UNIVERSITY OF PADOVA

Department of Mathematics "Tullio Levi-Civita"

Bachelor's Thesis in Mathematics

ON THE SEQUENCING OF FINITE GROUPS

Supervisor
Prof. Andrea Lucchini
University of Padova

Bachelor Candidate
Candidate David Milazzo

Student ID
2079200

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1 Introduction

You and your friend are playing your favorite game: finding a knight's tour on a regular chessboard. That is, determining a sequence of knight moves such that each square is visited exactly once. However, since you have played this game so often, it is starting to feel a bit boring. To spice things up, your friend suggests a similar but more challenging variant: instead of hopping around a chessboard, we hop around the elements $x_1, ..., x_n$ of a finite group $G = (x_1, ..., x_n)$, and instead of the steps $x_i^{-1}x_{i+1}$ being the knight's moves, they are required to be all different. Thus, all non-identity elements of G occur as steps. Informally, this is what the problem of *sequencing* a finite group is about. Gordon introduced this in 1961 [13] while working with latin squares, settling the abelian case. Subsequently, the *sequencing* of a finite group became a problem on its own, and still to this day there is not a definitive answer of which (non-abelian) groups are non-sequenceable.

Definition 1.1. *A non-trivial finite group G of order n is said to be sequenceable if its elements can be arranged in a sequence $(a_1, a_2, \ldots, a_n)$ in such a way that the partial products $(b_1, b_2, \ldots, b_n)$, where $b_i = a_1a_2 \cdots a_i$, are distinct. The sequence $(a_1, a_2, \ldots, a_n)$ is called a sequencing for G. The sequence $(b_1, b_2, \ldots, b_n)$ is called a basic directed terrace. For any element $g \in G$, the sequence $(gb_1, gb_2, \ldots, gb_n)$ is called a directed terrace.*

We observe that if $(a_1, a_2, \ldots, a_n)$ is a sequencing for G then $a_1 = e$, where e is the identity of G. That is because if $a_i = e$ for some $i \neq 1$ then $b_{i-1} = b_i$

As mentioned before, Gordon introduced the problem of finding sequencing of finite groups while searching for conditions on which complete latin squares exist. Recall the definitions:

Definition 1.2. *A latin square of order n is an $n \times n$ array defined on a set X with n elements such that every element of X appears once in each row and once in each column. The notation $L = (l_{ij})$ represents a latin square L with l_{ij} in the ith row and jth column.*

Definition 1.3. *An $n \times n$ latin square is said to be row complete if every pair $\{x, y\}$ of distinct elements of X occurs exactly once in each order in adjacent horizontal cells. A latin square is said to be column complete if every pair $\{x, y\}$ of distinct elements of X occurs exactly once in each order in adjacent vertical cells. If a latin square is both row complete and column complete then it is said to be complete.*

Let us show why the problem of finding a sequencing of a group is strictly correlated with finding a complete Latin square.

Theorem 1.4. *[13] Let G be a sequenceable group and $(a_1, a_2, \ldots, a_n)$ be a sequencing with associated basic directed terrace $(b_1, b_2, \ldots, b_n)$. Then $L = (l_{ij})$, where $l_{ij} = b_i^{-1}b_j$ for $1 \leq i, j \leq n$, is a complete latin square.*

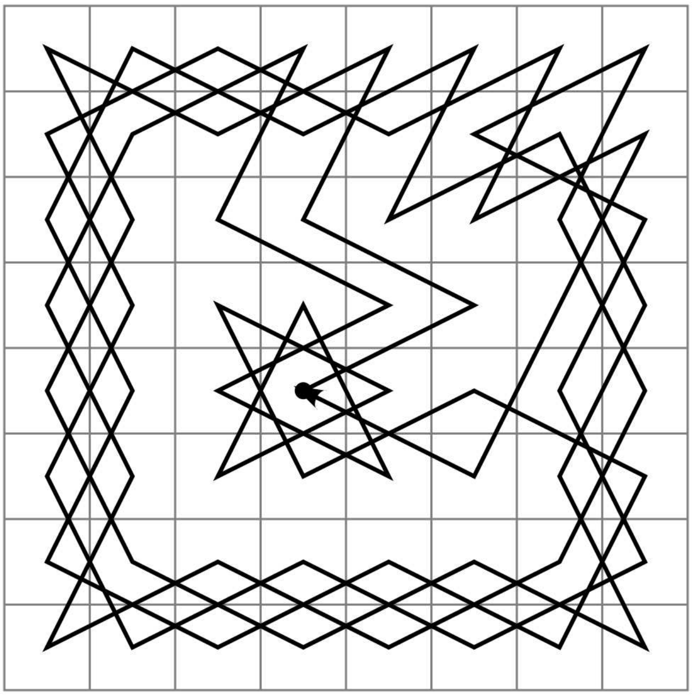
Proof. Suppose $l_{ij} = l_{ik}$ for some $1 \leq i, j, k \leq n$. Then $a_i^{-1}a_j = a_i^{-1}a_k$, giving $a_j = a_k$. Therefore $j = k$ and L has no repeated entries in any row. Similarly, L has no repeated entries in any column. Therefore L is a latin square.

To show that L is row complete we need $a_i^{-1}a_j = x$ and $a_i^{-1}a_{j+1} = y$ to have a unique solution for i and j given any ordered pair (x, y) of distinct elements of G.

Inverting both sides of the first equation and post-multiplying by the second gives $a_j^{-1}a_{j+1} = x^{-1}y$, that is $b_{j+1} = x^{-1}y$, uniquely determining j. Now $a_i^{-1}a_j = x$ uniquely determines i, and L is row complete.

An analogous argument shows that L is also column complete. Therefore L is a complete latin square. □

We will now look in detail at the problem of sequencing abelian groups, giving the result that Gordon proved in 1961, completely classifying the abelian case.



Knight's tour on a classical 8×8 *chessboard*

2 Sequenceable Groups

2.1 Abelian Groups

In this section, we will write the groups additively. We first recall the definition of a binary group.

Definition 2.1. *A binary group is a group with a unique element of order 2. If G is a binary group, we denote its unique subgroup of order 2 by $\Lambda(G)$.*

Theorem 2.2. *[13] A finite abelian group G is sequenceable if and only if G is a binary group.*

Proof ($\Rightarrow$). Suppose that G is sequenceable and let $(a_1, a_2, \cdots, a_n)$ be an ordering of the elements of G with $a_1, a_1a_2, \cdots, a_1a_2\cdots a_n$ all distinct. Let $b_i = a_1a_2\cdots a_i$. It is immediately seen that $a_1 = b_1 = e$, the identity element of G; for if $a_i = e$ for some $i > 1$, then $b_{i-1} = b_i$, contrary to the assumption. Hence $b_n \neq e$, so the sum of all the elements of G is not the identity. More precisely, we find that the sum of the elements of G coincides with the sum of the elements of order 2: if a is an element of order not equal to 2 then we match a with its inverse $-a$. The elements of order 2 with zero form a subgroup of G which is isomorphic to a vector space of dimension n over the field with two elements. The sum of the elements of this vector space is zero except when $n = 1$. In this case, then G has only one element of order 2 and then the 2-Sylow of G is cyclic. This implies that G has the form $A \times B$ with A cyclic of order 2^k $(k > 0)$ and B of odd order.

Proof ($\Leftarrow$). Suppose that $G = A \times B$, with A and B as above. We then show that G is sequenceable by constructing an ordering $(a_1, a_2, \cdots, a_n)$ of its elements with distinct partial products. Using the fundamental theorem of finitely generated abelian groups, we know that every finite abelian group G is isomorphic to a group of the form $G \cong \mathbb{Z}_{p_1^{k_1}} \times \mathbb{Z}_{p_2^{k_2}} \cdots \times \mathbb{Z}_{p_n^{k_n}}$ where the p_i are primes not necessarily distinct and $k_i > 0$. Using the Chinese remainder theorem, which states that $\mathbb{Z}_{jk} \cong \mathbb{Z}_j \times \mathbb{Z}_k$ if and only if j and k are coprime, we can write an equivalent version of the fundamental theorem of finitely generated abelian groups that states that every finite abelian group G is isomorphic to a group of the form $G \cong \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2} \times \cdots \times \mathbb{Z}_{k_m}$ where k_1 divides k_2, which divides k_3 and so on up to k_m. Applying this result to $G = A \times B$ as before we see that G has a basis of the form $(c_0, c_1, \cdots, c_m)$, where c_0 is of order 2^k, and where the orders $\delta_1, \delta_2, \cdots, \delta_m$ of $c_1, c_2, \cdots, c_m$ are odd positive integers each of which divides the next, that is $\delta_i|\delta_{i+1}$ for $0 < i < m$. If j is any positive integer, then there exist unique integers $j_0, j_1, \cdots, j_m$ such that

$$
\begin{gathered}
j \equiv j_0 \pmod{\delta_1\delta_2\cdots\delta_m} \\
j_0 = j_1 + j_2\delta_1 + j_3\delta_1\delta_2 + \cdots + j_m\delta_1\cdots\delta_{m-1} \\
0 \leq j_1 < \delta_1 \\
0 \leq j_2 < \delta_2 \\
\vdots \\
0 \leq j_m < \delta_m.
\end{gathered} \tag{2.1}
$$

The proof of the existence and uniqueness of this expansion is analogous to the expansion of an integer in powers of a number base: by induction, we can write j as $j = \tilde{j}_0 + t\delta_1 \cdots \delta_{m-1}$ with $\tilde{j}_0 = j_1 + \cdots + j_{m-1}\delta_1 \cdots \delta_{m-2}$ and t an integer. By dividing t by δ_m, we can write t as $q\delta_m + j_m$, with $0 \leq j_m < \delta_m$ and q a positive integer. Substituting back, we have $j = \tilde{j}_0 + (q\delta_m + j_m)\delta_1 \cdots \delta_{m-1} = \tilde{j}_0 + (j_m\delta_1 \cdots \delta_{m-1}) + q\delta_1 \cdots \delta_m$. Thus $j \equiv \tilde{j}_0 + j_m\delta_1 \cdots \delta_{m-1} \pmod{\delta_1\delta_2 \cdots \delta_m}$. But since $\tilde{j}_0 + (j_m\delta_1 \cdots \delta_{m-1}) = j_1 + \cdots + j_m\delta_1 \cdots \delta_{m-1}$ with $0 \leq j_m < \delta_m$, we have obtained the desired $(j_1, \cdots, j_m)$.

We are now in a position to define the desired sequencing of G. It is convenient to define the products $b_1, b_2, \cdots, b_n$ directly, to prove that they are all distinct, and then to verify that the corresponding a_i, calculated from the formula $a_1 = e, a_i = b_{i-1}^{-1}b_i$, are all distinct. If i is of the form $2j+1$ with $0 \leq j < n/2$, let

$$b_{2j+1} = c_0^{-j}c_1^{-j_1}c_2^{-j_2} \cdots c_m^{-j_m},$$

where $j_1, j_2, \cdots, j_m$ are the integers defined before. On the other hand, if i is of the form $2j+2$ with $0 \leq j < n/2$, let

$$b_{2j+2} = c_0^{j+1}c_1^{j_1+1}c_2^{j_2+1} \cdots c_m^{j_m+1}.$$

The elements $b_1, b_2, \cdots, b_n$ thus defined are all distinct: if $b_s = b_t$ with $s = 2u+1, t = 2v+1$, then

$$\begin{aligned} u &\equiv v \pmod{2^k} \\ u_1 &\equiv v_1 \pmod{\delta_1} \\ &\vdots \\ u_m &\equiv v_m \pmod{\delta_m}. \end{aligned} \tag{2.2}$$

From the inequalities in (2.1) we conclude that $u_1 = v_1, \cdots, u_m = v_m$. Hence $u_0 = v_0$, so that $u \equiv v \pmod{\delta_1 \cdots \delta_m}$; coupled with the first equations of (2.2), this gives $u \equiv v \pmod{n}$, which implies $u = v$. Similarly $b_{2u+2} = b_{2v+2}$ implies $u = v$, so that the "even" b's are distinct.

Next, suppose that

$$b_{2u+1} = b_{2v+2}.$$

Then

$$\begin{aligned} -u &\equiv v + 1 \pmod{2^k} \\ -u_1 &\equiv v_1 + 1 \pmod{\delta_1} \\ &\vdots \\ -u_m &\equiv v_m + 1 \pmod{\delta_m} \end{aligned}$$

or equivalently,

$$\begin{aligned} u + v + 1 &\equiv 0 \pmod{2^k} \\ u_1 + v_1 + 1 &\equiv 0 \pmod{\delta_1} \\ &\vdots \\ u_m + v_m + 1 &\equiv 0 \pmod{\delta_m}. \end{aligned} \tag{2.3}$$

Since $0 < u_i + v_i + 1 \leq 2(\delta_i - 1) + 1 < 2\delta_i$, we must have $u_i + v_i + 1 = \delta_i$. Reasoning similarly for $i = 2, \cdots, m$, we obtain

$$\begin{aligned} u_1 + v_1 + 1 &= \delta_1 \\ u_2 + v_2 + 1 &= \delta_2 \\ &\vdots \\ u_m + v_m + 1 &= \delta_m. \end{aligned}$$

Multiplying the $(i+1)$st equation of this system by $\delta_1\delta_2 \cdots \delta_i$ $(1 \leq i < m)$ and adding it, we get $u_0 + v_0 + 1 = \delta_1 \cdots \delta_m$, which implies $u + v + 1 \equiv 0 \pmod{\delta_1 \cdots \delta_m}$. Combining this with the first equation in (2.3), we find that $u + v + 1 \equiv 0 \pmod{n}$, which, due to inequality $0 < u + v + 1 < n$, is impossible. Hence $b_1, b_2, \cdots, b_n$ are all distinct.

Next, we calculate $a_1, a_2, \cdots, a_n$. If $i = 2j + 2$ $(0 \leq j < n/2)$, then

$$a_i = b_{i-1}^{-1} b_i = c_0^{2j+1} c_1^{2j_1+1} \cdots c_m^{2j_m+1}.$$

These are all different by the same argument as above. If $i = 2j + 1$, and $j_1 \neq 0$, then

$$a_i = c_0^{-2j} c_1^{-2j_1-1} c_2^{-2j_2-1} \cdots c_m^{-2j_m-1}.$$

If $i = 2j + 1$ and $j_1 = 0$, but $j_2 \neq 0$, then $a_i = c_0^{-2j} c_1^{-2j_1} c_2^{-2j_2-1} \cdots c_m^{-2j_m-1}$, while if $j_1 = j_2 = 0$ but $j_3 \neq 0$, then $a_i = c_0^{-2j} c_1^{-2j_1} c_2^{-2j_2} \cdots c_m^{-2j_m-1}$, etc. These a_i's are obviously distinct from each other by the same reasoning as before. Due to the exponent of c_0, they are also distinct from a_i with i even. $\square$

2.2 Dihedral Groups

Let $n \geq 3$. We describe the dihedral group D_{2n}, of order $2n$, as the set of ordered pairs (x, ϵ) with $x \in \mathbb{Z}_n$ and $\epsilon \in \mathbb{Z}_2$ and multiplication defined by:

$$(x, 0)(y, \delta) = (x + y, \delta)$$

$$(x, 1)(y, \delta) = (x - y, 1 + \delta).$$

Before giving a full characterization of the sequencing of dihedral groups, we need to study the cases of D_6 and D_8. Gordon [13] found that these two groups are not sequenceable and gives an idea on how to prove it: we need to take the *abelianization* of our group G, i.e. the quotient group G/G' where G' is the *commutator* subgroup of G. We will now give an explicit application of this idea.

Lemma 2.3. *[11] Suppose G is a sequenceable group with sequencing $(a_1, \ldots, a_n)$ and distinct partial products $(b_1, \ldots, b_n)$. Let H be a normal subgroup of G and let*

$$\varphi : G \longrightarrow G/H$$

be the natural homomorphism defined by $\varphi(g) = \overline{g} = gH$. Write

$$G/H = \{\overline{x}_1, \ldots, \overline{x}_r\}, \text{ where } r = \frac{n}{h} \ (|H| = h).$$

Then each $\overline{x}_i$, for $1 \leq i \leq r$, must occur h times in both sequences $(\overline{a}_1, \ldots, \overline{a}_n)$ and $(\overline{b}_1, \ldots, \overline{b}_n)$.

Proof. Each $a_i \in G$ gets assigned to a specific $\overline{x}_j \in G/H$. Thus, if $\varphi(a_i) = \overline{a}_i = \overline{x}_j$ for $1 \leq i \leq n$ and $1 \leq j \leq r$, then a_i is replaced by $\overline{x}_j$ in the sequence $(\overline{a}_1, \ldots, \overline{a}_n)$. The same reasoning applies to the sequence of mapped partial products. Since there are h elements in G represented by each $\overline{x}_j$, and each $\overline{x}_j$ is distinct, there must be $n = rh$ replacements, h of which there are $\overline{x}_j$ for $1 \leq j \leq r$. □

Definition 2.4. *Let G be a group of order n and let $H \trianglelefteq G$ of order h. Let*

$$G/H = \{\overline{x}_1, \ldots, \overline{x}_r\} \text{ with } r = \frac{n}{h}.$$

A sequence α of length n consisting of elements of G/H is called a quotient sequencing of G/H if each $\overline{x}_i$, for $1 \leq i \leq r$, occurs h times in both α and $P(\alpha)$, where $P(\alpha)$ denotes the sequence of partial products of α.

Using the above definition, Lemma 2.3 basically tells us that the image under φ of a sequencing of G is a quotient sequencing of G/H. Taking H as the commutator subgroup of G, we now have an easy way (when the order of G is small) to check if a group is sequenceable by studying its abelianization.

Lemma 2.5. *[15] Let D_{2n} be the dihedral group of order $2n$. If n is odd, the abelianization of D_{2n} is $\mathbb{Z}_2$, if n is even, the abelianization is $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. A dihedral group is generated by two reflections s_1, s_2. For D_{2n}, we have the relation $(s_1 s_2)^n = 1$. The element $s_1 s_2 s_1 s_2$ is always a commutator, which means that in the abelianization we have $([s_1][s_2])^2 = 1$ and $([s_1][s_2])^n = 1$.

If n is odd, this means $[s_1] = [s_2]$ (since the order of $[s_1 s_2]$ must divide both n and 2) and the abelianization is $\mathbb{Z}_2$. If n is even, then $[s_1]$ and $[s_2]$ are distinct generators of order 2 that commute, so the abelianization is $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

Lemma 2.6. *D_6 is not sequenceable.*

Proof. Let $D_6 = \langle r, s \mid r^3 = s^2 = 1,\ srs = r^{-1} \rangle$. D_6 has elements:

$$D_6 = \{1,\ r,\ r^2,\ s,\ sr,\ sr^2\}$$

Take $D_6' = \{1, r, r^2\}$ the commutator subgroup of D_6. Then $D_6' \trianglelefteq D_6$ and as proved in 2.5:

$$D_6/D_6' \cong \mathbb{Z}_2$$

Let $\varphi : D_6/D_6' \to \mathbb{Z}_2$ be the abelianization homomorphism. Explicitly we have:

$$\begin{aligned} \varphi(1) &= 0, \\ \varphi(r) &= 0, \\ \varphi(r^2) &= 0, \\ \varphi(s) &= 1, \\ \varphi(sr) &= 1, \\ \varphi(sr^2) &= 1. \end{aligned}$$

Let now be $(a_1, \cdots, a_6)$ a permutation of the elements of D_6 and $(b_1, \cdots, b_6)$ the sequence of the correspondent partial products. As seen in 2.5, a quotient sequencing of D_6/D_6' must have three 0's and three 1's in each of its sequence of elements $(\varphi(a_1), \cdots, \varphi(a_6))$ and partial products $(\varphi(b_1), \cdots, \varphi(b_6))$. Moreover, if D_6 is to have a sequencing, we must have a 0 as the first element of both sequences ($\varphi(a_1) = \varphi(b_1) = 0$) since $a_1 = b_1 = 1$ and $\varphi(1) = 0$) and $b_6 = 1$ since $\varphi(b_1) + \cdots + \varphi(b_6) = 1$. The problem is then to find all possible sequences of 0's and 1's that could satisfy these conditions. By adding the fact that in the sequences of the partial products $(\varphi(b_1), \cdots, \varphi(b_6))$ we can only have three changes from 0 to 1 and vice-versa (corresponding to the three elements of D_6 that are not in the kernel of φ) we can easily determine that there are four distinct possibilities for quotient sequencings of D_6/D_6'. They are

$$0,0,1,0,1,1; \quad 0,0,1,1,1,0; \quad 0,1,0,1,0,1; \quad 0,1,1,0,1,0 \tag{1}$$

with corresponding sequences of partial products:

$$0,0,1,1,0,1; \quad 0,0,1,0,1,1; \quad 0,1,1,0,0,1; \quad 0,1,0,0,1,1.$$

Now we need to take each of the sequences in (1) and project them back to D_6. Let us illustrate how this works for the first quotient sequencing in the list. We begin by taking the first 0 to be the identity 1. For the second elements we may choose either r or r^2 since an automorphism on D_6 can take $r \mapsto r^2$. Let's project the second back to r. For similar reasons, we may take the first 1 to be s. So far we have the sequence $\{1, r, s\}$. The fourth term must be r^2, and up to this point the partial products in D_6 will be $\{1, r, sr^2, sr\}$. Now, when we add the fifth term, either sr^2 or sr, one of the partial product is repeated (either r or 1). Therefore this particular quotient sequence cannot project back to a sequencing of D_6. A similar argument for the remaining three quotient sequencings demonstrates that D_6 is not sequenceable. □

Lemma 2.7. D_8 *is not sequenceable.*

Proof. We proced in the same way we did in 2.6. Let $D_8 = \langle r, s \mid r^4 = 1 = s^2,\ srs = r^{-1}\rangle$. Its elements are:

$$D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$$

Take $D_8' = \{1, r^2\}$ the commutator subgroup of D_8. Then $D_8' \trianglelefteq D_8$ and as proved in 2.5:

$$D_8/D_8' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Let $\varphi : D_8/D_8' \to \mathbb{Z}_2 \times \mathbb{Z}_2$ be the abelianization homomorphism. Explicitly we have:

$$\begin{aligned}
\varphi(1) &= (0,0)\\
\varphi(r) &= (1,0)\\
\varphi(r^2) &= (0,0)\\
\varphi(r^3) &= (1,0)\\
\varphi(s) &= (0,1)\\
\varphi(sr) &= (1,1)\\
\varphi(sr^2) &= (0,1)\\
\varphi(sr^3) &= (1,1)
\end{aligned}$$

Let now be $(a_1, \cdots, a_8)$ a permutation of the elements of D_8 and $(b_1, \cdots, b_8)$ the sequence of the correspondent partial products.

As seen in 2.5, a quotient sequencing of D_8/D_8' must have two copies of $(0,0)$, two copies of (0,1), two copies of (1,0) and two copies of (1,1) in each of its sequence of elements $(\varphi(a_1), \cdots, \varphi(a_8))$ and partial products $(\varphi(b_1), \cdots, \varphi(b_8))$. Moreover, if D_8 is to have a sequencing, we must have $(0,0)$ as the first element of both sequences ($\varphi(a_1) = \varphi(b_1) = (0,0)$ since $a_1 = b_1 = 1$ and $\varphi(1) = (0,0)$) and $b_8 = (0,0)$ since $\varphi(b_1)+\cdots+\varphi(b_8) = (0,0)$. The problem is then to find all possible sequences of elements that could satisfy these conditions. It is convenient to approach the problem coordinate-wise. For each coordinate, in the sequence we can have four changes from 0 to 1 and vice-versa (corresponding to the six elements of D_8 that are not in the kernel of φ). With this conditions, we can easily determine that for each coordinate we have exactly nine distinct possibilities for quotient sequencing of D_8/D_8'. They are

$$\begin{matrix}
0,0,1,1,1,0,0,1; & 0,0,1,0,1,1,0,1; & 0,0,1,0,0,1,1,1;\\
0,1,1,0,1,0,0,1; & 0,1,0,1,0,1,0,1; & 0,1,0,0,1,0,1,1;\\
0,1,0,0,1,1,1,0; & 0,1,0,1,1,0,1,0; & 0,1,1,1,0,0,1,0;
\end{matrix} \tag{1}$$

with corresponding sequences of partial products

$$\begin{matrix}
0,0,1,0,1,1,1,0; & 0,0,1,1,0,1,1,0; & 0,0,1,1,1,0,1,0;\\
0,1,0,0,1,1,1,0; & 0,1,1,0,0,1,1,0; & 0,1,1,1,0,0,1,0;\\
0,1,1,1,0,1,0,0; & 0,1,1,0,1,1,0,0; & 0,1,0,1,1,1,0,0.
\end{matrix}$$

Now we need to take two sequences from (1), one for the first coordinate and one for the second. This gives us a total of 81 cases. Of this 81 cases not all of them are actually valid. For example, let us take the first sequence for the first coordinate and the second sequence for the second coordinate. This gives us the following:

$$\begin{gathered}(0,0)\\(0,0)\\(1,1)\\(1,0)\\(1,1)\\(0,1)\\(0,0)\\(1,1)\end{gathered}$$

Of course this is not a valid sequence since we have the element $(0,0)$ repeated three times. Let us illustrate now how the things work with a valid sequence by taking the fourth sequence for the first coordinate and the fifth sequence for the second. We obtain the following:

$$\begin{gathered}(0,0)\\(1,1)\\(1,0)\\(0,1)\\(1,0)\\(0,1)\\(0,0)\\(1,1)\end{gathered}$$

Let us now project them back to D_8. We begin by taking the first (0,0) to be the identity 1. For the second element, $(1,1)$, we may choose either sr or sr^3 since an automorphism on D_8 can take $sr \mapsto sr^3$ and vice-versa. Let's project the second back to sr. For similar reasons, we may take the element $(1,0)$ to be r or r^3. Let's choose r. Now observe that once we choose that the second element of our sequencing is sr and the third is r the only automorphism that fixes this two elements is the identity. For this reason, since the element $(0,1)$ can be either s or sr^2, we need to study each choice separately. If we choose sr^2, our sequence of elements would be $(1, sr, r, sr^2)$ with its corresponding partial products being $(1, sr, sr^2, 1)$. Since we have a repetition this could not be a valid sequencing. We are then forced to choose s. Now our sequence of elements is $(1, sr, r, s)$ with its sequence of partial products $(1, sr, sr^2, r^2)$ For the fifth element we have $(1,0)$ which has to be projected back to r^3, so our sequence is now $(1, sr, r, s, r^3)$ and the partial products are $(1, sr, sr^2, r)$. Now our sixth element is $(0,1)$ which projects back to sr^2 but then the partial product sr is repeated, therefore this quotient sequence cannot project back to a sequencing of D_8.

A similar argument for the remaining sequencing demonstrates that D_8 is not sequenceable. □

It is worth observing that this exact argument can be applied to prove that the quaternion group Q_8 is not sequenceable. The reasoning is exactly the same since we have $Q_8/Q_8' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ In fact, Keedwell conjectures that D_6, D_8 and Q_8 are the only non-abelian non-sequenceable groups.

We can now state two theorems that fully resolve the case of dihedral groups. The demonstrations are originally given by Isbell [16], Li [19] and Anderson [1] and they explicitly construct a sequence for each dihedral group D_n, dividing the problem into different cases based on the congruence of n modulo 4

Theorem 2.8. *[16] The dihedral groups D_{2n}, of order $2n$, are sequenceable for all n, where $n \neq 3$ (D_6 is not sequenceable) and $n \neq 4k$.*

Theorem 2.9. *[19] The dihedral groups D_{2n} are sequenceable when $n = 4k$, except when $n = 4$.*

Anderson in his work [1] used a computer search to prove that all dihedral groups D_{2n} with $5 \leq n \leq 50$ are sequenceable, and this result is used in both theorems to cover the small cases. We can now give a full classification of sequenceable dihedral groups with their respective sequences.

Table 1: $k \equiv 0 \pmod 4, k \geq 4$

Sequencing	No. of terms
(a) (0,0)	1
(b) (0,1), (1,1), (2,1), ..., (2k-2,1)	$2k-1$
(c) $(4k-2, 1)$	1
(d) $(2k-1, 1), (2k, 1), (2k+1, 1), \ldots, (4k-3, 1)$	$2k-1$
(e) (2k,0)	1
(f) $[(4k-3, 0), (5, 0), (4k-7, 0), (9, 0), \ldots, (2k-3, 0)$	$k-2$
(g) $(2k+2, 0)$	1
(h) $(2k-1, 0), (2k+3, 0), (2k-5, 0), (2k+7, 0), \ldots, (3, 0)$	$k-1$
(i) $(4k-2, 0)$	1
(j) $(4k-1, 1)$	1
(k) $(2, 0), (4k-4, 0), (6, 0), (4k-8, 0), \ldots, (k-2, 0)$	$k/2-1$
(l) (1,0)	1
(m) $(3k-4, 0), (k+6, 0), (3k-8, 0), (k+10, 0), \ldots, (2k-2, 0)$	$k/2-2$
(n) (3k,0)	1
(o) $(2k+1, 0)$	1
(p) $(k+2, 0)$	1
(q) $(2k-4, 0), (2k+6, 0), (2k-8, 0), (2k+10, 0), \ldots, (3k-2, 0)$	$k/2-2$
(r) $(4k-1, 0)$	1
(s) $(k, 0), (3k+2, 0), (k-4, 0), (3k+6, 0), \ldots, (4, 0)$	$k/2-1$

Table 2: $k \equiv 1 \pmod 4, k \geq 5$

Sequencing	No. of terms
(a) (0,0)	1
(b) (0,1), (1,1), (2,1), ..., (2k-2,1)	$2k-1$
(c) $(4k-2,1)$	1
(d) $(2k-1,1),(2k,1),(2k+1,1),\dots,(4k-3,1)$	$2k-1$
(e) (2k,0)	1
(f) $[(4k-3,0),(5,0),(4k-7,0),(9,0),\dots,(2k-1,0)$	$k-1$
(g) $(2k+2,0)$	1
(h) $(2k-3,0),(2k+5,0),(2k-7,0),(2k+9,0),\dots,(3,0)$	$k-2$
(i) $(4k-2,0)$	1
(j) $(4k-1,1)$	1
(k) $(2,0),(4k-4,0),(6,0),(4k-8,0),\dots,(k,0)$	$(k-3)/2$
(l) (1,0)	1
(m) $(3k-3,0),(k+5,0),(3k-7,0),(k+9,0),\dots,(2k-4,0)$	$(k-5)/2$
(n) $(3k+1,0)$	1
(o) $(2k+1,0)$	1
(p) $(k+1,0)$	1
(q) $(2k-2,0),(2k+4,0),(2k-6,0),(2k+8,0),\dots,(3k-1,0)$	$(k-1)/2$
(r) $(4k-1,0)$	1
(s) $(k-1,0),(3k+3,0),(k-5,0),(3k+7,0),\dots,(4,0)$	$(k-3)/2$

Table 3: $k \equiv 2 \pmod 4, k > 6$

Sequencing	No. of terms
(a) (0,0)	1
(b) (0,1),(1,1),(2,1),...,(2k-2,1)	$2k-1$
(c) $(4k-2,1)$	1
(d) $(2k-1,1),(2k,1),(2k+1,1),\dots,(4k-3,1)$	$2k-1$
(e) (2k,0)	1
(f) $(4k-3,0),(5,0),(4k-7,0),(9,0),\dots,(2k-3,0)$	$k-2$
(g) $(2k+2,0)$	1
(h) $(2k-1,0),(2k+3,0),(2k-5,0),(2k+7,0),\dots,(3,0)$	$k-1$
(i) $(4k-2,0)$	1
(j) $(4k-1,1)$	1
(k) $(2,0),(4k-4,0),(6,0),(4k-8,0),\dots,(k,0)$	$k/2$
(l) $(4k-1,0)$	1
(m) $(3k-2,0),(k+4,0),(3k-6,0),(k+8,0),\dots,(2k-2,0)$	$k/2-1$
(n) $(3k,0)$	1
(o) $(2k+1,0)$	1
(p) $(k+2,0)$	1
(q) $(2k-4,0),(2k+6,0),(2k-8,0),(2k+10,0),\dots,(3k-4,0)$	$k/2-3$
(r) (1,0)	1
(s) $(k-2,0),(3k+4,0),(k-6,0),(3k+8,0),\dots,(4,0)$	$k/2-2$

Table 4: $k \equiv 3 \pmod 4, k > 7$

Sequencing	No. of terms
(a) (0,0)	1
(b) (0,1),(1,1),(2,1),...,(2k-2,1)	$2k-1$
(c) $(4k-2,1)$	1
(d) $(2k-1,1),(2k,1),(2k+1,1),\ldots,(4k-3,1)$	$2k-1$
(e) (2k,0)	1
(f) $(4k-3,0),(5,0),(4k-7,0),(9,0),\ldots,(2k-1,0)$	$k-1$
(g) $(2k+2,0)$	1
(h) $(2k-3,0),(2k+5,0),(2k-7,0),(2k+9,0),\ldots,(3,0)$	$k-2$
(i) $(4k-2,0)$	1
(j) $(4k-1,1)$	1
(k) $(2,0),(4k-4,0),(6,0),(4k-8,0),\ldots,(k-1,0)$	$(k-1)/2$
(l) $(4k-1,0)$	1
(m) $(3k-1,0),(k+3,0),(3k-5,0),(k+7,0),\ldots,(2k-4,0)$	$(k-3)/2$
(n) $(3k+1,0)$	1
(o) $(2k+1,0)$	1
(p) $(k+1,0)$	1
(q) $(2k-2,0),(2k+4,0),(2k-6,0),(2k+8,0),\ldots,(3k-3,0)$	$(k-3)/2$
(r) (1,0)	1
(s) $(k-3,0),(3k+5,0),(k-7,0),(3k+9,0),\ldots,(4,0)$	$(k-5)/2$

2.3 Binary Groups

It is now our aim to study whether a binary group can have a particular kind of sequence, called *symmetric sequence.*

Definition 2.10. *Let G be a binary group of order $2n$ and let z be its unique element of order 2. A sequencing a of G is said to be symmetric if it is of the form*

$$a = (e, a_2, a_3, \cdots, a_n, z, a_n^{-1}, \cdots, a_3^{-1}, a_2^{-1}).$$

This type of sequencing was first introduced informally by Lucas [20] in the context of his work on Latin squares. A systematic study, including the first significant results, was later conducted by Bailey and Praeger [8] and by Nilrat [21].

To determine whether a binary group G admits a symmetric sequencing, we study the quotient $G/\Lambda(G)$ (or a similar quotient with $\Lambda(G)$ in the denominator), where $\Lambda(G)$ denotes the unique subgroup of order 2.

The idea is similar to the one used in Lemma 2.6: we analyze the quotient sequencings and then attempt to project them back to G in order to construct a symmetric sequencing.

To do this, we first need to understand the structure of binary groups and their quotient groups $G/\Lambda(G)$. Let us begin by proving a basic fact about binary groups.

Lemma 2.11. *Let G be a binary group of order $2n$ and let $H \subseteq G$ be a subgroup of even order. Then H is a binary group.*

Proof. By Cauchy's theorem, H has at least one element of order 2. By definition, if a is an element of order 2 in H, then it is also an element of order 2 in G. If H had more than one such element, we would reach a contradiction. □

From this it follows that, in particular, each 2-Sylow subgroup of a binary group is binary.

Definition 2.12. *The generalized quaternion group is*

$$Q_{2n} = \langle u, v \mid u^{2n-1} = e,\ v^2 = u^{2n-2},\ vuv^{-1} = u^{-1} \rangle.$$

Burnside [10] has completely classified binary 2-groups: they are cyclic or generalized quaternion groups. The Sylow 2-subgroups of $G/\Lambda(G)$ have the form $S/\Lambda(S)$ for Sylow 2-subgroups S of G. The quotient $S/\Lambda(S)$ is either cyclic if S is cyclic, or dihedral according if S is a generalized quaternion group. Glauberman [7] also shows, with a cohomological argument, that if H is a finite group with cyclic or dihedral Sylow 2-subgroups, then there is a unique binary group G with $G/\Lambda(G) \cong H$. Therefore there is a correspondence between the classification of binary groups and that of groups with cyclic or dihedral Sylow 2-subgroups. This classification is given by two important theorems: the Gorenstein-Walter Theorem [14] and the Burnside's transfer Theorem [10].

Explicitly, the classification is as it follows: let H be a finite group with 2-Sylow subgroup P. Let $O(H)$ be the maximal normal subgroup of odd order of H. Then:

1. if P is cyclic, then $H/O(H) \cong P$
2. if P is dihedral, then $H/O(H)$ is isomorphic to the alternating group A_7, or a subgroup of $\mathrm{P\Gamma L}_2(q)$ containing $\mathrm{PSL}_2(q)$ for q an odd prime power or to P.

If G is a soluble binary group of order $2n$, we take $H = G/(O(G)\Lambda(G))$ with 2-Sylow subgroup P. Then H is isomorphic to A_4, S_4, a cyclic or dihedral 2-group or the elementary abelian 2-group of order 4. Explicitly we have the following cases:

- if P is a cyclic 2-group we have $H/O(H) \cong P$ from 1. ;
- if P is a dihedral 2-group we can have $H/O(H) \cong P$ from 2. ;
- since G is soluble, then H is soluble. If P is a dihedral 2-group, we study the remaining cases from 2. We cannot have $H \cong A_7$ since A_7 is not soluble. $\mathrm{PSL}_2(q)$ and the groups between it and $P\Gamma L_2(q)$ are not solvable if q is an odd prime, with the exception of $q = 3$, and it is known that $\mathrm{PSL}_2(3) \cong A_4$ and that $\mathrm{P\Gamma L}_2(3) \cong S_4$;
- the last case appears when the Sylow 2-subgroup of G is Q_8: we have $H = G/(O(G)\Lambda(G)) = Q_8/\Lambda(G) = Q_8/\langle -1 \rangle \cong V_4$, where V_4 is the abelian 2-group of order 4.

We now give the following definition:

Definition 2.13. *Let G be a group of order n. A 2-sequencing of G is a sequence of elements $(e, a_2, a_3, \ldots, a_n)$, not necessarily distinct, such that:*

- *the associated partial products $e, ea_2, ea_2a_3, \ldots, ea_2 \cdots a_n$ are all distinct;*
- *if $g \in G$ and $g \neq g^{-1}$ then*

$$|\{i : 2 \leq i \leq n, a_i \in \{g, g^{-1}\}\}| = 2,$$

- *if $g \in G$ and $g = g^{-1}$ then*

$$|\{i : 1 \leq i \leq n, a_i = g\}| = 1.$$

It is worth noting that a sequencing for G is also a 2-sequencing for G: the first condition is met by definition of sequencing; the second and the third conditions are met because in a sequencing of G each element appears exactly once: if $g \neq g^{-1}$ then there are two appearances and if $g = g^{-1}$ then there is one.

After various intermediate results, Anderson and Ihrig showed that:

Theorem 2.14. *[4] If G is a binary group and $G/O(G)\Lambda(G)$ has a 2-sequencing then G has a symmetric sequencing.*

As showed by Anderson [1], A_4 and S_4 admit sequencing and therefore possess 2-sequencings as well. Moreover, we have established that cyclic and dihedral 2-groups admit sequencings (see Theorem 2.9). As we stated, the scenario where $G/O(G)\Lambda(G) \cong V_4$, with V_4 being the elementary abelian 2-group of order 4, occurs exclusively when examining binary groups whose Sylow 2-subgroup is Q_8. We know that Q_8 itself is not sequenceable, however, Anderson and Leonard [6] prove that groups of the form $Q_8 \times B$, where B is a non-trivial abelian group of odd order, admit symmetric sequencings. Such groups are Hamiltonian; that is, they are non-abelian groups in which every subgroup is normal. These are the only Hamiltonian binary groups. Anderson and Ihrig [4] prove that every soluble binary group G satisfying $G/O(G)\Lambda(G) \cong V_4$, with $G \neq Q_8$, admits symmetric sequencings. This result combined with Theorem 2.14 now gives us the following theorem that settles the case for soluble binary groups.

Sequencing	Partial products
()	()
(3 4)	(3 4)
(2 3)	(2 4 3)
(2 4 3)	(2 3 4)
(1 2)	(1 3 4 2)
(2 3 4)	(1 3 2 4)
(2 4)	(1 3 2)
(1 2 4 3)	(2 4)
(1 2)(3 4)	(1 4 3 2)
(1 2 4)	(2 3)
(1 3 2)	(1 2)
(1 3 2 4)	(1 3)(2 4)
(1 4 3 2)	(1 2 3 4)
(1 3 4)	(1 4 2 3)
(1 4 2 3)	(1 2)(3 4)
(1 3)(2 4)	(1 4)(2 3)
(1 3 4 2)	(1 2 4 3)
(1 2 3 4)	(1 4 2)
(1 4)(2 3)	(1 2 3)
(1 4 2)	(1 4 3)
(1 4)	(1 3)
(1 4 3)	(1 4)
(1 3)	(1 3 4)
(1 2 3)	(1 2 4)

Sequencing of S_4

Sequencing	Partial products
()	()
(1 3 4)	(1 3 4)
(1 2)(3 4)	(1 2 3)
(2 3 4)	(1 2) (3 4)
(1 4 3)	(1 3 2)
(2 4 3)	(1 3)(2 4)
(1 2 3)	(1 4 2)
(1 4 2)	(1 2 4)
(1 4)(2 3)	(2 3 4)
(1 3)(2 4)	(1 4 3)
(1 3 2)	(2 4 3)
(1 2 4)	(1 4)(2 3)

Sequencing of A_4

Theorem 2.15. *[4] All finite soluble binary groups, except* Q_8*, have symmetric sequencings.*

The problem of finding symmetric sequencing for non-soluble binary groups was studied by Anderson and Ihring [5]. Their research demonstrates that to find all possible sequencings for insoluble binary groups, you only need to find 2-sequencings for three specific types of groups: A_7, $\mathrm{PSL}_2(q)$, and $\mathrm{P\Gamma L}_2(q)$, where q is an odd prime power greater than 3. That is coherent with the classification mentioned before. The authors further show that this construction contains no redundancy: obtaining a 2-sequencing of each of these groups gives us symmetric sequencings for an infinite collection of insoluble binary groups, with these collections being disjoint. A noticeable example of this is given by Anderson [2] that, by giving the sequencing of $PSL(2,5) \cong A_5$, shows that such infinite collections of sequenceable insoluble binary groups exist.

2.4 Groups of odd order

As Gordon noted [13] and as we have seen in 2.6, studying the quotient G/H of some group G over a normal subgroup H and its corresponding *quotient sequencing* is a good way to extract information about the existence and nature of a sequencing of G. By applying this technique, Wang [22] and Keedwell [18] found the sequencing of some non-abelian groups of odd order which have a cyclic normal subgroup with prime index. Let us take $G/H \cong C_p$ for some odd prime p. Let α be a primitive root of p such that $\alpha/(\alpha-1)$ is also a primitive root of p. Wang proves that such α exists and gives us the quotient sequencing of G/H:

Lemma 2.16. *[22] Let G be a group of order n and let H be a normal subgroup of index p in G. Let α be a primitive root of p such that $\alpha/(\alpha-1)$ is also a primitive root of p. Let $m = n/p$. The following is a quotient sequencing of G/H:*

$$\underbrace{0, 0, \ldots, 0}_{m-1 \text{ times}}$$

Followed by a 1 and then $m-2$ copies of $(p-1)$-tuples:

$$\alpha^{p-2} - 1, \alpha^{p-3} - \alpha^{p-2}, \cdots, \alpha^2 - \alpha^3, \alpha - \alpha^2, 1 - \alpha$$

then ending with the following $2(p-1)$ elements:

$$\alpha^{p-2} - 1, \alpha^{p-3} - \alpha^{p-2}, \cdots, \alpha^2 - \alpha^3, \alpha - \alpha^2, 0,$$

$$\frac{\alpha^2}{\alpha-1} - \alpha, \frac{\alpha^3}{(\alpha-1)^2} - \frac{\alpha^2}{\alpha-1}, \cdots, \frac{\alpha^{p-2}}{(\alpha-1)^{p-3}} - \frac{\alpha^{p-3}}{(\alpha-1)^{p-4}}, (\alpha-1) - \frac{\alpha^{p-2}}{(\alpha-1)^{p-3}}, (1-\alpha).$$

Let us now define the following.

Definition 2.17. *Let G be a finite group of order n. An R-sequencing of G is a sequence $(e, a_2, \cdots, a_n)$ of its elements such that the partial products $(e, a_2, a_2a_3, \cdots, a_2a_3\cdots a_{n-1})$ are distinct and $ea_2a_3\cdots a_{n-1}a_n = e$*

Wang and Keedwell proceed as follows: starting with a group G satisfying $G/H \cong C_p$ (as previously defined), they first construct an R-sequencing for the cyclic group C_p. The initial $p-1$ elements of this sequencing are then used to form the first $p-1$ elements of the sequencing for G. The remaining elements are filled in a manner that ensures compatibility with the quotient sequencing described in Lemma 2.16. By doing this, Wang proves the following theorem, which settles the case for non-abelian groups of order p^n as defined before.

Theorem 2.18. *[22] Let G be a non-abelian group of order p^{n+2}, with p an odd prime and n a positive integers. If G has an element of order p^{n+1} then it is sequenceable.*

It is worth noting that such a group exists: for example, let us take $p = 3$ and $n = 1$. The group:

$$C_9 \rtimes C_3 = \langle a, b \mid a^9 = b^3 = e, bab^{-1} = a^4 \rangle$$

has order 27 and has an element of order 9

Wang and Keedwell also investigated a different kind of group of odd order: those with an order of pq, with $p < q$ odd primes. It is easy to show that there exists a non-abelian group G of order pq if and only if $q = 2pn + 1$ for some positive integer n: in fact, that is simply the semidirect product $G = C_q \rtimes C_p$. Keedwell settled the case whenever 2 is a primitive root of p.

Theorem 2.19. *[18] Let G be a group of order pq with $p < q$ odd primes. If 2 is a primitive root of p, then G is sequenceable.*

Similarly to what we have seen before, Keedwell found the exact sequencing of G by studying the quotient sequencing of G/H, where H is the unique subgroup of order q. Let us now give an explicit example by taking $p = 5$ and $q = 11$ as reported in [18].

Example 2.1. The explicit sequencing of the non-abelian group

$$G = C_{11} \rtimes C_5 = \langle a, b \mid a^{11} = b^5 = 1, bab^{-1} = a^3 \rangle$$

of order 55 is the following.

Sequencing	Partial products	Sequencing (contd.)	Partial products (contd.)
e	e	ba	b^4a^2
a^{10}	a^{10}	b^3a^2	b^2a
a^9	a^8	b^4a	ba^5
a^7	a^4	b^2a^8	b^3a^9
a^3	a^7	ba^2	b^4a^7
a^6	a^2	b^3a^4	b^2a^6
a	a^3	b^4a^2	ba^4
a^2	a^5	b^2a^5	b^3a^8
a^4	a^9	ba^4	b^4a^6
a^8	a^6	b^3a^8	b^2a^5
b	ba^7	b^4a^4	ba^2
b^2a^3	b^3	b^2a^{10}	b^3a^6
ba^9	b^4a^9	ba^8	b^4a^4
b^3a^7	b^2a^8	b^3a^5	b^2a^3
b^4a^9	ba^8	b^4a^8	ba^9
b^2a^6	b^3a	b^2a^9	b^3a^2
ba^7	b^4a^{10}	ba^5	b^4
b^3a^3	b^2a^9	b^3a^{10}	b^2a^{10}
b^4a^7	ba^{10}	b^4a^5	ba
b^2a	b^3a^3	b^2a^7	b^3a^5
ba^3	b^4a	ba^{10}	b^4a^3
b^3a^6	b^2	b^3a^9	b^2a^2
b^4a^3	ba^3	a^5	b^2a^7
b^2a^2	b^3a^7	b^2	b^4a^8
ba^6	b^4a^5	b^4	b^3a^{10}
b^3a	b^2a^4	b^3	ba^6
b^4a^6	b	b^4a^{10}	a
b^2a^4	b^3a^4		

The question whether non-abelian groups of order pq, where 2 is not a primitive root of p, are sequenceable remains open. Wang, using similar arguments as he did in Lemma 2.16, proves the following result that covers some cases where 2 is not a primitive root of p.

Theorem 2.20. *[22] Let G be a non-abelian group of order pq, with $p < q$ odd primes. If there exists r such that $r^p \equiv 1 \bmod q$ and $(r - r^{1-\epsilon})/(r - r^{1-\epsilon} - 1)$ is a primitive root of q, where ϵ is a primitive root of p such that $\epsilon/(\epsilon - 1)$ is also a primitive root of p; then G is sequenceable.*

This result, for example, proves that the group $G = C_{29} \rtimes C_7$ of order 203 is sequenceable. In fact:

- $q = 4p + 1$, so G is a non-abelian group of order 203;
- choose $r = 16 \bmod 29$. Then $r^p = r^7 = 16^7 = 1 \bmod 29$;
- choose $\epsilon = 3 \bmod 7$. Then ϵ is a primitive root of p and $\epsilon/(\epsilon - 1) = \frac{3}{2} = 3 \cdot 2^{-1} = 3 \cdot 4 = 5 \bmod 7$, which is a primitive root of p;
- $(r - r^{1-\epsilon})/(r - r^{1-\epsilon} - 1) = \frac{16-16^{1-3}}{16-16^{1-3}-1} = \frac{16-16^5}{16-16^5-1} = \frac{16-23}{16-23-1} = \frac{-7}{-8} = \frac{22}{21} = 22 \cdot 21^{-1} = 22 \cdot 18 = 19 \bmod 29$ is a primitive root of q

It is worth noticing that in this case it is necessary to use Theorem 2.20 since 2 is not a primitive root of p: $2^3 = 1 \bmod 7$.

Anderson also expanded on the results of Keedwell by studying the congruence modulo 4 of p and q. Let us now define a particular type of sequencing.

Definition 2.21. *Let G be a group of order n. A sequencing $(e, a_2, \cdots, a_n)$, is said to be a starter-translate sequencing, or st-sequencing, if both sets $\{a_2, a_4, \cdots, a_{n-1}\}$ and $\{a_3, a_5, \cdots .a_n\}$ contain precisely one of g and g^{-1} for each $g \in G\backslash\{e\}$.*

Anderson [3] proves the following theorems.

Theorem 2.22. *Let G, H be groups with st-sequencing. Then $G \times H$ also has st-sequencing*

Theorem 2.23. *Let G be a group of order pq with $p < q$ odd primes with 2 being a primitive root of p. Then G has st-sequencing if and only if $p, q \equiv 3 \bmod 4$*

Combined, these two theorems significantly expand the range of odd integers n for which a sequenceable group of order n can be found, since we can now easily say that the direct products of some type of groups that we know are sequenceable for Theorem 2.19 are also sequenceable.

3 R-Sequencings

In this last chapter we now turn to a more detailed discussion of R-sequenceability by summarizing the classes of groups that are currently known to admit an R-sequencing. As in the case of ordinary sequencing, no complete classification is available: it remains an open problem to determine precisely which abelian and non-abelian groups possess an R-sequencing.

The biggest contribution for the abelian case came from Friedlander, Gordon and Miller, that in [12] prove that the following groups are R-sequenceable:

Theorem 3.1. *The following types of abelian group admits an R-sequencing:*

1. C_n*, with* n *an odd integer;*
2. *Abelian groups of odd order with cyclic Sylow 3-subgroups;*
3. C_3^n*, with* $n \geq 2$*;*
4. $C_2 \times C_{4n}$*, with* $n \geq 1$*;*
5. *Abelian groups with Sylow 2-subgroups* C_n^2*, with* $n = 2$ *or* $n \geq 4$*;*
6. *Abelian groups with Sylow 2-subgroups* $C_2 \times C_{2^n}$*, with* n *an odd integer.*

In [23] Wang expands this list with the following:

Theorem 3.2. $C_3 \times C_{3n}$ *is R-sequenceable for all* $n \geq 1$*.*

We will now provide some examples of R-sequencing groups of some of the types discussed above.

Example 3.1. An R-sequencing of C_{11} is:

R-sequencing	Partial products
e	e
a	a
a^3	a^4
a^5	a^9
a^4	a^2
a^8	a^{10}
a^9	a^8
a^{10}	a^7
a^7	a^3
a^2	a^5
a^6	e

Example 3.2. An R-sequencing of $C_3 \times C_5 \cong C_{15}$ is:

R-sequencing	Partial products
e	e
b	b
b^2	b^3
b^4	b^2
a	ab^2
b^3	a
ab	a^2b
ab^3	b^4
a^2b^3	a^2b^2
a^2b^2	ab^4
ab^4	a^2b^3
a^2	ab^3
ab^2	a^2
a^2b	ab
a^2b^4	e

Example 3.3. An R-sequencing of C_2^3 is:

R-sequencing	Partial products
e	e
a^2b	a^2b
a	b
b	b^2
ab	a
ab^2	a^2b^2
a^2	ab^2
b^2	ab
a^2b^2	e

Before giving some results about the R-sequenceability of non-abelian groups, let us introduce the following notation:

Definition 3.3. *The dicyclic group Q_{4n} is defined as:*

$$Q_{4n} = \langle a, b : a^{2n} = e, b^2 = a^n, ab = ba^{-1} \rangle$$

It is worth noticing that if n is a power of 2, then the dicyclic group Q_{4n} is a generalized quaternion group, as defined in Definition 2.12.

In [17] Keedwell proved the following theorem that settles the case of the R-sequenceability of dihedral groups.

Theorem 3.4. *The dihedral group D_{2n} of order $2n$ is R-sequenceable if and only if n is even.*

Keedwell proves this by explicitly giving an R-sequencing of a generic dihedral group $D_{2n} = \langle a, b : a^n = e = b^2, ab = ba^{-1}\rangle$ that is obtained by studying the quotient sequencing with the group $H = \langle a : a^n = e\rangle = \{e, x\}$:

$$\underbrace{1, \cdots, 1}_{m-\text{ times}} \times \underbrace{1, \cdots, 1}_{m-\text{ times}} \times \underbrace{x, \cdots x}_{(2m-1)-\text{ times}}$$

where $n = 2m$.

In the same paper, Keedwell also proved the following:

Theorem 3.5. *The non-abelian groups of order pq, where p and q are odd primes, $p < q$ and p has 2 as a primitive root, are R-sequenceable.*

Keedwell gives an explicit construction of an R-sequencing by using the exact same method he used in Theorem 2.19. This condition was removed by Wang and Leonard in [23]. Let us give an example with the same group we used in Theorem 2.19.

Example 3.4. An R-sequencing of $= C_{11} \rtimes C_5 = \langle a, b \mid a^{11} = b^5 = 1, bab^{-1} = a^3\rangle$ is:

R-sequencing	Partial products	R-sequencing (contd.)	Partial products (contd.)
e	e	b^2a^4	b^3a^2
a^2	a^2	ba	b^4a^7
a^4	a^6	b^3a^9	b^2
a^8	a^3	b^4a^6	ba^6
a^5	a^8	b^2a^8	b^3a^7
a^{10}	a^7	ba^2	b^4a
a^9	a^5	b^3a^7	b^2a
a^7	a	b^4a	ba^5
a^3	a^4	b^2a^5	b^3a^6
a^6	a^{10}	ba^4	b^4
b	ba^8	b^3a^3	b^2a^3
b^2a^3	b^3a^9	b^4a^2	ba^3
ba^9	b^4a^3	b^2a^{10}	b^3a^4
b^3a^4	b^2a^8	ba^8	b^4a^9
b^4a^{10}	ba^9	b^3a^6	b^2a^7
b^2a^6	b^3a^{10}	b^4a^4	ba^{10}
ba^7	b^4a^4	b^2a^9	b^3
b^3a^8	b^2a^6	ba^5	b^4a^5
b^4a^9	b	b^3a	b^2a^4
b^2a	b^3a	b^4a^8	ba^2
ba^3	b^4a^6	b^2a^7	b^3a^3
b^3a^5	b^2a^2	ba^{10}	b^4a^8
b^4a^7	ba^4	b^3a^2	b^2a^9
b^2a^2	b^3a^5	a	b^2a^{10}
ba^6	b^4a^{10}	b^2	b^4a^2
b^3a^{10}	b^2a^5	b^4	b^3a^8
b^4a^3	ba	b^3	ba^7
		b^4a^5	e

In the same paper [23] where Wang and Leonard removed the root condition, they also proved the following theorem that fully classify the R-sequenceability for dicyclic groups:

Theorem 3.6. *The dicyclic group Q_{4n} is R-sequenceable if and only if n is an even integer greater than 2.*

Example 3.5. An R-sequencing of $Q_8 = \langle a, b : a^8 = e, b^2 = a^4, ab = ba^{-1} \rangle$ is:

R-sequencing	Partial products
e	e
a	a
a^4	a^5
a^5	a^2
b	ba^6
a^6	ba^4
a^7	ba^3
a^2	ba^5
a^3	b
ba^7	a^3
ba^4	ba
ba^3	a^6
ba^5	ba^7
ba^2	a^7
ba	ba^2
ba^6	e

The last useful result is given to us by Bedford that, in [9], concludes that the two non abelian groups of order 27, $C_3^2 \rtimes C_3$ and $C_9 \rtimes C_3$, are R-sequenceable.

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