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Equilibria in a large Lotka-Volterra model for complex ecosystems: a random matrix perspective

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Introduction

Ecosystems usually consist of a large number of species that interact with each other through various mechanisms, such as predation, competition or mutualism. Understanding these interactions is fundamental for ecological analysis, since they shape ecosystem dynamics. In this context, random matrices play a crucial role, as they are essential to model species interactions in large ecosystems where the exact knowledge of each pairwise interaction is often out of reach. Random Matrix Theory was first introduced in ecological studies in the 1970s by Robert May, who employed it to analyse the local stability of large ecosystems. His pioneering work [16] initiated the complexity-stability debate in ecology: contrary to empirical observations, May's result showed that increased complexity implies decreased stability. In the investigation of this problem, the Lotka-Volterra model plays a key role, since it reflects ecosystem complexity while maintaining mathematical tractability. In this thesis we analyse the properties of the equilibria of a system of Lotka-Volterra equations linearly coupled via a random matrix. Our aim is to predict the properties of the equilibria of such system from the statistics of the spectrum of the interaction matrix. We focus in particular on two key properties: stability, i.e. how likely an ecosystem at equilibrium would return to it after a perturbation, and feasibility, that is the existence of an equilibrium where all species survive.
The thesis follows primarily the work presented in two papers [1] and [8], and is organised as outlined below:
Chapter 1: In the first chapter we present May's work, showing how he included random matrices in the study of local stability. We see, in particular, that he modeled at random the so-called community matrix, which reflects the effect of each species on another in the neighborhood of an equilibrium. May's paradox then emerged from the analysis of the eigenvalues of this matrix, performed using tools from Random Matrix Theory. To extend May's local analysis and investigate the relation between complexity and stability in large ecosystems, we introduce the Lotka-Volterra system of differential equations coupled through a random matrix. We describe in particular the elliptic model, a random model characterized by cross-diagonal correlations through which we can represent different types of species interactions. The features

of such matrix depend on the parameters involved in the formulation of the model. In the following chapters we analyse how these parameters influence the global behavior of the ecosystems.
Chapter 2: In this chapter we address the problem of stability. In the study of dynamical systems, the most common notion of stability is the Lyapunov stability, whose definition we recall in the first part of this chapter. The core of the chapter is then focused on presenting and proving some important results concerning necessary or sufficient conditions for stability. In particular, the most relevant theorem establishes a set of parameter values that guarantees the existence of a unique globally stable equilibrium for the elliptic Lotka-Volterra system.
Chapter 3: In the final chapter, we focus on the issue of feasibility in cases where the elliptic Lotka-Volterra system admits a unique equilibrium. We state and prove a key result which establishes a phase transition from non-feasibility to feasibility, whenever certain parameter conditions are satisfied. Then, in the case where feasibility is not reached, we analyse some properties of the surviving species at equilibrium through simulations and heuristic arguments.

Chapter 1

Complex systems in ecology

An ecosystem is a community of living organisms interacting with each other and with non-living components. Particularly important are the interactions among species (such as predation, competition, and mutualism), which can influence their survival, growth, and behavior, and play a pivotal role in shaping the structure and dynamics of ecosystems. In this thesis, we focus specifically on ecosystems composed of a large number of interacting species. As they usually present a high degree of interconnectedness and interdependence among species, such ecosystems can be regarded as archetypal complex dynamical systems. These systems can be modeled by a set of coupled differential equations of the form:

$$\frac{dx_i}{dt} = x_i \phi_i(x_1, \ldots, x_N), \tag{1.1}$$

where, in an ecological application, N denotes the number of species, x_i is the abundance of species $i \in \{1, \ldots, N\}$ and

$$\phi_i = \frac{1}{x_i} \frac{dx_i}{dt}$$

is the so-called net growth rate, which holds all the sources of growth and mortality of each species.
In the first section of this chapter we present Robert May's work, which represents the first attempt to apply results from complex dynamical systems to ecology. In the second section we define the Lotka-Volterra model which will be our main object of study in the next chapters.

1.1 Robert May's work

Robert May was the first eminent proponent of applying results from complex dynamical systems to ecosystems. His main interest was to clarify the relation between stability and complexity in ecosystems with a large number of interacting species. In order to do that, he considered a system with N functions $\boldsymbol{n} : t \mapsto (n_i(t))_{i\in[N]}$, each of which represents the abundance of species i at time t. The vector $\boldsymbol{n}$ satisfies a system of first order nonlinear differential equations of the form:

$$\frac{dn_i}{dt} = F_i(\boldsymbol{n}) \tag{1.2}$$

Then, May assumed the existence of an equilibrium $\boldsymbol{n}^* = (n_i^*)_{i\in[N]}$. In its neighborhood, a solution n_i can be written as $n_i = n_i^* + \varepsilon_i(t)$. Here the stability of the system (1.2) boils down to the stability of the linear system

$$\frac{d\varepsilon}{dt} = J(\boldsymbol{n}^*)\varepsilon,$$

where $J := J(\boldsymbol{n}^*)$ is the so-called community-matrix, defined as:

$$J_{ij} := \frac{\partial F_i}{\partial n_j}(\boldsymbol{n}^*).$$

The entries J_{ij} reflect the effect of a slight increase in the population j on the growth rate of population i. It is known that in such a system the equilibrium is stable if and only if all the eigenvalues of J have negative real parts. Thus, evaluating stability in an equilibrium requires a complete knowledge of the community matrix J, which is often out of reach. May's key idea was therefore to model J as a random matrix in order to estimate its eigenvalues using results from Random Matrix Theory (RMT). More precisely, May modeled J as:

$$J = -I + M \tag{1.3}$$

where I is the unit matrix, $M_{ii} = 0$ and for $i \neq j$, M_{ij} are independent identically distributed centered random variables with variance V and with a distribution independent from N. Notice, in particular, that May chose to replace the self-interaction coefficients J_{ii} by -1, meaning that each species, if disturbed from equilibrium, would return to it with a characteristic damping time equal to 1.

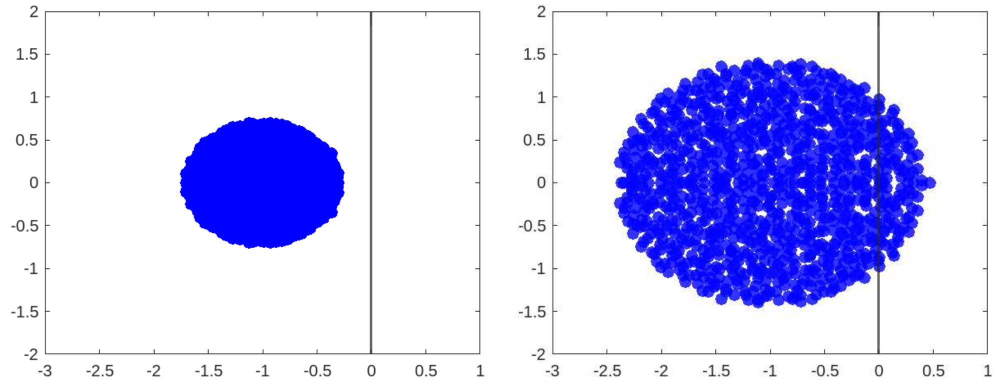
Ginibre in [14] proved that for N large enough, there is no eigenvalues of matrix J outside the disk centered at $(-1, 0)$ with radius $\sqrt{NV}$. Relying on this result, May stated the following phase transition:

Proposition 1.1. *If the matrix J is given by* (1.3)*, the equilibrium is stable with high probability if*

$$V < \frac{1}{N}$$

and unstable with high probability if

$$V > \frac{1}{N}.$$



(a) *Stability of the spectrum* (b) *Instability of the spectrum*

Figure 1.1: Spectrum of the community matrix $J = -I + M$ for $N = 1000$ species. The entries M_{ij} for $i \neq j$ are independent normal centered variables with variance $V = \frac{1}{2N} < \frac{1}{N}$ in (a) and $V = \frac{2}{N} > \frac{1}{N}$ in (b).

In Figure 1.1 we illustrate the spectrum of the community matrix J for different values of V.

Taking inspiration by Ashby and Gardner's work [11], May introduced an additional parameter C, called connectance, which expresses the probability that any pair of species will interact. In other words, this means that each species has an effect on another species with probability C. The matrix M of (1.3) is replaced by a matrix $\widetilde{M}$ equal to:

$$\widetilde{M} := \Delta_{ER} \circ M = ([\Delta_{ER}]_{ij} M_{ij})$$

where Δ_{ER} is the adjacency matrix of the Erdös-Renyi graph meaning that each entry of Δ_{ER} has probability C to be equal to 1 and 0 otherwise. Notice that the matrix $\widetilde{M}$ differs from matrix M only in terms of variance, as $\mathrm{Var}(\widetilde{M}_{ij}) = CV$, for $i \neq j$. Thus, using this new model, the phase transition can be stated as:

Proposition 1.2. *If C is the connectance of the model, the equilibrium is stable with high probability if*

$$CV < \frac{1}{N}$$

and unstable with high probability if

$$CV > \frac{1}{N}.$$

This result reveals a sharp transition from stability to instability as either the number of species N, or the connectance C, or variance V, exceeds a critical value. Hence in order to be stable, an ecosystem cannot be too complex, where complexity is represented by the product NCV. This is the so-called May's paradox, as it seems to be in contrast with empirical observations: ecosystems with a large number of species that are strongly connected appear to exist and persist for long times. In fact, this paradox seems to be only apparent. First of all, the real world is not a random system, but nature represents a small and special part of parameter space. Then, the loss of stability does not necessarily imply the extinction of species: the ecosystem may still persist in a steady out-of-equilibrium state. However, after that, theoretical and empirical work on this topic developed in order to question May's result and to further explore the relation between complexity and stability. A key limitation of May's approach is that it provides only a local analysis of stability, based on the linearization of dynamics near an equilibrium whose existence is assumed, but not proved. Local stability can only describe the behavior of the system in the neighborhood of an equilibrium point, whereas real ecosystems might operate far from equilibrium. Furthermore, May did not provide much insight into the dynamics of general ecosystems, besides showing that they can exhibit a transition to instability. The aim of this thesis is to provide a more complete analysis of ecosystem dynamics, focusing, in particular, on the following fundamental questions:

- Does an equilibrium exist?
- Is it unique?
- Is it stable?
- Is it feasible, i.e. no species go extinct?

Such questions can be addressed through the study of the following Lotka-Volterra system of differential equations:

$$\frac{dx_i}{dt} = x_i(r_i - x_i + (\mathit{\Gamma}\boldsymbol{x})_i), \quad i \in [N],$$

which we will describe in detail in the next section. In this system, the equilibria properties depend on the structure of the interaction matrix $\mathit{\Gamma}$, which, inspired by May's work, is assumed to be random. In our work, we model the matrix $\mathit{\Gamma}$ using the elliptic model, defined by

$$\mathit{\Gamma}_{ij} = \frac{A_{ij}}{\alpha_N \sqrt{N}} + \frac{\mu}{N}\mathbf{1}_N\mathbf{1}_N^\top,$$

where A is a random matrix with prescribed cross-diagonal correlations (details provided in the next section). This model is particularly useful for modeling

realistic ecosystem interactions while maintaining mathematical tractability. Nevertheless, this is only a possible formulation of the matrix. In the literature, many works focus on more involved models, which can give a better description of real ecosystems. Notable examples are the Erdös-Renyi model, useful when the only parameter of interest is the average number of interactions for a given species; and the Stochastic Block Model, which considers the existence, within the ecosystem, of groups of species sharing the same connection patterns. These models, however, are not the focus of our study.
By studying the elliptic Lotka-Volterra system, we aim to address the questions outlined above, focusing in particular on how the different parameters involved in the model formulation influence the dynamics of the ecosystem. Upon completion of this analysis, we will have established:

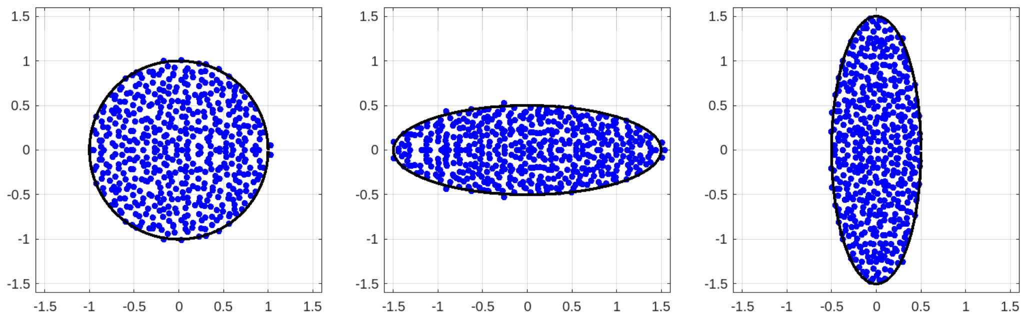
- a set of parameter values that guarantee the existence, the uniqueness and the global stability of an equilibrium for a sufficiently large number of species N;
- in the case where the uniqueness is assured, a threshold value of α_N above which, for large N and for $\boldsymbol{r} = \mathbf{1}$, feasibility happens with high probability.

1.2 Lotka-Volterra systems

Our main object of study is the following Lotka-Volterra system of differential equations:

$$\frac{dx_i}{dt} = x_i(r_i - x_i + (\Gamma \boldsymbol{x})_i) \tag{1.4}$$

where $i \in [N] := \{1, \ldots, N\}$ and $\boldsymbol{x} = (x_1, \ldots, x_N)$. The parameter N represents the number of species, $x_i = x_i(t)$ is a dimensionless quantity in relation with the abundance of species i, r_i represents the intrinsic growth of species i and $\Gamma = \Gamma_{ij}$ is a $N \times N$ matrix reflecting the interaction effect of species j on the growth of species i.
It is clear that if we define $\phi_i^{LV}(\boldsymbol{x}) := r_i - x_i + (\Gamma \boldsymbol{x})_i$ the equation (1.4) follows the general form of complex dynamical systems (1.1).
Our interest lies in the study of large ecosystem, i.e. systems with $N \gg 1$. In such instances, the precise knowledge of the interaction matrix Γ is often out of reach. Thus, inspired by May's work, an interesting alternative is to model Γ as a random matrix in order to use RMT. The statistical properties of the entries may then reflect a partial knowledge of the ecological interaction network. In the following, we describe a possible random model for the matrix Γ.



(a) $\gamma = 0$ (b) $\gamma = 0.5$ (c) $\gamma = -0.5$

Figure 1.2: Spectrum of 500×500 matrices defined as in (1.5) with standard Gaussian entries, $\mu = 0$ and (a) $\gamma = 0$, (b) $\gamma = 0.5$, (c) $\gamma = -0.5$. The solid line represents the boundary of the support of the circular law in (a) and the elliptic distribution with parameter γ in (b) and (c).

The elliptic model

In this model the interaction matrix Γ takes the following form:

$$\Gamma_{ij} = \frac{A_{ij}}{\sqrt{N}} + \frac{\mu}{N}\mathbf{1}_N\mathbf{1}_N^\top, \tag{1.5}$$

where N is the number of species, μ is a fixed real number, $\mathbf{1}_N$ stand for the $N \times 1$ vector of one and $A = (A_{ij})_{i,j\in[N]}$ is a random matrix satisfying:

i) for $i \leq j$, A_{ij} are independent identically distributed centered variables with unit variance,

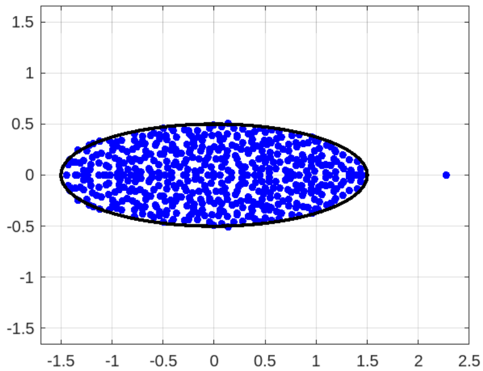
ii) for $i < j$, the vector (A_{ij}, A_{ji}) is a standard bivariate vector, independent from the remaining random variables, with covariance $\mathrm{cov}(A_{ij}, A_{ji}) = \gamma \in [-1, 1]$.

In the case where $\mu = 0$ and $|\gamma| \neq 1$, the spectrum of Γ converges toward the uniform distribution on the ellipse:

$$\mathcal{E}_\gamma = \left\{ z \in \mathbb{C}, \frac{\mathrm{Re}^2(z)}{(1+\gamma)^2} + \frac{\mathrm{Im}^2(z)}{(1-\gamma)^2} \leq 1 \right\},$$

see Figure 1.2 (b) and (c).
If $\mu > 1$ and does not belong to $\mathcal{E}_\gamma$, then it has been shown in [19] that there exists an extra random eigenvalue of Γ which will converge to $\mu + \frac{\gamma}{\mu}$ as $N \to \infty$, see Figure 1.3.
We can also consider the elliptic model with an extra normalization term α_N that may or may not depend on N:

$$\Gamma_{ij} = \frac{A_{ij}}{\alpha_N\sqrt{N}} + \frac{\mu}{N}\mathbf{1}_N\mathbf{1}_N^\top. \tag{1.6}$$



(a) $\gamma = 0.5$ $\mu = 2$

Figure 1.3: Spectrum of 500×500 matrix Γ with standard Gaussian entries, $\gamma = 0.5$ and $\mu = 2$. The solid line represents the boundary of the support of the limiting spectral distribution for an elliptic model. As expected there's an extra eigenvalue near $2 + \frac{\gamma}{2}$.

If $\alpha_N = \alpha$ is fixed, it only tunes the variance of the entries since $\mathrm{var}(\Gamma_{ij}) = \frac{1}{\alpha^2 N}$. Otherwise, if $\alpha_N \to \infty$ as $N \to \infty$, it has the effect of asymptotically squeezing to zero the spectral radius $\rho(\Gamma) := \max\{|\lambda(\Gamma)|, \lambda(\Gamma) \in \mathbb{C}$ eigenvalue of $\Gamma\}$ of the interaction matrix Γ.
This model is particularly interesting for its flexibility. By tuning the correlation parameter γ, it can describe different types of ecological communities, going from the symmetric case ($\gamma = 1$), known as the Wigner model, to the antisymmetric one ($\gamma = -1$). In particular, positive correlations are used to model mutualistic interactions while negative ones model predator/prey interactions. A special case arises when $\gamma = 0$ and $\mu = 0$, that corresponds to the model with independent and identically distributed entries in which the reciprocal interactions are uncorrelated, see Figure 1.2 (a). In this case the circular law (cf. [4]) asserts that the spectrum of Γ converges toward the uniform distribution on the disk of radius 1.
In May's work the stability of system's equilibria depends on the product NCV. Similarly, in the Lotka-Volterra model dynamics are governed by some parameters such as the normalization term α_N, the covariance γ and the bias μ. Our goal in the following chapters will be to investigate how these parameters can affect the behavior of the corresponding dynamical system. In particular, as we wrote in the previous section, we aim to address the following fundamental questions: does an equilibrium exist? Is it unique? Is it stable? Is it feasible?

Chapter 2

Stability

In the study of ecosystems an interesting issue concerns the stability of equilibria: what happens if we perturb a solution $\boldsymbol{x}$ at equilibrium? Will it return to the equilibrium? In this chapter we aim to address such questions through the analysis of the Lotka-Volterra system.
In the first section we introduce some preliminary notions that will be crucial in the following analysis of stability. In the second section we present three theorems that supply necessary or sufficient conditions for the stability of the system.

2.1 Preliminary Notions

In this section, we introduce some preliminary notions that are essential in the analysis of stability. We begin by observing the non-negativity of Lotka-Volterra system's solutions, which is a fundamental property in ecological models. Then, we define the key concepts of stability and uninvadability. Finally, we introduce the Linear Complementarity Problem, a mathematical tool useful for the proofs of the theorems presented in the next section.

Non-negativity of solutions

We first observe an important property of the solutions of system (1.4):

Remark 2.1. *If $\boldsymbol{x}(t=0) = \boldsymbol{x}_0 > 0$ componentwise, then $\boldsymbol{x}(t) > 0$ for $t > 0$.*

This observation follows from the existence and uniqueness of solutions theorem for differential equations, whose assumptions are satisfied by the Lotka-Volterra system (1.4). Define

$$\mathbb{R}^N_+ = \{\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : x_i \geq 0 \ for \ i = 1, \dots, N\}. \tag{2.1}$$

For each $i \in [N]$, the constant solution $x_i(t) \equiv 0$ is the unique solution of the i-th equation of (1.4) satisfying $x_i(0) = 0$. This implies that the coordinate hyperplane $x_i = 0$ is invariant, in the sense that any solution which starts in it remains there for all time for which it is defined. Consequently, the boundary of $\mathbb{R}_+^N$ is invariant and so is $\operatorname{int}\mathbb{R}_+^N$. In particular, this means that if $x_i(0) > 0$ then $x_i(t) > 0$ for all $t \geq 0$ and for all $i \in [N]$.
Observation 2.1 ensures, in particular, that all solutions and consequently all equilibria of the system have biological meaning, as they remain non-negative for all time.

Notions of stability and uninvadability

When we talk about stability, we always refer to Lyapunov stability, that is the most common notion of stability in dynamical systems. We now recall the definition:

Definition 2.1. *The equilibrium $\boldsymbol{x}^*$ is said to be:*

- *Lyapunov stable if for any neighborhood U of $\boldsymbol{x}^*$, there exists a neighborhood W of $\boldsymbol{x}^*$ such that*

$$\boldsymbol{x}(0) \in W \implies \boldsymbol{x}(t) \in U \text{ for all } t \geq 0.$$

- *Asymptotically stable, if and only if it is stable and the neighborhood W can be chosen so that*

$$\boldsymbol{x}(0) \in W \implies \boldsymbol{x}(t) \xrightarrow[t\to\infty]{} \boldsymbol{x}^*.$$

- *Globally stable if it is stable and*

$$\forall \boldsymbol{x}(0) \in (0,\infty)^N,\ \boldsymbol{x}(t) \xrightarrow[t\to\infty]{} \boldsymbol{x}^*.$$

Another fundamental concept is the notion of uninvadability:

Definition 2.2. *An equilibrium $\boldsymbol{x}^* = (x_i^*)$ of (1.1) is called uninvadable if it satisfies one of the following conditions:*

a) $\phi(\boldsymbol{x}^*) = 0$ *and* $x_i^* > 0$

b) $x_i^* = 0$ *and* $\phi(\boldsymbol{x}^*) \leq 0$

In the first case the species x_i^* is said to survive, while in the second it is said to vanish.

Linear Complementarity Problem

The Linear Complementarity Problem (LCP), which is part of the theory of mathematical programming, is important because it is closely connected to the problem of finding out uninvadable equilibrium points of system (1.4).
Given a $N \times N$ matrix M and a vector $q \in \mathbb{R}^N$, we say that the $LCP(M, q)$ admits a solution $(z, w) \in \mathbb{R}^N \times \mathbb{R}^N$ if there exist two such vectors satisfying the following set of constraints:

$$\begin{cases} w & = Mz + q \geq 0, \\ z & \geq 0, \\ w^\top z & = 0. \end{cases}$$

In this case, we simply write $z \in LCP(M, q)$ since w can be inferred from z. Noting that $z, w \geq 0$, the condition $w^\top z = \sum_{i=1}^N w_i z_i = 0$ implies that $w_i z_i = 0$ for each $i = 1, \dots, N$. Hence, a solution $\boldsymbol{x}$ of $LCP(M, q)$ satisfies $x_i[q_i + (M\boldsymbol{x})_i] = 0$ for $i = 1, \dots, N$.
Thus, setting $M = I - \mathit{\Gamma}$, $q = -\boldsymbol{r}$, the system above is equivalent to:

$$\begin{cases} y_i = -r_i + [(I - \mathit{\Gamma})\boldsymbol{x}]_i & \geq 0, \\ x_i & \geq 0, \qquad for\ i \in [N] \\ x_i(-r_i + [(I - \mathit{\Gamma})\boldsymbol{x}]_i) & = 0. \end{cases}$$

A solution $\boldsymbol{x}^* \in LCP(I - \mathit{\Gamma}, -\boldsymbol{r})$ of this system correspond exactly to an uninvadable equilibrium $\boldsymbol{x}^* = (x_i^*)_{i \in [N]}$ of (1.4).

2.2 Results on Lyapunov stability

Now that we have defined the necessary tools for analysing stability, we can present three main results.
The first theorem concerns a general complex dynamical system and establishes a link between the stability of an equilibrium and the Linear Complementarity Problem.

Theorem 2.1. *Consider the system*

$$\frac{dx_i}{dt} = x_i \phi_i(x_1 \dots, x_N), \quad i \in [N], \tag{2.2}$$

with all ϕ_i continuous. If an equilibrium point $\boldsymbol{x}^* \geq 0$ of (2.2) *is stable, then $\phi_i(x_1^*, \dots, x_N^*) \leq 0,\ i \in [N]$.*

Proof. Let $\boldsymbol{x}^*$ be an equilibrium. If $x_i^* > 0$ then necessarily $\phi_i(\boldsymbol{x}^*) = 0$. Thus, if $\boldsymbol{x}^* > 0$ the theorem is barely true. Hence, let us suppose that exists i such that $x_i^* = 0$ and define $I = \{i \in [N] : x_i^* = 0\}$. If the theorem is not true, then there exists $i \in I$ such that $\phi_i(\boldsymbol{x}^*) > 0$. Since $\phi_i(\boldsymbol{x})$ is continuous, for some $\varepsilon > 0$, $\phi_i(\boldsymbol{x}) > \delta > 0$ for $\boldsymbol{x} \in S_\varepsilon = \{\boldsymbol{x} \in \mathbb{R}_+^N | |\boldsymbol{x} - \boldsymbol{x}^*| \leq \varepsilon\}$, where $\mathbb{R}_+^N$ is defined as in (2.1). Let $\Omega = S_\varepsilon \cap \operatorname{int} \mathbb{R}_+^N$. Then

$$x_i \phi_i(\boldsymbol{x}) > 0 \quad \text{for } \boldsymbol{x} \in \Omega \tag{2.3}$$

For $\boldsymbol{x} \in \Omega$, $\boldsymbol{x} > 0$ and also its time derivative along a solution of (2.2) is positive by (2.3).
Let us take an initial value $\boldsymbol{x}^0$ of (2.2) in the neighborhood Ω of $\boldsymbol{x}^*$ and let $\Omega(x_i^0) = \Omega \cap \{\boldsymbol{x} \in \mathbb{R}_+^N | x_i \geq x_i^0\} \subset \Omega$. If the solution $\boldsymbol{x}(t)$ of (2.2) with initial value $\boldsymbol{x}^0$ belongs to $\Omega(x_i^0)$, then

$$\begin{aligned} x_i(t) &= x_i^0 + \int_0^t \frac{dx_i(t)}{dt}\Big|_{(2.2)} dt \\ &\geq x_i^0 + \alpha(x_i^0) t \end{aligned} \tag{2.4}$$

where

$$\alpha(x_i^0) = \min_{\boldsymbol{x} \in \Omega(x_i^0)} \{x_i \phi_i(\boldsymbol{x})\} > 0. \tag{2.5}$$

In particular, this implies that

$$x_i(t) > x_i^0 \quad \text{for all } t > 0. \tag{2.6}$$

By (2.4) e (2.5) and the boundedness of $x_i(t)$ on $\Omega(x_i^0)$, $\boldsymbol{x}(t)$ intersect the boundary of $\Omega(x_i^0)$ ($\operatorname{bd} \Omega(x_i^0)$) at some finite time $t^* > 0$. If $\boldsymbol{x}(t^*) \in \operatorname{bd} \Omega(x_i^0)$ but $\boldsymbol{x}(t^*) \notin \operatorname{bd} S_\varepsilon$, then $x_i(t^*) = x_i^0$ which contradicts (2.6). So $\boldsymbol{x}(t^*) \in \operatorname{bd} S_\varepsilon$. Hence $\boldsymbol{x}^*$ is not stable and this is absurd. □

Notice that this result implies that an equilibrium $\boldsymbol{x}^*$ of (1.4) can be stable only if it is uninvadable, i.e. it have to be a solution of $LCP(I - \mathit{\Gamma}, -\boldsymbol{r})$.
However, this theorem does not guarantee the existence of an equilibrium. We have already observe that an equilibrium point $\boldsymbol{x}^*$ of (1.4), if it exist, is a non-negative point satisfying

$$x_i^*(r_i - x_i^* + (\mathit{\Gamma} \boldsymbol{x}^*)_i) = 0, \quad i \in [N],$$

but existence and uniqueness of the equilibrium may not be guaranteed. The following theorem by Takeuchi and Adachi [20] provides a sufficient condition for the existence of a unique equilibrium that is globally stable, based on properties of the matrix $\mathit{\Gamma} - I$ of the model (1.4). Before stating and proving the theorem, we need to introduce some useful classes of matrix and to present two auxiliary lemmas.

Definition 2.3. *Suppose that A is a $N \times N$ real matrix. A is said to be :*

- *Volterra-Lyapunov stable (VL-stable) if there exists a $N \times N$ positive definite diagonal matrix W such that $WA + A^\top W$ is negative definite*
- *a negative quasi-definite matrix if and only if $u^\top Bu$ is negative for any vector $u \in \mathbb{R}^N$, $u \neq 0$*
- *a P-matrix if and only if its principal minors are all positive.*

Lemma 2.1. *If A is VL-stable then $-A$ is a P-matrix.*

Proof. If A is VL-stable then $\boldsymbol{x}^\top(WA + A^\top W)\boldsymbol{x} < 0$ for $\boldsymbol{x} \in \mathbb{R}^N$, $\boldsymbol{x} \neq 0$, i.e.

$$\begin{aligned}
0 > \boldsymbol{x}^\top WA\boldsymbol{x} + \boldsymbol{x}^\top A^\top W\boldsymbol{x} &= \sum_{i=1}^N x_i w_i \sum_{k=1}^N a_{ik}x_k + \sum_{i=1}^N x_i \sum_{k=1}^N a_{ik}^{(\top)} w_k x_k \\
&= \sum_{i=1}^N x_i w_i \sum_{k=1}^N a_{ik}x_k + \sum_{k=1}^N w_k x_k \sum_{i=1}^N a_{ik}^{(\top)} x_i \\
&= \sum_{i=1}^N x_i w_i \sum_{k=1}^N a_{ik}x_k + \sum_{k=1}^N x_k w_k \sum_{i=1}^N a_{ki} x_i \\
&= 2\sum_{i=1}^N x_i w_i \sum_{k=1}^N a_{ik}x_k \\
&= 2\boldsymbol{x}^\top WA\boldsymbol{x}
\end{aligned} \tag{2.7}$$

Hence, WA is negative quasi-definite by Definition 2.3. Thus, $-WA$ is a P-matrix [18]. The principal minors of $-WA$ are equal to the product of the corresponding principal minors of $-A$ and W, since W is a diagonal matrix. Therefore, every principal minor of $-A$ is positive, that is, $-A$ is a P-matrix, because W is positive definite. □

Lemma 2.2. *Suppose that A is a $N \times N$ matrix and $b \in \mathbb{R}^N$. Then $LCP(A, b)$ has a unique solution for each $b \in \mathbb{R}^N$ if and only if A is a P-matrix.*

Proof. The proof of this lemma is omitted and can be found in [17]. □

Now we can state the theorem:

Theorem 2.2. *Consider the system* (1.4)*:*

$$\frac{dx_i}{dt} = x_i(r_i - x_i + (\Gamma\boldsymbol{x})_i) = x_i(r_i + [(-I + \Gamma)\boldsymbol{x}]_i)$$

and assume that the matrix $\Gamma - I$ is VL-stable. Then there exist a unique equilibrium point $\boldsymbol{x}^*$ solution of the $LCP(I - \Gamma, -\boldsymbol{r})$ and this equilibrium is globally stable.

Proof. Let $A = \Gamma - I$. If A is VL-stable, then $-A$ is a P-matrix. So, by Lemma 2.1 and Lemma 2.2 $LCP(-A, -\boldsymbol{r})$ has a unique solution $\boldsymbol{x}^*$ for each $\boldsymbol{r} \in \mathbb{R}^N$ which is an uninvadable equilibrium point of (1.4). Now we need to prove that this equilibrium is globally stable.
Let I be a subset of $M = \{1, \dots, N\}$ satisfying $x_i^* = 0$ for any $i \in I$ and $J = M - I$. Define the set $\mathbb{R}_I^N$ as
$\mathbb{R}_I^N = \{\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}_+^N | x_i \geq 0 \text{ for } i \in I \text{ and } x_j > 0 \text{ for } j \in J\}$.
Now, consider the following continuously differentiable function:

$$V(x) = \sum_{j \in J} w_j \left[x_j - x_j^* - x_j^* \log \left(\frac{x_j}{x_j^*} \right) \right] + \sum_{i \in I} w_i x_i,$$

where the w_i $(i = 1, \dots, N)$ are positive constant numbers which are yet unspecified. Further, define a compact subset Ω of $\mathbb{R}_I^N$ such that:

$$\Omega(L) = \{\boldsymbol{x} \in \mathbb{R}_I^N | V(\boldsymbol{x}) \leq L(\boldsymbol{x}(0))\}$$

where $L(\boldsymbol{x}(0))$ is a positive constant number which depends on an initial value $\boldsymbol{x}(0)$ and it satisfies $L(\boldsymbol{x}(0)) \geq V(\boldsymbol{x}(0))$. Clearly $V(\boldsymbol{x}^*) = 0$ and now we prove that $V(\boldsymbol{x}) > 0$ for $\boldsymbol{x} \in \Omega$, $\boldsymbol{x} \neq \boldsymbol{x}^*$. It is obvious that $\sum_{i \in I} w_i x_i \geq 0$ and $w_j > 0$ for $j \in J$. Hence, we just need to show that $f(x_j) := x_j - x_j^* - x_j^* \log \left(\frac{x_j}{x_j^*} \right) > 0$ for each $j \in J$, $\boldsymbol{x} \neq \boldsymbol{x}^*$. We study the derivative of $f(x_j)$:

$$f'(x_j) = 1 - \frac{x_j^*}{x_j}.$$

Since $x_j > 0$ it is clear that $f'(x_j) > 0$ if $x_j > x_j^*$, $f'(\boldsymbol{x}) = 0$ only if $x_j = x_j^*$ and $f'(x_j) < 0$ otherwise. Hence, x_j^* is such that $f(x_j^*) = 0$ and it is the unique point of minimum of $f(x_j)$. This proves what we want.
Now, in order to use the second Lyapunov's Theorem, we study the time derivative of $V(\boldsymbol{x}(t))$ along a solution of (1.4):

$$\begin{aligned}
\dot{V}(\boldsymbol{x}(t))|_{(1.4)} =& \sum_{j \in J} w_j \left(1 - \frac{x_j^*}{x_j} \right) \dot{x}_j + \sum_{i \in I} w_i \dot{x}_i \\
=& \sum_{j \in J} w_j \left(1 - \frac{x_j^*}{x_j} \right) x_j \left(r_j + \sum_{k=1}^{N} a_{jk} x_k \right) + \sum_{i \in I} w_i x_i \left(r_i + \sum_{k=1}^{N} a_{ik} x_k \right) \\
=& \sum_{j \in J} w_j (x_j - x_j^*) \sum_{k=1}^{N} a_{jk} x_k + \sum_{i \in I} w_i x_i \sum_{k=1}^{N} a_{ik} x_k \\
&+ \sum_{j \in J} w_j (x_j - x_j^*) r_j + \sum_{i \in I} w_i x_i r_i
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in J} w_j(x_j - x_j^*) \sum_{k=1}^{N} a_{jk}(x_k - x_k^*) + \sum_{i\in I} w_i x_i \sum_{k=1}^{N} a_{ik}(x_k - x_k^*) \\
&\quad + \sum_{j\in J} w_j(x_j - x_j^*)\left(r_j + \sum_{k=1}^{N} a_{ik}x_k^*\right) + \sum_{i\in I} w_i x_i \left(r_i + \sum_{k=1}^{N} a_{jk}x_k^*\right) \\
&= \sum_{i=1}^{N} w_i(x_i - x_i^*) \sum_{k=1}^{N} a_{ik}(x_k - x_k^*) \\
&\quad + \sum_{i\in I} w_i x_i \left(r_i + \sum_{k=1}^{N} a_{jk}x_k^*\right) \qquad (2.8)
\end{aligned}
$$

Now, the second term of the right hand side in (2.8) is nonpositive, since $\boldsymbol{x}^*$ is un uninvadable equilibrium. The first term, as we have seen in (2.7) can be written as:

$$
\sum_{i=1}^{N} w_i(x_i - x_i^*) \sum_{k=1}^{N} a_{ik}(x_k - x_k^*) = \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^*)^\top (WA + A^\top W)(\boldsymbol{x} - \boldsymbol{x}^*)
$$

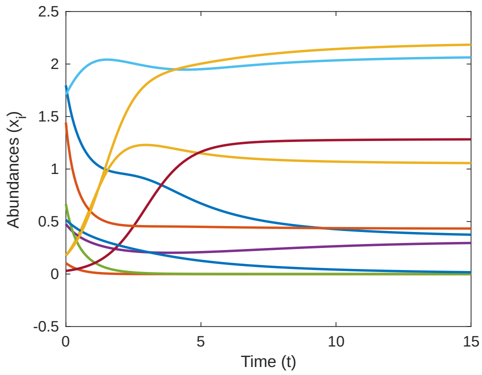
where $W = diag(w_1, \ldots, w_N)$. Since A is VL-stable, this term is negative definite. Therefore, $\dot{V}(\boldsymbol{x}(t))|_{(1.4)}$ is negative in Ω besides $\boldsymbol{x} = \boldsymbol{x}^*$ and it vanishes only at $\boldsymbol{x} = \boldsymbol{x}^*$. Hence, by the second theorem of Lyapunov, $\boldsymbol{x}^*$ is asymptotically stable with respect to Ω. The union of all sets $\Omega(L)$ as $L \to +\infty$ is equal to $\mathbb{R}_I^N$. Consequently, $\boldsymbol{x}^*$ is stable with respect to $\mathbb{R}_I^N$ and every solution converges to $\boldsymbol{x}^*$ as $t \to +\infty$ if $\boldsymbol{x}(0) \in \mathbb{R}_I^N$. This complete the proof of the theorem. □

This result provides a condition on matrix $\Gamma - I$ of a general LV system ensuring existence, uniqueness and global stability of the equilibrium. However, it is usually difficult to determine when this matrix satisfies the hypothesis, i.e. is VL-stable. The next theorem establishes a set of parameter values for which the matrix $\Gamma - I$, defined as in the elliptic model (1.6), is VL-stable. In Figure 2.1, we present simulations illustrating the convergence of $\boldsymbol{x}$ to the equilibrium $\boldsymbol{x}^*$ under these parameter values.

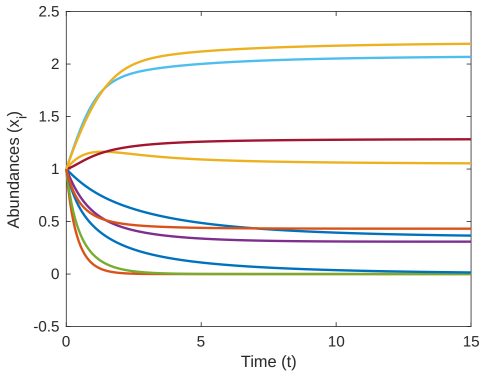
Theorem 2.3. *Let Γ be defined as in* (1.6) *with entries with a fourth finite moment. Let $\alpha_N = \alpha$ be fixed and let $(\gamma, \alpha, \mu) \in \mathcal{A}$, where $\mathcal{A}$ is*

$$
\mathcal{A} = \Bigg\{ (\gamma, \alpha, \mu) \in (-1, 1) \times (0, \infty) \times \mathbb{R}, \\
\alpha > \sqrt{2(1+\gamma)}, \mu < \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 - \tfrac{2(1+\gamma)}{\alpha^2}} \Bigg\}.
$$

Then almost surely, there exists N large enough such that the system (1.4) *admits a unique globally stable equilibrium $\boldsymbol{x}^*$.*



(a) *Initial condition drawn in* $(0, 2)$



(b) *Initial condition equal to* 1

Figure 2.1: Representation of the dynamics of a system with $N = 10$ species. We consider two distinct initial conditions for a fixed matrix $\mathit{\Gamma}$ with parameters $\gamma = 0$, $\alpha = 2$, $\mu = 0$. Simulations show that the abundances converge in both cases toward the unique globally stable equilibrium $\boldsymbol{x}^*$, as predicted by Theorem 2.3.

Proof. We want to prove that the matrix $\mathit{\Gamma} - I$ is VL-stable in order to use Theorem 2.2. Therefor, we study

$$\mathit{\Gamma} - I + \mathit{\Gamma}^\top - I = \left(\frac{A + A^\top}{\alpha\sqrt{N}} + \frac{2\mu}{N}\mathbf{1}\mathbf{1}^\top \right) - 2I.$$

We will rely on the following condition:

$$(\mathit{\Gamma} + \mathit{\Gamma}^\top) - 2I \text{ is negative definite } \Leftrightarrow \lambda_{max}(\mathit{\Gamma} + \mathit{\Gamma}^\top) < 2 \tag{2.9}$$

Notice that $\frac{(A+A^\top)}{\alpha}$ is a symmetric matrix with independent centered entries with variance $\frac{2(1+\gamma)}{\alpha^2}$ above the diagonal and by assumption, its entries have finite fourth moment. In this case, it is known that the largest eigenvalue of the normalized matrix almost surely converges to the right edge of the support of the semi-circular law (see [2]):

$$\lambda_{max}\left(\frac{A + A^\top}{\alpha\sqrt{N}}\right) \xrightarrow[N\to\infty]{a.s.} \frac{2\sqrt{2(1+\gamma)}}{\alpha}. \tag{2.10}$$

Suppose that $(\gamma, \alpha, \mu) \in \mathcal{A}$. In this case, since $\alpha > \sqrt{2(1+\gamma)}$,

$$\frac{\sqrt{1+\gamma}}{\alpha\sqrt{2}} < \frac{1}{2} < \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2(1+\gamma)}{\alpha^2}}.$$

We consider three subcases:

(i) $\mu = 0$,

(ii) $\mu \leq \frac{\sqrt{1+\gamma}}{\alpha\sqrt{2}}$,

(iii) $\mu \in \left(\frac{\sqrt{1+\gamma}}{\alpha\sqrt{2}}, \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2(1+\gamma)}{\alpha^2}} \right)$.

In the centered case (i), for (2.10), it is clear that condition (2.9) asymptotically occurs whenever $\alpha > \sqrt{2(1+\gamma)}$.
In order to study subcases (ii) and (iii), we denote by $P = \frac{2\mu}{N}\mathbf{1}\mathbf{1}^\top$ and by $H = \frac{A+A^\top}{\alpha\sqrt{N}}$. We are interested in the top eigenvalue of the symmetric matrix $H + P$. Notice that P admits a unique non zero eigenvalue 2μ. Based on a result by Capitaine et al. [7, Theorem 2.1], we have:

$$\lambda_{max}(H+P) \xrightarrow[N\to\infty]{a.s.} \begin{cases} 2\mu + \frac{1+\gamma}{\alpha^2\mu} & \text{if } \mu > \frac{\sqrt{1+\gamma}}{\sqrt{2}\alpha} \\ \frac{2\sqrt{2(1+\gamma)}}{\alpha} & \text{else.} \end{cases}$$

Consider subcase (ii), then

$$\lambda_{max}(H+P) \xrightarrow[N\to\infty]{a.s.} \frac{2\sqrt{2(1+\gamma)}}{\alpha}$$

which is lower than 2 since $(\gamma, \alpha, \mu) \in \mathcal{A}$. Hence in this case condition (2.9) is eventually satisfied.
We finally consider subcase (iii). In this case,

$$\lambda_{max}(H+P) \xrightarrow[N\to\infty]{a.s.} 2\mu + \frac{1+\gamma}{\alpha^2\mu}.$$

We shall prove that

$$2\mu + \frac{1+\gamma}{\alpha^2\mu} < 2$$

which is equal to

$$2\alpha^2\mu^2 - 2\alpha^2\mu + 1 + \gamma < 0. \tag{2.11}$$

Therefore, we study the polynomial $P(X) = 2\alpha^2X^2 - 2\alpha^2X + 1 + \gamma$. P's discriminant is positive if $\alpha > \sqrt{2(1+\gamma)}$ and P' roots are:

$$\mu^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{2(1+\gamma)}{\alpha^2}}.$$

Since $P\left(\frac{\sqrt{1+\gamma}}{\alpha\sqrt{2}}\right) < 0$ and $P(X) < 0$ for $X \in (\mu^-, \mu^+)$, then $\frac{\sqrt{1+\gamma}}{\alpha\sqrt{2}} \in (\mu^-, \mu^+)$. In particular, condition (2.11) is fulfilled for $\mu \in \left(\frac{\sqrt{1+\gamma}}{\alpha\sqrt{2}}, \mu^+\right)$, which is exactly subcase (iii). Hence a.s $\lambda_{max}(H+P)$ is eventually strictly lower than 2.
We have proved that for $(\gamma, \alpha, \mu) \in \mathcal{A}$, matrix $\Gamma - I$ is VL-stable. Thus, we can rely on Theorem 2.2 to conclude.

□

Chapter 3

Feasibility

Besides stability, another interesting question concerns feasibility, that is the condition for which no species vanishes at the equilibrium.

Definition 3.1. *An equilibrium* $\boldsymbol{x}^* = (x_i^*)_{i\in[N]}$ *of* $\frac{dx_i}{dt} = x_i\phi_i(x_1, \ldots, x_N)$ *is called feasible if* $x_i^* > 0$ *for* $i \in [N]$.

In this chapter, we investigate the existence of a unique feasible equilibrium of the elliptic Lotka-Volterra system. In the first section, we analyse how the probability of observing feasibility depends on the parameter α. In particular, we establish a sharp phase transition from non-feasibility to feasibility around the threshold value $\alpha_N^* = \sqrt{2\log(N)}$. The second section is devoted to the proof of this result. Finally, in the last section, we examine the case where a unique equilibrium exists but feasibility is not reached, focusing on the number of survivors and the individual distribution of the abundance of a given surviving species.

3.1 Feasibility for different α values

In this section, we first show that a feasible equilibrium is unlikely to occur when α is fixed. Therefore, in the second part, we analyse feasibility in the case where α_N tends to infinity.

No feasibility if α is fixed

Assume that $\boldsymbol{x}^* = (x_i^*)_{i\in[N]}$ is a feasible equilibrium of the Lotka-Volterra system $\frac{dx_i}{dt} = x_i(r_i - x_i + (\Gamma\boldsymbol{x})_i)$. Then it satisfies:

$$(I - \Gamma)\boldsymbol{x}^* = \boldsymbol{r}. \tag{3.1}$$

Let $\Gamma = \frac{A_{ij}}{\alpha_N \sqrt{N}}$, with $\gamma = \mathrm{cov}(A_{ij}, A_{ji}) = 0$ (i.i.d. model). The spectral radius $\rho\big(\frac{A}{\sqrt{N}}\big)$ converges to 1 a.s. (see [13]). Hence, for every $\alpha > 0$, $\rho(\Gamma) < 1$ eventually, for $N \to \infty$, and the matrix $(I - \Gamma)$ is a.s. invertible for large N. As a consequence, we can represent the equilibrium as:

$$\boldsymbol{x}^* = \left(I - \frac{A}{\alpha\sqrt{N}}\right)^{-1} \boldsymbol{r}.$$

Consider the simpler case where $\boldsymbol{r} = \mathbf{1}$ and let α be fixed. Extending a result by Geman and Hwang [12], Dougoud et al. [9] have proved that x_i^* are asymptotically i.i.d. Gaussian random variables:

$$x_i^* \xrightarrow[N\to\infty]{\mathcal{L}} \mathcal{N}(1, \sigma_\alpha^2) \quad \text{for all} \quad i = 1, \dots, N,$$

where $\sigma_\alpha^2 = \frac{1}{4\alpha^2 - 1}$ depends on α and $\xrightarrow{\mathcal{L}}$ stands for the convergence in distribution. Since every x_i^* is identically normally distributed, $\mathbb{P}(x_i^* > 0) < 1$ for every $1 \leq i \leq N$. Hence, by independence of the x_i^*:

$$\mathbb{P}(\min_{i\in[N]} x_i^* > 0) = \prod_{i\in[N]} \mathbb{P}(x_i^* > 0) \xrightarrow[N\to\infty]{} 0$$

which means that the event $\boldsymbol{x}^* > 0$ is very unlikely to happen.

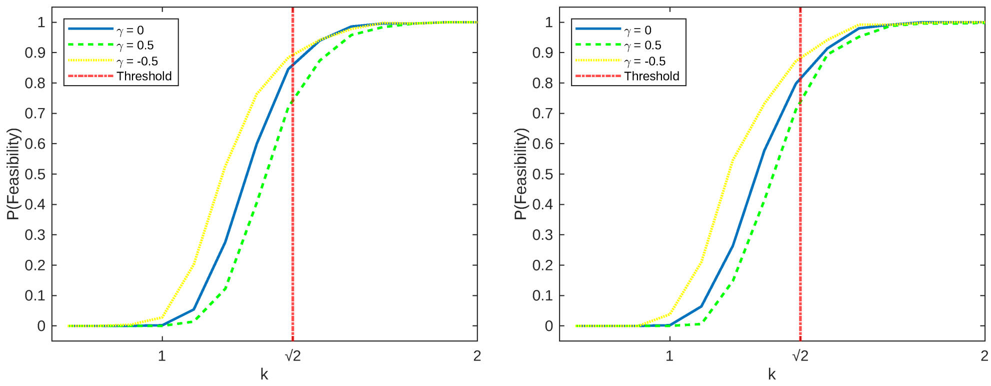
Feasibility if α_N growth to infinity

In the case where $\alpha_N \to \infty$, the following result shows that there exists a threshold value $\alpha_N^* = \sqrt{2\log(N)}$ above which feasibility occurs with probability growing to 1. Conversely, below the threshold there is no feasibility with very high probability. We assume that $\boldsymbol{r} = \mathbf{1}$ and that the entries of matrix Γ are Gaussian pairwise correlated random variables.

Theorem 3.1. *Assume that matrix Γ is given by model (1.6) where A_{ij} are standard Gaussian entries and for $i < j$ the vector (A_{ij}, A_{ji}) is a standard bivariate Gaussian vector. Let $\alpha_N \xrightarrow[N\to\infty]{} \infty$ and denote by $\alpha_N^* = \sqrt{2\log N}$. If $\mu \neq 1$ and $\boldsymbol{r} = \mathbf{1}$ then (3.1) almost sure eventually admits a unique solution $\boldsymbol{x} = (x_i)_{i\in[N]}$ and:*

(1) If $\mu < 1$ and $\exists\, \varepsilon > 0$ such that, for N large enough, $\alpha_N \geq (1+\varepsilon)\alpha_N^*$ then $\mathbb{P}\{\min_{i\in[N]} x_i > 0\} \xrightarrow[N\to\infty]{} 1$.

(2) If $\mu > 1$ or $\exists\, \varepsilon > 0$ such that, for N large enough, $\alpha_N \leq (1-\varepsilon)\alpha_N^*$ then $\mathbb{P}\{\min_{i\in[N]} x_i > 0\} \xrightarrow[N\to\infty]{} 0$.



(a) *Gaussian entries* (b) *Uniform entries*

Figure 3.1: Transition towards feasibility for the centered elliptic model with Gaussian entries and with uniform $[-\sqrt{3}, \sqrt{3}]$ entries for three distinct values $\gamma = 0$ (i.i.d. model), $\gamma = 0.5$ and $\gamma = -0.5$. For each k on the x-axis, we simulate 500 matrices of size $n = 500$ and compute the solution $\boldsymbol{x}$ of Theorem 3.1 at the scaling $\alpha_N(k) = k\sqrt{\log N}$. Then we plot the proportion of feasible solutions obtained for the 500 simulations. The vertical line corresponds to the critical scaling $\alpha_N^* = \sqrt{2\log N}$ for $k = \sqrt{2}$

Remark 3.1. *In the case where $\boldsymbol{r} \neq 1$, there is not a sharp phase transition at $\alpha_N^* = \sqrt{2\log N}$, but there's a wider region, called transition buffer, $[\alpha_{min}^*, \alpha_{max}^*]$ from non feasibility to feasibility.*

Remark 3.2. *Simulations support the idea that the theorem remains true even if Γ's entries are no longer Gaussian (see Figure 3.1). However an effective proof requires some arguments based on the Gaussianity of the entries.*

The next section is devoted to the proof of Theorem 3.1.

3.2 Proof of Theorem 3.1

Let Γ be defined as in Theorem 3.1. We first analyse the centered case ($\mu = 0$) and then the non centered one ($\mu \neq 0$).

The centered case: $\mu = 0$

Let $\mu = 0$ and $\boldsymbol{r} = \mathbf{1}$. Then we focus on the equation:

$$\left(I - \frac{A}{\alpha_N\sqrt{N}}\right)\boldsymbol{x} = \mathbf{1}. \tag{3.2}$$

Let $Q_N = \left(I - \frac{A}{\alpha_N\sqrt{N}}\right)^{-1}$. The following results from random matrix theory prove that Q_N is a.s. eventually well-defined.

Random matrix theory
In the sequel we use the following notations: if $\boldsymbol{v}$ is a vector then $\|\boldsymbol{v}\|$ stands for its euclidian norm; if A is a matrix then $\|A\| = \sqrt{\rho(A^\top A)}$ stands for its spectral norm and $\|A\|_F = \sqrt{\sum_{ij}|M_{ij}|^2}$ for its Frobenius norm.

Definition 3.2. *A Wigner matrix is a Hermitian random matrix whose entries above the diagonal are i.i.d. complex random variables with fixed variance and whose diagonal elements are i.i.d. real random variables.*

Lemma 3.1. *Let A a $N \times N$ matrix with i.i.d. standard Gaussian entries for $i \le j$ and (A_{ij}, A_{ji}) a standard bivariate Gaussian vector with covariance γ for $i < j$, then the following estimate holds true: almost surely,*

$$\limsup_{N\to\infty} \left\| \frac{A}{\sqrt{N}} \right\| \le \sqrt{2}(\sqrt{1+\gamma} + \sqrt{1-\gamma}) \le 2\sqrt{2}$$

Proof. We can decompose matrix $\frac{A}{\sqrt{N}}$ as linear combination of Hermitian Wigner matrices:

$$\frac{A}{\sqrt{N}} = \frac{A + A^\top}{2\sqrt{N}} - i\frac{[i(A - A^\top)]}{2\sqrt{N}}, \qquad (i^2 = -1).$$

Both matrices $W_1 = \frac{A+A^\top}{2\sqrt{N}}$ and $W_2 = \frac{i(A-A^\top)}{2\sqrt{N}}$ are Wigner matrices. $\sqrt{N}W_1$ and $\sqrt{N}W_2$ have off-diagonal variances ($i < j$):

$$\mathrm{var}\left(\left[\frac{A + A^\top}{2}\right]_{ij}\right) = \frac{1+\gamma}{2} \quad \text{and} \quad \mathrm{var}\left(\left[\frac{i(A - A^\top)}{2}\right]_{ij}\right) = \frac{1-\gamma}{2}.$$

Hence, the classical estimate of the asymptotic spectral norm of a Wigner matrix [2, Theorem 5.1] leads to

$$\limsup_{N\to\infty} \left\| \frac{A}{\sqrt{N}} \right\| \le \limsup_{N\to\infty} \|W_1\| + \limsup_{N\to\infty} \|W_2\| = 2\left(\sqrt{\frac{1+\gamma}{2}} + \sqrt{\frac{1-\gamma}{2}}\right)$$

and clearly $\sqrt{2}(\sqrt{1+\gamma} + \sqrt{1-\gamma}) \le 2\sqrt{2}$ for $|\gamma| \le 1$. □

As a consequence $\limsup_{N\to\infty}\left\|\frac{A}{\sqrt{N}}\right\|$ is a.s. bounded. Then

$$\left\|\frac{A}{\alpha_N\sqrt{N}}\right\| \xrightarrow[N\to\infty]{a.s} 0$$

and Q_N is a.s. eventually well-defined. Hence the solution $\boldsymbol{x}$ of (3.2) writes $\boldsymbol{x} = Q_N\mathbf{1}$.
Denote now by $\boldsymbol{e}_i$ the i-th canonical vector of $\mathbb{R}^N$. The following representation holds true (in the sequel we shall often write Q instead of Q_N):

$$\begin{aligned} x_i &= \boldsymbol{e}_i^\top x = \boldsymbol{e}_i^\top Q\mathbf{1} = \sum_{k=0}^{\infty}\boldsymbol{e}_i^\top\left(\frac{A}{\alpha_N\sqrt{N}}\right)^k\mathbf{1}, \\ &= 1 + \frac{1}{\alpha_N}\boldsymbol{e}_i^\top\left(\frac{A}{\sqrt{N}}\right)\mathbf{1} + \frac{1}{\alpha_N^2}\boldsymbol{e}_i^\top\left(\frac{A}{\sqrt{N}}\right)^2 Q\mathbf{1}. \end{aligned}$$

Denote by

$$\begin{aligned} Z_{i,N}(A) &= \boldsymbol{e}_i^\top\left(\frac{A}{\sqrt{N}}\right)\mathbf{1} = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} A_{ij} \quad \text{and} \\ R_{i,N}(A) &= \boldsymbol{e}_i^\top\left(\frac{A}{\sqrt{N}}\right)^2 Q\mathbf{1}. \end{aligned}$$

Notice that the $Z_{i,N}$'s are standard Gaussian with covariance

$$\mathrm{cov}(Z_{i,N}, Z_{j,N}) = \frac{1}{N}\mathrm{cov}(A_{ij}, A_{ji}) = \frac{\gamma}{N}, \quad i \neq j.$$

Let $M_N = \max_{i\in[N]} Z_{i,N}$ and $\hat{M}_N = \min_{i\in[N]} Z_{i,N}$. We notice that (3.3) yields (we often drop index N in the following):

$$\begin{cases} \min_{i\in[N]} x_i \geq 1 + \frac{1}{\alpha_N} + \frac{1}{\alpha_N^2}\min_{i\in[N]} R_i(A) \\ \min_{i\in[N]} x_i \leq 1 + \frac{1}{\alpha_N}\hat{M} + \frac{1}{\alpha_N^2}\max_{i\in[N]} R_i(A) \end{cases}.$$

which we can rewrite:

$$\min_{i\in[N]} x_i \geq 1 + \frac{\alpha_N^*}{\alpha_N}\left(\frac{\hat{M}+\beta_N^*}{\alpha_N^*} - \frac{\beta^*}{\alpha_N^*} + \frac{\min_{i\in[N]} R_i(A)}{\alpha_N^*\alpha_N}\right) \tag{3.6}$$

and

$$\min_{i\in[N]} x_i \leq 1 + \frac{\alpha_N^*}{\alpha_N}\left(\frac{\hat{M}+\beta_N^*}{\alpha_N^*} - \frac{\beta^*}{\alpha_N^*} + \frac{\max_{i\in[N]} R_i(A)}{\alpha_N^*\alpha_N}\right) \tag{3.7}$$

The following results show that $\frac{\hat{M}+\beta_N^*}{\alpha_N^*} = o(1)$.

Extreme Value Theory (EVT) and the normal comparison lemma
Let $(\mathcal{Z}_i)_{i\in[\mathbb{N}]}$ be a sequence of i.i.d. standard Gaussian random variables and denote:

$$\begin{cases} \mathcal{M}_N = \max_{i\in[N]} \mathcal{Z}_i \\ \hat{\mathcal{M}}_N = \min_{i\in[N]} \mathcal{Z}_i \end{cases} \quad \alpha_N^* = \sqrt{2\log N} \quad \beta_N^* = \alpha_N^* - \frac{1}{2\alpha_N^*}\log(4\pi\log N).$$

For $x \in \mathbb{R}$, let $G(x) = e^{-e^{-x}}$ be the Gumbel cumulative distribution function. Then classical EVT results (see [15, Theorem 1.5.3]) yield that for every $x \in \mathbb{R}$,

$$\mathbb{P}\{\alpha_N^*(\mathcal{M}_N - \beta_N^*) \le x\} \xrightarrow[N\to\infty]{} G(x), \quad \mathbb{P}\{\alpha_N^*(\hat{\mathcal{M}}_N + \beta_N^*) \ge -x\} \xrightarrow[N\to\infty]{} G(x). \tag{3.8}$$

Now, let $(Z_{i,N})_{i\in[N]}$ be a Gaussian vector whose components are $\mathcal{N}(0,1)$ with covariance $\text{cov}(Z_{i,N}, Z_{j,N}) = \frac{\gamma}{N}$, $|\gamma| \le 1$, $i \ne j$.
Denote by

$$M_N = \max_{i\in[N]} Z_{i,N}, \quad \hat{M}_N = \min_{i\in[N]} Z_{i,N}.$$

We shall prove the counterpart of (3.8) with the help of Normal Comparison Lemma (NCL):

Theorem 3.2. *Suppose that $(\xi_i, i \in [N])$ is a Gaussian vector where the ξ_i's are standard normal variables, with covariance matrix $\Lambda^1 = \left(\Lambda^1_{ij}\right)$. Similarly, let $(\eta_i, i \in [N])$ be a Gaussian vector where the η_i's are standard normal, with covariance matrix $\Lambda^0 = \left(\Lambda^0_{ij}\right)$. Denote by $\gamma_{ij} = \max\left\{|\Lambda^0_{ij}|, |\Lambda^1_{ij}|\right\}$ and let $(u_i, i \in [N])$ be real numbers. Then:*

$$\begin{aligned} &|\mathbb{P}\{\xi_j \le u_j, j \in [N]\} - \mathbb{P}\{\eta_j \le u_j, j \in [N]\}| \\ &\qquad \le \frac{1}{2\pi} \sum_{1\le i<j\le N} |\Lambda^1_{ij} - \Lambda^0_{ij}|(1-\gamma_{ij}^2)^{-\frac{1}{2}} \exp\left(- \frac{\frac{1}{2}(u_i^2 + u_j^2)}{1+\gamma_{ij}} \right). \end{aligned}$$

Corollary 3.1. *Recall the definition of $(Z_{i,N})_{i\in[N]}$, M_N and $\hat{M}_N$ above. Then*

$$\begin{aligned} \mathbb{P}\{\alpha_N^*(M_N - \beta_N^*) \le x\} &\xrightarrow[N\to\infty]{} G(x), \\ \mathbb{P}\{\alpha_N^*(\hat{M}_N + \beta_N^*) \ge -x\} &\xrightarrow[N\to\infty]{} G(x). \end{aligned}$$

Proof. Let $\gamma_{ij} = \frac{|\gamma|}{N}$ and $u_N(x) = \frac{x}{\alpha_N^*} + \beta_N^*$. We apply the NCL to $(\mathcal{Z}_i)_{i\in[N]}$

and $(Z_{i,N})_{i\in[N]}$:

$$\begin{aligned}
&|\mathbb{P}\{\alpha_N^*(\mathcal{M}_N-\beta_N^*)\le x\}-\mathbb{P}\{\alpha_N^*(M_N-\beta_N^*)\le x\}| \\
&= |\mathbb{P}\{\mathcal{M}_N\le \frac{x}{\alpha_N^*}+\beta_N^*\}-\mathbb{P}\{M_N\le \frac{x}{\alpha_N^*}+\beta_N^*\}| \\
&= |\mathbb{P}\{\mathcal{Z}_i\le u_N(x), i\in[N]\}-\mathbb{P}\{Z_{i,N}\le u_N(x), i\in[N]\}| \\
&\le \frac{1}{2\pi}\frac{N(N-1)}{2}\frac{|\gamma|}{N}\Bigg(1-\frac{\gamma^2}{N^2}\Bigg)^{-\frac{1}{2}}\exp\Bigg(-\frac{u_N^2(x)}{1+\frac{|\gamma|}{N}}\Bigg) \\
&\le KN\exp\Bigg(-\frac{u_N^2(x)}{1+\frac{1}{N}}\Bigg).
\end{aligned}$$

Now eventually, for $N\to\infty$:

$$u_N(x)=\alpha_N^*\Bigg(\frac{x}{(\alpha_N^*)^2}+\frac{\beta_N^*}{\alpha_N^*}\Bigg)=\alpha_N^*(1+o(1))\ge c\alpha_N^* \quad \text{for any } c<1.$$

Hence

$$\begin{aligned}
N\exp\Bigg(-\frac{u_N^2(x)}{1+\frac{1}{N}}\Bigg) &\le N\exp\Bigg(-\frac{c^2(\alpha_N^*)^2}{1+\frac{1}{N}}\Bigg) \\
&= N\exp\Bigg(-\log N\frac{2c^2}{1+\frac{1}{N}}\Bigg) \\
&= N^{-\left(\frac{2c^2}{1+\frac{1}{N}}-1\right)}.
\end{aligned}$$

For a well-chosen c sufficiently close to one, this last term goes to zero as $N\to\infty$ and this concludes the proof for M_N. The proof for $\hat{M}_N$ is similar with minor modifications. □

Hence, as a consequence of this corollary, we have $\frac{\hat{M}+\beta_N^*}{\alpha_N^*}=o(1)$ and, by (3.6) and (3.7):

$$\begin{aligned}
\min_{i\in[N]} x_i &\le 1+\frac{\alpha_N^*}{\alpha_N}\Bigg(-1+o(1)+\frac{\max_{i\in[N]}R_i(A)}{\alpha_N^*\alpha_N}\Bigg) \\
\min_{i\in[N]} x_i &\ge 1+\frac{\alpha_N^*}{\alpha_N}\Bigg(-1+o(1)+\frac{\min_{i\in[N]}R_i(A)}{\alpha_N^*\alpha_N}\Bigg).
\end{aligned}$$

The proof in the centered case follows then from the following result:

Lemma 3.2. *Let $R_{i,N}(A)$ be defined as in* (3.5) *and recall that* $\alpha_N \xrightarrow[N\to\infty]{} \infty$*, then:*

$$\frac{\max_{i\in[N]} R_{i,N}(A)}{\alpha_N\sqrt{2\log N}} \xrightarrow[N\to\infty]{\mathcal{P}} 0 \quad \textit{and} \quad \frac{\min_{i\in[N]} R_{i,N}(A)}{\alpha_N\sqrt{2\log N}} \xrightarrow[N\to\infty]{\mathcal{P}} 0.$$

This implies, in particular, that $\min_{i\in[N]} x_i = 1 - \frac{\alpha_N^*}{\alpha_N}$ for $N \to \infty$. Thus, $\min_{i\in[N]} x_i > 0$, i.e. $\boldsymbol{x} = (x_i)_{i\in[N]}$ is a feasible equilibrium, if and only if $\alpha_N > \alpha_N^*$. For $\varepsilon > 0$ this is equivalent to $\alpha_N \geq (1+\varepsilon)\alpha_N^*$, which is the statement of the theorem.
Before proving this lemma, we need to present some auxiliary results, which are necessary for its proof.

Truncated version of $R_{i,N}(A)$
We first introduce a truncated version of $R_{i,N}(A)$. Let $\eta \in (0,1)$ and $\varphi : \mathbb{R}^+ \to [0,1]$ a smooth function satisfying:

$$\varphi(x) = \begin{cases} 1 & \text{if} \quad x \in [0, 2\sqrt{2}+\eta] \\ 0 & \text{if} \quad x \geq 4 \end{cases},$$

decreasing from 1 to 0 gradually as x goes from $2\sqrt{2}+\eta$ to 4. Let

$$\varphi_N = \varphi\left(\left\|\frac{A}{\sqrt{N}}\right\|\right).$$

Notice that $\mathbb{P}\{\varphi_N < 1\} = \mathbb{P}\left\{\left\|\frac{A}{\sqrt{N}}\right\| > 2\sqrt{2}+\eta\right\} \xrightarrow[N\to\infty]{} 0$ by Lemma 3.1. Hence, we can introduced the truncated version

$$\widetilde{R}_{i,N} = \varphi_N R_{i,N} \tag{3.11}$$

which differs from $R_{i,N}$ with vanishing probability. To prove Lemma 3.2 we can work with $\widetilde{R}_{i,N}$ instead of $R_{i,N}$.
The next step is showing that $\widetilde{R}_i(A)$ can be expressed as a Lipschitz function of i.i.d. $\mathcal{N}(0,1)$ entries.

Lipschitzianity of $\widetilde{R}_k(A)$ and uniform estimation of $\mathbb{E}(\widetilde{R}_k(A))$

Lemma 3.3. *Let $\widetilde{R}_i$ defined by* (3.11) *and M a $N \times N$ matrix. Then the function*

$$M \mapsto \widetilde{R}_i(M) = \boldsymbol{e}_i^\top \left(\frac{M}{\sqrt{N}}\right)^2 \left(I - \frac{M}{\alpha_N\sqrt{N}}\right)^{-1} \mathbf{1}$$

is K-Lipschitz, i.e.

$$|\widetilde{R}_i(M) - \widetilde{R}_i(L)| \leq K\|M - L\|_F$$

where M, L are $N \times N$ matrices and K a is a constant independent from i and N.

Lemma 3.4. *Consider the linear function $\Omega : \mathbb{R}^{N\times N} \to \mathbb{R}^{N\times N}$ defined by*

$$\Omega_{ii}(X) = X_{ii} \quad \text{and} \quad \begin{cases} \Omega_{ij}(X) = \sqrt{\frac{1+\gamma}{2}}X_{ij} + \sqrt{\frac{1-\gamma}{2}}X_{ji} \\ \Omega_{ji}(X) = \sqrt{\frac{1+\gamma}{2}}X_{ij} - \sqrt{\frac{1-\gamma}{2}}X_{ji} \end{cases} \quad \text{for} \quad i < j.$$

Then

1. *$\|\Omega(X)\|_F \leq K_\gamma \|X\|_F$ where $K_\gamma = 2\sqrt{1 + |\gamma|}$ hence Ω is K_γ-Lipschitz.*
2. *If matrix $X = (X_{ij})$ has i.i.d $\mathcal{N}(0, 1)$ entries, then $A = \Omega(X)$ has i.i.d $\mathcal{N}(0, 1)$ entries on and above the diagonal and each vector (A_{ij}, A_{ji}) is a standard bivariate Gaussian vector with covariance γ for $i < j$.*

It follows from these two lemmas that $\widetilde{R}_i(A) = \widetilde{R}_i(\Omega(X))$ is $K \times K_\gamma$-Lipschitz. By applying Tsirelson-Ibragimov-Sudakov inequality [5, Theorem 5.5], we now bound the quantity $\mathbb{E}\max_{i\in[N]}(\widetilde{R}_i(A) - \mathbb{E}\widetilde{R}_i(A))$.

Proposition 3.1. *Let K the Lipschitz constant of Lemma 3.3 and $K_\gamma = 2\sqrt{1 + |\gamma|}$. Then*

$$\mathbb{E}\max_{i\in[N]} \big(\widetilde{R}_i(A) - \mathbb{E}\widetilde{R}_i(A)\big) \leq 2KK_\gamma\sqrt{\log N}.$$

Details of the proof are similar to those in [3] and are thus omitted.
The following proposition provides a uniform estimate for $\mathbb{E}\widetilde{R}_i(A)$.

Proposition 3.2. *The following estimate $\mathbb{E}\widetilde{R}_i(A) = \mathcal{O}(1)$ holds true, uniformly for $i \in [N]$.*

Proof. The proof is omitted and can be found in [8]. □

We are now in position to prove Lemma 3.2.

Proof of Lemma 3.2. We prove the first assertion of Lemma 3.2:

$$\frac{\max_{i\in[N]} R_{i,N}(A)}{\alpha_N\sqrt{2\log N}} \xrightarrow[N\to\infty]{\mathcal{P}} 0.$$

Note that $\mathbb{E}\widetilde{R}_i(A) = \mathbb{E}\widetilde{R}_1$ (see [8, Proof of Proposition 3.9]). Since $\max_{i\in[N]} \widetilde{R}_i(A) - \widetilde{R}_1(A) \geq 0$, Markov inequality yields:

$$\begin{aligned}
\mathbb{P}\left\{ \frac{\max_{i\in[N]} \widetilde{R}_i - \widetilde{R}_1}{\alpha_N \sqrt{2\log N}} \geq \varepsilon \right\} &\leq \frac{\mathbb{E}(\max_{i\in[N]} \widetilde{R}_i - \widetilde{R}_1)}{\varepsilon \alpha_N \sqrt{2\log N}} \\
&= \frac{\mathbb{E}(\max_{i\in[N]}(\widetilde{R}_i - \mathbb{E}\widetilde{R}_i + \mathbb{E}\widetilde{R}_1) - \widetilde{R}_1)}{\varepsilon \alpha_N \sqrt{2\log N}} \\
&= \frac{\mathbb{E}(\max_{i\in[N]}(\widetilde{R}_i - \mathbb{E}\widetilde{R}_i))}{\varepsilon \alpha_N \sqrt{2\log N}} \\
&\leq \frac{\sqrt{2} K K_\gamma}{\varepsilon \alpha_N},
\end{aligned}$$

where the last inequality follows from Proposition 3.1.
Since $\alpha_N \xrightarrow[N\to\infty]{} \infty$, $\frac{\sqrt{2}KK_\gamma}{\varepsilon\alpha_N} \xrightarrow[N\to\infty]{} 0$ and this implies that

$$\frac{\max_{i\in[N]} \widetilde{R}_i(A) - \widetilde{R}_1(A)}{\alpha_N \sqrt{2\log N}} \xrightarrow[N\to\infty]{\mathcal{P}} 0. \tag{3.12}$$

It remains to prove that

$$\frac{\widetilde{R}_1(A)}{\alpha_N \sqrt{2\log N}} \xrightarrow[N\to\infty]{\mathcal{P}} 0. \tag{3.13}$$

Once this fact is established, by (3.12) we have

$$\frac{\max_{i\in[N]} \widetilde{R}_i(A)}{\alpha_N \sqrt{2\log N}} \xrightarrow[N\to\infty]{\mathcal{P}} 0$$

and then

$$\begin{aligned}
\mathbb{P}\left\{ \left| \frac{\max_{i\in[N]} R_i(A)}{\alpha_N \sqrt{2\log N}} \right| > \varepsilon \right\} &\leq \mathbb{P}\left\{ \max_{i\in[N]} R_i(A) \neq \max_{i\in[N]} \widetilde{R}_i(A) \right\} \\
&\quad + \mathbb{P}\left\{ \left| \frac{\max_{i\in[N]} \widetilde{R}_i(A)}{\alpha_N \sqrt{2\log N}} \right| > \frac{\varepsilon}{2} \right\} \\
&= \mathbb{P}\{\varphi_N < 1\} + \mathbb{P}\left\{ \left| \frac{\max_{i\in[N]} \widetilde{R}_i(A)}{\alpha_N \sqrt{2\log N}} \right| > \frac{\varepsilon}{2} \right\},
\end{aligned}$$

where the latter term goes to 0 as $N \to \infty$.
Let us establish (3.13). By Proposition 3.2, $\mathbb{E}\widetilde{R}_1(A) = \mathcal{O}(1)$ hence

$$\frac{\mathbb{E}\widetilde{R}_1(A)}{\alpha_N \sqrt{2\log N}} \xrightarrow[N\to\infty]{} 0. \tag{3.14}$$

Recall that $\widetilde{R}_1(A)$ is $K \times K_\gamma$-Lipschitz by Lemma 3.4. Then applying Poincare's inequality [5, Theorem 3.20] to $\widetilde{R}_1(A)$, we can bound its variance by $(KK_\gamma)^2$. Then Chebyshev inequality yields:

$$\mathbb{P}\left(\left|\frac{\widetilde{R}_1(A) - \mathbb{E}\widetilde{R}_1(A)}{\alpha_N\sqrt{2\log N}}\right| > \varepsilon\right) \leq \frac{\mathrm{var}(\widetilde{R}_1(A))}{2\varepsilon^2\alpha_N^2 \log N} \leq \frac{(KK_\gamma)^2}{2\varepsilon^2\alpha_N^2\log N} \xrightarrow[N\to\infty]{} 0.$$

This and (3.14) yield (3.13). This concludes the proof of the first assertion. Proof of the second assertion of Lemma 3.2

$$\frac{\min_{i\in[N]} R_{i,N}(A)}{\alpha_N\sqrt{2\log N}} \xrightarrow[N\to\infty]{\mathcal{P}} 0$$

can be done similarly. □

The non centered case

Denote $\boldsymbol{u} = \frac{1}{\sqrt{N}}\mathbf{1}$. The spectrum of $I - \mu\boldsymbol{u}\boldsymbol{u}^\top$ is $\{1-\mu, 1\}$, where the eigenvalue 1 has multiplicity $N-1$. Hence, if $\mu \neq 1$, then $I - \mu\boldsymbol{u}\boldsymbol{u}^\top$ is invertible. As a consequence also $I - \frac{A}{\alpha_N\sqrt{N}} - \mu\boldsymbol{u}\boldsymbol{u}^\top$ is eventually invertible as $\left\|\frac{A}{\alpha_N\sqrt{N}}\right\| \to 0$ a.s.

Denote by $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ the vectors solutions of respectively the centered and non centered systems, i.e:

$$\boldsymbol{x} = \mathbf{1} + \frac{A}{\alpha_N\sqrt{N}}\boldsymbol{x} \quad \text{and} \quad \tilde{\boldsymbol{x}} = \mathbf{1} + B\tilde{\boldsymbol{x}} = \mathbf{1} + \left(\frac{A}{\alpha_N\sqrt{N}} + \mu\boldsymbol{u}\boldsymbol{u}^\top\right)\tilde{\boldsymbol{x}}.$$

Then

$$x = Q\mathbf{1} \quad \text{and} \quad \tilde{\boldsymbol{x}} = (I-B)^{-1}\mathbf{1}.$$

By Woodbury matrix identity for rank one perturbation, we have:

$$(I-B)^{-1} = (Q^{-1} - \mu\boldsymbol{u}\boldsymbol{u}^\top)^{-1} = Q + Q\left(\frac{\mu\boldsymbol{u}\boldsymbol{u}^\top}{1-\mu\boldsymbol{u}^\top Q\boldsymbol{u}}\right)Q.$$

and

$$\tilde{\boldsymbol{x}} = \frac{Q\mathbf{1}(1-\mu\boldsymbol{u}^\top Q\boldsymbol{u}) + \mu Q\boldsymbol{u}\boldsymbol{u}^\top Q\mathbf{1}}{1-\mu\boldsymbol{u}^\top Q\boldsymbol{u}} = \frac{\boldsymbol{x}}{1-\mu\boldsymbol{u}^\top Q\boldsymbol{u}}.$$

Hence $\tilde{\boldsymbol{x}} > 0$ if $\boldsymbol{x} > 0$ and $1 - \mu\boldsymbol{u}^\top Q\boldsymbol{u} > 0$. For what we have seen in the centered case $\boldsymbol{x} > 0$ if $\alpha_N \geq (1+\varepsilon)\alpha_N^*$. For the denominator we rely on the fact that $\|Q - I\| \xrightarrow[N\to\infty]{} 0$ a.s. As a consequence

$$\boldsymbol{u}^\top Q\boldsymbol{u} \xrightarrow[N\to\infty]{a.s.} 1.$$

Hence $1 - \mu \boldsymbol{u}^\top Q \boldsymbol{u}$ is eventually positive only if $\mu < 1$.
In conclusion if $\mu < 1$ and $\alpha_N \geq (1+\varepsilon)\alpha_N^*$ then eventually $\tilde{\boldsymbol{x}}$ has positive components. If $\mu > 1$ or $\alpha_N \leq (1-\varepsilon)\alpha_N^*$ this is no longer the case. This concludes the proof of Theorem 3.1.

3.3 What happens when feasibility fails?

In the previous sections, we have established conditions under which there exists a unique feasible equilibrium. Here, we examine the case where a unique equilibrium exists but feasibility is not reached. Using simulations and heuristic arguments, we analyse two main aspects: the number of survivors and the distribution of the abundance of a given surviving species.

Number of surviving species

In the case where a unique equilibrium exists, it is interesting to estimate the proportion of surviving species $p = p(\alpha)$ as a function of parameter α. To address this question, Bunin in [6], relying on the cavity method, and Galla in [10], using generating functional techniques, provide heuristics applicable to the elliptic model. Here we focus on the i.i.d. model.
Given the random equilibrium $\boldsymbol{x}^*$, we introduce the following quantities:

$$\mathcal{S} = \{i \in [N], x_i^* > 0\}, \quad \hat{p} = \frac{|\mathcal{S}|}{N}, \quad \hat{m}^2 = \frac{1}{|\mathcal{S}|} \sum_{i \in [N]} (x_i^*)^2.$$

Denote by $Z \sim \mathcal{N}(0,1)$ and by Φ the cumulative Gaussian distribution function:

$$\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du.$$

In what follows, we compare the theoretical result given by the next conjecture to the values obtained by simulations.

Conjecture 3.1. *Let $\alpha \in (\sqrt{2}, \sqrt{2\log(N)})$ and assume that Γ follows the i.i.d. model. The following system of two equations and two unknowns (p, m)*

$$m\sqrt{p}\Phi^{-1}(1-p) + \alpha = 0, \tag{3.15}$$

$$1 + \frac{2m\sqrt{p}}{\alpha}\mathbb{E}\left(Z | Z > -\frac{\alpha}{m\sqrt{p}}\right) + \frac{m^2 p}{\alpha^2}\mathbb{E}\left(Z^2 | Z > -\frac{\alpha}{m\sqrt{p}}\right) = m^2 \tag{3.16}$$

admits a unique solution (p^*, m^*) and the following convergence holds

$$\hat{p} \xrightarrow[N\to\infty]{a.s.} p^* \quad \text{and} \quad \hat{m} \xrightarrow[N\to\infty]{a.s.} m^*.$$

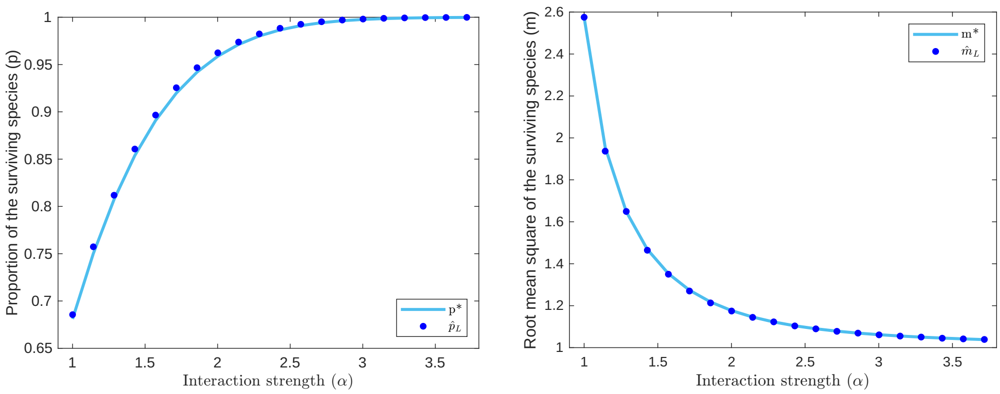


Figure 3.2: The theoretical proportion of surviving species $p^*(\alpha)$ (on the left) and second moment $m^*(\alpha)$ (on the right) obtained by solving the equations (3.15)-(3.16) are compared to the empirical Monte-Carlo counterpart $(\hat{p}_L(\alpha), \hat{m}_L(\alpha))$. Notice that simulations show a remarkable matching with the heuristics even for $\alpha \in (1, \sqrt{2}]$.

Remark 3.3. *Notice that, by Theorem 2.3, the condition $\alpha > \sqrt{2}$ is sufficient to guarantee the existence of a unique equilibrium. However, as we observe in the following simulations, such condition might be relaxed.*

We investigate Conjecture 3.1 through numerical simulations. We fix $N = 1000$ and draw L independent realizations of matrix $A^{(i)}$. We specifically choose $L = 500$, for $\alpha \in (\sqrt{2}, \sqrt{2\log(1000)})$ and $L = 100$ for $\alpha \in (1, \sqrt{2}]$. We then compute the corresponding equilibria $\boldsymbol{x}^{(i)}(\alpha)$ and their related quantities $(\hat{p}^{(i)}(\alpha), \hat{m}^{(i)}(\alpha))$. We finally compare the empirical Monte-Carlo averages:

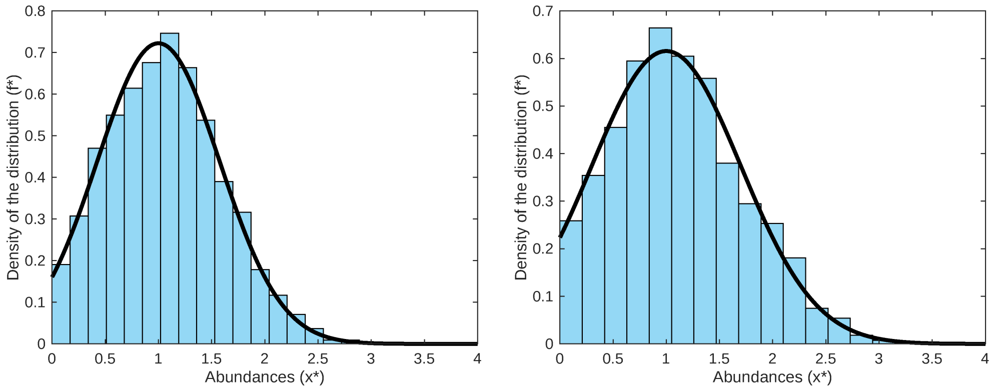
$$\hat{p}_L(\alpha) = \frac{1}{L}\sum_{i=1}^{L} \hat{p}^{(i)}(\alpha) \quad \text{and} \quad \hat{m}_L(\alpha) = \frac{1}{L}\sum_{i=1}^{L} \hat{m}^{(i)}(\alpha)$$

to their theoretical counterparts $p^*(\alpha), m^*(\alpha)$, solutions of (3.15) and (3.16). In Figure 3.2, we see that the matching is remarkable, even for $\alpha < \sqrt{2}$.

Single species distribution

Another interesting question concerns the distribution of a given abundance x_i^* where index i corresponds to a surviving species. As we have done before, we compare the theoretical results given by heuristics with simulations.

Conjecture 3.2. *Let $\alpha \in (\sqrt{2}, \sqrt{2\log(N)})$ and let $i \in \mathcal{S}$, i.e. i corresponds to a surviving species. Let p^*, m^* be the solutions of* (3.15) *and* (3.16) *and*



(a) *Gaussian entries*   (b) *Uniform entries*

Figure 3.3: The solid line represents the theoretical distribution f^* as given by Conjecture 3.2. The histogram is built by solving the LCP problem with an interaction matrix of size $N = 2000$. On the left, the entries are Gaussian $\mathcal{N}(0,1)$ and the interaction strength is fixed to $\alpha = 2$. On the right, the entries are uniform $\mathcal{U}([-\sqrt{3}, \sqrt{3}])$ and the parameter set to $\alpha = 2$.

$Z \sim \mathcal{N}(0,1)$. *Then the distribution of* x_i^* *is a truncated Gaussian:*

$$\mathcal{L}(x_i^*) = \mathcal{L}\left(1 + \frac{m^*\sqrt{p^*}}{\alpha} Z \,\middle|\, Z > -\frac{\alpha}{m^*\sqrt{p^*}}\right).$$

Otherwise stated, x_i^* *admits the following density:*

$$f^*(v) = \frac{1_{(v>0)}}{\Phi(-\delta)} \frac{\delta}{\sqrt{2\pi}} \exp\left(-\frac{\delta^2(v-1)^2}{2}\right)$$

where

$$\delta = \frac{\alpha}{m^*\sqrt{p^*}}$$

and Φ *stands for the cumulative Gaussian distribution.*

In Figure 3.3, we illustrate the matching between the theoretical density f^* given in Heuristics 3.2 and a histogram of a given equilibrium $\boldsymbol{x}^*$. Notice, in particular, that the theoretical distribution matches even with non-Gaussian entries.

Conclusions

Lotka-Volterra systems of differential equations are widely used in theoretical ecology to model the dynamics of complex ecosystems. This thesis focused on a large Lotka-Volterra system, with the aim of investigating the relationship between complexity (i.e., how numerous, connected, and strongly interacting species are) and stability in large ecosystems. This relationship has become central in theoretical ecology since Robert May's pioneering work in the 1970s [16], which revealed a fundamental paradox: while he predicted that increased complexity leads to decreased stability, empirical observations suggest a positive correlation between them. May obtained this result by studying the stability of ecosystems in the neighborhood of an equilibrium. Since stability depends on the community matrix, whose exact structure is often unavailable, his fundamental idea was to model this matrix randomly. This made it possible to use Random Matrix Theory results to calculate its eigenvalues and analyse ecosystem stability. However, May's analysis was limited to local stability around an equilibrium. In this thesis, following [1], we introduced the Lotka-Volterra system to extend his approach and investigate the global dynamics of ecosystems. Following May's methodology, the interaction matrix was modeled randomly, adopting in particular the elliptic model. Through the analysis of the statistics of the spectrum of such matrix, some important properties of the system's equilibria were investigated: existence, uniqueness, stability and feasibility.

Concerning stability, we provided three main results [8, 20] (all notation and parameters involved in what follows are defined in Chapter 1 and 2). The first result, Theorem 2.1, established a link between the Linear Complementarity Problem (LCP) and the stability of a general complex dynamical system: for an equilibrium $\boldsymbol{x}^*$ to be stable, it must be a solution of $LCP(I-\Gamma, -\boldsymbol{r})$. However, it does not guarantee the existence and uniqueness of an equilibrium. Hence, Theorem 2.2 proved that under specific conditions on the matrix $\Gamma - I$ of the Lotka-Volterra system, a unique equilibrium exists and a Lyapunov function can be constructed to ensure its global stability. Finally, Theorem 2.3 provided explicit parameter values, $\alpha > \sqrt{2(1+\gamma)}$ and $\mu < \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{2(1+\gamma)}{\alpha^2}}$, which ensure, for sufficiently large N, the existence, uniqueness and global stability

of the equilibrium.
Then, the issue of feasibility was addressed. The analysis first demonstrated that feasibility is very unlikely to occur when α is fixed [9]. Therefore, following [8], it addressed the case where α_N tends to infinity for the elliptic Lotka-Volterra model with normal pairwise correlated entries and $\boldsymbol{r} = \mathbf{1}$. Under the condition that a unique equilibrium exists, a sharp phase transition around the threshold value $\alpha_N^* = \sqrt{2\log(N)}$ was established by Theorem 3.1. Above the threshold, feasibility occurs with probability approaching 1, while below the threshold, it fails with high probability. To conclude, simulations and heuristic arguments were presented to examine the case where a unique equilibrium exists but is not feasible. In particular, the analysis estimated the proportion of surviving species, showing how it increases with α, and described the distribution of the abundance of a given surviving species, which turned out to be a truncated Gaussian.

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