UNIVERSITÀ DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Path integral of topological defects in superfluids and superconductors

Thesis supervisor

Prof. Luca Salasnich

Candidate

Manuel Dell'Anna

Academic Year 2024/2025

Indice

1 Superfluidity **3**
- 1.1 Bosonic superfluidity . . . 3
- 1.2 Landau criterion . . . 7
- 1.3 Fermionic superfluid and superconductivity . . . 8
- 1.4 BCS theory . . . 10
- 1.5 BCS path integral . . . 13
- 1.6 Mean field treatment . . . 17
- 1.7 D=2 case . . . 21
  - 1.7.1 2-loop expansion . . . 28
- 1.8 Vortex solution . . . 32

2 Vortices EFT **41**
- 2.1 Vortices energy . . . 44
- 2.2 Gran canonical ensemble of vortices . . . 47

3 BKT transition **51**
- 3.1 Renormalized stiffness . . . 51
- 3.2 Renormalization group approach . . . 54
- 3.3 Path integral approach . . . 55
- 3.4 Flow analysis . . . 57

4 Conclusions **61**
- Appendix . . . 67

Introduction

Quantum fluids—both superfluids and superconductors—represent one of the most striking manifestations of collective quantum behavior in condensed-matter physics. In these systems, a macroscopic wavefunction describes the entire fluid, leading to phenomena such as dissipationless flow in superfluids and perfect diamagnetism in superconductors. At the heart of this thesis lies the study of topological defects—vortices—and the way these excitations govern the loss of long-range coherence in two-dimensional geometries. Although one often imagines a vortex as little more than a swirling eddy, in a quantum fluid each vortex carries a quantized "twist" of the phase, which in turn has profound consequences for the system's thermodynamic and transport properties.

The primary goal of this work is to develop a field-theoretic, path-integral description of vortices in both neutral bosonic superfluids and charged superconducting films, culminating in a clear exposition of the Berezinskii–Kosterlitz–Thouless (BKT) transition in two dimensions. In contrast to more pedestrian accounts that simply treat vortices as classical point charges in a logarithmic potential, our emphasis will be on how the vortex degrees of freedom emerge naturally from the underlying order-parameter field, and how one can systematically integrate out smooth phase fluctuations to isolate the physics of discrete, quantized defects.

In Chapter 1, we begin by reviewing the fundamentals of superfluidity in a bosonic system and superconductivity in a fermionic system, with an eye toward a unified path-integral framework. We first recall how the low-energy dynamics of a weakly interacting Bose gas can be described by a single complex field whose amplitude and phase encode density and velocity fluctuations. Without delving into every algebraic detail, we will explain how the "phase-only" action emerges at low temperatures, and why it encapsulates both the phonon-like collective modes and the possibility of topological excitations. We then turn to fermionic superfluidity and superconductivity, reviewing the standard mean-field approach to pairing and showing why phase fluctuations become especially important in two dimensions, where true long-range order is forbidden by general theorems.

Chapter 2 is devoted to the construction of single-vortex solutions in two dimensions—first for a neutral superfluid, then for a thin superconducting film. In each case, we discuss how the kinetic energy of the order-parameter field leads to a logarithmically divergent energy, justifying the picture of a vortex as a "charged" object in a two-dimensional Coulomb gas. We also point out the differences introduced by the long-range electromagnetic coupling in the superconducting case, where screening and magnetic penetration depth play a role. After building intuition from a single vortex, we show how to introduce a grand-canonical ensemble of vortices and antivortices within the path-integral formalism. By carefully separating the smooth (spin-wave) phase fluctuations from the singular, multivalued part, one arrives at an effective description in which vortex configurations interact with each other and carry an entropic cost of creation.

Building on that framework, Chapter 3 undertakes a renormalization-group analysis of the two-dimensional Coulomb-gas representation of vortices. We explain how, at low temperature, vortex–antivortex pairs remain bound, preserving algebraic order in the phase field, whereas above a certain critical temperature these pairs unbind, driving a sudden collapse of superfluid stiffness. Although the BKT transition has become a textbook staple, our goal is to provide a self-contained derivation that highlights the central role of the path-integral representation and connects smoothly with microscopic parameters. We also discuss finite-size effects, which are crucial for experiments on thin films or trapped ultracold gases, and comment on how one can extract the universal jump in stiffness from numerical simulation or direct measurement.

Chapter 4 offers a summary of some experimental evidence of this transition and sketches possible extensions of the present work. Among the directions we mention are the inclusion of weak disorder (which tends to pin vortices and shift the transition temperature), the coupling to dynamic electromagnetic fields in the superconducting setting (which can modify both the vortex core energy and long-distance interactions), and the generalization to multi-component condensates systems, where exotic vortex textures may appear.
In two Appendices, we collect several technical derivations that support the main text: Appendix A reviews the Thomas–Fermi approximation for inhomogeneous Bose gases, and Appendix B presents a hydrodynamic derivation of the force acting on a moving vortex in the presence of background flow. By tracing the emergence of vortex degrees of freedom directly from the microscopic order parameter, we hope to give a coherent and pedagogical account of why two-dimensional superfluids and superconductors undergo a topological transition, and how that physics can be observed and quantified in real-world experiments. Although this thesis does not contain original research results in the experimental or numerical sense, it aims to clarify and reproduce key theoretical results on vortex dynamics and the Berezinskii–Kosterlitz–Thouless (BKT) transition that are often presented in a compact or opaque manner in the literature. Particular attention has been devoted to the derivation of the effective vortex action and the renormalization group (RG) flow equations, with the goal of making the underlying assumptions and intermediate steps fully transparent. In this sense, the thesis can be viewed as a pedagogically motivated reconstruction of nontrivial results, with an emphasis on internal consistency and clear connection to physical observables.

1 Superfluidity

The study of Superfluidity and in general of the Bose-Einstein condensation mechanism poses its roots in the pioneering works of Bose and Einstein in 1924-1925 [13][21][22] where for the first time were predicted the arising of a macroscopic occupation of the lowest energy number below a characteristic critical temperature. Fourteen years later the theoretical prediction [30], superfluidity has been observed experimentally in $^4He - II$, but the achievement of the true Bose-Einstein condensation has been possible with cold alkali atoms systems only in 1995. Only then condensation has been observed first by Carl Wieman and Eric Cornell group of the University of Colorado at Boulder [18], but few months later a group led by Wolfgang Ketterle at the MIT obtained better results, with higher densities [5]. In order to observe this condensation mechanism it was necessary to reach near to absolute zero temperatures, but also low densities, so that the system does not solidify. This has been possible through the combined use of laser cooling techniques and evaporative cooling techniques [31]. The relevance of the study of cold atoms systems is due to a large variety of aspects [17], among them there is the strong interplay between experimental and theoretical results.
One of the key aspects of the success of cold atoms systems is constituted by the possibility of tuning the interaction strength between atoms by the means of a Feshbach resonance [14], which provides a very useful tool in the test of our theories. At its heart, superfluidity is the state of matter in which a fluid can flow without any viscosity, meaning it moves without friction. As mentioned, this behavior was first observed in liquid helium-4 when it was cooled below its critical temperature—around 2.17 K, known as the lambda point. In this state, helium-4 not only flows without resistance but also displays other remarkable effects, such as climbing the walls of a container (the fountain effect) and sustaining perpetual, undamped currents in closed loops.[1][37][50] This effects are due to the rising of a coherent macroscopic wave function, from this the term "condensation" , that describes the entire fluid. The coherence among the particles is what allows them to march in step, eliminating the usual sources of friction and dissipative processes present in classical fluids.

From a theoretical standpoint, the phenomenon is often analyzed using a two-fluid model, where the liquid is considered as a mixture of a frictionless superfluid component and a normal component that carries thermal excitations and in this project we will see how this description naturally emerges from loops effects. Additionally, the Landau criterion provides insight into the stability of superfluid flow by linking the critical velocity for dissipation to the spectrum of excitations in the fluid.

1.1 Bosonic superfluidity

In the following we introduce the field theory point of view associated to the phenomenon of superfluidity, which rely on a given gran canonical ensamble of neutral, spinless particles.

Our system is then described in terms of a coherent state path integral

$$\mathcal{Z} = \int \mathcal{D}\psi\, \mathcal{D}\psi^*\, e^{-\frac{S_E[\psi,\psi^*]}{\hbar}} = e^{-\beta\Omega(T,\mu)} \tag{1.1}$$

where $S_E[\psi, \psi^*]$ is the Euclidean action of the system, and the integral is taken over all possible field configurations. For a weakly interacting Bose gas, the effective Euclidean action can be written as:

$$S_E = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\ \psi^*(\mathbf{r},\tau) \left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right) \psi(\mathbf{r},\tau) + \frac{g}{2}\left|\psi(\mathbf{r},\tau)\right|^4 . \tag{1.2}$$

where D is the spacial dimension, $\mu = \mu(T)$ is the chemical potential of our system and g parametrizes the strength of the interaction that is assumed to be small to characterize the weakly interacting Bose gas. Notice that our action is invariant under global $U(1)$ transformation, that is under

$$\begin{aligned} \psi &\to e^{i\alpha}\psi \\ \psi^* &\to \psi^* e^{-i\alpha} \end{aligned} \tag{1.3}$$

with $\alpha \in \mathbb{R}$. By the Noether theorem, we know that this symmetry imposes the conservation of the particle number in our system.
By considering the potential of our theory

$$V(\psi, \psi) = -\mu|\psi|^2 + \frac{g}{2}|\psi|^4 \tag{1.4}$$

through minimization condition we understand that, for $\mu(T) > 0$ there will exist infinitely many ground states of the form

$$\psi_0(\phi) = \sqrt{\frac{\mu}{g}}e^{i\phi} \tag{1.5}$$

At this point is possible to interpret this results in terms of a condensation procedure as long as $\mu(T) < 0$ the mean field solution is $|\psi_0| = 0$ reflecting the fact that no condensation has occurred. Conversely, below a certain critical temperature T_c such that $\mu > 0$ the ground state becomes macroscopically occupied developing a vev as before and breaking the $U(1)$ symmetry. Now we understand that in any dimension $D \geq 3$ connecting one of these ground state to another would require, in the thermodynamic limit, an infinite amount of energy. Thus we can expect the system to select one (e.g. the one with $\phi = 0$) effectively breaking the $U(1)$ symmetry introduced before. This means that, allowing thermal or quantum fluctuation to act, we can parametrize our field configuration as

$$\psi(\mathbf{r},\tau) = \sqrt{\rho(\mathbf{r},\tau)}e^{i\theta(\mathbf{r},\tau)} \tag{1.6}$$

[1] With $\langle\psi(\mathbf{r},\tau)\rangle \neq 0$ as long as $\mu \geq 0$, where the radial and phase fluctuations are treated as small. Notice that actually,this procedure is well defined as long as $D \geq 3$.
In this range indeed we can expect phase-fluctuations to be small enough to not destroy the coherence of the ground state allowing the SSB to happen as we will explain later on in this project. Inserting the latter parametrization in the original action we obtain

$$S = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r}\ \ i\hbar\ \rho\ \partial_\tau\theta + \frac{\hbar^2}{2m}\frac{1}{4\rho}(\nabla\rho)^2 + \frac{\hbar^2\rho}{2m}(\nabla\theta)^2 - \mu\rho + \frac{g}{2}\rho^2 \tag{1.7}$$

[1]this transformation is called Madelung transformation

where we recognize the feature of the phase $\theta(\mathbf{r},\tau)$ of being a Goldstone mode i.e. constant fluctuations $\theta = const$ do not require any energy cost. Expanding the amplitude as

$$\rho = |\psi_0|^2 + \delta\rho \tag{1.8}$$

with $|\psi_0|^2$ as in (1.5), we obtain

$$S \approx \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r} \;\; i\hbar\, \delta\rho\, \partial_\tau\theta + \frac{\hbar^2}{2m}\frac{1}{4|\psi_0|^2}(\nabla\delta\rho)^2 + \frac{\hbar^2|\psi_0|^2}{2m}(\nabla\theta)^2 + \frac{\hbar^2\delta\rho}{2m}(\nabla\theta)^2 + \frac{g}{2}\delta\rho^2 \tag{1.9}$$

Where the mixed initial term has the canonically "momentum $\times\partial_\tau$(coordinate)" structure indicative of the two canonical conjugates coordinates.
Considering the classical e.o.m. associated to this action

$$\begin{aligned} &\partial_\tau \frac{\partial\mathcal{L}}{\partial(\partial_\tau\theta)} + \nabla\frac{\partial\mathcal{L}}{\partial(\nabla\theta)} - \frac{\partial\mathcal{L}}{\partial\theta} = 0 \\ &\partial_\tau \frac{\partial\mathcal{L}}{\partial(\partial_\tau\delta\rho)} + \nabla\frac{\partial\mathcal{L}}{\partial(\nabla\delta\rho)} - \frac{\partial\mathcal{L}}{\partial\delta\rho} = 0 \end{aligned} \tag{1.10}$$

we would obtain the following equations

$$\begin{aligned} &i\hbar\, \partial_\tau\delta\rho \;\; + \nabla\cdot\left(\frac{\hbar^2}{m}(|\psi_0|^2 + \delta\rho)\nabla\theta\right) = 0 \\ &i\hbar\partial_\tau\theta \;\; + \frac{\hbar^2}{2m}(\nabla\theta)^2 + g\delta\rho \;\; - \frac{\hbar^2}{m}\frac{1}{4|\psi_0|^2}\nabla\cdot\nabla\delta\rho = 0 \end{aligned} \tag{1.11}$$

To appreciate the physical consequences of the set of equations notice that if we define the following velocity field and density function

$$\mathbf{v}(\mathbf{r},\tau) = \frac{\hbar}{m}\nabla\theta(\mathbf{r},\tau) \tag{1.12}$$

$$n(\mathbf{r},\tau) = |\psi_0|^2 + \delta\rho(\mathbf{r},\tau) \tag{1.13}$$

we can write the previous equations as

$$\begin{aligned} &i\;\partial_\tau n \;\; + \nabla\cdot(n\mathbf{v}) = 0 \\ &i\;\partial_\tau(m\mathbf{v}) \;\; + \nabla\left(\frac{1}{2}m\mathbf{v}^{\mathbf{2}} + gn \;\; - \mu - \frac{\hbar^2}{m}\frac{1}{4|\psi_0|^2}\nabla\cdot\nabla n\right) = 0 \end{aligned} \tag{1.14}$$

which are the (Euclidean version of) the Navier-Stokes equations for a zero viscosity

$$\nabla\cdot\mathbf{v} = 0 \tag{1.15}$$

fluid.
In particular, in the low velocity limit

$$\mathbf{v} \approx 0 \tag{1.16}$$

there exists a steady state solutions $\partial_\tau\delta\rho = 0$, $\partial_\tau\mathbf{v} = 0$ to these equations with non-vanishing current flow given by

$$\begin{aligned} &\delta\rho = 0 \\ &\nabla\cdot\mathbf{J} = 0 \end{aligned} \tag{1.17}$$

with $\mathbf{J} = n\mathbf{v}$.

To understand the phenomenology of supercurrent flow from a microscopic perspective, note that steady state current flow in normal environments is prevented by the mechanism of energy dissipation, i.e. particles constituting the current flow scatter off imperfections inside the system, thereby converting part of their energy into the creation of elementary excitations, but for this system this mechanism seems to be automatically prevented.
We postpone the reason of why this is possible to the next section, in the mean time notice that the divergenceless flow is strictly connected to the presence of a Goldstone mode in our theory, indeed, if θ were to have a mass term, no steady and divergenceless solution would appear.
For this reason, let us try to extract the effective goldstone boson theory bu integrating out amplitude fluctuations, obtaining

$$S_{\text{eff}}[\theta] = \int d\tau \int d^D\mathbf{r}\ \frac{\hbar^2}{2g}(\partial_\tau\theta)^2 + \frac{\hbar^2}{2m}|\psi_0|^2(\nabla\theta)^2 = \int d\tau \int d^D\mathbf{r}\frac{K}{2}(\partial_\tau\theta)^2 + \frac{J}{2}(\nabla\theta)^2 \tag{1.18}$$

Which describes a Goldstone theory which exactly resembles (at tree level) the equations of motion (1.17), the general form of the coupling of the operator $(\nabla\theta)^2$ will be related to the superfluid density via

$$J = \frac{\hbar^2}{m}n_s \tag{1.19}$$

and we can see that, at tree level, the superfluid density naively coincides with the condensed part of the system

$$n_s^{tree} = |\psi_0|^2 \tag{1.20}$$

In general this association will not hold, indeed as we will see shortly, not only $\mathbf{k} = 0$ particles will participate to the superfluid flow. Furthermore, we can be a little more quantitative about the energy of the excitations in this kind of model, looking at (1.9) and passing in Fourier space, we can put this action in a the following form

$$S = \frac{1}{2}\sum_{\omega_n,\mathbf{k}} \begin{pmatrix}\delta\rho_{\omega_n,\mathbf{k}} & \theta_{\omega_n,\mathbf{k}}\end{pmatrix} \begin{pmatrix} g + \frac{\hbar^2}{2m}\frac{|\mathbf{k}|^2}{2|\psi_0|^2} & -\hbar\omega_n \\ \hbar\omega_n & \frac{\hbar^2}{2m}|\mathbf{k}|^2 \end{pmatrix} \begin{pmatrix}\delta\rho_{-\omega_n,-\mathbf{k}} \\ \theta_{-\omega_n,-\mathbf{k}}\end{pmatrix} \tag{1.21}$$

Where the quadratic matrix coincides with the inverse of the Green's function in Fourier space

$$G^{-1}(\omega_n,\mathbf{k}) = \begin{pmatrix} g + \frac{\hbar^2}{2m}\frac{|\mathbf{k}|^2}{2|\psi_0|^2} & -\hbar\omega_n \\ \hbar\omega_n & \frac{\hbar^2}{2m}|\mathbf{k}|^2 \end{pmatrix} \tag{1.22}$$

The procedure to extract the spectrum of the excitations in this system requires to compute the poles of the Green function $G(\omega_n,\mathbf{k})$, which in this case means that we have to compute the zeros of the determinant of the previous matrix, yielding

$$\hbar^2\omega_n^2 + E^2(\mathbf{k}) = 0 \tag{1.23}$$

where, upon Wick rotation $\omega_n \to -i\omega$, we understand that the excitations have the following dispersion relation

$$E(\mathbf{k}) = \sqrt{\frac{\hbar^2}{2m}\mathbf{k}^2\left(2g|\psi_0|^2 + \frac{\hbar^2}{2m}\mathbf{k}^2\right)} \tag{1.24}$$

which is known as Bogolioubov spectrum.

A particularity of this spectrum is that, at variance of the single particle dispersion, this is in the long wavelength limit

$$E(\mathbf{k}) \approx \hbar v_s|\mathbf{k}| \tag{1.25}$$

where we indicated

$$v_s = \sqrt{\frac{g}{m}}|\psi_0| \tag{1.26}$$

This fact seems to be counterintuitive: on one hand we just proved that, below a certain condensation temperature, this system can sustain superfluid flow and on the other hand, from the gapless dispersion relation, we understand that in principle it is always possible to create an excitation with sufficient enough energy (as E($\mathbf{k}$)$\rightarrow$0 in the long wave lengths limit) that will serve as dissipative effect in our system.
This paradox has been resolved by Landau, which gave in a pragmatic way, an explanation of how superfluid flow can be sustained regardless of a gapless dispersion relation.

1.2 Landau criterion

In was in 1941 when Lev Landau in his seminal works [34][35] laid the foundations in the understanding of the excitation spectrum of the form given in (1.26) during his studies with liquid helium-4 below its λ point.
Landau proposed a microscopic theory- known as two fluid model- that explained superfluidity by introducing the concept of elementary excitations—phonons at low momentum and rotons at higher momentum. His groundbreaking insight was to derive a criterion (now known as the Landau criterion) for superfluidity, which states that if the superfluid's velocity is below a certain critical value, then the creation of excitations (and hence energy dissipation) is forbidden, enableing therefore the appearance of a steady flow.
Normally one interprets this kind of behavior as a consequence of excitations whose energy is so high that the energy stored in the current flow it is not enough to create them. But this would be the case of a dispersion relation gapped ($\omega_{\mathbf{k}} \rightarrow const$ as $\mathbf{k} \rightarrow 0$) and certainly does not apply in the superfluid case.
To heuristically explain how dissipationless flow can occurs in the case of a superfluid, imagine we have a fluid element of mass M flowing through a pipe with velocity $\mathbf{V}$ with respect the laboratory frame of reference $\mathcal{S}$.
in this reference we would say that the energy of this fluid element would be

$$E^{<}_{\mathcal{S}} = \frac{1}{2}M\mathbf{V}^2 \tag{1.27}$$

Now performing a Galilean transformation into the rest frame of the fluid $\mathcal{S}'$, we would see the walls moving with velocity $-\mathbf{V}$.
If we suppose that the frictional forces release an excitation of momentum -$\mathbf{p}$ and energy $\omega_{\mathbf{p}}$ into the fluid i.e.

$$E^{>}_{\mathcal{S}'} = \omega_{\mathbf{p}} \tag{1.28}$$

going back into $\mathcal{S}$ we would have

$$E^{>}_{\mathcal{S}} = \frac{1}{2}M\mathbf{V}^2 - \mathbf{p}\cdot\mathbf{V} + \omega_{\mathbf{p}} \tag{1.29}$$

at this point we understand that since the excitation energy is provided by the fluid itself, energy balance should hold

$$E^{<}_{\mathcal{S}} = E^{>}_{\mathcal{S}} \tag{1.30}$$

leading to

$$\omega_{\mathbf{p}} = \mathbf{p}\cdot\mathbf{V} \tag{1.31}$$

and we notice that being $\mathbf{p} \cdot \mathbf{V} \leq |\mathbf{p}||\mathbf{V}|$ the energy conservation condition is always met if

$$|\mathbf{V}| \geq \frac{\omega_{\mathbf{p}}}{|\mathbf{p}|} \geq \min_{\mathbf{p}} \frac{\omega_{\mathbf{p}}}{\mathbf{p}} \tag{1.32}$$

Notice that if $\omega_{\mathbf{p}} \sim |\mathbf{p}|^2$ the walls, no matter how slow they move, will always have enough energy to create excitations compatible with energy conservation, but if

$$\min_{|\mathbf{p}|} \frac{\omega_{\mathbf{p}}}{|\mathbf{p}|} = v_c \neq 0 \tag{1.33}$$

there exists a limit velocity v_s of the walls such that, for $|\mathbf{V}| < v_s$ they cannot create excitations in the system i.e. the motion is non dissipative! This is the famous Landau criterion that in the superfluid case, being the respective dispersion relation linear in $|\mathbf{p}|$, indeed holds with

$$v_s = \sqrt{\frac{g}{m}|\psi_0|^2} \tag{1.34}$$

Finally resolving the apparent paradox we encountered in the previous section.

As we pointed out, also gapped excitation can sustain superfluid behavior and actually, regardless of this *condensation* mechanism that one naively could relate to a bosonic system, also fermoinic systems can sustain a superfluid response, which, as we will see shortly, turn out to have exhibit exactly a gapped dispersion relation. In the following sections, we will introduce the concept of fermionic superfluidity, trying to be more quantitative, and later on a model of prominent importance both for its experimental application and theoretical investigation: the BCS model.

1.3 Fermionic superfluid and superconductivity

Superconductivity is one of the most fascinating and versatile topic in physics. Indeed, it has the feature of being multifaceted from the theoretical investigation point of view: the fundamental mechanisms that characterize this phenomenon are somewhat ubiquitous, yet the physical consequences and applications change with the context in which they appear. The same mathematical tools that one uses to describe superconductivity indeed enable us to understand the emergence in our universe of particles with electric charges or to understand why particles acquires mass or, again, why that piece of stone your family sent you from a vacation sticks of your refrigerator! Furthermore, superconductors have a wide range of modern technology applications that go from the refining of imagine techniques for medical purposes, quantum computing to their applications in laser and sensors technologies.

It was Heike Kamerlingh Onnes in 1911 [42] the first physicist to observe one of the *two* main features characterizing this kind of materials: the ideal conductors behavior. He observed, during an experiment involving liquid Mercury that once cooled under a characteristic critical temperature, its resistance would drop abruptly to zero,this signaling the presence of a frictionless electric flow. The second property characterizing a superconductor is its feature as a perfect diamagnet once it is cooled down below a certain critical temperature i.e. their tendency to expell all magnetic flux from its interior. This feature is known as Meissner effect and it is important to emphatize that this effect separates superconductor from ideal conductors. Next we show one of the most spectacular manifestation of the field -aversion tendency of superconductors: the magnetic levitation.

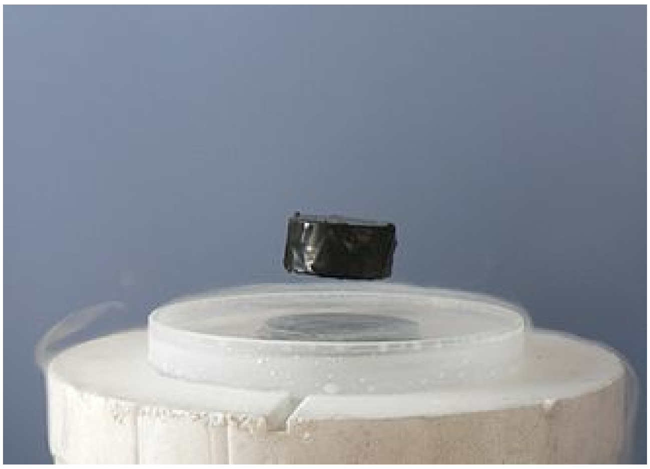

Figure 1.1: Magnetic levitation

This can be done by putting the superconductor into a magnetic field and cooling it down below its critical temperature: the superconductor will expell the magnetic field and thus levitate

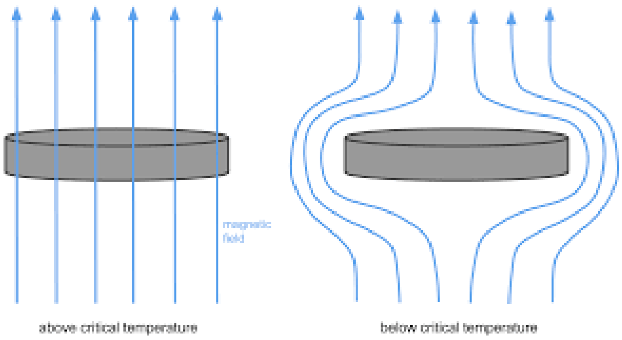

Figure 1.2: Schematic representation on the Meissner effect

The first attempt to explain from the phenomenological point of view superconductivity was done by the London brothers (Fritz and Heinz) around 1935 and they explained with two *ad hoc* equations the two effects characterizing superconducting behviour.[38] They postulated the existence of *super* carries with density n_s whose super current $\mathbf{J}_s$ satisfied the following equations

- *Zero resistivity*

$$\frac{d\mathbf{J}_s}{dt} = \frac{n_s e^2}{m_e}\mathbf{E} \tag{1.35}$$

- *Perfect diamagnet*

$$\nabla \times \mathbf{J}_s = -\frac{n_s e^2}{m_e}\mathbf{B} \tag{1.36}$$

the first equation can be somewhat justified by postulating a frictionless flow through the Drude model, while the second states the diamagnetic response of the microscopic magnetization of this super current $\mathbf{M} = \nabla \times \mathbf{J}_s$ against the external magnetic field $\mathbf{B}$.
Notice that, if we postulate the first equation we immediately obtain, implementing in it the Lenz law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.37}$$

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{J}_s + \frac{n_s e^2}{m}\mathbf{B}) = 0 \tag{1.38}$$

and we see at this point the phenomenological spirit of these equations in spite of which we *impose*

$$\nabla \times \mathbf{J}_s + \frac{n_s e^2}{m}\mathbf{B} \overset{!}{=} 0 \tag{1.39}$$

in order to explain the Meissner effect. Indeed, using the Ampère law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_s \tag{1.40}$$

from (1.36) we obtain

$$\nabla^2 \mathbf{B} = \frac{1}{\lambda_L}\mathbf{B} \tag{1.41}$$

where $\lambda_L = \frac{m_e}{\mu_0 n_s e^2}$ is the London penetration depth,showing that the magnetic field decays exponentially inside the superconductor, whose typical values are

$$\lambda_L \approx 10^3 \text{Å} \tag{1.42}$$

for $n_s \approx 10^{29} m^{-3}$.

1.4 BCS theory

The development of a microscopic theory explaining superconductive behavior came almost 50 years after the discovery of this phenomenon when J.Bardeen, L.N. Cooper and J.R.Schrieffer explained superconductivity in terms of an ordered state of conduction electrons in metal driven by the presence of an effective attractive interaction between electrons mediated by the phonons inside the metal.[7][16] To explain qualitatively the mechanism of this process, one focuses on the existence of *two* time scales: the inverse of the Fermi frequency $\hbar\epsilon_F^{-1}$ related to electrons and the inverse of Debye frequency ω_D^{-1} related to phonon oscillations. When an electron passes in the vicinity of the ions of the metal, it distorts them from the equilibrium position, causing an accumulation of charges which take $w_D^{-1} \gg \hbar\epsilon_F^{-1}$ seconds to relax back into their equilibrium position. This means that, long after the first electron has passed, another electron may benefit from the distorted ion potential. The effect of this retardation mechanism is an attractive interaction between the two electrons which could create a 'bound' state called Cooper pair. To characterize the interaction, we suppose the Hamiltonian of our system to be of the following form

$$\hat{H} = \hat{H}_{el} + \hat{H}_{ph} + \hat{H}_{el-el} \tag{1.43}$$

where

$$\hat{H}_{el} = \int d^D\mathbf{r} \sum_{\sigma=\uparrow\downarrow} \hat{\psi}^\dagger_\sigma(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m} + U(\mathbf{r})\right) \hat{\psi}_\sigma(\mathbf{r}) \tag{1.44}$$

where

$$\hat{\psi}_\sigma(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \hat{c}_{\mathbf{k}\sigma} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.45}$$

is the fermionic field operator and σ is the spin quantum number.

$$U(\mathbf{r}) = \sum_{I=1}^{N} V_{el-ion}(\mathbf{r} - \mathbf{R}^I(t)) \tag{1.46}$$

indicate the interaction between one electron with the N ions of the system which are supposed to have equilibrium positions $\left\{\mathbf{R}^I_{eq}\right\}_{I=1\ldots N}$ corresponding to an ensamble of Bravais lattice vectors. Our ions are supposed to perform small oscillations around their equilibrium position

$$\mathbf{R}^I = \mathbf{R}^I_{eq} + \mathbf{u}(\mathbf{R}^I_{eq}, t) \tag{1.47}$$

with $\mathbf{u}(\mathbf{R}^I_{eq}, t) \approx 0$.
Now a priori one should consider also the ions as quantum particles endowed with a standard kinetic term $-\frac{\hbar^2}{2M}\nabla^2$, with M the mass of our ion, and possibly also all the interactions between the ions. But because in general $M \gg m$, we are led to assume that, rather than the ion itself, its oscillation modes $\mathbf{u}(\mathbf{R}^I_{eq}, t)$ would be quantized. In first approximation we can suppose harmonic oscillation modes of the form

$$H_{ph} = \sum_I \frac{1}{2} M |\dot{\mathbf{u}}(\mathbf{R}^I_{eq}, t)|^2 + \frac{1}{2}\sum_{I,J} \mathbf{u}(\mathbf{R}^I_{eq}, t) \cdot \hat{D}(\mathbf{R}^I_{eq} - \mathbf{R}^J_{eq})\mathbf{u}(\mathbf{R}^J_{eq}, t) \tag{1.48}$$

where $\hat{D}(\mathbf{R}^I_{eq} - \mathbf{R}^J_{eq})$ is a symmetric tensor characterizing the harmonic interaction. Now if we expand in normal modes

$$\mathbf{u}(\mathbf{r}, t) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}\in 1^{st}BZ,\ \lambda} \mathbf{e}(\mathbf{k}, \lambda)\ Q_{\mathbf{k},\lambda}(t)\ e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.49}$$

with $\{\mathbf{e}(\mathbf{k}, \lambda)\}_{\lambda=1,2\ldots D}$ being eigenvectors of the so called dynamical matrix $\hat{D}(\mathbf{k}) = \sum_{\mathbf{R}} \hat{D}(\mathbf{R})e^{-i\mathbf{k}\cdot\mathbf{R}}$ with eigenvalues $\omega_{\mathbf{k},\lambda}$ we end up with the familiar hamiltonian

$$H_{ph} = \sum_{\mathbf{k}\in 1^{st}BZ,\lambda} \frac{1}{2} M |\dot{Q}_{\mathbf{k},\lambda}|^2 + \frac{1}{2} M \omega^2_{\mathbf{k},\lambda} |Q_{\mathbf{k},\lambda}|^2 \tag{1.50}$$

which describes N coupled harmonic oscillators that can be quantized straight away

$$\hat{H}_{ph} = \sum_{\mathbf{k}\in 1^{st}BZ,\lambda} \frac{|\hat{P}_{\mathbf{k},\lambda}|^2}{2M} + \frac{1}{2} M \omega^2_{\mathbf{k},\lambda} |\hat{Q}_{\mathbf{k},\lambda}|^2 \tag{1.51}$$

with

$$[\hat{Q}_{\mathbf{k}\lambda}, \hat{P}_{\mathbf{k}'\lambda'}] = i\hbar \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda\lambda'} \tag{1.52}$$

performing now the change of variables

$$\hat{Q}_{\mathbf{k},\lambda} = \sqrt{\frac{\hbar}{2M\omega_{\mathbf{k},\lambda}}} \left(\hat{a}_{\mathbf{q},\lambda} + \hat{a}^\dagger_{-\mathbf{q},\lambda}\right) \tag{1.53}$$

$$\hat{P}_{\mathbf{k},\lambda} = i\sqrt{\frac{M\hbar\omega_{\mathbf{k},\lambda}}{2}} \left(\hat{a}_{\mathbf{q},\lambda} - \hat{a}^\dagger_{-\mathbf{q},\lambda}\right) \tag{1.54}$$

we obtain the standard form

$$\hat{H}_{ph} = \sum_{\mathbf{k},\lambda} \hbar\omega_{\mathbf{k},\lambda} \left(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + \frac{1}{2}\right) \tag{1.55}$$

Let us focus now on the electron-ions potential

$$U(\mathbf{r}) = \sum_I V_{el-ion}(\mathbf{r} - \mathbf{R}_I(t)) \tag{1.56}$$

where

$$\mathbf{R}_I = \mathbf{R}^I_{eq} + \mathbf{u}(\mathbf{R}^I_{eq}, t)$$

Exploiting the fact that we have small oscillations around the equilibrium positions, we perform an expansion around these to the first order in the *displacement* $\mathbf{u}(\mathbf{R}^I_{eq}, t)$

$$U(\mathbf{r} - \mathbf{R}^I(t)) \approx U(\mathbf{r} - \mathbf{R}^I_{eq}) - \nabla_{\mathbf{r}} U(\mathbf{r} - \mathbf{R}^I_{eq}) \ \cdot \mathbf{u}(\mathbf{R}^I_{eq}, t) \tag{1.57}$$

Now, the first term would contribute determining the bands structure of electrons in the solid taking into account the adiabatic response of electrons to the electric field of the ion frozen in their equilibrium positions. This would give rise to a potential

$$U_{band}(\mathbf{r}) = \sum_I V_{el-ion}(\mathbf{r} - \mathbf{R}^I_{eq}) \tag{1.58}$$

that, once endowed with symmetry properties compatible with the underlying crystalline structure of our solid, will became the core of the study of band theory. The second term

$$U_{el-ph}(\mathbf{r}) = -\sum_I \nabla_{\mathbf{r}} V_{el-ion}(\mathbf{r} - \mathbf{R}^I_{eq}) \ \cdot \mathbf{u}(\mathbf{R}^I_{eq}, t) \tag{1.59}$$

is a non-adiabatic effect that describes in first approximation the interaction between electron and oscillations modes of our ions (phonons). Focusing of the latter, we consider the continuum limit

$$\sum_I \to \int \frac{d^D \mathbf{R}_{eq}}{\Omega} \tag{1.60}$$

where Ω is the volume of the primitive cell, extending this to the expansion before we get

$$-\sum_I \nabla_{\mathbf{r}} V_{el-ion}(\mathbf{r} - \mathbf{R}^I_{eq}) \ \cdot \mathbf{u}(\mathbf{R}^I_{eq}, t) \to -\frac{1}{\Omega} \int d^D \mathbf{R}_{eq} \nabla_{\mathbf{r}} V_{el-ion}(\mathbf{r} - \mathbf{R}_{eq}) \ \cdot \mathbf{u}(\mathbf{R}_{eq}, t) \ . \tag{1.61}$$

Now, if we consider the Fourier transform of our quantities

$$\begin{aligned} V_{el-ion}(\mathbf{r} - \mathbf{R}_{eq}) &= \frac{1}{V} \sum_{\mathbf{k}} V_{el-ion}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r} - \mathbf{R}_{eq})} \\ \mathbf{u}(\mathbf{R}_{eq}, t) &= \frac{1}{\sqrt{N}} \sum_{\mathbf{q}} \mathbf{u}(\mathbf{q}, t) e^{i\mathbf{q}\cdot\mathbf{R}_{eq}} \end{aligned} \tag{1.62}$$

where we shorthand indicated

$$\mathbf{u}(\mathbf{q}, t) = \sum_\lambda \mathbf{e}(\mathbf{k}, \lambda) \ Q_{\mathbf{k},\lambda}(t) \tag{1.63}$$

we obtain

$$U_{el-ph}(\mathbf{r}) = -\frac{1}{\Omega} \sum_{\mathbf{k},\mathbf{q}} \int d^D \mathbf{R}_{eq} \left(\frac{1}{V} V_{el-ion}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r} - \mathbf{R}_{eq})} \right) i\mathbf{k}\cdot \left(\frac{1}{\sqrt{N}} \mathbf{u}(\mathbf{q}, t) e^{i\mathbf{q}\cdot\mathbf{R}_{eq}} \right) \tag{1.64}$$

Performing the integration over $\mathbf{R}_{eq}$

$$U_{el-ph}(\mathbf{r}) = -\frac{1}{\Omega} \sum_{\mathbf{k}} V_{el-ion}(\mathbf{k}) \ \left(\frac{i\mathbf{k}}{\sqrt{N}} \cdot \mathbf{u}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}} \right) \tag{1.65}$$

Now if we make the reasonable assumption that our potential is short ranged i.e. we assume that an electron only perturbs the ions in his immediate vicinity

$$V_{el-ion}(\mathbf{k}) \approx V_{el-ion}(0) \tag{1.66}$$

we obtain the so-called *Fröhlich deformation potential*

$$U_{el-ph}(\mathbf{r}) \approx \lambda \nabla_{\mathbf{r}} \cdot \mathbf{u}(\mathbf{r},t) \tag{1.67}$$

Putting everything together

$$\hat{H} = \int d^D\mathbf{r} \sum_{\sigma=\uparrow\downarrow} \hat{\psi}^\dagger_\sigma(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m} + U_{band}(\mathbf{r}) + U_{el-ph}(\mathbf{r}) \right) \hat{\psi}_\sigma(\mathbf{r}) + \sum_{\mathbf{k},\lambda} \hbar\omega_{\mathbf{k},\lambda} \left(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + \frac{1}{2} \right) + \hat{H}_{el-el} \tag{1.68}$$

For what concerns the electron-phonon interaction part, we overall have, considering (1.49) and (1.53)

$$\sum_{\sigma=\uparrow\downarrow} \int d^D\mathbf{r}\ \hat{\psi}^\dagger_\sigma(\mathbf{r}) \left(\lambda \nabla_{\mathbf{r}} \cdot \mathbf{u}(\mathbf{r}) \right) \hat{\psi}_\sigma(\mathbf{r}) = \sum_{\mathbf{k}\sigma,\ \mathbf{q}} g(\mathbf{q})(\hat{a}_{\mathbf{q},1} + \hat{a}^\dagger_{-\mathbf{q},1})\hat{c}^\dagger_{\mathbf{k}+\mathbf{q},\sigma}\hat{c}_{\mathbf{k}\sigma} \tag{1.69}$$

with

$$g(\mathbf{q}) = i\lambda \sqrt{\frac{\hbar}{2MN\omega_{\mathbf{q}}}} q_1 \tag{1.70}$$

where we used the fact that for an isotropic crystal out of the three polarization vectors $\mathbf{e}(\mathbf{q},\lambda)\ \ \lambda = 1 \dots D$ in (1.49) only the one parallel to $\mathbf{q}$, say $\lambda = 1$, contributes to the sum and we identified

$$q_1 = \mathbf{q} \cdot \mathbf{e}(\mathbf{q},1) \tag{1.71}$$

Putting ourselves in the quasi-free electrons regime, that is $U_{band}(\mathbf{r}) \approx 0$ and considering the fact that at sufficiently low temperatures T the Coulomb electron-electron interaction $\hat{H}_{el-el}$ is exponentially screened at distances larger that $\lambda_{TF} \approx 1$Å (Thomas-Fermi screening) we are left with

$$\hat{H} = \int d^D\mathbf{r} \sum_{\sigma=\uparrow\downarrow} \hat{\psi}^\dagger_\sigma(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m} + U_{el-ph}(\mathbf{r}) \right) \hat{\psi}_\sigma(\mathbf{r}) + \sum_{\mathbf{k},\lambda} \hbar\omega_{\mathbf{k},\lambda} \left(\hat{a}^\dagger_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda} + \frac{1}{2} \right) \tag{1.72}$$

Actually, the fact of taking advantage of the Thomas-Fermi screening effect can be also justified a-posteriori considering the typical lengths Λ of Cooper pairs which amounts to $\Lambda \approx 1000$Å enabling us to effectively neglet the Coulomb interaction.

1.5 BCS path integral

Having at hand (1.72) we are ready to set up the gran canonical partition function for our system

$$\mathcal{Z} = \int \mathcal{D}\left[\psi,\psi^*\right] \mathcal{D}[\phi^*,\phi] e^{-\frac{1}{\hbar}S_E[\psi^*,\psi,\phi^*,\phi]} \tag{1.73}$$

where we indicated with ψ and ϕ respectively the eigenvalues of the fermionic annihilation operators $\hat{c}$ and the phononic annihilation operators $\hat{a}$ in the corresponding coherent state. The Euclidean action S_E is

$$S_E = S_{el} + S_{ph} + S_{el-ph} \tag{1.74}$$

with

$$S_{\text{fermions}} = \int_0^{\beta\hbar} d\tau \sum_{\mathbf{k}\ \sigma} \psi^*_{\mathbf{k}\sigma}(\tau) \left(\hbar\frac{\partial}{\partial\tau} - \mu + \frac{\hbar^2\mathbf{k}^2}{2m} \right) \psi_{\mathbf{k}\sigma}(\tau) \tag{1.75}$$

with μ the chemical potential of our fermions and our field $\psi_{\mathbf{k}\sigma}(\tau)$ satisfying the anti-periodic boundary condition

$$\psi_{\mathbf{k}\sigma}(0) = -\psi_{\mathbf{k}\sigma}(\beta\hbar) \tag{1.76}$$

for every $\mathbf{k}$ and σ.

$$S_{ph} = \int_0^{\beta\hbar} d\tau \sum_{\mathbf{q}\in 1^{st}BZ,\ \lambda} \phi^*_{\mathbf{q}\lambda}(\tau)\left(\hbar\frac{\partial}{\partial\tau} + \hbar\omega_{\mathbf{q}\lambda}\right)\phi_{\mathbf{q}\lambda}(\tau) \tag{1.77}$$

with $\phi_{\mathbf{q}\lambda}(\tau)$ satisfying periodic boundary condition

$$\phi_{\mathbf{q}\lambda}(0) = \phi_{\mathbf{q}\lambda}(\beta\hbar) \tag{1.78}$$

and

$$S_{el-ph} = \int_0^{\beta\hbar} d\tau \sum_{\mathbf{k}\sigma,\ \mathbf{q}} g(\mathbf{q})(\phi_{\mathbf{q},1}(\tau) + \phi^*_{-\mathbf{q},1}(\tau))\psi^*_{\mathbf{k}+\mathbf{q},\sigma}(\tau)\psi_{\mathbf{k}\sigma}(\tau) \tag{1.79}$$

At this point, having a quadratic action in the phonon field, we are able to integrate the latter out to see the effect of the electron-phonon interaction in terms of a interacting theory of only electrons.
To integrate out the phonon field we first Fourier transform our fields

$$\psi_{\mathbf{k}\sigma}(\tau) = \frac{1}{\sqrt{\beta\hbar}}\sum_{\omega_n}\psi_{\mathbf{k}\sigma n}e^{-i\omega_n\tau} \tag{1.80}$$

where, thanks to the anti-periodic boundary condition

$$\omega_n = \frac{(2n+1)\pi}{\beta\hbar} \tag{1.81}$$

and

$$\phi_{\mathbf{q}\lambda}(\tau) = \frac{1}{\sqrt{\beta\hbar}}\sum_{\Omega_m}\phi_{\mathbf{q}\lambda m}e^{-i\Omega_m\tau} \tag{1.82}$$

with, thanks to the periodic boundary condition

$$\Omega_m = \frac{2m\pi}{\beta\hbar} \tag{1.83}$$

plugging these transforms back in the original Euclidean action

$$S_{\text{fermions}} = \sum_{\mathbf{k}\sigma n}\psi^*_{\mathbf{k}\sigma n}\left(-i\hbar\omega_n - \mu + \frac{\hbar^2\mathbf{k}^2}{2m}\right)\psi_{\mathbf{k}\sigma n} \tag{1.84}$$

$$S_{phonons} = \sum_{\mathbf{q}\in 1^{st}BZ,\lambda,m}\phi^*_{\mathbf{q}\lambda m}\left(-i\hbar\Omega_m + \hbar\omega_{\mathbf{q}\lambda}\right)\phi_{\mathbf{q}\lambda m} \tag{1.85}$$

$$S_{el-ph} = \frac{1}{\sqrt{\beta\hbar}}\sum_{\mathbf{k}\sigma,\mathbf{q}}\sum_{\Omega_m,\omega_n} g(\mathbf{q})(\phi_{\mathbf{q},1,m} + \phi^*_{-\mathbf{q},1,-m})\psi^*_{\mathbf{k}+\mathbf{q},\sigma,n+m}\psi_{\mathbf{k}\sigma n} \tag{1.86}$$

We notice at this point that the only phonon polarization that is relevant in the ϕ-integration procedure is the longitudinal one $\lambda = 1$. The transversal polarizations, entering the partition function only in the kinetic term, will give a constant factor contribution that will not contribute to the electronic Green functions. For this reason in the following we will drop the summation and the subscript λ, implicitly considering $\lambda = 1$. To further lighten the notation we define

$$J_{-\mathbf{q},-m} = \frac{1}{\sqrt{\beta\hbar}}g(\mathbf{q})\sum_{\mathbf{k}\sigma\omega_n}\psi^*_{\mathbf{k}+\mathbf{q},\sigma,n+m}\psi_{\mathbf{k}\sigma n} = \frac{1}{\sqrt{\beta\hbar}}g(\mathbf{q})\rho_{-\mathbf{q}-m} \tag{1.87}$$

remebering the definition of electron density

$$\rho_{\mathbf{q}m} = \sum_{\mathbf{k}\sigma\omega_n} \psi^*_{\mathbf{k},\sigma,n}\psi_{\mathbf{k}+\mathbf{q},\sigma,n+m} \tag{1.88}$$

notice that

$$\rho^*_{\mathbf{q}m} = \rho_{-\mathbf{q}-m} \tag{1.89}$$

and, recalling (1.70)

$$g(\mathbf{q})^* = g(-\mathbf{q}) \tag{1.90}$$

remembering that reality condition of the displacement field (1.49) require

$$\mathbf{e}^*(\mathbf{q}, 1) = \mathbf{e}(-\mathbf{q}, 1) \tag{1.91}$$

so that, overall

$$J^*_{\mathbf{q}m} = J_{-\mathbf{q},-m} \tag{1.92}$$

At this point, we would like to have a more comfortable, symmetric form of our quadratic action in order to directly applying the famous identity

$$\int \mathcal{D}[\phi^*\phi] e^{-\phi^\dagger \cdot A\phi - J^\dagger\cdot\phi - \phi^\dagger\cdot J} \propto e^{J^\dagger \cdot A^{-1}J} \tag{1.93}$$

with

$$S_{\text{eff}} = -J^\dagger \cdot A^{-1}J \tag{1.94}$$

being the effective action resulting from the integration over phonons. To apply the former formula, notice that at this point we have an action that looks like this

$$S_{ph} + S_{el-ph} = \sum_{\mathbf{q},m} \phi^*_{\mathbf{q}m}\left(-i\hbar\Omega_m + \hbar\omega_{\mathbf{q}}\right)\phi_{\mathbf{q}m} + \sum_{\mathbf{q},m}(\phi_{\mathbf{q},m} + \phi^*_{-\mathbf{q},-m})J_{-\mathbf{q},-m} \tag{1.95}$$

but now, implementing (1.92) we end up with

$$S_{phonons} + S_{el-ph} = \sum_{\mathbf{q},m} \phi^*_{\mathbf{q}m}\left(-i\hbar\Omega_m + \hbar\omega_{\mathbf{q}}\right)\phi_{\mathbf{q}m} + J^*_{\mathbf{q},m}\phi_{\mathbf{q},m} + \phi^*_{\mathbf{q},m}J_{\mathbf{q},m} \tag{1.96}$$

That exactly resembles (1.93), thus obtaining, after the integration procedure, an effective action of the form

$$S_{\text{eff}} = -\sum_{\mathbf{q},\Omega_m} \frac{J^*_{\mathbf{q}m}J_{\mathbf{q}m}}{-i\hbar\Omega_m + \hbar\omega_{\mathbf{q}}} = -\frac{1}{2}\sum_{\mathbf{q},\Omega_m} \frac{2\omega_{\mathbf{q}}}{\hbar(\Omega_m^2 + \omega_{\mathbf{q}}^2)} J^*_{\mathbf{q}m}J_{\mathbf{q}m} \tag{1.97}$$

Going back to the expression in term of the fermionic fields $\psi_{\mathbf{k}\sigma n}$

$$S_{\text{eff}} = -\frac{1}{2}\sum_{\mathbf{q},\Omega_m} \frac{2\omega_{\mathbf{q}}}{\hbar(\Omega_m^2 + \omega_{\mathbf{q}}^2)} \frac{1}{\beta\hbar}|g(\mathbf{q})|^2 \rho_{\mathbf{q},m}\rho_{-\mathbf{q},-m} \tag{1.98}$$

And we understand at this point that the effect of the phonon-electron interaction sums up to an effective electron-electron interaction.
To study the real time t interaction and thus see the retardation effect we mentioned above, we would have to analytically continue the latter through a Wick rotation

$$\tau \to it \tag{1.99}$$

which, in terms of Mastubara frequencies Ω_m and real frequencies ω is

$$\Omega_m \to -i\omega + \eta \tag{1.100}$$

where η is a regulator that should be sent to 0^+ at the end of calculation. Performing this Wick rotation we obtain

$$S_{\text{eff}} = \frac{1}{2\beta\hbar^2} \sum_{\mathbf{q},\Omega_m} |g(\mathbf{q})|^2 \frac{\omega_{\mathbf{q}}}{(\omega^2 - \omega_{\mathbf{q}}^2)} \rho_{\mathbf{q},m}\rho_{-\mathbf{q},-m} \tag{1.101}$$

Now, this retarded interaction depends on the accessible phonons frequencies $\omega_{\mathbf{q}}$ which introduce a time scale

$$\tau_{ph} \propto \frac{1}{\omega_{\mathbf{q}}} \tag{1.102}$$

that tells us what is the typical relaxation time of the lattice after being deformed by an incoming electron. The energy ω is released by the electrons to the phonon (or viceversa).
It is important to keep in mind that at sufficiently low temperature T not all phonon states are accessible, indeed, and that is the very core of the Debye assumption, we know that there exists a frequency cut off ω_D, called the Debye frequency, with the property that every phonon state will have a frequency $\omega_{\mathbf{q}} \lesssim \omega_D$ and these states will be populated with density $D(\omega) \propto \omega^2$. Consequently, being the density quadratic in the frequencies, the higher the frequencies will be much more populated thus enabling us to set

$$\omega_{\mathbf{q}} \approx \omega_D \tag{1.103}$$

Note that in general for our monoatomic harmonic crystal the dispersion relation ω_k is acoustic in the long-wavelength limit

$$\omega_{\mathbf{k}} \propto |\mathbf{q}| \quad as \quad |\mathbf{q}| \to 0 \tag{1.104}$$

But in the Debye approximation we take it as an equality because we can expect that at low temperature T only acoustic phonons can be present, thus obtaining

$$\omega_{\mathbf{q}} = v_s|\mathbf{q}| \approx \omega_D \tag{1.105}$$

where we indicated with v_s the sound velocity for our crystal.

Again for low enough T only electron near the Fermi surface [2] with energy $E \approx \epsilon_F$ will interact absorbing or emitting a phonon, thus we can see that if the energy exchanged ω is such that

$$\omega \ll \omega_D \tag{1.106}$$

the effective electron-electron interaction is indeed attractive. Now in this limit, which is called *static* limit we can indeed set

$$V_{\text{eff}}(\omega, \mathbf{q}) \approx \frac{1}{2\beta\hbar^2}|g(\mathbf{q})|^2 \frac{\omega_{\mathbf{q}}}{(\omega^2 - \omega_{\mathbf{q}}^2)} \approx -\frac{1}{2\beta\hbar^2}\frac{|g(\mathbf{q})|^2}{\omega_{\mathbf{q}}^2} \approx const \tag{1.107}$$

where we made use of (1.104) to qualitatively obtain

$$g(\mathbf{q}) \sim |\mathbf{q}|^2 \sim \omega_q^2 \approx \omega_D^2 \tag{1.108}$$

now, setting this constant to

$$\begin{gathered} V_{\text{eff}}(\omega,\mathbf{q}) \approx -g \\ g > 0 \end{gathered} \tag{1.109}$$

we finally end up with the so-called BCS Hamiltonian

[2] That is a consequence of the Fermi-Dirac distribution shape

$$\hat{H} = \int d^3\mathbf{r} \sum_{\sigma=\uparrow\downarrow} \hat{\psi}^\dagger_\sigma(\mathbf{r}) \left(-\frac{\hbar^2\nabla^2}{2m}\right) \hat{\psi}_\sigma(\mathbf{r}) - g \int d^3\mathbf{r}\ \hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r})\ \hat{\psi}^\dagger_\downarrow(\mathbf{r})\hat{\psi}_\downarrow(\mathbf{r}) \tag{1.110}$$

with partition function

$$\mathcal{Z} = \int \mathcal{D}\left[\psi, \psi^*\right] e^{-\frac{1}{\hbar}S_E[\psi^*,\psi]} \tag{1.111}$$

$$S_E = \int_0^{\beta\hbar} d\tau \int d^3\mathbf{r} \sum_\sigma \psi^*_\sigma(\mathbf{r},\tau) \left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu\right) \psi_\sigma(\mathbf{r},\tau) - g\ \psi^\dagger_\uparrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau)\ \psi^\dagger_\downarrow(\mathbf{r},\tau)\psi_\downarrow(\mathbf{r},\tau) \tag{1.112}$$

1.6 Mean field treatment

We would like now to investigate the possibility for a bound state to rise as a consequence of this attractive interaction.
In a quantum framework this eventuality is not always met, meaning that at variance of the classical case, as a consequence of the uncertainty principle, an attractive interaction does not always sustain a ground state made up of bound states. We could nevertheless study the fate of these pairs approaching this problem perturbatively in the interaction strength g and trying, for example, to evaluate the 4-point electronic Green function. Retaining only "ring" diagrams and considering the homogeneous and static pairs one can see that at some temperature T^* (at least for $D \geq 3$) this 4-point Green function has a singularity signaling the breaking of perturbation theory around the ground state of free electrons and the need of finding another ground state that takes into account the strong binding of Cooper pairs. Because of the previous discussion we need non-perturbative methods to study the dynamic of Cooper pairs. Clearly a straightforward path integration is not possible due to the presence of a quartic interaction. To decouple this complicated interaction we make use of the so called *Hubbard-Stratanovich* transformation which relays on the introduction of an auxiliary complex field $\Delta(\mathbf{r},\tau)$ as follows

$$e^{g\int d^D\mathbf{r}\ \psi^\dagger_\uparrow(\mathbf{r})\psi_\uparrow(\mathbf{r})\ \psi^\dagger_\downarrow(\mathbf{r})\psi_\downarrow(\mathbf{r})} = \int \mathcal{D}[\Delta,\Delta^*] e^{-\int d^D\mathbf{r}\int_0^{\beta\hbar} d\tau\left[\frac{|\Delta|^2}{g} - \left(\Delta^*\psi_\downarrow\psi_\uparrow + \Delta\psi^\dagger_\uparrow\psi^\dagger_\downarrow\right)\right]} \tag{1.113}$$

for the structure of the bilinear decoupled $\hat{c}\hat{c}$ and $\hat{c}^\dagger\hat{c}^\dagger$ the HS transformation in this framework is often called Cooper channel. Plugging this identity into the BCS partition function we have

$$\mathcal{Z} = \int \mathcal{D}[\Delta,\Delta^*]\mathcal{D}\left[\psi,\psi^*\right] e^{-\frac{1}{\hbar}S_{\text{eff}}[\psi^*,\psi,\Delta,\Delta^*]} \tag{1.114}$$

with

$$\begin{aligned} S_{\text{eff}} = &\int_0^{\beta\hbar} d\tau \int d^D\mathbf{r} \sum_{\sigma=\uparrow,\downarrow} \psi^*_\sigma(\mathbf{r},\tau)\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right)\psi_\sigma(\mathbf{r},\tau) \\ &+ \frac{|\Delta(\mathbf{r},\tau)|^2}{g} - \left[\Delta^*(\mathbf{r},\tau)\psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau) + \Delta(\mathbf{r},\tau)\psi^\dagger_\uparrow(\mathbf{r},\tau)\psi^\dagger_\downarrow(\mathbf{r},\tau)\right]. \end{aligned} \tag{1.115}$$

and we notice that thanks to this transformation we have obtained an action that is quadratic in the fermionic fields. We are thus able to integrating them out by introducing the so called

Nambu spinors

$$\Psi(\mathbf{r},\tau) = \begin{pmatrix} \psi_\uparrow(\mathbf{r},\tau) \\ \psi_\downarrow^\dagger(\mathbf{r},\tau) \end{pmatrix}, \quad \Psi^\dagger(\mathbf{r},\tau) = \begin{pmatrix} \psi_\uparrow^\dagger(\mathbf{r},\tau) & \psi_\downarrow(\mathbf{r},\tau) \end{pmatrix} .$$

Then, the effective action takes the quadratic form

$$S_{\text{eff}} = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r} \ -\Psi^\dagger(\mathbf{r},\tau)\hat{\mathcal{G}}_\Delta^{-1}\Psi(\mathbf{r},\tau) + \frac{|\Delta(\mathbf{r},\tau)|^2}{g} . \tag{1.116}$$

where we recognize

$$\hat{\mathcal{G}}_\Delta^{-1}(\mathbf{r},\tau) = \begin{pmatrix} -\hbar\partial_\tau + \frac{\hbar^2\nabla^2}{2m} + \mu & \Delta(\mathbf{r},\tau) \\ \Delta^*(\mathbf{r},\tau) & -\hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu \end{pmatrix} \tag{1.117}$$

as the inverse of the propagator of the fermionic operator $\Psi(\mathbf{r},\tau)$ which in this representation is called *Gor'kov Grenn function.*

exploiting now the following identity valid for fermionic fields

$$\int \mathcal{D}[\psi^\dagger,\psi] e^{\psi^\dagger \cdot \hat{A}\psi} = e^{\log\left(\det \hat{A}\right)} \tag{1.118}$$

we obtain

$$\mathcal{Z} = \int \mathcal{D}[\Delta,\Delta^*] e^{-\frac{1}{\hbar}S_\Delta} \tag{1.119}$$

where

$$S_\Delta = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r} \frac{|\Delta(\mathbf{r},\tau)|^2}{g} - \hbar \log\left(\det \hat{\mathcal{G}}_\Delta^{-1}\right) \tag{1.120}$$

To understand the role of the auxiliary field Δ we can initially try to study the saddle point equation associated to this action

$$\begin{gathered} \frac{\delta S_\Delta}{\delta \bar{\Delta}} = 0 \\ \Delta(\mathbf{r},\tau) = \Delta_0 \end{gathered} \tag{1.121}$$

That is, considering $\langle \Delta(\mathbf{r},\tau)\rangle = \Delta_0$ uniform solution of (1.121) neglecting radial and phase fluctuations, obtaining the famous BCS gap equation

$$\frac{\Delta_0}{g} - \hbar \ Tr(\hat{\mathcal{G}} \ \partial_{\bar{\Delta}} \hat{\mathcal{G}}^{-1}) = 0 \tag{1.122}$$

Now exploiting the identities

$$\det\left(\log \hat{A}\right) = Tr(\log \hat{A}) \tag{1.123}$$

$$\partial \log\left(\hat{A}\right) = \hat{A}^{-1}\partial\hat{A} \tag{1.124}$$

and using the fact that

$$\partial_{\bar{\Delta}}\hat{\mathcal{G}}^{-1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.125}$$

we get, passing in Mastubara frequencies ω_n and momentum $\mathbf{q}$ domain

$$\frac{\Delta_0}{g} = \hbar\, Tr \begin{pmatrix} -\hbar\frac{\partial}{\partial\tau} + \frac{\hbar^2\nabla^2}{2m} + \mu & \Delta_0 \\ \bar{\Delta}_0 & -\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu \end{pmatrix}^{-1}_{21} = \frac{k_b T}{V}\hbar \sum_{\omega_n \mathbf{k}} \frac{\Delta_0}{\hbar^2\omega_n^2 + \xi_{\mathbf{k}}^2 + |\Delta_0|^2} \tag{1.126}$$

having indicated

$$\xi_{\mathbf{k}} = \frac{\hbar^2\mathbf{k^2}}{2m} - \mu \tag{1.127}$$

performing the standard Matsubara summation we get

$$\frac{1}{g} = \frac{1}{V}\sum_{\mathbf{k}} \frac{\tanh(\beta E_{\mathbf{k}})}{2E_{\mathbf{k}}} \tag{1.128}$$

with

$$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta_0^2} \tag{1.129}$$

notice that this equation has to be simultaneously solved with the *number equation*

$$N = -\frac{1}{\beta}\left(\frac{\log\mathcal{Z}}{\partial\mu}\right)_{V,T} = \sum_{\mathbf{k}}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right)\right) \tag{1.130}$$

Employing the latter equation one is, in principle, able to extract from (1.128) and (1.130) the uniform order parameter Δ_0 and the chemical potential μ as a function of $T, V, g(a_s^D)$ where a_s^D is the scattering length that we will define in the following.[3].

BCS-BEC crossover

We emphasize the fact that the interaction strength, once the ultraviolet divergencies are removed with the help of scattering theory, will be expressed in terms of the D-dimensional scattering length which, through Feshbach resonance, can be tuned to a certain value.
This variation in the scattering length (and consequently in the interaction strength) can lead to different regimes that describe very different physics ;

- In the case $a_s^D < 0$ with $|k_F\, a_s^D| \ll 1$ we find ourselves in the BCS regime where $\mu \approx E_F$
- in the case $a_s^D > 0$ we are in the BEC regime where the chemical potential is negative $\mu \approx -\frac{E_b}{2}$ with $E_b = \frac{\hbar}{ma_s^2}$, reflecting the fact the now the system is well described by a gas of molecular dimers (two fermions molecules).
- In the case $|a_s^D| \to \infty$ all the information on the interaction is lost and the system in now described by a Fermi gas with modified chemical potential $\mu \approx 0.59E_F$.

What we described is known as BCS-BEC crossover which although being first suggested in 1969 by David Eagles and in 1980 by Anthony Laggett [36][20], found experimental confirmations only in the last twenty years witnin the study of ultracold atoms and superconducting system, in particular, it has been observed in 2004 studying two-fermionic components of ^{40}K and ^{6}Li atoms.[51][45]

[3] It is worth emphasizing that in the derivation of the BCS thoery in previous section we neglected this phenomenon and regarded g as a metal-dependent coupling.

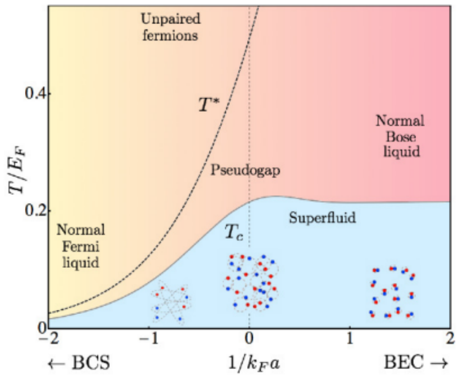

Figure 1.3: BCS-BEC crossover as a function of the scattering lenght a_s

Without entering the details of such interesting phenomenon, let us focus on the BCS regime $\mu \approx \epsilon_F$ in order to highlight the main feature of the mean field approach, which is well defined whenever there exists a broken phase $\langle \Delta(\mathbf{r},\tau)\rangle \neq 0$ that is, for $D \geq 3$.
Within this regime, performing the standard summation over Mastubara frequencies and employing the fact that the electrons energy can take value in the range $[\epsilon_F - \hbar\omega_D, \epsilon_F + \hbar\omega_D]$ we obtain

$$\frac{1}{g\nu} = \int_0^{\hbar\omega_D} d\xi \frac{\tanh\left(\frac{\lambda(\xi)}{2k_bT}\right)}{\lambda(\xi)} \tag{1.131}$$

where ν is the density of states at the Fermi surface and

$$\lambda(\xi) = \sqrt{\xi^2 + |\Delta_0|^2} \tag{1.132}$$

Solving the gap equation for low temperature $T \ll \Delta_0$ i.e. when $\tanh\left(\frac{\lambda(\xi)}{2k_bT}\right) \approx 1$ we obtain the value of Δ_0 at zero temperature

$$\Delta_0(T=0) = 2\hbar\omega_D e^{-\frac{1}{g\nu}} \tag{1.133}$$

corresponding with the energy required to completely break an Cooper pair.

Furthermore, imposing $\Delta_0 = 0^+$ we get an equation for the critical temperature T^* at which the superconducting phase transition takes place

$$T^* = \mathcal{O}(1) \times \frac{\hbar\omega_D}{k_b} e^{-\frac{1}{g\nu}} \tag{1.134}$$

At this point we have understood the role of the field Δ as a order parameter signaling the superconducting phase transition marked by a temperature T^* near which the modulus of the field can be assumed small $|\Delta| \approx 0$, specifically, the temperature dependence of the modulus of the field Δ_0 can be found to be

$$\Delta_0 \approx \mathcal{O}(1) \times \sqrt{T^*(T^* - T)} \tag{1.135}$$

explicitly signaling the presence of a second order phase transition; in particular it can be shown that the vanishing of the order parameter occurs with a diverging derivative. In this spirit, we can try to extract an effective theory around T^* from (1.149) by expanding it around $\Delta(\mathbf{r},\tau) \approx 0$. Before continuing, notice that in the previous discussion we naively focused on the mean field approach which did not take into account the role of radial or phase fluctuations

of the field $\Delta(\mathbf{r},\tau)$, incidentally, noticing that in this setting the role of dimension is irrelevant we particularized the study to the three-dimensional case.
In general, however, there could be major differences in the treatment of fluctuations. As we have highly stressed, for $D \leq 2$ it will be crucial to account for thermal fluctuations of our field, which, according to the Mermin Wagner theorem, could be very large even at low temperature, endangering therefore all the previous discussion.
We will therefore show in the following, what happens in the $D = 2$ case.

1.7 D=2 case

As we have already stressed enough, in the 2 dimensional case, things gets tricky due to the fact that phase fluctuations are greatly enhanced by thermal fluctuations forbidding the breaking of the $U(1)$ symmetry as prescribed by the Mermin-Wagner theorem.
In particular, notice that if we expand the logarithmic term in (1.149) and parametrize our order parameter as

$$\Delta(\mathbf{r},\tau) = (\Delta_0 + \sigma(\mathbf{r},\tau))e^{i\theta(\mathbf{r},\tau)} \tag{1.136}$$

we would obtain, up to a modification of the stiffness, a Goldstone action of the form given in (1.18).
As a consequence of the dimension in which we are working in the correlation associated to the Goldstone mode $\theta(\mathbf{r},\tau)$ goes like

$$\left\langle(\theta(\mathbf{r}) - \theta(0))^2\right\rangle \sim \log\left(\frac{|\mathbf{r}|}{a}\right) \tag{1.137}$$

and consequently

$$\left\langle e^{i(\theta(\mathbf{r})-\theta(0))}\right\rangle = e^{-\langle(\theta(\mathbf{r})-\theta(0))\rangle^2} \tag{1.138}$$

[4] goes like a power law!
Indeed, computing the four point correlation function of fermionic fields of the form

$$\rho_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = \langle\bar{\psi}_\uparrow(\mathbf{r}_1,0)\bar{\psi}_\downarrow(\mathbf{r}_2,0)\psi_\downarrow(\mathbf{r}_3,0)\psi_\uparrow(\mathbf{r}_4,0)\rangle \tag{1.139}$$

One can show that this exhibits off-diagonal long-range order (ODLRO) at $T = 0$ [46] and algebraic long-range order (ALRO) for $0 < T < T_{\mathrm{BKT}}$. In particular, introducing the center-of-mass coordinates of the two Cooper pairs,

$$\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2}, \qquad \mathbf{R}' = \frac{\mathbf{r}_3 + \mathbf{r}_4}{2},$$

and their relative separations, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{r}' = \mathbf{r}_4 - \mathbf{r}_3$, one can rewrite the four-point correlation as (considering the limit $|\mathbf{R} - \mathbf{R}'| \to +\infty$)

$$\begin{aligned}\rho_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) &\simeq F^*(\mathbf{r})\,F(\mathbf{r}')\left\langle e^{\,i\big[\theta(\mathbf{R},0)-\theta(\mathbf{R}',0)\big]}\right\rangle \\ &\simeq F^*(\mathbf{r})\,F(\mathbf{r}')\,\exp\!\Big(-\tfrac{1}{2}\big\langle[\theta(\mathbf{R},0)-\theta(\mathbf{R}',0)]^2\big\rangle\Big) \\ &\simeq F^*(\mathbf{r})\,F(\mathbf{r}')\left(\frac{R_0}{|\mathbf{R}-\mathbf{R}'|}\right)^{k_BT/(8\pi J)},\end{aligned} \tag{1.140}$$

[4] Note that this relation holds as long as the expectation value is performed over a gaussian action.

where

$$R_0 \simeq \xi$$

is the coherence-length scale of phase fluctuations as in (1.218), and

$$F(\mathbf{r}') = \langle \sigma(\mathbf{r}') \rangle = \frac{1}{L^2} \sum_{\mathbf{k}} \frac{\Delta_0}{2E_k} \tanh\left(\tfrac{\beta E_k}{2}\right) e^{i\mathbf{k}\cdot\mathbf{r}'}$$

is the mean-field Cooper-pair wavefunction. The quasicondensate density of atoms in the 2D superfluid is then

$$n_0 = 2 \int d^2\mathbf{r}' \left| F(\mathbf{r}') \right|^2 = \frac{\Delta_0^2}{2L^2} \sum_{\mathbf{k}} \frac{\tanh^2(\beta E_k/2)}{E_k^2}.$$

Thus we notice a remarkable difference with the three-dimensional case where the latter correlation, for $T < T^*$ goes like a *constant*, i.e. we have a long ranger order signaling the fact that phase fluctuations are difficult to create thermally, thus enabling us to set the phase to a constant and considering *small* fluctuations around it.
The previous digression servers as a warning to the fact that in the 2 dimensional case we ought to be careful with the treatment of the phase of the order parameter, indeed the power law decay signals the fact that the true condensation temperature for which $\langle \Delta \rangle \neq 0$ is $T^* = 0$. In the following we will extract the effective Goldstone action we naively mentioned before writing the order parameter in (1.149) as (1.136) and later on we will understand which phase configurations will be more likely to be created thermally. Let us start by rewriting the action we derived in the previous section by means of an Hubbard-Stratanovich transformation

$$S_{\text{eff}} = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r} \; -\Psi^\dagger(\mathbf{r},\tau)\hat{\mathcal{G}}_\Delta^{-1}\Psi(\mathbf{r},\tau) + \frac{|\Delta(\mathbf{r},\tau)|^2}{g}. \tag{1.141}$$

with, again

$$\Psi(\mathbf{r},\tau) = \begin{pmatrix} \psi_\uparrow(\mathbf{r},\tau) \\ \psi_\downarrow^\dagger(\mathbf{r},\tau) \end{pmatrix}, \quad \Psi^\dagger(\mathbf{r},\tau) = \begin{pmatrix} \psi_\uparrow^\dagger(\mathbf{r},\tau) & \psi_\downarrow(\mathbf{r},\tau) \end{pmatrix}. \tag{1.142}$$

being the *Nambu spinors*. As previously done, we integrate out the fermionic fields extracting an effective action of the gap field $\Delta(\mathbf{r},\tau)$

$$S_\Delta = \int_0^{\beta\hbar} d\tau \int d^D\mathbf{r} \frac{|\Delta(\mathbf{r},\tau)|^2}{g} - \hbar \log\left(\det \hat{\mathcal{G}}_\Delta^{-1}\right) \tag{1.143}$$

with

$$\hat{\mathcal{G}}_\Delta^{-1}(\mathbf{r},\tau) = \begin{pmatrix} -\hbar\partial_\tau + \frac{\hbar^2\nabla^2}{2m} + \mu & \Delta(\mathbf{r},\tau) \\ \Delta^*(\mathbf{r},\tau) & -\hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu \end{pmatrix} \tag{1.144}$$

Writing the gap field as in (1.136) we could in principle extract an effective field theory in terms of the phase and the radial field, but being the radial fluctuations massive, and thus difficult to create thermally, it is reasonable to assume it to be small $\sigma(\mathbf{r},\tau) \approx 0$ in first approximation, leaving us with

$$\hat{\mathcal{G}}_\Delta^{-1}(\mathbf{r},\tau) = \begin{pmatrix} -\hbar\partial_\tau + \frac{\hbar^2\nabla^2}{2m} + \mu & \Delta_0 \, e^{i\theta(\mathbf{r},\tau)} \\ \Delta_0 \, e^{-i\theta(\mathbf{r},\tau)} & -\hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu \end{pmatrix} \tag{1.145}$$

Now we would like to bring the phase dependence on the diagonal rather than the anti-diagonal, in this spirit we exploit the following identities

$$\det\left(\log \hat{\mathcal{G}}_\Delta^{-1}\right) = Tr(\log \hat{\mathcal{G}}_\Delta^{-1}) \tag{1.146}$$

and

$$Tr\left(\log\hat{\mathcal{G}}_{\Delta}^{-1}\right) = Tr\left(\log\left(U\hat{\mathcal{G}}_{\Delta}^{-1}U^{\dagger}\right)\right) \tag{1.147}$$

with

$$U = \exp\left(-\frac{i\theta(\mathbf{r},\tau)}{2}\tau_3\right) \tag{1.148}$$

Notice that if we were to deal with a $U(1)$ Gauge-invariant theory, this would be interpreted as a Gauge transformation acting also on the electro-magnetic field; the factor $\frac{1}{2}$ is essential in order to maintain the $U(1)$ invariance of the operators $\Delta^*\psi\psi$ and $\Delta\psi^\dagger\psi^\dagger$.
Thus we end up with the following action

$$\frac{S_\Delta}{\hbar} = \beta L^2\frac{|\Delta_0|^2}{g} - Tr\log\left(\hat{\mathcal{G}}_{\Delta_0}^{-1} - \hat{\Sigma}_\theta\right) \tag{1.149}$$

with

$$\hat{\mathcal{G}}_{\Delta_0}^{-1}(\mathbf{r},\tau) = \begin{pmatrix} -\hbar\partial_\tau + \frac{\hbar^2\nabla^2}{2m} + \mu & \Delta_0 \\ \Delta_0 & -\hbar\partial_\tau - \frac{\hbar^2\nabla^2}{2m} - \mu \end{pmatrix} \tag{1.150}$$

and

$$\hat{\Sigma}_\theta = \tau_3\left(\frac{i\hbar}{2}\partial_\tau\theta(\mathbf{r},\tau) + \frac{\hbar^2}{8m}(\nabla\theta(\mathbf{r},\tau))^2\right) - \hat{I}\left(i\frac{\hbar^2}{4m}\nabla^2\theta(\mathbf{r},\tau) + i\frac{\hbar^2}{2m}\nabla\theta(\mathbf{r},\tau)\cdot\nabla\right) \tag{1.151}$$

Rearranging the logarithmic term

$$\frac{S_\Delta}{\hbar} = \beta L^2\frac{|\Delta_0|^2}{g} - Tr\log\left(\hat{\mathcal{G}}_{\Delta_0}^{-1} - \hat{\Sigma}_\theta\right) = \beta L^2\frac{|\Delta_0|^2}{g} - Tr\log\left(\hat{\mathcal{G}}_{\Delta_0}^{-1}\right) - Tr\log\left(\hat{I} - \hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta\right) \tag{1.152}$$

where $\hat{\mathcal{G}}_{\Delta_0}(\mathbf{r},\tau)$ satisfy

$$\Big[-\hbar\frac{\partial}{\partial\tau}\hat{I} + \tau_3\left(\frac{\hbar^2}{2m}\nabla^2 + \mu\right) + \Delta_o\tau_1\Big]\hat{\mathcal{G}}_{\Delta_0}(\mathbf{r},\tau) = \delta(\tau)\delta^2(\mathbf{r}) \tag{1.153}$$

Notice that the first two terms of (1.152) are nothing else than the mean field term that we derive in the previous treatment pointing out the consequences of the spontaneous symmetry breaking deriving the gap equation through a minimization condition; the third term pops out entirely because of the phase field, so it is pretty natural to identify it as the effective action of the Goldstone mode

$$\frac{S_\theta}{\hbar} = -Tr\log\left(\hat{I} - \hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta\right) \tag{1.154}$$

Notice now that, due to the existence of the underlying $U(1)$ symmetry in our theory, the phase $\theta(\mathbf{r},\tau)$ enters our action only through derivatives terms which are invariant under phase shift, moreover, since the low energy dynamics of the phase field is mainly governed by its long-wavelength fluctuations, we can expand the latter logarithm and retain only the leading derivatives term.
In the literature this is know as *gradient expansion* [6] and it is a really common feature of low-energy effective field theories. With this in mind, we expand to the second order the phase action, remembering the correspondence

$$\frac{S_\theta}{\hbar} = \beta\Omega_\theta \tag{1.155}$$

$$\beta \Omega_\theta = -Tr \log\left(\hat{I} - \hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta\right) = Tr \sum_{k=1}^{\infty} \frac{(\hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta)^k}{k} = Tr(\hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta) + \frac{1}{2}Tr(\hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta\hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta) + o(\Sigma_\theta^3) \tag{1.156}$$

Before dive ourselves into the calculations, we express the Green function (1.153) in Fourier space $(i\omega_n, \mathbf{k})$

$$\hat{\mathcal{G}}_{\Delta_0}(i\omega_n, \mathbf{k}) = -\frac{i\hbar\omega_n\hat{I} + \xi_{\mathbf{k}}\tau_3 - \Delta_0\tau_1}{\hbar^2\omega_n^2 + E_{\mathbf{k}}^2} \tag{1.157}$$

and the phase $\theta(\mathbf{r}, \tau)$ dependent operator Σ_θ as

$$\hat{\Sigma}_\theta = \tau_3\hat{O}_1 - \hat{I}\hat{O}_2 \tag{1.158}$$

Furthermore notice that from the gradient expansion we would like to extract information about the kinetic terms $\sim (\nabla\theta)^2$ and $\sim (\partial_\tau\theta)^2$ of the phase field; to this end, we notice that the only term of $\hat{\Sigma}_\theta$ which contributes at the linear level to the kinetic operators is the one proportional to $\hat{O}_1$, so that

$$Tr(\hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta)_{|kin} = Tr(\hat{\mathcal{G}}_{\Delta_0}\tau_3)\hat{O}_1 = Tr\left(\sum_{\mathbf{k},\omega_n}\hat{\mathcal{G}}_{\Delta_0}(i\omega_n, \mathbf{k})\tau_3\right)\hat{O}_1 \tag{1.159}$$

At this point we understand that the previous sum decays, for $\omega_n \to \infty$ as $\sim \frac{1}{\omega_n}$ so we need to regularize it.
We can do it employing the time ordering of our fields into our calculations, which eventually leads to a modification of the Green's function given by

$$\hat{\mathcal{G}}_{\Delta_0}(i\omega_n, \mathbf{k}) \to \hat{\mathcal{G}}_{\Delta_0}(i\omega_n, \mathbf{k})\exp\left\{\left(i\omega_n 0^+\tau_3\right)\right\} \tag{1.160}$$

leading to the following regularized summation

$$Tr(\hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta)^{reg}_{kin} = \left(\sum_{\mathbf{k},\omega_n} -\frac{i\hbar\omega_n(e^{i\omega_n 0^+} - e^{-i\omega_n 0^+}) + 2\xi_{\mathbf{k}}}{\hbar^2\omega_n^2 + E_{\mathbf{k}}^2}\right)\hat{O}_1 \tag{1.161}$$

where we performed the trace in the matrices indices.
Now, notice that the summation over Matsubara frquencies can be split into three parts, with corresponding three grand potential terms

$$\frac{\Omega_1}{L^2} = \frac{1}{\beta L^2}\left(\sum_{\omega_n} -\frac{i\hbar\omega_n e^{i\omega_n 0^+}}{\hbar^2\omega_n^2 + E_{\mathbf{k}}^2}\right)\hat{O}_1 \tag{1.162}$$

$$\frac{\Omega_2}{L_2} = \frac{1}{\beta L^2}\left(\sum_{\omega_n} \frac{i\hbar\omega_n e^{-i\omega_n 0^+}}{\hbar^2\omega_n^2 + E_{\mathbf{k}}^2}\right)\hat{O}_1 \tag{1.163}$$

$$\frac{\Omega_3}{L^2} = \frac{1}{\beta L^2}\left(\sum_{\omega_n} -\frac{2\xi_{\mathbf{k}}}{\hbar^2\omega_n^2 + E_{\mathbf{k}}^2}\right)\hat{O}_1 \tag{1.164}$$

for what concerns (1.162) we realize that

$$\frac{1}{\beta L^2}\left(\sum_{\omega_n} -\frac{i\hbar\omega_n e^{i\omega_n 0^+}}{\hbar^2\omega_n^2 + E_{\mathbf{k}}^2}\right)\hat{O}_1 = -\frac{1}{L^2}\sum_{z=i\hbar\omega_n} Res\left(h_1(z)\frac{z e^{\frac{z}{\hbar}0^+}}{z^2 - E_{\mathbf{k}}^2}\right) \tag{1.165}$$

with

$$h_1(z) = \frac{1}{e^{\beta z} + 1} \tag{1.166}$$

and having used

$$Res(h_1(z))_{|i\hbar\omega_n} = -\frac{1}{\beta} \tag{1.167}$$

Now we realize

$$\sum_{z=i\hbar\omega_n} Res\left(h_1(z)\frac{ze^{\frac{z}{\hbar}0^+}}{z^2 - E_{\mathbf{k}}^2}\right) = \frac{1}{2\pi i}\oint_{\gamma_1} h_1(z)\frac{ze^{\frac{z}{\hbar}0^+}}{z^2 - E_{\mathbf{k}}^2}\,dz \tag{1.168}$$

where γ_1 is a circuit that embraces the imaginary axis anti clockwise like (a) in (1.4). If we inflate γ_1 like in (b) in (1.4) so that it avoids the poles of $\frac{ze^{\frac{z}{\hbar}0^+}}{z^2-E_{\mathbf{k}}^2}$ and more importantly,choosing $h_1(z)$ so that it controls the divergent behavior of $e^{\frac{z}{\hbar}0^+}$ for $\mathrm{Re}\{(z)\} \to +\infty$, the resulting integral over the new inflated γ_2 will take into account only the residues in the poles of $\frac{ze^{\frac{z}{\hbar}0^+}}{z^2-E_{\mathbf{k}}^2}$ because, thanks to the careful choice of $h_1(z)$ the integral on the other pieces of γ_2 which extend at ∞ is 0..

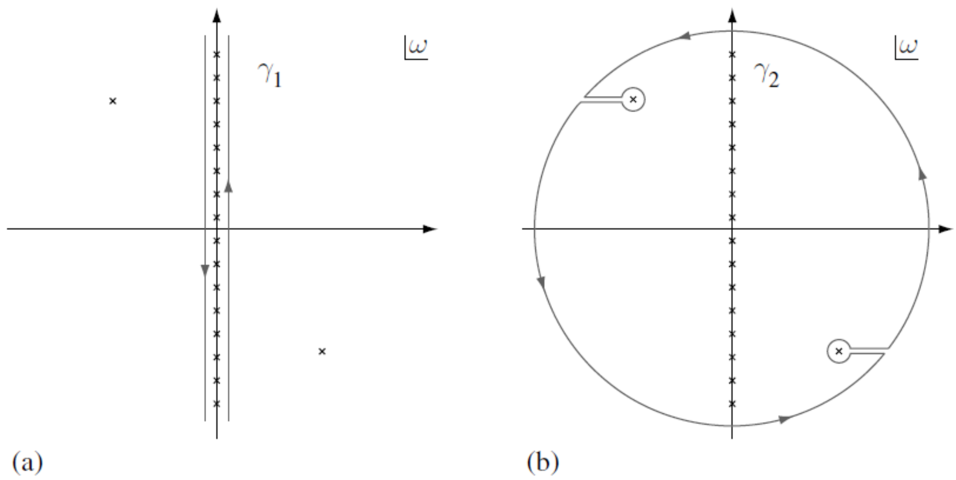

Figure 1.4: Adapted from [2]

Thus, we conclude

$$\sum_{z=i\hbar\omega_n} Res\left(h_1(z)\frac{ze^{\frac{z}{\hbar}0^+}}{z^2 - E_{\mathbf{k}}^2}\right) = -\sum_{z=\pm E_{\mathbf{k}}} Res\left(h_1(z)\frac{ze^{\frac{z}{\hbar}0^+}}{z^2 - E_{\mathbf{k}}^2}\right) = -\frac{1}{2} \tag{1.169}$$

and finally

$$\frac{1}{\beta L^2}\left(\sum_{\omega_n} -\frac{i\hbar\omega_n e^{i\omega_n 0^+}}{\hbar^2\omega_n^2 + E_{\mathbf{k}}^2}\right)\hat{O}_1 = \frac{1}{L^2} \times \frac{1}{2}\hat{O}_1 \tag{1.170}$$

For what concerns (1.163), the argument is pretty much the same, except for the choice of the weighting function. Indeed, due to the different pathological behavior of the exponential term $e^{-\frac{z}{\hbar}0^+}$ which diverges for $\mathrm{Re}(z) \to -\infty$ we ought to choose a weighting function which controls it, like the function

$$h_2(z) = \frac{1}{e^{-\beta z} + 1} \tag{1.171}$$

which satisfies

$$Res(h_2(z))_{|z=i\hbar\omega_n} = \frac{1}{\beta} \tag{1.172}$$

Thus, following the previous procedure we conclude

$$\frac{1}{\beta L^2}\left(\sum_{\omega_n}\frac{i\hbar\omega_n e^{-i\omega_n 0^+}}{\hbar^2\omega_n^2+E_{\mathbf{k}}^2}\right)\hat{O}_1=\frac{1}{L^2}\times\frac{1}{2}\hat{O}_1 \tag{1.173}$$

The third piece (1.164) is instead more easy to deal with, because the sum over ω_n converges for this reason it is not glued to a regulating factor. Moreover it does not have pathological behavior at $|z|=+\infty$ so the choice of the weighting function does not spoil the final result. Following the previous step is straightforward to show that, using for example as weighting function $h_2(z)$ which satisfy (1.171)

$$\frac{1}{\beta L^2}\left(\sum_{\mathbf{k},\omega_n}-\frac{2\xi_{\mathbf{k}}}{\hbar^2\omega_n^2+E_{\mathbf{k}}^2}\right)\hat{O}_1=-\frac{1}{L^2}\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right)\hat{O}_1 \tag{1.174}$$

In fact

$$\begin{aligned}\frac{1}{\beta L^2}\left(\sum_{\mathbf{k},\omega_n}-\frac{2\xi_{\mathbf{k}}}{\hbar^2\omega_n^2+E_{\mathbf{k}}^2}\right)&=\frac{1}{L^2}\sum_{z=i\hbar\omega_n}Res\left(h_2(z)\frac{2\xi_{\mathbf{k}}}{z^2-E_{\mathbf{k}}^2}\right)\\&=-\frac{1}{L^2}\sum_{z=\pm E_{\mathbf{k}}}Res\left(h_2(z)\frac{2\xi_{\mathbf{k}}}{z^2-E_{\mathbf{k}}^2}\right)=-\frac{1}{L^2}\frac{\xi_{\mathbf{k}}}{E_k}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right)\end{aligned} \tag{1.175}$$

Putting everything together we end up with

$$\frac{\Omega_{kin}}{L^2}=\frac{1}{L^2}\sum_{\mathbf{k}}\left(1-\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right)\right)\hat{O}_1=\frac{1}{L^2}\frac{\hbar^2}{8m}\sum_{\mathbf{k}}\left(1-\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right)\right)(\nabla\theta)^2 \tag{1.176}$$

up to total derivative terms. We now realize that the term inside the brackets is nothing else than the number equation given in (1.130). For what concerns the quadratic term in the original expansion

$$\frac{1}{2}Tr(\hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta\hat{\mathcal{G}}_{\Delta_0}\hat{\Sigma}_\theta) \tag{1.177}$$

we notice that the term proportional to the kinetic operator $(\nabla\theta)^2$ would be the one that goes like $\sim\hat{O}_2\hat{O}_2$, that is

$$\frac{1}{2}Tr(\hat{\mathcal{G}}_{\Delta_0}\hat{I}\hat{O}_2\hat{\mathcal{G}}_{\Delta_0}\hat{I}\hat{O}_2) \tag{1.178}$$

Thanks to the fact that, up to total derivative terms

$$\hat{O}_2\hat{O}_2\sim-\frac{\hbar^4}{4m^2}\nabla\theta\cdot\nabla\left(\nabla\theta\cdot\nabla\right)\sim-\frac{\hbar^4}{4m^2}\sum_{i,j}\partial_i\theta\;\partial_j\theta\;\partial_i\partial_j \tag{1.179}$$

and passing in Fourier space

$$\frac{1}{\beta L^2}\frac{1}{2}Tr(\hat{\mathcal{G}}_{\Delta_0}\hat{I}\hat{O}_2\hat{\mathcal{G}}_{\Delta_0}\hat{I}\hat{O}_2)\sim-\frac{1}{2}\frac{1}{\beta L^2}\frac{\hbar^4}{4m^2}Tr\Big[\sum_{\mathbf{k},\omega_n}\sum_{i,j}\left(\frac{i\hbar\omega_n\hat{I}+\xi_{\mathbf{k}}\tau_3-\Delta_0\tau_1}{\hbar^2\omega_n^2+E_{\mathbf{k}}^2}\right)^2k_ik_j\Big]\partial_i\theta\;\partial_j\theta \tag{1.180}$$

Performing the trace over the matrix indices and summing over the i,j indices we end up with

$$\frac{1}{\beta L^2}\frac{1}{2}Tr(\hat{\mathcal{G}}_{\Delta_0}\hat{I}\hat{O}_2\hat{\mathcal{G}}_{\Delta_0}\hat{I}\hat{O}_2)\sim-\frac{1}{2}\frac{1}{\beta L^2}\frac{\hbar^2}{4m}\Big[\sum_{\mathbf{k},\omega_n}\frac{-2\hbar^2\omega_n^2+2\xi_{\mathbf{k}}^2+2\Delta_0^2}{(\hbar^2\omega_n^2+E_{\mathbf{k}}^2)^2}\frac{\hbar^2}{2m}|\mathbf{k}|^2\Big](\nabla\theta)^2 \tag{1.181}$$

having used the property

$$\sum_{\mathbf{k}} f(|\mathbf{k}|)k_i k_j = \frac{1}{2}\sum_{\mathbf{k}} f(|\mathbf{k}|)|\mathbf{k}|^2 \tag{1.182}$$

for a generic function rotationally invariant $f(|\mathbf{k}|)$.

Now, the summation over Matsubara frequencies does not need regulating factors: everything here converges!
Exploiting the following identities which can be proved adapting the previous discussion about weighting function

$$\frac{1}{\beta}\sum_{\omega_n}\frac{1}{\hbar^2\omega_n^2+E_{\mathbf{k}}^2} = \frac{1}{2E_{\mathbf{k}}}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right) \tag{1.183}$$

$$\frac{1}{\beta}\sum_{\omega_n}\frac{1}{(\hbar^2\omega_n^2+E_{\mathbf{k}}^2)^2} = \frac{1}{4E_{\mathbf{k}}^3}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right) - \frac{1}{4E_{\mathbf{k}}^2}\tanh'\left(\frac{\beta E_{\mathbf{k}}}{2}\right) \tag{1.184}$$

we end up with

$$\frac{1}{\beta L^2}\frac{1}{2}Tr(\hat{\mathcal{G}}_{\Delta_0}\hat{I}\hat{O}_2\hat{\mathcal{G}}_{\Delta_0}\hat{I}\hat{O}_2) \sim -\frac{1}{2}\frac{\hbar^2}{4mL^2}\sum_{\mathbf{k}}\frac{\mathbf{k}^2}{2m}\ \tanh'\left(\frac{\beta E_{\mathbf{k}}}{2}\right)(\nabla\theta)^2 \tag{1.185}$$

To conclude, we notice that the contribution to the coupling of the kinetic operator $(\partial_\tau\theta)^2$ will stem from the term of the gradient expansion

$$\frac{1}{2}Tr(\hat{\mathcal{G}}_{\Delta_0}\tau 3\hat{\mathcal{G}}_{\Delta_0}\tau_3)\hat{O}_1\hat{O}_1 \tag{1.186}$$

with

$$\hat{O}_1\hat{O}_1 = -\frac{\hbar^2}{4}(\partial_\tau\theta)^2 + o(\partial^3) \tag{1.187}$$

Computing the trace similarly to what done previously we obtain

$$\frac{1}{\beta L^2}\frac{1}{2}Tr(\hat{\mathcal{G}}_{\Delta_0}\tau_3\hat{\mathcal{G}}_{\Delta_0}\tau_3)\hat{O}_1\hat{O}_1 = \frac{1}{2}\frac{\hbar^2}{4L^2}\sum_{\mathbf{k}}\left(\frac{\Delta_0^2}{E_{\mathbf{k}}^3}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right) + \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^2}\tanh'\left(\frac{\beta E_{\mathbf{k}}}{2}\right)\right)(\partial_\tau\theta)^2 \tag{1.188}$$

Finally, plugging Eqs(1.176), (1.185), (1.188) into the original gradient expansion (1.156) we arrive at an effective field theory of the form

$$\frac{\Omega}{L^2} = \int_0^{\beta\hbar} d\tau\int_{L^2} d^2\mathbf{r}\left[\frac{K}{2}(\partial_\tau\theta(\mathbf{r},\tau))^2 + \frac{J}{2}(\nabla\theta(\mathbf{r},\tau))^2\right] \tag{1.189}$$

where

$$K = \frac{\hbar^2}{4L^2}\sum_{\mathbf{k}}\left(\frac{\Delta_0^2}{E_{\mathbf{k}}^3}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right) + \frac{\xi_{\mathbf{k}}^2}{E_{\mathbf{k}}^2}\tanh'\left(\frac{\beta E_{\mathbf{k}}}{2}\right)\right) \tag{1.190}$$

and

$$J = \frac{\hbar^2}{4mL^2}\sum_{\mathbf{k}}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right) - \frac{\hbar^2\mathbf{k}^2}{2m}\tanh'\left(\frac{\beta E_{\mathbf{k}}}{2}\right)\right) \tag{1.191}$$

where we can appreciate the fermionic normal part of the superfluid density in (1.191).
Summarizing, up until now we retained only the leading quadratic order terms of the expansion (1.156), deriving the depletion of the superfluid density caused by the existence of fermionic quasiparticles in the spectrum.
However, it is possible to acknowledge the presence in the stiffness of also bosonic excitations that would eventually depletes the superfluid density; but to see this we have to consider also 1-loop corrections to the stiffness δJ_B.

1.7.1 2-loop expansion

To derive the corrections to the stiffness we have to employ the perturbative effects of the order $n = 3, 4$ in the initial gradient expansion. [9]
To see this, notice that in complete generality we can write, consodering the third and the fourth order terms in (1.156):

$$e^{-\beta\Omega} = \int \mathcal{D}\theta \;\; e^{-\beta(\Omega_G+\Omega_3+\Omega_4)} \tag{1.192}$$

where

$$\Omega_G = L^2 \int_0^{\beta\hbar} d\tau \int_{L^2} d^2\mathbf{r} \left[\frac{K}{2}(\partial_\tau \theta(\mathbf{r},\tau))^2 + \frac{J}{2}(\nabla\theta(\mathbf{r},\tau))^2 \right] \tag{1.193}$$

$$\Omega_3 = \frac{1}{3}\frac{1}{\beta} Tr \left(\hat{\mathcal{G}}_0 \hat{\Sigma}_\theta \hat{\mathcal{G}}_0 \hat{\Sigma}_\theta \hat{\mathcal{G}}_0 \hat{\Sigma}_\theta \right) \tag{1.194}$$

$$\Omega_4 = \frac{1}{4}\frac{1}{\beta} Tr \left(\hat{\mathcal{G}}_0 \hat{\Sigma}_\theta \hat{\mathcal{G}}_0 \hat{\Sigma}_\theta \hat{\mathcal{G}}_0 \hat{\Sigma}_\theta \hat{\mathcal{G}}_0 \hat{\Sigma}_\theta \right) \tag{1.195}$$

Note that (1.192) can be eventually rewritten as

$$e^{-\beta\Omega} = \mathcal{Z}_G \left\langle e^{-\beta(\Omega_3+\Omega_4)} \right\rangle_G \tag{1.196}$$

which eventually becomes

$$\Omega = -\frac{1}{\beta}\log \mathcal{Z}_G - \frac{1}{\beta}\log \left\langle e^{-\beta(\Omega_3+\Omega_4)} \right\rangle_G \tag{1.197}$$

Now, a very know result which goes under the name of "Linked cluster expansion" shows that

$$\log \left\langle e^{-\beta(\Omega_3+\Omega_4)} \right\rangle_G = \sum_n \frac{(-1)^n}{n!} \beta^n \left\langle (\Omega_3 + \Omega_4)^n \right\rangle_G^{CONN.} \tag{1.198}$$

Where the sum is performed over the "connected" part of the diagrams.
To understand the diagrammatic interpretation of the previous expansion, we give the following Feynman rules:

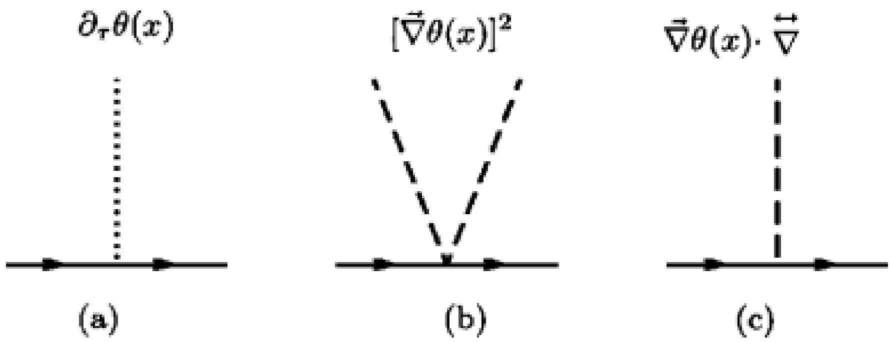

Figure 1.5: Source [9]

justified by the form of the fermionic self energy previously defined

$$\hat{\Sigma}_\theta = \tau_3 \left(\frac{i\hbar}{2}\partial_\tau\theta(\mathbf{r},\tau) + \frac{\hbar^2}{8m}(\nabla\theta(\mathbf{r},\tau))^2 \right) - \hat{I}\left(i\frac{\hbar^2}{4m}\nabla^2\theta(\mathbf{r},\tau) + i\frac{\hbar^2}{2m}\nabla\theta(\mathbf{r},\tau)\cdot\nabla \right) \tag{1.199}$$

Notice that $\overleftrightarrow{\nabla}$ is a short cut to indicate the term proportional to $\hat{I}$ in Σ_θ.
From the previous figure, we understand that the vertex (a) with an incoming dotted line stems for the

$$\Sigma_1^B = \tau_3 \left(\frac{i\hbar}{2} \partial_\tau \theta \right) \tag{1.200}$$

in Σ_θ.
The vertex (b) with one incoming dashed line and one outgoing dashed line stems for the term

$$\Sigma_2^B = \tau_3 \left(\frac{\hbar^2}{8m} (\nabla\theta)^2 \right) \tag{1.201}$$

Finally the (c) with one incoming dashed line stems for the term

$$\Sigma_1^F = -\hat{I} \left(i\frac{\hbar^2}{4m} \nabla^2\theta(\mathbf{r},\tau) + i\frac{\hbar^2}{2m} \nabla\theta(\mathbf{r},\tau) \cdot \nabla \right) \tag{1.202}$$

From the previous Feynman rules we also understand that each solid black line corresponds to a Fermionic propagator $\mathcal{G}_0(\omega_n, \mathbf{k})$. From this rules, the diagrams of the 3rd order interaction Ω_3 that contributes to a correction of the stiffness are

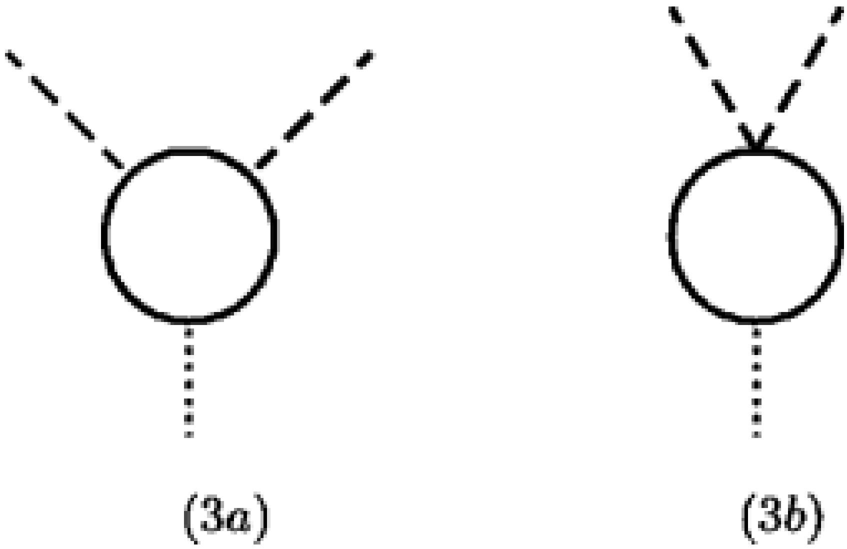

Figure 1.6: 3rd order interactions

while for what concerns the fourth order interaction Ω_4 we have

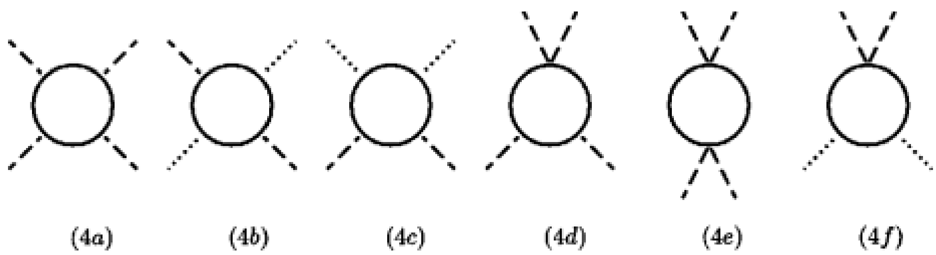

Figure 1.7: 4th order interactions

The terms $\Sigma_1^B + \Sigma_2^B$ are analogous to those one obtains deriving the phase only action for purely bosonic system, like in (1.18), and we will thus refer to them as *bosonic*.

The term Σ_1^F will be addressed as *fermionic* because of the fact that while computing the stiffness J its contribution generates fermionic quasi-particle excitations.
In terms of this classifications, we notice the first order expression of the stiffness (1.176) previously derived can be depicted as

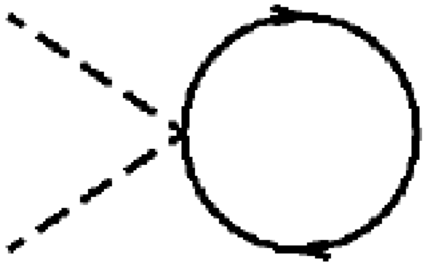

Figure 1.8: 1st order expansion

whereas (1.185) can be depicted as

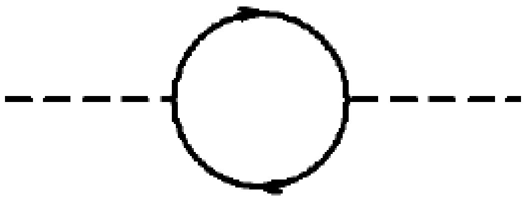

Figure 1.9: 2nd order expansion

Finally the compressibility K in (1.188) also has a diagrammatic representation given by the following

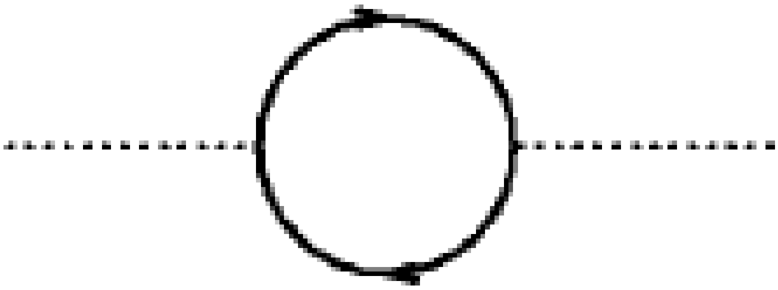

Figure 1.10: Compressibility diagrammatic depiction

Now to extract the 1-loop correction to the stiffness we ought to contract two θ lines from each 4th order diagram and the same for two 3rd order diagrams, as one would explicitly obtain from the expansion (1.198). Before doing so, notice that for what concern the 4th-order diagrams, not all of them will contribute the same to the correction of the stiffness.
In fact, from the previous classification of the Σ_θ terms, we are able to distinct three kinds of corrections

i *Fermionic corrections* like the one given by a contraction of the 4a diagram in Fig.(1.7).

ii *Bosonic corrections* which arise from the contraction of two 3b in Fig. (1.6) diagrams or one 4e diagram.

iii Mixed corrections which stem from the rest of the diagrams.

At the end of the day one is able to see that out of all the possible 2-loops diagrams contributing to the correction of the stiffness, only the following diagrams contribute

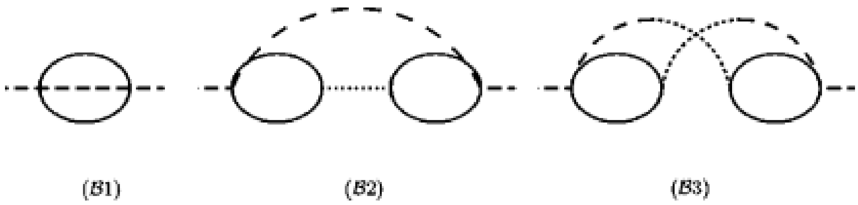

Figure 1.11: 2-loops corrections

while the remaining diagrams

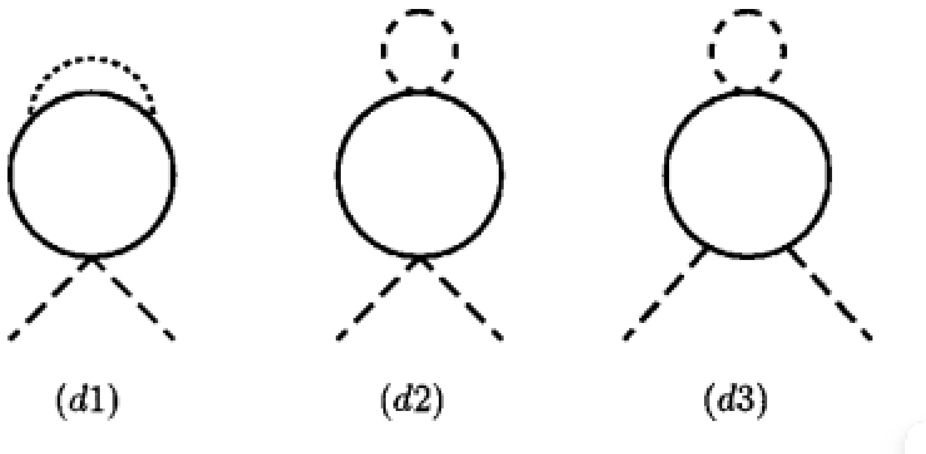

Figure 1.12

are compensated by the fact that during the 1-loop summation given by the expansion (1.198), also the chemical potential gets shifted $\mu \to \mu + \delta\mu$, which in return causes the stiffness to have one more correction due exactly to this shift

$$\delta J_\mu = \frac{\partial J}{\partial \mu}\delta\mu \tag{1.203}$$

the previous correction exactly compensates the one generated by the diagrams in Fig. (1.12) if one *fixes* the density of particles.
Thanks to the particle number constraints we can focus only on the corrections given by the diagrams in Fig.(1.11).
Notice that, from this diagrammatic interpretation it is straightforward to see that

$$\mathcal{B}_1 \sim \frac{K}{m^2}(\nabla\theta)^2 \left\langle(\nabla\theta)\right\rangle^2 \tag{1.204}$$

where the propagator in momentum space is given by

$$\langle\theta(\Omega_n, \mathbf{k})\theta(-\Omega_n, -\mathbf{k})\rangle = \frac{1}{K\Omega_n^2 + J\mathbf{k}^2} \tag{1.205}$$

and

$$\mathcal{B}_2, \mathcal{B}_3 \sim \frac{K^2}{m^2}(\nabla\theta)^2 \left\langle (\partial_\tau\theta)^2 (\nabla\theta)^2 \right\rangle \tag{1.206}$$

Performing all the tedious calculations which are similar to the one previously performed one obtains [5]

$$\delta K_B = \frac{\hbar^4}{8m^2}\frac{1}{2L^2}\sum_{\mathbf{k}} \mathbf{k}^2 n'_B(\hbar\omega_{\mathbf{k}}) \tag{1.207}$$

where $n_B(z)$ is the Bose-Einstein distribution and finally one can appreciates the depletion of the superfluid density caused by bosonic collective excitations with dispersion relation

$$\omega_{\mathbf{k}} = \sqrt{\frac{J}{K}}|\mathbf{k}| \tag{1.208}$$

which are crucial if one wishes to study the BCS-BEC crossover.
For our purposes it serves to show that the superfluid density can be depleted also by a bosonic contribution associated with the formation of bosonic dimers, from (1.19)

$$n_s(T) = \frac{1}{4L^2}\sum_{\mathbf{k}} \left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\tanh\left(\frac{\beta E_{\mathbf{k}}}{2}\right) - \frac{\hbar^2\mathbf{k}^2}{4m} n'_F(\beta E_{\mathbf{k}}) - \frac{\hbar^2\mathbf{k}^2}{4m} n'_B(\hbar\omega_{\mathbf{k}})\right) \tag{1.209}$$

Up until now, we made are calculations trying to extract the Goldstone action of our system, knowing well that it is a crucial procedure to study superfluidity.
We now face the problem of working in two spatial dimension, where, as pointed out early, no true long range order can rises.
With this in mind, we would like to understand which kind of phase-configurations will be dominant in our system and we will approach this problem taking a little step back, returning to the Bosonic theory of the first section.

1.8 Vortex solution

Stemming from the similarity of the Goldstone effective field theory between a purely bosonic system (1.18) and a fermionic one (1.189) we can study the dominant phase configurations that will prevent the $U(1)$ to be broken in the traditional sense from one of them, knowing that the same solutions will hold for the other one.
In the present treatment,we choose to consider the saddle point equation (a.k.a. Gross-Pitaevskii equation [25][44]) associated with the bosonic action introduced at the beginning of this project

$$-\hbar\frac{\partial\psi(\mathbf{r},\tau)}{\partial\tau} = \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi(\mathbf{r},\tau) + g|\psi(\mathbf{r})|^2\psi(\mathbf{r},\tau) \tag{1.210}$$

This equation admits particular static solutions $\hbar\frac{\partial\psi(\mathbf{r},\tau)}{\partial\tau} = 0$ which satisfy

$$\left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi(\mathbf{r}) + g|\psi(\mathbf{r})|^2\psi(\mathbf{r}) = 0 \tag{1.211}$$

with $\psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}e^{i\theta(\mathbf{r})}$ being a solution of this second-order(with respect to spatial coordinates), expressed through the Madelung trasformation.

[5] Aside from the fact that in these ones, being two loops diagrams, one usually sends the internal lines momentum to zero compatible a long wavelengths approximation

Such a solution has to be a smooth function of r, meaning that the gradient of the phase $\theta(r)$ of such field is a well defined function of r at any point where $|\psi(\mathbf{r})| \neq 0$.
[49] As opposed to its gradient the field $\theta(\mathbf{r})$ itself is defined only up to a multiple of 2π ; as a result it can feature topological vortex defects (vortices).
Vortices are thus characterized by a set of points,at which $\theta(r)$ is fundamentally ill defined, such that, as mentioned before the circuitation of $\nabla\theta(\mathbf{r})$ around any closed loop $\mathcal{C}$ that enclose the vortex, is proportional to a non-zero winding number q :

$$\Delta\theta = \oint_{\mathcal{C}} \nabla\theta \cdot d\mathbf{l} = 2\pi q \quad q \in \mathbb{Z} \tag{1.212}$$

For vortex defects to have finite energy, the modulus of our solution has to be 0 at the point of $\nabla\theta(r)$ singularity ; the winding number q is called topological charge of the vortex and is independent of the shape of $\mathcal{C}$ by continuity, being itself an integer.
Note that by the same argument, vortices in 3D cannot have loose ends i.e. they have to be closed loops or terminate at the boundaries of our system.
It will be thus used the term "lines" for vortices in 3D and "points" for those in 2D.
Let's study a point-vortex solution which,in polar coordinates, (ρ, ϕ), are characterized by a phase of the form:

$$\theta(\rho, \phi) = q\ \phi \quad q \in \mathbb{Z} \tag{1.213}$$

that is

$$\psi(\rho, \phi) = \sqrt{n(\rho, \phi)}e^{iq\phi} \quad q \in \mathbb{Z} \tag{1.214}$$

being ϕ the azimuthal angle and ρ the distance from the origin (where the vortex is placed).
Let's suppose for the sake of simplicity that the topological charge is $q = 1$ and that the density profile has radial symmetry i.e. it does not depend of ϕ ; with these assumption our single-vortex solution is:

$$\psi(\rho) = \sqrt{n(\rho)}e^{i\phi} \tag{1.215}$$

For the reasons stated before, the solution is not diverging at $\rho = 0$, implying:

$$n(\rho) \approx \rho^2 \quad as \quad \rho \to 0 \tag{1.216}$$

Where instead we can expect for $n(\rho)$ to approach the uniform value away from the vortex:

$$n(\rho) \approx n_s = \frac{N}{V} \quad as \quad \rho \to \infty \tag{1.217}$$

By dimensional argument, we have that the size of the vortex core, that is the region for which the density $n(\mathbf{r})$ is significantly different from the uniform value is of the order of

$$\xi = \sqrt{\frac{\hbar^2}{2m\mu}} \tag{1.218}$$

called healing length, where μ is fixed by a minimum condition away from the vortex.
For this reason let's parametrize our solution as

$$\psi(\rho, \phi) = \sqrt{n_s}\ f(x)e^{i\phi} \tag{1.219}$$

with $x = \frac{\rho}{\xi}$; plugging the latter in (1.210) we obtain:

$$\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} - 2\left(f^2 - 1 + \frac{1}{2x}\right) f = 0 \tag{1.220}$$

with boundary conditions

$$\begin{aligned} f(0) &= 0 \\ f(\infty) &= 1 \end{aligned} \tag{1.221}$$

Studying the previous equation for density profile near the vortex, that is for distances $\rho \ll \xi$, we find that the term $\frac{1}{2x}$ is much larger than the other one in the parenthesis, yielding the following approximated equation:

$$x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx} - f = 0 \tag{1.222}$$

whose solution, keeping in mind that $f(0) = 0$, is $f \propto x$.
To find the asymptotic behavior in the opposite limit $\rho \gg \xi$ we take the ansatz $f = 1 + g$ with $g = o(\frac{1}{x})$ to satisfy the boundary condition at ∞.
In this way one obtains:

$$\frac{d^2 g}{dx^2} + \frac{1}{x}\frac{dg}{dx} - 2\left(g^2 + 2g + \frac{1}{2x}\right)(1+g) = 0 \tag{1.223}$$

noting that

$$\frac{d^2 g}{dx^2} \sim \frac{1}{x}\frac{dg}{dx} \sim o(\frac{1}{x^3}) \tag{1.224}$$

at the leading order in $o(\frac{1}{x})$ we find

$$g = -\frac{1}{4x^2} \tag{1.225}$$

showing indeed that our solution approaches the equilibrium one following a power law:

$$f(x) \approx 1 - \frac{1}{4x} \quad x \gg 1 \tag{1.226}$$

As for the phase, we have, in cilindrical coordinates:

$$\begin{aligned} &\theta(\rho, \phi) = \phi \\ &\nabla\theta = \frac{1}{\rho}\frac{\partial}{\partial\phi}\theta \;\; e_\phi = \frac{1}{\rho} \;\; \mathbf{e}_\phi \end{aligned} \tag{1.227}$$

so that (18) become:

$$\oint_{\mathcal{C}} \nabla\theta \cdot d\mathbf{l} = 2\pi \tag{1.228}$$

where $\mathcal{C}$ is a loop that goes around $\rho = 0$.
In Fig(1.13) are plotted profiles density for different values of winding number q: Note that, for general q, the profile density approaches the core as $f(x) \sim x^q$, which indeed solve (1.222) where the last term gets modified due to $q \neq 1$ and becomes $-q^2 f(x)$.

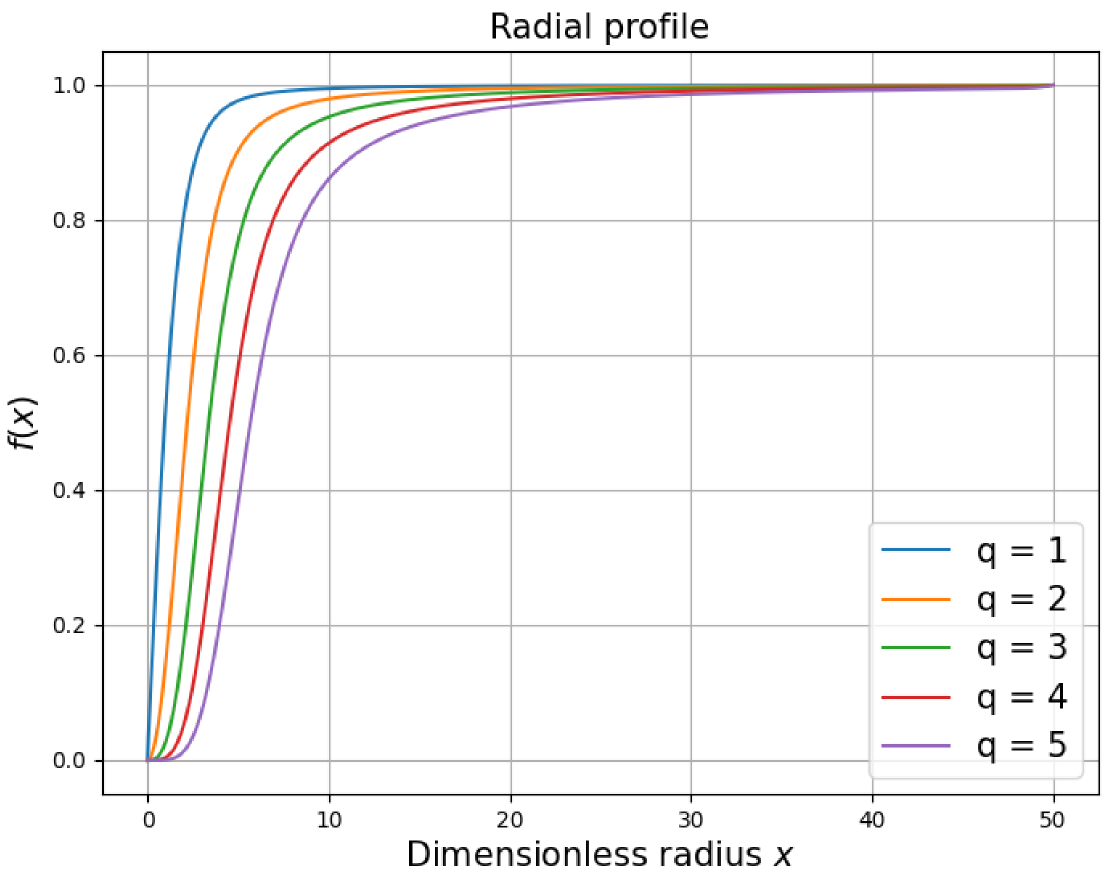

Figure 1.13: Profiles density $f(x)$ for different values of q

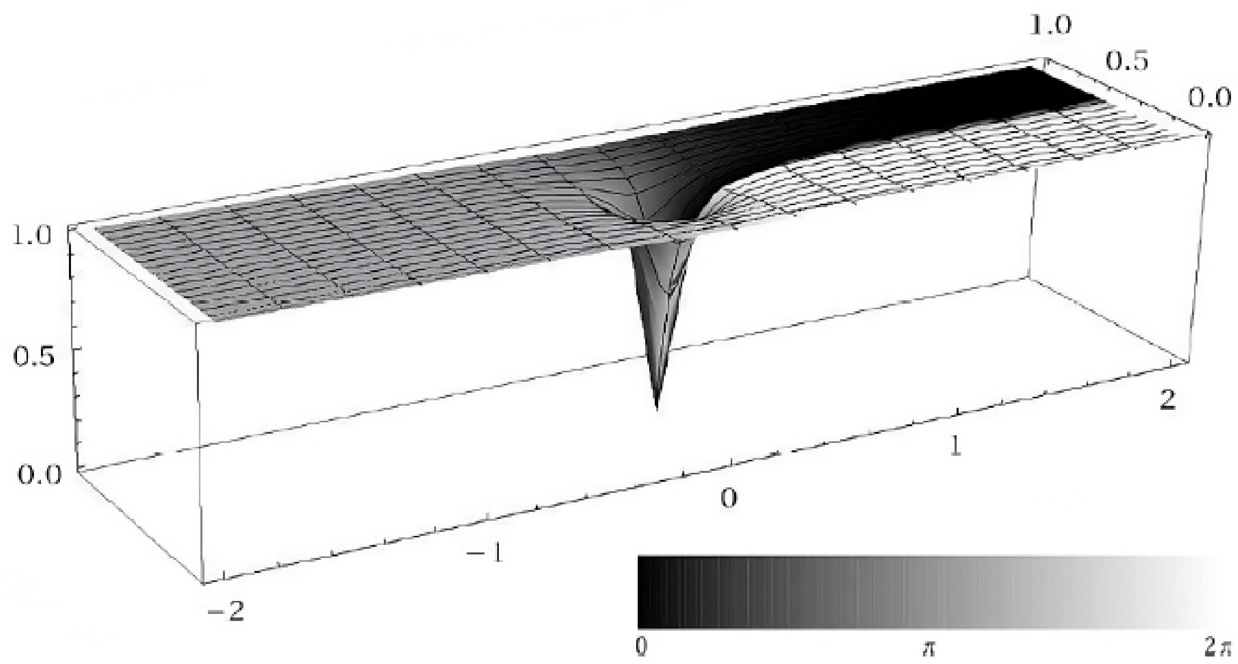

Figure 1.14: Illustration of a single vortex on a superfluid strip. The vertical axis shows the amplitude of the wavefunction $\psi(\mathbf{r})$, while the grayscale indicates the phase profile. Although the phase has a cut in the plane, the overall wavefunction is not multi valued. The amplitude variation is concentrated in the core, which we can ignore by imposing a cutoff ξ Image adapted from

Vortex core energy

We are now in position to calculate the partition function

$$\mathcal{Z} = \int \mathcal{D}\left[\psi, \psi^*\right] e^{-\frac{1}{\hbar}S_E[\psi^*,\psi]} \approx e^{-\frac{1}{\hbar}S_E[\psi_c^*,\psi_c]} \tag{1.229}$$

with

$$S_E[\psi^*, \psi] = \int_0^{\beta\hbar} d\tau \int_V d^3\mathbf{r}\ \hbar\ \psi^*(\mathbf{r},\tau)\frac{\partial\psi(\mathbf{r},\tau)}{\partial\tau} + \frac{\hbar^2}{2m}|\nabla\psi(\mathbf{r},\tau)|^2 - \mu|\psi(\mathbf{r},\tau)|^2 + \frac{g}{2}|\psi(\mathbf{r},\tau)|^2 \tag{1.230}$$

Now through the saddle point approximation we compute S_E for $\psi_c(\mathbf{r},\tau)$ of the form (in polar coordinates) given in (1.219) which describes as before discussed a (τ independent) vortex solution centered at $\rho = 0$ and with winding number $q = 1$, yielding:

$$S_E = 2\pi\beta\hbar \int d\rho\ \rho \left[\frac{\hbar^2}{2m}n_s\left(\left(\frac{df}{d\rho}\right)^2 + \frac{f^2}{\rho^2}\right) - \mu\ n_s f^2 + \frac{g}{2}n_s^2 f^4\right] \tag{1.231}$$

now through the matching :

$$e^{-\frac{S_E}{\hbar}} = e^{-\beta\Omega} \tag{1.232}$$

we extrapolate the grand potential of the given vortex

$$\Omega(\mu, g) = 2\pi \int d\rho\ \rho \left[\frac{\hbar^2}{2m}n_s\left(\left(\frac{df}{d\rho}\right)^2 + \frac{f^2}{\rho^2}\right) - \mu\ n_s f^2 + \frac{g}{2}n_s^2 f^4\right] \tag{1.233}$$

Now we see that this grand potential describe the feature of the vortex solution of being an excited solution of our classical equations of motion, in particular far away from the vortex, according also to the boundary conditions (1.221), this potential describes also the thermodynamic properties of our uniform solution $n_s = \frac{N}{V}$. In other words, Ω has two contributions:

$$\Omega = \Omega_{bulk} + \Omega_{vortex} \tag{1.234}$$

where

$$\Omega_{bulk} = 2\pi \int d\rho\ \rho \left[-\mu\ n_s + \frac{g}{2}n_s^2\right] \tag{1.235}$$

and, through a minimization condition

$$\frac{\delta \Omega_{bulk}}{\delta n_s} = 0 \tag{1.236}$$

we obtain

$$\mu = g n_s \tag{1.237}$$

thus yielding

$$\Omega_{bulk} = 2\pi \int d\rho\ \rho \left(-\frac{g}{2}n_s^2\right) \tag{1.238}$$

Conversely, the single vortex contribution to the thermodynamic potential reads

$$\Omega_{vortex} = \Omega - \Omega_{bulk} = 2\pi \int d\rho\ \rho \left[\frac{\hbar^2}{2m}n_s\left(\left(\frac{df}{d\rho}\right)^2 + \frac{f^2}{\rho^2}\right) + \frac{g n_s^2}{2}(1-f^2)^2\right] \tag{1.239}$$

Performing now the change of variable

$$\begin{aligned} x &= \frac{\rho}{\xi} \\ \xi &= \sqrt{\frac{\hbar^2}{2\mu m}} = \sqrt{\frac{\hbar^2}{2 g n_s^2 m}} \end{aligned} \tag{1.240}$$

we finally obtain the thermodynamic potential of a vortex in terms of a dimensionless integral

$$\Omega_{vortex} = \Omega - \Omega_{bulk} = \pi\ n_s \frac{\hbar^2}{2m} \int dx\ x \left[2\left(\frac{df}{dx}\right)^2 + 2\frac{f^2}{x^2} + (1-f^2)^2\right] \tag{1.241}$$

Notice that the integral is divergent in the infrared limit due to the second term in the brackets which scales as

$$\Omega_{div} = \pi \frac{\hbar^2 n_s}{m} \log\left(\frac{R}{\xi}\right) \tag{1.242}$$

where we introduced R as an infrared cutoff describing the size of our system. On the other hand, in the ultraviolet limit, the former integral converges due to the boundary condition $f(0)$ as in (1.222), and we can identify the core free energy component of the vortex as

$$E_c = \Omega_{vortex} - \Omega_{div} = \frac{\pi \hbar^2 n_s}{2m} \int_0^\infty dx\ x \left[2\left(\frac{df}{dx}\right)^2 + (1-f^2)^2\right] \tag{1.243}$$

Note that, thanks to the subtraction, the integral can be extended over all possible (positive) values of x although it takes its major contribution for $x \lesssim 1$, just as what one should expect, and its values can be computed, plugging in the numerical result of $f(x)$ which satisfy (1.220) yielding

$$E_c = \Omega_{vortex} - \Omega_{div} = \Omega - \Omega_{bulk} - \Omega_{div} \simeq 1.76\ \frac{\pi \hbar^2 n_s}{2m} \tag{1.244}$$

Which slightly differs from the value 1.56 obtained in [39]. To conclude, in the following we numerically evaluate the core energy for general winding number q, remembering that the dependence on the winding number comes from solving the differential equation (1.220) for general winding number q, the results are shown in Figure (1.15)

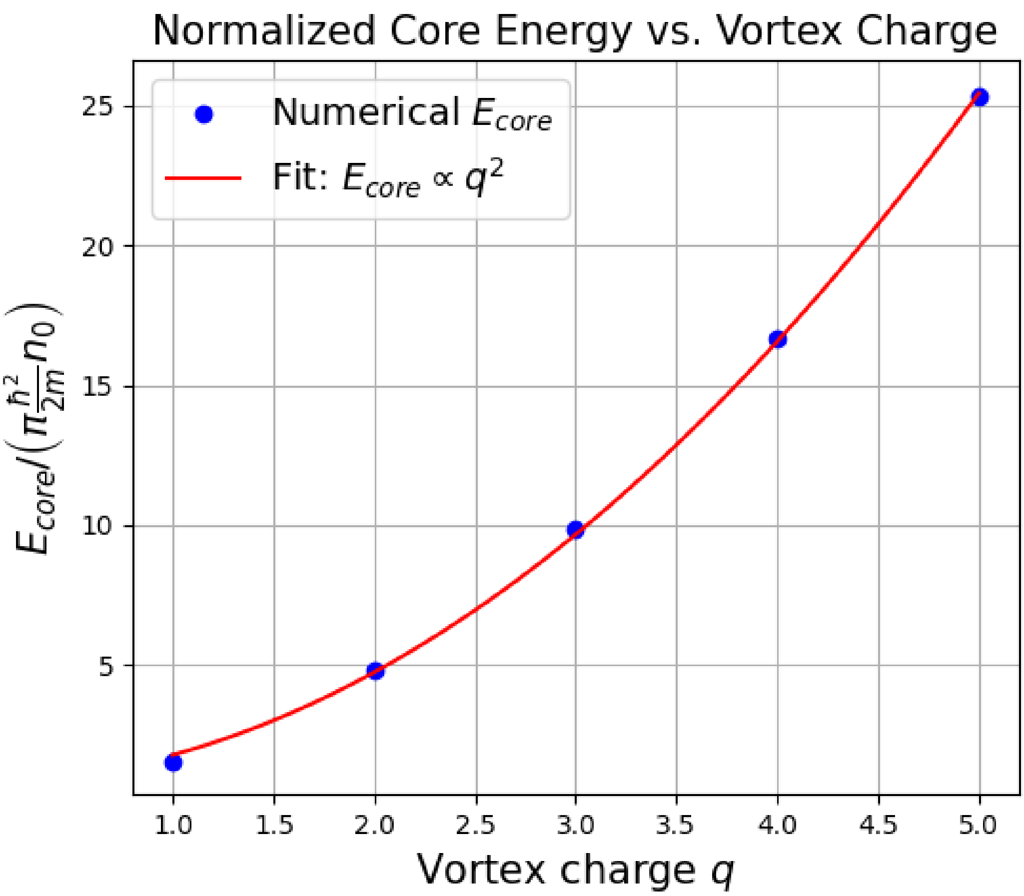

Figure 1.15: E_c vs q

In the figure we can appreciate the quadratic dependence $\propto q^2$ which will be crucial later on in the discussion of an ensemble of point-vortex defects.

Single vortex partition function

Let us now point out that in the general case of a single vortex solution placed at $\mathbf{r} = \mathbf{r}_1$ and with winding number q_1, the approximation (1.229) becomes

$$\mathcal{Z} = \int \mathcal{D}\left[\psi, \psi^*\right] e^{-\frac{1}{\hbar} S_E[\psi^*, \psi]} \approx \sum_{\mathbf{r}_i, q_i} e^{-\frac{1}{\hbar} S_E[\psi_c^*, \psi_c]} = \sum_{\mathbf{r}_1, q_1} e^{-\beta \Omega^1_{vortex}} \tag{1.245}$$

where

$$\Omega^1_{vortex} = \Omega^1 - \Omega^1_{bulk} = \pi\, n_s \frac{\hbar^2}{2m} \int dx\; x \left[2\left(\frac{df(q_1)}{dx}\right)^2 + 2q_i^2 \frac{f(q_1)^2}{x^2} + (1 - f(q_1)^2)^2 \right] \tag{1.246}$$

with apex "1" stemming for the single vortex in our system and

$$x = \frac{\rho - \rho_1}{\xi} \tag{1.247}$$

emphasizing that f depends on q_i through the GPE.
Actually, using (1.243), the former partition function becomes a particular case of the gran canonical partition function with $N = 1$:

$$\mathcal{Z} = \sum_{\mathbf{r}_1, q_1} e^{-\beta(E_c^1 + \Omega_{div}^1)} = \sum_{\mathbf{r}_1, q_1} z^1 \mathcal{Z}_1 \tag{1.248}$$

with

$$z = e^{-\beta E_c^1} \tag{1.249}$$

the fugacity of our vortex and

$$\mathcal{Z}_1 = e^{-\beta \Omega_{div}^1} \tag{1.250}$$

Notice also that, in the thermodynamic limit, $\mathcal{Z}_1$ would be identically 0 due to logarithmically divergent term (1.242), nonetheless, if we naively identify

$$\mathcal{Z} = \sum_{\mathbf{r}_1, q_1} e^{-\beta(E_c + \Omega_{div})} = e^{-\beta F} \tag{1.251}$$

with

$$F = E - TS \tag{1.252}$$

the free energy of our single vortex at finite temperature T, which, through the *sum* over $\mathbf{r_1}$ and considering $q_1 = 1$ because it is the main contribution in the sum, takes into account also the gain in entropy in creating a vortex anywhere in our system. Passing indeed in the continuum

$$\sum_{\mathbf{r}_1} \to \int \frac{d^2\mathbf{r}_1}{\xi^2} \tag{1.253}$$

$$F \simeq \pi \frac{\hbar^2 n_s}{m} \log\left(\frac{R}{\xi}\right) - T\ k_b \log\left(\frac{\pi R^2}{\pi \xi^2}\right) = \left(\pi \frac{\hbar^2 n_s}{m} - 2\ T\ k_b\right) \log\left(\frac{R}{\xi}\right) \tag{1.254}$$

Signaling a particular temperature

$$T_{BKT} = \frac{\pi \hbar^2 n_s}{2 k_b m} \tag{1.255}$$

above which, the creation of a single vortex is actually favored, meaning that adding a vortex in the system will lower the free energy $\Delta F \leq 0$.
The temperature T_{BKT} signals the presence of a particular phase transition that we will discuss in detail later on.
Of course, to inspect the possibility of a larger number of vortices rising due to thermal fluctuations, one has to generalize (1.229) allowing the presence of multiple vortices in the solution of the GPE equation possibly interacting with each other. This will be the purpose of the next chapter.

2 Vortices EFT

In the previous discussion we derived the field configuration together with the corresponding free energy of a single vortex close and far away from the vortex core.
The aim of this section is instead to construct an effective field theory describing an ensemble of interacting (static) vortex defects which are treated as if they were point-like, i.e. we will, at first, neglect the effect related to finite-sized cores.
This can be effectively done by considering the following energy functional:

$$E[\theta] = \frac{\hbar^2 n_s}{2m} \int d^2\mathbf{r}\ (\nabla\theta)^2 \tag{2.1}$$

which can be derived from the kinetic term of (1.230) by setting $\psi(\mathbf{r}) = \sqrt{n_s} e^{i\theta(\mathbf{r})}$.
Notice that when the phase field $\theta(\mathbf{r})$ describes a single vortex as in (1.227) the previous energy function is nothing but the vortex contribution to the energy far away from the core.
In order to generalize the discussion to an arbitrary number of vortices at fixed position, we recall the definition of superfluid velocity :

$$\mathbf{v} = \frac{\hbar}{m}\nabla\theta \tag{2.2}$$

in terms of which the energy functional becomes:

$$E[\mathbf{v}] = \frac{1}{2} m n_o \int d^2\mathbf{r}\ \mathbf{v}^2 \tag{2.3}$$

Now, a generic vector field $\mathbf{v}(\mathbf{r})$ can always be decomposed in the following way:

$$\mathbf{v} = \mathbf{v}_{sw} + \mathbf{v}_v \tag{2.4}$$

such that

$$\begin{aligned} \nabla \times \mathbf{v}_{sw} &= 0 \\ \nabla \cdot \mathbf{v}_v &= 0 \end{aligned} \tag{2.5}$$

thus, if one would like to describe vortices configuration, these have to be necessary related ,see (1.212), to the component $\mathbf{v}_v$; conversely, the component $\mathbf{v}_{sw}$ describes the contribution of the *spin waves-like* thermal fluctuations, making evident the parallelism of our treatment with the low-temperature limit of the XY model.
[10] To employ now the presence of multiple point-vortices placed at $\mathbf{r}_i$ with winding numbers q_i, notice first that a generalization of the single vortex case of winding number q placed at $(0,0)$ [1]

$$\theta_v(\mathbf{r}) = q \arctan\left(\frac{y}{x}\right) \quad q \in \mathbb{Z} \tag{2.6}$$

[1] cfr with (1.213)

is

$$\theta_v(\mathbf{r}) = \sum_i q_i \arctan\left(\frac{y-y_i}{x-x_i}\right) \tag{2.7}$$

Consequently (1.228) becomes :

$$\oint_{\mathcal{C}} \nabla\theta_v \cdot d\mathbf{l} = 2\pi \sum_i q_i \tag{2.8}$$

It is possible to write at this point (2.8) in a local form, recalling that $\mathbf{v} = \frac{\hbar}{m}\nabla\theta$ and making use of Stokes theorem we obtain:

$$\oint_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{l} = \oint_{\mathcal{C}} \mathbf{v}_v \cdot d\mathbf{l} = \frac{2\pi\hbar}{m} \sum_i q_i \tag{2.9}$$

$$\int_{\mathcal{A}} \nabla \times \mathbf{v} \cdot d\mathbf{A} = \int_{\mathcal{A}} \nabla \times \mathbf{v}_v \cdot d\mathbf{A} = \frac{2\pi\hbar}{m} \sum_i q_i \tag{2.10}$$

$$\begin{aligned} \nabla \times \mathbf{v}_v &= \frac{2\pi\hbar}{m} \sum_i q_i \, \delta^2(\mathbf{r} - \mathbf{r_i})\mathbf{u}_z \\ \mathbf{r_i} &= (x_i, y_i) \end{aligned} \tag{2.11}$$

where $\mathcal{A}$ is a surface such that $\partial\mathcal{A} = \mathcal{C}$ is a path enclosing all the vortices and $\mathbf{u}_z$ is the unit vector along the z-direction.
The quantity $\mathbf{w} := \nabla \times \mathbf{v}_v$ it is called vorticity and (2.9) states that it is concentrated on the vortices.
It is now possible to solve the second equation in (2.5) and (2.11) using an electro-static analogy. Indeed, the equations

$$\begin{aligned} \nabla \cdot \mathbf{v}_v &= 0 \\ \nabla \times \mathbf{v}_v &= \frac{2\pi\hbar}{m} \sum_i q_i \, \delta^2(\mathbf{r} - \mathbf{r_i})\mathbf{u}_z \end{aligned} \tag{2.12}$$

can be put in a more comfortable form setting:

$$\mathbf{v}_v = \mathbf{u}_z \times \mathcal{E} \tag{2.13}$$

where $\mathcal{E}$ is an electric-like field; with this definition (2.12) become:

$$\nabla \cdot \mathbf{v}_v = -(\nabla \times \mathcal{E}) \cdot \mathbf{u}_z = 0 \tag{2.14}$$

$$\nabla \times \mathbf{v}_v = \nabla \cdot \mathcal{E} \; \mathbf{u}_z = \frac{2\pi\hbar}{m}\rho(\mathbf{r}) \; \mathbf{u}_z \tag{2.15}$$

where we introduced the quantity:

$$\rho(\mathbf{r}) = \sum_i q_i \delta^2(\mathbf{r} - \mathbf{r_i}) \tag{2.16}$$

The term *electric-like* is justified by the fact that trough the identifications:

$$\frac{2\pi\hbar}{m} \leftrightarrow \frac{1}{\epsilon_s} \tag{2.17}$$

$$q_i \leftrightarrow q_i e \tag{2.18}$$

we can map our problem into the problem of find the electrostatic field $\mathcal{E}$ generated by a charge distribution $\rho(\mathbf{r})$.
Equivalently, one can see that equations (2.14) and (2.15) could be mapped into the magnetostatic problem of finding the magnetic field $\mathbf{B}$ generated by an electric current $\mathbf{j}(\mathbf{r})$ through the identifications:

$$\begin{aligned} &\mathbf{v}_v \leftrightarrow \mathbf{B} \\ &\frac{2\pi\hbar}{m} \leftrightarrow \mu_0 \\ &\sum_i q_i\ \delta^2(\mathbf{r}-\mathbf{r_i})\mathbf{u}_z \leftrightarrow \mathbf{j}(\mathbf{r}) \end{aligned} \tag{2.19}$$

We will nonetheless stick with the electrostatic analogy because in the present case, working in 2D, our vortices are point-like, making so easier the parallelism with electric point charges.
In dimensions > 2 , the shape of the defects solutions will be that of a closed loop, making the analogy of the magnetostatic more comfortable.

In the setting of the electrostatic analogy,using equation (2.14),it is possible to rewrite our field in terms of a psedo-electrostatic potential $V(\mathbf{r})$ as

$$\mathcal{E} = -\nabla V \tag{2.20}$$

and inserting this definition into (2.15) we find the following Poisson-like equation:

$$\nabla^2 V(\mathbf{r}) = -\frac{2\pi\hbar}{m}\rho(\mathbf{r}) \tag{2.21}$$

with solution

$$V(\mathbf{r}) = \frac{2\pi\hbar}{m}\int d^2\mathbf{r}\ G(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}') \tag{2.22}$$

where $G(\mathbf{r})$ is the 2-dimensional Green's function satisfying

$$\nabla^2 G(\mathbf{r}) = -\delta^2(\mathbf{r}) \tag{2.23}$$

The formal solution of (2.23) is

$$G(\mathbf{r}) = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{\mathbf{k}^2} = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi \int dk \frac{e^{ikr\cos(\phi)}}{k} \tag{2.24}$$

Now we address the criticalities of such Green's function; first, notice that it has both ultraviolet ($|\mathbf{k}| \to +\infty$) and infrared ($|\mathbf{k}| \to 0$) divergencies.
To handle the first one, we impose a cutoff on the oscillation modes $k_{cutoff} \sim \frac{1}{\xi}$ with ξ given in (1.218) that will serves as upper bound of the second integral in (2.23), conversely the infrared divergence is cured subtracting the contribution at $|\mathbf{r}| = 0$.
With these adjustments (2.23) becomes:

$$G(\mathbf{r}) - G(0) = \frac{1}{(2\pi)^2}\int_0^{2\pi} d\phi \int_0^{\frac{1}{\xi}} dk \frac{e^{ikr\cos(\phi)}-1}{k} = \frac{1}{2\pi}\int_0^{\frac{1}{\xi}} \frac{dk}{k}(J_0(kr)-1) \tag{2.25}$$

where $J_0(kr)$ is the zeroth Bessel function.
Writing the integral in terms of the dimensionless variable $x = kr$ we finally obtain [47]:

$$G(\mathbf{r}) - G(0) = \frac{1}{2\pi}\left(\int_0^1 dx\ \frac{J_0(x)-1}{x}\right) + \frac{1}{2\pi}\left(\int_1^{\frac{r}{\xi}} dx \frac{J_0(x)-1}{x}\right) \tag{2.26}$$

2.1 Vortices energy

Considering now the energy functional (2.3), using (2.5) we obtain:

$$E[\mathbf{v_{sw}}, \mathbf{v}_v] = \frac{1}{2} m n_s \int d^2\mathbf{r}\ (\mathbf{v}_{sw}^2 + \mathbf{v}_v^2 + 2\mathbf{v}_{sw} \cdot \mathbf{v}_v) \tag{2.27}$$

the mixed term, using the irrotational condition of $\mathbf{v}_{sw}$ in (2.5) and (2.20),become:

$$E_{mix}[\mathbf{v}_{sw}, \mathbf{v}_v] = m n_s \int d^2\mathbf{r}\ \mathbf{v}_{sw} \cdot \mathbf{v}_v = m n_s \int d^2\mathbf{r}\ \nabla\theta_{sw} \cdot \mathbf{v}_v = m n_s (\theta_{sw}\mathbf{v}_v)|_{\partial S} = 0 \tag{2.28}$$

where we used the fact that $\nabla \cdot \mathbf{v}_v = 0$ and the fact that the boundary of our system is, in the thermodynamic limit, at spatial infinity.
So our energy functional is made up of two separate terms:

$$E[\mathbf{v}_{sw}, \mathbf{v}_v] = E_{sw}[\mathbf{v}_{sw}] + E_v[\mathbf{v}_v] \tag{2.29}$$

Considering only the vortices contribution, using the electrostatic analogy above, we have:

$$E_v[\mathbf{v}_v] = \frac{1}{2} m n_s \int d^2\mathbf{r}\ \mathbf{v}_v \cdot \mathbf{v}_v = \frac{1}{2} m n_s \int d^2\mathbf{r}\ \mathcal{E}^2 = -\frac{1}{2} m n_s \int d^2\mathbf{r}\ V(\mathbf{r})\nabla^2 V(\mathbf{r}) \tag{2.30}$$

using (2.21) and (2.22)

$$E_v[\mathbf{v}_v] = \frac{1}{2} m n_s \left(\frac{2\pi\hbar}{m}\right)^2 \int d^2\mathbf{r}\ d^2\mathbf{r}'\ \rho(\mathbf{r}) G(\mathbf{r} - \mathbf{r}') \rho(\mathbf{r}') \tag{2.31}$$

employing (2.16)

$$E_v[\mathbf{v}_v] = \frac{1}{2} m n_s \left(\frac{2\pi\hbar}{m}\right)^2 \sum_{i,j} q_i q_j G(\mathbf{r}_i - \mathbf{r}_j) \tag{2.32}$$

We now have to make use of the regularization (2.26), to this end let's rewrite our energy functional, to do so let's naively add and subtract the (divergent) quantity $\frac{1}{2} m n_s \left(\frac{2\pi\hbar}{m}\right)^2 \left(\sum_i q_i\right)^2 G(0)$:

$$E_v[\mathbf{v}_v] = \frac{1}{2} m n_s \left(\frac{2\pi\hbar}{m}\right)^2 \sum_{i\neq j} q_i q_j (G(\mathbf{r}_i - \mathbf{r}_j) - G(0)) + \frac{1}{2} m n_s \left(\frac{2\pi\hbar}{m}\right)^2 \left(\sum_i q_i\right)^2 G(0) \tag{2.33}$$

Now, if the total winding number of the system $\sum_i q_i$ were not to be 0, we would experience a logarithmic divergent term that would kill the Boltzmann factor $e^{-\beta H_v}$.
To prevent this to happen, we impose an analog of electric neutrality of our system $\sum_i q_i = 0$, leaving us with:

$$E_v[\mathbf{v}_v] = \frac{1}{2} m n_s \left(\frac{2\pi\hbar}{m}\right)^2 \sum_{i\neq j} q_i q_j (G(\mathbf{r}_i - \mathbf{r}_j) - G(0)) \tag{2.34}$$

Before continuing, let's make some comments about the regularization condition(2.26).
it is immediate to see that, the first integral of (2.26) sums up to:

$$\int_0^1 dx\ \frac{J_0(x) - 1}{x} = -\gamma \tag{2.35}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant, while the second integral in the limit $|\mathbf{r}_i - \mathbf{r}_j| \gg \xi$ is:

$$\int_1^{\frac{r}{\xi}} dx\ \frac{J_0(x) - 1}{x} \sim \ln 2 - \int_1^{\frac{r}{\xi}} dx\ \frac{1}{x} \sim -\log\frac{r}{\xi} \tag{2.36}$$

And it is really evident that once inserted this results into (2.34) we obtain

$$E_v[\mathbf{v}_v] = -\frac{1}{2}mn_s\left(\frac{2\pi\hbar}{m}\right)^2\sum_{i\neq j} q_iq_j\frac{\gamma}{2\pi} - \frac{1}{2}mn_s\left(\frac{2\pi\hbar}{m}\right)^2\sum_{i\neq j} q_iq_j\ \frac{1}{2\pi}\log\left(\frac{|\mathbf{r}_i-\mathbf{r}_j|}{\xi}\right) \tag{2.37}$$

Using now the neutrality condition $\sum_{i\neq j} q_iq_j = -\sum_i q_i^2$ we're finally left with:

$$E_v[\mathbf{v}_v] = +\gamma\frac{\pi\hbar^2 n_s}{m}\sum_i q_i^2 - \frac{1}{2}\frac{2\pi\hbar^2 n_s}{m}\sum_{i\neq j} q_iq_j\log\left(\frac{|\mathbf{r}_i-\mathbf{r}_j|}{\xi}\right) \tag{2.38}$$

Where we factored out $\frac{1}{2}$ to acknowledge the double counting in the summation.
Identifying

$$\mu_v \equiv \gamma\frac{\pi\hbar^2 n_s}{m} \tag{2.39}$$

with the chemical potential of the vortex i.e. the energy required to add a vortex in our system, we are left with

$$\begin{aligned} E_v[\mathbf{v}_v] &= \mu_v\sum_i q_i^2 - \frac{1}{2}\frac{2\pi\hbar^2 n_s}{m}\sum_{i\neq j} q_iq_j\log\left(\frac{|\mathbf{r}_i-\mathbf{r}_j|}{\xi}\right) \\ &= \mu_v\sum_i q_i^2 + \frac{1}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i\neq j} q_iq_j\ G_{\text{reg}}(\mathbf{r}_i-\mathbf{r}_j) \end{aligned} \tag{2.40}$$

We can appreciate that $\mu_v \simeq E_c(q=1)$ where E_c is the vortex core energy in (1.243) which indeed in principle should depend only on a non-electrostatic contribution. Finally, we recover the quadratic dependence on the winding number q for *each* vortex, which was previously shown numerically in (1.15).

The XY model case

To conclude, the regularization condition (2.26) is not suitable for every system at study.
For example in system like the XY model,where we deal with a large number of spins attached at points on a 2D lattice with lattice constant ξ and whose interaction is ferromagnetic-like:

$$H = -J\sum_{i,j}\cos(\theta_i-\theta_j) \tag{2.41}$$

where $J > 0$ and θ_i represent the angle formed by the spin on the i-th site with the x-axis and the sum over spins is performed with the nearest neighbor. The partition function of this system is given by

$$\mathcal{Z}_{XY} = \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_N \quad e^{-\beta H} \tag{2.42}$$

Now, at very low temperature, we can expect the spins to fluctuate very little with respect to the ground state in which all spins are aligned in the same direction; consequently

$$\theta_i - \theta_j \simeq \delta\theta_i \tag{2.43}$$

where i, j are n.n, so that

$$\cos(\delta\theta_i) \simeq 1 - \frac{1}{2}(\delta\theta(\mathbf{r}_i))^2 \tag{2.44}$$

and we understand that, in the continuum limit, we recover, using

$$(\delta\theta(\mathbf{r_i}))^2 \approx a^2 \left(\frac{\partial\theta(\mathbf{r}_i)}{\partial\hat{l}}\right)^2 \tag{2.45}$$

with $\hat{l} = x, y$

$$H \simeq \frac{J}{2}\int d^2\mathbf{r}\ (\nabla\theta(\mathbf{r}))^2 \tag{2.46}$$

which has the same form of (2.1).
The core difference of the XY model with our model of a superfluid is that in the former case the theory is actually discretized, meaning again that the spins are placed on the sites of a square lattice of spacing ξ, as a consequence one has to discretize the first integral in (2.26) in order to properly define the energy required to create a vortex at
scales $\sim \xi$, i.e. at $x \sim 1$ so that the result differs because the Fourier transform of the Green function $G(\mathbf{k})$ is no more $\frac{1}{\mathbf{k}^2}$ as in 2.23 but becomes:

$$G(\mathbf{k}) = \frac{\xi^2}{4 - 2\cos(k_x\xi) - 2\cos(k_y\xi)} \tag{2.47}$$

so that the *core contribution* in (2.26) changes:

$$\begin{aligned} G(|\mathbf{r}| = \xi) - G(0) &= \int_{|\mathbf{k}|\leq\frac{1}{\xi}} \frac{d^2\mathbf{k}}{(2\pi)^2}\ (e^{i\mathbf{k}\cdot\mathbf{r}} - 1)G(\mathbf{k}) \\ &= \int_{|\mathbf{k}|\leq\frac{1}{\xi}} \frac{d^2\mathbf{k}}{(2\pi)^2}\ \frac{(e^{i\mathbf{k}\cdot\mathbf{r}} - 1)\xi^2}{4 - 2\cos(k_x\xi) - 2\cos(k_y\xi)} \end{aligned} \tag{2.48}$$

Now exploiting the fact that

$$\text{Im}(G(|\mathbf{r}| = \xi) - G(0)) = \int_{|\mathbf{k}|\leq\frac{1}{\xi}} \frac{d^2\mathbf{k}}{(2\pi)^2}\ \frac{i\sin(\mathbf{k}\cdot\mathbf{r})\xi^2}{4 - 2\cos(k_x\xi) - 2\cos(k_y\xi)} = 0 \tag{2.49}$$

being $\sin(\mathbf{k}\cdot\mathbf{r})$ odd, we're left with:

$$G(|\mathbf{r}| = \xi) - G(0) = \int_{|\mathbf{k}|\leq\frac{1}{\xi}} \frac{d^2\mathbf{k}}{(2\pi)^2}\ \frac{(\cos(\mathbf{k}\cdot\mathbf{r}) - 1)\xi^2}{4 - 2\cos(k_x\xi) - 2\cos(k_y\xi)} \tag{2.50}$$

Remembering now that our Green's function depends only on $|\mathbf{r}| = \xi$ we can write:

$$G(|\mathbf{r}| = \xi) = \frac{1}{2}\left(G(\xi\mathbf{u}_x) + G(\xi\mathbf{u}_y)\right) \tag{2.51}$$

obatining:

$$G(|\mathbf{r}| = \xi) - G(0) = \frac{1}{2}\int_{|\mathbf{k}|\leq\frac{1}{\xi}} \frac{d^2\mathbf{k}}{(2\pi)^2}\ \frac{(\cos(k_x\xi) + \cos(k_y\xi) - 2)\xi^2}{4 - 2\cos(k_x\xi) - 2\cos(k_y\xi)} = -\frac{1}{4} \tag{2.52}$$

Resulting in a difference between the chemical potential of a vortex μ_v (2.39) in a superfluid system that amounts to[10]:

$$\mu_{XY} = \frac{\pi}{2\gamma}\mu_v \tag{2.53}$$

2.2 Gran canonical ensemble of vortices

Analogously to the single vortex case (1.245), canonical partition function of an ensemble of M vortices with fixed charges $q_1, q_2 \dots q_M$ such that, see (2.33), $\sum_{i=1}^{M} q_i = 0$, generalizing as in (1.245) the saddle point condition, becomes:

$$\mathcal{Z} = \sum_{M=0}^{\infty} \sum_{\mathbf{r}_1,\mathbf{r}_2,\dots\mathbf{r}_M} \frac{1}{\prod_{\alpha=1}^{M} N_{q_\alpha}!} e^{-\beta E_v} = \sum_{M=0}^{\infty} \sum_{\mathbf{r_1},\mathbf{r_2},\dots\mathbf{r_M}} \frac{1}{\prod_{\alpha=1}^{M} N_{q_\alpha}!} z^{\sum_{i=1}^{M} q_i^2} e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i\neq j} q_i q_j G_{\text{reg}}(\mathbf{r}_i-\mathbf{r}_j)} \tag{2.54}$$

where $z = e^{-\beta\mu_v}$ is the fugacity of our vortex and $N_{q_\alpha}!$ stems for the combinations of vortices with same charge q_α. Dividing by these combinatorial factors prevents us by over counting the configurations through the *sum* over the positions $\mathbf{r_1}, \mathbf{r}_2, \dots \mathbf{r_M}$, being the configuration in which a vortex with charge q_1 is in $\mathbf{r_1}$ and an another one with the same charge in $\mathbf{r}_2$ indistinguishable from the obtained through the swapping of the vortices.[2]
At this point we note that, being the core energy $\propto q_i^2$ and exploiting the astonishing fact that, for $A, B \geq 0$

$$(A+B)^2 \geq A^2 + B^2 \tag{2.55}$$

a configuration with charges $|q_i| = 1$ would be thermically more favorable than a configuration with charges $|q_i| > 1$ so that in (2.54) we can restrict ourselves to the configuration with $\frac{M}{2}$ vortices with $q = +1$ and $\frac{M}{2}$ vortices with $q = -1$, in order to respect also the neutrality condition.
We also stress the fact that, thanks to (2.33), a non-neutral configuration would give a 0 contribution to the partition function in the thermodynamic limit.
With these adjustments, (2.54) becomes

$$\mathcal{Z} = \sum_{M=0}^{\infty} \sum_{\mathbf{r}_1,\mathbf{r}_2,\dots\mathbf{r}_M} \frac{1}{(\frac{M}{2})!(\frac{M}{2})!} e^{-\beta E_v} = \sum_{M=0}^{\infty} \sum_{\mathbf{r_1},\mathbf{r_2},\dots\mathbf{r_M}} \frac{1}{(\frac{M}{2})!(\frac{M}{2})!} z^M e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i\neq j} \epsilon_i\epsilon_j G_{\text{reg}}(\mathbf{r}_i-\mathbf{r}_j)} \tag{2.56}$$

with $\epsilon_1 = \epsilon_2 = \dots \epsilon_{\frac{M}{2}} = +1$ and $\epsilon_{\frac{M}{2}+1} = \epsilon_{\frac{M}{2}+2} = \dots \epsilon_M = -1$.
We thus recognize the gran canonical form of our $\mathcal{Z}$

$$\mathcal{Z} = \sum_{M=0}^{\infty} z^M \mathcal{Z}_M \tag{2.57}$$

with

$$\mathcal{Z}_M = \sum_{\mathbf{r_1},\mathbf{r_2},\dots\mathbf{r_M}} \frac{1}{(\frac{M}{2})!(\frac{M}{2})!}\ e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i\neq j} \epsilon_i\epsilon_j G_{\text{reg}}(\mathbf{r}_i-\mathbf{r}_j)} \tag{2.58}$$

Next, realizing that $G_{\text{reg}}(\mathbf{r}) = G(\mathbf{r}) - G(0)$ is nothing else than the (regularized) kernel of the ∇^2 operator in two dimension, we exploit the following identity, where ϕ is a scalar neutral field :

$$e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i\neq j} \epsilon_i\epsilon_j G_{\text{reg}}(\mathbf{r}_i-\mathbf{r}_j)} = \mathcal{N}\int \mathcal{D}\phi\ e^{-(\int d^2\mathbf{r}\ \frac{\mathcal{K}}{2}(\nabla\phi)^2 + \mathcal{J}(\mathbf{r})\phi(\mathbf{r}))} \tag{2.59}$$

with

$$\mathcal{J}(\mathbf{r}) = i\sum_{i=1}^{M} \epsilon_i\ \delta^2(\mathbf{r}-\mathbf{r}_i) \tag{2.60}$$

$$\mathcal{K} = \frac{1}{4\pi^2\beta J} \qquad J = \frac{\hbar^2 n_s}{m} \tag{2.61}$$

[2] They would be distinguishable if the vortices had names like Luca and Paolo or other quantum numbers that could make them distinguishable.

and $\mathcal{N}$ is an irrelevant multiplicative factor.
Indeed, taking the Fourier transform of our fields, the kinetic and the "interaction" term become:

$$\begin{aligned}\int d^2\mathbf{r}\ \frac{\mathcal{K}}{2}(\nabla\phi(\mathbf{r}))^2 &= \frac{1}{2}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\ |\mathbf{k}|^2\phi(-\mathbf{k})\phi(\mathbf{k}) \\ \int d^2\mathbf{r}\ \mathcal{J}(\mathbf{r})\phi(\mathbf{r}) &= \frac{1}{2}\Big(\int d^2\mathbf{k}\ \mathcal{J}(\mathbf{k})\phi(-\mathbf{k}) + \mathcal{J}(-\mathbf{k})\phi(\mathbf{k})\Big)\end{aligned} \tag{2.62}$$

making the change of variable

$$\phi(\mathbf{k}) = \tilde{\phi}(\mathbf{k}) - \frac{\mathcal{J}(\mathbf{k})}{\mathcal{K}|\mathbf{k}|^2} \tag{2.63}$$

we obtain:

$$\int \mathcal{D}\phi\ e^{-(\int d^2\mathbf{r}\ \frac{\mathcal{K}}{2}(\nabla\phi)^2+\mathcal{J}(\mathbf{r})\phi(\mathbf{r}))} = \Big(\int \mathcal{D}\tilde{\phi}e^{-\int d^2\mathbf{r}\ \frac{\mathcal{K}}{2}(\nabla\tilde{\phi})^2}\Big)\ e^{+\frac{1}{2\mathcal{K}}\int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{\mathcal{J}(-\mathbf{k})\mathcal{J}(\mathbf{k})}{|\mathbf{k}|^2}} \tag{2.64}$$

Now, going back to the real 2-dimensional space

$$e^{+\frac{1}{2\mathcal{K}}\int d^2\mathbf{k}\frac{\mathcal{J}(-\mathbf{k})\mathcal{J}(\mathbf{k})}{|\mathbf{k}|^2}} = e^{+\frac{1}{2\mathcal{K}}\int d^2\mathbf{r}\ d^2\mathbf{r}'\mathcal{J}(\mathbf{r})G(\mathbf{r}-\mathbf{r}')\mathcal{J}(\mathbf{r}')} \tag{2.65}$$

where we recognize

$$G(\mathbf{r}-\mathbf{r}') = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\frac{e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{|\mathbf{k}|^2} \tag{2.66}$$

as the kernel of ∇^2 operator in 2-dimension and

$$\int \mathcal{D}\tilde{\phi}\ \ e^{-\int d^2\mathbf{r}\ \frac{\mathcal{K}}{2}(\nabla\tilde{\phi})^2} \equiv \frac{1}{\mathcal{N}} \tag{2.67}$$

is just a multiplicative factor.
Now, choosing $\mathcal{J}(\mathbf{r})$ and $\mathcal{K}$ as in (2.60), (2.61), we end up with:

$$\int \mathcal{D}\phi\ e^{-(\int d^2\mathbf{r}\ \frac{1}{8\pi^2\beta J}(\nabla\phi)^2+i\sum_i \epsilon_i\phi(\mathbf{r}_i))} = \frac{1}{\mathcal{N}}\ e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i,j}\epsilon_i\epsilon_j G(\mathbf{r}_i-\mathbf{r}_j)} \tag{2.68}$$

Finally, exploiting the fact that we have a neutral system, see (2.32),(2.33), we end up with:

$$\frac{1}{\mathcal{N}}\ e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i,j}\epsilon_i\epsilon_j G(\mathbf{r}_i-\mathbf{r}_j)} = \frac{1}{\mathcal{N}}\ e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i,\neq j}\epsilon_i\epsilon_j G_{\text{reg}}(\mathbf{r}_i-\mathbf{r}_j)} \tag{2.69}$$

and finally

$$\int \mathcal{D}\phi\ e^{-(\int d^2\mathbf{r}\ \frac{1}{8\pi^2\beta J}(\nabla\phi)^2+i\sum_i \epsilon_i\phi(\mathbf{r}_i))} = \frac{1}{\mathcal{N}}\ e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i,j}\epsilon_i\epsilon_j G_{\text{reg}}(\mathbf{r}_i-\mathbf{r}_j)} \tag{2.70}$$

Which is exactly the initial statement.
This form of our partition would be handy when we will focus of the RG equations of our model; for the time being, notice that we can make one last time use of the neutrality condition to put the previous Path-integral in a more comfortable form. Following the steps that led to (2.33), we can see that the quantity [3]:

$$\int \mathcal{D}\phi\ e^{-(\int d^2\mathbf{r}\ \frac{1}{8\pi^2\beta J}(\nabla\phi)^2+i\sum_i q_i\ \ \phi(\mathbf{r}_i))} = e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i,j}q_i q_j G(\mathbf{r}_i-\mathbf{r}_j)} \tag{2.71}$$

[3]Note how we did not use subscript "reg" for $G(\mathbf{r}-\mathbf{r}')$

is 0 for a configuration $q_1, q_2, \dots q_M$ such that $\sum_i q_i \neq 0$.
Using this fact, we see that :

$$\int \mathcal{D}\phi \sum_{q_1,q_2,\dots q_M=\pm 1} e^{-(\int d^2\mathbf{r} \; \frac{1}{8\pi^2\beta J}(\nabla\phi)^2 + i\sum_i q_i \; \phi(\mathbf{r}_i))} = \frac{M!}{(\frac{M}{2})!(\frac{M}{2})!} e^{-\frac{\beta}{2}\frac{(2\pi\hbar)^2 n_s}{m}\sum_{i\neq j}\epsilon_i\epsilon_j G_{\mathrm{reg}}(\mathbf{r}_i-\mathbf{r}_j)} \tag{2.72}$$

with $\epsilon_1 = \epsilon_2 = \dots \epsilon_{\frac{M}{2}} = +1$ and $\epsilon_{\frac{M}{2}+1} = \epsilon_{\frac{M}{2}+2} = \dots \epsilon_M = -1$.

Let us stress the fact the in the sum on the LHS, the configurations $q_1, q_2, \dots q_M$ such that $\sum_i q_i \neq 0$, such as, for example the one with $q_1 = q_2 = \dots q_M = +1$, give a 0 contribution; so that on the RHS we only see the contribution of the neutral configuration $\epsilon_1 = \epsilon_2 = \dots \epsilon_{\frac{M}{2}} = +1$ and $\epsilon_{\frac{M}{2}+1} = \epsilon_{\frac{M}{2}+2} = \dots \epsilon_M = -1$ multiplied by the number of possible neutral configurations $\frac{M!}{(\frac{M}{2})!(\frac{M}{2})!}$.

Plugging (2.72) back into (2.56), we obtain, neglecting the multiplicative factor $\mathcal{N}$

$$\mathcal{Z} = \sum_{M=0}^{\infty} \sum_{\mathbf{r}_1,\mathbf{r}_2,\dots\mathbf{r}_M} \int \mathcal{D}\phi \sum_{q_1,q_2,\dots q_M=\pm 1} \frac{1}{M!} \; z^M \; e^{-(\int d^2\mathbf{r} \; \frac{1}{8\pi^2\beta J}(\nabla\phi)^2 + i\sum_i q_i \; \phi(\mathbf{r}_i))} \tag{2.73}$$

Performing the summation over $q_1, q_2, \dots q_M = \pm 1$

and identifying

$$\sum_{\mathbf{r}_1,\mathbf{r}_2,\dots\mathbf{r}_M} \equiv \int \frac{d^2\mathbf{r}_1}{\xi^2} \int \frac{d^2\mathbf{r}_2}{\xi^2} \cdots \int \frac{d^2\mathbf{r}_M}{\xi^2} \tag{2.74}$$

where the integrals region is extended over $|\mathbf{r}_i - \mathbf{r}_j| \geq \xi \;\forall\; i \neq j$, we finally get:

$$\mathcal{Z} = \int \mathcal{D}\phi \; e^{-\int d^2\mathbf{r} \; \left[\frac{1}{8\pi^2\beta J}(\nabla\phi(\mathbf{r}))^2 - \frac{2z}{\xi^2}\cos(\phi(\mathbf{r}))\right]} \tag{2.75}$$

where we used the identity

$$\frac{e^{i\phi} + e^{-i\phi}}{2} = \cos(\phi) \tag{2.76}$$

Notice that incidentally this partition function coincides with the partition function of the so called Sine-Gordon model, proving that the latter, our model for a superfluid and the XY-model, belong to the same universality class.

3 BKT transition

We now face the most striking feature of bidimensional superfluid system: the Berezinski-Thouless-Kosterlitz transition, first studied in [11][33] in 1972 and 1973 respectively. To heuristically understand the rising of this phenomenon, we naively study both the energetic and entropic effect of the presence of a vortex in a bidimensional system and we showed pedagogically (1.254) that, due to the fact that in 2D the energetic cost of the creation of a vortex and the entropic gain have the same logarithmic functional form, a critical temperature T_{BKT} marks the transition between a state in which it is not thermally convenient to add a vortex in the system and a state in which actually we have proliferation of them. To better interpret this result in terms of the partition function derived previously,note that at low temperatures the fugacity z is really small, so the system contains only bound pairs of vortices with opposite topological charge, forming dipoles. As the temperature increases, an increasing number of dipoles form with progressively larger separations. Above a critical temperature, these dipoles dissociate, leading to the emergence of a plasma of free vortices. This transition may be interpreted as a change from a dielectric (or insulating) phase to a conducting (or plasma) phase composed of unbound topological charges (vortices). These two phases exhibit distinct physical properties, enabling their identification—for instance, through the behavior of an external electric field $\mathbf{E}$ within the system. In most superfluid systems it is not always straightforward to define a direct analogue of the electric field; nevertheless, the underlying idea remains applicable. In a dielectric medium, the interaction between two test charges is modified by the presence of the medium, typically described by the introduction of a dielectric constant ϵ, which accounts for polarization effects.

In the following shall consider two test vortices and analyze their interaction in the presence of additional vortices, assumed to be organized into dipolar configurations an doing so,we will demonstrate that the interaction between the test vortices is renormalized, leading to the emergence of an effective stiffness J_{eff}, and that this mechanism underlies the aforementioned phase transition. Alternative derivations are available in the literature.

3.1 Renormalized stiffness

Let us consider two vortices with charge of opposite sign, located at positions $\mathbf{x}_1$ and $\mathbf{x}_2$. We aim to determine the effective interaction between these test charges in the presence of additional vortices situated between them. We will approach this problem perturbatively in the fugacity z, thereby assuming that only a small number of vortices are present in the system. To compute the effective interaction, we first evaluate the partition function associated with the configuration containing the two test vortices, with respect to the configuration without this additional dipole, as shown in [2]

$$\mathcal{Z} = \frac{\int \frac{d^2x_1 d^2x_2}{\xi^4}\, z^2 e^{-C(\mathbf{x}_1-\mathbf{x}_2)} \sum_{\{k_i\}} z^N \int \prod_{i=1}^N \frac{d^2r_i}{\xi^2}\, e^{-\left[\sum_{i<j} C(\mathbf{r}_i,\mathbf{r}_j)+\sum_i (C(\mathbf{r}_i-\mathbf{x}_1)+C(\mathbf{r}_i-\mathbf{x}_2))\right]}}{\sum_{\{k_i\}} z^N \int \prod_{i=1}^N \frac{d^2r_i}{\xi^2}\, e^{-\sum_{i<j} C(\mathbf{r}_i,\mathbf{r}_j)}} \tag{3.1}$$

where, to ease the notation, Cft with (2.56)

$$C(\mathbf{x}-\mathbf{y}) = \beta\frac{(2\pi\hbar)^2 n_s}{m} G_{\text{reg}}(\mathbf{x}-\mathbf{y}) \tag{3.2}$$

To extract information from the previous partition function, we work perturbatively in the small fugacity z therefore expanding around $z \approx 0$; of course this approximation holds for $T \lesssim T_{BKT}$ where we know that adding a vortex is not convenient. We stress the fact that we seek to *integrate out* all the vortices except the ones located in $\mathbf{x}_{1,2}$ to extract the effective vortex-antivortex interaction

$$\mathcal{Z} = \int \frac{d^2x_1 d^2x_2}{\xi^4}\, z^2 e^{-C_{\text{eff}}(\mathbf{x}_1-\mathbf{x}_2)} \tag{3.3}$$

Expanding (3.1) and equating to (3.3) we obtain

$$\begin{aligned} e^{-C_{\text{eff}}(\mathbf{x}_1-\mathbf{x}_2)} \approx & \frac{e^{-C(\mathbf{x}_1-\mathbf{x}_2)}\left(1+z^2\int\frac{d^2r_1d^2r_2}{\xi^4}e^{-[C(\mathbf{r}_1-\mathbf{r}_2)+D(\mathbf{x}_1,\mathbf{x}_2,\mathbf{r}_1,\mathbf{r}_2)]}+\mathcal{O}(z^4)\right)}{1+z^2\int\frac{d^2r_1d^2r_2}{\xi^4}e^{-C(\mathbf{r}_1-\mathbf{r}_2)}+o(z^4)} \\ \approx & e^{-C(\mathbf{x}_1-\mathbf{x}_2)}\left[1+z^2\int\frac{d^2r_1d^2r_2}{\xi^4}e^{-C(\mathbf{r}_1-\mathbf{r}_2)}\left(e^{-D(\mathbf{x}_1,\mathbf{x}_2,\mathbf{r}_1,\mathbf{r}_2)}-1\right)+o(z^4)\right] \end{aligned} \tag{3.4}$$

where we used the extraordinary result $(1+\epsilon)^{-1} \approx 1-\epsilon+o(\epsilon^2)$.
Notice that from the physical point of view, we are considering the dipole-dipole contribution to effective interaction, parametrized by the the term

$$\begin{aligned} D =& C(\mathbf{r}_1-\mathbf{x}_1)-C(\mathbf{r}_2-\mathbf{x}_1)-C(\mathbf{r}_1-\mathbf{x}_2)+C(\mathbf{r}_2-\mathbf{x}_2) \\ =& C\left(\mathbf{R}+\frac{\mathbf{r}}{2}-\mathbf{x}_1\right)-C\left(\mathbf{R}-\frac{\mathbf{r}}{2}-\mathbf{x}_1\right)-C\left(\mathbf{R}+\frac{\mathbf{r}}{2}-\mathbf{x}_2\right)+C\left(\mathbf{R}-\frac{\mathbf{r}}{2}-\mathbf{x}_2\right) \\ \approx& 2\mathbf{r}\cdot\left(\nabla_{\mathbf{r}}C\left(\mathbf{R}+\frac{\mathbf{r}}{2}-\mathbf{x}_1\right)\Big|_{\mathbf{r}=0}-\nabla_{\mathbf{r}}C\left(\mathbf{R}+\frac{\mathbf{r}}{2}-\mathbf{x}_2\right)\Big|_{\mathbf{r}=0}\right)+o(r^3) \\ =& \mathbf{r}\cdot\nabla_{\mathbf{R}}\left(C(\mathbf{R}-\mathbf{x}_1)-C(\mathbf{R}-\mathbf{x}_2)\right)+o(r^3) \end{aligned} \tag{3.5}$$

where we adopted the change of variables $(\mathbf{x}_1,\mathbf{x}_2)\to(\mathbf{R},\mathbf{r})$ with $\mathbf{R}$ being the coordinate of the center of mass of the dipole and $\mathbf{r}$ the separation vector of the dipole and expanded around $\mathbf{r}\approx 0$ with the idea of evaluating the leading order effects in the vicinity of the dipole.
Substituting the latter expression into (3.4) and expanding the exponential up to the second order in $\mathbf{r}$ we obtain

$$\int\frac{d^2R\,d^2r}{\xi^4}\,e^{-C(r)}\left[\mathbf{r}\cdot\nabla_{\mathbf{R}}\left(C(\mathbf{R}-\mathbf{x}_1)-C(\mathbf{R}-\mathbf{x}_2)\right)+\frac{1}{2}\left(\mathbf{r}\cdot\nabla_{\mathbf{R}}\left(C(\mathbf{R}-\mathbf{x}_1)-C(\mathbf{R}-\mathbf{x}_2)\right)\right)^2\right] \tag{3.6}$$

for symmetry reason, the first term vanishes being odd under $\mathbf{r}\to-\mathbf{r}$. For what concerns the second term, we notice that passing in polar coordinates and considering the angular part, we will have to deal with

$$\int_0^{2\pi} d\theta\ \cos^2(\theta) = \frac{1}{2} \tag{3.7}$$

so, at the end of the day we are left with

$$\int r\,dr\,\frac{r^2}{\xi^4}e^{-C(r)}\int d^2R\,\frac{1}{4}\left|\nabla_{\mathbf{R}}\left(C(\mathbf{R}-\mathbf{x}_1)-C(\mathbf{R}-\mathbf{x}_2)\right)\right|^2 \tag{3.8}$$

We then evaluate the integral over $\mathbf{R}$ by integrating by parts, recalling that

$$\nabla^2 C(\mathbf{r}-\mathbf{r}_1) = 4\pi^2\beta J\delta^{(2)}(\mathbf{r}-\mathbf{r}_1)$$

as defined in equation (2.23). The surface term resulting from integration by parts can be neglected, as it corresponds to configurations in which the internal dipole is located at the system boundaries and thus does not interact with the test vortices. Furthermore, the term $C(\mathbf{0})$ may be omitted by noting that $\mathbf{x}_1$ and $\mathbf{x}_2$ cannot be separated by distances shorter than ξ. The integral over $\mathbf{R}$ in equation (3.8) simplifies to:

$$\begin{aligned}
&\int d^2\mathbf{R}\,|\nabla_{\mathbf{R}}\left(C(\mathbf{R}-\mathbf{x}_1)-C(\mathbf{R}-\mathbf{x}_2)\right)|^2 \\
&= -\int d^2\mathbf{R}\,\left(C(\mathbf{R}-\mathbf{x}_1)-C(\mathbf{R}-\mathbf{x}_2)\right)\nabla^2\left(C(\mathbf{R}-\mathbf{x}_1)-C(\mathbf{R}-\mathbf{x}_2)\right) \\
&= -4\pi^2\beta J\int d^2\mathbf{R}\,\left(C(\mathbf{R}-\mathbf{x}_1)-C(\mathbf{R}-\mathbf{x}_2)\right)\left[\delta^{(2)}(\mathbf{R}-\mathbf{x}_1)-\delta^{(2)}(\mathbf{R}-\mathbf{x}_2)\right] \\
&= -4\pi^2\beta J\left[-2C(\mathbf{x}_1-\mathbf{x}_2)+2C(\mathbf{0})\right] \\
&= 8\pi^2\beta J\,C(\mathbf{x}_1-\mathbf{x}_2)
\end{aligned} \tag{3.9}$$

It is now possible to rewrite the expression in equation (3.3), which defines C_{eff}, as follows:

$$e^{-C_{\text{eff}}(\mathbf{x}_1-\mathbf{x}_2)} \approx e^{-C(\mathbf{x}_1-\mathbf{x}_2)}\left[1+2\pi^2\beta J z^2 C(\mathbf{x}_1-\mathbf{x}_2)\int\frac{r^3}{\xi^4}e^{-C(r)}\left(\frac{r}{\xi}\right)^{-2\pi\beta J}dr+o(z^4)\right] \tag{3.10}$$

$$= e^{-C(\mathbf{x}_1-\mathbf{x}_2)}\left[1+2\pi^2\beta J z^2 C(\mathbf{x}_1-\mathbf{x}_2)\int\frac{r^3}{\xi^4}\left(\frac{r}{\xi}\right)^{-2\pi\beta J}dr+o(z^4)\right] \tag{3.11}$$

$$\approx \exp\left[-C(\mathbf{x}_1-\mathbf{x}_2)+2\pi^2\beta J z^2 C(\mathbf{x}_1-\mathbf{x}_2)\int\frac{r^3}{\xi^4}\left(\frac{r}{\xi}\right)^{-2\pi\beta J}dr+o(z^4)\right] \tag{3.12}$$

Recognizing the expression in square brackets as the second-order expansion of an exponential, the resulting effective interaction C_{eff} takes the form:

$$C_{\text{eff}}(\mathbf{x}_1-\mathbf{x}_2) = C(\mathbf{x}_1-\mathbf{x}_2)\left[1-2\pi^2\beta J g^2\int\frac{r^3}{\xi^4}\left(\frac{r}{\xi}\right)^{-2\pi\beta J}dr\right] \tag{3.13}$$

The functional form of the effective interaction is identical to that of the non-normalized one C, only the constant J is replaced by an effective stiffness J_{eff} such that one can write C_{eff} as:

$$C_{\text{eff}}(\mathbf{x}_1-\mathbf{x}_2) = 2\pi\beta J_{\text{eff}}\ln\left(\frac{|\mathbf{x}_1-\mathbf{x}_2|}{\xi}\right) \tag{3.14}$$

By reintroducing the integration limits, J_{eff} is calculated from J as:

$$\beta J_{\text{eff}} = \beta J - 2\pi^2(\beta J)^2 z^2\int_{\xi}^{+\infty}dr\,r^{(3-2\pi\beta J)}\xi^{(2\pi\beta J-4)}+o(z^4) \tag{3.15}$$

The integral in r diverges as $r\to+\infty$ if $2\pi\beta J\leq 4$, and otherwise converges; once again, one finds the critical temperature $T_c=\frac{\pi J}{k_B}$, which separates the two behaviors. If the integral diverges, one cannot use the perturbative method, since in that case the perturbative correction becomes large regardless of z. To resolve this difficulty, Eq.(3.15) must be studied through the renormalization group; this calculation was first carried out by José et al. in 1977 [29].

3.2 Renormalization group approach

First, we divide the integral in Eq (3.15) as follows:

$$\int_{\xi}^{+\infty} = \int_{\xi}^{b\xi} + \int_{b\xi}^{+\infty} \tag{3.16}$$

With $b > 1$, this way we separate the contribution of dipoles of length between ξ and $b\xi$ from the others. The renormalized stiffness is now given by:

$$K_{\text{eff}} = \underbrace{\left[K - 2\pi^2 K^2 z^2 \xi^{(2\pi K-4)} \int_{\xi}^{b\xi} dr\, r^{(3-2\pi K)}\right]}_{K(b)} - 2\pi^2 K^2 z^2 \xi^{(2\pi K-4)} \int_{b\xi}^{+\infty} dr\, r^{(3-2\pi K)} \tag{3.17}$$

We rename $\beta J = K$ and define $K(b)$ as the stiffness renormalized by only the dipoles of maximum length $b\xi$. The value of $K(b)$ in the limit $b \to +\infty$ thus corresponds to K_{eff}.

The second step consists of rescaling the integration variable r in the second integral in (3.17), so as to obtain the same integration limits as in the original integral, that is ξ to $+\infty$. We perform the change of variable $r \to br$, obtaining:

$$K_{\text{eff}} = K(b) - 2\pi^2 K^2 z^2 b^{4-2\pi K} \xi^{(2\pi K-4)} \int_{\xi}^{+\infty} dr\, r^{(3-2\pi K)} + o(z^4) \tag{3.18}$$

Since $K(b) = K + o(z^2)$, we can rewrite this expression, to the same order in z, as:

$$K_{\text{eff}} = K(b) - 2\pi^2 K(b)^2 z(b)^2 \xi^{(2\pi K(b)-4)} \int_{\xi}^{+\infty} dr\, r^{(3-2\pi K(b))} + o(z^4) \tag{3.19}$$

This expression is identical to that found in (3.15). K_{eff} is a quantity that appears only on macroscopic scales, and Eqs.(3.15) and (3.19) allow us to calculate K_{eff} from a microscopic model. According to (3.15), the initial model is considered at a rescaled level by a factor b, where everything occurring at distances smaller than $b\xi$ is neglected. (3.15) shows that by replacing the original parameters with $K(b)$ and $z(b)$, the system retains the same macroscopic properties, i.e., the same K_{eff}.

The new parameters therefore represent how the original ones, K and z, are modified when the system is studied at different scales. How this happens for K has already been discussed. The parameter z is modified because, at larger scales, dipoles of smaller length disappear. The number of vortices changes, but in the grand canonical ensemble this number is linked to z, which will also be modified. The renormalized fugacity $z(b)^2$ can be associated with the number of dipoles with separation greater than $b\xi$.

By repeating this procedure multiple times, the system is studied at progressively larger scales. $K(b)$ and $z(b)$ can be obtained by iterating the following substitutions, which follow from (3.17):

$$\begin{aligned} K(b) &= K - 2\pi^2 K^2 z^2 \frac{b^{4-2\pi K} - 1}{4 - 2\pi K} \\ z(b) &= z b^{\pi K - 2} \end{aligned} \tag{3.20}$$

from which we can derive the differential form by taking the limit $b \to 1$.
Remembering that $K(\ln b)_{|b=1} = K$ and the same goes for z we obtain:

$$\begin{cases} K(\ln b) = K - 2\pi^2 K^2 z^2 \dfrac{e^{\ln(b)(4-2\pi K)} - 1}{4 - 2\pi K} \\ z(\ln b) = z e^{\ln(b)(\pi K - 2)} \end{cases} \xrightarrow{\ln(b)\approx 0} \begin{cases} \dfrac{dK}{d\ln b} = -2\pi^2 K^2 z^2 \ln b \\ \dfrac{dz}{d\ln b} = z(\pi K - 2)\ln b \end{cases} \tag{3.21}$$

The obtained equations express the variations of K and z as a result of an infinitesimal rescaling.

Rewriting the equation for K in terms of K^{-1}, the previous system becomes:

$$\begin{cases} (K^{-1})' = 2\pi^2 z^2 \\ z' = (2 - \pi K) y \end{cases} \tag{3.22}$$

The equation obtained by Kosterlitz for K^{-1} has a coefficient $4\pi^3$ in front of y^2, unlike what is obtained here. However, this difference is irrelevant for the results that will follow.

3.3 Path integral approach

We start from the partition function describing a 2D Coulomb gas or equivalently a sine-Gordon field theory:

$$\mathcal{Z} = \int \mathcal{D}\phi \, \exp\left(- \int d^2\mathbf{r} \left[\frac{1}{8\pi^2 \beta J} (\nabla\phi)^2 - \frac{2z}{\xi^2} \cos(\phi(\mathbf{r})) \right] \right),$$

where the quadratic term is the Gaussian (free) part, and the cosine represents the interaction due to vortex-antivortex pairs with fugacity z. Define:

$$S_0[\phi] = \int d^2\mathbf{r} \, \frac{1}{8\pi^2 \beta J} (\nabla\phi)^2, \qquad S_{\text{int}}[\phi] = - \int d^2\mathbf{r} \, \frac{2z}{\xi^2} \cos(\phi(\mathbf{r})).$$

We split the field into slow and fast components:

$$\phi(\mathbf{r}) = \phi_<(\mathbf{r}) + \phi_>(\mathbf{r}),$$

where $\phi_>$ contains modes with momenta $\Lambda/b < |\mathbf{k}| < \Lambda$, and $\phi_<$ contains the remaining low-momentum modes.

We integrate out the fast modes to obtain an effective action for $\phi_<$:

$$\mathcal{Z} = \int \mathcal{D}\phi_< \, e^{-S_0[\phi_<]} \underbrace{\int \mathcal{D}\phi_> \, e^{-S_0[\phi_>]} e^{-S_{\text{int}}[\phi_< + \phi_>]}}_{\equiv e^{-S_{\text{eff}}[\phi_<]}}.$$

We expand the interaction term in a cumulant (linked cluster) expansion:

$$S_{\text{eff}}[\phi_<] = -\log \left\langle e^{-S_{\text{int}}[\phi_< + \phi_>]} \right\rangle_>,$$

where $\langle \cdots \rangle_>$ denotes expectation with respect to the Gaussian measure over $\phi_>$. Expanding to second order in z:

$$S_{\text{eff}}[\phi_<] = \langle S_{\text{int}}[\phi_< + \phi_>] \rangle_> - \frac{1}{2} \left(\langle S_{\text{int}}^2[\phi_< + \phi_>] \rangle_> - \langle S_{\text{int}}[\phi_< + \phi_>] \rangle_>^2 \right) + \cdots.$$

To compute the first term, use:

$$\langle \cos(\phi_< + \phi_>) \rangle_> = \cos(\phi_<) e^{-\frac{1}{2}\langle \phi_>^2 \rangle}.$$

Since $\phi_>$ is Gaussian:

$$\langle \phi_>^2 \rangle = \int_{\Lambda/b<|\mathbf{k}|<\Lambda} \frac{d^2k}{(2\pi)^2} \frac{4\pi^2 \beta J}{k^2} = 2\pi\beta J \ln b.$$

Then, the renormalized cosine term becomes:

$$\frac{2z}{\xi^2} \cos(\phi_<) \to \frac{2z}{\xi^2} b^{-\pi\beta J} \cos(\phi_<).$$

Of course, exactly how it has been done previously, to compare the two effective action at the same energy scale Λ we have to rescale our coordinates

$$\mathbf{r} \to b\mathbf{r} \tag{3.23}$$

This shows that under coarse-graining, the fugacity flows as:

$$z(b) = z\, b^{2-\pi\beta J} \qquad \Rightarrow \frac{dz}{d\ell} = z(2 - \pi K),$$

where $K = \beta J$ and z is a dimensionless fugacity.

For the second-order term (Gaussian average of $\cos(\phi_< + \phi_>) \cos(\phi_< + \phi_>)$), the cumulant contributes a correction to the stiffness using the identity

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)] \tag{3.24}$$

with

$$A = \phi_<(\mathbf{x}) + \phi_>(\mathbf{x}) \tag{3.25}$$

$$B = \phi_<(\mathbf{y}) + \phi_>(\mathbf{y}) \tag{3.26}$$

And computing an expansion around $\mathbf{x} \approx \mathbf{y}$, exactly how has been done previously, yields a correction to K of the form:

$$\frac{dK^{-1}}{d\ell} = 2\pi^2 z^2.$$

Putting everything together, the RG flow equations become:

$$\boxed{\begin{aligned} \frac{dK^{-1}}{d\ell} &= 2\pi^2 z^2, \\ \frac{dz}{d\ell} &= z(2 - \pi K). \end{aligned}} \tag{3.27}$$

These are the celebrated KT RG equations, describing how the effective stiffness K and vortex fugacity z evolve under scale transformations.

3.4 Flow analysis

Following the flow line with initial data K and z for $\ln b \to +\infty$, at constant temperature, the macroscopic parameters K_{eff} and z_{eff} renormalized by the vortices are obtained. The initial data for the superfluid film must satisfy the relation

$$z = \exp(-\beta E_{\text{cor}}) = \exp\left(-\frac{K}{J} E_{\text{cor}}\right),$$

where J is the widely discussed stiffness, is initially fixed. As the temperature varies, the points lie along the curve shown in Figure (3.1).

Since J is proportional to the density of the superfluid phase, a new density ρ_s^R, renormalized by vortices, is associated with J_{eff} and is observable on macroscopic scales.

At low temperatures, the initial data for the superfluid film lie in region I. The flow lines (see Figure (3.1)) in this region tend toward a fixed point at $z = 0$ and $K^{-1} > \frac{\pi}{2}$, which leads to $J_{\text{eff}} > \frac{2}{\beta\pi}$, i.e., a non-zero superfluid phase density.

At high temperatures, the system lies in region II, where the flow lines tend toward increasing values of K^{-1}, hence $J_{\text{eff}} = 0$, and consequently no superfluid phase exists.

The critical temperature is the one for which the initial data line intersects the separatrix between regions I and II. This value turns out to be slightly lower than the one predicted through naive analysis previously. From Figure (3.1) it can be seen that the point where this intersection occurs has a value of K^{-1} smaller than $\frac{\pi}{2}$, which would be the value if T_c were that predicted in (1.255).

The separatrix, associated with the initial data at $T = T_c$, leads to the critical value $K_c = \frac{2}{\pi}$.

Recalling that $(z(b))^2$, and thus $(z_{\text{eff}})^2$, can be associated with the number of dipoles of length greater than a certain scale, it follows that at low temperature $z_{\text{eff}} = 0$, and only short dipoles are present in the system, this corresponds to the dielectric phase where actually the formation of single vortices is disfavoured.

At high temperature, the flow lines tend to increasing z, and thus z_{eff} is non-zero. In this case, a number of large-separation dipoles will be present, leading to a configuration more similar to a plasma where actually we witness a proliferation of vortices which now can be excited thermally.

By studying the flow equations near the critical point $\left(\frac{\pi}{2}, 0\right)$, one can show that for $T \to T_c^-$, the behavior of K_{eff} is, where a is a constant (see Figure (3.1)):

$$K_{\text{eff}} = \frac{2}{\pi}\left(1 + a\sqrt{T_c - T}\right) \tag{3.28}$$

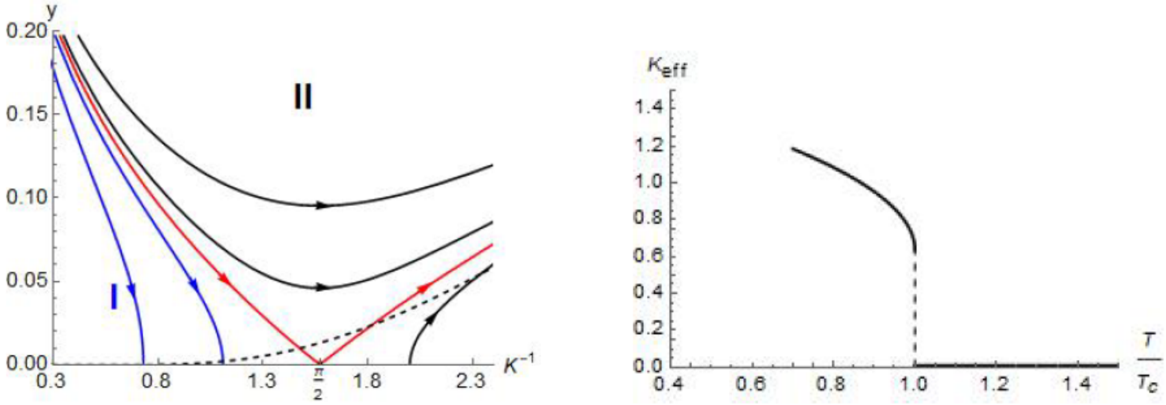

Figure 3.1: RHS: RG flow
LHS: behavior of $K_{\text{eff}}(T)$

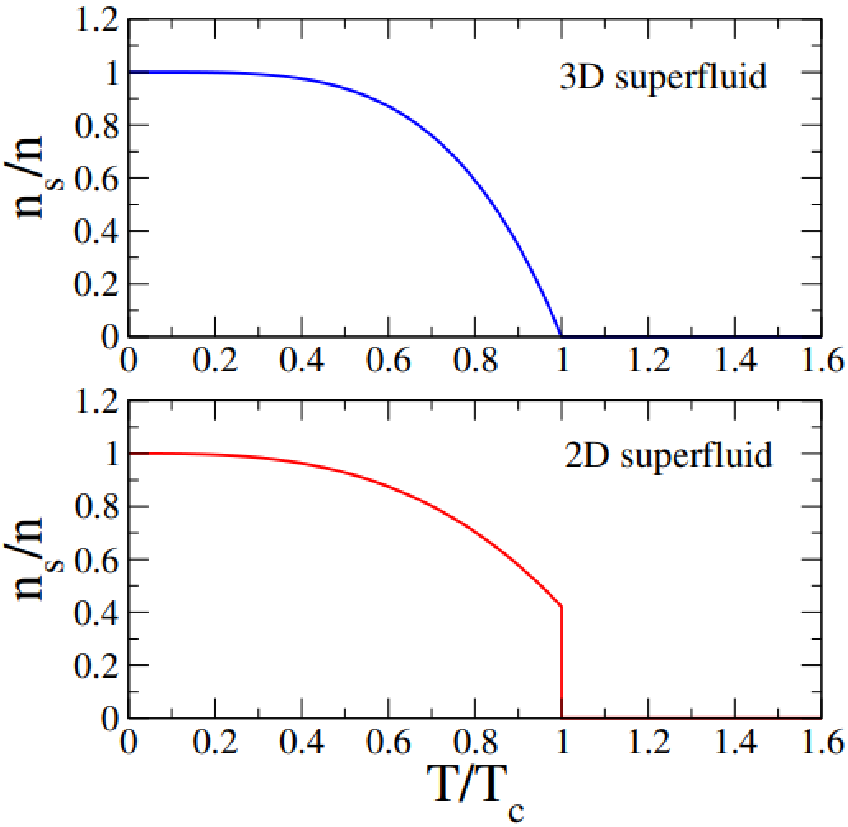

Figure 3.2: Difference of behavior of superfluid density between 3D and 2D

Actually we can estimate the way the correlation length ξ diverges in proximity of T_{BKT}. Notice indeed that, in proximity of the fixed point, the system (3.27) is equivalent to the following system (up to multiplicative constants)

$$\begin{aligned} \frac{dx}{dl} &= -y^2 \\ \frac{dy}{dl} &= -xy \end{aligned} \tag{3.29}$$

where $2 - \pi K = -x \approx 0$ and $z^2 \propto y^2 \approx 0$.
We can easily solve these differential equations by noticing that they can be rewritten as

$$x\,\frac{dx}{d\ell} - y\,\frac{dy}{d\ell} = 0, \tag{3.30}$$

from which it follows that

$$x^2 - y^2 = A^2. \tag{3.31}$$

This relation describes the renormalization-group flow in the vicinity of the fixed point $(x, y) = (0, 0)$ in the transformed $x-y$ plane. The resulting flow lines are hyperbolas. Their symmetry axis is

$$y = 0 \quad \text{if} \quad A^2 > 0,$$

which corresponds to region (I), or

$$x = 0 \quad \text{if} \quad A^2 < 0,$$

which corresponds to region (II). The critical separatrix between the two behaviors is given by $A^2 = 0$.

Approaching the critical point $A \to 0^+$, the original RG equation for x can be rewritten as

$$\frac{dx}{d\ell} = -\,x^2.$$

The solution of the previous equation is

$$x(\ell) = \frac{1}{\ell + c},$$

where c is a constant determined by the initial condition $x(0) = x_0$. Along the critical line, x ultimately flows to zero but in an extremely slow (logarithmic) fashion, since $\ell = \ln(\xi'/\xi)$ is the logarithm of the rescaled correlation length.

In the regime $A^2 > 0$, one finds that x (and hence K) flows to a finite value. This corresponds to the low-temperature phase, characterized by a nonzero superfluid stiffness and vanishing vortex fugacity z. Indeed, substituting $x^2 = y^2 + A^2$ into the flow equation for y gives

$$\frac{dy}{d\ell} = -\,y\,\sqrt{y^2 + A^2},$$

whose solution is

$$y(\ell) = \frac{A}{\sinh\!\big(A\ell + \operatorname{arsinh}(A/y_0)\big)} \;\longrightarrow\; 0 \quad (\ell \to \infty).$$

Following the same method, the corresponding solution for x is

$$x(\ell) = \frac{A}{\tanh\!\big(A\ell + \operatorname{arsinh}(A/y_0)\big)} \;\longrightarrow\; A \quad (\ell \to \infty).$$

Thus, in the low-temperature phase the superfluid stiffness tends to a finite limit, while the vortex coupling vanishes under coarse graining.

The opposite regime, $A^2 < 0$, corresponds to temperatures $T > T_{\mathrm{BKT}}$. Here the stiffness x goes to zero, and one can define the correlation length as the scale ℓ^* at which $x(\ell^*) = 0$. Writing $-A^2 = C^2 > 0$ and using $y^2 = x^2 + C^2$, the flow equation for x becomes

$$\frac{dx}{d\ell} = -\big(x^2 + C^2\big) \quad \Longrightarrow \quad \frac{x(l)}{C} = \tan\!\big(-C\ell + \arctan(x_0/C)\big).$$

From this expression one sees that x vanishes at the scale satisfying

$$\arctan\bigl(x_0/C\bigr) = C\,\ell^*.$$

Near the transition $x_0 \approx y_0$, so

$$C^2 = y_0^2 - x_0^2 = (y_0 - x_0)(y_0 + x_0) \simeq 2\,y_0\,(y_0 - x_0).$$

Since at the transition $x = y$, the difference $y_0 - x_0$ is, to leading order, proportional to the reduced temperature

$$t \;=\; \frac{T - T_{\mathrm{BKT}}}{T_{\mathrm{BKT}}}.$$

Thus one finds

$$C \;=\; \alpha\,\sqrt{t},$$

where α is a constant of order unity.

Furthermore, in the limit $t \ll 1$ we have $\arctan(x_0/C) \simeq \pi/2$. From the RG solution it then follows that

$$C\,\ell^* \sim O(1) \quad\Longrightarrow\quad \ell^* = \frac{b}{\sqrt{t}}.$$

Since $\ell^* = \ln(\xi'/\xi)$, the correlation length diverges as

$$\frac{\xi'}{\xi} = \exp\!\Bigl(\frac{b}{\sqrt{t}}\Bigr).$$

The nonuniversal constant b depends on the microscopic details of the model. This exponential divergence of ξ as $t \to 0^+$ is one of the hallmark signatures of the BKT transition, in contrast to the power-law divergence seen in conventional critical phenomena.

4 Conclusions

At $T = T_c$, BKT theory predicts a discontinuity of the renormalized stiffness from 0 to J_c, independently of the particular system considered, hence the so-called universal jump of the BKT. For the superfluid film, the discontinuity concerns the renormalized density ρ_s^R, as already discussed. Recalling the value of J_c and the relation between stiffness and density, the discontinuity in the density is given by:

$$J_c = \frac{2\,k_B\,T_c}{\pi} = \left(\tfrac{\hbar}{m}\right)^2 \rho_s^R\big(T_c^-\big) \quad\Longrightarrow\quad \rho_s^R\big(T_c^-\big) = \left(\tfrac{m}{\hbar}\right)^2 \frac{2\,k_B}{\pi}\,T_c = \big(3.491\times 10^{-9}\,\mathrm{g\,cm^{-2}\,K^{-1}}\big)\,T_c\,. \tag{4.1}$$

The critical temperature depends on the system considered and Eq. (4.1) predicts that $\rho_s^R(T_c^-)$ is proportional to T_c. In 1980 Bishop and Reppy verified this behavior by studying several helium films with different T_c, finding a proportionality constant equal to 0.96 times the theoretical prediction (see Figure (4.1)). The theory also predicts that ρ_s^R approaches the behavior shown in (3.28) as $T \to T_c^-$; however, this is difficult to verify experimentally, because the measurement of the superfluid density is obtained by probing the film's response to external dynamic perturbations. In that case, the dynamics of quantized vortices—which has so far been ignored—becomes important. The theory developed up to now, commonly referred to as "static," can be corrected by taking into account dynamical effects via the Ambegaokar dynamical theory [4] , which Bishop and Reppy also confirmed in the same experiment [12].

Another difficulty in studying the BKT transition concerns the finite dimensions of the experimental apparatus. By studying system (3.27), one indeed derives K_{eff} from the limit of the renormalization-group flow lines as $\ln b \to +\infty$, i.e., by considering the system at an infinite scale. If the system has finite dimensions, one can follow the flow lines only up to the physical size of the apparatus, and it is from that point that K_{eff} is obtained.

In this case, the effect is that the discontinuity at T_c disappears, because K_{eff} tends to zero continuously. In situations where the total number of particles in the system is relevant, it is possible to rewrite the Kosterlitz relations as a function of the particle number N instead of the scale factor b. One starts from system (3.27) and recalls that, after rescaling the system, the core radius of each vortex becomes $b\,\xi$. If one requires that, within the disk of radius $b\,\xi$, that is, inside the vortex core, there are N particles present, then, by introducing the total particle density n, one can perform the following substitutions that lead to a rewriting of system (3.27) as:

$$N = \pi\, b^2\, \xi^2\, n\Big|_{N_0=\pi\,\xi^2 n} \longrightarrow \ln(b) = \tfrac{1}{2}\ln\!\big(\tfrac{N}{N_0}\big) \longrightarrow \begin{cases} \dfrac{dK(N)}{d\ln N} = -4\pi^3\,K(N)^2\,y(N)^2, \\[2mm] \dfrac{dy(N)}{d\ln N} = y(N)\big(\pi\,K(N) - 2\big). \end{cases} \tag{4.2}$$

One obtains the Kosterlitz relations parameterized by the particle number N present below a certain scale; now it is N that determines that scale. The pair $\big(K(N),\, y(N)\big)$ can be interpreted as the renormalized parameters due to vortices in a system of N particles in the same way that $\big(K(b),\, y(b)\big)$ represent the renormalized parameters in a system of linear size $b\,\xi$. Equations (4.2) can be used in a Monte Carlo simulation where one considers a large but finite number N of particles. Equations similar to (4.2) were employed by Pilati, Giorgini, and Prokof'ev [43]in a simulation with $N = 10^5$ particles in two dimensions interacting via a hard-sphere potential. They measured the superfluid phase density ρ_s as a function of temperature for systems of different sizes (i.e., different particle numbers), extrapolating the behavior to the thermodynamic limit $N \to +\infty$. The obtained result confirms the presence of the BKT transition in the system, with the universal jump predicted by the static theory and the correct transition temperature T_c calculated for the system under consideration (see Figure (4.2)). Finally, we present a recent result obtained by Christodoulou *et al.* [15], who studied the second-sound velocity in a two-dimensional ultracold atomic gas confined by an optical trap. The experiment appears to confirm the static BKT theory, according to which the second-sound velocity varies with temperature in a manner similar to that of the superfluid phase density.

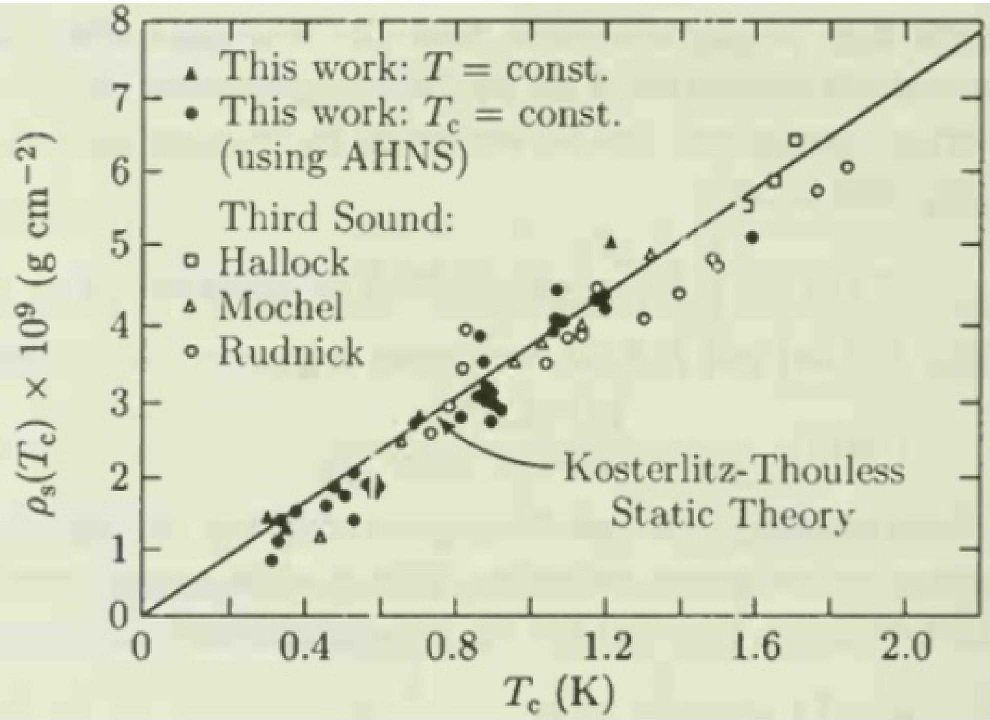

Figure 4.1: $\rho_s(T_c^-)$ values stemming from measeurements on several kinds of superfluid films. Source [12]

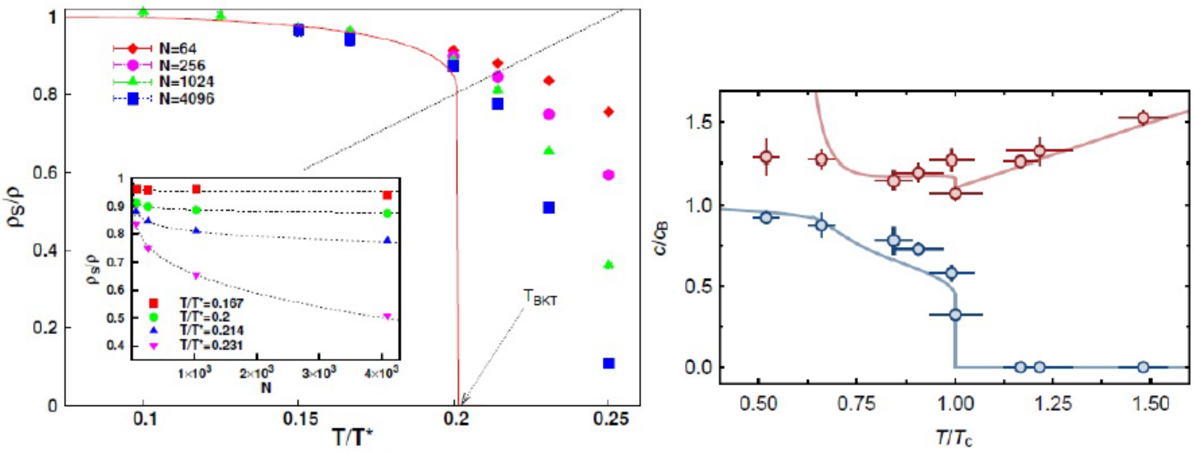

Figure 4.2: On the left: results obtained from the simulations of Pilati, Giorgini, and Prokof'ev. The colored points represent the behavior of the superfluid density when the system contains N particles. The solid red line shows the behavior extrapolated to the thermodynamic limit $N \to +\infty$. On the right: results from the experiment by Christodoulou *et al.* in the study of the sound velocity in a two-dimensional ultracold atomic gas. Shown in blue are the experimental data for the second-sound velocity; the solid (blue) line represents the behavior expected from BKT theory for an infinite system . Images adapted from [43][15]

Over the past decade, a series of experiments have unequivocally demonstrated that also two-dimensional fermionic superfluids undergo a phase-coherence transition driven by the Berezinskii–Kosterlitz–Thouless (BKT) mechanism, in which the unbinding of vortex–antivortex pairs destroys quasi-long-range order. In ultrathin superconducting films of NbN, Mondal *et al.* measured the superfluid density $\rho_s(T)$ by means of a contactless mutual-inductance technique and observed a pronounced downturn in ρ_s at a temperature T_{BKT} that is consistent with the universal Nelson–Kosterlitz jump, similar to (4.1)

$$\rho_s(T^-_{\mathrm{BKT}}) \;=\; \frac{2\,m^2}{\pi\hbar^2}\,k_B\,T_{\mathrm{BKT}}\,, \qquad K_s \;=\; \frac{\hbar^2\,\rho_s}{m^2\,k_B\,T} \;=\; \frac{2}{\pi} \quad \text{at } T = T_{\mathrm{BKT}}\,, \tag{4.3}$$

where now m is the Cooper-pair mass [40]. Although in practice the NbN film exhibits some broadening of the transition—attributable to disorder and finite-size effects—the sharpness of the ρ_s collapse remains far more abrupt than what conventional Ginzburg–Landau theory would predict, as shown in (4.3). To account for the detailed temperature dependence of ρ_s below T_{BKT}, Benfatto *et al.* introduced a renormalization-group analysis in which the vortex-core energy μ_{core} enters as a key parameter, showing that μ_{core} scales with the superconducting gap rather than with ρ_s itself [8]. This theoretical framework not only reproduces the observed smearing of the superfluid-density jump but also explains why the resistive tail above T_{BKT} follows a characteristic exponential form,

$$R(T) \;\sim\; R_0 \exp\!\left[-b\,(T/T_{\mathrm{BKT}} - 1)^{-1/2}\right], \tag{4.4}$$

as expected for thermally activated vortex–antivortex unbinding. The combination of high-precision measurements by the group led by Pratap Raychaudhuri at TIFR (India) and the detailed vortex-core–energy analysis by Benfatto's collaboration provided the first conclusive evidence of a BKT transition in a charged, fermionic superfluid [40, 8].

In a complementary context, Murthy *et al.* extended these ideas to neutral fermionic atoms by studying a two-dimensional gas of ^{6}Li in an optical trap [41]. Using a pair of parallel 2D clouds tuned via a Feshbach resonance into the strongly interacting (unitary) regime, they performed matter-wave interference after time-of-flight to extract a central-region visibility $V(T)$, analogous to the fringe-contrast c_0 used in bosonic experiments. As the temperature was lowered below a critical T_{BKT}, $V(T)$ exhibited an abrupt increase to values on the order of 0.2–0.3, indicating the sudden onset of quasi-long-range coherence. Simultaneously, *in situ* measurements of the equation of state allowed Murthy *et al.* to determine the superfluid fraction $\rho_s(T)$, which showed a jump at T_{BKT} consistent with

$$\rho_s(T_{\mathrm{BKT}}^-) = \frac{2\,m\,k_B\,T_{\mathrm{BKT}}}{\pi\hbar^2} \quad \text{(unitary Fermi gas),} \tag{4.5}$$

thus confirming that even in a strongly interacting neutral Fermi system, the phase-coherence transition is governed by the same vortex-unbinding mechanism [41].

Together, these experimental realizations—one in a solid-state NbN film (charged fermions) and one in an ultracold ^{6}Li gas (neutral fermions)—demonstrate the universality of the BKT transition in two-dimensional fermionic superfluids. The sharp collapse of $\rho_s(T)$ at T_{BKT}, the characteristic resistive tail above T_{BKT}, and the sudden appearance of high-visibility interference fringes all point to the unambiguous role of vortex–antivortex proliferation. The collaboration between L. Benfatto (providing the renormalization-group vortex analysis) and Indian experimentalists in the NbN work [40, 8], followed by the neutral-Fermi realization of Murthy *et al.* [41], has firmly established that the BKT paradigm applies equally to bosonic and fermionic pairs, charged or neutral, whenever the system is effectively two-dimensional.

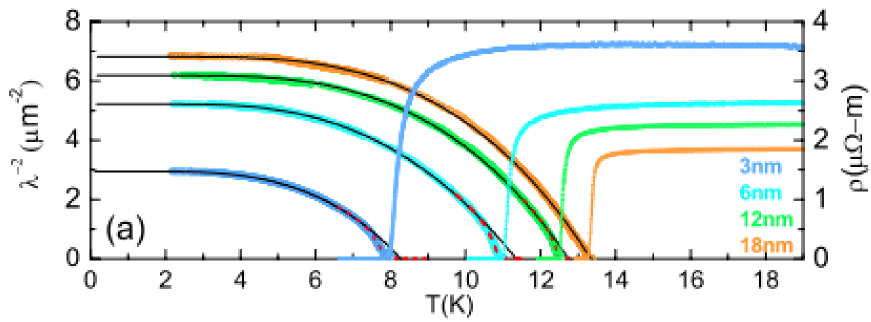

Figure 4.3: Temperature dependence of the penetration depth $\lambda^{-2}(T)$ and the resistivity $\rho(T)$ for four NbN films with different thickness. The solid (black) lines and dashed (red) lines correspond to the BCS and BKT fits of the $\lambda^{-2}(T)$ data, respectively.

To conclude, another experimental evidence of this Kosterlitz–Thouless transition and the associated unbinding of vortices have recently been observed in an atomic Bose gas by Hadzibabic *et al.* [28]. In their experiment, the proliferation of free vortices is directly imaged by allowing two two-dimensional clouds to expand and interfere with one another. The free vortices can then be counted individually by examining the number of defects, see Figure (4.4), in the interference pattern [26, 27]. The result is shown in Fig. 15.2. In an optical lattice, the same transition was also observed by Schweikhard *et al.* [48] using conventional absorption imaging of the vortex cores.

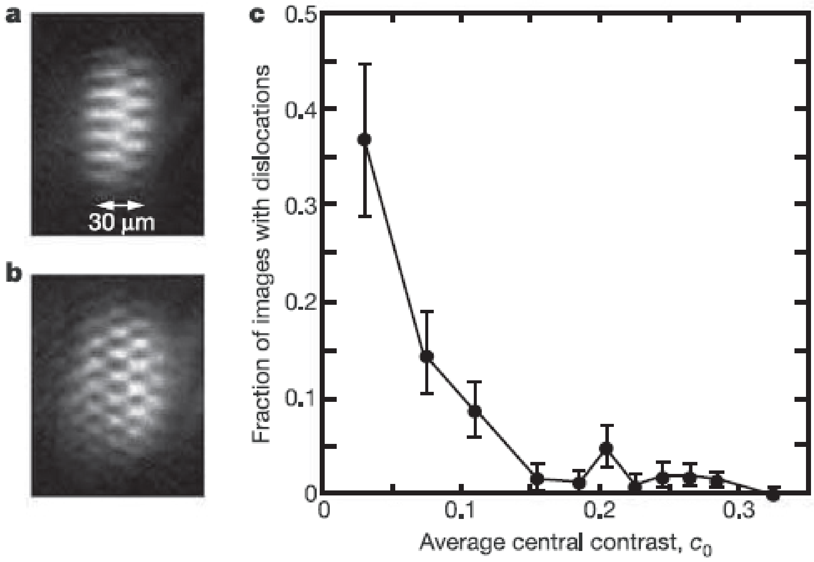

Figure 4.4: (a) Example of an interference pattern showing a sharp dislocation that is attributed to the presence of a free vortex in one of the interfering clouds. (b) Interference pattern showing several dislocations. (c) Fraction of images showing at least one dislocation, where the average central contrast c_0 is a measure of the degeneracy of the system. For a lower contrast, i.e. a higher temperature, there are free vortices, whereas for a higher contrast, i.e. a lower temperature, there are no free vortices.

Notice that actually, from a more rigorous study of this transition [24], it can be shown that in our two-dimensional Coulomb gas, at very low temperatures ($T \approx 0$), vortices form higher order multipoles effectively screening the long-range Coulomb interaction and preventing any free vortices thus forming an ordered state. As the temperature rises past a first threshold T_m, these multipolar aggregates break into bound dipole pairs, which remain the dominant excitations while partially reducing screening.

Finally, at the critical temperature T_c, dipole pairs dissociate completely into free vortices, marking the onset of the disordered phase.

$$\underbrace{\text{Bound Multipoles}}_{T\approx 0} \xrightarrow{T\uparrow} \underbrace{\text{Bound Dipoles}}_{T_m<T<T_c} \xrightarrow{T\uparrow} \underbrace{\text{Free Vortices}}_{T>T_c}$$

We conclude by noting that, while this thesis does not introduce new theoretical models or original data, it provides a detailed and self-contained derivation of several key results that are often presented in an obscure or implicit form in the original literature. In particular, the renormalization-group treatment of vortex unbinding and the mapping between the phase field and vortex degrees of freedom have been reconstructed from first principles using the path-integral formalism. In doing so, this work aims to bridge the gap between introductory reviews and more advanced theoretical papers, rendering technical arguments more accessible without compromising rigor.

With this foundation, the thesis may serve as a stepping stone for future studies, such as the introduction of anisotropic spin-orbit interactions, which can give rise to more exotic topological vortex structures [19]; or the investigation of BKT transitions in multicomponent superfluid systems, where richer defect structures can emerge, such as composite vortices, vortex molecules, and domain walls, along with the possible manifestation of new symmetries [32, 23].

Finally, a natural extension would be the exploration of non-equilibrium vortex dynamics, particularly in quenched systems where the binding of vortex-antivortex pairs may be suppressed. This setting connects closely to the Kibble–Zurek mechanism (KZM), which predicts the generation of topological defects during rapid phase transitions [3].

Appendix

Thomas-Fermi approximation

Making use, in the Gross-Pitaevskii equation, of the "backwards" Wick's rotation $\tau \to it$ we obtain its real-time version, introduicing a trapping potential $U(\mathbf{r})$

$$i\hbar\frac{\partial\ \psi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{r}) - \mu\right)\psi(\mathbf{r},\tau) + g|\psi(\mathbf{r},\tau)|^2\psi(\mathbf{r},\tau) \tag{6}$$

A particular solution of the latter can be found imposing time independence:

$$\frac{\partial\ \psi(\mathbf{r},t)}{\partial t} = 0 \tag{7}$$

and neglecting the kinetic term:

$$-\frac{\hbar^2\nabla^2}{2m}\psi(\mathbf{r},t) \sim 0 \tag{8}$$

These are the assumptions of the so called Thomas-Fermi approximation, which lead to a solution, for the density profile $n(r) = |\psi(\mathbf{r})|^2$, of the form:

$$n_{TF}(r) = \frac{\mu - U(\mathbf{r})}{g}\Theta(\mu - U(\mathbf{r})) \tag{9}$$

We can study the particular case when the trapping potential is harmonic $U(\mathbf{r}) = \frac{1}{2}m\omega^2\mathbf{r}^2$, integrating the density over the volume V, working at fixed particle number $N = \int_V n$, we obtain:

$$N = \frac{4\pi\mu}{3g}R_{TF}^3 - \frac{2\pi}{5}\frac{m\omega^2}{g}R_{TF}^2 \tag{10}$$

with $R_{TF} = \sqrt{\frac{2\mu}{m\omega^2}}$ with the property that

$$\frac{1}{2}m\omega^2 R_{TF}^2 = \mu \tag{11}$$

is called Thomas-Fermi radius.

Solving for the chemical potential :

$$\mu = \frac{1}{2}\hbar\omega\left(\frac{15Na_s}{a_H}\right)^{\frac{2}{5}} \tag{12}$$

where $a_H = \sqrt{\frac{\hbar}{m\omega}}$ is the characteristic lenght of the isotropic harmonic trapping potential and a_s is the scattering length.
Using (11) and (12) we obtain:

$$R_{TF} = \left(\frac{15N\hbar^2}{4m\omega^2 a_s}\right)^{\frac{1}{5}} \tag{13}$$

The Thomas-Fermi radius describes the size at which the condensate density drops significantly. In particular in the inner regions of the trap the density of the condensate is nearly constant, while at the edges , the density decreases such that the Thomas-Fermi radius defines a boundary beyond which the density becomes negligible.
Note that the latter approximation is valid is the high density regime, where effectively the momentum of the particle in the condensate is small.
Finally, this approximation is the starting point in one wishes to linearize (6) studying so the collective exitations around the equilibrium profile density $n_{TF}(\mathbf{r})$ as in the following.

Hydrodynamics description

The Gross-Pitaeskii equation can be casted in a very useful form that can shed light on the physical interpretation of its solutions,in fact, making use in (6) of the so-called Madelung transformation :

$$\psi(\mathbf{r},t) = \sqrt{n(\mathbf{r},t)}\; e^{i\theta(\mathbf{r},t)} \tag{14}$$

we obtain the following set of equations:

$$\mathbf{v}(\mathbf{r},t) = \frac{\hbar}{m}\nabla\theta(\mathbf{r},t) \tag{15}$$

$$\frac{\partial}{\partial t}n + \nabla\cdot(n\mathbf{v}) = 0 \tag{16}$$

$$\frac{\partial}{\partial t}\mathbf{v} = -\nabla\left(\frac{1}{2}mv^2 + U(\mathbf{r}) + gn - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n}}{\sqrt{n}}\right) \tag{17}$$

where we introduced (15) to rewrite the equation for the evolution of $\theta(\mathbf{r},t)$ as (17).
The physical meaning of the resulting equations is really straightforward: they describe the conservation of mass and linear momentum of a fluid that flows without friction (inviscid).
Note that the last equation tells us that our fluid is irrotational, that is, when the phase $\theta(r,t)$ is not singular, $\nabla\times\mathbf{v} = 0$.
it is worth emphatizing that the latter result is not valid in the presence of defects in our solution ; in that case, the single-valuedness of our solution $\psi(\mathbf{r},t)$ implies that around a closed contour its change in phase must be a multiple of 2π.

$$\varDelta\theta = \oint_{\mathcal{C}}\nabla\theta\cdot d\mathbf{l} = 2\pi q \quad q\in\mathbb{Z} \tag{18}$$

where $\mathcal{C}$ is a closed loop that could encircles one or more defects.
This shows that for general configurations of $\theta(r,t)$, we have that the circulation of $\mathbf{v}(r,t)$ is quantized:

$$\varGamma = \oint\mathbf{v}\cdot d\mathbf{l} = \frac{\hbar}{m}2\pi q \tag{19}$$

In 1941 Lev Landau, on the basis of previous important theoretical hints of Fritz London and Laszlo Tisza, developed the two-fluid model of superfluid ^4He.
At zero-temperature this model gives exactly equations (15) , (16), (17).
Its useful to study, in this hydrodynamics framework,small oscillations around the Thomas Fermi density profile introduced previously, that is, setting :

$$n(\mathbf{r},t) = n_{TF}(\mathbf{r}) + \delta n(\mathbf{r},t) \tag{20}$$

$$\theta(\mathbf{r},t) = \delta\theta(\mathbf{r},t) \tag{21}$$

one obtains:

$$\frac{\partial}{\partial t}\delta n + \nabla\cdot(n_{TF}\mathbf{v}) + h.o. \tag{22}$$

$$\frac{\partial}{\partial t}\mathbf{v} = -\nabla\left(U(r) + gn_{TF} + g\delta n - \frac{\hbar^2}{4m\; n_{TF}}\nabla\cdot(\nabla n_{TF} + \nabla\delta n)\right) + h.o. \tag{23}$$

$$\mathbf{v}(r,t) = \frac{\hbar}{m}\nabla\delta\theta(r,t) \tag{24}$$

Now, considering that for the Thomas-Fermi profile it holds:

$$U(\mathbf{r}) + gn_{TF} = \mu \tag{25}$$

$$\nabla\mu = 0 \tag{26}$$

$$\nabla^2 n_{TF} \approx 0 \tag{27}$$

and inserting the second equation into the first one after applying on the latter time-derivative on both sides, we obtain, at the lowest order:

$$m\frac{\partial^2 \delta n}{\partial t^2} - g\nabla n_{TF} \cdot \nabla\delta n - g\ n_{TF}\nabla^2\delta n + \frac{\hbar^2}{4m^2}\nabla^4\delta n = 0 \tag{28}$$

These equations are quite difficult to resolve due to the fact the our equilibrium density profile is not homogeneus $\nabla n_{TF} \neq 0$,we could try to find however particular solutions to these set of equations by setting a periodic fluctiation of the form $\delta n(\mathbf{r}, t) = \delta n(\mathbf{r})e^{i\omega t}$.
In order to grasp the basic physical concept of (28) let's consider instead the quite simpler case of no trapping potential $U(\mathbf{r}) = 0$, in this case we have that $n_s = \frac{N}{V}$ with $\mu = gn_s$ and (28) reduce to, considering that we have $\nabla n_s = 0$

$$\frac{\partial^2 \delta n}{\partial t^2} - \frac{g\ n_s}{m}\nabla^2\delta n + \frac{\hbar^2}{4m^2}\nabla^4\delta n = 0 \tag{29}$$

Taking the Fourirer transform one obtains the following dispersion relation:

$$\hbar\omega(k) = \sqrt{\frac{\hbar^2 k^2}{2m}\left(\frac{\hbar^2 k^2}{2m} + 2mc_B\right)} \tag{30}$$

This is the so called Bogoliubov spectrum and one can see that indeed it satisfies the Landau criterion for superfluidity where $v_c = \min_k\left(\frac{\omega_k}{k}\right) = c_B$ is the critical velocity under which an external particle entering the fluid cannot release any energy into the latter.
To be more specific, once quantized, this kind of exitations will describe quasi- particles or, in the context of superconductivity, Cooper's pairs, that will undergo a superlfluid transition.

Bibliografia

[1] J. F. Allen e A. D. Misener. “Flow of Liquid Helium II”. In: *Nature* 141 (1938), p. 75. DOI: 10.1038/141075a0.

[2] Alexander Altland e Ben D. Simons. *Condensed Matter Field Theory*. 2nd. Cambridge University Press, 2010. ISBN: 978-0521769754.

[3] E. Altman et al. “Nonequilibrium Phase Transition in a Two-Dimensional Driven Open Quantum System”. In: *Physical Review X* 5.4 (2015), p. 041028. DOI: 10.1103/PhysRevX.5.041028.

[4] V. Ambegaokar et al. “Dynamics of superfluid films”. In: *Physical Review B* 21.5 (1980), pp. 1806–1826. DOI: 10.1103/PhysRevB.21.1806.

[5] M. H. Anderson et al. “Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor”. In: *Science* 269 (1995), p. 5221.

[6] E. Babaev e H. Kleinert. “Nonperturbative XY-model approach to strong coupling superconductivity in two and three dimensions”. In: *Physical Review B* 59.18 (1999), pp. 12083–12086. DOI: 10.1103/PhysRevB.59.12083.

[7] John Bardeen, Leon N. Cooper e J. Robert Schrieffer. “Theory of Superconductivity”. In: *Physical Review* 108.5 (1957), pp. 1175–1204. DOI: 10.1103/PhysRev.108.1175.

[8] L. Benfatto, C. Castellani e T. Giamarchi. “Broadening of the Berezinskii–Kosterlitz–Thouless transition by inhomogeneity and finite-size effects: The case of thin superconducting films”. In: *Phys. Rev. B* 80 (2009), p. 214506.

[9] L. Benfatto, A. Toschi e S. Caprara. “Low-energy phase-only action in a superconductor: A comparison with the XY model”. In: *Physical Review B* 69 (2004), p. 184510. DOI: 10.1103/PhysRevB.69.184510.

[10] Lara Benfatto, Claudio Castellani e Thierry Giamarchi. “Berezinskii–Kosterlitz–Thouless Transition within the Sine-Gordon Approach: The Role of the Vortex-Core Energy”. In: *40 Years of Berezinskii–Kosterlitz–Thouless Theory*. A cura di Jorge V. José. World Scientific, 2013, pp. 161–199. DOI: 10.1142/9789814417648_0005.

[11] V. L. Berezinskii. “Destruction of long-range order in one-dimensional and two-dimensional systems possessing a continuous symmetry group. II. Quantum systems”. In: *Soviet Physics JETP* 34.3 (1972). Zh. Eksp. Teor. Fiz. 61, 1144 (1971), pp. 610–616.

[12] D. J. Bishop e J. D. Reppy. “A precision measurement of the superfluid density near the transition of a two-dimensional superfluid”. In: *Le Journal de Physique Colloques* 39 (1978). Presented at the 2nd International Conference on Collective Phenomena, Paris, 1978, pp. C6-339–C6-340.

[13] S. N. Bose. “Planck’s law and the light quantum hypothesis”. In: *Z. Phys.* 26 (1924), p. 178.

[14] C. Chin et al. “Feshbach resonances in ultracold gases”. In: *Rev. Mod. Phys.* 82 (2010), p. 1225.

[15] P. Christodoulou et al. “Observation of first and second sound in a Berezinskii–Kosterlitz–Thouless superfluid”. In: *arXiv preprint* (2020). arXiv: 2008.06044 [cond-mat.quant-gas]. URL: https://arxiv.org/abs/2008.06044.

[16] Leon N. Cooper. “Bound Electron Pairs in a Degenerate Fermi Gas”. In: *Physical Review* 104.4 (1956), pp. 1189–1190. DOI: 10.1103/PhysRev.104.1189.

[17] F. Dalfovo et al. “Theory of Bose-Einstein condensation in trapped gases”. In: *Rev. Mod. Phys.* 71.3 (1999), p. 463.

[18] K. B. Davis et al. “Bose-Einstein Condensation in a Gas of Sodium Atoms”. In: *Phys. Rev. Lett.* 75 (1995), p. 3969.

[19] J. P. A. Devreese, J. Tempere e C. A. R. Sá de Melo. “Effects of Spin-Orbit Coupling on the Berezinskii-Kosterlitz-Thouless Transition and the Vortex-Antivortex Structure in Two-Dimensional Fermi Gases”. In: *Physical Review Letters* 113.16 (2014), p. 165304. DOI: 10.1103/PhysRevLett.113.165304.

[20] D. M. Eagles. “Possible Pairing Without Superconductivity at Low Carrier Concentrations in Bulk and Thin-Film Superconducting Semiconductors”. In: *Physical Review* 186.2 (1969), pp. 456–463. DOI: 10.1103/PhysRev.186.456.

[21] A. Einstein. “Quantentheorie des einatomigen idealen Gases. Erste Abhandlung”. In: *Sitzungsber. Preuss. Akad. Wiss.* 1925 (1925), p. 3.

[22] A. Einstein. “Quantentheorie des einatomigen idealen Gases. Zweite Abhandlung”. In: *Sitzungsber. Preuss. Akad. Wiss.* 1925 (1925), p. 18.

[23] Koichiro Furutani, Andrea Perali e Luca Salasnich. “Berezinskii–Kosterlitz–Thouless Phase Transition with Rabi-Coupled Bosons”. In: *Physical Review A* 107.4 (2023), p. L041302. DOI: 10.1103/PhysRevA.107.L041302.

[24] G. Gallavotti e F. Nicolò. “The screening phase transitions in the two-dimensional Coulomb gas”. In: *Journal of Statistical Physics* 39 (1985), pp. 133–156. DOI: 10.1007/BF01007976.

[25] E. P. Gross. “Structure of a quantized vortex in boson systems”. In: *Nuovo Cim.* 20.3 (1961), pp. 454–477. DOI: 10.1007/BF02731494.

[26] Zoran Hadzibabic et al. “Berezinskii–Kosterlitz–Thouless crossover in a trapped atomic gas”. In: *Nature* 441 (2006), pp. 1118–1121. DOI: 10.1038/nature04851.

[27] Zoran Hadzibabic et al. “Defects in the interference pattern of two-dimensional Bose gases”. In: *Phys. Rev. Lett.* 95.20 (2005), p. 201201. DOI: 10.1103/PhysRevLett.95.201201.

[28] Zoran Hadzibabic et al. “Direct imaging of vortices in a two-dimensional Bose gas”. In: *Phys. Rev. Lett.* 93.18 (2004), p. 180403. DOI: 10.1103/PhysRevLett.93.180403.

[29] Jorge V. José et al. “Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model”. In: *Physical Review B* 16.3 (1977), pp. 1217–1241. DOI: 10.1103/PhysRevB.16.1217.

[30] P. Kapitza. “Viscosity of Liquid Helium below the λ-Point”. In: *Nature* 141 (1938), p. 74.

[31] W. Ketterle. “Nobel lecture: When atoms behave as waves: Bose-Einstein condensation and the atom laser”. In: *Rev. Mod. Phys.* 74 (2002), p. 1131.

[32] M. Kobayashi, M. Eto e M. Nitta. “Berezinskii-Kosterlitz-Thouless Transition of Two-Component Bose Mixtures with Intercomponent Josephson Coupling”. In: *Physical Review Letters* 123.7 (2019), p. 075303. DOI: 10.1103/PhysRevLett.123.075303.

[33] J. M. Kosterlitz e D. J. Thouless. “Ordering, metastability and phase transitions in two-dimensional systems”. In: *Journal of Physics C: Solid State Physics* 6.7 (1973), pp. 1181–1203. DOI: 10.1088/0022-3719/6/7/010.

[34] L.D. Landau. “Theory of the superfluidity of helium II”. In: *Physical Review* 60 (1941). Short English-language note summarizing the J. Phys. (USSR) result, pp. 356–358.

[35] L.D. Landau e E.M. Lifshitz. *Statistical Physics — Part 2*. 2nd. Contains a detailed discussion of the Landau criterion for superfluidity. Oxford: Pergamon Press, 1980.

[36] A. J. Leggett. “Diatomic Molecules and Cooper Pairs”. In: *Modern Trends in the Theory of Condensed Matter*. A cura di Andrzej Pekalski e Jerzy Przystawa. Vol. 115. Lecture Notes in Physics. Springer, 1980, pp. 13–27. DOI: 10.1007/BFb0120843.

[37] Anthony J. Leggett. *Quantum Liquids: Bose Condensation and Cooper Pairing in Condensed-Matter Systems*. Oxford University Press, 2006. ISBN: 9780198526438.

[38] Fritz London e Heinz London. “The Electromagnetic Equations of the Supraconductor”. In: *Proceedings of the Royal Society A* 149.866 (1935), pp. 71–88. DOI: 10.1098/rspa.1935.0048.

[39] Petter Minnhagen e Mats Nylén. “Charge density of a vortex in the Coulomb-gas analogy of superconducting films”. In: *Physical Review B* 31.9 (1985), p. 5768.

[40] Mintu Mondal et al. “Role of the vortex-core energy on the Berezinskii–Kosterlitz–Thouless transition in thin films of NbN”. In: *Phys. Rev. Lett.* 107 (2011), p. 217003.

[41] P. A. Murthy et al. “Observation of the Berezinskii–Kosterlitz–Thouless phase transition in an ultracold Fermi gas”. In: *Phys. Rev. Lett.* 115 (2015).

[42] Heike Kamerlingh Onnes. “The resistance of pure mercury at helium temperatures”. In: *Communications from the Physical Laboratory at the University of Leiden* 120b (1911). First observation of superconductivity.

[43] S. Pilati, S. Giorgini e N. Prokof’ev. “Critical temperature of interacting Bose gases in two and three dimensions”. In: *Physical Review Letters* 100.14 (2008), p. 140405. DOI: 10.1103/PhysRevLett.100.140405.

[44] L. P. Pitaesvkii. “Vortex Lines in an Imperfect Bose Gas”. In: *Soviet Physics JETP* 13.2 (1961).

[45] C. A. Regal, M. Greiner e D. S. Jin. “Observation of Resonance Condensation of Fermionic Atom Pairs”. In: *Physical Review Letters* 92.4 (2004), p. 040403. DOI: 10.1103/PhysRevLett.92.040403.

[46] Luca Salasnich, P. A. Marchetti e F. Toigo. “Superfluidity, sound velocity, and quasi-condensation in the two-dimensional BCS-BEC crossover”. In: *Physical Review A* 88.5 (2013), p. 053612. DOI: 10.1103/PhysRevA.88.053612. URL: https://doi.org/10.1103/PhysRevA.88.053612.

[47] Adriaan M. J. Schakel. *Boulevard of Broken Symmetries: Effective Field Theories of Condensed Matter*. World Scientific, 2008. ISBN: 978-9812813909. URL: https://books.google.it/books/about/Boulevard_of_Broken_Symmetries.html?id=OAo-UoBdBkEC.

[48] Volker Schweikhard, Sheng-Jen Tung e Eric A. Cornell. "Vortex proliferation in the Berezinskii–Kosterlitz–Thouless regime on a two-dimensional lattice of Bose–Einstein condensates". In: *Phys. Rev. Lett.* 99.3 (2007), p. 030401. DOI: 10.1103/PhysRevLett.99.030401.

[49] H. T. C. Stoof, K. B. Gubbels e D. B. M. Dickerscheid. *Ultracold Quantum Fields*. Theoretical and Mathematical Physics. Springer, 2009. ISBN: 978-1-4020-9768-2. DOI: 10.1007/978-1-4020-9769-9.

[50] David R. Tilley e John Tilley. *Superfluidity and Superconductivity*. 3rd. Institute of Physics Publishing, 1990.

[51] M. W. Zwierlein et al. "Condensation of Pairs of Fermionic Atoms Near a Feshbach Resonance". In: *Physical Review Letters* 92.12 (2004), p. 120403. DOI: 10.1103/PhysRevLett.92.120403.