Università degli Studi di Padova

DIPARTIMENTO MATEMATICA
DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA"

Master Thesis in Mathematics

Rolling manifolds: an approach through geometric control

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Academic Year 2024/2025
21 February 2025

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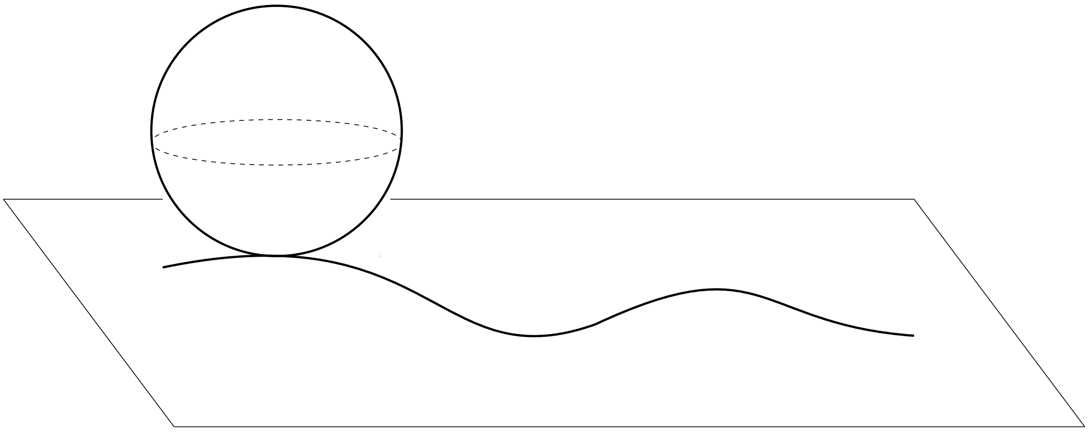
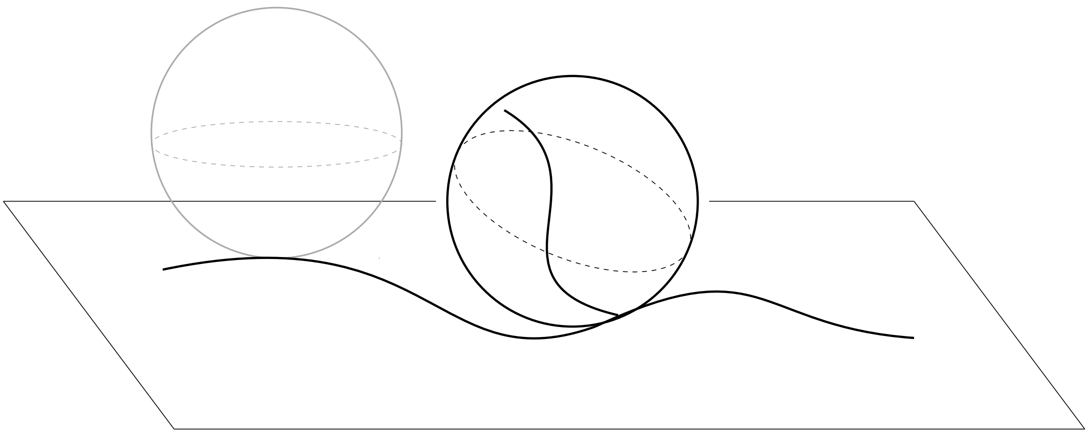
Introduction

Take a ball and put it on the table. Draw an arbitrary curve on the plane and now roll the ball along the curve.



If we impose no instantaneous twist and slip, the motion of the ball along the curve draws a corresponding unique curve on the surface.



Observing the motion, we also see the orientation of the ball changing along the path.
Have you ever wondered if you can reach the final point of the curve with the starting orientation? Or if you can reach the final point with a desired orientation of your ball?

What we want to do in this work is study this kind of motion. The ball and the table are two rigid bodies. These two objects are represented by a sphere and a plane: two smooth, connected

and oriented manifolds. Moreover, in our setting, to measure angles, we endow these manifolds with a metric, so we assume that they are Riemannian.
The motion of a manifold running on another manifold is simply called rolling and we place it in a setting where this is possible; therefore, the manifolds are supposed to touch and there is no deformation during the motion.
In particular, in this work we consider the rolling with two conditions; we require no instantaneous slip or twist. That is, we don't want a manifold to slide on the other and we don't want neither of them to rotate on itself. We call these conditions constraints, and we refer to them as no slipping and no twisting.
These precise constraints will shape the problem and they will provide relevant consequences. The theory in this work follows an intrinsic approach, which means that we consider references only on the objects of the problem and not in the ambient space, the space where the problem is immersed. We work on the tangent spaces and use references on them, is the coordinate-free approach historically developed after Cartan's moving frame method. The approach to intrinsic theory is gradual; indeed, the work is divided into four main parts, and we go through a first 2-dimensional example to the 2-dimensional developed theory and from an n-dimensional example to the general theory.

In the first chapter, an easy-to-imagine example is presented: the ball rolling on the plane. This is an example where both manifolds are 2-dimensional, and it is an introduction to the setting and to the tools we use. It is important to notice that the description of the first example is not intrinsic; there we use the 3-dimensional framework.

In the second chapter, we step into the intrinsic description. Here is fully explained what it means to have a reference in a precise point of the manifold, what is the meaning of contact point and why do we imagine overlapped tangent planes. This is also the chapter where the concept of curvature first appears: the possibility of reaching a certain configuration depends on the geometry of the manifolds, and here we explain this.

In the third chapter, we go through the first generalization. In this work, for completeness, we want to describe the problem not only in the easy-to-imagine dimension. Therefore, here we consider a n-dimensional sphere rolling on a n-dimensional plane. As a generalization of the example seen in the first chapter, here we lose the intrinsic approach, but we step in a wider context.

In the fourth chapter, we are ready to restore the intrinsic approach in the n-dimensional setting, and a general theorem is proven.

An important tool that we have used is the tool of connection. Thanks to it, we approach the problem intrinsically and we can "move" references from one tangent space to another. In particular, in the fourth chapter, the use of this tool becomes more involved; therefore, we have dedicated the last chapter to the explanation of the facts we use about connections.

The first formulation of the 2-dimensional theorem from an intrinsic point of view is due to Agrachev and Sachkov; see [2]. There, all the 2-dimensional theory is developed with the Hamiltonian formalism related to symplectic mechanics. We maintain the intrinsic approach while avoiding the Hamiltonian language, so translating back to the simple Riemannian language.
For the n-dimensional case we developed the argument following as principal reference the article of Grong, [6]. However, other works, such as [5], [4], [8], and [9], face the problem via an intrinsic approach through slightly different strategies.
This work aims to give a viewable description of the rolling problem, driving the intuition of an easy-to-image problem to the mathematical details that lead to what can be thought as a generalization of the classical Frobenius theorem.

Chapter 1

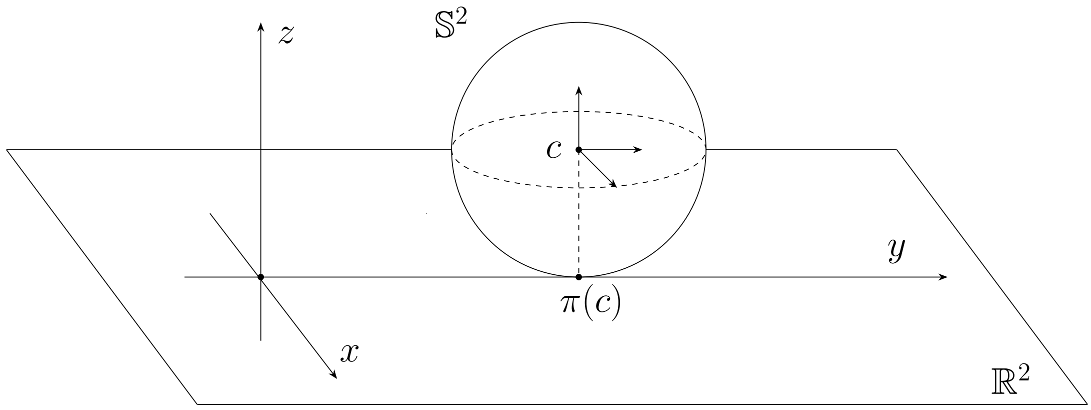
A 2 dimensional example: The sphere $\mathbb{S}^2$ rolling on the plane $\mathbb{R}^2$

The first step to understand the problem of rolling manifolds is to display a practical example. Imagine again a ball rolling on a plane, this is the example from which we want to develop the setting, the notation, and the ideas. So we are in $\mathbb{R}^3$, the ball is the sphere $\mathbb{S}^2$ rolling on a plane represented by the manifold $\mathbb{R}^2$. We want to study the motion of rolling with two constraint: without slipping or twisting.

The ambient space of the problem is $\mathbb{R}^3$ and as the approach here is not yet intrinsic, we define a reference in $\mathbb{R}^3$. We fix the third coordinate of the center c of the sphere equal to one and image the sphere rolling on the plane $z = 0$;

$$c = (x, y, 1) \in \mathbb{R}^3.$$

An arbitrary configuration of the system is described by the projection $\pi(c)$ of the center c



on the plane $z = 0$ and the orientation of the sphere. We call $\pi(c) \in \mathbb{R}^2$ the point of contact between the manifolds.
The orientation of the sphere is given by an orthonormal frame attached to the center c, which remain fixed relative to the sphere during the motion. The relative orientation of this frame with respect to the standard orthonormal frame in $\mathbb{R}^3$ is represented by a matrix $M \in SO(3)$, where we have chosen positive determinant because we want to preserve the orientation.

The space of all possible configurations hence is given by

$$\mathcal{Q} = \mathbb{R}^2 \times SO(3);$$

a point of contact and an orientation.
This space $\mathcal{Q}$ is indeed a Lie group, since it is product of Lie groups, thus it is induced a Lie algebra and every tangent space is characterized in a precise way (more details are presented in A.1). Thanks to that we have this description of the tangent space at a point $q = (\pi(c), R) \in \mathcal{Q}$

$$T_q\mathcal{Q} = \mathbb{R}^2 \times R \cdot \mathfrak{so}(3).$$

We recall that the Lie algebra $\mathfrak{so}(3)$ is identified with the group of skew-symmetric matrices skew(3). The group $SO(3)$ represents rotations in $\mathbb{R}^3$, while the matrices in $\mathfrak{so}(3)$ are not properly rotation matrices, but since they are the differential of rotations matrices we call them infinitesimal rotation matrices, representing infinitely small rotation.
Let $\{e_x, e_y, e_z\}$ be the following basis of the Lie algebra $\mathfrak{so}(3)$,

$$e_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

the first matrix e_x represents the *infinitesimal* clockwise rotation around the x-axis, and similarly e_y and e_z represent the *infinitesimal* clockwise rotation around the y-axis and around the z-axis respectively.

Now, since we have some constraints, no-slipping and no-twisting, we try to understand what the allowed movements of the system are. If we want to move the sphere along the x-direction, we cannot simply translate it along the axis because in that case we are sliding the sphere. In other words, we cannot slide the sphere without changing its relative orientation. So rolling the sphere, without slipping, along the x direction in the plane also corresponds to a clockwise rotation around the y-axis in terms of the orientation. Therefore the vector field

$$X_1 = \frac{\partial}{\partial x} + R \cdot e_y \in Vec(\mathcal{Q})$$

corresponds exactly to the description of this movement in the tangent space of the configuration space $\mathcal{Q}$, in particular $X_1 \in T_q\mathcal{Q}$ with $q = (\pi(c), M)$ contact point.

The same holds if we want to move the sphere along the y-direction: rolling without slipping along the y-axis corresponds, again, to a clockwise rotation around the x-axis. That is, this other movement is described by

$$X_2 = \frac{\partial}{\partial y} + R \cdot e_x \in Vec(\mathcal{Q}).$$

Since in the motion twisting is not allowed, that in this setting corresponds to a rotation around the z-axis, the e_z movement is not permitted. In other words, the admissible movements of the system are represented by the vector fields

$$X_1 \text{ and } X_2,$$

but what about the combination of these two directions? The idea behind combing directions is the Lie bracket; the bracket $[X_1, X_2]$ is again a vector field in $\mathcal{Q}$ and if it is independent, as a vector, from the previous two, we can interpret it as a new *infinitesimal* movement of the system. We present a proper justification for this later, in the next Chapter 2. So, for now, following this idea, we compute the Lie bracket to understand all the possible movements of the system

$$\begin{aligned}[X_1, X_2] &= [\frac{\partial}{\partial x} + R \cdot e_y, \frac{\partial}{\partial y} + R \cdot e_x] \\ &= R \cdot [e_y, e_x] = -R \cdot [e_x, e_y],\end{aligned}$$

that is, computing the commutator of matrices,

$$[e_x, e_y] = e_x e_y - e_y e_x = e_z$$

exactly the rotation around the z-axis. The Lie bracket $[X_1, X_2]$ produces an independent direction from the previous ones, the counter-clockwise rotation around the z-axis

$$X_3 := -R \cdot e_z,$$

that corresponds to the non-admissible movement of twisting. This means that starting from a point $q \in \mathcal{Q}$ and moving along the combinations of X_1 and X_2 using the Lie bracket, we can reach a configuration of the system that is the same as the one we would obtain if the twist at q were admitted. We can iterate this procedure computing all the possible Lie brackets.
Considering that the commutators of matrices are

$$\begin{aligned}[e_y, e_z] &= e_x, \\ [e_z, e_x] &= e_y;\end{aligned}$$

we obtain

$$\begin{aligned}[X_1, X_3] &= [\frac{\partial}{\partial x} + R \cdot e_y, -R \cdot e_z] = -R \cdot e_x, \\ [X_2, X_3] &= [\frac{\partial}{\partial y} + R \cdot e_x, -R \cdot e_z] = R \cdot e_y.\end{aligned}$$

Moreover the vector fields $X_1, X_2, X_3, X_4 := [X_1, X_3], X_5 := [X_2, X_3]$ are linearly independent therefore

$$\text{span}\{X_1, X_1, X_2, X_3, X_4, X_5\}|_q = T_q\mathcal{Q}.$$

We find out that combining all the directions using the Lie bracket, we recover all the principal

movements of the system at the *infinitesimal* level, also the nonadmissible ones.
For example, if we want to slide the sphere in the x direction respecting the constraints, we first roll in the x direction, following X_1 and then we "twist" it back to the original orientation, i.e., we follow the Lie bracket combination $-X_5$

$$\frac{\partial}{\partial x} = X_1 - X_5.$$

The fact that the two admissible directions X_1 and X_2 enable us to recover all possible movements is the core idea behind a controllable system.
We develop this theory further in the following Chapters.

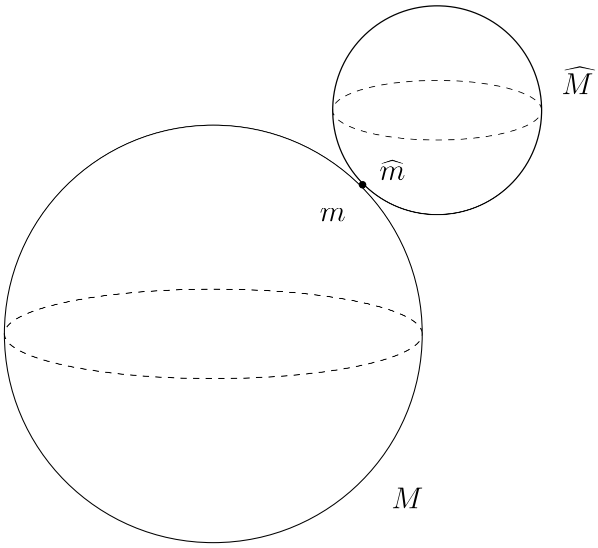
Chapter 2

The two dimensional case

Now we are ready to develop the general theory of the two-dimensional case. The bodies are represented by the two-dimensional manifolds M and $\widehat{M}$, which are smooth, connected, and oriented. We underline that M and $\widehat{M}$ are general manifolds, but to encourage imagination, we show pictures of two spheres of different radius.
Here we step into the intrinsic description, so we have no references in $\mathbb{R}^3$.

To describe the motion, we endow the manifolds with a metric structure. In fact M and $\widehat{M}$ are Riemannian manifolds where the metric is defined, as usual, as a family $\langle \cdot, \cdot \rangle|_m$ of scalar products on the tangent spaces T_mM, smoothly depending on $m \in M$, and analogously on $\widehat{M}$.

The two manifolds are assumed to touch and then roll on each other. As the manifolds represent rigid bodies, there isn't any deformation and every moment of the rolling involves exactly one point $m \in M$ and one point $\widehat{m} \in \widehat{M}$, which we call the points of contact.
We are building a geometric model from a natural setting and it is clear that if we consider



manifolds with positive curvature we can assume that contact between them is admissible.

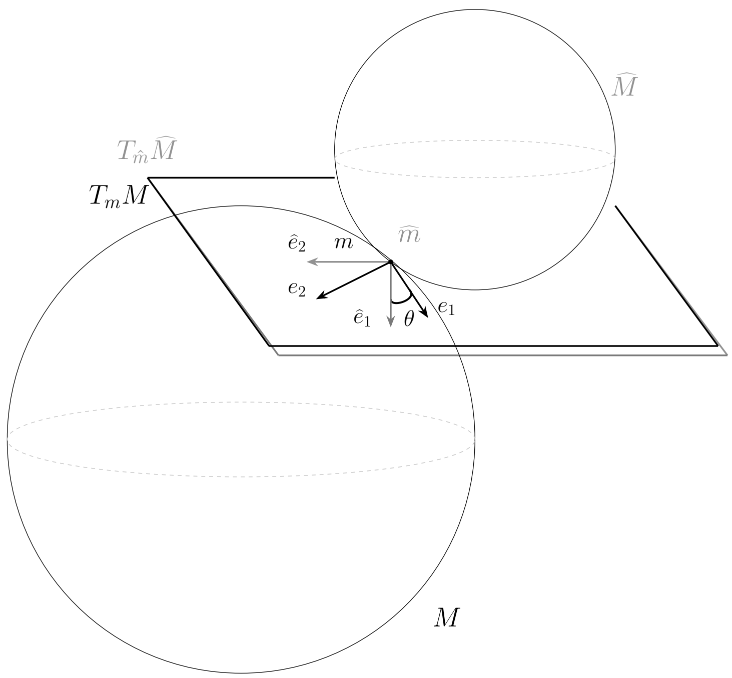
Before studying the motion of rolling, we have to clarify the starting setting. The points of contact are unique up to rotations of one manifold on itself, so we can start thinking that a

5-tuple of the kind

$$(m, \widehat{m}, \theta), \qquad m \in M, \widehat{m} \in \widehat{M}, \theta \in \mathbb{S}^1$$

describes the system. In the next lines we will give a precise meaning to the angle θ.

The angle θ has the same role of the orthogonal matrix of Chapter 1, it describes the orientation. However, this time, it isn't the orientation of a manifold in the ambient space, it is the relative orientation between the two objects.
We define this relative orientation using references on the tangent spaces $T_m M$ and $T_{\hat{m}}\widehat{M}$ of the point of contact. The intuitive idea is that at the contact points m and $\widehat{m}$, the tangent planes are "overlapped" and we measure how much a reference frame in $T_m M$ is rotated relative to a reference frame in $T_{\hat{m}}\widehat{M}$. We measure it with an isometry that is represented by the angle θ.
In their respective tangent spaces $T_m M$ and $T_{\hat{m}}\widehat{M}$ we can choose a reference frame, that is,



a choice of a basis: $\{e_1(m), e_2(m)\}$ a basis for $T_m M$ and $\{\widehat{e}_1(\widehat{m}), \widehat{e}_2(\widehat{m})\}$ a basis for $T_{\hat{m}}\widehat{M}$. We chose them orthonormal, i.e.:

$$\langle e_i, e_j \rangle_M = \delta_{ij}, \qquad \langle \widehat{e}_i, \widehat{e}_j \rangle_{\widehat{M}} = \delta_{ij}.$$

The isometry that measures the relative orientation is a map of this kind

$$R : T_m M \to T_{\hat{m}}\widehat{M};$$

it rotates the frame $\{e_1, e_2\}$ onto $\{\widehat{e}_1, \widehat{e}_2\}$

$$e_1 \mapsto \widehat{e}_1$$
$$e_2 \mapsto \widehat{e}_2;$$

and it is represented by this matrix

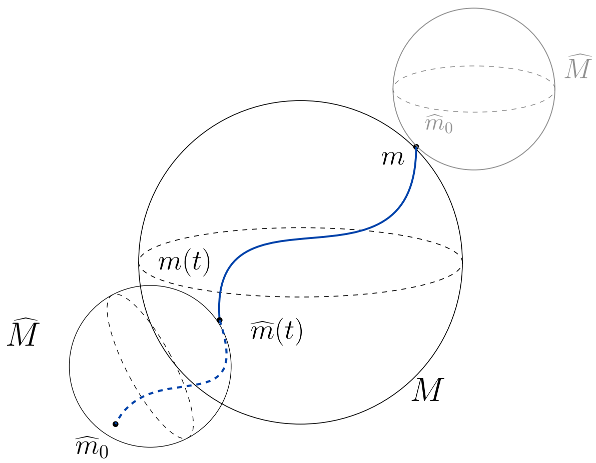
$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where θ is the aforementioned characterizing angle: the angle of rotation between the frame $\widehat{e}_1, \widehat{e}_2$ and the frame Re_1, Re_2 at the points of contact $m, \widehat{m}$.

Now that we have defined a stationary configuration, we characterize the motion of rolling. Since a 5-tuple $(m, \widehat{m}, \theta)$ represents a moment of the rolling, now we consider it varies in time;

$$(m(t), \widehat{m}(t), \theta(t)) \qquad m(t) \in M,\ \widehat{m}(t) \in \widehat{M};$$

are the trajectories of the points of contact on the manifolds and $\theta(t)$ is the variation of the angle. Here we always suppose that the movement is smooth in order to have smooth curves. The motion can change the relative orientation between the manifolds and we want to measure



this variation to characterize the curve of the motion. To measure it, we need to relate tangent spaces at different points of the same manifold because we need to keep a trace of the starting orientation. The connection, which gives us the parallel transport, is the tool that permits us to put in relation tangent spaces at different points of the same manifold.

Fix a path $q(t) = (m(t), \widehat{m}(t), \theta(t)) \in \mathcal{Q}$, with $t \in [0, T]$; we choose two frames at the starting contact points $m(0)$ and $\widehat{m}(0)$:

$$e_0 = \{e_1, e_2\}\Big|_{m(0)} \qquad \text{and} \qquad \widehat{e}_0 = \{\widehat{e}_1, \widehat{e}_2\}\Big|_{\widehat{m}(0)}.$$

These frames have a relative orientation described by the angle $\theta(0)$ since the starting configuration is $(m(0), \widehat{m}(0), \theta(0))$. Then, we parallel transport the vector field e_1 along $m(t)$ and $\widehat{e}_1$ along $\widehat{m}(t)$; notice that it is sufficient to compute only one parallel transport since it is an isometry, and we recover the other vector field of the frame due to the orthogonality condition. Now, the relative orientation at the final configuration $q(T)$ is given by the angle $\theta(T)$ between the parallel transported frames. In particular, we are using the parallel transport to "move" the

references from the tangent space $T_{m(0)}M$ to $T_{m(T)}M$ and similarly on $\widehat{M}$.
This holds because there exists a unique smooth vector field X along $m(t)$ such that $X|_{m(0)} = e_1|_{m(0)}$ and parallel along $m(t)$ with respect to the chosen connection; for more details, see Appendix A.1.

Idea. *To figure it out, we can think of $M = \mathbb{R}^n$ and ∇ as just the standard directional derivative; in this setting, we obtain an intuitive picture of what parallel transport and a parallel vector field are.*
This setting corresponds to the choice $\Gamma^k_{i,j} = 0$. Let $\{e_1, \ldots, e_n\}$ be a local frame and let $Y(\gamma(t))$ a vector field defined along the smooth curve $\gamma(t) : [0,T] \longrightarrow M$

$$Y(\gamma(t)) = \sum_{i=1}^{n} y_i(t) e_i|_{\gamma(t)}.$$

Let $Y(\gamma(t))$ be parallel with respect to the connection ∇ chosen before. The parallel condition $\nabla_{\dot\gamma(t)}Y$ in this case translate into (see Lemma A.1)

$$\dot y_k(t) = 0 \qquad \forall\, k = 1, \ldots, n$$

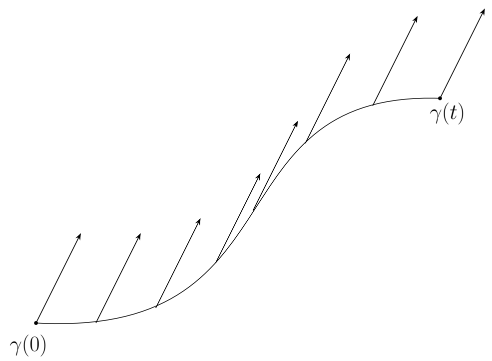
therefore

$$y_k(t) \equiv y_i(0) \qquad \forall\, k = 1, \ldots, n$$

that is, the vector field $Y(\gamma(t))$ has in every point the same value

$$Y(\gamma(t)) = \sum_{i=1}^{n} y_i(0) e_i|_{\gamma(t)} \quad \forall t \in [0,T]\,.$$

In this sense the vector corresponding to the value of the vector field at time $t = 0$ is transported



in a parallel way along the curve $\gamma(t)$ and the vector field $Y(\gamma(t))$ is parallel with respect to ∇, parallel to itself along all the points of the curve.

Notation. *From now on we will use both the coordinates and the vectorial notation for vector fields, i.e., if $\{e_1, e_2\}$ is a local frame and X is a vector field*

$$X = \sum_{i=1}^{2} X_i e_i = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

Once we can move the reference frames that we fixed at time zero along $m(t)$ and $\widehat{m}(t)$ respectively, the variation of the angle $\theta(t)$ can be measured at every point and the relative orientation of the system is well defined.
The choice of connection we make here is one compatible with the metric: the Levi-Civita connection.

At this time, we can define the *state space of the system.* Every 5-tuple of the form

$$q = (m_1, m_2, \widehat{m}_1, \widehat{m}_2, \theta)$$

is a state of the system and can be viewed as a point in the set of all configurations, what we call the *configuration space*

$$\mathcal{Q} = \{R : T_m M \to T_{\widehat{m}} \widehat{M} \mid \widehat{m}, m, R \text{ isometry}\};$$

a 5-dimensional manifold; choosing local coordinates (m_1, m_2) on M and $(\widehat{m}_1, \widehat{m}_2)$ on $\widehat{M}$, we obtain local coordinates $(m_1, m_2, \widehat{m}_1, \widehat{m}_2, \theta)$ on $\mathcal{Q}$.

Remark. *Here we deal with orientation-preserving isometries, that is, with the determinant equal to one and we remark that isometries are linear maps which preserve the Riemannian structure, namely if R is an orientation-preserving isometry*

$$\langle v,\, w\rangle\big|_M = \langle R(v),\, R(w)\rangle\big|_{\widehat{M}} \qquad \forall v, w \in T_m M\ .$$

Lastly, it is useful to denote the projections from $\mathcal{Q}$ to M and $\widehat{M}$:

$$\pi(q) = m, \quad \widehat{\pi}(q) = \widehat{m}, \quad R : T_m M \to T_{\widehat{m}} \widehat{M}, \qquad q \in \mathcal{Q},\, m \in M,\, \widehat{m} \in \widehat{M}.$$

This concludes the description of the problem's setting, but we recall that at the beginning we have imposed two constraints on the motion: no slipping and no twisting. These constraints translate into admissibility conditions of the movement with a precise geometric meaning. We recall, for example, Chapter 1, where the constraints allowed only two movements. So, the next step is to translate the constraints in order to build the distribution that describes the admissible movements of the system.

2.1 No slipping

If one manifold slides on the other, suppose $\widehat{M}$ on M, the point of contact on $\widehat{M}$ does not change, meaning that you are not following any curve on $\widehat{M}$, in particular

$$\widehat{m}(t) \equiv \widehat{m}(0)$$

if the sliding starts from the initial time. The consequence is that the velocity of the contact points is different $\dot{\widehat{m}}(t) = 0 \neq \dot{m}(t)$, therefore, to avoid slipping, we ask the velocity of the

contact points to be the same with respect to both manifolds.
We want that the velocities of the trajectories correspond one to another through the isometry, namely

$$R(t)\dot{m}(t) = \dot{\widehat{m}}(t). \tag{2.1}$$

2.2 No twisting

The angle θ measures the relative orientation between the manifold, if we allow twisting, the angle θ changes at the same contact point providing two different orientations for two same points' configurations.

To explicit this constraint we recur again to the notion of parallel transport, so in the following lines we refer to the connection on M as ∇ and to the connection on $\widehat{M}$ as $\widehat{\nabla}$. We recall that a vector field X defined along a curve γ in M is said to be *parallel* with respect to ∇, if $\nabla_{\dot{\gamma}}X = 0$ along γ. The no twisting condition is given by

$$\nabla_{\dot{m}(t)}X(t) = 0 \quad \Leftrightarrow \quad \widehat{\nabla}_{\dot{\widehat{m}}(t)}(R(t)X(t)) = 0, \tag{2.2}$$

namely we are asking that *the image through R of a parallel vector field along $m(t)$ is again a vector field parallel along $\widehat{m}(t)$*, i.e.,

$$q(t)(\text{vector field parallel along } m(t)) = \text{vector field parallel along } \widehat{m}(t).$$

The idea behind this condition is: if a vector field $X(t) \in T_{m(t)}M$ keeps a parallel direction wrt ∇ along the curve $m(t)$, we want that also the image through the isometry $q(t)X(t) \in T_{\widehat{m}(t)}\widehat{M}$ keeps a parallel direction wrt the other connection $\widehat{\nabla}$ along $\widehat{m}(t)$.This permits that the angle θ doesn't change at the same contact point, so it guarantees no twisting.

In some sense with these constraints the isometry $R(t)$ establishes the movement of the manifold $\widehat{M}$, because the velocities of the trajectory curves and the directions of the vector fields along the curves correspond one to the other at every point exactly thought $R(t)$.

In particular, these conditions permit us to draw completely the path of the motion $q(t)$. In the next section, we see that the velocity of a curve $q(t) \in \mathcal{Q}$ with all admissibility conditions is described by a rank 2 distribution Δ. The dimension of the distribution comes out to be 2 because the admissibility conditions tie up the velocities of the curves $m(t)$ to those of $\widehat{m}(t)$ and with the variation of the angle $\theta(t)$. This follows the intuition because we are rolling on a two-dimensional space, so two are the independent vector fields on this space.

Restoring again the ideas of the previous Chapter; we have seen what the configuration space is, now we want to understand to what movements in the principal directions correspond.

2.3 Building of the distribution Δ

Now we start the description of the admissible motions by explicitly detailing all the relations imposed by the constraints.

We have said that a curve $q(t)$ in $\mathcal{Q}$ describes the whole motion, the velocity of such a curve is a 5-tuple of the kind

$$\dot{q}(t) = (\underbrace{\dot{m}_1(t), \dot{m}_2(t)}_{\dot{m}(t)}, \underbrace{\dot{\widehat{m}}_1(t), \dot{\widehat{m}}_2(t)}_{\dot{\widehat{m}}(t)}, \dot{\theta}(t)).$$

As introduced before we are using orthonormal frames: $\{e_1, e_2\}_M$ on M and $\{\widehat{e}_1\,\widehat{e}_2\}_{\widehat{M}}$ on $\widehat{M}$, with structure's functions c_i and $\widehat{c}_i$ defined in this way

$$\begin{aligned} [e_1, e_2] &= c_1 e_1 + c_2 e_2, \qquad && c_i \in \mathcal{C}^\infty(M) \\ [\widehat{e}_1, \widehat{e}_2] &= \widehat{c}_1 \widehat{e}_1 + \widehat{c}_2 \widehat{e}_2, \qquad && \widehat{c}_i \in \mathcal{C}^\infty(\widehat{M}). \end{aligned}$$

We decompose in these frames the velocities of the trajectories in M and in $\widehat{M}$ in this way, where $a_i, \widehat{a}_i \in \mathbb{R}$ for $i = 1, 2$ since we suppose that the manifolds move with constant speed,

$$\begin{aligned} \dot{m} &= a_1 e_1 + a_2 e_2, \\ \dot{\widehat{m}} &= \widehat{a}_1 \widehat{e}_1 + \widehat{a}_2 \widehat{e}_2. \end{aligned}$$

From the no-slipping condition (2.2) we get the relations:

$$\begin{pmatrix} \widehat{a}_1 \\ \widehat{a}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 \cos\theta - a_2 \sin\theta \\ a_1 \sin\theta - a_2 \cos\theta \end{pmatrix}.$$

Notation. *We write e_i without specifying $e_i(m)$ when the point we are considering is clear and we recall that $\theta = \theta(t)$ even if we do not always write the time dependence.*

Now we want to obtain a similar expression for $\dot{\theta}$ using the no twisting condition. We consider $X(t)$ a smooth vector field in M, defined along $m(t)$

$$X(m(t)) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

saying that $X(t)$ is parallel along the trajectory $m(t)$ means imposing

$$\begin{aligned} 0 = \nabla_{\dot{m}} X|_{m(t)} &= \sum_{k=1}^{2} \left(\dot{f}_k(t) + \sum_{i,j=1}^{2} a_i(t) f_j(t) \Gamma_{ij}^k(m(t)) \right) e_k \\ &= (\dot{f}_1 + a_1 f_2 c_1 + a_2 f_2 c_2) e_1 + \dot{f}_2 - a_1 f_1 c_1 - a_2 f_1 c_2) e_2 \end{aligned}$$

that is

$$\nabla_{\dot{m}(t)} X(t) = 0 \iff \begin{cases} \dot{f}_1 = -a_1 f_2 c_1 - a_2 f_2 c_2 \\ \dot{f}_2 = a_1 f_1 c_1 + a_2 f_1 c_2 \end{cases} .$$

We obtain a similar system for the condition $\widehat{\nabla}_{\dot{\widehat{m}}(t)}(q(t)X(t)) = 0$:

$$q(t)X(t)) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} \cos\theta f_1 - \sin\theta f_2 \\ \sin\theta f_1 + \cos\theta f_2 \end{pmatrix}$$

$$\nabla_{\dot{\widehat{m}}(t)} q(t)X(t) = 0 \iff \begin{cases} \dot{g}_1 = -\widehat{a}_1 g_2 c_1 - \widehat{a}_2 g_2 c_2 \\ \dot{g}_2 = \widehat{a}_1 g_1 c_1 + \widehat{a}_2 g_1 c_2 \end{cases} .$$

Now computing the expression $\dot{g}_i$ and substituting the expression of $\dot{f}_i$ in the second system we obtain

$$\begin{cases} 0 = -\dot{\theta} f_1 \sin\theta - f_2 c_2 a_2 \cos\theta - f_2 c_1 a_1 \cos\theta - \dot{\theta}\cos\theta f_2 - f_1 c_1 a_1 \sin\theta - f_1 c_2 a_2 \sin\theta + \\ \quad + f_1 \widehat{a}_1 \widehat{c}_1 \sin\theta + f_2 \widehat{c}_1 \widehat{a}_1 \cos\theta + f_1 \widehat{c}_2 \widehat{a}_2 \sin\theta + f_2 \widehat{c}_2 \widehat{a}_2 \cos\theta = A \\ 0 = -\dot{\theta} f_1 \cos\theta - f_2 c_2 a_2 \sin\theta - f_2 c_1 a_1 \sin\theta - \dot{\theta}\sin\theta f_2 + f_1 c_1 a_1 \cos\theta + f_1 c_2 a_2 \cos\theta + \\ \quad - f_1 \widehat{a}_1 \widehat{c}_1 \cos\theta + f_2 \widehat{c}_1 \widehat{a}_1 \sin\theta + f_1 \widehat{c}_2 \widehat{a}_2 \cos\theta + f_2 \widehat{c}_2 \widehat{a}_2 \sin\theta = B \end{cases} \tag{2.3}$$

and finally we recover the expression of $\dot{\theta}$ by combining the equations in (2.3) in this way $\cos\theta A + \sin\theta B$, obtaining:

$$\begin{aligned} \dot{\theta} &= -c_1 a_1 - c_2 a_2 + \widehat{c}_1 \widehat{a}_1 + \widehat{c}_2 \widehat{a}_2 \\ &= a_1(-c_1 + \widehat{c_1}\cos\theta + \widehat{c}_2 \sin\theta) + a_2(-c_2 - \widehat{c_1}\sin\theta + \widehat{c}_2\cos\theta) . \end{aligned}$$

All expression of velocities depends on two parameters, that is, a_1, a_2 the two velocity coefficients of the trajectory $m(t)$. Therefore we observe that the movement in the e_1 direction, given by a_1 reflects a movement in the $\widehat{e}_1$ and $\widehat{e}_2$ direction and also in the variation of the angle $\frac{\partial}{\partial\theta}$; similarly for the movement in the e_2 direction,

$$\begin{gathered} \dot{m} = a_1 e_1 + a_2 e_2 \\ \dot{\widehat{m}} = a_1(\cos\theta e_1 + \sin\theta e_2) + a_2(-\sin\theta e_1 + \cos\theta e_2) \\ \dot{\theta} = a_1(-c_1 + \widehat{c_1}\cos\theta + \widehat{c}_2\sin\theta)\frac{\partial}{\partial\theta} + a_2(-c_2 + \widehat{c_1}\sin\theta + \widehat{c}_2\cos\theta)\frac{\partial}{\partial\theta}. \end{gathered}$$

This is the same thing as saying that admissible velocities are described by a rank two distributions on the manifold $\mathcal{Q}$. We call the distribution Δ, spanned by

$$\Delta = \operatorname{span}\{X_1, X_2\}$$

where

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ \cos\theta \\ \sin\theta \\ -c_1 + \widehat{c_1}\cos\theta + \widehat{c_2}\sin\theta \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ \sin\theta \\ \cos\theta \\ -c_2 + \widehat{c_1}\sin\theta + \widehat{c_2}\cos\theta \end{pmatrix}$$

are vector fields in $\mathcal{Q}$.

2.4 Controllability

With the construction of Δ we can interpret the problem of rolling as a control problem, of which we want to study the set of reachable configuration

$$\dot{q}(t) = v_1(t)X_1(q(t)) + v_2(t)X_2(q(t)) \qquad q(t) \in \mathcal{Q}. \tag{2.4}$$

Starting from a point $q_0 \in \mathcal{Q}$ and switching different values of v_1 and v_2 time to time, what other points in $\mathcal{Q}$ can we reach?

The parameters v_i, called controls, are piecewise constant functions that take values in $\{0,1\}$. These controls tell us at every time which flow is following the point. The idea is that v_1 and v_2 are like two buttons that correspond to one vector field, and that have to be pressed one at a time, they live in

$$U = \{(v_1, v_2) \in \mathbb{R}^2 : v_1 = 1 \Rightarrow v_2 = 0 \quad \& \quad v_2 = 1 \Rightarrow v_1 = 0\}.$$

Indeed, the parameter v_1 controls the vector field X_1 and v_2 controls X_2. Each choice of couples in U corresponds to a control strategy and each control strategy permits to reach different points on the manifold $\mathcal{Q}$.

Our aim is to understand what the biggest possible region of points that we can reach varying among all the control strategies.
At every time the velocity of the point is either X_1 or X_2 therefore, as already said, a point q moves along the flows of these vector fields. Now denoting ϕ_u the flow of the control problem (2.4) at time t we define the *reachable set* from a point $q_0 \in \mathcal{Q}$ for a time $t \geq 0$ as follows:

$$\mathcal{A}_{q_0}(t) = \left\{\phi_u^t(q_0) \mid u \in U, q_0 \in \mathcal{Q}\right\} = \left\{\phi_{u_k}^{\tau_k} \circ \ldots \circ \phi_{u_1}^{\tau_1}(q_0) \mid \tau_i \geq 0, \sum_{i=1}^{k} \tau_i = t, u_i \in U, k \in \mathbb{N}\right\}$$
$$= \left\{\phi_{X_{i_k}}^{\tau_k} \circ \ldots \circ \phi_{X_{i_1}}^{\tau_1}(q_0) \mid \tau_i \geq 0, \sum_{i=1}^{k} \tau_i = t, i_j \in \{1,2\}, k \in \mathbb{N}\right\}.$$

The key point is that we are composing the two flows, so this expression underlines that the reachable points may depend on the commuting property of the flows.
Following the flow of the vector field X_i is what we have called in the previous example (1) following the direction given by X_i. Therefore, changing flow means changing the direction, and

we want to understand if this has a consequence on the reachable set. We mentioned that this is related to the Lie bracket; in this precise sense:

Lemma 2.1. *Let M be a Riemannian manifold; $X, Y \in Vec(M)$ and $m \in M$ we have*

$$\phi_Y^{-t} \circ \phi_X^{-t} \circ \phi_Y^{t} \circ \phi_X^{t}(m) = m + t^2[X,Y](m) + o(t).$$

The Lie bracket measures the possibility that the flow, following the directions $X, Y, -X$ and $-Y$ has to return to the point m.

Therefore, if the flows of the two vector fields commute, that is, $[X,Y] = 0$, then we return to the same point where we have started moving, otherwise we end up in another point. If this is the case, we can define a new infinitesimal direction different from X, Y.

Summing up, if the flows commute, the set of reachable points does not depend on the number of times we change the flow. While in the non-commutative case, more number of switches corresponds to more points that can be reached: suppose that we can move along vector fields $\pm X_1$ and $\pm X_2$. Then, in the non-commutative case, we can move in the new infinitesimal direction $\pm [X_1, X_2]$. The new direction is in general linearly independent from the initial ones $\pm X_1, \pm X_2$; otherwise we have an *involutive* distribution and we have no new direction. In the case of a new direction obtained we can iterate the process, and again if linearly independent, we add one more infinitesimal direction $\pm [X_1, [X_1, X_2]]$. In the same way, we can also obtain $\pm [X_2, [X_1, X_2]]$.
Iterating this procedure with all the new directions, we obtain the Lie bracket of arbitrarily high order with a sufficiently large number of switches.

In general, given a distribution $\Delta = \text{span}\{X_1, \dots, X_k\}$ on the manifold $\mathcal{Q}$ we call the *Lie Algebra of the distribution* the space spanned by all the possible combinations of directions, i.e.,

$$Lie\Delta = \text{span}\{[X_1, [\dots, [X_{k-1}, X_k], \dots]] : X_i \in \Delta, k \in \mathbb{N}\}$$

and its evaluation at a point $q \in \mathcal{Q}$

$$Lie_q\Delta = \text{span}\{[X_1, [\dots, [X_{k-1}, X_k], \dots]]\Big|_q : X_i \in \Delta, k \in \mathbb{N}\} \subseteq T_q\mathcal{Q}.$$

In particular, we call a distribution *bracket generating* if $Lie_q\Delta = T_q\mathcal{Q}$ and in this case the dimension of the Lie algebra is equal to the dimension of the manifold on which the distribution is defined.
The previous intuition regarding the Lie bracket tries to justify the way in which we will compute the reachable set, the main fact is that

$$\mathcal{A}_{q_0} \cong Lie_{q_0}\Delta$$

so to understand the dimension of that set it is sufficient to work with the brackets of the distribution. This result is the core of the orbit theorem and we present a statement which is suitable for our purpose in the appendix; see A.3.

Remark. *In our case, there is no distinction between the reachable set of the system $\mathcal{A}_q$ and the orbit of the distribution $\mathcal{O}_q$, compare again A.3; therefore, from now on $\mathcal{A}_q = \mathcal{O}_q$ and we will refer to it as the reachable set of the system.*

In this context, to keep trace of the number of iterated brackets, it is useful to introduce the following notation:

$$\Delta^2 = \Delta + [\Delta, \Delta]$$
$$\Delta^{k+1} = \Delta^k + [\Delta, \Delta^k].$$

We find that a distribution on $\mathcal{Q}$ is called *bracket generating at a point* q if $\dim Lie_q\Delta = \dim \mathcal{Q}$, while we call the *step* of the distribution Δ at a point q the smallest integer k such that

$$\Delta_q^k = Lie_q\Delta.$$

In our setting, following this idea, the next step is to understand how many new directions we can obtain combining our initial ones X_1 and X_2. These vector fields live in a 5-dimensional manifold; therefore, the best outline we hope to obtain is to generate another 3 new directions. In the case of this Chapter, if the Lie algebra is of dimension 5 it means that we can reach all the points in the manifold, because

$$\dim Lie_q\Delta = 5 = T_q\mathcal{Q}$$

and we call the system completely controllable.

In this work, we investigate the sufficient conditions that permit us to obtain a completely controllable system, and the step at which this happens. Studying the Lie algebra generated by the directions X_1 and X_2 we figure out that the conditions are dictated by the curvature of the objects we are considering, as one may expect looking at the natural model.
The following is the main theorem that outlines the relation between the dimension of the reachable set and the curvature of the two manifolds. We denote $k, \widehat{k}$ the Gaussian curvature of the Riemannian manifolds M and $\widehat{M}$ respectively, and we consider the lift of these from the manifolds to $\mathcal{Q}$ in this sense:

$$k(q) = k(\pi(q)), \quad \widehat{k}(q) = \widehat{k}(\widehat{\pi}(q)),\ q \in \mathcal{Q}.$$

The formulas used in the statement and in the proof of the theorem are recalled in B.1.3.

2.5 Sufficient conditions for controllability

Theorem 2.1. *The reachable set $\mathcal{O}_q$ from a point $q_0 \in \mathcal{Q}$ is an immersed smooth connected submanifold of $\mathcal{Q}$ and it holds that*

- *if $k - \widehat{k}|_q \equiv 0 \quad \forall q \in \mathcal{O}_{q_0}$ then $\dim \mathcal{O}_{q_0} = 2$;*

•• *if* $k - \widehat{k}|_q \not\equiv 0 \quad \forall q \in \mathcal{O}_{q_0}$ *then* $\dim \mathcal{O}_{q_0} = 5$.

In particular, $\dim \mathcal{O}_q = 5$ *if and only if* $k - \widehat{k} \not\equiv 0$ *on* $\mathcal{O}_q$.

Proof. The orbit theorem (see A.3) assures us that the reachable set of the system from every point $q \in \mathcal{Q}$ is an immersed smooth connected submanifold of $\mathcal{Q}$. We underline again that the orbit theorem permits us to determinate the size of reachable set just by computing the brackets of Δ. Therefore it is only left to show that the dimension is either 2 or 5.
These are the principal directions deduced before:

$$\begin{aligned} X_1 &= e_1 + \cos\theta\,\widehat{e}_1 + \sin\theta\,\widehat{e}_2 + A\partial_\theta, \\ X_2 &= e_2 - \sin\theta\,\widehat{e}_1 + \cos\theta\,\widehat{e}_2 + B\partial_\theta; \end{aligned}$$

where A and B are functions that depends on the geometry of the manifolds,

$$\begin{aligned} A = A(m, \widehat{m}, \theta) &= -c_1 + \widehat{c}_1 \cos\theta + \widehat{c}_2 \sin\theta, \\ B = B(m, \widehat{m}, \theta) &= -c_2 - \widehat{c}_2 \sin\theta + \widehat{c}_2 \cos\theta. \end{aligned}$$

We start computing the Lie algebra, the iterated Lie brackets of the vector fields X_1, X_2. The first new direction is

$$X_3 := [X_1, X_2];$$

we compute it by:

$$X_3 = [e_1 + \cos\theta\,\widehat{e}_1 + \sin\theta\,\widehat{e}_2 + A\partial_\theta, e_2 - \sin\theta\,\widehat{e}_1 + \cos\theta\,\widehat{e}_2 + B\partial_\theta] = \tag{2.5}$$

$$= [e_1, e_2] + \cancel{[e_1, -\sin\theta\,\widehat{e}_1]} + \cancel{[e_1, \cos\theta\,\widehat{e}_2]} + [e_1, B\partial_\theta] + \tag{2.6}$$

$$+ \cancel{[\cos\theta\,\widehat{e}_1, e_2]} + \cancel{[\cos\theta\,\widehat{e}_1, -\sin\theta\,\widehat{e}_1]} + [\cos\theta\,\widehat{e}_1, \cos\theta\,\widehat{e}_1] + [\cos\theta\,\widehat{e}_1, B\partial_\theta] + \tag{2.7}$$

$$+ \cancel{[\sin\theta\,\widehat{e}_2, e_2]} + [\sin\theta\,\widehat{e}_2, -\sin\theta\,\widehat{e}_1] + \cancel{[\sin\theta\,\widehat{e}_2, -\sin\theta\,\widehat{e}_1]} + [\sin\theta\,\widehat{e}_2, B\partial_\theta] + \tag{2.8}$$

$$+ [A\partial_\theta, e_2] + [A\partial_\theta, -\sin\widehat{e}_1] + [A\partial_\theta, \cos\widehat{e}_2] + [A\partial_\theta, B(\theta)\partial_\theta]. \tag{2.9}$$

In the next computations it appears an expression which represents the difference of curvatures. The functions of curvatures k and $\widehat{k}$ are expressed through structure functions since we are working with orthonormal frames. We have obtained these expressions from straightforward computations in B.2:

$$\begin{aligned} k &= e_1(c_2) - e_2(c_1) - c_1^2 - c_2^2 \\ \widehat{k} &= \widehat{e}_1(\widehat{c}_2) - \widehat{e}_2(\widehat{c}_1) - \widehat{c}_1^2 - \widehat{c}_2^2. \end{aligned}$$

Now we compute every piece of the previous formula (2.9) and from now on we don't write anymore the zero Lie Brackets:

$$\begin{aligned}
[e_1, e_2] =& c_1 e_1 + c_2 e_2 \\
[e_1, B\partial_\theta] =& -[e_1, c_2\partial_\theta] = -e_1(c_1)\partial_\theta \\
[\cos\theta\,\widehat{e}_1, \cos\theta\,\widehat{e}_2] =& \cos^2(\widehat{e}_1(\widehat{e}_2) - \widehat{e}_2(\widehat{e}_1)) \\
[\cos\theta\,\widehat{e}_1, B\partial_\theta] =& \cdots = B\sin\theta\widehat{e}_1 + \cos\theta(-\sin\theta\widehat{e}_1(\widehat{c}_1) + \cos\theta\widehat{e}_1(\widehat{c}_2))\partial_\theta \\
[\sin\theta\,\widehat{e}_2, -\sin\theta\,\widehat{e}_1] =& -\sin\theta^2(\widehat{e}_2(\widehat{e}_1) + \widehat{e}_1(\widehat{e}_2)) \\
[\sin\theta\,\widehat{e}_2, B\partial_\theta] =& \cdots = B\cos\theta\widehat{e}_2 + \sin\theta(-\sin\theta\widehat{e}_2(\widehat{c}_1) + \cos\theta\widehat{e}_2(\widehat{c}_2))\partial_\theta \\
[A\partial_\theta, e_2] =& -[c_1\partial_\theta, e_2] = e_2(c_1)\partial_\theta \\
[A\partial_\theta, -\sin\widehat{e}_1] =& \cdots = -A\cos\theta\widehat{e}_1 + \sin\theta(\cos\theta\widehat{e}_1(\widehat{c}_1) - \sin\theta\widehat{e}_1(\widehat{c}_2))\partial_\theta \\
[A\partial_\theta, \cos\widehat{e}_2] =& \cdots = -A\sin\theta\widehat{e}_2 + \cos\theta(\cos\theta\widehat{e}_2(\widehat{c}_1) + \sin\theta\widehat{e}_2(\widehat{c}_2))\partial_\theta \\
[A\partial_\theta, B(\theta)\frac{\partial}{\partial\sigma}] =& \cdots = A(-\widehat{c}_1\cos\theta - \widehat{c}_2\sin\theta)\partial_\theta - B(-\widehat{c}_1\sin\theta - \widehat{c}_2\cos\theta)\partial_\theta
\end{aligned}$$

Now simplifying the terms:

$$\begin{aligned}
X_3 =& c_1e_1 + c_2e_2 + \\
&+(\widehat{c}_1 + B\sin\theta - A\cos\theta)\widehat{e}_1 + (\widehat{c}_2 - B\cos\theta - A\sin\theta)\widehat{e}_2 + \\
&+[-e_1(c_2) + e_2(c_1) + \widehat{e}_1(\widehat{c}_2) - \widehat{e}_2(\widehat{c}_1) - \widehat{c}_1^2 - \widehat{c}_2^2 + c_1A + c_1^2 + c_2B + c_2^2]\partial_\theta \\
=& c_1e_1 + c_2e_2 + (c_1\cos\theta - c_2\sin\theta)\widehat{e}_1 + (c_1\sin\theta + c_2\cos\theta)\widehat{e}_2 + (c_1A + c_2B + (k - \widehat{k})\partial_\theta)
\end{aligned}$$

We can write this expression in a compact way, useful for the next computations:

$$X_3 = \begin{pmatrix} c_1 \\ c_2 \\ \widehat{c}_1 + B\sin\theta - A\cos\theta \\ \widehat{c}_2 - B\cos\theta - A\sin\theta \\ c_1A + c_2B + (k - \widehat{k})\partial_\theta \end{pmatrix} = c_1X_1 + c_2X_2 + (k - \widehat{k})\partial_\theta.$$

Up to now we have obtained the distribution at *step* 2, with the following computation we are performing *step* 3. We continue with the computations of the next Lie Brackets,

$$X_4 := [X_1, X_3] = [X_4, c_1X_1] + [X_1, c_2X_2] + [X_1, (\widehat{k} - k)\partial_\theta]$$

where

$$\begin{aligned}
[X_4, c_1X_1] =& X_1(c_1X_1) - c_1X_1(X_1) = X_1(c_1)X_1 \\
[X_1, c_2X_2] =& X_1(c_2X_2) + c_2[X_1, X_2] = X_1(c_2X_2) + c_2X_3 \\
[X_1, (\widehat{k} - k)\partial_\theta] =& X_1(\widehat{k} - k)\partial_\theta + (\widehat{k} - k)[X_1, \partial_\theta]\,.
\end{aligned}$$

$$X_5 := [X_2, X_3] = [X_2, c_1X_1] + [X_2, c_2X_2] + [X_2, (\widehat{X} - k)\partial_\theta]$$

where

$$
\begin{aligned}
[X_2, c_1 X_1] =& X_2(c_1) X_1 - c_1 X_3 \\
[X_2, c_2 X_2] =& X_2(c_2) X_2 \\
[X_2, (\widehat{k} - k)\partial_\theta] =& X_2(\widehat{k} - k)\partial_\theta + (\widehat{k} - k)[X_2, \partial_\theta] \, .
\end{aligned}
$$

We have concluded *step* 3, if the vector fields obtained are independent, we reach at this step the bracket generating property. So now we check what is the dimension of the space spanned by $\{X_1, X_2, X_3, X_4, X_5\}$

$$
\begin{pmatrix}
1 & 0 & c_1 & X_1(c_1) + c_1 c_2 & X_2(c_1) + c_1^2 \\
0 & 1 & c_2 & X_1(c_1) + c_2^2 & X_2(c_2) + c_1 c_2 \\
0 & 0 & \widehat{k} - k & c_2(\widehat{k} - k) + X_1(\widehat{k} - k) & -c_1(\widehat{k} - k) + X_2(\widehat{k} - k) \\
0 & 0 & 0 & (\widehat{k} - k) & 0 \\
0 & 0 & 0 & 0 & (\widehat{k} - k)
\end{pmatrix} .
$$

The determinant is identically $(\widehat{k} - k)^3$, therefore, if $\widehat{k} - k|_{\mathcal{O}} \not\equiv 0$ the dimension of $\mathcal{O}$ is 5, while if $\widehat{k} - k|_{\mathcal{O}} \equiv 0$ then $span(X_1, \ldots, X_5) = span(X_1, X_2)$ and so the dimension is 2. This concludes all implications in the statement.

□

The difference of curvatures appears as a coefficient in front of the vector field $\frac{\partial}{\partial \theta}$ in the first computation; it is exactly the term that decides whether the new direction X_3 is linearly independent from the others or not.

Looking at the theorem, we see that we have only two possible cases, when the dimension is 5, the system is completely free to reach every possible configuration, while in the other case it is not. When the system is not completely controllable, we have less freedom: we can choose freely our initial configuration, we start the motion, but we cannot decide completely the final configuration. We have only 2 degrees of freedom out of five, meaning that if, for example, we want to choose the final contact point $m(t)$ on M, the other $\widehat{m}(t)$ is automatically determined as the final relative orientation $\theta(t)$.

Chapter 3

An n dimensional example: The sphere $\mathbb{S}^n$ rolling on the plane $\mathbb{R}^n$

In this Chapter, we generalize in a n-dimensional context the example seen in Chapter 1. We consider the unit n-dimensional sphere $\mathbb{S}^n$ rolling without slipping or twisting along the n-dimensional plane $\mathbb{R}^n$. The manifolds are both immersed in $\mathbb{R}^{n+1}$ and to describe the problem we use references in this space; here, the approach is no more intrinsic.

We endow the sphere $\mathbb{S}^n$ with an orthonormal frame $\mathcal{E} = \{e_1, \dots, e_{n+1}\}$ attached to the centre c. The frame during the motion remains fixed with respect to the sphere, it describes the sphere's relative orientation. The state configuration of the system is the manifold

$$\mathcal{Q} = \mathbb{R}^n \times SO(n+1)$$

since the projection of the centre, $\pi(c) =: x \in \mathbb{R}^n$, the contact point, and an arbitrary orientation of the sphere $R \in SO(n+1)$ describe a configuration of the system. The manifold $\mathcal{Q}$ is also in this case a Lie group, thus it is induced a Lie algebra and every tangent space is characterized analogously to the 2-dimensional case.

Therefore, we have that, the velocity at point $q = (x, R) \in \mathcal{Q}$ is described by a tuple in

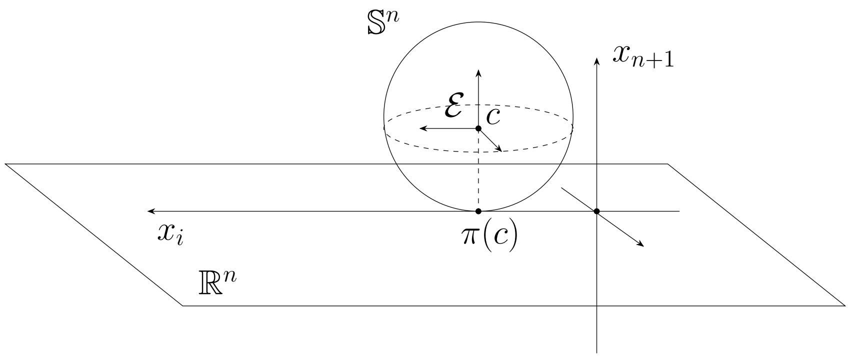
$$T_q\mathcal{Q} = \mathbb{R}^n \times R \cdot \mathfrak{so}(n+1).$$

We denote a basis of the skew-symmetric matrices $\mathfrak{so}(n+1)$ with

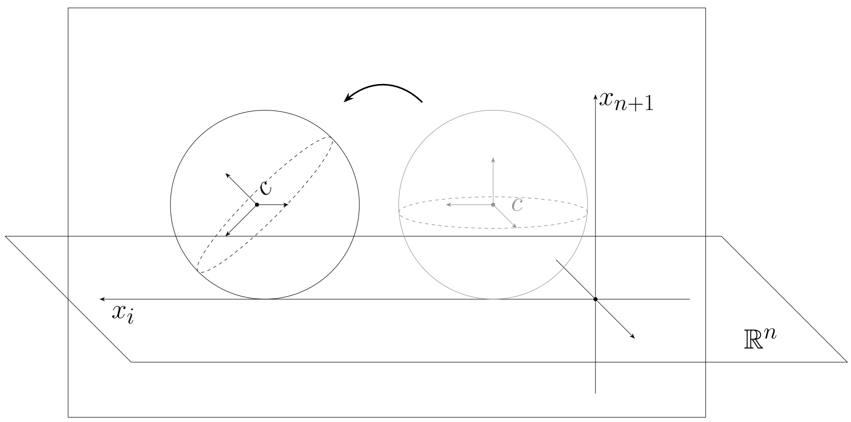
$$e_{i,j} := e_i \otimes e_j - e_j \otimes e_i \qquad \text{for } 1 \leq j < i \leq n+1$$

where we use the tensor notation, in particular $e_{i,j}$ is a matrix with a $+1$ in position (i, j) and a -1 in the anti-symmetric position (j, i). We fix the coordinates in $\mathbb{R}^{n+1}$ in such a way that sphere $\mathbb{S}^n$ is rolling on the plane defined by equation $x_{n+1} = 0$, i.e., the $n + 1$ coordinate of its center is always equal to one.

We can roll the sphere in n independent directions along the plane x_{n+1} and every movement



corresponds to a rotation in $SO(n)$. If we roll the sphere in the x_i direction, where $i = 1, \ldots, n$ along the plane; we obtain an infinitesimal rotation of the sphere in the 2-dimensional plane in $\mathbb{R}^{n+1}$ defined by the plane that contains the x_i and x_{n+1} directions.



Therefore for every direction x_i we have the vector field

$$X_i = \frac{\partial}{\partial x_i} - R \cdot e_{i,n+1} \qquad i = 1, \ldots, n$$

that describes the movement in that direction, in particular $X_i \in T_q\mathcal{Q}$ with $q = (x, R)$ contact point. Now we check if the distribution Δ spanned by all the directions is bracket generating;

$$\Delta = \text{span}\{X_1, \ldots, X_n\}.$$

We have that

$$\begin{aligned} [X_i, X_j] &= [\frac{\partial}{\partial x_i} - R \cdot e_{i,n+1}, \frac{\partial}{\partial x_j} - R \cdot e_{j,n+1}] \\ &= [R \cdot e_{i,n+1}, R \cdot e_{j,n+1}] = R \cdot [e_{i,n+1}, e_{j,n+1}] \qquad \forall i, j; \\ &= R \cdot e_{i,j} \end{aligned}$$

since the matrices commutator is

$$\begin{aligned}[e_{i,n+1}, e_{j,n+1}] &= e_{i,n+1}e_{j,n+1} - e_{j,n+1}e_{i,n+1} \\ &= -e_i \otimes e_j + e_j \otimes e_i \qquad\qquad \forall i, j. \\ &= e_{i,j}\end{aligned}$$

Therefore we have $n(n-1)$ new directions

$$X_{i+j} = [X_i, X_j] = R \cdot e_{i,j} \qquad \forall i < j$$

among which $\frac{n(n-1)}{2}$ are independent because $X_{i+j} = -X_{j+i}$. We iterate the computation (step 3)

$$[X_{i+j}, X_k] = R \cdot [e_{i,j}, e_{k,n+1}] \qquad \forall i < j, k. \tag{3.1}$$

In particular,

$$[e_{i,j}, e_{k,n+1}] = \delta_{jk} e_{i,h} - \delta_{ik} e_{j,h}$$

that gives
if $k \neq i$ and $k \neq j$

$$[e_{i,j}, e_{k,n+1}] = 0,$$

if $k = i$

$$[e_{i,j}, e_{i,n+1}] = -e_{j,n+1},$$

if $k = j$

$$[e_{i,j}, e_{j,n+1}] = e_{i,n+1}.$$

From this step, we obtain another n independent directions allowing us to reach the dimension of the configuration space. Indeed, summing all the independent directions, we get

$$n + \frac{n(n-1)}{2} + n = \frac{n(n+3)}{2}$$

exactly the dimension of $\mathcal{Q}$. We can conclude that the distribution Δ is bracket generating, so the system is completely controllable.

Chapter 4

The n dimensional case

Now we restore the intrinsic approach, but in a wider context. The aim of this Chapter is to generalize the theory of Chapter 2. We study the same problem, but we do not fix a precise manifold or a precise dimension anymore.

The objects of rolling are M and $\widehat{M}$ two n-dimensional connected and oriented Riemannian manifolds. In this Chapter, as in Chapter 2, the setting is symmetric, there is no preferred manifold rolling on the other one, and also in this sense, the approach here is intrinsic.

The configuration space $\mathcal{Q}$ is conceived in the same way,

$$\mathcal{Q} = \{R \in SO(T_m M, T_{\hat{m}} \widehat{M}) : m \in M, \hat{m} \in \widehat{M}\}$$

the isometry R represents as before the relative orientation between the manifolds and a point $q \in \mathcal{Q}$ a configuration. As in Chapter 2 we build a distribution Δ in $\mathcal{Q}$ which describes the velocities of the motion and again the admissible conditions will permit us to link the velocities of the two trajectory curves and the relative orientation of the manifolds.

One can observe that the setting is exactly the same and we will go over again the same idea of the 2-dimensional case; the difficulty comes out with computations. Our goal is to find sufficient conditions to understand when the distribution Δ is bracket generating, but one can immediately understand that it is not possible to compute everything by hand as before.

Therefore, we present this generalization because the idea is the same, but the solution proposed to simplify the procedure is interesting. Before giving further explanation, we go again through the building of the distribution.

4.1 The distribution Δ

The configuration space of the system is

$$\mathcal{Q} = \{R \in SO(T_m M, T_{\hat{m}} \hat{M}) : m \in M, \widehat{m} \in \widehat{M}\} \tag{4.1}$$
$$= \{R : T_m M \to T_{\hat{m}} \widehat{M} : m \in M, \widehat{m} \in \widehat{M}, R \in SO(n)\}; \tag{4.2}$$

and a state of the system or a configuration q this time is a $\frac{n(n+3)}{2}$-uple, indeed

$$q \in \mathcal{Q} \cong M \times \widehat{M} \times SO(n).$$

We denote with $\overline{\pi}$ the following projection

$$\overline{\pi} : \mathcal{Q} \to M \times \widehat{M}.$$

Let $q = (m, \widehat{m}, R) \in \mathcal{Q}$ be a configuration, the tangent space at q is

$$\begin{aligned} T_q \mathcal{Q} &= T_m M \times T_{\hat{m}} \widehat{M} \times T_R SO(n) \\ &\cong T_m M \times T_{\hat{m}} \widehat{M} \times R \cdot \mathfrak{so}(n). \end{aligned}$$

We denote by $e = \{e_1, \dots, e_n\}$ an orthonormal frame for $T_m M$, $\widehat{e} = \{\widehat{e}_1, \dots, \widehat{e}_n\}$ an orthonormal frame for $T_{\hat{m}} \widehat{M}$ and $\{e_{i,j} : 1 \leq i < j \leq n\}$ is the usual basis of the anti-symmetric matrices $\mathfrak{so}(n)$, we recall that

$$e_{i,j} := e_i \otimes e_j - e_j \otimes e_i.$$

In order to have local references, we define these vector fields

$$W_{i,j} := R \cdot e_{i,j} \qquad \text{for } 1 \leq i < j \leq n, \tag{4.3}$$

these are vector fields tangent to $SO(n)$ at the point R and can be considered a *local left invariant basis* of $\ker \overline{\pi}_*$.

Before writing the distribution, we recall the two admissibility conditions.

4.1.1 No slipping

The no slipping condition is

$$\dot{\widehat{m}}(t) = R(t) \dot{m}(t). \tag{4.4}$$

It describes the $T_{\hat{m}} \widehat{M}$ components of the vector fields in the distribution.

4.1.2 No twisting

If the no slipping condition reads identically, the no twisting condition needs an adjustment in order to be read in a more suitable way for this case.

Let $X(m(t))$ a vector field on M defined along the curve $m(t)$, the no twisting condition is

$$R(t)(\nabla_{\dot{m}(t)}X(m(t))) = \widehat{\nabla}_{R\dot{m}(t)}R(t)X(m(t)) \tag{4.5}$$

that is equivalent to the one we have presented before 2.2

$$\nabla_{\dot{m}(t)}X(m(t)) = 0 \Leftrightarrow \widehat{\nabla}_{\dot{\widehat{m}}(t)}R(t)X(m(t)) = 0;$$

the horizontal vector fields correspond one to other through the isometry R. This condition in particular permits to describe the $T_RSO(n)$ components of the vector fields in the distribution, in this sense: let $q(t)$ be a rolling, then

$$\dot{q}(t) = \dot{m}(t) + \dot{\widehat{m}}(t) + \dot{R}(t)$$

and we want to write $\dot{R}(t)$ in the left invariant basis of given by the $W_{i,j}$ in 4.3. We recall that

$$R(t)e_j = \sum_{i=1}^{n} R_{ij}(t)\widehat{e}_i, \qquad \text{and} \qquad R^{-1}(t)\widehat{e}_i = \sum_{j=1}^{n} R_{ij}(t)e_j$$

for orthonormal frames e and $\widehat{e}$.

Notation. *In the next computations, we use two similar but different expressions, namely*

$$\begin{aligned} e_{ij} &= e_i \otimes e_j \\ e_{i,j} &= e_i \otimes e_j - e_j \otimes e_i. \end{aligned}$$

Moreover, from now on we avoid writing the curve along which the vector field is defined, or we simply use a lighter notation when it is necessary:

$$X(m(t)) = X|_{m(t)} = X.$$

Condition 4.5 holds if and only if

$$R(\nabla_{\dot{m}(t)}e_j|_{m(t)}) = \widehat{\nabla}_{R\dot{m}(t)}R(e_j|_{m(t)}) \qquad \forall j = 1, \dots, n$$

if and only if

$$\left\langle R\nabla_{e_k}e_j|_{m(t)} - \widehat{\nabla}_{Re_k}Re_j|_{m(t)}, \widehat{e}_i|_{\widehat{m}(t)} \right\rangle = 0 \qquad \forall i, j = 1, \dots, n.$$

Computing the second term of the left hand side

$$\begin{aligned} \widehat{\nabla}_{Re_k}Re_j &= \widehat{\nabla}_{Re_k}\left(\sum_i R_{ij}\widehat{e}_i\right) \\ &= \sum_i \dot{R}_{ij}\widehat{e}_i + \sum_i R_{ij}\widehat{\nabla}_{Re_k}\widehat{e}_i \end{aligned}$$

we obtain

$$\begin{aligned}
0 &= \left\langle R\nabla_{e_k} e_j(m(t)) - \widehat{\nabla}_{Re_k} Re_j(m(t)), \widehat{e}_i \right\rangle \\
&= \left\langle \nabla_{e_k} e_j, R^{-1}\widehat{e}_i \right\rangle - \left\langle \sum_{k=1}^{n} \dot{R}_{kj} \widehat{e}_k, \widehat{e}_i \right\rangle - \left\langle \sum_{k=1}^{n} R_{kj} \nabla_{Re_k} \widehat{e}_k, \widehat{e}_i \right\rangle \\
&= \sum_{l=1}^{n} R_{il} \left\langle \nabla_{e_k} e_j, e_l \right\rangle - \dot{R}_{ij} - \sum_{l=1}^{n} R_{lj} \left\langle \nabla_{Re_k} \widehat{e}_l, \widehat{e}_i \right\rangle
\end{aligned}$$

for every $i, j = 1, \ldots, n$. Therefore we have an expression for $\dot{R}_{ij}$

$$\begin{aligned}
\sum_{i,j=1}^{n} \dot{R}_{ij} e_{ij} &= \sum_{i,j=1}^{n} \left(\sum_{l=1}^{n} R_{il} \left\langle \nabla_{e_k} e_j, e_l \right\rangle - \sum_{l=1}^{n} R_{lj} \left\langle \nabla_{Re_k} \widehat{e}_l, \widehat{e}_i \right\rangle \right) e_{ij} \\
&= \sum_{j,l=1}^{n} \left\langle \nabla_{e_k} e_j, e_l \right\rangle R \cdot e_{lj} - \sum_{i,l=1}^{n} \left\langle \nabla_{Re_k} \widehat{e}_l, \widehat{e}_i \right\rangle e_{il} \cdot R \\
&= \sum_{s,r=1}^{n} \left\langle \nabla_{e_k} e_s, e_r \right\rangle R \cdot e_{rs} - \sum_{i,j,r,s=1}^{n} R_{ir} R_{js} \left\langle \nabla_{Re_k} \widehat{e}_j, \widehat{e}_i \right\rangle R \cdot e_{rs} \\
&= \sum_{r,s=1}^{n} \left(\left\langle \nabla_{e_k} e_s, e_r \right\rangle - \left\langle \nabla_{Re_k} \sum_{s=1}^{n} R_{js} \widehat{e}_j, \sum_{i=1}^{n} R_{ir} \widehat{e}_i \right\rangle \right) R \cdot e_{rs} \\
&= \sum_{r,s=1}^{n} \left(\left\langle \nabla_{e_k} e_s, e_r \right\rangle - \left\langle \nabla_{Re_k} Re_s, Re_r \right\rangle \right) R \cdot e_{rs} \\
&= \sum_{i,j=1}^{n} \left(\left\langle \nabla_{e_k} e_j, e_i \right\rangle - \left\langle \nabla_{Re_k} Re_j, Re_i \right\rangle \right) R \cdot e_{ij}
\end{aligned}$$

and writing it in the $W_{i,j}$ basis we have

$$\begin{aligned}
\sum_{i,j=1}^{n} \dot{R}_{ij} e_{ij} &= \sum_{i,j=1}^{n} \left(\left\langle \nabla_{e_k} e_j, e_i \right\rangle - \left\langle \nabla_{Re_k} Re_j, Re_i \right\rangle \right) R \cdot e_{ij} \\
&= \sum_{i<j}^{n} \left(\left\langle \nabla_{e_k} e_j, e_i \right\rangle - \left\langle \nabla_{Re_k} Re_j, Re_i \right\rangle \right) R \cdot e_{i,j} \\
&= \sum_{i<j}^{n} \left(\left\langle \nabla_{e_k} e_j, e_i \right\rangle - \left\langle \nabla_{Re_k} Re_j, Re_i \right\rangle \right) W_{i,j}.
\end{aligned}$$

Summing up, the distribution Δ, in the above basis, is spanned by n linearly independent vector fields

$$E_k := e_k + Re_k + \sum_{i<j}^{n} \left(\left\langle \nabla_{e_k} e_j, e_i \right\rangle - \left\langle \nabla_{Re_k} Re_j, Re_i \right\rangle \right) W_{i,j} \qquad \text{for } k = 1, \ldots, n; \tag{4.6}$$

i.e.,

$$\Delta = \{E_1, \ldots, E_n\}.$$

The distribution is built and now to understand if the system is controllable, we have to check if the distribution Δ is bracket generating. As we were saying, this requires a lot of computations,

and to simplify the procedure and to point out the relation with the curvature, we change perspective.
This means that we look at our problem from another space. Consider the frame bundle of the tangent bundle TM, we call it $F(M)$, and similarly consider $F(\widehat{M})$, the frame bundle on $T\widehat{M}$. The configuration space $\mathcal{Q}$ can be thought of as

$$\mathcal{Q} \cong F(M) \times F(\widehat{M})/SO(n);$$

which corresponds to the identification of tangent planes through an isometry $R \in SO(n)$. Given a point of contact $(m, \hat{m}, R)$, if there is an isometry that maps one frame on $T_m M$ to another frame on $T_{\hat{m}} \widehat{M}$, we identify the two frames. The change in perspective consists of working in a larger space: the space without this identification

$$F(M) \times F(\widehat{M}).$$

When we work in $\mathcal{Q}$ we are computing up to isometries, now we remove the identification between the tangent spaces. Indeed the space in which we have worked up to now can be seen as the target space of the projection:

$$F(M) \times F(\widehat{M}) \xrightarrow{\Pi} F(M) \times F(\widehat{M})/SO(n).$$

In particular, $\Pi(f, \hat{f}) = R$ if and only if $\hat{f} = R \cdot f$ for all f frame in M and for all $\hat{f}$ frame in $\widehat{M}$,

$$\begin{aligned} F(M) \times F(\widehat{M}) &\xrightarrow{\Pi} F(M) \times F(\widehat{M})/SO(n) \\ (f, \hat{f}) &\longmapsto R = \hat{f} \cdot f. \end{aligned}$$

The next step is finding the corresponding distribution to the rolling one in this bigger space, but before looking at this correspondence, let's understand in which spaces we are working.

4.2 The principal $SO(n)$-bundles $F(M)$ and $F(\widehat{M})$

Let us explain $F(M)$, the frame bundle on M. For each $m \in M$, a frame at m corresponds to the choice of a basis of $T_m M$. We can consider a frame f as a linear map in $GL(\mathbb{R}^n, T_m M)$. In particular, if f is an orthonormal frame, $f \in SO(\mathbb{R}^n, T_m M)$.

We call $F_m(M) := SO(\mathbb{R}^n, T_m M)$ the space of orthonormal frames of $T_m M$. There is a natural action of $SO(n)$ on $F_m(M)$ given by right composition

$$\begin{aligned} R_A : F(M) &\to F(M) \\ f &\mapsto f \cdot A; \end{aligned}$$

by rotating the frame, we remain within the space: $f \cdot A \in SO(\mathbb{R}^n, T_m M)$ for every $A \in SO(n)$ and for every $f \in SO(\mathbb{R}^n, T_m M)$.

Endowed with this structure $F(M)$ becomes a principal bundle

$$\tau : F(M) \longrightarrow M \text{ with right action } F(M) \times SO(n) \to F(M)$$
$$(f, A) \mapsto f \cdot A.$$

A point in this bundle is given by the couple (f, m) where f is the frame and m is the point where the frame is attached, sometimes if the point of attachment is clear, we write only f as a point in $F(M)$. The fibers of this bundle $F_m(M) = \tau^{-1}(m)$ for $m \in M$ are the set of all orthonormal frames with which we can endow the tangent space T_mM.

In order to be able to restore the setting of the problem, we want to provide a connection for this bundle. We remind that the constraints translate with the connection and that we have already chosen one on the manifold M. There is one choice of connection compatible, in a certain sense, with the Levi-Civita connection that we have given to M. In fact, we are in a special case, the connection on M induces the connection on $F(M)$ and vice versa; we explain this better in the last Chapter 5.
We call this connection $\widetilde{\nabla}$.

Considering this fiber bundle $\tau : F(M) \to M$, we recall some terminology of principal fiber bundles.
We call the space V_f of tangent vectors to the fiber through f, the *vertical subspace*

$$V_f := ker \tau_*|_f = T_f F(M).$$

Choosing a principal connection on $\tau : F(M) \to M$ means choosing a subspace H_f of $T_fF(M)$ for each $f \in F(M)$ such that

$$T_f F(M) = V_f \oplus H_f, \quad H_{A \cdot f} = (R_A)_* H_f \qquad \forall f \in F(M), A \in SO(n).$$

We call this subspace H_f the *horizontal subspace* of $T_fF(M)$.

Each fiber $F_m(M)$ for every $m \in M$ is isomorphic by definition to the group

$$F_m(M) \cong SO(n)$$

and there is an isomorphism between each vertical subspace and the Lie algebra of the group, $\mathfrak{so}(n)$

$$V_f := T_f F_m(M) \cong \mathfrak{so}(n).$$

Another important thing to remark is that $\tau_* : T_fF(M) \to T_{\tau(f)}M$ is an isomorphism when restricted to H_f and in this way we have the following identification

$$T_f F(M) = V_f \oplus H_f \cong \mathfrak{so}(n) \oplus T_{\tau(f)} M.$$

Given a frame $f = \{f_1, \dots, f_n\} \in F_m(M)$, thanks to the connection, we can lift the vector fields $f_1, \dots, f_n$ to a local basis of H_f. Denote a local basis for $\mathfrak{so}(n)$ as $\{e_{i,j} : 1 \leq j < i \leq n\}$ and let Γ_{ij}^k the Christoffel symbols related to the frame f, given by the Levi-Civita connection on M, namely:

$$\Gamma_{ij}^k = \langle \nabla_{f_i} f_j, f_k \rangle.$$

The horizontal lifts of the vector fields f_i, for $i = 1, \dots, n$ with respect to $\widetilde{\nabla}$ are given by

$$f_i^* = f_i - \sum_{\alpha<\beta} \Gamma_{i\beta}^{\alpha} e_{\alpha,\beta} \qquad \forall i = 1, \dots, n. \tag{4.7}$$

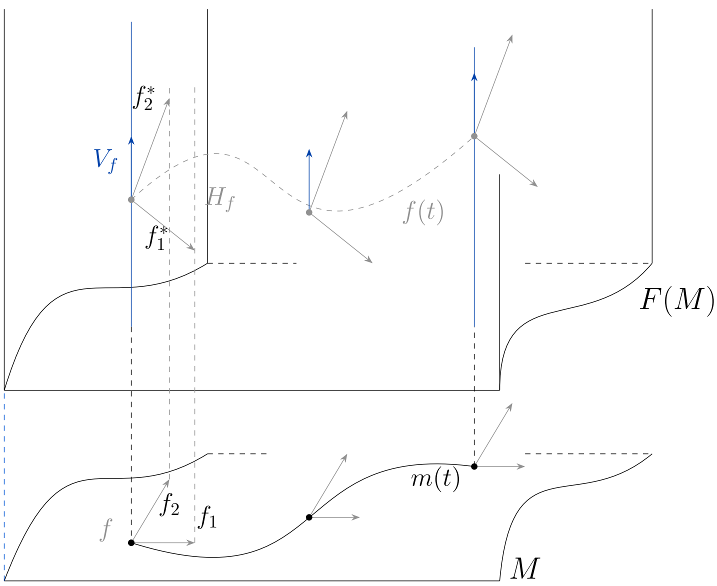
We recall that given a vector field $f_i(t)$ on M, its horizontal lift is the unique vector field $f_i^* \in \mathrm{Vec}(F(M))$ such that

$$f_i^*\Big|_f \in H_f \quad \text{and} \quad \tau_* f_i^* = f_i.$$

The process through which we arrive at the formulas in 4.7 is not trivial and we describe it in Chapter 5. In any case here it is important to understand in which sense the two connections are related. Notice that in the lifts we are using the Christoffel symbols of the connection on M, also in this sense they correspond. We have that

$$\begin{aligned} \nabla_{\dot m(t)} f_i = 0 \quad \forall i &\Leftrightarrow \widetilde{\nabla}_{\dot f(t)} (\text{tangent vectors to } f(t)) = 0; \\ \nabla_{\dot m(t)} f_i = 0 \quad \forall i &\Leftrightarrow \widetilde{\nabla}_{\dot f(t)} f_i^* = 0 \quad \forall i; \end{aligned}$$

meaning that we have the possibility of transporting in a parallel way the upper vector fields along a curve of frames in the very same way in which we transport the vector fields down along the curve $m(t) = \tau(f(t))$.



Remark. *Given a frame, the Christoffel symbols related to that frame are functions on the base manifold. The Christoffel symbols given by the Levi-Civita connection ∇ are functions on M*

$$\Gamma_{ij}^k : M \to \mathbb{R} \qquad \forall i, j, k;$$

by saying that we are using the same Christoffel symbols on $F(M)$, which are functions on $M \times SO(n)$, we mean considering the same functions but defined on a wider domain. Let $\overline{\Gamma}_{ij}^k$ be the C. symbols on $F(M)$

$$\overline{\Gamma}_{ij}^k : F(M) \to \mathbb{R} \qquad \forall i, j, k.$$

It holds

$$\overline{\Gamma}_{ij}^k = \overline{\Gamma}_{ij}^k\Big|_M = \Gamma_{ij}^k \qquad \forall i, j, k.$$

and we no longer distinguish them.

We recall that the horizontal lifts are invariant by the right action R_A, that is, for every $A \in SO(n)$

$$\tau(f_i^*|_{f'}) = \tau(R_A \cdot f_i^*|_f) = \tau(f_i^*|_f).$$

Given a connection, for each $f \in F(M)$ it remains defined a 1-form ω_f on $F(M)$ with values in V_f the tangent space of the fiber. The form ω_f is determined by

$$\omega_f(X) = \xi \Leftrightarrow X - \xi \in H_f \qquad \forall X \in T_f F(M);$$

in particular

$$\ker(\omega_f) = H_f.$$

When there is no risk of confusion we write only ω avoiding the pedicle.

We define also the *tautological one-form* $\theta = (\theta_1, \dots, \theta_n)$ on $F(M)$, as the 1-form with values in $\mathbb{R}^n$ such that

$$\theta_j|_f = \tau^*(\flat f_j)$$

where $\flat : TM \to T^*M$ is the isomorphism induced by the Riemannian metric.
In particular, the horizontal lifts satisfy

$$\theta_j(f_i^*) = \langle f_j, \tau_* f_i \rangle = \delta_{i,j}.$$

The description of $F(\widehat{M})$ is analogous.

Now that we have clarified the wider context in which we want to make the computations, we have to understand what rolling looks like.

4.3 The rolling distribution in $F(M) \times F(\widehat{M})$

We have to readapt the description of the rolling to this space so we start again by translating the constraints.
Let $(f(t), \widehat{f}(t))$ be a curve in $F(M) \times F(\widehat{M})$ such that $\Pi(f(t), \widehat{f}(t)) = R(t)$, we are wondering if $R(t)$ is rolling without slipping or twisting what are the conditions that describe the motion on the upper curve? Are these conditions also sufficient?

The two constraints of the motion have a precise translation in $F(M) \times F(\widehat{M})$ through the aforementioned forms: ω and θ of $F(M)$; $\widehat{\omega}$ and $\widehat{\theta}$ of $F(\widehat{M})$. As we are now working on the Cartesian product $F(M) \times F(\widehat{M})$, from now on, we consider all of these as forms in this space.

Now we give a local description of the space $F(M) \times F(\widehat{M})$, it will be useful also later. We first choose a pair of local sections (e, U) of $F(M)$ and $(\widehat{e}, \widehat{U})$ on $F(\widehat{M})$. Let $(m, \widehat{m}) \in U \times \widehat{U}$, then any other frame $(f, \widehat{f}) \in F_m(M) \times F_{\hat{m}}(\widehat{M})$ can be written in this form

$$f_j = \sum_{i=1}^{n} f_{ij} e_i|_m$$
$$\widehat{f}_j = \sum_{i=1}^{n} \widehat{f}_{ij} \widehat{e}_i|_{\widehat{m}}.$$

For what we have said before we have that

$$T_m M \times T_{\hat{m}} \widehat{M} \times f \cdot \mathfrak{so}(n) \times \hat{f} \cdot \mathfrak{so}(n) \cong H_f \times H_{\hat{f}} \times V_f \times V_{\hat{f}}.$$

We choose a local left basis of V_f and $V_{\hat{f}}$ in this way

$$w_{i,j} := f \cdot e_{i,j} \qquad \forall i < j; \tag{4.8}$$
$$\widehat{w}_{i,j} := \widehat{f} \cdot e_{i,j} \qquad \forall i < j \tag{4.9}$$

and this comes up to be the local description of the tangent spaces which we use:

$$T_f F(M) = \operatorname{span}\{\underbrace{f_1^*, \ldots, f_n^*}_{H_f}, \underbrace{w_{i,j} : i < j}_{V_f}\}$$
$$T_{\hat{f}} F(\widehat{M}) = \operatorname{span}\{\underbrace{\widehat{f}_1^*, \ldots, \widehat{f}_n^*}_{H_{\hat{f}}}, \underbrace{\widehat{w}_{i,j} : i < j}_{V_{\hat{f}}}\}.$$

We know that $ker(\omega) = H_f$ and $\ker(\widehat{\omega}) = H_{\hat{f}}$ but considering them as forms on $F(M) \times F(\widehat{M})$

we have that

$$\ker(\omega) = \operatorname{span}\{H_f, H_{\hat{f}}, V_{\hat{f}}\}$$
$$\ker(\widehat{\omega}) = \operatorname{span}\{H_{\hat{f}}, H_f, V_f\}.$$

Similarly for θ and $\widehat{\theta}$ as forms on $F(M) \times F(\widehat{M})$,

$$(\theta - \widehat{\theta})(f_i^* + \widehat{f}_j^*) = \theta(f_i^*) - \widehat{\theta}(\widehat{f}_j^*) = 0 \text{ if and only if } i = j$$

therefore

$$ker(\theta - \widehat{\theta}) = \operatorname{span}\{V_f, V_{\hat{f}}, f_i^* + \widehat{f}_i^* : i = 1, \dots n\}.$$

4.3.1 No slipping

It holds that

$$R(t) \text{ is a rolling without slipping } \Leftrightarrow (f(t), \widehat{f}(t)) \text{ is horizontal to } \ker(\theta - \widehat{\theta}).$$

Let's understand why.

Remark. *Where we say that a curve $\gamma(t)$ is horizontal to a distribution D if $\dot{\gamma}(t) \in D|_{\gamma(t)}$.*

The rolling condition on $\mathcal{Q}$ says that $R(t)$ is a rolling without slipping if and only if, see 4.4

$$\dot{\hat{m}}(t) = R(t) \cdot \dot{m}(t) \quad \text{with} \quad R(t) \in SO(T_m M, T_{\hat{m}} \widehat{M}).$$

Since $R = \Pi(f, \widehat{f}) = \widehat{f} \cdot f^T$ we have

$$\dot{\hat{m}}(t) = \left(\widehat{f}(t) \cdot f(t)^T\right) \cdot \dot{m}(t) \quad \text{with} \quad f(t) \in SO(\mathbb{R}^n, T_m M)$$
$$\text{and} \quad \widehat{f}(t) \in SO(\mathbb{R}^n, T_{\hat{m}} \widehat{M});$$

this leads to

$$\widehat{f}(t)^T \cdot \dot{\hat{m}}(t) = f(t)^T \cdot \dot{m}(t).$$

We can break this expression in

$$\widehat{f}_i^T \cdot \dot{\hat{m}} = f_i^T \cdot \dot{m} \qquad \forall i$$

and since now these are equations between maps with values in $\mathbb{R}^n$ and in particular between matrices, we have

$$\widehat{f}_i = f_i \qquad \forall i.$$

Therefore the no slipping condition is translated in n equations and the vector fields in $F(M) \times F(\widehat{M})$ that satisfy these equations are vector fields in $\langle V_f, V_{\hat{f}} \rangle$ or of the kind $f_i^* + \widehat{f}_i^*$ so exactly the vector fields in $ker(\theta - \widehat{\theta})$.

4.3.2 No twisting

It holds that

$$R(t) \text{ is a rolling without twisting } \Leftrightarrow (f(t), \widehat{f}(t)) \text{ is horizontal to } \ker(\omega) \cap \ker(\widehat{\omega}).$$

The condition 4.5 tells us that the isometry R preserves the parallelism. The frames that correspond through the isometry R are transported in a parallel way with respect to the connection, so if $\Pi(f, \widehat{f}) = R$ we have

$$\widetilde{\nabla}_{\dot{f}(t)} f_i^* = 0 \quad \forall i$$
$$\widetilde{\nabla}_{\dot{\widehat{f}}(t)} \widehat{f}_i^* = 0 \quad \forall i$$

and thanks to the choice of the connection $\widetilde{\nabla}$ this corresponds to

$$\nabla_{\dot{m}(t)} f_i(t) = 0 \qquad \forall i = 1, \dots, n$$
$$\nabla_{\dot{\widehat{m}}(t)} \widehat{f}_i(t) = 0 \qquad \forall i = 1, \dots, n.$$

Therefore asking $R(t)$ to be a rolling without twisting is asking every vector field $f_i(t)$ to be parallel along $m(t)$ and every vector field $\widehat{f}_i(t)$ to be parallel along $\widehat{m}(t)$, and this means asking them to be in the distribution which defines the connection

$$f_i^* \in H_f = \ker \omega$$
$$\widehat{f}_i^* \in H_{\hat{f}} = \ker \widehat{\omega}.$$

Summing up, we have understood that if $(f(t), \widehat{f}(t))$ is a curve in $F(M) \times F(\widehat{M})$ such that $\Pi(f(t), \widehat{f}(t)) = R(t)$ then both constraints impose:

$$\begin{array}{c} R(t) \text{ is a rolling} \\ \text{no slipping or twisting} \end{array} \Leftrightarrow \begin{array}{c} (f(t), \widehat{f}(t)) \text{ is horizontal to} \\ \ker(\theta - \widehat{\theta}) \cap \ker(\omega) \cap \ker(\widehat{\omega}). \end{array}$$

We call $\mathcal{D}$ the upper rolling distribution.

4.4 The correspondence between the distributions

We recall that the relation between the configurations' space and the frame bundles is given by

$$F(M) \times F(\widehat{M}) \xrightarrow{\Pi} F(M) \times F(\widehat{M})/SO(n).$$

The distributions we have built with the constraints have indeed a precise correspondence through the map Π, given by the following theorem.

This gives precisely the idea that we are looking at the same object but in a wider context.

Theorem 4.1. *Let Δ be the rolling distribution in $\mathcal{Q}$ and let*

$$\mathcal{D} := \ker\omega \cap \ker\widehat{\omega} \cap \ker(\theta - \widehat{\theta});$$

be the rolling distribution in $F(M) \times F(\widehat{M})$; it holds that

$$\Pi_*\mathcal{D} = \Delta$$

and the map is a linear isomorphism on every fiber.

Proof. We prove this theorem in two steps: first, in section 4.4.1, we build the distribution $\mathcal{D}$ in coordinates, and then, in 4.4.2, we study the comparison between the two distributions.
To keep trace of the dimension at every step we call $s := \dim(SO(n)) = \dim(\mathfrak{so}(n))$ and n still refers to the dimension of M and to the dimension of $\widehat{M}$.

4.4.1 The distribution $\mathcal{D}$

We work again with local coordinates, so again, we choose a pair of local sections (e, U) of $F(M)$ and $(\widehat{e}, \widehat{U})$ on $F(\widehat{M})$. Let $(m, \widehat{m}) \in U \times \widehat{U}$, then any other frame $(f, \widehat{f}) \in F_m(M) \times F_{\widehat{m}}(\widehat{M})$ can be written in this form

$$f_j = \sum_{i=1}^{n} f_{ij} e_i|_m$$
$$\widehat{f}_j = \sum_{i=1}^{n} \widehat{f}_{ij} \widehat{e}_i|_{\widehat{m}}.$$

The Cartesian product of the frame bundles is a $2n + 2s$-dimensional space and we use these local coordinates

$$F(M) \times F(\widehat{M}) \cong M \times \widehat{M} \times SO(n) \times SO(n);$$
$$(f, \widehat{f}) \mapsto (m, \widehat{m}, f, \widehat{f})$$

the tangent space at a point $(m, \widehat{m}, f, \widehat{f})$ is

$$T_m M \times T_{\widehat{m}}\widehat{M} \times T_f SO(n) \times T_{\widehat{f}} SO(n) \cong T_m M \times T_{\widehat{m}}\widehat{M} \times f \cdot \mathfrak{so}(n) \times \hat{f} \cdot \mathfrak{so}(n)$$

and following what we have written above, we have this local description of the tangent spaces:

$$T_f F(M) = \text{span}\{\underbrace{f_1^*, \ldots, f_n^*}_{H_f}, \underbrace{w_{i,j} : i < j}_{V_f}\}$$
$$T_{\widehat{f}} F(\widehat{M}) = \text{span}\{\underbrace{\widehat{f}_1^*, \ldots, \widehat{f}_n^*}_{H_{\hat{f}}}, \underbrace{\widehat{w}_{i,j} : i < j}_{V_{\hat{f}}}\}.$$

We have already shown that:

$$\ker(\omega) = \operatorname{span}\{H_f, H_{\hat{f}}, V_{\hat{f}}\}$$
$$\ker(\widehat{\omega}) = \operatorname{span}\{H_{\hat{f}}, H_f, V_f\}$$
$$ker(\theta - \widehat{\theta}) = \operatorname{span}\{V_f, V_{\hat{f}}, f_i^* + \widehat{f}_i^* : i = 1, \dots n\}.$$

Now computing the intersections we have that the following equations describe the distribution in the $2n+2s$-dimensional space

$$\begin{cases} V_f = 0 \text{ (s equations)} \\ V_{\hat{f}} = 0 \text{ (s equations)} \\ f_i^* - \widehat{f}_i^* = 0 \text{ (n equations)} \end{cases}$$

and therefore the distribution has rank n and

$$\mathcal{D} = \operatorname{span}\{f_i^* + \widehat{f}_i^* : i = 1, \dots, n\}.$$

For the next step, it is useful to write the explicit expression of the lifted vector fields f_j^* and $\widehat{f}_j^*$ in the local coordinates.
It is practical to introduce also the analogue “right” vector fields, right with respect to 4.9

$$\widetilde{w}_{i,j} := e_{i,j} \cdot f.$$

The relation between the $w_{i,j}$ and the $\widetilde{w}_{i,j}$ is

$$\widetilde{w}_{l,k} = \sum_{\alpha,\beta} f_{l\alpha} f_{k\beta} w_{\alpha,\beta} \tag{4.10}$$

and it is proven in the appendix, see A.4; it plays a role in the next computations.
Now we recover the local expression of the lifts of the vector fields f_i with respect to $\widetilde{\nabla}$. We underline that the bold Christoffel symbols $\boldsymbol{\Gamma}$ refer to the frame f, while from the fourth line on we refer to Γ as the Christoffel symbols of the frame e:

$$\begin{aligned} f_i^* &= f_i - \sum_{\alpha<\beta} \boldsymbol{\Gamma}_{\mathbf{i}\beta}^{\alpha} w_{\alpha,\beta} \\ &= f_i - \frac{1}{2} \sum_{\alpha,\beta} \boldsymbol{\Gamma}_{\mathbf{i}\beta}^{\alpha} w_{\alpha,\beta} \\ &= \sum_s f_{si} e_s - \frac{1}{2} \sum_{\alpha,\beta} \langle \nabla_{\mathbf{f_i}} \mathbf{f}_\beta, \mathbf{f}_\alpha \rangle w_{\alpha,\beta} \\ &= \sum_s f_{si} e_s - \frac{1}{2} \sum_{\alpha,\beta} \sum_{l,s,k} f_{l\alpha} f_{si} f_{k\beta} \Gamma_{sk}^{l} w_{\alpha,\beta} = \\ &= \sum_s f_{si} e_s - \frac{1}{2} \sum_s f_{si} \sum_{\alpha,\beta} \sum_{l,k} f_{l\alpha} f_{k\beta} \Gamma_{sk}^{l} w_{\alpha,\beta}. \end{aligned}$$

Now using 4.10 we get

$$\begin{aligned}\sum_s f_{si}\left[e_s - \frac{1}{2}\sum_{\alpha,\beta}\sum_{l,k} f_{l\alpha}f_{k\beta}\Gamma^l_{sk}w_{\alpha,\beta}\right] &= \sum_s f_{si}\left[e_s - \frac{1}{2}\sum_{l,k}\Gamma^l_{sk}\widetilde{w}_{l,k}\right] \\ &= \sum_s f_{si}\left[e_s - \sum_{\alpha<\beta}\Gamma^\alpha_{s\beta}\widetilde{w}_{\alpha,\beta}\right] \\ &= \sum_s f_{si}\left[e_s - \sum_{\alpha<\beta}\Gamma^\alpha_{s\beta}\widetilde{w}_{\alpha,\beta}\right].\end{aligned}$$

Similarly we have

$$\sum_s \widehat{f}_{si}\left[\widehat{e}_s - \sum_{\alpha<\beta}\widehat{\Gamma}^\alpha_{s\beta}\widetilde{\widehat{w}}_{\alpha,\beta}\right].$$

Therefore a generating element of $\mathcal{D}$ is

$$f^*_i + \widehat{f}^*_i = \sum_s f_{si}\left[e_s - \sum_{\alpha<\beta}\Gamma^\alpha_{s\beta}\widetilde{w}_{\alpha,\beta}\right] + \sum_s \widehat{f}_{si}\left[\widehat{e}_s - \sum_{\alpha<\beta}\widehat{\Gamma}^\alpha_{s\beta}\widetilde{\widehat{w}}_{\alpha,\beta}\right].$$

4.4.2 Comparison of the two distributions

We recall that $\Delta = \{E_1, \ldots, E_n\}$ with

$$E_k := e_k + Re_k + \sum_{i<j}^{n}\left(\langle\nabla_{e_k}e_j, e_i\rangle - \langle\nabla_{Re_k}Re_j, Re_i\rangle\right)W_{i,j} \qquad \text{for } k = 1, \ldots, n;$$

In order to compare the basis of the two distributions we rewrite the last term of the above equation using also here a "locally right invariant" basis, namely

$$\widetilde{W}_{\beta,\gamma} := e_{\beta,\gamma}\cdot R \qquad \text{for } \beta < \gamma;$$

the relation between the two basis is analogous to 4.10

$$\widetilde{W}_{i,j} = \sum_{l,s} R_{il}R_{js}W_{l,s} \tag{4.11}$$

and it is proven in the appendix, see A.4. Now we rewrite the expression using these relations:

$$
\begin{aligned}
\sum_{i<j}^{n} \langle \nabla_{Re_k} Re_j, Re_i \rangle W_{i,j} &= \frac{1}{2}\sum_{i,j}^{n} \langle \nabla_{Re_k} Re_j, Re_i \rangle W_{i,j} \\
&= \frac{1}{2}\sum_{i,j}^{n} \left\langle \sum_{\alpha,\beta} R_{\alpha k} R_{\beta j} \nabla_{\widehat{e}_\alpha} \widehat{e}_\beta, \sum_{\gamma} R_{\gamma i} \widehat{e}_\gamma \right\rangle W_{i,j} \\
&= \frac{1}{2}\sum_{i,j}^{n} \sum_{\alpha,\beta,\gamma} R_{\alpha k} R_{\beta j} R_{\gamma i} \left\langle \nabla_{\widehat{e}_\alpha} \widehat{e}_\beta, \widehat{e}_\gamma \right\rangle W_{i,j} \\
&= \frac{1}{2}\sum_{i,j}^{n} \sum_{\alpha,\beta,\gamma} R_{\alpha k} R_{\beta j} R_{\gamma i} \widehat{\Gamma}^{\gamma}_{\alpha\beta} W_{i,j} \\
&= \frac{1}{2} \sum_{\alpha,\beta,\gamma} R_{\alpha k} \widehat{\Gamma}^{\gamma}_{\alpha\beta} \widetilde{W}_{\gamma,\beta} \\
&= \sum_{\alpha} R_{\alpha k} \sum_{\gamma<\beta} \widehat{\Gamma}^{\gamma}_{\alpha\beta} \widetilde{W}_{\gamma,\beta} \\
&= \sum_{s} R_{sk} \sum_{i<j} \widehat{\Gamma}^{i}_{sj} \widetilde{W}_{i,j}.
\end{aligned}
$$

Therefore we have this rewriting for the expression of E_k:

$$
E_k = e_k + Re_k + \sum_{i<j}^{n} \Gamma^{i}_{kj} W_{i,j} - \sum_{s} R_{sk} \sum_{i<j} \widehat{\Gamma}^{i}_{sj} \widetilde{W}_{i,j}. \tag{4.12}
$$

Now that we have clarified all the basis we use, let's understand the action of the map; locally it is given by

$$
\begin{aligned}
\Pi : M \times \widehat{M} \times SO(n) \times SO(n) &\longrightarrow M \times \widehat{M} \times SO(n) \\
\left(m, \widehat{m}, f, \widehat{f}\right) &\mapsto (m, \widehat{m}, R) \text{ s.t. } R = \widehat{f} \cdot f^T.
\end{aligned}
$$

In order to understand the action on the tangent space, here we write two lemmas.

Lemma 4.1.

$$
\Pi_*(\widetilde{w}_{\alpha,\beta}) = -W_{\alpha,\beta}.
$$

Proof. Consider a curve of frames $(F(t), \widehat{F}(t)) \in SO(n) \times SO(n)$ such that

$$
\begin{aligned}
(F(0), \widehat{F}(0)) &= (f, \widehat{f}) \\
(\dot{F}(0), \dot{\widehat{F}}(0)) &= (\widetilde{w}_{i,j}, 0)
\end{aligned}
$$

then we can compute

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \Pi(F(t), \widehat{F}(t)) &= \frac{d}{dt}\Big|_{t=0} \dot{\widehat{F}}(0) \cdot F(0)^T + \widehat{F}(0) \cdot \dot{F}(0)^T \\
&= 0 + \widehat{f} \cdot (e_{\alpha,\beta} \cdot f)^T \\
&= \widehat{f} \cdot f^T \cdot e_{\alpha,\beta}^T \\
&= - R \cdot e_{\alpha,\beta} = -W_{\alpha,\beta}
\end{aligned}
$$

therefore $-W_{\alpha,\beta}$ is the tangent vector to $\Pi((F(t),\widehat{F}(t))$ at $\Pi((F(0),\widehat{F}(0)) = \widehat{f}\cdot f^T = R$. □

Lemma 4.2.

$$\Pi_*(\widetilde{\widetilde{w}}_{\alpha,\beta}) = \widetilde{W}_{\alpha,\beta}.$$

Proof. Consider a curve of frames $(F(t),\widehat{F}(t)) \in SO(n)\times SO(n)$ such that

$$\begin{aligned}(F(0),\widehat{F}(0)) &= (f,\widehat{f})\\ (\dot{F}(0),\dot{\widehat{F}}(0)) &= (0,\widetilde{\widetilde{w}}_{i,j})\end{aligned}$$

then we can compute

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\Pi(F(t),\widehat{F}(t)) = \frac{d}{dt}\Big|_{t=0} & \dot{\widehat{F}}(0)\cdot F(0)^T + \widehat{F}(0)\cdot\dot{F}(0)^T\\ =& e_{\alpha,\beta}\cdot\widehat{f}\cdot f^T + 0\\ =& e_{\alpha,\beta}\cdot R\\ =& \widetilde{W}_{\alpha,\beta}\end{aligned}$$

therefore $\widetilde{W}_{\alpha,\beta}$ is the tangent vector to $\Pi((F(t),\widehat{F}(t))$ at $\Pi((F(0),\widehat{F}(0)) = \widehat{f}\cdot f^T = R$. □

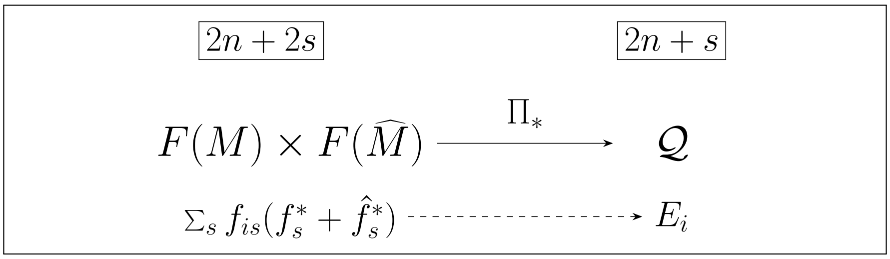
The map Π projects a $2n+2s$ dimensional space on a $2n+s$ space and on the tangent space we have

$$\Pi_*:\quad \begin{aligned} e_i &\longmapsto e_i\\ \widehat{e}_i &\longmapsto \widehat{e}_i\\ \widetilde{w}_{\alpha,\beta} &\longmapsto -W_{\alpha,\beta}\\ \widetilde{\widetilde{w}}_{\alpha,\beta} &\longmapsto \widetilde{W}_{\alpha,\beta}\end{aligned}\quad ;$$

it's clear that $\ker(\Pi_*) \cong \mathrm{span}\{\widetilde{\widetilde{w}}_{\alpha,\beta} : \alpha<\beta\} \cong \mathfrak{so}(n)$, since the vector fields $\widetilde{W}_{\alpha,\beta}$ are linear combinations of the $W_{\alpha,\beta}$, see 4.11.
In particular the relation between an element E_k of the basis of Δ and an element $f_j^* + \widehat{f}_j^*$ of the basis of $\mathcal{D}$ is

$$\begin{aligned}\Pi_*(\sum_j f_{kj}\left(f_j^*+\widehat{f}_j^*\right)) &= \sum_j f_{kj}\Pi_*\left(\sum_s f_{sj}\left[e_s - \sum_{\alpha<\beta}\Gamma^\alpha_{s\beta}\widetilde{w}_{\alpha\beta}\right] + \sum_s \widehat{f}_{sj}\left[\widehat{e}_s - \sum_{\alpha<\beta}\widehat{\Gamma}^\alpha_{s\beta}\widetilde{\widetilde{w}}_{\alpha\beta}\right]\right)\\ &= \sum_j f_{kj}\left(\sum_s f_{sj}\left[e_s + \sum_{\alpha<\beta}\Gamma^\alpha_{s\beta}W_{\alpha\beta}\right] + \sum_s \widehat{f}_{sj}\left[\widehat{e}_s - \sum_{\alpha<\beta}\widehat{\Gamma}^\alpha_{s\beta}\widetilde{W}_{\alpha\beta}\right]\right)\\ &= \sum_s \delta_{ks}\left[e_s + \sum_{\alpha<\beta}\Gamma^\alpha_{s\beta}W_{\alpha\beta}\right] + \sum_{s,j}\widehat{f}_{sj}f_{kj}\left[\widehat{e}_s - \sum_{\alpha<\beta}\widehat{\Gamma}^\alpha_{s\beta}\widetilde{W}_{\alpha\beta}\right]\\ &= \left[e_k + \sum_{\alpha<\beta}\Gamma^\alpha_{k\beta}W_{\alpha\beta}\right] + \sum_s R_{sk}\left[\widehat{e}_s - \sum_{\alpha<\beta}\widehat{\Gamma}^\alpha_{s\beta}\widetilde{W}_{\alpha\beta}\right]\\ &= e_k + R\cdot e_k + \sum_{\alpha<\beta}\Gamma^\alpha_{k\beta}W_{\alpha\beta} - \sum_s R_{sk}\sum_{\alpha<\beta}\widehat{\Gamma}^\alpha_{s\beta}\widetilde{W}_{\alpha\beta}\\ &= E_k.\end{aligned}$$

This expression also shows that the map is injective once the fiber is fixed. Therefore, Π is a



diffeomorphism in every fiber. This concludes the proof of the theorem. □

4.5 Considerations

The main advantage of this point of view, since $\text{Lie}_q\Delta$ and $\text{Lie}_q\mathcal{D}$ turn out to be equivalent, is the simplification of the effective calculations. We recall that we are interested in understanding the dimension of set $\mathcal{O}_{q_0}$ of reachable points by rolling starting from a point q_0, where the set $\mathcal{O}_{q_0}$ is, as we have already defined in Chapter 2, the Lie algebra of the distribution Δ.

Notice that, as in the 2-dimensional case, step 3 is the minimum step at which we can stop the iteration in the computations of the Lie bracket; indeed,

$\dim M\&\widehat{M}$	STEP 1	STEP 2	STEP 3	$\dim \mathcal{Q}$
2	2	$2+1$	$3+2$	5
n	n	$n+\frac{n(n-1)}{2}<d$	$\frac{n(n+1)}{2}+\frac{n(n^3+6n^2-5n+2)}{8}>d$	$\frac{n(n+3)}{2}=d.$

A direct consequence showing the equivalence between $\text{Lie}_q\Delta$ and $\text{Lie}_q\mathcal{D}$ is the following.

Corollary 4.1. $\Pi_*\mathcal{D}^k_q=\Delta^k_q$ *for any* $q\in Q, k\in\mathbb{N}$.

Proof. Since everything is related to a point $q\in\mathcal{Q}$, we fix local coordinates as we have already done before.
From theorem 4.1 we have that

$$\Pi_*\mathcal{D}=\Delta$$

therefore we only have to prove that

$$\Pi_*([\mathcal{D},\mathcal{D}])=[\Delta,\Delta]$$

and the rest follows by definition. The theorem shows also that Π is a diffeomorphism in each fiber; therefore,

$$\begin{aligned}[E_i,E_j]&=\left[\Pi_*\sum_{s=1}^{n}f_{is}\left(f_s^*+\widehat{f}_s^*\right),\Pi_*\sum_{s=1}^{n}f_{js}\left(f_s^*+\widehat{f}_s^*\right)\right]\\&=\Pi_*\left[\sum_{s=1}^{n}f_{is}\left(f_s^*+\widehat{f}_s^*\right),\sum_{s=1}^{n}f_{js}\left(f_s^*+\widehat{f}_s^*\right)\right]\end{aligned}$$

and since $\sum_{s=1}^{n} f_{is}\left(f_s^* + \widehat{f}_s^*\right)$ is a local basis for $\mathcal{D}$ we can conclude. □

4.6 Computing the brackets in $\mathcal{D}$

Our goal now is to compute $[f_i^* + \widehat{f}_i^*, f_j^* + \widehat{f}_j^*]$ but since $\left[f_i^*, \widehat{f}_j^*\right] = 0$, computations of brackets of $\mathcal{D}$, can be reduced to mostly computing brackets of sections corresponding to vector fields in each of the manifolds involved.

We recall that we want to compute the brackets in $\mathcal{D}$ and then project the result on $\mathcal{Q}$, here is a recap of the situation.

$F(M) \times F(\widehat{M})$	$F(M) \times F(\widehat{M})/SO(n) \cong \mathcal{Q}$
$\mathcal{D} = \{f_i^* + \widehat{f}_i^* : i = 1, \cdots, n\}$	$\Delta = \{E_i : i = 1, \cdots, n\}$
$f_i^* = \sum_s f_{si}\left[e_s - \sum_{\alpha<\beta} \Gamma_{s\beta}^{\alpha} \widetilde{w}_{\alpha\beta}\right]$	$E_k = e_k + Re_k + \sum_{i<j}^{n} \Gamma_{kj}^{i} W_{i,j} - \sum_s R_{sk} \sum_{i<j} \widehat{\Gamma}_{sj}^{i} \widetilde{W}_{i,j}.$
$\widehat{f}_i^* = \sum_s \widehat{f}_{si}\left[\widehat{e}_s - \sum_{\alpha<\beta} \widehat{\Gamma}_{s\beta}^{\alpha} \widetilde{\widehat{w}}_{\alpha\beta}\right]$	

In order to compute the bracket in $\mathcal{D}$, we introduce some tensor tools; the importance of these objects will be clear from Lemma 4.4. In particular, we introduce an object that permits to associate a vector field on $F(M)$ to a tensor on M.

Consider that we will apply the followings to the curvature tensor R on M, a 4-tensor antisymmetric in the first two arguments

$$R(f_i, f_j, f_h, f_k) := \langle R(f_i, f_j) f_h, f_k \rangle$$

where

$$\begin{aligned} R(f_i, f_j) &= \nabla_{f_i} \nabla_{f_j} - \nabla_{f_j} \nabla_{f_i} - \nabla_{[f_i, f_j]} \\ &= f_i^* f_j^* - f_i^* f_j^* - [f_i, f_j]^*; \end{aligned}$$

a recap on these formulas is presented in B.1.3.

Let T be a l-tensor on M, we can associate to it the functions, which evaluate the tensor in a frame,

$$\begin{aligned} \mathscr{E}_{i_1,\ldots i_l}(T) : F(M) &\mapsto \mathbb{R} \\ f &\mapsto T\left(f_{i_1}, \ldots, f_{i_l}\right) \quad , \quad 1 \leq i_j \leq n. \end{aligned}$$

If $l = 2 + g$, and T is antisymmetric in the first two arguments, we can define vector fields on $F(M)$ by

$$\mathscr{W}_{i_1,\dots,i_g}(T) = \sum_{\alpha<\beta} \mathscr{E}_{\alpha,\beta,i_1,\dots,i_g}(T) w_{\alpha,\beta},$$

recall that $w_{\alpha,\beta}$ is a local basis of V_f, so these are vertical vector fields.
The curvature tensor R on M is exactly of the desired type, therefore we obtain these vector fields on M

$$\mathscr{W}_{i,j}(R) = \sum_{\alpha<\beta} R(f_\alpha, f_\beta, f_i, f_j) w_{\alpha,\beta}.$$

If $\widehat{T}$ is a tensor on $\widehat{M}$, we define $\mathscr{E}_{i_1,\dots i_l}(\widehat{T})$ and $\mathscr{W}_{i_1,\dots,i_g}(\widehat{T})$ analogously.

We have these properties.

Lemma 4.3. *Let f_k^* be a generator of H_f.*

- *For any l-tensor T,*
$$f_k^* \mathscr{E}_{i_1,\dots,i_l}(T) = \mathscr{E}_{i_1,\dots,i_l,k}(\nabla T)$$
- *For any $2 + l$-tensor T, that is antisymmetric in the first two arguments*
$$[f_k^*, \mathscr{W}_{i_1,\dots,i_l}(T)] = \mathscr{W}_{i_1,\dots,i_l,k}(\nabla T) - \sum_{s=1}^{n} \mathscr{E}_{s,k,i_1,\dots,i_l}(T) f_s^*$$

This Lemma is proven in the appendix, since it is a tensorial computation, see A.6.
Consider it in our context, applied to the curvature tensor R. We have

- $l = 4$, $f_k^*(R(f_\alpha, f_\beta, f_i, f_j)(T)) = \mathscr{E}_{\alpha,\beta,i,j,k}(\nabla R) = \nabla R(f_\alpha, f_\beta, f_i, f_j, f_k)$;
- $2 + l = 4$, $[f_k^*, \mathscr{W}_{i,j}(R)] = \mathscr{W}_{i,j,k}(\nabla R) - \sum_{s=1}^{n} \mathscr{E}_{s,k,i,j}(R) f_s^*$.

Now we are ready for the computation of the brackets.

Lemma 4.4. *Let f_k^* be a generator of H_f.*

- $\left[f_i^*, f_j^*\right] = -\mathscr{W}_{ij}(R)$ *for* $i, j = 1 \dots n$.
- $[f_k^*, [f_i^*, f_j^*]] = -\mathscr{W}_{i,j,k}(\nabla R) - \sum_{s=1}^{n} \mathscr{E}_{s,k,i,j}(R) f_s^*$

Proof. The first bracket is a consequence of the structures' equations. The structure equations are recalled in the appendix in A.5.
Indeed, we have that

$$T_f F(M) = H_f \oplus V_f, \quad \ker\theta = V_f, \quad \ker\omega = H_f$$

and since $\ker\theta \cap \ker\omega$ only contains the zero section, we can show the equality in the above equation by evaluating the left and right hand side by θ and ω and see if it produces the same

result.
Evaluating by θ we have, on the right hand side

$$\theta[f_i^*, f_j^*] = -d\theta\left(f_i^*, f_j^*\right) = (\omega(f_i^*) \cdot \theta(f_j^*) - \omega(f_j^*) \cdot \theta(f_i^*)) = 0$$

because $\omega(f_i^*) = 0$ since they are horizontal vector fields. On the left hand side

$$\theta\left(\mathscr{W}_{i,j}(R)\right) = 0$$

since it is a vertical vector field.
Evaluating by ω we have, on the right side

$$\omega\left(\left[f_i^*, f_j^*\right]\right) = -\Omega\left(f_i^*, f_j^*\right) = -\sum_{\alpha<\beta} R\left(f_\alpha, f_\beta, f_i, f_j\right) e_{\alpha,\beta}.$$

While on the other side

$$\omega\left(-\mathscr{W}_{i,j}(R)\right) = \omega\left(-\sum_{\alpha<\beta} R(f_\alpha, f_\beta, f_i, f_j) w_{\alpha,\beta}\right) = -\sum_{\alpha<\beta} R(f_\alpha, f_\beta, f_i, f_j) e_{\alpha,\beta}.$$

The second iteration $[f_k^*, [f_i^*, f_j^*]$ is just a rewriting of Lemma 4.3.

□

4.7 Projection on $\mathcal{Q}$

We could continue the computation, but it is hard to determine which brackets actually give us independent vector fields; therefore, as in the lower-dimensional case, we investigate sufficient conditions.
We have already noticed, with the previous lemma, how the curvature controls the coefficients. The next step is the projection of the results on the configuration space $\mathcal{Q}$, but in order to give a meaning to the curvatures of M and $\widehat{M}$ in this space, we introduce a tool that measures in $\mathcal{Q}$ their difference.

Remark. *We have different projection maps:*

$$\Pi : F(M) \times F(\widehat{M}) \to \mathcal{Q};$$
$$\pi : \mathcal{Q} \to M;$$
$$\widehat{\pi} : \mathcal{Q} \to \widehat{M}.$$

We want to project the previous results through Π.

Let R and $\widehat{R}$ be the curvature tensor on M and $\widehat{M}$ respectively. We define a 4-tensor in Δ in this way

$$\overline{R} = \pi^*(R) - \widehat{\pi}^*(\widehat{R}).$$

We recall that by basic facts of tensor algebra, $\overline{R}$ may also be seen as a bilinear map

$$\overline{R} \in \mathrm{Bil}(\bigwedge^2 \Delta \times \bigwedge^2 \Delta, \mathbb{R}).$$

Similarly, we define $\overline{\nabla R}$ a 5 -tensor on Δ, by $\pi^*(\nabla R) - \widehat{\pi}^*(\nabla \widehat{R})$.
Now given the results of the previous section and knowing how the projection Π acts on tangent space, we can finally write the brackets of Δ.

Notation. *Now we denote by q the isometry $f^T \cdot \widehat{f}$ in order not to be confused with the curvature tensor symbol R.*

Lemma 4.5. *Let (e, U) and $(\widehat{e}, \widehat{U})$ be two local sections of $F(M)$ and $F(\widehat{M})$, respectively. Then on $Q|_{U \times \widehat{U}}$, in terms of the notation already introduced, we have*

$$\Delta^2 = \Delta \oplus \mathrm{span}\left\{\sum_{\alpha<\beta} \overline{R}\left(E_\alpha, E_\beta, E_i, E_j\right) W_{\alpha,\beta}\right\}_{i,j=1}^{n}$$

$$\Delta^3 = \Delta^2 + \mathrm{span}\left\{\sum_{\alpha<\beta} \overline{\nabla R}\left(E_\alpha, E_\beta, E_i, E_j, E_k\right) W_{\alpha,\beta} + qR\left(e_i, e_j\right) e_k \right.$$
$$\left. - \widehat{R}\left(qe_i, qe_j\right) qe_k + \sum_{1\le\alpha<\beta\le n} \left\langle qe_\alpha, \nabla_{qR(e_i,e_j)e_k - \widehat{R}(qe_i,qe_j)qe_k} qe_\beta \right\rangle\right\}_{i,j,k=1}^{n}$$

Proof. The expression of Δ^2 follows from Lemma 4.4 because

$$[\mathcal{D}, \mathcal{D}] = \mathrm{span}\left\{\left[f_i^* + \widehat{f}_i^*, f_j^* + \widehat{f}_j^*\right]\right\}\Big|_{i,j=1}^{n}$$
$$= \mathrm{span}\left\{-\sum_{\alpha<\beta}\left(R\left(f_\alpha, f_\beta, f_i, f_j\right) w_{\alpha,\beta} + \widehat{R}\left(\widehat{f}_\alpha, \widehat{f}_\beta, \widehat{f}_i, \widehat{f}_j\right) \widehat{w}_{\alpha,\beta}\right)\right\}_{i,j=1}^{n}$$

that projected through Π gives,

$$[\Delta, \Delta] = \mathrm{span}\left\{\sum_{\alpha<\beta}\left(R\left(e_\alpha, e_\beta, e_i, e_j\right) - \widehat{R}\left(\widehat{e}_\alpha, \widehat{e}_\beta, \widehat{e}_i, \widehat{e}_j\right)\right) W_{\alpha,\beta}\right\}_{i,j=1}^{n}$$
$$= \mathrm{span}\left\{\sum_{\alpha<\beta} \overline{R}\left(E_\alpha, E_\beta, E_i, E_j\right) W_{\alpha,\beta}\right\}_{i,j=1}^{n}.$$

The expression for Δ^3 follows from the same Lemma, after these two observations.

$$\sum_{s=1}^{n}\left(\mathscr{E}_{skij}(R) f_s^* + \mathscr{E}_{skij}(\widehat{R}) \widehat{f}_s^*\right) = -\sum_{s=1}^{n}\left(\mathscr{E}_{ijks}(R) f_s^* + \mathscr{E}_{ijks}(\widehat{R}) \widehat{f}_s^*\right)$$
$$= \sum_{s=1}^{n}\left(\mathscr{E}_{ijks}(R) - \mathscr{E}_{ijks}(\widehat{R})\right) \widehat{f}_s^* \quad (\mathrm{mod}\, \mathcal{D}).$$

It holds, that

$$\Pi_*\left(\sum_{s,\mu,\lambda,l}^{n} f_{i\mu}f_{j\lambda}f_{kl}\left(\mathscr{E}_{\mu\lambda ls}(R)-\mathscr{E}_{\mu\lambda ls}(\widehat{R})\right)\widehat{f}_s^*\right)$$

is equal to

$$qR(e_i,e_j)e_k-\widehat{R}(qe_i,qe_j)qe_k-\sum_{\alpha<\beta}^{n}\left\langle qe_\alpha,\nabla_{qR(e_i,e_j)e_k}qe_\beta\right\rangle W_{\alpha\beta}-\sum_{\alpha<\beta}^{n}\left\langle qe_\alpha,\nabla_{R(qe_i,qe_j)qe_k}qe_\beta\right\rangle W_{\alpha\beta}$$

Indeed, since $f_s^*=\sum_j f_{js}\left[e_j-\sum_{\alpha<\beta}\Gamma^\alpha_{s\beta}\widetilde{w}_{\alpha\beta}\right]$, we have

$$\begin{aligned}
\Pi_*\left(\sum_{s,\mu,\lambda,l}^{n} f_{i\mu}f_{j\lambda}f_{kl}(\mathscr{E}_{\mu\lambda ls}(R))\right)\left(\sum_y \widehat{f}_{ys}\widehat{e}_y\right) &= \Pi_*\left(\sum_{s,\mu,\lambda,l,y}^{n} f_{i\mu}f_{j\lambda}f_{kl}\widehat{f}_{ys}(\mathscr{E}_{\mu\lambda ls}(R))\widehat{e}_y\right)\\
&= \Pi_*\left(\sum_{s,\mu,\lambda,l,y}^{n} f_{i\mu}f_{j\lambda}f_{kl}\widehat{f}_{ys}\langle R(f_\mu,f_\lambda)f_l,f_s\rangle\widehat{e}_y\right)\\
&= \Pi_*\left(\sum_{s,y}^{n}\widehat{f}_{ys}\langle R(e_i,e_j)e_k,f_s\rangle\widehat{e}_y\right)\\
&= \Pi_*\left(\sum_{s,y,h}^{n}\widehat{f}_{ys}f_{hs}\langle R(e_i,e_j)e_k,e_h\rangle\widehat{e}_y\right)\\
&= \sum_{s,y}^{n}\widehat{f}_{ys}f_{ks}R(e_i,e_j)\widehat{e}_y\\
&= \sum_{y}^{n} q_{yk}R(e_i,e_j)\widehat{e}_y\\
&= q(R(e_i,e_j)e_k).
\end{aligned}$$

Similarly

$$
\begin{aligned}
\Pi_*\left(-\sum_{s,\mu,\lambda,l}^{n} f_{i\mu}f_{j\lambda}f_{kl}(\mathscr{E}_{\mu\lambda ls}(\widehat{R}))\right)\left(\sum_y \widehat{f}_{ys}\widehat{e}_y\right) &= \Pi_*\left(-\sum_{s,\mu,\lambda,l,y}^{n} f_{i\mu}f_{j\lambda}f_{kl}\widehat{f}_{ys}\langle \widehat{R}(\widehat{f}_\mu,\widehat{f}_\lambda)\widehat{f}_l,\widehat{f}_s\rangle\widehat{e}_y\right)\\
&= \Pi_*\left(-\sum_{s,y}^{n}\widehat{f}_{ys}\langle\widehat{R}(qe_i,qe_j)qe_k,\widehat{f}_s\rangle\widehat{e}_y\right)\\
&= \Pi_*\left(-\sum_{s,y}^{n}\widehat{f}_{ys}\langle\widehat{R}(qe_i,qe_j)\sum_l^n q_{lk}\widehat{e}_l,\widehat{f}_s\rangle\widehat{e}_y\right)\\
&= \Pi_*\left(-\sum_{s,y}^{n}\sum_l^n q_{lk}\widehat{f}_{ys}\langle\widehat{R}(qe_i,qe_j)\widehat{e}_l,\sum_t^n \widehat{f}_{ts}\widehat{e}_t\rangle\widehat{e}_y\right)\\
&= \Pi_*\left(-\sum_{s,y}^{n}\sum_l^n q_{lk}\widehat{f}_{ys}\widehat{f}_{ls}\widehat{R}(qe_i,qe_j)\widehat{e}_y\right)\\
&= \Pi_*\left(-\sum_{s,y}^{n} f_{ks}\widehat{f}_{ys}\widehat{R}(qe_i,qe_j)\widehat{e}_y\right)\\
&= -q(\widehat{R}(qe_i,qe_j)e_k).
\end{aligned}
$$

The third term is given by

$$
\begin{aligned}
&\Pi_*\left(\sum_{s,\mu,\lambda,l}^{n} -f_{i\mu}f_{j\lambda}f_{kl}\mathscr{E}_{\mu\lambda ls}(\widehat{R})\right)\left(-\sum_y^n \widehat{f}_{ys}\sum_{\alpha<\beta}\widehat{\Gamma}^{\alpha}_{y\beta}\widetilde{\widehat{w}}_{\alpha\beta}\right) =\\
&= \Pi_*\left(\sum_{s,\mu,\lambda,l,y}^{n}\sum_{\alpha<\beta} f_{i\mu}f_{j\lambda}f_{kl}\widehat{f}_{ys}\langle\widehat{R}(\widehat{f}_\mu,\widehat{f}_\lambda)\widehat{f}_l,\widehat{f}_s\rangle\widehat{\Gamma}^{\alpha}_{y\beta}\widetilde{\widehat{w}}_{\alpha\beta}\right)\\
&= \Pi_*\left(\sum_{s,y}^{n}\sum_{\alpha<\beta}\sum_l^n q_{lk}\widehat{f}_{ys}\widehat{f}_{ls}\widehat{R}(qe_i,qe_j)\widehat{\Gamma}^{\alpha}_{y\beta}\widetilde{\widehat{w}}_{\alpha\beta}\right)\\
&= \Pi_*\left(\sum_{s,y}^{n}\sum_{\alpha<\beta} f_{ks}\widehat{f}_{ys}\widehat{R}(qe_i,qe_j)\widehat{\Gamma}^{\alpha}_{y\beta}\widetilde{\widehat{w}}_{\alpha\beta}\right)\\
&= \Pi_*\left(\sum_{y}^{n}\sum_{\alpha<\beta} q_{yk}\widehat{R}(qe_i,qe_j)\langle\nabla_{\widehat{e}_y}\widehat{e}_\beta,\widehat{e}_\alpha\rangle\widetilde{\widehat{w}}_{\alpha\beta}\right)\\
&= \Pi_*\left(\sum_{\alpha<\beta}\widehat{R}(qe_i,qe_j)\langle\nabla_{qe_k}\widehat{e}_\beta,\widehat{e}_\alpha\rangle\widetilde{\widehat{w}}_{\alpha\beta}\right)\\
&= \Pi_*\left(\sum_{\alpha<\beta}\langle\nabla_{\widehat{R}(qe_i,qe_j)qe_k}\widehat{e}_\beta,\widehat{e}_\alpha\rangle\widetilde{\widehat{w}}_{\alpha\beta}\right)\\
&= -\sum_{\alpha<\beta}\langle\nabla_{\widehat{R}(qe_i,qe_j)qe_k}\widehat{e}_\beta,\widehat{e}_\alpha\rangle W_{\alpha,\beta}\\
&= \cdots\\
&= -\sum_{\alpha<\beta}\left\langle\nabla_{R(qe_i,qe_j)qe_k}qe_\beta,qe_\alpha\right\rangle W_{\alpha,\beta};
\end{aligned}
$$

similarly the last term. □

4.8 Sufficient conditions for controllability

Now every element of our setting is generalized and developed. We see that the difference of curvature coefficients controls some directions.
The scheme is analogous to the one of the 2-dimensional case. Indeed, the theorem that links the geometry of the rigid bodies and controllability in this n-dimensional case appears as a rewriting of the main theorem of Chapter 2.

Theorem 4.2. *For $q \in Q$, let k_q denote the Gaussian curvature of M at $\pi(q)$, and let $\widehat{k}_q$ denote the Gaussian curvature of $\widehat{M}$ at $\widehat{\pi}(q)$, and similarly for $q_0 \in \mathcal{Q}$. Then,*

- *if $k - \widehat{k}\Big|_q \equiv 0 \quad \forall q \in \mathcal{O}_{q_0}$ on then $\dim \mathcal{O}_{q_0} = 2$;*

•• *if $k - \widehat{k}\Big|_q \equiv 0 \quad \forall q \in \mathcal{O}_{q_0}$ then $\dim \mathcal{O}_{q_0} = 5$.*

Furthermore we can express the same concept also in terms of sectional curvature if we define a function $\overline{\sec}_q$ on 2-dimensional planes L in Δ_q by the formula

$$\overline{\sec}_q(L) = \sec_{\pi(q)}(\pi_* L) - \widehat{\sec}_{\widehat{\pi}(q)}(\widehat{\pi}_* L)\,.$$

Here $\sec_m$ and $\widehat{\sec}_{\widehat{m}}$ denote the respective sectional curvatures of M and $\widehat{M}$ at the indicated points, for more details see B.1.3.

Theorem 4.3. *Let $q, q_0 \in Q$ and let $\overline{R}$ be the 4-tensor defined above. Then,*

- *if $\overline{R}|_q \equiv 0 \quad \forall q \in \mathcal{O}_{q_0}$ then $\dim \mathcal{O}_{q_0} = n$;*

•• *if $\overline{R}_q$ is non degenerate, then Δ is bracket generating of step 3 at q.*

While in terms of sectional curvature, we have

- *if $\overline{\sec}_q \equiv 0 \quad \forall q \in \mathcal{O}_{q_0}$ then $\dim \mathcal{O}_{q_0} = n$;*

•• *If $\overline{\sec}_q \neq 0$, then Δ is bracket generating of step 3 at q.*

Proof. • The first point is a direct consequence of Lemma 4.5, indeed $\Delta^3 = \Delta^2 = \Delta$.

•• Let $\overline{\pi}(q) = (m, \widehat{m})$, where $\overline{\pi} : Q \to M \times \widehat{M}$ is the projection.

Notation. *Here again we denote the isometry with q, instead of the previous given name R, in order to avoid confusion with the curvature tensor. So, in this proof q stands for the configuration $q \in \mathcal{Q}$ and it represents also the isometry between the tangent space.*

Consider local sections (e, U) and $(\widehat{e}, \widehat{U})$ of $F(M)$ and $F(\widehat{M})$ and introduce the respective local coordinates.
Since $\overline{R}|_q$ is non degenerate, we have that

$$\operatorname{span}\left\{ \sum_{\alpha<\beta} \overline{R}\left(E_\alpha, E_\beta, E_i, E_j\right) W_{\alpha,\beta} \right\}_{i,j=1}^{n} = \operatorname{span}\left\{W_{i,j}\right\}_{i,j=1}^{n}$$

and this adds $\frac{n(n-1)}{2}$ new directions. At the third step, again since $\overline{R}|_q$ is non degenerate, we have that

$$\operatorname{span}\left\{\left(q(R(e_i, e_j)\, e_k) - \widehat{R}(qe_i, qe_j)\, qe_k\right)\right\}_{i,j,k=1}^n = \operatorname{span}\{qe_i\}_{i=1}^n\,,$$

and from Lemma 4.5 we can finally conclude that

$$\begin{aligned}\Delta_q^3 &= \operatorname{span}\{E_j|_q, qe_i|_m, W_{\alpha,\beta}|_q\}_{i,j,\alpha,\beta=1}^n\\ &= \operatorname{span}\{e_i|_m, \widehat{e}_i|_{\widehat{m}}, W_{\alpha,\beta}|_q\}_{i,\alpha,\beta=1}^n = T_q\mathcal{Q}.\end{aligned}$$

The formulation in terms of sectional curvature it's just a consequence of the fact that $\overline{R}$ is a bilinear map. A bilinear map is degenerate if the determinant of the associated matrix is zero. Chosen an orthonormal basis $\{v_j\}$ of $T_mM, m = \pi(q)$. The map $\overline{R}$ is a $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ matrix; if the determinant is non-zero, then we have local controllability at q

$$\det\left(R(v_\alpha, v_\beta, v_i, v_j) - \widehat{R}(qv_\alpha, qv_\beta, qv_i, qv_j)\right),$$

$1 \le \alpha < \beta \le n$ are row indices, $\quad 1 \le i < j \le n$ are column indices. For example, if $n = 3$, the aspect of the first matrix in the difference is,

$$\begin{pmatrix} R(v_1,v_2,v_1,v_2) & R(v_1,v_2,v_1,v_3) & R(v_1,v_2,v_1,v_4) & \cdots \\ R(v_1,v_3,v_1,v_2) & R(v_1,v_3,v_1,v_3) & \cdots & \\ R(v_1,v_4,v_1,v_2) & \cdots & & \\ R(v_2,v_3,v_1,v_2) & & & \\ R(v_2,v_4,v_1,v_2) & & & \\ R(v_3,v_4,v_1,v_2) & \cdots & & \end{pmatrix}$$

where we notice that the diagonal elements are (the opposite of) the sectional curvatures. Therefore, we can reformulate in terms of sectional curvature as in the statement.
Let's see the proof of that part of statement.

- If $\overline{\sec}_q \equiv 0$ it means that $\overline{\sec}_q(L) = 0\ \forall L \subseteq \Delta_q$ then $\overline{R}_q = 0$ and then $\dim \mathcal{O}_{q_0} = n$ by the previous part of the theorem.

•• If $\overline{\sec}_q > 0$ (resp. $\overline{\sec}_q < 0$) for every L, then $-\overline{R}_q$ (resp. $\overline{R}_q$) in non-degenerate as a bilinear map on $\bigwedge^2 \Delta_q$ and this proves the other implications.
Indeed, look at the case $\overline{\sec}_q > 0$ and pick the already known basis $E_1, \dots, E_n$ of Δ_q. Let

$$L_{ij} := \operatorname{span}\{E_i, E_j\}$$

In this basis, we have

$$\begin{aligned}0 < \overline{\sec}_q(L_{ij}) &= \sec_m(e_i, e_j) - \widehat{\sec}_m(e_i, e_j)\\ &= -R(e_i, e_j, e_i, e_j) + \widehat{R}(e_i, e_j, e_i, e_j)\\ &= -\overline{R}(E_i, E_j, E_i, E_j)\end{aligned}$$

that is

$$\overline{R}(E_i, E_j, E_i, E_j) < 0 \qquad \forall i, j.$$

The case $\overline{\sec}_q < 0$ is analogous.

□

Notice that we are talking about local controllability; all the conditions are formulated in terms of a fixed point. However, if they hold for all points, they naturally become sufficient conditions for complete controllability.
This result is the final generalization. We can recover all the previous examples as simple consequences.

4.8.1 Example of the sphere rolling on a plane

Let M be the sphere $\mathbb{S}^n \subseteq \mathbb{R}^n + 1$ of radius r while let $\widehat{M}$ be a n-dimensional plane.
The sphere has constant sectional curvature

$$\sec_m \equiv \frac{1}{r^2} \qquad \forall m \in \mathbb{S}^n;$$

while the plane has constant sectional curvature equal to zero, therefore

$$\overline{sec}_q \equiv \frac{1}{r^2} \qquad \forall q \in \mathcal{Q}.$$

The rolling distribution Δ is bracket generating at step 3 at every point of the configuration space $\mathcal{Q}$, that is, the system is completely controllable, as shown by hand in Chapter 3.

4.8.2 Example of two sphere

Another easy example to recover is the case where M and $\widehat{M}$ are two spheres, of radius r and $\widehat{r}$ respectively. The difference of sectional curvatures is

$$\overline{sec}_q \equiv \frac{1}{r^2} - \frac{1}{\widehat{r}^2} \qquad \forall q \in \mathcal{Q},$$

so we have a controllable system if and only if the two radii are different.

Chapter 5

About connections

In Chapter 4 we have strongly used the fact that a connection on a manifold M induces a connection on the corresponding frame bundle $F(M)$. With frame bundle $F(M)$ of the manifold we mean properly the bundle of frames of the tangent bundle TM of M that we have already introduced.

$$\pi : TM \longrightarrow M$$
$$\tau : F(M) \longrightarrow M$$

In this Chapter we give an intuition of the process by which a connection ∇ on the vector bundle TM over M gives rise to an horizontal distribution on the associated frame bundle $F(M)$. In particular, we give an explanation of the lifts formulas presented in 4

$$f_i^* = f_i - \sum_{\alpha<\beta} \Gamma^{\alpha}_{i\beta} e_{\alpha,\beta} \qquad \forall i = 1, \dots, n. \tag{5.1}$$

Detailed introduction to the concept of connection is presented in the appendix B.1.1.

Consider a Riemannian manifold M on which we choose a connection ∇, here we choose the Levi-Civita one, but this choice has not important consequences for the main considerations. The fact that the connection on $F(M)$ is already written when we choose a connection on M can be seen, but the expression is not so easy to recover.

It can be seen because, as in the tangent bundle TM, the following holds.

Proposition 5.1. *Let ∇ be a linear connection. Let $\gamma : [0, T] \to M$ be a smooth curve and $v_0 \in T_{\gamma(0)}M$. There exists a unique smooth vector field V along γ such that $V|_{\gamma(0)} = v_0$ and V parallel along γ with respect to ∇.*

There is also a unique way to parallel transport a chosen frame $e_{\gamma(0)} \in T_{\gamma(0)}M$ along the curve $\gamma(t)$. We have already used this idea to characterize the rolling motion, described in 2. In terms of the frame bundle $F(M)$, this means that every curve in M has a unique lift to $F(M)$ starting at $e_{\gamma(0)}$ and representing parallel frames along the curve. This kind of lift is called a horizontal lift. The initial tangent vector of a horizontal lift at $e_{\gamma(0)}$ is a horizontal vector at $e_{\gamma(0)}$. The main point is that the horizontal vectors at a point $e_{\gamma(0)}$ of $F(M)$ form a subspace of the tangent space of the frame bundle $T_{e_{\gamma(0)}}F(M)$ that defines the horizontal distribution needed to define a connection on $F(M)$.

Now following this reasoning, we provide the formulas following these definitions; a *parallel frame* along the curve $\gamma(t)$ is a set of vector fields $\{e_1(t), \ldots, e_n(t)\}, t \in [0, c]$, such that

- each field is parallel along the curve: $\nabla_{\dot\gamma(t)} e_i(t)$ for $i = 1, \ldots, n$
- (for each $t \in [0, c]$ they form a basis of $T_{\gamma(t)}M$);

a **lift** of the curve $\gamma(t)$is a curve $\gamma^*(t)$ in $F(M)$ such that

- $\gamma(t) = \tau(\gamma^*(t))$;

a **horizontal lift** of the curve $\gamma(t)$is a curve $\gamma^*(t)$ in $F(M)$ such that

- $\gamma(t) = \tau(\gamma^*(t))$;
- $\gamma^*(t)$ is a parallel frame along c.

Parallel transport is a linear isomorphism between tangent spaces; therefore, if a set of parallel sections forms a basis in $\gamma(0)$, then it gives a basis for every $T_{\gamma(t)}M$, that is, given a point $(m, f) = f_m \in F(M)$ and a curve $\gamma(t)$, such that $\gamma(0) = m$, there is a unique horizontal lift $\gamma^*(t)$ such that $\gamma^*(0) = f_m$. This shows the existence and uniqueness of a horizontal lift in the frame bundle.

Now in terms of vector fields we show by formulas what is the induced connection on $F(M)$. Let $\pi : TM \to M$ the tangent bundle endowed with the Levi-Civita connection ∇, let $X \in T_mM$ a vector field and $e = \{e_1, \ldots, e_n\}$ an orthonormal frame of T_mM, the vector field is written

$$X = \sum_i^n X_i e_i.$$

We know that is well defined the covariant derivative

$$\nabla_X e_j = \nabla_{\sum X_i e_i} e_j = \sum_i X_i \nabla_{e_i} e_j = \sum_{i,k} X_i \Gamma_{ij}^k e_k \tag{5.2}$$

this expression can be written using the form-expression, in this way

$$\nabla_X e_j = \sum_k ((\omega_e)_{kj}(X)) e_k \tag{5.3}$$

where the $\mathbb{R}^{2n}$-valued 1-form ω_e encode the linearity of the connection. This is just a compact way to write the connection, indeed it holds

$$(\omega_e)_{kj}(X) = \sum_i X_i \Gamma_{ij}^k.$$

The matrix of 1-forms is antisymmetric, since the Christoffel symbols are, therefore, we can write it using the usual basis of $skew(n)$: $\{e_{i,j} : i < j\}$. It turns out that

$$\omega_e = \sum_{k<j} \left(\sum_i X_i \Gamma_{ij}^k \right) e_{k,j}.$$

We recall that the Christoffel symbols Γ_{ij}^k are related to the chosen frame e, as the matrix ω_e is. Now we try to build the correspondence with a form ω that acts on $F(M)$, we have to understand how ω acts on a vector field $\tilde{Y}$ in $F(M)$. Given a point $p = (m, e) \in F(M)$, a vector field in the frame bundle is an object

$$\tilde{Y} \in T_p F(M) \cong T_m M \times e \cdot \mathfrak{so}(n)$$

so ω act on it like ω_e in the $T_m M$ component and like the dual object in the other component. In particular

$$\tilde{Y} = \sum_i \tilde{\tilde{Y}}_i e_i + \sum_{i<j} V_{ij} w_{i,j} \tag{5.4}$$

and ω is the $skew(n)$-valued 1-form that acts in such a way

$$\omega(\tilde{Y})_{ij} = (\omega_e(\tilde{\tilde{Y}}))_{ij} + V_{i,j}.$$

Now that we have defined the, in some sense, extension ω of the form ω_e we can define the horizontal lift to $F(M)$ of a vector field X in TM.
We say that $X^* \in F(M)$ is the horizontal lift of the vector field $X \in TM$ if

- $\tau_*(X^*) = X$
- $\omega(X^*) = 0$;

these two conditions permit us to recover a local expression of a horizontal lift, let us see in the following lines.
Let X be the vector field we want to lift

$$X = \sum_i X_i e_i$$

let $\tilde{Y}$ a vector field in $F(M)$, then $\tilde{Y}$ is the horizontal lift of X if and only if

- from the first condition: $\tilde{\tilde{Y}}_i = X_i \quad \forall i$
- from the second condition:

$$\begin{aligned} 0 = \omega(\tilde{Y}) = \omega(X + \sum_{i<j} V_{i,j} w_{i,j}) \Leftrightarrow (\omega(\tilde{Y}))_{ij} &= (\omega_e(X))_{ij} + V_{ij} \\ &= \sum_k X_k \Gamma_{ij}^k + V_{ij}. \end{aligned}$$

Therefore the expression for the horizontal lift is

$$X^* = X - \sum_k X_k \Gamma_{ij}^k w_{i,j}. \tag{5.5}$$

Chapter A

Appendix

A.1 Tangent space on Lie groups

If G is a Lie group then its tangent space to the identity $e \in G$ is a vector space which is called *Lie algebra* and denoted by $\mathfrak{g}$, namely

$$\mathfrak{g} := T_e G$$

Since the group operations are smooth, the left translation $L_g : G \to G$ is a diffeomorphism for all $g \in G$, and therefore $(L_g)_* : T_e G \to T_g G$ is also a diffeomorphism. We recall that the Lie algebra $T_e G$ is isomorphic as vector space to the Lie sub-algebra of left invariant vector fields $\mathrm{Vec}_L(G)$

$$\mathrm{Vec}_L(G) \longrightarrow T_e G$$
$$X \mapsto X_e$$

hence one can recover the tangent space to every point just by applying the differential of the left translation

$$T_g G = L_{g*}(T_e G) = L_{g*}(\mathfrak{g}). \tag{A.1}$$

A.1.1 $G = O(n)$

The orthogonal group

$$O(n) := \{\ M \in GL(n) : MM^T = I\}$$

represents the isometries of a n-dimensional space. It is a Lie group of dimension $\frac{n(n-1)}{2}$ indeed

$$O(n) = f^{-1}(I)$$

where f is the submersion

$$f : \mathrm{GL(n)} \to \mathrm{Symm}(n).$$
$$A \mapsto AA^T$$

The matrices in $O(n)$ have determinant ± 1 therefore the space is not connected, it is the union of the two disjoint subsets $SO(n)_{\pm}(n) = \{R \in O(n) : \det R = \pm 1\}$. The special orthogonal group $SO(n) := SO(n)_+$ is a subgroup of $O(n)$ and an open subset of it; it represents the isometries that preserves the orientation. The Lie algebra of $O(n)$ is by definition

$$\mathfrak{o}(n) := T_I O(n) = \ker Df(I)$$

and considering that

$$\begin{aligned} Df(A) : T_A GL(n) &\longrightarrow T_{AA^T} Symm(n) \\ M &\mapsto MA^T + AM^T \end{aligned}$$

we have

$$\mathfrak{o}(n) = \left\{ M \in GL(n) : M + M^T = 0 \right\} =: \text{ skew } (n)$$

equipped with the matrix commutator. Since $T_I SO(n) = T_I O(n)$, because $SO(n)$ open subset, we know also the Lie algebra of $SO(n)$

$$\mathfrak{so}(n) = \text{skew}(n).$$

In particular, looking at A.1 the tangent space at every point is

$$T_M SO(n) = (L_M)_* (\mathfrak{so}(n)) = (L_M)_* \text{skew}(n) = M \cdot \text{skew}(n) \quad \forall M \in SO(n). \tag{A.2}$$

Remark. *It holds for all $M \in SO(n)$, for all $E, F \in \mathfrak{so}(n)$ that*

$$[M \cdot E, M \cdot F] = M \cdot [E, F]$$

where the first is a Lie bracket between left invariant vector fields while the second bracket is the commutator of matrices.

A.2 Differentiation along curves and parallel transport

The results presented here hold for a general linear connection.

Lemma A.1. *Given a linear connection ∇, a regular curve $\gamma : [0, T] \to M$ (with $\dot\gamma \neq 0$) and a smooth vector field Y defined only along γ, it is well defined the vector field $\nabla_{\dot\gamma} Y$ along γ and if we consider a local frame $e_1, \ldots, e_n$ and write*

$$\dot\gamma(t) = \sum_{i=1}^{n} v_i(t) e_i|_{\gamma(t)}, \quad Y(\gamma(t)) = \sum_{j=1}^{n} Y_j(\gamma(t)) e_j|_{\gamma(t)} = \sum_{j=1}^{n} y_j(t) e_j|_{\gamma(t)} \tag{A.3}$$

then the coordinate formula of the vector field is:

$$\nabla_{\dot\gamma} Y|_{\gamma(t)} = \sum_{k=1}^{n} \left(\dot y_k(t) + \sum_{i,j=1}^{n} v_i(t) y_j(t) \Gamma_{ij}^k(\gamma(t)) \right) e_k|_{\gamma(t)} \tag{A.4}$$

Proof. Firstly

$$\dot y_j(t) = \frac{d}{dt} y_j(t) = \frac{d}{dt} Y_j(\gamma(t)) = \sum_i e_i(Y_j)\Big|_{\gamma(t)} (\dot\gamma(t))^i = \sum_i e_i(Y_j)\Big|_{\gamma(t)} v_i(t)$$

and so

$$\sum_j \dot y_j(t) e_j\Big|_{\gamma(t)} = \sum_{i,j} v_i(t) e_i(Y_j)\Big|_{\gamma(t)} e_j\Big|_{\gamma(t)}.$$

Then differentiating

$$\begin{aligned}\nabla_{\dot\gamma} Y\Big|_{\gamma(t)} &= \sum_i \sum_j v_i(t) \left(e_i(Y_j)\Big|_{\gamma(t)} \right) e_j\Big|_{\gamma(t)} + \sum_{i,j,k} v_i(t) y_j(t) \Gamma_{ij}^k(\gamma(t)) e_k\Big|_{\gamma(t)} \\ &= \sum_k \dot y_k(t) e_k\Big|_{\gamma(t)} + \sum_{i,j,k} v_i(t) y_j(t) \Gamma_{ij}^k(\gamma(t)) e_k\Big|_{\gamma(t)} \\ &= \sum_k \left(\dot y_k + \sum_{i,j} v_i(t) y_j(t) \Gamma_{ij}^k(\gamma(t)) \right) e_k\Big|_{\gamma(t)}.\end{aligned}$$

□

Proposition A.1. *Let ∇ be a linear connection. Let $\gamma : [0,T] \to M$ be a smooth curve and $v_0 \in T_{\gamma(0)}M$. There exists a unique smooth vector field V along γ such that $V|_{\gamma(0)} = v_0$ and V parallel along γ with respect to ∇.*

Proof. We have to solve the non autonomous linear system of differential equations

$$\dot y_k(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^k(\gamma(t)) \dot\gamma_i(t) y_j(t) = 0, \quad k = 1, \dots, n$$

which can be written setting $A_{k,j}(t) = \sum_{i=1}^n \Gamma_{ij}^k(\gamma(t))\dot\gamma_i(t)$ as

$$\dot y_k(t) + \sum_{j=1}^{n} A_{k,j}(t) y_j(t) = 0, \quad k = 1, \dots, n$$

Since the differential equation is linear then the solution is global, i.e., defined on $[0,T]$. □

A.3 Orbit Theorem, Nagano-Sussmann

Let Δ be a distribution on $\mathcal{Q}$ of complete and smooth vector fields

$$\Delta\Big|_q = \operatorname{span}\{X_1, \dots, X_h\}\Big|_q.$$

We have defined the attainable set from a point $q_0 \in \mathcal{Q}$

$$\mathcal{A}_{q_0}(t) = \left\{ \phi^{\tau_k}_{X_{i_k}} \circ \ldots \circ \phi^{\tau_1}_{X_{i_1}}(q_0) \mid \tau_i \geq 0, \sum_{i=1}^{k} \tau_i = t, i_j \in \{1, \ldots, h\}, k \in \mathbb{N} \right\}.$$

Now consider a larger set; the orbit of the distribution Δ through a point $q_0 \in \mathcal{Q}$:

$$\mathcal{O}_{q_0} = \left\{ \phi^{\tau_k}_{X_{i_k}} \circ \ldots \circ \phi^{\tau_1}_{X_{i_1}}(q_0) \mid \tau_i \in \mathbb{R}, i_j \in \{1, \ldots, h\}, k \in \mathbb{N} \right\}.$$

In an orbit $\mathcal{O}_{q_0}$, movement along vector fields X_i is allowed both forward and backward, while in an attainable set $\mathcal{A}_{q_0}$ only forward motion is possible, see $\tau \geq 0$. Although, if the distribution Δ is symmetric, that is, $X_i \in \Delta \Rightarrow -X_i \in \Delta$, then the attainable set from a point q_0 coincides with the orbit of that point:

$$\mathcal{O}_{q_0} = \mathcal{A}_{q_0} \qquad \forall q_0 \in M.$$

In general, orbits have a simpler structure than attainable sets, and this is described in this theorem.
In this work, we use the setting which gives us the simplest description; therefore, here we write a version of the Orbit Theorem which is not the more general but which is suitable for our purpose; if not specified, we are working under the same hypothesis of this theorem.

Theorem A.1. *Let $\Delta \subset \operatorname{Vec} M$ and $q_0 \in M$. Then $\mathcal{O}_{q_0}$ is a connected immersed sub-manifold of M and if Δ is analytic, then $T_q\mathcal{O}_{q_0} = Lie_q\Delta$*

A.4 Relationship between left and right invariant basis of $\mathfrak{so}(n)$

Here we prove the relation between the left invariant vector fields $W_{i,j} := R \cdot e_{i,j}$ and the right analogous $\widetilde{W}_{i,j} := e_{i,j} \cdot R$, with $R \in SO(n)$ and $e_{i,j} = e_i \otimes e_j - e_j \otimes e_i$.
This relation is the same that holds between $w_{i,j} := f \cdot e_{i,j}$ and $\widetilde{w}_{i,j} := e_{i,j} \cdot f$ so we prove it just for the first ones.

Notation. *For any matrix $A \in \mathbb{R}^{n^2}$ and any pair $i, j \in \{1, \ldots, n\}$, the product $A(e_i \otimes e_j)$ is a matrix in $\mathbb{R}^{n^2}$. Moreover, we have the following expression:*

$$[A(e_i \otimes e_j)]_{l,k} = \sum_m A_{l,m}(e_i \otimes e_j)_{m,k} = A_{l,i}\, \delta_{j,k}, \tag{A.5}$$

where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ otherwise. We can rewrite (A.5) in a more compact form as:

$$A(e_i \otimes e_j) = (A \cdot e_i) \otimes e_j = A_{\cdot,i} \otimes e_j, \tag{A.6}$$

where $A_{\cdot,i}$ denotes the i-th column of the matrix A.
Similarly, we obtain the following expression for the right multiplication:

$$(e_i \otimes e_j)A = e_i \otimes (e_j \cdot A) = e_i \otimes A_{j,\cdot}, \tag{A.7}$$

where $A_{j,\cdot}$ denotes the j-th row of the matrix A.

Now we can compute $W_{i,j}$ and $\widetilde{W}_{i,j}$. We use A.6

$$W_{i,j} = R[e_i \otimes e_j - e_j \otimes e_i] = R_{\cdot,i} \otimes e_j - R_{\cdot,j} \otimes e_i \tag{A.8}$$

and using A.7

$$\widetilde{W}_{i,j} = e_i \otimes R_{j,\cdot} - e_j \otimes R_{\ i,\cdot} \tag{A.9}$$

We can prove that

$$\widetilde{W}_{i,j} = \sum_{l,s} R_{i,l} R_{j,s} W_{l,s} \tag{A.10}$$

indeed it holds

$$\begin{aligned}
\sum_{l,s} R_{i,l} R_{j,s} W_{l,s} &= \sum_{l,s} R_{i,l} R_{j,s} \left[R_{\cdot,l} \otimes e_s - R_{\cdot,s} \otimes e_l\right] \\
&= \sum_{l} R_{i,l} \left(\sum_{s} R_{j,s} R_{\cdot,l} \otimes e_s\right) - \sum_{s} R_{j,s} \left(\sum_{l} R_{i,l} R_{\cdot,s} \otimes e_l\right) \\
&= \sum_{l} R_{i,l} \left(R_{\cdot,l} \otimes \sum_{s} R_{j,s} e_s\right) - \sum_{s} R_{j,s} \left(R_{\cdot,s} \otimes \sum_{l} R_{i,l} e_l\right) \\
&= \sum_{l} R_{i,l} \left(R_{\cdot,l} \otimes R_{j,\cdot}\right) - \sum_{s} R_{j,s} \left(R_{\cdot,s} \otimes R_{i,\cdot}\right) \\
&= \left(\sum_{l} R_{i,l} R_{\cdot,l}\right) \otimes R_{j,\cdot} - \left(\sum_{s} R_{j,s} R_{\cdot,s}\right) \otimes R_{i,\cdot} \\
&= e_i \otimes R_{j,\cdot} - e_j \otimes R_{i,\cdot}
\end{aligned}$$

where in the last line we have used the orthogonality of R. Indeed,

$$\left(\sum_{l} R_{i,l} R_{\cdot,l}\right)_k = \sum_{l} R_{i,l} R_{k,l} = \sum_{l} R_{i,l} R^T_{l,k} = (RR^T)_{i,k} = \delta_{i,k}.$$

A.5 Curvature form and structure equation in a Principal Bundle

Here we write and prove the structure equations for a Riemannian manifold endowed with the Levi-Civita connection. Everything here is written in suitable terms for our work: structure equations for the frame bundle $F(M)$.
Objects involved: θ, Ω, ω

- ω=connection form: $\mathfrak{so}(n)$-valued 1-form on $F(M)$.
  By definition we have $\ker(\omega) = H$.

- $\theta =$ tautological one-form: $\mathbb{R}^n$-valued 1-form on $F(M)$.

$$\theta = (\theta_1, \dots, \theta_n) \quad \text{s.t.} \quad \theta_j(v) = \langle f_j, \tau_* v\rangle \quad \forall v \in \mathrm{Vec}(F(M)).$$

  By definition we have $\ker(\tau_*) = V \subseteq \ker\theta$.

- $\Omega =$ curvature form: $\mathfrak{so}(n)$-valued 2-form on $F(M)$.

$$\Omega_{ij}(v_1, v_2) = R(\tau_* v_1, \tau_* v_2, f_j, f_i) \qquad v_1, v_2 \in \mathrm{Vec}(F(M)),$$

  where R is the Riemannian curvature tensor.

Remark. *To define the wedge product of a pair of Lie algebra-valued differential forms, we substitute the standard product with the Lie bracket, to obtain another Lie algebra-valued form. Let ω a $\mathfrak{g}$-valued p-form and η a $\mathfrak{g}$-valued q-form, their wedge product $\omega \wedge \eta$ is given by*

$$\omega \wedge \eta\,(v_1, \dots, v_{p+q}) = \frac{1}{p!q!}\sum_\sigma \mathrm{sgn}(\sigma)\left[\omega\left(v_{\sigma(1)}, \dots, v_{\sigma(p)}\right), \eta\left(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)}\right)\right]$$

where the v_i's are tangent vectors. For example, if ω and η are Lie-algebra-valued one forms, then one has

$$\omega \wedge \eta\,(v_1, v_2) = \left[\omega\,(v_1), \eta\,(v_2)\right] - \left[\omega\,(v_2), \eta\,(v_1)\right].$$

Theorem A.2 (1st Structure Equation)**.** *Let $X, Y \in \mathrm{Vec}(F(M))$ it holds*

$$d\theta(X,Y) = -(\omega(X)\cdot\theta(Y) - \omega(Y)\cdot\theta(X)) \text{ matrix expression}$$
$$d\theta_i(X,Y) = -\sum_k \omega_{ik}\wedge\theta_k(X,Y) \text{ component wise expression}$$

Compact expression:

$$d\theta = -\omega\wedge\theta.$$

Theorem A.3 (2nd Structure Equation)**.** *Let $X, Y \in \mathrm{Vec}(F(M))$ it holds*

$$d\omega(X,Y) = -[\omega(X), \omega(Y)] + \Omega(X,Y) \text{ matrix expression}$$
$$d\omega_{ij}(X,Y) = -\sum_k \omega_{ik}\wedge\omega_{kj}(X,Y) + \Omega_{ij}(X,Y) \text{ component wise expression}$$

Compact expression:

$$d\omega = -\omega\wedge\omega + \Omega$$

Corollary A.1. *Let $X, Y \in \mathrm{Vec}(F(M))$, it holds*

$$\theta([X,Y]) = -d\theta(X,Y).$$

Corollary A.2. *If $X, Y \in H$ then*

$$\omega([X,Y]) = -\Omega(X,Y).$$

Here we report only the proof of the second structure equation theorem A.3, since the technique is rather similar for all the statements in this section.

Proof. We choose a local section (e, U) of $F(M)$ and we consider a frame $f \in F_m M$ for $m \in U$. Local coordinates are given as in the proof of Theorem 4.1 .
Every vector of $F(M)$ is a sum of a vertical vector and a horizontal vector. Since both sides of the above equality are bilinear and skew-symmetric in X and Y, it is sufficient to verify the equality in the following three special cases.

- X and Y are horizontal:
  The $\ker(\omega_f) = H_f$ so in this case, $\omega(X) = \omega(Y) = 0$ and the equality, using Cartan' formula, reduces to

$$\begin{aligned} d\omega_{i,j}(X,Y) &= X\omega_{i,j}(Y) - Y\omega_{i,j}(X) - \omega_{i,j}([X,Y]) \\ &= -\langle [\tau_* X, \tau_* Y]_{ver} f_i, f_j \rangle \\ &= -\langle R(\tau_* X, \tau_* Y) f_i, f_j \rangle \\ &= -\Omega_{j,i}(X,Y) \\ &= \Omega_{i,j}(X,Y). \end{aligned}$$

- X and Y are vertical:
  Let $X = A^*$ and $Y = B^*$ at f, where $A, B \in \mathfrak{so}(n)$. Here A^* and B^* are the fundamental vector fields corresponding to A and B respectively. By Cartan's formula, we have

$$\begin{aligned} d\omega\left(A^*, B^*\right) &= A^*\left(\omega\left(B^*\right)\right) - B^*\left(\omega\left(A^*\right)\right) - \omega\left(\left[A^*, B^*\right]\right) \\ &= A^*(B) - B^*(A) - [A,B] \\ &= -\left[\omega\left(A^*\right), \omega\left(B^*\right)\right] \end{aligned}$$

  since $\omega\left(A^*\right) = A, \omega\left(B^*\right) = B$ are constant element, therefore differentiating they go to zero and $[A^*, B^*] = [A,B]^*$. On the other hand,

$$\Omega\left(A^*, B^*\right) = 0.$$

- X is horizontal and Y is vertical:
  Let $Y = A^*$ at f, where $A \in \mathfrak{so}(n)$. The right hand side of the equality vanishes,

$$-[\omega(X), \omega(A^*)] + \Omega(X, A^*) = -[0, A] + R(\tau_*(X), \tau_*(A^*)) = 0.$$

we have to show that $d\omega\left(X,A^{*}\right)=0$. Again by Cartan's formula , we have

$$\begin{aligned}d\omega\left(X,A^{*}\right)&=X\left(\omega\left(A^{*}\right)\right)-A^{*}(\omega(X))-\omega\left(\left[X,A^{*}\right]\right)\\&=-\omega\left(\left[X,A^{*}\right]\right);\end{aligned}$$

we want to prove that $[X,A^{*}]$ is horizontal, if this is the case, also the left hand side vanishes and the equality is proven.It holds that

$$[X,A^{*}]=\lim_{t\to 0}\frac{1}{t}\left[R_{a_t}(X)-X\right]$$

and if X is horizontal, so is $R_{a_t}(X)$. Thus $[X,A^{*}]$ is horizontal.

□

A.6 Tensorial algebra

Here we provide the computation to prove the following Lemma.

Lemma A.2. *Let f_k^{*} be a generator of H_f.*

- *For any l-tensor T,*

$$f_k^{*}\mathscr{E}_{i_1,\ldots,i_l}(T)=\mathscr{E}_{i_1,\ldots,i_l,k}(\nabla T)$$

- *For any $2+l$-tensor T, that is antisymmetric in the first two arguments*

$$[f_k^{*},\mathscr{W}_{i_1,\ldots,i_l}(T)]=\mathscr{W}_{i_1,\ldots,i_l,k}(\nabla T)-\sum_{s=1}^{n}\mathscr{E}_{s,k,i_1,\ldots,i_l}(T)f_s^{*}$$

Proof. We use a local representation of f_i^{*}

$$f_i^{*}=f_i-\sum_{\alpha<\beta}\mathbf{\Gamma}^{\alpha}_{\mathbf{i}\beta}w_{\alpha,\beta}$$

Since $w_{\alpha\beta}$ is just the representation of $e_{\alpha\beta}$ in local coordinates, we put ourself in the simpler case

$$f_i^{*}=f_i-\sum_{\alpha<\beta}\mathbf{\Gamma}^{\alpha}_{\mathbf{i}\beta}e_{\alpha,\beta};$$

we recall that $\mathbf{\Gamma}^{\alpha}_{\mathbf{i}\beta}=\langle f_\alpha,\nabla_{f_i}f_\beta\rangle$. First we write the matrix-vector product

$$\left(e_{\alpha\beta}f_i\right)_k=\delta_{\beta,k}f_{\alpha i}-\delta_{\alpha,k}f_{\beta i}\qquad k=1,\ldots,n$$

with this compact notation

$$e_{\alpha\beta}f_i=\delta_{\beta,j}f_\alpha-\delta_{\alpha,j}f_\beta$$

where j on the right-hand-side is a free index and this writing means that if $\beta = j$ then we put the α-th component of f_i in the i-th component of the resulting vector and if $\alpha = j$ then in the i-th component we put the β-th component of f_i.
This leads to

$$\begin{aligned}\sum_{\alpha<\beta} \langle f_\alpha, \nabla_{f_k} f_\beta \rangle e_{\alpha,\beta} f_i &= \frac{1}{2} \sum_{\alpha,\beta} \mathbf{\Gamma}^{\alpha}_{\mathbf{k}\beta} (\delta_{\beta,i} f_\alpha - \delta_{\alpha,i} f_\beta) \\ &= \frac{1}{2} \sum_{\alpha,\beta} \mathbf{\Gamma}^{\alpha}_{\mathbf{k}\beta} (\delta_{\beta,i} f_\alpha) - \frac{1}{2} \sum_{\alpha,\beta} \mathbf{\Gamma}^{\alpha}_{\mathbf{i}\beta} (\delta_{\beta,i} f_\alpha) \\ &= \frac{1}{2} \sum_{\alpha} \mathbf{\Gamma}^{\alpha}_{\mathbf{ki}} f_\alpha + \frac{1}{2} \sum_{\beta} \mathbf{\Gamma}^{\beta}_{\mathbf{ki}} f_\beta \\ &= \sum_{\alpha} \mathbf{\Gamma}^{\alpha}_{\mathbf{ki}} f_\alpha \\ &= \sum_{\alpha} \langle f_\alpha, \nabla_{f_k} f_i \rangle f_\alpha = \nabla_{f_k} f_i.\end{aligned}$$

Then the conclusion follows from

$$\begin{aligned}f_k^* \mathscr{E}_{i_1,\dots,i_l}(T) &= f_k T(f_{i_1}, \dots, f_{i_l}) - \sum_{\alpha<\beta} \mathbf{\Gamma}^{\mathbf{i}}_{\mathbf{k}\beta} e_{\alpha\beta} T(f_{i_1}, \dots, f_{i_l}) \\ &= f_k T(f_{i_1}, \dots, f_{i_l}) - \sum_{\alpha<\beta} \mathbf{\Gamma}^{\mathbf{i}}_{\mathbf{k}\beta} T(e_{\alpha\beta} f_{i_1}, \dots, f_{i_l}) - \sum_{\alpha<\beta} \mathbf{\Gamma}^{\mathbf{i}}_{\mathbf{k}\beta} T(f_{i_1}, e_{\alpha\beta} f_{i_2}, \dots, f_{i_l}) \\ &\quad - \dots - \sum_{\alpha<\beta} \mathbf{\Gamma}^{\mathbf{i}}_{\mathbf{k}\beta} T(f_{i_1}, f_{i_2}, \dots, e_{\alpha\beta} f_{i_l}) \\ &= f_k T(f_{i_1}, \dots, f_{i_l}) - T(\nabla_{f_k} f_{i_1}, \dots, f_{i_l}) - T(f_{i_1}, \nabla_{f_k} f_{i_2}, \dots, f_{i_l}) \\ &\quad - \dots - T(f_{i_1}, f_{i_2}, \dots, \nabla_{f_k} f_{i_l}) \\ &= \nabla T(f_{i_l}, \dots, f_{i_l}, f_k) \\ &= \mathcal{E}_{i_l,\dots,i_l,k}(\nabla T).\end{aligned}$$

The second part of the statement is given below.
Recall that $\theta_j(f_k^*) = \delta_{j,k}$ and $\omega(f_k^*) = 0$, while $\theta_j(e_{h,\lambda}) = 0$ and $\omega(e_{h,\lambda}) = w_{h,\lambda}$. Hence, we have, using the previous notation

$$[f_k^*, e_{h,\lambda}] = \delta_{k,h} f_\lambda^* - \delta_{k,\lambda} f_h^*.$$

We conclude that

$$\begin{aligned}
[f_k^*, \mathscr{W}_{i_1,\dots,i_l}(T)] &= \frac{1}{2}\sum_{h,\lambda} f_k^*\left(\mathscr{E}_{h,\lambda,i_1,\dots,i_l}\right) e_{h,\lambda} + \frac{1}{2}\sum_{h,\lambda=1}^{n} \mathscr{E}_{h,\lambda,i_1,\dots,i_l}\,[f_k^*, e_{h,\lambda}] \\
&= \sum_{h<\lambda} f_k^*\left(\mathscr{E}_{h,\lambda,i_1,\dots,i_l}\right) e_{h,\lambda} + \frac{1}{2}\sum_{h,\lambda=1}^{n} \mathscr{E}_{h,\lambda,i_1,\dots,i_l}(\delta_{k,h} f_\lambda^* - \delta_{k,\lambda} f_h^*) \\
&= \mathscr{W}_{i_1,\dots,i_l,k}(\nabla T) + \frac{1}{2}\sum_{h,\lambda=1}^{n} \mathscr{E}_{h,\lambda,i_1,\dots,i_l}\delta_{k,h} f_\lambda^* - \frac{1}{2}\sum_{h,\lambda=1}^{n} \mathscr{E}_{h,\lambda,i_1,\dots,i_l}\delta_{k,\lambda} f_h^* \\
&= \mathscr{W}_{i_1,\dots,i_l,k}(\nabla T) + \frac{1}{2}\sum_{\lambda=1}^{n} \mathscr{E}_{k,\lambda,i_1,\dots,i_l} f_\lambda^* - \frac{1}{2}\sum_{h}^{n} \mathscr{E}_{h,k,i_1,\dots,i_l} f_h^* \\
&= \mathscr{W}_{i_1,\dots,i_l,k}(\nabla T) - \sum_{s} \mathscr{E}_{s,k,i_1,\dots,i_l} f_s^*
\end{aligned}$$

since T is antisymmetric in the first coordinates.

□

Chapter B

Appendix on Curvature

B.1 Riemannian curvature

B.1.1 Connection

In our work, we endow the vector bundle TM and the frame bundle $F(M)$ with connections. Here we report the definitions for a general vector bundle $\pi : F \to M$. Let $F_m = \pi^{-1}(m)$ denote the fiber of the bundle at the point $m \in M$. Choosing a connection ∇ on a vector bundle F, as we already said, means choosing in every point a complement to the tangent space of a fiber. So, if we call the subspace tangent to the fiber the vertical space $\mathcal{V}_f$

$$\mathcal{V}_f := \ker \pi_*\big|_f = T_f F_{\pi(f)} \qquad \forall f \in F$$

we have the splitting of the tangent space to F in the vertical and in the horizontal parts

$$T_f F = \mathcal{V}_f \oplus \mathcal{H}_f;$$

and choosing a connection means choosing a proper horizontal space $\mathcal{H}$ for every $f \in F$. The collection of these subspaces for every point forms, respectively, the vertical and the horizontal distribution. Given $v \in Vec(M)$ a vector field on M we define the *horizontal lift* of v as the unique vector field $v^* \in Vec(F)$ such that

$$v^*\Big|_f \in \mathcal{H}_f \quad \& \quad \pi_* v^* = v \qquad \forall f \in F.$$

Remark. *In literature there are more than one way to denote the horizontal lifts. We have chosen this one even if this expression does not carry the symbol of the connection. If necessary we use also the following equivalent notation*

$$v^* = \nabla_v.$$

Notice that the lift is unique because the differential of π

$$\pi_*\Big|_f : T_f F \to T_{\pi(f)} M$$

is a surjective map with the kernel equal to the vertical subspace, so it becomes an isomorphism when we restrict it to the horizontal space. We underline that the choice of connection determines the expression of the lifts.
The tool of connection gives us the possibility to put in relation fibers of different points in the manifold with the concept of *parallel transport*. Indeed, let $\gamma : [0,T] \to M$ a smooth curve on the manifold and the vector field $X(t) \in Vec(M)$, defined in a neighbourhood of the curve, that represents an extension of velocity field of the curve

$$\dot{\gamma}(t) = X(\gamma(t)) \qquad \forall t \in [0,T];$$

we can consider the vector field given by its lift $\nabla_{\dot{\gamma}(t)}$ or equivalently, following the previous notation

$$X^*(t).$$

The parallel transport along the curve γ is the map P defined by the flow of the non-autonomous vector field $X^*(t)$; let $0 \leq t_0 < t_1 \leq T$, the parallel transport from the fiber of $\gamma(t_0)$ to the fiber of $\gamma(t_1)$ is

$$P^{\gamma}_{t_0,t_1} : F_{\gamma(t_0)} \to F_{\gamma(t_1)}.$$

This is a general approach to the definition of connection which focuses on the geometric visualization, but it isn't the only one.
If we consider the case of an affine connection, connection on the tangent bundle TM, we also step into the concept of differential operator. Connection becomes also a tool to differentiate vector fields on manifolds, this other aspect is highlighted with the following definition of connection. In [**tu2017differential**], for example, is shown as an affine connection on a manifold M induces a unique covariant derivative of vector fields along a smooth curve in M. This in some sense generalizes the derivative of a vector field along a smooth curve in $\mathbb{R}^n$. In the next lines, we try to give an intuition about this and to see how the two definitions correspond. This also permits us to present a way of describing a connection in local coordinates, using the so-called Christoffel symbols.

An affine connection on TM is a map

$$\nabla : \mathrm{Vec}(M) \times \mathrm{Vec}(M) \to \mathrm{Vec}(M)$$

bilinear and such that

- $\nabla_{fX}(Y) = f\nabla_X Y \qquad \forall f \in \mathcal{C}^\infty(M)$
- $\nabla_X(gY) = (Xg)Y + g\nabla_X Y \qquad \forall g \in \mathcal{C}^\infty(M)$

Given a local frame $v_1, \dots, v_n$ we define the Christoffel symbols of ∇ associated to the frame as the set of functions Γ_{ij}^k satisfying

$$\nabla_{v_i} v_j = \sum_{k=1}^{n} \Gamma_{ij}^k v_k;$$

the Christoffel symbols encodes the linearity of the connection.
In this way, we recover the idea of differentiating along a curve because given a regular curve $\gamma : [0, T] \to M$ (with $\dot{\gamma} \neq 0$) and a smooth vector field Y defined only along γ, the vector field

$$\nabla_{\dot{\gamma}} Y$$

along γ is well defined. In this context, we define what it means to be parallel: we say that a vector field Y along a smooth curve $\gamma : [0, T] \to M$ is *parallel* with respect to ∇ if

$$\nabla_{\dot{\gamma}} Y = 0$$

along γ. From A.2 we know that there exists a unique smooth vector field V along γ such that $V|_{\gamma(0)} = v_0$ and V parallel along γ with respect to ∇. The map which associates v_0 with $V|_{\gamma(0)}$ is exactly the parallel transport of v_0 along γ

$$P_{0,t}^{\gamma} : T_{\gamma(0)} M \to T_{\gamma(t)} M$$
$$v_0 \mapsto v_t := V|_{\gamma(t)}$$

B.1.2 Curvature of a connection

After these observations on connections, we proceed with the related definitions. Given an element $W \in T_f F = \mathcal{V}_f \oplus \mathcal{H}_f$, we denote by W_{hor} its projection on the horizontal subspace at the point f and analogously W_{ver} is the projection on the vertical subspace.

The Lie bracket of two vertical vector fields is again a vertical vector field, because the pushforward commutes with the bracket and vertical vector fields are in the kernel of π_*; let $W, Z \in \mathcal{V}$

$$\pi_*([W^*, Z^*]) = [\pi_*(W^*), \pi_*(Z^*)] = [0, 0] = 0.$$

The curvature operator is a way of measuring how far we are from having this property also for two horizontal vector fields.

Let $X, Y \in \mathrm{Vec}(M)$ the *curvature of the connection* ∇ is the operator

$$R(X, Y) := [\nabla_X, \nabla_Y]_{ver}.$$

Notice that, given a vector field $W \in T_f F$, its horizontal part coincides, by definition, with the lift of its projection. In particular

$$W_{hor} = \pi_* W = \nabla_{\pi_* W};$$

therefore for $X, Y \in \mathrm{Vec}(M)$

$$\pi_*([\nabla_X, \nabla_Y]) = [\pi_* \nabla_X, \pi_* \nabla_Y] = [X, Y],$$
$$[\nabla_X, \nabla_Y]_{hor} = \nabla_{[X,Y]}.$$

Now through previous identities we rewrite the curvature $R(X, Y)$ in a more classical way

$$\begin{aligned} R(X,Y) &= [\nabla_X, \nabla_Y] - [\nabla_X, \nabla_Y]_{hor} \\ &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \\ &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}. \end{aligned}$$

Notice that if $R(X, Y) = 0$ for every X, Y then the horizontal distribution is involutive

$$[\nabla_X, \nabla_Y] = \underbrace{[\nabla_X, \nabla_Y]_{ver}}_{R(X,Y)} + [\nabla_X, \nabla_Y]_{hor} \qquad \forall \nabla_X, \nabla_Y \in \mathcal{H}$$

and the converse is also true.

B.1.3 Riemannian curvature

If the manifold M is endowed with a Riemannian metric we can speak about compatibility with the connection and we consider a well known fact that on a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ there exists a unique affine connection that is compatible with the metric and with zero torsion. In this context we can write a scalar value of the curvature that recovers the idea of Gaussian curvature. In this case the curvature tensor is often written in the $(3, 1)$ form, known as the *Riemannian curvature tensor*. Let $X, Y, Z, W \in Vec(M)$

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W \rangle.$$

We recall some properties:

- R is a tensor, i.e., the value of $R(X, Y)$ at a point depends only on the value of X and Y at the point itself;
- skew-symmetric: $R(X, Y) = -R(Y, X)$; therefore $R(X, Y, Z, W)$ is skew symmetric in the first two entries;
- R is $C^\infty(M)$-linear in every variable.

The Gaussian curvature, concept related to a 2-dimensional Riemannian manifold M, can be expressed in these terms in the following way. The Gaussian curvature k_q at a point $q \in M$ is

$$k_q = \langle R_q(X, Y)Y, X \rangle$$

for any orthonormal basis X, Y for the tangent plane T_qM; R_q is the value of the curvature tensor computed at the point q. The definition is well defined, i.e. is independent of the choice of orthonormal basis.

B.2 Curvature of a 2D-Riemannian manifold

Here we recover the expression of the Riemannian curvature tensor avoiding the Hamiltonian formalism.

Let $(M, \langle\, , \,\rangle)$ be a Riemannian manifold of dimension two. Let X_1, X_2 be a local orthonormal frame with structure's functions $c_1, c_2 \in \mathcal{C}^\infty(M)$:

$$\langle X_i, X_j \rangle = \delta_{ij}, \quad [X_1, X_2] = c_1 X_1 + c_2 X_2.$$

In the next computation we use the Levi-Civita connection ∇.

The Christoffel symbols of this connection are completely described in the terms of the functions' structure if we use an orthonormal frame, in particular using the Koszul formula one can prove this relations (here provided for a general n-dim manifold):

$$X_1, \dots, X_n \text{ local orthonormal frame such that } [X_i, X_j] = \sum_{k=1}^n c_{ij}^k X_k$$

$$\text{then the Christoffel symbols associated } \nabla_{X_i} X_j = \sum_{k=1}^n \Gamma_{ij}^k X_k \text{ are}$$

$$\Gamma_{ij}^k = \frac{1}{2}(c_{ij}^k - c_{jk}^i + c_{ki}^j).$$

In particular we have that $c_{ii}^k = 0 \ \forall k$ and $\Gamma_{ij}^k = -\Gamma_{ik}^j$.

Remark. *This description of the Christoffel symbols depends on the choice of the frame, if we chose, for example, a coordinate frame, i.e., $X_i = \frac{\partial}{\partial x_i}$ we obtain the description of the symbols in terms of the metric, is an equivalent description.*

In the 2-dim case we use the following identifications:

$$c_{21}^1 = -c_{12}^1 = -c_1, \qquad c_{21}^2 = -c_{12}^2 = -c_2,$$

in order to have

$$[X_1, X_2] = c_{12}^1 X_1 + c_{12}^2 X_2 = c_1 X_1 + c_2 X_2$$
$$[X_2, X_1] = c_{21}^1 X_1 + c_{21}^2 X_2 = -c_1 X_1 - c_2 X_2.$$

There are 2^3 total indices, 2^2 non zero and only 2 independent:

$$\Gamma_{ii}^i = 0, \quad \Gamma_{ij}^j = 0,$$
$$\Gamma_{12}^1 = c_1 = -\Gamma_{11}^2,$$
$$\Gamma_{22}^1 = c_2 = -\Gamma_{21}^2.$$

Now we can compute the Riemann curvature tensor:

$$\begin{aligned}R(X_1, X_2)X_2 &= [\nabla_{X_1}, \nabla_{X_2}]X_2 - \nabla_{[X_1,X_2]}X_2\\ &= \nabla_{X_1}\nabla_{X_2}X_2 - \nabla_{X_2}\nabla_{X_1}X_2 - \nabla_{[X_1,X_2]}X_2.\end{aligned}$$

Remember that $\nabla_{X_i}X_j = \sum_{k=1}^{n}\Gamma_{ij}^{k}X_k$, so

$$\begin{aligned}\nabla_{X_1}X_1 &= 0 + \Gamma_{11}^{2}X_2 = -c_1X_2, \qquad &\nabla_{X_1}X_2 = \Gamma_{12}^{1}X_1 + 0 = c_1X_1\\ \nabla_{X_2}X_1 &= 0 + \Gamma_{21}^{2}X_2 = -c_2X_2, \qquad &\nabla_{X_2}X_2 = \Gamma_{22}^{1}X_1 + 0 = c_2X_1.\end{aligned}$$

Then putting them in the previous expression we have

$$\begin{aligned}R(X_1, X_2)X_2 &= \nabla_{X_1}(c_2X_1) - \nabla_{X_2}(c_1X_1) - c_1\nabla_{X_1}X_2 - c_2\nabla_{X_2}X_2\\ &= \nabla_{X_1}(c_2)X_1 + c_2(-c_1X_2) - \nabla_{X_2}(c_1)X_1 + c_1(c_2X_2) - c_1^2X_1 - c_2^2X_1\\ &= X_1(c_2)X_1 - c_1c_2X_2 - X_2(c_1)X_1 + c_1c_2X_2 - c_1^2X_1 - c_2^2X_1\\ &= X_1(c_2)X_1 - X_2(c_1)X_1 - c_1^2X_1 - c_2^2X_1\end{aligned}$$

and computing the inner product we obtain the scalar value of the curvature written depending on the functions' structure.

$$\langle R(X_1, X_2)X_2, X_1\rangle = X_1(c_2) - X_2(c_1) - c_1^2 - c_2^2. \tag{B.1}$$

B.3 Sectional Curvature

Let M be a Riemannian manifold and p a point in M. If L is a 2-dimensional subspace of the tangent space T_qM, then we define the sectional curvature of L to be

$$\sec(L) = \langle R\left(e_1, e_2\right) e_2, e_1\rangle$$

for any orthonormal basis e_1, e_2 of L. Just as in the definition of the Gaussian curvature, the right-hand side is independent of the orthonormal basis.
If X, Y is an arbitrary basis for the 2-plane L, then the sectional curvature of L is also given by

$$\sec(X, Y) = \frac{R(X, Y, Y, X)}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle}.$$

B.4 Curvature of the sphere

This is a known fact, but just to use the tools we have in this work, we compute the curvature of the sphere $\mathbb{S}^2$ with the Riemann curvature tensor. Consider a 2-dimensional sphere of radius

R, we use polar coordinates to parametrize it, $(\theta, \phi) \in [0, 2\pi]$

$$\begin{aligned} x &= R\cos\theta\sin\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\phi. \end{aligned}$$

The pullback of the standard Euclidean metric of $\mathbb{R}^3$ gives us the metric on the sphere:

$$R^2\sin^2\phi d\theta^2 + R^2 d\phi^2.$$

Now we write it in the matrix form g, followed by the inverse

$$g = R^2\begin{pmatrix}\sin^2\phi & 0\\ 0 & 1\end{pmatrix}, \qquad g^{-1} = \frac{1}{R^2\sin^2\phi}\begin{pmatrix}1 & 0\\ 0 & \sin^2\phi\end{pmatrix}.$$

The next step is to compute the Christoffel symbols. We derive them from the metric using the following classical formula, this is the general formula of the Christoffel symbols of the Levi Civita connection associated with the coordinate frame $\{X_i = \frac{\partial}{\partial x_i}, i = 1, \ldots, \dim M\}$

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{m=1}^{n} g^{mk}\left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m}\right).$$

From the symmetry of the metric, we have that $\Gamma_{ij}^k = \Gamma_{ji}^k \quad \forall i, j, k$. In our case $X_1 = \frac{\partial}{\partial\theta}$ and $X_2 = \frac{\partial}{\partial\phi}$ and we have 8 symbols among which 6 independents and since our metric is diagonal we obtain

$$\begin{aligned} &\Gamma_{11}^1 = 0 = \Gamma_{22}^2;\\ &\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{\cos\phi}{\sin\phi};\\ &\Gamma_{12}^2 = \Gamma_{21}^2 = 0;\\ &\Gamma_{11}^2 = -\sin\phi\cos\phi;\\ &\Gamma_{22}^1 = 0. \end{aligned}$$

Now we can write the derivations $\nabla_{X_i}X_j = \sum_k \Gamma_{ij}^k X_k$;

$$\begin{aligned} &\nabla_{X_1}X_1 = \Gamma_{11}^2 X_2;\\ &\nabla_{X_2}X_2 = 0;\\ &\nabla_{X_1}X_2 = \Gamma_{12}^1 X_1;\\ &\nabla_{X_2}X_1 = \Gamma_{21}^1 X_1. \end{aligned}$$

Finally the sectional curvature

$$\sec(X_1, X_2) = \frac{R(X_1, X_2, X_2, X_1)}{||X_1||^2||X_2||^2 - \langle X_1, X_2\rangle};$$

$$R(X_1, X_2, X_2, X_1) = \langle R(X_1, X_2)X_2, X_1\rangle = \langle -\nabla_{X_2}\nabla_{X_1}X_2, X_1\rangle = \langle X_1, X_1\rangle = R^2 \sin^2\phi;$$
$$||X_1||^2 = R^2 \sin^2\phi; \quad ||X_2||^2 = R^2; \quad \langle X_1, X_2\rangle = 0.$$

Therefore we obtain a constant expression for every point:

$$\sec(X_1, X_2) = \frac{R^2 \cancel{\sin^2\phi}}{R^4 \cancel{\sin^2\phi}} = \frac{1}{R^2}.$$

And since $\text{span}\{X_1, X_2\}|_x = T_xM \quad \forall x \in M$, the curvature is equal to the sectional curvature at every point.

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