conjecture for abelian extensions of imaginary quadratic fields

The main article analysed in this thesis [BCG23] proposes a Conjecture that relates to Stark's Conjecture for extensions $K(\mathfrak{f})/K$ where K is a complex cubic field. Given the importance of narrow ray class fields among the abelian extensions of a certain field, one may hope that, thanks to the contribution of the main article, a proof of Stark's Conjecture will be reached for all the abelian extensions $\widetilde{K}/K$ of a complex cubic field K.
As we saw, many of the results created by the authors of [BCG23] try to mimic those established for the case of abelian extensions of imaginary quadratic fields, a case in which Stark's Conjecture is actually proved. Since most of the tools for this proof were already given throughout this thesis, it is convenient to present now a proof of this result; we will follow [Tat84] (Ch. 4) and [Iye20]. The proof will not be complete as we'll need to assume facts from CM[1] theory and other branches of Algebraic Geometry.
Let K/k be an abelian extension where k is an imaginary quadratic number field. Let S be a set of places as it was taken in Section 4.2: we ask that it contains all archimedean places as well as places that ramify in K and a place that splits completely in K. We previously saw as a condition that $|S| \geq 2$, but assume for simplicity that $|S| \geq 3$. As saw in Remark 19, the case $|S| = 2$ would require a different but similar treatment, because of its particular definition of $U^{(v)}$.
Recall the definition of the partial zeta function associated to the set of places S:

$$\zeta(s,\sigma) = \zeta_{S,k}(s,\sigma) = \sum_{(\mathfrak{a},S)=1,\mathfrak{a}\in[\sigma]} N(\mathfrak{a})^{-s}$$

where $[\sigma]$ is the class of ideals that correspond to σ under the Artin map. Recall the notations $e := |\mu(K)|$, $G := Gal(K/k)$. We restate the claim to allow us to work without recalling $U^{(v)}$ (in the case $|S| \geq 3$) and $U^{ab}_{K/k}$:

Conjecture 7. *Let K/k be an abelian extension and S a set of places like above. Fix a place v of S that splits completely in K and one of its lifts w in K; then there exists a unit $\varepsilon \in K^\times$ such that:*

i) $|\varepsilon|_{w'} = 1$ *for all places of K, $w' \nmid v$;*

ii) $K(\varepsilon^{1/e})/k$ *is an abelian extension;*

[1] Complex Multiplication

iii) $\zeta'(0,\sigma) = -\frac{1}{e}\log|\varepsilon^\sigma|_w$*, for all* $\sigma \in G$.

The Conjecture is usually referred to as $\mathrm{St}(K/k,S)$. This notation could be misleading as we saw in Remark 18 that the Conjecture doesn't depend on S, however it is kept for historical reasons. What we aim to prove is:

Theorem 4. *If k is a quadratic imaginary number field, then $St(K/k,S)$ is true for all abelian extensions K/k.*

Proof. First of all we want to reduce the claim to the case of narrow ray class field extensions: to do so, a result form [Tat84] (Ch. IV, Prop 3.5) says that for a chain of abelian extensions $k \subset L' \subset L$, if $\mathrm{St}(L/k)$ is true than also $\mathrm{St}(L'/k)$ is true. A proof of this result would require to introduce too many technical elements and is therefore skipped.
Let

$$\mathfrak{m} := \Bigg(\prod_{\mathfrak{p}\in S\setminus\{\infty\}} \mathfrak{p}\Bigg)^n$$

where we multiply only the places of S corresponding to prime ideals and not the archimedeans. From a known result of Class Field Theory we get that for n large enough, $K \subset k(\mathfrak{m})$, where $k(\mathfrak{m})$ is the narrow ray class field of k of conductor $\mathfrak{m}$, that recall is an abelian extension of k unramified outside the primes that don't divide $\mathfrak{m}$. One easily sees that an S that satisfy the needed conditions for K (at the beginning of Section 4.2), satisfies the conditions also for $k(\mathfrak{m})$ (since the only places that ramify in $k(\mathfrak{m})$ are those which lie over places of k that belong in S by definition of $\mathfrak{m}$; the other conditions are trivial). By the result recalled from Tate at the beginning of the proof, proving the Conjecture for $k(\mathfrak{m})/k$ would prove it for K/k; we will then further assume, without loss of generality, that $K = k(\mathfrak{m})$.
Furthermore, recalling that $w_\mathfrak{m} := |\{\zeta \in \mu(k) \mid \zeta \equiv 1 \mod \mathfrak{m}\}|$, we can assume, by taking n in the definition of $\mathfrak{m}$ large enough, that $w_\mathfrak{m} = 1$.
We now plan on using Kronecker's second Limit Formula, which was proven in the case of imaginary quadratic fields in Section 3.4. The definition of the Siegel-Ramachandra invariant in the current case is:

$$g_\mathfrak{m}([\mathfrak{b}]) = g_\mathfrak{m}(\mathfrak{b}) = g_\mathfrak{m}(\sigma_\mathfrak{b}) := G(1;\mathfrak{m}\mathfrak{b}^{-1})^f$$

for every class $[\mathfrak{b}] \in \mathscr{C}l_k(\mathfrak{m})$, and where f is the smallest positive integer in $\mathfrak{m}\cap\mathbb{Z}$. Recall we can also define this invariant as associated to an element of the Galois group, say $\sigma_\mathfrak{b} \in G = Gal(K/k) = Gal(k(\mathfrak{m})/k)$ thanks to the identification between G and $\mathscr{C}l_k(\mathfrak{m})$ provided by the Artin map. Kronecker's Limit Formula in the case of imaginary quadratic number fields, Corollary 1, says that (in our formulation, cf. Remark 17):

$$\zeta_S'(\sigma_\mathfrak{b},0) = \zeta_S'(\mathfrak{b},0) = -\frac{1}{6fw_\mathfrak{m}}\log|g_\mathfrak{m}(\mathfrak{b})| \tag{A.1}$$

We will need some results on $g_\mathfrak{m}(\mathfrak{b})$:

a) $g_\mathfrak{m}([\mathfrak{b}])$ doesn't depend on the choice of $\mathfrak{b} \in [\mathfrak{b}]$; for that reason we can just write $g_\mathfrak{m}(\mathfrak{b}),\ g_\mathfrak{m}(\sigma_\mathfrak{b})$;

b) $g_\mathfrak{m}([\mathfrak{b}]) \in k(\mathfrak{m})$;

c) $g_\mathfrak{m}(\mathfrak{b}) = \sigma_\mathfrak{b}(g_\mathfrak{m}(1))$;

d) $|g_\mathfrak{m}(\mathfrak{b})|_{w'} = 1$ for $w' \nmid v$;

e) The extension $k(g_\mathfrak{m}(\mathfrak{b})^{1/6f})/k$ is abelian.

Proof of property a). A different element in the class $[\mathfrak{b}]$ would be $\lambda\mathfrak{b}$ for $\lambda \in (1+L)$, $L := \mathfrak{m}\mathfrak{b}^{-1}$ (thanks to Remark 10 adapted to our current notation; recall we don't have the + signs in the quadratic imaginary case, for there is no real embedding, cf. Remark 14). We then need to prove

$$\forall \lambda \in (1+L),\, L := \mathfrak{m}\mathfrak{b}^{-1}, \quad g_{\mathfrak{m}}(\mathfrak{b}) = g_{\mathfrak{m}}(\lambda\mathfrak{b}) \quad \text{i.e.} \quad G(1;\mathfrak{m}\mathfrak{b}^{-1})^f = G(1;\lambda^{-1}\mathfrak{m}\mathfrak{b}^{-1})^f$$

Recall $G(z;L) := e^{-6z\eta(z;L)}\sigma(z;L)^{12}\Delta(L)$. A first question to ponder is what happens to the functions $\sigma(z;L), \Delta(L)$ when we substitute L with $\lambda^{-1}L$:
Recall $\Delta(L) = g_2^3 - 27g_3^2$ with $g_k := \sum_{\gamma\in L\setminus\{0\}} \gamma^{-2k}$. So $\Delta(\lambda^{-1}L) = g_2'^3 - 27g_3'^2$ with e.g.

$$g_2' = \sum_{\gamma'\in\lambda^{-1}L\setminus\{0\}} \gamma'^{-2\cdot 2} = \sum_{\gamma\in L\setminus\{0\}} (\lambda^{-1}\gamma)^{-4} = \lambda^4 \sum_{\gamma\in L\setminus\{0\}} \gamma^{-4} = \lambda^4 g_2$$

Analogously $g_3' = \lambda^6 g_3$, so that we get

$$\Delta(\lambda^{-1}L) = g_2'^3 - 27g_3'^2 = \lambda^{12}g_2^3 - 27\lambda^{12}g_3^2 = \lambda^{12}\Delta(L)$$

For $\sigma(z;L)$, recall its definition:

$$\sigma(z;L) = z\prod_{\gamma\in L\setminus\{0\}}\left(1-\frac{z}{\gamma}\right)e^{\frac{z}{\gamma}+\frac{1}{2}\left(\frac{z}{\gamma}\right)^2}$$

from which one clearly sees that $\sigma(\lambda^{-1}z;\lambda^{-1}L) = \lambda^{-1}\sigma(z,L)$.
Recall moreover one of the properties of the η function regarding its relationship with the σ function (see [Sha87] Ch.II, Section 2 Eq.3): for $w \in L, z \in \mathbb{C}$ we have

$$\sigma(z+w;L) = \pm e^{\eta(w;L)\left(z+\frac{w}{2}\right)}\sigma(z;L) \tag{A.2}$$

The claim $G(1;L)^f = G(1;\lambda^{-1}L)^f$ is then equivalent, by definition and properties of Δ, σ above, to:

$$G(1;L)^f = G(1;\lambda^{-1}L)^f \Leftrightarrow e^{-6f\eta(1;L)}\sigma(1;L)^{12f}\Delta(L)^f = e^{-6f\eta(1;\lambda^{-1}L)}\sigma(1;\lambda^{-1}L)^{12f}\Delta(\lambda^{-1}L)^f$$

$$\Leftrightarrow e^{-6f\eta(1;L)}\sigma(1;L)^{12f}\Delta(L)^f = e^{-6f\eta(1;\lambda^{-1}L)}(\lambda^{-1})^{12f}\sigma(\lambda;L)^{12f}\lambda^{12f}\Delta(L)^f \Leftrightarrow$$

by erasing $(\lambda^{-1})^{12f}\lambda^{12f}$ on the RHS and the discriminant from both sides, and moreover applying Eq. A.2 to $\lambda := 1+w, w \in L$ (note that the $\pm$ sign in Eq. A.2 goes away thanks to the 12th power):

$$\Leftrightarrow e^{-6f\eta(1;L)}\sigma(1;L)^{12f} = e^{-6f\eta(1;\lambda^{-1}L)}\left[e^{\eta(w;L)\left(1+\frac{w}{2}\right)}\sigma(1;L)\right]^{12f} \Leftrightarrow$$

$$\Leftrightarrow e^{-6f\eta(1;L)}\sigma(1;L)^{12f} = e^{-6f\eta(1;\lambda^{-1}L)}e^{6f\eta(w;L)(2+w)}\sigma(1;L)^{12f}$$

We can now simplify $\sigma(1;L)^{12f}$ and we remain with an exponential equation; recalling the complex exponential has period $2\pi i$, the equality to check now becomes a congruence modulo $2\pi i$. More precisely our claim becomes:

$$-6f\eta(1;L) \equiv -6f\eta(1;\lambda^{-1}L) + 12f\eta(w;L) + 6wf\eta(w;L) \mod 2\pi i \tag{A.3}$$

We will prove that

$$-f\eta(1;L) \equiv -f\eta(1;\lambda^{-1}L) + 2f\eta(w;L) + wf\eta(w;L) \mod 2\pi i \tag{A.4}$$

which is a stronger condition: if this holds, then also Eq. A.3 holds, simply by multiplying both sides by 6. We use throughout that f is real so it can enter and exit the argument of η by $\mathbb{R}$-linearity of the latter. Moreover since $f, w \in L$ we have that:

$$f\eta(w;L) - w\eta(f;L) \equiv 0 \mod 2\pi i \tag{A.5}$$

This is a generalization of Legendre relation (see [Sha87] Ch. II, Section 2, Eq. 3) that says that if ω_1, ω_2 are the generators of L, $\omega_1\eta(\omega_2;L) - \omega_2\eta(\omega_1;L) = \pm 2\pi i$.
One can prove easily, directly from definition of the Weierstrass-ζ function (Def. 10), that for all $\omega \in \mathbb{C}$

$$\zeta(\omega;\lambda^{-1}L) = \lambda\zeta(\lambda\omega;L)$$

By definition of η (always Def. 10), for $\omega \in \lambda^{-1}L, z \in \mathbb{C}$

$$\eta(\omega;\lambda^{-1}L) = \zeta(z+\omega;\lambda^{-1}L) - \zeta(z;\lambda^{-1}L) = \lambda[\zeta(\lambda z + \lambda\omega;L) - \zeta(\lambda z;L)] =$$

$$= \lambda\eta(\lambda\omega;L)$$

as $\lambda\omega \in L$. Let's now see how this helps us conclude: first note that $f \in \lambda^{-1}L$, indeed we now prove equivalently that $\lambda f \in L$; if we call again ω_1, ω_2 the generators of L, note that since f is the least positive integer in $\mathfrak{m}$, it can be chosen as one of the generators, wlog $f = \omega_2$. Then since $w \in L$, there exist integers a', b' such that $w = fa' + \omega_1 b' \Rightarrow f\lambda = f(1+w) = f + f^2a' + f\omega_1 b' = f(1+fa') + \omega_1(fb')$ with $1 + fa', fb' \in \mathbb{Z}$ so that $f\lambda \in L$.
The claim (Eq. A.4) with $\eta(f;\lambda^{-1}L) = \lambda\eta(\lambda f;L)$ for what just said, becomes (all congruences are intended modulo $2\pi i$)

$$-f\eta(1;L) \equiv -\lambda\eta(\lambda f;L) + 2f\eta(w;L) + wf\eta(w;L) \equiv -\eta(f;L) - \eta(fw;L) - w\eta(f;L) -$$

$$-w\eta(fw;L) + 2f\eta(w;L) + wf\eta(w;L) \equiv -\eta(f;L) - w\eta(f;L) + f\eta(w;L)$$

where we substituted $\lambda = 1 + w$ and then simplified the common terms. Moreover by erasing on both sides of the congruence $-f\eta(1;L) = -\eta(f;L)$, our claim becomes

$$0 \equiv -w\eta(f;L) + f\eta(w;L) \mod 2\pi i$$

which is true by Eq. A.5. Then our original claim is also true and the proof of fact $a)$ is finished. □

Properties $b), e)$ come from the theory of Complex Multiplication: since k is a quadratic imaginary number field, there exist elliptic curves (call E one of them) which have CM by $\mathscr{O}$ (i.e. such that $\mathrm{End}(E) \cong \mathscr{O}$) with $\mathscr{O}$ and order in k. We will now just sketch the motivation of these two results, based on some well-known facts. Recall the notation $\Gamma_N := \left\{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N\right\}$.

Facts 1 (From CM).

- *Narrow ray class fields such as $k(\mathfrak{m})$ are obtained from k by adding the j-invariant of an elliptic curve E with CM by an order $\mathscr{O}$ such that $[\mathscr{O}_k : \mathscr{O}] = f$ (in the case of $k(\mathfrak{m})$), where f is again the smallest positive integer in $\mathfrak{m} \cap \mathbb{Z}$ and $\mathscr{O}_k$ is the ring of integers of k.*
- *Thanks to the proof of property a), one can see that the function $G(1;L)^f$ is modular of level Γ_f; this implies (taking $L = \mathfrak{m}\mathfrak{b}^{-1}$) that $g_{\mathfrak{m}}(\mathfrak{b})$ and $g_{\mathfrak{m}}(\mathfrak{b})^{1/6f}$ are both values of certain modular functions at a certain point $\theta \in k \cap \mathscr{H}$.*

- *If f is a modular function of level Γ_N and $\theta \in k \cap \mathscr{H}$, then $f(\theta) \in k(N\mathscr{O}_k)$, the narrow ray class field of conductor the ideal $N\mathscr{O}_k \subset \mathscr{O}_k$. In particular this tells us that the extension $k(f(\theta))/k$ is abelian.*

Property $e)$ above, i.e. abelianity of the extension $k(g_{\mathfrak{m}}(\mathfrak{b})^{1/6f})/k$ comes by combining the second and third facts.
For property $b)$: $g_{\mathfrak{m}}([\mathfrak{b}]) \in k(f\mathscr{O}_k)$ comes from the third fact. From that, by definition of f and taking $L = \mathfrak{m}\mathfrak{b}^{-1}$, we have that $g_{\mathfrak{m}}([\mathfrak{b}])$ actually lies in $k(\mathfrak{m})$.
Property $c)$ comes from a reciprocity law in CM theory that associates to the Siegel-Ramachandra invariant an elliptic curve and allows to prove the result by working on the latter. However we will take it without proof.
We won't prove even property $d)$, as it comes from Algebraic Theory of elliptic units.
Let's go back to the proof of Stark's Conjecture, with the help of the properties $a) - e)$.
Claim: $g_{\mathfrak{m}}(1)^e$ admits a $(6f)$-th root contained in $k(\mathfrak{m})$, call it u.[2]
Proof: We plan to use a result from [Sta80] (Lemma 6) which says the following: for an extension K/k abelian, $\delta \neq 0$ a number such that $\alpha = \delta^n \in K$, if $K(\delta)/K$ is normal and $Gal(K(\delta)/K)$ is contained in the center of $Gal(K(\delta)/k)$ and if K contains exactly e roots of unity, then $\alpha^{(e,n)}$ is an n-th power in K.
In our case $k(\mathfrak{m})/k$ is an abelian extension, we want to use $n := 6f$ and $\alpha := g_{\mathfrak{m}}(1)$. Note that such α belongs to $k(\mathfrak{m})$ by combining properties $b), c)$ above: $g_{\mathfrak{m}}(1) = \sigma_{\mathfrak{b}}^{-1}(g_{\mathfrak{m}}(\mathfrak{b}))$ and since $g_{\mathfrak{m}}(\mathfrak{b}) \in k(\mathfrak{m})$, then so does $g_{\mathfrak{m}}(1)$.
In our case $\delta = g_{\mathfrak{m}}(1)^{1/6f}$. Using property $e)$ for the class $[1] \in \mathscr{C}l_k(\mathfrak{m})$, we get that $k(g_{\mathfrak{m}}(1)^{1/6f})/k = k(\delta)/k$ is abelian. Thus, since both $k(\mathfrak{m})/k$ and $k(\delta)/k$ are abelian, then so is also $k(\mathfrak{m})(\delta)/k$, in particular it is also Galois. This implies that $k(\mathfrak{m})(\delta)/k(\mathfrak{m})$ is also Galois, hence normal, and that the center of $Gal(k(\mathfrak{m})(\delta)/k)$ is the whole group, hence $Gal(k(\mathfrak{m})(\delta)/k(\mathfrak{m}))$ is surely contained in it.
We can then apply the Lemma from [Sta80], which tells us that $g_{\mathfrak{m}}(1)^{(e,6f)}$ admits a $(6f)$-th root in $k(\mathfrak{m})$. To conclude we just need to prove that $(e,6f) = e$, i.e. that e divides $6f$. As we saw in the proof of Proposition 7, if $k(\mathfrak{m})$ contains a p-th root of unity with p coprime with $\mathfrak{m}$, then p is either 2 or 3 (recall in that proof we had a condition $p \leq 4$ which imposed $p \in \{2,3\}$ as 4 isn't prime; in this case, since we're in a quadratic imaginary case, the condition is directly $p \leq 3$), in particular $p \mid 6$. If $k(\mathfrak{m})$ contains a p-th root of unity with p not coprime with $\mathfrak{m}$, then p must divide f. In either case $p \mid 6f$ so that $e \mid 6f$ and we're done. ■
This claim allows us to conclude: applying it together with $c)$, and recalling that we can assume $w_{\mathfrak{m}} = 1$, we get from Eq. A.1

$$\zeta_S'(\sigma_{\mathfrak{b}}, 0) = -\frac{1}{6fw_{\mathfrak{m}}} \log|g_{\mathfrak{m}}(\mathfrak{b})| = -\frac{1}{6f} \log|\sigma_{\mathfrak{b}}(g_{\mathfrak{m}}(1))| =$$

$$= -\frac{1}{6ef} \log|\sigma_{\mathfrak{b}}(g_{\mathfrak{m}}(1)^e)| = -\frac{1}{e} \log|\sigma_{\mathfrak{b}}(u)|$$

for all $\sigma_{\mathfrak{b}} \in G$, which proves $iii)$ of Conjecture 7 with $\varepsilon := u$. (Recall that Tate doesn't write explicitly the dependence that ζ has on S, but his $\zeta'(0,\sigma)$ is precisely our $\zeta_S'(\sigma_{\mathfrak{b}}, 0)$.)
Condition $i)$ corresponds to $d)$ recalling the relation between u and $g_{\mathfrak{m}}(\mathfrak{b})$ via $c)$, invariance of the norm under Galois action and multiplicativity of the norm, while for condition $ii)$ note that $K(u^{1/e}) = k(\mathfrak{m})(u^{1/e}) = k(\mathfrak{m})(g_{\mathfrak{m}}(1)^{1/6f})$ is an abelian extension of k thanks to property $e)$. □

The case of abelian extension of imaginary quadratic fields is one of the few in which Stark's Conjecture has already been proved. We once again invite the curious reader to take a look to Tate's book on the subject [Tat84].

[2] Both articles [Tat84] and [Iye20] that we're following for this proof say that $g_{\mathfrak{m}}(1)^e$ admits a $(12f)$-th root in $k(\mathfrak{m})$, consistently with their notation (cf. Remark 17).

We conclude with the hope that the next case of this beautiful conjecture to be proved is that of abelian extensions of complex cubic fields, maybe with the help of the new Conjecture formulated by the authors of [BCG23].

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