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Mean Field Games with common noise: modeling systemic financial decarbonization

Relatrice:
Prof.ssa Giorgia Callegaro

Laureanda:
Alice Magnanini
2074382

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Abstract

Mean Field Games (MFG) theory, introduced by J.-M. Lasry and P.-L. Lions in 2006, offers a powerful analytical framework for modeling strategic interactions among a continuum of rational agents. Inspired by mean field techniques in statistical physics, MFG theory distinguishes itself by capturing the behavior of decision-makers who optimize individual objectives in response to the evolving collective state.

Recent developments have extended MFG models into the realm of green finance, particularly in analyzing the strategic behavior of firms and investors under climate-related risks. A notable contribution in this area is the work of P. Tankov and P. Lavigne (2023), "Decarbonization of Financial Markets: A Mean-Field Game Approach", which introduces a rigorous MFG framework for modeling decarbonization dynamics in financial markets. Their model establishes the existence and uniqueness of an optimal stochastic discount factor, offering significant insights into the pricing of assets and the decarbonization dynamics in the presence of climate risk.

This thesis provides an accessible introduction to the foundational concepts of Mean Field Games theory, presents a detailed exposition and proof of the main results from Tankov and Lavigne's model, and provides a qualitative discussion of the simulation outcomes.

Contents

Introduction

The theory of Mean Field Games (MFG) was first introduced in 2006 by J.-M. Lasry and P.-L. Lions through a series of lectures at the Collège de France. In these lectures, the authors laid the conceptual groundwork for this emerging field by analyzing simplified models that illustrated the core mechanisms of strategic interaction in large populations. Their ideas were later formalized in a trilogy of seminal papers [1, 2, 3], which rigorously developed the mathematical principles of MFG theory and defined a robust analytical framework for its application.

MFG theory draws inspiration from classical mean field methods in statistical mechanics and particle physics, where systems with a large number of interacting entities, typically particles, are approximated by their average behavior. However, the key innovation in MFG theory lies in replacing these passive particles with strategic agents: rational decision-makers who individually optimize certain objectives, while simultaneously influencing, and being influenced by, the statistical behavior of the entire population. In this context, agents are typically assumed to be infinitesimal in size, forming a continuum, and their interactions are governed by the "mean field", a macroscopic signal summarizing the distribution of states across the population.

MFG models rest on several core assumptions. The continuum of agents captures the idea that in large populations, individual influence is negligible, enabling analysis via stochastic differential equations. Agent anonymity means the game's outcome is unchanged under player permutations, simplifying the framework. Most importantly, agents interact through the mean field rather than direct pairwise exchanges, responding to the overall population behavior. These assumptions allow MFG theory to bridge complex, finite-player games and their infinite-player limits, offering insights into macroscopic dynamics and equilibrium behavior.

This powerful abstraction has made MFG theory a valuable tool across a variety of domains, including economics, network theory, crowd dynamics, and more recently, quantitative finance. In finance, MFG provides a natural framework for analyzing systems composed of a large number of heterogeneous, strategically interacting agents, be they investors, firms, or institutions. The ability to model collective dynamics while preserving individual optimization makes MFG particularly well-suited to studying systemic risk, market impact, and optimal control under uncertainty.

A particularly promising area for the application of MFG theory is green finance, which encompasses financial practices aimed at promoting environmentally sustainable outcomes. As global awareness of the climate crisis intensifies, traditional carbon-intensive sectors face increasing pressure, while low-emission and energy-efficient enterprises attract growing interest and capital inflow. Policy incentives, investor sentiment, and regulatory frameworks are all evolving in response to the climate emergency, creating a complex and dynamic landscape that demands new modeling approaches. One of the central challenges is to account for both the stochastic nature of climate-related financial risks and the strategic behavior of economic agents navigating the transition to a low-carbon economy.

A significant step in this direction has been made by P. Tankov and P. Lavigne in their recent work [4], where they apply a mean field game framework to model the decarbonization of financial markets. Their model features two distinct classes of investors: green investors and regular investors, who differ in their beliefs about climate risk. It also includes a continuum of firms that optimize their carbon emission strategies by balancing industrial output with environmental penalties. The core technical contribution lies in establishing a fixed-point argument that guarantees the existence and uniqueness of an optimal stochastic discount factor. This result enables the precise characterization of the representative firm's optimal emission strategy and its associated value process.

This thesis contributes to the interdisciplinary development of MFG applications in green finance by providing a corrected version of the proof of [4, Theorem 5], which establishes the existence and uniqueness of a solution to the fixed-point problem arising in the mean field market model constructed. In particular, Chapter 1 introduces the general framework of MFG theory, including both the classical formulation and the extension to MFGs with common noise. The aim is to equip the reader with the foundational concepts and tools necessary to understand both the theory itself and the subsequent analysis of the model under consideration. In Chapter 2, we present a detailed exposition of the model proposed in [4], explaining the rationale behind the modeling choices and outlining the construction of the fixed-point argument, along with the novel MFG-based approach introduced to solve it. Finally, we present the corrected version of the proof of the theorem, structured through a sequence of lemmas that clarify and complete the original argument, and we provide a qualitative discussion of the simulation outcomes.

Chapter 1

The Mean Field Games formulation

1.1 Fundamentals of stochastic calculus

The purpose of this section is to introduce the preliminary concepts needed to develop and understand the probabilistic theory of mean field games. The subsections are designed to be as self-contained as possible and focus on the simplest cases. In Sections 1.2 and 1.3, we will extensively build upon these notions, using them in a more comprehensive form where the ideas presented separately here are integrated into a unified framework.

1.1.1 Weak convergence and Wasserstein metrics

Throughout this section, let $(\mathcal{X}, d)$ denote a metric space and equip $\mathcal{X}$ with the Borel σ-field $\mathcal{B}(\mathcal{X})$. Let $\mathcal{P}(\mathcal{X})$ denote the set of Borel probability measures on $\mathcal{X}$, and $C_b(\mathcal{X})$ the set of bounded continuous real-valued functions on $\mathcal{X}$. Recall that, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we say that X *is a $\mathcal{X}$-valued random variable* if $X : \Omega \to \mathcal{X}$ is $(\mathcal{F}, \mathcal{B}(\mathcal{X}))$-measurable.

Definition 1.1 (Weak convergence on $\mathcal{P}(\mathcal{X})$)**.** Given a probability measure $\mu \in \mathcal{P}(\mathcal{X})$ and a sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\mathcal{X})$, then we say that μ_n *converges weakly to* μ, or $\mu_n \rightharpoonup \mu$, if

$$\lim_{n\to\infty} \int_{\mathcal{X}} f d\mu_n = \int_{\mathcal{X}} f d\mu, \quad \forall f \in C_b(\mathcal{X}).$$

Equivalently, given a sequence $(X^k)_{k\in\mathbb{N}}$ of $\mathcal{X}$-valued random variables, we say that X^k *converges weakly* to another $\mathcal{X}$-valued random variable X, or

$X^k \rightharpoonup X$, if

$$\lim_{k\to\infty} \mathbb{E}[f(X^k)] = \mathbb{E}[f(X)], \quad \forall f \in C_b(\mathcal{X}).$$

Definition 1.2 (Empirical measure)**.** Given a finite set $(X^i)_{i\in\{1,\dots,n\}}$ of i.i.d. $\mathcal{X}$-valued random variables, we say that the *empirical measure of* $(X^i)_i$ is the $\mathcal{P}(\mathcal{X})$-valued random variable $\hat{\mu}^n$ defined as:

$$\hat{\mu}^n := \frac{1}{n}\sum_{i=1}^n \delta_{X^i}$$

where $\delta_{X^i(\omega)}(A) = 1$ if $X^i(\omega) \in A$, and 0 otherwise.

Many results in mean field game theory are formulated in terms of the weak convergence of empirical measures. Therefore, it is natural to search for a metric on $\mathcal{P}(\mathcal{X})$ that is compatible with weak convergence, so that $\mathcal{P}(\mathcal{X})$ can be regarded as a metric space, and convergence with respect to this metric corresponds somehow to weak convergence.

Assume $(\mathcal{X}, d)$ separable, so that $\mathcal{B}_{\mathcal{X}\times\mathcal{X}} = \mathcal{B}_{\mathcal{X}} \otimes \mathcal{B}_{\mathcal{X}}$. For $\mu, \nu \in \mathcal{P}(\mathcal{X})$, let $\Pi(\mu,\nu)$ denote the set of Borel probability measures π on $\mathcal{X}\times\mathcal{X}$ such that $\pi(A\times\mathcal{X}) = \mu(A)$ and $\pi(\mathcal{X}\times A) = \nu(A)$ for every Borel set $A \subseteq \mathcal{X}$ (that is, π has first marginal μ and second marginal ν). For $p \geq 1$, let $\mathcal{P}^p(\mathcal{X})$ denote the set probability measures $\mu \in \mathcal{P}(\mathcal{X})$ such that $\int_{\mathcal{X}} d(x,x_0)^p \mu(dx) < +\infty$ for some (and so, by the triangle inequality, for every) $x_0 \in \mathcal{X}$.

Definition 1.3. The p-Wasserstein metric on $\mathcal{P}^p(\mathcal{X})$ relative to the metric d is defined by:

$$\mathcal{W}_{\mathcal{X},p}(\mu,\nu) := \Big(\inf_{\pi\in\Pi(\mu,\nu)} \int_{\mathcal{X}\times\mathcal{X}} d(x,y)^p \pi(dx,dy)\Big)^{\frac{1}{p}} = \Big(\inf_{X\sim\mu, Y\sim\nu} \mathbb{E}[d(X,Y)^p]\Big)^{\frac{1}{p}}$$

If the space $\mathcal{X}$ is clear, we will write $\mathcal{W}_p$ instead of $\mathcal{W}_{\mathcal{X},p}$. The following results hold:

Theorem 1.1. *If $(\mathcal{X}, d)$ is complete and separable, then $\mathcal{W}_p$ defines a metric on $\mathcal{P}^p(\mathcal{X})$. Moreover, $(\mathcal{P}^p(\mathcal{X}), \mathcal{W}_p)$ is a complete and separable metric space.*

Proof. See [5, pp. 207-208, Theorem 7.3] and [6, p. 372]. □

Theorem 1.2. *Assume $(\mathcal{X}, d)$ separable. Suppose $(X^i)_{i=1,\dots,n}$ are i.i.d. $\mathcal{X}$-valued random variables with law $\mu \in \mathcal{P}^p(\mathcal{X})$, $p \geq 1$, and let $\hat{\mu}^n$ be the empirical measure of $(X^i)_i$. Then $\mathcal{W}_p(\hat{\mu}^n,\mu) \to 0$ a.s., and also $\mathbb{E}[\mathcal{W}_p^p(\hat{\mu}^n,\mu)] \to 0$.*

Proof. The result follows from [5, pp. 212-218, Theorem 7.12]. See [7, p. 15, Corollary 2.14] for a proof. □

To avoid the limitations of having metrics defined only on subsets of $\mathcal{P}(\mathcal{X})$, consider the new metric $\bar{d}(x,y) \coloneqq 1 \wedge d(x,y)$ for $x,y \in \mathcal{X}$. Then $\bar{d}$ generates on $\mathcal{X}$ the same topology as d, and since $\bar{d}$ is bounded, the p-Wasserstein metric $\overline{\mathcal{W}}_p$ on $\mathcal{P}^p(\mathcal{X})$ relative to $\bar{d}$ can be legitimately extended to $\mathcal{P}(\mathcal{X})$. For $\mu, (\hat{\mu}^n)_n \in \mathcal{P}(\mathcal{X})$, $\overline{\mathcal{W}}_p(\hat{\mu}^n, \mu) \to 0$ if and only if $\hat{\mu}^n \rightharpoonup \mu$ (see [7, p. 16]). We can conclude that, if we work with a bounded metric on $\mathcal{X}$, then for every $p \geq 1$ the corresponding p-Wasserstein metric $\overline{\mathcal{W}}_p$ provides a metric on $\mathcal{P}(\mathcal{X})$ that is compatible with the weak convergence.

1.1.2 SDE of McKean-Vlasov type

Throughout this section, let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, supporting an i.i.d. sequence $(\xi_i)_{i\in\mathbb{N}}$ of $\mathbb{R}^d$-valued $\mathcal{F}_0$-measurable random variables and a sequence $(W^i)_{i\in\mathbb{N}}$ of independent m-dimensional $\mathbb{F}$-Brownian motions. A stochastic process $(X_t)_{t\geq 0}$ is a collection of random variables, all defined on the same probability space and indexed by time, taking values in a common measurable space. A stochastic process $(X_t)_{t\geq 0}$ is said to be *adapted* to the filtration $\mathbb{F}$ if X_t is $\mathcal{F}_t$-measurable for any $t \geq 0$.

Assumption 1.1. Suppose $\mathbb{E}[|\xi_1|^2] < \infty$. Assume $b : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \to \mathbb{R}^{d\times m}$ are Lipschitz, in the sense that $\exists L > 0$ s.t.

$$|b(x,m) - b(x',m')| + |\sigma(x,m) - \sigma(x',m')| \leq L(|x-x'| + \mathcal{W}_2(m,m'))$$

where we equip $\mathbb{R}^d$ with the Euclidean norm and $\mathbb{R}^{d\times m}$ with the Frobenius norm, both denoted by $|\cdot|$, and $\mathcal{W}_2$ is the 2-Wasserstein metric on $\mathcal{P}^2(\mathbb{R}^d)$ relative to the Euclidean distance on $\mathbb{R}^d$. We use the 2-Wasserstein metric here because, unlike Wasserstein distances induced by other exponents, $\mathcal{W}_2$ defines a Riemannian distance (see [8, p. 139]).

Proposition 1.3. *Under Assumption 1.1, the system of n interacting particles $(X^{n,1}, \dots, X^{n,n})$, where each $X^{n,i}$, $i = 1, \dots, n$ is driven by a stochastic differential equation of the form:*

$$\begin{cases} dX_t^{n,i} = b(X_t^{n,i}, \hat{\mu}_t^n)dt + \sigma(X_t^{n,i}, \hat{\mu}_t^n)dW_t^i, \quad X_0^i = \xi_i \\ \hat{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}, \end{cases} \tag{1.1}$$

is well-posed and admits a unique strong solution, for each $n \in \mathbb{N}$.

The precise definition of strong solutions will be provided later. Note that the coefficients b and σ, which represent the *drift* and *volatility* terms of the SDEs, respectively, are identical for each particle. Moreover, the dependence of particle i on the other particles $k \neq i$ occurs solely through the empirical measure $\hat{\mu}_t^n$. The Equation (1.1) is an example of SDE of *McKean-Vlasov type*, that is, a SDE in which the law of the solution shows up in the coefficients of the SDE.

Mean field theory emerges from the study of the limit $n \to \infty$ of n-particle interacting systems. As $n \to \infty$, the interaction becomes negligible, since the contribution of each particle is of order $1/n$, due to the symmetry induced to the system by the coefficients b, σ. Then, if $\hat{\mu}_t^n$ converges in some sense to some $\hat{\mu}_t \in \mathcal{P}^2(\mathbb{R}^d)$, we should expect that the dynamics at time t of a representative particle Y will be in the form of a SDE of McKean-Vlasov type depending on $\hat{\mu}_t$.

We now introduce a framework for constructing a simple McKean–Vlasov type SDE. Fix a time horizon $T > 0$ and let $\mathcal{C}^d := C([0,T];\mathbb{R}^d)$ denote the set of continuous $\mathbb{R}^d$-valued functions, equipped with the Borel σ-field $\mathcal{B}(\mathcal{C}^d)$ and the supremum norm $\|x\| := \sup_{t\in[0,T]} |x_t|$. Consider $C([0,T];\mathcal{P}^2(\mathbb{R}^d))$, i.e., the set of time-continuous measure flows $[0,T] \ni t \mapsto \mu_t \in \mathcal{P}^2(\mathbb{R}^d)$. Given $\mu \in \mathcal{P}^2(\mathcal{C}^d)$, we can define μ_t as the image of μ through the map $\mathcal{C}^d \ni x \mapsto x_t \in \mathbb{R}^d$. Then

$$\mathcal{P}^2(\mathcal{C}^d) \ni \mu \mapsto (\mu_t)_{t\in[0,T]} \in C([0,T];\mathcal{P}^2(\mathbb{R}^d))$$

is a surjection that, given a $\mathcal{C}^d$-valued random variable X, maps $\mu = \mathcal{L}(X)$ to its time-t marginals $\mu_t = \mathcal{L}(X_t)$. By Theorem 1.1, $(\mathcal{P}^2(\mathcal{C}^d), \mathcal{W}_{\mathcal{C}^d,2})$ and $(\mathcal{P}^2(\mathbb{R}^d), \mathcal{W}_{\mathbb{R}^d,2})$ are complete and separable metric space. Moreover, if we equip $C([0,T];\mathcal{P}^2(\mathbb{R}^d))$ with the supremum distance induced by $\mathcal{W}_{\mathbb{R}^d,2}$, i.e., $\sup_{t\in[0,T]} \mathcal{W}_{\mathbb{R}^d,2}(\mu_t,\nu_t)$, then the surjection becomes continuous and 1-Lipschitz, since

$$\sup_{t\in[0,T]} \mathcal{W}_{\mathbb{R}^d,2}(\mu_t,\nu_t) \leq \mathcal{W}_{\mathcal{C}^d,2}(\mu,\nu) \quad \forall \mu,\nu \in \mathcal{P}^2(\mathcal{C}^d).$$

Let W be a m-dimensional $\mathbb{F}$-Brownian motion and ξ a $\mathcal{F}_0$-measurable $\mathbb{R}^d$-valued random variable. The simplest case of a McKean–Vlasov SDE is given by the following:

$$\begin{cases} dY_t = b(Y_t,\mu_t)dt + \sigma(Y_t,\mu_t)dW_t, \quad t \in [0,T], \quad Y_0 = \xi, \\ \mu = \mathcal{L}(Y). \end{cases} \tag{1.2}$$

Definition 1.4. A *strong solution* for the McKean-Vlasov type SDE (1.2) is a pair (Y,μ) where Y is a time-continuous $\mathbb{F}$-adapted $\mathbb{R}^d$-valued stochastic process (i.e., a $\mathcal{C}^d$-valued random variable), and $\mu \in \mathcal{P}^2(\mathcal{C}^d)$ such that both equations in (1.2) hold simultaneously.

Theorem 1.4. *Suppose Assumption 1.1 holds; then there exists a unique strong solution of the McKean-Vlasov SDE* (1.2).

Theorem 1.5. *Consider the n-particle system* (1.1) *under Assumption 1.1. Consider the McKean-Vlasov SDE* (1.2) *with coefficients b, σ and initial condition ξ with the same law as ξ_i, and let $(Y, \hat{\mu})$ be its unique strong solution. Let $\hat{\mu}^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{n,i}}$ denote the (lifted) empirical measure. Then*

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{W}^2_{\mathcal{C}^d,2}(\hat{\mu}^n, \hat{\mu})] = 0$$

and, for a fixed $k \in \mathbb{N}$, we have the weak convergence

$$(X^{n,1}, \ldots, X^{n,k}) \rightharpoonup (Y^1, \ldots, Y^k) \tag{1.3}$$

where $Y^1, \ldots, Y^k$ are independent copies of the solution Y of the considered McKean-Vlasov SDE.

Remark 1.1. The choice of the first k particles in (1.3) is inconsequential. Indeed, the n-particle system is *exchangeable*, in the sense that any permutation $(X^{n,\pi(1)}, \ldots, X^{n,\pi(n)})$ of $(X^{n,1}, \ldots, X^{n,n})$ is equal in distribution by the uniqueness of solution of (1.1). We should then interpret the limit (1.3) as $X^{n,i}$, $i = 1, \ldots, n$, become asymptotically i.i.d. as $n \to \infty$. The exchangeability of particles plays a crucial role in the theory of dynamic interacting systems, as it greatly simplifies analysis and allows for solution methods. In the MFG framework, this concept appears as the anonymity of agents.

1.1.3 Stochastic optimal control

Throughout this section, fix a finite time horizon $T > 0$ and consider the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ supporting a d-dimensional $\mathbb{F}$-Brownian motion W, with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, $\mathcal{F}_t = \sigma(W_s : 0 \le s \le t)$. Suppose to have one particle, and that it is capable of making rational decisions. To emphasize this assumption, we will henceforth refer to it as 'agent'. The agent chooses actions from a subset $A \subseteq \mathbb{R}^k$, which we suppose closed. We make the following assumptions:

Assumption 1.2.

1. $b : \mathbb{R}^d \times A \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times A \to \mathbb{R}^{d\times m}$ are measurable and uniformly Lipschitz in x, that is, $\exists K > 0$ such that, $\forall x, x' \in \mathbb{R}^d$,

$$|b(x,a) - b(x',a)| + |\sigma(x,a) - \sigma(x',a)| \le K|x - x'|.$$

2. $f : \mathbb{R}^d \times A \to \mathbb{R}$, $g : \mathbb{R}^d \to \mathbb{R}$ are continuous and bounded from above.

The agent selects a *control process* $\psi = (\psi_t)_{t\in[0,T]}$, an A-valued stochastic process that influences the dynamics of its state process X, which now evolves according to the following SDE:

$$dX_t = b(X_t, \psi_t)dt + \sigma(X_t, \psi_t)dW_t, \quad X_0 = x \in \mathbb{R}^d. \tag{1.4}$$

If ψ is $\mathbb{F}$-progressively measurable and satisfies $\mathbb{E}[\int_0^T |\psi_t|^2 dt] < +\infty$, then it is said to be *admissible*. Let $\mathbb{A}$ denote the set of admissible controls. Moreover, if ψ satisfies:

$$\mathbb{E}\Big[\int_0^T (|b(0,\psi_t)|^2 + |\sigma(0,\psi_t)|^2)dt\Big] < +\infty,$$

then ψ is said to be an *open loop*, meaning that the system lacks of a feedback mechanism: the agent controls its own state with a predetermined strategy, without any self-correction or adjustment based on the system's output. Open-loop systems are generally more tractable; indeed, classical theory guarantees that the SDE (1.4) admits a unique strong solution (see [7, p. 65]). The goal of the agent is to solve the following optimization problem:

$$\operatorname*{argmax}_{\psi} J(\psi), \quad J(\psi) = \mathbb{E}\Big[\int_0^T f(X_t,\psi_t)dt + g(X_T)\Big]$$

for ψ that belongs to (some subset of) $\mathbb{A}$. $J(\psi)$ is referred to as the *objective function* and f, g are called *running* and *terminal* functions respectively.

1.2 n-player stochastic differential games

Fix a finite time horizon $T > 0$ and consider the filtered probability space $(\Omega^{(n)}, \mathcal{F}^{(n)}, \mathbb{F}^{(n)}, \mathbb{P}^{(n)})$ supporting n independent m-dimensional $\mathbb{F}^{(n)}$-Brownian motions $W^1, \dots, W^n$, with $\mathbb{F}^{(n)} = (\mathcal{F}_t^{(n)})_{t\in[0,T]}$, $\mathcal{F}_t^{(n)} = \sigma(W_s^i : 0 \le s \le t, i = 1, \dots, n)$. Suppose there are n players $i = 1, \dots, n$. Player i chooses an admissible control $\psi^i = (\psi_t^i)_{t\in[0,T]} \in \mathbb{A}_i$ with values in some closed subset A_i of an Euclidean space; the d-dimensional state process $X^i = (X_t^i)_{t\in[0,T]}$ of each player is influenced by *all* the controls chosen by the players, i.e.,

$$dX_t^i = b(X_t, \vec{\psi}_t)dt + \sigma(X_t^i, \vec{\psi}_t)dW_t^i, \quad X_0^i = x_i \tag{1.5}$$

where $\vec{\psi}_t = (\psi_t^1, \dots, \psi_t^n) \in \prod_{i=1}^n A_i$ is the vector of time t-controls and

$$b : \mathbb{R}^d \times \prod_{i=1}^n A_i \to \mathbb{R}^d, \qquad \sigma : \mathbb{R}^d \times \prod_{i=1}^n A_i \to \mathbb{R}^{d\times m}$$

are, respectively, the drift and volatility functions. We assume that these coefficients are common to all players and satisfy Assumptions 1.1 and 1.2(1), ensuring that the SDE (1.5) admits a strong solution for all $i = 1, \dots, n$. The objective of player i is to solve the following optimization problem:

$$\operatorname*{argmax}_{\psi^i} J^i(\vec{\psi}), \quad J^i(\vec{\psi}) = \mathbb{E}^{(n)}\Big[\int_0^T f^i(X_t^{n,i}, \vec{\psi}_t)dt + g^i(X_T^{n,i})\Big]$$

where f^i, g^i are running and terminal functions respectively that we assume to satisfy Assumption 1.2(2), for every $i = 1, \ldots, n$. The n-player stochastic differential game consists in each player simultaneously optimizing their own objective function, while accounting for the strategies of all other players. We formulate the following notion of solution for the game:

Definition 1.5. A vector of admissible controls $\vec{\hat{\psi}} = (\hat{\psi}^1, \ldots, \hat{\psi}^n) \in \prod_{i=1}^n \mathbb{A}_i$ is called a *Nash equilibrium* for the n-player stochastic differential game if:

$$\forall i = 1, \ldots, n \;\; \forall \varphi^i \in \mathbb{A}_i, \qquad J^i(\vec{\hat{\psi}}) \geq J^i((\vec{\hat{\psi}}^{-i}, \varphi^i))$$

where we make use of the standard game-theoretic abbreviation:

$$(\psi^{-1}, \varphi) \coloneqq (\psi^1, \ldots, \psi^{i-1}, \varphi, \psi^{i+1}, \ldots, \psi^n)$$

The existence and uniqueness (or potential lack thereof) of Nash equilibria in n-player stochastic differential games, as well as the characteristics of the associated optimal strategy profiles, are heavily influenced by the players' information structures and the nature of their allowable actions. Various formulations of Nash equilibrium have been proposed to account for these differences in information and action types, each tailored to specific modeling assumptions.

1.3 Stochastic differential mean field games

Mean field games can be viewed as the limiting case of controlled n-particle systems governed by the McKean–Vlasov equation, as $n \to \infty$. Heuristically, in the same setting of Section 1.2, we start by considering a stochastic differential game with n players, each one controlling its private state $X_t^{n,i} \in \mathbb{R}^d$ by taking an action ψ_t^i in a common closed convex subset A of an Euclidean space, taking into consideration the distribution of the totality of states through the empirical measure $\hat{\mu}_t^n$. Modeling this scenario requires the use of McKean–Vlasov type SDEs, alongside tools from stochastic optimal control and finite-player stochastic differential games theory. We suppose that the dynamics of the private states are given by Itô's SDEs of the form:

$$\begin{cases} dX_t^{n,i} = b(X_t^{n,i}, \hat{\mu}_t^n, \psi_t^i)dt + \sigma(X_t^{n,i}, \hat{\mu}_t^n, \psi_t^i)dW_t^i, \quad X_0^i = \xi_i, \\ \hat{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}, \end{cases} \tag{1.6}$$

for some drift and volatility functions b, σ, common to all agents, satisying:

$$b : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times A \to \mathbb{R}^d, \qquad \sigma : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times A \to \mathbb{R}^{d\times m}.$$

If we suppose that the running and terminal functions f, g are common to all agents, then the optimization problem of player i becomes:

$$\operatorname*{argmax}_{\psi^i} J^i(\vec{\psi}), \quad J^i(\vec{\psi}) = \mathbb{E}^{(n)}\Big[\int_0^T f(t, X_t^{n,i}, \hat{\mu}_t^n, \psi_t^i)dt + g(X_T^{n,i}, \hat{\mu}_T^n)\Big] \quad (1.7)$$

As $n \to \infty$, an appropriate controlled analogue of Theorem 1.5 justifies focusing on a single *representative agent* in place of the full system. Consider a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, supporting a m-dimensional $\mathbb{F}$-Brownian motion W.

Definition 1.6. A *(stochastic differential) mean field game problem* is articulated in the following way:

1. For each fixed deterministic measure flow $\mu = (\mu_t)_{t\in[0,T]} \in C([0,T]; \mathcal{P}^2(\mathbb{R}^d))$ solve the stochastic optimal control problem faced by the representative agent:

$$\operatorname*{argmax}_{\psi} J^\mu(\psi), \quad J^\mu(\psi) = \mathbb{E}\Big[\int_0^T f(t, Y_t^\mu, \mu_t, \psi_t)dt + g(Y_T^\mu, \mu_T)\Big], \quad (1.8)$$

   where ψ belongs to (some subset of) $\mathbb{A}$, under the dynamical constraint:

$$dY_t^\mu = b(Y_t^\mu, \mu_t, \psi_t)dt + \sigma(Y_t^\mu, \mu_t, \psi_t)dW_t, \quad Y_0^\mu = \xi. \quad (1.9)$$

   Let $\hat{\psi}$ denote the optimal solution to (1.8), and let $Y^{\mu,\hat{\psi}}$ denote the solution to (1.9) for $\psi = \hat{\psi}$.

2. Find a flow $\mu = (\mu_t)_{t\in[0,T]}$ such that $\mathcal{L}(Y_t^{\mu,\hat{\psi}}) = \mu_t$ for all $t \in [0,T]$.

It is evident that the MFG problem described above can be formulated as a fixed-point problem. Specifically, one can define a mapping Φ such that:

$$\begin{aligned} \Phi : C([0,T]; \mathcal{P}^2(\mathbb{R}^d)) &\to C([0,T]; \mathcal{P}^2(\mathbb{R}^d)) \\ \mu = (\mu_t)_{t\in[0,T]} &\mapsto \Phi(\mu) = (\mathcal{L}(Y_t^{\mu,\hat{\psi}}))_{t\in[0,T]} \end{aligned}$$

and search for fixed points of Φ.

Definition 1.7. We say that $\mu = (\mu_t)_{t\in[0,T]} \in C([0,T]; \mathcal{P}^2(\mathbb{R}^d))$ is a *mean field equilibrium* (MFE) if $\mu = \Phi(\mu)$, i.e., μ is a fixed point of Φ.

Remark 1.2. Given a measure flow μ, the optimal control $\hat{\psi}$ may not be unique. Hence, in certain models it becomes more appropriate to define a set-valued mapping Φ as:

$$\Phi(\mu) = \{(\mathcal{L}(Y_t^{\mu,\hat{\psi}}))_{t\in[0,T]} : \hat{\psi} \text{ is optimal for (1.8)-(1.9) } \}$$

and search for $\mu \in \Phi(\mu)$; within this context, we say that $(\mu, \hat{\psi})$ is a MFE.

The literature (see, for example, [9]) offers several theorems that establish the connection between solutions of mean field games and those of the corresponding finite-player games. In particular, under suitable assumptions on the system's coefficients, initial conditions, and the nature of the controls, it can be shown that the mean field equilibrium approximates the Nash equilibrium of the n-player game, with the approximation becoming increasingly accurate as n tends to infinity. These results are typically formulated as follows: under the aforementioned assumptions, there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$, converging to 0 as $n \to \infty$, such that the strategies $(\vec{\hat{\psi}}_n = (\hat{\psi}^1, \dots, \hat{\psi}^n))_{n\in\mathbb{N}}$ constructed as function of the solution of the MFG, satisfy, for every $n \in \mathbb{N}$:

$$\forall i = 1, \dots, n, \;\; \forall \varphi^i \in \mathbb{A}_i, \quad J^i(\vec{\hat{\psi}}_n) \geq J^i((\vec{\hat{\psi}}_n^{-i}, \varphi^i)) - \varepsilon_n. \tag{1.10}$$

Such $\vec{\hat{\psi}}_n$ is referred to as an *ε_n-approximate open-loop Nash equilibrium.*

1.4 Incorporation of common noise

In the finite-player stochastic differential games and mean field games discussed so far, the randomness influencing agents' dynamics has been modeled using independent Brownian motions. However, this assumption is limiting in frameworks where aggregate shocks–random events that simultaneously impact all agents–cannot be neglected. Such situations frequently arise in economic and financial settings, where firms and investors must consider not only the decisions of other agents but also unexpected macroeconomic events that affect the entire system. These effects can be incorporated into the differential games framework by introducing what is known as *common noise.*

Consider an n-player stochastic differential game in which each agent solves the same optimization problem as in Equation (1.7), but with private state dynamics governed by stochastic differential equations of the form:

$$\begin{cases} dX_t^{n,i} = b(X_t^{n,i}, \hat{\mu}_t^n, \psi_t^i)dt + \sigma(X_t^{n,i}, \hat{\mu}_t^n, \psi_t^i)dW_t^i + \sigma^0(X_t^{n,i}, \hat{\mu}_t^n, \psi_t^i)dW_t^0, \quad X_0^i = \xi_i, \\ \hat{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}, \end{cases} \tag{1.11}$$

where $W^1, \dots, W^n, b, \sigma$ are defined as usual; W^0 is a m_0-dimensional $\mathbb{F}^{(n)}$-Brownian motion such that $W^0, W^1, \dots, W^n$ are independent; $\sigma^0 : \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \times A \to \mathbb{R}^{d\times m_0}$ is the volatility function associated to W^0; $\xi_1, \dots, \xi_n$ are $\mathcal{F}_0^{(n)}$-measurable initial states, where each $\mathcal{F}_t^{(n)}$, $t \in [0, T]$, is intended to account also for W^0, i.e., $\mathcal{F}_t^{(n)} = \sigma(W_s^i, 0 \leq s \leq t, i = 0, 1, \dots, n)$. The Brownian motions $W^1, \dots, W^n$ are referred to as *idiosyncratic noises* since each W^i influences only the dynamics of agent i, whereas W^0 is termed *common noise* because it simultaneously affects the states of all agents.

The introduction of the common noise term significantly complicates the problem, as in the limit $n \to \infty$ the solutions become stochastic rather than deterministic, since the equilibrium distribution of the population must still feel the influence of the common noise.

Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ supporting a $\mathcal{F}_0$-measurable initial state ξ and two independent Brownian motions W and W^0, of dimensions m, m_0 respectively. Let $\mathbb{F}^0 = (\mathcal{F}^0_t)_{t\in[0,T]}$ denote the filtration generated by the common noise only, i.e., $\mathcal{F}^0_t = \sigma(W^0_s : 0 \le s \le t)$.

Definition 1.8. A *(stochastic differential) mean field game with common noise problem* is articulated in the following way:

1. For each fixed $\mathbb{F}^0$-adapted $\mathcal{P}(\mathbb{R}^d)$-valued process $\mu = (\mu_t)_{t\in[0,T]}$ solve the stochastic optimal control problem faced by the representative agent:

$$\operatorname*{argmax}_{\psi} J^\mu(\psi), \quad J^\mu(\psi) = \mathbb{E}\Big[\int_0^T f(t, Y^\mu_t, \mu_t, \psi_t)dt + g(Y^\mu_T, \mu_T)\Big], \tag{1.12}$$

   where ψ belongs to (some subset of) $\mathbb{A}$, under the dynamical constraint:

$$dY_t = b(Y_t, \mu_t, \psi_t)dt + \sigma(Y_t, \mu_t, \psi_t)dW_t + \sigma^0(Y_t, \mu_t, \psi_t)dW^0_t, \quad Y_0 = \xi. \tag{1.13}$$

   Let $\hat{\psi}$ denote the optimal solution to Equation (1.12), and let $Y^{\mu,\hat{\psi}}$ denote the solution to Equation (1.13) for $\psi = \hat{\psi}$.

2. Find a flow $\mu = (\mu_t)_{t\in[0,T]}$ such that $\mathcal{L}(Y^{\mu,\hat{\psi}}_t | \mathcal{F}^0_t)(dx, \cdot) = \mu_t$ for all $t \in [0, T]$, i.e., a fixed point of the mapping:

$$\mu = (\mu_t)_{t\in[0,T]} \mapsto \Phi(\mu) = (\mathcal{L}(Y^{\mu,\hat{\psi}}_t | \mathcal{F}^0_t))_{t\in[0,T]}.$$

We refer to such a measure μ as the MFE of the considered mean field game with common noise. Following the terminology of Carmona et al. (see [10, p. 3]), we refer to the MFE μ as a *strong solution* in the common noise context since it is adapted to the filtration generated by W^0. More general and precise definitions of *strong solvability* and the corresponding strong solutions can be found in [9, Section 2.2]. Intuitively, strong solvability holds when the MFG problem admits at least one strong solution for every choice of the filtration $\mathbb{F}^0$. The most relevant cases occur when $\mathbb{F}^0$ is the filtration generated by the common noise or its usual augmentation. Conversely, if the MFG problem with common noise admits solutions only for specific filtrations $\mathbb{F}^0$ rather than all possible ones, it lies within the *weak solvability* regime (see [9, Section 2.3]).

Chapter 2

MFG model for decarbonization of financial markets

Notation. Fix a time horizon $T > 0$. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a pair of independent Brownian motions (W, W^0), where W is $\mathbb{R}^m$-valued and represents idiosyncratic noise, and W^0 is $\mathbb{R}^{m_0}$-valued and represents common noise. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ denote the $\mathbb{P}$-complete natural filtration generated by (W, W^0) and $\mathbb{F}^0 = (\mathcal{F}^0_t)_{t\in[0,T]}$ the $\mathbb{P}$-complete natural filtration generated by W^0 only.

Let $\mathbb{E}$ denote the expectation under the real-world probability $\mathbb{P}$; for every other probability $\mathbb{Q}$, the expectation under $\mathbb{Q}$ will be denoted by $\mathbb{E}_{\mathbb{Q}}$. Moreover, for every σ-field $\mathcal{G}$, filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ and parameter $p \in \mathbb{R}^+$, let:

- $L^0(\mathcal{G})$ denote the space of $\mathbb{R}$-valued $\mathcal{G}$-measurable random variables;
- $L^p(\mathcal{G}) \coloneqq \{X \in L^0(\mathcal{G}) \text{ s.t. } \mathbb{E}[|X|^p] < +\infty\}$;
- $L^p_+(\mathcal{G}) \coloneqq \{X \in L^p(\mathcal{G}) \text{ s.t. } X \geq 0 \ \mathbb{P}\text{-a.s.}\}$;
- $L^0(\mathbb{G})$ denote the space of $\mathbb{R}$-valued $\mathbb{G}$-progressively measurable random processes (i.e., $X(s,\omega) : [0,t]\times\Omega \to \mathbb{R}$ is $\mathcal{B}([0,t])\times\mathcal{F}_t$-measurable for every $t \in [0,T]$, which is a stronger requirement than mere adaptedness);
- $L^p(\mathbb{G}) \coloneqq \{(X_t)_{t\in[0,T]} \in L^0(\mathbb{G}) \text{ s.t. } \mathbb{E}[\int_0^T |X_t|^p dt] < +\infty\}$;

Finally, let $\mathbb{1} \in L^0(\mathcal{F}_T)$ be the random variable equal to 1 $\mathbb{P}$-a.s.; note that $\mathbb{1} \in L^p(\mathcal{F}_T)$ for every p.

2.1 Mean field market model

We aim to model a large financial market in which numerous small firms, potentially climate-conscious, interact with both traditional and green-oriented investors. This interaction is described through a stochastic framework.

We assume a complete, arbitrage-free financial market in which a risk-free asset and a risky asset are traded. Additionally, we assume that the risk-free asset is available in unlimited supply. This setting guarantees the existence of a unique, strictly positive equilibrium stochastic discount factor (SDF) process (see [11, p. 192]). A stochastic process $(\xi_t)_{t\in[0,T]}$ is called a SDF process on the finite continuous time interval $[0,T]$ if (i) $\xi_0 = 1$, (ii) $\xi_t > 0$ for every $t \in [0,T]$, and (iii) $(\xi_t)_{t\in[0,T]}$ is an $\mathbb{F}^0$-martingale and $(S_t\xi_t)_{t\in[0,T]}$ is a $\mathbb{F}$-martingale, where S is the risky asset price (see [11, p. 324]).

According to the new classical economic theory (see, for example, [12]), we assume that supply equals demand for any traded good or service, resulting in neither surplus nor shortage. When this balance is achieved, the market is said to "clear". If all relevant information is fully known and there are no frictions preventing price adjustments, prices will continuously adapt to maintain market clearing.

We study a mean field financial market where shares of a continuum of firms are traded. Each firm independently determines its dynamic and stochastic emission strategy based on its private information, as well as on market-wide risk factors and decarbonization trends. Crucially, firms do not observe the individual actions of other small firms; instead, their decisions are influenced by aggregate market dynamics. To determine its emission levels, the representative firm optimizes an objective function that balances financial and environmental performance. Financial performance is captured by the market value of the firm's shares, which depends on the stochastic discount factor, thereby introducing interdependence among firms through the financial market. Environmental performance is penalized in proportion to carbon emissions, and this penalty is modeled as a stochastic process to reflect the uncertainty surrounding climate transition risks.

2.2 Construction of a fixed point problem in SDF space

Let $\xi \in L^1_+(\mathcal{F}^0_T)$ and define $\xi_t := \mathbb{E}[\xi|\mathcal{F}_t]$ for all $t \in [0,T]$. Suppose that $(\xi_t)_{t\in[0,T]}$ is an SDF process. Note that by the tower property of conditional expectation, $(\xi_t)_{t\in[0,T]}$ is also an $\mathbb{F}$-martingale. Throughout this chapter, we will refer to the random variable ξ simply as the stochastic discount factor, or SDF. In our framework, the SDF is both a driver and an outcome of the interaction among market participants. It plays a central role: agents compute prices based on it. Since the existence and uniqueness of a strictly positive equilibrium SDF process is ensured by theory, our goal is to construct a method that characterizes it explicitly using a fixed-point argument.

The fixed-point problem is constructed in the following three steps:

- In Subsection 2.2.1 we define the representative firm's problem: given a fixed terminal SDF ξ, the firm seeks the optimal emission control ψ^ξ that maximizes its objective function. This objective depends on ξ and accounts for both the firm's asset price and the costs associated with emission intensity.

- In line with the MFG framework, we incorporate the distribution of the state process into the fixed point formulation. Here, the state process is the value-per-share process of the representative firm, which depends on its emission control. In Subsection 2.2.2, for each admissible control ψ, we compute the corresponding terminal value V_T^ψ and define m^ψ as the conditional distribution of V_T^ψ, that is, the distribution of firm-level outcomes across the population, given the realization of the common noise.

- To close the loop, in Subsection 2.2.3 we introduce a mapping that assigns to each admissible conditional distribution m an associated terminal stochastic discount factor ξ^m. To construct this mapping, we define the investors' problem as a wealth maximization problem and impose the market clearing condition. This yields an explicit expression for the optimal ξ^m corresponding to each conditional distribution m.

2.2.1 The representative firm's problem

Let $(V_t)_{t\in[0,T]}$ be the value-per-share process of the representative firm, and suppose that its dynamics is given by the following SDE:

$$dV_t = V_t(\mu_t dt + \sigma_t dW_t + \sigma_t^0 dW_t^0) + c_t\psi_t dt, \quad V_0 = v \tag{2.1}$$

where $v \in L^0(\mathcal{F}_0),\ v > 0$ $\mathbb{P}$-a.s., and:

- $(\mu_t)_{t\in[0,T]} \in L^0(\mathbb{F})$ is $\mathbb{R}$-valued and represents the market's drift term;
- $(\sigma_t)_{t\in[0,T]} \in L^0(\mathbb{F})$ is $\mathbb{R}^m$-valued and represents the idiosyncratic volatility term;
- $(\sigma_t^0)_{t\in[0,T]} \in L^0(\mathbb{F})$ is $\mathbb{R}^{m_0}$-valued and represents the volatility associated to the common noise;

- $(c_t)_{t\in[0,T]} \in L^0(\mathbb{F})$ is $\mathbb{R}$-valued, satisfies $c > 0$ $\mathbb{P}$-a.s., and is interpreted as being inversely proportional to the emission intensity of production. Accordingly, it is referred to as the "emission efficacy" process: its values are low for carbon-intensive firms and high for environmentally efficient, "green" firms that generate substantial economic value with relatively low emissions;
- $(\psi_t)_{t\in[0,T]} \in L^2(\mathbb{F})$ is $\mathbb{R}$-valued and represents the firm's instantaneous emissions. It can be viewed as the firm's emission control or emission strategy, as the firm retains full discretion over its emission rate at each point in time, based on its information and beliefs.

We require that the coefficients satisfy:

$$\int_0^T (|\mu_t| + |\sigma_t|^2 + |\sigma_t^0|^2 + |c_t|^2)dt < +\infty \quad \mathbb{P}\text{-a.s.} \tag{2.2}$$

Under this assumption, the stochastic exponential $(\mathcal{E}_t)_{t\in[0,T]}$, defined for every $t \in [0,T]$ as

$$\mathcal{E}_t := \exp\Big(\int_0^t \sigma_s dW_s + \int_0^t \sigma_s^0 dW_s^0 + \int_0^t \Big(\mu_s - \frac{|\sigma_s|^2}{2} - \frac{|\sigma_s^0|^2}{2}\Big)ds\Big), \tag{2.3}$$

is well defined.

Lemma 2.1. *For any emission control $\psi \in L^2(\mathbb{F})$, there exists a unique solution of Equation* (2.1) *given by*

$$V_t^\psi = \mathcal{E}_t v + \int_0^t \mathcal{E}_{s,t} c_s \psi_s ds, \quad V_0^\psi = v \tag{2.4}$$

where $(\mathcal{E}_t)_{t\in[0,T]}$ is the stochastic exponential defined in (2.3) *and $\mathcal{E}_{s,t} := \frac{\mathcal{E}_t}{\mathcal{E}_s}$.*

Proof. Note that Equation (2.1) can be written in the form of a stochastic exponential equation, i.e., $V_t = H_t^\psi + \int_0^t V_s dK_s$ where:

$$H_t^\psi := V_0 + \int_0^t c_s \psi_s ds, \quad dK_s := \mu_s ds + \sigma_s dW_s + \sigma_s^0 dW_s^0. \tag{2.5}$$

Recall that a semimartingale is any $\mathbb{R}$-valued process that can be decomposed as sum of a local martingale and a càdlàg adapted process of locally bounded variation (see [13, Chapter II, Theorem 9]). Hence, since integrals in Equation (2.5) are well-defined, then H^ψ is a semimartingale and K is a continuous semimartingale, which can be forced to satisfy $K_0 = 0$. Then, the existence and uniqueness of a solution V^ψ to Equation (2.1), which depends on ψ, are consequences of [13, Chapter V, Theorem 7]. Moreover, by [13, Chapter V, Theorem 52], this solution V^ψ is given explicitly by Equation (2.4). □

Given a SDF $\xi \in L^1_+(\mathcal{F}^0_T)$, the firm can compute its share price $S^{\xi,\psi}_t$ at any time $t \in [0,T]$ by discounting the terminal value-per-share V^{ψ}_T:

$$S^{\xi,\psi}_t = \frac{1}{\xi_t}\mathbb{E}[\xi V^{\psi}_T|\mathcal{F}_t]. \tag{2.6}$$

Following the approach proposed by Nordhaus (see [14, p. 277]) and later adopted by De Angelis et al. (see [15, p. 20]), the economic impact of climate change can be effectively approximated using a quadratic climate damage function. Accordingly, the objective function of the representative firm is given by:

$$J[\xi](\psi) = S^{\xi,\psi}_0 - \mathbb{E}\Big[\int_0^T \frac{\alpha_t\psi_t^2}{2}dt|\mathcal{F}_0\Big]. \tag{2.7}$$

The second term represents the direct or indirect costs linked to emissions, with the $\mathbb{R}$-valued stochastic process $(\alpha_t)_{t\in[0,T]} > 0$ $\mathbb{P}$-a.s. satisfying $\alpha^{-1} \in L^2(\mathbb{F})$ capturing the future intensity of these costs. Each firm is thus faced with a trade-off: on one hand, maximizing financial performance through its share price; on the other, limiting environmental damage by minimizing the penalty associated with emissions. The representative firm's problem is then defined as follows:

$$\underset{\psi\in L^2_\alpha(\mathbb{F})}{\operatorname{argmax}} J[\xi](\psi), \quad J[\xi](\psi) \text{ as in Equation (2.7)} \tag{P_f}$$

where $L^2_\alpha(\mathbb{F}) := \{\psi \in L^2(\mathbb{F}), \psi\alpha^{\frac{1}{2}} \in L^2(\mathbb{F})\}$, so that both terms in Equation (2.7) are well defined and finite. The following lemma establishes the existence and uniqueness of a solution ψ^ξ to (P_f) in $L^2_\alpha(\mathbb{F})$, and provides an explicit expression for the optimal control ψ^ξ as a function of the stochastic discount factor ξ.

Lemma 2.2. *Assume $\mathbb{E}[\xi\mathcal{E}_T] < +\infty$, and define the control process ψ^ξ by:*

$$\psi^\xi_t := \frac{c_t}{\alpha_t}\mathbb{E}[\xi\mathcal{E}_T|\mathcal{F}_t] \tag{2.8}$$

for any $t \in [0,T]$. If $\psi^\xi \in L^2_\alpha(\mathbb{F})$, then ψ^ξ is the unique solution to (P_f).

Proof. Plugging into $J[\xi](\psi)$ the formulas (2.6), (2.4) for $S^{\xi,\psi}_0$, V^ψ_T respectively, and recalling that $\xi_0 = 1$, we obtain:

$$J[\xi](\psi) = \mathbb{E}[\xi\mathcal{E}_T v|\mathcal{F}_0] + \mathbb{E}\Big[\int_0^T \Big(\xi\mathcal{E}_{t,T}c_t\psi_t - \frac{\alpha_t\psi_t^2}{2}\Big)dt\Big|\mathcal{F}_0\Big]. \tag{2.9}$$

Hence, the second term of the right-hand side of Equation (2.9) becomes:

$$\mathbb{E}\Big[\int_0^T \Big(c_t\mathbb{E}[\xi\mathcal{E}_{t,T}|\mathcal{F}_t]\psi_t - \frac{\alpha_t\psi_t^2}{2}\Big)dt\Big|\mathcal{F}_0\Big] = \mathbb{E}\Big[\int_0^T \Big(\alpha_t\psi_t\psi_t^\xi - \frac{\alpha_t\psi_t^2}{2}\Big)dt\Big|\mathcal{F}_0\Big], \tag{2.10}$$

and by simple algebraic manipulations, we note that it is equal to

$$\mathbb{E}\Big[\int_0^T \frac{\alpha_t}{2}(|\psi_t^\xi|^2 - |\psi_t - \psi_t^\xi|^2)dt\Big|\mathcal{F}_0\Big], \tag{2.11}$$

which is clearly maximized by $\psi = \psi^\xi$ only. Since maximizing $J[\xi](\psi)$ is equivalent to maximizing Equation (2.11), then we conclude that ψ^ξ is the optimal emission strategy, i.e., ψ^ξ is the unique solution to (P_f). $\square$

Remark 2.1. Tankov and Lavigne offer an interpretation of the expression for ψ^ξ by rewriting it as $\psi^\xi = \xi_t \times \frac{1}{\alpha_t} \times \frac{c_t}{\xi_t}\mathbb{E}[\xi\mathcal{E}_T|\mathcal{F}_t]$. This decomposition highlights three distinct effects. The first term, ξ_t, is the stochastic discount factor at time t, reflecting market conditions. Since the SDF is perfectly negatively correlated with the market (see [16, p. 27]), it can be interpreted as the inverse of the market portfolio. Consequently, in a declining market (high ξ_t), firms tend to increase emissions to offset reduced growth through higher production, while in a rising market (low ξ_t), they reduce emissions. The second term, $\frac{1}{\alpha_t}$, captures regulatory or environmental penalties: stronger penalties naturally discourage emissions. The final term represents the market value, at time t, of the future wealth generated by one unit of emissions at time t.

2.2.2 Incorporation of the collective behavior

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\mathcal{G}$ a σ-field in $\mathcal{F}$. For any $A \in \mathcal{F}$, the Radon-Nikodym theorem guarantees the existence of a $\mathcal{G}$-measurable random variable $\mathbb{P}(A|\mathcal{G}) : \Omega \to \mathbb{R}$, uniquely defined up to sets of probability zero, such that $\int_G \mathbb{P}(A|\mathcal{G})(\omega)d\mathbb{P}(\omega) = \mathbb{P}(A \cap G)$ for any $G \in \mathcal{G}$ (see [17, p. 430]). $\mathbb{P}(\cdot|\mathcal{G})$ is called the conditional probability given $\mathcal{G}$. Now let X be a $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. For each $B \in \mathcal{B}(\mathbb{R})$ and $\omega \in \Omega$, define a Markov transition kernel between $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ by:

$$\mathcal{L}(X|\mathcal{G})(B, \omega) := \mathbb{P}(X^{-1}(B)|\mathcal{G})(\omega) \quad \mathbb{P}\text{-a.s.} \tag{2.12}$$

Since X is $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$-valued, then for any fixed $\omega \in \Omega$, $\mathcal{L}(X|\mathcal{G})(\cdot, \omega)$ is a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (see [17, p. 439]). In this case, the conditional expectation of X given $\mathcal{G}$ can be written as:

$$\mathbb{E}[X|\mathcal{G}](\omega) = \int_{-\infty}^{+\infty} x\mathcal{L}(X|\mathcal{G})(dx, \omega) \quad \mathbb{P}\text{-a.s.} \tag{2.13}$$

We call the mapping $\Omega \ni \omega \mapsto \mathcal{L}(X|\mathcal{G})(dx, \omega) \in \mathcal{P}(\mathbb{R})$ the conditional distribution for X given $\mathcal{G}$.

Recall that we are considering a market in which only the shares of a single representative firm are traded. In the spirit of MFGs, we aim to capture collective behavior by incorporating the distribution (law) of the state process of this representative asset. Specifically, we are interested in the conditional law of the terminal value per share of the representative firm, given the realization of the common noise:

$$m^{\psi} := \mathcal{L}(V_T^{\psi} \mid \mathcal{F}_T^0)(dx, \cdot). \tag{2.14}$$

This distribution is no longer deterministic, but itself a random object. Note that the SDF does not play a role in the definition of m^{ψ}, since the terminal value V_T^{ψ} depends solely on the admissible emission control ψ, and not on the SDF ξ. Thus, Equation (2.14) combined with Equation (2.4) can be interpreted as a mapping that associates to each admissible emission strategy ψ the corresponding conditional distribution of the terminal value per share m^{ψ}. This formulation implicitly assumes that the infinitely many firms represented by the single representative firm all adopt the same emission strategy ψ.

2.2.3 The investors' problems and the market clearing

Investors' beliefs. We consider a market populated by two types of investors: *regular* and *green*. Since we are within the MFG framework, we model each class by a representative investor. These investors differ in their beliefs regarding the representative firm's future cash flow news. At any time $t \in [0, T]$, regular investors have access to information about the firm's cash flows and emission strategy over the interval $[0, t]$. However, they do not incorporate expectations about future cash flow news into their decision-making. As a result, it is reasonable to assume that the probability measure $\mathbb{P}^r$ governing the regular investor's asset allocation coincides with the real-world probability measure $\mathbb{P}$ (see [15, p. 12]). Then, for a generic asset D, the regular investor's dividend forecast at any time $t \in [0, T]$ is given by:

$$\mathbb{E}^{\mathbb{P}^r}[D_T|\mathcal{F}_t] = \mathbb{E}[D_T|\mathcal{F}_t] = D_t. \tag{2.15}$$

In contrast, the green investor incorporates the potential financial impact of climate-related external factors on the firm in which they are investing. As noted by De Angelis et al. in [15, pp. 12-13], these externalities can be negative, such as physical climate risks or regulatory penalties, or positive, such as a forward-looking climate strategy or limited exposure to environmental risks. In addition to the information available to the regular investor, we assume that the green investor has full knowledge of the financial implications of these climate-related externalities. As a result, the green investor bases their asset allocation not only on past and present cash flow and emissions information, but also on anticipated cash flow news over the remaining horizon $[t, T]$. The green investor's expectations are evaluated under a distinct probability measure $\mathbb{P}^g$, defined on the same Ω, which reflects their climate-conscious beliefs. Under this measure, the dividend forecast at time t is given by:

$$\mathbb{E}^{\mathbb{P}^g}[D_T|\mathcal{F}_t] = D_t + \int_t^T h_s(\psi_s)ds \tag{2.16}$$

where $h_s(\cdot)$ is a function that depends on the firm's emission strategy ψ_s (recall that $\psi \in L^2(\mathbb{F})$). This function is assumed to be decreasing in ψ_s, capturing the idea that higher emissions reduce the perceived value for the green investor. It may also vary with time s to account for changing sensitivity to emissions over the investment horizon.

In our model, the probability measure $\mathbb{P}^g$ associated with the green investor is defined via a change of measure from the real-world probability $\mathbb{P}$. Specifically, the Radon-Nikodym derivative that relates $\mathbb{P}^g$ to $\mathbb{P}$ is given by:

$$\frac{d\mathbb{P}^g}{d\mathbb{P}} = Z, \qquad Z := \exp\Big(\int_0^T \lambda_t dW_t^0 - \frac{1}{2}\int_0^T |\lambda_t|^2 dt\Big) \tag{2.17}$$

where $\lambda \in L^0(\mathbb{F}_0)$ satisfies Novikov's condition $\mathbb{E}[\exp(\frac{1}{2}\int_0^T |\lambda_t|^2 dt)] < +\infty$. Assuming that the green investor forms expectations according to Equation (2.16), the process λ can be expressed as $\lambda_s = (\sigma_s^0)^{-1} h_s(\psi_s)$ (see [15, p. 14]).

Investors' problems. We assume that both regular and green investors have Constant Absolute Risk Aversion (CARA) preferences, meaning their degree of risk aversion remains constant regardless of wealth levels. Let γ^r and γ^g denote the risk aversion coefficients of the regular and green investors, respectively. The global risk aversion γ^* is then given by $\frac{1}{\gamma^*} = \frac{1}{\gamma^r} + \frac{1}{\gamma^g}$. Given initial wealth levels w^r and w^g for the regular and green investors, and subject to individual budget constraints, each investor seeks to maximize the expected utility of terminal wealth, with exponential utility functions. This leads to the following optimization problems:

$$\sup_{W \in \mathcal{W}} 1 - \mathbb{E}[e^{-\gamma^r W}] \text{ s.t. } \mathbb{E}[\xi W] \leq w^r \tag{P_r}$$

$$\sup_{W \in \mathcal{W}} 1 - \mathbb{E}^{\mathbb{P}^g}[e^{-\gamma^g W}] \text{ s.t. } \mathbb{E}[\xi W] \leq w^g \tag{P_g}$$

where $\mathcal{W} := \{W \in \mathcal{F}_T^0 : \mathbb{E}[\xi|W|] < +\infty\}$ is the set of admissible terminal wealths, given a SDF ξ satisfying $\mathbb{E}[\xi] = 1$.

Lemma 2.3. *The optimal terminal wealths for the regular and green investors are given by:*

$$W^r = w^r - \frac{1}{\gamma^r}\ln\xi + \frac{1}{\gamma^r}\mathbb{E}[\xi\ln\xi], \quad W^g = w^g - \frac{1}{\gamma^g}\ln(\xi/Z) + \frac{1}{\gamma^g}\mathbb{E}[\xi\ln(\xi/Z)]. \tag{2.18}$$

Proof. Recall Fenchel's duality formula for the negative entropy:

$$e^y = \sup_{x>0} xy - x(\ln x - 1), \tag{2.19}$$

for all $y \in \mathbb{R}$. In particular, $e^y \geq xy - x(\ln x - 1)$ for all $x > 0,\ y \in \mathbb{R}$, and equality holds for $x = y$. For every $\lambda > 0$, by replacing $x = \lambda\xi(\omega)Z^{-1}(\omega)$ and $y = -\gamma^g W(\omega)$ for $\omega \in \Omega$ and $W \in \mathcal{W}$, we obtain:

$$Z(\omega)e^{-\gamma^g W(\omega)} \geq -\lambda\gamma^g\xi(\omega)W(\omega) - \lambda\xi(\omega)\Big(\ln\Big(\lambda\xi(\omega)Z^{-1}(\omega)\Big) - 1\Big) \tag{2.20}$$

which holds $\mathbb{P}$-a.s. By taking expectations, and recalling that $\mathbb{E}[\xi] = 1$ and $\mathbb{E}[\xi W] \leq w^g$, we obtain that:

$$\begin{aligned}\mathbb{E}[Ze^{-\gamma^g W}] &\geq -\lambda\gamma^g\mathbb{E}[\xi W] - \lambda\mathbb{E}[\xi\ln(\lambda\xi/Z)] + \lambda\mathbb{E}[\xi] \\ &\geq \lambda\Big(-\gamma^g w^g - \mathbb{E}[\xi\ln(\xi/Z)]\Big) - \lambda(\ln\lambda - 1).\end{aligned} \tag{2.21}$$

Since the choice of $\lambda > 0$ is arbitrary, then, by applying Fenchel's duality, we obtain $\mathbb{E}[Ze^{-\gamma^g W}] \geq \lambda^*$, where $\lambda^* := \exp(-\gamma^g W^g - \mathbb{E}[\xi\ln(\xi/Z)])$. Now, let:

$$W^g := -\frac{1}{\gamma^g}\ln\left(\frac{\lambda^*\xi}{Z}\right), \quad W^r := -\frac{1}{\gamma^r}\ln(\lambda^*\xi). \tag{2.22}$$

$\mathbb{E}[\xi|W^g|] < +\infty$ since $x \mapsto x\ln x$ is bounded for $x < 1$. Hence, $W^g \in \mathcal{W}$. Moreover, evaluating for W^g yields $\mathbb{E}[Ze^{-\gamma^g W^g}] = \lambda^*$. Thus, we obtain that:

$$W^g = \operatorname*{argmax}_{W\in\mathcal{W}} 1 - \mathbb{E}^{\mathbb{P}^g}[e^{-\gamma^g W}], \quad \mathbb{E}[\xi W^g] \leq w^g, \tag{2.23}$$

that is, W^g is the solution to the green investor's problem (P_g). The solution W^r to the regular investor's problem (P_r) can be derived analogously by setting $Z = 1$ $\mathbb{P}$-a.s. $\square$

The market clearing condition. Since we have supposed the market in equilibrium, the following market clearing condition must hold:

$$W^r + W^g = \mathbb{E}[V_T^\psi|\mathcal{F}_T^0] + K \tag{MC}$$

where K is a constant. Equation (MC) should be interpreted as requiring that the total allocations of all investors at time T equal the total terminal value of the traded assets. This total value consists of the average value of the infinitely many firms, represented by a single representative firm having a prescribed emission strategy ψ, plus a constant K which ensures market equilibrium given the assumption of an unlimited supply of the risk-free asset.

Combining (2.22) and (MC), we can solve for ξ:

$$\xi = \tilde{K} \exp\Big(\rho \ln Z - \gamma^* \mathbb{E}[V_T^\psi | \mathcal{F}_T^0]\Big) \tag{2.24}$$

where $\tilde{K} := \frac{\exp(-\gamma^* K)}{\lambda^*}$ and $\rho := \frac{\gamma^r}{\gamma^r + \gamma^g}$. Imposing that $\mathbb{E}[\xi] = 1$, we obtain:

$$\xi = \frac{\exp\Big(\rho \ln Z - \gamma^* \mathbb{E}[V_T^\psi | \mathcal{F}_T^0]\Big)}{\mathbb{E}\Big[\exp\Big(\rho \ln Z - \gamma^* \mathbb{E}[V_T^\psi | \mathcal{F}_T^0]\Big)\Big]} \tag{2.25}$$

Recall that, from Equations (2.13) and (2.14), we can express $\mathbb{E}[V_T^\psi | \mathcal{F}_T^0]$ in terms of m^ψ. Based on this, the market clearing condition defines a mapping I that assigns to each admissible conditional law m a SDF $\xi^m = I(m)$, where:

$$I(m) := \frac{\exp\Big(\rho \ln Z - \gamma^* \int_{\mathbb{R}} xm\Big)}{\mathbb{E}\Big[\exp\Big(\rho \ln Z - \gamma^* \int_{\mathbb{R}} xm\Big)\Big]}. \tag{2.26}$$

Remark 2.2. To simplify the analysis of how green and regular investors' wealth affects equilibrium variables, and without loss of generality, we assume that both investor types have identical relative risk aversions. That is, $\gamma^R = \gamma^r w^r = \gamma^g w^g$. Under this assumption, $\rho = \frac{w^g}{w^r + w^g} \in (0, 1)$ can be interpreted as the current proportion of green investors in the population.

2.2.4 On the structure of the problem

Let $\Xi \subseteq L^1_+(\mathcal{F}_T^0)$ be the subset of the stochastic discount factors ξ satisfying:

(i) $\mathbb{E}[\xi] = 1$, from the definition of SDF;

(ii) $\mathbb{E}[\xi \ln \xi] < +\infty$, $|\mathbb{E}[\xi \ln Z]| < +\infty$ from imposing that W^r, W^g are admissible, i.e., belong to $\mathcal{W}$ and satisfy the budget costraints, in the investors' problems;

(iii) $\mathbb{E}[\xi \mathcal{E}_T v] < +\infty$, that is equivalent to ask that $v < +\infty$ $\mathbb{P}$-a.s. and $\mathbb{E}[\xi \mathcal{E}_T] < +\infty$, from the computations of $S_0^{\xi,\psi}$ and ψ^ξ in the representative firm's problem;

(iv) $\mathbb{E}[\int_0^T \frac{c_t^2}{\alpha_t^2}\mathbb{E}[\xi\mathcal{E}_{t,T}|\mathcal{F}_t]^2 dt] < +\infty$, $\mathbb{E}[\int_0^T \frac{c_t^2}{\alpha_t}\mathbb{E}[\xi\mathcal{E}_{t,T}|\mathcal{F}_t]^2 dt] < +\infty$ from imposing that $\psi^\xi \in L^2(\mathbb{F})$ and $\psi^\xi\alpha^{1/2} \in L^2(\mathbb{F})$ respectively.

Recall that $\mathcal{P}(\mathbb{R})$ denotes the set of Borel probability measures on $\mathbb{R}$. Any conditional law m arising from an emission strategy, as defined in Equation (2.14), is a function $\Omega \to \mathcal{P}(\mathbb{R})$. Standard MFG theory approaches the resulting fixed-point problem in the following way:

1. For each fixed $m \in \mathcal{P}(\mathbb{R})^\Omega$ and each $\mathbb{F}^0$-adapted $\mathcal{P}(\mathbb{R})$-valued process $\mu = (\mu_t)_{t\in[0,T]}$ with terminal condition $\mu_T = m$, compute ξ^m as described in Subsection 2.2.3. Then, using ξ^m, determine ψ^{ξ^m} as outlined in Subsection 2.2.1, solving the optimization problem subject to the dynamical constraint. Let $V^\mu = (V_t^\mu)_{t\in[0,T]}$ denote the solution to Equation (2.1) corresponding to the control ψ^{ξ^m} and the measure flow μ;

2. Find a flow $\mu = (\mu_t)_{t\in[0,T]}$ satisfying $\mu_T = m$ and $\mathcal{L}(V_t^\mu \mid \mathcal{F}_t^0)(dx,\cdot) = \mu_t$ for all $t \in [0,T]$.

Due to the presence of the common noise, the natural space in which one searches for the fixed point following the standard mean field games approach is $[C([0,T],\mathcal{P}(\mathbb{R}))]^\Omega$, being seen as the image of $C([0,T],\mathcal{P}(\mathbb{R})^\Omega)$ through the map:

$$\left(\left(\omega \mapsto \mu_t(\omega)\right)_{t\in[0,T]}\right) \longmapsto \left(\omega \mapsto (\mu_t(\omega))_{t\in[0,T]}\right). \tag{2.27}$$

Since Ω is not necessarily finite, this natural space is excessively large, making it very challenging to identify compact subsets that are left invariant under the transformations of interest (see [10, p. 2]). Therefore, it is necessary to adopt a different approach to tackle the fixed-point problem.

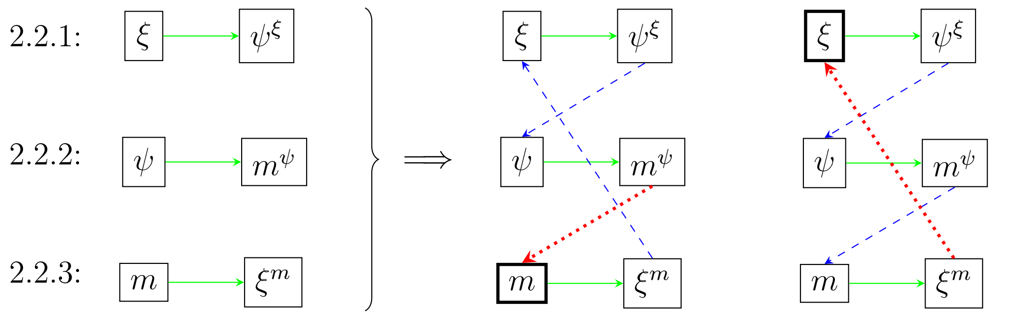
We leverage the fact that full knowledge of the flow μ is not required to solve our problem, as the only relevant object is the terminal random measure μ_T. Moreover, given the limited information available about the space of conditional laws, it is natural to reformulate the problem on the space Ξ of the admissible stochastic discount factors. The new approach proceeds as follows:

1. For each fixed SDF $\xi \in \Xi$, compute ψ^ξ as described in Subsection 2.2.1 via the optimization problem. Then, using ψ^ξ, determine m^{ψ^ξ} as outlined in Subsection 2.2.2.

2. Find $\xi \in \Xi$ such that the SDF defined by Equation (2.26), when evaluated at m^{ψ^ξ}, coincides with ξ, i.e. solve $\xi = I(m^{\psi^\xi})$.

It is clear that if such a fixed point $\bar{\xi}$ exists, then the corresponding optimal emission strategy $\bar{\psi}$ and the equilibrium conditional distribution of firms $\bar{m}$ are uniquely determined by Equations (2.8) and (2.14).

The diagram below illustrates the known relationships between SDFs, emission strategies, and conditional laws, represented by solid green arrows. It also depicts both the standard and the proposed approaches. The dashed blue arrows indicate deliberate choices, where the output of a previous computation is used as the input for the next. In contrast, the red dotted arrows highlight constraints required to complete the fixed-point loop.



a) Known relations *b) Standard MFG approach* *c) Proposed approach*

2.3 Existence and uniqueness of equilibrium SDF

We will make use of the following auxiliary functions: the negative entropy $h : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{+\infty\}$ and the indicator function $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R} \cup \{+\infty\}$ on the set $\{0\}$, defined by:

$$h(x) := \begin{cases} x(\ln x - 1) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \end{cases} \qquad \chi(x) := \begin{cases} +\infty & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Furthermore, we define an entropic functional $H : L^1_+(\mathcal{F}^0_T) \to \mathbb{R} \cup \{+\infty\}$ and a linear quadratic functional $L : L^1_+(\mathcal{F}^0_T) \to \mathbb{R} \cup \{+\infty\}$ by:

$$H(\xi) := \mathbb{E}[Z^\rho h(\xi/Z^\rho)], \qquad L(\xi) := \gamma^* \mathbb{E}\Big[\xi \mathcal{E}_T v + \int_0^T \frac{\alpha_t}{2} |\psi^\xi_t|^2 dt\Big].$$

Finally, we define a functional $G : L^1_+(\mathcal{F}^0_T) \to \mathbb{R} \cup \{+\infty\}$ that will play the role of potential of the fixed-point problem:

$$G(\xi) := H(\xi) + L(\xi) + \chi(\mathbb{E}[\xi] - 1). \tag{2.28}$$

We are now in a position to state the main result.

Theorem 2.4. *Assume that λ satisfies the Novikov's condition, $\alpha^{-1} \in L^2(\mathbb{F})$ and $\mathbb{E}[|\ln Z|] < +\infty$. Assume that the parameters $\alpha, c, \sigma, \sigma^0, \mu$ are chosen such that $L(\mathbb{1}) < +\infty$. Then, there exists a unique solution $\bar{\xi} \in \Xi$ to the fixed-point problem $\xi = I(m^{\psi^\xi})$. This solution satisfies $\bar{\xi} > 0$ $\mathbb{P}$-a.s., and it is characterized as:*

$$\bar{\xi} = \underset{\xi \in L^1_+(\mathcal{F}^0_T)}{\operatorname{argmin}} G(\xi). \tag{2.29}$$

The proof of Theorem 2.4 is carried out through a series of lemmas, each of which holds under the assumptions of the theorem.

Lemma 2.5. *Let $\bar{G} := \inf_{\xi \in L^1_+(\mathcal{F}^0_T)} G(\xi)$. Then, $G(\mathbb{1})$ and $\bar{G}$ are finite. Moreover, G is strictly convex and weakly lower semicontinuous.*

Lemma 2.6. *Define $\mathcal{C} := \{X \in L^1_+(\mathcal{F}^0_T) \text{ s.t. } \bar{G} \leq G(X) \leq G(\mathbb{1})\}$. Then, for every $X \in \mathcal{C}$, we have $X \in \Xi$; thus, $\mathcal{C} \subseteq \Xi$, and the minimizer of G can be sought over $\mathcal{C}$ instead of Ξ.*

Lemma 2.7. *There exists a unique minimizer $\bar{\xi}$ of G in $\mathcal{C}$.*

Lemma 2.8. *The minimizer $\bar{\xi}$ satisfies $\bar{\xi} > 0$ $\mathbb{P}$-a.s.*

Lemma 2.9. *The following are equivalent:*

$$\bar{\xi} \text{ is solution to } \xi = I(m^{\psi^\xi}) \iff \bar{\xi} = \underset{\xi \in L^1_+(\mathcal{F}^0_T)}{\operatorname{argmin}} G(\xi)$$

Once these lemmas are established, the conclusion of Theorem 2.4 follows naturally. Specifically, Lemma 2.9 shows that the fixed-point problem is equivalent to a minimization problem. The existence and uniqueness of a solution to this minimization problem are then demonstrated in Lemma 2.7, using the direct method in the calculus of variations. This argument relies on the technical properties developed in Lemmas 2.5 and 2.6. Finally, Lemma 2.8 proves the positivity of the solution, a necessary condition for it to qualify as a stochastic discount factor.

We proceed to prove the lemmas.

Proof of Lemma 2.5. Firstly, observe that $\mathbb{1} \in L^1_+(\mathcal{F}^0_T)$ and that

$$G(\mathbb{1}) = H(\mathbb{1}) + L(\mathbb{1}) + \chi(\mathbb{E}[\mathbb{1}] - 1) = -\rho\mathbb{E}[\ln Z] - 1 + L(\mathbb{1}) < +\infty \tag{2.30}$$

by the assumptions. Hence, $\bar{G} < +\infty$. On the other hand, for any $x, y \in \mathbb{R}^+$ it holds that:

$$\inf_{x>0} y^\rho h(x/y^\rho) = -\sup_{x>0} \Big(x \ln y^\rho - x(\ln x - 1)\Big) = -e^{\ln y^\rho} = -y^\rho \tag{2.31}$$

where we have used Fenchel's duality. By replacing $y = Z(\omega)$, $x = \xi(\omega)$ for $\omega \in \Omega$ and $\xi \in L^1_+(\mathcal{F}^0_T)$, we obtain $Z^\rho h(\xi/Z^\rho) \geq -Z^\rho$ $\mathbb{P}$-a.s. Hence, $H(\xi) \geq \mathbb{E}[-Z^\rho] \geq -\mathbb{E}[Z]^\rho$ by Jensen's inequality for $\rho < 1$. By the Novikov's condition, the process $(Z_t)_{t\in[0,T]}$, where

$$Z_t := \exp\Big(\int_0^t \lambda_s dW^0_s - \frac{1}{2}\int_0^t |\lambda_s|^2 ds\Big), \tag{2.32}$$

is a martingale. Thus, $\mathbb{E}[Z_T|\mathcal{F}^0_0] = Z_0$ $\mathbb{P}$-a.s. Since $Z = Z_T$, by applying the law of total expectation we obtain:

$$\mathbb{E}[Z] = \mathbb{E}[Z_T] = \mathbb{E}[\mathbb{E}[Z_T|\mathcal{F}^0_0]] = \mathbb{E}[Z_0] = 1. \tag{2.33}$$

In summary, $H(\xi) \geq -1$. Moreover, for each $\xi \in L^1_+(\mathcal{F}^0_T)$ such that $\mathbb{E}[\xi] = 1$, it holds that $L(\xi) \geq 0$ since $\gamma^* > 0$ and $|\psi^\xi|^2 \geq 0$, $V > 0$, $\mathcal{E}_T > 0$, $\alpha > 0$, all of them in the $\mathbb{P}$-a.s. sense. In conclusion, for every $\xi \in L^1_+(\mathcal{F}^0_T)$:

$$G(\xi) = H(\xi) + L(\xi) + \chi(\mathbb{E}[\xi] - 1) \geq -1. \tag{2.34}$$

Hence, $G(\mathbb{1})$ and $\bar{G}$ are finite.

Observe that H is strictly convex since h is strictly convex and expectations preserve convexity. Furthermore, L is strictly convex in ξ, being of linear-quadratic form. The linear component is trivially convex, and the strict convexity of the quadratic term follows from the positive definiteness of the 2-norm, together with the fact that both integration and expectation preserve convexity. Moreover, the mapping $\xi \mapsto \chi(\mathbb{E}[\xi] - 1)$ is trivially convex. In conclusion, G is strictly convex, as it is the sum of a convex function and strictly convex functions.

Let $\xi \in L^1_+(\mathcal{F}^0_T)$, and let $(\xi^n)_{n\in\mathbb{N}} \subseteq L^1{+}(\mathcal{F}^0_T)$ be a sequence that converges strongly to ξ. By Markov's inequality, convergence in L^1 implies convergence in probability. Moreover, convergence in probability ensures the existence of a subsequence that converges to ξ almost surely. Therefore, without loss of generality, we may assume that $\xi^n \to \xi$ $\mathbb{P}$-a.s., as $n \to +\infty$. First of all, this implies that $\mathbb{E}[\xi^n] \to \mathbb{E}[\xi]$, hence:

$$\chi(\mathbb{E}[\xi] - 1) \leq \liminf_{n\to+\infty} \chi(\mathbb{E}[\xi^n] - 1). \tag{2.35}$$

Secondly, this yields to $Z^\rho h(\xi^n/Z^\rho) \to Z^\rho h(\xi/Z^\rho)$ $\mathbb{P}$-a.s., and by taking expectations we obtain $H(\xi^n) \to H(\xi)$. Recalling that H is lower bounded by a constant, we can apply the shifted version of Fatou's lemma and obtain:

$$\mathbb{E}[\liminf_{n\to+\infty} H(\xi^n)] \leq \liminf_{n\to+\infty} \mathbb{E}[H(\xi^n)].$$

Using the fact that $\liminf_n H(\xi^n) = \lim_n H(\xi^n) = H(\xi)$, and noting that H is itself an expectation, we obtain the inequality:

$$H(\xi) \leq \liminf_{n\to+\infty} H(\xi^n). \tag{2.36}$$

On the other hand, for the same sequence $(\xi^n)_{n\in\mathbb{N}}$ let $\eta^n := \inf_{k\geq n} \xi^k$. Clearly $\eta^n \to \xi$ as $n \to +\infty$. Then, by Fatou's lemma on $(\eta^n)_{n\in\mathbb{N}}$, we obtain:

$$\psi_t^\xi = \frac{c_t}{\alpha_t}\mathbb{E}[\xi\mathcal{E}_{t,T}|\mathcal{F}_t] \leq \liminf_{n\to+\infty} \frac{c_t}{\alpha_t}\mathbb{E}[\eta^n\mathcal{E}_{t,T}|\mathcal{F}_t] = \liminf_{n\to+\infty} \psi_t^{\eta^n}. \tag{2.37}$$

By using that expectations preserve monotonicity, and by applying Fatou's lemma, we obtain:

$$L(\xi) \leq L(\liminf_{n\to+\infty} \eta^n) \leq \liminf_{n\to+\infty} L(\eta^n) \leq \liminf_{n\to+\infty} L(\xi^n) \tag{2.38}$$

where in the last inequality we used the definition of η^n. In conclusion, G is sum of strongly lower semicontinuous functions, thus G is itself strongly lower semicontinuous. Moreover, since G is strictly convex, then, by [18, Corollary 3.9], G is weakly lower semicontinuous. $\square$

Proof of Lemma 2.6. Since both $G(\mathbb{1})$ and $\bar{G}$ are finite, the set $\mathcal{C}$ is well-defined. Moreover, if $\xi \in \mathcal{C}$, then it must satisfy $\mathbb{E}[\xi] = 1$; otherwise, the indicator function forces $G(\xi)$ to be infinite. We now show that there exists a constant $C > 0$ such that, for every $\xi \in \mathcal{C}$, the following inequalities hold:

$$\begin{gathered} a)\ \mathbb{E}[\xi \ln \xi] \leq C, \qquad b)\ \mathbb{E}[\xi\mathcal{E}_T v] \leq C, \qquad c)\ |\mathbb{E}[\xi \ln Z]| \leq C, \\ d)\ \mathbb{E}\Big[\int_0^T \frac{c_t^2}{\alpha_t}\mathbb{E}[\xi\mathcal{E}_{t,T}|\mathcal{F}_t]^2 dt\Big] \leq C, \qquad e)\ \mathbb{E}\Big[\int_0^T \frac{c_t^2}{\alpha_t^2}\mathbb{E}[\xi\mathcal{E}_{t,T}|\mathcal{F}_t]^2 dt\Big] \leq C. \end{gathered} \tag{2.39}$$

a) Let $\xi \in \mathcal{C}$. Observe that $H(\xi) = \mathbb{E}[\xi \ln \xi - \xi \ln Z^\rho - \xi]$. Since ξ and Z are independent, and $\mathbb{E}[\xi] = 1$, then $H(\xi) = \mathbb{E}[\xi \ln \xi] - \mathbb{E}[\ln Z^\rho + 1]$. Moreover, since $L(\xi) \geq 0$, then $H(\xi) \leq H(\xi) + L(\xi) \leq G(\mathbb{1})$. On the other hand, $G(\mathbb{1}) = -\mathbb{E}[\ln Z^\rho + 1] + L(\mathbb{1})$. Combining these relations, we obtain:

$$\mathbb{E}[\xi \ln \xi] - \mathbb{E}[\ln Z^\rho + 1] = H(\xi) \leq G(\mathbb{1}) = -\mathbb{E}[\ln Z^\rho + 1] + L(\mathbb{1}).$$

Thus, $\mathbb{E}[\xi \ln \xi] \leq L(\mathbb{1}) < +\infty$.

b,d) Let $\xi \in \mathcal{C}$. Recall that $H(\xi) \geq -1$. It follows that $L(\xi) \leq G(\mathbb{1}) - H(\xi) \leq G(\mathbb{1}) + 1$. Recall that $L(\xi) \geq 0$, and that $L(\xi) = \gamma^*\mathbb{E}[\xi\mathcal{E}_T V] + \gamma^*\mathbb{E}[\int_0^T \frac{c_t^2}{\alpha_t}\mathbb{E}[\xi\mathcal{E}_{t,T}|\mathcal{F}_t]^2 dt]$ is sum of two non-negative terms, due to the non-negativeness of $\gamma^*, |\psi^\xi|^2, v, \mathcal{E}_T$, and α (understood in the $\mathbb{P}$-a.s. sense where necessary). We conclude that each one of these two terms needs to be upper bounded, at least by the same upper bound for $L(\xi)$:

$$\mathbb{E}[\xi \mathcal{E}_T v] < G(\mathbb{1}) + 1, \quad \mathbb{E}\Big[\int_0^T \frac{c_t^2}{\alpha_t} \mathbb{E}[\xi \mathcal{E}_{t,T} | \mathcal{F}_t]^2 dt\Big] < G(\mathbb{1}) + 1. \tag{2.40}$$

c) Let $\xi \in \mathcal{C}$. Recall that $H(\xi) \leq G(\mathbb{1})$. This implies:

$$\rho \mathbb{E}[\xi \ln Z] \geq -G(\mathbb{1}) + \mathbb{E}[\xi \ln \xi] - 1. \tag{2.41}$$

Observe that, for every $x, y \in \mathbb{R}^+$, it holds that $x \ln y \leq x \ln x + y - x$ (this follows easily by considering the real function $f(y) := x \ln y - y$ and studying its monotonicity and convexity properties). By replacing $y = Z(\omega)$, $x = \xi(\omega)$ for $\omega \in \Omega$, we obtain $\xi \ln Z \leq \xi \ln \xi + Z - \xi$ $\mathbb{P}$-a.s. By taking expectations, and combining this result with Equation (2.41), we obtain:

$$\rho \mathbb{E}[\xi \ln Z] \geq -G(\mathbb{1}) + \mathbb{E}[\xi \ln Z - Z + \xi] - 1 = -G(\mathbb{1}) + \mathbb{E}[\xi \ln Z] - 1.$$

Recalling that $\rho \in (0,1)$, we obtain $\mathbb{E}[\xi \ln Z] \leq \frac{G(\mathbb{1})+1}{1-\rho} < +\infty$.

On the other hand, from $-H(\xi) \geq -G(\mathbb{1})$, it follows that:

$$\rho \mathbb{E}[\xi \ln Z] \geq -G(\mathbb{1}) + \mathbb{E}[\xi \ln \xi - \xi] = -G(\mathbb{1}) + \mathbb{E}[h(\xi)] \geq -G(\mathbb{1}) - 1. \tag{2.42}$$

For the last inequality, we used that $h(\xi) \geq -1$ $\mathbb{P}$-a.s., as implied by Equation (2.31).

e) Since $\frac{1}{\alpha} \in L^2(\mathbb{F})$, the result follows directly by combining this with inequality (d).

Let $C := \max\{L(\mathbb{1}), G(\mathbb{1}) + 1, \frac{G(\mathbb{1})+1}{1-\rho}, \frac{G(\mathbb{1})+1}{\rho}\}$. Then, for every $\xi \in \mathcal{C}$, the conditions in (2.39) are satisfied. Hence, $\mathcal{C} \subseteq \Xi$. □

Proof of Lemma 2.7. The proof follows the direct method in the calculus of variations. By Lemma 2.6, $\sup_{\xi \in \mathcal{C}} \mathbb{E}[\xi \ln \xi] \leq C < +\infty$. Since $x \mapsto x \ln x$ is convex, and

$$\lim_{x \to 0} x \ln x = 0, \quad \text{and} \quad \lim_{x \to \infty} \frac{x \ln x}{x} = +\infty, \tag{2.43}$$

we can apply the De la Vallée-Poussin theorem [19, Chapter II, Theorem T22] on $\mathcal{C} \subseteq L^1_+(\mathcal{F}^0_T)$, and conclude that $\mathcal{C}$ is uniformly integrable. Then, by the Dunford-Pettis theorem [19, Chapter II, Theorem T23], $\mathcal{C}$ is weakly relatively compact. Finally, by the Eberlein-Šmulian Theorem [20, p. 430, Theorem 1], $\mathcal{C}$ is weakly sequentially relatively compact.

Let now $(\xi^n)_{n \in \mathbb{N}}$ be a minimizing sequence in $\mathcal{C}$, i.e. $\lim_{n \to +\infty} G(\xi^n) = \bar{G}$. Since $\mathcal{C}$ is weakly sequentially relatively compact, there exists $\bar{\xi} \in \mathcal{C}$ such that $(\xi^n)_{n\in\mathbb{N}}$ weakly converges to $\bar{\xi}$. Finally, using that G is weakly lower semicontinuous, we obtain:

$$\bar{G} \leq G(\bar{\xi}) \leq \liminf_{n\to+\infty} G(\xi^n) \leq \lim_{n\to+\infty} G(\xi^n) = \bar{G}. \tag{2.44}$$

Hence, $\bar{\xi}$ is a minimizer of G over $\mathcal{C}$. Moreover, since G is strictly convex, $\bar{\xi}$ is the unique minimizer. $\square$

Proof of Lemma 2.8. Assume by contradiction that there exists $A \in \mathcal{F}_T^0$ such that $p_A := \mathbb{P}(A) > 0$ and $\bar{\xi} = 0$ on A. For $\varepsilon \in (0,1)$, define:

$$\xi_\varepsilon := (1-\varepsilon)\bar{\xi} + \frac{\varepsilon}{p_A}\mathbf{1}_A \tag{2.45}$$

where $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, 0 otherwise. Clearly, $\xi_\varepsilon \in L^1_+(\mathcal{F}_T^0)$ for every ε, since they are linear combinations of $\bar{\xi}, \mathbf{1}_A \in L^1_+(\mathcal{F}_T^0)$. One one hand, if $\omega \notin A$, then $\bar{\xi}(\omega) > 0$ and $\xi_\varepsilon(\omega) = (1-\varepsilon)\bar{\xi}(\omega) > 0$. This implies that:

$$\begin{aligned}
&\int_{\Omega\setminus A}\Big[\xi_\varepsilon(\omega)h\Big(\frac{\xi_\varepsilon(\omega)}{Z(\omega)^\rho}\Big) - \bar{\xi}(\omega)h\Big(\frac{\bar{\xi}(\omega)}{Z(\omega)^\rho}\Big)\Big]d\mathbb{P}(\omega) = \\
&\quad = (1-\varepsilon)\ln(1-\varepsilon)\int_{\Omega\setminus A}\bar{\xi}(\omega)d\mathbb{P}(\omega) - \varepsilon\int_{\Omega\setminus A}\bar{\xi}(\omega)h\Big(\frac{\bar{\xi}(\omega)}{Z(\omega)^\rho}\Big)d\mathbb{P}(\omega) = O(\varepsilon).
\end{aligned}$$

For the last equality we used the fact that the first integral is bounded by $\mathbb{E}[\bar{\xi}] = 1 < +\infty$, and the second by $H(\bar{\xi}) \leq G(\bar{\xi}) < +\infty$. On the other hand, if $\omega \in A$, then $\bar{\xi}(\omega) = 0$ while $\xi_\varepsilon(\omega) = \frac{\varepsilon}{p_A} > 0$. By definition of h, this implies that:

$$\begin{aligned}
&\int_A\Big[\xi_\varepsilon(\omega)h\Big(\frac{\xi_\varepsilon(\omega)}{Z(\omega)^\rho}\Big) - \bar{\xi}(\omega)h\Big(\frac{\bar{\xi}(\omega)}{Z(\omega)^\rho}\Big)\Big]d\mathbb{P}(\omega) = \\
&\quad = \int_A \frac{\varepsilon}{p_A}\Big(\ln\varepsilon - \ln p_A - \ln Z(\omega)^\rho - 1\Big)d\mathbb{P}(\omega) - 0 = \varepsilon\ln\varepsilon + O(\varepsilon).
\end{aligned}$$

Moreover, since L is linear-quadratic in the SDF, then $L(\xi_\varepsilon) - L(\bar{\xi}) = O(\varepsilon)$ as $\varepsilon \to 0$. Combining these results, we obtain that:

$$G(\xi_\varepsilon) - G(\bar{\xi}) = \Big(H(\xi_\varepsilon) - H(\bar{\xi})\Big) + \Big(L(\xi_\varepsilon) - L(\bar{\xi})\Big) = \varepsilon\ln\varepsilon + O(\varepsilon). \tag{2.46}$$

Since the right-hand side becomes negative for sufficiently small ε, this contradicts the optimality of $G(\bar{\xi})$. $\square$

Prior to proving Lemma 2.9, we prove the following auxiliary result.

Lemma 2.10. *Let $\xi \in \mathcal{C}$. For every $\eta \in L^1(\mathcal{F}_T^0)$ such that $\mathbb{E}[\eta] = 0$ and $|\eta| \leq \frac{1}{2}\xi$ $\mathbb{P}$-a.s., the following equation holds:*

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda}\Big(G(\xi + \lambda\eta) - G(\xi)\Big) = \mathbb{E}\Big[\eta\Big(\delta H(\xi) + \delta L(\xi)\Big)\Big] \tag{2.47}$$

where $\lambda \in (0, 1)$, $\delta H(\xi) := \ln(\xi/Z^\rho)$ *and* $\delta L(\xi) := \gamma^* \mathbb{E}[V_T^{\psi^\xi}|\mathcal{F}_T^0]$.

Proof of Lemma 2.10. Observe that $\mathbb{E}[\xi + \lambda\eta] = \mathbb{E}[\xi] = 1$. It follows that:

$$\frac{1}{\lambda}\Big(G(\xi+\lambda\eta)-G(\xi)\Big) = \frac{1}{\lambda}\Big(H(\xi+\lambda\eta)-H(\xi)\Big)+\frac{1}{\lambda}\Big(L(\xi+\lambda\eta)-L(\xi)\Big). \tag{2.48}$$

Each term is analyzed individually. For H, its definition yields:

$$\frac{1}{\lambda}\Big(H(\xi + \lambda\eta) - H(\xi)\Big) = \frac{1}{\lambda}\mathbb{E}\Big[Z^\rho\Big(h\Big(\frac{\xi + \lambda\eta}{Z^\rho}\Big) - h\Big(\frac{\xi}{Z^\rho}\Big)\Big)\Big]. \tag{2.49}$$

Let $(x, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ such that $y \in [-x/\lambda, x/\lambda]$. Hence, $x \pm \lambda y \in [0, 2x]$. By the fundamental theorem of calculus, we have:

$$\frac{1}{\lambda}\Big(h(x + \lambda y) - h(x)\Big) = y\int_0^1 \ln(x + t\lambda y)dt \tag{2.50}$$

for $t \in [0, 1]$. Using the admissible ranges of y and t, we find that $x + t\lambda y \geq (1-t)x$ and $x+t\lambda y \leq 2x$. Using the monotony of the logarithm, this implies:

$$\ln x + \ln(1 - t) \leq \ln(x + t\lambda y) \leq \ln x + \ln 2. \tag{2.51}$$

Combining these results, by recalling that $\int_0^1 \ln(1-t)dt = -1$ and by replacing $x = \xi(\omega)/Z(\omega)^\rho$ and $y = \eta(\omega)/Z(\omega)^\rho$ for $\omega \in \Omega$, we obtain:

$$\mathbb{E}[\eta\ln(\xi/Z^\rho) - \eta] \leq \frac{1}{\lambda}\mathbb{E}\Big[Z^\rho\Big(h\Big(\frac{\xi + \lambda\eta}{Z^\rho}\Big) - h\Big(\frac{\xi}{Z^\rho}\Big)\Big)\Big] \leq \mathbb{E}[\eta\ln(\xi/Z^\rho) + \eta\ln 2]. \tag{2.52}$$

Recalling the properties of η (i.e., $\mathbb{E}[\eta] = 0$ and $|\eta| \leq \frac{1}{2}\xi$ $\mathbb{P}$-a.s.), we obtain:

$$\begin{aligned}\frac{1}{\lambda}\mathbb{E}\Big[Z^\rho\Big(h\Big(\frac{\xi + \lambda\eta}{Z^\rho}\Big) - h\Big(\frac{\xi}{Z^\rho}\Big)\Big)\Big] &= \\ = \mathbb{E}[\eta\ln(\xi/Z^\rho)] \leq \frac{1}{2}\mathbb{E}[\xi\ln(\xi/Z^\rho)] &\leq \frac{1}{2}(G(\mathbb{1}) + 1)\end{aligned} \tag{2.53}$$

where the last inequality follows from Equation (2.41). Hence, we conclude that Equation (2.49) admits an integrable bound, so by dominated convergence we have that:

$$\lim_{\lambda \to 0^+} \frac{1}{\lambda}\Big(H(\xi + \lambda\eta) - H(\xi)\Big) = \mathbb{E}[\eta\ln(\xi/Z^\rho)]. \tag{2.54}$$

Now we focus on the functional L. By definition we have that:

$$\frac{1}{\lambda}\Big(L(\xi+\lambda\eta)-L(\xi)\Big)=\frac{\gamma^*}{\lambda}\mathbb{E}\Big[\lambda\eta\mathcal{E}_T v+\int_0^T\frac{\alpha_t}{2}\Big(|\psi_t^{\xi+\lambda\eta}|^2-|\psi_t^{\xi}|^2\Big)dt\Big]. \tag{2.55}$$

Observe that $|\psi_t^{\xi+\lambda\eta}|^2=|\psi_t^{\xi}|^2+2\lambda\psi_t^{\xi}\psi_t^{\eta}+\lambda^2|\psi_t^{\eta}|^2=|\psi_t^{\xi}+\lambda\psi_t^{\eta}|^2$. We obtain that the right-hand side of Equation (2.55) becomes:

$$\gamma^*\mathbb{E}\Big[\eta\mathcal{E}_T v+\int_0^T\alpha_t\psi_t^{\xi}\psi_t^{\eta}dt\Big]+\frac{\lambda\gamma^*}{2}\mathbb{E}\Big[\int_0^T\alpha_t|\psi_t^{\eta}|^2dt\Big] \tag{2.56}$$

The last term converges to zero as $\lambda\to 0^+$. The first term instead becomes:

$$\gamma^*\mathbb{E}\Big[\eta\mathcal{E}_T v+\eta\int_0^T\mathcal{E}_{t,T}c_t\psi_t^{\xi}dt\Big]+\gamma^*\mathbb{E}\Big[\int_0^T\alpha_t\psi_t^{\xi}\psi_t^{\eta}dt-\eta\int_0^T\mathcal{E}_{t,T}c_t\psi_t^{\xi}dt\Big]. \tag{2.57}$$

Here, we added and subtracted the same quantity. Again, we consider the two terms separately. By applying the law of total expectation on the first term with respect to $\mathcal{F}_T^0$, we obtain that it is equal to:

$$\gamma^*\mathbb{E}\Big[\mathbb{E}\Big[\eta\mathcal{E}_T v+\eta\int_0^T\mathcal{E}_{t,T}c_t\psi_t^{\xi}dt\Big|\mathcal{F}_T^0\Big]\Big]=\mathbb{E}[\eta\gamma^*\mathbb{E}[V_T^{\psi^{\xi}}|\mathcal{F}_T^0]] \tag{2.58}$$

since η is $\mathcal{F}_T^0$-measurable. Finally, by applying the law of total expectation on the second term with respect to $\mathcal{F}_0$, we obtain that it is equal to:

$$\gamma^*\mathbb{E}\Big[\mathbb{E}\Big[\int_0^T\alpha_t\psi_t^{\xi}\psi_t^{\eta}dt-\eta\int_0^T\mathcal{E}_{t,T}c_t\psi_t^{\xi}dt\Big|\mathcal{F}_0\Big]\Big]=0 \tag{2.59}$$

by definition of $\psi_t^{\eta}=\frac{c_t}{\alpha_t}\mathbb{E}[\eta\mathcal{E}_{t,T}|\mathcal{F}_t]$ as in Lemma 2.2. Combining the results, we obtain:

$$\lim_{\lambda\to 0^+}\frac{1}{\lambda}\Big(L(\xi+\lambda\eta)-L(\xi)\Big)=\mathbb{E}[\eta\gamma^*\mathbb{E}[V_T^{\psi^{\xi}}|\mathcal{F}_T^0]]. \tag{2.60}$$

In conclusion, Equation (2.47) follows directly from Equations (2.54) and (2.60). □

Proof of Lemma 2.9. We start by showing the direct implication. Let $\bar{\xi}\in\Xi$ be the solution of the fixed-point problem $\xi=I(m^{\psi^{\xi}})$, i.e.,

$$\bar{\xi}=\frac{\exp\Big(\rho\ln Z-\gamma^*\mathbb{E}[V_T^{\psi^{\bar{\xi}}}|\mathcal{F}_T^0]\Big)}{\mathbb{E}\Big[\exp\Big(\rho\ln Z-\gamma^*\mathbb{E}[V_T^{\psi^{\bar{\xi}}}|\mathcal{F}_T^0]\Big)\Big]}. \tag{2.61}$$

For every $\xi \in L^1_+(\mathcal{F}^0_T)$ such that $\mathbb{E}[\xi] = 1$, by convexity of H and L we have:

$$G(\xi) - G(\bar{\xi}) \geq \mathbb{E}[(\xi - \bar{\xi})(\delta H(\bar{\xi}) + \delta L(\bar{\xi}))]. \tag{2.62}$$

Plugging Equation (2.61) into Equation (2.62), we obtain that:

$$G(\xi) - G(\bar{\xi}) \geq \ln \mathbb{E}\Big[\exp\Big(\rho \ln Z - \gamma^* \mathbb{E}[V_T^{\psi^\xi} | \mathcal{F}^0_T]\Big)\Big] \mathbb{E}[\xi - \bar{\xi}]. \tag{2.63}$$

Since $\mathbb{E}[\xi - \bar{\xi}] = 0$, then $G(\xi) - G(\bar{\xi}) \geq 0$. Hence, $\bar{\xi}$ is a minimizer of G.

Now, we show the inverse implication. Let $\bar{\xi}$ be a minimizer of G in $L^1_+(\mathcal{F}^0_T)$. Since the choice of η is arbitrary in $N \coloneqq \{\eta \in L^1(\mathcal{F}^0_T) \text{ s.t. } \mathbb{E}[\eta] = 0,\ |\eta| \leq \frac{1}{2}\xi \ \mathbb{P}\text{-a.s.}\}$, by Lemma 2.10 we obtain:

$$\inf_{\eta \in N} \mathbb{E}\Big[\eta\Big(\delta H(\bar{\xi}) + \delta L(\bar{\xi})\Big)\Big] \geq 0. \tag{2.64}$$

Let $f(\bar{\xi}) \coloneqq \delta H(\bar{\xi}) + \delta L(\bar{\xi})$, $a \coloneqq \mathbb{E}[f(\bar{\xi})]$, and $A \coloneqq \{\omega \in \Omega : f(\bar{\xi}(\omega)) > a\}$. Assume by contradiction that $p_A \coloneqq \mathbb{P}(A) > 0$. We obtain that:

$$\begin{aligned} \mathbb{E}[\eta f(\bar{\xi})] &\leq \int_A \eta(\omega) f(\bar{\xi}(\omega)) d\mathbb{P}(\omega) + a \int_{\Omega \setminus A} \eta(\omega) d\mathbb{P}(\omega) = \\ &= \int_A \eta(\omega)\Big(f(\bar{\xi}(\omega)) - a\Big) d\mathbb{P}(\omega) + a\mathbb{E}[\eta]. \end{aligned} \tag{2.65}$$

The last term vanishes as $\mathbb{E}[\eta] = 0$. Choose $\eta \in N$ such that $\eta(\omega) < 0$ for all $\omega \in A$. Since $f(\bar{\xi}(\omega)) - a > 0$ for $\omega \in A$, it follows that the integral is negative, which contradicts Equation (2.64). Hence, $p_A = 0$ and necessarily $f(\bar{\xi})$ is constant and $\mathbb{P}$-a.s. equal to a, i.e.:

$$\ln \bar{\xi} - \rho \ln Z + \mathbb{E}[V^{\psi^{\bar{\xi}}} | \mathcal{F}^0_T] = a \quad \mathbb{P}\text{-a.s.} \tag{2.66}$$

The value of a can be found by imposing that $\mathbb{E}[\bar{\xi}] = 1$, leading to the fixed-point equation (2.61). □

2.4 Examples and conclusions

In this section, we briefly illustrate the example proposed by Tankov and Lavigne to demonstrate the practical applicability of the model described.

Let (W, W^0) be a pair of independent Brownian motions. We assume that $W^0 \coloneqq (W^{0,1}, W^{0,2})$ is 2-dimensional for computational reasons. Moreover, we make the following assumptions:

- The emission efficacy $(c_t)_{t\in[0,T]}$ is constant in time for each firm, i.e., $c_t := C$ for any time t, where C is an $\mathcal{F}_0$-measurable random variable that is large for green, carbon efficient firms and small for brown firms;
- The emission penalty $(\alpha_t)_{t\in[0,T]}$ is the same for all firms, is $\mathbb{F}^0$-measurable and takes the form of a stochastic process defined by $\alpha_t := \exp(\gamma W_t^{0,2} - \frac{\gamma^2}{2}t)$, where $\gamma \in \mathbb{R}^+$ is a positive constant that measures the uncertainty associated to future emission penalty, i.e., γ is a climate transition risk parameter. Moreover, without loss of generality, by normalizing C we can assume that $\alpha_0 = 1$, so that α is a martingale under $\mathbb{P}$ with constant expectation, modeling the evolution of a market in which all firms operate including climate-related externalities;
- The Radon-Nikodym derivative between $\mathbb{P}^g$ and $\mathbb{P}$ is defined by $Z := \exp(\lambda W_T^{0,2} - \frac{\lambda^2}{2}T)$, where $\lambda \in \mathbb{R}^+$ is a positive constant. Thus, $W_t^{0,g} := W_t^{0,2} - \lambda t$ is a Brownian motion under $\mathbb{P}^g$, and $\alpha_t = \exp(\gamma W_t^{0,g} + \gamma\lambda t - \frac{\gamma^2}{2}t)$ has increasing expectation. This models the green investors' concern about costs and penalties associated to carbon emissions growth; in fact, λ can be interpreted as the environmental concern of the green investor, since it modulates the strength of this effect through Z;
- The drift μ and the volatilities σ, σ^0 are assumed to be constant. Moreover, we let $\overline{\mathcal{E}}_t := \exp(\sigma^0 W_t^{0,1} + (\mu - (\sigma^0)^2/2)t)$.

The fixed point equation then writes:

$$\begin{aligned} \xi &= \frac{\exp\Big(-\gamma^* V_T^\xi + \rho(\lambda W_T^{0,2} - \lambda^2 T/2)\Big)}{\mathbb{E}\Big[\exp\Big(-\gamma^* V_T^\xi + \rho(\lambda W_T^{0,2} - \lambda^2 T/2)\Big)\Big]}, \\ V_T^\xi &:= \overline{v}\overline{\mathcal{E}}_T + \int_0^T \overline{C^2} e^{-\gamma W_t^{0,2} + \gamma^2 t/2} \overline{\mathcal{E}}_{t,T} \mathbb{E}[\xi \overline{\mathcal{E}}_{t,T} | \mathcal{F}_t^0] dt, \end{aligned} \tag{2.67}$$

where $\overline{C^2} := \mathbb{E}[C^2]$ and $\overline{v} := \mathbb{E}[v]$. This problem can be addressed using the fixed-point algorithm proposed by Tankov and Lavigne. The algorithm is formally stated in [4, Algorithm 6] and described in detail on [4, pp. 27–28]. A proof of its convergence is provided in [4, Theorem 7] and further discussed in [4, Section 5.2]. Since a detailed analysis of the algorithm and its convergence lies beyond the scope of this thesis, they will not be discussed here.

Additional outputs. Beyond the previously listed quantities, the algorithm also computes the following economically relevant measures:

- The total average emissions, given by

$$\overline{\Psi}_T := \int_0^T \mathbb{E}[\psi_t | \mathcal{F}_T^0] dt = \int_0^T \mathbb{E}[C] e^{-\gamma W_t^{0,2} + \gamma^2 t/2} E[\xi \overline{\mathcal{E}}_{t,T} | \mathcal{F}_T^0] dt; \tag{2.68}$$

- The expected emissions of the representative firm at date t, given by

$$\mathbb{E}[\psi_t|\mathcal{F}_0] = C\mathbb{E}\Big[\xi e^{-\gamma W_t^{0,2}+\gamma^2 t/2}\overline{\mathcal{E}}_{t,T}\Big]; \tag{2.69}$$

- The initial stock price of the representative firm, given by

$$S_0^\xi = v\mathbb{E}[\xi\overline{\mathcal{E}}_T] + C^2\mathbb{E}\Big[\int_0^T e^{-\gamma W_t^{0,2}+\gamma^2 t/2}\mathbb{E}[\xi\overline{\mathcal{E}}_{t,T}|\mathcal{F}_t]^2 dt\Big]. \tag{2.70}$$

  The first term reflects the portion of the market price from firm's initial value, while the second captures the value generated through emissions.

The authors implemented the algorithm to illustrate how climate risk and the proportion and concern of green investors affect decarbonization dynamics and firms' share prices. The parameters ρ (green investor proportion), γ (volatility of emissions penalty), and λ (environmental concern of green investors) vary across tests, while all other parameters are held constant (see Table 2.1). A detailed analysis of the results is provided in [4, Section 6]; here, we offer a brief overview, as a full exposition falls outside the scope of this thesis. The figures and table are taken from [4, pp. 30–33].

Impact of climate risk on decarbonization dynamics. Assuming no green investors ($\rho = 0,\ \lambda = 0$), the impact of climate risk is studied by varying γ (see Figure 2.1). As climate uncertainty grows, firms and markets increase their emissions due to investors diversifying their asset allocations, which reduces companies' incentives to lower their carbon footprints. This result was also noted in [15], where weak or uncertain climate policies fail to incentivize emission reductions.

Impact of green investors on the decarbonization dynamics. As ρ increases (indicating an higher fraction of green investors) while keeping λ and γ fixed, a significant reduction in carbon emissions is observed (see Figure 2.2). Conversely, as λ increases (indicating stronger environmental preferences) while holding ρ and γ constant, the expected emissions decrease and the distribution of emissions narrows and shifts to the left, indicating a wider spread of lower emission outcomes (see Figure 2.3).

Impact of green investors on share prices. We exploit the fact that the initial stock price (2.70) is sum of two different terms indicating respectively the sensitivity to the firm value and the sensitivity to the emission efficacy. Varying γ, λ, ρ (see Table 2.2), we find that sensitivity to firm value remains relatively stable, due to the weak dependence on ξ as $\mathbb{E}[\xi] = 1$. In contrast, sensitivity to emission efficacy changes significantly: it rises with climate uncertainty, as firms emit more to generate value. Green firms, however, create more value per unit of emissions and face less climate risk, widening the price spread with brown firms. This sensitivity declines as the share of green investors increases, since strong environmental preferences lead all firms to emit less and pay lower dividends.

Variable	Value	Description
T	5	Time horizon, in years
γ^*	0.5	Global risk aversion parameter
σ^0	10%	Volatility of the common noise term of firm's value's dynamics
μ	5%	Drift of firm's value's dynamics
$\overline{v}$	1	Average initial firm's value
$\overline{C^2}$	1	Average squared emission efficacy of production
$\overline{C}$	0.7	Average emission efficacy of production

Table 2.1: Values of the constant parameters used in the simulations.

γ	λ	ρ	Sensitivity to firm's value	Sensitivity to emissions
		Impact of γ		
0.15	0	0.5	1.2102 ± 0.0004	6.260 ± 0.011
0.3	0	0.5	1.2112 ± 0.0003	6.928 ± 0.014
0.45	0	0.5	1.2128 ± 0.0003	8.143 ± 0.010
		Impact of λ		
0.03	0	0.5	1.2112 ± 0.0003	6.928 ± 0.014
0.3	0.2	0.5	1.2103 ± 0.0003	6.834 ± 0.017
0.3	0.4	0.5	1.2115 ± 0.0003	6.730 ± 0.019
		Impact of ρ		
0.3	0.4	0.0	1.2112 ± 0.0003	6.928 ± 0.014
0.3	0.4	0.25	1.2108 ± 0.0005	6.825 ± 0.020
0.3	0.4	0.5	1.2115 ± 0.0003	6.730 ± 0.019
0.3	0.4	0.75	1.2119 ± 0.0004	6.710 ± 0.020
0.3	0.4	1.0	1.2113 ± 0.0003	6.595 ± 0.021

Table 2.2: The two components of the price formula (2.70). The standard errors quantify the Monte Carlo error only and were computed by running the test 10 times.

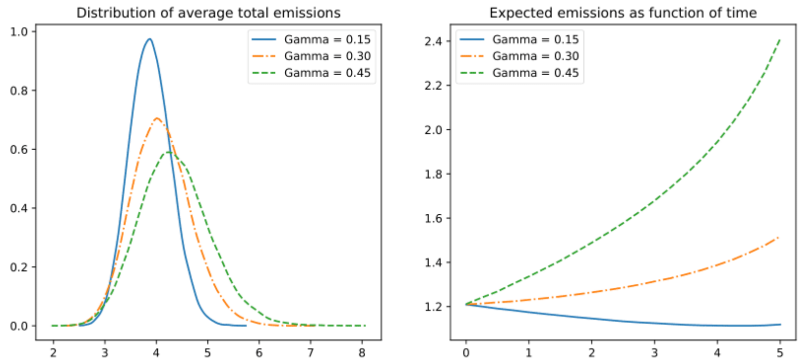


Figure 2.1: Distribution of total average emissions (left) and expected emissions of the representative company per unit of time (right), for different values of the volatility of emission penalty γ.

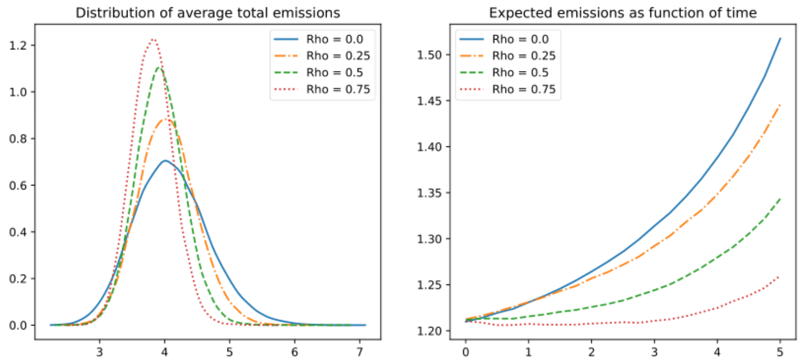


Figure 2.2: Distribution of total average emissions (left) and expected emissions of the representative company per unit of time (right), for different values of the proportion of green investors ρ.

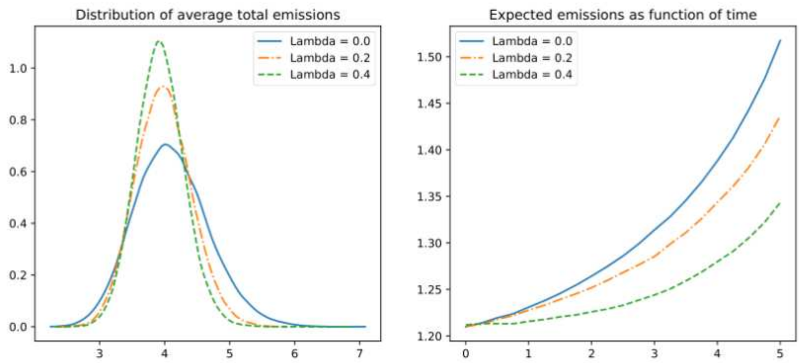


Figure 2.3: Distribution of total average emissions (left) and expected emissions of the representative company per unit of time (right), for different values of the environmental concern of green investors λ.

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