UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia “Galileo Galilei”

Master Degree in Physics

Final Dissertation

Effects of Non-Standard Neutrinos on Gravitational Waves

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Academic Year 2025/2026

“Laws of physics are invariant under notation.”

— Unknown

Abstract

It is well known that the primordial stochastic background of gravitational waves (GWs) can be damped due to the free-streaming of neutrinos during the propagation of GWs through cosmic structures. However, for some frequencies of the GWs an opposite process can take place due to free-streaming neutrinos, with an enhancement of the amplitude of GWs. This thesis aims first of all at reviewing such a phenomenon (that has direct implications for the GWs that can be measured through Cosmic Microwave Background anisotropies). It will then focus on scenarios of non-standard neutrinos (typically considered both in particle physics and cosmology as a possible solution to some experimental data anomalies). In particular, the aim is to understand whether the enhancement or further damping takes place and, if so, if it can place at the frequencies of Pulsar Timing Arrays (PTA) collaborations, given the importance of their first measurement of a stochastic gravitational wave background in 2023. This analysis can allow us to understand whether such measurements can put constraints on non-standard neutrino physics. We incorporate realistic thermodynamic evolution, including Standard Model particle de-relativization and phase transitions. Using numerical solutions of the tensor perturbation equation, we compute the primordial gravitational waves (PGWs) transfer function. The effects are evaluated across frequencies relevant to PTA experiments.

Contents

Listing of figures

Listing of tables

Listing of acronyms

SR .... Special Relativity

E.o.M. .... Equation of Motion

w.r.t. .... With Respect To

GR .... General Relativity

TT .... Traceless-Transverse

GWs .... Gravitational Waves

FLRW .... Friedmann–Lemaître–Robertson–Walker

NR .... Non-relativistic

CMB .... Cosmic Microwave Background

SM .... Standard Model

FRW .... Friedmann–Robertson–Walker

SSB .... Spontaneous Symmetry Breaking

GUT .... Grand Unified Theory

R.H.S. .... Right Hand Side

KG .... Klein Gordon

EFE .... Einstein Field Equation

2PCF .... 2 Point Correlation Function

PGWs .... Primordial Gravitational Waves

V-A .... Vector - Axial

CC .... Charged Current

NC .... Neutral Current

IVB .... Interacting Vector Boson

EFT .... Effective Field Theory

EM .... Electromagnetic

LH .... Left Handed

LH .... Right Handed

CKM .... Cabibbo–Kobayashi–Maskawa

PMNS .... Pontecorvo–Maki–Nakagawa–Sakata

BBN .... Big Bang Nucleosynthesis

EW .... Electroweak

R.D. .... Radiation Dominated

M.D. .... Matter Dominated

d.o.f. .... Degrees of Freedom

CνB .... Cosmic Neutrino Background

WI .... Weak Interaction

QFT .... Quantum Field Theory

QGP .... Quark Gluon Plasma

1
Introduction

The primordial stochastic background of gravitational waves (PGWs) offers a powerful probe of the early universe, carrying information from energy scales and epochs otherwise inaccessible to electromagnetic observations. These tensor perturbations, generated during inflation, propagate nearly freely across cosmic history. However, their evolution is not entirely trivial: interactions with the surrounding Standard Model particles, particularly through the anisotropic stress of free-streaming neutrinos, can leave measurable imprints in the gravitational wave spectrum observable today.

The central aim of this thesis is to investigate how standard and non-standard neutrino physics affects the evolution of PGWs across cosmological time, especially for the frequencies accessible to current and near-future experiments such as Pulsar Timing Arrays (PTAs). Special attention is given to the damping effects induced by free-streaming neutrinos, and whether these effects could be modified by secret neutrino interactions (SνI) mediated by a massive scalar. These scenarios are motivated by theoretical extensions of the Standard Model that also address some experimental anomalies.

To this end, we perform both analytic and numerical computations of the PGW transfer function $\chi(u)$ by solving the tensor perturbation equation, taking into account the temperature-dependent scale factor derived from realistic evolution of the effective degrees of freedom $g_*(T)$ and $g_{*s}(T)$. Our analysis incorporates physical processes such as the de-relativization of SM particles, electron-positron annihilation, QCD and electroweak phase transitions, and the onset of neutrino free-streaming, as well as the presence of additional neutrino self-interactions.

The content of the thesis is organized as follows:

- **Chapter 2** reviews the foundations of General Relativity and the role of gravitational waves as tensor perturbations on a cosmological background.

- **Chapter 3** introduces the inflationary paradigm, generation of tensor modes, the reheating era, and analytic solutions for the evolution of PGWs in radiation- and matter-dominated eras.

- **Chapter 4** covers the neutrinos described within the Standard Model, their decoupling, and the effects of free-streaming, as well as theoretical motivations and phenomenology of secret neutrino interactions.

- **Chapter 5** presents the main numerical work of the thesis, where we solve for the PGW transfer function with realistic inputs, analyze the impact of SM particle de-relativization, and study the damping signatures of neutrino free-streaming and non-standard interactions.

- **Chapter 6** concludes with a discussion of the main results and outlines possible future directions for applying this framework to constrain new physics with gravitational wave observations.

2

General Relativity

2.1 Introduction

We will start by introducing the basic concepts from special relativity (SR). Every inertial frame is physically equivalent and the speed of light is constant in all frames. We define an ”event” as a point in spacetime, while we associate every frame with a set of spacetime coordinates. Convention in Cartesian coordinates is $x^{\mu} = (ct, x, y, z)$.

Spacetime interval between two events [1] is

$$\Delta s^2 = \eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu} \tag{2.1}$$

with $\Delta x^{\mu} = x^{\mu}_B - x^{\mu}_A$. Infinitesimally, this becomes $ds^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu}$. One can classify type of spacetime intervals as

$$\begin{aligned} \Delta s^2 &> 0 \ , \ space - like \\ \Delta s^2 &< 0 \ , \ time - like \\ \Delta s^2 &= 0 \ , \ light - like/null \end{aligned} \tag{2.2}$$

Classical trajectories in SR becomes world-lines which are paths in spacetime, characterized by affine parameter $\lambda \in R \to x^{\mu}(\lambda) \in R^4$ so the evolution is given by $\dot{x}^{\mu} = \frac{dx^{\mu}(\lambda)}{d\lambda}$

2.2 Dynamics of Point Particles

The metric contains the information about gravity. The equation of motion (E.o.M.) should respect the followings:

- They need to be invariant under coordinate transformations.

- For flat metric, E.o.M. should represent straight lines.

- E.o.M. should reduce to Newtonian one in proper limit.

For a free particle, 4-acceleration should be null $\alpha^\mu = 0$ in other words $\frac{d^2x^\mu}{d\tau^2} = \frac{du^\mu}{d\tau} = 0$ where τ in this case is the proper time (which depends on the trajectory). In the instantaneous rest frame $dx^i = 0$

$$-ds^2 = d\tau^2 = -\eta_{\mu\nu}dx^\mu(\lambda)dx^\nu(\lambda) = -\dot{x}^\mu\dot{x}_\mu d\lambda^2 \tag{2.3}$$

Here, dots are with respect to (w.r.t.) parameter λ. One can now express $\frac{d}{d\tau} = \frac{1}{\sqrt{-\dot{x}^\mu\dot{x}_\mu}}\frac{d}{d\lambda}$, which gives the four-acceleration

$$\alpha^\mu = \frac{d}{d\lambda}\left(\frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\right) = 0 \tag{2.4}$$

One can as well see this through the principle of action

$$S = -m\int_{initial}^{final} d\tau = -m\int d\lambda\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \tag{2.5}$$

The variation of the action gives

$$\begin{aligned}\delta S = 0 = -m\int_{in.}^{fin.} d\lambda\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} &= m\int_{in.}^{fin.} d\lambda\frac{\eta_{\mu\nu}\dot{x}^\nu\delta\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\\ = m\int_{in.}^{fin.} d\lambda\delta x^\mu\eta_{\mu\nu}&\underbrace{\frac{d}{d\lambda}\left(\frac{\dot{x}^\nu}{\sqrt{-\dot{x}^\mu\dot{x}_\mu}}\right)}_{=0} + (surface\ \ term)\end{aligned} \tag{2.6}$$

In the last step, integration by parts has been used. For a massless particle, proper time is not well-defined and one can not set $m = 0$ in the action. One can use the following trick $ds^2 =$

$h(\lambda)d\lambda^2 = -e^2(\lambda)d\lambda^2$, then one gets

$$\tilde{S} = -\frac{1}{2}\int d\lambda\sqrt{-h(\lambda)}\left(h^{-1}\dot{x}^\mu\dot{x}_\mu + m^2\right) = \frac{1}{2}\int d\lambda\left(e^{-1}\dot{x}^\mu\dot{x}_\mu - m^2 e\right) \tag{2.7}$$

From the variation, one obtains

$$\delta\tilde{S} = \frac{1}{2}\int d\lambda\delta e\left(-\frac{1}{e^2}\dot{x}^\mu\dot{x}_\mu - m^2\right) = 0 \tag{2.8}$$

Hence $e = \frac{\sqrt{-\dot{x}^\mu\dot{x}_\mu}}{m}$. For massive particles we have $e = \frac{1}{m}$ since $u^\mu u_\mu = -1$, while for massless particles we have $me = \sqrt{-\dot{x}^\mu\dot{x}_\mu} = 0$ so $g_{\mu\nu}dx^\mu dx^\nu = 0$. Four momentum is given by $p^\mu = e^{-1}\dot{x}^\mu$.

In curved spacetime, action with a general metric becomes

$$\tilde{S} = -\frac{1}{2}\int d\lambda(e^{-1}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu - m^2 e) \tag{2.9}$$

where $e = \frac{\sqrt{-g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}}{m}$. E.o.M for $x^\mu(\lambda)$ from the variation of the action is

$$\begin{aligned}\delta\tilde{S} = 0 &= \frac{1}{2}\int d\lambda e^{-1}\delta[g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu] = \frac{1}{2}\int d\lambda e^{-1}[2g_{\mu\nu}(x)\dot{x}^\nu\delta\dot{x}^\mu + \partial_\mu g_{\nu\rho}(x)\delta x^\mu\dot{x}^\nu\dot{x}^\rho] \\ &= \int d\lambda e^{-1}\left[g_{\mu\nu}(x)\delta\dot{x}^\mu\dot{x}^\nu + \frac{1}{2}\delta x^\mu\partial_\mu g_{\nu\rho}(x)\dot{x}^\nu\dot{x}^\rho\right] \\ &= -\int d\lambda\delta x^\mu\underbrace{\left[\frac{d}{d\lambda}(e^{-1}g_{\mu\nu}\dot{x}^\nu) - \frac{1}{2}e^{-1}\partial_\mu g_{\nu\rho}\dot{x}^\nu\dot{x}^\rho\right]}_{=0}\end{aligned} \tag{2.10}$$

We have exploited $\delta\dot{x}^\mu = \frac{d}{d\lambda}\delta x^\mu$. From the term in brackets, one obtains the geodesic equation

$$\begin{aligned}0 &= \frac{d}{d\lambda}(e^{-1}g_{\mu\nu}\dot{x}^\nu) - \frac{1}{2}e^{-1}\partial_\mu g_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = g_{\mu\nu}\ddot{x}^\nu + \partial_\rho g_{\mu\nu}\dot{x}^\rho\dot{x}^\nu - \frac{1}{2}\partial_\mu g_{\nu\rho}\dot{x}^\nu\dot{x}^\rho \\ &= g_{\mu\sigma}\left[\ddot{x}^\sigma + \frac{1}{2}g^{\sigma\tau}\left(-\partial_\tau g_{\nu\rho} + \partial_\rho g_{\nu\tau} + \partial_\nu g_{\rho\tau}\right)\dot{x}^\rho\dot{x}^\nu\right]\end{aligned} \tag{2.11}$$

Hence, the term in brackets produces the geodesic equation as

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0 \tag{2.12}$$

Where the Christoffel symbols are defined as

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\rho} \left(g_{\rho\sigma,\nu} + g_{\nu\sigma,\rho} - g_{\nu\rho,\sigma} \right) \tag{2.13}$$

with the following property that they are symmetric under the exchange of lower indices $\Gamma^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\rho\nu}$. In the most generic case, lower indices can be symmetric or anti-symmetric. Torsion is defined in terms of the connection [2] as follows

$$T^{\mu}_{\nu\rho} \equiv \Gamma^{\mu}_{\nu\rho} - \Gamma^{\mu}_{\rho\nu} \tag{2.14}$$

The connection is not uniquely defined but there is one special connection called Levi-Civita connection with properties:

$$\begin{aligned} T^{\mu}_{\nu\rho} &= 0 \qquad (\textit{torsion free}) \\ \nabla_{\mu} g_{\nu\rho} &= 0 \qquad (\textit{metric coonnection}) \end{aligned} \tag{2.15}$$

2.3 Curvature and Gravity

In this section, we will find a way to characterize honest gravitational field in a way that for a given metric $g_{\mu\nu}(x)$ there is no inertial frame $\hat{x}^{\mu}$ in which $ds^2 = g_{\mu\nu}(x) dx^{\mu} dx^{\nu} = \eta_{\mu\nu} d\hat{x}^{\mu} d\hat{x}^{\nu}$.

Now consider two independent infinitesimal displacements $\delta_1 x^{\mu}$ and $\delta_2 x^{\mu}$ where we parallel transport a vector field V^{μ} from point A to D through 1) $A \xrightarrow{\delta_1 x^{\mu}} B \xrightarrow{\delta_2 x^{\mu}} D$ and 2) $A \xrightarrow{\delta_2 x^{\mu}} C \xrightarrow{\delta_1 x^{\mu}} D$

$$\begin{aligned} V^{\mu}_{D}|_{1\to 2} &= V^{\mu}_{B} - \Gamma^{\mu}_{\nu\rho}(x_B) V^{\mu}_{B} \delta_2 x^{\rho} \\ &= V^{\mu}_{A} - \Gamma^{\mu}_{\nu\rho}(x_A) V^{\mu}_{A} \delta_1 x^{\rho} - \Gamma^{\mu}_{\nu\rho}(x_A) V^{\mu}_{A} \delta_2 x^{\rho} - \partial_{\sigma} \Gamma^{\mu}_{\nu\rho}(x_A) V^{\nu}_{A} \delta_1 x^{\sigma} \delta_2 x^{\rho} \\ &+ \Gamma^{\mu}_{\nu\rho}(x_A) \Gamma^{\nu}_{\alpha\sigma}(x_A) V^{\alpha}_{A} \delta_1 x^{\sigma} \delta_2 x^{\rho} \end{aligned} \tag{2.16}$$

In the derivation above, we used $V^{\nu}_{B} = V^{\nu}_{A} - \Gamma^{\nu}_{\alpha\sigma} V^{\alpha} \delta_1 x^{\sigma}$ and $\Gamma^{\mu}_{\nu\rho}(x_B) = \Gamma^{\mu}_{\nu\rho}(x_A) + \partial_{\sigma} \Gamma^{\mu}_{\nu\rho}(x_A) \delta_1 x^{\sigma}$. Then we obtain

$$V^{\mu}_{D}|_{1\to 2} - V^{\mu}_{D}|_{2\to 1} = -R^{\mu}_{\nu\rho\sigma}(x_A) V^{\nu}_{A} \delta_1 x^{\rho} \delta_2 x^{\sigma} \tag{2.17}$$

with the Riemann curvature tensor being

$$R^{\mu}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\sigma,\rho} - \Gamma^{\mu}_{\nu\rho,\sigma} + \Gamma^{\mu}_{\alpha\rho} \Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\alpha\sigma} \Gamma^{\alpha}_{\nu\rho} \tag{2.18}$$

It is purely local since we considered infinitesimal difference, this tells us how much spacetime is curved. $R^{\mu}_{\nu\rho\sigma} \neq 0 \leftrightarrow$ *spacetime is curved*, $R^{\mu}_{\nu\rho\sigma} = 0 \leftrightarrow$ *spacetime is locally flat*. Since $[R^{\mu}_{\nu\rho\sigma}] = L^{-2}$, $(R^{\mu}_{\nu\rho\sigma})^{-1/2}$ gives an estimate of length scale above which spacetime can not be approximated by a flat one.

The properties of the Riemann curvature tensor are

$$\begin{aligned} &R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} \quad , \quad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \quad , \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \\ &R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 \end{aligned} \tag{2.19}$$

Another important tensor we will encounter is the Ricci tensor given as

$$R_{\mu\nu} = R^{\rho}_{\mu\rho\nu} \tag{2.20}$$

Note that the Ricci tensor is symmetric $R_{\nu\mu} = R^{\rho}_{\nu\rho\mu} = g^{\rho\tau} R_{\tau\nu\rho\mu} = g^{\rho\tau} R_{\rho\mu\tau\nu} = R^{\tau}_{\mu\tau\nu} = R_{\mu\nu}$ and finally the Ricci scalar is given by

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}_{\mu} \tag{2.21}$$

2.4 Einstein Field Equations

Our guess for the action is almost in the usual form but one has to identify a volume element $dVol_n$ in a coordinate-independent way to integrate over our spacetime. In n-dimensional metric space, natural coordinate independent volume element is

$$dV \equiv dVol_n \equiv d^n x \sqrt{|\det g|} \equiv d^n x \sqrt{|g|} \tag{2.22}$$

One can see the invariance of the volume through

$$d\tilde{Vol}_n(\tilde{x}) = d^n \tilde{x} \sqrt{|\det \tilde{g}(\tilde{x})|} = d^n x \sqrt{|\det g(x)|} \tag{2.23}$$

Where we have exploited the following:

$$d^4 \tilde{x} = d^4 x \left| \det \frac{\partial \tilde{x}^{\mu}}{\partial x^{\nu}} \right| \qquad , \qquad \det \tilde{g} = \left(\frac{\partial x}{\partial \tilde{x}} \right)^2 \det g \tag{2.24}$$

The latter is obtained from the transformation of the metric tensor

$$\tilde{g}_{\mu\nu} = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma} \tag{2.25}$$

Now we have the tools to define Einstein-Hilbert action [1] as

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R \tag{2.26}$$

Since the action has to be dimensionless, one expects $[\kappa] = M^{-2} \propto M_{pl}^{-2}$. To derive the Einstein equations, take the variation of the S_{EH}

$$\delta S_{EH} = \delta\left(\int d^4x\sqrt{-g}g^{\mu\nu}R_{\mu\nu}\right) = \int d^4x(\underbrace{\delta\sqrt{-g}R}_{1} + \sqrt{-g}\delta g^{\mu\nu}R_{\mu\nu} + \underbrace{\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu}}_{2}) \tag{2.27}$$

We are going to use the following identity $\log(\det g) = \log(\Pi_a\lambda_a) = \sum_a \log\lambda_a = Tr(\log g)$ in 1

$$\delta\det g = \delta[e^{\log\det g}] = \delta e^{Tr(\log g)} = e^{Tr(\log g)}Tr(\delta\log g) = \det g Tr(g^{-1}\delta g) = -\det g g_{\mu\nu}\delta g^{\mu\nu} \tag{2.28}$$

So, finally the variation in 1 becomes

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{2.29}$$

While the variation in 2 will be derived from $\delta R_{\mu\nu} = \delta R^\rho_{\mu\rho\nu}$. Let us first do it for Riemann tensor

$$\begin{aligned}\delta R^\rho_{\mu\sigma\nu} &= \delta\left[\partial_\sigma\Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} - (\sigma\leftrightarrow\nu)\right]\\ &= \partial_\sigma\delta\Gamma^\rho_{\mu\nu} + \delta\Gamma^\rho_{\lambda\sigma}\Gamma^\lambda_{\mu\nu} + \Gamma^\rho_{\lambda\sigma}\delta\Gamma^\lambda_{\mu\nu} - \partial_\nu\delta\Gamma^\rho_{\mu\sigma} - \delta\Gamma^\rho_{\lambda\nu}\Gamma^\lambda_{\mu\sigma} - \Gamma^\rho_{\lambda\nu}\delta\Gamma^\lambda_{\mu\sigma}\\ &= \nabla_\sigma\delta\Gamma^\rho_{\mu\nu} - \nabla_\nu\delta\Gamma^\rho_{\mu\sigma}\end{aligned} \tag{2.30}$$

This gives us the variation in Ricci tensor

$$\delta R_{\mu\nu} = \nabla_\rho\delta\Gamma^\rho_{\mu\nu} - \nabla_\nu\delta\Gamma^\sigma_{\mu\sigma} \tag{2.31}$$

Putting it all together, one obtains

$$\delta S_{EH} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \delta g^{\mu\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \tag{2.32}$$

Where the term inside the parenthesis is the Einstein tensor $G_{\mu\nu}$ and $\kappa = 8\pi G$ so Einstein equation is

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.33}$$

with $T_{\mu\nu}$ is being the energy-momentum tensor. Full decomposition can include also a term $\propto \frac{\Lambda}{\kappa} \int d^4x \sqrt{-g} \subset S_{EH}$, so the complete Einstein field equation taking into account the dark energy (cosmological constant in this case) becomes [3]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{2.34}$$

where speed of light is taken in natural units $c = 1$.

2.5 Gravitational Waves

If we perturb the gravitational field by some event, perturbation propagates along space in the form of gravitational waves.

One can derive linearized Einstein equations in empty space (in the absence of energy-momentum tensor), expanding around Minkowski spacetime in weak field approximation [4] $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$

$$\begin{aligned} R_{\nu\rho} &= R^{\mu}_{\nu\mu\rho} \simeq \partial_\mu \Gamma^\mu_{\nu\rho} - \partial_\rho \Gamma^\mu_{\nu\mu} + \mathcal{O}(h^2) \\ &\simeq \frac{1}{2} \partial^\sigma \left(\partial_\nu h_{\sigma\rho} + \partial_\rho h_{\sigma\nu} - \partial_\sigma h_{\nu\rho} \right) - \frac{1}{2} \partial_\rho \big(\partial_\nu h^\mu_\mu + \underbrace{\partial^\mu h_{\mu\nu} - \partial^\mu h_{\nu\mu}}_{=0} \big) \\ &\simeq \frac{1}{2} \left(\partial_\nu \partial^\sigma h_{\sigma\rho} + \partial_\rho \partial^\sigma h_{\sigma\nu} - \partial_\nu \partial_\rho h - \Box h_{\nu\rho} \right) = 0 \end{aligned} \tag{2.35}$$

Performing $\nu \leftrightarrow \mu$ and $\rho \leftrightarrow \nu$, we obtain

$$\Box h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\sigma h_{\sigma\nu} - \partial_\nu \partial^\sigma h_{\sigma\mu} = 0 \tag{2.36}$$

To remove the gauge-freedom, use gauge-fixing condition, the de-Donder gauge [5]. Then 2.36

simplifies into

$$\Box h_{\mu\nu} = 0 \tag{2.37}$$

which is the wave equation in case of gravity.

2.5.1 Physical Effects of Gravitational Waves

One can see the effect in the TT-gauge for GWs traveling along the z-direction [1]

$$ds^2 \approx \eta_{\mu\nu} dx^\mu dx^\nu + h_+(t-z)(dx^2 - dy^2) + 2h_\times(t-z)dxdy \tag{2.38}$$

In the TT-gauge $x^0(t) = t, x^i(t) = const.$ describes time-like geodesics. In the absence of source, single point particle will not be affected by GWs. Geodesic equation then reads

$$\frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{dt}\frac{dx^\rho}{dt} = \Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\nu}(2\partial_0 h_{0\nu} - \partial_\nu h_{00}) = 0 \tag{2.39}$$

So particles are at rest in TT-gauge. However, physical distance between two geodesics will change. The best way to understand GW is via geodesic deviation equation (it takes simple form in freely falling frame). Consider two nearby geodesics and two particles A and B separated by εS^μ. Relative acceleration is

$$\hat{A}^\alpha = \frac{d^2\hat{S}^\alpha}{dt^2} = \hat{R}^\alpha_{00\beta}\hat{S}^\beta \tag{2.40}$$

where freely-falling basis $\hat{e}^\alpha_\mu = \delta^\alpha_\mu + \mathcal{O}(h)$ and $\hat{e}^\alpha = \hat{e}^\alpha_\mu dx^\mu \simeq \delta^\alpha_\mu dx^\mu + \mathcal{O}(h)$ while $\hat{R}^\alpha_{00\beta} \simeq R^\alpha_{00\beta} + \mathcal{O}(h^2)$. Using Riemann tensor $R_{a00b} \simeq \frac{1}{2}\ddot{h}_{ab}$, we get

$$\begin{aligned}
\frac{d^2\hat{S}^0}{dt^2} &= 0 \\
\frac{d^2\hat{S}^3}{dt^2} &= 0 \\
\frac{d^2\hat{S}^1}{dt^2} &= R^1_{001}\hat{S}^1 + R^1_{002}\hat{S}^2 = \frac{1}{2}\ddot{h}_+\hat{S}^1 + \frac{1}{2}\ddot{h}_\times\hat{S}^2 \\
\frac{d^2\hat{S}^3}{dt^2} &= R^2_{001}\hat{S}^1 + R^2_{002}\hat{S}^2 = \frac{1}{2}\ddot{h}_\times\hat{S}^1 - \frac{1}{2}\ddot{h}_+\hat{S}^2
\end{aligned} \tag{2.41}$$

Which gives the following solutions:

$$
\begin{aligned}
\hat{S}^1 &= \hat{S}_0^1 + \frac{1}{2}h_+\hat{S}_0^1 + \frac{1}{2}h_\times\hat{S}_0^2 \\
\hat{S}^2 &= \hat{S}_0^2 + \frac{1}{2}h_\times\hat{S}_0^1 - \frac{1}{2}h_+\hat{S}_0^2
\end{aligned}
\tag{2.42}
$$

Where the second and the third terms are the perturbations $\Delta\vec{S}$

- Assume $h_\times = 0, h_+ \neq 0$

This produces

$$
\begin{aligned}
\Delta\hat{S}^1 &= \frac{1}{2}h_+\hat{S}_0^1 \\
\Delta\hat{S}^2 &= -\frac{1}{2}h_+\hat{S}_0^2
\end{aligned}
\tag{2.43}
$$

so fluctuations ΔS are proportional to initial position $\hat{S}_0^i$.

- Assume $h_+ = 0, h_\times \neq 0$

This produces

$$
\begin{aligned}
\Delta\left(\hat{S}^1 + \hat{S}^2\right) &= \frac{1}{2}h_\times\left(\hat{S}_0^1 + \hat{S}_0^2\right) \\
\Delta\left(\hat{S}^1 - \hat{S}^2\right) &= -\frac{1}{2}h_\times\left(\hat{S}_0^1 - \hat{S}_0^2\right)
\end{aligned}
\tag{2.44}
$$

Therefore

$$
\begin{aligned}
\Delta\hat{S}^1 &= \frac{1}{2}h_\times\hat{S}_0^2 \\
\Delta\hat{S}^2 &= \frac{1}{2}h_\times\hat{S}_0^1
\end{aligned}
\tag{2.45}
$$

These effects are visualized as [6]

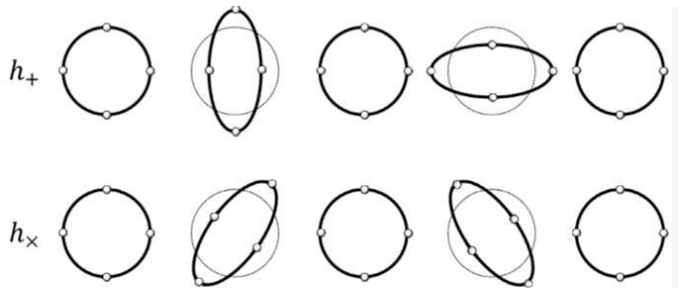

Figure 2.1: Physical effects of gravitational waves.

where for h_+, the second deformation is due to $h_+ < 0$ while the fourth deformation is due to $h_+ > 0$.

This chapter introduces the fundamentals of General Relativity, derivation of Einstein Field Equations, and the physics of gravitational waves (GWs). It sets the theoretical foundation for how gravitational waves propagate in the universe.

3

Inflationary Cosmology

3.1 Introduction

The Friedmann–Lemaître–Robertson–Walker (FLRW) metric for an expanding universe is[1]

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\Omega^2\right] \tag{3.1}$$

The solid angle is $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ and $k = +1, -1, 0$ implies closed, open and flat universe respectively. There is no time translation invariance because the universe has to evolve.

The Friedmann equations derived from the Einstein field equation combined with equation of state are [1]

$$\begin{aligned}
H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \\
\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \\
\dot{\rho} &+ 3H(\rho + p) = 0 \\
p &= w\rho
\end{aligned} \tag{3.2}$$

with the following

$$w : \begin{cases} 0 & \text{N.R. matter} \\ 1/3 & \text{Radiation} \\ -1 & \text{Dark energy} \end{cases}$$

The cosmological constant Λ is very small at early times so one can neglect it. Evolution of the scale factor and the energy density are given by

$$a(t) = a_* \left(\frac{t}{t_*}\right)^{\frac{2}{3(1+w)}} \quad , \quad \rho = \rho_* \left(\frac{a}{a_*}\right)^{-3(1+w)} \tag{3.3}$$

Which gives $\rho \propto a^{-3}$ in matter-dominated, $\rho \propto a^{-4}$ in radiation-dominated and $\rho \propto const.$ in dark energy-dominated eras.

Before diving into the discussion of inflation, let us first briefly talk about some concepts. Physical distances are those which incorporates the effect of the cosmic expansion on the co-moving scales $\lambda_{phys.} = a(t)\lambda_{comov.}$. Comoving scales are constant during the evolution of the universe.

The cosmological horizon/Hubble horizon is defined to be the maximum distance from which we received a light signal within the age of the universe. These regions are the boundaries of causally connected patches of spacetime

$$d_H(t) = a(t) \int_0^t \frac{cdt'}{a(t')} \tag{3.4}$$

From the metric given, since photons follow null geodesics $ds^2 = 0$, comoving distance becomes

$$l = \int_0^t \frac{cdt'}{a(t')} \tag{3.5}$$

If $d_H(t)$ exists and finite, it is called the particle horizon. Hubble radius instead, is the distance from which objects in the universe are moving away from us at the speed of light given as

$$R_c(t) = \frac{c}{H(t)} = \frac{3}{2}(1+w)t = \frac{1+3w}{2} d_H(t) \tag{3.6}$$

In the last step, we correlated Hubble radius with the Hubble horizon. Comoving Hubble radius is $r_H(t) = R_c(t)/a(t)$.

3.2 Problems of Hot Big-Bang & Cosmic Inflation as a Solution

- Horizon Problem

Without inflation, comoving Hubble radius always grows. So as time passes, larger and larger scales can cross the horizon. However, from the first CMB observation [7] we found out that very far even the opposite regions in the universe, share the same temperature without having been in causal connection before. According to the hot Big-Bang model, only the small patches in the figure below could have had the same temperature. This is called the Horizon problem. [8], presented in the figure below [9]

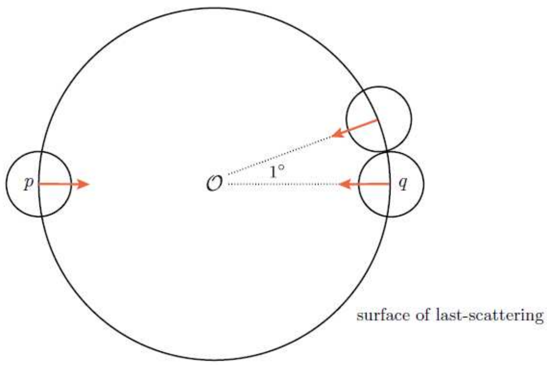

Figure 3.1: Regions of causally connected patches of universe in distant sky predicted by hot Big-Bang model.

Consider now the FLRW metric in conformal time given by

$$ds^2 = a^2(\tau)\left[-d\tau^2 + d\chi^2 + f_k(\chi)\, d\Omega^2\right] \tag{3.7}$$

Photons follow null geodesics, so the lines in conformal coordinates with 45° angle in Minkowski diagrams are light-like $d\chi = \pm d\eta \equiv \pm d\tau$. $\tau = 0$ is the singularity in the hot Big-Bang model. Let p and q be two photons from CMB: if we follow back their geodesics, we see that they could not have interacted with each other back in time due to the singularity at $\tau = 0$ [9].

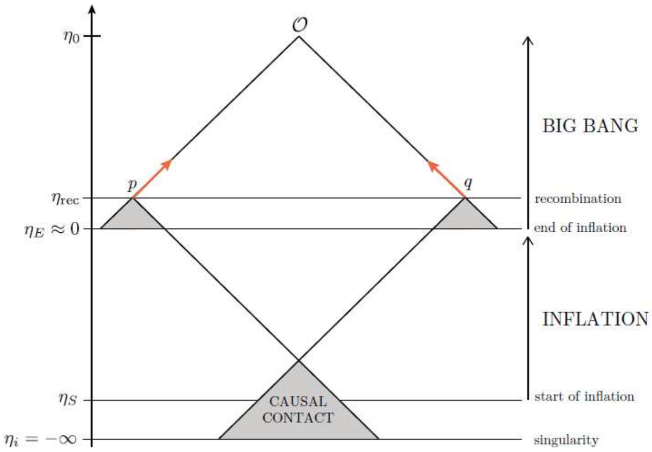

Figure 3.2: How inflation solves the horizon problem.

However, inflation pushes this singularity to infinitely back in time $\tau = -\infty$, which allows all photons we have observed in any direction in the sky to be in causal connection in the past. In other words, inflation makes $r_H(t)$ to decrease, causing a lot of scales λ to exit it. Once inflation ends, $r_H(t)$ grows again so scales λ re-enter the horizon, this time with a specific feature that they have been in causal connection when the scales were inside the horizon.

So far we have explained the solution but let us now present the details. Decreasing of co-moving Hubble radius $r_H(t) = 1/\dot{a}$ during inflation as shown in the figure below [10], implies $\dot{r}_H(t) < 0$, this would give

$$\dot{r}_H(t) = -\frac{\ddot{a}}{\dot{a}^2} < 0 \tag{3.8}$$

meaning $\ddot{a} > 0$ so the universe has to expand in an accelerating way during inflation. In other words, having exploited Friedmann equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \tag{3.9}$$

given $p = w\rho$, to have an acceleration one should have $w < -1/3$. Remember cosmological constant has $w = -1$, however it never ends. Inflation has a much more dynamic feature than that of cosmological constant.

One quantity that is important in our discussion is the number of e-folds, which tells us how much the universe expanded

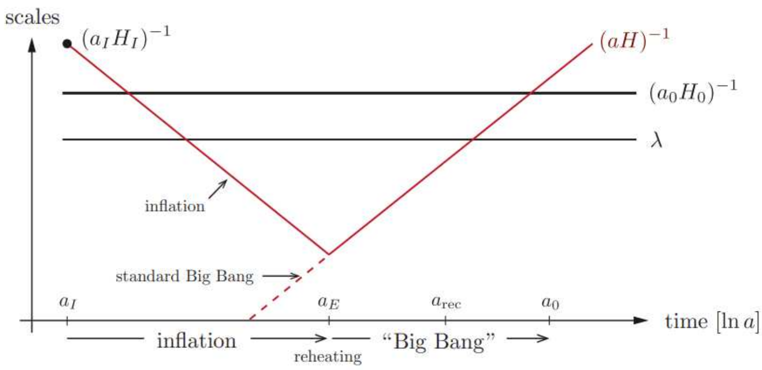

Figure 3.3: Evolution of the comoving Hubble radius.

Number of e-folds is

$$N = \int_{t_i}^{t_f} H(t)dt = \ln \frac{a_f}{a_i} \rightarrow e^N = \frac{a_f}{a_i} \tag{3.10}$$

To solve the horizon problem, we require $r_H(t_i) \geq r_H(t_0)$

$$\frac{a_f}{a_i}\frac{1}{H_i} \geq \frac{a_f}{a_0}\frac{1}{H_0} \rightarrow e^N \geq \frac{a_f}{a_0}\frac{H_i}{H_0} \tag{3.11}$$

Using the scale factor-temperature relation $T \propto a^{-1}$, one obtains

$$N \geq \ln \frac{T_0}{H_0} + \ln \frac{H_i}{T_f} \tag{3.12}$$

The first term is known values of the temperature and Hubble parameter today $T_0 = 10^{-13} GeV$ and $H_0 = 10^{-42} GeV$. The second term is model dependent since it captures the T_f reheating temperature (when inflation ends).

Assuming quasi-de-Sitter inflation ($H \approx const.$), we can understand the effect of the second term on the number of e-folds. Exploiting the Friedmann equation at the end of inflation when the universe was radiation dominated $H^2(t_i) = H^2(t_f)$

$$H^2(t_i) = H^2(t_f) = \frac{8\pi}{3}\Big(\frac{\pi^2}{30} g_* \frac{T_f^2}{M_{pl}^2}\Big) \tag{3.13}$$

which gives $H_{inf} \simeq T_f^2/M_{pl}$. Equation 3.12 becomes

$$N \geq \underbrace{\ln \frac{T_0}{H_0}}_{\approx 67} + \underbrace{\ln \frac{T_f}{M_{pl}}}_{\approx[-11,0]} \tag{3.14}$$

where the latter is set by observation and the fact that reheating can take place at most around the Planck scale.

- Flatness/Fine-Tuning Problem

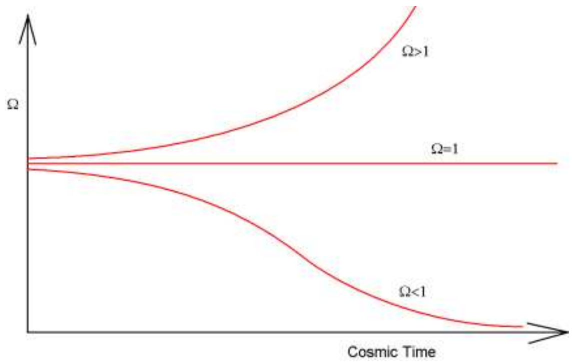

Figure 3.4: Evolution of the density parameter.

This problem of the conventional hot Big-Bang model can be seen through the Friedmann equation with the curvature contributions by

$$1 = \frac{8\pi G}{3H^2}\rho - \frac{k}{a^2H^2} = \frac{\rho}{\rho_c} - \Omega_k \tag{3.15}$$

We know from observations that $|\Omega_k| = |\Omega(t_0) - 1| < 10^{-3}$. Also notice that $\Omega_k = kr_H^2(t) = \frac{k}{a^2H^2}$. For a matter dominated universe, $|1 - \Omega| \sim t^{2/3}$, for radiation dominated universe $|1 - \Omega| \sim t$ so if Ω is closer to 1 today, it had to be much closer to 1 in the past. We used $\Omega = \Omega_m + \Omega_\Lambda$. If $\Omega \gg 1$, universe would have collapsed on itself before making galaxies, while If $\Omega \ll 1$, it would have expanded so rapidly that structures would not have formed.

If the universe is almost flat now, it had to be extremely flat at early times, which requires very fine-tuned initial conditions to explain the flatness we observe today. This is what we call the flatness problem. [8] From Friedmann equations, we know $\rho_c a^2 - \rho a^2 = -3k/8\pi G$, this can be put in terms of density parameter $(\Omega^{-1} - 1)\rho a^2 = -3k/8\pi G = const.$ in time.

Since r_H decreases very fast during inflation, density parameter asymptotically approaches to 1. Inflation solves the fine-tuning problem if

$$\frac{1-\Omega_i^{-1}}{1-\Omega_0^{-1}} \geq 1 \tag{3.16}$$

meaning deviations at the beginning are much more than today

$$\frac{\Omega_i^{-1}-1}{\Omega_0^{-1}-1} = \frac{\rho_0 a_0^2}{\rho_i a_i^2} = \frac{\rho_0 a_0^2}{\rho_{eq} a_{eq}^2}\frac{\rho_{eq} a_{eq}^2}{\rho_f a_f^2}\frac{\rho_f a_f^2}{\rho_i a_i^2} = \underbrace{\left(\frac{a_0}{a_{eq}}\right)^{-1}\left(\frac{a_{eq}}{a_f}\right)^{-2}}_{X^{-1}}\left(\frac{a_f}{a_i}\right)^{-(1+3w_{inf})} \geq 1 \tag{3.17}$$

where we have decomposed different epochs and used energy density scale factor relation. Using the $T-a$ relationship [11] in X, we obtain

$$X = \frac{a_0 a_{eq}}{a_f^2} = \frac{a_{eq}}{a_0}\frac{a_0^2}{a_f^2} = (1+z_{eq})^{-1}\left(\frac{T_f}{T_{pl}}\right)^2\left(\frac{T_{pl}}{T_0}\right)^2 = 10^{60}\left(\frac{T_f}{T_{pl}}\right)^2 \tag{3.18}$$

3.17 gives the number of e-folds to be

$$e^{-N(1+3w_{inf})} \geq X \longrightarrow N \geq \frac{\ln X}{|1+3w_{inf}|} \tag{3.19}$$

Assuming quasi-de-Sitter inflation $w_{inf} = -1$, it sets $N_{min} \sim [50, 70]$. Inflation has an attractor mechanism so for any given initial conditions, inflationary expansion straightens out the curvature $[\Omega^{-1}(t_0)-1] = 10^{-N}\cdot[\Omega_i^{-1}-1]$. One could have already noticed that the horizon and flatness problems are related so once the horizon problem is solved, thr flatness problem is automatically solved and vice-versa.

- Unwanted Relics

In the early universe at high temperatures, topological defects can be produced [12] which might contribute to the density parameter today, which in fact over-close the universe we observe today. This is called the unwanted relics problem.

These relics are usually a consequence of symmetry breaking of some underlying symmetries. One can characterize them by their dimension: magnetic monopoles are $0D$, cosmic strings are $1D$, domain walls are $2D$ and so on. Taking domain walls as an example, one can show how inflation could solve the problem.

This relic let us call X, has a number density n_X. If these objects were inside the horizon when they were produced, we have correlation length of field φ; $\xi < d_H(t) = 2ct$ with $d_H = H^{-1}$ and $H(t) = 1/2t$ in radiation era, therefore the number density becomes

$$n_X \leq H^3 \approx g_*^{3/2} \frac{T^6}{M_{pl}^3} \tag{3.20}$$

We exploited $H \approx g_*^{1/2} \frac{T^2}{M_{pl}}$. The number density in photons is $n_\gamma \propto g_\gamma T^3$. So, the ratio of the relic number density to that of photons becomes

$$\left.\frac{n_X}{n_\gamma}\right|_{T_{GUT}} \leq g_*^{3/2} \left(\frac{T_{GUT}}{M_{pl}}\right)^3 \approx 10^{-9} \simeq \eta = \frac{n_B}{n_\gamma} \tag{3.21}$$

where η is the baryon asymmetry parameter. Therefore, $n_{0,X} = n_{0,b}$. To have a quantitative result, density parameter for these relics with a mass $m_X \approx 10^{16} GeV$ and baryonic density of $\Omega_{0,b} \approx 0.05$ [13] reads

$$\Omega_{0,X} = \frac{\rho_{0X}}{\rho_{0c}} = \frac{n_{0X} m_X}{\rho_{0c}} = \frac{m_X}{m_b} \frac{n_{0b} m_b}{\rho_{0c}} = \Omega_{0,b} \frac{m_X}{m_b} \approx 10^{15} \tag{3.22}$$

We now see how they would over-close the universe. However, incorporating inflation dilutes the number density of these relics exponentially as $n_X \propto a^{-3Ht}$. Therefore, we do not have unwanted relics anymore.

3.3 Dynamics of Inflation

One can consider the dynamics of different fields associated with inflation. We will focus on the scalar field for simplicity and because: if we assume a ground state $\langle\varphi\rangle = \langle 0|\varphi|0\rangle = const.$, this would be wrong since inflation has to end at some point in time $\langle 0|\varphi|0\rangle = f(t) \neq const.$, so taking a vector field A^μ, one has $\langle 0|A^\mu|0\rangle \neq 0$ which would violate rotational invariance (isotropy) which contradicts with the cosmological principle. Same goes for a spinor field ψ. However, we can have a fermionic condensate $\psi\bar{\psi}$.

Considering a real scalar field, the action in full generality is

$$S_{tot} = S_{GR} + S_\varphi[\varphi, g_{\mu\nu}] + S_m = \int d^4x \sqrt{-g} \Big(\frac{R}{16\pi G} + \mathcal{L}_\varphi[\varphi, g_{\mu\nu}] + \sum_{fields} \mathcal{L}_{fields} \Big) \tag{3.23}$$

The last term captures all the Standard Model (SM) interactions, which are negligible at very early times. The Lagrangian of a real scalar field is

$$\mathcal{L}_\varphi = -\frac{1}{2}g^{\mu\nu}\varphi_{;\mu}\varphi_{;\nu} - V(\varphi) \tag{3.24}$$

Inflaton potential $V(\varphi)$ might contain self interactions $\frac{\lambda}{4}\varphi^4$, mass term $\frac{1}{2}m_\varphi^2$ and more terms where inflaton could couple to other fields. One might as well have terms like $\xi\varphi^2 R$ which can modify GR, however, this is beyond the scope of this chapter.

One can split the field into background inflaton field (homogeneous and isotropic scalar field in FRW) and the quantum fluctuation of $\varphi(\vec{x}, t)$ around the classical trajectory[14] as

$$\varphi(\vec{x}, t) = \langle 0|\varphi(\vec{x}, t)|0\rangle + \delta\varphi(\vec{x}, t) = \varphi_0(t) + \delta\varphi(\vec{x}, t) \tag{3.25}$$

This splitting will enable us to do perturbation theory if $\langle\delta\varphi^2(\vec{x}, t)\rangle \ll \varphi_0^2(t)$. Applying perturbation theory is valid because from CMB observations we know that $\delta\varphi$ produces $\frac{\delta T}{T} \sim 10^{-5}$ [15]. Energy-momentum tensor for inflaton field is given by

$$\begin{aligned} T^\varphi_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta S_\varphi}{\delta g^{\mu\nu}} = -\frac{2}{\sqrt{-g}}\left[\frac{\partial(\sqrt{-g}\mathcal{L}_\varphi)}{\partial g^{\mu\nu}} + \partial_\alpha\frac{\partial(\sqrt{-g}\mathcal{L}_\varphi)}{\partial g^{\mu\nu}_{,\alpha}} - \cdots\right] \\ &= -2\frac{\partial\mathcal{L}_\varphi}{\partial g^{\mu\nu}} + \mathcal{L}_\varphi g_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi + g_{\mu\nu}\left[-\frac{1}{2}g^{\alpha\beta}\varphi_{;\alpha}\varphi_{;\beta} - V(\varphi)\right] \end{aligned} \tag{3.26}$$

where the $\partial_\alpha(\cdots)$ is a surface term from integration by parts. Therefore we have energy density and isotropic pressure of inflaton field as

$$\begin{aligned} T^0_0 &= -\left[\frac{1}{2}\dot{\varphi}_0^2 + V(\varphi_0)\right] = -\rho_\varphi(t) \\ T^i_j &= \left[\frac{1}{2}\dot{\varphi}_0^2 - V(\varphi_0)\right]\delta^i_j = p_\varphi(t)\delta^i_j \end{aligned} \tag{3.27}$$

So, $T^\varphi_{\mu\nu}$ is the energy-momentum tensor of a perfect fluid. For inflation, negative isotropic pressure is required, meaning $p_\varphi < 0$. This is achieved by the slow-roll condition $V(\varphi_0) \gg \frac{1}{2}\dot{\varphi}_0^2$. In this case, $p_\varphi \approx -V(\varphi_0) \approx -\rho_\varphi$, hence $w_\varphi = -1$. Instead if we had $\frac{1}{2}\dot{\varphi}_0^2 \gg V(\varphi_0)$, we would have $w_\varphi = 1$ so $\rho \propto a^{-3(1+w_\varphi)} = a^{-6}$.

The equation of motion describing the evolution of the scalar field $\Box\varphi(\vec{x}, t) = -\frac{\partial V}{\partial\varphi}$ comes from $\frac{\delta S_\varphi}{\delta\varphi} = 0$. The D'Alambertian in curved spacetime with a metric tensor $g_{\mu\nu} = diag(1, -a^2, -a^2, -a^2)$

is

$$\Box\varphi(\vec{x},t) = \varphi^{;\mu}_{;\mu} = \frac{1}{\sqrt{-g}}\left(g^{\mu\nu}\sqrt{-g}\varphi_{;\mu}\right)_{;\nu} = -\frac{\partial V}{\partial\varphi} \tag{3.28}$$

So, this becomes

$$\Box\varphi(\vec{x},t) = \ddot{\varphi}(\vec{x},t) + 3H\dot{\varphi}(\vec{x},t) - \frac{\nabla^2}{a^2}\varphi(\vec{x},t) = -\frac{\partial V}{\partial\varphi} \tag{3.29}$$

The inverse metric components $g^{00} = 1$ and $g^{ij} = -\delta^{ij}/a^2$ have been used, which reproduces the Klein-Gordon equation for inflaton field in FRW metric with a source term encapsulated by the derivative of the potential. In the second term, φ field feels the friction due to the expansion of the universe. Define $\varphi_0(t) \equiv \varphi(t)$, we end up with

$$\ddot{\varphi} + 3H\dot{\varphi} = -\frac{\partial V}{\partial\varphi} \equiv V' \tag{3.30}$$

From the Friedmann equation, we obtain

$$H^2 = \frac{8\pi G}{3}\rho_\varphi - \frac{k}{a^2} = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\varphi}_0^2 + V(\varphi_0)\right) - \frac{k}{a^2} \tag{3.31}$$

Two conditions to realize inflation for a long enough time while solving the shortcomings of hot Big-Bang model are

– First slow-roll condition: $\frac{1}{2}\dot{\varphi}_0^2 \ll V(\varphi_0)$. This gives us $H^2 \approx \frac{8\pi G}{3}V(\varphi)$

– Second slow-roll condition: $\ddot{\varphi} \ll 3H\dot{\varphi}$. This gives us $\dot{\varphi} \approx -\frac{V'}{3H}$

Let us introduce the slow-roll parameters.[14]

- ε is related to the flatness of the inflationary potential

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} \ll 1 \tag{3.32}$$

The relation of ε to slow-roll condition is as follows

$$\frac{d}{dt}H^2(t) = 2H\dot{H} = \frac{8\pi G}{3}\left(\dot{\varphi}\ddot{\varphi} + V'\dot{\varphi}\right) = -8\pi GH\dot{\varphi}^2 \tag{3.33}$$

3.30 is plugged in for $\ddot{\varphi}$. From which it is obtained $\dot{H} = -4\pi G\dot{\varphi}^2$. Finally, it becomes

$$\varepsilon = -\frac{\dot{H}}{H^2} = 4\pi G\frac{\dot{\varphi}^2}{H^2} \simeq \frac{3}{2}\frac{\dot{\varphi}^2}{V(\varphi)} \tag{3.34}$$

In order to have 3.32, we need $\dot{\varphi}^2 \ll V(\varphi)$ hence the first slow-roll condition.

In other words, slow roll parameter is

$$\varepsilon \approx \frac{3}{2}\frac{\dot{\varphi}^2}{V(\varphi)} = \frac{3}{2}\frac{1}{V}\frac{V'^2}{9H^2} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \tag{3.35}$$

So, to have $\varepsilon \ll 1$, one requires potential to be flat enough $V' \ll V$.

- η is related to the duration of inflation

$$\eta \equiv -\frac{\ddot{\varphi}}{H\dot{\varphi}} \tag{3.36}$$

We require $|\eta| \ll 1$. Exploiting the second slow-roll condition with $\ddot{\varphi} = -\frac{V''\dot{\varphi}}{3H} + \frac{\dot{H}V'}{3H^2}$ the second slow roll parameter becomes

$$\eta = \frac{V''}{3H^2} - \frac{V'}{3H\dot{\varphi}}\frac{\dot{H}}{H^2} = \eta_V - \varepsilon \tag{3.37}$$

where $\eta_V = \frac{V''}{3H^2} = \frac{1}{8\pi G}\frac{V''}{V}$.

One can write the second derivative of the scale factor as

$$\ddot{a} = \frac{d}{dt}(aH) = \dot{a}H + a\dot{H} = aH^2\left(1 + \frac{\dot{H}}{H^2}\right) = aH^2(1-\varepsilon) \tag{3.38}$$

So we have $\ddot{a} > 0$ if $\varepsilon < 1$, since $\varepsilon \ll 1$ we have accelerated expansion. When we deal with quantum fluctuations, we will use expansions on slow-roll parameters ε and η, which will enter in observables. Exponential expansion $\ddot{a}$ can be sub-classified through

$$\begin{cases} \dot{H} < 0, \dot{H} < H^2 & \text{Sub-exponential inflation} \\ \dot{H} = 0, H = const & \text{de-Sitter inflation} \\ \dot{H} > 0 & \text{Super-exponential inflation} \end{cases}$$

3.4 Inflationary Models

- Large Field Models

These models are usually monotonic and they have power law potentials of the form $V(\varphi) \propto \varphi^{\alpha}$ [14], the field slowly rolls down the potential and drives inflation. Chaotic inflation and natural inflation are of this type of models [14]. Large-field models predict significant primordial gravitational waves, leading to a potentially observable tensor-to-scalar ratio $r \sim [0.01 - 0.1]$. Inflaton field oscillates after inflation, leading naturally to energy transfer to other fields (reheating).

Writing the first slow-roll parameter, one obtains

$$\varepsilon \simeq \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = \alpha^2 \frac{M_{pl}^2}{\varphi^2} \tag{3.39}$$

In order to have $\varepsilon \ll 1$, we require $\varphi \gg M_{pl}$. Variation/excursion of the field during inflation becomes

$$\Delta\varphi = \int_{\varphi_{CMB}}^{\varphi_{end}} d\varphi = \int_{t_{CMB}}^{t_{end}} \dot{\varphi} dt = \int_{t_{CMB}}^{t_{end}} \frac{\dot{\varphi}}{H} H dt = \frac{\dot{\varphi}}{H}\int_{t_{CMB}}^{t_{end}} H dt = \frac{\dot{\varphi}}{H} N_{CMB} \tag{3.40}$$

Remember $\varepsilon = -\frac{\dot{H}}{H^2} = 4\pi G \frac{\dot{\varphi}^2}{H^2}$ which gives $\frac{\dot{\varphi}}{H} = \frac{\sqrt{\varepsilon}}{2\sqrt{\pi G}} = \sqrt{\varepsilon} M_{pl}$. One can take N_{CMB} from $N_{min} = [50, 70]$

So, the excursion of the field is

$$\Delta\varphi \simeq \sqrt{\varepsilon} N_{CMB} M_{pl} \tag{3.41}$$

In these models, $\varepsilon \approx N_{CMB}^{-1}$ so in other words $\Delta\varphi \gg M_{pl}$ hence large field model. This might seem like we are probing regime of quantum gravity and possibly bending the laws of physics, however fields can be as large as they could, since they are not observables.

- Small Field Models

The potential is typically flattened near the top. They have potentials in the form of [14] $V(\varphi) \propto V_0 \left[1 - \left(\frac{\varphi}{\mu}\right)^p\right] + \cdots$ with $p > 2$ and $\varphi < \mu < M_{pl}$. New inflation and hybrid inflation are of this type of models. Small-field models generally predict negligible primordial gravitational waves $r \ll 0.01$ making them harder to detect.

Slow-roll parameter becomes

$$\varepsilon \simeq \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = \frac{1}{16\pi G}\left(\frac{p\varphi^{p-1}}{\mu^p - \varphi^p}\right)^2 \xrightarrow{\mu \gg \varphi} \frac{1}{16\pi G}\left(\frac{p\varphi^{p-1}}{\mu^p}\right)^2 = \frac{p^2\varphi^{2p} M_{pl}^2}{\mu^{2p}\varphi^2} \tag{3.42}$$

Observe that as $\varepsilon \to 0$, then $\varphi \to 0$ small field model $\Delta\varphi \ll M_{pl}$.

■ R^2/Starobinsky -Inflation

It arises as a modification to general relativity by adding a term proportional to Ricci scalar squared[16]. So, Einstein-Hilbert action becomes

$$S = \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g} \left(R + \frac{R^2}{6M^2} \right) \tag{3.43}$$

M refers to the energy scale of inflation. R^2 term naturally leads to inflation without introducing additional fields. R^2 term can be re-written in terms of a scalar field via conformal transformation where the equivalence is given by inserting a potential

$$V(\varphi) \propto M_{pl}^2 M^2 \left(1 - e^{-\frac{2\varphi}{\sqrt{3}M_{pl}}} \right)^2 \tag{3.44}$$

where the Weyl transformation is $g_{\mu\nu} \to e^{-2\omega} g_{\mu\nu}$ where $\omega = \varphi/\sqrt{3}M_{pl}$

It predicts a tensor to scalar ratio $r \approx \frac{10}{N_{CMB}^2} \approx 0.003$ meaning very weak primordial gravitational waves. Spectral index $n_s \approx 1 - \frac{1}{N_{CMB}} = 0.965$ which is in excellent agreement with CMB observation from Planck.

■ Natural Inflation

It is an interesting model where the inflaton field is a pseudo-Nambu-Goldstone boson, arising from spontaneous symmetry breaking. The potential for this inflationary model is

$$V(\varphi) = V_0 \left[1 - \cos \frac{\varphi}{\mu} \right] \tag{3.45}$$

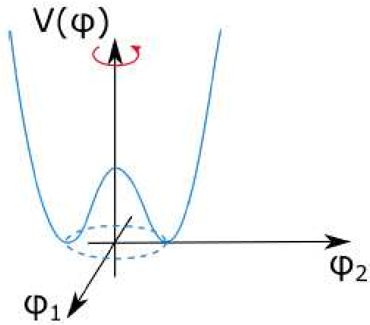

Figure 3.5: Potential of natural inflation.

μ is analogous to f axion-like decay constant. The potential for natural inflation is plotted in the figure above. [17] For small field models $\varphi < \mu < M_{pl}$, the potential can be expanded as $V \propto V_0 \left[1 - \left(\frac{\varphi}{\mu}\right)^2\right]$. We start with $V = const$, add dynamics in inflation by non-perturbative processes which breaks the symmetry. If the symmetry is exact, it implies $m_\varphi^2 = 0$ so potential is exactly flat and inflation never ends. Usually symmetry is broken slightly and small mass is generated $m_\varphi^2 \neq 0$, hence the field becomes psuedo-Nambu-Goldstone boson. These systems have a shift symmetry $\varphi \to \varphi + C$ which caa be realized by global $U(1)$ symmetry.

It predicts low level of B-modes polarization, large axion decay constant in contradiction to particle physics predictions. However, it predicts consistent spectral index with CMB data.

- Old Inflation

At $T = T_c$, there is a first order phase transition which causes spontaneous symmetry breaking from $G_{GUT} \to SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$[8]. At $T = 0$, the potential $V(\varphi) = \frac{\lambda}{4}\left(\varphi^2 - \sigma^2\right)^2$ which is invariant under $\varphi \to -\varphi$ but the vacuum breaks the symmetry. Symmetry is present at high temperatures, and broken by phase transitions.

At $T \neq 0$, generic potential is

$$V(\varphi, T) = V(\varphi) + \gamma|\varphi|^3 T + \alpha^2 \varphi^2 T^2 + \beta T^4 \tag{3.46}$$

with $[\alpha, \beta, \gamma] = 0$. At high temperatures, the third term dominates since the last term only gives a vertical shift. In the figure below, $\varphi = 0$ is the false vacuum while $\varphi = \pm\sigma$ is the true vacuum. The dashed line represents the minimum of the potential before spontaneous symmetry breaking $V = \lambda\sigma^4/4$.

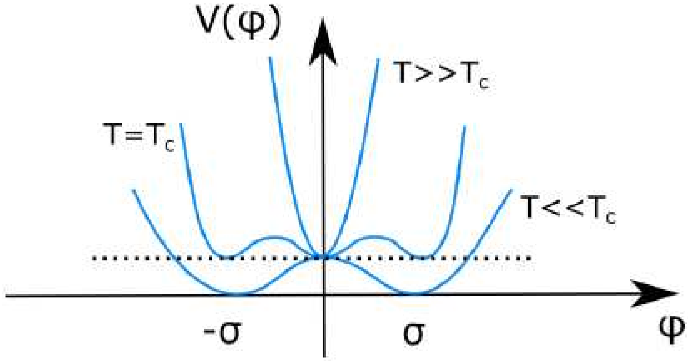

Figure 3.6: Potential of old inflation.

The potential for old inflation is plotted in the figure above. [17] At high temperatures, parity invariance is fully recovered with $\varphi = 0$ being the stable configuration. As T drops, the potential changes its form, while at $T = T_c$ there appear 3 minimas resulting the same energy (degeneracy). Lowering T below the critical temperature, $\varphi = 0$ is no longer stable configuration (becomes the false vacuum), the field can pass to the true vacuum by quantum tunneling or it stays at $\varphi = 0$ until $T = 0$ then it reaches the true vacuum.

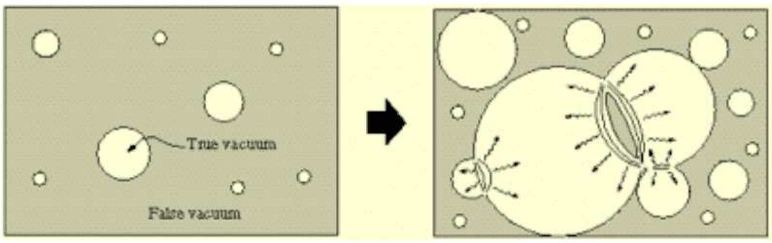

Figure 3.7: First order phase transitions proceeded by bubble nucleation.

When the field goes from false vacuum to true vacuum, it undergoes first-order phase transition which is proceeded by bubble nucleation. Hoowever, this inflationary model has a problem: bubbles can not merge to cover the whole universe because the space is expanding exponentially. This is called the "graceful-exit problem"[8].

Due to the phase transition, there is latent heat $|\Delta V| = |V(\varphi = 0) - V(\varphi = \pm\sigma)|$ once the phase transition is over. This reheats the universe, reheating takes place in bubbles so thermalization process is inefficient. Through quantum tunneling, we can avoid graceful-exit problem but with a cost of overproduction of cosmic defects.

- New Inflation

This model was introduced to solve the graceful-exit problem based on SSB[18]. It is a second-order phase transition where the inflaton field slowly rolls down its potential. (no bubble nucleation). Compared to the old inflation, this model has the feature that the field starts around the top of an almost flat potential, it slowly-rolls instead of quantum tunneling so there is a smooth transition which leaves a homogeneous universe.

When the slow roll ends, the field starts to oscillate around the true vacua, leading to the reheating phase. However, this model also has a problem: it describes energy density fluctuations of $\delta\rho/\rho \propto \lambda^{1/2}$ with $\lambda \sim \mathcal{O}(1)$ being the coupling constant of the inflaton field with thermal plasma, to have φ in thermal equilibrium. Whereas, CMB predicts the fluctuations to be of the order $\delta\rho/\rho \propto 10^{-5}$. To have the observed CMB prediction, one could try solving this problem by bringing the coupling constant down to $\lambda \sim 10^{-10}$. However, then our field becomes almost non-interacting, thus violating the requirement of having φ in thermal equilibrium.

The potential that drives inflation is given by the Coleman-Weinberg potential obtained from GUT symmetry breaking to SM is

$$V(\varphi) = \frac{B\sigma^4}{2} + B\varphi^4 \left[\ln\frac{\varphi^2}{\sigma^2} - \frac{1}{2}\right] \simeq \frac{B\sigma^4}{2} - \frac{\lambda}{4}\varphi^4 \tag{3.47}$$

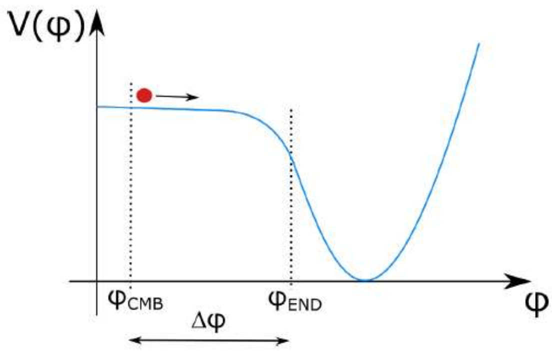

Figure 3.8: Coleman-Weinberg potential.

The potential for new inflation is plotted in the figure above. [17] where $\lambda = \left|4B\ln\frac{\varphi^2}{\sigma^2}\right|$, $B \propto \alpha_{GUT}^2 = 10^{-3}$ and $\sigma \propto T_c = 10^{15}GeV$. Energy density fluctuations within the given potential becomes

$$\zeta \sim \frac{\delta\rho}{\rho}\bigg|_{N_{CMB}} \approx -\frac{H^2}{\dot{\varphi}}\bigg|_{N_{CMB}} = \frac{3H^3}{V'}\bigg|_{N_{CMB}} = \frac{3}{\lambda}\left(\frac{H}{\varphi}\right)^3 \tag{3.48}$$

We have exploited $\dot{\varphi} \approx -V'/3H$ and $V' \approx -\lambda\varphi^3$. For inflation to last long enough, from the calculation of number of e-folds $N \simeq \frac{3H^2}{2\lambda}\frac{1}{\varphi_i^2}$, one can express ζ as

$$\zeta \approx \lambda^{1/2} N_{CMB}^{3/2} = 10^2 \tag{3.49}$$

which is in clear contradiction with CMB.

■ Chaotic inflation

It was proposed[19] to address the thermalization problem of new inflation. Unlike new inflation discussed before, which required the universe to cool before inflation could start, chaotic inflation allows inflation to begin at arbitrarily high energies, which makes it more natural to fit in the GUT framework. The field slowly rolls down, naturally driving inflation without the need of phase transition from SSB.

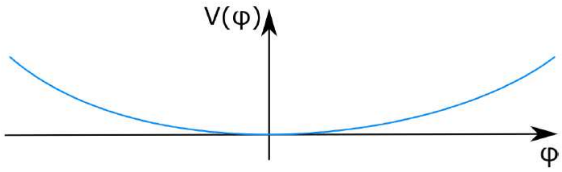

Figure 3.9: Chaotic inflation potential.

The potential for chaotic inflation is plotted in the figure above. [17]. Consider inflation with potential

$$V(\varphi) = \frac{\lambda}{4}\varphi^4 \tag{3.50}$$

with a constraint from quantum gravity $V \leq M_{pl}^4$ gives us $-M_{pl}/\lambda^{1/4} \leq \varphi \leq M_{pl}/\lambda^{1/4}$. Since the coupling can be arbitrarily small which would cause a very flat potential where inflaton field can slow-roll. This would yield regions where the field $\varphi > M_{pl}$. In this model, energy density perturbations $\delta\rho/\rho \sim \lambda^{1/2}$ with the choice of $\lambda \approx 10^{-10}$ can account for CMB data.

■ Hybrid inflation

The key idea is that inflation is driven by one field φ inflaton while a second field ψ is responsible for ending inflation.

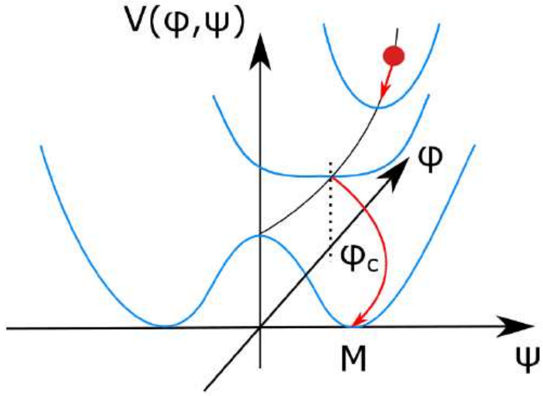

Figure 3.10: Hybrid model of inflation potential.

The potential for hybrid inflation is plotted in the figure above. [17].The potential in this case is

$$V(\varphi,\psi) = \frac{1}{2}m^2\varphi^2 + \frac{\lambda}{4}\left(\psi - M^2\right)^2 + \frac{1}{2}\lambda'\varphi^2\psi^2 \tag{3.51}$$

defining $\varphi_c = \sqrt{\lambda/\lambda'}M$ from $m_\psi^2 = \frac{\partial^2 V}{\partial \psi^2} = \lambda'\varphi^2 - \lambda M^2 = 0$ as the point where the m_ψ changes sign.

$$\begin{cases} \varphi > \varphi_c, m_\psi^2 > 0 & \text{equilibrium stable, maintaining inflation} \\ \varphi < \varphi_c, m_\psi^2 < 0 & \text{system is unstable, adding dynamics to } \varphi \end{cases}$$

After φ_c, inflaton is sitting on an unstable configuration, where the system chooses one of the true minimas $\psi = \pm M$.

3.5 Cosmic No-Hair Principle

Does inflation start from a very general initial conditions? Do we still end up with FLRW?

- Thanks to the cosmic no-hair principle, yes!

Space-times which are homogeneous but not isotropic are called Bianchi models. So Bianchi-1 universe has

$$ds^2 = -dt^2 + a_1^2dx^2 + a_2^2dy^2 + a_3^2dz^2 \tag{3.52}$$

The mean scale factor is $\bar{a} \propto V^{1/3}$ with $V = a_1a_2a_3$, Friedmann equation reads

$$\bar{H}^2 = \frac{1}{9}\left(\frac{\dot{V}}{V}\right)^2 = \frac{8\pi G}{3}\rho + F(a_i) \tag{3.53}$$

$F(a_i) \propto \bar{a}^{-p}$ captures the effects of anisotropies. Energy densities would be diluted $\rho_m \propto \bar{a}^{-3}$, $\rho_r \propto \bar{a}^{-4}$, $F \propto \bar{a}^{-p}$ with $p > 2$. Let us see if φ provides inflation in this model, assume $\rho_\varphi = \rho_\Lambda = const$. Each contribution to the energy density decreases quite fast, at some point ρ_φ will dominate

$$\bar{H}^2 = \frac{8\pi G}{3}\rho_\varphi \tag{3.54}$$

which gives $\bar{a} \propto e^{Ht}$ so $\rho \propto e^{-Ht}$ our metric asymptotically reaches to FRW. However, there is an exception to this: highly positively curved universe $k > 0$, universe does not have enough time to inflate.

Cosmic no-hair theorem: If cosmological space-time obeys the Einstein field equations with $\Lambda > 0$, then space-time asymptotically becomes de-Sitter universe in the future[20].

What if φ reaches the true minimum before the potential starts to dominate due to anisotropies?

3.53 and 3.30 tell us that anisotropies will increase the mean Hubble friction term for φ so we will have even more slow-roll. Therefore, it will not reach the true minimum before $V(\varphi)$ dominates.

3.6 Reheating

During inflationary epoch, the temperature of the universe drops exponentially $T \propto e^{-Ht}$, so at the end of inflation, temperature would be too small to allow thermalization of particles. Therefore, right after the inflationary epoch, we need another epoch to reheat the universe until the start of the radiation era. After the end of slow-roll, inflaton field starts to oscillate around its minima with a frequency of ω and it decays into relativistic particles with a decay rate of Γ_φ. This is the mechanism behind reheating epoch post-inflation. Taking into consideration this effect, E.o.M. becomes

$$\ddot{\varphi} + \left(3H + \Gamma_\varphi\right)\dot{\varphi} + V'(\varphi) = 0 \tag{3.55}$$

One can solve the evolution equation describing the energy density of inflaton via $\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$

$$\dot{\rho_\varphi} + 3H\dot{\varphi}^2 = -\Gamma_\varphi\dot{\varphi}^2 \tag{3.56}$$

where we have multiplied the 3.55 by $\dot{\varphi}$ and used the $\dot{\rho_\varphi}$. From above equation, we see that it behaves like N.R. matter.

Since the time of oscillation is smaller then the time of expansion, one can average over a period of oscillation, resulting $\langle V\rangle_{period} = \langle \frac{1}{2}\dot{\varphi}^2\rangle$, so $\rho_\varphi = \langle\dot{\varphi}^2\rangle_{period}$. Finally the evolution equation encapsulating the inflaton decay becomes[21]

$$\dot{\rho_\varphi} + 3H\rho_\varphi = -\Gamma_\varphi \rho_\varphi \tag{3.57}$$

where the minus sign implies the energy transfer from inflaton field to SM particles. Solution to this differential equation is

$$\rho_\varphi = \rho_{osc}\left(\frac{a}{a_{osc}}\right)^{-3} \exp\left[-\Gamma_\varphi H(t - t_{osc})\right] \tag{3.58}$$

where $\rho_{osc} = M^4$ is the initial condition, t_{osc} is when the oscillations start. Around $t \approx t_{osc}$, inflaton decays become inefficient, so $\rho_\varphi \propto a^{-3}$. As the field decays and more time passes, at some point in time when $t_{decay} \sim \frac{1}{\Gamma_\varphi}$, decays become very efficient. How does the energy density for radiation evolve as the inflaton field decays into relativistic particles? The equation one needs to solve is the following

$$\dot{\rho_R} + 4H\rho_R = \Gamma_\varphi \rho_\varphi \tag{3.59}$$

The change in the sign on the R.H.S. is the result of the production of relativistic particles. One can quantify the oscillation time since $t_{osc} \simeq H_{osc}^{-1}$ with $H_{osc}^2 = \frac{8\pi G}{3} M^4$. Therefore, one obtains

$$t_{osc} \sim \frac{M_{pl}}{M^2} \frac{1}{\left(\frac{8\pi}{3}\right)^{1/2}} \tag{3.60}$$

From t_{osc} to t_{decay}, universe is dominated by oscillating φ (equivalent to N.R. matter) $\rho_\varphi \propto a^{-3}$, $a \propto t^{2/3}$ and $H \approx \frac{2}{3t}$. Equation 3.59 reads

$$\dot{\rho_R} + \frac{8}{3t}\rho_R = \Gamma_\varphi \rho_{osc}\left(\frac{a}{a_{osc}}\right)^{-3} = \Gamma_\varphi M^4 \left(\frac{t}{t_{osc}}\right)^{-2} \simeq \left(\frac{8\pi}{3}\right)^{-1} \Gamma_\varphi M_{pl}^2 \frac{1}{t^2} \tag{3.61}$$

The solution to the homogeneous part with the ansatz $\rho_R \propto At^\alpha$ gives $\alpha = -8/3$ so $\rho_R \propto At^{-8/3}$. The particular solution with $\rho_R \propto Bt^\beta$ substituted in 3.61 gives $\beta = -1$ and $B = \frac{9}{40\pi}\Gamma_\varphi M_{pl}^2$. So, the full solution is $\rho_R = At^{-8/3} + Bt^{-1}$. One can fix the pre-factor A by the fact that at the time of oscillation $\rho_R(t_{osc}) = 0$ which gives $A = -\frac{9}{40\pi}\Gamma_\varphi M_{pl}^2 t_{osc}^{-5/3}$. By exploiting

$a \propto t^{2/3}$ and t_{osc} given previously, evolution of the energy density for radiation becomes

$$\rho_R \simeq \frac{3}{5}\Gamma_\varphi M_{pl} M^2 \left(\frac{a}{a_{osc}}\right)^{-3/2}\left[1-\left(\frac{a}{a_{osc}}\right)^{-5/2}\right] \tag{3.62}$$

One can observe how ρ_R grows from 0 to $\rho_{R,max} \approx \Gamma_\varphi M_{pl} M^2$. Energy density for radiation will deviate from the conventional $\rho_R \propto a^{-4}$ scaling for $a \gg a_{osc}$. Instead we have $\rho_R \propto a^{-3/2}$. These particles interact with each other and thermalize, maximum temperature $T_{max} \propto g_*^{-1/4}\rho_{R,max}^{1/4} \simeq g_*^{-1/4}\Gamma_\varphi^{1/4} M_{pl}^{1/4} M^{1/2}$.

In inflationary phase, total entropy increases because the inflaton decays and produces new particles with more degrees of freedom. We know that $S = sa^3$ where $s \propto T^3 = \rho_R^{3/4} \approx a^{-9/8}$, exploiting $T \propto \rho_R^{1/4}$. So, the total entropy goes as $S \propto a^{15/8}$.

When the decays become more efficient, energy density in inflaton will decrease very fast and the one of radiation will start dominating. One can calculate the temperature at this reheating epoch, since it is radiation dominated $H = \frac{1}{2t} \approx \frac{\Gamma_\varphi}{2}$.

$$H^2\Big|_{t=\Gamma_\varphi^{-1}} = \frac{\Gamma_\varphi^2}{4} = \frac{8\pi}{M_{pl}^2}\left(\frac{\pi^2}{30} g_* T^4\right)\Bigg|_{t=\Gamma_\varphi^{-1}} \tag{3.63}$$

which leads to the reheating temperature $T_{rh} \approx g_*^{-1/4}(\Gamma_\varphi M_{pl})^{1/2}$.

3.7 Cosmological Perturbations From Inflation

In this chapter, we will discuss how to generate cosmological perturbations[22] on large scales $\lambda \gg H^{-1}$ starting from small scale quantum fluctuations. From 3.29, perturbing it linearly, one reaches

$$\begin{aligned} \delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \frac{\nabla^2}{a^2}\delta\varphi &= -V''\delta\varphi \\ \ddot{\varphi}_0 + 3H\dot{\varphi}_0 - \underbrace{\frac{\nabla^2}{a^2}\varphi_0}_{=0} &= -V' \end{aligned} \tag{3.64}$$

Since $H \approx const.$, applying a time derivative on the second line, one gets

$$(\dot{\varphi}_0)^{\cdot\cdot} + 3H(\dot{\varphi}_0)^{\cdot} = -V''(\varphi_0)\dot{\varphi}_0 \tag{3.65}$$

Notice that $\delta\varphi$ and $\dot{\varphi}_0$ have the same E.o.M. apart from the Laplacian term. Comparing the effectiveness of the term $3H\delta\dot{\varphi}$ with $\frac{\nabla^2}{a^2}\delta\varphi$. Since $\partial/\partial t \approx H$, one can write down the second term $\sim 3H^2\delta\varphi$. We obtain $H^2\delta\varphi$ and $\frac{k^2}{a^2}\delta\varphi$ in Fourier space. On large scales $\lambda_{phys} = a\lambda \sim \frac{a}{k} \gg H^{-1}$, Laplacian term can be neglected.

Since at large scales, two equations of motions are identical, their solution may not be independent: one must check the Wronskian $W(\delta\varphi, \dot{\varphi}_0) = \delta\dot{\varphi}\dot{\varphi}_0 - \delta\varphi\ddot{\varphi}_0$. Having exploited the 3.65 and 3.64 in large scale approximation, we get

$$\dot{W} = \delta\ddot{\varphi}\dot{\varphi}_0 - \delta\varphi\dddot{\varphi}_0 = -3HW \tag{3.66}$$

Assuming quasi de-Sitter inflation $H \approx const.$, one gets $W \propto e^{-3Ht}$ so they are asymptotically dependent. After coarse graining, it becomes

$$\delta\varphi(\vec{x}, t) = \delta t(\vec{x})\dot{\varphi}_0(t) \tag{3.67}$$

Therefore, the full scalar field is

$$\varphi(\vec{x}, t) = \varphi_0(t) + \delta\varphi(\vec{x}, t) = \varphi_0(t) - \delta t(\vec{x})\dot{\varphi}_0(t) = \varphi_0(t - \delta t(\vec{x})) \tag{3.68}$$

This tells us that the field φ at large scales will have the same value as φ_0 everywhere but at slightly different times.

3.7.1 Approximated Solutions

How to solve the following equation

$$\delta\ddot{\varphi} + 3H\delta\dot{\varphi} - \frac{\nabla^2}{a^2}\delta\varphi = -V''\delta\varphi \tag{3.69}$$

In Fourier space the solution is

$$\delta\varphi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} \delta\varphi_{\vec{k}}(t) \tag{3.70}$$

Note that $\delta\varphi_{\vec{k}} = \delta\varphi^*_{-\vec{k}}$ and $\delta\varphi(\vec{x}, t) = \delta\varphi^*(\vec{x}, t)$ since it is observable. We used plane waves because we assumed spatially flat $k = 0$ FLRW metric. In general, the solutions are in the form of Helmholtz equation.

Quantize our classical in expanding universe, first rescale the field $\delta\hat{\varphi}(\vec{x},\tau) = a\delta\varphi(\vec{x},\tau)$

$$\delta\hat{\varphi}(\vec{x},\tau) = \int \frac{d^3k}{(2\pi)^3}\left[a_{\vec{k}}u_{\vec{k}}(\tau)e^{i\vec{k}\vec{x}} + a^{\dagger}_{\vec{k}}u^*_{\vec{k}}(\tau)e^{-i\vec{k}\vec{x}}\right] \tag{3.71}$$

$a^{\dagger}_{\vec{k}}$ and $a_{\vec{k}}$ being the creation and annihilation operators respectively: $\langle 0|a^{\dagger}_{\vec{k}} = 0$ and $a_{\vec{k}}|0\rangle = 0$ for any $\vec{k}$. Vacuum state for $\delta\varphi$ is non-interacting because KG is linearized. Normalization condition is $u_{\vec{k}}(\tau)u^*_{\vec{k}'}(\tau) - u^*_{\vec{k}}(\tau)u_{\vec{k}'}(\tau) = i$, while the canonical quantization condition for operators are $[a_{\vec{k}}, a_{\vec{k}'}] = 0 = [a^{\dagger}_{\vec{k}}, a^{\dagger}_{\vec{k}'}]$ and the non-vanishing one is $[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = (2\pi)^3\hbar\delta^3(\vec{k} - \vec{k}')$.

In flat space-time, $u_{\vec{k}'}(\tau) = \frac{e^{-i\omega_k\tau}}{\sqrt{2\omega_k}}$ where $\omega_k = \sqrt{k^2+m^2}$. In Minkowski spacetime, the natural vacuum is the one where the positive-frequency modes behave as plane waves. In curved spacetime, $u_{\vec{k}}$ is not necessarily a plane wave; there is an ambiguity in the definition of vacuum state. Bunch-Davies vacuum[23] is defined as the quantum state that looks like the Minkowski vacuum at early times (high energy) when modes are deep inside the horizon:

$$u_{\vec{k}}(\tau) = \frac{e^{-i\omega_k\tau}}{\sqrt{2\omega_k}} \approx \frac{e^{-ik\tau}}{\sqrt{2k}} \tag{3.72}$$

so at early times, for small scales, $u_{\vec{k}}(\tau)$ approaches the one in flat spacetime. We have exploited $\omega_k \approx k$ for $k \gg aH$ (small scales).

Now, first pass from the KG equation in proper time to conformal time using $H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{a'}{a}$, $\dot{\varphi} = \frac{\varphi'}{a(\tau)}$ and $\ddot{\varphi} = \frac{\varphi''}{a^2} - \varphi'\frac{a'}{a^3}$. So, 3.69 reads

$$\delta\varphi'' + 2\frac{a'}{a}\delta\varphi' - \nabla^2\delta\varphi = -a^2\frac{\partial^2 V}{\partial\varphi^2}\delta\varphi \tag{3.73}$$

Then rescale the field as mentioned before which leads to the disappearance of the Hubble friction term seen as

$$\delta\hat{\varphi}'' - \frac{a''}{a}\delta\hat{\varphi} - \nabla^2\delta\hat{\varphi} = -a^2\frac{\partial^2 V}{\partial\varphi^2}\delta\hat{\varphi} \tag{3.74}$$

In Fourier space using $|\delta\hat{\varphi}_{\vec{k}}| = |u_{\vec{k}}| = a|\delta\varphi_{\vec{k}}|$, perturbations reads

$$u''_{\vec{k}}(\tau) + \left[k^2 - \frac{a''}{a} + a^2\frac{\partial^2 V}{\partial\varphi^2}\right]u_{\vec{k}}(\tau) = 0 \tag{3.75}$$

In a time-varying background, there is no unique way to define the vacuum state, there are different $u_{\vec{k}}$ that solves the above-mentioned equation but they are related to each other by Bogoliubov transformation[24, 25]

$$u_{\vec{k}}(\tau) = \alpha_k v_k(\tau) + \beta_k v_k^*(\tau) \tag{3.76}$$

where α_k and β_k are time independent but k-dependent coefficients, called Bogoliubov coefficients. If we have the same normalization for u and v, then $|\alpha_k|^2 - |\beta_k|^2 = 1$

$$\delta\varphi(\vec{x}, \tau) = \int \frac{d^3k}{(2\pi)^3} \left[b_{\vec{k}} v_{\vec{k}}(\tau) e^{i\vec{k}\vec{x}} + b^\dagger_{\vec{k}} v^*_{\vec{k}}(\tau) e^{-i\vec{k}\vec{x}} \right] \tag{3.77}$$

We get 2 vacuum associated to $u_{\vec{k}} \leftrightarrow |0_{(a)}\rangle$ and $v_{\vec{k}} \leftrightarrow |0_{(b)}\rangle$ that gives $a_{\vec{k}}|0_{(a)}\rangle = 0$ and $b_{\vec{k}}|0_{(b)}\rangle = 0$

$$\begin{aligned} b_{\vec{k}} &= \alpha_{\vec{k}} a_{\vec{k}} + \beta^*_{\vec{k}} a^\dagger_{\vec{k}} \\ b^\dagger_{\vec{k}} &= \alpha^*_{\vec{k}} a^\dagger_{-\vec{k}} + \beta_{\vec{k}} a_{\vec{k}} \end{aligned} \tag{3.78}$$

We choose $\alpha_{\vec{k}} = 1$ and $\beta_{\vec{k}} = 0$ based on the argument of number density of b-type particles on a-type particles $\langle_{(a)}0|N_b|0_{(a)}\rangle = |\beta_{\vec{k}}|^2 = 0$.

Now, let's solve 3.75: consider a massless scalar field $m^2_\varphi = \frac{\partial^2 V}{\partial\varphi^2} = 0$ in the presence of de-Sitter $H = const.$, hence 3.75 simplifies to

$$u''_{\vec{k}}(\tau) + \left[k^2 - \frac{a''}{a} + a^2 \right] u_{\vec{k}}(\tau) = 0 \tag{3.79}$$

To solve this equation in sub-horizon and super-horizon regimes, re-express the second term in brackets in pure de-Sitter. Through $d\tau = \frac{dt}{a}$, we obtain $\tau = -\frac{1}{aH}$ so $a(\tau) = -\frac{1}{\tau H}$, finally it becomes $\frac{a''}{a} = \frac{2}{\tau^2} = 2(aH)^2 = \frac{2}{r_H^2}$

- Sub-horizon regime ($k \gg aH$)

Since in this regime, $\lambda_{phys} \ll H^{-1} \rightarrow k \gg aH$, one can write the E.O.M. as

$$u''_{\vec{k}} + k^2 u_{\vec{k}} = 0 \tag{3.80}$$

with the solution being $u_{\vec{k}} = \frac{1}{\sqrt{2k}} e^{-ik\tau}$, solution for the field perturbation is

$$\delta\varphi_{\vec{k}} = \frac{1}{a\sqrt{2k}} e^{-ik\tau} \tag{3.81}$$

and $\left|\delta\varphi_{\vec{k}}\right| = \left|\frac{1}{a\sqrt{2k}}\right|$. In this limit, gravitational amplification mechanism helps density fluctuations to grow.

- Super-horizon regime ($k \ll aH$)

Since in this regime, $\lambda_{phys} \gg H^{-1} \to k \ll aH$, one can write the E.O.M. as

$$u''_{\vec{k}} - \frac{a''}{a} u_{\vec{k}} = 0 \tag{3.82}$$

with the solution being $u_{\vec{k}}(\tau) = B(k)a(\tau) + C(k)a^{-1}(\tau)$ which has the growing and decaying modes respectively. This gives $\delta\varphi_{\vec{k}} = B(k) + C(k)a^{-3}(\tau) \simeq B(k) = const.$ in time. Fluctuations over super-horizon scales are frozen. This happens because there is no causal process that can affect the evolution of fluctuations when $\lambda_{phys} \gg H^{-1}$ so micro-physics can not operate. $B(k)$ is set by the condition that at horizon crossing scales the fluctuations of a given k on sub-horizon and super-horizon has to match

$$|\delta\varphi_{\vec{k}}| = \left.\frac{1}{a\sqrt{2k}}\right|_{k=aH} = |B(k)| = \frac{H}{\sqrt{2k^3}} \tag{3.83}$$

3.7.2 Exact Solutions in Quasi de-Sitter Spacetime

We know that $\frac{\ddot{a}}{a} = H^2(1-\varepsilon)$ where $\varepsilon = -\frac{\dot{H}}{H^2}$, using $\frac{\ddot{a}}{a} = H^2(1-\varepsilon) = \frac{1}{a^2}\frac{d}{d\tau}\left(\frac{a'}{a}\right)$ which gives $\frac{a''}{a} = a^2H^2(2-\varepsilon)$. So, equation 3.75 becomes

$$u''_{\vec{k}}(\tau) + \left[k^2 - a^2H^2(2-\varepsilon) + a^2\frac{\partial^2 V}{\partial\varphi^2}\right] u_{\vec{k}}(\tau) = 0 \tag{3.84}$$

Consider massless scalar field $m_\varphi^2 = 0$ in quasi de-Sitter spacetime with $\varepsilon = -\frac{\dot{H}}{H^2} \ll 1$ and at lowest order conformal time is $\tau = -\frac{1}{aH(1-\varepsilon)}$, whereas in pure de-Sitter it was $\varepsilon = 0$ and $\tau = -\frac{1}{aH}$.

Expanding the second term in brackets, one reaches

$$\frac{a''}{a} = a^2H^2(2-\varepsilon) = \frac{2-\varepsilon}{\tau^2(1-\varepsilon)^2} \approx \frac{2}{\tau^2}\left(1 + \frac{3}{2}\varepsilon + \cdots\right) \tag{3.85}$$

One gets the Bessel-type equation given by

$$u''_{\vec{k}}(\tau) + \left[k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right] u_{\vec{k}}(\tau) = 0 \tag{3.86}$$

where $\nu^2 = \frac{9}{4} + 3\varepsilon$ so $\nu \approx \frac{3}{2} + \varepsilon$ is exploited. The solution is

$$u_{\vec{k}}(\tau) = \sqrt{-\tau}\Big[C_1(k)H_\nu^{(1)}(-k\tau) + C_2(k)H_\nu^{(2)}(-k\tau)\Big] \tag{3.87}$$

- On sub-horizon scales ($k \gg aH$)

For $k/aH \to \infty$, we require plane wave solution $u_{\vec{k}}(\tau) \approx \frac{e^{-ik\tau}}{\sqrt{2k}}$, change the variable $x \equiv -k\tau$ which gives the following Henkel function of first type

$$H_\nu^{(1)} \approx \sqrt{\frac{2}{\pi x}} e^{i\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)} \xrightarrow{x \gg 1} \sqrt{\frac{2}{\pi}} \frac{e^{ix}}{\sqrt{x}} \tag{3.88}$$

From the asymptotic behavior of Henkel functions, we choose $C_2 = 0$ and $C_1 = \frac{\sqrt{\pi}}{2} e^{i(\nu+1/2)\pi/2}$ that gives

$$u_{\vec{k}}(\tau) \simeq \sqrt{-\tau} C_1 \frac{e^{-ik\tau}}{\sqrt{-k\tau}} \tag{3.89}$$

- On super-horizon scales ($k \ll aH$)

$$H_\nu^{(1)}(x) \xrightarrow{x \ll 1} x^{-\nu} \tag{3.90}$$

and $u_k(\tau) \simeq \sqrt{-\tau}(-k\tau)^{-\nu}$ while the fluctuations are

$$|\delta\varphi_k| = \frac{|u_k|}{a} = -H\tau|u_k| \approx \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH}\right)^{3/2-\nu} \tag{3.91}$$

We have exploited $\tau = -1/aH$ so there exist a k-dependence in the fluctuations. Note that in pure de-Sitter $\varepsilon = 0$ and $\nu = 3/2$. In quasi de-Sitter space-time $|\delta\varphi_k| \propto k^{-\varepsilon}$, general prediction of inflationary models.

Now let us investigate the case where the scalar field has a small mass $m_\varphi^2 \ll H^2$ in quasi de-Sitter spacetime. Remember that $\eta_V = \frac{1}{3}\frac{m_\varphi^2}{H^2} \ll 1$ and $\varepsilon = -\frac{\dot{H}}{H^2} \ll 1$ so 3.75 becomes

$$u_k'' + \left[k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right] u_k = 0 \tag{3.92}$$

where $\nu^2 = \frac{9}{4} + 3\varepsilon - 3\eta_V$ so $\nu \approx \frac{3}{2} + \varepsilon - \eta_V$.

$$|\delta\varphi_k| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{3/2-\nu} \tag{3.93}$$

It is worth noting that before when we derived the d'Alambertian in FLRW background, we perturbed the scalar field $\varphi = \varphi_0 + \delta\varphi$ but not the metric (which we should have perturbed). Perturbation of the field $\delta\varphi$ will give perturbations of $T_{\mu\nu}$ which will cause perturbations on the metric $g_{\mu\nu} \to g^0_{\mu\nu} + \delta g_{\mu\nu}$ which will impact the evolution equation of the inflaton field. To resolve this inconsistency, introduce gauge-invariant perturbation Sasaki-Mukhanov variable Q_φ which takes into account both perturbations of the field and metric tensor.

$$Q_\varphi = \delta\varphi + \frac{\dot{\varphi}}{H}\hat{\phi} \tag{3.94}$$

$\hat{\phi}$ is related to the scalar perturbations from the spatial part of the metric tensor. Without diving into the details of gauge-invariant perturbations, the E.O.M. becomes

$$\hat{Q}_\varphi''(\tau) + \left[k^2 - \frac{a''}{a} + a^2\frac{\partial^2 V}{\partial\varphi^2}\right]\hat{Q}_\varphi(\tau) = 0 \tag{3.95}$$

with the amplitude of gauge invariant perturbations

$$|\hat{Q}_\varphi| = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{3/2-\nu} \tag{3.96}$$

3.7.3 Gravitational Waves From Inflation

These GWs arise from quantum fluctuations in spacetime itself during inflation[22], quantum fluctuations occur in inflaton field which induce perturbations of the metric. These fluctuations include tensor perturbations which correspond to gravitational waves. They consist of signals coming from every direction in the sky unlike GWs from astrophysical origins which are localized. We will present the theoretical predictions while the observational constraints [26] will also be given as comparison.

Perturbed FLRW metric with $k = 0$, neglecting the scalar and vector perturbations, is

$$ds^2 = -a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij}(\vec{x}, \tau))dx^i dx^j\right] \tag{3.97}$$

where the spatial part of the tensor perturbations satisfy traceless-transverse condition $h^i_i = 0$ and $\partial^i h_{ij}$. Note that since the metric tensor is symmetric, tensor perturbations should also be symmetric $h_{ij} = h_{ji}$.

Perturbing EFE gives at linear order, one gets

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2(\tau)}h_{ij} = 16\pi G\Pi_{ij} \tag{3.98}$$

which is the propagating wave equation in expanding universe[11]. Π_{ij} is the anisotropic stress tensor. The homogeneous wave equation with vanishing source term gives in Fourier space the following

$$h_{ij}(\vec{x}, \tau) = \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}} h_\lambda(\vec{k}, \tau)\varepsilon^\lambda_{ij}(\vec{k}) \tag{3.99}$$

where $\varepsilon^\lambda_{ij}(\vec{k})$ is the polarization tensor with normalization $\varepsilon_{ij}(\vec{k}, \lambda)\varepsilon^{*ij}(\vec{k}', \lambda') = \delta_{\lambda\lambda'}$ and $\varepsilon_{ij}(-\vec{k}, \lambda) = \varepsilon^*_{ij}(\vec{k}, \lambda)$.

h_{ij} has 9 d.o.f $-$ $h_{ij} = h_{ji}$ (3 d.o.f) $-$ TT-gauge (4 d.o.f) $=$ 2 d.o.f. so GWs have 2-polarizations $\lambda = (+, \times)$. Equation of motion in Fourier space reads

$$\ddot{h}_\lambda + 3H\dot{h}_\lambda + \frac{k^2}{a^2}h_\lambda = 0 \tag{3.100}$$

which is the E.O.M. of a minimally coupled scalar field.

- Sub-horizon scales $k \gg aH$

GWs freely stream, experiences the redshift and their amplitude dilutes. $h_{+,\times} \sim e^{ik\tau}/a(\tau)$

- Super-horizon scales $k \ll aH$

Whereas in this regime $|h_{+,\times}| \propto \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{\frac{3}{2}-\nu} = \frac{H}{\sqrt{2k^3}}\left(\frac{k}{aH}\right)^{-\varepsilon}$ as shown before in the super-horizon scale where there exist a decaying and a constant mode.

We will present the 2-point correlation function ($\sim$ probability of finding two points at a given distance), but first define the Fourier transform of a field

$$\delta(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\delta_{\vec{k}}(t)e^{i\vec{k}\vec{x}} \tag{3.101}$$

So 2PCF evaluated at a distance $\vec{r}$ away is

$$\xi(\vec{r}) = \langle\delta(\vec{x}+\vec{r},t)\delta(\vec{x},t)\rangle = \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}e^{i\vec{k}\vec{x}}e^{i\vec{k}'(\vec{x}+\vec{r})}\langle\delta_{\vec{k}}\delta_{\vec{k}'}\rangle = \int \frac{d^3k'}{(2\pi)^3}e^{i\vec{k}'\vec{r}}P(k') \tag{3.102}$$

So P is the Fourier transform of ξ. We have used $\langle\delta_{\vec{k}}\delta_{\vec{k}'}\rangle = (2\pi)^3P(k)\delta^3(\vec{k}+\vec{k}')$. Notice that the power spectrum P depends on the amplitude of $|\vec{k}|$ due to isotropy and Dirac-delta encapsulates the homogeneity.

2PCF evaluated at the same point is

$$\xi(0) = \langle\delta^2(\vec{x},t)\rangle = \int \frac{d^3k}{(2\pi)^3}P(k) = \frac{1}{2\pi^2}\int_0^\infty \frac{dk}{k}k^3P(k) = \int_0^\infty \frac{dk}{k}\Delta(k) \tag{3.103}$$

The dimensionless power spectrum is

$$\Delta(k) = \frac{k^3}{2\pi^2}P(k) \tag{3.104}$$

Exploiting $P_\varphi(k) = |\delta\varphi_k|^2 = \frac{|u_k|^2}{a^2}$ so dimensionless power spectrum becomes

$$\Delta_{\delta\varphi}(k) = \frac{k^3}{2\pi^2}\frac{|u_k|^2}{a^2} \tag{3.105}$$

On super-horizon scales using 3.91, $\Delta_{\delta\varphi}(k)$ gives

$$\Delta_{\delta\varphi}(k) = \left(\frac{H}{2\pi}\right)^2\left(\frac{k}{aH}\right)^{3-2\nu} \tag{3.106}$$

with $\nu = \frac{3}{2} + 3\varepsilon - \eta_V$. From the dimensionless power spectrum, one can define the spectral index $n_s \equiv n(k)$ that describes the shape of the power spectrum, which tells us the scale dependence of scalar perturbations.

$$n(k) - 1 = \frac{d\ln\Delta(k)}{d\ln k} \tag{3.107}$$

$$\Delta(k) = \Delta(k_*) \left(\frac{k}{k_*}\right)^{n-1} \tag{3.108}$$

with k_* being a reference scale.

$n_s = 1$ implies purely scale-invariant spectrum, which means that perturbations have the same amplitude at all length scales. However from CMB observations we know that $n_s \approx 0.965$ so there is no exact scale dependence[13].

Inflationary models predict tiny variations from exact scale invariance $n_s \neq 1$. This means something like inflation must have happened in the past. $n_s \neq 1$ produces $n_s - 1 = 3 - 2\nu = 2\eta_V - 6\varepsilon$ which matches the CMB observation. $n_s < 1$ means a red-tilted spectrum due to the slow-roll dynamics, so the power of perturbations decreases in smaller scales. $n_s > 1$ means a blue-tilted spectrum so the power of perturbations increases in smaller scales.

Another important prediction is the power spectrum of tensor modes (GWs from inflation) which produces $\Delta_h(k) \propto \frac{16}{\pi M_{pl}^2} H^2 \left(\frac{k}{aH}\right)^{-2\varepsilon}$. Since PGWs on super-horizon scales are constant, at horizon crossing $\Delta_h(k)\big|_{k=aH} \approx \frac{16}{\pi M_{pl}^2} H^2$. From which tensor spectral index is given as

$$n_T = \frac{d \ln \Delta_h(k)}{d \ln k} = -2\varepsilon < 0 \tag{3.109}$$

So n_T is red-tilted, the amplitude decreases in smaller scales.

Amplitude of GWs considering the energy density from inflation is

$$\Delta_h \propto \frac{H^2}{M_{pl}^2} \simeq \frac{\rho_\varphi}{M_{pl}^4} \approx \frac{V_{inf}}{M_{pl}^4} \approx \frac{E_{inf}^4}{M_{pl}^4} \tag{3.110}$$

So, detecting the tensor spectrum of inflation would give us the energy scale of inflation. Another prediction of inflation is the prediction of B-modes from the CMB[27] (smoking gun for inflation).

Tensor to scalar ratio $r = \frac{\Delta_h}{\Delta_\varphi}$ measures the relative strength of primordial gravitational waves (tensor perturbations) compared to the density fluctuations (scalar perturbations). Exploiting $\Delta_h = \frac{16H^2}{\pi M_{pl}^2}$ and $\Delta_{\frac{\delta\rho}{\rho}} = \left(\frac{H^2}{2\pi\dot{\varphi}}\right)^2$ and the slow-roll parameter $\varepsilon = 4\pi G \left(\frac{\dot{\varphi}}{H}\right)^2$ one gets

$$r = \frac{64\pi}{M_{pl}^2} \left(\frac{\dot{\varphi}}{H}\right)^2 = 16\varepsilon \tag{3.111}$$

The energy scale of inflation is given by $E_{inf} \approx V^{1/4} = 10^{16} GeV \left(\frac{r}{0.01}\right)^{1/4}$ where the normalization with 10^{-2} is chosen because the the sensitivity of next generation experiment is of this order. From the CMB anisotropies, we have the upper bound for the tensor-to-scalar ratio $r < 0.03$. Since the spectral index n_s is related to the slow-roll parameters, measuring n_s and r allows us to constrain inflationary models.

The consistency relation is expressed as

$$r = -8n_T \tag{3.112}$$

This tells us that if we measure both r and n_T, we can check if single-field slow-roll inflation is correct since any deviation/violation of this would indicate physics beyond single-field slow-roll inflationary models. Notice that observation of r would give us information regarding the excursion of the field since $\Delta\varphi \approx \varepsilon^{1/2} N_{CMB} M_{pl} = r^{1/2} N_{CMB} M_{pl}$.

In total, we have 4 observables: 1 power spectrum for scalar perturbations and 1 power spectrum for tensor perturbations, 1 scalar spectral index and 1 tensor spectral index.

In this chapter, we explore the inflationary paradigm as the source of primordial gravitational waves, discussing different models of inflation, slow-roll dynamics, tensor perturbations, and the reheating phase that connects inflation to standard Big Bang model.

4

Neutrino Physics

4.1 Introduction

First hints of the cracks in our understanding of particle physics due to neutrinos dates back to $1900 - 1930$s when physicists were studying radioactive decays, namely β-decay ${}^{A}_{Z}\mathrm{X} \rightarrow {}^{A}_{Z}\mathrm{Y} + e^{-}$ with the given mass number A and atomic numbers Z of mother nucleus X and product nucleus Y. They concluded that all electrons should carry the same energy, however this conclusion was quite contradictory with the experimental results. What they have observed was a continuous spectrum[28]. In addition to energy-momentum conservation, it was discovered that the total angular momentum was not conserved.

In 1930 Wolfgang Pauli proposed a particle with extremely small mass, no electric charge and spin-$\frac{1}{2}$ properties to account for the observed anomalies [29]. Due to the difficulties of detecting such particle, "I have done a terrible thing," Pauli said. "I have postulated a particle that cannot be detected." Neutrinos were detected two decades later by Clyde L. Cowan and Frederick Reines[30].

In 1934, Fermi's theory describes the weak interaction as a four-fermion interaction[31]. to explain the β decay. In terms of Lagrangian, it is expressed as

$$\mathcal{L}_{Fermi} = \frac{G_F}{\sqrt{2}} \left[\bar{p}\gamma^{\mu} n\right] \left[\bar{e}\gamma_{\mu}\nu\right] + h.c. \tag{4.1}$$

which later has been successful in describing possible other decays among which some of the most famous ones are the muon and charged pion decays with final-state neutrinos.

Wu experiment in 1957 [32], which studied the beta decay of cobalt-60, ${}^{60}_{28}\mathrm{Co} \to {}^{60}_{Z7}\mathrm{Ni} + e^- + \bar{\nu}_e + 2\gamma$ showed that electrons were emitted preferentially in a direction opposite to the nuclear spin, which indicated that nature treats the left and right differently. Therefore, parity was violated in weak interaction.

Fermi's theory could not explain this effect. So, the structure of the Lagrangian had to be modified. To accommodate the observations, V-A (Vector minus Axial-vector) structure was proposed by Feynman and Gell-Mann [33], so that the weak interaction has a chiral structure (which distinguishes between left- and right-handed components of fields). This chiral behavior exhibits itself in the current terms $j^\mu \propto \bar{u}_{\nu_e} \left(\gamma^\mu - \gamma^\mu\gamma_5\right) u_e$ namely by vector - axial vector (V-A).

V-A has the shortcoming of describing only charged-current (CC) processes while later this was extended to include neutral-current (NC) processes with explicit vector and axial-vector couplings. The theory was non-renormalizable, and unitarity was violated when $\sqrt{s} \leq 300GeV$, s stands for the Mandelstam variable $s = (p_1 + p_2)^2$, where p_1 and p_2 are the four-momenta of the incoming particles. These problems (except the renormalizability) were solved by the interacting vector boson (IVB) theory [34]where the vector bosons were the massive W and Z.

Despite the fact that IVB theory and Fermi theory have a different structure, their matching at low energies can be seen through the contraction of massive gauge boson M_X where $X = W^\pm, Z$ as the following diagrams show

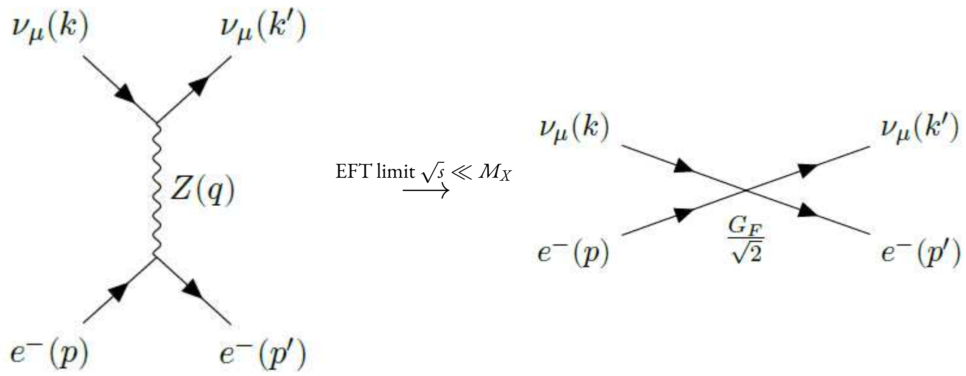

Figure 4.1: Feynman diagrams for neutral current processes involving neutrinos is equivalent to 4-Fermi interaction at the effective field theory limit.

Neutrinos are quite interesting in so many ways, first of all they are a few orders of magnitude lighter than the lightest particles in SM. Even though within SM they are predicted to be massless, we know that they have a mass due to the neutrino oscillation phenomena. There are 4 fundamental forces in nature (electromagnetic force, strong nuclear force, weak nuclear force and gravity) among which neutrinos are only sensitive to weak interaction (and extremely weakly to gravity). As understood by their name, they do not carry an electric charge. Despite the fact that they only interact through the weak interaction, their interaction cross section is extremely small which makes them very hard to detect. In fact, billions of solar neutrinos produced by the sun pass through our bodies every second without us noticing.

Neutrinos within weak interaction can participate charged-current (CC) or neutral-current (NC) interactions by exchanging $W^{\pm}$ and Z bosons respectively. In contrast to electromagnetic interaction where the force carrier particles photons are massless which makes EM interaction range infinite, weak gauge bosons are massive therefore they have a finite range of interaction.

Neutrinos just like other fermions have half-integer spins, however it was soon observed that all neutrinos have their spin anti-parallel to the direction of propagation (momentum) hence they are left-handed (LH), while all the anti-neutrinos are right-handed. Even after 70 years, why all neutrinos are LH is still an open question.

All SM fermions are Dirac particles, while presumably neutrinos are Majorana particles, this feature arises from the fact that SM fermions have 4 states (matter, antimatter each with 2 spin states) however ν and $\bar{\nu}$ are the two different spin states of a 2 state. In general (Dirac type), fermion mass term is coupled to LH and RH fields[35]

$$-\mathcal{L}_D = D(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) = D\bar{\psi}\psi \tag{4.2}$$

given $\psi = \psi_L + \psi_R$. In contrast, Majorana mass term couples the fields of the same type

$$-\mathcal{L}_M = A\bar{\psi}_L\psi_L + B\bar{\psi}_R\psi_R + h.c. \tag{4.3}$$

Neutrinos being chargless might seem quite natural, however the Lagrangian with Majorana mass term has far-reaching consequences on neutrino charge. Since under $U(1)$ global symmetry (holds also for gauge symmetry), the fields transform as $\psi_{L,R} \rightarrow e^{i\alpha}\psi_{L,R}$, this would imply Majorana mass term would violate the conservation of electric charge by 2 units which as we know is not allowed. Therefore, they have to be charge neutral as Majorana particles.

The standard model Lagrangian after spontaneous symmetry breaking [35] is

$$
\begin{aligned}
\mathcal{L}_{SM}^{SSB} &= -\frac{1}{4}G^{q}_{\mu\nu}G^{\mu\nu}_{a} - \frac{1}{4}\left|\partial_\mu A_\nu - \partial_\nu A_\mu - ie(W^-_\mu W^+_\nu - W^+_\mu W^-_\nu)\right|^2 \\
&- \frac{1}{2}\left|\partial_\mu W^+_\nu - \partial_\nu W^+_\mu - ie(W^+_\mu A_\nu - W^+_\nu A_\mu) + ig' c_w (W^+_\mu Z_\nu - W^+_\nu Z_\mu)\right|^2 \\
&- \frac{1}{4}\left|\partial_\mu Z_\nu - \partial_\nu Z_\mu - ig' c_w (W^-_\mu W^+_\nu - W^+_\mu W^-_\nu)\right|^2 \\
&+ \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) - \frac{1}{2}M_h^2 h^2 - \lambda v h^3 - \frac{\lambda}{4}h^4 - \left(1+\frac{h}{v}\right)^2\left[M_W^2 W^{\mu +} W^-_\mu + \frac{1}{2}M_Z^2 Z^\mu Z_\mu\right] \\
&+ \sum_\lambda i\left[\bar{e}^\lambda \partial\!\!\!/ e^\lambda + \bar{\nu}^\lambda_L \partial\!\!\!/ \nu^\lambda_L + \bar{u}^\lambda\left(\partial\!\!\!/ + ig_s\frac{\lambda_a}{2}A\!\!\!/^a\right)u^\lambda + \bar{d}^\lambda\left(\partial\!\!\!/ + ig_s\frac{\lambda_a}{2}A\!\!\!/^a\right)d^\lambda\right] \\
&- \sum_{\lambda,\lambda'}\frac{g}{2\sqrt{2}}\left[W^+_\mu\left(\bar{u}^\lambda\gamma^\mu(1-\gamma_5)V^{\lambda\lambda'}_{CKM}d^{\lambda'}\right) + \left(\bar{\nu}^\lambda\gamma^\mu(1-\gamma_5)e^{\lambda'}\right) + h.c.\right] \\
&- \sum_f\sum_\lambda\left[\frac{g}{c_w}\left[Z_\mu \bar{f}^\lambda\gamma^\mu\left(\frac{T^f_{3L} - 2s_w^2 Q_f}{2} - \frac{T^f_{3L}}{2}\gamma_5\right)f^\lambda\right] - e\left[A_\mu \bar{f}^\lambda \gamma_\mu Q_f f^\lambda\right]\right] \\
&- \sum_{f\neq\nu}\sum_\lambda\left(1+\frac{h}{v}\right)m_f^\lambda \bar{f}^\lambda f^\lambda
\end{aligned}
\tag{4.4}
$$

where $f = e, \nu, u, d$ while electric charge in terms of the weak coupling and Weinberg angle $e = gs_w = g'c_w = gg'/\sqrt{g^2+g'^2}$. The masses of the gauge bosons and fermions are given by $M_h^2 = 2\lambda v^2, M_W^2 = g^2v^2/4, M_Z^2 = (g^2+g'^2)v^2/4$ and $m_f^\lambda = y_f^\lambda v/\sqrt{2}$.

Charged fermions within SM acquire their mass through Yukawa interactions of the form $\mathcal{L}_{Yuk} \sim y\bar{\psi}_L\phi\psi_R + h.c.$, since neutrinos lack the RH partners, Yukawa coupling term that would generate a mass through the Higgs v.e.v can not be constructed for neutrinos. This is explicitly shown in the last line of 4.4.

Analogous to quark-mixing through the CKM matrix [36, 37], there is a somewhat similar effect in the neutrino sector known as flavor mixing. Each flavor eigen-state can be written as a combination of mass eigen-states via Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix[38, 39, 40].

$$
\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = U_{PMNS}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix} = \begin{pmatrix}U_{e1} & U_{e2} & U_{e3}\\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3}\\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\\ \nu_3\end{pmatrix}
$$

It is interesting to see that we were able to make predictions on neutrino properties through cosmology; upper bound to the number of neutrino flavors were predicted by the Big Bang nucleosynthesis (BBN)[41]. Neutrino to photon energy density ratio between the time of electron-positron annihilation and the non-relativistic transition of neutrinos simply tanslates to $\frac{\rho_\nu}{\rho_\gamma} = \frac{7}{8} N_\nu \left(\frac{T_\nu}{T_\gamma}\right)^4$ where $\frac{T_\nu}{T_\gamma} = \left(\frac{4}{11}\right)^{1/3}$ which sets $N_\nu = 3$.

The same value within the experimental bound is also calculated through the total decay rate of Z-boson [35]

$$\Gamma_Z = \Gamma_{had} + \Gamma_{e\bar{e}} + \Gamma_{\mu\bar{\mu}} + \Gamma_{\tau\bar{\tau}} + \Gamma_{inv} = \Gamma_{vis} + \Gamma_{inv} \tag{4.5}$$

where the hadronic width is given by $\Gamma_{had} = \sum_{q \neq t} \Gamma_{q\bar{q}}$ with the final quark states, while invisible width can be expressed as $\Gamma_{inv} = N_\nu \Gamma_{\nu\bar{\nu}} = 166 MeV \cdot N_\nu$, experimentally $\Gamma_Z \approx 2500 MeV$ and $\Gamma_{vis} \approx 2000 MeV$ which sets $N_\nu = 2.994 \pm 0.012$. [42]

The strongest bound for neutrino mass comes in fact from cosmology; observations of the CMB, combined with lensing and baryon acoustic oscillations data sets set an upper bound of $0.12 eV$ at 95% confidence level. To give a rough estimate, one can compute the energy density parameter in neutrinos. When neutrinos are NR, their energy density will evolve $\rho_\nu \approx \sum m_i n_i$, number densities for each species is more or less identical therefore one can write down $\sum m_\nu = m_1 + m_2 + m_3$. This gives us $\Omega_\nu = \rho_\nu / \rho_c$, resulting in $\sum m_\nu \leq 37 eV$.

4.2 Beyond Standard Model Neutrino Interactions

As explained in the next chapter in detail, neutrinos decouple around MeV temperature and since then they have been free-streaming in the universe. However, the decoupling time can be modified by beyond standard model physics which would directly impact on the amplitude of GWs. In this chapter, we will study such phenomena.

Consider a secret neutrino interaction mediated by a massive scalar field with the given interaction Lagrangian [43]

$$\mathcal{L}_{int} = g_\nu \phi \bar{\nu} \nu \tag{4.6}$$

with the g_ν being the dimensionless coupling constant.

Associated Feynman diagram for such a process is given by

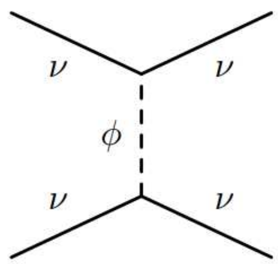

Figure 4.2: Feynman diagram for secret neutrino interaction mediated by a scalar field.

When the temperature of neutrinos falls below the mass of the mediator scalar field, one can exploit effective field theory approach by integrating out the heavy field.

$$\sigma \propto \left[-\frac{g_\nu^2}{(s-M_\phi^2)}\right]^2 \xrightarrow{\sqrt{s}\approx T_\nu \ll M_\phi} \left[\frac{g_\nu^2}{M_\phi^2}\right]^2 = G_\nu^2 \tag{4.7}$$

Therefore, in EFT limit one obtains a 4-Fermion interaction with a dimensional coupling constant $G_\nu = g_\nu^2/M_\phi^2$

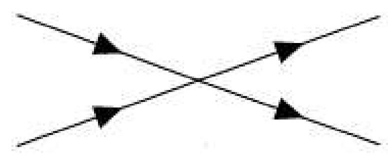

Figure 4.3: Feynman diagram for secret neutrino interaction in effective field theory limit.

Incorporating the detailed derivation given for calculating the cross section for interactions mediated by massive bosons in 5, thermally averaged neutrino-neutrino cross section becomes

$$\langle \sigma_{\nu\nu} v \rangle \propto G_\nu^2 T_\nu^2 \sim \left(\frac{G_\nu}{G_F}\right)^2 \langle \sigma_{\nu\nu}^{SM} v \rangle \tag{4.8}$$

this is valid as long as neutrinos are relativistic. T_ν refers to the temperature of neutrino bath, v is the velocity of neutrinos and Fermi constant $G_F = 1.166 \cdot 10^{-11} MeV^{-2}$. This shows how effectively secret neutrino interactions increase for increasing G_ν relative to the standard model.

In the early universe, self interactions makes the neutrino medium opaque with an opacity [44]

$$\dot{\tau}_\nu \equiv \frac{d}{d\tau}\tau_\nu = -a(\tau)\xi G_\nu^2 T_\nu^5 \propto \Gamma_\nu \tag{4.9}$$

which is proportional to the interaction rate. $\xi \sim \mathcal{O}(1)$ constant depending on the neutrino interactions.

If $G_\nu \gg G_F$, one can neglect the contributions from electroweak physics to the opacity. Define an effective coupling $G_{eff} \equiv \sqrt{\xi}G_\nu = \sqrt{\xi}g_\nu^2/M_\phi^2$. Opacity of ν-medium defines the neutrino visibility function (probability distribution function in terms of redshift when neutrinos start free-streaming)

$$\tilde{g}_\nu(z) \equiv -\dot{\tau}_\nu e^{-\tau_\nu} \tag{4.10}$$

note that with effective coupling constant, the opacity becomes $\dot{\tau}_\nu = -aG_{eff}^2 T_\nu^5$. As it has been studied in the literature, introducing new interactions in neutrino sector with various G_{eff}, seem to push the peak of the neutrino visibility function to lower redshift (late at times).

We will keep our discussion general, so the decays or annihilation of massive scalar particle will not be considered since their effects are very model dependent. However, one can study these scenarios and understand better the reheating of the neutrino sector within the model in consideration.

Considering both the effect of beyond standard model interactions and EW-interactions, total interaction rate for neutrinos is $\Gamma_\nu \approx (G_\nu^2 + G_F^2)T^5$, given $H^2 \approx T^2/M_{pl}$, one obtains the decoupling temperature

$$T_{\nu-dec} \approx \left[M_{pl}(G_\nu^2 + G_F^2)\right]^{-1/3} \tag{4.11}$$

Therefore, one concludes that

- If $G_\nu \gg G_F$, $T_{\nu-dec}$ will be dominated by secret neutrino interactions so ν free-streaming time can be delayed much beyond the weak decoupling time.

- If $G_\nu \ll G_F$, $T_{\nu-dec}$ will be dominated by weak interactions.

In this chapter, we review the role of neutrinos in particle physics and cosmology, focusing on their decoupling and the free-streaming behavior. Then it explores non-standard neutrino interactions mediated by a new scalar particle. It presents the motivation for such models, current constraints, and how they alter the decoupling and free-streaming of neutrinos, potentially impacting cosmological observables.

5 Primordial Gravitational Waves

5.1 Introduction

Primordial gravitational waves (PGWs) are ripples in spacetime generated in the early universe, offering a unique probe into high-energy physics beyond the reach of conventional particle accelerators. These waves are predicted by inflationary cosmology, where quantum fluctuations in spacetime were stretched to macroscopic scales during the accelerated expansion of the universe. Unlike electromagnetic radiation, PGWs interact extremely weakly with matter, allowing them to carry information about the universe's earliest moments. Their detection would provide strong evidence for inflation, constrain the energy scale of the early universe, and offer insights into fundamental physics, including quantum gravity and potential deviations from general relativity.

In addition to their role as a key observational test of inflation, PGWs could also shed light on phase transitions in the early universe, such as those associated with grand unified theories (GUTs) or electroweak symmetry breaking. The spectrum of PGWs carries imprints of the primordial perturbations and the physics governing the hot Big Bang era. Different inflationary models predict varying tensor-to-scalar ratios, making PGW measurements crucial for distinguishing between competing theories. Furthermore, alternative sources such as topological defects and cosmic strings could also generate PGWs, providing a broader window into the physics of the early cosmos.

Current observational efforts, such as Cosmic Microwave Background (CMB) B-mode polarization measurements, pulsar timing arrays [45, 46, 47, 48], and future space-based interferometers [49] aim to detect these primordial signals and refine our understanding of the early universe. As technology advances, the detection of PGWs [13, 26] would mark a groundbreaking achievement in cosmology, offering a direct glimpse into the universe at energy scales far beyond those accessible in terrestrial experiments.

Let us dive into the details of PGWs: for tensor perturbations on an isotropic, uniform, spatially-flat background space-time, the metric is given by

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j \tag{5.1}$$

where h_{ij} represents the perturbation in the spatial part of the metric. The conditions to be satisfied in order to identify the GWs perturbation modes are simply given by the transverse and traceless conditions as follows:

$$\partial^i h_{ij} = 0 \qquad and \qquad h^i_i = 0 \tag{5.2}$$

The first one is the transversality condition which removes the longitudinal degrees of freedom, meaning perturbations do not change the distances along the direction of propagation. The second one, traceless condition is the sum of the diagonal terms in space components of the perturbation. Tracelessness implies that perturbations do not change the overall volume of space. Therefore, if gravitational waves stretch space in one direction, it must simultaneously compress space in the perpendicular direction in such a way that overall volume remains unchanged.

The transverse traceless (TT) gauge simplifies the equation of motion for gravitational waves by eliminating redundant degrees of freedom.

The metric can be written in terms of conformal time

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right] \tag{5.3}$$

where h is a small perturbation with respect to the background metric. By using the relation between coordinate time and the conformal time $d\tau = \frac{dt}{a(t)}$ where $\tau(t) = \int_0^t \frac{dt'}{a(t')}$. As it is obvious now the scale factor in the above equation $a(\tau) \equiv a(t(\tau))$ is used.

To obtain the equation of motion for the perturbation, we require the perturbed metric $g_{\mu\nu}$

to satisfy Einstein equation to $\mathcal{O}(h)$ i.e. we linearize Einstein equations.

$$\delta G_{\mu\nu} = 8\pi G \delta T_{\mu\nu} \tag{5.4}$$

where the Einstein equations are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{5.5}$$

and the Christoffel symbols are given by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (g_{\sigma\nu,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}) \tag{5.6}$$

with the following equation seperating it into background metric and first order corrections due to $h_{\mu\nu}$ terms

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{(0)\lambda}_{\mu\nu} + \delta\Gamma^{\lambda}_{\mu\nu} \tag{5.7}$$

To get the inverse of the metric $g^{\mu\nu}$, use the following ansatz $g^{\mu\nu} = \frac{1}{a^2(\tau)}(\eta^{\mu\nu} + \delta g^{\mu\nu})$ and the identity $g_{\mu\alpha} g^{\alpha\nu} = \delta^{\nu}_{\mu}$. Hence

$$\begin{aligned} g_{\mu\alpha} g^{\alpha\nu} = \delta^{\mu}_{\nu} &= a^2(\tau)(\eta_{\mu\alpha} + h_{\mu\alpha}) \frac{1}{a^2(\tau)} (\eta^{\alpha\nu} + \delta g^{\alpha\nu}) \\ &= \eta_{\mu\alpha}\eta^{\alpha\nu} + \eta_{\mu\alpha}\delta g^{\alpha\nu} + h_{\mu\alpha}\eta^{\alpha\nu} + h_{\mu\alpha}\delta g^{\alpha\nu} = \delta^{\mu}_{\nu} + \eta_{\mu\alpha}\delta g^{\alpha\nu} + h_{\mu\alpha}\eta^{\alpha\nu} \end{aligned} \tag{5.8}$$

where we neglect the last term because we work to linear order in $h_{\mu\nu}$. Therefore we get

$$\eta_{\mu\alpha}\delta g^{\alpha\nu} = -h_{\mu\alpha}\eta^{\alpha\nu} \to \eta^{\mu\beta}\eta_{\mu\alpha}\delta g^{\alpha\nu} = -\eta^{\mu\beta} h_{\mu\alpha}\eta^{\alpha\nu} \to \delta g^{\beta\nu} = -h^{\beta\nu} \tag{5.9}$$

The inverse metric tensor becomes

$$g^{\mu\nu} = \frac{1}{a^2(\tau)} (\eta^{\mu\nu} - h^{\mu\nu}) \tag{5.10}$$

In Friedman–Robertson-Walker (FRW)

$$g_{\mu\nu} = a^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}) = \bar{g}_{\mu\nu} + a^2 h_{\mu\nu} \;\; and \;\; g^{\mu\nu} = \frac{1}{a^2(\tau)}(\eta^{\mu\nu} - h^{\mu\nu}) = \bar{g}^{\mu\nu} - \frac{1}{a^2} h^{\mu\nu} \tag{5.11}$$

obtaining $\bar{g}_{\mu\nu} = a^2 \eta_{\mu\nu}$, $\bar{g}^{\mu\nu} = \frac{1}{a^2}\eta^{\mu\nu}$, $\delta g_{\mu\nu} = a^2 h_{\mu\nu}$ and $\delta g^{\mu\nu} = -\frac{1}{a^2} h^{\mu\nu}$

Christoffel symbols can be written as $\Gamma^{\lambda}_{\mu\nu} = \bar{\Gamma}^{\lambda}_{\mu\nu} + \delta\Gamma^{\lambda}_{\mu\nu}$ with the perturbed part being

$$\delta\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\nabla_{\mu}a^2h_{\nu\sigma} + \nabla_{\nu}a^2h_{\mu\sigma} - \nabla_{\sigma}a^2h_{\mu\nu}\right) \approx \frac{1}{2}\eta^{\lambda\sigma}\left(\nabla_{\mu}h_{\nu\sigma} + \nabla_{\nu}h_{\mu\sigma} - \nabla_{\sigma}h_{\mu\nu}\right) \quad (5.12)$$

while the barred notation refers to the background. Writing explicitly the perturbation to the Einstein equation, one gets

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} + \delta\left(-\frac{1}{2}g_{\mu\nu}R\right) = 8\pi G\delta T_{\mu\nu} \quad (5.13)$$

The variation of Ricci tensor

$$\delta R_{\mu\nu} = \delta R^{\rho}_{\mu\rho\nu} \quad (5.14)$$

can be derived from the Riemann tensor

$$\begin{aligned}\delta R^{\rho}_{\mu\sigma\nu} &= \delta\left[\partial_{\sigma}\Gamma^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\lambda}_{\mu\nu} - (\sigma \leftrightarrow \nu)\right] \\ &= \partial_{\sigma}\delta\Gamma^{\rho}_{\mu\nu} + \delta\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\rho}_{\lambda\sigma}\delta\Gamma^{\lambda}_{\mu\nu} - \partial_{\nu}\delta\Gamma^{\rho}_{\mu\sigma} - \delta\Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\rho}_{\lambda\nu}\delta\Gamma^{\lambda}_{\mu\sigma} \\ &= \nabla_{\sigma}\delta\Gamma^{\rho}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\rho}_{\mu\sigma}\end{aligned} \quad (5.15)$$

Therefore

$$\begin{aligned}\delta R^{\lambda}_{\mu\lambda\nu} &= \delta R_{\mu\nu} = \nabla_{\lambda}\delta\Gamma^{\lambda}_{\mu\nu} - \nabla_{\nu}\delta\Gamma^{\lambda}_{\mu\lambda} \\ &= \nabla_{\lambda}\left(\frac{1}{2}\bar{g}^{\lambda\sigma}\left(\nabla_{\mu}a^2h_{\nu\sigma} + \nabla_{\nu}a^2h_{\mu\sigma} - \nabla_{\sigma}a^2h_{\mu\nu}\right)\right) - \nabla_{\nu}\left(\frac{1}{2}\bar{g}^{\lambda\sigma}\left(\nabla_{\mu}a^2h_{\lambda\sigma} + \nabla_{\lambda}a^2h_{\mu\sigma} - \nabla_{\sigma}a^2h_{\mu\lambda}\right)\right) \\ &\sim \frac{1}{2}\nabla_{\lambda}\left(\frac{1}{a^2}\eta^{\lambda\sigma}\left(-\nabla_{\sigma}a^2h_{\mu\nu}\right)\right)\end{aligned} \quad (5.16)$$

This term's structure makes it easier to identify the contributions to the overall term of $\delta R_{\mu\nu}$ that relates to the divergence of perturbations while from the $\nabla_{\nu}(\cdots)$ there are no dynamic terms in a sense of $\sim \nabla_{\nu}\nabla^{\nu}h$, so

$$\delta R_{\mu\nu} \sim -\frac{1}{2}\nabla^{\sigma}\nabla_{\sigma}h_{\mu\nu} + \textit{other terms} \quad (5.17)$$

Now considering the second term in the perturbed Einstein equation, one obtains

$$
\begin{aligned}
-\frac{1}{2}g_{\mu\nu}R &= -\frac{1}{2}(\bar{g}_{\mu\nu} + a^2 h_{\mu\nu})(\bar{R} + \delta R) \\
&= \underbrace{-\frac{1}{2}\bar{g}_{\mu\nu}\bar{R}}_{\textit{background term}} \underbrace{-\frac{1}{2}\bar{g}_{\mu\nu}\delta R - \frac{1}{2}a^2 h_{\mu\nu}\bar{R}}_{\textit{1st order perturbation terms}} - \frac{1}{2}a^2 h_{\mu\nu}\delta R
\end{aligned}
\tag{5.18}
$$

Variation of it takes the form

$$
\delta\left(-\frac{1}{2}g_{\mu\nu}R\right) = -\frac{1}{2}\bar{g}_{\mu\nu}\delta R - \frac{1}{2}a^2 h_{\mu\nu}\bar{R} \tag{5.19}
$$

while the Ricci scalar is

$$
\begin{aligned}
R = g^{\mu\nu}R_{\mu\nu} &= \left(\bar{g}^{\mu\nu} - \frac{1}{a^2}h^{\mu\nu}\right)\left(\bar{R}_{\mu\nu} + \delta R_{\mu\nu}\right) = \bar{R} + \delta R \\
&= \underbrace{\bar{g}^{\mu\nu}\bar{R}_{\mu\nu}}_{\textit{background term}} + \underbrace{\bar{g}^{\mu\nu}\delta R_{\mu\nu} - \frac{1}{a^2}h^{\mu\nu}\bar{R}_{\mu\nu}}_{\textit{1st order perturbation terms}} - \frac{1}{a^2}h^{\mu\nu}\delta R_{\mu\nu}
\end{aligned}
\tag{5.20}
$$

so that we get

$$
\delta R = -\frac{1}{a^2}h^{\mu\nu}\bar{R}_{\mu\nu} + \frac{1}{a^2}\eta^{\mu\nu}\delta R_{\mu\nu} = 0 \tag{5.21}
$$

Therefore perturbed Einstein equation becomes

$$
\begin{aligned}
\delta G_{\mu\nu} &= \delta R_{\mu\nu} + \delta\left(-\frac{1}{2}g_{\mu\nu}R\right) = 8\pi G\delta T_{\mu\nu} \\
&= -\frac{1}{2}\nabla^{\sigma}\nabla_{\sigma}h_{\mu\nu}
\end{aligned}
\tag{5.22}
$$

Considering the spatial part

$$
\delta G_{ij} = -\frac{1}{2}\nabla^{\nu}\nabla_{\nu}h_{ij} = -\frac{1}{2}h_{ij\ ;\nu}^{\ \ ;\nu} = 8\pi G\delta T_{ij} \tag{5.23}
$$

where the semicolon denotes the covariant derivatives. We have now the evolution equation for the spatial part of the perturbations.

$$
-\frac{1}{2}h_{ij\ ;\nu}^{\ \ ;\nu} = 8\pi G\Pi_{ij} \tag{5.24}
$$

$\Pi_{ij}(t,\vec{x})$ is the anisotropic part of the stress tensor, where p stands for the isotropic pressure and the saptial part of the energy-momentum tensor is $T_{ij} = pg_{ij} + a^2\Pi_{ig}$. For perfect fluid $\Pi_{ij} = 0$, not true in general, since amplitude of gravitational waves are affected by the anisotropic stress e.g. when neutrinos are freely streaming. The L.H.S. of the 5.24

$$h_{ij\ ;\nu}^{\ \ ;\nu} = \bar{g}^{\mu\nu}\left(h_{ij\ ,\ \mu\nu} - \Gamma^{\alpha}_{\mu\nu}h_{ij\ ,\ \alpha}\right) \tag{5.25}$$

To calculate the Christoffel symbol in the abovementioned equation, neglect the perturbation h_{ij} in the metric with coordinate time 5.1 since it would yield $\sim h_{ij}h_{ij\ ,\ \alpha}$ hence higher order corrections. From the definition of Christoffel symbols in 5.6

$$\Gamma^0_{ij} = \frac{1}{2}g^{0\sigma}(g_{\sigma j,i} + g_{i\sigma,j} - g_{ij,\sigma}) = \frac{1}{2}g^{00}(-g_{ij,0}) = \dot{a}a\delta_{ij} \tag{5.26}$$

$$\Gamma^i_{j0} = \Gamma^i_{0j} = \frac{1}{2}g^{i\sigma}(g_{\sigma 0,j} + g_{j\sigma,0} - g_{j0,\sigma}) = \frac{1}{2}g^{i\sigma}(g_{j\sigma,0} - g_{j0,\sigma}) = \frac{1}{2}g^{ik}(g_{jk,0}) = \frac{\dot{a}}{a}\delta^i_j \tag{5.27}$$

Using $\bar{g}^{00} = -1$ and $\bar{g}^{ij} = \frac{1}{a^2}\delta^{ij}$, equation 5.25 becomes

$$\begin{aligned} h_{ij\ ;\nu}^{\ \ ;\nu} &= \bar{g}^{00}(h_{ij\ ,\ 00} - \Gamma^{\alpha}_{00}h_{ij\ ,\ \alpha}) + \bar{g}^{mn}(h_{ij\ ,\ mn} - \underbrace{\Gamma^{\alpha}_{mn}h_{ij\ ,\ \alpha}}_{=\Gamma^0_{mn}h_{ij,0}}) \\ &= -\ddot{h}_{ij} + \frac{1}{a^2}\nabla^2 h_{ij} - \frac{1}{a^2}\delta^{mn}(\dot{a}a\delta_{mn}\dot{h}_{ij}) \end{aligned} \tag{5.28}$$

Therefore

$$h_{ij\ ;\nu}^{\ \ ;\nu} = -\ddot{h}_{ij} + \frac{1}{a^2}\nabla^2 h_{ij} - 3\frac{\dot{a}}{a}\dot{h}_{ij} \tag{5.29}$$

In Fourier space, 5.24 becomes

$$\ddot{h}_{\lambda,\vec{k}} + 3\frac{\dot{a}}{a}\dot{h}_{\lambda,\vec{k}} + \frac{k^2}{a^2}h_{\lambda,\vec{k}} = 16\pi G\Pi_{\lambda,\vec{k}} \tag{5.30}$$

The second term highlights the effect of the expansion of the universe. Derivatives denoted with "·" are with respect to coordinate time.

Using the metric with conformal time

$$\Gamma^0_{ij} = \frac{1}{2}g^{0\sigma}(g_{\sigma j,i} + g_{i\sigma,j} - g_{ij,\sigma}) = \frac{1}{2}g^{00}(-g_{ij,0}) = \frac{a'}{a}\delta_{ij} \tag{5.31}$$

$$\Gamma^i_{0j} = \frac{1}{2} g^{i\sigma} (g_{\sigma j,0} + g_{0\sigma,j} - g_{0j,\sigma}) = \frac{1}{2} g^{ik} (g_{kj,0} + g_{0k,j} - g_{0j,k}) = \frac{1}{2a^2} \delta^{ik} (2a' a \delta_{kj}) = \frac{a'}{a} \delta^i_j \quad (5.32)$$

$$\Gamma^0_{00} = \frac{1}{2} g^{0\sigma} (g_{\sigma 0,0} + g_{0\sigma,0} - g_{00,\sigma}) = \frac{1}{2} g^{00} g_{00,0} = \frac{a'}{a} \quad (5.33)$$

Finally the evolution equation for tensor perturbations reads

$$\begin{aligned}
h_{ij\ ;\nu}^{\ \ ;\nu} &= \bar{g}^{00} \left(h_{ij\ ,\ 00} - \Gamma^\alpha_{00} h_{ij\ ,\ \alpha} \right) + \bar{g}^{mn} \big(h_{ij\ ,\ mn} - \underbrace{\Gamma^\alpha_{mn} h_{ij\ ,\ \alpha}}_{=\Gamma^0_{mn} h_{ij,0}} \big) \\
&= -\ddot{h}_{ij} + \frac{1}{a^2} \nabla^2 h_{ij} - \frac{1}{a^2} \delta^{mn} (\dot{a} a \delta_{mn} \dot{h}_{ij}) \\
&= -\frac{1}{a^2} h''_{ij} - \frac{1}{a^2} \left(-\frac{a'}{a} h'_{ij} \right) + \frac{1}{a^2} \delta^{mn} \left(h_{ij,mn} - \Gamma^0_{mn} h_{ij,0} \right) \\
&= -\frac{1}{a^2} h''_{ij} + \frac{a'}{a^3} h'_{ij} + \frac{1}{a^2} \nabla^2 h_{ij} - 3 \frac{a'}{a^3} h'_{ij}
\end{aligned} \quad (5.34)$$

Passing to Fourier space as before

$$h_{ij\ ;\nu}^{\ \ ;\nu} = -\frac{1}{a^2} h''_{ij} + \frac{1}{a^2} \nabla^2 h_{ij} - \frac{2}{a^2} \frac{a'}{a} h'_{ij} \quad (5.35)$$

we obtain

$$h''_{\lambda,\vec{k}} + 2 \frac{a'}{a} h'_{\lambda,\vec{k}} + k^2 h_{\lambda,\vec{k}} = 16\pi G a^2 \Pi_{\lambda,\vec{k}} \quad (5.36)$$

where $\lambda = +, \times$ denotes the polarization states.

As one can notice, this is the massless Klein-Gordon equation for a plane wave in an expanding universe with a source term. In fact in an expanding universe with FRW metric $ds^2 = a^2(-d\tau^2 + d\vec{x}^2)$, $\sqrt{-g} = a^4$, $g^{00} = -\frac{1}{a^2}$ and $g^{ij} = -\frac{1}{a^2}\delta^{ij}$. The D'Alambertian operator in curved spacetime in the absence of source term is

$$\begin{aligned}
\Box\phi &= \frac{1}{\sqrt{-g}} \left(g^{\mu\nu} \sqrt{-g} \phi_{,\mu} \right)_{,\nu} = \frac{1}{a^4} \left(g^{00} a^4 \phi_{,0} \right)_{,0} + \frac{1}{a^4} \left(g^{ij} a^4 \phi_{,i} \right)_{,j} \\
&= -\frac{1}{a^2} \phi'' - \frac{2}{a^2} \frac{a'}{a} \phi' + \frac{1}{a^2} \nabla^2 \phi = 0
\end{aligned} \quad (5.37)$$

In Fourier space, it takes the same form as 5.36 without the source term

$$\phi_k'' + 2\frac{a'}{a}\phi_k' + k^2\phi_k = 0 \tag{5.38}$$

Each polarization state of the wave behaves as a massless (because there is no $\sim h^2$ term appearing), minimally coupled, real scalar field.

We will solve this equation to understand how the tensor modes behave but first some concepts have to be presented,

The so called comoving Hubble radius $r_H(t)$ is defined as

$$r_H(t) = \frac{R_H(t)}{a(t)} = \frac{c}{aH(t)} = \frac{1}{aH} = \frac{1}{\dot{a}} \tag{5.39}$$

Hubble radius/horizon $R_H(t)$ can be defined as the distance at which the recession velocity of objects due to expansion of the universe is equal to the speed of light. It determines the region beyond which causal interactions are not possible due to rapid expansion of the universe. Inside the Hubble radius $d < R_H$, objects causally interact, while outside the Hubble radius $d > R_H$, objects are causally not connected.

Not to confuse this concept with particle horizon which is defined as the maximum distance from which light emitted since the beginning of the universe could have reached an observer by present time

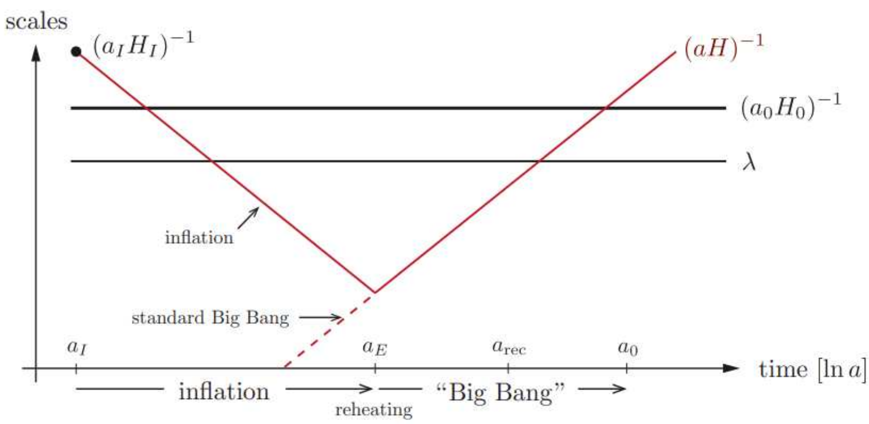

Figure 5.1: Evolution of the comoving Hubble radius as a function of time.

Above-mentioned figure [50] describes the evolution of comoving Hubble radius with time. During inflation (accelerated expansion) r_H decreases as $r_H = \frac{1}{e^{Ht}H}$, if one takes H being constant. This means fluctuations on different scales are stretched out of Hubble radius. After the end of inflation, r_H increases, this allows those fluctuations that were stretched out to re-enter the horizon. If fluctuations survive when $\lambda > r_H$, we can generate cosmological fluctuations at large scales starting from microscopic scales.

At large scales $\lambda_{physical} = a\lambda \sim a/k > H^{-1}$ we require $k \ll aH$ (super-horizon scales). Note $\lambda \sim 2\pi/k$. Effect of anisotropic stress outside the horizon can be ignored since this term is given by a causal mechanism that vanishes outside the horizon.

If h evolves on a timescale similar to the expansion of the universe $\frac{\partial}{\partial t}h \sim Hh$. Using $dt = ad\tau$, one obtains $h' = aHh$. So comparing the second and third terms in 5.36,

$$\frac{2Hh'}{k^2 h} \sim \frac{aH^2 h}{k^2 h} = \frac{a^2 H^2}{ak^2} = \frac{1}{a}\frac{a^2 H^2}{k^2} \gg 1 \tag{5.40}$$

where $\frac{a^2H^2}{k^2} \gg 1$ on super-horizon scales. Hence second term dominates the third term and 5.36 becomes

$$h''_{\lambda,\vec{k}} + 2\mathcal{H}h'_{\lambda,\vec{k}} \approx 0 \tag{5.41}$$

where $\mathcal{H}$ stands for conformal Hubble parameter. To solve the differential equation, it can be re-arranged

$$\frac{h''_{\lambda,\vec{k}}}{h'_{\lambda,\vec{k}}} \approx -2\frac{a'}{a} \longrightarrow \frac{d}{d\tau}\ln h'_{\lambda,\vec{k}} = -2\frac{d}{d\tau}\ln a \tag{5.42}$$

First integrating over τ then exponentiating we get $h'_{\lambda,\vec{k}} = \frac{B}{a^2}$ which leads to

$$h_{\lambda,\vec{k}}(\tau) = A + B\int^{\tau}\frac{d\tau'}{a^2(\tau')} \tag{5.43}$$

A and B are integration constants. Ignoring the second term that is a decaying mode, one finds out that $h(\tau)_{\lambda,\vec{k}}$ remains constant outside the horizon. General solution for $h(\tau)_{\lambda,\vec{k}}$ at any time can be written as

$$h_{\lambda,\vec{k}}(\tau) \equiv h^{prim}_{\lambda,\vec{k}}T(\tau,\vec{k}) \tag{5.44}$$

$h^{prim}_{\lambda,\vec{k}}$ being the primordial GW mode that left the horizon during inflation while $T(\tau,k)$ is the transfer function that describes the sub-horizon evolution of GW modes after the modes entered the horizon. As we have just proved above, during inflation, modes that exit the horizon freeze in amplitude $T(\tau,k) \approx const.$, After inflation, as the universe expands, these modes re-enter horizon and their behavior change $T(\tau,k)$ with T being an oscillating function. The k-dependence in transfer function characterizes the scale of perturbations: large k-values mean small wavelengths/scales, small k-values mean large wavelengths/scales. It shows how different scales evolve differently depending on their relation to Hubble radius at a given time. Horizon crossing occurs when $k = aH$.

Conformal time or scale factor dependence in transfer function appears because the behavior of $T(\tau, k)$ changes due to changing a in radiation-dominated, matter-dominated and dark energy-dominated eras.

The normalization of $T(\tau, k)$ such that; $T(\tau, k) \to 1$ as $k \to 0$ where perturbations are unaltered from their initial conditions.

Let us now decompose the tensor perturbations

$$h_{ij}(\tau, \vec{x}) = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} h_\lambda(\tau; \vec{k}) e^{i\vec{k}\cdot\vec{x}} \varepsilon^{\lambda}_{ij} \tag{5.45}$$

where the polarization tensor $\varepsilon^{\lambda}_{ij}$ satisfies transverse-traceless conditions. The 2 point correlation function of GWs $\Delta^2_h(k)$ is defined as

$$\langle h_{ij}(\tau, \vec{x}) h^{ij}(\tau, \vec{x} + \vec{r}) \rangle = \sum_{\lambda,\lambda'} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle h_{\lambda,\vec{k}} h_{\lambda',\vec{k}'} \rangle e^{i(\vec{k}+\vec{k}')\vec{x}} e^{i\vec{k}'\vec{r}} \varepsilon^{\lambda}_{ij}(k) \varepsilon^{ij}_{\lambda'}(k') \tag{5.46}$$

at $\vec{r} = 0$, one finds

$$\begin{aligned} \langle h_{ij}(\tau, \vec{x}) h^{ij}(\tau, \vec{x}) \rangle &= \sum_{\lambda,\lambda'} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle h_{\lambda,\vec{k}} h_{\lambda',\vec{k}'} \rangle \varepsilon^{\lambda}_{ij}(k) \varepsilon^{ij}_{\lambda'}(k') e^{i(\vec{k}+\vec{k}')\vec{x}} \\ &= \sum_{\lambda,\lambda'} \int \frac{d^3k}{(2\pi)^3} \langle h_{\lambda,\vec{k}} h_{\lambda',-\vec{k}} \rangle \varepsilon^{\lambda}_{ij}(k) \varepsilon^{ij}_{\lambda'}(-k) \end{aligned} \tag{5.47}$$

Using $h^*_\lambda(\tau; \vec{k}) = h_\lambda(\tau; -\vec{k})$ and $\varepsilon^{\lambda}_{ij}(k) \varepsilon^{ij}_{\lambda'}(k) = 2\delta_{\lambda\lambda'}$ where 2 emphasizes two degrees of freedom for GWs leads to

$$\begin{aligned} \langle h_{ij}(\tau, \vec{x}) h^{ij}(\tau, \vec{x}) \rangle &= \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} \langle h_{\lambda,\vec{k}} h^*_{\lambda,\vec{k}} \rangle \cdot 2 = \sum_{\lambda} \int \frac{2}{(2\pi)^3} 4\pi k^2 dk \langle h_{\lambda,\vec{k}} h^*_{\lambda,\vec{k}} \rangle \\ &= \int \frac{dk}{k} \frac{2k^3}{2\pi^2} \sum_{\lambda} \langle |h_{\lambda,\vec{k}}(\tau)|^2 \rangle = \int \frac{dk}{k} \Delta^2_h(k) \end{aligned} \tag{5.48}$$

Therefore, one obtains

$$\Delta_h^2(\tau,k) = \frac{2k^3}{2\pi^2}\sum_\lambda \langle |h_{\lambda,\vec{k}}(\tau)|^2\rangle \tag{5.49}$$

Using the general solution for $h_{\lambda,\vec{k}}$ at any time $h_{\lambda,\vec{k}}(\tau) = h^{prim.}_{\lambda,\vec{k}} T(\tau,k)$. We can account for the time evolution of the power spectrum

$$\Delta_h^2(\tau,k) = \frac{2k^3}{2\pi^2}\sum_\lambda \langle |h^{prim.}_{\lambda,\vec{k}}|^2\rangle [T(\tau,k)]^2 = \Delta^2_{h,prim.}[T(\tau,k)]^2 \tag{5.50}$$

with

$$\Delta^2_{h,prim.} = \frac{2k^3}{2\pi^2}\sum_\lambda \langle |h^{prim.}_{\lambda,\vec{k}}|^2\rangle \quad and \quad \Delta_h^2(\tau,k) = \Delta^2_{h,prim.}[T(\tau,k)]^2 \tag{5.51}$$

To get the amplitude of GWs from de-Sitter inflation; remember 5.36 and take $H = const.$ (in pure de-Sitter inflation), neglecting the source term and rescaling the tensor modes $\hat{h}_{\lambda,\vec{k}} = a(\tau)h_{\lambda,\vec{k}}$ thus $h_{\lambda,\vec{k}} = a^{-1}(\tau)\hat{h}_{\lambda,\vec{k}}$ which gets rid of the Hubble friction term;

$$h'_{\lambda\vec{k}} = \frac{d}{d\tau}h_{\lambda\vec{k}} = \frac{1}{a}\hat{h}'_{\lambda,\vec{k}} - \frac{a'}{a^2}\hat{h}_{\lambda,\vec{k}} \tag{5.52}$$

$$h''_{\lambda\vec{k}} = \frac{d}{d\tau}h'_{\lambda\vec{k}} = \frac{1}{a}\hat{h}''_{\lambda,\vec{k}} - 2\frac{a'}{a^2}\hat{h}'_{\lambda,\vec{k}} - \frac{a''}{a^2}\hat{h}_{\lambda,\vec{k}} + 2\frac{a'^2}{a^3}\hat{h}_{\lambda,\vec{k}} \tag{5.53}$$

The rescaled tensor mode evolution equation, neglecting the stress tensor and combining the above-mentioned equations, one gets

$$\hat{h}''_{\lambda,\vec{k}} - \frac{a''}{a}\hat{h}_{\lambda,\vec{k}} + k^2\hat{h}_{\lambda,\vec{k}} = 0 \tag{5.54}$$

Define $|\hat{h}_{\lambda,\vec{k}}| = a|h_{\lambda,\vec{k}}| = |u^h_{\vec{k}}|$. Above equation becomes

$$u^{h''}_{\vec{k}}(\tau) + \left[k^2 - \frac{a''}{a}\right]u^h_{\vec{k}}(\tau) = 0 \tag{5.55}$$

when $H = const.$ by de-Sitter inflation: $a(t) = e^{Ht}$, $dt = ad\tau \to \tau = -1/aH$ and scale factor in conformal time $a(\tau) = -1/H\tau$. Exploiting the last relationship, the third term in the above equation simplifies to $a''/a = 2/\tau^2 = 2(aH)^2$. Now one can study the evolution in super-horizon and sub-horizon scales.

On super-horizon scales ($k \ll aH$), one can neglect the second term compared to the third term

$$u_{\vec{k}}^{h''}(\tau) = \frac{a''}{a} u_{\vec{k}}^{h}(\tau) \tag{5.56}$$

which produces the following solution to the differential equation

$$u_{\vec{k}}^{h}(\tau) = B(k)a(\tau) \tag{5.57}$$

and tensor modes in super-horizon scales remain constant in time

$$|h_{\lambda,\vec{k}}| = \frac{|u_{\vec{k}}^{h}|}{a} = B(k) = const. \tag{5.58}$$

However, on sub-horizon scales ($k \gg aH$), the second term will dominate the third term

$$u_{\vec{k}}^{h''}(\tau) + k^2 u_{\vec{k}}^{h}(\tau) = 0 \tag{5.59}$$

The solution for the above equation when $k \gg aH$ is

$$|h_{\lambda,\vec{k}}| = \frac{1}{a\sqrt{2k}} \tag{5.60}$$

Because on super-horizon scales. modes freeze, one can set $B(k)$ on super-horizon scale by the value at sub-horizon at horizon crossing $k = aH$

$$|h_{\lambda,\vec{k}}| = \left.\frac{1}{a\sqrt{2k}}\right|_{k=aH} = B(k) = \frac{H}{\sqrt{2k^3}} \tag{5.61}$$

The primordial power spectrum becomes

$$\Delta^2_{h,prim.} = \frac{2k^3}{2\pi^2} \sum_{\lambda} \langle |h_{\lambda,\vec{k}}^{prim.}|^2 \rangle = \frac{16}{\pi} \left(\frac{H_{inf}}{M_{pl}} \right)^2 \tag{5.62}$$

M_{pl} stands for Planck mass while H_{inf} is the Hubble parameter in inflationary epoch.

5.2 Stress-Energy Tensor For Gravitational Waves

This tensor is not directly derived from field equations but it is instead defined to describe the energy, momentum and stress that are carried by the waves. In the weak field limit, the effective stress-energy tensor $T^{GW}_{\mu\nu}$ for GW[51] is given by

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi G}\langle \partial_\mu h_{\alpha\beta}\partial_\nu h^{\alpha\beta}\rangle \tag{5.63}$$

where $\langle\cdots\rangle$ is an average over the wavelengths of GWs to smooth out the rapid oscillations. $h_{\alpha\beta}$ represents the transverse-traceless part of GW perturbations, which are the physical degrees of freedom. Energy density of GWs is given by T^{GW}_{00}, thus

$$\rho_h(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle \tag{5.64}$$

Passing from coordinate time to conformal time through the usual relationship, one obtains

$$\rho_h(\tau) = \frac{1}{32\pi G a^2}\langle h'_{ij}(\tau,\vec{x})h'^{ij}(\tau,\vec{x})\rangle \tag{5.65}$$

Relative spectral energy density $\Omega_h(\tau,k)$ is given by the Fourier transform of energy density $\tilde{\rho}_h(\tau) \equiv \frac{d\rho_h}{dlnk}$ divided by the critical density of the universe $\rho_{cr}(\tau)$, $\Omega_h(\tau,k) \equiv \frac{\tilde{\rho}_h(\tau,k)}{\rho_{cr}(\tau)}$. Let us now provide a full derivation for the relative spectral density $\Omega_h(k)$:

The Ricci tensor with a given metric can be expanded in perturbations h

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \mathcal{O}(h^3) \tag{5.66}$$

where superscript 1 and 2 denotes 1st and 2nd order in h. As Einstein equation is non-linear, in general the background Ricci tensor $\bar{R}_{\mu\nu}$ is non-linear in $h_{\mu\nu}$. Linear term in 5.66 must obey the vacuum Einstein equation ($T_{\mu\nu} = 0$), so

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{5.67}$$

Barred terms refer to background as before

$$\begin{aligned} R_{\mu\nu} &= \bar{R}_{\mu\nu} + \delta R_{\mu\nu} \\ R &= \bar{R} + \delta R \\ g_{\mu\nu} &= \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2\eta_{\mu\nu} + a^2 h_{\mu\nu} \end{aligned} \tag{5.68}$$

So that the background equation reads

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}g_{\mu\nu} = 0 \tag{5.69}$$

Background metric in FRW is given by 5.1 without the perturbation term h_{ij}. From 5.26 and 5.27, The terms contributing to the background Riemann tensor are

$$\bar{R}^0_{i0j} = \partial_0\bar{\Gamma}^0_{ij} - \partial_j \underbrace{\bar{\Gamma}^0_{i0}}_{=0} + \underbrace{\bar{\Gamma}^0_{\mu 0}}_{=0} \bar{\Gamma}^\mu_{ij} - \bar{\Gamma}^0_{kj}\bar{\Gamma}^k_{00} = a\ddot{a}\delta_{ij} \tag{5.70}$$

$$\bar{R}^i_{jkl} = \bar{\Gamma}^i_{\mu k}\bar{\Gamma}^\mu_{jl} - \bar{\Gamma}^i_{\mu l}\bar{\Gamma}^\mu_{jk} = \bar{\Gamma}^i_{0k}\bar{\Gamma}^0_{jl} - \bar{\Gamma}^i_{0l}\bar{\Gamma}^0_{jk} = \dot{a}^2(\delta^i_k\delta_{jl} - \delta^i_l\delta_{jk}) \tag{5.71}$$

and $\bar{R}^0_{ijk} = 0$ and $\bar{R}^i_{jk0} = 0$ due to isotropy. Ricci tensor non-vanishing terms are

$$\bar{R}_{00} = \bar{R}^i_{0i0} + \bar{R}^0_{000} = \bar{g}^{ij}\bar{R}_{j0i0} = \bar{g}^{ij}\bar{R}_{0j0i} = \bar{g}^{ij}\bar{g}_{00}\bar{R}^0_{joi} = -3\frac{\ddot{a}}{a} \tag{5.72}$$

$$\bar{R}_{ij} = \bar{R}^0_{i0j} + \bar{R}^k_{ikj} = (2\dot{a}^2 + a\ddot{a})\delta_{ij} \tag{5.73}$$

while $\bar{R}_{0i} = \bar{R}_{i0} = 0$. In the above expression $\bar{g}^{ij} = \delta^{ij}/a^2$ and $\bar{g}_{00} = -1$ are used.

The background Ricci scalar is

$$\bar{R} = \bar{g}_{00}\bar{R}_{00} + \bar{g}^{ij}\bar{R}_{ij} = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right) \tag{5.74}$$

The perturbed Einstein equation reads

$$\delta R_{\mu\nu} - \frac{1}{2}\delta R\bar{g}_{\mu\nu} - \frac{1}{2}\bar{R}\delta g_{\mu\nu} = 0 \tag{5.75}$$

In TT-gauge these will give the equation for GW perturbations that simplifies to

$$h''_{ij} + 2\frac{a'}{a}h'_{ij} - \nabla^2 h_{ij} = 0 \tag{5.76}$$

Background equation: governs the evolution of the scale factor $a(\tau)$. Perturbed equation: governs the evolution of $h_{\mu\nu}$ describing GW. We get $R^{(1)}_{\mu\nu} = 0$, this is an equation for the propagation of GW. Vacuum field equation $R_{\mu\nu} = 0$ from 5.66 gives us

$$\bar{R}_{\mu\nu} + \langle R^{(2)}_{\mu\nu} \rangle = 0 \tag{5.77}$$

which shows that stress–energy in gravitational waves create background curvature[51].

$R_{\mu\nu}$ represents the curvature of spacetime and its behavior can be affected by coarse graining. Coarse graining in cosmology refers to the process of averaging out small scale inhomogeneities to focus on large scale behavior of $g_{\mu\nu}$ and $R_{\mu\nu}$. In a universe that is homogenous and isotropic on large scales, coarse graining means averaging out small scale structures to focus on the larger, more unifirm behavior of the universe.

If we average out the perturbations over large scales, we smooth out small scale variations. For Ricci tensor

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \textit{ additional terms due to } h_{\mu\nu} \tag{5.78}$$

Using 5.66 and 5.77, one gets $\bar{R}_{\mu\nu} = -\langle R^{(2)}_{\mu\nu} \rangle$, up to second order in $h_{\mu\nu}$

$$R^{(1)\ non-lin.}_{\mu\nu} + R^{(2)}_{\mu\nu} - \langle R^{(2)}_{\mu\nu} \rangle = 0 \tag{5.79}$$

Note that $R^{(1)\ non-lin.}_{\mu\nu}$ represents the non-linear correction to the propagation of $h_{\mu\nu}$. In fact $R^{(1)}_{\mu\nu} = 0$ gives $\quad h_{\mu\nu} \to h_{\mu\nu} + j_{\mu\nu}$ with $j_{\mu\nu} \sim \mathcal{O}(h^2)$

$$R^{(1)}_{\mu\nu} = \frac{1}{2} \langle h_{\alpha\mu|\nu}{}^{\alpha} + h_{\alpha\nu|\mu}{}^{\alpha} - h_{\mu\nu|\alpha}{}^{\alpha} - h_{|\nu\nu} \rangle \tag{5.80}$$

"|" notation is used for covariant derivative with respect to the background metric $\bar{g}_{\mu\nu}$. $h_{\mu\nu}$ is homogeneous solution describing free evolutions of GWs decoupled from sources, while $j_{\mu\nu}$ is the particular solution desribing the inhomogeneous part of the solution which arises from specific sources or stress-energy anisotropies. Hence $R^{(1)}_{\mu\nu}$ takes into account both free GW and their interaction with sources like matter perturbation.

5.77 represents how stress energy in GWs creates the background curvature. 2nd order effects encapsulates how the presence of perturbations like GWs modify the overall curvature of spacetime. GWs can be treated as perturbation in metric. Energy density of these waves contribute to 2nd order Ricci tensor. Remember $R^{(2)} \sim \mathcal{O}(h^2)$. Energy and momentum of GWs can cause the universe to behave differently than predicted by background model alone.

Einstein equation in vacuum then becomes

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}g_{\mu\nu} = 8\pi G T^{GW}_{\mu\nu} \tag{5.81}$$

where the definition of energy-momentum tensor for gravitational waves is

$$T^{GW}_{\mu\nu} = \frac{1}{8\pi G}\left(\langle R^{(2)}_{\mu\nu}\rangle - \frac{1}{2}\bar{g}_{\mu\nu}\langle R^{(2)}\rangle\right) \tag{5.82}$$

$\langle\cdots\rangle$ like before denotes an average over several wavelengths. It tells us how back reaction from energy density of GWs would affect the expansion of the background universe. Effective energy momentum tensor obtained here is not the same as the one obtained by Noether current which can be written as

$$T^{Noether}_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S^{(2)}}{\delta g^{\mu\nu}} \tag{5.83}$$

$S^{(2)}$ is the second order perturbations in Einstein-Hilbert action. These two definitions of energy-momentum tensor matches deep inside the horizon. On deep super-horizon limit (when $\lambda_{GW} \gg r_H$), expansion of the universe becomes more significant compared to oscillation frequency of GWs, which leads to a scenario where the dynamics of the wave is effectively frozen in time. However, when GWs are within the horizon (sub-horizon), their behavior become more complex due to the oscillations and interactions. The two definitions might not coincide because in this regime, oscillatory behavior of the waves influences their energy density and momentum.

Since $\langle R^{(2)}\rangle = 0$,

$$T^{GW}_{\mu\nu} = -\frac{1}{8\pi G}\langle R^{(2)}_{\mu\nu}\rangle \tag{5.84}$$

As derived in the literature [51]

$$\begin{aligned} R^{(2)}_{\mu\nu} = \frac{1}{2}\Bigg[&\frac{1}{2}h_{\alpha\beta|\mu}h^{\alpha\beta}{}_{|\nu} + h^{\alpha\beta}(h_{\alpha\beta|\mu\nu} + h_{\mu\nu|\alpha\beta} - h_{\alpha\mu|\nu\beta} - h_{\alpha\nu|\mu\beta}) + h_{\nu}{}^{\alpha|\beta}((h_{\alpha\mu|\beta} - h_{\beta\mu|\alpha})) \\ &+ \left(h^{\alpha\beta}{}_{|\beta} - \frac{1}{2}h^{|\alpha}\right)(h_{\alpha\mu|\nu} + h_{\alpha\nu|\mu} - h_{\mu\nu|\alpha})\Bigg] \end{aligned} \tag{5.85}$$

The energy -momentum tensor appearing in 5.84 becomes

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi G}\langle h_{\alpha\beta|\mu}h^{\alpha\beta}{}_{|\nu}\rangle = \frac{1}{32\pi G}\langle h_{\alpha\beta,\mu}h^{\alpha\beta}{}_{,\nu}\rangle + \mathcal{O}(h^3) \tag{5.86}$$

In linear theory, one neglects higher order terms in $T^{GW}_{\mu\nu}$. The energy density of GWs as a function of coordinate time, ρ_h is defined by the 00-component of $T^{GW}_{\mu\nu}$.

$$\rho_h(t) \equiv T^{GW}_{00} = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle \tag{5.87}$$

Note that h_{ij} is in TT-gauge. There are only 2 independent polarizations for GWs. Let propagation direction be in z-direction, then we can write

$$h_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Additionally we know that $h_{0\mu} = 0$: GWs are spatial perturbations, and $h_{z\mu} = 0$ since the wave is propagating along the z-direction which means no perturbations along z-direction. We also know that $h_{\mu\nu} = h_{\nu\mu}$. In fact, by traceless condition $h^\mu_\mu = 0$, one can set the diagonal terms as $h_{11} = h_+ = -h_{22}$ and $h_{12} = h_\times$. Therefore,

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The energy density is expressed as

$$\rho_h(t) = \frac{1}{32\pi G}\langle \dot{h}_{ij}\dot{h}^{ij}\rangle = \frac{1}{32\pi G}\langle \dot{h}^2_{11} + 2\dot{h}^2_{12} + \dot{h}^2_{22}\rangle = \frac{1}{16\pi G}\langle \dot{h}^2_+ + \dot{h}^2_\times\rangle \tag{5.88}$$

and as a function of conformal time it's written as

$$\rho_h(\tau) = \frac{1}{16\pi G a^2}\langle h'^2_+ + h'^2_\times\rangle \tag{5.89}$$

Remembering 5.45, which can be decomposed as

$$h_{ij}(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} h_{+,\vec{k}}\ e^{i\vec{k}\cdot\vec{x}}\varepsilon^+_{ij} + \int \frac{d^3k}{(2\pi)^3} h_{\times,\vec{k}}\ e^{i\vec{k}\cdot\vec{x}}\varepsilon^\times_{ij} \tag{5.90}$$

as it is implemented in 5.88

$$\rho_h(\tau) = \frac{1}{16\pi G a^2}\int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\langle (h'_{+,\vec{k}}h'_{+,\vec{k}} + h'_{\times,\vec{k}'}h'_{\times,\vec{k}'})e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}\rangle \tag{5.91}$$

where integral is performed over $d^3\vec{x}$. Having used $h^*_\lambda(\tau;\vec{k}) = h_\lambda(\tau;-\vec{k})$ and $\varepsilon^\lambda_{ij}(k)\varepsilon^{ij}_{\lambda'}(k) = 2\delta_{\lambda\lambda'}$, the energy density for tensor modes reads

$$\rho_h(\tau) = \frac{1}{16\pi G a^2}\int \frac{d^3k}{(2\pi)^3}\left[|h'_{+,\vec{k}}(\tau)|^2 + |h'_{\times,\vec{k}'}(\tau)|^2\right] \tag{5.92}$$

It's worth noting that $\langle\cdots\rangle$ over wavelengths that can be written as the ensemble in k-space $\langle h'_{\lambda,\vec{k}}h'_{\lambda',\vec{k}'}\rangle = (2\pi)^3\delta_{\lambda\lambda'}\delta^3(\vec{k}+\vec{k}')|h'_{\lambda,\vec{k}}|^2$.

It is reasonable to assume that primordial GWs are unpolarized, implying

$$|h_{+,\vec{k}}|^2 = |h_{\times,\vec{k}}|^2 \tag{5.93}$$

To be able to express ρ_h in terms of transfer function and primordial amplitude, exploiting 5.44 and 5.62

$$\begin{aligned}\rho_h(\tau) &= \frac{1}{16\pi G a^2}\int \frac{1}{\pi^2}|h'_\lambda|^2 k^2 dk = \frac{1}{16\pi G a^2}\int \frac{1}{\pi^2}|h^{prim.}_{\lambda,k}|^2 k^2 dk[T']^2 \\ &= \frac{1}{32\pi G a^2}\int \frac{dk}{k}\Delta^2_{h,prim.}[T']^2 = \frac{1}{32\pi G a^2}\int dlnk\Delta^2_{h,prim.}[T']^2\end{aligned} \tag{5.94}$$

$|h^{prim.}_k|$ is being the amplitude of GWs outside the horizon (super-horizon). Inside the horizon

$$T \propto \frac{h_{\lambda,k}(\tau)}{h^{prim}_{\lambda,k}} \propto \frac{a(\tau_{in})}{a(\tau)}e^{\pm ik\tau} \tag{5.95}$$

where τ_{in} refers to in-horizon time.

$$T' \propto \frac{a(\tau_{in})}{a(\tau)}e^{\pm ik\tau}\left(-\frac{a'}{a} \pm ik\right) \tag{5.96}$$

On sub-horizon $k \gg aH$, since $\frac{a'}{a} \propto \tau^{-1} \propto aH$, thus $k \gg aH$ is equivalent to saying $|k| \gg \left|\frac{a'}{a}\right|$, and the oscillations dominate on sub-horizon scales. Leading term for the $[\bar{T'}]^2 \propto a^{-2} \propto \tau^{-2}$ in radiation-era and $[\bar{T'}]^2 \propto a^{-2} \propto \tau^{-4}$ in matter era therefore this shows us that $\rho_h \propto a^{-4}$ implying that gravitons behave like radiation, in agreement with the fact that gravitons are massless and relativistic. ρ_{cr} is defined by the universe with a spatially-flat geometry $k = 0$. From the Friedmann equations $H^2(t) = \frac{8\pi G}{3}\rho_{cr}(t) \rightarrow \rho_{cr}(t) = \frac{3H^2(t)}{8\pi G}$. The relative spectral density becomes

$$\begin{aligned}\Omega_h(\tau, k) &= \frac{\tilde{\rho}_h(\tau, k)}{\rho_{cr}(\tau)} = \frac{1}{\rho_{cr}}\frac{d\rho_h(\tau)}{dlnk} = \frac{1}{32\pi G a^2}\frac{\Delta^2_{h,prim}}{\rho_{cr}(\tau)}[T'(\tau, k)]^2 \\ &= \frac{\Delta^2_{h,prim}}{12H^2(\tau)a^2}[T'(\tau, k)]^2\end{aligned} \tag{5.97}$$

5.94 is exploited in the last step of first line. Deep inside the horizon $k \gg aH$, $|T'|^2 \approx k^2|T|^2$

$$\Omega_h(\tau, k) = \frac{\Delta^2_{h,prim}}{12H^2(\tau)a^2(\tau)}k^2[T(\tau, k)]^2 \tag{5.98}$$

The transfer functions are typically Bessel-like functions

$$T(x) = \frac{1}{x^n}\left[Aj_n(x) + By_n(x)\right] \;\; and \;\; T'(x) \equiv \frac{dT(x)}{d\tau} = -\frac{k}{x^n}\left[Aj_{n+1}(x) + By_{n+1}(x)\right] \tag{5.99}$$

with $x \equiv k\tau$, where y_n and j_n stands for spherical Bessel and spherical Neumann functions respectively. The following identities hold for $z_n(x)$ that could be spherical Bessel, spherical Neumann, Bessel and Neumann functions

$$\frac{d}{dx}\left[\frac{z_n(x)}{x^n}\right] = -\frac{z_{n+1}(x)}{x^n} \quad and \quad \frac{d}{dx}\left[x^{n+1}z_n(x)\right] = x^{n+1}z_{n-1}(x) \tag{5.100}$$

spherical Bessel and spherical Neumann functions have the relationship

$$y_n(c) = (-1)^{n+1}j_{-n-1}(x) \tag{5.101}$$

For $x \gg 1$, they can be expressed as

$$j_n(x) \approx \frac{\sin(x - n\pi/2)}{x} \quad and \quad y_n(x) \approx -\frac{\cos(x - \pi/2)}{x} \tag{5.102}$$

One can observe that for $n = even, j_n(x) \approx \pm j_0(x) \;\; and \;\; y_n(x) \approx \pm y_0(x)$, while for $n = odd$, $j_n(x) \approx \pm y_0(x) \;\; and \;\; y_n(x) \approx \pm j_0(x)$. 1st and 2nd kinds of spherical Henkel functions are written as combination of spherical Bessel and spherical Neumann functions

$$h_n^{(1)}(x) = j_n(x) + iy_n(x) \;\; and \;\; h_n^{(2)}(x) = j_n(x) - iy_n(x) \tag{5.103}$$

One can see the following plot that was calculated in [52],

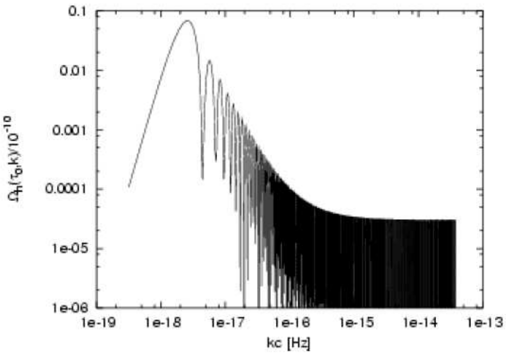

Figure 5.2: Primordial gravitational wave spectrum as a function of comoving wave number

One can see the prediction that the PGW spectrum is scale invariant for large wavenumbers[52]. Figure 5.2 is the spectrum where the changes in effective relativistic degrees of freedom are not taken into account, neither does the effect of neutrino free-streaming. Dark energy contribution is not considered in this figure, and $\Omega_r h^2 = 4.15 \cdot 10^{-5}$ with $h = 0.7$. Low frequency modes/large scales are the ones that re-entered the horizon at late times, that's why $\Omega_h(\tau, k)$ is relatively higher at lower frequencies since they experience less redshift. Note that physical frequency of GW $f_0 = kc/2\pi$.

5.3 Analytic Solutions for Tensor Mode Fluctuations

Now let's discuss the analytic solutions to 5.36 with a vanishing anisotropic stress. Exploiting the form of the general solution in 5.44 and

$$a = \begin{cases} -(H\tau)^{-1} & \text{Inflation} \\ \tau & \text{R.D.} \\ \tau^2 & \text{M.D.} \end{cases}$$

Equation 5.36 reads

$$\text{Differential equations:} \begin{cases} h'' - \frac{2}{\tau}h' + k^2 h = 0 & \text{Inflation} \\ h'' + \frac{2}{\tau}h' + k^2 h = 0 & \text{R.D.} \\ h'' + \frac{4}{\tau}h' + k^2 h = 0 & \text{M.D.} \end{cases}$$

The form of the spherical Bessel differential equation is $y'' + 2y'/x + (1 - n(n+1)/x^2)y = 0$. Above differential equations can be written over T since $h_k^{prim.}$ terms can be canceled out.

- In radiation-dominated era

Use $x \equiv k\tau$, replace $\frac{d}{d\tau} = k\frac{d}{dx}$. First differential equation is re-written as

$$k^2 T'' + \frac{2}{x}k^2 T' + k^2 T = 0 \rightarrow T'' + \frac{2}{x}T' + T = 0 \tag{5.104}$$

which is the spherical Bessel differential equation with $n = 0$. Prime notation is used as the derivative with respect to τ and x for simplicity. Therefore, the solution is

$$\begin{aligned} T(k\tau) &= \frac{1}{(k\tau)^n}\left[Aj_n(k\tau) + By_n(k\tau)\right] = Aj_0 + By_0 \\ &= A\frac{\sin(k\tau)}{k\tau} + B\frac{\cos(k\tau)}{k\tau} = \frac{\sin(k\tau)}{k\tau} \equiv j_0(k\tau) \end{aligned} \tag{5.105}$$

in the last step, boundary conditions are implemented since $T \rightarrow 1$ as $k \rightarrow 0$. Our coefficients are $B = 0 \;\; A = 1$. Therefore

$$h_{\vec{k}}(\tau) = h_{\vec{k}}^{prim}\left[j_0(k\tau)\right] \tag{5.106}$$

- In matter-dominated era

It's not as straightforward to solve the differential equation as in the radiation era, first re-write the spherical Bessel differential equation with y_l which turns into

$$y_l'' + \frac{2}{x}y_l' + \left(1 - \frac{l(l+1)}{x^2}\right)y_l = 0 \tag{5.107}$$

define a function $y_l(x) = x^l Y_l(x)$ so 5.107 becomes

$$Y_l'' + \frac{2(L+1)}{x}Y_l' + Y_l = 0 \tag{5.108}$$

Our differential equation in this era

$$T'' + \frac{4}{\tau}T' + T = 0 \tag{5.109}$$

5.109 resembles to spherical Bessel differential equation 5.107 with $l = 1$. Transfer function is

$$\begin{aligned} T(k\tau) &= \frac{1}{(k\tau)^n}\left[Aj_n(k\tau) + By_n(k\tau)\right] = \frac{1}{k\tau}\left[Aj_1 + By_1\right] \\ &= \frac{1}{k\tau}\left[A\frac{1}{k\tau}\left(\frac{\sin(k\tau)}{k\tau} - \cos(k\tau)\right) - B\frac{1}{k\tau}\left(\frac{\cos(k\tau)}{k\tau} + \sin(k\tau)\right)\right] \\ &= \frac{3j_1(k\tau)}{k\tau} \end{aligned} \tag{5.110}$$

In the last step, coefficients A and B are set by $T \to 1$ as $k \to 0$, and trigonometric functions are expanded, we get $A = 3$ and $B = 0$. Therefore

$$h_{\vec{k}}(\tau) = h_{\vec{k}}^{prim}\left[\frac{3j_1(k\tau)}{k\tau}\right] \tag{5.111}$$

- In inflationary era

$$h'' - \frac{2}{\tau}h' + k^2 h = 0 \xrightarrow{x=k\tau} T'' - \frac{2}{x}T' + T = 0 \tag{5.112}$$

Define $T = x^l Y$, one obtains

$$Y'' + \frac{2(l-1)}{x}Y' + \left(1 + \frac{l^2 - 3l}{x^2}\right)Y = 0 \tag{5.113}$$

with $l = 2$, it resembles to spherical Bessel differential equation with $n = 1$. Transfer function

$$\begin{aligned} T &= x^l Y_n = x^2 Y_1 = x\left[Aj_1(x) + By_1(x)\right] \\ &= \left[A\left(\frac{\sin(k\tau)}{k\tau} - \cos(k\tau)\right) - B\left(\frac{\cos(k\tau)}{k\tau} + \sin(k\tau)\right)\right] \end{aligned} \tag{5.114}$$

Since $T = -k\tau\left[j_1(k\tau) - iy_1(k\tau)\right] = -k\tau\left[h_1^{(2)}(k\tau)\right]$, and $\left(h_{\vec{k}}^{prim}\right)^2 = \frac{8\pi}{k^3}\left(\frac{H_{inf}}{M_{pl}}\right)^2$. Our solution reads

$$h_{\vec{k}}(\tau) = h_{\vec{k}}^{prim}T = -\sqrt{8\pi G k}\frac{\tau}{a}\left[h_1^{(2)}(k\tau)\right]\alpha(\vec{k}) = \frac{\sqrt{8\pi G}}{\sqrt{ka}}\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau}\alpha(\vec{k}) \tag{5.115}$$

with 2nd kind Henkel function being $h_1^{(2)} = -\frac{1}{k\tau}\left(1 - \frac{i}{k\tau}\right)e^{-ik\tau}$.

$$|\vec{k}| : \begin{cases} k > k_{eq}, & \tau_{h.c.} < \tau_{eq} \\ k < k_{eq}, & \tau_{h.c.} > \tau_{eq} \end{cases}$$

$k > k_{eq}$ means smaller scales so modes that entered the horizon in radiation era, while $k < k_{eq}$ means larger scales so modes that entered the horizon in matter era. τ_{eq} and $\tau_{h.c.}$ are at matter-radiation equality and horizon crossing respectively. Horizon crossing time is given by $k\tau_{h.c.} = 1$. Apparently, $|h_{\vec{k}}(\tau)|^2$ for R.D. and inflationary epochs do not depend on time at super-horizon scales $|k\tau| \ll 1$. One can see this through the equations 5.105 and 5.115 as $|h_{\vec{k}}|^2 = |h_{\vec{k}}^{prim}|^2$ and $|h_{\vec{k}}|^2 = \frac{8\pi G}{ka^2}\left(1 + \frac{1}{k^2\tau^2}\right) = \frac{8\pi G}{k^3}H^2(k^2\tau^2+1) \approx \frac{8\pi G}{k^3}H^2$ with $H = const.$.

Dimensionless power spectrum from 5.62 with the tensor mode fluctuations defined in 5.115

$$\Delta_h^2 = \frac{4k^3}{2\pi^2}|h_k^{inf}|^2 = \frac{16}{\pi}\left(\frac{H_{inf}}{M_{pl}}\right)^2(k^2\tau^2+1) \approx \frac{16}{\pi}\left(\frac{H_{inf}}{M_{pl}}\right)^2 \tag{5.116}$$

Here we see again the important prediction of inflation that the dimensionless power spectrum is nearly scale invariant (almost independent of k). From Friedmann equations during inflation

$$H_{inf}^2 = \frac{8\pi}{3M_{pl}^2}\rho_{inf} \approx \frac{8\pi}{3M_{pl}^2}V(\phi) \tag{5.117}$$

for inflation negative isotropic pressure is required $p_{inf} = \frac{1}{2}\dot{\phi}^2 - V(\phi) < 0$ therefore $V(\phi) > \frac{1}{2}\dot{\phi}^2$. 5.116 exploiting the Hubble parameter during inflation gives the primordial dimensionless power spectrum

$$\Delta_{h,prim}^2 \approx 10\frac{V(\phi)}{M_{pl}^2} \tag{5.118}$$

which is sensitive to the shape of the inflaton potential. Evolution of the GW amplitude depends on the behavior of the transfer function on the modes and when they entered the horizon.

$$\begin{aligned} T(\tau < \tau_{eq}, k > k_{eq}) &= j_0(k\tau) \quad R.D \\ T(\tau > \tau_{eq}, k > k_{eq}) &= \frac{\tau_{eq}}{\tau}\left[A(k)j_1(k\tau) + B(k)y_1(k\tau)\right] \quad R.D \\ T(\tau, k < k_{eq}) &= 3\frac{j_1(k\tau)}{k\tau} \quad M.D \end{aligned} \tag{5.119}$$

while the time derivatives very straightforwardly calculated to be

$$\begin{aligned} T'(\tau < \tau_{eq}, k > k_{eq}) &= -kj_1(k\tau) \quad R.D \\ T'(\tau > \tau_{eq}, k > k_{eq}) &= -k\frac{\tau_{eq}}{\tau}\left[A(k)j_2(k\tau) + B(k)y_2(k\tau)\right] \quad R.D \\ T'(\tau, k < k_{eq}) &= -3\frac{j_2(k\tau)}{\tau} \quad M.D \end{aligned} \tag{5.120}$$

k-dependent coefficients A and B are determined by the matching condition of T and T' at matter-radiation equality $\tau = \tau_{eq}$ [52].

Finally, the relative spectral density 5.97 for the above-mentioned regimes with $H^2 = H_{eq}^2\left(\frac{a_{eq}}{a}\right)^4$ in R.D. and, $H^2 = H_0^2\left(\frac{a_0}{a}\right)^3$ in M.D. becomes

$$\begin{aligned} \Omega_h(\tau < \tau_{eq}, k > k_{eq}) &= \frac{\Delta_{h,prim}^2 a^2}{12H_{eq}^2 a_{eq}^4}k^2\left[j_1(k\tau)\right]^2 \\ \Omega_h(\tau > \tau_{eq}, k > k_{eq}) &= \frac{\Delta_{h,prim}^2 a}{12H_0^2 a_0^3}k^2\frac{\tau_{eq}^2}{\tau^2}\left[A(k)j_2(k\tau) + B(k)y_2(k\tau)\right]^2 \\ \Omega_h(\tau > \tau_{eq}, k < k_{eq}) &= \frac{\Delta_{h,prim}^2 a}{12H_0^2 a_0^3}k^2\left[3\frac{j_2(k\tau)}{\tau}\right]^2 \end{aligned} \tag{5.121}$$

Last two refers to the matter dominated epoch for modes that entered the horizon before and after τ_{eq}, while the first one refers to the radiation dominated epoch for modes that entered the horizon before τ_{eq}. $k_{eq} \approx 10^{-15}$Hz

5.4 Statistical Mechanics In Expanding Universe

Number density will decrease as the universe expands, however comoving number density changes only if number changing interactions take place and it is not affected by the expansion. Phase space distribution $f_i(\vec{x}, \vec{p}, t) \rightarrow f_i(E, T)$, where i denotes for particle species, having exploited the homogeneity of the universe, phase space distribution function can not depend on $\vec{x}$ and it can not depend on $\vec{p}$ by isotropy but depends on $|\vec{p}|$ which by dispersion relation is related to energy $E^2 = p^2 + m_i^2$. Since time and temperature are related, T can be used as the evolution variable.

Evolutions of f_i are described by the Boltzmann equations. Early enough, initial conditions are those of equilibrium

$$f_i : \begin{cases} \frac{1}{e^{\frac{E_i-\mu_i}{T}}+1} & \text{Fermions} \\ \frac{1}{e^{\frac{E_i-\mu_i}{T}}-1} & \text{Bosons} \\ \frac{1}{e^{\frac{E_i-\mu_i}{T}}} & \text{Maxwell-Boltzmann} \end{cases}$$

For Maxwell-Boltzmann distribution, fermions and bosons behave similarly. In the early universe at high temperatures, the chemical potential μ_i can be neglected.

Number density, energy density and pressure are[53]

$$\begin{cases} n_i = g_i \int \frac{d^3p}{(2\pi)^3} f_i \\ \rho_i = g_i \int \frac{d^3p}{(2\pi)^3} E f_i \\ p_i = g_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E} f_i \end{cases}$$

For particles in equilibrium

- Relativistic (T$\gg$ m)

$$n_i = g_i \int \frac{d^3p}{(2\pi)^3} \frac{1}{e^{\frac{E_i-\mu_i}{T}} \pm 1} = \frac{g_i}{2\pi^2} \int dp \frac{p^2}{e^{p/T} \pm 1} = g_{i,eff} \frac{\zeta(3)}{\pi^2} T^3 \tag{5.122}$$

Because we work in relativistic regime $E \approx p$ with $g_{i,eff} = g_i$ for bosons and $g_{i,eff} = \frac{3}{4} g_i$ for fermions. Polar coordinates are used since cosmological principle holds, by exploiting isotropy.

$$\rho_i = g_i \int \frac{d^3p}{(2\pi)^3} \frac{p}{e^{\frac{E_i-\mu_i}{T}} \pm 1} = \frac{g_i}{2\pi^2} \int dp \frac{p^3}{e^{p/T} \pm 1} = g^{(\rho)}_{i,eff} \frac{\pi^2}{30} T^4 \tag{5.123}$$

with $g^{(\rho)}_{i,eff} = g_i$ for bosons and $g^{(\rho)}_{i,eff} = \frac{7}{8} g_i$ for fermions.

$$p_i = \frac{1}{3}\rho_i = g^{(\rho)}_{i,eff} \frac{\pi^2}{90} T^4 \tag{5.124}$$

- Non-relativistic (m$\gg$T)

$$\begin{aligned} n_i &= \frac{g_i}{2\pi^2} \int dE \sqrt{E^2 - m^2} E e^{-(E-\mu_i)/T} = \frac{g_i}{2\pi^2} e^{\mu_i/T} m_i^2 T K_2 \left(\frac{m_i}{T}\right) \\ &=\approx g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} e^{\frac{\mu - m_i}{T}} \end{aligned} \tag{5.125}$$

The integral over energy obtained as $\frac{d^3p}{(2\pi)^3} = \frac{1}{2\pi^2}p^2dp = \frac{1}{2\pi^2}\sqrt{E^2 - m^2}EdE$. At late times, the chemical potential can not be neglected. K_2 is the Bessel function $K_2(x) \approx e^{-x}\sqrt{\pi/2}x^{-1/2}$ expanded in non-relativistic limit. This is the Maxwell-Boltzmann distribution for number density. Energy density in this limit by counting each particle by their rest mass becomes

$$\rho_i \approx m_i n_i \tag{5.126}$$

while the pressure as analogous to ideal gas relationship $pV = nT$

$$p_i \approx n_i T \tag{5.127}$$

One can see that these relations yield to $\frac{p_i}{\rho_i} = \frac{T}{m_i} \approx 0 = w$ which is the typical value for non-relativistic particles. For a thermal bath of particles in equilibrium, the total energy density is

$$\rho(T) = \sum \rho_i(T) = \frac{\pi^2}{30} g_*(T) T^4 = \frac{\pi^2}{30}\left(\sum \delta g_{i*}\right)(T_i)T^4 \tag{5.128}$$

where $g_*(T)$ is the effective relativistic degrees of freedom (d.o.f.), T is the temperature of the thermal bath, with $\delta g_{i*}(T \gg m) = \frac{30}{\pi^2 T^4} g_i \int \frac{dE}{2\pi^2} \frac{\sqrt{E^2-m^2}E^2}{e^{E/T}\pm 1}$ which gives g_i for bosons and $\frac{7}{8}g_i$ for fermions. Therefore, the effective relativistic degrees of freedom reads

$$g_*(T \gg m_i) = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^4 \tag{5.129}$$

When $T \ll m_i$, contributions of species-i is exponentially suppressed. Considering the particles in standard model of particle physics with the top quark being the heaviest, effective d.o.f. contributing to the energy density is $g_*(T \gg m_{top})|_{SM} = 106.75$ (when all the SM particles are relativistic which is the case in the early universe.). This number is calculated according to the helicity states given in the table below.

Particle	Rest Mass [MeV]	Helicity States g_i
γ	0	2
$\nu, \bar{\nu}$	0	$3 \cdot 2$
e^-, e^+	0.51	$2 \cdot 2$
μ^-, μ^+	106	$2 \cdot 2$
π^-, π^+	135	2
π^0	140	1
g	0	$8 \cdot 2$
$u, \bar{u}$	5	$3 \cdot 2 \cdot 2$
$d, \bar{d}$	9	$3 \cdot 2 \cdot 2$
$s, \bar{s}$	115	$3 \cdot 2 \cdot 2$
$c, \bar{c}$	$1.3 \cdot 10^3$	$3 \cdot 2 \cdot 2$
τ^+, τ^-	$1.8 \cdot 10^3$	$2 \cdot 2$
$b, \bar{b}$	$4.4 \cdot 10^3$	$3 \cdot 2 \cdot 2$
W^+, W^-	$80 \cdot 10^3$	$3 \cdot 2$
Z	$91 \cdot 10^3$	3
H	$125 \cdot 10^3$	1
$t, \bar{t}$	$175 \cdot 10^3$	$3 \cdot 2 \cdot 2$
SUSY	$\sim 10^6$	~ 110

Table 5.1: Helicity states of the standard model particles and their rest mass.

Entropy density is defined as entropy per volume

$$s = \frac{p + \rho}{T} \approx \frac{4\rho}{3T} = \frac{2\pi^2}{45} g_{*s}(T) T^3 \tag{5.130}$$

while entropy $S(T) = s(T)a^3(T) = const$, in other words: $g_{*s}^{1/3}(T)Ta = const$. This relation is also known as the conservation of entropy (not true in case of inflation). Entropy density however is not conserved in expanding universe. In 5.130 we exploited the fact that relativistic particles dominate s therefore $p = \frac{\rho}{3}$.

Note that g_{*s} is different than g_*, however

$$g_{*s}(T) \overset{\text{relativ.}}{\approx} \frac{3}{4}\left(g_*(T) + \frac{1}{3} g_*(T)\right) = g_*(T) \tag{5.131}$$

Finally, notice that g_{*s} changes when the mass of the particle species drops below the temperature of the thermal bath (when they become non-relativistic, they stop contributing).

Usually, the energy density is given by $\rho \propto a^{-4}$ during the radiation era; however, this is not exactly true. In fact, this was the reason why $\Omega_h(k)$ was scale invariant for $k > k_{eq}$. As discussed already, particle de-relativizations will modify the behavior of energy density ρ since it is sensitive to how many relativistic species we had in the universe at a given epoch. From the entropy conservation, one gets $T \propto g_{*s}^{-1/3}(T)a^{-1}(T)$, so 5.128 in radiation era becomes $\rho(T) \propto g_*(T)g_{*s}^{-4/3}(T)a^{-4}(T)$.

We calculated how the effective relativistic degrees of freedom evolve as the universe cools down. The full standard model particle species, their de-relativizations, EW SSB, QCD PT, neutrino decoupling, electron-positron annihilation and many more effects have been taken into account.

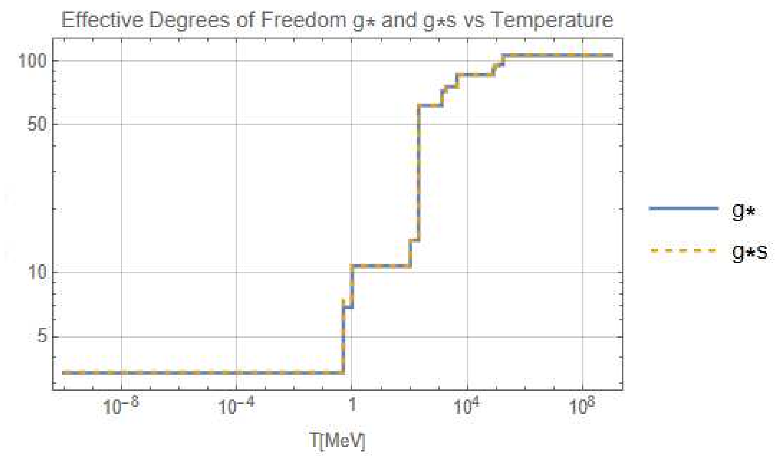

Figure 5.3: Evolution of the effective relativistic degrees of freedom with respect to temperature as the universe cools down.

The evolution of the above plot g_* and g_{*s} is analyzed w.r.t τ since in 5.132, Hubble parameter evolves as a function of conformal time.

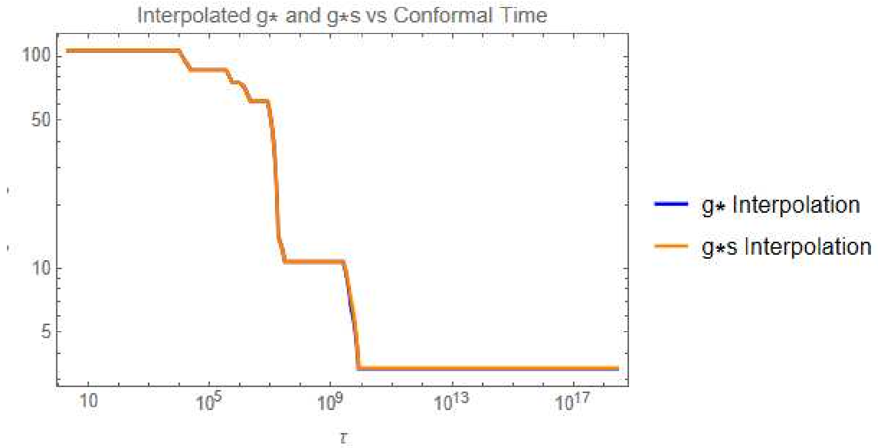

Figure 5.4: Evolution of the effective relativistic degrees of freedom with respect to conformal time [seconds] as the universe cools down.

The effective relativistic degrees of freedom is computed by summing the contributions of all Standard Model particles, and dropping each species from the sum once the temperature falls below its mass threshold. This approach captures the de-relativization of particles as the universe cools. To obtain a smooth behavior suitable for numerical calculations, the resulting function is then interpolated, providing a continuous approximation of the effective relativistic degrees of freedom across the relevant temperature range.

The equation of motion for tensor mode fluctuations 5.36 have a dependence on the Hubble friction term which encapsulates the effect of the changes of relativistic d.o.f. From Friedmann equation $H^2 = \frac{8\pi G}{3}(\rho_r + \rho_m) = \frac{H_0^2}{\rho_c(t_0)}(\rho_r + \rho_m)$

$$\begin{aligned} H &= H_0\sqrt{\frac{\rho_r(\tau)}{\rho_c(\tau_0)} + \frac{\rho_m(\tau)}{\rho_c(\tau_0)}} = H_0\sqrt{\frac{\rho_r(\tau)}{\rho_r(\tau_0)}\frac{\rho_r(\tau_0)}{\rho_c(\tau_0)} + \frac{\rho_m(\tau)}{\rho_m(\tau_0)}\frac{\rho_m(\tau_0)}{\rho_c(\tau_0)}} \\ \frac{a'(\tau)}{a(\tau)^2} &= H_0\sqrt{\left(\frac{g_*}{g_{*0}}\right)\left(\frac{g_{*s}}{g_{*s0}}\right)^{-4/3}\left(\frac{a}{a_0}\right)^{-4}\Omega_r + \Omega_m\left(\frac{a}{a_0}\right)^{-3}} \end{aligned} \tag{5.132}$$

where we have exploited $H(\tau_0) = \frac{H(t_0)}{a_0} = H(t_0)$ and $\rho_c(\tau_0) = \rho_c(t_0)$ while the critical density today is defined to be $\rho_c(t_0) = 3H_0^2/8\pi G$.

In the most generic way, relativistic d.o.f are calculated by[52]

$$g_{*,i}(T) = g_i \frac{15}{\pi^4} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}}{e^u \pm 1} u^2 du \quad , \quad g_{*s,i}(T) = g_i \frac{15}{\pi^4} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}}{e^u \pm 1} \left(u^2 - \frac{x_i^2}{4} \right) du \tag{5.133}$$

given by dimensionless parameters $x_i = m/T$ defining the de-relativization and $u = E/T$ with energy obtained from dispersion relation. The number of helicity states are given in table 5.1. Chemical potential μ_i is assumed to be negligible. From the expression of number density, we get

$$g_{*n,i}(T) = g_i \frac{1}{2\zeta(3)} \int_{x_i}^{\infty} \frac{(u^2 - x_i^2)^{1/2}}{e^u \pm 1} u du \tag{5.134}$$

the expression $\zeta(3)$ being the Riemann-zeta function. Effective number of relativistic degrees of freedom by temperature-weighted sum of particles in consideration is

$$g_*(T) = \sum_i g_{*,i}(T) \left(\frac{T_i}{T} \right)^4 \quad , \quad g_{*s}(T) = \sum_i g_{*s,i}(T) \left(\frac{T_i}{T} \right)^3 \quad , \quad g_{*n}(T) = \sum_i g_{*n,i}(T) \left(\frac{T_i}{T} \right)^3 \tag{5.135}$$

which takes into account that particle species might have a different thermal distribution than that of thermal bath (photons). This can happen when particles decouple, let's quantify our description of what we mean by particle decoupling.

It is assumed that the interaction rate Γ between particles and anti-particles is relatively faster compared to the expansion rate of the universe H to keep thermal equilibrium, however interactions are assumed to be weak enough to be able to treat them as ideal gas.

$$\begin{aligned} N_{collis} &= \int_{t_1}^{t_2} dt \Gamma(t) = -\int_{T_2}^{T_1} \frac{dt}{dT} \Gamma(T) = \int_{T_2}^{T_1} \frac{dT}{T} \frac{\Gamma(T)}{H(T)} \\ &\approx \frac{\Gamma(T)}{H(T)} \ln \left(\frac{T + \Delta T}{T} \right) = \frac{\Gamma(T)}{H(T)} \end{aligned} \tag{5.136}$$

conservation of entropy has been exploited above, meaning $Ta = const$ since g_{*s} is constant at high temperatures. This gives us $\frac{dT}{dt} = -HT$. Note that $t_1 < t_2 \rightarrow T_1 > T_2$. Decoupling is quantified by[12]

$$\begin{aligned} \frac{\Gamma}{H} &\geq 1 \quad \text{equilibrium} \\ \frac{\Gamma}{H} &< 1 \quad \text{decoupled} \end{aligned} \tag{5.137}$$

One of the most prominent example of particle species departing from the thermal equilibrium is neutrinos, where their temperature is determined by the entropic relativistic degrees of freedom $\frac{T_\nu}{T_\gamma} = \left(\frac{g_{*s}(t_e+\varepsilon)}{g_{*s}(t_e-\varepsilon)}\right)^{1/3} = \left(\frac{2}{2+\frac{7}{8}\cdot 2\cdot 2}\right)^{1/3} = \left(\frac{4}{11}\right)^{1/3}$, here t_e marks the time when the temperature of the universe was at the electron mass. This is the famous prediction for neutrino temperature $T_\nu \approx 1.96K$ for the cosmic neutrino background ($C\nu B$), whereas cosmic microwave background (CMB) temperature is $T_\gamma \approx 2.73K$.

Neutrinos are cooler than photons because they missed the heating from electron-positron annihilation, which deposited the energy to photons. Before the neutrino decoupling, neutrinos and photons shared the same temperature even though ν and γ can not interact within the SM, but the thermal equilibrium was maintained by the $e^-\gamma$ through EM interactions, and $e^-\nu$ through WI so photons and neutrinos kept the same temperature.

To compute the decoupling temperature, one can use the jargon of QFT to express the thermally averaged interaction rate

$$\Gamma = n\langle\sigma v\rangle \approx \sigma n \tag{5.138}$$

the cross section then

$$\sigma \sim \alpha^2|\text{propagator}|^2 \cdot \frac{q^4}{E^2} \tag{5.139}$$

q is the momentum of the particles in the interaction. Two different cases will be investigated depending on the nature of the gauge bosons:

- Mediated by massless gauge bosons,

propag. $\sim -i\frac{g^{\mu\nu}}{q^2}$, cross section goes like $\sigma \propto \frac{\alpha^2}{T^2}$, since number density goes proportional to temperature as $n \propto T^3$, 5.138 reads $\Gamma \approx \alpha^2 T$, exploiting 5.137 and [1] gives us the decoupling temperature $T_{dec} = \alpha^2 T$.

- Mediated by massive gauge bosons,

propag. $\sim \frac{-ig^{\mu\nu}+q^\mu q^\nu/M_X^2}{q^\mu q^\nu - M_X^4} \approx i\frac{g^{\mu\nu}}{M_X^2}$, cross section goes like $\sigma \propto \frac{\alpha^2}{M_X^4}T^2 = G_X^2 T^2$, where $q \approx E \approx T$, interaction rate $\Gamma \approx G_X^2 T^5$ which leads to the conclusion $T_{dec} = (G_X^2 M_{pl})^{-1/3} \approx$

[1] Hubble parameter to quantify expansion rate is given $H^2 = \frac{8\pi G}{3}\rho_R = \frac{8\pi}{M_{pl}^2}\left(\frac{\pi^2}{30}g_* T^4\right)$.

$\left(\frac{M_X}{100GeV}\right)^{4/3} MeV$. which for neutrinos mediated by $W^{\pm}, Z$-bosons of the masses respectively 91GeV and 80GeV, results with a decoupling temperature of $T_{dec} \approx MeV$.

$g_*(T)$ and $g_{*s}(T)$ has kept a constant value since $T < 0.1MeV$ to the present day $T_0 = 10^{-4}eV$. These values $g_{*0} = g_\gamma \left(\frac{T_\gamma}{T_\gamma}\right)^4 + g_\nu \left(\frac{T_\nu}{T_\gamma}\right)^4 = 2 + \frac{7}{8} \cdot 6 \left(\frac{4}{11}\right)^{4/3} = 3.3626$ while $g_{*s0} = g_\gamma \left(\frac{T_\gamma}{T_\gamma}\right)^3 + g_\nu \left(\frac{T_\nu}{T_\gamma}\right)^3 = 2 + \frac{7}{8} \cdot 6 \left(\frac{4}{11}\right) = 3.90909$.

Evolution of the effective relativistic degrees of freedom parametrized by g_* and g_{*s} affect the expansion history of the universe, if they were to be constant we would observe a constant $a'(\tau)$ in radiation era.

5.5 Effects of the Relativistic Degrees of Freedom

Due to the changes in the effective relativistic degrees of freedom, PGW spectrum will be modified and no longer be scale-invariant at $k > k_{eq}$. What we will see is a spectrum with a rich feature encapsulating the important events (e.g. quark-gluon phase transition, e^-e^+ annihilation, neutrino free-streaming) in the history of the universe.

Relative spectral density for the PGW modes k that entered the horizon before the matter-radiation equality $\tau_{hc} < \tau_{eq}$ (in radiation era), from 5.97 becomes

$$\Omega_h(\tau_0, k > k_{eq}) = \Omega_h(\tau_{hc}, k)\Omega_{r0} \left[\frac{g_{*s}(T_{hc})}{g_{*s0}}\right]^{-4/3} \left[\frac{g_*(T_{hc})}{g_{*0}}\right] \tag{5.140}$$

where subscript "0" refers to today. This relation tells us that the modes that entered the horizon earlier suffer from a larger suppression due to the fact that g_* and g_{*s} are higher at earlier times. However, at late times $T \leq 0.1MeV$, g_* and g_{*s} are almost the same as calculated before. Therefore, the relative spectral density is dampened by a factor of $\left[\frac{g_*}{g_{*0}}\right]^{-1/3}$, which implies that the modes that entered the horizon in the matter era are unaffected since the effective relativistic degrees of freedom do not evolve in matter era.

GW spectrum is subjected to 20% and 30% suppression due to two different, yet very important epochs in the cosmic history, namely e^-e^+ annihilation and QGP phase transition respectively. In our discussion, QGP phase transition is assumed to be instantaneous which is a valid approximation[52].

One interesting feature of detecting PGW is that it would provide a fascinating playground for constraining new physics. For instance, if supersymmetry is unbroken around TeV scale, number of relativistic d.o.f. in SM would at least be doubled. So, PGW spectrum would be suppressed by 20%[52]. Unlike CMB which gives us information from 380.000 years after the Big-Bang, gravitational waves could shed light on the epoch of hot Big-Bang (reheating) which corresponds to extremely high frequencies k_{rh}. In the opposite frequency range of GW spectrum (extremely low frequencies $< 10^{-18}Hz$), since dark energy dominates over matter and radiation, the modes that entered the horizon at this epoch will be dampened by the accelerated expansion of the universe.

One has to be careful while talking about the frequencies because what we mean by k is the comoving wave-number (kc in units of s^{-1}) which is related to the frequency today by $2\pi f_0 = kc/a_0$. In order to relate the frequency of GWs to the temperature at horizon-crossing, exploit the horizon crossing condition $k_{hc} = a_{hc}H_{hc}$ and the Friedmann equation at horizon crossing

$$H_{hc}^2 = \frac{8\pi G}{3}\rho_{hc} = \frac{8\pi G}{3}\left(\frac{\pi^2}{30}g_{*hc}T_{hc}^4\right) = \frac{8\pi^3 G}{90}g_{*hc}T_{hc}^4 \tag{5.141}$$

Entropy conservation $g_{*s,hc}a_{hc}^3T_{hc}^3 = g_{*s,0}a_0^3T_0^3$ provides us the relation between the scale factor at horizon crossing and today

$$\frac{a_{hc}}{a_0} = \left(\frac{g_{*s,0}}{g_{*s,hc}}\right)^{1/3}\frac{T_0}{T_{hc}} = 8\cdot 10^{-14}\left(\frac{1GeV}{T_{hc}}\right)\left(\frac{100}{g_{*s,hc}}\right)^{1/3} \tag{5.142}$$

given $T_0 = 2.35\cdot 10^{-13}GeV$. The frequency today is written as

$$f_0 = 1.65\cdot 2\pi\cdot 10^{-7}\left(\frac{T_{hc}}{1GeV}\right)\left(\frac{g_{*s,hc}}{100}\right)^{-1/3}\left(\frac{g_{*,hc}}{100}\right)Hz \tag{5.143}$$

where the conventional definition of frequency $f_{hc} = \frac{k_{hc}}{2\pi a_{hc}} = \frac{H_{hc}}{2\pi}$ has been used to express $f_0 = f_{hc}\frac{a_{hc}}{a_0}$.

5.6 Neutrino Free Streaming

Until now, even though our discussion presented us a rich environment to understand the spectrum of GWs in the presence of changing relativistic d.o.f., our analysis is missing a crucial piece: effect of non-vanishing anisotropic stress. However, free-streaming of relativistic neutrinos significantly contribute to the anisotropic stress by damping the PGW amplitude. [54]

As one expects Π_k depends on the fraction of total energy density in neutrinos. As discussed in the previous chapter, unlike the quark-gluon phase transition where the spontaneous transition is a valid approximation, for neutrino decoupling this is not true. Meaning that neutrino decoupling do not occur at a sharply defined point in time, instead the decoupling occurs over a extended period of time. Instantaneous decoupling gives rise to an oscillatory feature, however this effect is artificial and it would be smoothed out considering a thick decoupling surface.

First, we need to solve the Boltzmann equation separately for neutrinos taking into account the neutrino decoupling. Before the decoupling, neutrinos are in thermal equilibrium with the rest of the universe, however after the decoupling once they start free streaming, they satisfy the collisionless Boltzmann equation (Vlasov equation)

$$\frac{dF(x,P)}{dt} = 0 \quad with \quad F(x,P) = \bar{F}(P) + \delta F(x,P) \tag{5.144}$$

where $F(x,P)$ is the distribution function. The average distribution function for relativistic neutrinos is $\bar{F}(P^0) = g_\nu(e^{P^0/T}+1)^{-1}$ given the helicity states for neutrinos.

Four vector notation will be used throughout this chapter $P^\mu \equiv \frac{dx^\mu}{d\lambda}$ with the time-component of the four-momentum being $P^0 = \sqrt{g_{ij}P^iP^j}$.

Note that the time component of the 4-momentum (energy of neutrinos) will acquire a minus sign when it's lowered/raised by the metric tensor. 3-momentum $P^i = C\gamma^i P_0$ is related to the energy through on-shell condition, so P^μ will have only 3-independent components

$$\begin{aligned} -P_0^2 + a^2\delta_{ij}P^iP^j + a^2 h_{ij}P^iP^j = 0 \\ a^2C^2 + a^2C^2 h_{ij}\gamma^i\gamma^j = 1 \end{aligned} \tag{5.145}$$

where γ^i are the directional cosines with the following properties $\gamma^i = \gamma_i$ and $\delta_{ij}\gamma^i\gamma^j = 1$. This gives us $C = \pm\frac{1}{a}\frac{1}{\sqrt{1+h_{ij}\gamma^i\gamma^j}}$, therefore the momentum of neutrinos become up to $\mathcal{O}(h^2)$

$$P^i = C\gamma^i P_0 \approx \pm\frac{\gamma^i P_0}{a}\left(1 - \frac{1}{2}h_{jk}\gamma^j\gamma^k\right) + \mathcal{O}(h^2) \tag{5.146}$$

Consider 5.144

$$\frac{dF(t,x^i,\gamma^i,P^0)}{dt} = \frac{\partial F}{\partial t} + \frac{dx^i}{dt}\frac{\partial F}{\partial x^i} + \frac{dP^0}{dt}\frac{\partial F}{\partial P^0} + \frac{d\gamma^i}{dt}\frac{\partial F}{\partial \gamma^i} = 0 \tag{5.147}$$

where the last term is negligible in linear perturbation theory since it goes up to second order.

It can be demonstrated by exploiting the geodesic equation written in terms of four velocity

$$\begin{aligned}\frac{du^\mu}{d\lambda}+\Gamma^\mu_{\alpha\beta}u^\alpha u^\beta=0\\ \dot{\gamma}^j\equiv\frac{d\gamma^j}{dt}=-\Gamma^i_{jk}\gamma^j\gamma^k=-\frac{1}{2a}h_{jk,i}\gamma^j\gamma^k\end{aligned}\tag{5.148}$$

the relation for directional cosines $\gamma_i=u_i/|u|$ and Christoffel symbols for space components $\Gamma^i_{jk}\approx\frac{1}{2a}h_{jk,i}$ has been used in the second line. Since $\frac{\partial F}{\partial\gamma^i}\sim\mathcal{O}(h)$, the last term is of higher order and discarding it for our purpose is valid.

2nd term in 5.147 is of 1st order in perturbations, using the definition of four-momentum, we get $\frac{dx^i}{dt}=\frac{dx^i}{d\lambda}\frac{d\lambda}{dt}=\frac{P^i}{P^0}$, exploiting 5.146, momentum of neutrinos becomes

$$\frac{dx^i}{dt}\frac{\partial F}{\partial x^i}=\frac{P^i}{P^0}\frac{\partial F}{\partial x^i}\approx\frac{\gamma^i}{a}\frac{\partial F}{\partial x^i}\tag{5.149}$$

For the 3rd term in 5.147, get use of the geodesic equation in 5.148 to get the evolution for the energy of neutrinos up to first order in perturbations gives

$$\frac{dt}{d\lambda}\frac{dP^0}{dt}=P^0\frac{dP^0}{dt}=\frac{1}{2}\left[g_{00,0}(P^0)^2-g_{ij,0}P^iP^j\right]=-\frac{\dot{a}}{a}(P^0)^2-\frac{1}{2}a^2\dot{h}_{ij}P^iP^j\tag{5.150}$$

So, energy of neutrinos as they propagate through the expanding universe with gravitational waves is described by

$$\frac{1}{P^0}\frac{dP^0}{dt}=-\frac{\dot{a}}{a}-\frac{1}{2}\frac{\partial h_{ij}}{\partial t}\gamma^i\gamma^j\tag{5.151}$$

where the 3-momentum of neutrinos is plugged in the above expression up to zeroth order since the last term is already linear in perturbations. There are two very important conclusions one can draw from this equation:

- The first term highlights the redshift in the energy of neutrinos due to the expansion of the universe.
- The second term captures the effect of neutrinos on gravitational waves: neutrinos lose energy if $\frac{\partial h_{ij}}{\partial t}>0$, and gain energy if $\frac{\partial h_{ij}}{\partial t}<0$.

Energy gain from GWs to neutrinos would imply a damping of the GW amplitude. In contrast, loss of energy in neutrinos is transferred to the GWs hence it would cause an amplification in the GW amplitude.

Putting back these findings in the Vlasov equation 5.147 with the tensor type perturbation in the distribution function δF at first order one gets

$$\left.\frac{dF}{dt}\right|_{1st\ order} = \frac{\partial \delta F}{\partial t} + \frac{\gamma^j}{a}\frac{\delta F}{\partial x^i} - P^0 \frac{\partial \delta F}{\partial P^0}\frac{\dot{a}}{a} - P^0 \frac{\partial \bar{F}}{\partial P^0}\frac{1}{2}\frac{\partial h_{ij}}{\partial t}\gamma^i\gamma^j = 0 \tag{5.152}$$

Performing Fourier transform, it becomes

$$\frac{\partial f_k}{\partial t} - \frac{\dot{a}}{a}P^0\frac{\partial f_k}{\partial P^0} + \frac{ik\mu}{a}f_k = P^0\frac{\partial \bar{F}}{\partial P^0}\frac{1}{2}\frac{\partial h_k}{\partial t} \tag{5.153}$$

with $\mu \equiv \gamma^j k_i/k$ and the gradient in momentum space $\partial_{x_i} = ik_i$.

Fourier transform is performed over

$$h_{ij}(t,\vec{x}) = \sum_{\lambda=+,\times}\int \frac{d^3k}{(2\pi)^3} h_{\lambda,k}(t) Q^{\lambda}_{ij}(\vec{x}) \tag{5.154}$$

So that

$$\delta F = \sum_{\lambda=+,\times}\int \frac{d^3k}{(2\pi)^3} f_{\lambda,k}(t,P^0,\mu)\gamma^i\gamma^j Q^{\lambda}_{ij}(\vec{x}) \tag{5.155}$$

One has to note that 5.153 is the generalization of the tensor perturbations we expressed before, but in a generic metric. Tensor harmonics Q^{λ}_{ij} arise as a solution to tensor Helmholtz equation $Q^{\lambda}_{ij|a}{}^{|a}(\vec{x}) + k^2 Q^{\lambda}_{ij}(\vec{x}) = 0$ with the following properties

$$\begin{aligned} Q^{\lambda}_{ij} &= Q^{\lambda}_{ji} && symmetric \\ \gamma^{ij}Q^{\lambda}_{ij} &= 0 && traceless \\ Q^{\lambda}_{ij}{}^{\,|\,j} &= 0 && divergence-free \end{aligned} \tag{5.156}$$

where the $\gamma^{ij} = a^2 g^{ij}$ and the covariant derivative written with respect to the γ^{ij} is implemented by "|". To be able to solve the differential equation, absorb the Hubble friction term by introducing a comoving momentum $q^{\mu} \equiv aP^{\mu}$, define $q \equiv q^0$ and pass from coordinate time to conformal time. Vlasov equation at first order will become

$$\frac{\partial f_k}{\partial \tau} + ik\mu f_k = q\frac{\partial \bar{F}}{\partial q}\frac{1}{2}\frac{\partial h_k}{\partial \tau} \tag{5.157}$$

given $f_k = f_k(\tau, q, \mu)$. Solution for the distribution function is

$$f_k(\tau, q, \mu) = e^{-i\mu k(\tau - \tau_{\nu-dec})} f_k(\tau_{\nu-dec}, q, \mu) + \frac{q}{2}\frac{\partial \bar{F}}{\partial q}\int_{\tau_{\nu-dec}}^{\tau} d\tau' h'_k(\tau') e^{-i\mu k(\tau-\tau')} \tag{5.158}$$

- Before neutrino decoupling, neutrinos are strongly interacting with the rest of the universe and keeping their thermal equilibrium. Because of their strong interaction, perturbations in the distribution function that will contribute to the anisotropic stress get suppressed.
- After neutrino decoupling, neutrinos start free-streaming (they propagate without interacting effectively with the thermal bath). Therefore, gravitational waves can induce anisotropies in the distribution of free-streaming neutrinos.

Let us derive the explicit form of the anisotropic stress from 5.24 and using the definition of the energy-momentum tensor in the same page. Stress energy tensor of neutrinos becomes

$$\delta T_{ij}^{(\nu)} = a^2 \Pi_{ij}^{(\nu)} = a^2 \sum_{\lambda=+,\times} \int \frac{d^3k}{(2\pi)^3} \Pi_{\lambda,k} Q_{ij}^{\lambda}(\vec{x}) \tag{5.159}$$

Knowing that

$$T_{ij}^{(\nu)} = \frac{1}{\sqrt{-g}} \int \frac{d^3q}{q^0} q_i q_j F(q) = a^{-4} \int \frac{d^3q}{q^0} q_i q_j F(q) \tag{5.160}$$

perturbing the spatial component of the energy-momentum tensor for neutrinos gives

$$\delta T_{ij}^{(\nu)} = a^{-4} \int \frac{d^3q}{q^0} \left[\bar{q}_i \bar{q}_j \delta F + (\delta q_i \bar{q}_j + \bar{q}_i \delta q_j)\bar{F}\right] \tag{5.161}$$

Combining the comoving energy of neutrinos $q^0 = q$, perturbation in the comoving momentum is $\delta q^i = -\frac{1}{2}\frac{\gamma^i q}{a} h_{jk}\gamma^j\gamma^k$ and using $\bar{q}^i = a^{-1}q\gamma^i$, $\bar{q}_i = aq\gamma^i$ since $\gamma_i = \gamma^i$, 5.161 re-arranges into

$$\begin{aligned}\delta T_{ij}^{(\nu)} &= a^{-4} \int \frac{d^3q}{q^0} [\bar{q}_i \bar{q}_j] \delta F = a^{-2} \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3q}{q^0} q^2 \gamma^i \gamma^j \gamma^l \gamma^m f_{\lambda,k} Q_{lm}^{\lambda}(\vec{x}) \\ &= a^2 \int \frac{d^3k}{(2\pi)^3} \Pi_{\lambda,k} Q_{ij}^{\lambda}(\vec{x})\end{aligned} \tag{5.162}$$

After exploiting the identity between directional cosines and Kronecker deltas and performing an integration by parts, anisotropic stress is

$$\Pi_k = -4\bar{\rho}_\nu(\tau) \int_{\tau_{\nu-dec}}^{\tau} d\tau' \left[\frac{j_2[k(\tau-\tau')]}{[k(\tau-\tau')]^2}\right] h_k'(\tau') \tag{5.163}$$

unperturbed energy density for neutrinos is given by $\bar{\rho}_\nu(\tau) = a^{-4}\int d^3q q\bar{F}(q)$.

Integro-differential equation describing the evolution of the tensor mode fluctuations considering the effect of anisotropic stress from free-streaming neutrinos becomes

$$h_k''(\tau)+2\left[\frac{a'}{a}\right]h_k'(\tau)+k^2h_k(\tau) = -24\frac{\bar{\rho}_\nu(\tau)}{\bar{\rho}(\tau)}\left[\frac{a'}{a}\right]^2 \int_{\tau_{\nu-dec}}^{\tau} d\tau' \left[\frac{j_2[k(\tau-\tau')]}{[k(\tau-\tau')]^2}\right] h_k'(\tau') \tag{5.164}$$

in the above equation, Friedmann equation $8\pi G\bar{\rho}/3 = (a'/a^2)^2$ has been used. Fraction of total energy density in neutrinos

$$f_\nu(\tau) = \frac{\bar{\rho}_\nu(\tau)}{\bar{\rho}(\tau)} = \frac{\Omega_\nu\left(\frac{a_0}{a}\right)^4}{\Omega_m\left(\frac{a_0}{a}\right)^3 + (\Omega_\nu+\Omega_\gamma)\left(\frac{a_0}{a}\right)^4} = \frac{\Omega_\nu}{(\Omega_\nu+\Omega_\gamma)\left(1+\frac{a(\tau)}{a_{eq}}\right)} = \frac{f_\nu(0)}{1+\frac{a(\tau)}{a_{eq}}} \tag{5.165}$$

By numerically solving the 5.132 differential equation, we obtained how the fraction of total energy density in neutrinos changes as the universe expands

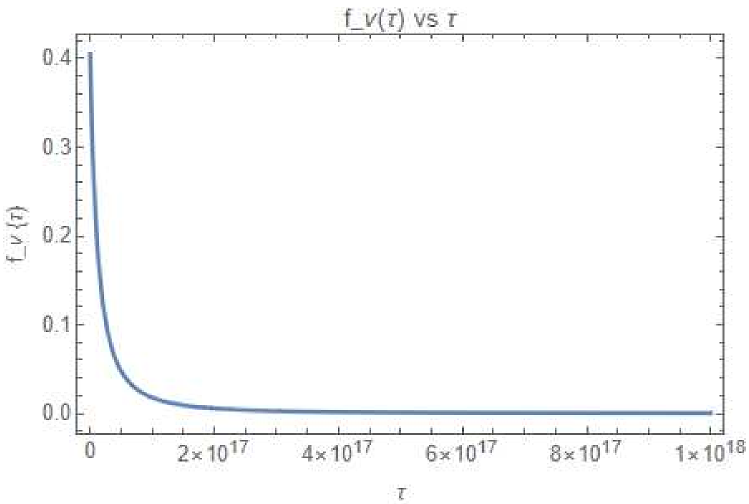

Figure 5.5: Evolution of the fraction of total energy density in neutrinos w.r.t conformal time [second].

where we have exploited the relation at matter-radiation equality $\Omega_{m0}a_{eq}^{-3} = \Omega_{r0}a_{eq}^{-4}$ to replace matter density parameter, while

$$f_\nu(0) = \frac{\Omega_\nu}{\Omega_\nu + \Omega_\gamma} = \frac{g_{*\nu}}{g_{*\nu} + g_{*\gamma}\left(\frac{11}{4}\right)^{1/3}} = 0.40523 \tag{5.166}$$

The latter result is obtained by using the energy densities for ρ_ν and ρ_γ with the temperature for neutrinos $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$ where the relativistic d.o.f. are $g_{*\gamma} = 2$ and $g_{*\nu} = 5.25$. Since in matter era $f_\nu \to 0$, Einstein-Vlasov equation resembles the one of homogeneous wave equation in expanding universe so the damping effect is insignificant in this era.

After separating the amplitude of cosmological fluctuations with a term defined at the end of inflation $h_\lambda(0) = h_{\lambda,\vec{k}}(\tau_{end})$ (since they are frozen until re-entering the horizon), the equation we have to solve numerically can be re-expressed in the variable χ defined by $h_\lambda(u) \equiv h_\lambda(0)\chi(u)$ it the n assumes the following form

$$\chi''(u)+2\left[\frac{a'(u)}{a}\right]\chi'(u)+\chi(u) = -24 f_\nu(u)\left[\frac{a'(u)}{a}\right]^2 \int_{u_{\nu-dec}}^{u} dU \left[\frac{j_2[u-U]}{[u-U]^2}\right]\chi'(U) \tag{5.167}$$

with initial conditions $\chi(0) = 1$ and $\chi'(0) = 0$, where $u = k\tau$.

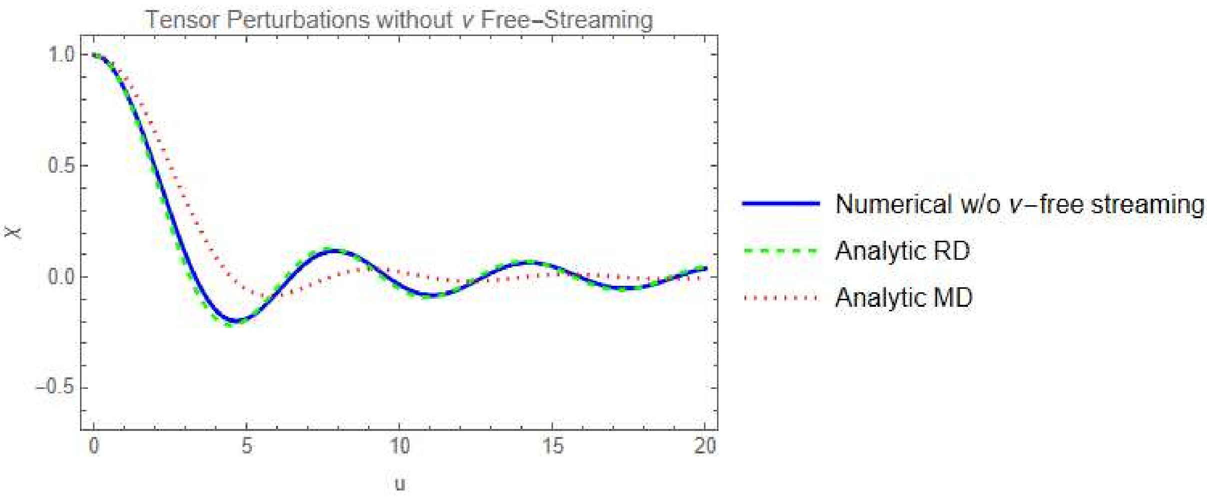

Figure 5.6: Analytic solution for the tensor perturbations in radiation-dominated and matter dominated epochs. Numerical solution for the tensor perturbations with the g_* and g_{*s} effects but without ν free-streaming.

We presented in figure above 5.6 how the tensor perturbations evolve for the modes in which Pulsar Timing Arrays (PTA) are sensitive ($k = 10^{-7} \sim f = 10$ns) in 1) radiation-dominated and 2) matter-dominated epochs without considering the changes in the effective d.o.f. Finally the above integro-differential equation 5.167 is solved numerically, taking into account the changes in the effective relativistic degrees of freedom but not the neutrino free-streaming effect

In the absence of anisotropic stress, the evolution of tensor perturbations in a Friedmann–Lemaître–Robertson–Walker (FLRW) universe admits analytic solutions. During the radiation-dominated (RD) era, the solution is given by a spherical Bessel function $\chi(k\tau) = j_0(k\tau)$. In the matter-dominated (MD) era, the solution takes the form $\chi(k\tau) = 3j_1(k\tau)/k\tau$

Neutrinos affect the evolution of primordial gravitational waves (GWs) through their anisotropic stress after decoupling, leading to damping of tensor modes that enter the horizon during the radiation-dominated era. This is most evident in the behavior of the time derivative of the tensor perturbation h'_{ij}. Now our results also take into account the effect of neutrino free-streaming on GWs in addition to the effects of the SM particle species' changing effective relativistic degrees of freedom on the expansion history of the universe

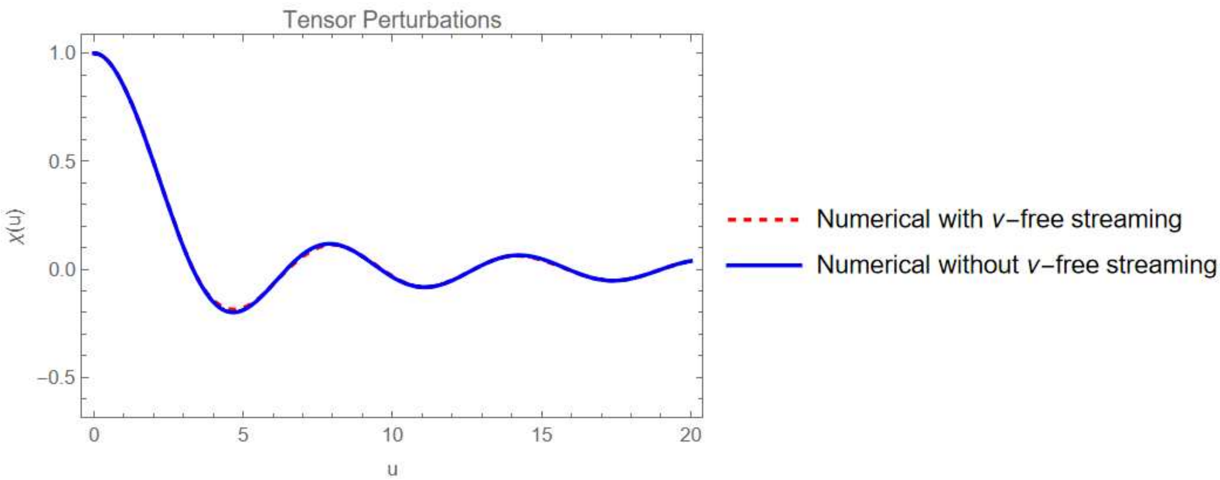

Figure 5.7: Numerical solution for the tensor perturbations considering the changes in g_*, g_{*s} together with the ν free-streaming.

Finally, from the numerical solution to the integro-differential equation above, we calculated $\frac{\partial h}{\partial u}$ since 5.151 explicitly shows whether the perturbations are dampened / amplified

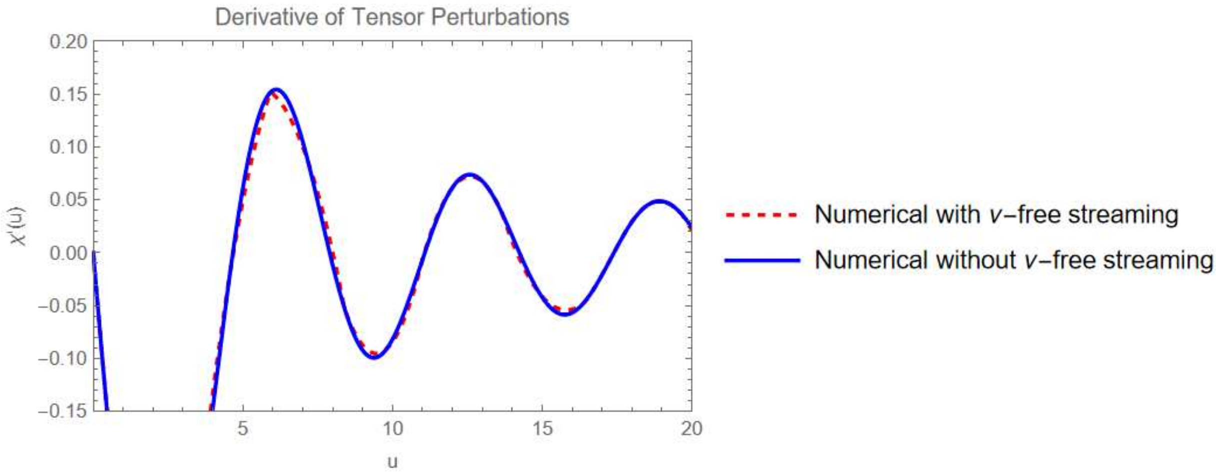

Figure 5.8: Derivative of the tensor perturbation calculated from the solution for χ.

In terms of the modes entering the horizon, one can conclude that

- Lower k-modes re-enter the horizon late at time, therefore the numerical solution for $\chi(u)$ is expected to be close to the one of analytic solution in homogeneous case in matter era.
- Higher-k modes re-entering in radiation era after ν-decoupling will have their amplitude $\chi(u)$ dampened compared to that of analytic solution in homogeneous case in radiation era.

Given the neutrino decoupling time, one can get the mode which entered the horizon at the ν-decoupling through $k\tau_{\nu-dec} = 1$. By analyzing the behavior of $\chi'(u)$ for neutrino decoupling time for various modes (depending on where the peak of χ' is), one can observe that GWs are dampened/amplified by neutrino free-streaming.

Let us now consider the secret neutrino interactions and how they would impact the tensor perturbations. As it was discussed in 4

$$\mathcal{L}_{int} = g_\nu \phi \bar{\nu} \nu \tag{5.168}$$

with the g_ν being the dimensionless coupling constant at vertex.

Associated Feynman diagram for such a process is

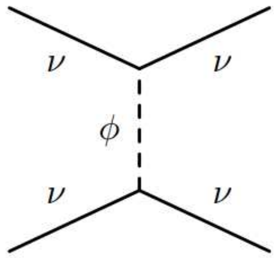

Figure 5.9: Feynman diagram for secret neutrino interaction mediated by a scalar field.

In the figure below[55], neutrino coupling g, and mediator mass M of ϕ is plotted, where diagonal dotted contours are shown for the values of the dimensionful coupling G. The blue-shaded regions are excluded by astrophysical and cosmological considerations. The pink-dashed lines indicate the flavor-dependent limits based on meson and lepton decays. The red-shaded region is excluded based on measurement of Z-boson decay. The gray-shaded region is the non-perturbative regime. The orange lines are contours for different initial neutrino energies, for which the IceCube is sensitive to νSI.

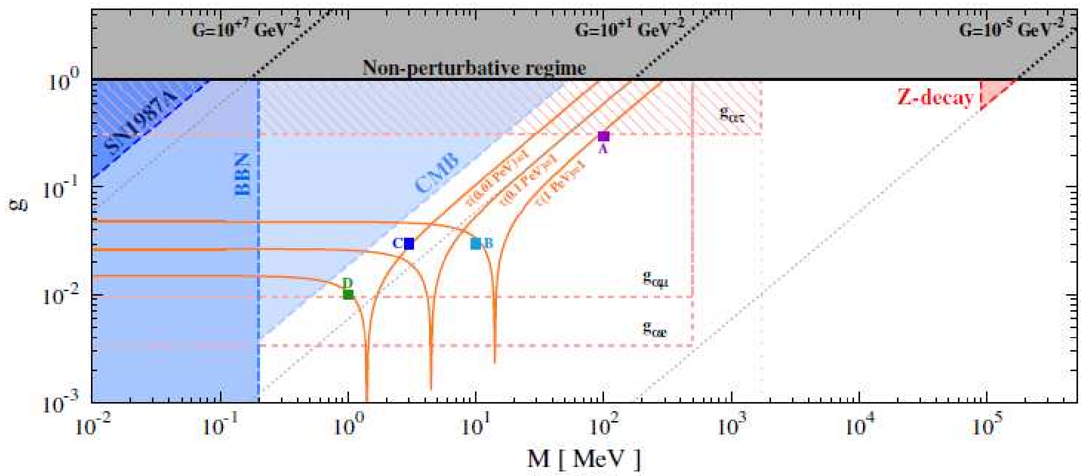

Figure 5.10: Present constraints and future sensitivity to νSI in terms of neutrino coupling, g, and mediator mass,

Considering the effect of beyond standard model interactions and EW-interactions, total interaction rate for neutrinos is $\Gamma_\nu \approx (G_\nu^2 + G_F^2)T^5$, given $H^2 \approx T^2/M_{pl}$, one obtains the decoupling temperature $T_{\nu-dec} \approx [M_{pl}(G_\nu^2+G_F^2)]^{-1/3}$. Within the constraints from cosmology and particle physics data, allowed regime of g_ν and M is shown above. The allowed range for the effective coupling is $G_\nu \in [10^{-9} - 10^{-2}]GeV^{-2}$.

Fermi constant $G_F = 1.166 * 10^{-5} GeV^{-2}$, so for effective coupling of secret neutrino interactions (SνI) much weaker than this, the decoupling of the neutrinos would be determined solely by standard model interactions. However, for stronger coupling from non-standard neutrino interactions with $G_\nu = 10^{-2} GeV^{-2}$, the neutrino decoupling temperature, time and conformal time are pushed to $T_\nu = 10^{-2} MeV$, $t_\nu = 10^3 s$ and $\tau_\nu \approx 10^{12} s$ respectively. We observe that the SνI delays the decoupling time hence it shortens the free-streaming time of neutrinos. Although secret neutrino interactions introduce new physics beyond the Standard Model, their influence on the evolution of tensor perturbations—particularly on the amplitude of primordial gravitational waves—appears to be modest. This is because the dominant contribution to the damping of gravitational waves during the radiation-dominated era arises from the anisotropic stress sourced by the free-streaming neutrino background. In the integro-differential equation governing the evolution of tensor modes, it is the fractional energy density of neutrinos relative to the total radiation content that determines the strength of the damping term. Our results show that this fractional density remains the primary factor even in the presence of additional neutrino interactions mediated by a massive scalar.

In this chapter, we presented a detailed study of tensor perturbations, starting from analytic solutions of the tensor mode equation in radiation and matter domination. We derive the energy-momentum tensor for gravitational waves and analyze how the evolution of effective relativistic degrees of freedom and neutrino free-streaming dampens tensor modes. Finally, we investigate the effect of secret neutrino interactions on the GW transfer function, focusing on potential signatures in the PTA frequency range.

6
Conclusion

Throughout this thesis work, we performed a detailed theoretical and numerical study of the evolution of primordial gravitational waves (PGWs) across cosmic history, focusing on the impact of neutrino free-streaming, beyond Standard Model neutrino interaction, temperature-dependent thermodynamic evolution of the universe, and the de-relativization of Standard Model particles for the frequencies which PTAs are sensitive to. The main results, which contribute both scientifically and methodologically, are listed below:

Evolution of the primordial gravitational waves: PGW transfer function $\chi(u)$ is numerically computed, solving the tensor perturbation equation with anisotropic stress sourced by neutrinos, using a temperature-dependent scale factor $a(T)$ derived from interpolated $g_*(T)$ and $g_{*s}(T)$. This approach captures the time-varying effects of all SM particle species becoming non-relativistic, rather than assuming constant behavior, as it is often done in simplified treatments.

Impact of particle de-relativization: The successive de-relativization of SM particles including electrons, muons, quarks, neutrinos and gauge bosons, leads to measurable features in the PGW spectrum. Each transition alters the Hubble rate and slows down the expansion relative to the radiation-dominated ideal case, thereby modifying the damping profile of sub-horizon tensor modes.

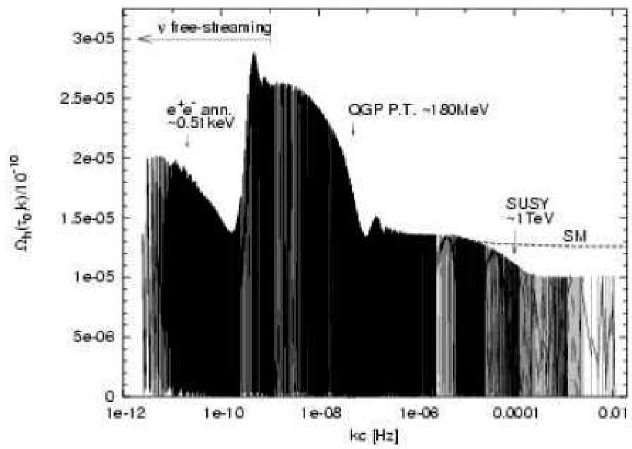

Figure 6.1: The primordial gravitational wave spectrum at present, $\Omega_h(\tau_0, k)/10^{-10}$, as a function of the comoving wavenumber, k (or kc in units of Hertz). is plotted.

Electron-positron annihilation: Around $T \sim 0.5$ MeV, electrons and positrons annihilate as the universe cools below their rest mass threshold. This process transfers energy primarily to the photon bath, since neutrinos have already decoupled by this time (at $T \sim 1$ MeV) and they are no longer in thermal equilibrium with the photon-electron-positron plasma. As a result, the temperature of the photons increases relative to the neutrinos, leading to the well-known relation $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$.

This redistribution of entropy reduces the effective number of relativistic degrees of freedom, $g_*(T)$ and $g_{*s}(T)$, causing a slight change in the expansion rate of the universe. Consequently, this event imprints a characteristic feature on the gravitational wave transfer function. Moreover, since gravitational waves couple to the total anisotropic stress, this photon-neutrino temperature split alters the energy density fractions and therefore impacts the damping amplitude.

Signatures of the QCD phase transition: Around $T \sim 150 MeV$, the strong interaction undergoes a QCD phase transition, leading to a rapid change in the number of effective degrees of freedom. This produces a localized feature in the GW spectrum as seen in figure 6.1, which could be observable in future detectors sensitive to the relevant frequencies.

Electroweak symmetry breaking (EWSB): At higher temperatures, around $T \sim 100 GeV$, the electroweak phase transition occurs, reducing the relativistic degrees of freedom due to the acquisition of mass by W, Z, and Higgs bosons. This phase also contributes a gentle modulation in the transfer function, relevant at higher frequencies.

Neutrino free-streaming: The damping of PGWs by neutrino anisotropic stress was clearly observed in the numerical solutions, especially at frequencies corresponding to horizon entry during the radiation era. The damping is more efficient for modes entering the horizon after neutrino decoupling.

Non-standard neutrino interactions: Scenarios with secret neutrino interactions in the early universe mediated by a massive scalar field have been explored. The results show that it is the fractional energy density of neutrinos relative to the total radiation content that determines the strength of the damping term.

The evolution of the PGW for the radiation-dominated analytic solution compared to the numerical solution that takes into account the de-relativization of SM particles and important phase transitions in early universe (without the neutrino free-streaming) gives a damping of 9.09% at early times, and once the oscillations stabilize at late times (considering the long-term effect of the damping on the amplitude) it corresponds to 8.62% damping,

Comparing the numerical solution that takes into account the neutrino free-streaming, with the analytic radiation-dominated solution, this gives 8.45% of suppression of the amplitude at late times, while the damping effect at the early times is of 13.7%. The presence of additional neutrino interactions mediated by a massive scalar field is found to have a subdominant effect on the damping of primordial gravitational waves. The strength of the damping remains primarily governed by the fractional energy density of neutrinos relative to the total radiation content, indicating that the overall back-reaction is largely insensitive to the specific interaction details in this regime.

Our results demonstrate a significant effect of gravitational waves transferring energy into the neutrino sector, as evidenced by the damping of the derivatives of the tensor perturbations. This damping reflects the back-reaction from neutrino anisotropic stress, confirming the role of free-streaming neutrinos in absorbing gravitational wave energy during the early universe.

Future Developments

We list here some possible future developments that could benefit from some of the results analyzed and obtained in the thesis.

1. Inclusion of phase transitions beyond the Standard Model: The methods developed here can be applied to study early phase transitions predicted by BSM theories (e.g., in supersymmetry). Such transitions could produce their own gravitational wave backgrounds or modify the damping and enhancement of primordial GWs through changes in $g_*(T)$ and $g_{*s}(T)$.

2. Probing axion-like particles through PGWs: Light bosonic fields such as axions, which are candidates for dark matter, can decouple early in the thermal history. Their presence increases the effective number of relativistic species and modifies the anisotropic stress history of the universe. One could study how such species influence the PGW transfer function. Precise modeling may allow gravitational wave observations to constrain axion cosmology and the thermal history of hidden sectors.

3. Time-varying degrees of freedom as a cosmological diagnostic: The thesis results demonstrate that realistic modeling of the Standard Model particle content including their derelativization, leaves imprints on the GW spectrum. This insight can be inverted: future GW measurements could help reconstruct the thermal history of the universe, constraining both known and hypothetical transitions.

4. Model building implications for inflation and reheating: Finally, since the primordial GW amplitude also depends on the inflationary energy scale and the reheating history, combining the refined transfer function with inflationary models could help narrow down the inflationary potential and differentiate between reheating scenarios.

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Acknowledgments

I would like to express my deepest gratitude to my supervisor, Prof. Nicola Bartolo and my co-supervisor Prof. Massimiliano Lattanzi, for their valuable guidance, support, and encouragement throughout this research. Their insights and expertise have been instrumental in shaping this work.

I am also grateful to my colleagues and friends at the University of Padova for their thought-provoking discussions and continuous support. Finally, I extend my appreciation to my family for their support and patience during this journey.