UNIVERSITÀ DEGLI STUDI DI PADOVA

Dipartimento di Fisica e Astronomia "Galileo Galilei"

Master Degree in Physics

Final Dissertation

Topological string theory from the worldsheet

A Batalin-Vilkovisky perspective

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Academic Year 2024/2025

Abstract

The Batalin–Vilkovisky (BV) formalism is a powerful mathematical framework for studying field theories. Originally developed as a homological method for quantizing gauge theories, it has been reinterpreted in recent years from the perspective of derived deformation theory.
In this thesis, we first explore these mathematical structures and their role in the study of perturbative field theories. We then apply these techniques to a concrete model: topological string theory. From the worldsheet perspective, topological string theory is a two-dimensional sigma model coupled to topological gravity. However, an alternative approach was initiated by Bershadsky, Cecotti, Ooguri, and Vafa, who proposed a formulation as a quantum field theory living on the target space. More recently, Costello and Li, using the BV formalism and techniques similar to those used in this thesis, provided a precise formulation of the target-space BCOV model.
Our goal is to connect these two perspectives – worldsheet and target space – by constructing a model for topological string theory from the worldsheet and comparing it with the BCOV theory as formulated by Costello and Li.

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Introduction

In recent years, the interplay between the most formal corners of Mathematics and Theoretical Physics has deepened into a rich and evolving dialogue. Physics is providing new insights into mathematics, while mathematics is offering physics new tools and modern techniques to understand the structures underlying theoretical physics.
One of the most fruitful areas of physics where this interplay is unfolding is Quantum Field Theory. Once regarded primarily as a powerful computational tool, it is now revealing a far richer structure – one that deeply benefits from, and contributes to, fields such as algebraic geometry, category theory, and algebraic topology.

In the context of perturbative field theories, the Batalin–Vilkovisky (BV) formalism stands as a remarkable example of this interplay. Originally developed by Batalin and Vilkovisky (whose foundational work was initiated in [BV81]) as a homological approach to quantizing gauge theories, the formalism has been reinterpreted in more recent years from the perspective of derived deformation theory (cf. [CG16, CG21]).
The BV formalism offers a unified framework for describing gauge theories: it encodes fields, symmetries, equations of motion, and relations between equations (such as the Bianchi identity) within a single complex. This can be constructed purely from physical reasoning. At the same time, derived deformation theory becomes relevant because a BV theory has precisely the algebraic structure that describes formal moduli problems. In particular, the moduli space in question is that of solutions to the equations of motion: a BV theory controls the infinitesimal deformations of a chosen solution, describing the formal neighbourhood of the moduli space at that point.
The first goal of this work is then to offer an introduction to these techniques, highlighting the connection between the BV formalism and derived geometry.

The BV formalism provides us with the tools and framework we want to use. Once we understand how powerful they are, our goal becomes applying these techniques to a concrete model – namely, topological string theory.
Topological string theory can be viewed from the worldsheet perspective, as a two-dimensional sigma model from a Riemann surface to a complex manifold (typically a Calabi–Yau threefold), coupled to topological gravity on the worldsheet.
Bershadsky, Cecotti, Ooguri, and Vafa (BCOV) [BCOV94], on the other end, initiated a different approach. Focusing on topological strings in a B-twist, they analyzed correlation functions by using worldsheet techniques, and they identified two central structures: the tt^*-equations and the holomorphic anomaly equations. Crucially, BCOV demonstrated that these structures admit a natural reinterpretation from the target space point of view, in particular in terms of the Kodaira-Spencer theory of gravity. This theory describes deformations of complex structures

on the Calabi-Yau manifold and provides a target space quantum field theory that captures the full content of the B-model. In this way, BCOV proposed a shift in perspective – from a worldsheet theory to a theory defined on the target Calabi-Yau space.
More recently, Costello and Li [CL12] formulated a rigorous version of the BCOV theory as a perturbative quantum field theory on the target space, using the Batalin–Vilkovisky formalism. Their construction generalizes the original BCOV model from Calabi-Yau threefolds to Calabi-Yau manifolds of arbitrary dimension.
The second goal of this work is then to connect these two perspectives – worldsheet and target space – by constructing a model from the worldsheet and comparing it with the work of Costello and Li on the BCOV theory.
To construct our model, we start by a $\mathcal{N} = 2$ superconformal field theory and couple it to gravity by gauging a supersymmetric extension of conformal symmetry. In our framework, gauging is interpreted as coupling two formal moduli problems: one describing the field theory and the other describing its symmetry. This perspective makes the coupling to gravity natural – gauging conformal symmetry corresponds to moving within the moduli space of Riemann surfaces. After performing the B-twist on the coupled model, we recover the familiar interpretation of a topological string theory as a topological conformal field theory coupled with topological gravity. However, we will see that this is not sufficient to recover the description of Costello and Li. In particular, to get gravitational descendants, one must take into account the global structure of the gauge group – that is, our topological gravity sector.

The thesis is divided into three chapters.
In Chapter 1, we provide an extensive introduction to the BV formalism, with particular emphasis on how derived geometric structures play a role in the description of classical field theories.
Chapter 2 is dedicated to the connection between BV theories and formal moduli problems, where we clarify why the BV formalism naturally describes only *perturbative* field theories. This discussion also leads us to the treatment of symmetries and how they are encoded in this framework.
Chapter 3 is where we construct our model for topological string theory, but first we review the construction by Costello and Li with which we aim to compare. Along the way, we highlight the key structural differences between a topological field theory and a topological string theory.

Chapter 1

The Batalin-Vilkovisky formalism

The Batalin–Vilkovisky (BV) formalism is a powerful mathematical framework for studying the perturbative structure of classical and quantum field theories, particularly gauge theories. It was originally developed by Batalin and Vilkovisky – whose foundational work was initiated in [BV81] – as a homological method for the quantization of gauge theories, generalizing the BRST approach.
In this thesis, we adopt the perspective developed by Costello in [Cos11], and further elaborated by Costello and Gwilliam in [CG16, CG21], where the BV formalism is reinterpreted through the lens of derived geometry.

In this chapter, we introduce the foundational aspects of the BV formalism, with a particular emphasis on its interplay with derived geometric structures.
In Section 1.1, we begin by outlining the basic data that define a classical field theory, and explain how derived algebraic geometry arises in this context. In Sections 1.2, 1.3, and 1.4 we develop the formalism through the derived constructions necessary to describe BV theories. We also include a practitioner's guide in Section 1.5, both because this was how we initially learned the formalism, and because it collects all the practical steps needed to construct a BV theory in one place. Finally, Section 1.6 is dedicated to worked-out examples.

1.1 Classical field theory and derived geometry

We start our exploration of the Batalin-Vilkovisky formalism by introducing the notations and conventions we will use when talking about a classical field theory. In particular, we focus on field theories that admit a Lagrangian or Hamiltonian formulation. So, what are the basic ingredients of a classical field theory?

First of all, we need to define our spacetime and the kind of geometric structure it possesses. Throughout, we will work with a smooth manifold M of finite dimension d. The manifold may come equipped with various geometric structures, such as a complex structure, a conformal structure, or a Riemannian structure.

Next, we need to specify the space of fields. The off-shell space of fields, which we denote by $\mathscr{E}(M)$, is given by the sections of some natural (graded) vector bundle over M. For example, the fields of a complex scalar field theory are just smooth complex-valued functions:

$$\mathscr{E}(M) = \Gamma(M, M \times \mathbb{C}) = C^{\infty}(M, \mathbb{C}) \tag{1.1}$$

In other theories, the space of fields might look very different – for instance, it could be the space of connections on a principal G-bundle, as in the case of Chern-Simons theory or Yang-Mills theory, or it could be the space of metrics, as in a theory of gravity.

Moreover, since we are considering field theories in the Lagrangian/Hamiltonian formalism, we have a smooth[1] function on the space of fields – the action functional – which encodes the dynamics of our theory:

$$\mathrm{S} : \mathscr{E}(M) \to \mathbb{K} \tag{1.2}$$

where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.

Together, these are the basic data of a classical field theory. The action functional determines a subspace of the space of fields, called the on-shell fields, i.e., the field configurations that satisfy the equations of motion. By Hamilton's principle, these correspond to the critical points of S, given by the vanishing of its exterior differential: $d\mathrm{S} = 0$. We refer to this subspace as the critical locus of the action functional S:

$$\mathrm{Crit}(\mathrm{S})(M) = \{\phi \in \mathscr{E}(M) : d\mathrm{S}(\phi) = 0\} \subseteq \mathscr{E}(M)\ . \tag{1.3}$$

Thus, the study of a classical field theory reduces to the study of the critical locus of its action functional.
However, this is not the full story: in a gauge theory, for instance, there is usually a local gauge symmetry acting on the space of fields, but still leaving the action functional S invariant. This introduces a large amount of redundancy in our description, and therefore, the physically relevant space is the quotient of the critical locus by the gauge symmetries.

All this discussion still needs to be refined. The first refinement we want to make is about the space of fields. More precisely, we aim to capture the intrinsic locality of fields – the idea that all physical observations can be made by a system of local observers. From physical intuitions, we want it to be true that locally defined field configurations on regions $U_i \subseteq M$ that agree on their overlaps $U_i \cap U_j$, "glue" together into a uniquely defined global field configuration on the union $\cup_i U_i$.
This notion of locality and consistent gluing is precisely formalized by the concept of a sheaf[2]. In other words, the space of fields naturally forms a sheaf over the spacetime manifold M. This means that for any open set $U \subseteq M$, we associate to U the space of fields over that open set $U \mapsto \mathscr{E}(U)$, and these spaces are compatible w.r.t. gluing.
Notice that we are intentionally being vague about the target category at this stage. In fact, the space of fields in a classical field theory is more precisely a sheaf of L_∞ algebras, also known as a local Lie algebra. We will slowly build up to this notion and fully appreciate it only in Chapter 2.

This is not enough: we also want to consider the observables of a classical field theory. A natural framework for this is to think of a geometric space – such as the critical locus of an action functional S – as a locally ringed space[3]. Concretely, every geometric space can be described as a topological space equipped with a structure sheaf, that is, a sheaf of functions over that space.
This perspective is very powerful: the same underlying topological space can come equipped with different structure sheaves. This is what we will do in the following: we will work with

[1]The smooth structure on the space of fields can be constructed using the language of differentiable vector spaces, as in [CG16], but we will not need these technical details later on.
[2]See Appendix A for basic definitions on sheaves.
[3]See Appendix E for more details.

the sheaf of functions over the critical locus and then modify it to get better-behaved, *derived* models.

In particular, we adopt the following convention: the functions on a given space will always be modeled as the symmetric algebra generated by the linear dual over that space. In other words:

$$\mathscr{O}(\mathscr{E}) := \mathrm{Sym}^*_{\mathbb{K}}(\mathscr{E}^\vee) \tag{1.4}$$

Remark. We mentioned the observables of a classical field theory, but then we just talked about the sheaf of functions over a given space. With our conventions – and when the space in question is the space of fields – these two concepts essentially coincide, and we will often use these terms interchangeably. However, it is important to emphasize that the space of observables forms a *cosheaf*, rather than a sheaf. This difference comes from the fact that, when defining observables, we focus on compactly supported sections of the structure sheaf, which form naturally a cosheaf. In this thesis, actually, we will be even more restrictive: we will just focus on observables supported at a single point. This is formalized by asking for the *costalk* of the cosheaf of observables at a point.

We have now given all the conventions for a classical field theory, and it is clear what we want to study: the critical locus of the action functional, modulo gauge equivalences – that is, the moduli space of solutions to the equations of motion. However, this is a highly non-trivial space and could be very difficult to study with standard techniques. This is where derived algebraic geometry comes in hand.

In fact, when studying these spaces, we can encounter two main difficulties.

1. The critical locus can be described as the intersection of two subspaces (as we will explicitly see in Section 1.2). However, this intersection could be non-transverse, making it difficult to study with classical tools. To address this problem, we should consider the so-called *derived* intersection.

2. The second challenge concerns the quotient of the critical locus by the space of local gauge symmetries. These symmetries can act non-freely on the space of fields, destroying, for instance, its smooth structure. Of course, this is a problem: how can we even define the equations of motion as a system of partial differential equations if the underlying space lacks a smooth structure? To overcome this, we replace the naive quotient with a better-behaved model known as the *stacky* quotient (this will be done in Section 1.3).

1.2 The derived critical locus

For simplicity, we consider the space of fields $\mathscr{E}$ to be finite-dimensional – that is, for all open sets $U \subseteq M$, $\mathscr{E}(U)$ is finite-dimensional. This assumption is not very physical: in general, the space of fields in a field theory is infinite-dimensional. However, we will not worry much about defining all the notions in the infinite-dimensional setting, although this can be done rigorously (see [CG21], §4.2).

In what follows, keep in mind that the spaces of fields we consider carry the structure of a sheaf, and that our arguments can be made rigorous at two different levels: the functorial level, reasoning directly at the level of sheaves, and the concrete level, reasoning in terms of a fixed section of the sheaf. Clearly, the functorial approach is more general and abstract, but we will mostly adopt the latter approach because it is simpler to work with it. Accordingly, when we write $\mathscr{E}$ to denote the space of fields, what we typically mean is $\Gamma(M, \mathscr{E}) = \mathscr{E}(M)$, its space of global sections.

We are interested in studying the critical locus of the action functional S. Since $\mathrm{S} \in C^\infty(\mathscr{E}, \mathbb{K})$, its exterior differential is a section of the cotangent bundle of $\mathscr{E}$ – that is, $d\mathrm{S} \in \Omega^1(\mathscr{E}, \mathbb{K})$. In other words, it defines a map $\mathscr{E} \to T^*\mathscr{E}$, and its image lies in the graph of dS.
Thus, the critical locus of S can be described as the fiber product of $\mathscr{E}$ with itself over $T^*\mathscr{E}$, where one copy maps into the cotangent bundle via dS, and the other via the zero section $0 : \mathscr{E} \to T^*\mathscr{E}$. Explicitly:

$$\mathrm{Crit}(\mathrm{S}) = \mathrm{graph}(d\mathrm{S}) \times_{T^*\mathscr{E}} \mathscr{E} \tag{1.5}$$

This is summarized by the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Crit}(\mathrm{S}) = \mathscr{E} \times_{T^*\mathscr{E}} \mathscr{E} & \longrightarrow & \mathscr{E} \\ \downarrow & & \downarrow{\scriptstyle 0} \\ \mathscr{E} & \xrightarrow{\ d\mathrm{S}\ } & T^*\mathscr{E} \end{array}$$

However, this space is in general complicated and problems could arise because the intersection is non-transverse[4], as we remarked in Section 1.1.

To address this, we replace the critical locus by the so-called *derived* critical locus, denoted by $\mathrm{Crit}^{\mathrm{h}}(\mathrm{S})$. As we know from Section 1.1, every geometric space is described not just by its underlying topological space, but also by its structure sheaf – the sheaf of functions over that space. Concretely, what we do is replacing this structure sheaf with a better-behaved, derived version:

$$\begin{array}{c} \mathscr{O}(\mathrm{Crit}(\mathrm{S})) = \mathscr{O}(\mathrm{graph}(d\mathrm{S})) \otimes_{\mathscr{O}(T^*\mathscr{E})} \mathscr{O}(\mathscr{E}) \\ \Big\downarrow \text{derived version} \\ \mathscr{O}(\mathrm{Crit}^{\mathrm{h}}(\mathrm{S})) = \mathscr{O}(\mathrm{graph}(d\mathrm{S})) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*\mathscr{E})} \mathscr{O}(\mathscr{E}) \end{array}$$

The derived tensor product $\otimes^{\mathbb{L}}$ arises as the left derived functor of the ordinary tensor product functor. Basically, we apply the same procedure that we would apply to compute the Tor groups, but we stop before taking cohomology. Let us briefly recall how this construction works in our case (for a general discussion on derived functors, see Appendix C):

1. First, we choose a resolution of $\mathscr{O}(\mathrm{graph}(d\mathrm{S}))$ by a complex of projectives (or free) $\mathscr{O}(T^*\mathscr{E})$-modules, call it $\mathscr{K}^* \xrightarrow{\epsilon} \mathscr{O}(\mathrm{graph}(d\mathrm{S}))$ (alternatively, one could choose a resolution of $\mathscr{O}(\mathscr{E})$: in the end the results are quasi-isomorphic).

2. Next, we apply the tensor product functor $- \otimes_{\mathscr{O}(T^*\mathscr{E})} \mathscr{O}(\mathscr{E})$ to $\mathscr{K}^*$. Since this functor is right exact, we can simply cut the term with $\mathscr{O}(\mathrm{graph}(d\mathrm{S}))$ off because the information is already contained in cohomology (specifically, in the H^0 term).

3. The derived tensor product is what is left:

$$\mathscr{O}(\mathrm{graph}(d\mathrm{S})) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*\mathscr{E})} \mathscr{O}(\mathscr{E}) = \mathscr{K}^* \otimes_{\mathscr{O}(T^*\mathscr{E})} \mathscr{O}(\mathscr{E}). \tag{1.6}$$

Remark. After performing this modification, we are no longer dealing with an ordinary manifold, but with a new kind of geometric object: a derived manifold or differential graded (dg) manifold. We can think of it as a locally ringed space where the structure sheaf is allowed to be a sheaf of nonpositively graded dg commutative algebras, rather than just ordinary algebras. In this setting, to recover the classical topological space underlying a derived manifold (also called

[4] Why the derived intersection is better than the usual one? See the introduction of [Lur09] for more details.

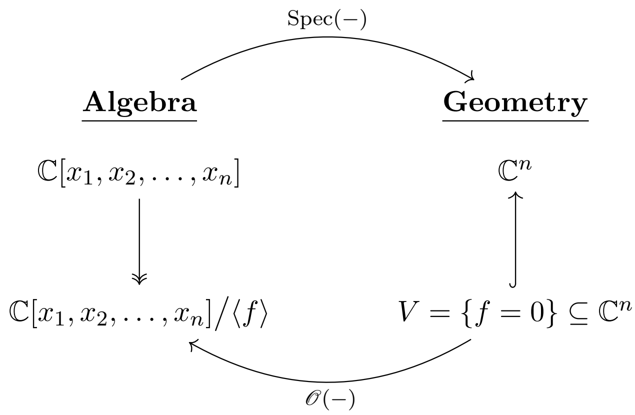
truncation), we take the spectrum of the degree-zero cohomology of the dg structure sheaf. In the case of the derived critical locus, this means:

$$\mathrm{Crit}(\mathrm{S}) = \mathrm{Spec}(H^0(\mathscr{O}(\mathrm{graph}(d\mathrm{S})) \otimes^{\mathbb{L}}_{\mathscr{O}(T^*\mathscr{E})} \mathscr{O}(\mathscr{E}))) \tag{1.7}$$

Digression: the Koszul resolution

The complex we will use to resolve $\mathscr{O}(\mathrm{graph}(d\mathrm{S}))$ is the Koszul complex. We mainly follow the construction done in [Gwi12], but first, to better understand how it works in an easier but instructive example, we retreat ourselves to the algebraic world.

Suppose we are interested in a subspace V of $\mathbb{C}^n$ cut out by a single polynomial equation $\{f = 0\}$. If f is the equation of motion of an object, we can really think of this subspace as the critical locus of f.
Rather than studying this subspace geometrically, we can equivalently study its algebra of functions. The situation is depicted below (recall that quotients in algebra, are subspaces in geometry; see Appendix E for a review of basic concepts in Algebraic Geometry):



We now want to replace the quotient $\mathbb{C}[x_1, x_2, \ldots, x_n]/\langle f\rangle$ with the simplest possible resolution of it: a free resolution, i.e., every term in the complex should be a free module.
The goal is to recover the object we want to resolve as the H^0 term in cohomology; we should not impose the equation of motion on the nose, but it should be encoded in the differential in some way. To do this, we introduce a parameter ξ of cohomological degree -1, which allows us to construct the following free resolution (that in this case is really a short exact sequence):

$$\begin{array}{ccccccccc} \underline{-2} & & \underline{-1} & & \underline{0} & & & & \\ 0 & \longrightarrow & \xi \cdot \mathbb{C}[x_1, x_2, \ldots, x_n] & \xrightarrow{f\frac{\partial}{\partial\xi}} & \mathbb{C}[x_1, x_2, \ldots, x_n] & \longrightarrow & \frac{\mathbb{C}[x_1, x_2, \ldots, x_n]}{\langle f\rangle} & \longrightarrow & 0 \end{array}$$

First of all, observe that $f\frac{\partial}{\partial\xi}$ indeed squares to zero, so it defines a valid differential. If we denote by $\mathbb{C}[x_1, x_2, \ldots, x_n; \xi] := \xi \cdot \mathbb{C}[x_1, x_2, \ldots, x_n] \oplus \mathbb{C}[x_1, x_2, \ldots, x_n]$, then the Koszul complex can be written as:

$$\mathscr{K}^* = (\mathbb{C}[x_1, x_2, \ldots, x_n; \xi], d = f\frac{\partial}{\partial\xi}) \tag{1.8}$$

Computing the degree zero cohomology we find $H^0(\mathscr{K}^*) = \mathbb{C}[x_1, x_2, \ldots, x_n]/\langle f\rangle$, as expected.

We can summarize the core idea behind this construction with the following slogan[5]:

Replace a complicated object by pasting together boring ones.

This captures the essence of what a resolution does: it expresses a complex or singular object in terms of simpler, well-understood pieces glued together in a structured way. Whenever we resolve an object, this should be the guiding philosophy to keep in mind.

Now, what if we consider a subspace cut out by $k < n$ equations $\{f_1 = f_2 = \cdots = f_k = 0\}$ instead? We can try something similar: add a generator in odd degree for every equation of motion.

$$\begin{array}{ccccccc} \underline{-1} & & \underline{0} & & & & \\ \bigoplus_{i=1}^{k} \xi_i \cdot \mathbb{C}[x_1, x_2, \ldots, x_n] & \xrightarrow{\sum_{i=0}^{k} f_i \frac{\partial}{\partial \xi_i}} & \mathbb{C}[x_1, x_2, \ldots, x_n] & \longrightarrow & \frac{\mathbb{C}[x_1,x_2,\ldots,x_n]}{\langle f_1,f_2,\ldots,f_k\rangle} & \longrightarrow & 0 \end{array}$$

Computing H^0, we indeed recover the desired quotient. However, this is not yet a free resolution! Recall that a free resolution must have trivial cohomology in every degree except from degree zero. In this case, the kernel in degree -1 is non-vanishing: it contains all elements of the form $\alpha_{ij} = f_i\xi_j - f_j\xi_i$.
This reveals that there are non-trivial relations among the equations f_i. To kill this kernel, we need to add one more term in the complex; then, by repeating this process degree by degree, we continue adding terms until we get the free resolution we were looking for. Denoting $\mathbb{C}[x_1, x_2, \ldots, x_n] = R$, the full complex is given by:

$$0 \longrightarrow \xi_i\xi_j \ldots \xi_k \cdot R \xrightarrow{d} \bigoplus^{\binom{k}{k-1}} \xi_i\xi_j \ldots \xi_{k-1} \cdot R \xrightarrow{\quad d \quad} \ldots$$
$$\ldots \xrightarrow{d} \bigoplus^{\binom{k}{2}} \xi_i\xi_j \cdot R \xrightarrow{\quad d \quad} \bigoplus^{\binom{k}{1}} \xi_i \cdot R \xrightarrow{\quad d \quad} R \longrightarrow 0$$

Here, with the notation "$\bigoplus^{\binom{k}{l}}$" we want to highlight that in degree $-k+l$, the direct sum has $\binom{k}{l}$ summands – in other words, the rank of the complex in degree $-l$ is $\mathrm{rk}(\mathscr{K}^{-l}) = \binom{k}{l}$.
The differential d is defined as above, and it satisfies $d^2 = 0$ because of the Koszul sign rule. This complex is a free resolution: each term is a free module, and the cohomology vanishes in all negative degrees.

However, there is an important caveat: for this to indeed define a free resolution, the elements $f_1, f_2, \ldots, f_k$ of $\mathbb{C}[x_1, \ldots, x_n]$ should form a regular sequence.

Definition 1 ([sta25], Definition 10.68.1)**.** *Let R be a ring. A sequence of elements $f_1, f_2, \ldots, f_k$ of R is called a regular sequence if f_i is a nonzero-divisor on $R/\langle f_1, \ldots, f_{i-1}\rangle$ for each $i = 1, \ldots, k$.*

With this definition in hand, we can state the following theorem.

Theorem 1 ([Wei94], §4, Corollary 4.5.5)**.** *If $\{f_1, f_2, \ldots, f_k\} = \mathbf{f}$ is a regular sequence in R, the Koszul complex is a free resolution of $R/\langle f_1, \ldots, f_k\rangle$. That is, the following sequence is exact:*

$$0 \to \wedge^n R^n \to \wedge^{n-1} R^n \to \cdots \to \wedge^2 R^n \to \wedge R^n \to R^n \xrightarrow{\cdot \mathbf{f}} R \to R/\langle \mathbf{f}\rangle \to 0$$

[5] I didn't come up with this slogan by myself, but rather I heard Ingmar Saberi saying it countless times, and I honestly think it is very effective.

Notice that the resolution we wrote above for $\mathbb{C}[x_1,\dots,x_n]/\langle f_1,\dots,f_k\rangle$ is the same as the one in the theorem by the antisymmetry of the wedge product.

Remember that we introduced all this machinery with the goal of resolving $\mathscr{O}(\text{graph}(d\text{S}))$ using a Koszul complex. Since $\mathscr{O}(\text{graph}(d\text{S})) = \mathscr{O}(T^*\mathscr{E})/\langle X - dS(X)\rangle$, the Koszul resolution is explicit given by (note that we have already omitted the final term $\mathscr{O}(\text{graph}(d\text{S}))$ in the complex, as its information is in the zeroth cohomology):

$$0 \longrightarrow \Gamma(\wedge^n T\mathscr{E}) \otimes_{\mathscr{O}(\mathscr{E})} \mathscr{O}(T^*\mathscr{E}) \longrightarrow \Gamma(\wedge^{n-1} T\mathscr{E}) \otimes_{\mathscr{O}(\mathscr{E})} \mathscr{O}(T^*\mathscr{E}) \longrightarrow \dots$$

$$\dots \longrightarrow \Gamma(\wedge^2 T\mathscr{E}) \otimes_{\mathscr{O}(\mathscr{E})} \mathscr{O}(T^*\mathscr{E}) \longrightarrow \Gamma(T\mathscr{E}) \otimes_{\mathscr{O}(\mathscr{E})} \mathscr{O}(T^*\mathscr{E}) \longrightarrow \mathscr{O}(T^*\mathscr{E}) \longrightarrow 0$$

Why do we choose $\Gamma(T\mathscr{E})$ to construct the Koszul complex? The reason is that we want a vector bundle of rank $n = \dim(\mathscr{E})$, since we are imposing precisely n equations of motion. In order to recover $\mathscr{O}(\text{graph}(d\text{S}))$ as the H^0 term of the Koszul complex, we define the differential as follows:

$$\begin{aligned} \Gamma(T\mathscr{E}) \otimes_{\mathscr{O}(\mathscr{E})} \mathscr{O}(T^*\mathscr{E}) &\to \mathscr{O}(T^*\mathscr{E}) \\ X \otimes 1 &\mapsto X - d\text{S}(X) \end{aligned} \tag{1.9}$$

for each vector field X, and then we extend it in the standard way as a Koszul complex.

Next, to find $\mathscr{O}(\text{Crit}^{\text{h}}(\text{S}))$, we should apply the tensor product functor $- \otimes_{\mathscr{O}(T^*\mathscr{E})} \mathscr{O}(\mathscr{E})$ to this resolution.
Using the fact that $\mathscr{O}(\mathscr{E})$ inherits its $\mathscr{O}(T^*\mathscr{E})$-module structure from the map $\mathscr{E} \xrightarrow{0} T^*\mathscr{E}$, and performing the obvious identifications, we get the sequence:

$$\mathscr{O}(\text{Crit}^{\text{h}}(\text{S})) = \mathscr{K}^* \otimes_{\mathscr{O}(T^*\mathscr{E})} \mathscr{O}(\mathscr{E}) = \Gamma(\wedge^n T\mathscr{E}) \to \dots \to \Gamma(\wedge^2 T\mathscr{E}) \to \Gamma(T\mathscr{E}) \to \mathscr{O}(\mathscr{E})$$

where the differential acts on vector fields by $X \mapsto -d\text{S}(X) = -\iota_{d\text{S}}(X)$.
As we already remarked before, notice that now the structure sheaf of the critical locus is no longer a sheaf of commutative algebras, but becomes a sheaf of nonpositively graded dg commutative algebras.

There are two important points to highlight:

1. Since
$$\mathscr{O}(\text{Crit}^{\text{h}}(\text{S})) = (\text{Sym}^*_{\mathscr{O}(\mathscr{E})}(T[1]\mathscr{E}), -\iota_{d\text{S}}) = (\mathscr{O}(T^*[-1]\mathscr{E}), -\iota_{d\text{S}}), \tag{1.10}$$
we see that observables on the derived critical locus can be interpreted as the observables on the (-1)-shifted cotangent bundle, equipped with non-trivial differential. The underlying graded vector space is called the space of *polyvector fields*.
The (-1)-shifted cotangent bundle comes with a canonical symplectic structure of degree -1, which induces a Poisson bracket on the space of observables of degree +1. In particular, the differential $-\iota_{d\text{S}}$, can be rewritten as the bracket with the action functional S, i.e., $-\iota_{d\text{S}} = \{\text{S}, -\}$.
This is a general feature of a BV theory: the space of BV fields is always equipped with a pairing of degree -1.

2. From the explicit expression for $\mathscr{O}(\text{Crit}^{\text{h}}(\text{S}))$, we see that this is actually the Koszul complex associated to $\mathscr{O}(\mathscr{E})/\langle \mathbf{d}\text{S}^i\rangle$. Therefore, if the equations of motion form a regular sequence in $\mathscr{O}(\mathscr{E})$, we get a free resolution. However, this is not typically the case: in

general, the complex has non-vanishing cohomology in negative degrees! This is a sign that there are relations among the equations themselves (think about Bianchi identities in a gauge theory).

We have already come a long way! However, we now clearly see that we need to take care of this negative degree cohomology. Noether's second theorem relates these relations among the equations of motion to the action of local gauge symmetries on the space of fields. Our next step will be to quotient out these gauge symmetries, but, of course, in a derived way.

1.3 Taking care of gauge symmetries

So, how do we do it? Given a local gauge symmetry $\mathscr{G}$ acting on the space of fields, we promote $\mathscr{E}$ to the so-called *stacky* quotient $[\mathscr{E}/\mathscr{G}]$. In Chapter 2, it will become clearer why we call it *stacky*; the idea is that instead of identifying gauge equivalent field configurations we keep them distinct, but somehow remembering that there is a gauge transformation relating them.

For our purposes, the stacky quotient can be understood as the derived version of taking invariants. To be physically meaningful, an observable should be gauge-invariant. However, since we are working in a derived setting, we do not ask for strictly invariant observables; rather, we consider the derived invariants. We will recover strictly gauge-invariant observables in degree-zero cohomology.

Suppose we have a Lie algebra of gauge symmetries $\mathfrak{g}$ acting on $\mathscr{O}(\mathscr{E})$, making $\mathscr{O}(\mathscr{E})$ into a $\mathfrak{g}$-module. Asking for gauge-invariant observables, means restricting our attention to the submodule:

$$\mathscr{O}(\mathscr{E})^{\mathfrak{g}} = \{f \in \mathscr{O}(\mathscr{E}) \mid g \cdot f = 0,\ \forall g \in \mathfrak{g}\} \tag{1.11}$$

The operation of taking invariants can be made into a functor: it is just the identity on morphisms. In Appendix D, we proved that the functor of invariants is left exact: it is then possible to construct its right derived functors. However, we already know (see again Appendix D), what is an appropriate complex in this case: the Chevalley-Eilenberg cochains. Let us recall the definition:

Definition 2 ([CG16], Appendix A, Definition 3.1.2)**.** *The Chevalley-Eilenberg cochains of the $\mathfrak{g}$-module M are given by the complex:*

$$C^*(\mathfrak{g}, M) = (\mathrm{Sym}^*_{\mathbb{K}}(\mathfrak{g}^\vee[-1]) \otimes_{\mathbb{K}} M, d_{\mathrm{CE}}) \tag{1.12}$$

where the differential encodes the linear dual to the bracket of $\mathfrak{g}$ on itself and on M. Fixing a linear basis $\{e_k\}$ for $\mathfrak{g}$ and denoting $\{e^k\}$ the dual basis, we have:

$$d_{\mathrm{CE}}(e^k \otimes m) = \sum_i e^k \wedge e^i \otimes_{\mathbb{K}} e_i \cdot m - \sum_{i<j} e^k([e_i, e_j]) e^i \wedge e^j \otimes_{\mathbb{K}} m \tag{1.13}$$

and d_{CE} is extended to the rest of the complex as a derivation of cohomological degree 1 (using the Leibniz rule repeatedly to reduce to the explicit formula above).

Thus, we define the observables on the stacky quotient to be the Chevalley-Eilenberg cochains, i.e., the derived version of invariants. More precisely, we have[6]:

$$\mathscr{O}([\mathscr{E}/\mathscr{G}]) := (\mathscr{O}(\mathscr{E})^{\mathfrak{g}})^{\mathrm{h}} = C^*(\mathfrak{g}, \mathscr{O}(\mathscr{E})) = (\mathrm{Sym}^*_{\mathbb{K}}(\mathfrak{g}^\vee[-1]) \otimes \mathscr{O}(\mathscr{E}), d_{\mathrm{CE}}) \tag{1.14}$$

[6]Here and in what follows, the cohomological shift indicated by square brackets denotes a shift to the left. More precisely, for a $\mathbb{Z}$-graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$, the shifted graded vector space $V[k]$, $\forall k \in \mathbb{Z}$ has components $(V[k])_i := V_{i+k}$.

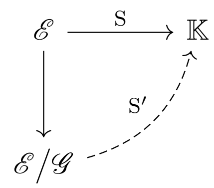
Since $\mathscr{O}(\mathscr{E}) = \mathrm{Sym}^*_{\mathbb{K}}(\mathscr{E}^\vee)$, observables on the stacky quotient can be interpreted as observables on the graded manifold $\mathscr{E} \oplus \mathfrak{g}[1]$ equipped with the CE differential:

$$\mathscr{O}([\mathscr{E}/\mathscr{G}]) = (\mathscr{O}(\mathfrak{g}[1] \oplus \mathscr{E}), d_{\mathrm{CE}}) \tag{1.15}$$

Just like when computing observables on the derived critical locus, observables on the stacky quotient form a sheaf of dg commutative algebras.
Notice that this is the usual BRST procedure we do in physics: the elements of $\mathfrak{g}[1]$ correspond exactly to the ghost fields.

1.4 The BV-BRST complex

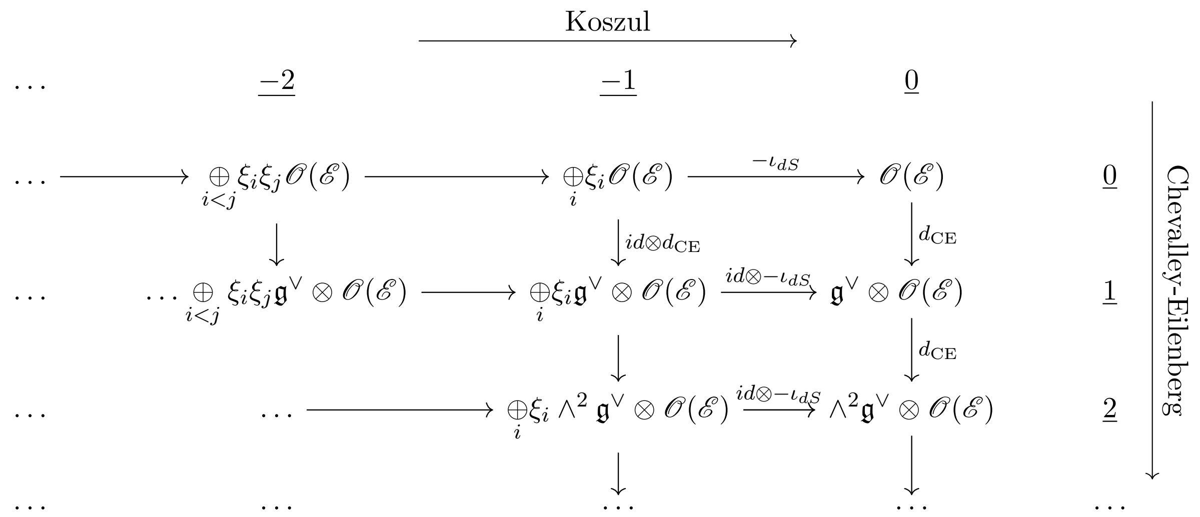
Now we should put all together: we want to construct a derived model for gauge-invariant observables on the critical locus.
Suppose that the original action S is gauge-invariant. In that case, it descends to the stacky quotient, defining the same action there.



The critical locus of this action is then given by the following intersection:

$$\mathrm{Crit}(\mathrm{S}') = \mathrm{graph}(d\mathrm{S}') \times_{T^*([\mathscr{E}/\mathscr{G}])} [\mathscr{E}/\mathscr{G}] \tag{1.16}$$

To model this intersection correctly, we must take into account the derived intersection (using the Koszul resolution) and the derived model for the stacky quotient (via the Chevalley-Eilenberg complex). Putting these ingredients together, we obtain the following double complex (here, the ξ_i are anticommuting sections of the tangent bundle $T\mathscr{E}$):



Finally, observables are obtained by taking the totalization of this double complex – i.e., summing over the antidiagonals with differential given by the alternating sum of the component differentials. Specifically, we get:

$$(\mathscr{O}(T^*[-1](\mathfrak{g}[1] \oplus \mathscr{E})), -\iota_{dS} + d_{\mathrm{CE}}) \tag{1.17}$$

These are the observables of our BV theory. In particular, the degree-zero cohomology of this cochain complex yields on-shell gauge-invariant observables, as we wanted.
Crucially, we can package the differential as the bracket with a single action functional S_{BV}, such that bracketing with S_{BV} encodes both the bracket with the original action, and the action of the Chevalley-Eilenberg differential.
This is possible because the odd vector field dual to the CE differential is compatible with the symplectic structure we have on the (-1)-shifted cotangent bundle. Hence, the BV-BRST complex of observables for a gauge theory can be rewritten as:

$$(\mathscr{O}(T^*[-1](\mathfrak{g}[1] \oplus \mathscr{E})), \{\mathrm{S}_{\mathrm{BV}}, -\}) \tag{1.18}$$

1.5 Practitioner guide

Let us now see, in practice, how to construct the BV-BRST complex of a theory and the associated BV action.
We assume the original action S is given, as well the action of the Chevalley-Eilenberg differential d_{CE} on observables. Our goal is to construct a BV action S_{BV} so that bracketing with it on the space of observables reproduces both the bracket with the original action, and the action of d_{CE}.

Step 1: constructing the BRST space of fields. We start by defining the BRST space of fields. This is constructed by enlarging the space of physical fields $\mathscr{E}$ with the symmetry data: we place the Lie algebra of symmetries $\mathfrak{g}$ in degree -1.

$$\mathscr{F} = \mathfrak{g}[1] \oplus \mathscr{E} \tag{1.19}$$

We call the elements of $\mathfrak{g}[1]$ ghosts.
On observables, the action of d_{CE} encodes both the linear dual to the bracket of $\mathfrak{g}$ and the module structure on $\mathscr{O}(\mathscr{E})$. For our purposes, as observables, it is sufficient to consider pointwise observables, i.e., fields evaluated at a point $x \in M$, where M is our spacetime manifold. Then, considering $c \in \mathfrak{g}$ and $\psi \in \mathscr{E}$, we have:

$$\begin{aligned} d_{\mathrm{CE}}(c(x)) &= [c(x), c(x)]_{\mathfrak{g}} \\ d_{\mathrm{CE}}(\psi(x)) &= \delta_{\mathrm{BRST}}(\psi(x)) \end{aligned} \tag{1.20}$$

where δ_{BRST} gives the module structure.

Step 2: constructing the BV space of fields. In order to define the BV action, we should also add antifields. The full space of BV fields is given by the (-1)-shifted cotangent bundle of the BRST field space:

$$\mathscr{F}_{\mathrm{BV}} := T^*[-1]\mathscr{F} = \mathscr{F} \oplus \mathscr{F}^\vee[-1] = \mathfrak{g}[1] \oplus \mathscr{E} \oplus \mathscr{E}^\vee[-1] \oplus \mathfrak{g}^\vee[-2] \tag{1.21}$$

This space comes equipped with a canonical symplectic structure of cohomological degree -1. It allows us to pair fields with antifields (the elements of $\mathscr{E}^\vee[-1]$), and ghosts with antighosts (the elements of $\mathfrak{g}^\vee[-2]$). We denote antifields and antighosts using a dagger, e.g., $\psi^\dagger$ for the antifield corresponding to ψ.

Step 3: constructing the BV action. We are now ready to define the BV action, which takes the form:

$$\mathrm{S}_{\mathrm{BV}} = \mathrm{S} + \sum_{\psi \in \mathscr{F}} \int_M \psi^\dagger(x) d_{\mathrm{CE}} \psi(x) \tag{1.22}$$

The first thing to check is whether this indeed defines a differential on observables. It does, if it satisfies the so-called *classical master equation*:

$$\{S_{BV}, S_{BV}\} = 0 \tag{1.23}$$

Let us unpack the content of this bracket on observables:

- Acting on $c(x)$ or $\psi(x)$, we recover exactly the BRST transformations as given in Eq. (1.20).
- Acting on antifields or antighost, we get the equations of motion for the fields and the ghost, respectively.

An important feature is that the space of BV fields is itself a cochain complex. The linear part of the differential $\{S_{BV}, -\}$ of $\mathscr{O}(\mathscr{F}_{BV})$ is dual to a differential on the space of fields itself!
Meanwhile, the quadratic part of the differential on observables is dual to a Lie bracket on the space of fields (more precisely, to ensure that the Lie bracket is of degree zero, on $\mathscr{F}_{BV}[-1]$). This is part of a more general story: we will see that the space of BV fields carries the structure of a L_∞-algebra.

1.6 Examples

Chern-Simons theory

Let us consider non-abelian Chern-Simons theory on a three-dimensional manifold M. In this case, physical fields are connections of a principal G-bundle over M, i.e., one forms valued in the Lie algebra $\mathfrak{g}$ of G. We denote them by $A \in \Omega^1(M, \mathfrak{g})$.
The classical action takes the form:

$$S_{CS} = \mathrm{tr} \int_M A(x) \wedge dA(x) + \frac{2}{3} A(x) \wedge [A(x), A(x)] \tag{1.24}$$

where the Lie bracket that appears is the wedge product on forms and the usual Lie bracket on the Lie algebra.

What are the ghosts of the theory, or, in other words, what is the space of gauge transformations? The action is gauge invariant whenever $A \sim A + dc + [c, A]$, with $c \in \Omega^0(M, \mathfrak{g})$.
Therefore, the space of BRST fields is given by:

$$\mathscr{F} = \Omega^0(M, \mathfrak{g})[1] \oplus \Omega^1(M, \mathfrak{g}) \tag{1.25}$$

Th action of the Chevalley-Eilenberg differential on observables is given by:

$$\begin{aligned} d_{CE}(c(x)) &= [c(x), c(x)] \\ d_{CE}(A(x)) &= dc(x) + [c(x), A(x)] \end{aligned} \tag{1.26}$$

Let us now construct the BV theory explicitly. The total space of BV fields is obtained by taking the (-1)-shifted cotangent bundle of $\mathscr{F}$.
To describe this space concretely, we need to identify the duals of the field components, which requires specifying the (-1)-shifted symplectic structure. In this case, it is just given by the wedge product on differential forms, the trace on the Lie algebra and integration on M.
Therefore, the dual spaces are given by:

- $(\Omega^1(M, \mathfrak{g}))^\vee = \Omega^2(M, \mathfrak{g})$;

- $(\Omega^0(M,\mathfrak{g}))^\vee = \Omega^3(M,\mathfrak{g})$

Putting all together, we get $\mathscr{F}_{\mathrm{BV}} = T^*[-1]\mathscr{F}$:

$$\begin{array}{cccc} \underline{-1} & \underline{0} & \underline{1} & \underline{2} \\ \Omega^0(M,\mathfrak{g}) & \Omega^1(M,\mathfrak{g}) & \Omega^2(M,\mathfrak{g}) & \Omega^3(M,\mathfrak{g}) \end{array}$$

The BV action is constructed just by following the general procedure we described above (see Eq. (1.22)). In this case, we get:

$$\mathrm{S}_{\mathrm{BV}} = \mathrm{S}_{\mathrm{CS}} + \mathrm{tr}\int_M c^\dagger(x)\wedge d_{\mathrm{CE}}(c(x)) + A^\dagger \wedge d_{\mathrm{CE}}(A(x)) \tag{1.27}$$

Let us explicitly see how the bracket with S_{BV} in $\mathscr{O}(T^*[-1]\mathscr{F})$ acts on generators:

$$\begin{aligned} \{\mathrm{S}_{\mathrm{BV}}, c(x)\} &= \frac{\partial \mathrm{S}_{\mathrm{BV}}}{\partial c^\dagger(x)} = d_{\mathrm{CE}}(c(x)) = \underset{\sim\sim\sim}{[c(x),c(x)]} \\ \{\mathrm{S}_{\mathrm{BV}}, A(x)\} &= \frac{\partial \mathrm{S}_{\mathrm{BV}}}{\partial A^\dagger(x)} = d_{\mathrm{CE}}(A(x)) = \underline{dc(x)} + \underset{\sim\sim\sim}{[c(x),A(x)]} \\ \{\mathrm{S}_{\mathrm{BV}}, A^\dagger(x)\} &= \frac{\partial \mathrm{S}_{\mathrm{BV}}}{\partial A(x)} = \underline{dA(x)} + \underset{\sim\sim\sim}{[A(x),A(x)]} + \underset{\sim\sim\sim}{[A^\dagger(x),c(x)]} \\ \{\mathrm{S}_{\mathrm{BV}}, c^\dagger(x)\} &= \frac{\partial \mathrm{S}_{\mathrm{BV}}}{\partial c(x)} = \underline{dA^\dagger(x)} + \underset{\sim\sim\sim}{[c^\dagger(x),c(x)]} + \underset{\sim\sim\sim}{[A(x),A^\dagger(x)]} \end{aligned} \tag{1.28}$$

We see that bracketing with $A^\dagger(x)$ – the antifield associated to $A(x)$ – produces a term that recovers the equation of motion for $A(x)$. The linear part of the BV differential (that generates the underlined underlined terms in Eq. (1.28)) is dual to a differential on fields; in this case, it is simply the de Rham differential. Instead, the quadratic part of the differential (that generates the wavy underlined terms in Eq. (1.28)) is dual to a Lie bracket on fields.
Therefore, the space of BV fields acquires the structure of a differential graded Lie algebra, with underlying cochain complex given by the usual de Rham complex, and Lie bracket given by the wedge product on differential forms and the usual Lie bracket on $\mathfrak{g}$:

$$\begin{array}{ccccccc} \underline{-1} & & \underline{0} & & \underline{1} & & \underline{2} \\ \Omega^0(M,\mathfrak{g}) & \xrightarrow{d} & \Omega^1(M,\mathfrak{g}) & \xrightarrow{d} & \Omega^2(M,\mathfrak{g}) & \xrightarrow{d} & \Omega^3(M,\mathfrak{g}) \end{array}$$

Figure 1.1: The BV complex of Chern-Simons theory

Notice this pattern: a quadratic term in the BV action corresponds to a differential on fields; a cubic term, instead, corresponds to a Lie bracket on fields.

Yang-Mills theory

Now, consider non-abelian Yang-Mills theory on a d-dimensional Riemannian manifold M. As in the case of Chern-Simons theory, the physical fields are given by connections on a principal G-bundle over M and the ghost fields are given by functions on M valued in the Lie algebra $\mathfrak{g}$. Since the Lie bracket on the ghosts and the module structure on the fields are the same as in the Chern–Simons case, the Chevalley–Eilenberg differential acts on observables in exactly the

same way (see Eq. (1.26)).
The classical action of Yang-Mills theory is given by:

$$\mathrm{S}_{\mathrm{YM}} = \mathrm{tr} \int_M F_A \wedge * F_A \tag{1.29}$$

where $F_A = dA + [A, A]$ is the curvature of the connection A, and $*$ is the Hodge star operator associated with the chosen metric on M.

In this case, the total space of BV fields is:

$$\begin{array}{cccc} \underline{-1} & \underline{0} & \underline{1} & \underline{2} \\ \Omega^0(M, \mathfrak{g}) & \Omega^1(M, \mathfrak{g}) & \Omega^{d-1}(M, \mathfrak{g}) & \Omega^d(M, \mathfrak{g}) \end{array}$$

Since the symmetry structure is unchanged w.r.t. the Chern-Simons case, the BV action S_{BV} is constructed analogously by adding terms of the same form of Eq. (1.27) (be careful that now $A^\dagger$ and $c^\dagger$ are, respectively, $d-1$ and d differential forms).
Therefore, also the brackets with the BV action are formally the same as in Eq. (1.28); the only one that changes is the one with the antifield $A^\dagger(x)$ because of the bracketing with the classical action:

$$\{\mathrm{S}_{\mathrm{BV}}, A^\dagger(x)\} = \underline{d * dA(x)} + \underset{\sim\sim\sim}{[A(x), A(x)]} + \underset{\sim\sim\sim}{[A^\dagger(x), c(x)]} + \underline{[*A(x), [A(x), A(x)]]} \tag{1.30}$$

Notice that in this case, the differential $\{\mathrm{S}_{\mathrm{BV}}, -\}$ acting on observables includes not only linear or quadratic terms, but also a cubic part – namely, the one that generates the last term in Eq. (1.30). This is a consequence of having a quartic term in the action, specifically expanding $F_A \wedge *F_A$ in terms of A.
This cubic part of the differential is dual to a 3-ary bracket on the space of fields: we see how, in this case, the space of BV fields is no longer a differential graded Lie algebra, but also gets higher brackets. This reveals that, actually, the space of BV fields carries the structure of a L_∞-algebra.
Explicitly, the cochain complex of BV fields for Yang-Mills theory is given by:

$$\begin{array}{ccccccc} \underline{-1} & & \underline{0} & & \underline{1} & & \underline{2} \\ \Omega^0(M, \mathfrak{g}) & \xrightarrow{d} & \Omega^1(M, \mathfrak{g}) & \xrightarrow{d*d} & \Omega^{d-1}(M, \mathfrak{g}) & \xrightarrow{d} & \Omega^d(M, \mathfrak{g}) \end{array}$$

Figure 1.2: The BV complex of Yang-Mills theory

The operations on this complex are:

- A Lie bracket, given by the wedge product on differential forms and the usual Lie bracket on the Lie algebra $\mathfrak{g}$.
- A 3-ary bracket of the form:

$$l_3(A, A, A) = [*A, [A, A]] \tag{1.31}$$

To ensure the correct cohomological gradings for the brackets – we want the usual Lie bracket on fields to be an operation of cohomological degree 0 – we should shift the space of BV fields by 1 to the right. This means that the space that has a L_∞-algebra structure is actually $\mathscr{F}_{\mathrm{BV}}[-1]$. This algebraic structure will be introduced and defined in more details in the next chapter.

Chapter 2

Formal moduli problems and symmetries

The Batalin-Vilkovisky formalism allows one to describe the perturbative behaviour of a classical field theory. In Chapter 1, we saw how to construct the BV-BRST complex of a classical field theory concretely, given appropriate input data. In particular, through specific choices, we were able to hide the inherently perturbative nature of the BV description. For example, in the case of Chern-Simons theory, physical fields are given by flat connections on a principal G-bundle, where G is the Lie group with Lie algebra $\mathfrak{g}$. However, to write down the BV-BRST complex as we did in Figure 1.1, we made the simplifying choices of working with the trivial G-bundle and expanding around the trivial flat connection[1].

In this chapter, we first want to make the perturbative nature of the BV formalism evident. Specifically, we will interpret the space of BV fields as governing the moduli of deformations around a fixed point in the moduli space of solutions to the equations of motion. This perspective is developed in Section 2.1, where we provide a brief introduction to formal moduli problems. Though not fully rigorous, this discussion aims to highlight key ideas and motivations that will appear repeatedly in this thesis.
Next, in Section 2.2, we describe how to treat symmetries in the BV formalism. We will see how the formal moduli problem perspective becomes crucial when interpreting the gauging of a symmetry.
In the last section (2.3), we present a concrete example where we apply all the technology developed troughout the chapter.

2.1 Perturbative field theory as derived deformation theory

In Chapter 1, we have seen in some concrete examples how the space of BV fields possesses a higher algebraic structure. For instance, in the case of Yang-Mills theory, in addition to a differential and a Lie bracket, we noticed the presence of a 3-ary bracket. This is part of a more general story: a BV theory describes the formal deformations around a specific solution of the equations of motion, and such a formal moduli problem is always controlled by a higher algebraic structure, specifically a L_∞ algebra.
In this section, we will give all the relevant definitions to clarify this statement.

[1] We will make explicit these choices in Chapter 3, Section 3.5.

Let us start by giving the definition of L_∞ algebra.

Definition 3 ([CG21], Definition A.1.1.2)**.** *A L_∞ algebra is a $\mathbb{Z}$-graded vector space L equipped with multilinear operations, called higher Lie brackets:*

$$l_n := [-, \dots, -]_n : L^{\otimes n} \to L, \quad n \in \mathbb{N}_{\geq 1}$$

A bracket l_n is a map of degree $2-n$ to L and satisfies, for $x_1, \dots, x_n \in L$ with $x_i \in L_{|x_i|}$ homogeneous:

1. *Graded antisymmetry: for all $n \geq 2$,*

$$l_n(x_1, \dots, x_i, x_{i+1}, \dots, x_n) = -(-1)^{|x_i||x_{i+1}|} l_n(x_1, \dots, x_{i+1}, x_i, \dots, x_n)$$

2. *Graded Jacobi identity: for all $n \geq 1$,*

$$0 = \sum_{k=1}^{n} (-1)^k \sum_{\epsilon \in \mathrm{Unshuff(k,n-k)}} \mathrm{sgn}^{(|x_1|,\dots,|x_n|)}(\epsilon) l_{n-k+1}(l_k(x_{i_1}, \dots x_{i_k}), x_{j_1}, \dots, x_{j_{n-k}})$$

Here, ϵ goes through all unshuffles, i.e. the permutations of the form:

$$\begin{pmatrix} 1 & 2 & \dots & k & k+1 & \dots & n \\ i_1 & i_2 & \dots & i_k & j_1 & \dots & j_{n-k} \end{pmatrix}$$

with $i_1 < \dots < i_k$ and $j_1 < \dots < j_{n-k}$. $\mathrm{sgn}^{(|x_1|,\dots,|x_n|)}(\epsilon)$ is instead the graded antisymmetric sign of the permutation: exchanging x_1 with x_2, you gain the sign $-(-1)^{|x_1||x_2|}$.

The definition does not look very friendly, but for small n' s, it just encodes usual definitions and it is very tractable. Let us see some explicit examples.

- A L_∞ algebra concentrated in degree zero is just an ordinary Lie algebra.
- If all the brackets l_n vanish for $n > 1$, then our L_∞ algebra is just a cochain complex with differential l_1. The fact that $l_1^2 = 0$ comes from the graded Jacobi identity for $n = 1$.
- A L_∞ algebra where all the higher brackets l_n vanish for $n > 2$ is a differential graded (dg) Lie algebra. The Leibniz rule follows from the Jacobi identity.
- Explicitly, for $n = 1, 2, 3$, the Jacobi identities read:

$$\begin{aligned} l_1^2 &= d^2 = 0 \\ d[x_1, x_2] &= [dx_1, x_2] - (-1)^{|x_1||x_2|}[x_1, dx_2] \\ ([[x_1, x_2], x_3] \pm \text{permutations}) &= ([dx_1, x_2, x_3] \pm \text{permutations}) \pm d[x_1, x_2, x_3] \end{aligned} \tag{2.1}$$

  where $\pm$ is the graded antisymmetric sign of the underlying permutation. We see that the usual Jacobi identity (the one for $n = 3$) is satisfied up to an exact term: this is why L_∞ algebras have the interpretation of homotopy algebras[2]. In particular, this means that the relations of a usual dg Lie algebra are not *strictly* satisfied, but are satisfied up to homotopy terms.

[2] This name comes from the fact that if you have a Lie algebra (which can be seen as a cochain complex with no differential and concentrated in degree zero) and you want to transfer this algebraic structure to a cochain complex quasi-isomorphic to the first one, then you get an algebraic structure where the relations are only satisfied up to homotopy. This technique is called homotopy transfer, see [Val14] for an introduction to the subject.

This describes the algebraic structure underlying the space of fields of a BV theory. More precisely, the object that carries this algebraic structure is the shift of the space of fields by one to the right, which ensures that the brackets have the correct degree. In the BV formalism, we place physical fields in degree zero, so that in degree-zero cohomology lies the physics. In the L_∞ algebra grading, instead, physical fields are placed in degree one.
As already remarked in Section 1.6, the arity of the brackets on fields depends directly to the polynomial degree of the terms in the action: bracketing on observables with a quadratic term in the action is dual to a differential on the space of fields, bracketing with a cubic term is dual to a Lie bracket on the space of fields, and so on.
Since the space of BV fields is actually a sheaf, we need to refine our definition accordingly. This refinement is captured by the notion of a local Lie algebra.

Definition 4 ([CG21], Definition 3.1.3.1)**.** *Let M be a smooth manifold. A local Lie algebra on M is a graded vector bundle L on M, equipped with polydifferential operators:*

$$l_n : \mathcal{L}^{\otimes n} \to \mathcal{L}, \quad n \in \mathbb{N}_{\geq 1}$$

of cohomological degree $2 - n$, that make its sheaf of sections $\mathcal{L}$ into a sheaf of L_∞ algebras on M.

The condition that $\mathcal{L}$ arises as smooth sections of a graded vector bundle is there because we want $\mathcal{L}$ to be a *fine* sheaf[3]. This is helpful because it means that we do not have to worry if we place $\mathcal{L}$ on some complicated global geometries – the nice properties of fine sheaves already take care of these issues.

Up to this point, we have described the structure of the space of BV fields, but not why such a structure arises. The underlying reason is profound and lies in the moduli-theoretic nature of the problem that the BV formalism aims to describe. In what follows, we will present an intuitive overview of this powerful idea, while omitting most of the technical details. This choice is justified by the scope of this thesis: for our purposes, it suffices to work with a presentation of a field theory as a local Lie algebra. However, from a conceptual and interpretative point of view, the following discussion remains highly significant.

As discussed in Chapter 1, the essential goal in studying a field theory is to describe the moduli space of solutions to the equations of motion, modulo gauge equivalences. This space is complicated and we have seen how to deal with that: to cure possible non-transverse intersections, one passes to the derived critical locus; instead, to take care of gauge symmetries, one considers a derived model for the stacky quotient. In the end, after making all these modifications, the moduli space of interest is no longer "just a scheme[4]" – rather it is a derived stack.

To gain some intuition about derived stacks, recall that a scheme – apart from the definition in terms of locally ringed space (see Appendix E) – can be equivalently defined in terms of its functor of points[5]. This is a special[6] functor from the opposite category of affine schemes $\mathrm{Aff}^{\mathrm{op}}$ (which is equivalent to the category of commutative algebras CAlg, via the Spec functor and

[3]A sheaf $\mathcal{F}$ on a topological space M is fine if for every locally finite open cover $\{U_i\}$ of M, $\mathcal{F}$ admits a partition of unity subordinate to $\{U_i\}$. A fine sheaf is always soft; this means that you can always extend local sections to global ones.

[4]For this discussion, we retreat to the algebraic world, but remember that all our spaces are equipped with a smooth structure.

[5]See [EH06] for the functor-of-points approach to schemes. See, instead, [Vez11] for a brief introduction to derived stacks.

[6]"Special" here means that has to satisfy additional conditions. The most important one, which must also hold in the derived setting, is some version of Zariski/Étale descent: these functors are sheaves.

the global section functor) to Sets. It assigns to each test algebra the set of algebra maps from the structure sheaf of the scheme we are interested in. We can think of it as follows: we can recover the space we are studying by knowing how other spaces map into it.
One way of interpreting this functorial description is that the points of a scheme form a set.

Going from the critical locus to the *derived* critical locus (as in Section 1.2), mirrors going from a scheme to a derived scheme. In the functor-of-points approach, this shift can be described by enlarging the source category[7]: from commutative algebras we pass to nonpositively differential graded commutative algebras $\mathrm{CAlg}^{\mathrm{dg}}_{\leq 0}$. This is what we have done when describing intersections: replacing the tensor product with the derived tensor product, naturally yields dg commutative algebras concentrated in nonpositive degrees (see Eq. (1.6)).

Now, we have another step to take. The moduli space of solutions to the equations of motion must be considered up to gauge equivalences. A standard way to do that is by taking a quotient, identifying solutions related by gauge transformations. However, this loses important structure: we want to remember not just which solutions are equivalent, but also the data of the equivalences themselves. In other words, we want that the points of our moduli space do not form just a set, but rather a groupoid – a collection of objects (solutions) together with all invertible morphisms (gauge equivalences) between them.
In the functor-of-points perspective, this corresponds to enlarge the target category from Sets to the category of groupoids Grpds. This is what we call a stack.

To arrive at the notion of a derived stack, we need to take a small further conceptual step. Since also gauge equivalences have redundancies, instead of quotienting these out, we continue to remember them, along with equivalences between equivalences, and so on. The target category that does this job is the category of simplicial sets SSets.

This discussion is elegantly captured by the following diagram, which originally appeared in [Vez11], but we took the version of [GW25].

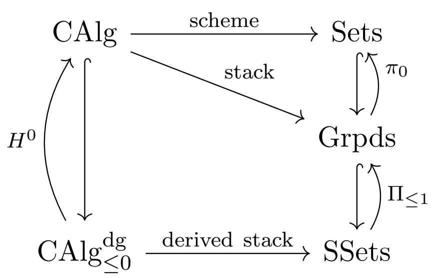


Figure 2.1: A derived stack

A derived stack of this type can describe the entire moduli space. In practice, we restrict ourselves to perturbation theory. That is, we choose a solution ϕ to the equations of motion and study formal deformations around it. In other words, we choose a point of the moduli space and we study a formal neighbourhood of it. "Formal" means that we do not care about convergence issues. Technically, this is done by restricting our source category to the subcategory of Artinian dg algebras.

Definition 5 ([CG21], Definition A.2.2.1)**.** *An Artinian dg algebra A is a nonpositively graded dg commutative algebra over $\mathbb{C}$ such that each component is finite-dimensional and has an unique*

[7]More precisely, this requires working within the framework of ∞-categories, as derived geometry is naturally formulated in higher-categorical terms.

maximal ideal $\mathfrak{m}$*, closed under the differential, and satisfying* $A/\mathfrak{m} \cong \mathbb{C}$*.*

Why does this correspond to probing the formal neighbourhood of a point? An example of Artin algebra is given, for instance, by $A = \mathbb{C}[\epsilon]/\epsilon^n$, where ϵ is a variable of degree zero. The unique maximal ideal is generated by ϵ and we recall that having a unique maximal ideal geometrically corresponds to having just one single point. The image of this point selects the solution ϕ. However, the algebra A also encodes infinitesimal "thickenings" around ϕ: its structure allows us to consider a one-parameter family of points infinitesimally close to ϕ, up to order ϵ^n.
In other words, maps from Artinian algebras correspond to probing formal neighbourhoods around ϕ, capturing infinitesimal deformations. This motivates the definition of a formal moduli problem as a functor from the category of Artinian dg algebras to SSets (with some more technical conditions, see [Lur11], Definition 0.0.8).

Our aim is then to describe a formal moduli problem, or, more precisely, a sheaf of formal moduli problems. In fact, if we choose a solution ϕ on the entire spacetime manifold M, then on each open set $U \subseteq M$, there is a formal moduli problem describing deformations of the restriction $\phi|_U$. The space of BV fields does the job, and this is because of the fundamental theorem of derived deformation theory ([Lur11]).

There is an equivalence of $(\infty, 1)$*-categories between the category of dg Lie algebras and the category of formal pointed moduli problems*[8].

Essentially, this means that every formal moduli problem is controlled by a L_∞ algebra. This explains why the space of BV fields naturally carries the structure of a L_∞ algebra: we are describing some formal neighbourhood in the moduli space of solutions to the equations of motion. More precisely, the correct structure is that of a local Lie algebra, which also takes into account the sheaf-theoretic nature of the space of fields.

We emphasize that in the BV formalism, we focus on variational formal moduli problems, i.e. those arising from equations of motion that come from varying an action functional. In [PTVV13], it is shown that variational formal moduli problems carry a natural (-1)-shifted symplectic structure. This explains why a model for the space of BV fields is given by the (-1)-shifted cotangent bundle.
Notice that not all formal moduli problems arise in this way; nevertheless, all of them are always controlled by a L_∞ algebra.
Now that we have developed the necessary background, we can offer an alternative definition of a classical field theory: it is a sheaf of (-1)-shifted symplectic formal moduli problems.

2.2 Symmetries in BV

In this section, we explore how to treat symmetries in the BV formalism. We begin by describing how symmetries are formulated in the derived setting: the key idea is that symmetries are modeled by local Lie algebras, which often have a direct geometric interpretation in deformation theory. Next, we examine what it means for a symmetry to act on the space of BV fields, and we interpret gauging a symmetry as the coupling of two formal moduli problems.

2.2.1 Symmetries as local Lie algebras

In classical field theory, infinitesimal symmetries are described by a Lie algebra. In the BV formalism, however, we aim to model symmetries as local Lie algebras, in keeping with the

[8]This formulation is the one given in [CG21], §3.2.1.

derived nature of the framework. The key idea is to construct a resolution of the classical Lie algebra of symmetries – this resolution is what defines the local Lie algebra in the derived setting. When we take the degree-zero cohomology of the global sections of this local Lie algebra, we recover the original classical Lie algebra of symmetries.
We now present a few examples to show how this construction works in practice.

Example 1: conformal symmetry. Let (M, g) be a Riemannian manifold, where g is the chosen Riemannian metric. Conformal symmetries are described infinitesimally by the Lie algebra of conformal Killing vector fields – that is, vector fields that preserve the metric up to a local rescaling. Denoting by $\mathrm{Vect}(M)$ the space of smooth vector fields on M, and by $\Gamma(M, \mathrm{Sym}^2 T^*M)$ the space of smooth symmetric two-index tensors on M, a model for the corresponding local Lie algebra is given by the two-term complex:

$$
\begin{array}{ccc}
\underline{0} & & \underline{1} \\[2em]
\mathrm{Vect}(M) & \xrightarrow{L} & \Gamma(M, \mathrm{Sym}^2 T^*M) \\
 & \overset{g}{\nearrow} & \\
C^\infty(M) & &
\end{array}
$$

where $L(X) = L_X g$ is the Lie derivative of g along X, and the diagonal map is just given by rescaling $f \mapsto f \cdot g$.
The Lie bracket is defined by saying that $\mathrm{Vect}(M)$ acts everywhere by Lie derivative, and that $[f, \alpha] = f \cdot \alpha$ for $f \in C^\infty(M)$ and $\alpha \in \Gamma(M, \mathrm{Sym}^2 T^*M)$.
The degree-zero cohomology consists of pairs (X, f) such that:

$$
L_X g + f \cdot g = 0 \tag{2.2}
$$

which is precisely the condition for X to be a conformal Killing vector field.
This local Lie algebra also has a clear deformation-theoretic interpretation. It describes the formal moduli problem of deformations of conformal structures around the background metric g. These deformations are encoded in the degree-one cohomology, which consists of symmetric two-index tensors, modulo those perturbations arising from infinitesimal diffeomorphisms ($L_X g$) and Weyl rescalings ($f \cdot g$). In other words, degree-one cohomology describes *genuine* deformations of the conformal structure.

Example 2: symplectic vector fields. Given a symplectic manifold (M, ω), the local Lie algebra of symplectic vector fields – those vector fields preserving the symplectic structure – can be modeled by the two-term complex:

$$
\begin{array}{ccc}
\underline{0} & & \underline{1} \\[1em]
\mathrm{Vect}(M) & \xrightarrow{L} & \Omega^2(M)
\end{array}
$$

where $L(X) = L_X \omega$ is the Lie derivative of the symplectic structure along X. In this case, the degree-zero cohomology of the global sections on M recovers the Lie algebra of symplectic vector fields, whereas the degree-one cohomology classes represent perturbations around the chosen symplectic form ω.

Example 3: the Witt algebra. The Witt algebra is the Lie algebra of holomorphic vector fields on the punctured complex plane: $\mathrm{Vect}^{\mathrm{hol}}(\mathbb{C}^\times)$. To obtain its local version, we need to resolve the holomorphic constraint. A standard method is by using the Dolbeault resolution. On each open subset $U \subset \mathbb{C}^\times$ (locally isomorphic to $\mathbb{C}$), the local Witt algebra is given by:

$$\begin{array}{ccc} \underline{0} & & \underline{1} \\ \Omega^{0,0}(\mathbb{C}, T^{1,0}\mathbb{C}) & \xrightarrow{\bar{\partial}} & \Omega^{0,1}(\mathbb{C}, T^{1,0}\mathbb{C}) \end{array}$$

Here, $\bar{\partial}$ is the Dolbeault differential (also known as antiholomorphic de Rham operator) that just acts on Dolbeault forms, while $T^{1,0}\mathbb{C}$ is the holomorphic tangent bundle.
The Lie bracket is defined as the $\Omega^{0,*}$-linear extension of the Lie bracket of vector fields.
This local Lie algebra has a rich geometric interpretation that will be fundamental in what's coming next. In particular, it is the dg Lie algebra that governs deformations of complex structures. This is a well-known fact and goes back to the foundational work of Kodaira and Spencer (see [KS58]).

2.2.2 The action of a local Lie algebra on the space of fields

By now, we have seen that both the space of fields and the symmetries can be described by local Lie algebras. In this section, we will see how one local Lie algebra can act on another. When the space of fields carries an action of a local Lie algebra $\mathcal{L}$, we say $\mathcal{L}$ is a symmetry of the theory. We will then discuss what it means to gauge such a symmetry and explain the deformation-theoretic interpretation of this process. The main reference for this section is [CG21], §12.2.

We begin by recalling the classical notion of an action of a Lie algebra $\mathfrak{g}$ on another Lie algebra $\mathfrak{h}$. This is given by a map of Lie algebras:

$$\mathfrak{g} \longrightarrow \mathrm{Der}(\mathfrak{h}) \tag{2.3}$$

where $\mathrm{Der}(\mathfrak{h})$ denotes the Lie algebra of derivations of $\mathfrak{h}$. Given such an action, one can then construct the associated semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$, which lives in the following short exact sequence:

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \ltimes \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow 0 \tag{2.4}$$

Conversely, given such a SES, one can reconstruct the action of $\mathfrak{g}$ on $\mathfrak{h}$, up to equivalences.
The idea is to extend this definition to the setting of local Lie algebras. We need two refinements: one related to the sheaf nature of a local Lie algebra, and one related to the fact that a local Lie algebra has the structure of a L_∞ algebra, and not just of a Lie algebra.

Definition 6 ([CG21], Definition 12.2.1.1)**.** *An action of a L_∞ algebra $\mathfrak{g}$ on a L_∞ algebra $\mathfrak{h}$ is a L_∞ algebra structure on $\mathfrak{g} \oplus \mathfrak{h}$, which we denote $\mathfrak{g} \ltimes \mathfrak{h}$, with the property that the maps in the short exact sequence of vector spaces of Eq. (2.4) are strict maps of L_∞ algebras.*

This is equivalent ([CG21], Lemma 12.2.1.2) to give a L_∞ algebra map – also called homotopy-coherent map – of the form:

$$\rho : \mathfrak{g} \rightsquigarrow \mathrm{Der}(\mathfrak{h}) \tag{2.5}$$

where now $\mathrm{Der}(\mathfrak{h})$ denotes the L_∞ algebra of derivations of $\mathfrak{h}$.
In the cases relevant for this thesis, the space of fields on which a symmetry acts has just the structure of a (sheaf of) cochain complex, meaning it is a L_∞ algebra with a trivial Lie bracket and vanishing higher brackets. In such cases, $\mathrm{Der}(\mathfrak{h})$ simplifies to the complex of endomorphisms

End($\mathfrak{h}$), i.e. the complex of linear maps that preserve the cochain structure. The differential of End($\mathfrak{h}$), is given by:

$$d_{\mathrm{End}(\mathfrak{h})}(\phi) := d_{\mathfrak{h}} \circ \phi - (-1)^{|\phi|} \phi \circ d_{\mathfrak{h}}, \quad \forall \phi \in \mathrm{End}(\mathfrak{h}) \tag{2.6}$$

where $d_{\mathfrak{h}}$ is the differential of $\mathfrak{h}$ (the 1-ary bracket).
Since this is the primary case of interest, we just present the relevant definitions and constructions accordingly.
Let us break explicitly down the definition of a homotopy-coherent map. ρ is a collection of graded antisymmetric maps $(\rho^{(0)}, \rho^{(1)}, \rho^{(2)}, \dots)$, such that:

- $\rho^{(0)} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{h})$ is a map of degree 0;
- $\rho^{(1)} : \mathfrak{g}^{\otimes 2} \longrightarrow \mathrm{End}(\mathfrak{h})$ is a map of degree -1;
- $\rho^{(2)} : \mathfrak{g}^{\otimes 3} \longrightarrow \mathrm{End}(\mathfrak{h})$ is a map of degree -2;
- and so on ...

These maps are subject to conditions that generalize the requirements for a *strict* dg Lie algebra map. In particular, the lowest component $\rho^{(0)}$ must be a cochain map, meaning it commutes with the differentials. However, it is not required to preserve the Lie bracket exactly – it may do so only up to homotopy. This is precisely where $\rho^{(1)}$ comes in: it encodes the first homotopy correction to the failure of $\rho^{(0)}$ to preserve the Lie bracket. Similarly, higher components $\rho^{(n)}$ for $n \geq 2$ account for higher homotopies.
Let us now make explicit the condition satisfied by $\rho^{(0)}$. For every $g_1, g_2 \in \mathfrak{g}$:

$$[\rho^{(0)}(g_1), \rho^{(0)}(g_2)]_{\mathrm{End}(\mathfrak{h})} - \rho^{(0)}([g_1, g_2]_{\mathfrak{g}}) = \partial \rho^{(1)}(g_1, g_2) \tag{2.7}$$

This equation shows that $\rho^{(0)}$ preserves the Lie bracket up to an exact term, i.e the difference on the LHS is homotopically equivalent to zero.
The exact term on the right-hand side is defined as:

$$\partial \rho^{(1)}(g_1, g_2) := d_{\mathrm{End}(\mathfrak{h})} \rho^{(1)}(g_1, g_2) - (-1)^{|\rho^{(1)}|} \rho^{(1)}(d_{\mathfrak{g}^{\otimes 2}}(g_1, g_2)) \tag{2.8}$$

where the differential on the tensor product $\mathfrak{g}^{\otimes 2}$ is given by:

$$d_{\mathfrak{g}^{\otimes 2}}(g_1, g_2) := d_{\mathfrak{g}} g_1 \otimes g_2 + (-1)^{|g_1|} g_1 \otimes d_{\mathfrak{g}} g_2 \tag{2.9}$$

The higher components of ρ must satisfy additional compatibility conditions, but the underlying idea remains the same: the L_∞ algebra structure must be preserved at each level, up to a homotopic correction. These conditions ensure that ρ defines an homotopy-coherent map. However, we do not write all the relations since in the examples relevant for this thesis we are lucky enough to work only with strict dg Lie algebra maps.

The final refinement needed to define an action of a local Lie algebra on another local Lie algebra is to account for the fact that we are working with sheaves. The action must then be compatible with the sheaf structure.

Definition 7 ([CG21], Definition 12.2.2.1)**.** *Let $\mathcal{L}$ and $\mathcal{M}$ be local Lie algebras on M. Then an action of $\mathcal{L}$ on $\mathcal{M}$ is a local Lie algebra structure on $\mathcal{L} \oplus \mathcal{M}$, which we denote $\mathcal{L} \ltimes \mathcal{M}$, such that the short exact sequence of maps of sheaves:*

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{L} \ltimes \mathcal{M} \longrightarrow \mathcal{L} \longrightarrow 0 \tag{2.10}$$

consists of maps of L_∞ algebras.

In classical field theory, the space of BV fields has more structure than just that of a local Lie algebra: it is also equipped with a (-1)-shifted symplectic structure. Importantly, the local Lie algebra structure is not defined directly on the space of fields $\mathcal{M}$, but rather on its shift $\mathcal{M}[-1]$, to ensure the correct cohomological gradings of the brackets. As a result, the (-1)-shifted symplectic structure corresponds to a nondegenerate pairing of degree -3 on $\mathcal{M}[-1]$.
This to say, if now we consider an action of a local Lie algebra on another local Lie algebra – where the latter describes a classical field theory – we must also require that the action be compatible with the symplectic structure.

Definition 8 ([CG21], Definitions 12.2.2.2 & 12.2.2.3)**.** *Suppose that $\mathcal{M}$ has an invariant pairing $\langle -, - \rangle$. An action of $\mathcal{L}$ on $\mathcal{M}$ preserves the pairing if for any compactly supported sections $\{\alpha_1, \dots, \alpha_r\}$ of $\mathcal{L}$ and $\{\beta_1, \dots, \beta_s\}$ of $\mathcal{M}$, the expression:*

$$\langle l_{r+s}(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_{s-1}), \beta_s \rangle$$

is graded totally antisymmetric under permutations of β_i.

If the pairing of $\mathcal{M}$ has degree -3 – that is, encodes the data of a classical field theory – and the action of $\mathcal{L}$ preserves the pairing, we call $\mathcal{L}$ a symmetry of the classical field theory $\mathcal{M}$.

The compatibility of the action with the pairing is essential, as it allows us to incorporate new terms into the local action functional that encode the interaction between the symmetry and the fields. This will be illustrated explicitly in the example in Section 2.3. However, the underlying idea is clear: in the BV formalism, the entire local Lie algebra structure of the space of fields is encoded in the action functional itself (recall that bracketing the action functional with a field recovers all the L_∞ algebra structure; see Section 1.6). Now, if we have a symmetry of our BV theory, we can also encode the action of the symmetry on the space of fields by adding new terms to the action.

Suppose now we have a symmetry $\mathcal{L}$ that acts on the space of fields $\mathcal{M}$ of our theory. What does it mean to gauge this symmetry in our formalism?[9]
As we said in Definition 7, having a symmetry corresponds to choosing a local Lie algebra structure on the semidirect product $\mathcal{L} \ltimes \mathcal{M}$, which is also compatible with the pairing. In the language of formal moduli problems, the semidirect product corresponds to a fibration: the base is described by the background symmetry $\mathcal{L}$, and the fiber describes the theory $\mathcal{M}$.
Before gauging the symmetry, we study the formal moduli space controlled by $\mathcal{L} \ltimes \mathcal{M}$ on a chosen basepoint on $\mathcal{L}$, focusing on deformations along the fiber. In other words, we fix a point in the base (a background symmetry configuration) and consider equivariant deformations within the fiber, i.e., within the space of BV fields $\mathcal{M}$. If we think of the base as being the moduli space of conformal structures, this amounts to fixing a specific conformal structure: we look at the formal moduli problem controlled by $\mathcal{M}$ keeping the conformal structure fixed.

After gauging the background symmetry, we are no longer restricted to deformations along the fiber alone. Instead, we study deformations around the chosen basepoint in the total space. A schematic illustration of this process is in Figure 2.2.

In more practical terms, how do we gauge the background symmetry $\mathcal{L}$? The idea is to promote $\mathcal{L}$ to a variational formal moduli problem – that is, to construct a new BV theory $\hat{\mathcal{L}}$ equipped with a nondegenerate invariant pairing of cohomological degree -3 (or equivalently -1, if we

[9] I'm very thankful to Ingmar Saberi for having explained this subtle point. Also, Figure 2.2 is stolen from one of his blackboards.

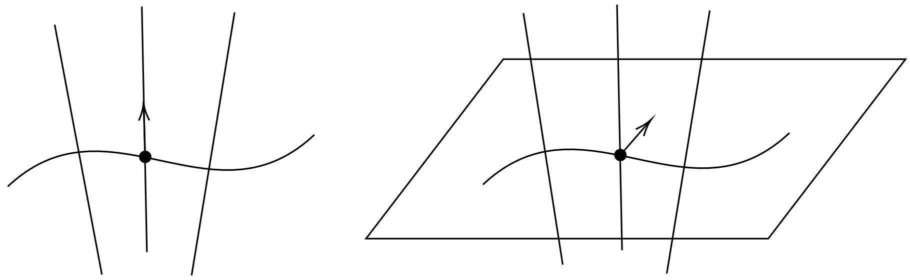


Figure 2.2: Before gauging $\longrightarrow$ after gauging

consider the BV grading) that maps to $\mathcal{L}$. This allows us to write down an action functional for $\hat{\mathcal{L}}$, and by bracketing the action functional with the elements in $\hat{\mathcal{L}}$, we recover the full structure of $\mathcal{L}$ as a local Lie algebra.
Once promoted to a variational formal moduli problem, $\hat{\mathcal{L}}$ can be interpreted as describing the formal moduli space of solutions to the equations of motion defined by this action functional.
A standard way to construct such a theory is to consider the associated BF theory[10]:

$$\hat{\mathcal{L}} = \mathcal{L} \oplus \mathcal{L}^![-3] \tag{2.11}$$

In the end, gauging a symmetry means studying the coupled formal moduli problem given by the semidirect product $\hat{\mathcal{L}} \ltimes \mathcal{M}$.

2.3 Example: the holomorphic bosonic string

In this section, we construct the holomorphic bosonic string and interpret the model from the perspective of derived geometry (a detailed exposition can be found in [GW17]). This example brings together many of the key concepts developed in this chapter. In particular, it shows how gauging a symmetry works in the BV formalism. Moreover, it will serve as our starting point for constructing a topological string theory from the worldsheet perspective.

Let us consider the free holomorphic $\beta\gamma$ system. The physical fields of this theory are holomorphic functions:

$$\gamma : \Sigma \longrightarrow V \tag{2.12}$$

where Σ is a Riemann surface and V a complex vector space. These fields satisfy the equation of motion $\bar{\partial}\gamma = 0$, so the moduli space of solution is the space of holomorphic maps from Σ to V, denoted by $\mathrm{Map}^{\mathrm{hol}}(\Sigma, V)$.
The γ fields can be understood as the chiral components of a massless scalar field ϕ. To directly relate ϕ to γ, recall that the 2d massless scalar field theory is a theory of harmonic real functions, i.e., functions satisfying the Laplace equation $\Delta\phi = 0$. However, if the underlying manifold comes equipped with a complex structure, then every complex-valued harmonic function splits into the sum of a holomorphic and an antiholomorphic function. Thus, if we complexify the scalar field ϕ, it can be written as a pair $(\gamma, \bar{\gamma})$, where:

[10] $\mathcal{L}^!$ is the so-called shriek dual of $\mathcal{L}$. Recall that $\mathcal{L}$ emerges as sections of a vector bundle L over M. Then, $\mathcal{L}^!$ are sections of the vector bundle $L^! = L^\vee \otimes \mathrm{Dens}(M)$ – that is, the linear dual of L twisted by the bundle of densities on M. For simplicity, we always fix an orientation and think of densities as top forms on M. Even if we did not stress this subtle point at all before, this is the correct notion of duals we should consider, and not just the linear dual. However, in concrete examples, we always took duals w.r.t. integration, and this results in the same thing.

1. γ is the chiral (holomorphic) component;
2. $\bar{\gamma}$ is the antichiral (antiholomorphic) component.

In what follows, we just consider the chiral sector of the theory.

The BV formalism tells us to introduce also antifields β, which, in this case, are holomorphic (1,0)-forms on Σ with values in $V^\vee$. The corresponding chiral action functional is given by:

$$\mathrm{S}_{\text{chiral}} = \int_\Sigma \langle \beta, \bar{\partial}\gamma \rangle_V \tag{2.13}$$

where the pairing $\langle -, - \rangle_V$ is defined by the evaluation pairing on V and the wedge product on differential forms.
Notice that the equations of motion for the antifields $\bar{\partial}\beta = 0$ impose the holomorphicity on β. Thus, the moduli space of classical solutions consists of holomorphic maps and holomorphic (1,0)-forms. Since we are working in a derived setting, we are not just interested in the *strict* moduli space but in a derived enhancement thereof:

$$T^*[-1]\mathbb{R}\mathrm{Map}^{\mathrm{hol}}(\Sigma, V) = \mathbb{R}\Gamma(\Sigma, \mathscr{O}^{\mathrm{hol}}) \oplus \mathbb{R}\Gamma(\Sigma, \Omega^1_{\mathrm{hol}}) \tag{2.14}$$

where $\mathscr{O}^{\mathrm{hol}}$ is the sheaf of holomorphic functions, Ω^1_{hol} is the sheaf of holomorphic one forms and $\mathbb{R}\Gamma$ are the derived global sections, meaning that we take a resolution of these sheaves before applying the global section functor (recall that this functor is left exact: $\mathbb{R}$ stands for *right* derived functor). In particular, a standard resolution in such cases is the Dolbeaut resolution. Hence, the global sections of the space of BV fields are given by the following complex:

$$\mathscr{F}_{\text{chiral}}(\Sigma) = (\Omega^{0,*}(\Sigma, V) \oplus \Omega^{1,*}(\Sigma, V^\vee), \bar{\partial}) \tag{2.15}$$

Or, more explicitly:

$$\begin{array}{ccc} \underline{0} & & \underline{1} \\[2ex] \Omega^{0,0}_{\gamma_0}(\Sigma, V) & \xrightarrow{\bar{\partial}} & \Omega^{0,1}_{\gamma_1}(\Sigma, V) \\[2ex] \Omega^{1,0}_{\beta_0}(\Sigma, V^\vee) & \xrightarrow{\bar{\partial}} & \Omega^{1,1}_{\beta_1}(\Sigma, V^\vee) \end{array}$$

Figure 2.3: The BV complex of the holomorphic $\beta\gamma$ system

We now want to exhibit a natural symmetry that the holomorphic $\beta\gamma$ system possesses, gauge it, and see how we can interpret the resulting gauged symmetry as *holomorphic gravity.*
The symmetry we consider is given by the local Lie algebra controlling deformations of complex structures; see Example 3 of Section 2.2.1. The global sections of this dg Lie algebra are:

$$\mathcal{L}^{\mathrm{hol}}(\Sigma) = (\Omega^{0,0}_{l_0}(\Sigma, T^{1,0}\Sigma) \xrightarrow{\bar{\partial}} \Omega^{0,1}_{l_1}(\Sigma, T^{1,0}\Sigma)) \tag{2.16}$$

where the Lie algebra structure comes from the $\Omega^{0,*}(\Sigma)$-extension of the Lie bracket of vector fields.
To show that $\mathcal{L}^{\mathrm{hol}}$ is a symmetry of the free holomorphic $\beta\gamma$ system, we must exhibit a homotopy-coherent map:

$$\rho : \mathcal{L}^{\mathrm{hol}} \rightsquigarrow \mathrm{End}(\mathscr{F}_{\text{chiral}}[-1]) \tag{2.17}$$

The shift of the complex of fields by $[-1]$ appears because we consider the L_∞ grading.
As explained in Section 2.2.2, we know the conditions that a homotopy-coherent map ρ and its components must satisfy. In practice, we can proceed with the following strategy:

1. Start by guessing the zero degree component $\rho^{(0)}$ of the homotopy-coherent map.
2. Next, check if $\rho^{(0)}$ is a cochain map and a Lie algebra map.
3. If $\rho^{(0)}$ turns out to be a cochain map but not a strict Lie algebra map, then we must look for homotopic corrections.

In our case, there is a very natural guess: the action of $\mathcal{L}^{\text{hol}}$ on the fields of the holomorphic $\beta\gamma$ system is given by the $\Omega^{0,*}$-extension of the Lie derivative $L = [\partial, \iota]$ where ι is the usual contraction of vector fields. Explicitly on a local patch:

$$(\rho^{(0)}(l_0))(\gamma_0 + \beta_0) = l_0(z,\bar{z})\frac{\partial \gamma_0(z,\bar{z})}{\partial z} + \frac{\partial(l_0(z,\bar{z})\beta_0(z,\bar{z}))}{\partial z}dz \tag{2.18}$$

$$(\rho^{(0)}(l_0))(\gamma_1 + \beta_1) = l_0(z,\bar{z})\frac{\partial \gamma_1(z,\bar{z})}{\partial z}d\bar{z} + \frac{\partial(l_0(z,\bar{z})\beta_1(z,\bar{z}))}{\partial z}dz \wedge d\bar{z} \tag{2.19}$$

For the moment, we do not guess the action of $\rho^{(0)}(l_1)$ on fields: we will find it by imposing on $\rho^{(0)}(l_0)$ the condition of being a cochain map.
Specifically, $\rho^{(0)}$ is a cochain map if the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{L}^{\text{hol}} & \xrightarrow{\quad\bar{\partial}\quad} & \mathcal{L}^{\text{hol}} \\
\Big\downarrow{\scriptstyle \rho^{(0)}} & & \Big\downarrow{\scriptstyle \rho^{(0)}} \\
\text{End}(\mathscr{F}_{\text{chiral}}[-1]) & \xrightarrow{d_{\text{End}}} & \text{End}(\mathscr{F}_{\text{chiral}}[-1])
\end{array}$$

In this case, if $\alpha \in \text{End}(\mathscr{F}_{\text{chiral}}[-1])$ and $m \in \mathscr{F}_{\text{chiral}}[-1]$, the differential d_{End} of Eq. (2.6) reads as:

$$d_{\text{End}}(\alpha(m)) = \bar{\partial}(\alpha(m)) - (-1)^{|\alpha|}\alpha(\bar{\partial}(m)) \tag{2.20}$$

Let us now impose the commutativity of the diagram described above. This condition must hold for every element $l \in \mathcal{L}^{\text{hol}}$ and for every field $\gamma, \beta \in \mathscr{F}_{\text{chiral}}[-1]$. We begin by analyzing the case of a fixed element $l_0 \in \mathcal{L}^{\text{hol}}$.

- l_0, γ_0

$$\begin{aligned}
(\rho^{(0)}(\bar{\partial}l_0))(\gamma_0) &= \bar{\partial}(\rho^{(0)}(l_0)(\gamma_0)) - \rho^{(0)}(l_0)(\bar{\partial}\gamma_0) \\
&= \frac{\partial l_0(z,\bar{z})}{\partial \bar{z}}\frac{\partial \gamma_0(z,\bar{z})}{\partial z}d\bar{z}
\end{aligned} \tag{2.21}$$

- l_0, β_0

$$\begin{aligned}
(\rho^{(0)}(\bar{\partial}l_0))(\beta_0) &= \bar{\partial}(\rho^{(0)}(l_0)(\beta_0)) - \rho^{(0)}(l_0)(\bar{\partial}\beta_0) \\
&= \frac{\partial}{\partial z}\left(\frac{\partial l_0(z,\bar{z})}{\partial \bar{z}}\beta_0(z,\bar{z})\right)d\bar{z} \wedge dz
\end{aligned} \tag{2.22}$$

- l_0, γ_1

$$\begin{aligned}
(\rho^{(0)}(\bar{\partial}l_0))(\gamma_1) &= \bar{\partial}(\rho^{(0)}(l_0)(\gamma_1)) \\
&= \bar{\partial}\left(l_0(z,\bar{z})\frac{\partial \gamma_1(z,\bar{z})}{\partial \bar{z}}d\bar{z}\right) = 0
\end{aligned} \tag{2.23}$$

- l_0, β_1 is similar to the previous case.

From these explicit relations, we can see that the action of $(\rho^{(0)}(l_1))$ should again be given by a $\Omega^{0,*}$-extension of the Lie derivative. Notice that, since l_1 has cohomological degree $+1$ and $\rho^{(0)}$ is a degree zero map, it follows that $\rho^{(0)}(l_1)$ is a degree $+1$ endomorphism on $\mathscr{F}_{\text{chiral}}[-1]$. Moreover, it is easy to verify that if we take $\rho^{(0)}(l_1)$ to be the Lie derivative, then the condition that $\rho^{(0)}$ is a cochain map is satisfied (just need to check conditions very similar to the one we wrote explicitly).

Now that we know the action of $\rho^{(0)}$ on all elements, we need to check if it is a strict Lie algebra map. This means that the following should be satisfied (see Eq. (2.7)):

$$\forall \alpha, \delta \in \mathcal{L}^{\text{hol}},\ [\rho^{(0)}(\alpha), \rho^{(0)}(\delta)]_{\text{End}} - \rho^{(0)}([\alpha, \delta]_{\mathcal{L}^{\text{hol}}}) = 0 \tag{2.24}$$

where $[-,-]_{\text{End}}$ is just the graded commutator.
If this is not the case, at least this should be true homotopically, i.e. the difference above should be an exact term. But in our specific case, the map is a strict Lie algebra map. To see how the computation works, we leave an example here:

$$\begin{aligned}&([\rho^{(0)}(l_0), \rho^{(0)}(l_0')]_{\text{End}})(\gamma_0) - \rho^{(0)}([l_0, l_0']_{\mathscr{L}^{\text{hol}}})(\gamma_0) = \\ &= (\rho^{(0)}(l_0)\rho^{(0)}(l_0') - \rho^{(0)}(l_0')\rho^{(0)}(l_0))(\gamma_0) - \rho^{(0)}((l_0\partial_z l_0' - l_0'\partial_z l_0)\partial_z)(\gamma_0) = 0\end{aligned} \tag{2.25}$$

We could expect this result because the usual Lie derivative satisfies $[L(X), L(Y)] - L_{[X,Y]} = 0$, and we are just considering a linear extension of the Lie derivative. So, we conclude that $\mathcal{L}^{\text{hol}}$ is a symmetry of the holomorphic $\beta\gamma$ system[11].

We now want to gauge this background symmetry and study the coupled formal moduli problem. To do this, we first give a description in terms of fields of this symmetry, identifying a (-1)-shifted symplectic formal moduli problem that maps into $\mathcal{L}^{\text{hol}}$. As explained in Section 2.2.1, one standard choice is taking the associated BF theory of $\mathcal{L}^{\text{hol}}$. In BV grading, the complex of global sections is given by:

$$\hat{\mathcal{L}}(\Sigma) = (\mathcal{L}^{\text{hol}}[1] \oplus (\mathcal{L}^{\text{hol}})^![-2])(\Sigma) = (\Omega^{0,*}(\Sigma, T^{1,0}\Sigma)[1] \oplus \Omega^{1,*}(\Sigma, T^{*1,0}\Sigma)[-1], \bar{\partial}) \tag{2.26}$$

Gauging means studying the semidirect product of the two formal moduli problems: $\hat{\mathcal{L}} \ltimes \mathscr{F}_{\text{chiral}}$. The total space of fields is given explicitly as in Figure 2.4.

The action that encodes all the relations of the coupled formal moduli problem is given by:

$$\mathrm{S}_{\text{tot}} = \int_\Sigma \langle \beta, \bar{\partial}\gamma \rangle_V + \langle b, \bar{\partial}c \rangle_T + \langle b, [c, c]_{\mathcal{L}^{\text{hol}}} \rangle_T + \langle \beta, L_c\gamma \rangle_V \tag{2.27}$$

where $\langle -,- \rangle_T$ is the pairing given by the wedge product on forms and the evaluation pairing on $T^{1,0}\Sigma$. To explicitly see how this action encodes both the action of $\mathcal{L}^{\text{hol}}$ on $\mathscr{F}_{\text{chiral}}$ and the dg Lie algebra structure of $\mathcal{L}^{\text{hol}}$, we simply compute the relevant brackets. Specifically, we have:

$$\begin{aligned}\{\mathrm{S}_{\text{tot}}(x), \gamma(x)\} &= \bar{\partial}\beta + L_c\beta \qquad & \{\mathrm{S}_{\text{tot}}(x), \beta(x)\} &= \bar{\partial}\gamma + L_c\gamma \\ \{\mathrm{S}_{\text{tot}}(x), c(x)\} &= \bar{\partial}c + [c, c]_{\mathcal{L}^{\text{hol}}} \qquad & \{\mathrm{S}_{\text{tot}}(x), b(x)\} &= \bar{\partial}b + [c, b]_{\mathcal{L}^{\text{hol}}}\end{aligned} \tag{2.28}$$

[11] As explained in Section 2.2.2, for a map to define a symmetry of the theory, it must also be compatible with the symplectic structure on the space of fields. However, if we succeed in constructing an extended action functional that includes the interaction between the symmetry and the fields, and that satisfies the classical master equation (Eq. (1.23)), then this compatibility with the symplectic pairing is automatically ensured. In our example, this is the case: the action that includes the interaction (and more) will be given in Eq. (2.27).

$\underline{-1}$	$\underline{0}$	$\underline{1}$	$\underline{2}$
	$\big(\Omega^{0,0}(\Sigma, V)$	$\xrightarrow{\bar\partial} \Omega^{0,1}(\Sigma, V)\big)_\gamma$	
	$\big(\Omega^{1,0}(\Sigma, V^\vee)$	$\xrightarrow{\bar\partial} \Omega^{1,1}(\Sigma, V^\vee)\big)_\beta$	
$\big(\Omega^{0,0}(\Sigma, T^{1,0}\Sigma)$	$\xrightarrow{\bar\partial} \Omega^{0,1}(\Sigma, T^{1,0}\Sigma)\big)_c$		
		$\big(\Omega^{1,0}(\Sigma, T^{*1,0}\Sigma)$	$\xrightarrow{\bar\partial} \Omega^{1,1}(\Sigma, T^{*1,0}\Sigma)\big)_b$

Figure 2.4: The BV complex of the holomorphic bosonic string

The equations of motion, thus, are now given by:

$$\begin{aligned} \bar\partial\gamma + L_c\gamma = 0 \qquad & \bar\partial\beta + L_c\beta = 0 \\ \bar\partial c + \frac{1}{2}[c, c]_{\mathcal{L}^{\text{hol}}} = 0 \qquad & \bar\partial b + [c, b]_{\mathcal{L}^{\text{hol}}} = 0 \end{aligned} \tag{2.29}$$

The equation for c is the familiar Maurer-Cartan equation, which encodes a deformation of the complex structure. Concretely, now we still ask γ and β to be holomorphic, but with respect to the deformed complex structure $\bar\partial + L_c$.

From the perspective of derived geometry, this theory admits a straightforward interpretation, as highlighted in [GW17].
The first moduli space of interest was the moduli space of holomorphic maps from Σ to V, and by writing the corresponding BV theory we were studying the moduli of deformations around a fixed solution γ. Upon gauging the background symmetry, we allow the complex structure to vary infinitesimally. In doing so, we start moving within the moduli space of Riemann surfaces, since a complex structure endows the surface with the structure of a Riemann surface.
As highlighted in Section 2.2.2 (see in particular Figure 2.2), studying the coupled formal moduli problem corresponds to a fibration, where the fiber is described by the original theory, and, in this case, the base is the moduli space of Riemann surfaces. We are considering the following moduli problem:

$$\text{Map}^{\text{hol}}(-, V) \tag{2.30}$$

where the Riemann surface is not fixed a priori. A solution to the equations of motion determines a point $(\gamma,\, c)$ of this moduli space, and by writing the coupled formal moduli problem we are studying deformations (now in this total moduli space) around that point. More precisely, we are considering, as BV prescribes, the derived enhancement of the cotangent theory, but this is "just" to have a (-1)-shifted symplectic structure on the total space of BV fields.
This is why gauging $\mathcal{L}^{\text{hol}}$ can be interpreted as gauging "holomorphic" gravity: we see that the moduli space of Riemann surfaces arises, and this is a typical feature when considering gravity as a dynamical degree of freedom. Moreover, in one complex dimension, a complex structure determines a conformal structure, which is essentially the datum of a metric (up to diffeomorphism and Weyl rescaling).

Chapter 3

Worldsheet versus target

The aim of this chapter is to connect two different points of view on topological string theory: the one from the worldsheet and the one from the target. A topological string theory can be seen as a two-dimensional field theory: a sigma model from a Riemann surface to a complex manifold (usually a Calabi-Yau 3-fold) coupled to topological gravity on the worldsheet. On the other end, Bershadsk-Cecotti-Ooguri-Vafa [BCOV94] initiated a different approach: they started thinking about topological string theory as a quantum field theory on the target space, that for a Calabi-Yau 3-fold, means considering a six-dimensional quantum field theory.
How do we connect these two different perspectives? There is an obvious connection we can try to check: the observables of a topological string theory from the worldsheet perspective should be the fields from the target point of view.

In the following, we will construct a topological string theory from the worldsheet and we will compare our construction with the work of Costello-Li on the BCOV model that they initiated in [CL12]. Along the way, we will also try to mark the subtle difference between a topological field theory and a topological string theory, with the topological $\beta\gamma$ systems and the topological Landau-Ginzburg model as our main examples.

3.1 The topological $\beta\gamma$ system

We start by analyzing the topological $\beta\gamma$ system, which will constitute our matter system when constructing a model for topological string theory. This theory describes constant maps from a Riemann surface Σ to a compact Calabi-Yau manifold X of dimension n. From the physical point of view, this corresponds to the B-twist of n chiral superfields: recall that after performing the B-twist of a two-dimensional sigma model, the path integral localizes exactly on constant maps. In other words, only constant configurations contribute to the quantum theory (see, for example, [HKK$^+$03], 16).
For simplicity, we consider $\Sigma = \mathbb{C}$ and X to be a complex vector space. This linear setting simplifies computations, though the discussion extends naturally (with minor adjustments) to general Riemann surfaces and non-linear targets.

In the BV formalism, we can realize the topological $\beta\gamma$ system as the (-1)-shifted cotangent theory of the moduli space of constant maps (more precisely, we take its derived enhancement); the global sections of the complex of BV fields are therefore:

$$\mathscr{F}_{\beta\gamma}(\mathbb{C}) = (T^*[-1](\Omega^*(\mathbb{C}, X)), d) = (\Omega^*(\mathbb{C}, X) \oplus \Omega^*(\mathbb{C}, X^\vee)[1], d) \tag{3.1}$$

Or, more explicitly:

$$\begin{array}{cccc} \underline{-1} & \underline{0} & \underline{1} & \underline{2} \\ & \big(\Omega^0(\mathbb{C},X) \xrightarrow{d} & \Omega^1(\mathbb{C},X) \xrightarrow{d} & \Omega^2(\mathbb{C},X)\big)_\gamma \\ \big(\Omega^0(\mathbb{C},X^\vee) \xrightarrow{d} & \Omega^1(\mathbb{C},X^\vee) \xrightarrow{d} & \Omega^2(\mathbb{C},X^\vee)\big)_\beta & \end{array}$$

Figure 3.1: The BV complex of the topological $\beta\gamma$ system

Notice that here we adopt the convention of total forms: γ can be any field in the first row and β any field in the second row.

The space of BV fields has just the structure of a cochain complex: there is no Lie bracket on fields and no higher L_∞-brackets. Thus, the BV action should be quadratic: we are considering a free theory.
Since the theory is designed to encode the moduli space of constant maps, the equations of motion should impose the condition for a field to be (locally) constant; the appropriate free action functional is therefore:

$$\mathrm{S}_{\text{free}} = \int_{\mathbb{C}} \langle \beta, d\gamma \rangle_X \tag{3.2}$$

Here, $\langle -,- \rangle_X$ denotes the natural pairing on BV fields of cohomological degree -1. In this case, it is given by the wedge product on differential forms and the evaluation pairing between $X^\vee$ and X. We then get a density on $\mathbb{C}$ that we can integrate to obtain the action functional. Explicitly:

$$\langle \beta, \gamma \rangle_X = \beta_0 \wedge \gamma_2 + \beta_1 \wedge \gamma_1 + \beta_2 \wedge \gamma_0 \tag{3.3}$$

where the lower indices stand for the form degrees.

Let us now focus on the observables of this theory. As usual, they form a cochain complex where the differential is given by the bracket with the action functional. In this case, the differential acts on generators of observables (evaluation of fields at a point $x \in \mathbb{C}$) as follows:

$$\begin{aligned} \{\mathrm{S}_{free}, \gamma_0(x)\} &= 0 & \{\mathrm{S}_{free}, \beta_0(x)\} &= 0 \\ \{\mathrm{S}_{free}, \gamma_1(x)\} &= d\gamma_0(x) & \{\mathrm{S}_{free}, \beta_1(x)\} &= d\beta_0(x) \\ \{\mathrm{S}_{free}, \gamma_2(x)\} &= d\gamma_1(x) & \{\mathrm{S}_{free}, \beta_2(x)\} &= d\beta_1(x) \end{aligned} \tag{3.4}$$

These are the so-called descendant relations we have in the usual B-model, and are specific of TQFTs of cohomological type.

Actually, these are all the brackets we need to consider. Indeed, we are focusing on the costalk of observables at a point $x \in \mathbb{C}$ (since the theory is topological, the specific choice of the point is not important). A model for the costalk is given by the algebra of functions on the jets of fields at that point[1]. In practice, this means that point-like observables are spanned by evaluations of fields and their derivatives at the chosen point.

However, because the theory is topological, taking cohomology w.r.t. the de Rham differential – equivalently, the bracket with the free BV action – makes all higher derivatives cohomologically

[1] See [CG21], §10, for more details.

trivial. In other words, all higher-order Taylor coefficients are exact, and thus the only physically interesting point observables come from evaluating the "zeroth" Taylor coefficient of some field at a point – that is, the BV fields at a point. This justifies ignoring derivatives.

Furthermore, in this case, the costalk at a point is enough to capture all observables. Indeed, since the theory is topological, observables have the structure of a so-called *locally constant factorization algebra*[2]. This means that, for any open set $U \subseteq \mathbb{C}$, the cochain complex of observables on U is quasi-isomorphic to the observables on any other open set $V \supset U$. In particular, observables at a point are quasi-isomorphic to those defined on any open neighbourhood of that point. This is why we can just consider the costalk, but keep in mind that this is not usually the case.

Now, let us compute the cohomology of point-like observables, which computes the physically relevant ones. In general, we begin with a cosheaf of cochain complexes that describes observables, and taking the costalk at a point induces a differential on point-like observables. We then compute cohomology with respect to this differential.
However, in this case, the description is greatly simplified: since we are considering a free theory, the fields form just a cochain complex. This means that all the homological information is already encoded in the differential on fields. As a result, we can work at the level of fields (taking cohomology there) and then dualize to obtain the cohomology of observables. Moreover, as already discussed, we should not worry about derivatives of fields.
Putting all together, the cohomology of observables at a point $x \in \mathbb{C}$ is given by:

$$\lim_{\substack{\longleftarrow \\ x\in U}} H^*(\mathrm{Obs}(\mathscr{F}_{\beta\gamma})) := H^*(\mathrm{Obs}_x(\mathscr{F}_{\beta\gamma})) \simeq \mathbb{C}[\gamma_0, \beta_0] \tag{3.5}$$

Notice that this cohomology is generated by γ_0, a variable of cohomological degree 0, and β_0, of cohomological degree +1. We recover the usual physical description of the local observables of the B-model as done, for example, in [BBZB$^+$20], §4.2.1:

$$\mathbb{C}[\gamma_0, \beta_0] \simeq H^*_{\bar\partial}(\Omega^{0,*}(X, \wedge^* T^{1,0}X)) \tag{3.6}$$

Thus, the complex of observables of the topological $\beta\gamma$ system can be identified with the complex of polyvector fields on the target space X. These are anti-holomorphic differential forms on X with values in the exterior powers of the holomorphic tangent bundle $\wedge^* T^{1,0}X$:

$$\mathrm{PV}^{i,j}(X) = \Omega^{0,j}(X, \wedge^i T^{1,0}X) \tag{3.7}$$

That is, a polyvector field of bi-degree (i, j) is a $(0, j)$-form on X with values in $\wedge^i T^{1,0}X$.
Thus, we have:

$$\mathrm{Obs}_x(\mathscr{F}_{\beta\gamma}) \simeq (\mathrm{PV}^{*,*}(X), \bar\partial) \tag{3.8}$$

where the differential $\bar\partial$ acts on forms, leaving holomorphic vector fields invariant.
From a target space perspective, this complex would play the role of our BV complex for the space of fields.

[2] In this thesis, we do not need the technology of (pre)factorization algebras, but when talking about observables of a perturbative field theory, it is worth noting they have such a structure. For a definition of locally constant factorization algebras, see [CG16], §6.

3.2 The topological Landau-Ginzburg model

What happens now if we turn on a superpotential[3]? Let $W \in \mathrm{Sym}^{\geq 2}(X^\vee)$ be a polynomial on X that is at least quadratic. We additionally require this polynomial to have non-degenerate and isolated critical points, and to be quasi-homogeneous.
Turning on the superpotential modifies the action as follows:

$$\mathrm{S} = \mathrm{S}_{\text{free}} + \mathrm{S}_{\text{int}} = \mathrm{S}_{\text{free}} + \int_{\mathbb{C}} [W(\gamma)]_{\text{top}} \tag{3.9}$$

where with $[W(\gamma)]_{\text{top}}$ we mean that we are selecting only the two-form part of $W(\gamma)$. Indeed, the superpotential acts on the X part of our fields γ, leaving us with just differential forms: to have a well-defined term in the action, we should be able to integrate it over the worldsheet. Explicitly, denoting $\{e_i\}_{i=1}^n$ a basis of X, we have that $W \in \mathbb{C}[e^1, \ldots, e^n]$, where e^i defines the dual basis. The γ fields can be expanded as:

$$\gamma_0 = \gamma_0^i e_i \qquad \gamma_1 = \gamma_1^i e_i \qquad \gamma_2 = \gamma_2^i e_i \tag{3.10}$$

where $\gamma_j^i \in \Omega^j(\mathbb{C})$ and the sum over repeated indices is understood.
Therefore, the interaction term in the action is of the form:

$$\mathrm{S}_{\text{int}} = \frac{1}{2} \int_{\mathbb{C}} (\partial_i W(\gamma_0)\gamma_2^i + \partial_i \partial_j W(\gamma_0)\gamma_1^i \wedge \gamma_1^j) \tag{3.11}$$

Adding this term to the free action the brackets that get modified with respect to Eq. (3.4) are the ones involving the β fields, becoming:

$$\{\mathrm{S}, \beta_0(x)\} = \frac{1}{2} W'(\gamma_0(x)) \tag{3.12}$$

$$\{\mathrm{S}, \beta_1(x)\} = d\gamma_0(x) + W''(\gamma_0(x))\gamma_1(x) \tag{3.13}$$

$$\{\mathrm{S}, \beta_2(x)\} = d\gamma_1(x) + \frac{1}{2}(W''(\gamma_0(x))\gamma_2(x) + W'''(\gamma_0(x))(\gamma_1(x))^2) \tag{3.14}$$

We are being a bit sloppy about the indices relative to the basis of X, for example, we should have written $\{\mathrm{S}, \beta_0^i(x)\} = \frac{1}{2}\partial_i W(\gamma_0(x))$, but in this way the difference with respect to the brackets of the topological $\beta\gamma$ system is more evident.

Now, let us consider the complex of observables at a point $x \in \mathbb{C}$. The differential is given by bracketing with the action and is composed of two parts: one coming from the free action and one coming from the interaction term. The differential associated to the free part is just the de Rham differential. This results in a filtration on the local observables by form degrees and, correspondingly, on a spectral sequence induced by this filtration. On the E_0 page, we take cohomology with respect to the free part of the differential. The E_1 page can thus be identified with the physical observables at a point of the topological $\beta\gamma$ system, but now with a non-zero differential:

$$\mathrm{Obs}_x = (\mathbb{C}[\gamma_0, \beta_0], \{W, -\}) \tag{3.15}$$

If we take cohomology with respect to this differential, we get the chiral Jacobi ring, as expected from the Landau-Ginzburg model:

$$\mathrm{Obs}_x = \mathbb{C}[\gamma_0]/\langle W'(\gamma_0)\rangle = \mathrm{Jac}(W) \tag{3.16}$$

[3]For a similar description, but in the holomorphic context see [SW20].

From the target space perspective, adding the superpotential to the topological $\beta\gamma$ system modifies the differential on the complex of fields (the local observables from the perspective we had up to now) into:

$$(\mathrm{PV}^{*,*}(X), \bar{\partial} + \iota_{dW}) \tag{3.17}$$

Notice that, when adding the superpotential, the complex of local observables becomes $\mathbb{Z}/2$-graded. We will discuss the reason behind this change in the grading later.

We can give a more geometric interpretation of these topological models. As we already said, the topological $\beta\gamma$ system is the cotangent theory to the moduli space of constant maps $\mathbb{R}\mathrm{Map}^{\mathrm{const}}(\mathbb{C}, X)$. We can also look at it from a different perspective: as a sigma model from the dg manifold $\mathbb{C}_{\mathrm{dR}}$, equipped with de Rham forms on $\mathbb{C}$ (which resolve locally constant functions on $\mathbb{C}$) as dg ring of functions, to X. Therefore, the $\beta\gamma$ system can be seen as the cotangent theory of the following sigma model:

$$T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{dR}}, X) \tag{3.18}$$

These represents the fields of our BV theory, and we can rewrite them as:

$$T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{dR}}, X) = \Omega^*(\mathbb{C}) \otimes X \oplus \Omega^*(\mathbb{C}) \otimes X^\vee[1] = \Omega^*(\mathbb{C}) \otimes (X \oplus X^\vee[1]) \tag{3.19}$$

and since $T^*[1]X = X \oplus X^\vee[1]$, we get that the topological $\beta\gamma$ system is a sigma model with target the derived manifold $T^*[1]X$. Explicitly:

$$T^*[-1]\mathrm{Map}(\mathbb{C}_{\mathrm{dR}}, X) = \mathrm{Map}(\mathbb{C}_{\mathrm{dR}}, T^*[1]X) \tag{3.20}$$

Here, $T^*[1]X$ is equipped with the standard ring of functions, i.e. symmetric powers of the linear dual, with no differential. What happens when we turn on the superpotential is that the ring of functions of $T^*[1]X$ gets deformed into:

$$(\mathscr{O}(T^*[1]X), \{W, -\}) \tag{3.21}$$

Notice that this is precisely the ring of functions of the derived critical locus of W (see Section 1.2). The only difference is in the grading: for the standard Koszul resolution, we consider the (-1)-shifted cotangent bundle. In turn, they are the same if we take the gradings modulo 2. This also explains a problem we did not stress before: if we look at the interaction term in the action (Eq. (3.11)), it is not of cohomological degree zero. But it is still of even degree, therefore modulo 2, we recover the usual grading for the action functional. Moreover, this explains why we take the complex of local observables (Eq. (3.17)) to be $\mathbb{Z}/2$-graded.

All in all, we can interpret the topological $\beta\gamma$ system with a superpotential turned on as a sigma model whose target is the derived critical locus of W:

$$\mathrm{Map}(\mathbb{C}_{\mathrm{dR}}, \mathrm{Crit}^{\mathrm{h}}(W)) \tag{3.22}$$

This recovers the familiar physical description of B-twisted Landau-Ginzburg models: the path integral localizes to constant maps into the stationary points of the superpotential ([HKK$^+$03], §16.4.2)!

If we think about the observables of this system, since we are considering a theory of constant maps, the space of observables is "as large as" the target. The moduli space encoded by these observables is their geometric incarnation, i.e. a smooth enhancement of Spec. In this case,

since W has isolated critical points, the underlying manifold (i.e. the spectrum of the degree zero cohomology) looks just like a "fat" point, i.e. topologically a point but with a dg ring of functions over it.
Interestingly, as we will see later, coupling this theory to topological gravity affects the size of this moduli space.

3.3 Coupling with topological gravity: target space perspective

What happens now if we couple the topological systems we discussed above with topological gravity? This is the standard procedure to construct a topological string theory, as was done, for example, in the seminal work of Dijkgraaf-Verlinde-Verlinde in [DVV91, DVV90]. In their approach, the matter sector is taken to be a two-dimensional topological conformal field theory (TCFT), obtained by performing the B-twist of a $\mathcal{N} = 2$ superconformal field theory.
The coupling with topological gravity is obtained by promoting the worldsheet metric to a dynamical field, but in such a way that it becomes exact with respect to some nilpotent fermionic charge. The term *topological* gravity then mirrors the terminology used by Witten in [Wit88] for topological quantum fields theories, where after twisting a supersymmetric theory, the stress energy tensor turns out to be exact.
In [DVV91, DVV90], they take a worldsheet perspective, and in Section 3.4 we follow their philosophy to formulate topological string theory using the BV formalism. Now, instead, we describe what happens when we couple a topological field theory with topological gravity from the target point of view, following the description given in [CL12] and [CL20].

Let us consider the topological $\beta\gamma$ system as described in Section 3.1. Recall that the observables of a BV theory from a worldsheet perspective are the fields from the target point of view. In particular, the net effect on the complex of fields of coupling with topological gravity is the following:

$$\begin{array}{c}(\mathrm{PV}^{*,*}(X), \bar\partial) \\ \Big\downarrow{\scriptstyle \text{coupling to topological gravity}} \\ (\mathrm{PV}^{*,*}(X)[|u|], \bar\partial + u\partial_\Omega)\end{array}$$

Here, u is a variable of cohomological degree $+2$ that represents gravitational descendants, while the divergence operator ∂_Ω is constructed as follows:

1. First notice that if we equip the algebra of forms $\Omega^{*,*}(X)$ with the differential $\bar\partial$, this becomes a differential bi-graded module for the differential bi-graded algebra $(\mathrm{PV}^{*,*}(X), \bar\partial)$. In particular, the module structure is given by the unique $\Omega^{0,*}(X)$-linear extension of the contraction map:
$$\mathrm{PV}^{k,0}(X) \otimes \Omega^{i,0}(X) \to \Omega^{i-k,0}(X) \tag{3.23}$$
$$\alpha \otimes \omega \mapsto \alpha \vee \omega \tag{3.24}$$

2. Let us choose a holomorphic volume form $\Omega \in \Omega^{n,0}(X)$. Since X is a Calabi-Yau manifold, this is always possible. In particular, contracting with the chosen volume form yields an isomorphism:
$$\mathrm{PV}^{i,j}(X) \cong \Omega^{n-i,j}(X) \tag{3.25}$$

3. The operator ∂_Ω on $\mathrm{PV}^{*,*}(X)$ corresponds to the holomorphic de Rham differential ∂ on $\Omega^{*,*}(X)$ under the isomorphism $\mathrm{PV}^{i,j}(X) \cong \Omega^{n-i,j}(X)$. More explicitly, it is defined as:

$$\partial_\Omega : \mathrm{PV}^{i,j}(X) \to \mathrm{PV}^{i-1,j}(X) \tag{3.26}$$

by the formula:

$$(\partial_\Omega \alpha) \vee \Omega = \partial(\alpha \vee \Omega) \tag{3.27}$$

Notice that the cohomological degree of an element $\alpha \in \mathrm{PV}^{i,j}(X)$ is $|\alpha| = i + j$, therefore ∂_Ω is an operator of degree -1.

We see that, at the level of the complex of fields, coupling with topological gravity means turning on gravitational descendants and modifying the differential. But why is this the case? The reason was already highlighted in [DVV90], but is made more precise in [Get94b].

Passing from a topological field theory to a topological string theory changes the definition of physical observables. If, for a topological field theory of cohomological type as the B-model we are considering, the physical observables are just given by some cohomology classes, when considering a topological string theory we should consider equivariant cohomology with respect to a S_1-action. In fact, the complex $(\mathrm{PV}^{*,*}(X)[|u|], \bar{\partial} + u\partial_\Omega)$ is the complex whose cohomology is the equivariant cohomology of $(\mathrm{PV}^{*,*}(X), \bar{\partial})$.
It remains to show where the equivariance condition comes from: to do that, we first review how equivariant cohomology is defined.

Digression: equivariant cohomology

In order to get a better understanding, it is easier to see how equivariant cohomology works in the case of the de Rham complex (we follow mainly [Get94b]).

Consider a manifold M equipped with a S_1-action. If Y is the vector field that generates the circle action, then the de Rham complex $(\Omega^*(M), d)$ carries an action of the Lie derivate $L_Y : \Omega^*(M) \to \Omega^*(M)$. From the Cartan homotopy formula, we know that:

$$L_Y = [d, \iota_Y] \tag{3.28}$$

where ι_Y is the contraction by Y.

In the case of equivariant cohomology, we do not want to consider just closed differential forms but also invariant forms under the action of the Lie derivative. Therefore, we could define equivariant cohomology as the de Rham cohomology of the basic subcomplex $\Omega^*(M)_{\text{basic}}$ – that is, differential forms on which L_Y and ι_Y vanish.
This is almost correct: the problem is that this does not give a well-defined cohomology theory; indeed, it does not satisfy, for instance, the Mayer-Vietoris sequence[4]. The idea is then to define a complex that has the same cohomology of the basic subcomplex.

Let us denote by $\mathfrak{h}$ the Lie algebra of the circle group. We define the Weil complex of $\mathfrak{h}$ as:

$$\mathcal{W}(\mathfrak{h}) := (\mathrm{Sym}^*_{\mathbb{K}}(\mathfrak{h}^\vee[-1] \oplus \mathfrak{h}^\vee[-2]), \delta) \tag{3.29}$$

In our case, the Weil complex has two generators: ω in degree $+1$, and u in degree $+2$. The differential δ acts on these generators as $\delta\omega = u$ and $\delta u = 0$, and is then extended as a derivation.

[4]Every well-defined cohomology theory shares some properties that are collected in the Eilenberg-Steenrod axioms. In particular, the axiomatic approach allows one to prove results, such as the Mayer–Vietoris sequence, that are common to all cohomology theories satisfying the axioms. For more on the axiomatic approach, see, for instance, [Hat02], §2.3.

Very importantly, the Weil complex is contractible, i.e. homotopy equivalent to the zero complex and therefore the inclusion of complexes $(\Omega^*(M), d) \hookrightarrow (\mathcal{W}(\mathfrak{h}) \otimes \Omega^*(M), \delta + d)$ induces an isomorphism of cohomology. To prove this, it is sufficient to consider the contracting homotopy h that acts on the Weil complex as:

$$hu^n = n\omega u^{n-1} \qquad h\omega u^n = 0 \tag{3.30}$$

Notice that these are the only terms we have in the Weil complex: indeed, since ω is of degree $+1$, and therefore an anticommuting variable, we cannot have more than one power of ω.

Now, if we consider the derivation ι on $\mathcal{W}(\mathfrak{h})$ that acts on generators by $\iota\omega = 1$ and $\iota u = 0$, the action of $\iota + \iota_Y$ on $\mathcal{W}(\mathfrak{h}) \otimes \Omega^*(M)$ is exact. So, as a definition of equivariant cohomology, we can take the cohomology of the basic subcomplex of $\mathcal{W}(\mathfrak{h}) \otimes \Omega^*(M)$, on which L_Y and $\iota + \iota_Y$ vanish.
More explicitly, the kernel of $\iota + \iota_Y$ is given by:

$$\ker(\iota + \iota_Y) = \{\alpha - \omega(\iota_Y \alpha) \mid \alpha \in \Omega^*(M)[u]\} \tag{3.31}$$

Taking an element of the above kernel and applying the differential, we get:

$$\begin{aligned}(\delta + d)(\alpha - \omega(\iota_Y\alpha)) &= d\alpha - u(\iota_Y\alpha) - \omega d(\iota_Y\alpha) - \omega\iota_Y(d\alpha) + \omega\iota_Y(d\alpha) \\ &= (1 + \omega\iota_Y)(d - u\iota_Y)\alpha - \omega L_Y\alpha\end{aligned} \tag{3.32}$$

Thus, we can identify the basic subcomplex with the following one:

$$\{\Omega^*(M)(0)[u], d - u\iota_Y\} \tag{3.33}$$

where the differential forms in $\Omega^*(M)(0)$ are the ones invariant under the action of the Lie derivative, i.e. the ones satisfying $L_Y\alpha = 0$.

This construction may be defined on any complex $\mathcal{C}$, with operators ι of degree -1 and L such that $L = [d, \iota]$ and $\iota^2 = 0$. In such cases, the equivariant cohomology $H^*_{S_1}(\mathcal{C})$ is the cohomology of the differential $d - u\iota$ on $\mathcal{C}(0)[u]$, where $\mathcal{C}(0)$ is now the kernel of L.

In summary, one way of computing equivariant cohomology is to construct a new complex such that the cohomology is the equivariant cohomology of the original one.

We now explain why one needs to compute equivariant cohomology for the local observables of the $\beta\gamma$ system when coupling to topological gravity, and why $(\mathrm{PV}^{*,*}(X)[|u|], \bar{\partial} + u\partial_\Omega)$ is the right complex.

As clarified in [CL20], not all local operators of the topological field theory (the $\beta\gamma$ system) can be modifications of the closed-string background, but we should only consider operators that are invariant under worldsheet reparametrizations.
Since the worldsheet theory is a TFT, small reparametrizations act trivially up to homotopy on the space of local operators. To formalize this, consider inserting a local operator and cutting out a small disk $D \subset \mathbb{C}$ around the insertion point. The group of orientation-preserving diffeomorphism of the disk $\mathrm{Diff}_+(D)$ acts as the group of local reparametrizations of the worldsheet. Therefore, asking for invariant physical operators under worldsheet reparametrizations means taking the equivariant cohomology with respect to $\mathrm{Diff}_+(D)$. But it is a classical result of Smale [Sma59] that $\mathrm{Diff}_+(D)$ is homotopically equivalent to the circle: invariance under local

worldsheet reparametrizations translates into taking S^1-equivariant cohomology of the complex of local observables.

Concretely, the action of $\mathrm{Diff}_+(D)$ on the observables $\mathrm{Obs}_x(\mathscr{F}_{\beta\gamma})$ (as given in Eq. (3.8)) is given by a map $\rho : \mathfrak{h} \to \mathrm{End}(\mathrm{Obs}_x(\mathscr{F}_{\beta\gamma}))$, where $\mathfrak{h}$ is the Lie algebra of the circle. Since $\mathfrak{h}$ is one-dimensional, and denoting by Y a generator of $\mathfrak{h}$, we can just consider the endomorphism given by:

$$\rho_Y : \mathrm{Obs}_x(\mathscr{F}_{\beta\gamma}) \to \mathrm{Obs}_x(\mathscr{F}_{\beta\gamma}) \tag{3.34}$$

But since small reparametrizations should act trivially up to homotopy, ρ_Y must be homotopically trivial – i.e., there must exist a degree -1 operator ι such that:

$$\rho_Y = [\bar{\partial}, \iota] \tag{3.35}$$

This reproduces precisely the situation we were in when defining equivariant cohomology. In particular, in our case, $\iota = \partial_\Omega$ and therefore $\rho_Y = 0$.

Putting all together, we see how the complex for computing equivariant cohomology of the local observables of the B-model is indeed given by:

$$(\mathrm{PV}^{*,*}(X)[|u|], \bar{\partial} + u\partial_\Omega) \tag{3.36}$$

This is the complex defined by Costello and Li to describe the BV fields of the BCOV model.

We have just seen how Costello and Li justify the equivariance condition, and we derived the complex of fields we would like to compare with the observables of our worldsheet model for topological string theory. However, the physical significance of this condition is perhaps more evident in the work of Dijkgraaf-Verlinde-Verlinde in [DVV91, DVV90], and again emphasized in [EKYY93]. Now, we briefly review their construction of a topological string theory to show the physical consequences of this equivariance condition.

To construct a topological string theory, the standard procedure is to couple a topological conformal field theory with topological gravity. In [DVV90] §5.1, topological gravity is obtained by the familiar BRST gauge-fixing procedure in physics. In particular, worldsheet gravity is introduced as a metric on the worldsheet with its superpartner. Therefore, the symmetries to gauge are super-diffeomorphism invariance and super-Weyl rescaling.

In order to gauge fix super-diffeomorphism invariance, the condition chosen is the superconformal gauge: the metric (and its superpartner) should be proportional to the flat metric up to a conformal factor. Instead, to gauge fix super-Weyl transformations, the condition chosen is that the curvature of the metric on the worldsheet is given by some fixed background curvature, while the curvature of the superpartner is just zero.
To implement these fixing conditions through the BRST procedure, we should add ghost fields and, in particular, the topological gravity sector of the action is constructed out of these ghost fields. The superconformal gauge is implemented through the familiar $\beta\gamma$ and bc systems of superstring theory, whereas the condition chosen to fix super-Weyl transformations is implemented through the so-called Liouville fields.

We were very fast in presenting this construction: the reason is that the details are not important for our purposes. What is important for us is that all the sectors of topological string theory (matter + ghosts + Liouville) have the structure of a TCFT. This means that the state space of every sector is a topological Virasoro module (as defined in [Get94b] §2), which means that it is acted on by the topological Virasoro algebra. This algebra is generated by a stress energy tensor $L(z)$ of spin 2, an anomalous $U(1)$ current $J(z)$, and two fermionic currents $G^+(z)$ and $G^-(z)$ that have spin 2 and spin 1 respectively. In particular, $L(z)$ generates the Witt algebra, and we can think of the topological Virasoro algebra as the twist (for our purposes, we always consider the B-twist) of a $\mathcal{N} = 2$ superconformal algebra done by modifying the stress energy tensor by mixing it with the $U(1)$ current. If we expand every current in Fourier modes, we get the explicit commutation relations ($n, m \in \mathbb{Z}$):

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} & [J_m, J_n] &= dm\delta_{m+n,0} \\
[L_m, G^+_n] &= (m-n)G^+_{m+n} & [J_m, G^+_n] &= -G^+_{m+n} \\
[L_m, G^-_n] &= -nG^-_{m+n} & [J_m, G^-_n] &= G^-_{m+n} \\
[G^+_m, G^-_n] &= L_{m+n} + nJ_{m+n} + \frac{1}{2}dm(m+1)\delta_{m+n,0} \\
[L_m, J_n] &= -nJ_{m+n} - \frac{1}{2}dm(m+1)\delta_{m+n,0} \\
[G^+_m, G^+_n] &= 0 & [G^-_m, G^-_n] &= 0
\end{aligned} \tag{3.37}$$

Here, d is the central charge of the untwisted $\mathcal{N} = 2$ superconformal algebra. This is why $J(z)$ is an anomalous $U(1)$ current: it is coupled to a background charge.

The state space of a TCFT is a complex, where the grading is given by J_0, and we call the differential Q_{tot}. This differential is composed of two parts: the supercharge coming from the supersymmetry transformations that after the twist Q_S becomes a scalar and the BRST charge Q_{BRST} that implements the symmetries. In this case, the scalar supercharge is the integral of G^-, while to construct the BRST charge, we should integrate a combination of the stress energy tensor and its superpartner, i.e. the BRST current.

Let us give some details (for more, see [DVV90] §5.1 & §5.2). The conserved currents of the ghost sector are given by:

$$\begin{aligned}
L_{gh} &= c\partial b + 2\partial c b + \gamma\partial\beta + 2\partial\gamma\beta & J_{gh} &= bc + 2\beta\gamma \\
G^+_{gh} &= c\partial\beta + 2\partial c\beta & G^-_{gh} &= b\gamma
\end{aligned} \tag{3.38}$$

Here, b and c are the anticommuting ghosts of spin 2 and -1, while β and γ are their commuting supersymmetric partners of the same spin.
Instead, the Liouville sector is realized in terms of free fields (π and χ are just Lagrange multipliers to implement the gauge fixing condition, while ψ and φ are supersymmetric partners to each other). The conserved currents are given by:

$$\begin{aligned}
L_L &= \partial\pi\partial\varphi + \partial^2\pi + \partial\chi\partial\psi & J_L &= \psi\partial\chi + \partial\varphi \\
G^+_L &= \partial\chi\partial\varphi + \partial^2\chi & G^-_L &= \psi\partial\pi + \partial\psi
\end{aligned} \tag{3.39}$$

The supercharges are then given by:

$$\begin{aligned}
Q_{S,TOT} &= \oint G^-_m + G^-_{gh} + G^-_L \\
Q_{BRST,TOT} &= \oint c[L_{m+L} + \frac{1}{2}L_{gh}] + \gamma[G^+_{m+L} + \frac{1}{2}G^+_{gh}]
\end{aligned} \tag{3.40}$$

where with the lower index m we mean the conserved currents of the matter system.

In the end, the physical states belong to the cohomology of $Q_{S,TOT} + Q_{BRST,TOT}$, but with an important additional condition: they should also be annihilated by L_0 (the zero mode of the energy stress tensor of the matter TCFT) and b_0, that is, the zero mode of the ghost b. This is the equivariance condition we were talking about[5]! Without this condition, there are no non-trivial pure topological gravity states (the ones constructed out of the ghost and Liouville sectors) and therefore the topological gravity sector completely decouples from the matter sector. Instead, the non-triviality of the interaction between matter and topological gravity is precisely encoded by this condition.

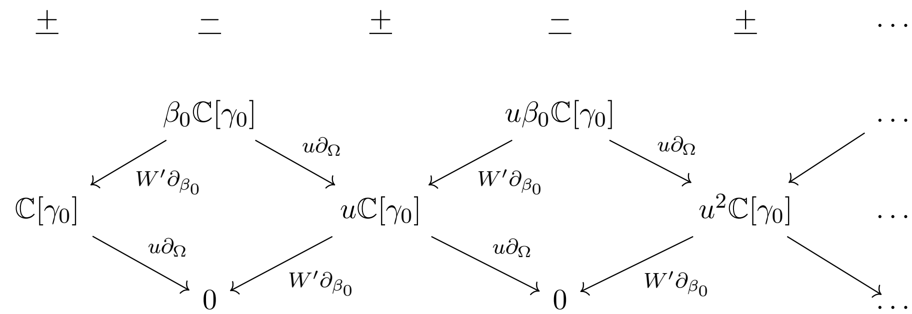
Why does it resemble an S^1-equivariant cohomology condition? Well, looking at the explicit commutation relations in Eq. (3.37), we see that for the matter system, $L_0 = [G_0^-, G_0^+]$. However, since G_0^- is our differential and G_0^+ is an operator of degree -1, we recover Eq. (3.28). The same happens when including the other sectors to get the condition for b_0 (for a rigorous treatment, see [Get94b] §2 & §3).

The fact that the topological gravity sector does not decouple from the matter sector can also be seen by an explicit computation. Let us consider the topological Landau-Ginzburg model (as described in Section 3.2) now coupled with topological gravity. The complex of fields from the target point of view is given by:

$$(\mathrm{PV}^{*,*}(X)[|u|], \bar{\partial} + u\partial_\Omega + \iota_{dW}) \tag{3.41}$$

Recall that when the superpotential is introduced, the BV complex becomes $\mathbb{Z}/2$-graded.

Let us now compute cohomology: we will do it in steps using the same spectral sequence argument given in Section 3.2. If we first compute cohomology with respect to the $\bar{\partial}$ operator and use the identification of Eq. (3.6), we get the remaining complex:



Now, let us take cohomology w.r.t. the $u\partial_\Omega$ operator. After doing the identification of Eq. (3.6), ∂_Ω has the common form of a BV Laplacian, that is, $\partial_\Omega = \frac{\partial^2}{\partial\gamma_0\partial\beta_0}$. The explicit action on elements is then given by:

$$\beta_0\gamma_0^n \mapsto u\gamma_0^{n-1} \qquad u\beta_0 \mapsto 0 \qquad u\gamma_0 \mapsto 0 \tag{3.42}$$

[5] Notice that throughout this discussion we just considered the chiral sector of the state space. To complete the discussion, we should also consider the antichiral sector. In the end, it does not add much; just the equivariance condition on a state $|v\rangle$ would read $(L_0 - \bar{L}_0)|v\rangle = (b_0 - \bar{b}_0)|v\rangle = 0$, where the bar indicates the antichiral operators.

Taking cohomology, we then get the following:

$$
\begin{array}{cccccc}
\underline{+} & \underline{-} & \underline{+} & \underline{-} & \underline{+} & \cdots \\
 & \beta_0 & & u\beta_0 & & \cdots \\
 & \swarrow W'\partial_{\beta_0} & & \swarrow W'\partial_{\beta_0} & & \\
\mathbb{C}[\gamma_0] & & 0 & & 0 & \cdots \\
 & \swarrow W'\partial_{\beta_0} & & \swarrow W'\partial_{\beta_0} & & \\
 & 0 & & 0 & & \cdots
\end{array}
$$

If we now take cohomology w.r.t. the differential induced by turning on the superpotential, we get the chiral jacobi ring, but not only! This is what we were hinting at the end of Section 3.2: the size of the moduli space encoded by the local observables of topological Landau-Ginzburg model changes after coupling with topological gravity.

3.4 Topological string theory from the worldsheet

In this section, we try to give a worldsheet perspective on topological string theory, and our goal is to check if the observables of our theory agree with the fields defined by Costello and Li (see Eq. (3.36)).
Our construction is very similar to the one in [DVV91, DVV90] in spirit: as a matter system, we too consider a TCFT that emerges as the B-twist of a superconformal field theory, and then we couple this theory with topological gravity.
In our approach, however, the coupling with worldsheet gravity will emerge naturally as the coupling of two formal moduli problems: *making the metric a dynamical variable* translates into gauging a background symmetry that acts on the matter system.

This section is divided as follows: in 3.4.1, we describe the field content of the untwisted matter system and its symmetry; in 3.4.2, we perform the topological twist on the coupled system, and in 3.4.3, we finally describe the full theory.

3.4.1 The matter system and its symmetry

Let us first describe the matter system of interest. We consider the theory of holomorphic maps from a super Riemann surface to a complex vector space X of dimension n. For simplicity, we define our theory on $\mathbb{C}^{1|1}$, where the second upper index indicates the number of odd directions. Our matter system will then be the cotangent theory to the moduli space of holomorphic maps from $\mathbb{C}^{1|1}$ to X, which we denote as $\mathrm{Map}^{\mathrm{hol}}(\mathbb{C}^{1|1}, X)$.
Here, we treat the odd direction of $\mathbb{C}^{1|1}$ purely algebraically. This means that if we have a map $\gamma : \mathbb{C}^{1|1} \to X$ and we introduce a parameter ε of cohomological degree $+1$, we can look at γ as a function on $\mathbb{C}$ with values in $X \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$. With this convention:

$$\mathrm{Map}^{\mathrm{hol}}(\mathbb{C}^{1|1}, X) = \mathrm{Map}^{\mathrm{hol}}(\mathbb{C}, X \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]) \tag{3.43}$$

From the physical point of view, we are considering n chiral superfields. We could also add the antichiral superfields, described by antiholomorphic maps, but this does not add much to the present discussion.

The BV fields are given by a local Lie algebra that describes a formal neighbourhood of the moduli space (3.43). In particular, very similarly to what we did in Section 2.3, we want to

$$
\begin{array}{ccccccc}
\underline{-1} & & \underline{0} & & \underline{1} & & \underline{2} \\
 & & \Omega^{0,0}(\mathbb{C},V) & \xrightarrow{\bar\partial} & \Omega^{0,1}(\mathbb{C},V) & & \\
 & & & & \varepsilon\Omega^{0,0}(\mathbb{C},V) & \xrightarrow{\bar\partial} & \varepsilon\Omega^{0,1}(\mathbb{C},V) \\
 & & \Omega^{1,0}(\mathbb{C},V^\vee) & \xrightarrow{\bar\partial} & \Omega^{1,1}(\mathbb{C},V^\vee) & & \\
\varepsilon^\vee\Omega^{1,0}(\mathbb{C},V^\vee) & \xrightarrow{\bar\partial} & \varepsilon^\vee\Omega^{1,1}(\mathbb{C},V^\vee) & & & &
\end{array}
\quad
\begin{array}{l}
\left.\vphantom{\begin{array}{c}a\\a\end{array}}\right\}\gamma \\
\left.\vphantom{\begin{array}{c}a\\a\end{array}}\right\}\beta
\end{array}
$$

Figure 3.2: The BV complex of the matter system $\mathcal{M}$

describe the derived enhancement of the cotangent theory to this moduli space, and to resolve the holomorphic constraint we take the Dolbeaut resolution. The global sections of the complex of BV fields are then given by (the complex is also explicitly depicted in Figure 3.2):

$$\mathcal{M}(\mathbb{C}) = (T^*[-1](\Omega^{0,*}(\mathbb{C},X)[\varepsilon]),\bar\partial) = (\Omega^{0,*}(\mathbb{C},X)[\varepsilon] \oplus \Omega^{1,*}(\mathbb{C},X^\vee)[\varepsilon^\vee],\bar\partial) \tag{3.44}$$

This is really the holomorphic $\beta\gamma$ system we described in Section 2.3, now considering also the superpartners to each field.
Notice that the space of fields is just a cochain complex: it does not have a Lie bracket or higher brackets. Also in this case, then, we are studying a free theory.

The moduli space we want to describe, from a field theory perspective, should arise as a variational problem: the equations of motion of the fields should impose the holomorphicity condition. Therefore, the action is given by:

$$\int_{\mathbb{C}} \langle\beta,\bar\partial\gamma\rangle_X \tag{3.45}$$

where $\langle -,-\rangle_X$ is the pairing of cohomological degree -1 given by the wedge product on forms and the evaluation pairing on $X\otimes_{\mathbb{C}}\mathbb{C}[\varepsilon]$ (the explicit form is very similar to Eq. (3.3)).

Let us now see how gravity enters the game. As mentioned earlier, the guiding idea is to gauge a background symmetry that acts on the total space of fields $\mathcal{M}$, much like we did in the case of the holomorphic bosonic string (see Section 2.3).
Before carrying this out, however, we need to clarify three things: what the relevant symmetry is, how it acts on the matter system, and why this symmetry can be interpreted as gravity. In our description of the symmetry, we closely follow [SW23].

The matter system admits an action of $\mathrm{Vect}(1|1)$, our notation for the Lie algebra of $\mathcal{N}=2$-extended holomorphic vector fields on the punctured plane $\mathbb{C}^\times = \mathbb{C}\setminus\{0\}$. The reason why we call them "$\mathcal{N}=2$-extended" is that $\mathrm{Vect}(1|1)$ is isomorphic (as shown in [KL88]) to $K(1|2)$, the Lie algebra of contact vector fields defined on $\mathbb{C}^{1|2}$.
In what follows, we only need to deal with $\mathrm{Vect}(1|1)$, so we do not spell out all the properties of $K(1|2)$. However, this identification is conceptually important: a theory is said to be $\mathcal{N}=k$ supersymmetric if it admits an action of $K(1|k)$. Thus, our matter system is a $\mathcal{N}=2$ supersymmetric theory.
More precisely, we say that it is $\mathcal{N}=2$ superconformal (echoing the description of [DVV91]), and

the reason is that Vect(1|1) $\cong K(1|2)$ plays the role that the classical Virasoro or Witt algebra (see Example 3, Section 2.2.1) plays in the bosonic case. Just as the Witt algebra consists of holomorphic vector fields on the punctured plane, here we consider a natural supersymmetric extension.
As a vector space, Vect(1|1) can be thought of as:

$$\Gamma^{\mathrm{hol}}(\mathbb{C}^{1|1}, T\mathbb{C}^{1|1}) := \Gamma^{\mathrm{hol}}(\mathbb{C}^{\times}, T^{1,0}\mathbb{C}^{\times})[\varepsilon] \oplus \mathscr{O}^{\mathrm{hol}}(\mathbb{C}^{\times}) \otimes_{\mathbb{C}} \mathrm{Der}(\mathbb{C}[\varepsilon]) \tag{3.46}$$

where $\mathscr{O}^{\mathrm{hol}}(\mathbb{C}^{\times})$ denotes holomorphic functions on $\mathbb{C}^{\times}$, and with "Der" we mean derivations (schematically: $\mathrm{Der}(\mathbb{C}[\varepsilon]) = \mathbb{C}[\varepsilon]\partial_{\varepsilon}$).

To be compatible with the BV formalism, we want to think of Vect(1|1) as a local Lie algebra, as explained in Section 2.2.1. This involves resolving the holomorphic constraints using the Dolbeaut resolution and treating it as a sheaf of dg Lie algebras. Taking global sections of this local Lie algebra over $\mathbb{C}^{\times}$, and restricting to degree zero cohomology recovers the classical Lie algebra of Eq. (3.46).
As a local Lie algebra, we describe Vect(1|1) over open sets $U \cong \mathbb{C}$ – and, by abuse of notation, simply write them as defined on $\mathbb{C}$. The resulting complex is:

$$\mathscr{L} = (\Omega^{0,*}(\mathbb{C}, T^{1,0}\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]) \bowtie (\Omega^{0,*}(\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Der}(\mathbb{C}[\varepsilon])) \tag{3.47}$$

Here, " $\bowtie$ "denotes the direct sum of the two differential graded vector spaces but equipped with a different Lie algebra structure; indeed, we also want to consider the action of a summand on the other one. Concretely, the dg Lie algebra structure can be described as follows:

- the differential is $\bar{\partial}$ on both summands of the above decomposition;
- the Lie bracket on $\Omega^{0,*}(\mathbb{C}, T^{1,0}\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$ is obtained from tensoring the usual Lie bracket on vector fields with the graded commutative product on $\mathbb{C}[\varepsilon]$;
- the Lie bracket on the other summand is given by tensoring the wedge product on differential forms with the Lie bracket on derivations on $\mathbb{C}[\varepsilon]$;
- the remaining brackets are through the Lie derivate of holomorphic vector fields on $\mathbb{C}$ and the action of $\mathrm{Der}(\mathbb{C}[\varepsilon])$ on $\mathbb{C}[\varepsilon]$.

Diagrammatically, we can represent $\mathscr{L}$ as in Figure 3.3, where, by abuse of notation, $\mathscr{O}^{\mathrm{hol}}(\mathbb{C}) = (\Omega^{0,*}(\mathbb{C}), \bar{\partial})$. Notice that in this case, we are adopting the L_∞ convention for the cohomological degree.

$\underline{-1}$	$\underline{0}$	$\underline{1}$
	$\mathscr{O}^{\mathrm{hol}}(\mathbb{C})\partial_z$	
$\mathscr{O}^{\mathrm{hol}}(\mathbb{C})\partial_{\varepsilon}$		$\mathscr{O}^{\mathrm{hol}}(\mathbb{C})\varepsilon\partial_z$
	$\mathscr{O}^{\mathrm{hol}}(\mathbb{C})\varepsilon\partial_{\varepsilon}$	

Figure 3.3: The underlying complex of Vect(1|1) as a local Lie algebra

How does this symmetry act on the matter system $\mathcal{M}$? A possible guess is by an $\Omega^{0,*}(\mathbb{C})$-linear extension of the Lie derivative of holomorphic vector fields on $\mathbb{C}$ and by the natural action of

$\mathrm{Der}(\mathbb{C}[\varepsilon])$ on $\mathbb{C}[\varepsilon]$. To be more explicit, let us give two examples (all the other cases are very similar).

I. If $\gamma \in \Omega^{0,0}(\mathbb{C}, V) \subset \mathcal{M}$ and $c_z \in (\Omega^{0,1}(\mathbb{C}, T^{1,0}\mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]) \subset \mathscr{L}$, given $\{e_i\}_{i\in\mathbb{N}}$ a basis of V, locally on $\mathbb{C}$ we have: $\gamma = \gamma^i(z, \bar{z})e_i$ and $c_z = \varepsilon c_z(z, \bar{z})d\bar{z} \otimes_{\mathbb{C}} \partial_z$. The action of c_z on γ then is:

$$L_{c_z}\gamma := \varepsilon c_z(z, \bar{z})\frac{\partial\gamma^i(z, \bar{z})}{dz}d\bar{z} \otimes_{\mathbb{C}} e_i \in \varepsilon\Omega^{0,1}(\mathbb{C}, V) \subset \mathcal{M} \tag{3.48}$$

II. If $\gamma \in \varepsilon\Omega^{0,0}(\mathbb{C}, V) \subset \mathcal{M}$ and $c_\varepsilon \in (\Omega^{0,1}(\mathbb{C}) \otimes_{\mathbb{C}} \mathrm{Der}(\mathbb{C}[\varepsilon])) \subset \mathscr{L}$, we locally have that: $\gamma = \varepsilon\gamma^i(z, \bar{z})e_i$ and $c_\varepsilon = \varepsilon c_\varepsilon(z, \bar{z})d\bar{z} \otimes_{\mathbb{C}} \partial_\varepsilon$. Thus:

$$c_\varepsilon \cdot \gamma := \varepsilon c_\varepsilon(z, \bar{z})\gamma^i(z, \bar{z})d\bar{z} \otimes_{\mathbb{C}} e_i \in \varepsilon\Omega^{0,1}(\mathbb{C}, V) \subset \mathcal{M} \tag{3.49}$$

It is possible to check that this action of $\mathscr{L}$ on $\mathcal{M}$ gives a homotopy-coherent map $\rho : \mathscr{L} \rightsquigarrow \mathrm{End}(\mathcal{M}[-1])$, which in this case is just a strict dg Lie algebra map. The calculations are very similar to the ones we did in Section 2.3. We conclude that $\mathscr{L}$ is really a symmetry of our matter system.

As explained in Section 2.2.2, gauging the local Lie algebra $\mathscr{L}$ means considering the new formal moduli problem of the coupled system $\mathscr{L} \ltimes \mathcal{M}$. Since $\mathscr{L}$ is a supersymmetric extension of the Witt algebra – which governs deformations of complex structures – its interpretation closely parallels that given for the holomorphic bosonic string in Section 2.3: *making the metric a dynamical variable* corresponds to moving in the moduli space of Riemann surfaces. However, unlike the holomorphic bosonic string, with $\mathscr{L}$ we will also introduce superpartners to each gauged field: these are needed to perform the twist and obtain topological gravity.

Before giving a description of $\mathscr{L}$ in terms of fields and making explicit the coupling terms in the action functional of the full theory, in 3.4.2 we explain what performing the topological twist actually means.

3.4.2 Performing the topological twist

At the level of a BV theory, performing a twist means deforming the BV differential. In the language of formal moduli problems, it is clear why: we need to select an element of cohomological degree +1, that satisfies the Maurer-Cartan (MC) equation – that is, a Maurer-Cartan element. The latter is responsible for deformations of the underlying dg Lie algebra structure.
In our case, we first apply this twist to the symmetry algebra $\mathscr{L}$, and since we consider a coupled system, the matter system $\mathcal{M}$ gets twisted accordingly.
Coupling the twisted $\mathscr{L}$ with the twisted $\mathcal{M}$ can then be interpreted as *coupling the matter system with topological gravity*, recovering the description of [DVV91].

Looking at the explicit form of $\mathscr{L}$ (see Figure 3.3), we see that we do not have many possibilities to choose a MC element: it is natural to consider $\varepsilon\partial_z$. In fact, this is a degree +1 element and satisfies the MC equation, which in this case is just $[\varepsilon\partial_z, \varepsilon\partial_z] = 0$. To compare with the usual physics literature on topological twist, $\varepsilon\partial_z$ is the fermionic nilpotent supercharge with respect to which we perform the twist.

How does $\varepsilon\partial_z$ act on a generic element of $\mathscr{L}$? Schematically, if $f(z)$ is a holomorphic function:

$$\begin{aligned} [\varepsilon\partial_z, f(z)\partial_\varepsilon] &= \frac{\partial f(z)}{dz}\varepsilon\partial_\varepsilon + f(z)\partial_z \qquad & [\varepsilon\partial_z, f(z)\partial_z] &= \frac{\partial f(z)}{dz}\varepsilon\partial_z \\ [\varepsilon\partial_z, f(z)\varepsilon\partial_\varepsilon] &= f(z)\varepsilon\partial_\varepsilon \qquad & [\varepsilon\partial_z, f(z)\varepsilon\partial_z] &= 0 \end{aligned} \tag{3.50}$$

Thus, Figure 3.3 gets the following arrows:

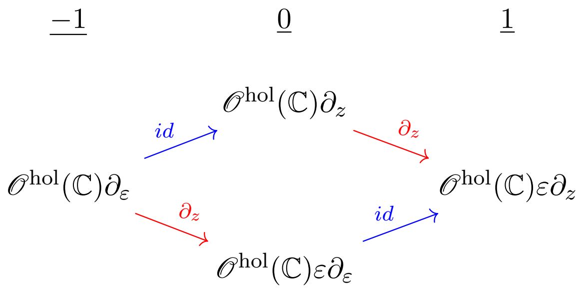


Figure 3.4: Performing the topological twist on $\mathscr{L}$

We denote the blue arrows by id and the red arrows by ∂_z because of the action of $[\varepsilon\partial_z, -]$ on the holomorphic function $f(z)$.

As remarked above, what really happens in the twist is a deformation of the original differential in the BV complex. In this case, the Dolbeaut operator $\bar{\partial}$, gets modified as:

$$\bar{\partial} \to \bar{\partial} + [\varepsilon\partial_z, -] \tag{3.51}$$

Since we are considering a coupled system (even though we did not make the coupling explicit yet), we should also perform the topological twist on the matter system. This modifies $\mathcal{M}$ (originally as in Figure 3.2) as follows:

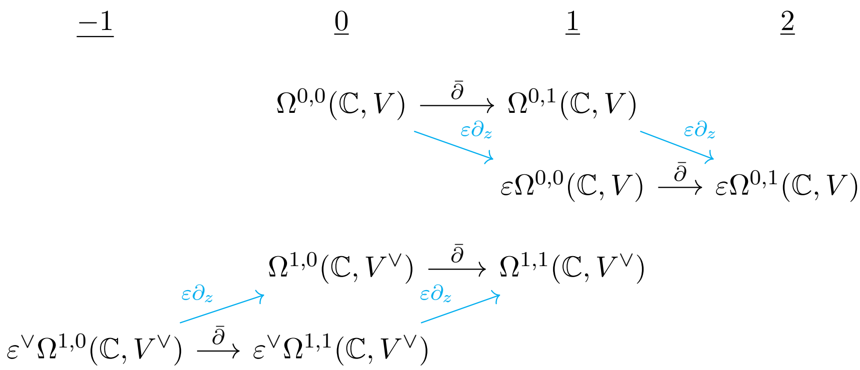


Figure 3.5: Performing the topological twist on $\mathcal{M}$

Notice that, under the identifications $\varepsilon = dz$ and $\varepsilon^\vee = \partial_z$, the operator $\varepsilon\partial_z$ becomes ∂, the holomorphic part of the de Rham differential. Thus, performing the topological twist on $\mathcal{M}$ corresponds to deforming the differential $\bar{\partial}$ to the full de Rham differential: $\bar{\partial} \mapsto \bar{\partial} + \partial = d$.
Taking the totalization of the complexes in Figure 3.5, we get two copies of the de Rham complex, which recovers the BV description of the topological $\beta\gamma$ system, as described in Section 3.1 (see Figure 3.1).
We see that, after the twist, the matter system becomes a topological field theory. Since it arises from the twist of a $\mathcal{N} = 2$ superconformal field theory, it defines a topological conformal field theory, as described in [DVV91].

3.4.3 The full theory

We are now ready to describe the full theory. In the BV approach, given an appropriate action functional, all the relations among the fields are encoded in the BV differential $\{S_{BV}, -\}$. Before writing the explicit form of the total action, we need to give a description of $\mathscr{L}$ in terms of fields and antifields. This will allow us to have an action associated with $\mathscr{L}$ that reflects all its structure as a dg Lie algebra.

To this end, we construct the BF theory associated with $\mathscr{L}$, mimicking what we have done in the case of the holomorphic bosonic string in Section 2.3. Thus, the total space of fields is given by the following shifted cotangent bundle: $\mathscr{L}[1] \oplus \mathscr{L}^![-2]$. Here, $\mathscr{L}^!$ denotes the shriek dual of $\mathscr{L}$, and those shifts are made to ensure the correct BV grading.

The total BV complex is depicted in Figure 3.6. Notice that here we are already making the identifications we mentioned in 3.4.2 ($\varepsilon = dz$ and $\varepsilon^\vee = \partial_z$), allowing us to write the complex more concisely.

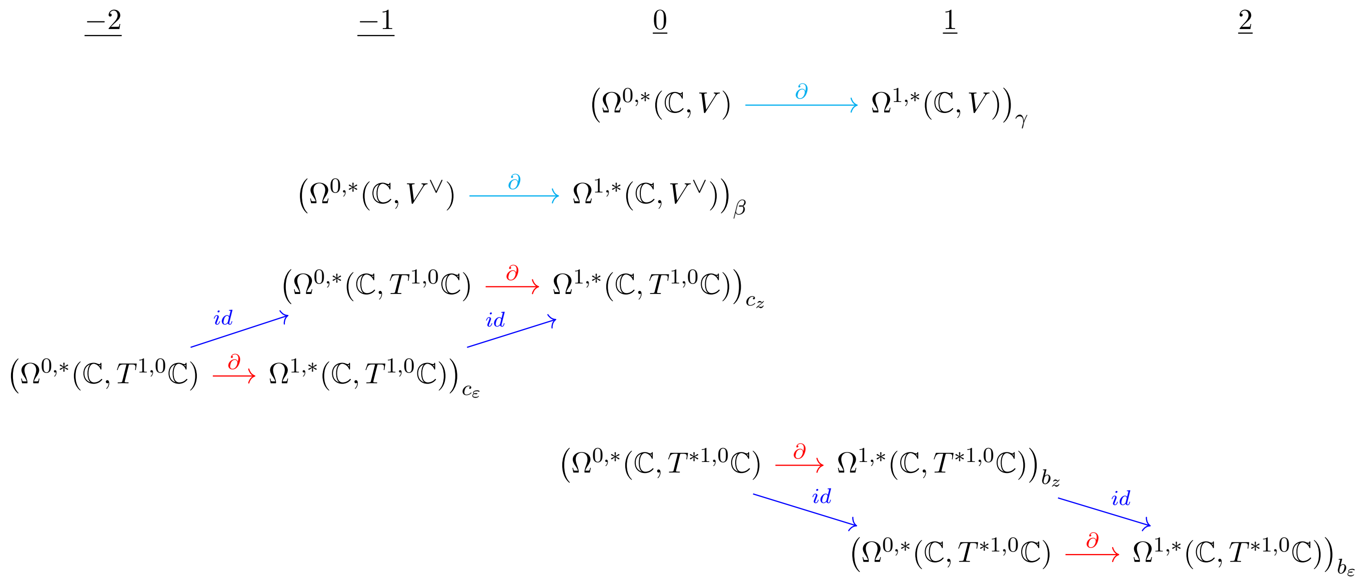


Figure 3.6: The BV complex of topological string theory

We are now in a position of giving the explicit form of the full BV action. This action has three terms: one reflecting the relations of the twisted $\mathcal{M}$ (that is, the topological $\beta\gamma$ system), one the relations of the twisted $\mathscr{L}$, and the last one expliciting the coupling between the TCFT and topological gravity.
Recalling the action of $\mathscr{L}$ on $\mathcal{M}$, given by equations (3.48), (3.49) and similar ones, and introducing the notation $\langle -, - \rangle_T$ for the pairing between holomorphic vector fields and holomorphic differential forms (extended by the wedge product between differential forms), we get the following action:

$$S_{BV} = S_{\mathcal{M}} + S_{\mathscr{L}} + S_{\mathscr{L} \ltimes \mathcal{M}} \tag{3.52}$$

where the three terms are explicitly given by:

- $S_{\mathcal{M}} = \int_{\mathbb{C}} \langle \beta, (\bar\partial + \partial)\gamma \rangle_V = \int_{\mathbb{C}} \langle \beta, d\gamma \rangle_V$
- $S_{\mathscr{L}} = S_{\mathscr{L},\text{free}} + S_{\mathscr{L},\text{int}}$ and:

  · $\mathrm{S}_{\mathscr{L},\text{free}} = \int_{\mathbb{C}} \langle b_z, dc_z\rangle_T + \langle b_\varepsilon, dc_\varepsilon\rangle_T$

  · $\mathrm{S}_{\mathscr{L},\text{int}} = \int_{\mathbb{C}} \langle b_z, [c_z, c_z]\rangle_T + \langle b_\varepsilon, [c_\varepsilon, c_\varepsilon]\rangle_T + \langle b_z, c_\varepsilon \cdot c_z\rangle_T + \langle b_\varepsilon, c_z \cdot c_\varepsilon\rangle_T$

- $\mathrm{S}_{\mathscr{L}\ltimes\mathcal{M}} = \int_{\mathbb{C}} \langle \beta, L_{c_z}\gamma\rangle_V + \langle \beta, c_\varepsilon \cdot \gamma\rangle_V$

If we now look at the cohomology of local observables in this model for topological string theory, we notice something strange: there are no gravitational descendants. That is, the physical local observables are the same as in the usual B-model.
After performing the topological twist, indeed, the gravity sector of our model gets the blue arrows we denoted by “identity”(see Figure 3.6). Taking cohomology with respect to those arrows makes the whole sector trivial.
The crucial observation here is that we are accounting only for the formal moduli problem. Gravitational descendants come from the global structure of the gauge group, as we try to motivate in the next section.

3.5 Gravitational descendants from the worldsheet

In this section, we motivate why, from the worldsheet perspective, gravitational descendants come from the global structure of the gauge group. We will do it by analogy with other models constructed in the literature for topological string theory, and then we will see how to implement them in our model.

First, what is the global structure of the gauge group in our model? Recall from Section 3.4, that our approach to constructing a topological string theory involves starting with a matter system – specifically a $\mathcal{N} = 2$ superconformal theory – and then gauging a background symmetry to account for gravity. This symmetry is taken to be a supersymmetric extension of the Witt algebra.
After performing the topological twist, the twisted system is interpreted as topological string theory: a topological conformal field theory coupled with topological gravity, where the latter arises from gauging the twisted symmetry. Ultimately, the relevant symmetry algebra after twisting is the *real* part of the Witt algebra – that is, the Lie algebra of real smooth vector fields on the circle.
This is because, to perform the twist, we first complexify the matter system, and, therefore, also the symmetry has to be taken complex (recall the discussion in Section 2.3, where the relation between a massless scalar field and its chiral components was explained). However, after the twist, the structure reduces to de Rham-type data. Therefore, the global gauge group underlying our model is essentially given by the real vector fields on the circle rather than by the holomorphic vector fields.

Given that the Lie algebra we are considering is then $\mathrm{Vect}(S^1)$, what is its global structure? In other words, does there exist a Lie group whose Lie algebra is given by $\mathrm{Vect}(S^1)$?
The answer is yes and is given by the orientation-preserving diffeomorphism group of the circle $\mathrm{Diff}_+(S^1)$ (see [Sch08], §5.1). Being an infinite dimensional Lie group, it is very difficult to treat it as it is. However, we can make a tremendous simplification: $\mathrm{Diff}_+(S^1)$ is homotopically equivalent to S^1 itself, as we now show.

Proposition 1. *The space of orientation-preserving diffeomorphisms of the circle* $\mathrm{Diff}_+(S_1)$ *has the same homotopy type of the circle itself.*

Proof. First, let us denote with $\mathrm{Diff}_+(S_1)_p$ the orientation-preserving diffeomorphisms of the

circle that leave invariant the point p, and with $\text{Isom}_+(S_1)$ the orientation-preserving isometries of the circle. $\text{Isom}_+(S_1)$ is homeomorphic to $U(1)$: every orientation-preserving isometry from the circle to itself is given by a rigid rotation.
Notice that every $\phi \in \text{Diff}_+(S_1)$ can be written as a composition of $f \in \text{Diff}_+(S_1)_p$ and $g \in \text{Isom}_+(S_1)$. Indeed, suppose $\phi(p) = p'$ and consider the rigid rotation $g : p \mapsto p'$. Then, since every isometry is invertible, we can form the composition $f = g^{-1} \circ \phi$ to obtain an element $f \in \text{Diff}_+(S_1)_p$. Hence, we have $\phi = g \circ f$, and because of the generality of the diffeomorphisms, we see that:

$$\text{Diff}_+(S_1) = \text{Diff}_+(S_1)_p \ \text{Isom}_+(S_1)$$

Now, since $\text{Isom}_+(S_1)$ is homeomorphic to $U(1)$, it suffices to show that $\text{Diff}_+(S_1)_p$ is contractible to prove the claim. Without loss of generality, we take $p = 0$.
Notice that we can identify the circle with the unit interval $I = [0, 1]$ where the two endpoints are identified, i.e. $S^1 \cong [0, 1]/(0 \sim 1)$. In this way, every $f \in \text{Diff}_+(S_1)_0$ lifts uniquely to a orientation-preserving diffeomorphism of the unit interval $\tilde{f}$ fixing the endpoints. Therefore, we identify $\text{Diff}_+(S_1)_0$ with the space:

$$\mathcal{D} = \{\tilde{f} \in C^\infty(I, I) \mid \tilde{f}(0) = 0,\ \tilde{f}(1) = 1,\ \tilde{f}'(x) > 0\ \forall x \in (0, 1)\}$$

where the condition $\tilde{f}' > 0$ is needed to ensure smooth invertibility.
Now, we wish to prove that $\mathcal{D}$ is contractible to a point. In particular, every $\tilde{f} \in \mathcal{D}$ can be deformed to the identity $id_{\mathcal{D}} = x$. The deformation retract is given by:

$$\begin{aligned} \gamma : [0, 1] \times \mathcal{D} &\to \mathcal{D} \\ (t, \tilde{f}) &\mapsto \gamma_t(\tilde{f})(x) = (1 - t)\tilde{f}(x) + tx \end{aligned}$$

To finish the proof, we just need to verify that at each $t \in [0, 1]$, γ_t stays in $\mathcal{D}$. This is straightforward, in fact, $\gamma_t(0) = 0$ and $\gamma_t(1) = 1$. Moreover:

$$\gamma_t'(x) = (1 - t)\tilde{f}'(x) + t$$

and since $\tilde{f}' > 0$ and $t \in [0, 1]$, $\gamma_t'(x) > 0\ \forall x \in (0, 1)$.
Therefore, $\mathcal{D}$, and hence also $\text{Diff}_+(S^1)_0$, is contractible. This proves the proposition.

□

In conclusion, at least homotopically, the global structure of the gauge group is just $U(1)$!
There are some hints in the literature that go in this direction. In [Wit90] §2.2, Witten highlights how topological gravity can be constructed as the theory of a two dimensional spin connection, i.e. a $SO(2)$ gauge field ω. This can also be interpreted as gauging Lorentz symmetry on the worldsheet. But $SO(2)$ is isomorphic to $U(1)$ as Lie groups, and so we see how the global gauge group is $U(1)$. In particular, Witten constructs this theory by introducing the BRST multiplet (ω, ψ, ϕ) where ω, ψ and ϕ have ghost number 0, 1 and 2 respectively. A BRST transformation acts on this multiplet as follows (we borrow the same notation of [Wit90], §2.2, Eq. (2.12)):

$$\begin{aligned} \delta_{BRST}\omega_\mu &= i\epsilon\psi_\mu \\ \delta_{BRST}\psi_\mu &= -\epsilon\partial_\mu\phi \\ \delta_{BRST}\phi &= 0 \end{aligned} \tag{3.53}$$

In our notation, this would correspond to the following diagram (notice that the cohomological degree is the opposite as the one in [Wit90], because Witten is working on the space of observables, while we are now presenting the space of fields):

$$\begin{array}{ccccc} \underline{-2} & & \underline{-1} & & \underline{0} \\ & & & & \Omega^1_\omega \\ & & & \overset{id}{\nearrow} & \\ \Omega^0_\phi & \xrightarrow{d} & \Omega^1_\psi & & \end{array}$$

As we can notice explicitly, this diagram does not form a cochain complex: the differential does not square to zero. But, as highlighted by Witten, it squares to zero up to a gauge transformation. In our language, we would then add a ghost field that makes the space of fields into a cochain complex:

$$\begin{array}{ccccc} \underline{-2} & & \underline{-1} & & \underline{0} \\ & & \Omega^0_c & \xrightarrow{d} & \Omega^1_\omega \\ & \overset{id}{\nearrow} & & \overset{id}{\nearrow} & \\ \Omega^0_\phi & \xrightarrow{d} & \Omega^1_\psi & & \end{array}$$

We see the similarity with the fields of our topological gravity sector in Figure 3.6.

Another equivalent formulation, mentioned in [Wit90,LWP88], is based on $SL(2,\mathbb{R})$ gauge fields. Still, $SL(2,\mathbb{R})$ is homotopically equivalent to $U(1)$. Therefore, we believe that assuming that the global structure of the gauge group is $U(1)$ is correct.

We have motivated why it is natural to consider $U(1)$ as the global structure of the topological gravity sector in our model. Now, we should explain how to account for this global structure at the level of the BV theory. To introduce the techniques needed, we first treat the case of a model where the geometric interpretation is clear: abelian Chern-Simons theory. Since the global gauge group in this case is also $U(1)$, we can construct gravitational descendants in our model by analogy with this example.

Digression: non-perturbative abelian Chern Simons theory

The BV complex describes a *formal* moduli problem, i.e. a neighbourhood around a chosen solution of the equations of motion in the moduli space. This is why we say that BV describes perturbation theory in field theory. What if we want to describe the entire derived stack of solutions or, in other words, what if we want to give a non-perturbative description of our moduli problem?
Of course, this is not always possible, but we will see in this section how we can construct a model for non-perturbative abelian Chern-Simons theory, and, by similar arguments, how we can implement the gravitational descendants in our model for topological string theory.

Recall from Section 1.6, the BV complex of Chern-Simons theory on a three-dimensional manifold M is just given by de Rham forms on M with values in a Lie algebra $\mathfrak{g}$ (see Figure 1.1), equipped with the de Rham differential. As a formal moduli problem, this describes deformations around the trivial flat connection on a trivial G-bundle, where G is the Lie group whose Lie algebra is $\mathfrak{g}$. But where do we see these choices[6]?

[6] For a careful treatment of this formal moduli problem, see [CG21], §3.3.1.

Let $P \to M$ be a principal G-bundle equipped with a chosen connection, and let $\mathfrak{g}_P := P \times_G \mathfrak{g}$ be the adjoint bundle. The connection induces a derivation ∇ of degree $+1$ acting on de Rham forms valued in $\mathfrak{g}_P$, and if the connection is flat $\nabla^2 = 0$. If we now want to describe the formal moduli problem describing a neighbourhood of this chosen connection, which we take to be flat, we write the following local Lie algebra (we are not careful about the correct gradings):

$$(\Omega^*(M, \mathfrak{g}_P), \nabla) \tag{3.54}$$

with a Lie bracket inherited from $\mathfrak{g}_P$ that is a bundle of Lie algebras.
If we take P to be the trivial principal bundle and look around the trivial flat connection, then we really get perturbative Chern-Simons theory: $(\Omega^*(M, \mathfrak{g}), d)$.

Now, what if we want to describe the entire moduli space of flat connections? In other words, we do not want to choose a specific connection on a specific principal bundle and look at deformations of that one, but we want to classify all principal bundles with flat connection, including the ones with non-trivial global topology.

Consider the case of abelian Chern-Simons theory, where $G = U(1)$ and $\mathfrak{g} = \mathbb{R}$. A model for the non-perturbative version is given by smooth Deligne cohomology. This is the hypercohomology[7] of the following complex of sheaves:

$$0 \longrightarrow \underline{U(1)} \xrightarrow{dlog} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3 \xrightarrow{d} 0$$

where $\underline{U(1)}$ is the sheaf of smooth locally constant functions with values in $U(1)$. If we think about Chern-Simons theory, in this case the ghosts are really maps into the global gauge group, not just into the Lie algebra.

There is a version of the Deligne complex that is more appropriate for our purposes. Notice that the short exact sequence of sheaves

$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow \underline{\mathbb{R}} \xrightarrow{\exp} \underline{U(1)} \longrightarrow 0$$

induces the following quasi-isomorphism of complex of sheaves:

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}} & \hookrightarrow & \underline{\mathbb{R}} & \longrightarrow & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 & \xrightarrow{d} & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \exp} & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \underline{U(1)} & \xrightarrow{dlog} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 & \xrightarrow{d} & 0 \end{array}$$

Therefore, we can take as definition of smooth Deligne cohomology the hypercohomology of the first complex above.

A way to compute this hypercohomology is through the Čech-Deligne double complex, where the vertical row is given by the Čech complex, while the horizontal row is the complex we just defined. In order to do that we need to choose a *good*[8] open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ of our manifold

[7] Hypercohomology is defined as the sheaf cohomology of a complex of sheaves. Specifically, when you resolve the complex, you get a double complex and the hypercohomology is the cohomology of the totalization of that double complex. In this section, we do not need to worry much about the formal definition, since we will give an explicit double complex to compute smooth Deligne cohomology.

[8] An open cover $\mathfrak{U} = \{U_i\}_{i \in I}$ is called a good open cover if each open set U_i and each intersection $U_{i_1 i_2 \dots i_n} = U_{i_1} \cap \dots \cap U_{i_n}$ is contractible. Any smooth manifold admits such a cover. This condition is needed so that computing Čech cohomology is the same as computing ordinary sheaf cohomology.

M and take the Čech complex on this cover.
Recall that the Čech complex of a presheaf $\mathcal{F}$ with respect to the good open cover $\mathcal{U}$ is defined as follows ([sta25], §20.9):

1. If we denote with $U_{i_0 \dots i_n} = U_{i_0} \cap \dots \cap U_{i_n}$ the $(n+1)$-fold intersection of elements of $\mathcal{U}$, we have that the degree n part of the Čech complex is given by:

$$\check{\mathcal{C}}^n(\mathcal{U}, \mathcal{F}) = \prod_{(i_0 \dots i_n) \in I^{n+1}} \mathcal{F}(U_{i_0 \dots i_n}) \tag{3.55}$$

2. Given an element $s \in \check{\mathcal{C}}^n(\mathcal{U}, \mathcal{F})$, we denote by $s_{i_0 \dots i_n}$ its value in $\mathcal{F}(U_{i_0 \dots i_n})$. The differential on this element s is given by:

$$\begin{aligned} \delta : \check{\mathcal{C}}^n(\mathcal{U}, \mathcal{F}) &\to \check{\mathcal{C}}^{n+1}(\mathcal{U}, \mathcal{F}) \\ \delta(s)_{i_0 \dots i_{n+1}} &= \sum_{j=0}^{n+1} (-1)^j s_{i_0 \dots \hat{i}_j \dots i_n+1}|_{U_{i_0 \dots i_{n+1}}} \end{aligned} \tag{3.56}$$

   It easy to see that $\delta^2 = 0$.

3. Therefore, the Čech complex of $\mathcal{F}$ with respect to the open cover $\mathcal{U}$ is $(\check{\mathcal{C}}^*(\mathcal{U}, \mathcal{F}), \delta)$.

Finally, we can give the explicit form of the Čech-Deligne double complex (the horizontal degree is the BV grading).

$$\begin{array}{cccccccccc}
\underline{3} & \underline{\mathbb{Z}}(\bigcup U_{ijkl}) & \hookrightarrow & \Omega^0(\bigcup U_{ijkl}) & \longrightarrow & \dots & & & & \\
 & \delta\uparrow & & \delta\uparrow & & \uparrow & & & & \\
\underline{2} & \underline{\mathbb{Z}}(\bigcup U_{ijk}) & \hookrightarrow & \Omega^0(\bigcup U_{ijk}) & \xrightarrow{d} & \Omega^1(\bigcup U_{ijk}) & \longrightarrow & \dots & & \\
 & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \uparrow & & \\
\underline{1} & \underline{\mathbb{Z}}(\bigcup U_{ij}) & \hookrightarrow & \Omega^0(\bigcup U_{ij}) & \xrightarrow{d} & \Omega^1(\bigcup U_{ij}) & \xrightarrow{d} & \Omega^2(\bigcup U_{ij}) & \longrightarrow & \dots \\
 & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \uparrow \\
\underline{0} & \underline{\mathbb{Z}}(\bigcup U_{i}) & \hookrightarrow & \Omega^0(\bigcup U_{i}) & \xrightarrow{d} & \Omega^1(\bigcup U_{i}) & \xrightarrow{d} & \Omega^2(\bigcup U_{i}) & \xrightarrow{d} & \Omega^3(\bigcup U_{i}) \\
 & & & & & & & & & \\
 & \underline{-2} & & \underline{-1} & & \underline{0} & & \underline{1} & & \underline{2}
\end{array}$$

Figure 3.7: The Čech-Deligne double complex

The smooth Deligne cohomology $\hat{H}^*_D(M)$ is the cohomology of the totalization of this double complex.

How can we see that this is a model for the entire moduli space of flat connections? The key fact is that the degree zero[9] Deligne cohomology is isomorphic to isomorphism classes of principal $U(1)$-bundles equipped with connection[10]. Although we do not go into the proof of this (see [Bry09], §2.2), we will now see some properties of $\hat{H}^0_D(M)$ that motivate this statement.

[9]In the literature, it is usually the degree one cohomology the one of interest, but we are adopting different conventions.

[10]Usually, the complex is truncated in horizontal degree 0. Since we want to model *flat* connections, in the computation of cohomology, we also add the horizontal degree 1.

The key observation is that the smooth Deligne cohomology includes both topological and geometric data. To see this, since we are interested in degree zero Deligne cohomology, let us look at cochains that belong to the diagonal of total degree zero of Figure 3.7. They consist of the data:

$$\mathcal{A} = (\{A_i\}_i, \{c_{ij}\}_{i,j}, \{\eta_{ijk}\}_{i,j,k}) \in \Omega^1(\bigcup U_i) \bigoplus \Omega^0(\bigcup U_{ij}) \bigoplus \Omega^0(\bigcup U_{ijk}, \mathbb{Z}) \tag{3.57}$$

A cochain of this type is called a cocycle if it is closed under the total differential (for a moment, we omit the flatness condition):

$$\begin{aligned} \delta(A)_{ij} = (A_i - A_j)|_{U_{ij}} &= dc_{ij} \\ \delta(c)_{ijk} &= -\eta_{ijk} \\ \delta(\eta)_{ijkl} &= 0 \end{aligned} \tag{3.58}$$

We can recover the closure condition just by following the images of the vertical/horizontal differential in the double complex, as in Figure 3.8.

$$\begin{array}{ccccc} \delta(\eta)_{ijkl} = 0 & & & & \\ \delta\uparrow & & & & \\ \eta_{ijk} & \hookrightarrow & \delta(c)_{ijk} = -\eta_{ijk} & & \\ & & \delta\uparrow & & \\ & & c_{ij} & \xrightarrow{d} & \delta(A)_{ij} = dc_{ij} \\ & & & & \delta\uparrow \\ & & & & A_i \end{array}$$

Figure 3.8: Deligne cocycles

Here, we explicitly see the geometric and topological data encoded in a cocycle. The A_i's are local connection one forms and, on intersections U_{ij}, they are related by gauge transformations c_{ij}. Notice that we are also accounting for the so called *large gauge transformations* that are those set of globally defined one forms with integral period that are closed but not exact under the de Rham differential. Indeed, since we are considering a good open cover, and therefore every intersection is contractible, on every intersection we can use the Poincarè lemma: even the restriction of a large gauge transformation on an intersection can be written as an exact term.

The topological data are encoded in the gauge transformations: these define the transition functions of a U(1) principal bundle as $\varphi_{ij} = \exp(2\pi i c_{ij})$. The cocycle condition for transition functions on a triple intersection ($\varphi_{ij}\varphi_{jk}\varphi_{ki} = 1$) holds as a consequence of $\delta(c)_{ijk}$ being $\mathbb{Z}$-valued. The set of φ_{ij} tells us about the topology of our bundle, in particular how to glue local patches together: we are allowing objects with non-trivial global topology to be described by local data.

Since on each intersection $d(A_i - A_j) = d^2 c_{ij} = 0$, the collection $\{dA_i\}$ can be identified with the set of local representatives of a closed two form, the curvature F. This is globally defined, and the conditions on c_{ij} and η_{ijk} imply that it has an integral period, so we recover the basic setting of Chern-Simons theory.

Therefore, the datum of a Deligne cocycle is equivalent to the datum of a curvature two form globally defined on our bundle. The fact that F has integral period, means that its de Rham

cohomology class $[F] \in H^2(M,\mathbb{R})$ is the image of a class $H^2(M,\mathbb{Z})$ under the coefficient morphism $\mathbb{Z} \hookrightarrow \mathbb{R}$. However, when we consider the morphism $H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{R})$, the torsion part of $H^2(M,\mathbb{Z})$ gets killed.
The crucial fact[11] is that we can also encode this torsion part of $H^2(M,\mathbb{Z})$ through the data of η_{ijk} and φ_{ij}, therefore, every class in $H^2(M,\mathbb{Z})$ can be represented by a class in Deligne cohomology. This is why Deligne cohomology is considered as a refinement of integral cohomology.
For our purposes, this is crucial: elements of $H^2(M,\mathbb{Z})$ are in one-to-one correspondence with isomorphism classes of $U(1)$-principal bundles. Thus, we see how topological non-trivial bundles are encoded in Deligne cohomology.
More precisely, this is a consequence of the following short exact sequence (first proven in [CS85]):

$$0 \longrightarrow \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) \longrightarrow \hat{H}^0_D(M) \longrightarrow H^2(M,\mathbb{Z}) \longrightarrow 0$$

where $\Omega^1_{\mathbb{Z}}(M)$ denotes one form with integral period. This short exact sequence makes precise what we were saying: $\hat{H}^0_D(M)$ contains both the topological data (contained in $H^2(M,\mathbb{Z})$) and the geometric data (the connection one form up to gauge equivalences).
To all this discussion, we then add the flatness condition, since we are just interested in flat connections because of the equation of motion of Chern-Simons theory.

In summary, we have seen that to model the moduli space of $U(1)$-principal bundles with flat connections, we need to modify the BV complex for abelian Chern-Simons theory to also capture the topological data. A model for this is given by smooth Deligne cohomology:

$$\begin{array}{ccccccccc} \underline{-2} & & \underline{-1} & & \underline{0} & & \underline{1} & & \underline{2} \\ \underline{\mathbb{Z}} & \hookrightarrow & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \end{array}$$

Figure 3.9: Non-perturbative abelian Chern-Simons theory

Returning to topological string theory, recall that we take the global structure of the topological gravity sector in our model to be precisely $U(1)$. By accounting for this global structure, we can make gravitational descendants appear. In particular, by analogy with abelian Chern-Simons theory, we modify the topological gravity sector as follows (compare to Figure 3.6):

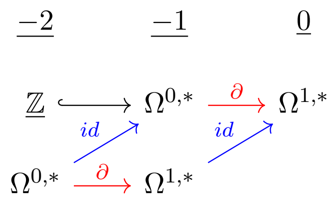


Figure 3.10: Gravitational descendants

Taking the cohomology of local observables in this modified model of topological string theory, we find that, in addition to the usual observables of the topological $\beta\gamma$ system, the topological gravity sector remains trivial – except for the appearance of a new variable in cohomological

[11] For a nice treatment of this subtle point, see [GT14].

degree $+2$.
This variable represents the gravitational descendant, aligning with the description given by Costello and Li, who identify gravitational descendants as degree $+2$ variables in the theory (see Section 3.3).

Conclusions

In this work, we have first explored the Batalin-Vilkovisky formalism and how it describes classical field theories, reinterpreting it in terms of derived deformation theory. We have then constructed a model for topological string theory from the worldsheet perspective and compared it with the target space approach developed by Costello and Li.

The first two chapters were primarily focused on establishing the formalism.

In the first chapter, we began from the very basics, outlining the data that define a classical field theory, and identifying the moduli space of solutions to the equations of motion as our main object of study. We then explained how certain constructions from derived geometry become essential when dealing with singular cases. We first examined the case of non-transverse intersections, which led us to consider the derived critical locus in place of the classical one. We then recognized that incorporating gauge symmetries into our framework requires accounting for the stacky quotient, a derived model for taking invariants. Bringing all of these elements together, in Section 1.4, we derived the explicit form of the BV-BRST complex, which later we applied to the study of Chern-Simons theory and Yang–Mills theory. In the latter example, we began to see that a BV theory naturally carries the structure of a L_∞ algebra.

In the second chapter, it became clear why the Batalin–Vilkovisky formalism naturally encodes perturbation theory. After precisely describing the structure of the space of fields of a BV theory – as that of a local Lie algebra (a sheaf of L_∞ algebras) – we introduced, in an intuitive and non-rigorous way, a functorial perspective for describing moduli spaces (and, more generally, spaces). In particular, we explored how to formulate a formal moduli problem, which corresponds to the formal neighbourhood of a fixed point in the moduli space of interest. By the fundamental theorem of derived deformation theory, we understood that a BV theory controls the deformations around a fixed solution in the moduli space of solutions to the equations of motion. This insight explains why it naturally describes perturbative field theories. From this perspective, we arrived at a new definition of a classical field theory: a sheaf of (-1)-shifted symplectic formal moduli problems.
We then turned our attention to describing symmetries within the BV formalism. We began by understanding how to express symmetries in terms of local Lie algebras. From there, we examined what it means for a symmetry to act on the space of fields and, when desired, how to gauge such a symmetry. Gauging was interpreted as the coupling of two formal moduli problems.
We concluded the chapter with the example of the holomorphic bosonic string, where we saw the full machinery developed so far put into practice.

The final chapter was devoted to the analysis of topological string theory, using the techniques developed in the first two chapters.

We began by describing in detail two topological field theories that could be later considered as matter systems in a topological string theory model: the topological $\beta\gamma$ and the topological Landau–Ginzburg model in two dimensions. The latter differs from the former by including a superpotential.
We then asked ourselves: what does it mean to couple these topological models to topological gravity? To address this, we adopted a target space perspective, aiming to recover the BCOV model as described by Costello and Li. A key insight emerged: the Costello–Li model computes the S^1-equivariant cohomology of the observables of the B-model. This equivariance condition turned out to be fundamental. By retracing the steps of Dijkgraaf, Verlinde and Verlinde in constructing a topological string theory, we saw that without the equivariance condition, the topological gravity sector completely decouples from the matter system. In other words, the equivariance condition is precisely what makes the interaction between the matter system and topological gravity non-trivial.
Up to this point of the work, we marked some structural differences between topological field theory and topological string theory. In particular, some distinctions are in the following table.

	Topological field theory	Topological string theory
Physical operators	cohomology	S^1-equivariant cohomology
Moduli space (when W turned on)	dimension ~ 0	dimension > 0
Algebraic structure of observables	Batalin-Vilkovisky algebra	Gravity algebra

Although we did not address the last row of the table directly in this thesis, we felt it was important to highlight it for completeness, as it reflects a well-known result by Getzler in his series of articles [Get94a, Get94b, Get95].

We then turned to the construction of our model for topological string theory from a worldsheet perspective. Specifically, we considered a $\mathcal{N} = 2$ superconformal field theory and gauged the symmetry acting on it – a supersymmetric extension of conformal symmetry. By performing a twist on the coupled system, we recovered the interpretation of topological string theory as a topological conformal field theory coupled to topological gravity.
When computing the local observables of our model, we expected to obtain a description analogous to that of Costello and Li. However, this was not the case: the cohomology of local observables in the topological gravity sector turned out to be trivial, and we were unable to recover the gravitational descendants.
In the final section, we addressed this issue. We inferred that gravitational descendants arise as a manifestation of the global structure of the gauge group – that is, the topological gravity sector in our case. Proceeding by analogy with the non-perturbative description of abelian Chern–Simons theory in terms of smooth Deligne cohomology, we were able to recover the gravitational descendants as degree $+2$ variables, consistent with the description given by Costello and Li.

There are still many questions and things that are not fully understood. Some future directions to continue this work are listed below.

1. Although we managed to recover the gravitational descendants as cohomological degree $+2$ variables by accounting for the global structure of the gauge group, we did not reconstruct the complex presented by Costello and Li. In particular, it remains unclear how to derive the component of the differential involving $u\partial_\Omega$ in Eq. (3.36). A possible speculative

direction would be to refine our model into one that computes cyclic cohomology, but at this stage, our understanding of such a construction is limited.

2. Another possible direction would be to study the factorization algebra of observables associated with our model and attempt to compute correlation functions, in order to explore further possible comparisons with the work of Costello and Li.

3. In this work, we did not address BV quantization and restricted ourselves to the classical BV formalism. An interesting direction for future research would be to attempt the quantization of our model for topological string theory, using the formalism and techniques developed by Costello in [Cos11].

With these possible future directions, we conclude this thesis. To those who have followed the work to this point – thank you for your attention. I hope you found something interesting along the way.
Best wishes for your explorations (in physics and in life!) ahead.

Appendix A

Sheaves: basic definitions

Sheaves and cosheaves are tools for organizing local data in a consistent global way. In this appendix, we recall some standard definitions and facts about sheaves and cosheaves that will be useful throughout the thesis. In particular, keep in mind that the space of fields in a classical field theory naturally forms a sheaf, whereas the observables have the structure of a cosheaf.
Our main references for this appendix are [Tu22], which provides a concise and accessible introduction to sheaves and sheaf cohomology – especially suitable for those who are meeting these notions for the first time – and [CG16], §A.4. We refer the reader to these two references for a more precise and example-driven treatment of sheaves and cosheaves. Here, we will limit ourselves to some basic definitions, just to have them readily available while reading this thesis.

Let M be a topological space, and let Opens_M denote the following category:

- the objects are open set $U \subseteq M$;
- the morphisms are inclusion of open sets. In other words, if $U \subset V$, there is a morphism from U to V.

A *presheaf* on a topological space M, valued in a category $\mathcal{C}$, is a contravariant functor:

$$\mathcal{F} : \mathrm{Opens}_M \to \mathcal{C} \tag{A.1}$$

That is, to each open set $U \subseteq M$, the presheaf assigns an object $\mathcal{F}(U) \in \mathcal{C}$, and to each inclusion of open sets $U \hookrightarrow V$, it assigns a restriction morphism $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, such that these morphisms are functorial: the identity inclusion induces the identity morphism, and composition of inclusions corresponds to composition of restriction maps.

A sheaf is a presheaf whose value on an open set is determined by its values on smaller open subsets. This idea is formalized in the following definition, where we restrict to the category of vector spaces – that is, $\mathcal{C} = \mathrm{Vect}_{\mathbb{K}}$ (just for simplicity).

Definition 9. *A presheaf of vector spaces on M is called a sheaf, if for any open cover $\{U_i\}_i$ of any open set $U \subseteq M$, the following sequence is exact:*

$$0 \longrightarrow \mathcal{F}(U) \xrightarrow{\ r\ } \bigoplus_i \mathcal{F}(U_i) \xrightarrow{\ \delta\ } \bigoplus_{i<j} \mathcal{F}(U_i \cap U_j) \tag{A.2}$$

Here, r is the restriction map, and δ is the Čech coboundary operator, defined by restricting sections in two ways – the first from $\mathcal{F}(U_i) \to \mathcal{F}(U_i \cap U_j)$, and the second from $\mathcal{F}(U_j) \to \mathcal{F}(U_i \cap U_j)$ – and taking their difference.

In order to develop more intuition about this definition, let us unpack it. Working at the level of sections, the exactness of the sequence in Eq. (A.2) corresponds to the following two conditions:

1. (uniqueness property) if $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$;
2. (gluing property) if $\{s_i \in \mathcal{F}(U_i)\}$ is a collection of sections such that:

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall i, j \tag{A.3}$$

   then there is a section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

These two properties make the conditions for consistent gluing of local data across an open cover explicit.

Example 1. Let $\mathcal{F}$ be the presheaf of constant real-valued functions on M – that is, for every opens set $U \subseteq M$ we associate a constant function on U. Then $\mathcal{F}$ is not a sheaf, because it does not satisfy the gluing property. Indeed, if U_1 and U_2 are disjoint open sets, and $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ have different values, there is no constant function s on $U_1 \cup U_2$ that restricts to s_1 on U_1 and to s_2 on U_2.

Example 2. If instead of constant functions we consider *locally* constant functions, then these form a sheaf. We denote the sheaf of locally constant real-valued functions as $\underline{\mathbb{R}}$. In the same way, if the functions are $\mathbb{Z}$-valued, the corresponding sheaf of locally constant $\mathbb{Z}$-valued functions is denoted by $\underline{\mathbb{Z}}$.

Another definition we want to introduce is that of the stalk of a sheaf. This concept resembles the notion of the germ of a function at a point.

Definition 10. *Let $\mathcal{F}$ be a sheaf on M. The stalk of $\mathcal{F}$ at a point $p \in M$, is given by its direct limit at p:*

$$\mathcal{F}_p := \varinjlim_{x \in U} \mathcal{F}(U) \tag{A.4}$$

These are some basic ingredients that we will need in this thesis, but the theory of sheaves is much more than these basic definitions.

Just for completeness, we sketch what is a cosheaf, i.e. the dual notion of a sheaf. While a presheaf is a contravariant functor from the category of open sets (with inclusions as morphisms) to some target category, a precosheaf is instead a covariant functor. That is, morphisms go in the direction of inclusions rather than restrictions.
A cosheaf is then a precosheaf that satisfies dual versions of the usual gluing conditions. Intuitively, this means that instead of gluing local data to get global data (as in a sheaf), we distribute global data to local pieces in a consistent way.
A standard example of a cosheaf is the cosheaf of compactly supported functions: when we extend a compactly supported function from a smaller open set to a larger one, we do so by extending it by zero outside its support.
For a comprehensive review of cosheaves (but also sheaves), we refer the reader to [Cur14].

Appendix B

Some categorical preliminaries

Exactness

In the following, we give some categorical notions that will be useful in the course of this thesis. Since we want to do homological algebra, we restrict our attention to working only with abelian categories. The exact definition of an abelian category is not important for us; just keep in mind that for abelian categories the notions of kernel and cokernel are always meaningful.

Definition 11. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. For every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathscr{A}$, F is called:*

- ***left exact*** *if the sequence $0 \to F(A) \to F(B) \to F(C)$ is exact in $\mathscr{B}$.*
- ***right exact*** *if the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact in $\mathscr{B}$.*
- ***exact*** *if it is both left and right exact, i.e. if it preserves exact sequences.*

Two important examples of right and left exact functors are the tensor product functor and the Hom functor.

1. Let M be an object in $\mathscr{A}$. Then $\mathrm{Hom}_{\mathscr{A}}(M, -)$ is a covariant functor from $\mathscr{A}$ to Ab:
   - *objects*: $\forall A \in \mathrm{Obj}(\mathscr{A})$, $\mathrm{Hom}_{\mathscr{A}}(M, A)$ is the set of morphisms from M to A;
   - *morphisms*: if $A \xrightarrow{f} B$, then $\mathrm{Hom}_{\mathscr{A}}(M, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathscr{A}}(M, B)$ where f_* is just the composition with f.

   Proposition 2. $\mathrm{Hom}_{\mathscr{A}}(M, -)$ *is a left exact functor.*

   Proof. Consider the short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathscr{A}$. We want to show that the sequence $0 \to \mathrm{Hom}_{\mathscr{A}}(M, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathscr{A}}(M, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathscr{A}}(M, C)$ is exact in Ab. If $\alpha \in \mathrm{Hom}_{\mathscr{A}}(M, A)$, then $f_*\alpha = f \circ \alpha$. Since f is injective, if $f_*\alpha = 0$ then α must be the zero morphism. Consequently, $\ker(f_*) = 0$ and this proves the injectivity of f_*.
   It remains to show that $\mathrm{Im}(f_*) = \ker(g_*)$. Since $g \circ f = 0$, we have $g_* f_*(\alpha) = g \circ f \circ \alpha = 0$, so $\mathrm{Im}(f_*) \subseteq \ker(g_*)$. Now, consider $\beta \in \mathrm{Hom}_{\mathscr{A}}(M, B)$ so that $g_*\beta = g \circ \beta = 0$. Then $\beta(M) \subseteq \ker(g) = \mathrm{Im}(f)$. But this means that we can write $\beta = f \circ \alpha$ for some $\alpha \in \mathrm{Hom}_{\mathscr{A}}(M, A)$ proving that $\ker(g_*) = \mathrm{Im}(f_*)$.
   □

   Proposition 3. $\mathrm{Hom}_{\mathscr{A}}(-, M)$ *is a left exact contravariant functor.*

Proof. Since $\mathrm{Hom}_{\mathscr{A}}(A, M) = \mathrm{Hom}_{\mathscr{A}^{op}}(M, A)$, we can just use the previous proposition applied to $\mathrm{Hom}_{\mathscr{A}^{op}}(M, -)$.

□

2. Let us now consider the tensor product functor over the ring R. We denote the category of right R-modules with module homomorphisms as $\mathrm{Mod_R}$.

Proposition 4. *Let M be a left R-module. Then $- \otimes_R M : \mathrm{Mod_R} \to \mathrm{Ab}$ is a right exact covariant functor.*

Proof. Consider the short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathrm{Mod_R}$. We want to show that $A \otimes_R M \xrightarrow{f \otimes id_M} B \otimes_R M \xrightarrow{g \otimes id_M} C \otimes_R M \to 0$ is exact in Ab.
It is clear that $g \otimes id_M$ is surjective because g is surjective, so the sequence is exact at $C \otimes_R M$.
Now, by functoriality of the tensor product, $\mathrm{Im}(f \otimes id_M) \subseteq \ker(g \otimes id_M)$. Indeed, $(g \otimes id_M) \circ (f \otimes id_M) = g \circ f \otimes id_M = 0$, since $g \circ f = 0$. Another way to say it is that $g \otimes id_M$ factors as in:

$$\begin{array}{ccc} B \otimes_R M & \xrightarrow{g \otimes id_M} & C \otimes_R M \\ \big\downarrow p & & \nearrow \exists \\ \frac{B \otimes_R M}{\mathrm{Im}(f \otimes id_M)} & & \end{array}$$

If we now show that the dashed arrow is an isomorphism, we have finished the proof. To do this let us construct an explicit inverse. Define:

$$C \times M \longrightarrow \frac{B \otimes_R M}{\mathrm{Im}(f \otimes id_M)}$$

$$(c, m) \longmapsto \overline{c} \otimes m + \mathrm{Im}(f \otimes id_M),$$

where $\overline{c} \in B$ is such that $g(\overline{c}) = c$. Given the exactness of the short exact sequence first mentioned this map is well-defined and R-bilinear. Therefore, it induces a map:

$$C \otimes_R M \longrightarrow \frac{B \otimes_R M}{\mathrm{Im}(f \otimes id_M)},$$

which is the required inverse.

□

Adjoint functors

Adjoint functors are very useful in proving the left and right exactness of functors, one of the conditions that becomes necessary to construct derived functors. In this section, we will review their definition and an important theorem that will be useful later; we loosely follow [Lei16] and [Wei94].

Roughly speaking, given $\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B}$ categories and functors, L is said to be left adjoint to R if, whenever $A \in \mathscr{A}$ and $B \in \mathscr{B}$, maps $L(A) \to B$ are essentially the same thing as maps $A \to R(B)$. Let us give the precise definition.

Definition 12. *Let $\mathscr{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{B}$ be categories and functors. We say that L is left adjoint to R, and R is right adjoint to L, and write $L \dashv R$, if:*

$$\mathrm{Hom}_{\mathscr{B}}(L(A), B) \cong_{\mathrm{Sets}} \mathrm{Hom}_{\mathscr{A}}(A, R(B)) \tag{B.1}$$

naturally in $A \in \mathscr{A}$ and in $B \in \mathscr{B}$.

But what does it mean naturally in this context? There must exist a set of bijections for all $A \in \mathscr{A}$ and for all $B \in \mathscr{B}$:

$$\tau_{AB} : \mathrm{Hom}_{\mathscr{B}}(L(A), B) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(A, R(B)) \tag{B.2}$$

so that $\forall f : A \to A'$ in $\mathscr{A}$ and $\forall g : B \to B'$ in $\mathscr{B}$ the following commutes:

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathscr{B}}(L(A'), B) & \xrightarrow{Lf^*} & \mathrm{Hom}_{\mathscr{B}}(L(A), B) & \xrightarrow{g_*} & \mathrm{Hom}_{\mathscr{B}}(L(A), B') \\
\downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\
\mathrm{Hom}_{\mathscr{A}}(A', R(B)) & \xrightarrow{f^*} & \mathrm{Hom}_{\mathscr{A}}(A, R(B)) & \xrightarrow{Rg_*} & \mathrm{Hom}_{\mathscr{A}}(A, R(B'))
\end{array}$$

where $(-)^*$ for morphisms means precomposition and $(-)_*$ indicates composition.
This does not seem very natural, but the fact that this diagram commutes is equivalent to say that τ_{AB} are the components of a natural isomorphism τ between the two following functors:

$$\begin{array}{ccc}
& \mathrm{Hom}_{\mathscr{B}}(L(-), -) & \\
& \nearrow \qquad\qquad \searrow & \\
\mathscr{A}^{op} \times \mathscr{B} & \Downarrow \tau & \mathrm{Sets} \\
& \searrow \qquad\qquad \nearrow & \\
& \mathrm{Hom}_{\mathscr{A}}(-, R(-)) &
\end{array}$$

As an important fact, a given functor may or may not have a left/right adjoint, but if it does, this is unique up to isomorphisms. In this uniqueness property lies the usefulness of adjoint functors: once we find an adjoint, we can call it *the* adjoint.

We will now state and prove a theorem that makes evident the usefulness of adjoint functors.

Theorem 2. *Let $L : \mathscr{A} \to \mathscr{B}$ and $R : \mathscr{B} \to \mathscr{A}$ be an adjoint pair of additive functors. Then L is right exact and R is left exact.*

Proof. Consider the short exact sequence $0 \to B' \to B \to B'' \to 0$ in $\mathscr{B}$. By naturality of τ, $\forall A \in \mathscr{A}$ we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathscr{B}}(L(A), B') & \longrightarrow & \mathrm{Hom}_{\mathscr{B}}(L(A), B) & \longrightarrow & \mathrm{Hom}_{\mathscr{B}}(L(A), B'') \\
& & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
0 & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(A, R(B')) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(A, R(B)) & \longrightarrow & \mathrm{Hom}_{\mathscr{A}}(A, R(B''))
\end{array}$$

Now, since $\mathrm{Hom}_{\mathscr{B}}(L(A), -)$ is a left exact functor, the first row is exact $\forall A \in \mathscr{A}$. But then also the second row is exact. By the Yoneda lemma (cfr. [Wei94] Lemma 1.6.11), $0 \to R(B') \to R(B) \to R(B'')$ must be exact, therefore every right adjont R is left exact. For the same reason $L^{op} : \mathscr{A}^{op} \to \mathscr{B}^{op}$, which is right adjoint, is left exact, showing that L is right exact.

□

As an application of this theorem, we could see in a different way how the tensor product functor is right exact, while the Hom functor is left exact. Indeed, given a left R-module M, we have the following adjunction (see [Wei94], Application 2.6.2):

$$- \otimes_R M \dashv \mathrm{Hom}_{\mathrm{Ab}}(M, -) \tag{B.3}$$

Appendix C

Resolutions and derived functors

The aim of this appendix is to explain the concept of derived functors, very useful computational tools in homological algebra. Our main reference for this part is [Wei94].

Projective, injective

Let us start with a few preliminaries needed to introduce the definition of derived functors.

Definition 13. *An object P in an abelian category $\mathscr{A}$ is* ***projective*** *if given a surjection $g : B \to C$ and a morphism $\gamma : P \to C$, then there exists at least one $\beta : P \to B$ such that $\gamma = g \circ \beta$; that is if it satisfies the following universal lifting property:*

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\exists\beta} & \downarrow{\scriptstyle\gamma} & & \\ B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

This means that every morphism out of P factors through a surjection. The definition of injective object is similar: every map into our object factors through an injective one.

Definition 14. *An object I in an abelian category $\mathscr{A}$ is* ***injective*** *if given an injection $f : A \to B$ and a morphism $\alpha : A \to I$, there exists at least one map $\beta : B \to I$ such that $\alpha = \beta \circ f$; that is if it satisfies the following universal lifting property:*

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \downarrow{\scriptstyle\alpha} & \swarrow_{\exists\beta} & \\ & & I & & \end{array}$$

As a remark, we say that an abelian category $\mathscr{A}$ has *enough projectives* if for every object A of $\mathscr{A}$, there is a surjection $P \to A$ with P projective. Similarly, $\mathscr{A}$ has *enough injectives* if for every object A in $\mathscr{A}$ there is an injection $A \to I$ with I injective.

Definition 15. *Let M be an object of $\mathscr{A}$. A* ***left resolution*** *of M is a complex (P_*, d) with $P_i = 0$ for $i < 0$, together with a map $\epsilon : P_0 \to M$ so that the augmented complex:*

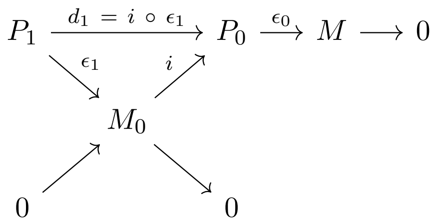
$$\ldots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M \to 0$$

is exact. If P_i is projective $\forall i$, we call it a ***projective resolution****.*

Dually, an injective resolution is a right resolution where every object is injective. Very importantly, the following lemma holds:

Lemma 1. *If an abelian category $\mathscr{A}$ has enough projectives, then every object M in $\mathscr{A}$ has a projective resolution.*

Proof. We prove this lemma by explicitly constructing a projective resolution. The important hypothesis here is that we are working with a category $\mathscr{A}$ that has enough projectives; for this reason, we can always find a surjection $P_0 \xrightarrow{\epsilon_0} M$ where P_0 is a projective object. Let us define $M_0 = \ker(\epsilon_0)$. Now, take another projective object P_1 and consider a surjection $P_1 \xrightarrow{\epsilon_1} M_0$. Let us define the map $d_1 : P_1 \to P_0$ as the composition of ϵ_1 with the inclusion of M_0 into P_0. The situation is represented in the following diagram:



Since $d_1(P_1) = M_0 = \ker(P_0)$ the sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon_0} M \to 0$ is exact in P_0. If we now proceed inductively with the same construction, we get that (P_*, d) is a projective resolution of M.

□

Of course the same holds for a category with enough injectives: it is always possible to explicitly construct an injective resolution.
The categories of left/right modules over a ring or over an associative algebra are examples of categories with enough projectives and injectives; the lemma proves that for every object we can then find a (projective/injective) resolution!

Derived functors

We are now in a position to give the construction of left and right derived functors. In particular, we will focus on left ones, but the same construction applies as well for right derived functors with the appropriate adjustments.

Let $F : \mathscr{A} \to \mathscr{B}$ be a right exact functor between two abelian categories, with the additional condition that $\mathscr{A}$ has enough projectives. Then, as we have seen before, every object M in $\mathscr{A}$ has a projective resolution. Let us choose one:

$$\dots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M \to 0$$

Now, let us apply to this exact sequence the right exact functor F:

$$\dots \xrightarrow{F(d)} F(P_2) \xrightarrow{F(d)} F(P_1) \xrightarrow{F(d)} F(P_0) \xrightarrow{F(\epsilon)} F(M) \to 0$$

We can now define what we mean by left derived functor.

Definition 16. *The i-th left derived functor L_iF $(i \geq 0)$ of F is defined for an object M in $\mathscr{A}$ as:*

$$L_iF(M) = H_i(F(P_*)) \tag{C.1}$$

i.e. the i-th homology of the chain $(F(P_*), F(d))$ constructed from a projective resolution of M.

Notice that since the functor F is right exact, we always have that $L_0F(M) = H_0(F(P_*)) \cong F(M)$. Indeed, $H_0(F(P_*)) = F(P_0)/[\mathrm{Im}(F(P_1) \xrightarrow{F(d)} F(P_0))]$ and, since F is right exact, the sequence $F(P_1) \xrightarrow{F(d)} F(P_0) \xrightarrow{F(\epsilon)} F(M) \to 0$ is exact. Applying the same construction we made proving Proposition 3, we can show there is an isomorphism between $F(P_0)/[\mathrm{Im}(F(P_1) \xrightarrow{F(d)} F(P_0))]$ and $F(M)$. This is why, after applying the functor F to the chosen projective resolution of M, we cut off $F(M)$: the information of $F(M)$ is already contained in homology.
As an important remark, notice that the construction is well defined: given another projective resolution (Q_*, q) of M, then there is a canonical isomorphism $L_iF(M) = H_i(F(P_*)) \xrightarrow{\cong} H_i(F(Q_*))$. This means that we can compute the derived functors of an object by choosing the resolution we prefer!

Let us state two important properties of left derived functors:

1. If M is projective, then $L_iF(M) = 0$ for $i \neq 0$. We say that M is an F-acyclic object, and a resolution constructed with objects that are all F-acyclics is called an F-acyclic resolution.

2. For every short exact sequence $0 \to A \to B \to C \to O$ in $\mathscr{A}$, if F is a right exact functor, then we have the following long exact sequence:

$$\cdots \to L_2F(C) \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0$$

   This is the same that happens in topology when we say we have a long exact sequence in homology.

Two important examples, that we will use repeatedly in this thesis are the derived functors of the tensor product functor and of the Hom functor. Specifically:

1. Given a left R-module M, recall that $- \otimes_R M : \mathrm{Mod}_R \to \mathrm{Ab}$ is right exact. Therefore, since the category Mod_R has enough projectives, we can construct the left derived functors. These are called the Tor groups of a right R-module P: $L_*(- \otimes_R M)(P) = \mathrm{Tor}_*^R(P, M)$.

2. Mod_R has also enough injectives, and since $\mathrm{Hom}_{\mathrm{Mod}_R}(M, -)$ for a given M is left exact, we can construct the associated right derived functors. These are called the Ext groups: $R^*(\mathrm{Hom}_{\mathrm{Mod}_R}(M, -))(P) = \mathrm{Ext}_R^*(M, P)$.

Appendix D

Lie algebra homology and cohomology

In this section, we review the construction of Lie algebra cohomology and homology. The techniques explained here are fundamental in dealing with gauge symmetries in the BRST construction (Section 1.3). We mainly follow [Wei94], §7 and the appendices of [CG16, CG21].

Lie algebras and their homological algebra

Definition 17. *Let $\mathbb{K}$ be a field of characteristic zero. A Lie algebra over $\mathbb{K}$ is a $\mathbb{K}$-vector space $\mathfrak{g}$ equipped with a bilinear map $[-,-] : \mathfrak{g} \otimes_{\mathbb{K}} \mathfrak{g} \to \mathfrak{g}$, called its Lie bracket, satisfying the following properties:*

(i) (anticommutativity). For all $x, y \in \mathfrak{g}$, we have:

$$[x, y] = -[y, x] \tag{D.1}$$

(ii) (Jacobi identity). For all $x, y, z \in \mathfrak{g}$, we have:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{D.2}$$

A homomorphism of Lie algebras is a linear map $f : \mathfrak{g} \to \mathfrak{h}$ that preserves the Lie bracket. We denote the category of Lie algebras over $\mathbb{K}$ together with their homomorphisms as $\mathrm{LieAlg}_{\mathbb{K}}$.

There is an obvious functor from the category of associative algebras (that we denote as $\mathrm{AssAlg}_{\mathbb{K}}$) to the category of Lie algebras; indeed, given any associative algebra A, we can construct a Lie algebra $\mathrm{Lie}(A)$ whose underlying vector space is the same as A and the Lie bracket is given by the commutator. We denote this functor as $\mathrm{Lie}(-)$.
We also need the notion of modules over a Lie algebra.

Definition 18. *Let $\mathfrak{g}$ be a Lie algebra. A (left) module over $\mathfrak{g}$ is a vector space M together with a bilinear map $\rho : \mathfrak{g} \otimes_{\mathbb{K}} M \to M$ (written as $x \otimes m \mapsto \rho(x \otimes m) = x \cdot m$) such that:*

$$[x, y] \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m) \qquad \forall x, y \in \mathfrak{g}, \forall m \in M \tag{D.3}$$

Equivalently, a $\mathfrak{g}$-module can also be described as a vector space M together with a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Lie}(\mathrm{End}_{\mathbb{K}}(M))$, that is just $x \mapsto \rho(x \otimes -)$, $\forall x \in \mathfrak{g}$.

To define the category of $\mathfrak{g}$-modules we also need a notion of morphism.

Definition 19. *A homomorphism of (left) $\mathfrak{g}$-modules is a linear map $f : M \to N$ compatible with the $\mathfrak{g}$-action:*

$$f(x \cdot m) = x \cdot f(m) \qquad \forall x \in \mathfrak{g}, \forall m \in M$$

We denote the category of $\mathfrak{g}$-modules together with the homomorphisms just defined as $\mathfrak{g}$-Mod.

Our goal is to define Lie algebra homology and cohomology as derived functors of two natural functors: the invariants and the coinvariants. Explicitly:

1. *invariants functor*

$$(-)^{\mathfrak{g}} : \mathfrak{g}\text{-Mod} \to \text{Vect}_{\mathbb{K}}$$
$$M \mapsto M^{\mathfrak{g}} = \{m \in M \mid x \cdot m = 0, \ \forall x \in \mathfrak{g}\}$$

   Therefore, for every $\mathfrak{g}$-module M, we take the maximal invariant submodule $M^{\mathfrak{g}}$. We can understand why they are called invariants by assuming that the Lie algebra $\mathfrak{g}$ we are considering comes from a Lie group G, and that we can recover the G-action on M by exponentiation of elements of $\mathfrak{g}$. Therefore, the invariant submodule $M^{\mathfrak{g}}$ is really the set of elements of M that are fixed by the group action.

2. *coinvariants functor*

$$(-)_{\mathfrak{g}} : \mathfrak{g}\text{-Mod} \to \text{Vect}_{\mathbb{K}}$$
$$M \mapsto M_{\mathfrak{g}} = M/\mathfrak{g}M$$

   Here, to every $\mathfrak{g}$-module M, we associate the largest quotient module of M on which $\mathfrak{g}$ acts trivially.

Lemma 2. *The invariants functor is a left exact functor, while the coinvariants functor is right exact.*

Proof. To prove this lemma, it suffices to show that the invariants and the coinvariants functors are respectively the right and the left adjoints to the trivial $\mathfrak{g}$-module functor. Then, by Theorem 2, we know that they are respectively left exact and right exact. The trivial $\mathfrak{g}$-module functor $T : \text{Vect}_{\mathbb{K}} \to \mathfrak{g}\text{-Mod}$ takes a vector space V and gives it a structure of $\mathfrak{g}$-module equipping it with the trivial $\mathfrak{g}$-action.

(i) For the invariants, given a vector space V and a $\mathfrak{g}$-module M, we have the natural isomorphism:

$$\text{Hom}_{\text{Vect}_{\mathbb{K}}}(V, M^{\mathfrak{g}}) \xrightarrow{\cong} \text{Hom}_{\mathfrak{g}-\text{Mod}}(T(V), M)$$
$$f \mapsto i \circ f$$
$$(g : V \to M^{\mathfrak{g}}) \leftarrow\!\shortmid (g : T(V) \to M)$$

where i is the inclusion of the submodule $M^{\mathfrak{g}}$ into M, and g is the same map on the right and on the left because, since it is a $\mathfrak{g}$-module homomorphism, $g(T(V)) = g(V)$ is an invariant submodule, and therefore $g(V) \subseteq M^{\mathfrak{g}}$ given the maximality of $M^{\mathfrak{g}}$.

(ii) In the case of the coinvariants, given a vector space V and a $\mathfrak{g}$-module M, we have the natural isomorphism:

$$\text{Hom}_{\mathfrak{g}-\text{Mod}}(M, T(V)) \xrightarrow{\cong} \text{Hom}_{\text{Vect}_{\mathbb{K}}}(M_{\mathfrak{g}}, V)$$
$$f \mapsto (\bar{f} : M_{\mathfrak{g}} \to V)$$

$$g \circ \pi \hookleftarrow g$$

where $\pi : M \to M/\mathfrak{g}M$ is the quotient map, and $\bar{f}$ is the induced map from f, since $\mathfrak{g}M \subseteq \ker(f)$.

□

Therefore, we can define Lie algebra cohomology as the right derived functor of invariants, and homology as the left derived functor of the functor of coinvariants:

Definition 20. *Let M be a $\mathfrak{g}$-module.*

1. $H_*(\mathfrak{g}, M) := L_*(-)_{\mathfrak{g}}(M)$ *are the homology groups of* $\mathfrak{g}$ *with coefficients in* M*. Notice that* $H_0(\mathfrak{g}, M) = M_{\mathfrak{g}}$*.*
2. $H^*(\mathfrak{g}, M) := R^*(-)^{\mathfrak{g}}(M)$ *are the cohomology groups of* $\mathfrak{g}$ *with coefficients in* M*. Notice that* $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$*.*

In order to have a well-defined construction, we are just missing a step: showing that the category $\mathfrak{g}$-Mod has enough projectives and injectives. To do this, we will take a small detour on the universal enveloping algebra construction, and we will also show how to realize these derived functors as Tor and Ext.

The universal enveloping algebra

In this section we review the construction of the universal enveloping algebra $U\mathfrak{g}$ associated to a Lie algebra $\mathfrak{g}$. The goal is to show that the category $\mathfrak{g}$-Mod is naturally isomorphic to the category $U\mathfrak{g}$-Mod, proving that $\mathfrak{g}$-Mod has enough projectives and injectives so that makes sense to construct the derived functors we were talking before.

Definition 21. *If M is any vector space, the tensor algebra $T(M)$ is the following graded associative algebra with unit generated by M:*

$$T(M) = \bigoplus_{n=0}^{\infty} M^{\otimes n} \tag{D.4}$$

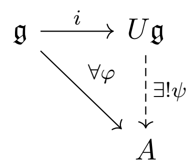
The product in $T(M)$ amounts to concatenation of terms.

Proposition 5. *$T(M)$ satisfies the following universal property: for every linear map $\varphi : M \to A$, where A is an associative algebra, there exists an unique associative algebra homomorphism $\psi : T(M) \to A$. In other words the following diagram commutes, where the map i is the evident inclusion:*

$$\begin{array}{ccc} M & \xrightarrow{\ i\ } & T(M) \\ & \searrow^{\forall \varphi} & \downarrow{\scriptstyle \exists ! \psi} \\ & & A \end{array}$$

This map ψ is simply given by: $\psi(1) = 1$, $\psi(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = \varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)$ $\forall x_1, x_2, \ldots, x_n \in M$.
Let us now give the definition of the universal enveloping algebra, through a similar universal property.

Definition 22. *Let* $\mathfrak{g}$ *be a Lie algebra. The universal enveloping algebra* $U\mathfrak{g}$ *of* $\mathfrak{g}$ *is defined to be any pair* $(U\mathfrak{g}, i)$ *where* $U\mathfrak{g}$ *is an associative algebra and* $i : \mathfrak{g} \to U\mathfrak{g}$ *is a Lie algebra homomorphism satisfying the universal property pictured in the diagram below: for any other associative algebra* A *and Lie algebra homomorphism* $\varphi : \mathfrak{g} \to A$*, there exists an unique associative algebra homomorphism* $\psi : U\mathfrak{g} \to A$ *so that the following diagram commutes.*



How to construct explicitly $U\mathfrak{g}$? Since $\varphi : \mathfrak{g} \to A$ is a linear map, it extends uniquely to an associative algebra homomorphism $\bar{\varphi} : T(\mathfrak{g}) \to A$. Moreover, φ is a Lie algebra homomorphism, so $\forall x, y \in \mathfrak{g}$ it satisfies the following:

$$\varphi([x, y]_{\mathfrak{g}}) = [\varphi(x), \varphi(y)]_{\mathrm{Lie}(A)} = \varphi(x)\varphi(y) - \varphi(y)\varphi(x) = \bar{\varphi}(x \otimes y - y \otimes x) \tag{D.5}$$

Therefore, $\bar{\varphi} : T(\mathfrak{g}) \to A$ has the elements $x \otimes y - y \otimes x - [x, y]_{\mathfrak{g}}$ in its kernel. If we denote by $\mathfrak{I}$ the two-sided ideal generated by all elements of this form, we have an unique and well-defined induced associative algebra homomorphism $\psi : T(\mathfrak{g})/\mathfrak{I} \to A$ because of the universal property of the quotient.

$U\mathfrak{g}$ is then defined as $T(\mathfrak{g})/\mathfrak{I}$, while i is the composition $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow U\mathfrak{g}$. From the relations imposed to define $U\mathfrak{g}$, it is clear that i is a Lie algebra homomorphism.

The universal property that satisfies $U\mathfrak{g}$ has very important consequences for us. Since the construction we did is natural in $\mathfrak{g}$, we can regard U as a functor $U : \mathrm{LieAlg}_{\mathbb{K}} \to \mathrm{AssAlg}_{\mathbb{K}}$, and the universal property tells us that we have an adjunction $U \dashv \mathrm{Lie}(-)$, indeed:

$$\mathrm{Hom}_{\mathrm{LieAlg}_{\mathbb{K}}}(\mathfrak{g}, \mathrm{Lie}(A)) \cong_{\mathrm{Sets}} \mathrm{Hom}_{\mathrm{AssAlg}_{\mathbb{K}}}(U\mathfrak{g}, A) \tag{D.6}$$

All this brought us to the following theorem.

Theorem 3. *If* $\mathfrak{g}$ *is a Lie algebra, then every left* $\mathfrak{g}$*-module is naturally a left* $U\mathfrak{g}$*-module, and conversely. The category* $\mathfrak{g}$-Mod *is naturally isomorphic to the category of* $U\mathfrak{g}$-Mod*.*

Proof. Recall that a $\mathfrak{g}$-module is a vector space M, together with a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Lie}(\mathrm{End}_{\mathbb{K}}(\mathrm{M}))$. Similarly, a $U\mathfrak{g}$-module is a vector space M together with an associative algebra homomorphism $U\mathfrak{g} \to \mathrm{End}_{\mathbb{K}}(\mathrm{M})$. But by adjointness we have:

$$\mathrm{Hom}_{\mathrm{LieAlg}_{\mathbb{K}}}(\mathfrak{g}, \mathrm{Lie}(\mathrm{End}_{\mathbb{K}(M)})) \cong_{\mathrm{Sets}} \mathrm{Hom}_{\mathrm{AssAlg}_{\mathbb{K}}}(U\mathfrak{g}, \mathrm{End}_{\mathbb{K}}(M)) \tag{D.7}$$

and the proof follows. □

Since the categories of modules over an associative algebra have enough projectives and injectives, it follows that also $\mathfrak{g}$-Mod has them: this is what we wanted to ensure the existence of derived functors. An useful computational tool is that we can realize these derived functors as Tor and Ext, because of the following corollary.

Corollary 1. *Let* M *be a* $\mathfrak{g}$*-module. Then:*

$$H_*(\mathfrak{g}, M) \cong \mathrm{Tor}_*^{U\mathfrak{g}}(\mathbb{K}, M) \tag{D.8}$$

$$H^*(\mathfrak{g}, M) \cong \mathrm{Ext}_{U\mathfrak{g}}^*(\mathbb{K}, M) \tag{D.9}$$

Proof. To show that two derived functors are isomorphic, we just need to show that the underlying functors are.

1. First, let us define the augmentation ideal. This is given by the two-sided ideal $\mathfrak{J} = \ker(\epsilon)$ of $U\mathfrak{g}$ generated by $i(\mathfrak{g})$, where $\epsilon : U\mathfrak{g} \to \mathbb{K}$ is the augmentation map defined by sending $i(\mathfrak{g})$ to zero. Then $\mathfrak{J} \cong \mathfrak{g}U\mathfrak{g}$ and $\mathbb{K} \cong U\mathfrak{g}/\mathfrak{J} = (U\mathfrak{g})_{\mathfrak{g}}$. Equipping $\mathbb{K}$ with the trivial right $U\mathfrak{g}$-action, we can consider:
$$\mathbb{K} \otimes_{U\mathfrak{g}} M = (U\mathfrak{g}/\mathfrak{J}) \otimes_{U\mathfrak{g}} M \cong M/\mathfrak{J}M = M/\mathfrak{g}M = M_{\mathfrak{g}} \tag{D.10}$$
where in the second step we used a general property of the tensor product between modules.

2. We can look at $\mathbb{K}$ as a left $U\mathfrak{g}$-module by equipping it with the trivial action of $\mathfrak{g}$ on it. Then:
$$\mathrm{Hom}_{U\mathfrak{g}-\mathrm{Mod}}(\mathbb{K}, M) = \mathrm{Hom}_{\mathfrak{g}-\mathrm{Mod}}(\mathbb{K}, M) \cong M^{\mathfrak{g}} \tag{D.11}$$
where the last isomorphism comes from the natural map $T : \mathrm{Hom}_{\mathfrak{g}-\mathrm{Mod}}(\mathbb{K}, M) \to M^{\mathfrak{g}}$ given by $g \mapsto g(1)$.

□

The Chevalley-Eilenberg complex

How to explicitly compute Lie algebra (co)homology? Let us first describe the procedure for computing homology. We saw in the previous section how to compute those groups using Tor, the left derived functor of the tensor product. Therefore, we can look for a projective resolution of $\mathbb{K}$ as right $U\mathfrak{g}$-module, and then just apply the algorithm we know to compute the Tor groups. The Chevalley-Eilenberg complex provides us with a standard resolution of $\mathbb{K}$. To motivate the construction of this complex, we start by noticing that $\mathbb{K}$ is the quotient of $U\mathfrak{g}$ by $\mathfrak{J}$, the two-sided ideal generated by $i(\mathfrak{g})$. $\mathfrak{J}$ was defined to be the kernel of the augmentation map $\epsilon : U\mathfrak{g} \to \mathbb{K}$ given by sending $i(\mathfrak{g})$ to zero. If we consider $d : \mathfrak{g} \otimes_{\mathbb{K}} U\mathfrak{g} \to U\mathfrak{g}$ to be just the product map, we have the following exact sequence:
$$\mathfrak{g} \otimes_{\mathbb{K}} U\mathfrak{g} \xrightarrow{d} U\mathfrak{g} \xrightarrow{\epsilon} \mathbb{K} \to 0$$
Now, we can add other terms to the sequence to provide the projective resolution we are looking for:
$$(\ldots \xrightarrow{d} \wedge^n \mathfrak{g} \otimes_{\mathbb{K}} U\mathfrak{g} \xrightarrow{d} \wedge^{n-1}\mathfrak{g} \otimes_{\mathbb{K}} U\mathfrak{g} \xrightarrow{d} \ldots \xrightarrow{d} \mathfrak{g} \otimes_{\mathbb{K}} U\mathfrak{g} \xrightarrow{d} U\mathfrak{g}) = V_*(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K}$$
where for a general n the differential $d : V_n(\mathfrak{g}) \to V_{n-1}(\mathfrak{g})$ is given by:
$$(x_1 \wedge \cdots \wedge x_n) \otimes_{\mathbb{K}} u \mapsto \sum_{i=1}^{n} (-1)^{i+1} x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \otimes_{\mathbb{K}} x_i u +$$
$$+ \sum_{1 \le i < j \le n} (-1)^{i+j} [x_i, x_j] \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n \otimes_{\mathbb{K}} u$$
The notation $\hat{x}_i$ indicates an omitted term.

Theorem 4. *$(V_*(\mathfrak{g}), d)$ is a chain complex.*

Proof. We need to show that d is actually a differential – that is, $d^2 = 0$. The computation is easy, but we need to be careful about the signs and the labelling of our elements. Explicitly, $d^2((x_1 \wedge \cdots \wedge x_n) \otimes_{\mathbb{K}} u)$ is given by:
$$\sum_{r<i} (-1)^{i+r} x_1 \wedge \cdots \wedge \hat{x}_r \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \otimes_{\mathbb{K}} x_r x_i u + \tag{D.12}$$

$$+\sum_{i<r}(-1)^{i+r+1}x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}x_rx_iu+ \tag{D.13}$$

$$+\sum_{r<j<i}(-1)^{i+j+r+1}[x_r,x_j]\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}x_iu+ \tag{D.14}$$

$$+\sum_{r<i<j}(-1)^{i+j+r}[x_r,x_j]\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}x_iu+ \tag{D.15}$$

$$+\sum_{i<r<j}(-1)^{i+j+r+1}[x_r,x_j]\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}x_iu+ \tag{D.16}$$

$$+\sum_{i<j}(-1)^{i+j+1}x_1\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}[x_i,x_j]u+ \tag{D.17}$$

$$+\sum_{r<i<j}(-1)^{i+j+r}[x_i,x_j]\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}x_ru+ \tag{D.18}$$

$$+\sum_{i<r<j}(-1)^{i+j+r+1}[x_i,x_j]\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}x_ru+ \tag{D.19}$$

$$+\sum_{i<j<r}(-1)^{i+j+r}[x_i,x_j]\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}x_ru+ \tag{D.20}$$

$$+\sum_{r<i<j}(-1)^{i+j+r}[[x_i,x_j],x_r]\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}u+ \tag{D.21}$$

$$+\sum_{i<r<j}(-1)^{i+j+r+1}[[x_i,x_j],x_r]\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}u+ \tag{D.22}$$

$$+\sum_{i<j<r}(-1)^{i+j+r}[[x_i,x_j],x_r]\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}u+ \tag{D.23}$$

$$+\sum_{r<s<i<j}(-1)^{i+j+r+s}[x_r,x_s]\wedge[x_i,x_j]\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_s\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}u+ \tag{D.24}$$

$$+\sum_{r<i<s<j}(-1)^{i+j+r+s+1}[x_r,x_s]\wedge[x_i,x_j]\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_s\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}u+ \tag{D.25}$$

$$+\sum_{r<i<j<s}(-1)^{i+j+r+s}[x_r,x_s]\wedge[x_i,x_j]\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge\hat{x}_s\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}u+ \tag{D.26}$$

$$+\sum_{i<r<s<j}(-1)^{i+j+r+s}[x_r,x_s]\wedge[x_i,x_j]\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_s\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}u+ \tag{D.27}$$

$$+\sum_{i<r<j<s}(-1)^{i+j+r+s+1}[x_r,x_s]\wedge[x_i,x_j]\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge\hat{x}_s\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}u+ \tag{D.28}$$

$$+\sum_{i<j<r<s}(-1)^{i+j+r+s}[x_r,x_s]\wedge[x_i,x_j]\wedge\cdots\wedge\hat{x}_i\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge\hat{x}_r\wedge\cdots\wedge\hat{x}_s\wedge\cdots\wedge x_n\otimes_{\mathbb{K}}u \tag{D.29}$$

Notice that:

1. (D.24)+(D.26)=0, (D.25)+(D.28)=0 and (D.26)+(D.27)=0;
2. (D.21)+(D.22)+(D.23)=0, because after relabelling we get the Jacobi identity: $[[x_i,x_j],x_r]-[[x_r,x_j],x_i]+[[x_r,x_i],x_j]=0$;
3. (D.12)+(D.13)=-(D.17);

4. (D.14)+(D.20)=0, (D.15)+(D.19)=0 and (D.16)+(D.18)=0.

Therefore, $(V_*(g), d)$ is a chain complex.

□

To use the technology of derived functors we need this complex to be a projective resolution of $\mathbb{K}$. The proof is long and technical, we refer the reader to [Wei94], Theorem 7.7.2.
Now, to compute $H_*(\mathfrak{g}, M)$ we need to tensor this resolution with $-\otimes_{U\mathfrak{g}} M$, where M is a $\mathfrak{g}$-module. Notice that:

$$(\wedge^n \mathfrak{g} \otimes_{\mathbb{K}} U\mathfrak{g}) \otimes_{U\mathfrak{g}} M \cong \wedge^n \mathfrak{g} \otimes_{\mathbb{K}} (U\mathfrak{g} \otimes_{U\mathfrak{g}} M) \cong \wedge^n \mathfrak{g} \otimes_{\mathbb{K}} M$$

Therefore $H_*(\mathfrak{g}, M)$ is the homology of the complex $C_*(\mathfrak{g}, M) := (\wedge^* \mathfrak{g} \otimes_{\mathbb{K}} M, d_{\mathrm{CE}})$, and $C_*(\mathfrak{g}, M)$ is called the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module M. There is another useful way to write this complex. If we look at $\mathfrak{g}$ as a $\mathbb{Z}$-graded vector space concentrated in degree zero, we can shift it in degree -1 (notation: $\mathfrak{g}[1]$) to write $\mathrm{Sym}^*_{\mathbb{K}}(\mathfrak{g}[1]) = \wedge^* \mathfrak{g}$, because the symmetric powers of a vector space concentrated in odd degree are the same as the exterior powers. Therefore: $C_*(\mathfrak{g}, M) = (\mathrm{Sym}^*_{\mathbb{K}}(\mathfrak{g}[1]) \otimes_{\mathbb{K}} M, d_{\mathrm{CE}})$.

Let us now describe what happens for cohomology. Recalling that $\mathrm{Hom}_{U\mathfrak{g}-\mathrm{Mod}}(-, M)$ is a left exact *contravariant*[1] functor, we can use the same projective resolution of $\mathbb{K}$. Moreover, notice that the collection of maps:

$$\mathrm{Hom}_{U\mathfrak{g}-\mathrm{Mod}}(\wedge^n \mathfrak{g} \otimes_{\mathbb{K}} U\mathfrak{g}, M) \to \mathrm{Hom}_{\mathrm{Vect}_{\mathbb{K}}}(\wedge^n \mathfrak{g}, M)$$
$$f \mapsto \overline{f} \text{ with } \overline{f}(x_1 \wedge \cdots \wedge x_n) = f((x_1 \wedge \cdots \wedge x_n) \otimes_{\mathbb{K}} 1)$$

is a quasi-isomorphism of chain complexes. If we denote as $\mathfrak{g}^\vee$ the linear dual of $\mathfrak{g}$, then $\mathrm{Hom}_{\mathrm{Vect}_{\mathbb{K}}}(\wedge^n \mathfrak{g}, M) = \mathrm{Sym}^n_{\mathbb{K}}(\mathfrak{g}^\vee[-1]) \otimes_{\mathbb{K}} M$. These are the vector spaces underlying our Chevalley-Eilenberg complex to compute cohomology. We summarize the two constructions in the following definition:

Definition 23 ([CG16], Appendix A, Definition 3.1.2). *1. The Chevalley-Eilenberg complex for Lie algebra homology of the $\mathfrak{g}$-module M is:*

$$C_*(\mathfrak{g}, M) = (\mathrm{Sym}^*_{\mathbb{K}}(\mathfrak{g}[1]) \otimes_{\mathbb{K}} M, d_{\mathrm{CE}}) \tag{D.30}$$

where the differential encodes the Lie bracket of $\mathfrak{g}$ and the module structure of M. Explicitly:

$$d_{\mathrm{CE}}(x_1 \wedge \cdots \wedge x_n \otimes m) = \sum_{i=1}^{n} (-1)^{i+1} x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \otimes_{\mathbb{K}} x_i \cdot m +$$
$$+ \sum_{1 \le i < j \le n} (-1)^{i+j} [x_i, x_j] \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n \otimes_{\mathbb{K}} m$$

We usually call this complex the CE chains.

2. The Chevalley-Eilenberg complex for Lie algebra cohomology of the $\mathfrak{g}$-module M is:

$$C^*(\mathfrak{g}, M) = (\mathrm{Sym}^*_{\mathbb{K}}(\mathfrak{g}^\vee[-1]) \otimes_{\mathbb{K}} M, d_{\mathrm{CE}}) \tag{D.31}$$

[1]This is a subtle point: usually when dealing with left exact *covariant* functors, we need an injective resolution of the object we are interested in to compute the corresponding right derived functors. But a left exact contravariant functor is a left exact covariant functor with source the opposite category. Therefore, we look for an injective resolution of our object in the opposite category, but this is the same as a projective resolution in the category we started with!

where the differential encodes the linear dual to the bracket of $\mathfrak{g}$ *on itself and on* M*. Fixing a linear basis* $\{e_k\}$ *for* $\mathfrak{g}$ *and denoting* $\{e^k\}$ *the dual basis, we have:*

$$d_{\mathrm{CE}}(e^k \otimes m) = \sum_i e^k \wedge e^i \otimes_{\mathbb{K}} e_i \cdot m - \sum_{i<j} e^k([e_i, e_j]) e^i \wedge e^j \otimes_{\mathbb{K}} M \tag{D.32}$$

and d_{CE} *is extended to the rest of the complex as a derivation of cohomological degree 1 (using the Leibniz rule repeatedly to reduce to the explicit formula above). We often call this complex the CE cochains.*

Appendix E

Basics of Algebraic Geometry

In this appendix we give some basic notions of algebraic geometry, mainly following [Vak24] and [sta25]. We start by recollecting some terminology on commutative rings, later we give all the definitions needed to arrive to define what a scheme is. As a matter of convention, all the rings we consider in the following will be assumed to be commutative, even if not specified.

Commutative rings

Let A be a commutative ring. An ideal $I \lhd A$ is called:

maximal, if I is proper and $I \subset J \lhd A$ implies $J = I$ or $J = A$.

prime, if I is proper and $xy \in I$ implies either $x \in I$ or $y \in I$.

radical, if $x^n \in I$ implies $x \in I$.

We also need the following; in a non-trivial ring A we call an element x:

a unit, if there exist y with $xy = 1$.

a zero-divisor, if there exists $y \neq 0$ with $xy = 0$.

nilpotent, if there exists $n > 0$ with $x^n = 0$.

And we call the ring A:

a field, if every non-zero element is a unit.

an integral domain, if it has no zero-divisors other than 0.

reduced, if it has no nilpotent elements other than 0.

An important characterization for maximal ideals is that I is maximal if and only if A/I is a field. Moreover, every maximal ideal is a prime ideal, and every prime ideal is also radical.
One useful property of prime ideals is that the preimage of a prime ideal under a ring homomorphism is again a prime ideal (this is not true for maximal ideals):

Lemma 3. *Let $f : A \to B$ be a ring homomorphism and $Q \lhd B$ a prime ideal. Then $f^{-1}(Q)$ is a prime ideal of A.*

Proof. Suppose $xy \in f^{-1}(Q)$. Then $f(xy) = f(x)f(y) \in Q$, because f is a ring homomorphism. Since Q is prime, this implies that either $f(x) \in Q$ or $f(y) \in Q$. Thus either $x \in f^{-1}(Q)$ or $y \in f^{-1}(Q)$, therefore $f^{-1}(Q)$ is prime. $\square$

This result will be important to consider Spec as contravariant functor.
We next explain what we mean by localization of a ring.

Definition 24. *Let A be a ring. A multiplicative subset $S \subseteq R$ is a collection of elements such that $1 \in A$ and it is close under multiplication.*

Given a multiplicative subset we can define a new ring $S^{-1}A$, called the localization of A at S. The elements of this new ring are of the form $\frac{a}{s}$ with $a \in A$, $s \in S$, and we identify two such elements if and only if:

$$\frac{a}{s} = \frac{a'}{s'} \Leftrightarrow \exists\, r \in S \;\text{ s.t. } r \cdot (as' - a's) = 0 \tag{E.1}$$

Sum and multiplication are defined precisely as you would define them in $\mathbb{Q}$; indeed $\mathbb{Q}$ can be constructed as the localization of $\mathbb{Z}$ at $\mathbb{Z}^{\times}$.
There are two different kinds of multiplicative subset that will be important to us:

1. If $f \in A$ and $S = \{1, f, f^2, \dots\}$, we denote the localization of A at S as A_f.
2. If $\mathfrak{p}$ is a prime ideal and $S = A/\mathfrak{p}$, we denote the localization of A at S as $A_{\mathfrak{p}}$.

Schemes

One of the ideas at the heart of Algebraic Geometry is that there is a deep correspondence between geometric spaces, i.e. topological spaces equipped with some geometric structure (as can be a smooth structure for a manifold), and the algebra of functions over those spaces. More precisely, the (sheaf of) functions over a space characterize completely the geometric structure it possesses.
In this appendix, we try to build some intuition about this correspondence, developing some language that will be useful in this thesis.

Given a topological space with a specific geometric structure, for example a smooth manifold, it is intuitive to think about from what kind of functions we can recover the geometric structure of the space. In the case of a smooth manifold, these should be the infinitely differentiable functions on it; or, in the case of a complex manifold, we can guess that they are the holomorphic functions. But, given an algebra A, what is its geometric incarnation? In other words, what is the geometric space whose algebra of functions is precisely A? Retreating to the algebraic world, this is the first step in understanding the definition of (affine) scheme.
We first understand this space as the sets of its point, then we equip it with a topology, and in the end we deal with local-to-global properties in promoting A to a sheaf of functions.

Let A be a commutative ring. We define the spectrum of A to be the set of its prime ideals and denote it Spec$A$. A prime ideal $\mathfrak{p}$ of A when considered as an element of Spec$A$ will be denoted $[\mathfrak{p}]$. We think of elements $a \in A$ as functions on Spec$A$, and their value at the point $[\mathfrak{p}]$ is a (mod. $\mathfrak{p}$). This means that if a lies in the prime ideal $\mathfrak{p}$, the value of a at $[\mathfrak{p}]$ is zero.

Example (the complex affine line). $\mathbb{A}^1_{\mathbb{C}} :=$ Spec$\mathbb{C}[x]$. What are the prime ideals of $\mathbb{C}[x]$? As $\mathbb{C}[x]$ is an integral domain, which means that there are no zero divisors other than 0, (0) is a prime ideal. Other prime ideals are of the form $(x-a)$ for $a \in \mathbb{C}$. Since the quotient by these ideals is a field, these are even maximal prime ideals:

$$0 \longrightarrow (x-a) \longrightarrow \mathbb{C}[x] \xrightarrow{f \mapsto f(a)} \mathbb{C} \longrightarrow 0$$

Notice that, since $\mathbb{C}$ is an algebraic closed field, we can factor every polynomial $f \in \mathbb{C}[x]$ as a product of powers of linear polynomials, by the division algorithm. Therefore, there are no other prime ideals than (0) and $(x-a), \forall a \in \mathbb{C}$.
The value of a polynomial function $f(x)$ at $[(x-a)]$ is precisely $f(a)$, while the value at $[(0)]$ is $f(x)$ mod. 0. Thus, we see that we can interpret $\text{Spec}\mathbb{C}[x]$ as the complex affine line $\mathbb{C}$: every ideal of the form $(x-a)$ corresponds to a point $a \in \mathbb{C}$, plus there is a so-called "generic point" $[(0)]$ that is contained in all other ideals, so that we can think about it near every point but located nowhere precisely: it is "generically on the line".
The upshot of this discussion is that we have found a geometric incarnation of the commutative algebra $\mathbb{C}[x]$; thus $\mathbb{C}[x]$ can be interpreted as the algebraic functions over the complex line.

Example (the complex affine plane). Consider $A = \mathbb{C}[x, y]$ and denote $\mathbb{A}^2_{\mathbb{C}} = \text{Spec}\mathbb{C}[x, y]$. What is the set of prime ideals of A? These can have three forms:

1. All the ideals of the type $(x-a, y-b)$, for every $a, b \in \mathbb{C}$; these ideals are maximal, indeed $\frac{\mathbb{C}[x,y]}{(x-a,y-b)} \cong \mathbb{C}$ via the evaluation map $f(x, y) \mapsto f(a, b)$.
2. Of course 0.
3. If $f(x, y)$ is an irreducible polynomial (like $y - x^2$), then $(f(x, y))$ is prime.

Let us try to visualize geometrically this set. In this case, we see that the maximal prime ideals correspond to the traditional points $(a, b) \in \mathbb{C}^2$. [0] is the generic point and, similarly to the case of the complex affine line, *lives everywhere but nowhere in particular*, because it is contained in all other prime ideals. The point $[(f(x, y))]$, where $f(x, y)$ is an irreducible polynomial, lies on the curve $f(x, y) = 0$ but nowhere in particular on it.
What is interesting to note here is that maximal ideals correspond to the smallest points; so that if one prime ideal contains another, the points have the opposite containment. Again, as in the case of the complex line, we recover all the complex plane with some bonus points: we can think of $\mathbb{C}[x, y]$ as the algebra of functions on the complex affine plane!

It is important for us (at least in the case of algebraic closed fields) to interpret maximal prime ideals as traditional points of affine space; this could be done because of the following theorem.

Theorem 5 (Hilbert's Weak Nullstellensatz, [Vak24], 3.2.5)**.** *If $\mathbb{K}$ is an algebraically closed field, then the maximal ideals of $\mathbb{K}[x_1, \ldots x_n]$ are precisely those ideals of the form $(x_1-a_1, \ldots, x_n-a_n)$, where $a_i \in \mathbb{K}$.*

In order to have some more intuitive geometric interpretation, we now see how two natural ways of getting new rings from old ones can be interpreted as subsets of $\text{Spec}A$.

1. *Quotients:* if I is an ideal of A, the prime ideals of A/I are in bijection with the prime ideals of A containing I. Therefore, we can picture $\text{Spec}(A/I)$ as a subset of $\text{Spec}A$. For instance, if $A = \mathbb{C}[x, y]$ and $I = (x-y)$, we can picture $\text{Spec}(\mathbb{C}[x, y]/(x-y))$ as the complex affine line inside the complex affine plane.
2. *Localizations:* here, as we explained above, we have two different flavors of localization.
   - The prime ideals of A_f are the prime ideals of A that do not contain f. Hence, $\text{Spec}A_f$ is the subset of $\text{Spec}A$ where f does not vanish.

- The prime ideals of $A_{\mathfrak{p}}$, instead, are all the prime ideals of A contained in $\mathfrak{p}$. Then Spec$A_{\mathfrak{p}}$ should be seen as a "shred of the space Spec$A$ near the subset corresponding to $\mathfrak{p}$".

Now that we have characterized Spec$A$ as a set, we proceed to define it as a topological space: this is always the starting point to have a geometric structure.

Definition 25. *Let S be a subset of A. The vanishing set of S is defined as the set of points on which all elements of S are zero:*

$$V(S) := \{[\mathfrak{p}] \in \mathrm{Spec}A : S \subset \mathfrak{p}\}. \tag{E.2}$$

The Zariski topology of Spec$A$ *is defined by declaring that $V(S)$ is closed for all S.*

This does not seem to be a very natural definition of the topology, but there is a nice base that has a natural interpretation. If $f \in A$, define the distinguished open set:

$$D(f) := \{[\mathfrak{p}] \in \mathrm{Spec}A : f \notin \mathfrak{p}\} \tag{E.3}$$

It is the locus where f does not vanish: we are taking the set of points for which f does not vanish to be open. By choosing these as a basis for the topology we are simply taking the weakest topology in which the sets that we expect to be open are in fact open.

Spec$A$ has now become a topological space, but we cannot think of giving it or recovering any interesting geometric structure from the functions A on it: geometric structures are defined by local data!
Consider for a moment a smooth manifold M: we usually give it the smooth structure by picking an atlas. Now, if we think of studying the associated algebra of $C^\infty(M)$ functions, we cannot hope to recover the atlas by just this algebra. In particular, we need a way to encode local data, and information on how to glue them together.
But local data and gluing conditions are exactly what sheaves are meant for! Here, we are grasping at the idea of locally ringed spaces; instead of giving a topological space a smooth structure by picking an atlas, we can just consider M as its set of points (and its topology) together with a sheaf of functions on M, and these data should locally look like $\mathbb{R}^n$ together with the sheaf of smooth functions $C^\infty_{\mathbb{R}^n}$ on it. The smooth structure on M can then be recovered from its sheaf of functions[1].
We arrived at the following definition:

Definition 26. *A locally ringed space is a pair $(X, \mathscr{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathscr{O}_X$ all of whose stalks are local rings, i.e. a ring with a unique maximal ideal.*

A morphism of locally ringed spaces $\pi : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ consists of a continuos map $\pi : X \to Y$ and a sheaf morphism $\pi^\sharp : \mathscr{O}_Y \to \pi_* \mathscr{O}_X$, such that this morphism induces local ring maps on stalks, i.e. it carries the maximal ideal into the maximal ideal.
Now we wish to make Spec$A$ into a locally ringed space, by giving it a structure sheaf. In principle, this means that we should assign a ring of functions on any open subset, but since we have described a basis for the Zariski topology, it is enough to define the structure sheaf on that basis. In particular, we do the following assignment: $\mathscr{O}_{\mathrm{Spec}A}(D(f)) := A_f$.

Definition 27. *The data* $(\mathrm{Spec}A, \mathscr{O}_{\mathrm{Spec}A})$ *of a locally ringed space defines what we call an affine scheme.*

[1]The proof that the two different definitions are equivalent is given, for example, in [Wed16], pages 78-79. Notice that there the author is considering premanifolds, but the proof still holds for manifolds (manifolds are just premanifolds where the underlying topological space is Hausdorff and 2[nd] countable).

It is easy to see that this is indeed a *locally* ringed space; in fact, the stalk at the point $[\mathfrak{p}] \in \mathrm{Spec}A$ is a local ring:

$$\mathscr{O}_{\mathrm{Spec}A,[\mathfrak{p}]} = \bigcup_{[\mathfrak{p}]\in D(f)} \mathscr{O}_{\mathrm{Spec}A}(D(f)) = \bigcup_{f\notin\mathfrak{p}} A_f = A_{\mathfrak{p}} \tag{E.4}$$

Moreover, since $\mathrm{Spec}A = D(1)$:

$$\Gamma(\mathrm{Spec}A, \mathscr{O}_{\mathrm{Spec}A}) = \mathscr{O}_{\mathrm{Spec}A}(\mathrm{Spec}A) = \mathscr{O}_{\mathrm{Spec}A}(D(1)) = A_1 = A \tag{E.5}$$

We see that A are the global sections of the structure sheaf, recovering the interpretation of A as the ring of functions over $\mathrm{Spec}A$.
We denote the category of affine schemes by Aff; objects in this category are obviously affine schemes, and morphisms are morphisms between locally ringed spaces.
Like a manifold is glued together from simpler spaces, a scheme is constructed by gluing together affine schemes.

Definition 28. *A scheme $(X, \mathscr{O}_X)$ is a locally ringed space such that $\forall x \in X$ there is an open neighbourhood $U \subseteq X$ of x such that $(U, \mathscr{O}_X|_U)$ is an affine scheme.*

Bibliography

[BBZB+20] Christopher Beem, David Ben-Zvi, Mathew Bullimore, Tudor Dimofte, and Andrew Neitzke. Secondary products in supersymmetric field theory. *Annales Henri Poincare*, 21(4):1235–1310, 2020.

[BCOV94] Michael Bershadsky, Sergio Cecotti, Hirosi Ooguri, and Cumrun Vafa. Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes. *Commun. Math. Phys.*, 165:311–428, 1994.

[Bry09] Jean-Luc Brylinski. *Loop Spaces, Characteristic Classes and Geometric Quantization.* Modern Birkhäuser Classics. Birkhäuser Boston, 2009.

[BV81] Igor A. Batalin and Grigory A. Vilkovisky. Gauge algebra and quantization. *Physics Letters B*, 102(1):27–31, 1981.

[CG16] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory, volume 1.* New Mathematical Monographs. Cambridge University Press, 2016.

[CG21] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory, volume 2.* New Mathematical Monographs. Cambridge University Press, 2021.

[CL12] Kevin Costello and Si Li. Quantum BCOV theory on Calabi-Yau manifolds and the higher genus B-model. 2012.

[CL20] Kevin Costello and Si Li. Anomaly cancellation in the topological string. *Adv. Theor. Math. Phys.*, 24(7):1723–1771, 2020.

[Cos11] Kevin Costello. *Renormalization and effective field theory.* Number 170. American Mathematical Soc., 2011.

[CS85] Jeff Cheeger and James Simons. Differential characters and geometric invariants. In *Geometry and Topology: Proceedings of the Special Year held at the University of Maryland, College Park 1983–1984*, pages 50–80. Springer, 1985.

[Cur14] Justin M. Curry. *Sheaves, cosheaves and applications.* PhD thesis, University of Pennsylvania, 2014.

[DVV90] Robbert Dijkgraaf, Herman L. Verlinde, and Erik P. Verlinde. Notes on topological string theory and 2-D quantum gravity. In *Cargese Study Institute: Random Surfaces, Quantum Gravity and Strings*, 11 1990.

[DVV91] Robbert Dijkgraaf, Herman L. Verlinde, and Erik P. Verlinde. Topological strings in d < 1. *Nucl. Phys. B*, 352:59–86, 1991.

[EH06] David Eisenbud and Joe Harris. *The geometry of schemes*, volume 197. Springer Science & Business Media, 2006.

[EKYY93] Tohru Eguchi, Hiroaki Kanno, Yasuhiko Yamada, and Sung-Kil Yang. Topological strings, flat coordinates and gravitational descendants. *Phys. Lett. B*, 305:235–241, 1993.

[Get94a] Ezra Getzler. Batalin-vilkovisky algebras and two-dimensional topological field theories. *Communications in mathematical physics*, 159(2):265–285, 1994.

[Get94b] Ezra Getzler. Two-dimensional topological gravity and equivariant cohomology. *Commun. Math. Phys.*, 163:473–489, 1994.

[Get95] Ezra Getzler. Operads and moduli spaces of genus 0 riemann surfaces. In *The moduli space of curves*, pages 199–230. Springer, 1995.

[GT14] Enore Guadagnini and Frank Thuillier. Path-integral invariants in abelian chern–simons theory. *Nuclear Physics B*, 882:450–484, 2014.

[GW17] Owen Gwilliam and Brian R. Williams. The holomorphic bosonic string. 11 2017.

[GW25] Owen Gwilliam and Brian R. Williams. Holomorphic field theories and higher algebra. *Bulletin of the London Mathematical Society*, 2025.

[Gwi12] Owen Gwilliam. *Factorization algebras and free field theories.* PhD thesis, Northwestern University, 2012.

[Hat02] Allen Hatcher. *Algebraic Topology.* Algebraic Topology. Cambridge University Press, 2002.

[HKK+03] Kentaro Hori, Sheldon H. Katz, Albrecht Klemm, Rahul Pandharipande, Rollin C. Thomas, Cumrun Vafa, Ravi Vakil, and Eric Zaslow. *Mirror symmetry*, volume 1 of *Clay mathematics monographs.* AMS, Providence, USA, 2003.

[KL88] Victor G. Kac and Johan W. van de Leur. On classification of superconformal algebras. In *Strings 88: A Superstring Workshop*, 1988.

[KS58] Kunihiko Kodaira and Donald C. Spencer. On deformations of complex analytic structures. *Annals of Mathematics*, 67(2):328–401, 1958.

[Lei16] Tom Leinster. Basic category theory, 2016.

[Lur09] Jacob Lurie. Derived algebraic geometry v: Structured spaces, 2009.

[Lur11] Jacob Lurie. Derived algebraic geometry x: Formal moduli problems. 2011.

[LWP88] Jose Manual Fernandez de Labastida, Edward Witten, and Mario Pernici. Topological gravity in two dimensions. *Nuclear Physics B, Particle Physics*, 310(3/4):611–624, 1988.

[PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publications mathématiques de l'IHÉS*, 117:271–328, 2013.

[Sch08] Martin Schottenloher. *A mathematical introduction to conformal field theory.* 2008.

[Sma59] Stephen Smale. Diffeomorphisms of the 2-sphere. *Proceedings of the American Mathematical Society*, 10(4):621–626, 1959.

[sta25] *The Stacks Project: an open source textbook and reference work on algebraic geometry.* 2025.

[SW20] Ingmar Saberi and Brian R. Williams. Twisted characters and holomorphic symmetries. *Lett. Math. Phys.*, 110(10):2779–2853, 2020.

[SW23] Ingmar Saberi and Brian R. Williams. Superconformal Algebras and Holomorphic Field Theories. *Annales Henri Poincare*, 24(2):541–604, 2023.

[Tu22] Loring W. Tu. Introduction to sheaf cohomology. *arXiv preprint arXiv:2206.07512*, 2022.

[Vak24] Ravi Vakil. *The Rising Sea.* Princeton University Press, 2024.

[Val14] Bruno Vallette. Algebra + homotopy= operad. *Symplectic, Poisson, and noncommutative geometry*, 62:229–290, 2014.

[Vez11] Gabriele Vezzosi. What is... a derived stack? *Notices of the American Mathematical Society*, 58(7):955–958, 2011.

[Wed16] Torsten Wedhorn. *Manifolds, Sheaves, and Cohomology.* Springer Studium Mathematik - Master. Springer Fachmedien Wiesbaden, 2016.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[Wit88] Edward Witten. Topological Quantum Field Theory. *Commun. Math. Phys.*, 117:353, 1988.

[Wit90] Edward Witten. On the structure of the topological phase of two-dimensional gravity. *Nuclear Physics B*, 340(2-3):281–332, 1990.