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On the Representation Theory of Incidence Algebras

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Introduction

Representation theory of finite dimensional algebras is a branch of mathematics that studies these abstract algebraic structures, aiming at a better understanding of their properties and their categories of modules. Even though it can be traced back further, its current state was highly influenced by the language of quivers introduced by Gabriel [Gab72a]. The quiver-theoretic approach that is widely used provides us a method for translating questions about algebraic structures into the language of linear algebra, a subject that is well understood.

An important question in representation theory of algebras regards the number of indecomposable modules over a given algebra. In general, algebras fall into three main categories of increasing complexity: finite type, where we only have a finite number of indecomposable modules, tame type, where we can parametrise all the families of indecomposables, and wild type, where such parametrisation is not possible. Drozd famously showed that any algebra, over an algebraically closed field, is either tame or wild, but never both [Dro80]. Hence, there is a dichotomy of algebras by representation type.

Gabriel classified finite-dimensional hereditary algebras with finitely many indecomposable modules in terms of Dynkin diagrams [Gab72a]. Around the same time, Loupias classified the representation-finite incidence algebras of posets [Lou06; Lou75]. This result was highly valuable in the study of the representation type of a large class of algebras, as one could often reduce to the case of posets. We will focus our attention on posets throughout this thesis, since they give us a more tractable class of algebras.

Some tilting theory (see [ASS06, Ch.VI]) will be necessary to better understand our results. Tilting theory gives a method of constructing a new algebra from another while carrying over known structure, simplifying the study of the representation theory of algebras. It paved the way for a staggering amount of progress, and it is still an important tool for modern representation theorists. In an effort to close gaps in the tilting theory when the underlying algebra is not hereditary, Adachi, Iyama and Reiten introduced τ-tilting theory [AIR14]. This new approach is relevant in representation theory because it extends tilting theory to a more general setting, provides a combinatorial and categorical framework for studying module categories, and builds deep connections to cluster theory and torsion theory. We introduce

τ-tilting theory in the first chapter of the thesis and we will use it in order to obtain a main result in the third chapter.

The aim of this work is to give a good description of the algebras arising from posets and to discuss an interesting equivalence that connects the representation type and the τ-tilting type of such algebras. Moreover we will prove some original results concerning particular infinite dimensional modules.

In Chapter 1 we give an overview of preliminary notions that we use in the thesis. We will introduce the theory of quivers and their representations, showing in particular that we can associate to a basic connected finite-dimensional algebra a certain quiver, reducing the study of an algebra to its associated quiver. We will then introduce the reflection functors of quiver representations, together with some of their important properties. These functors are really useful to consider different orientations for the underlying graph of a given quiver while maintaining the properties of indecomposable modules. Then we will continue the overview by presenting some concepts of tilting and τ-tilting theory. Starting from tilting modules, we will introduce the theory of concealed algebras (in particular tame concealed algebras) and after that, we will give the notion of τ-tilting finite algebra, that is an algebra admitting only a finite number of isomorphism classes of basic τ-tilting modules.

In Chapter 2 we will present the main protagonists of this thesis, the incidence algebras of posets. The *incidence algebra* $\mathcal{I}_{\mathbf{k}}(P)$ of a finite poset P will be defined as the bound path $\mathbf{k}$-algebra $\mathbf{k}Q_P/I_P$, with Q_P the *Hasse quiver* of P and I_P the ideal of $\mathbf{k}Q_P$ generated by the relations $\{p_1 - p_2\}_{(p_1,p_2)}$, where we index over all pairs of paths with the same start and endpoints. We will discuss some of their properties and the representation type of the incidence algebras using the tools developed by Loupias. We will study its reduction techniques, namely the subtraction (informally a subposet) and the contraction (informally an identification of adjacent elements). We will then present some particular classes of posets called *crucial*, *kind*, and *critical posets* together with some of their properties. We will see that kind posets coincide with representation-finite posets while crucial and critical posets are the same.

In Chapter 3 we will discuss a result of Erlend D. Børve, Jacob F. Grevstad and Endre S. Rundsveen. It is well known that representation-finite algebras are τ-tilting finite, but the converse does not hold in general. For incidence algebras of finite posets, however, there is an equivalence between τ-tilting finiteness and representation-finiteness:

Theorem 3.7. Let $\mathbf{k}$ be a field and let $(P, \leq)$ be a finite poset. The incidence algebra $\mathcal{I}_{\mathbf{k}}(P)$ is τ-tilting finite if and only if it is representation-finite.

The strategy of the proof will rely on three important results that we will have developed in these chapters: representation-infinite concealed $\mathbf{k}$-algebras are τ-tilting infinite (Lemma 1.62), the twelve classes of minimal representation-infinite incidence algebras provided by Loupias are tame concealed (Corollary 3.3) and the reduction techniques of Loupias preserves τ-tilting finiteness (Lemma 3.6).

In Chapter 4 we will show some original results. The twelve classes of minimal representation-infinite incidence algebras introduced before, by a result of W. Crawley-Boevey [Cra91], admit a *generic module*, that is a module with infinite length but finite endolength. Moreover, thanks to the previous theorem, such algebras are τ-tilting infinite. Hence, by a result of F. Sentieri [Sen23], they admit a brick that is not finitely generated, where a *brick* is a module whose endomorphism ring is a division ring. We can therefore ask whether such algebras admit a *generic brick*, i.e a generic module that is also a brick.

We will show that each class of algebras is *brick-continuous* (i.e. it admits an infinite family of bricks with the same dimension), explicitly describing the infinite families we have found. This process will utilise the concept of integral quadratic form together with its radical vectors. In particular we will use the Tits quadratic form associated to a quiver, that is defined through the number of arrows and relations of the quiver. By brick-continuity, thanks to a result of R. Bautista, E. Pérez, L. Salmerón [BPS24], for each class there exists a generic brick. We will directly compute it and thanks to the properties of reflection functors, we will prove the following result:

Theorem 4.22. The twelve families of minimally representation-infinite incidence $\mathbf{k}$-algebras given by critical posets in Table 2.1 (considering any orientation of the undirected edges), and also the opposites of these, are brick-continuous and admit a generic brick which can be explicitly constructed.

Lastly, since any representation-infinite incidence algebra can be reduced to one of the twelve classes through the reduction techniques of Loupias (Lemma 3.1), it is possible to find a generic brick for that algebra starting from the one of the class the algebra lies in. Indeed we will define some functors, which we will call *extension functors*, that applied to the generic brick that we found before give us as a result a generic brick for the representation-infinite algebra we started with. This will prove our final theorem:

Theorem 4.34. Any representation-infinite poset P admits a generic brick over $\mathcal{I}_{\mathbf{k}}(P)$ that can be explicitly computed.

1. Preliminary notions

In this first chapter we discuss some preliminary notions that will help the reader to understand the results that will be presented in the thesis.

Notation and conventions. An algebraically closed field $\mathbf{k}$ is fixed throughout. All modules over a finite-dimensional $\mathbf{k}$-algebra will be left modules. For a finite-dimensional $\mathbf{k}$-algebra Λ, let $\mathrm{Mod}(\Lambda)$ denote the category of left Λ-modules, and let $\mathrm{mod}(\Lambda)$ denote the subcategory of finite-dimensional Λ-modules.

1.1. Quivers

We will present the notions of quiver and of path algebras mainly following the definitions in [Sim23]. The reader may also consult [Ang17; ASS06] for further information.

Definition 1.1. A *quiver* $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of the following data

1. a set Q_0 of *vertices*,
2. a set Q_1 of *arrows* between vertices,
3. two maps $s, t\colon Q_1 \to Q_0$ called *source* and *target*, respectively, which associate to each arrow $\alpha\colon i \to j \in Q_1$ its source $s(\alpha) = i \in Q_0$ and its target $t(\alpha) = j \in Q_0$.

A quiver $Q = (Q_0, Q_1, s, t)$ is usually denoted briefly by $Q = (Q_0, Q_1)$ or even simply by Q.

Let Q be a quiver:

- Q is said to be *finite* if Q_0 and Q_1 are finite sets.
- The *underlying graph* of Q, denoted by $\overline{Q}$, is obtained from Q by forgetting the orientation of the arrows. The quiver Q is said to be *connected* if its underlying graph $\overline{Q}$ is a connected graph.

Remark 1.2. We define the concatenation of arrows in a quiver Q from right to left in the following sense: let $\alpha\colon i \to j$ and $\beta\colon j \to k$ be two arrows in Q. We have the following situation:

$$i \xrightarrow{\alpha} j \xrightarrow{\beta} k$$

and we denote such diagram with $\beta\alpha$.

Definition 1.3. Let Q be a quiver.

- A *path* of length $\ell \geq 1$ in Q with source i and target j (or, more briefly, from i to j) is a sequence $p = \alpha_\ell \dots \alpha_2\alpha_1$ (concatenation of arrows as defined in Remark 1.2) with $\alpha_k \in Q_1$ for all $1 \leq k \leq \ell$ such that $s(\alpha_1) = i$, $t(\alpha_k) = s(\alpha_{k+1})$ for each $1 \leq k \leq \ell - 1$, and finally $t(\alpha_\ell) = j$. It can be visualised as follows

  $$i = s(\alpha_1) \xrightarrow{\alpha_1} t(\alpha_1) = s(\alpha_2) \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_\ell} t(\alpha_\ell) = j$$

  In particular, p has length 1 if and only if $p \in Q_1$.

- We associate a path ε_i of length 0 to each vertex $i \in Q_0$, which is called the *trivial* or *stationary path* at i.

- If $s(\alpha_1) = t(\alpha_\ell)$, then $p = \alpha_\ell \dots \alpha_2\alpha_1$ is said to be an *oriented cycle*. An oriented cycle of length 1 is called a *loop*. An *acyclic* quiver is a quiver with no oriented cycles.

Definition 1.4. Let Q be a quiver. The path algebra $\mathbf{k}Q$ of Q is the $\mathbf{k}$-algebra whose underlying $\mathbf{k}$-vector space has as its basis the set of all paths of Q and such that the product of two basis elements is given by concatenation of paths, that is, given two paths $p = \alpha_\ell \dots \alpha_1$ and $p' = \beta_k \dots \beta_1$

$$pp' = \begin{cases} \alpha_\ell \dots \alpha_1\beta_k \dots \beta_1 & \text{if } t(\beta_k) = s(\alpha_1) \\ 0 & \text{otherwise} \end{cases}$$

Example 1.5. 1. Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

$\mathbf{k}Q$ has basis given by $\{\alpha^n \mid n \geq 0\}$, where $\alpha^0 := \varepsilon_1$ and multiplication given by $\alpha^n\alpha^m = \alpha^{n+m}$. The algebra $\mathbf{k}Q$ is isomorphic to the algebra $\mathbf{k}[x]$ of polynomials with one indeterminate.

2. Let Q be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

   $\mathbf{k}Q$ is generated by $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha, \beta, \beta\alpha\}$ and it is possible to see that it is isomorphic to the algebra of lower triangular 3×3 matrices.

Lemma 1.6. *The path algebra* $\mathbf{k}Q$ *satisfies the following properties:*

1. $\mathbf{k}Q$ *is an associative algebra.*
2. $\mathbf{k}Q$ *has an identity element* $1_{\mathbf{k}Q} = \sum_{i \in Q_0} \varepsilon_i$ *if and only if* Q_0 *is finite.*
3. $\mathbf{k}Q$ *is finite dimensional if and only if* Q *is finite and acyclic.*

Definition 1.7. Let Q be a finite quiver.

- The *arrow ideal* R_Q is the two-sided ideal of the path algebra $\mathbf{k}Q$ generated by all the arrows of Q.
- An *admissible ideal* I is a two-sided ideal of $\mathbf{k}Q$ such that there exists $m \geq 2$ for which $R_Q^m \subseteq I \subseteq R_Q^2$.

  Notice that if Q is acyclic, any ideal contained in R_Q^2 is admissible.
- If I is an admissible ideal of $\mathbf{k}Q$, the pair (Q, I) is said to be a *bound quiver*. The quotient algebra $\mathbf{k}Q/I$ is said to be the algebra of the bound quiver (Q, I) or, simply, a *bound quiver algebra*.

  The bound path algebra $\mathbf{k}Q/I$ is finite dimensional, since $R_Q^m \subseteq I$, and if Q is a connected quiver, it is a connected algebra (i.e. it is not the direct product of two algebras) because $I \subseteq R_Q^2$.
- A *relation* ρ is a linear combination $\rho = \sum_p \lambda_p p$ of paths in I all with length at least two and with the same start and same endpoints.

Remark 1.8 (c.f. [Ang17] Section 6.1)**.** Given a finite dimensional $\mathbf{k}$-algebra Λ, for the purpose of studying $\mathrm{Mod}(\Lambda)$ we may always assume that it is a connected algebra, i.e. it does not admit a non-trivial decomposition in a direct product of subalgebras.

Theorem 1.9. *Any finite dimensional* $\mathbf{k}$*-algebra* Λ *is Morita equivalent to a bound path algebra* $\mathbf{k}Q/I$*, that is*

$$\mathrm{Mod}(\Lambda) \cong \mathrm{Mod}(\mathbf{k}Q/I)$$

Proof. It follows directly from the fact that any finite dimensional algebra is Morita equivalent to its associated basic algebra [SY11, Ch.II, Theorem 6.16] and that basic algebras are isomorphic to bound path algebras [ASS06, Ch.II, Theorem 3.7]. □

Notation. For the purpose of studying $\mathrm{Mod}(\Lambda)$, by Remark 1.8 and the proof of Theorem 1.9 we may assume that any finite dimensional $\mathbf{k}$-algebra is basic and connected (c.f. [Ang17, Section 6.1]).

Therefore quivers provide a nice way to visualise finite dimensional algebras. Now, we explain how quivers can be used to visualise also modules and morphisms between modules.

1.1.1. Representations of quivers

Let Q denote a finite connected quiver and I an admissible ideal. Notice that if $I = 0$ is an admissible ideal then Q is acyclic.

Definition 1.10. A *representation* $M = (M_i, \varphi_\alpha)_{i\in Q_0,\, \alpha\in Q_1}$ (or simply $M = (M_i, \varphi_\alpha)$) of Q is given by:

1. $\mathbf{k}$-vector spaces M_i for all $i \in Q_0$
2. linear maps $\varphi_\alpha \colon M_{s(\alpha)} \to M_{t(\alpha)}$ for all $\alpha \in Q_1$

Let $p = \alpha_\ell \dots \alpha_1$ be a path in Q and $M = (M_i, \varphi_\alpha)$ be a representation of Q. We denote by φ_p the composition of linear maps

$$\varphi_p \coloneqq \varphi_{\alpha_\ell} \cdots \varphi_{\alpha_1}$$

and it is called the *evaluation* of M on the path p. Given a relation $\rho = \sum_p \lambda_p p$ in I we have $\varphi_\rho = \sum_p \lambda_p \varphi_p$.

A representation M is *finite dimensional* if M_i is finite dimensional $\forall i \in Q_0$. The *dimension vector* of M is the vector $\underline{\dim}(M) = (\dim M_i)_{i\in Q_0}$.

Definition 1.11. Let $M = (M_i, \varphi_\alpha)$ and $N = (N_i, \psi_\alpha)$ be two representations of Q. A *morphism* (of representations) $f \colon M \to N$ is a family $f = (f_i)_{i\in Q_0}$ of $\mathbf{k}$-linear maps $(f_i : M_i \to N_i)_{i\in Q_0}$ that are compatible with the structure maps of the representations, that is, for each arrow $\alpha \colon i \to j \in Q_1$, we have the following commutative diagram:

$$\begin{tikzcd}
M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\
N_i \arrow[r, "\psi_\alpha"'] & N_j
\end{tikzcd}$$

i.e. $f_j\varphi_\alpha = \psi_\alpha f_i$. The morphism $f = (f_i)_{i\in Q_0}$ is an *isomorphism* if each f_i is bijective.
Given two morphism $f\colon M \to M'$ and $g\colon M' \to M''$ of representations of Q, where $f = (f_i)_{i\in Q_0}$ and $g = (g_i)_{i\in Q_0}$, their *composition* is defined to be the family $gf = (g_i f_i)_{i\in Q_0}$ and it is a morphism from M to M''.

We have thus defined the category $\mathrm{Rep}_{\mathbf{k}}(Q)$ of representations of Q. We denote by $\mathrm{rep}_{\mathbf{k}}(Q)$ the full subcategory of $\mathrm{Rep}_{\mathbf{k}}(Q)$ consisting of the finite dimensional representations.

Definition 1.12. Let Q be a quiver and fix a pair $X = (X_i, \varphi_\alpha)$, $Y = (Y_i, \psi_\alpha)$ of representations of Q. We call X a *subrepresentation* of Y and write $X \subseteq Y$, if X_i is a subspace of Y_i for each vertex i and $\varphi_\alpha(x) = \psi_\alpha(x)$ for each arrow α and $x \in X_{s(\alpha)}$.
Given a morphism $f\colon X \to Y$, its *kernel* $\ker(f)$ is by definition the subrepresentation of X with $(\ker(f))_i = \ker(f_i)$ for each vertex i and $\mathbf{k}$-linear maps given by universal property of kernels. The *cokernel* $\operatorname{coker}(f)$ and the *image* $\operatorname{im}(f)$ are defined analogously. Note that f is a *monomorphism* if and only if $\ker(f) = 0$, while f is an *epimorphism* if and only if $\operatorname{coker}(f) = 0$. One defines addition and scalar multiplication for morphisms $X \to Y$ point-wise and that makes $\mathrm{Hom}(X, Y)$ into a vector space.

Definition 1.13. A *simple* representation is defined to be a non-zero representation with no proper subrepresentations.

Given a vertex i, let $S(i) = (S(i)_j, \zeta_\alpha)$ be the representation with

$$S(i)_j = \begin{cases} \mathbf{k} & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases} \quad \text{and} \quad \varphi_\alpha = 0$$

for $j \in Q_0$ and $\alpha \in Q_1$. This representation is simple.

Lemma 1.14. *Let Q be a quiver and $X = (X_j, \varphi_\alpha)$ be a representation of Q and suppose that i is a vertex with $X_i \neq 0$ and $\varphi_\alpha = 0$ for each arrow α starting at i. Then $S(i)$ is a subrepresentation of X.*

Proof. The assumption on X implies $\mathrm{Hom}(S(i), X) \cong X_i$. Indeed given any sequence of arrows of the following form $k \xrightarrow{\beta} i \xrightarrow{\alpha} j$ (if any in Q , otherwise we only consider $k \xrightarrow{\beta} i$ or $i \xrightarrow{\alpha} j$) we have that a morphism between $S(i)$ and X must be of the following form over this sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "f_k"'] & \mathbf{k} \arrow[r] \arrow[d, "f_i"'] & 0 \arrow[d, "f_j"'] \\
X_k \arrow[r, "\varphi_\beta"] & X_i \arrow[r, "\varphi_\alpha=0"] & X_j
\end{tikzcd}$$

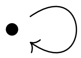
where the choice f_k and f_j is forced to be $f_k = 0 = f_j$. Therefore f_i is a $\mathbf{k}$-linear map between $\mathbf{k}$ and a $\mathbf{k}$-vector space, thus it only depends on the image of $1_{\mathbf{k}}$ in X_i. Any other map on the other vertices must be 0 by definition of $S(i)$, therefore we conclude. □

Corollary 1.15. *Suppose that Q is acyclic. Then for any simple representation S of Q, there exists a unique vertex i such that $S \cong S(i)$.*

Proof. Let $S = (S_j, \psi_\alpha)$ be a simple representation of Q. Then we must have $S_j = 0$ for all $j \in Q_0$ except for a unique $i \in Q_0$, for which $S_i \neq 0$. Indeed if there exist $h, k \in Q_0$ (with $h \neq k$) such that $S_h \neq 0$ and $S_k \neq 0$, then S will have two proper subrepresentations H having S_h in vertex h and 0 elsewhere and K having S_k in vertex k and 0 elsewhere, contradicting the fact that S is simple. Therefore by Lemma 1.14 $0 \neq S(i) \subseteq S$ but it cannot be proper, so we must conclude $S(i) \cong S$. □

On the other hand, there are additional simple representations if Q has oriented cycles.

Example 1.16 ([Kra08, Example 1.6.2])**.** The finite dimensional simple representations of the quiver



are parametrised by the monic irreducible polynomials over $\mathbf{k}$. More precisely, the representation corresponding to such a polynomial $\sum_{i=0}^{d} \lambda_i t^i$ of degree d is the pair (X, φ) consisting of the vector space $X = \mathbf{k}^d$ and the endomorphism $\varphi \colon X \to X$ with $\varphi(e_i) = e_{i+1}$ for $1 \leq i < d$ and $\varphi(e_d) = \sum_{i=1}^{d} -\lambda_{i-1} e_i$.

Definition 1.17. A representation $M = (M_i, \varphi_\alpha)$ of Q is said to be *bound by* I, or to be a *representation of* (Q, I), if $\varphi_\rho = 0$ for all $\rho \in I$.

We denote by $\mathrm{Rep}_{\mathbf{k}}(Q, I)$ (or by $\mathrm{rep}_{\mathbf{k}}(Q, I)$) the full subcategory of $\mathrm{Rep}_{\mathbf{k}}(Q)$ (or of $\mathrm{rep}_{\mathbf{k}}(Q)$, respectively) consisting of the representations of Q bound by I.

Notation. In the following, given a representation $M = (M_i, \varphi_\alpha)$ of a quiver Q, we may also write M_{ij} or φ_{ij} instead of φ_α, for $\alpha \colon i \to j \in Q_1$, when this does not create confusion, that is when there are no parallel arrows in Q.

Moreover if $I = 0$ is admissible then we have that $\mathrm{Rep}_{\mathbf{k}}(Q, 0) = \mathrm{Rep}_{\mathbf{k}}(Q)$ and $\mathrm{rep}_{\mathbf{k}}(Q, 0) = \mathrm{rep}_{\mathbf{k}}(Q)$.

Example 1.18.

1. An example of bound quiver (Q, I) is given by (the dashed line denotes a relation)

$$\begin{tikzcd}
& 4 & \\
2 \arrow[ur, "\beta"] & & 3 \arrow[ul, "\delta"'] \\
& 1 \arrow[ul, "\alpha"] \arrow[ur, "\gamma"'] \arrow[uu, dashed, no head] &
\end{tikzcd}$$

with $I = \langle \beta\alpha - \delta\gamma \rangle$. Since Q is acyclic and $I \subseteq R_Q^2$, it is an admissible ideal. A representation of this bound quiver is given for example by the following

$$\begin{tikzcd}
& \mathbf{k} & \\
\mathbf{k}^2 \arrow[ur, "(1\ 0)"] & & \mathbf{k}^2 \arrow[ul, "(0\ 1)"'] \\
& \mathbf{k} \arrow[ul, "\binom{1}{0}"] \arrow[ur, "\binom{0}{1}"'] \arrow[uu, dashed, no head] &
\end{tikzcd}$$

2. Let Q be the quiver

$$1 \xrightarrow{\alpha} 2$$

An example of representations M and N of Q and of a morphism f between them is given by the following

$$\begin{tikzcd}
M \arrow[d, "f"'] & & \mathbf{k} \arrow[r, "1"] \arrow[d, "1"'] & \mathbf{k} \arrow[d, "\binom{1}{0}"] \\
N & & \mathbf{k} \arrow[r, "\binom{1}{0}"'] & \mathbf{k}^2
\end{tikzcd}$$

Theorem 1.19 (Ch.III, Theorem 1.6 [ASS06]). *Let* $\mathbf{k}Q/I$*, where* Q *is a finite connected quiver and* I *is an admissible ideal of* $\mathbf{k}Q$*. There exists a* $\mathbf{k}$*-linear equivalence of categories*

$$F : \mathrm{Mod}(\mathbf{k}Q/I) \xrightarrow{\cong} \mathrm{Rep}_{\mathbf{k}}(Q, I)$$

that restricts to an equivalence of categories $F : \mathrm{mod}(\mathbf{k}Q/I) \xrightarrow{\cong} \mathrm{rep}_{\mathbf{k}}(Q, I)$

It follows that $\mathrm{Rep}_{\mathbf{k}}(Q, I)$ and $\mathrm{rep}_{\mathbf{k}}(Q, I)$ are abelian categories.

Definition 1.20. Let Λ be a finite dimensional $\mathbf{k}$-algebra. A Λ-module M has *finite length* if it admits a composition series (see [Ang17, Section 4.4])

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{m-1} \subseteq M_m = M \quad \text{with} \quad M_i/M_{i-1} \text{ simple modules}$$

The length m of any composition series of a module M is called the *length* of M, denoted by $\ell_\Lambda(M)$. If M is not of finite length it is said to be of *infinite length* and we set $\ell_\Lambda(M) = \infty$.

We have the following useful results:

Lemma 1.21 ([Ang17, Section 4.4]). *If F is any field (not necessarily algebraically closed) and M is an F-module (i.e. an F-vector space) we have that*

$$\ell_F(M) = \dim_F(M)$$

Theorem 1.22 (c.f. [ASS06, Ch.III, Corollary 3.6]). *Let $\Lambda \cong \mathbf{k}Q/I$ a finite dimensional $\mathbf{k}$-algebra, $M \in \mathrm{Mod}(\Lambda)$ and consider it also as a representation $M = (M_i, \varphi_{i,j}) \in \mathrm{Rep}_{\mathbf{k}}(Q, I)$ (by Theorem 1.19). We have that*

$$\ell_\Lambda(M) = \dim_{\mathbf{k}}(M) = \sum_{i \in Q_0} \dim_{\mathbf{k}}(M_i)$$

Definition 1.23. Given two representations $M = (M_i, \varphi_\alpha)$, $N = (N_i, \psi_\alpha)$ of Q, we can construct a new representation

$$M \oplus N := (M_i \oplus N_i, \begin{pmatrix} \varphi_\alpha & 0 \\ 0 & \psi_\alpha \end{pmatrix})$$

called the *direct sum* of M and N.

A representation (resp. a module) M is *indecomposable* if $M \neq 0$ and it cannot be written as a direct sum of two non-zero representations (resp. modules).

It is known that $\mathrm{mod}(\Lambda)$ is a *Krull-Remak-Schmidt category* (see [Ang17, Theorem 4.4.7]), i.e. every finitely generated module can be written in a unique way (up to isomorphisms and reordering) as the direct sum of indecomposables. Thanks to Theorem 1.19, $\mathrm{rep}_{\mathbf{k}}(Q, I)$ is a Krull-Remak-Schmidt category too.

Lemma 1.24 (c.f. [Kra08, Lemma 2.2.1]). *Let X be a finitely generated module over a finite dimensional $\mathbf{k}$-algebra (equivalently a finite dimensional representation) and f an endomorphism of X.*

1. *For large enough r, we have $X = \mathrm{im}(f^r) \oplus \ker(f^r)$.*
2. *If X is indecomposable, then f is either an automorphism or nilpotent.*

Proof. Since X is finitely generated over a finite dimensional $\mathbf{k}$-algebra, we may choose r large enough so that $\mathrm{im}(f^r) = \mathrm{im}(f^{r+1})$ and $\ker(f^r) = \ker(f^{r+1})$. In particular $(*)$ $\mathrm{im}(f^r) = \mathrm{im}(f^{2r})$ and $(**)$ $\ker(f^r) = \ker(f^{2r})$ hold.

- $X = \mathrm{im}(f^r) + \ker(f^r)$: let $x \in X$, then $x = x - f^r(y) + f^r(y)$ for $y \in X$ such that $f^{2r}(y) = f^r(x)$ (exists by $(*)$). We see that $x - f^r(y) \in \ker(f^r)$, indeed
$$f^r(x - f^r(y)) = f^r(x) - f^{2r}(y) = f^r(x) - f^r(x) = 0$$
- $\mathrm{im}(f^r) \cap \ker(f^r) = 0$: let $x \in \mathrm{im}(f^r) \cap \ker(f^r)$, then $x = f^r(y)$ for some $y \in X$ but now $0 = f^r(x) = f^{2r}(y)$. So $y \in \ker(f^{2r}) = \ker(f^r)$ by $(**)$, thus $x = f^r(y) = 0$.

We conclude that $X = \mathrm{im}(f^r) \oplus \ker(f^r)$ and we proved (1).

Now if X is indecomposable we must have one of the following:

- $\mathrm{im}(f^r) = 0$, so $X = \ker(f^r)$ and so f is nilpotent.
- $\ker(f^r) = 0$, so $X = \mathrm{im}(f^r)$, so f^r is an automorphism, but then the same holds for f. Indeed, since f^r is an automorphism, if $f(x) = f(y)$ then $f^r(x) = f^{r-1}(f(x)) = f^{r-1}(f(y)) = f^r(y)$ and so $x = y$ and moreover any $x \in X$ can be written as $x = f^r(y) = f(f^{r-1}(y))$ for some $y \in X$.

Thus we conclude that (2) holds. □

We recall that a ring is called *local* if the sum of two non-invertible elements is again a non-invertible element.

Proposition 1.25 (c.f. [Kra08, Proposition 2.2.2]). *A finitely generated module X over a finite dimensional $\mathbf{k}$-algebra (equivalently a finite dimensional representation) is indecomposable if and only if* $\mathrm{End}(X)$ *is local.*

Proof. Let X be indecomposable and $f, f' \in \mathrm{End}(X)$. Suppose $f + f'$ is invertible with inverse g such that $g(f+f') = 1_X$. If f is non-invertible then gf is non-invertible. Thus gf is nilpotent, say $(gf)^r = 0$, by Lemma 1.24. We obtain
$$(1_X - gf)(1_X + gf + \ldots + (gf)^{r-1}) = 1_X.$$
Therefore $gf' = 1_X - gf$ is invertible whence f' is invertible.

If $X = X_1 \oplus X_2$ with $X_i \neq 0$ for $i = 1, 2$, then we have idempotent endomorphisms ε_i of X with $\mathrm{im}(\varepsilon_i) = X_i$. Clearly, each ε_i is non-invertible but $1_X = \varepsilon_1 + \varepsilon_2$. □

The assumption on X to be a finitely generated module (equivalently finite dimensional representation) is necessary.

Example 1.26 ([Kra08, Example 2.2.3]). Let $k[x]$ denote the polynomial ring in one variable and consider the following indecomposable representation X of the quiver $\bullet \rightrightarrows \bullet$:
$$\begin{tikzcd}\mathbf{k}[x] \arrow[r, shift left, "1"] \arrow[r, shift right, "x"'] & \mathbf{k}[x]\end{tikzcd}$$
The endomorphism ring of X is isomorphic to $k[x]$, which is not local.

Definition 1.27. [ASS06; SS07] A finite dimensional algebra Λ is said to be of *finite representation type* (or *representation-finite*) if the category $\operatorname{mod}(\Lambda)$ has a finite number of isomorphism classes of indecomposable modules. Otherwise it is said to be of *infinite representation type* (or *representation-infinite*).

Moreover Λ is said to be:

- of *tame representation type* (shortly *tame*) if there is a classification of the isomorphism classes of the indecomposable modules in $\operatorname{mod}(\Lambda)$ in the sense that, for each integer $d \geq 1$, the indecomposable modules in $\operatorname{mod}(\Lambda)$ of dimension d form at most finitely many one-parameter families.
- of *wild representation type* (shortly *wild*) if, for any finite dimensional $\mathbf{k}$-algebra A, there exists an embedding functor $T\colon \operatorname{mod}(A) \to \operatorname{mod}(\Lambda)$.

It is clear by the definitions that an algebra of finite representation type is tame. Moreover Drozd [Dro80] showed that every finite dimensional $\mathbf{k}$-algebra is either tame or wild, and these two types of algebras are mutually exclusive (c.f.[SS07, Ch.XIX, Theorem 3.4]). This tame-wild dichotomy was only proved when the field $\mathbf{k}$ is algebraically closed.

Remark 1.28. A quiver Q (resp. bound quiver (Q, I)) is said to be of finite representation type, infinite representation type, tame or wild if its path algebra $\mathbf{k}Q$ (resp. bound path algebra $\mathbf{k}Q/I$) is of that type.

Example 1.29. Some examples of the different representation types of a quiver are given by:

- $1 \longrightarrow 2$ is of finite type;
- $1 \rightrightarrows 2$ is of tame type;
- $1 \mathrel{\substack{\rightarrow \\ \rightarrow \\ \rightarrow}} 2$ is of wild type.

1.1.2. Reflection functors

We present the notion of reflection functors, firstly introduced in [BGP73], following the paper of Henning Krause [Kra08]. These functors form a basic tool for classifying representations in terms of their dimension vectors.

Definition 1.30. A vertex i of Q is called a *sink* (resp. *source*) if there is no arrow in Q starting (resp. ending) at i.

Given any vertex i, the quiver $\sigma_i Q$ is obtained from Q by reversing all arrows which start or end at i.

Definition 1.31 ([Kra08, Section 3.2])**.** Let Q be a finite quiver.

- Let $n = \operatorname{card}(Q_0)$. The *Euler form* is the bilinear form

  $$\langle -,- \rangle \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z} \quad \text{with} \quad \langle x,y \rangle = \sum_{i\in Q_0} x_i y_i - \sum_{\alpha\in Q_1} x_{s(\alpha)} y_{t(\alpha)}.$$

  We obtain on $\mathbb{Z}^n$ the *symmetric Euler form* defining

  $$(x,y) \coloneqq \langle x,y \rangle + \langle y,x \rangle. \tag{1.1}$$

- For the set $\mathbb{Z}^n$ we use the *partial order* which is defined as follows:

  $$x \leq y \quad \Longleftrightarrow \quad x_i \leq y_i \quad \text{for all} \quad i$$

  and $x < y$ if $x_i \leq y_i$ and there exists i such that $x_i < y_i$.

- Suppose that Q has no *loops* (that is, arrows from a vertex to itself). The *reflection* with respect to a vertex i is the map $\sigma_i \colon \mathbb{Z}^n \to \mathbb{Z}^n$ defined by $\sigma_i(x) = x - \frac{2(x,e_i)}{(e_i,e_i)} e_i$ where e_i is the i-th coordinate vector.

It is easily checked that the σ_i are automorphisms of order two preserving the bilinear form $(-,-)$.

Definition 1.32 ([Kra08, Definition 3.3])**.** Let i be a vertex of Q. We define a pair of *reflection functors* $S_i^+, S_i^- \colon \operatorname{Rep}_{\mathbf{k}}(Q) \to \operatorname{Rep}_{\mathbf{k}}(\sigma_i Q)$. To this end fix representations $X = (X_j, \varphi_\alpha)$, $X' = (X'_j, \psi_\alpha)$ of Q and a morphism $f \colon X \to X'$.

(1) If the vertex i is a sink of Q, then we construct S_i^+ as follows. We define $S_i^+(X) = Y$ by letting $Y_j = X_j$ for a vertex $j \neq i$, and letting Y_i be the kernel of the map $\xi^+ = (\varphi_\alpha)$ in the following sequence

$$Y_i \xrightarrow{\check{\xi}} \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X_{s(\alpha)} \xrightarrow{\xi^+} X_i$$

where $\check{\xi}$ denotes the inclusion map of the kernel. For an arrow α in Q, we define the $\mathbf{k}$-linear maps η_α of the representation Y in the following way: $\eta_\alpha = \varphi_\alpha$ if $t(\alpha) \neq i$, and $\eta_\alpha \colon Y_i \to X_{s(\alpha)} = Y_{s(\alpha)}$ be the map $\check{\xi}$ followed by the canonical projection onto $X_{s(\alpha)}$ if $t(\alpha) = i$. For the morphism $S_i^+(f) = g$, let $g_j = (f)_j$ if $j \neq i$ and let $g_i \colon Y_i \to Y'_i$ be the restriction of the map

$$(f_{s(\alpha)}) \colon \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X_{s(\alpha)} \to \bigoplus_{\substack{\alpha\in Q_1\\ t(\alpha)=i}} X'_{s(\alpha)}$$

(2) If the vertex i is a source of Q, then we construct dually S_i^- as follows. We define $S_i^-(X) = Y$ by letting $Y_j = X_j$ for a vertex $j \neq i$, and letting Y_i be the cokernel of the map $\xi^- = (\varphi_\alpha)$ in the following sequence

$$X_i \xrightarrow{\xi^-} \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X_{t(\alpha)} \xrightarrow{\hat{\xi}} Y_i$$

where $\hat{\xi}$ denotes the canonical map onto the cokernel. For an arrow α in Q, we define the **k**-linear maps η_α of the representation Y in the following way: let $\eta_\alpha = \varphi_\alpha$ if $s(\alpha) \neq i$, and $\eta_\alpha \colon Y_{t(\alpha)} = X_{t(\alpha)} \to Y_i$ be the restriction of $\hat{\xi}$ to $X_{t(\alpha)}$ if $s(\alpha) = i$. For the morphism $S_i^-(f) = g$, let $g_j = f_j$ if $j \neq i$ and let $g_i \colon Y_i \to Y_i'$ be the map which is induced by

$$(f_{t(\alpha)}) \colon \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X_{t(\alpha)} \to \bigoplus_{\substack{\alpha \in Q_1 \\ s(\alpha)=i}} X'_{t(\alpha)}$$

(3) Let i be a sink of Q. Then we define a natural monomorphism

$$\iota_i X \colon S_i^- S_i^+(X) \to X \tag{1.2}$$

by letting $(\iota_i X)_j = 1_{X_j}$ for a vertex $j \neq i$, and letting $(\iota_i X)_i$ be the canonical map

$$(S_i^- S_i^+(X))_i = \operatorname{coker} \check{\xi} \cong \operatorname{im} \xi^+ \to X_i.$$

(4) Let i be a source of Q. Then we define a natural epimorphism

$$\pi_i X \colon X \to S_i^+ S_i^-(X) \tag{1.3}$$

by letting $(\pi_i X)_j = 1_{X_j}$ for a vertex $j \neq i$, and letting $(\pi_i X)_i$ be the canonical map

$$X_i \to \operatorname{im} \xi^- \cong \ker \hat{\xi} = (S_i^+ S_i^-(X))_i$$

Remark 1.33. The definition of reflection functors can be generalised to bound quivers (c.f. [APR79; BB06]). For the purpose of this thesis we will use the previous definition of reflection functors also on bound quivers (Q, I), assuming that we never change arrows involved in the relations in I. In this way all the following results still hold.

Lemma 1.34 ([Kra08, Lemma 3.3.1])**.** *For each vertex i, S_i^+ and S_i^- are functors.*

Lemma 1.35 ([Kra08, Lemma 3.3.2])**.** *Let X, X' be representations of Q and i be a vertex.*

(1) $S_i^\pm(X \oplus X') = S_i^\pm(X) \oplus S_i^\pm(X')$.

(2) $X = (S_i^- S_i^+ X) \oplus \operatorname{coker} \iota_i X$ *and* $X = (S_i^+ S_i^- X) \oplus \ker \pi_i X$.

(3) *If* $\operatorname{coker} \iota_i X = 0$, *then* $\underline{\dim}(S_i^+(X)) = \sigma_i(\underline{\dim}(X))$.

(4) *If* $\ker \pi_i X = 0$, *then* $\underline{\dim}(S_i^-(X)) = \sigma_i(\underline{\dim}(X))$.

Proof. (1) By definition $S_i^\pm$ is a functor satisfying $S_i^\pm(f+g) = S_i^\pm(f) + S_i^\pm(g)$ for any pair of parallel morphisms f, g, that is $S_i^\pm$ is an additive functor, so we conclude.

(2) The canonical map $\rho_i' \colon X_i \to \operatorname{coker} \xi^+$ has a section $\rho_i \colon \operatorname{coker} \xi^+ \to X_i$, that is, $\rho_i' \rho_i = 1_{\operatorname{coker} \xi^+}$. This gives a morphism $\rho \colon \operatorname{coker} \iota_i X \to X$ if we put $\rho_j = 0$ for $j \neq i$. It is clear that $\iota_i X \colon S_i^- S_i^+(X) \to X$ and $\rho \colon \operatorname{coker} \iota_i X \to X$ give a direct sum decomposition of X. The proof for $X = (S_i^+ S_i^-(X)) \oplus \ker \pi_i X$ is similar.

(3) If $\operatorname{coker} \iota_i X = 0$, then we have

$$\dim Y_i = \sum_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} \dim X_{s(\alpha)} - \dim X_i$$

and $\dim Y_j = \dim X_j$ for $j \neq i$. Thus $\underline{\dim}(Y) = \sigma_i(\underline{\dim}(X))$. The proof of (4) is similar. □

Note that the representations $\operatorname{coker} \iota_i X$ and $\ker \pi_i X$ are concentrated at the vertex i. Thus they are direct sums of copies of the simple representation $S(i)$.

Lemma 1.36 ([Kra08, Lemma 3.3.3])**.** *Let i be a sink and $X = (X_j, \varphi_\alpha)$ an indecomposable representation of Q. Then the following are equivalent:*

1. $X \ncong S(i)$.
2. $S_i^+(X)$ *is indecomposable.*
3. $S_i^+(X) \neq 0$.
4. $S_i^- S_i^+(X) \cong X$.
5. *The map* $(\varphi_\alpha) \colon \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} X_{s(\alpha)} \to X_i$ *is an epimorphism.*
6. $\sigma_i(\underline{\dim}(X)) > 0$.
7. $\underline{\dim}(S_i^+(X)) = \sigma_i(\underline{\dim}(X))$.

Proof. Apply Lemma 1.35. □

Remark 1.37. There is an analogue of Lemma 1.36 for a source of Q and the corresponding functor S_i^-.

The following theorem is a consequence and summarises the basic properties of the reflection functors.

Theorem 1.38 ([Kra08, Theorem 3.3.5]). *The functors* S_i^+ *and* S_i^- *induce mutually inverse bijections between the isomorphism classes of indecomposable representations of* Q *and the isomorphism classes of indecomposable representations of* $\sigma_i Q$*, with the exception of the simple representation* $S(i)$*, which is annihilated by these functors. Moreover,* $\underline{\dim}(S_i^{\pm}(X)) = \sigma_i(\underline{\dim}(X))$ *for every indecomposable representation* X *not isomorphic to* $S(i)$*.*

Therefore, reflection functors preserve the representation type of a quiver.

We will now study how reflection functors affect morphisms of representations.

Definition 1.39. Given a pair X, Y of representations, we define the *radical*

$$\mathrm{Rad}(X,Y) = \left\{ \phi \in \mathrm{Hom}(X,Y) \,\middle|\, \begin{array}{c} \tau\phi\sigma \text{ is non-invertible for every pair } Z \xrightarrow{\sigma} X \\ \text{and } Y \xrightarrow{\tau} Z \text{ with } Z \text{ indecomposable} \end{array} \right\}$$

We extend this definition recursively for each $n \geq 0$ as follows. Let $\mathrm{Rad}^0(X,Y) = \mathrm{Hom}(X,Y)$ and for $n > 0$ let $\mathrm{Rad}^n(X,Y)$ be the set of morphisms $\phi \in \mathrm{Hom}(X,Y)$ which admit a factorisation $\phi = \phi''\phi'$ with $\phi' \in \mathrm{Rad}(X,Z)$ and $\phi'' \in \mathrm{Rad}^{n-1}(Z,Y)$ for some representation Z. Note that $\mathrm{Rad}^1(X,Y) = \mathrm{Rad}(X,Y)$.

Lemma 1.40 ([Kra08, Lemma 2.3.1]). *Let* X, Y *be a pair of representations.*

(1) $\mathrm{Rad}(X,Y)$ *is a subspace of* $\mathrm{Hom}(X,Y)$.

(2) $\mathrm{Rad}(X, Y_1 \oplus Y_2) = \mathrm{Rad}(X,Y_1) \oplus \mathrm{Rad}(X,Y_2)$.

(3) $\mathrm{Rad}(X_1 \oplus X_2, Y) = \mathrm{Rad}(X_1,Y) \oplus \mathrm{Rad}(X_2,Y)$.

(4) *If* X *and* Y *are indecomposable, then* $\mathrm{Hom}(X,Y) \setminus \mathrm{Rad}(X,Y)$ *equals the set of isomorphisms* $X \to Y$.

Proof. (1) Let $\phi_1, \phi_2 \in \mathrm{Rad}(X,Y)$. Choose $\sigma \in \mathrm{Hom}(Z,X)$ and $\tau \in \mathrm{Hom}(Y,Z)$ with Z indecomposable. Then $\tau\phi_1\sigma$ and $\tau\phi_2\sigma$ are non-invertible, and therefore $\tau(\phi_1 + \phi_2)\sigma = \tau\phi_1\sigma + \tau\phi_2\sigma$ is non-invertible, since $\mathrm{End}(Z)$ is local by Proposition 1.25. Thus $\phi_1 + \phi_2$ belongs to $\mathrm{Rad}(X,Y)$.

(2) Let $Y = Y_1 \oplus Y_2$ and $\phi = (\phi_i) \in \mathrm{Hom}(X,Y)$ with $\phi_i \in \mathrm{Hom}(X,Y_i)$ for $i = 1,2$. Choose $\sigma \in \mathrm{Hom}(Z,X)$ and $\tau = (\tau_i) \in \mathrm{Hom}(Y,Z)$ with Z indecomposable and $\tau_i \in \mathrm{Hom}(Y_i,Z)$ for $i = 1,2$. Then $\tau\phi\sigma = \tau_1\phi_1\sigma + \tau_2\phi_2\sigma$.

If $\phi_i \in \mathrm{Rad}(X,Y_i)$ for $i = 1,2$, then $\tau_i\phi_i\sigma$ is non-invertible for $i = 1,2$, and therefore $\tau\phi\sigma$ is non-invertible, since $\mathrm{End}(Z)$ is local by Proposition 1.25. Thus ϕ belongs to $\mathrm{Rad}(X,Y)$. Conversely, let $\phi \in \mathrm{Rad}(X,Y)$ and fix $i \in \{1,2\}$. Then

$\phi_i \in \operatorname{Rad}(X, Y_i)$ because we can put $\tau_j = 0$ for $j \neq i$ and have that $\tau_i \phi_i \sigma = \tau \phi \sigma$ is non-invertible.

(3) Analogous to part (2).

(4) Let $\phi \in \operatorname{Hom}(X, Y) \setminus \operatorname{Rad}(X, Y)$. Choose $\sigma \in \operatorname{Hom}(Z, X)$ and $\tau \in \operatorname{Hom}(Y, Z)$ with Z indecomposable such that $\tau\phi\sigma$ is invertible. Then σ is invertible because X is indecomposable, and τ is invertible because Y is indecomposable. Thus ϕ is invertible. It is clear that an isomorphism $X \to Y$ does not belong to $\operatorname{Rad}(X, Y)$. □

Lemma 1.41 ([Kra08, Lemma 6.1.1]). *Let X, Y, Z be representations and $m, n \geq 0$.*

1. $\operatorname{Rad}^{n+1}(X, Y)$ *is a subspace of* $\operatorname{Rad}^n(X, Y)$.
2. *For each finite set of representations X_i and Y_j, we have*
$$\operatorname{Rad}^n\Big(\bigoplus_i X_i, \bigoplus_j Y_j\Big) = \bigoplus_{i,j} \operatorname{Rad}^n(X_i, Y_j).$$
3. *If $\phi \in \operatorname{Rad}^n(X, Y)$ and $\psi \in \operatorname{Rad}^m(Y, Z)$, then $\psi\phi \in \operatorname{Rad}^{n+m}(X, Z)$.*

Proof. It can be easily proven by induction using Lemma 1.40. □

Lemma 1.42 ([Kra08, Lemma 7.3.1]). *Let i be a sink and X, Y indecomposable representations not isomorphic to $S(i)$. Then S_i^+ induces isomorphisms*

$$\operatorname{Rad}^n(X, Y) \xrightarrow{\sim} \operatorname{Rad}^n(S_i^+(X), S_i^+(Y)) \quad \textit{for} \quad n \geq 0$$

In particular, S_i^+ induces an isomorphism

$$\operatorname{Hom}(X, Y) \xrightarrow{\sim} \operatorname{Hom}(S_i^+(X), S_i^+(Y))$$

Proof. We use the natural morphism $\iota_i Z \colon S_i^- S_i^+(Z) \to Z$ (1.2) which is defined for any representation Z; it is a split monomorphism by Lemma 1.35. Thus we can identify $S_i^- S_i^+(X) = X$ and $S_i^- S_i^+(Y) = Y$. Using this identification the inverse for $\operatorname{Hom}(X, Y) \to \operatorname{Hom}(S_i^+(X), S_i^+(Y))$ sends $\psi \in \operatorname{Hom}(S_i^+(X), S_i^+(Y))$ to $S_i^-(\psi)$. Now fix $\phi \in \operatorname{Hom}(X, Y)$. Clearly, ϕ is an isomorphism if and only if $S_i^+(\phi)$ is an isomorphism. Thus S_i^+ induces a bijection

$$\operatorname{Rad}^1(X, Y) \xrightarrow{\sim} \operatorname{Rad}^1(S_i^+(X), S_i^+(Y)).$$

Next we suppose $\phi \in \operatorname{Rad}^n(X, Y)$ and $n > 1$. Then ϕ admits a factorisation $\phi = \phi''\phi'$ with $\phi' \in \operatorname{Rad}^1(X, Z)$ and $\phi'' \in \operatorname{Rad}^{n-1}(Z, Y)$ for some representation Z. We know by induction that $S_i^+(\phi') \in \operatorname{Rad}^1(S_i^+(X), S_i^+(Z))$ and $S_i^+(\phi'') \in \operatorname{Rad}^{n-1}(S_i^+(Z), S_i^+(Y))$. Thus $S_i^+(\phi) \in \operatorname{Rad}^n(S_i^+(X), S_i^+(Y))$. The same argument shows that S_i^- maps $\operatorname{Rad}^n(S_i^+(X), S_i^+(Y))$ to $\operatorname{Rad}^n(X, Y)$. This establishes for all $n > 1$ the isomorphism

$$\operatorname{Rad}^n(X, Y) \xrightarrow{\sim} \operatorname{Rad}^n(S_i^+(X), S_i^+(Y))$$ □

Remark 1.43. There is an analogue of Lemma 1.42 for a source of Q and the corresponding functor S_i^-.

We recall the following definition:

Definition 1.44. Let Λ be a finite dimensional $\mathbf{k}$-algebra. A Λ-module B is called a *brick* if its endomorphism algebra $\operatorname{End}_\Lambda(B)$ is a division algebra over $\mathbf{k}$. If $B \in \operatorname{mod}(\Lambda)$ it is a brick if $\operatorname{End}_\Lambda(B) \cong \mathbf{k}$.

A similar definition holds for representations of a bound quiver (Q, I).

Lemma 1.45. *If $X \ncong S(i)$ is a brick then $S_i^\pm(X)$ is a brick.*

Proof. It follows immediately from Lemma 1.42 and Remark 1.43: indeed we have that $\operatorname{Hom}(X, Y) \cong \operatorname{Hom}(S_i^\pm(X), S_i^\pm(Y))$ and in particular if $Y = X$ it means that $\operatorname{End}(X) \cong \operatorname{End}(S_i^\pm(X))$. So if $X \ncong S(i)$ is a brick the same is true for $S_i^\pm(X)$. □

1.1.3. Auslander-Reiten translation

We will now introduce the concepts of Auslander-Reiten translation and Auslander-Reiten quiver that will be useful for a better understanding of next section on τ-tilting theory. The reader can find more details on the following definitions and results in [ASS06; Ang17; Sim23].

Definition 1.46. Let Λ be a finite dimensional $\mathbf{k}$-algebra. Given a left Λ-module M, we define the Λ-*dual functor* $(\text{-})^* = \operatorname{Hom}_\Lambda(\text{-}, \Lambda)\colon \operatorname{Mod}(\Lambda) \to \operatorname{Mod}(\Lambda^{\mathrm{op}})$ (where $\operatorname{Mod}(\Lambda^{\mathrm{op}})$ denotes the category of right Λ-modules) in the following way:

$$M^* \coloneqq \operatorname{Hom}_\Lambda(M, \Lambda)$$

We also define the *standard duality* functor $D(\text{-}) = \operatorname{Hom}_{\mathbf{k}}(\text{-}, \mathbf{k})\colon \operatorname{Mod}(\Lambda^{\mathrm{op}}) \to \operatorname{Mod}(\Lambda)$ as

$$D(M) \coloneqq \operatorname{Hom}_{\mathbf{k}}(M, \mathbf{k})$$

and the *Nakayama functor* $\nu\colon \operatorname{Mod}(\Lambda) \to \operatorname{Mod}(\Lambda)$ as

$$\nu(M) \coloneqq D(M^*)$$

It is clear that $(\text{-})^*$ can be restricted to a functor from $\operatorname{mod}(\Lambda)$ to $\operatorname{mod}(\Lambda^{\mathrm{op}})$, D from $\operatorname{mod}(\Lambda^{\mathrm{op}})$ to $\operatorname{mod}(\Lambda)$ and ν from $\operatorname{mod}(\Lambda)$ to $\operatorname{mod}(\Lambda)$.

Let $M \in \operatorname{mod}(\Lambda)$ and let

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

be a minimal projective presentation of M. Applying the Λ-dual functor on it, we obtain a minimal projective presentation

$$P_0^* \xrightarrow{p_1^*} P_1^* \to \operatorname{coker}(p_1^*) \to 0$$

Set $\operatorname{Tr}(M) \coloneqq \operatorname{coker}(p_1^*)$. The isomorphism class of $\operatorname{Tr}(M)$ does not depend on the choice of the projective presentation of M.

It is possible to see [Ang17, Lemma 7.3.1] that given a projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ of M, there is an exact sequence

$$0 \to D\operatorname{Tr}(M) \to \nu(P_1) \xrightarrow{\nu(p_1)} \nu(P_0) \to \nu(M) \to 0$$

Definition 1.47. We define the *Auslander-Reiten translation* τ (or *AR translation*) of a module M as

$$\tau(M) \coloneqq D\operatorname{Tr}(M) = \ker(\nu(p_1))$$

Definition 1.48. For $M, N \in \operatorname{mod}(\Lambda)$ we define

$$P(M,N) \coloneqq \{f \in \operatorname{Hom}_\Lambda(M,N) \mid f \text{ factors through a projective module}\}$$

It is possible to see that $P(M,N) \leq \operatorname{Hom}_\Lambda(M,N)$ is a subgroup.

We set $\underline{\operatorname{Hom}}_\Lambda(M,N) \coloneqq \operatorname{Hom}_\Lambda(M,N)/P(M,N)$. and let $\underline{\operatorname{mod}}(\Lambda)$ be the category with the same objects as $\operatorname{mod}(\Lambda)$ and morphisms $\underline{\operatorname{Hom}}_\Lambda(M,N)$. It is called the *stable category* of $\operatorname{mod}(\Lambda)$ modulo projectives.

Dually, we define

$$I(M,N) \coloneqq \{f \in \operatorname{Hom}_\Lambda(M,N) \mid f \text{ factors through an injective module}\}$$

It is possible to see that $I(M,N) \leq \operatorname{Hom}_\Lambda(M,N)$ is a subgroup.

We set $\overline{\operatorname{Hom}}_\Lambda(M,N) \coloneqq \operatorname{Hom}_\Lambda(M,N)/I(M,N)$. and let $\overline{\operatorname{mod}}(\Lambda)$ be the category with the same objects as $\operatorname{mod}(\Lambda)$ and morphisms $\overline{\operatorname{Hom}}_\Lambda(M,N)$. It is called the *costable category* of $\operatorname{mod}(\Lambda)$ modulo injectives.

Theorem 1.49 (Prop. 7.4.1 [Ang17])**.** *The AR translation define an equivalence*

$$\tau = D\operatorname{Tr}\colon \underline{\operatorname{mod}}(\Lambda) \to \overline{\operatorname{mod}}(\Lambda)$$

with inverse

$$\tau^{-1} = \operatorname{Tr} D\colon \overline{\operatorname{mod}}(\Lambda) \to \underline{\operatorname{mod}}(\Lambda)$$

Definition 1.50 ([Sim23])**.** A morphism $f\colon M \to N$ is said to be *irreducible* if:

- f is not a split monomorphism;

- f is not a split epimorphism;
- if $f = gh$, then h is a split monomorphism or g is a split epimorphism.

It is possible to see that an irreducible morphism is either injective or surjective, but not both.

Definition 1.51 ([Sim23])**.** The *Auslander-Reiten quiver* (or *AR-quiver*) $\Gamma(\mathrm{mod}(\Lambda))$ of $\mathrm{mod}(\Lambda)$ is defined by:

- the vertices of $\Gamma(\mathrm{mod}(\Lambda))$ are the isomorphism classes of indecomposable modules in $\mathrm{mod}(\Lambda)$;
- the arrows are the irreducible morphisms between the indecomposable modules.

Example 1.52. The AR-quiver of $\mathrm{mod}(\mathbf{k}Q)$, where Q is the quiver $1 \longrightarrow 2 \longrightarrow 3$, is given by

$$\begin{tikzcd}
 & & \mathbf{k} \xrightarrow{1} \mathbf{k} \xrightarrow{1} \mathbf{k} \arrow[dr] & & \\
 & 0 \rightarrow \mathbf{k} \xrightarrow{1} \mathbf{k} \arrow[ur] \arrow[dr] & & \mathbf{k} \xrightarrow{1} \mathbf{k} \rightarrow 0 \arrow[dr] & \\
0 \rightarrow 0 \rightarrow \mathbf{k} \arrow[ur] & & 0 \rightarrow \mathbf{k} \rightarrow 0 \arrow[ur] & & \mathbf{k} \rightarrow 0 \rightarrow 0
\end{tikzcd}$$

1.2. τ-tilting theory

We recall some definitions of τ-tilting theory, that was developed by Adachi-Iyama-Reiten [AIR14]. Let Λ be a finite-dimensional $\mathbf{k}$-algebra. We recall that:

- a Λ-module is *basic* if no indecomposable direct summand is repeated in its decomposition into indecomposable direct summands;
- a pair (M, P) of Λ-modules is *basic* if both M and P are basic as Λ-modules.

Definition 1.53. Let Λ be a finite-dimensional algebra over a field $\mathbf{k}$.

1. A module $M \in \mathrm{mod}(\Lambda)$ is said to be *rigid* if $\mathrm{Ext}^1_\Lambda(M,\, M) = 0$.
2. A module $M \in \mathrm{mod}(\Lambda)$ is said to be *τ-rigid* if $\mathrm{Hom}_\Lambda(M,\, \tau M) = 0$, where τ is the Auslander–Reiten translation of Λ.
   The set of isomorphism classes of basic τ-rigid Λ-modules is denoted by $\tau\text{-rigid}\,(\Lambda)$.

3. A module $M \in \mathrm{mod}(\Lambda)$ is said to be *τ-tilting* if M is τ-rigid and $|M| = |\Lambda|$, where $|X|$ is the number of non-isomorphic indecomposable direct summands of a left Λ-module X.
   The set of isomorphism classes of basic τ-tilting modules in $\mathrm{mod}(\Lambda)$ is denoted by $\tau\text{-tilt}\,(\Lambda)$.

4. A pair $(M,\, P)$ is *support τ-rigid* if M is a τ-rigid module and P is a finitely generated projective Λ-module such that $\mathrm{Hom}_\Lambda(P,\, M) = 0$.
   The set of isomorphism classes of basic τ-rigid pairs in $\mathrm{mod}(\Lambda)$ is denoted by $\tau\text{-rigid}\,\text{pair}\,(\Lambda)$.

5. A support τ-rigid pair $(M,\, P)$ is *support τ-tilting* if $|M| + |P| = |\Lambda|$.
   The set of isomorphism classes of basic support τ-tilting pairs in $\mathrm{mod}(\Lambda)$ is denoted by $\mathrm{s}\tau\text{-tilt}\,\text{pair}\,(\Lambda)$.

6. A module $M \in \mathrm{mod}(\Lambda)$ is *support τ-tilting* if there exists a projective left Λ-module P such that $(M,\, P)$ is a support τ-tilting pair.
   The set of isomorphism classes of basic support τ-tilting Λ-modules is denoted by $\mathrm{s}\tau\text{-tilt}\,(\Lambda)$.

7. A support τ-rigid pair $(N,\, Q)$ is a *direct summand* of the support τ-rigid pair $(M,\, P)$ if N is a direct summand of M and Q is a direct summand of P.

Definition 1.54 ([DIJ17])**.** We say that a finite-dimensional algebra Λ is *τ-tilting finite* if the set $\tau\text{-tilt}\,(\Lambda)$ is finite. If Λ is not τ-tilting finite, it is said to be *τ-tilting infinite.*

Theorem 1.55 (c.f. Corollary 2.9 [DIJ17])**.** *For a finite dimensional algebra Λ the following are equivalent:*

(1) Λ is τ-tilting finite;

(2) the set $\tau\text{-rigid}\,(\Lambda)$ is finite;

(3) the set $\mathrm{s}\tau\text{-tilt}\,(\Lambda)$ is finite.

Proof. (1) $\Rightarrow$ (2) Since Λ is τ-tilting finite, the set $\tau\text{-tilt}\,(\Lambda)$ is finite and consequently, up to isomorphism, we have only a finite number of direct summands of τ-tilting modules. Suppose by contradiction that the set $\tau\text{-rigid}\,(\Lambda)$ is infinite. Then by [AIR14, Theorem 0.2] any τ-rigid module is a direct summand of a τ-tilting module, but this leads to a contradiction. Therefore the set $\tau\text{-rigid}\,(\Lambda)$ must be finite.

(2) $\Rightarrow$ (3) Suppose by contradiction that the set $\mathrm{s}\tau\text{-tilt}\,(\Lambda)$ is infinite. By definition any support τ-tilting module is τ-rigid, therefore the set $\tau\text{-rigid}\,(\Lambda)$ cannot be finite, contradiction. Thus the set $\mathrm{s}\tau\text{-tilt}\,(\Lambda)$ is finite.

(3) $\Rightarrow$ (1) Suppose by contradiction that the set $\tau\text{-tilt}\,(\Lambda)$ is infinite. Any τ-tilting module M is support τ-tilting because the pair $(M, 0)$ is a support τ-tilting pair. Therefore we would have that the set $\text{s}\tau\text{-tilt}\,(\Lambda)$ is infinite, contradiction. Thus the set $\tau\text{-tilt}\,(\Lambda)$ is finite, i.e. Λ is τ-tilting finite. $\square$

Remark 1.56. For a finite-dimensional **k**-algebra Λ and a finite-dimensional Λ-module M, we denote by $\text{add}(M)$ the smallest additive subcategory of $\text{mod}(\Lambda)$ containing M.

Definition 1.57. Let Λ be a finite-dimensional **k**-algebra. We say that $T \in \text{mod}(\Lambda)$ is a *tilting* Λ*-module* if the following hold:

1. The Λ-module T is rigid.
2. $\text{pd}_\Lambda(T) \leq 1$, i.e. there exists a short-exact sequence in $\text{mod}(\Lambda)$
$$0 \longrightarrow P_1^T \longrightarrow P_0^T \longrightarrow T \longrightarrow 0$$
where the terms P_1^T and P_0^T are finite-dimensional projective Λ-modules.
3. There exists a short-exact sequence in $\text{mod}(\Lambda)$
$$0 \longrightarrow \Lambda \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$
where the terms T_0 and T_1 are in $\text{add}(T)$.

We recall that if a **k**-algebra Λ is representation-infinite, the set of indecomposable Λ-modules can be partitioned into three distinct classes:

- the *postprojective* Λ*-modules*, namely the Λ-modules M for which there exists an indecomposable finite-dimensional projective Λ-module P and a non-negative integer i such that $M \simeq \tau^{-i}P$.
- the *preinjective* Λ*-modules*, namely the Λ-modules M for which there exists an indecomposable finite-dimensional injective Λ-module I and a non-negative integer i such that $M \simeq \tau^{i}I$.
- the *regular* Λ*-modules*, that are neither postprojective nor preinjective.

Definition 1.58 ([BGR24])**.** Let Λ be a finite-dimensional **k**-algebra.

1. We say that Λ *has a postprojective component* if the Auslander–Reiten quiver of $\text{mod}(\Lambda)$ has an acyclic connected component in which every indecomposable module is isomorphic to a module of the form $\tau^{-i}P$ for some indecomposable finite-dimensional projective Λ-module P and some non-negative integer i.

2. We say that Λ *has a preinjective component* if the Auslander–Reiten quiver of $\mathrm{mod}(\Lambda)$ has an acyclic connected component in which every indecomposable module is isomorphic to a module of the form $\tau^i I$ for some indecomposable finite-dimensional injective Λ-module I and some non-negative integer i.

The path $\mathbf{k}$-algebra $\mathbf{k}Q$ of a quiver Q admit postprojective and preinjective components. For each connected component of Q there is a postprojective component in the AR-quiver of $\mathbf{k}Q$, and likewise for the preinjectives. Such a connected component Q' of Q yields an infinite postprojective (resp. preinjective) component if and only if Q' is a representation-infinite quiver.

Definition 1.59 ([BGR24, Definition 2.5])**.** Let Q be an acyclic quiver.

1. A *tilted* $\mathbf{k}$*-algebra of type* Q is isomorphic to $\mathrm{End}_{\mathbf{k}Q}(T)$ for some tilting $\mathbf{k}Q$-module T.

2. A *concealed* $\mathbf{k}$*-algebra of type* Q is a tilted $\mathbf{k}$-algebra isomorphic to $\mathrm{End}_{\mathbf{k}Q}(T)$ with T is a postprojective tilting $\mathbf{k}Q$-module.

3. Let $B = \mathrm{End}_{\mathbf{k}Q}(T)$ be a concealed $\mathbf{k}$-algebra of type Q. We say that B is *tame concealed* (resp. *wild concealed*) if Q is a tame and representation-infinite (resp. wild) quiver.

Remark 1.60. We recall that given a finite connected acyclic quiver Q with representation-infinite path algebra, Q is tame if and only if its underlying graph is an Euclidean diagram $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ (c.f. [SS07, Ch.XIX, Theorem 3.15]).

Following [ASS06; SS06; SS07], tame concealed algebras are also called *concealed algebras of Euclidean type* and they have been completely classified by Happel-Vossieck [HV83].

We also have the following result, justifying the name "tame concealed":

Theorem 1.61. *A tame concealed algebra is representation-infinite and tame.*

Proof. It follows directly from the classification of tame concealed algebras that they are representation-infinite (see [HV83][Rin06, Section 4.3, Proposition 7]). Moreover, by [SS07, Ch.XIX, Theorem 3.14] they are also tame. □

Let Q be a representation-infinite acyclic quiver, and let B be a concealed $\mathbf{k}$-algebra of type Q. Then B admits one postprojective component (resp. preinjective component) for every connected component of Q (c.f.[SS07, Ch.XX, Corollary 3.3]), and at least one which contains (countably) infinitely many indecomposable B-modules (c.f.[ASS06, Ch.VIII, Corollary 2.3]).

Lemma 1.62 (c.f. Remark 2.9 [Mou23])**.** *Let B be a tame concealed or a wild concealed* $\mathbf{k}$*-algebra. Then B is τ-tilting infinite.*

Proof. Tame concealed and wild concealed $\mathbf{k}$-algebras admit at least one infinite postprojective (and at least one infinite preinjective) component. It follows easily from directedness (c.f. [ASS06, Ch.VIII, Lemma 2.3]) that the indecomposable modules constituting postprojective or preinjective components are all τ-rigid (c.f. [ASS06, Ch.VIII, Lemma 2.7]). Thus, the postprojective (or preinjective) component of B provides a (countably) infinite set of τ-rigid B-modules. Therefore by Theorem 1.55 B is τ-tilting infinite. □

2. Incidence algebras

We introduce now the notion of incidence algebra of a given poset. Then we will study some of its properties in a quiver-theoretic way.

2.1. Basic definitions and properties

Definition 2.1. A *partially ordered set*, or *poset* for short, is a pair $(P, \leq)$, where P is a set and $\leq$ is a binary relation on P which is reflexive, transitive and anti-symmetric. When the binary relation $\leq$ is clear from context, or if the poset in question is arbitrary, we may omit $\leq$ from the notation, so a reference to the underlying set P is meant as reference to the entire structure.

A (closed) *interval* in a poset P is a subposet of the form

$$[a, b] \coloneqq \{x \mid a \leq x \leq b\} \subseteq P,$$

where $a, b \in P$. We denote the set of non-empty intervals in P by $\operatorname{int}(P)$.

A poset is *locally finite* if every interval therein is finite and it is *finite* if the set P is finite.

Definition 2.2 ([BGR24, Definition 1.1])**.** Let $(P, \leq)$ be a locally finite poset. Denote the set of functions from $\operatorname{int}(P)$ to $\mathbf{k}$ by $\mathbf{k}^{\operatorname{int}(P)}$, and endow this set with a $\mathbf{k}$-vector space structure using pointwise operations. The *incidence* $\mathbf{k}$-*algebra* of P has $\mathbf{k}^{\operatorname{int}(P)}$ as the underlying $\mathbf{k}$-vector space, and multiplication

$$\mathbf{k}^{\operatorname{int}(P)} \otimes_{\mathbf{k}} \mathbf{k}^{\operatorname{int}(P)} \xrightarrow{\ *\ } \mathbf{k}^{\operatorname{int}(P)}$$

given by $(f * g)([a, b]) \coloneqq \sum_{t \in [a,b]} f([t, b])g([a, t])$, that is well defined because each interval $[a, b]$ is finite. We denote the incidence $\mathbf{k}$-algebra of P by $\mathcal{I}_{\mathbf{k}}(P)$.

- $*$ is associative: indeed we have that

$$\begin{aligned}((f*g)*h)([a,b]) &= \sum_{t\in[a,b]} (f*g)([t,b])h([a,t]) \\ &= \sum_{t\in[a,b]} \left(\sum_{k\in[t,b]} f([k,b])g([t,k]) \right) h([a,t]) \\ &= \sum_{t\in[a,b]} \sum_{k\in[t,b]} f([k,b])g([t,k])h([a,t]) \\ &= \sum_{a\le t\le k\le b} f([k,b])g([t,k])h([a,t])\end{aligned}$$

  while

$$\begin{aligned}(f*(g*h))([a,b]) &= \sum_{t\in[a,b]} f([t,b])(g*h)([a,t]) \\ &= \sum_{t\in[a,b]} f([t,b]) \left(\sum_{k\in[a,t]} g([k,t])h([a,k]) \right) \\ &= \sum_{t\in[a,b]} \sum_{k\in[a,t]} f([t,b])g([k,t])h([a,k]) \\ &= \sum_{a\le k\le t\le b} f([t,b])g([k,t])h([a,k])\end{aligned}$$

  Therefore, up to changing the role of t and k in the second sum, we conclude that $((f*g)*h)([a,b]) = (f*(g*h))([a,b])$ for each interval $[a,b]$, thus they coincide as functions $(f*g)*h = f*(g*h)$.

- $\mathcal{I}_{\mathbf{k}}(P)$ has a unit: we define $1_{\mathcal{I}_{\mathbf{k}}(P)}\colon \operatorname{int}(P) \to \mathbf{k}$ as

$$1_{\mathcal{I}_{\mathbf{k}}(P)}([a,b]) := \begin{cases} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{cases}$$

  Indeed, given $f \in \mathbf{k}^{\operatorname{int}(P)}$, for each interval $[a,b]$ we have

$$(1_{\mathcal{I}_{\mathbf{k}}(P)} * f)([a,b]) = \sum_{t\in[a,b]} 1_{\mathcal{I}_{\mathbf{k}}(P)}([t,b])f([a,t]) = 1_{\mathcal{I}_{\mathbf{k}}(P)}([b,b])f([a,b]) = f([a,b])$$

  and also

$$(f*1_{\mathcal{I}_{\mathbf{k}}(P)})([a,b]) = \sum_{t\in[a,b]} f([t,b])1_{\mathcal{I}_{\mathbf{k}}(P)}([a,t]) = f([a,b])1_{\mathcal{I}_{\mathbf{k}}(P)}([a,a]) = f([a,b])$$

  Therefore $1_{\mathcal{I}_{\mathbf{k}}(P)} * f = f$ and $f * 1_{\mathcal{I}_{\mathbf{k}}(P)} = f$.

Example 2.3. Consider the poset $(\mathbb{N}, \leq)$ with the canonical order of the natural numbers. The intervals $\operatorname{int}(\mathbb{N})$ are given by $[n,m] = \{n, n+1, \dots, m-1, m\}$ for $n \leq m$ in $\mathbb{N}$. We have the incidence algebra $\mathcal{I}_{\mathcal{C}}(\mathbb{N})$ whose elements are functions $f\colon \operatorname{int}(\mathbb{N}) \to \mathcal{C}$. For example lets consider f_1 that sends $[1,3]$ and $[2,3]$ to i and the rest to 0 and f_2 that sends $[1,2]$ to $3-2i$ and the rest to 0. Then we have

$$\begin{aligned}(f_1 * f_2)([1,3]) &= \sum_{t\in[1,3]} f_1([t,3])f_2([1,t]) \\ &= f_1([1,3])f_2([1,1]) + f_1([2,3])f_2([1,2]) + f_1([3,3])f_2([1,3]) \\ &= i\cdot 0 + i\cdot(3-2i) + 0\cdot 0 \\ &= 2+3i\end{aligned}$$

Lemma 2.4. *Let P be a locally finite poset. The following are equivalent:*

(1) P is finite;

(2) $\operatorname{int}(P)$ is a finite set;

(3) $\mathcal{I}_{\mathbf{k}}(P)$ is finite dimensional.

In this case, one can present the $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(P)$ as the quotient of the free $\mathbf{k}$-algebra generated by $\operatorname{int}(P)$, denoted by $\mathbf{k}\langle\operatorname{int}(P)\rangle$, modulo the relations $\{[c,d][a,b] - \delta_{b,c}[a,d]\}$, where $\delta_{-,-}$ is the Kronecker delta symbol.

Proof. (1) ⇒ (2) Suppose P is finite with n elements, i.e. $P = \{x_1, \dots, x_n\}$. Then the possible intervals are $[x_i, x_j]$ with $i,j = 1,\dots,n$, and they are at most n^2 ("at most" since they may be empty, e.g. if two elements are not comparable) because we can consider the injective map (of sets) $\operatorname{int}(P) \to P \times P$ defined by $[a,b] \mapsto (a,b)$ and so $\operatorname{card}(\operatorname{int}(P)) \leq \operatorname{card}(P\times P) = n^2$. Therefore $\operatorname{int}(P)$ is a finite set.

(2) ⇒ (3) If $\operatorname{int}(P)$ is a finite set, the elements $\chi_{[x,y]}(\text{-})\colon \operatorname{int}(P) \to \mathbf{k}$, for $[x,y] \in \operatorname{int}(P)$, where

$$\chi_{[x,y]}([a,b]) := \begin{cases} 1 & \text{if } [x,y] = [a,b] \\ 0 & \text{otherwise} \end{cases}$$

give a finite basis for $\mathcal{I}_{\mathbf{k}}(P)$. Indeed, they clearly are linearly independent and any element $f \in \mathcal{I}_{\mathbf{k}}(P)$ can be written as the finite linear combination

$$f = \sum_{[a,b]\in\operatorname{int}(P)} f([a,b])\chi_{[a,b]}$$

Therefore $\mathcal{I}_{\mathbf{k}}(P)$ is finite dimensional.

(3) $\Rightarrow$ (1) Suppose by contradiction that P is infinite, so we have an infinite number of elements in P and for each $a \in P$ we can consider the interval $[a,a] = \{a\}$. We have an infinite number of such intervals and so we have an infinite number of elements $\chi_{[a,a]}$ and each finite set of this elements is linearly independent, since the only way to obtain the zero function with a finite linear combination of such elements is to have all zeroes as coefficients. Since we can take larger and larger finite sets of these elements, $\mathcal{I}_{\mathbf{k}}(P)$ cannot be finite dimensional, contradiction. Therefore P must be finite.

In this case we have a surjective morphism of algebras $\Phi \colon \mathbf{k}\langle \mathrm{int}(P)\rangle \to \mathcal{I}_{\mathbf{k}}(P)$ defined by $[x,y] \mapsto \chi_{[x,y]}(\text{-})$ and $[c,d][a,b] \mapsto \chi_{[c,d]} * \chi_{[a,b]}$. We consider the relations $\{[c,d][a,b] - \delta_{b,c}[a,d]\}$ and denote them by $\sim$. Such relations send the words $[c,d][a,b]$ with $b \neq c$ to the empty word while sending $[b,d][a,b]$ to $[a,d]$. Therefore the elements in $\mathbf{k}\langle \mathrm{int}(P)\rangle/\sim$ are linear combinations of intervals $[x,y]$, since any concatenation of intervals $[c,d][a,b]$ is either the empty word or $[a,d]$. With this setting, we have again a surjective morphism $\widetilde{\Phi} \colon \mathbf{k}\langle \mathrm{int}(P)\rangle/\sim \to \mathcal{I}_{\mathbf{k}}(P)$, defined as Φ, but it is also injective: indeed $\widetilde{\Phi}([x,y]) = \chi_{[x,y]} = \chi_{[a,b]} = \widetilde{\Phi}([a,b])$ implies $[x,y] = [a,b]$. Therefore $\mathcal{I}_{\mathbf{k}}(P) \cong \mathbf{k}\langle \mathrm{int}(P)\rangle/\sim$. □

Notation. In the following, all posets we consider will be finite.

Definition 2.5. The *Hasse quiver* of a poset P is the quiver where the vertices are the elements in P, and we have an arrow $x \to y$ if $x \leq y$ and no element lies strictly between x and y. We will denote this quiver by Q_P.

Proposition 2.6. *Let I_P denote the ideal of the path $\mathbf{k}$-algebra $\mathbf{k}Q_P$ generated by the relations $\{p_1 - p_2\}_{p_1,p_2}$, where we index over all pairs of paths with the same start and endpoints. Then we have an isomorphism:*

$$\mathcal{I}_{\mathbf{k}}(P) \cong \mathbf{k}Q_P/I_P$$

Proof. We have surjective morphism of algebras $\Psi \colon \mathbf{k}\langle \mathrm{int}(P)\rangle \to \mathbf{k}Q_P/I_P$ defined by $[x,y] \mapsto [p_{x,y}]$, where $[p_{x,y}]$ is the equivalence class of paths from x to y in $\mathbf{k}Q_P$ given by the quotient with I_P. In the same way as done in Lemma 2.4, considering the relations $\{[c,d][a,b] - \delta_{b,c}[a,d]\}$ and denoting them by $\sim$, we have a surjective morphism $\widetilde{\Psi} \colon \mathbf{k}\langle \mathrm{int}(P)\rangle/\sim \to \mathbf{k}Q_P/I_P$, defined as Ψ, that is also injective: indeed if $\widetilde{\Psi}([x,y]) = [p_{x,y}] = [p_{a,b}] = \widetilde{\Psi}([a,b])$ then the two paths must have same start and endpoints, implying than $x = a$ and $y = b$, i.e. $[x,y] = [a,b]$. Therefore $\mathbf{k}Q_P/I_P \cong \mathbf{k}\langle \mathrm{int}(P)\rangle/\sim \cong \mathcal{I}_{\mathbf{k}}(P)$ again by Lemma 2.4, so we conclude. □

Example 2.7. Let $n \geq 1$. A poset is of *type* $\mathbb{A}_n$ if it has n elements and the underlying unoriented graph of its Hasse quiver takes the form

$$\bullet \;\text{——}\; \bullet \;\text{——}\; \cdots \;\text{——}\; \bullet. \qquad (\mathbb{A}_n)$$

The incidence **k**-algebra of such a poset is isomorphic to the path algebra of the correspondingly oriented $\mathbb{A}_n$-quiver. Examples include the totally ordered sets, whose Hasse quivers are given by

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n. \tag{$\overrightarrow{\mathbb{A}}_n$}$$

Lemma 2.8 ([SO22, Lemma 1.2.5]). *A finite poset $P = \{x_1, \ldots, x_n\}$ can be labeled so that $x_i \leq x_j$ implies that $i \leq j$.*

Proof. We use induction on the cardinality of P. If P consists of a single element, then the result is obvious. Given $n \geq 2$, assume that every poset having fewer than n elements can be so labeled. Since P is finite, P has maximal elements (if not, for any element x there would exist y with $x \leq y$, so P should be infinite). Without loss of generality we may assume that x_n is a maximal element (up to re-indexing). The subposet $P' := P\ \{x_n\}$ has less than n elements, hence can be labeled as in the statement of the Lemma. It is clear that this labeling satisfies the desired property. □

We recall that $\mathbb{M}_n(\mathbf{k})$ is the set of all $n \times n$ square matrices with coefficients in **k**. It is a **k**-algebra with respect to the usual matrix addition and multiplication.

Let P be a poset, with $P = \{a_1, ..., a_n\}$. Consider the subset

$$\mathbf{k}P := \{M = (m_{ij}) \in \mathbb{M}_n(\mathbf{k}) \mid m_{hk} = 0 \text{ if } a_h \not\leq a_k\}$$

of $\mathbb{M}_n(\mathbf{k})$ consisting of all matrices $M = (m_{ij})$ such that $m_{ij} = 0$ if the relation $a_i \leq a_j$ does not hold in P. Using Lemma 2.8 it is possible to see that there is an isomorphism of algebras between $\mathbf{k}P$ and the subalgebra $\mathbb{U}_n(\mathbf{k})$ of the upper triangular matrices (see [SO22, Proposition 1.2.7]), so $\mathbf{k}P$ is a **k**-subalgebra of $\mathbb{M}_n(\mathbf{k})$. A basis of the **k**-vector space $\mathbf{k}P$ is given by the matrices E_{ij} with $a_i \leq a_j$ in P, where E_{ij} is the matrix with 1 in position (i, j) and 0 elsewhere.

Proposition 2.9 (c.f. [ASS06, Ch.I, Example 1.1 (d)]). *Let P be a finite poset. We have an isomorphism of **k**-algebras*

$$\mathcal{I}_{\mathbf{k}}(P) \cong (\mathbf{k}P)^{\mathrm{op}}$$

Proof. By Proposition 2.6 we have that $\mathcal{I}_{\mathbf{k}}(P) \cong \mathbf{k}Q_P/I_P$. We construct a map $f\colon \mathbf{k}Q_P \to (\mathbf{k}P)^{\mathrm{op}}$ assigning to each arrow $\alpha\colon a_i \to a_j \in (Q_P)_1$ in $\mathbf{k}Q_P$ the matrix E_{ij}. Given a non-zero path $p = \alpha_n \cdots \alpha_1$, we send it to the matrix $f(p)$ defined by$f(\alpha_1) \cdots f(\alpha_n)$, i.e. $f(p) = E_{s(\alpha_1),t(\alpha_1)} \cdots E_{s(\alpha_n),t(\alpha_n)}$ (notice the change of order). We extend the map f by **k**-linearity, that is $f(\sum_{h=0}^{m} \lambda_h p_h) = \sum_{h=0}^{m} \lambda_h f(p_h)$ (with $\lambda_h \in \mathbf{k}$). It is clear that is a surjective **k**-algebra morphism. Moreover its kernel is given by the linear combinations of paths with the same start and endpoints (thanks to the fact that $E_{ij}E_{hk} = \delta_{j,h}E_{ik}$) so $\ker(f) = I_P$. Therefore the map $\mathbf{k}Q_P/I_P \to (\mathbf{k}P)^{\mathrm{op}}$ is an isomorphism, thus $\mathcal{I}_{\mathbf{k}}(P) \cong \mathbf{k}Q_P/I_P \cong (\mathbf{k}P)^{\mathrm{op}}$ and we conclude. $\square$

2.2. Representation type of incidence algebras

We will now present some definitions and notions introduced by Loupias [Lou06; Lou75] that will be necessary to prove the main result of Chapter 3, Theorem 3.7.

Definition 2.10. Let P be a poset. It is said to be:

- *connected* if the bound quiver (Q_P, I_P) is connected.
- of *finite representation type* (or *representation-finite*) if the incidence algebra $\mathcal{I}_{\mathbf{k}}(P) \cong \mathbf{k}Q_P/I_P$ is an algebra of finite representation type.

The definition of a poset not representation-finite (i.e. representation-infinite), tame and wild is analogous.

Remark 2.11. For incidence algebras of posets, the choice of field does not affect the representation type. Indeed, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(P)$ of a poset P is representation-finite for some field $\mathbf{k}$ precisely when it is representation-finite for all fields $\mathbf{k}$ [RV87, Corollary 4].

Theorem 2.12. *Let P be a poset, $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_s$, where the P_i are the connected components of P (i.e. disjoint connected subposets of P with the induced order). Then there is an equivalence of categories*

$$\mathrm{Rep}_{\mathbf{k}}(Q_P, I_P) \cong \mathrm{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1}) \times \mathrm{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2}) \times \cdots \times \mathrm{Rep}_{\mathbf{k}}(Q_{P_s}, I_{P_s})$$

that restricts to an equivalence

$$\mathrm{rep}_{\mathbf{k}}(Q_P, I_P) \cong \mathrm{rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1}) \times \mathrm{rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2}) \times \cdots \times \mathrm{rep}_{\mathbf{k}}(Q_{P_s}, I_{P_s})$$

Moreover P is representation-finite if and only if $P_1, \dots, P_s$ are representation-finite.

Proof. The Hasse quiver Q_P of P will have as connected components the Hasse quivers $Q_{P_1}, \dots, Q_{P_s}$ of $P_1, \dots, P_s$ and each Hasse quiver will have its ideal I_{P_i} for each $i = 1, \dots, s$.

The equivalence of categories

$$\mathrm{Rep}_{\mathbf{k}}(Q_P, I_P) \cong \mathrm{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1}) \times \mathrm{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2}) \times \cdots \times \mathrm{Rep}_{\mathbf{k}}(Q_{P_s}, I_{P_s})$$

is given by sending a representation $M \in \operatorname{Rep}_{\mathbf{k}}(Q_P, I_P)$ to its "connected components" $(M_{|P_1}, \dots, M_{|P_s})$, where $M_{|P_i}$ is the restriction of the representation M considering only the vertices of P_i. The inverse is defined sending an element $(M_1, \dots, M_s) \in \operatorname{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1}) \times \cdots \times \operatorname{Rep}_{\mathbf{k}}(Q_{P_s}, I_{P_s})$ to the representation $M = M_1 \sqcup \cdots \sqcup M_s \in \operatorname{Rep}_{\mathbf{k}}(Q_P, I_P)$ that is defined in the following way: for each $x \in P$ there exists a unique P_i such that $x \in P_i$ (since they are disjoint). Then we set $M_x \coloneqq (M_i)_x$. If x, y in P, we have the following cases:

- $x \in P_i$, $y \in P_j$ with $i \neq j$, then x and y are not comparable (by definition of P_k), therefore there is no linear map between them in M;
- $x \leq y$ (resp. $x \geq y$) in a (unique) P_i, then $M_{xy} \coloneqq (M_i)_{xy}$ (resp. $M_{yx} \coloneqq (M_i)_{yx}$).

The restriction

$$\operatorname{rep}_{\mathbf{k}}(Q_P, I_P) \cong \operatorname{rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1}) \times \operatorname{rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2}) \times \cdots \times \operatorname{rep}_{\mathbf{k}}(Q_{P_s}, I_{P_s})$$

is clear because if M is finite dimensional, so are its restrictions $M_{|P_i}$ for all i, and a finite disjoint union of finite dimensional representations is again finite dimensional.

Now, assume P is representation-finite but $\exists P_k$ that is not representation-finite by contradiction. Up to re-indexing we may suppose it is P_1. Then P_1 has an infinite number of non-isomorphic indecomposable representations $\{N_j\}_{j \in J}$ (with J an infinite indexing set). Thus, by the equivalence, we an infinite number of non-isomorphic indecomposable representations of P given by $(N_j, 0, \dots, 0)$ for any $j \in J$. So we get a contradiction.

Conversely, if $P_1, \dots, P_s$ are of finite representation type, let $\{X_{i,1}, \dots, X_{i,n_i}\}$ be the set of representatives of the isomorphism classes of indecomposable representations of P_i. Then P has $n_1 n_2 \cdots n_s$ isomorphism classes of indecomposables given by $(X_{1,j_1}, \dots, X_{s,j_s})$ (with $1 \leq j_i \leq n_i$) through the equivalence. So P is representation-finite. $\square$

Example 2.13. Consider the following poset $P \coloneqq P_1 \sqcup P_2$

$$\begin{tikzcd}
 & 2 & 4 & \\
P_1 & 1 \arrow[u] & 3 \arrow[u] & P_2
\end{tikzcd}$$

Each P_i is representation-finite with three classes of non isomorphic indecomposables given by

$$\begin{tikzcd}
 & \mathbf{k} & & \mathbf{k} & & 0 \\
X_{i,1} & 0 \arrow[u] & X_{i,2} & \mathbf{k} \arrow[u, "1"] & X_{i,3} & \mathbf{k} \arrow[u]
\end{tikzcd}$$

As stated in the Theorem 2.12 we expect $9 = 3 \cdot 3$ non isomorphic indecomposable representations for P that are given by the following:

$$\begin{tikzcd}
 & \mathbf{k} & \mathbf{k} & & \mathbf{k} & \mathbf{k} & & \mathbf{k} & 0 \\
(X_{1,1}, X_{2,1}) & 0 \arrow[u] & 0 \arrow[u] & (X_{1,1}, X_{2,2}) & 0 \arrow[u] & \mathbf{k} \arrow[u, "1"] & (X_{1,1}, X_{2,3}) & 0 \arrow[u] & \mathbf{k} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathbf{k} & \mathbf{k} & & \mathbf{k} & \mathbf{k} & & \mathbf{k} & 0 \\
(X_{1,2}, X_{2,1}) & \mathbf{k} \arrow[u, "1"] & 0 \arrow[u] & (X_{1,2}, X_{2,2}) & \mathbf{k} \arrow[u, "1"] & \mathbf{k} \arrow[u, "1"] & (X_{1,2}, X_{2,3}) & \mathbf{k} \arrow[u, "1"] & \mathbf{k} \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 & \mathbf{k} & & 0 & \mathbf{k} & & 0 & 0 \\
(X_{1,3}, X_{2,1}) & \mathbf{k} \arrow[u] & 0 \arrow[u] & (X_{1,3}, X_{2,2}) & \mathbf{k} \arrow[u] & \mathbf{k} \arrow[u, "1"] & (X_{1,3}, X_{2,3}) & \mathbf{k} \arrow[u] & \mathbf{k} \arrow[u]
\end{tikzcd}$$

Notation. From now on we may suppose that any poset P is connected.

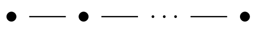
Remark 2.14. As a consequence, thanks to Proposition 2.6 and Theorem 1.19, a module over the incidence algebra $\mathcal{I}_{\mathbf{k}}(P)$ can be regarded as a representation of the bound quiver (Q_P, I_P). In particular

$$\mathrm{Mod}(\mathcal{I}_{\mathbf{k}}(P)) \cong \mathrm{Rep}_{\mathbf{k}}(Q_P, I_P) \qquad \text{and} \qquad \mathrm{mod}(\mathcal{I}_{\mathbf{k}}(P)) \cong \mathrm{rep}_{\mathbf{k}}(Q_P, I_P)$$

Notation. Throughout the thesis undirected edges in a quiver indicate that the arrows may go in either direction.

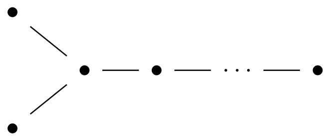
Some known posets of finite representation type are given by the following (see [Gab72a]):

$\mathbb{A}_n$



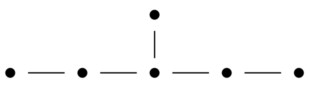
$(n \geq 1)$

$\mathbb{D}_n$

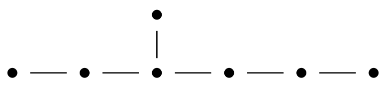


$(n \geq 4)$

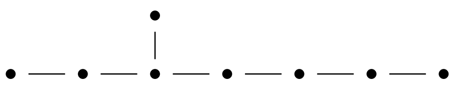
$\mathbb{E}_6$



$\mathbb{E}_7$

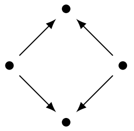


$\mathbb{E}_8$

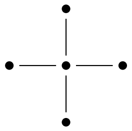


It is known that the following posets are representation-infinite and tame (see [Gab72a]):

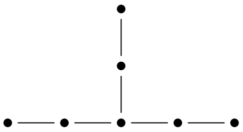
$\widetilde{\mathbb{A}}_3$



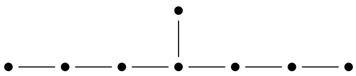
$\widetilde{\mathbb{D}}_4$



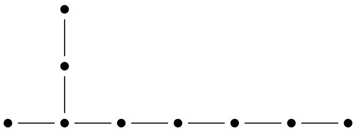
$\widetilde{\mathbb{E}}_6$



$\widetilde{\mathbb{E}}_7$

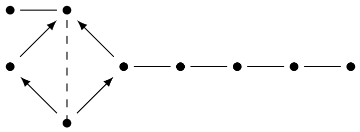


$\widetilde{\mathbb{E}}_8$



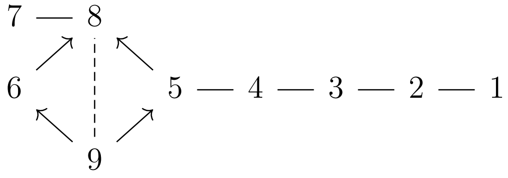
Let us consider some more posets.

Proposition 2.15 ([Lou75, Proposition 1])**.** *The poset* $\mathbb{R}_1$

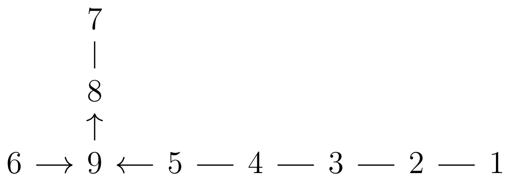


is of infinite representation type.

Proof. Let $\mathcal{C}$ be the full subcategory of $\operatorname{rep}_{\mathbf{k}}(Q_{\mathbb{R}_1}, I_{\mathbb{R}_1})$ of the representations V of $\mathbb{R}_1$ (considered with the following notation)



with $\ker(V_{96}) \cap \ker(V_{95}) = \{0\}$. We will see that it is equivalent to the category $\mathcal{D}$ of (finite dimensional) representations W of $\widetilde{\mathbb{E}}_8$ (considered with the following notation)



such that $\mathrm{im}(W_{69}) + \mathrm{im}(W_{59}) = W_9$.

Indeed we can define the functor $F\colon \mathcal{C} \to \mathcal{D}$ setting

$$F(V)_i = \begin{cases} V_i & \text{if } i = 1, \dots, 8 \\ V_6 \sqcup_{V_9} V_5 & \text{if } i = 9 \end{cases}$$

where $V_6 \sqcup_{V_9} V_5$ denotes the pushout of V_6 and V_5 under V_9. The $\mathbf{k}$-linear maps of the representation $F(V)$ are given by V or by the definition of pushout and its universal property.

Its inverse is given by the functor $G\colon \mathcal{D} \to \mathcal{C}$ defined as

$$G(W)_i = \begin{cases} W_i & \text{if } i = 1, \dots, 8 \\ W_6 \times_{W_9} W_5 & \text{if } i = 9 \end{cases}$$

where $W_6 \times_{W_9} W_5$ denotes the pullback of W_6 and W_5 over W_9. The $\mathbf{k}$-linear maps of the representation $G(W)$ are given by W or by the definition of pullback and its universal property.

The definition of F and G on morphisms between two representations is clear and derive from identity and universal property of pushout and pullback respectively.

We have that F and G are inverse to each other since for $i = 1, \dots, 8$ they are the identity while for $i = 9$ we have the following commutative diagrams

$$(1) \quad \begin{tikzcd} V_9 \arrow[r, "V_{96}"] \arrow[d, "V_{95}"'] & V_6 \arrow[d, "f_1"] \\ V_5 \arrow[r, "f_2"'] & V_6 \sqcup_{V_9} V_5 \end{tikzcd} \qquad \begin{tikzcd} W_6 \times_{W_9} W_5 \arrow[r, "g_1"] \arrow[d, "g_2"'] & W_6 \arrow[d, "W_{69}"] \\ W_5 \arrow[r, "W_{59}"'] & W_9 \end{tikzcd} \quad (2)$$

From (1) we have the exact sequence

$$0 \xrightarrow{(*)} V_9 \xrightarrow{(\,V_{96}, V_{95}\,)} V_6 \oplus V_5 \xrightarrow{\binom{f_1}{-f_2}} V_6 \sqcup_{V_9} V_5 \to 0$$

where the right part of the sequence is exact due to pushout properties while it is exact on $(*)$ thanks to the assumption $\ker(V_{96}) \cap \ker(V_{95}) = \{0\}$. So V_9 is the pullback of the diagram (1) and so $GF(V)_9 = V_9$.

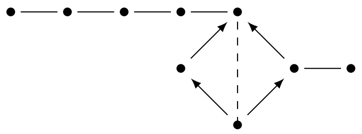
From (2) we have the exact sequence

$$0 \to W_6 \times_{W_9} W_5 \xrightarrow{(\,g_1, g_2\,)} W_6 \oplus W_5 \xrightarrow{\binom{W_{69}}{-W_{59}}} W_9 \xrightarrow{(**)} 0$$

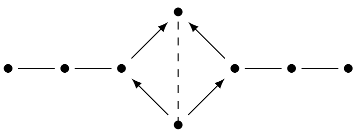
where the left part of the sequence is exact due to pullback properties while $(**)$ is exact thanks to the assumption $\operatorname{im}(W_{69}) + \operatorname{im}(W_{59}) = W_9$. So W_9 is the pushout of the diagram (2) and so $FG(W)_9 = W_9$.

Now $\mathcal{D}$ has an infinite number of non isomorphic indecomposable representations (see the representations shown in [SS06, Table 2.20 (d)]), so the same holds for $\mathcal{C}$ and then for $\operatorname{rep}_{\mathbf{k}}(Q_{\mathbb{R}_1}, I_{\mathbb{R}_1})$ too. So $\mathbb{R}_1$ is of infinite representation type. □

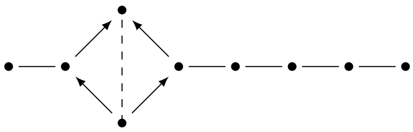
Proposition 2.16 ([Lou75, Proposition 1]). *The posets*



$\mathbb{R}_2$



$\mathbb{R}_3$



$\mathbb{R}_4$

are of infinite representation type.

Proof. The proof apply the same ideas of Proposition 2.15 to these posets, linking $\mathbb{R}_2$ to $\widetilde{\mathbb{E}}_8$, $\mathbb{R}_3$ to $\widetilde{\mathbb{E}}_7$ and $\mathbb{R}_4$ to $\widetilde{\mathbb{E}}_8$ through functors analogous to the ones previously defined. □

Lemma 2.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be two Krull-Remak-Schmidt categories, $F\colon \mathcal{C} \to \mathcal{D}$ an additive functor that admits an additive retraction $G\colon \mathcal{D} \to \mathcal{C}$ (i.e. $GF \cong 1_{\mathcal{C}}$). If $\mathcal{C}$ has an infinite number of isomorphism classes of indecomposable objects then the same is true for $\mathcal{D}$.*

Proof. Let C be an indecomposable object of $\mathcal{C}$. Let $F(C) = \bigoplus_i D_i$, with D_i indecomposables in $\mathcal{D}$. Applying G to $F(C)$ we have that

$$C \cong GF(C) = G(\bigoplus_i D_i) \overset{(*)}{=} \bigoplus_i G(D_i)$$

where $(*)$ holds since G is additive. But C was indecomposable in $\mathcal{C}$, therefore we must have that $G(D_i) = 0$ for all i except one, that we can assume to be D_1, and so $G(D_1) \cong C$. The isomorphism class of D_1 is completely determined by the isomorphism class of C, i.e. $C \cong C'$ if and only if $D_1 \cong D_1'$. Indeed, let C and C' be indecomposables, and use the previous process to find D_1 and D_1' such that $G(D_1) \cong C$ and $G(D_1') \cong C'$.

($\Leftarrow$) If $D_1 \cong D_1'$ then $C \cong G(D_1) \cong G(D_1') \cong C'$ (since functors preserves isomorphisms) and we conclude.

($\Rightarrow$) If $C \cong C'$ then we have

$$F(C) = D_1 \oplus \left(\bigoplus_i D_i \right) \cong D'_1 \oplus \left(\bigoplus_j D'_j \right) = F(C')$$

Since we are working with Krull-Remak-Schmidt categories, the decomposition is unique up to isomorphism and reordering. Now if $D_1 \cong D'_1$ we conclude. If not, then $D_1 \cong D'_j$ for some $j \neq 1$ but then we would have $C \cong G(D_1) \cong G(D'_j) = 0$ by the choice of D'_1 as done before. But if we assume that C and C' are non zero objects we get a contradiction (the case of the zero object is trivial). Thus we must have $D_1 \cong D'_1$.

Therefore we have an injection from the set of isomorphism classes of indecomposable objects of $\mathcal{C}$ to the set of isomorphism classes of indecomposable objects of $\mathcal{D}$. Thus, if $\mathcal{C}$ has an infinite number of isomorphism classes of indecomposable objects then it is the same for $\mathcal{D}$. $\square$

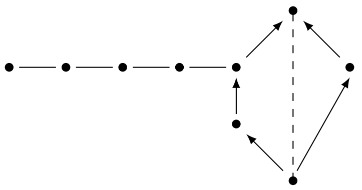
We will need the following result from [Gab72b]:

Lemma 2.18 ([Gab72b, Theorem 1]). *The category $\mathcal{G}$ whose objects are filtered (finite dimensional) vector spaces A with the following filtration*

$$(A, \mathcal{F}_A) = \begin{tikzcd}
A_4 \arrow[r, hook] & A_3 \arrow[r, hook] & A_2 \arrow[r, hook] & A_1 \arrow[r, hook] & A \arrow[dd, dash, dashed] & & \\
& & & Y \arrow[ur, hook] & & X \arrow[ul, hook] & B \arrow[l, hook'] \\
& & & & C \arrow[ul, hook] \arrow[ur, hook] & &
\end{tikzcd}$$

admits an infinite number of isomorphism classes of indecomposable objects.

Proposition 2.19 ([Lou75, Proposition 3]). *The poset $\mathbb{R}_5$*



is of infinite representation type.

Proof. Consider the category $\mathcal{G}$ of Lemma 2.18 and take an object $(A, \mathcal{F}_A)$

$$\begin{tikzcd}
A_4 \arrow[r, hook, "\alpha_4"] & A_3 \arrow[r, hook, "\alpha_3"] & A_2 \arrow[r, hook, "\alpha_2"] & A_1 \arrow[r, hook, "\alpha_1"] & A \arrow[dd, dash, dashed] & & \\
& & & Y \arrow[ur, hook, "\eta_1"] & & X \arrow[ul, hook, "\delta"'] & B \arrow[l, hook', "\gamma"'] \\
& & & & C \arrow[ul, hook, "\eta_0"] \arrow[ur, hook, "\zeta_0"'] & &
\end{tikzcd}$$

We define the functor $F\colon \mathcal{G} \to \operatorname{rep}_{\mathbf{k}}(Q_{\mathbb{R}_5}, I_{\mathbb{R}_5})$ setting

$$F((A,\mathcal{F}_A)) := \begin{tikzcd}
& & & & & A/B \arrow[ddd, dash, dashed] & \\
A_4 \arrow[r, "\alpha_4"] & A_3 \arrow[r, "\alpha_3"] & A_2 \arrow[r, "\alpha_2"] & A_1 \arrow[r, "\alpha_1"] & A \arrow[ur, "\beta"] & & X \arrow[ul, "\zeta_1"'] \\
& & & & Y \arrow[u, "\eta_1"] & & \\
& & & & & C \arrow[ul, "\eta_0"] \arrow[uur, "\zeta_0"'] &
\end{tikzcd}$$

where β is the cokernel of $\delta\gamma$, i.e.

$$0 \to B \xrightarrow{\delta\gamma} A \xrightarrow{\beta} A/B \to 0$$

and $\zeta_1 := \beta\delta$, while the other maps are the same. With these definitions it is clear that the relation $\beta\eta_1\eta_0 = \zeta_1\zeta_0$ holds: indeed, in $(A, \mathcal{F}_A)$ we have that $\eta_1\eta_0 = \delta\zeta_0$, therefore

$$\beta\eta_1\eta_0 = \beta\delta\zeta_0 = \zeta_1\zeta_0$$

The functor F is additive and admits a retraction $G\colon \operatorname{rep}_{\mathbf{k}}(Q_{\mathbb{R}_5}, I_{\mathbb{R}_5}) \to \mathcal{G}$ defined as follows: given a representation M of $\mathbb{R}_5$ of the form

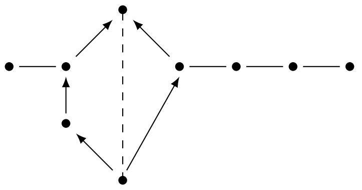
$$\begin{tikzcd}
& & & & & D \arrow[ddd, dash, dashed] & \\
A_4 \arrow[r, "\alpha_4"] & A_3 \arrow[r, "\alpha_3"] & A_2 \arrow[r, "\alpha_2"] & A_1 \arrow[r, "\alpha_1"] & A \arrow[ur, "\beta"] & & Z \arrow[ul, "\zeta_1"'] \\
& & & & Y \arrow[u, "\eta_1"] & & \\
& & & & & C \arrow[ul, "\eta_0"] \arrow[uur, "\zeta_0"'] &
\end{tikzcd}$$

we define $G(M)$ as

$$\begin{tikzcd}
\operatorname{im}(\alpha_1\alpha_2\alpha_3\alpha_4) \arrow[r] & \operatorname{im}(\alpha_1\alpha_2\alpha_3) \arrow[r] & \operatorname{im}(\alpha_1\alpha_2) \arrow[r] & \operatorname{im}(\alpha_1) \arrow[r] & A \arrow[dd, dash, dashed] & & \\
& & & \operatorname{im}(\eta_1) \arrow[ur] & & \beta^{-1}(\operatorname{im}(\zeta_1)) \arrow[ul] & \ker(\beta) \arrow[l] \\
& & & & \operatorname{im}(\eta_1\eta_0) \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

where the maps are the canonical inclusions given by the sequence of morphisms and the relation $\beta\eta_1\eta_0 = \zeta_1\zeta_0$. The relation in $G(M)$ holds since we are working with inclusion maps. Thanks to the fact that the maps in the object $(A, \mathcal{F}_A)$ are all inclusions, it is clear that $GF((A, \mathcal{F}_A)) \cong (A, \mathcal{F}_A)$. Therefore by Lemma 2.17 we conclude that $\mathbb{R}_5$ is of infinite representation type. □

Proposition 2.20 ([Lou75, Proposition 4])**.** *The poset* $\mathbb{R}_6$



is of infinite representation type.

Proof. We consider the functor $F\colon \operatorname{rep}_{\mathbf{k}}(Q_{\widetilde{\mathbb{D}}_4}) \to \operatorname{rep}_{\mathbf{k}}(Q_{\mathbb{R}_6}, I_{\mathbb{R}_6})$ defined in the following way: given a representation M of $\widetilde{\mathbb{D}}_4$ (with the given orientation)

$$\begin{tikzcd}
 & M_1 & \\
M_4 & M_0 \arrow[l, "\varphi_4"] \arrow[r, "\varphi_2"] \arrow[u, "\varphi_1"] \arrow[d, "\varphi_3"] & M_2 \\
 & M_3 &
\end{tikzcd}$$

we define $F(M)$ as the following representation of $\mathbb{R}_6$

$$\begin{tikzcd}
 & & & \bigoplus_{i=2}^4 M_i & & & & \\
M_2 \oplus M_3 \arrow[r, "\alpha"] & M_2 \oplus M_3 \oplus M_3 \oplus M_4 \arrow[urr, "\zeta"] & & & \bigoplus_{i=1}^4 M_i \arrow[ul, "\delta"'] & \bigoplus_{i=2}^4 M_i \arrow[l, "\beta_1"'] & M_3 \oplus M_4 \arrow[l, "\beta_2"'] & M_4 \arrow[l, "\beta_3"'] \\
 & \bigoplus_{i=2}^4 M_i \arrow[u, "\xi_1"] & & & & & & \\
 & & & M_0 \arrow[uuu, dash, dashed] \arrow[ull, "\xi_0"] \arrow[uur, "\gamma"'] & & & &
\end{tikzcd}$$

where the maps are

$$\begin{aligned}
&\xi_0(x_0) = (\varphi_2(x_0), \varphi_3(x_0), \varphi_4(x_0))\\
&\xi_1(x_2, x_3, x_4) = (x_2, 0, x_3, x_4)\\
&\zeta(x_2, x_3, y_3, x_4) = (x_2, x_3 + y_3, x_4)\\
&\alpha(x_2, x_3) = (x_2, x_3, 0, 0)\\
&\gamma(x_0) = (\varphi_1(x_0), \varphi_2(x_0), \varphi_3(x_0), \varphi_4(x_0))\\
&\delta(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4)\\
&\beta_1, \beta_2, \beta_3 \text{ are the canonical inclusions.}
\end{aligned}$$

With these definitions it is clear that the relation $\zeta\xi_1\xi_0 = \delta\gamma$ holds: indeed for any $x_0 \in M_0$ we have

$$\begin{aligned}
\zeta\xi_1\xi_0(x_0) &= \zeta\xi_1(\varphi_2(x_0), \varphi_3(x_0), \varphi_4(x_0)) = \zeta(\varphi_2(x_0), 0, \varphi_3(x_0), \varphi_4(x_0)) \\
&= (\varphi_2(x_0), \varphi_3(x_0), \varphi_4(x_0)) = \delta((\varphi_1(x_0), \varphi_2(x_0), \varphi_3(x_0), \varphi_4(x_0)) \\
&= \delta\gamma(x_0)
\end{aligned}$$

The functor F is additive and admits a retraction $G\colon \operatorname{rep}_{\mathbf{k}}(Q_{\mathbb{R}_6}, I_{\mathbb{R}_6}) \to \operatorname{rep}_{\mathbf{k}}(Q_{\widetilde{\mathbb{D}}_4})$ defined as follows: given a representation N of $\mathbb{R}_6$ of the form

$$\begin{tikzcd}
& & D & & & & \\
A_1 \arrow[r, "\alpha"] & A \arrow[ur, "\zeta"] & & B \arrow[ul, "\delta"'] & B_1 \arrow[l, "\beta_1"'] & B_2 \arrow[l, "\beta_2"'] & B_3 \arrow[l, "\beta_3"'] \\
& X \arrow[u, "\xi_1"] & & & & & \\
& & C \arrow[ul, "\xi_0"] \arrow[uur, "\gamma"'] \arrow[uuu, dash, dashed] & & & &
\end{tikzcd}$$

we define $G(N)$ as the representation

$$\begin{tikzcd}
& G(N)_1 & \\
G(N)_4 & G(N)_0 \arrow[l, "\psi_4"'] \arrow[u, "\psi_1"] \arrow[r, "\psi_2"] \arrow[d, "\psi_3"] & G(N)_2 \\
& G(N)_3 &
\end{tikzcd}$$

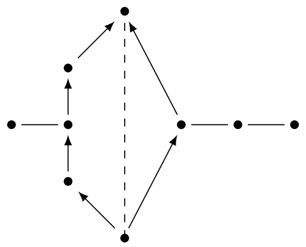
where we define ("can." means that we consider the canonical mapping):

$$\begin{aligned}
&G(N)_0 = C \\
&G(N)_1 = B/\operatorname{im}(\beta_1) && \text{and} \quad \psi_1\colon C \xrightarrow{\gamma} B \xrightarrow{can.} G(N)_1 \\
&G(N)_2 = D/\operatorname{im}(\delta\beta_1\beta_2) && \text{and} \quad \psi_2\colon C \xrightarrow{\delta\gamma} D \xrightarrow{can.} G(N)_2 \\
&G(N)_3 = D/(\zeta(\operatorname{im}(\alpha) \cap \operatorname{im}(\xi_1)) + \operatorname{im}(\delta\beta_1\beta_2\beta_3)) && \text{and} \quad \psi_3\colon C \xrightarrow{\delta\gamma} D \xrightarrow{can.} G(N)_3 \\
&G(N)_4 = A/(\operatorname{im}(\alpha) + \ker(\zeta)) && \text{and} \quad \psi_4\colon C \xrightarrow{\xi_1\xi_0} A \xrightarrow{can.} G(N)_4
\end{aligned}$$

Therefore by Lemma 2.17 we conclude that $\mathbb{R}_5$ is of infinite representation type.

□

Proposition 2.21 ([Lou75, Proposition 5]). *The poset* $\mathbb{R}_7$



is of infinite representation type.

Proof. We consider the functor $F\colon \operatorname{rep}_{\mathbf{k}}(Q_{\widetilde{\mathbb{D}}_4}) \to \operatorname{rep}_{\mathbf{k}}(Q_{\mathbb{R}_7}, I_{\mathbb{R}_7})$ defined in the following way: given a representation M of $\widetilde{\mathbb{D}}_4$

$$\begin{tikzcd}
 & M_1 & \\
M_4 & M_0 \arrow[l, "\varphi_4"'] \arrow[u, "\varphi_1"] \arrow[r, "\varphi_2"] \arrow[d, "\varphi_3"] & M_2 \\
 & M_3 &
\end{tikzcd}$$

we define $F(M)$ as the following representation of $\mathbb{R}_7$

$$\begin{tikzcd}
 & & M_2 \oplus M_3 & & & \\
 & \bigoplus_{i=2}^{4} M_i \arrow[ur, "can."] & & & & \\
M_2 \oplus M_3 \arrow[r, "\alpha"] & M_2 \oplus M_3 \oplus M_3 \oplus M_4 \arrow[u, "\zeta"] & & \bigoplus_{i=1}^{3} M_i \arrow[uul, "can."'] & M_2 \oplus M_3 \arrow[l, "can."'] & M_3 \arrow[l, "can."'] \\
 & \bigoplus_{i=2}^{4} M_i \arrow[u, "\xi_1"] & & & & \\
 & & M_0 \arrow[ul, "\xi_0"] \arrow[uur, "\gamma"'] \arrow[uuuu, dash, dashed] & & &
\end{tikzcd}$$

where "can." means that we consider the canonical mapping, the maps ξ_0, ξ_1, ζ and α are defined as in Proposition 2.20 and $\gamma(x_0) := (\varphi_1(x_0), \varphi_2(x_0), \varphi_3(x_0))$. The relation holds since, for any $x_0 \in M_0$, we have that $\zeta\xi_1\xi_0(x_0) = (\varphi_2(x_0), \varphi_3(x_0), \varphi_4(x_0))$, whose projection on $M_2 \oplus M_3$ is $(\varphi_2(x_0), \varphi_3(x_0))$, the same as the projection of $\gamma(x_0)$.

F has a retraction $G\colon \operatorname{rep}_{\mathbf{k}}(Q_{\mathbb{R}_7}, I_{\mathbb{R}_7}) \to \operatorname{rep}_{\mathbf{k}}(Q_{\widetilde{\mathbb{D}}_4})$ defined in the following way:

given a representation N of $\mathbb{R}_7$

$$\begin{tikzcd}
& & E & & & \\
& D \arrow[ur, "\chi"] & & & & \\
A_1 \arrow[r, "\alpha"] & A \arrow[u, "\zeta"] & & B \arrow[uul, "\delta"'] & B_1 \arrow[l, "\beta_1"'] & B_2 \arrow[l, "\beta_2"'] \\
& X \arrow[u, "\xi_1"] & & & & \\
& & C \arrow[ul, "\xi_0"] \arrow[uur, "\gamma"'] \arrow[uuuu, dash, dashed] & & &
\end{tikzcd}$$

we define $G(N)$ as the representation

$$\begin{tikzcd}
& G(N)_1 & \\
G(N)_4 & G(N)_0 \arrow[l, "\psi_4"] \arrow[r, "\psi_2"] \arrow[u, "\psi_1"] \arrow[d, "\psi_3"] & G(N)_2 \\
& G(N)_3 &
\end{tikzcd}$$

where we define ("can." means that we consider the canonical mapping):

$$\begin{aligned}
&G(N)_0 = C \\
&G(N)_1 = B/\operatorname{im}(\beta_1) && \text{and} \quad \psi_1 \colon C \xrightarrow{\gamma} B \xrightarrow{can.} G(N)_1 \\
&G(N)_2 = E/\operatorname{im}(\delta\beta_1\beta_2) && \text{and} \quad \psi_2 \colon C \xrightarrow{\delta\gamma} E \xrightarrow{can.} G(N)_2 \\
&G(N)_3 = E/\chi\zeta(\operatorname{im}(\alpha) \cap \operatorname{im}(\xi_1)) && \text{and} \quad \psi_3 \colon C \xrightarrow{\delta\gamma} E \xrightarrow{can.} G(N)_3 \\
&G(N)_4 = A/(\operatorname{im}(\alpha) + \ker(\zeta)) && \text{and} \quad \psi_4 \colon C \xrightarrow{\xi_1\xi_0} A \xrightarrow{can.} G(N)_4
\end{aligned}$$

Therefore by Lemma 2.17 we conclude that $\mathbb{R}_7$ is of infinite representation type. □

Remark 2.22. In the last Propositions 2.15, 2.16, 2.19, 2.20, 2.21 we considered just some particular orientations of the posets. Anyway, these results holds for any possible orientation of the undirected edges of the posets thanks to the properties of reflection functors (see Theorem 1.38 and Remark 1.33).

We can then summarise the known facts, the previous results and the remark into the following theorem:

Theorem 2.23. *The posets in Table 2.1 are of infinite representation type.*

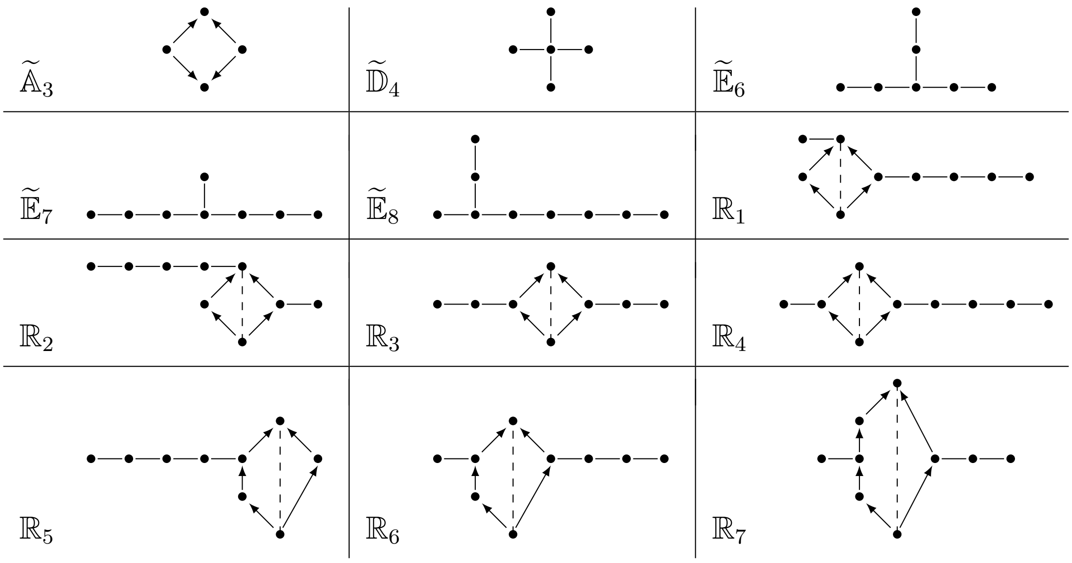


Table 2.1: Hasse quivers of the respective posets. Undirected edges indicate that the arrows may go in either direction.

Definition 2.24. Let P be a poset. The *opposite poset* of P, denoted $(P^{\mathrm{op}}, \preccurlyeq)$ or simply P^{op}, is the poset with the reverse order of P, that is if $x \leq y$ in P then $y \preccurlyeq x$ in P^{op}.

Remark 2.25. The categories $\mathrm{rep}_{\mathbf{k}}(Q_P, I_P) \cong \mathrm{mod}(\mathcal{I}_{\mathbf{k}}(P))$ and $\mathrm{rep}_{\mathbf{k}}(Q_{P^{\mathrm{op}}}, I_{P^{\mathrm{op}}}) \cong \mathrm{mod}(\mathcal{I}_{\mathbf{k}}(P^{\mathrm{op}}))$ are equivalent by duality of vector spaces. Therefore P is representation-finite if and only if P^{op} is representation-finite.

Definition 2.26 ([Lou06]). A poset P is said to be *crucial* if the Hasse quiver of P or P^{op} is one of those in Table 2.1.

2.2.1. Reduction techniques

Loupias devises two reduction procedures of incidence algebras that preserve representation-finiteness.

Definition 2.27 ([BGR24, Section 2.3]). Let P be a poset.

- A subposet of P (with the induced order) is also called a *subtraction* of P.
- A poset L is a *contraction* (or a *contracted poset*) of P if there exists a surjective order-preserving map $P \to L$ (called *contraction*) with connected fibers.

  An *elementary contraction* $c\colon P \to R$ is a contraction for which there exists exactly one element $x \in R$ such that $|c^{-1}(x)| = 2$ and $|c^{-1}(r)| = 1$ for all $r \in R \setminus \{x\}$.

- We say that a poset P *can be reduced to* L if it is the case that L can be obtained from P by subtraction and contraction in a finite number of steps.

Informally, an elementary contraction identifies two adjacent elements in P. Thus, there are as many elementary contractions as there are arrows in the Hasse quiver of the poset P.

If a poset P can be reduced to L, we may assume that each step of the reduction process either removes a single vertex or contracts an arrow in the Hasse quiver.

Example 2.28. (1) Let P be the poset

$$\begin{tikzcd}
 & c \arrow[dddr, dash, dashed, bend right] & d & \\
 & & b \arrow[ul] \arrow[u] \arrow[ddd, dash, dashed, bend left] & \\
x_2 \arrow[uur] & & & y_2 \arrow[ul] \\
x_1 \arrow[u] & & & y_1 \arrow[u] \\
 & & a \arrow[ull] \arrow[uuu] \arrow[ur] &
\end{tikzcd}$$

Consider the subposet P' with vertices $\{a, x_2, y_2, c, d\}$ and with the induced order. It has the following Hasse quiver

$$\begin{tikzcd}
 & c & d \\
x_2 \arrow[ur] & & y_2 \arrow[ul] \arrow[u] \\
 & a \arrow[ul] \arrow[ur] \arrow[uu, dash, dashed] &
\end{tikzcd}$$

and it is a subtraction of P.

(2) Let P be the poset

$$\begin{tikzcd}
a_2 & a_1 \arrow[l] & a \arrow[l] & & & \\
 & d \arrow[ur] & & b \arrow[ul] & b_1 \arrow[l] & b_2 \arrow[l] \\
 & & c \arrow[ul] \arrow[ur] \arrow[uu, dash, dashed] & & &
\end{tikzcd}$$

It can be contracted to the poset

$$\begin{tikzcd}
a_2 & & \\
a_1 \arrow[u] & & \\
\{a,b\} \arrow[u] & b_1 \arrow[l] & b_2 \arrow[l] \\
d \arrow[u] & & \\
c \arrow[u] & &
\end{tikzcd}$$

through the elementary contraction of a and b.

Lemma 2.29. *Every contraction of finite posets can be written as a composite of elementary contractions.*

Proof. Let $c\colon P \to Q$ be a contraction. We define a non-negative integer

$$w(c) = \sum_{q \in Q} \left(|c^{-1}(q)| - 1 \right).$$

Clearly, we have that $w(c) = 0$ if and only if c is the identity map, and $w(c) = 1$ if and only if c is elementary.

We prove our claim by induction on $w(c)$. The base cases $w(c) = 0$ and $w(c) = 1$ are trivial to establish. Let $\ell \geq 2$. Suppose that we can prove the claim for all contractions d with $w(d) < \ell$ and let c be a contraction such that $w(c) = \ell$. Fix an element $q \in Q$ with $|c^{-1}(q)| > 1$. We define a poset Q' whose underlying set is $(Q \setminus \{q\}) \sqcup |c^{-1}(q)|$ in which $x \leq_{Q'} y$ if

- $x \leq_Q y$ in $Q \setminus \{q\}$
- $x \leq_P y$ in $c^{-1}(q) \subseteq P$
- $x \leq_Q q$ and $y \in c^{-1}(q)$,
- $q \leq_Q y$ and $x \in c^{-1}(q)$.

We can then factorise c as $P \xrightarrow{c_1} Q' \xrightarrow{c_2} Q$ where

$$c_1(z) = \begin{cases} c(z) & \text{if } z \in P \setminus c^{-1}(q) \\ z & \text{if } z \in c^{-1}(q) \end{cases} \qquad c_2(z) = \begin{cases} q & \text{if } z \in c^{-1}(q) \\ z & \text{if } z \in Q \setminus \{q\} \end{cases}$$

By definition it is clear that $w(c_1)$ and $w(c_2)$ are less than ℓ. Therefore we conclude the proof by induction. □

Remark 2.30. We recall that given a finite dimensional $\mathbf{k}$-algebra Λ, a module $M \in \text{mod}(\Lambda)$ is indecomposable if and only if $\text{End}_\Lambda(M)$ is local (Proposition 1.25).

If Λ and Λ' are two finite dimensional $\mathbf{k}$-algebras, $M' \in \text{mod}(\Lambda')$ indecomposable and $F\colon \text{mod}(\Lambda') \to \text{mod}(\Lambda)$ a fully faithful functor, then $F(M')$ is indecomposable in $\text{mod}(\Lambda)$. Moreover, two indecomposable modules M and M' are isomorphic in $\text{mod}(\Lambda')$ if and only if $F(M)$ and $F(M')$ are isomorphic in $\text{mod}(\Lambda)$.

Indeed F fully faithful means that $\forall X, Y \in \text{mod}(\Lambda')$ we have

$$\text{Hom}_{\Lambda'}(X, Y) \cong \text{Hom}_\Lambda(F(X), F(Y))$$

and in particular

$$\text{End}_{\Lambda'}(X) = \text{Hom}_{\Lambda'}(X, X) \cong \text{Hom}_\Lambda(F(X), F(X)) = \text{End}_\Lambda(F(X))$$

So M' indecomposable implies that $\text{End}_{\Lambda'}(M') \cong \text{End}_\Lambda(F(M'))$ is local. Thus we also have that $F(M')$ is indecomposable.

Moreover $M \cong M'$ implies $F(M) \cong F(M')$ since functors preserves isomorphisms and the converse holds because F is fully faithful. Indeed, let g be an isomorphism $F(M) \xrightarrow{\cong} F(M')$ with inverse $g^{-1}\colon F(M') \xrightarrow{\cong} F(M)$. Since F is fully faithful there exists a unique $M \xrightarrow{f} M'$ such that $F(f) = g$ and a unique $M' \xrightarrow{h} M$ such that $F(h) = g^{-1}$. Now

$$F(fh) = F(f)F(h) = gg^{-1} = 1_{F(M')} = F(1_{M'})$$

and also

$$F(hf) = F(h)F(f) = g^{-1}g = 1_{F(M)} = F(1_M)$$

By faithfulness of F we conclude that $fh = 1_{M'}$ and $hf = 1_M$, i.e. $h = f^{-1}$ and so f is an isomorphism.

Lemma 2.31 ([Lou06, Propositions 1.2 and 1.3])**.** *Let P be a poset such that the incidence $\mathbf{k}$-algebra $\mathcal{I}_\mathbf{k}(P)$ is representation-finite.*

1. *Any subposet of P has a representation-finite incidence $\mathbf{k}$-algebra.*
2. *Any contraction of P has a representation-finite incidence $\mathbf{k}$-algebra.*

Proof. If L is a subposet of P, we can find a $\mathbf{k}$-algebra isomorphism

$$\mathcal{I}_\mathbf{k}(L) \simeq e\mathcal{I}_\mathbf{k}(P)\,e$$

where e is the idempotent $\sum_{i\in L} \varepsilon_i$ and the terms ε_i in the sum are primitive idempotents corresponding to stationary paths of the elements in L.

Indeed, by Proposition 2.9, we have $\mathcal{I}_{\mathbf{k}}(P) \cong (\mathbf{k}P)^{\mathrm{op}}$ and so we have a matrix $E_L \in \mathbf{k}P$ corresponding to the idempotent e defined as above. It is easy to see that if $M = (m_{ij}) \in \mathbf{k}P$ and $\widetilde{M} := E_L M E_L$, then $\widetilde{M} = (\widetilde{m}_{ij})$ is an $n \times n$ matrix in $\mathbf{k}P$ such that $\widetilde{m}_{ij} = 0$ whenever $i \in P \setminus L$ or $j \in P \setminus L$. This shows that $E_L(\mathbf{k}P)E_L$ is the $\mathbf{k}$-vector subspace of $\mathbf{k}P$ consisting of matrices $\widetilde{M} \in \mathbf{k}P$ with $\widetilde{m}_{ij} = 0$ whenever $i \in P \setminus L$ or $j \in P \setminus L$. Therefore there is a $\mathbf{k}$-algebra isomorphism $E_L(\mathbf{k}P)E_L \cong \mathbf{k}L$ (c.f. [ASS06, Ch.I, Example 6.7]). Thanks to Proposition 2.9 we conclude $\mathcal{I}_{\mathbf{k}}(L) \cong e\mathcal{I}_{\mathbf{k}}(P)\, e$.

Consequently, we have a functor

$$\mathrm{mod}(\mathcal{I}_{\mathbf{k}}(L)) \simeq \mathrm{mod}(e\mathcal{I}_{\mathbf{k}}(P)\, e) \xhookrightarrow{\mathrm{Hom}_{e\mathcal{I}_{\mathbf{k}}(P)e}(e\mathcal{I}_{\mathbf{k}}(P), -)} \mathrm{mod}(\mathcal{I}_{\mathbf{k}}(P)) \tag{2.1}$$

called *idempotent embedding functor*, that is fully faithful (see [ASS06, Ch.I, Theorem 6.8]).

If R is a contraction of P, we have, by definition, an epimorphism of posets $P \twoheadrightarrow R$. Since the module category $\mathrm{mod}(\mathcal{I}_{\mathbf{k}}(P))$ is equivalent to the category $\mathrm{rep}_{\mathbf{k}}(Q_P, I_P)$, and similarly $\mathrm{mod}(\mathcal{I}_{\mathbf{k}}(R)) \cong \mathrm{rep}_{\mathbf{k}}(Q_R, I_R)$, one can induce a functor

$$\mathrm{mod}(\mathcal{I}_{\mathbf{k}}(R)) \cong \mathrm{rep}_{\mathbf{k}}(Q_R, I_R) \hookrightarrow \mathrm{rep}_{\mathbf{k}}(Q_P, I_P) \cong \mathrm{mod}(\mathcal{I}_{\mathbf{k}}(P)) \tag{2.2}$$

directly from the map $P \twoheadrightarrow R$. Since this is an epimorphism with connected fibers, it follows that this functor is fully faithful [Lou75, Proposition 8].

The existence of the fully faithful functors displayed in (2.1) and (2.2) is all one needs in order to prove the Lemma: indeed, if by contradiction L (resp. R) is representation-infinite, by Remark 2.30, the functor (2.1) (resp. (2.2)) will give an infinite number of non-isomorphic indecomposables, contradicting the fact that P is representation-finite. □

Remark 2.32. The condition that a retraction has connected fibers is necessary for the Lemma: consider the (Hasse quivers of) posets

$$P := \begin{tikzcd} & & d \arrow[dd, dash, dashed] & & \\ a_1 \arrow[r] & a \arrow[ur] & & b \arrow[ul] & b_1 \arrow[l] \\ & & c \arrow[ul] \arrow[ur] & & \end{tikzcd} \qquad \widetilde{\mathbb{D}}_4 = \begin{tikzcd} & x_5 & \\ x_3 \arrow[r] & x_4 \arrow[u] & x_2 \arrow[l] \\ & x_1 \arrow[u] & \end{tikzcd}$$

and we consider a morphisms of posets, i.e. a surjective order-preserving map, $f\colon P \to \widetilde{\mathbb{D}}_4$ defined by

$$f(c) = x_1 \quad f(b_1) = x_2 \quad f(a_1) = x_3 \quad f(a) = f(b) = x_4 \quad f(d) = x_5$$

We know that $\widetilde{\mathbb{D}}_4$ is representation-infinite but, using Theorem 3.2 that we will prove later, it is possible to see that P is representation-finite. This happens because the fiber $f^{-1}(x_4) = \{a, b\}$ is not connected.

Example 2.33. Let P be a poset of *type* $\widetilde{\mathbb{A}}_n$, where $n \geq 3$, namely a poset with Hasse quiver as shown below:

$$\begin{tikzcd}
 & a_1 \arrow[r, dash] & \cdots \arrow[r, dash] & a_\ell & \\
m_1 \arrow[ur] \arrow[dr] & & & & m_2 \arrow[ul] \arrow[dl] \\
 & a_{n-1} \arrow[r, dash] & \cdots \arrow[r, dash] & a_{\ell+1} &
\end{tikzcd}$$

As a matter of definition, posets of type $\widetilde{\mathbb{A}}_n$ have circular Hasse quivers. Note that there are at least two minimal elements, denoted here by m_1 and m_2. Such posets can be contracted onto a poset of type $\widetilde{\mathbb{A}}_3$, namely a poset of the following form:

$$\begin{tikzcd}
 & \{a_1, \ldots, a_\ell\} & \\
m_1 \arrow[ur] \arrow[dr] & & m_2 \arrow[ul] \arrow[dl] \\
 & \{a_{n-1}, \ldots, a_{\ell+1}\} &
\end{tikzcd}$$

Similarly, a poset of *type* $\widetilde{\mathbb{D}}_n$, whose underlying undirected graph is of the form

$$\begin{tikzcd}
a_1 \arrow[dr, dash] & & & b_1 \arrow[dl, dash] \\
 & d_1 \arrow[r, dash] & \cdots \arrow[r, dash] & d_{n-3} \\
a_2 \arrow[ur, dash] & & & b_2 \arrow[ul, dash]
\end{tikzcd}$$

where $n \geq 4$, can be contracted onto a poset of type $\widetilde{\mathbb{D}}_4$, namely a poset for which the underlying unoriented graph of the Hasse quiver admits the following shape:

$$\begin{tikzcd}
a_1 \arrow[dr, dash] & & b_1 \arrow[dl, dash] \\
 & \{d_1, \ldots, d_{n-3}\} & \\
a_2 \arrow[ur, dash] & & b_2 \arrow[ul, dash]
\end{tikzcd}$$

Therefore, thanks to Lemma 2.31, we conclude that the posets of type $\widetilde{\mathbb{A}}_n$ (with the above orientation of arrows) and of type $\widetilde{\mathbb{D}}_n$ are representation-infinite.

2.2.2. Kind and critical posets

Definition 2.34 ([Lou06])**.** A poset P is said to be *critical* (or $\mathcal{I}_{\mathbf{k}}(P)$ is said to be *minimally representation-infinite*) if it is representation-infinite and all of its proper subposets and non-trivial contractions have representation-finite incidence $\mathbf{k}$-algebras.

A poset P is said to be *kind* if none of its subposets or contracted posets is crucial.

As a direct consequence of Lemma 2.31 and Theorem 2.23, any poset of finite representation type is kind. It is also clear by the definition that any subposet or contracted poset of a kind poset is kind itself.

Definition 2.35. Given a poset P and two elements $a, b \in P$ such that $a \leq b$ we will define the *open interval* of a and b as

$$]a, b[\coloneqq \{x \mid a < x < b\} \subseteq P$$

Similarly we define $[a, b[$, $]a, b]$, etc...

We say that two elements $a, b \in P$ are *neighbours* if they are different ($a \neq b$), comparable ($a \leq b$ or $b \leq a$) and $]a, b[=]b, a[= \emptyset$.

We say that a subset C in a poset P is *convex* in P if for all $a, b \in C$ with $a \leq b$ we have $[a, b] \subseteq C$.

Lemma 2.36. *In a kind poset, every element has at most three neighbours.*

Proof. Let P be a kind poset and suppose by contradiction that an element a has at least four neighbours a_1, a_2, a_3, a_4. The only valid possibilities for the comparability of these four elements are (for $i \neq j$):

- a_i not comparable to a_j
- $a_i \leq a \leq a_j$

Indeed the cases $a_i \leq a_j \leq a$ or $a \leq a_i \leq a_j$ are not possible since these elements are neighbours of a. Therefore taking the subposet $P' = \{a, a_1, a_2, a_3, a_4\}$ it will has Hasse quiver of $\widetilde{\mathbb{D}}_4$ type, so it is crucial, contradicting the fact that P is kind. □

Definition 2.37. A poset P is said to be *linear* if it is of the form

$$\begin{tikzcd} e_1 \arrow[r, dash] & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \arrow[r, dash] & e_2 \end{tikzcd}$$

and e_1, e_2 are its *ends*.

Definition 2.38. Let L be a subposet of a poset P. We say that L is a *thread* if:

- L is linear and convex in P, i.e. it is of the form

$$a_0 \text{ ------ } a_1 \text{ ------ } \cdots \text{ ------ } a_{n-1} \text{ ------ } a_n \qquad (n \geq 1)$$

- a_i has only two neighbours in P, a_{i-1} and a_{i+1} for $1 \leq i \leq n-1$,
  a_n has only one neighbour in P, that is a_{n-1},
  a_0 has at least three neighbours in P (with one of them being a_1).
  We say that a_0 is the *bond point* of L and a_n is the *extremity* of L.

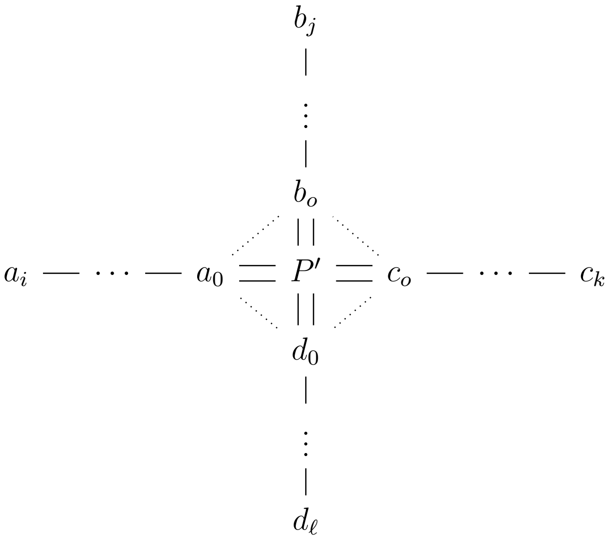
Proposition 2.39. *Let L be a thread in a poset P. The changing of the orientation of the arrows in L does not alter the nature of P, concerning the properties of kindness and representation-finiteness.*

Proof. The changing of the orientation of the arrows does not alter the kindness of the poset since if P contains a crucial subposet or contracted poset the same is true if we change the orientation of the arrows in the thread L.

The representation-finiteness is also preserved thanks to the properties of the reflection functors (see Theorem 1.38). □

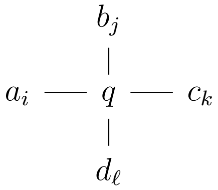
Lemma 2.40. *A kind poset has at most three threads.*

Proof. Let P be a kind poset and suppose by contradiction that it has at least four threads, L_a, L_b, L_c, L_d with bond points a_0, b_0, c_0, d_0 and lengths $i, j, k, \ell \geq 1$ respectively. The Hasse quiver of P will be of the following form



where P' indicates the subposet of P of all the elements not contained in the four threads L_a, L_b, L_c, L_d and the dotted lines suggest that some of the bond points may be comparable.

Now we consider the contraction that sends P', $\{a_0, \dots, a_{i-1}\}$, $\{b_0, \dots, b_{j-1}\}$, $\{c_0, \dots, c_{k-1}\}$, $\{d_0, \dots, d_{\ell-1}\}$ to one point q and that sends the extremities a_i, b_j, c_k, d_ℓ of the four threads to themselves. The contracted poset will be $\widetilde{\mathbb{D}}_4$



which is crucial, contradicting the fact that P is kind. □

Lemma 2.41. *Let P be a kind poset, H and K be two subposets of P such that:*

(1) $\mathrm{card}(H), \mathrm{card}(K) \geq 2$*;*

(2) $H \cup K$ *is connected;*

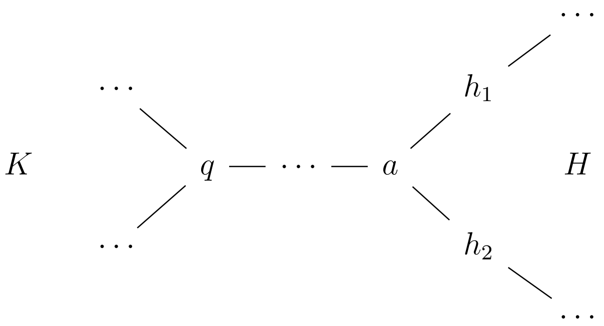
(3) $H \cap K = \{a\}$*;*

Then one of the two subposets H or K is linear with a as one of its ends.

Proof. If $\mathrm{card}(H) = \mathrm{card}(K) = 2$ the result is clear. So let us assume that $\mathrm{card}(H), \mathrm{card}(K) \geq 3$. We prove the statement by contradiction studying some cases:

1. K not linear, H linear but a is not one of its ends:

   we have that a has two neighbours h_1, h_2 in H (since H linear but a is not an end), only one neighbour in K (by kindness of P) and K must have a branch point q (i.e. a point with 3 neighbours) in some position (otherwise it would be linear). Therefore the subposet $H \cup K$ (which is connected) will have the following form



   Thus it can be contracted to $\widetilde{\mathbb{D}}_4$, contradicting the fact that P is kind.

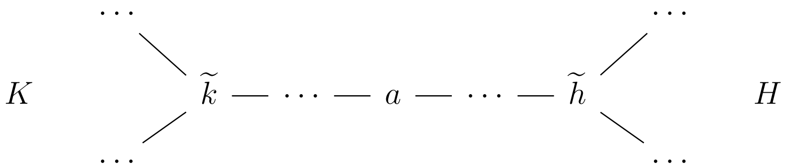
2. H not linear, K linear but a is not one of its ends (it is similar to the previous point).

3. K and H linear but a is not an end for both of them:

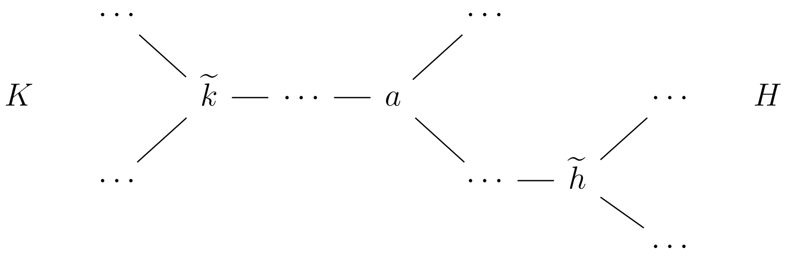
   a will have four neighbours, k_1, k_2 in K and h_1, h_2 in H, different to each other by hypothesis (3) and this contradicts the fact that P is kind by Lemma 2.36.

4. H and K not linear:

   in this case both H and K will have a branch point in some position, so we have a situation like the following



   or like



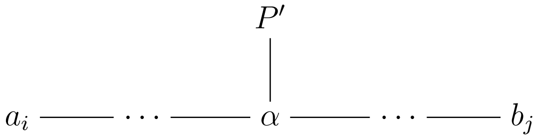
   (where the behaviour of H and K can be switched, i.e. we can denote K as H and viceversa). Therefore we can reduce it to $\widetilde{\mathbb{D}}_4$, which leads to a contradiction.

Then we conclude that one of the two subposets H or K is linear with a as one of its ends. ☐

Corollary 2.42. *If P is a kind poset, the bond points of two threads are the same if and only if P is equal to $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$.*

Proof. If P is equal to $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$ it is clear that it is a kind poset where the bond points of two (even three) threads are the same.

Suppose now P is kind with two threads L_1, L_2 having the same bond point α. The Hasse quiver of P will be



and the only possibility for P' is to be linear by Lemma 2.41 (taking $K = \{L_1, L_2\}$ and $H = \{P', \alpha\}$). Moreover, since P must be kind, one conclude that P is of the form $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$ (depending on the number of elements in the threads). $\square$

Lemma 2.43. *Let P be a kind poset, and let $x, y \in P$ be not comparable, i.e. $x \nleq y$ and $x \ngeq y$. Then the subposets*

$$L_{x,y} = \{z \in P \mid z \leq x,\, z \leq y\} \qquad U_{x,y} = \{z \in P \mid z \geq x,\, z \geq y\}$$

are either empty or linear intervals.

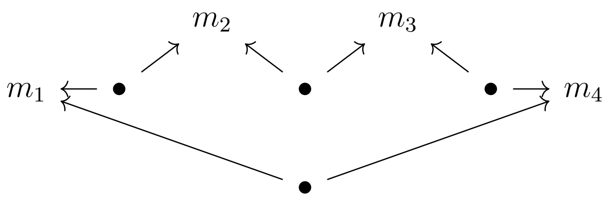
Proof. We give the proof for $L_{x,y}$ (for $U_{x,y}$ the proof is similar).
Suppose that $L_{x,y} \neq \emptyset$ and let $\operatorname{card}(L_{x,y}) = n$. If $n = 1$ the statement is clear, so let us consider $n \geq 2$. We see that any two elements of $L_{x,y}$ must be comparable: indeed if we have $a_1, a_2 \in L_{x,y}$ not comparable, the subposet $\{x, y, a_1, a_2\}$ would be $\widetilde{\mathbb{A}}_3$ contradicting the fact that P is kind.

Therefore $L_{x,y}$ is a total order with a minimum ℓ_1 and a maximum ℓ_n, so $L_{x,y} \subseteq [\ell_1, \ell_n]$. Moreover if $z \in [\ell_1, \ell_n]$ then $z \leq \ell_n \leq x, y$, so $z \in L_{x,y}$. Thus, we conclude that $L_{x,y} = [\ell_1, \ell_n]$ is a linear interval. $\square$

Lemma 2.44. *Let P be a kind poset with $p \leq 3$ threads, with P different from $\mathbb{A}_n$ and $\mathbb{D}_n$. The number n_1 (resp. n_2) of maximal (resp. minimal) points of P which do not belong to a thread is $\leq 3 - p$.*

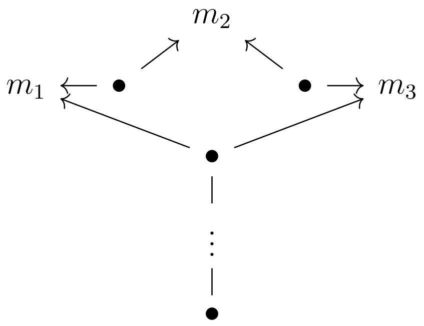
Idea of the proof. We just give an idea of the proof for the different cases with respect to the number $p \leq 3$ of threads. We will focus on the maximal points (for the minimal points the procedure is similar).

Case $p = 0$: suppose by contradiction that $n_1 > 3$. Then we will have four maximal points m_1, m_2, m_3, m_4 in the poset, which do not belong to a thread, with at least two neighbours, otherwise we would have a thread. It is possible to see that, with the hypothesis of kindness and that $P \neq \mathbb{A}_n, \mathbb{D}_n$, the poset has $\widetilde{\mathbb{D}}_4$ as a subposet or it can be reduced to



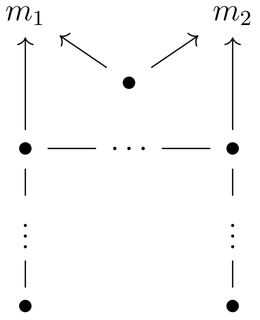
that can be contracted to $\widetilde{\mathbb{A}}_3$, contradicting the fact that P is kind.

Case $p = 1$: suppose by contradiction that $n_1 > 2$. Then we will have three maximal points m_1, m_2, m_3 in the poset, which do not belong to a thread, with at least two neighbours, otherwise we would have a thread. It is possible to see that, with the hypothesis of kindness and that $P \neq \mathbb{A}_n, \mathbb{D}_n$, the poset has $\widetilde{\mathbb{D}}_4$ as a subposet or it can be reduced to



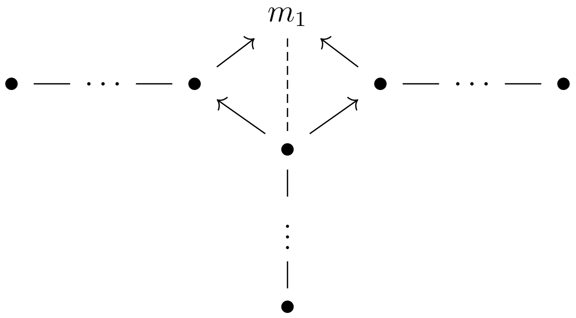
that can be contracted to $\widetilde{\mathbb{A}}_3$, contradicting the fact that P is kind.

Case $p = 2$: suppose by contradiction that $n_1 > 1$. Then we will have two maximal points m_1, m_2 in the poset, which do not belong to a thread, with at least two neighbours, otherwise we would have a thread. It is possible to see that, with the hypothesis of kindness and that $P \neq \mathbb{A}_n, \mathbb{D}_n$, the poset has $\widetilde{\mathbb{D}}_4$ as a subposet or it can be reduced to



that can be contracted to $\widetilde{\mathbb{A}}_3$ or $\widetilde{\mathbb{D}}_4$, contradicting the fact that P is kind.

Case $p = 3$: suppose by contradiction that $n_1 > 0$. Then we will have one maximal point m_1 in the poset, which does not belong to a thread, with at least two neighbours, otherwise we would have a thread. It is possible to see that, with the hypothesis of kindness and that $P \neq \mathbb{A}_n, \mathbb{D}_n$, the poset can be reduced to



that can be contracted to $\widetilde{\mathbb{D}}_4$, contradicting the fact that P is kind. □

Using these results, it is possible to classify all the kind posets and to show that they are all representation-finite (c.f. [Lou06; Lou75]). Therefore we have the following result:

Theorem 2.45 ([Lou06, Theorem 1.5]). *A poset P is representation-finite if and only if is kind.*

We omit the proof since it would need the classification of all the kind posets. Loupias classified the kind posets with $p = 3$ threads (see [Lou06, Proposition 2.8, 2.9]) and gave some ideas about the classification in the other cases, i.e. $p = 0, 1, 2$. The classification process in each case is based on the cardinality of the posets, checking if all the properties of kind posets that we introduced are satisfied and considering all the possible reduction that we can apply.

As a corollary of Theorem 2.45 we have the following result:

Corollary 2.46 ([Lou06, Theorem 1.4]). *A poset P is crucial if and only if is critical.*

Proof. The proper subposets and the proper contracted posets of the crucial posets are kind (by definition of kind posets) and thanks to Theorem 2.45 they are representation-finite. Thus every crucial poset is critical.

Conversely, if P is critical, it is representation-infinite so it is not kind by Theorem 2.45, therefore there exists a subposet or a contracted poset P' of P which is crucial. But P' cannot be a proper subposet or a non trivial contraction by definition of critical poset. Thus P' must be equal to P, showing that any critical poset is crucial. □

3. τ-tilting finiteness for incidence algebras

We will now discuss the equivalence between representation-finiteness and τ-tilting finiteness for incidence algebras, that was first shown in [BGR24]. Before that, we present some results that leads to the main theorem.

Lemma 3.1. *Any representation-infinite poset can be reduced to a critical poset (that has a minimally representation-infinite incidence algebra).*

Proof. Suppose P is a representation-infinite poset. If all of its proper subposets and non-trivial contractions have representation-finite incidence $\mathbf{k}$-algebras, it is a critical poset by Definition 2.34 so $\mathcal{I}_{\mathbf{k}}(P)$ is minimally representation-infinite. Otherwise, P will have at least one proper subposets or non-trivial contractions P' that is representation-infinite. The cardinality of P' is strictly less than the cardinality of P. Therefore by induction on the cardinality of the poset, P' can be reduced to a critical poset, thus the same holds for P, since we can consider the reduction techniques used from P to P' and from P' to a critical poset in order to reduce P to a critical poset. $\square$

We can now rephrase Theorem 2.45 and Corollary 2.46 into the following:

Theorem 3.2 ([BGR24, Theorem 2.11])**.** *A poset P has a representation-finite incidence $\mathbf{k}$-algebra if and only if it cannot be reduced to a crucial poset (i.e its Hasse quiver is not taken from the list in Table 2.1, nor from the list of the opposites of those). In other words, any minimally representation-infinite incidence $\mathbf{k}$-algebra falls into one of these twelve types.*

Proof. If P is representation-finite, by Theorem 2.45 it is kind and by definition it cannot be reduced to a crucial poset.

If P is representation-infinite, by Lemma 3.1 it can be reduced to a critical poset that is also crucial by Corollary 2.46. $\square$

Corollary 3.3 ([BGR24, Corollary 2.12])**.** *A poset P has a representation-infinite incidence $\mathbf{k}$-algebra if and only if it can be reduced to a poset L for which the incidence algebra $\mathcal{I}_{\mathbf{k}}(L)$ is tame concealed.*

Proof. Suppose that P has a representation-infinite incidence $\mathbf{k}$-algebra. Then by Theorem 3.2 P can be reduced to a crucial poset. Moreover the incidence $\mathbf{k}$-algebras of the crucial posets are all tame concealed. Indeed, all the Hasse quivers in Table 2.1 (and their opposites) appear on Happel-Vossieck's list of tame concealed algebras (see Remark 1.60), where we find that the incidence algebra of type $\mathbb{R}_i$, where $1 \leq i \leq 7$, is tame concealed of type $\widetilde{\mathbb{E}}$. This proves necessity.

Now suppose that a finite number of subtractions and contractions of P yields a poset L such that $\mathcal{I}_{\mathbf{k}}(L)$ is tame concealed. Since tame concealed algebras are representation-infinite (by Theorem 1.61), it follows from Lemma 2.31 that P has a representation-infinite incidence $\mathbf{k}$-algebra. This proves the converse. $\square$

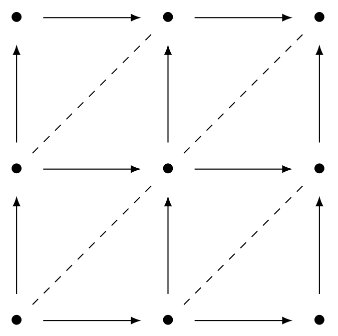
Example 3.4 ([BGR24, Example 2.13])**.** To better understand the previous results, we will now apply Theorem 3.2 to rediscover when the posets of the following form

$$\prod_{i=1}^{\ell} \overrightarrow{\mathbb{A}}_{n_i}$$

have representation-finite incidence $\mathbf{k}$-algebras, as originally shown by Leszczyński [Les94, Proposition 2.1 and Theorem 2.4]. We remind that the poset $\overrightarrow{\mathbb{A}}_{n_i}$ is totally ordered and has n_i elements. The elements of the poset $\prod_{i=1}^{\ell} \overrightarrow{\mathbb{A}}_{n_i}$ can be written as $(a_1, \dots, a_\ell)$ with $a_i \in \overrightarrow{\mathbb{A}}_{n_i}$, with $1 \leq a_i \leq n_i$, and the order relation is given by

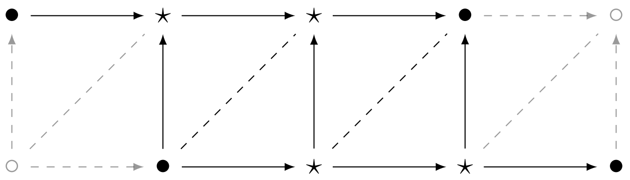
$$(a_1, \dots, a_\ell) \leq (b_1, \dots, b_\ell) \iff a_i \leq b_i \;\; \forall i$$

Firstly, we consider posets of the form $\overrightarrow{\mathbb{A}}_n \times \overrightarrow{\mathbb{A}}_m$, where $n \leq m$. If $n \geq 3$, we consider the poset $\overrightarrow{\mathbb{A}}_3 \times \overrightarrow{\mathbb{A}}_3$

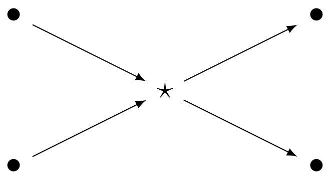


that clearly has a subposet of type $\widetilde{\mathbb{D}}_4$, and consequently $\overrightarrow{\mathbb{A}}_3 \times \overrightarrow{\mathbb{A}}_3$ is of infinite representation type. But $\overrightarrow{\mathbb{A}}_n \times \overrightarrow{\mathbb{A}}_m$ with $3 \leq n \leq m$ always contains $\overrightarrow{\mathbb{A}}_3 \times \overrightarrow{\mathbb{A}}_3$ as a subposet, so it is of infinite representation type too.

The poset $\overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_5$ also has a representation-infinite incidence **k**-algebra, since the subposet displayed by the solid nodes and arrows below



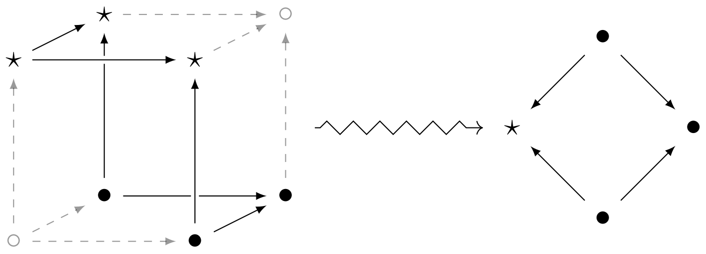
can be contracted onto the poset



which is of type $\widetilde{\mathbb{D}}_4$. A poset of the form $\overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_m$ can be reduced to $\overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_5$ as long as $m \geq 5$, whence all of these posets have representation-infinite incidence **k**-algebras.

One checks that $\overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_4$ is iterated tilted of type $\mathbb{E}_8$, so it has a representation-finite incidence **k**-algebra. If $n = 2$ and $m \leq 4$, one uses Theorem 3.2 to argue that $\overrightarrow{\mathbb{A}}_n \times \overrightarrow{\mathbb{A}}_m$ has a representation-finite incidence **k**-algebra. The remaining cases are those where $n = 1$, which yields the class of finite totally ordered posets $\overrightarrow{\mathbb{A}}_m$ and all of these certainly have a representation-finite incidence **k**-algebra.

The poset $\overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_2$ has a representation-infinite incidence **k**-algebra: indeed, the subposet obtained by removing the maximal and minimal elements can be contracted onto a poset of type $\widetilde{\mathbb{A}}_3$, as shown below.



In fact, by [CR19, Lemma 3.5] (see also [Len99, Example 18.6.2]) we have that $\overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_2$ has a tame incidence **k**-algebra.

More generally, if we take any poset $\prod_{i=1}^{\ell} \overrightarrow{\mathbb{A}}_{n_i}$ with $\ell \geq 3$ and $n_i \geq 2$ $\forall i$, it will always have a representation-infinite incidence **k**-algebra since it can clearly be reduced to $\overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_2 \times \overrightarrow{\mathbb{A}}_2$ taking a subtraction.

Therefore we conclude that $\prod_{i=1}^{\ell} \overrightarrow{\mathbb{A}}_{n_i}$ is representation-finite if and only if it is of the form $\overrightarrow{\mathbb{A}}_n \times \overrightarrow{\mathbb{A}}_m$ with $1 \in \{n, m\}$ or $\{n, m\} \in \big\{\{2,2\}, \{2,3\}, \{2,4\}\big\}$.

We need to prove a variation of Lemma 2.31 in order to obtain the main theorem of this chapter. The following theorem is well-known:

Theorem 3.5 ([DIJ17, Theorem 1.4])**.** *Let Λ be a finite dimensional $\mathbf{k}$-algebra. Then, Λ is τ-tilting finite if and only if there are only finitely many isomorphism classes of bricks in* $\operatorname{mod}(\Lambda)$*.*

We can now prove the following:

Lemma 3.6 (c.f. [Pla19, Corollary 2.2])**.** *Let Λ and Λ' be two finite dimensional $\mathbf{k}$-algebras and assume there exists a fully faithful functor $F\colon \operatorname{mod}(\Lambda') \to \operatorname{mod}(\Lambda)$. Then if Λ is τ-tilting finite then so is Λ'.*

As a consequence, if P is a poset such that the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(P)$ is τ-tilting finite, then

1. *any subposet of P has a τ-tilting finite incidence $\mathbf{k}$-algebra.*
2. *any contraction of P has a τ-tilting finite incidence $\mathbf{k}$-algebra.*

Proof. The first part of the result uses the same idea of Remark 2.30. If B is a brick in $\operatorname{mod}(\Lambda')$, then $F(B)$ is a brick in $\operatorname{mod}(\Lambda)$, since $\operatorname{End}_{\Lambda'}(B)$ is isomorphic to $\operatorname{End}_{\Lambda}(B)$. Moreover, two bricks B and B' in $\operatorname{mod}(\Lambda')$ are isomorphic if and only if $F(B)$ and $F(B')$ are isomorphic, since F is fully faithful.

If by contradiction Λ' is τ-tilting infinite then by Theorem 3.5 it admits infinitely many bricks, then so does Λ, contradicting the fact that Λ is τ-tilting finite. Thus, Λ' is τ-tilting finite too.

As a consequence, for any subposet (resp. contraction) P' of P we have the fully faithful functor presented in (2.1) (resp. (2.2)). Therefore, thanks to the first part of this result, since $\mathcal{I}_{\mathbf{k}}(P)$ is τ-tilting finite we conclude that also $\mathcal{I}_{\mathbf{k}}(P')$ is τ-tilting finite for any subtraction (resp. contraction) P' of P. $\square$

It is now time to state and prove the main theorem shown in [BGR24]:

Theorem 3.7 ([BGR24, Theorem 2.15])**.** *Let P be a finite poset. Then $\mathcal{I}_{\mathbf{k}}(P)$ is τ-tilting finite if and only if it is representation-finite.*

Proof. All representation-finite algebras are τ-tilting finite: indeed, by representation-finiteness there are only a finite number n of isomorphism classes of indecomposable modules, with representatives $\{M_1, \ldots, M_n\}$. Therefore we only have 2^n isomorphism classes of basic modules: indeed, we have a bijection

$$\{\text{isomorphism classes of basic modules}\} \xrightarrow{\cong} \mathcal{P}(\{1, \ldots, n\})$$

sending a basic module $B = \bigoplus_{i \in I \subseteq \{1,\ldots,n\}} M_i$ to the subset I and any subset $J \in \mathcal{P}(\{1, \ldots, n\})$ to the basic module $\bigoplus_{j \in J} M_j$ (with the empty set $\emptyset$ sent to the zero module). This bijection is well defined since the decomposition is unique up to re-ordering. Therefore we have $2^n = \operatorname{card}(\mathcal{P}(\{1, \ldots, n\}))$ isomorphism classes of basic modules. But now the set of isomorphism classes of basic τ-tilting modules is included in the set of isomorphism classes of basic modules, thus its cardinality must be $\leq 2^n$. So we conclude that the algebra is τ-tilting finite.

We need to prove that the converse assertion holds for incidence $\mathbf{k}$-algebras of finite posets. Suppose that P has a representation-infinite incidence $\mathbf{k}$-algebra. By Corollary 3.3, one can reduce P to a poset L such that the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(L)$ is tame concealed. Since tame concealed algebras are τ-tilting infinite, as we remarked in Lemma 1.62, it follows directly from Lemma 3.6 that P has a τ-tilting infinite incidence $\mathbf{k}$-algebra. ☐

A priori, the τ-tilting finiteness of an incidence algebra $\mathcal{I}_{\mathbf{k}}(P)$ depends both on the field $\mathbf{k}$ and the poset P. However, thanks to this Theorem 3.7, it only depends on P since by Remark 2.11, the representation-finiteness of incidence $\mathbf{k}$-algebras does not depend on the field.

The key insight leading to Theorem 3.7 was Corollary 3.3, where we showed that minimally representation-infinite incidence $\mathbf{k}$-algebras are tame concealed. Since any proper quotient of a tame concealed $\mathbf{k}$-algebra is representation-finite [Bon17, Proposition 6], we deduce that these incidence $\mathbf{k}$-algebras are *minimally τ-tilting infinite* $\mathbf{k}$*-algebras*, which is to state that they are τ-tilting infinite $\mathbf{k}$-algebras for which all proper quotients are τ-tilting finite (c.f.[MP23]).

Example 3.8 ([BGR24, Example 2.16])**.** Miyamoto and Wang have previously classified posets of the form $\overrightarrow{\mathbb{A}}_n \times \overrightarrow{\mathbb{A}}_m$ that have τ-tilting finite incidence $\mathbf{k}$-algebras [MW21]. In Example 3.4 it was stated that $\overrightarrow{\mathbb{A}}_n \times \overrightarrow{\mathbb{A}}_m$ is representation-finite if and only if

$$1 \in \{n, m\} \quad \text{or} \quad \{n, m\} \in \big\{\{2,2\}, \{2,3\}, \{2,4\}\big\} \tag{3.1}$$

By Theorem 3.7, it follows that the condition stated in (3.1) is necessary and sufficient for $\overrightarrow{\mathbb{A}}_n \times \overrightarrow{\mathbb{A}}_m$ to have a τ-tilting finite incidence $\mathbf{k}$-algebra.

4. Generic bricks over incidence algebras

In this chapter we will focus our study on representation-infinite posets. Firstly, we will give the following definitions:

Definition 4.1 ([BPS24, Introduction])**.** Let Λ be a finite dimensional $\mathbf{k}$-algebra.

- We say that the algebra Λ is *brick-infinite* if it admits an infinite family of non-isomorphic finite-dimensional bricks.
- The algebra Λ is called *brick-continuous* if it admits an infinite family of bricks with the same dimension.
- Given a module $G \in \operatorname{Mod}(\Lambda)$, the *endolength* $\operatorname{endol}(G)$ of G is the length of G when considered as an $\operatorname{End}_\Lambda(G)^{\mathrm{op}}$-module.
- A module $G \in \operatorname{Mod}(\Lambda)$ is called *generic module* if it is indecomposable, with infinite length as Λ-module, but with finite endolength.
- A generic module G is called a *generic brick* if it is also a brick, i.e. $\operatorname{End}_\Lambda(G)$ is a division algebra over $\mathbf{k}$.

For a finite dimensional $\mathbf{k}$-algebra Λ, we have the following result from F. Sentieri [Sen23]:

Theorem 4.2 ([Sen23, Theorem 4.14])**.** *A finite-dimensional $\mathbf{k}$-algebra Λ is τ-tilting finite if and only if every brick over Λ is finitely generated.*

We also have the following theorem from W. Crawley-Boevey [Cra91]:

Theorem 4.3 ([Cra91, Theorem 4.5])**.** *Λ has infinite representation type if and only if there is a generic Λ-module.*

Now let P be a representation-infinite poset. By definition, its incidence algebra $\mathcal{I}_{\mathbf{k}}(P)$ is representation-infinite and thanks to Theorem 3.5 we know that it is also τ-tilting-infinite. By the two previous results of Sentieri and Crawley-Boevey we have that $\mathcal{I}_{\mathbf{k}}(P)$ admits an infinite dimensional brick and a generic module in $\operatorname{Mod}(\mathcal{I}_{\mathbf{k}}(P))$

that in general may be different from each other. We will show that in our case we can find a module having both properties, that is a generic brick in $\mathrm{Mod}(\mathcal{I}_{\mathbf{k}}\,(P))$.

In order to understand if there is a generic brick in $\mathrm{Mod}(\mathcal{I}_{\mathbf{k}}\,(P))$ we will rely on the following result of R. Bautista, E. Pérez and L. Salmerón [BPS24]:

Theorem 4.4 ([BPS24, Corollary 1.3]). *Let Λ be a tame finite-dimensional algebra. Then, Λ is brick-continuous if and only if Λ admits a generic brick.*

However, to apply such theorem, we can only work with tame incidence algebras. We know that the minimally representation-infinite posets of Table 2.1 are tame concealed (by the proof of Corollary 3.3). Moreover, from Theorem 1.61, they are indeed tame, therefore we can apply Theorem 4.4 to guarantee the existence of a generic brick if we prove that they are brick-continuous.

For tame algebras we have the following result:

Theorem 4.5 ([Cra91, Theorem 4.4]). *If Λ has tame representation type and G is a generic Λ-module,then $\mathrm{End}_\Lambda(G)/\operatorname{Rad}(\mathrm{End}_\Lambda(G)) \cong \mathbf{k}(x)$ (where $\mathbf{k}(x)$ denotes the field of rational functions, i.e. the field of fractions of the polynomial ring $\mathbf{k}[x]$).*

We also have a useful corollary:

Corollary 4.6. *If Λ has tame representation type and G is a generic Λ-module, then G is a generic brick if and only if $\mathrm{End}_\Lambda(G) \cong \mathbf{k}(x)$.*

Proof. If G is a generic brick, $\mathrm{End}_\Lambda(G)$ is a division algebra over $\mathbf{k}$ and in this case $\operatorname{Rad}(\mathrm{End}_\Lambda(G)) = 0$ so by Theorem 4.5 we get that $\mathrm{End}_\Lambda(G) \cong \mathbf{k}(x)$.

Conversely, if $\mathrm{End}_\Lambda(G) \cong \mathbf{k}(x)$, G is a generic module that is also a brick, so it is a generic brick by definition. ☐

Remark 4.7. Notice that G generic module with $\mathrm{End}_\Lambda(G) \cong \mathbf{k}(x)$ implies G generic brick for any finite dimensional $\mathbf{k}$-algebra Λ of infinite representation type (not necessarily tame).

In this chapter, we will do the following:

(1) we prove that each of the twelve families of minimally representation-infinite posets is brick-continuous;

(2) we explicitly construct a generic brick for each of the twelve families;

(3) we compute a generic brick for any representation-infinite poset.

In order to obtain (3), we recall that any representation-infinite poset can be reduced to a minimally representation-infinite poset (see Lemma 3.1). Therefore we will define some special functors in Section 4.3, that we will call *extension functors*, "reversing" the reduction process. In this way we can apply them to the generic bricks that we found before to obtain a generic brick of any representation-infinite poset.

4.1. Tits form

We need the notions of quadratic form, in particular the Tits form of a quiver. We will follow the definitions in [Rin06], [Bon17], [Les03].

Definition 4.8 ([Rin06, Section 1.0])**.** A polynomial $q = q(x_1, \dots, x_n)$ in n variables with integral coefficients will be said to be an *integral quadratic form* provided it is of the form

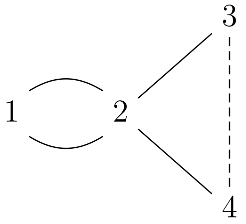
$$q = q(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} q_{ij} x_i x_j$$

with $q_{ij} \in \mathbb{Z}$. Evaluating q at n-tuples of integral numbers, we obtain a function $\mathbb{Z}^n \to \mathbb{Z}$ which also will be denoted by q. We have a partial ordering on $\mathbb{Z}^n$ defined in Definition 1.31. Given $z = (z_1, \dots, z_n) \in \mathbb{Z}^n$, it is said to be:

- *positive*, denoted by $z > 0$, provided that $z \neq 0$ and $z_i \geq 0$ for all i;
- *sincere* if $z_i \neq 0$ for all i.

We can associate a graph to an integral quadratic form. Given q we use $\{1, \dots, n\}$ as a set of vertices and, for $i < j$, we connect the vertex i with the vertex j with $-q_{ij}$ (undirected) edges if $q_{ij} < 0$ and by q_{ij} dashed edges if $q_{ij} > 0$.

For example,



describes the quadratic form

$$\sum_{i=1}^{4} x_i^2 - 2x_1x_2 - x_2x_3 - x_2x_4 + x_3x_4$$

Definition 4.9 ([Rin06, Section 1.1])**.** Given $q = q(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^2 + \sum_{i<j} q_{ij} x_i x_j$ an integral quadratic form in n variables, we define the corresponding symmetric bilinear form $\beta \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ in the following way: $\beta(x, y) = x^t B y$ with B a symmetric matrix with diagonal entries equal to 1, and off-diagonal entries in $\frac{1}{2}\mathbb{Z}$, namely $B = \frac{1}{2}(q_{ij})_{i,j}$ with $q_{ij} = q_{ji}$ for $i > j$, and $q_{ii} = 2$ for all i.

Note that $q(z) = \beta(z, z)$ for any z, and $q_{ij} = 2\beta(e_i, e_j)$, with $e_1, \dots, e_n \in \mathbb{Z}^n$ the canonical basis of $\mathbb{Z}^n$. Moreover the following equation holds (thanks to bilinearity and symmetry of β) for any $z, z' \in Z^n$, $a, b \in \mathbb{Z}$:

$$q(az + bz') = a^2 q(z) + b^2 q(z') + 2ab\beta(z, z') \tag{4.1}$$

For example the bilinear form β associated to the integral quadratic form $q = \sum_{i=1}^{4} x_i^2 - 2x_1x_2 - x_2x_3 - x_2x_4 + x_3x_4$ defined before, is given by the matrix

$$B = \frac{1}{2}\begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & -1 & -1 \\ 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}$$

So $\beta(x, y) = x^t By$ and it is clear that $\beta(x, x) = q(x)$.

Definition 4.10 ([Rin06, Section 1.0])**.** A quadratic form q in n variables with integral coefficients is said to be *positive semi-definite* if $q(z) \geq 0$ for all $z \in \mathbb{Z}^n$.

For a positive semi-definite quadratic form q, the elements $z \in \mathbb{Z}^n$ satisfying $q(z) = 0$ are called *radical vectors*. They form a subgroup $\mathrm{Rad}(q)$ of $\mathbb{Z}^n$ called the *radical of* q, and the rank of $\mathrm{Rad}(q)$ is called the *radical rank of* q.

In case 0 is the only radical element of q or, equivalently, the radical rank of q is 0, the form q is said to be *positive definite*.

$\mathrm{Rad}(q)$ is indeed a subgroup of $\mathbb{Z}^n$: if $x, y \in \mathrm{Rad}(q)$ with q positive semi-definite then, using (4.1), we have that

$$q(x + y) + q(x - y) = q(x) + q(y) + 2\beta(x, y) + q(x) + q(y) - 2\beta(x, y) = 0$$

since x and y are radical vectors. But now q is positive semi-definite, therefore we must have $q(x + y) = 0$ and $q(x - y) = 0$.

Lemma 4.11. *Let q be an integral quadratic form, β the corresponding symmetric bilinear form with matrix B. If q is positive semi-definite then we have that*

$$\mathrm{Rad}(q) = \ker(B)$$

Proof. ($\supseteq$) If $z \in \mathbb{Z}^n$ is in $\ker(B)$ (i.e. $Bz = 0$) then

$$q(z) = \beta(z, z) = z^t Bz = 0$$

Thus z is in the radical $\mathrm{Rad}(q)$ of q.

($\subseteq$) Let z be in the radical $\mathrm{Rad}(q)$ of q, so $q(z) = 0$. Since q is positive semi-definite, we have that β is a positive semi-definite bilinear form, i.e. B is a positive semi-definite matrix (all its eigenvalues are ≥ 0). By Cholesky factorisation (see [HJ12, Corollary 7.2.9]) we can write $B = L^t L$ with L a real lower triangular matrix with non-negative diagonal entries. Therefore we have

$$0 = q(z) = \beta(z, z) = z^t Bz = z^t L^t Lz = (Lz)^t Lz = \|Lz\|^2$$

with $\|\text{-}\|$ being the (real) Euclidean norm ($\|x\|^2 = x^t x = \sum_{i=1}^{n} x_i^2$). Then $\|Lz\|^2 = 0$, which implies that $Lz = 0$. Thus $Bz = L^t Lz = 0$, so $z \in \ker(B)$. □

Definition 4.12 ([Rin06, Section 1.0])**.** Given a quadratic form q in n variables, it is said to be *weakly positive* if $q(z) > 0$ for all positive $z \in \mathbb{Z}^n$.

We denote by q^t (with $1 \leq t \leq n$) the restriction of q to the hyper-plane defined by $z_t = 0$, thus

$$q^t = q(z_1, \ldots, z_{t-1}, 0, z_{t+1}, \ldots, z_n)$$

An integral quadratic form q in $n \geq 3$ variables is said to be *critical* provided q is not weakly positive, however all the forms q^t, $1 \leq t \leq n$, are weakly positive.

We have the following result due to Ovsienko:

Theorem 4.13 ([Rin06, Theorem 2])**.** *A critical quadratic form is positive semi-definite with radical rank* 1*, and with a sincere positive radical vector.*

Proof. Let q be a critical integral quadratic form in n variables, with $n \geq 3$. Since q is not weakly positive, there exists some positive vector $y = (y_1, \ldots, y_n) > 0$ with $q(y) \leq 0$. Since all q^t are weakly positive, $1 \leq t \leq n$, we must have $y_t > 0$ for all t: indeed if $y_t = 0$ for some t then $q^t(y) = q(y) \leq 0$, a contradiction. Thus, y is sincere. We choose such a y with $\sum_{i=1}^n y_i$ being minimal and we will say that y is minimal. Since y is minimal, for each i we have $q(y - e_i) > 0$ (otherwise it would contradict the minimality of y). Therefore we have

$$0 < q(y - e_i) = q(y) + q(e_i) - 2\beta(y, e_i)$$

and since $q(e_i) = 1$ for all i (it follows directly from the definition of quadratic form), we obtain

$$2\beta(y, e_i) - 1 < q(y) \Rightarrow 2\beta(y, e_i) \leq q(y)$$

(because we are working with integers). Therefore we have that

$$2q(y) = 2\beta(y, y) = \sum_{i=1}^n 2\beta(y, e_i) y_i \leq \sum_{i=1}^n q(y) y_i = q(y) \sum_{i=1}^n y_i$$

If we assume $q(y) < 0$, we can divide this inequality by $q(y)$ and obtain that $2 \geq \sum_{i=1}^n y_i \geq 3$ (the last inequality holds since $n \geq 3$ and y is sincere). Therefore $q(y) < 0$ leads to a contradiction, so we must have $q(y) = 0$. Combining this with $2\beta(y, e_i) \leq q(y)$ we conclude that $\beta(y, e_i) \leq 0$ for all i. However,

$$0 = q(y) = \beta(y, y) = \sum_{i=1}^n \beta(y, e_i) y_i$$

with $y_i > 0$, thus we must have $\beta(y, e_i) = 0$ for all i. In particular $\beta(y, z) = 0$ for all $z \in \mathbb{Z}^n$ (by bilinearity of β and the fact that e_i is a basis for $\mathbb{Z}^n$).

Now assume that there is $z \in \mathbb{Z}^n$ with $q(z) \leq 0$. Consider the ratios z_i/y_i (well defined since y is sincere positive) and we choose $1 \leq m \leq n$ such that $z_m/y_m \leq z_i/y_i$ for all i, thus $z_m y_i \leq z_i y_m$. Defining $x = y_m z - z_m y \in \mathbb{Z}^n$, we have that $x_m = 0$ and $x_i \geq 0$ for all i. Now

$$\begin{aligned} q(x) &= q(y_m z - z_m y) = q(y_m z) + q(z_m y) - 2\beta(y_m z, z_m y) \\ &= y_m^2 q(z) + z_m^2 q(y) - 2y_m z_m \beta(z, y) = y_m^2 q(z) \leq 0 \end{aligned}$$

that holds by (4.1) and for what we proved above. But $q(x) = q^m(x) \geq 0$ since $x_m = 0$ and q^m weakly positive. Thus $q^m(x) = 0$ that means $x = 0$, again since q^m weakly positive and $x \geq 0$. This shows that z and y are linearly dependent over $\mathbb{Z}$. Moreover, from $q(z) \leq 0$, we get

$$\begin{aligned} 0 \geq y_m^2 q(z) &= q(y_m z) = q(y_m z - z_m y + z_m y) = q(x - z_m y) \\ &= q(-z_m y) = z_m^2 q(y) = 0 \end{aligned}$$

so we must have $q(z) = 0$ (because y is sincere, so $y_m \neq 0$). Therefore, for any $z \in \mathbb{Z}^n$ we get that $q(z) \geq 0$, i.e. q is positive semi-definite. Since $q(y) = 0$ with q positive semi-definite we conclude that y is a sincere positive radical vector and by the reasoning above, any other radical vector is linearly dependent with y, i.e. $\mathrm{Rad}(q)$ has rank 1. ☐

Now we define some quadratic forms over algebras and quivers.

Definition 4.14 (c.f. [Bon17, Section 2.2])**.** Let Λ be a finite dimensional $\mathbf{k}$-algebra with finite global dimension ($\mathrm{gl.dim}(\Lambda) \leq m$), $\Lambda \cong \mathbf{k}Q/I$ for some finite quiver Q and admissible ideal I. Let $n = \mathrm{card}(Q_0)$, then we can define the *Euler characteristic* of Λ as the bilinear form

$$\langle \underline{\dim}(M), \underline{\dim}(M') \rangle := \sum_{i \geq 0} (-1)^i \dim_{\mathbf{k}} \mathrm{Ext}^i_\Lambda(M, M')$$

where M, M' are finitely generated Λ-modules. It is well defined since $\mathrm{gl.dim}(\Lambda) \leq m$, so $\mathrm{Ext}^i_\Lambda(M, M') = 0$ for $i \geq m+1$. We also define the *Euler quadratic form* of Λ as

$$\chi_\Lambda(\underline{\dim}(M)) := \langle \underline{\dim}(M), \underline{\dim}(M) \rangle$$

Remark 4.15. This form can be defined over the Grothendieck group $\mathbf{k}_0(\Lambda)$ of Λ that consists of all Λ-modules modulo all short exact sequences. $\mathbf{k}_0(\Lambda)$ can be identified with $\mathbb{Z}^n$ since the isomorphism classes of the simple Λ-modules form a of basis of $\mathbf{k}_0(\Lambda)$ and its elements can be identified with their dimension vector. For more details see [Rin06] and [ASS06, Ch.III, Section 3].

Definition 4.16 ([Les03, Introduction])**.** Consider an arbitrary bound quiver (Q, I) with $n = \operatorname{card}(Q_0)$ and Q acyclic. Let $\Lambda = \mathbf{k}Q/I$. The *Tits quadratic form* of Λ is the integral quadratic form $q_\Lambda \colon \mathbb{Z}^n \to \mathbb{Z}$ defined by the formula:

$$\begin{aligned} q_\Lambda(x) &:= \sum_{i\in Q_0} x_i^2 - \sum_{\alpha\in Q_1} x_{s(\alpha)}x_{t(\alpha)} + \sum_{i,j\in Q_0} r_{ij}x_ix_j \\ &= \sum_{i\in Q_0} x_i^2 - \sum_{i,j\in Q_0} d_{ij}x_ix_j + \sum_{i,j\in Q_0} r_{ij}x_ix_j \end{aligned}$$

where d_{ij} is the number of arrows from i to j and $r_{ij} = \operatorname{card}(R \cap \varepsilon_j\mathbf{k}Q\varepsilon_i)$, with R a minimal set of relations which generates I and $\varepsilon_j\mathbf{k}Q\varepsilon_i$ the vector space spanned by all the paths starting at i and ending in j.

We have the following result by K. Bongartz that will be useful in our study:

Theorem 4.17 ([Bon17, Proposition 2.2])**.** *Let $\Lambda = \mathbf{k}Q/I$ be a finite-dimensional $\mathbf{k}$-algebra with $\operatorname{gl.dim}(\Lambda) \leq 2$ and Q acyclic. Then χ_Λ and q_Λ coincide.*

Idea of the proof. It is possible to see that

- $\dim_{\mathbf{k}} \operatorname{Ext}^1_\Lambda(S(i), S(j)) = d_{ij}$ (c.f. [ASS06, Ch.III, Lemma 2.12]);
- $\dim_{\mathbf{k}} \operatorname{Ext}^2_\Lambda(S(i), S(j)) = r_{ij}$ (c.f. [Bon17, Section 1]).

From [Bon17, Proposition 2.2] we have that

$$\begin{aligned} \chi_\Lambda(\underline{\dim}(M)) = &\sum_{i\in Q_0} \underline{\dim}(M)_i^2 - \sum_{i,j\in Q_0} \dim_{\mathbf{k}} \operatorname{Ext}^1_\Lambda(S(i), S(j))\underline{\dim}(M)_i\underline{\dim}(M)_j \\ &+ \sum_{i,j\in Q_0} \dim_{\mathbf{k}} \operatorname{Ext}^2_\Lambda(S(i), S(j))\underline{\dim}(M)_i\underline{\dim}(M)_j \end{aligned}$$

that coincides with $q_\Lambda(\underline{\dim}(M))$ for any $M \in \operatorname{mod}(\Lambda)$ (more precisely $M \in \mathbf{k}_0(\Lambda)$). □

We want to study incidence algebras $\mathcal{I}_{\mathbf{k}}(P)$ of critical posets P. We recall that they are tame concealed by the proof of Corollary 3.3, in particular they are tilted algebras of Euclidean type, so the following result holds:

Theorem 4.18. *Let P be a critical poset, $\mathcal{I}_{\mathbf{k}}(P)$ its incidence algebra.*

1. $\operatorname{gl.dim}(\mathcal{I}_{\mathbf{k}}(P)) \leq 2$ *and any indecomposable $X \in \operatorname{mod}(\mathcal{I}_{\mathbf{k}}(P))$ has projective dimension or injective dimension less or equal than one.*
2. *Q_P is acyclic.*

(3) *The Tits form* $q_{\mathcal{I}_{\mathbf{k}}(P)}$ *(also denoted by* q_P *in the following) is critical. In particular it is positive semi-definite with radical rank* 1 *and with a sincere positive radical vector.*

Proof. (1) It is a direct application of [ASS06, Ch.VIII, Lemma 3.2] because the minimally representation-infinite incidence algebras $\mathcal{I}_{\mathbf{k}}(P)$ of the critical posets P are tilted of Euclidean type.

(2) It follows again by the fact that we are working with tilted algebras and for such algebras the associated quiver is acyclic (see [ASS06, Ch.VIII, Corollary 3.4]).

(3) By the proof of [Rin06, Proposition 4.3.7] we know that tame concealed algebras have a critical Euler quadratic form, therefore $\chi_{\mathcal{I}_{\mathbf{k}}(P)}$ (also denoted by χ_P) is critical. But now, since (1) and (2) hold, we can apply Theorem 4.17 and we have that $\chi_P = q_P$. Therefore the Tits form q_P is critical and by Theorem 4.13 we conclude that it is positive semi-definite with radical rank 1 and with a sincere positive radical vector. ☐

Remark 4.19. The last part of point (3) of the previous Theorem can be also proved differently, saying that each critical poset is tame concealed, in particular tilted of Euclidean type and for these algebras the result holds (see [Rin06, Proposition 4.2.8'] and c.f. [ASS06, Ch.VI, Proposition 4.5]).

4.2. Generic bricks of critical posets

We will now present the general construction of a generic brick for all the twelve families of critical posets in Table 2.1, i.e. for all the minimally representation-infinite incidence **k**-algebras. The idea we will apply is the following for all the different cases: let P be a critical poset,

- by Theorem 4.18 we know that q_P is critical, in particular it is positive semi-definite with radical rank 1 and with a sincere positive radical vector. We find $\mathrm{Rad}(q_P)$ using Lemma 4.11 through Gauss-Jordan elimination method on the matrix B_P (matrix associated to the Tits form q_P, i.e. $q_P(z) = z^t B_P z$, see Definition 4.9) and we denote by y_P the minimal sincere positive radical vector (in $\mathbb{Z}^n$);
- we define a family of representations $B(\lambda)_P$ with $\underline{\dim}(B(\lambda)_P) = y_P$ and we show that for $\lambda \in \mathbf{k}$ we obtain an infinite family of non-isomorphic bricks with the same dimension, proving that $\mathcal{I}_{\mathbf{k}}(P)$ is brick-continuous;
- we explicitly construct the generic brick G_P (that exists thank to Theorem 4.4) starting from the brick-continuous family $B(\lambda)_P$.

We will only consider some particular orientations of the critical posets thanks to the following result:

Lemma 4.20. *Let P_1 and P_2 be two poset with a common subposet P' such that their Hasse quiver is given by*

$$\begin{tikzcd} Q_{P'} \arrow[r, no head] & a_1 \arrow[r, no head] & a_2 \arrow[r, no head] & \cdots \arrow[r, no head] & a_n \end{tikzcd}$$

where the undirected edges denote that the arrows may go in either direction. Then there is a bijection between bricks in $\mathrm{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$ *and bricks in* $\mathrm{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$.

Moreover if we consider the opposite poset of P_1, of P_2 or both, we have a bijection of finite dimensional bricks.

Proof. Step 1: We show that given a representation X of the previous Hasse quiver we can obtain a representation of all the possible Hasse quivers obtained by considering every other orientation of the arrows. We prove the result by induction on the number n of elements a_i. If $n = 1$ we have the following situation

$$\begin{tikzcd} Q_{P'} \arrow[r, no head] & a_1 \end{tikzcd}$$

For each orientation of the undirected edge, a_1 is either a sink or a source, so we can apply its corresponding reflection functor on a representation X to change the orientation of the arrow.

Suppose the result holds for $n-1$ and consider the case with n elements with the following orientation

$$\begin{tikzcd} Q_{P'} \arrow[r] & a_1 \arrow[r, no head] & a_2 \arrow[r, no head] & \cdots \arrow[r, no head] & a_n \end{tikzcd}$$

Let $P'' := P' \cup \{a_1\}$ so we have

$$\begin{tikzcd} Q_{P''} \arrow[r, no head] & a_2 \arrow[r, no head] & \cdots \arrow[r, no head] & a_n \end{tikzcd}$$

and by induction we can reach every possible orientation of the arrows in the diagram in a representation X through a composition of reflection functors. In particular we can reach an orientation such that the arrow between a_1 and a_2 is of the form $a_1 \leftarrow a_2$. But then a_1 is a sink and so we can apply its corresponding reflection functor to change the orientation of the arrow.

For the case

$$\begin{tikzcd} Q_{P'} & a_1 \arrow[l] \arrow[r, no head] & a_2 \arrow[r, no head] & \cdots \arrow[r, no head] & a_n \end{tikzcd}$$

the reasoning is similar, since we can reach an orientation with $a_1 \to a_2$ and then a_1 is a source.

In any case we can change the orientation of all the arrows in a representation through a composition of reflection functors, so we conclude.

Step 2: We prove the result for the posets P_1 and P_2. These posets will have their own orientation that in general may be different. If $P_1 = P_2$ there is nothing to prove, so we assume that there is at least one arrow in Q_{P_1} that has a the opposite direction in Q_{P_2}. We prove the result by induction on the number m of reflection functors composed to link the representations of P_1 with the representations of P_2 as proved in Step 1. If $m = 1$ we have only applied one reflection functor $S^+_{a_i}$ (the case $S^-_{a_i}$ is analogous). By Theorem 1.38 we have a bijection of indecomposable representations given by the reflection functor, except for the simple representation $S(a_i)$. Moreover by Lemma 1.45 reflection functors send bricks to bricks. Therefore we have a bijection between bricks in $\mathrm{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$ and bricks in $\mathrm{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$ defined as follows: for $B_1 \in \mathrm{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$ brick we send it to $S^+_{a_i}(B_1)$, and the inverse is given by $S^-_{a_i}(B_2)$ for a brick $B_2 \in \mathrm{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$. The only element that we miss is the simple representation $S(a_i)_{P_1}$ that we send to $S(a_i)_{P_2}$. For the case $S^-_{a_i}$ the reasoning is the same. For $m \geq 2$ we apply the same process as before. Indeed we consider the first $m - 1$ reflection functors applied, that will give us the representation of a poset P_3 having the same properties of P_1 and P_2 with a different orientation. By induction we have a bijection between bricks in $\mathrm{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$ and bricks in $\mathrm{Rep}_{\mathbf{k}}(Q_{P_3}, I_{P_3})$ and applying the same reasoning as before we conclude the same for P_2 and P_3. So we get our result.

Step 3: We recall the definition of duality of representations: given $X \in \mathrm{Rep}_{\mathbf{k}}(Q, I)$ (for a bound quiver (Q, I)) we define $D(X) \in \mathrm{Rep}_{\mathbf{k}}(Q^{\mathrm{op}}, I')$ as the representation with $D(X)_i := D(X_i)$ in the vertices and linear maps $D(\varphi_\alpha)\colon D(X)_j \to D(X)_i$ for any $i, j \in Q_0$ and any arrow $\alpha\colon i \to j$, using the duality of vector spaces.

We consider the opposite poset P_2^{op} of P_2. By Remark 2.25, duality of finite dimensional representations is an equivalence (with DD equivalent to the identity functor thanks to the properties of duality of finite dimensional vector spaces), so D is fully faithful and by the proof of Lemma 3.6 it preserves bricks. Again by the properties of duality of vector spaces the dimension is also preserved. So we have a bijection of bricks between $\mathrm{rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$ and $\mathrm{rep}_{\mathbf{k}}(Q_{P_2^{\mathrm{op}}}, I_{P_2^{\mathrm{op}}})$. Using the previous steps we have a bijection of bricks between $\mathrm{rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$ and $\mathrm{rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$ (where the property of being finite dimensional is preserved by reflection functors thanks to point (7) of Lemma 1.36), and so we have a bijection between bricks in $\mathrm{rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$ and $\mathrm{rep}_{\mathbf{k}}(Q_{P_2^{\mathrm{op}}}, I_{P_2^{\mathrm{op}}})$. The cases P_1^{op} with P_2 and P_1^{op} with P_2^{op} are analogous, thus we conclude. □

Remark 4.21. Thanks to this result we can study only one particular orientation of the critical posets. Indeed if we find a brick-continuous family for one particular orientation of a critical poset, applying Lemma 4.20 we get brick-continuous families for any orientation of the undirected edges and all their opposite posets.

Our construction of the generic bricks for the critical posets will only depend on the brick-continuous family, so we can find a generic brick for all the possible orientations of the undirected edges of the critical posets and all the opposites of these.

We will now state a main theorem of this Chapter, whose complete and detailed proof can be found in Appendix A.

Theorem 4.22. *The twelve families of minimally representation-infinite incidence* $\mathbf{k}$*-algebras given by critical posets in Table 2.1 (considering any orientation of the undirected edges), and also the opposites of these, are brick-continuous and admit a generic brick which can be explicitly constructed.*

In particular we have the following generic bricks (considering a specific orientation for each critical poset):

($\widetilde{\mathbb{A}}_3$) *see Appendix A.1*

$$\begin{tikzcd}
 & \mathbf{k}(x) & \\
\mathbf{k}(x) \arrow[ur, "x"] \arrow[dr, "1"'] & & \mathbf{k}(x) \arrow[ul, "1"'] \arrow[dl, "1"] \\
 & \mathbf{k}(x) &
\end{tikzcd}$$

($\widetilde{\mathbb{D}}_4$) *see Appendix A.2*

$$\begin{tikzcd}
 & \mathbf{k}(x) \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & \\
\mathbf{k}(x) & \mathbf{k}(x)^2 \arrow[l, "(1\ 1)"] \arrow[d, "(x\ 1)"] & \mathbf{k}(x) \arrow[l, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] \\
 & \mathbf{k}(x) &
\end{tikzcd}$$

($\widetilde{\mathbb{E}}_6$) *see Appendix A.3*

$$\begin{tikzcd}
 & & \mathbf{k}(x) \arrow[d, "\phi_{54}"] & & \\
 & & \mathbf{k}(x)^2 \arrow[d, "\phi_{43,x}"] & & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"'] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"'] & \mathbf{k}(x)^3 & \mathbf{k}(x)^2 \arrow[l, "\phi_{63}"] & \mathbf{k}(x) \arrow[l, "\phi_{76}"]
\end{tikzcd}$$

($\widetilde{\mathbb{E}}_7$) *see Appendix A.4*

$$\begin{tikzcd}
 & & & \mathbf{k}(x)^2 \arrow[d, "\phi_{54,x}"] & & & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"] & \mathbf{k}(x)^3 \arrow[r, "\phi_{34}"] & \mathbf{k}(x)^4 & \mathbf{k}(x)^3 \arrow[l, "\phi_{64}"] & \mathbf{k}(x)^2 \arrow[l, "\phi_{76}"] & \mathbf{k}(x) \arrow[l, "\phi_{87}"]
\end{tikzcd}$$

$(\widetilde{\mathbb{E}}_8)$ *see Appendix A.5*

$$\begin{tikzcd}
 & & \mathbf{k}(x)^3 \arrow[d, red, "\phi_{96,x}"] & & & & & \\
\mathbf{k}(x)^2 \arrow[r, "\phi_{78}"] & \mathbf{k}(x)^4 \arrow[r, "\phi_{86}"] & \mathbf{k}(x)^6 & \mathbf{k}(x)^5 \arrow[l, "\phi_{56}"'] & \mathbf{k}(x)^4 \arrow[l, "\phi_{45}"'] & \mathbf{k}(x)^3 \arrow[l, "\phi_{34}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{23}"'] & \mathbf{k}(x) \arrow[l, "\phi_{12}"']
\end{tikzcd}$$

$(\mathbb{R}_1)$ *see Appendix A.6*

$$\begin{tikzcd}
 & & & & & \mathbf{k}(x)^2 \arrow[dd, dashed, no head] & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"] & \mathbf{k}(x)^3 \arrow[r, "\phi_{34}"] & \mathbf{k}(x)^4 \arrow[r, "\phi_{45}"] & \mathbf{k}(x)^5 \arrow[ur, red, "\phi_{59,x}"] & & \mathbf{k}(x)^3 \arrow[ul, "\phi_{89}"'] \\
 & & & & & \mathbf{k}(x)^4 \arrow[ul, "\phi_{65}"] \arrow[ur, "\phi_{68}"'] & \\
 & & & & & \mathbf{k}(x)^2 \arrow[u, "\phi_{76}"] &
\end{tikzcd}$$

$(\mathbb{R}_2)$ *see Appendix A.7*

$$\begin{tikzcd}
 & & & & \mathbf{k}(x) \arrow[dd, dashed, no head] & & \\
 & & & \mathbf{k}(x)^3 \arrow[ur, red, "\phi_{89,x}"] & & \mathbf{k}(x)^4 \arrow[ul, red, "\phi_{69,x}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{76}"'] \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"'] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"'] & \mathbf{k}(x)^3 \arrow[r, "\phi_{34}"'] & \mathbf{k}(x)^4 \arrow[r, "\phi_{45}"'] & \mathbf{k}(x)^5 \arrow[ul, "\phi_{58}"] \arrow[ur, "\phi_{56}"'] & &
\end{tikzcd}$$

$(\mathbb{R}_3)$ *see Appendix A.8*

$$\begin{tikzcd}
 & & & \mathbf{k}(x)^2 \arrow[dd, dashed, no head] & & & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"] & \mathbf{k}(x)^3 \arrow[ur, red, "\phi_{34,x}"] & & \mathbf{k}(x)^3 \arrow[ul, "\phi_{64}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{76}"'] & \mathbf{k}(x) \arrow[l, "\phi_{87}"'] \\
 & & & \mathbf{k}(x)^2 \arrow[ul, "\phi_{53}"] \arrow[ur, "\phi_{56}"'] & & &
\end{tikzcd}$$

$(\mathbb{R}_4)$ *see Appendix A.9*

$$\begin{tikzcd}
 & & \mathbf{k}(x)^3 \arrow[dd, dashed, no head] & & & & & \\
\mathbf{k}(x)^2 \arrow[r, "\phi_{87}"] & \mathbf{k}(x)^4 \arrow[ur, red, "\phi_{79,x}"] & & \mathbf{k}(x)^5 \arrow[ul, "\phi_{59}"'] & \mathbf{k}(x)^4 \arrow[l, "\phi_{45}"'] & \mathbf{k}(x)^3 \arrow[l, "\phi_{34}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{23}"'] & \mathbf{k}(x) \arrow[l, "\phi_{12}"'] \\
 & & \mathbf{k}(x)^3 \arrow[ul, "\phi_{67}"] \arrow[ur, "\phi_{65}"'] & & & & &
\end{tikzcd}$$

($\mathbb{R}_5$) *see Appendix A.10*

$$\begin{tikzcd}
 & & & & & \mathbf{k}(x)^3 \arrow[ddd, dash, dashed] & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"] & \mathbf{k}(x)^3 \arrow[r, "\phi_{34}"] & \mathbf{k}(x)^4 \arrow[r, "\phi_{45}"] & \mathbf{k}(x)^5 \arrow[ur, red, "\phi_{59,x}"] & & \mathbf{k}(x)^2 \arrow[ul, red, "\phi_{89,x}"'] \\
 & & & & \mathbf{k}(x)^3 \arrow[u, "\phi_{65}"] & & \\
 & & & & & \mathbf{k}(x) \arrow[ul, "\phi_{76}"] \arrow[uur, "\phi_{78}"'] &
\end{tikzcd}$$

($\mathbb{R}_6$) *see Appendix A.11*

$$\begin{tikzcd}
 & & \mathbf{k}(x)^2 \arrow[ddd, dash, dashed] & & & & \\
 & \mathbf{k}(x)^3 \arrow[ur, "\phi_{89}"] & & & & & \\
\mathbf{k}(x)^2 \arrow[r, "\phi_{76}"] & \mathbf{k}(x)^4 \arrow[u, "\phi_{68}"] & & \mathbf{k}(x)^4 \arrow[uul, red, "\phi_{49,x}"'] & \mathbf{k}(x)^3 \arrow[l, "\phi_{34}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{23}"'] & \mathbf{k}(x) \arrow[l, "\phi_{12}"'] \\
 & & \mathbf{k}(x)^3 \arrow[ul, "\phi_{56}"] \arrow[ur, "\phi_{54}"'] & & & &
\end{tikzcd}$$

($\mathbb{R}_7$) *see Appendix A.12*

$$\begin{tikzcd}
 & & \mathbf{k}(x)^2 \arrow[dddd, dash, dashed] & & & \\
 & \mathbf{k}(x)^3 \arrow[ur, "\phi_{69}"] & & & & \\
\mathbf{k}(x)^2 \arrow[r, "\phi_{76}"] & \mathbf{k}(x)^4 \arrow[u, "\phi_{68}"] & & \mathbf{k}(x)^3 \arrow[uul, red, "\phi_{39,x}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{23}"'] & \mathbf{k}(x) \arrow[l, "\phi_{12}"'] \\
 & \mathbf{k}(x)^3 \arrow[u, "\phi_{56}"] & & & & \\
 & & \mathbf{k}(x)^2 \arrow[ul, "\phi_{45}"] \arrow[uur, "\phi_{43}"'] & & &
\end{tikzcd}$$

4.3. Extension functors

We want to find a method to compute a generic brick for any representation-infinite poset. We recall that given a poset P, if it is representation-infinite it can be reduced to a critical poset (by Lemma 3.1) and for such posets we just computed a generic brick. If we can "reverse" this reduction process we may be able to compute a generic brick for all the representation-infinite posets. For this purpose, we will now define the "extension" functors of the contraction and of the subtraction and prove that they have the necessary properties.

4.3.1. Extension functor of contraction

We recall that any contraction of finite posets can be written as a composite of elementary contractions (Lemma 2.29) so we will focus our attention on elementary contractions.

Let P_1, P_2 be finite posets with $c\colon P_1 \to P_2$ an elementary contraction. If $\operatorname{card}(P_1) = n$ then $\operatorname{card}(P_2) = n-1$ (since c is elementary), so up to re-indexing the elements of the posets we may suppose that $P_1 = \{x_1, \dots, x_{n-1}, x_n\}$ with x_{n-1}, x_n neighbours and $P_2 = \{y_1, \dots, y_{n-1}\}$ and c defined by $x_i \mapsto y_i$ for $i = 1, \dots, n-2$ and $x_{n-1}, x_n \mapsto y_{n-1}$. Let $(Q_{P_1}, I_{P_1}), (Q_{P_2}, I_{P_2})$ be the Hasse quivers with relations of P_1 and P_2 respectively. We then define

$$E_c\colon \operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2}) \to \operatorname{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$$

in the following way: let $V = (V_{y_1}, \dots, V_{y_{n-1}}, \varphi_{y_i,y_j}) \in \operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$ and we set

$$E_c(V)_i \coloneqq \begin{cases} V_{y_i} & \text{if} \quad i = 1, \dots, n-1 \\ V_{y_{n-1}} & \text{if} \quad i = n \end{cases}$$

(where we set the notation $E_c(V)_i = E_c(V)_{x_i}$). We set

$$E_c(V)_{n-1} \xrightarrow{1} E_c(V)_n \quad \text{if } x_{n-1} \leq x_n \qquad \text{or} \qquad E_c(V)_n \xrightarrow{1} E_c(V)_{n-1} \quad \text{if } x_n \leq x_{n-1}$$

Now, since c is a contraction, it is order-preserving, so if $x_i \leq_{P_1} x_j$ then $c(x_i) \leq_{P_2} c(x_j)$. Consider $x_i \leq x_j$ with x_i, x_j neighbours and $(i,j) \notin \{(n-1,n),(n,n-1)\}$ (we already defined these cases). We have $c(x_i) \leq c(x_j)$:

- if $c(x_i), c(x_j)$ are neighbours (i.e. there is no $y \in P_2$ such that $c(x_i) \leq y \leq c(x_j)$) then in V we have the linear map $\varphi_{c(x_i),c(x_j)}\colon V_{c(x_i)} \to V_{c(x_j)}$ and we define

$$E_c(\varphi_{c(x_i),c(x_j)}) \coloneqq \varphi_{c(x_i),c(x_j)}\colon E_c(V)_i = V_{c(x_i)} \to V_{c(x_j)} = E_c(V)_j$$

- if if $c(x_i), c(x_j)$ are not neighbours, we list all the possible paths from if $c(x_i)$ to $c(x_j)$ (suppose there are m of them)

$$c(x_i) \leq z_{1,1} \leq \cdots \leq z_{a_1,1} \leq c(x_j)$$
$$\vdots$$
$$c(x_i) \leq z_{1,m} \leq \cdots \leq z_{a_m,m} \leq c(x_j)$$

  We define $\phi_{c(x_i),c(x_j)} \coloneqq \varphi_{z_{a_1,1},c(x_j)} \cdots \varphi_{c(x_i),z_{1,1}}$ that is a linear map $V_{c(x_i)} \to V_{c(x_j)}$. It is well defined because $V \in \operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_2)$ so a path starting at $c_(x_i)$ and ending in $c_(x_j)$ is equivalent to all such other paths, therefore

$$\varphi_{z_{a_1,1},c_(x_j)} \cdots \varphi_{c(x_i),z_{1,1}} = \cdots = \varphi_{z_{a_m,m},c_(x_j)} \cdots \varphi_{c(x_i),z_{1,m}}$$

So we define

$$E_c(V)_i = V_{c(x_i)} \xrightarrow{\phi_{c(x_i),c(x_j)}} V_{c(x_j)} = E_c(V)_j$$

Thanks to the definition of $E_c(V)$ on the K-linear maps it follows that the relations in I_{P_1} are preserved. Indeed if two equivalent paths are not affected by c we find the same relation in I_{P_2} and so we conclude. Otherwise if x_{n-1}, x_n are involved in the paths, c might reduce the length of one path, e.g.

$$\begin{tikzcd}
& c & & & & \\
& & x_n \arrow[ul] & & c & \\
b \arrow[uur] & & x_{n-1} \arrow[u] \arrow[r, "c"] & b \arrow[ur] & & y_{n-1} \arrow[ul] \\
& a \arrow[ul] \arrow[ur] \arrow[uuu, dash, dashed] & & & a \arrow[ul] \arrow[ur] \arrow[uu, dash, dashed] &
\end{tikzcd}$$

or completely eliminate a relation, e.g.

$$\begin{tikzcd}
& x_n & & y_{n-1} \\
b \arrow[ur] & & x_{n-1} \arrow[ul] \arrow[r, "c"] & b \arrow[u] \\
& a \arrow[ul] \arrow[ur] \arrow[uu, dash, dashed] & & a \arrow[u]
\end{tikzcd}$$

In any case, thanks to the definition of E_c on the linear maps, the relations in I_{P_1} are preserved, since in the first case only add the identity map in the relation and this process does not influence it

$$\begin{tikzcd}
& V_c & & & & \\
& & V_{y_{n-1}} \arrow[ul, "\varphi_{y_{n-1},c}"'] & & V_c & \\
V_b \arrow[uur, "\varphi_{b,c}"] & & V_{y_{n-1}} \arrow[u, "1"'] & V_b \arrow[l, "E_c"'] \arrow[ur, "\varphi_{b,c}"] & & V_{y_{n-1}} \arrow[ul, "\varphi_{y_{n-1},c}"'] \\
& V_a \arrow[ul, "\varphi_{a,b}"] \arrow[ur, "\varphi_{a,y_{n-1}}"'] \arrow[uuu, dash, dashed] & & & V_a \arrow[ul, "\varphi_{a,b}"] \arrow[ur, "\varphi_{a,y_{n-1}}"'] \arrow[uu, dash, dashed] &
\end{tikzcd}$$

while in the second case we use the composition of the maps we had and the identity map to get a well defined relation

$$\begin{tikzcd}
& V_{y_{n-1}} & & V_{y_{n-1}} \\
V_b \arrow[ur, "\varphi_{b,y_{n-1}}"] & & V_{y_{n-1}} \arrow[ul, "1"'] & V_b \arrow[l, "E_c"'] \arrow[u, "\varphi_{b,y_{n-1}}"'] \\
& V_a \arrow[ul, "\varphi_{a,b}"] \arrow[ur, "\varphi_{b,y_{n-1}}\varphi_{a,b}"'] \arrow[uu, dash, dashed] & & V_a \arrow[u, "\varphi_{a,b}"']
\end{tikzcd}$$

These examples can be easily generalised to any relation with paths of any length, so we conclude that $E_c(V) \in \operatorname{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$.

Let now define E_c on morphisms: let $f\colon V \to W$ be a morphism of representations in $\operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$. We define $E_c(f)\colon E_c(V) \to E_c(W)$ with components $E_c(f)_i\colon E_c(V)_i \to E_c(W)_i$ as

$$E_c(f)_i \coloneqq \begin{cases} f_{y_i} & \text{if} \quad i = 1, \dots, n-1 \\ f_{y_{n-1}} & \text{if} \quad i = n \end{cases}$$

If $x_{n-1} \leq x_n$ we have the commutative diagram

$$\begin{tikzcd}
E_c(V)_{n-1} = V_{n-1} \arrow[r, "1"] \arrow[d, "f_{y_{n-1}}"'] & V_{n-1} = E_c(V)_n \arrow[d, "f_{y_{n-1}}"] \\
E_c(W)_{n-1} = W_{n-1} \arrow[r, "1"'] & W_{n-1} = E_c(W)_n
\end{tikzcd}$$

A similar idea applies to the case $x_n \leq x_{n-1}$. Now we consider $x_i \leq x_j$ neighbours with $(i,j) \notin \{(n-1,n),(n,n-1)\}$. As before $c(x_i) \leq c(x_j)$. Notice that $\phi_{c(x_i),c(x_j)} = \varphi_{c(x_i),c_(x_j)}$ if $c(x_i)$ and $c(x_j)$ are neighbours. Moreover, thanks to the commutativity of the diagrams given by the morphism f we have that $f_{y_j}\phi_{c(x_i),c(x_j)} = \Psi_{c(x_i),c(x_j)} f_{y_i}$ for any $c(x_i) \leq c(x_j)$ (where W has linear maps ψ_{y_i,y_j} and $\Psi_{c(x_i),c(x_j)}$ is defined in W as $\phi_{c(x_i),c(x_j)}$ is in V, so we also have $\Psi_{c(x_i),c(x_j)} = \psi_{c(x_i),c(x_j)}$ if $c(x_i)$ and $c(x_j)$ are neighbours). So we may write

$$\begin{tikzcd}
E_c(V)_i \arrow[r, "\phi_{c(x_i),c(x_j)}"] \arrow[d, "E_c(f)_i"'] & E_c(V)_j \arrow[d, "E_c(f)_j"] \\
E_c(W)_i \arrow[r, "\Psi_{c(x_i),c(x_j)}"'] & E_c(W)_j
\end{tikzcd}$$

that commutes for any $x_i \leq x_j$ neighbours with $(i,j) \notin \{(n-1,n),(n,n-1)\}$ thanks to the definition of $E_c(f)$, so it is a morphism of representation in $\operatorname{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$.

Lemma 4.23. *E_c is an additive functor.*

Proof. We already seen that given $V, W \in \operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$ with $f\colon V \to W$ we have that $E_c(V), E_c(W) \in \operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$ and $E_c(f)\colon E_c(V) \to E_c(W)$ is a morphism of representations.

Let $1_V\colon V \to V$ be the identity on $V \in \operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$, then $E_c(1_V)_i = 1_{V_i}$ for any $i = 1, \dots, n-1$ and $E_c(1_V)_n = 1_{V_{n-1}}$, so we conclude that $E_c(1_V) = 1_{E_c(V)}$.

Given $V, W, Z \in \operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$ with morphisms $f\colon V \to W$, $g\colon W \to Z$ we have for $i = 1, \dots, n-1$

$$E_c(g \circ f)_i = (g \circ f)_i = g_i \circ f_i = E_c(g)_i \circ E_c(f)_i$$

and

$$E_c(g \circ f)_n = (g \circ f)_{n-1} = g_{n-1} \circ f_{n-1} = E_c(g)_n \circ E_c(f)_n$$

Therefore we conclude that $E_c(g \circ f) = E_c(g) \circ E_c(f)$, so $E_c\colon \operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2}) \to \operatorname{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1})$ is indeed a functor. It is also additive since given morphisms $f\colon V \to W$, $g\colon V \to W$ we have for $i = 1, \dots, n-1$

$$E_c(g + f)_i = (g + f)_i = g_i + f_i = E_c(g)_i + E_c(f)_i$$

and

$$E_c(g + f)_n = (g + f)_{n-1} = g_{n-1} + f_{n-1} = E_c(g)_n + E_c(f)_n$$

Therefore we conclude that $E_c(g + f) = E_c(g) + E_c(f)$. □

Definition 4.24. With the previous setting, we call the functor E_c the *extension functor* of the elementary contraction c.

We recall the following result:

Lemma 4.25 ([Jac85, Proposition 3.1])**.** *Given a finite dimensional* $\mathbf{k}$*-algebra* $\Lambda\,(\cong \mathbf{k}Q/I$ *for some bound quiver* $(Q, I))$*, a module* $M \in \operatorname{Mod}(\Lambda)$ *(equivalently a representation in* $\operatorname{Rep}_{\mathbf{k}}(Q, I)$*) is indecomposable if and only if* 0 *and* 1 *are the only idempotents in* $\operatorname{End}(M)$*.*

Theorem 4.26. *The extension functor* E_c *is fully faithful. In particular it preserves indecomposables and bricks.*

Proof. Given a morphism $f\colon V \to W$ with $V, W \in \operatorname{Rep}_{\mathbf{k}}(Q_{P_2}, I_{P_2})$ we can write it as $f = (f_1, \dots, f_{n-1})$. We consider

$$\Phi\colon \operatorname{Hom}(V, W) \to \operatorname{Hom}(E_c(V), E_c(W))$$

sending $f = (f_1, \dots, f_{n-1})$ to $\Phi(f) \coloneqq E_c(f) = (f_1, \dots, f_{n-1}, f_{n-1})$. It is clearly an injective homomorphism and we prove that is also surjective. Let $g = (g_1, \dots, g_{n-1}, g_n) \in \operatorname{Hom}(E_c(V), E_c(W))$. If $x_{n-1} \le x_n$ in P_1 we have

$$\begin{tikzcd}
E_c(V)_{n-1} = V_{n-1} \arrow[r, "1"] \arrow[d, "g_{n-1}"'] & V_{n-1} = E_c(V)_n \arrow[d, "g_n"] \\
E_c(W)_{n-1} = W_{n-1} \arrow[r, "1"'] & W_{n-1} = E_c(W)_n
\end{tikzcd}$$

so $g_{n-1} = g_n$. Therefore we may consider the morphism $f = (g_1, \dots, g_{n-1})$ (the commutativity of the diagrams for f follows immediately from the commutativity of the diagrams for g) in $\mathrm{Hom}(V, W)$ and we get that $\Phi(f) = g$. Thus we conclude that E_c is a fully faithful functor.

It follows that $\mathrm{End}(V) \cong \mathrm{End}(E_c(V))$ (taking $W = V$), so by Remark 2.30 and the proof of Lemma 3.6, E_c preserves bricks and finite dimensional indecomposables. Moreover E_c preserves any indecomposable thanks to Lemma 4.25: let M be an indecomposable representation of P_2 (not necessarily finite dimensional) and $E_c(M)$ the corresponding representation of P_1. Let $e \in \mathrm{End}(E_c(M))$ be an idempotent ($e^2 = e$). Since E_c is full there exist $f, g \in \mathrm{End}(M)$ such that $E_c(f) = e^2$ and $E_c(g) = e$. But now $E_c(f) = e^2 = e \circ e = E_c(g)E_c(g) = E_c(g^2)$ and $E_c(f) = e^2 = e = E_c(g)$. By faithfulness of E_c we must have $f = g^2 = g$. So g is an idempotent in $\mathrm{End}(M)$ but M is indecomposable, thus $g = 0, 1$. This implies that $e = E_c(g) = 0, 1$, so $E_c(M)$ is indecomposable too. □

4.3.2. Extension functor of subtraction

We recall that given a subtraction of a poset we may suppose to remove one vertex at the time (c.f. Definition 2.27), so we will focus our attention on subtractions that remove just one element. Our definition of the extension functor of the subtraction is analogous to the functor proposed in [Mit65, Lemma 9.3] and [Lou75, Proposition 7], and we will give more details about its properties.

Let P be a finite poset, P' a subposet of P with $n = \mathrm{card}(P) = \mathrm{card}(P') + 1$ and denote by $s \colon P' \to P$ the inclusion of P' in P as a subposet. Up to re-indexing the elements of the posets we may suppose that $P = \{1, \dots, n-1, n\}$ and $P' = \{1, \dots, n-1\}$, so the subtraction removed the vertex n. Let $(Q_P, I_P), (Q_{P'}, I_{P'})$ be the Hasse quivers with relations of P and P' respectively. We then define

$$R_s \colon \mathrm{Rep}_{\mathbf{k}}(Q_P, I_P) \to \mathrm{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})$$

in the following way: let $V = (V_1, \dots, V_n, \varphi_{i,j}) \in \mathrm{Rep}_{\mathbf{k}}(Q_P, I_P)$ and we set

$$R_s(V)_i \coloneqq V_i \quad \text{for} \quad i = 1, \dots, n-1$$

For $i \leq_P n$ or $n \leq_P j$ we have no arrows between i and n or n and j in P' (since we removed n) and so there is no $\mathbf{k}$-linear map in $R_s(V)$ associated to them. Now let $i \leq_P j$ with $i, j \neq n$:

(1) if they are neighbours we set $R_s(\varphi_{i,j}) = \varphi_{i,j}$.

(2) if they are not neighbours, it means that $]i, j[_P \neq \emptyset$.

   (*i*) If $n \notin]i, j[_P$ we apply (1) for any neighbour elements $i \leq k$, $h \leq j$ or $a \leq b$ (with $a, b, k, h \in]i, j[_P$).

(*ii*) Otherwise $n \in]i,j[_P$ and we choose i,j such that $]i,n[_P =]n,j[_P = \emptyset$. If $]i,j[_P = \{n\}$ it means that we have a unique sequence $i \leq n \leq j$ in P and we define

$$R_s(V)_i \xrightarrow{\varphi_{n,j}\varphi_{i,n}} R_s(V)_j$$

If there are other elements different from n in $]i,j[_P$ (i.e. $]i,j[_P \setminus \{n\} \neq \emptyset$) we apply (1) for any neighbour elements $i \leq k$, $h \leq j$ or $a \leq b$ (with $a,b,k,h \in]i,j[_P$).

The definition of $R_s(V)$ on linear maps preserves the relations in I_P. The paths not containing n remain the same in P', so do the relations involving them in $I_{P'}$. Removing n might remove completely a relation or shorten a path containing n but in any case the relation in $I_{P'}$ is preserved thanks to the definition of R_s. Therefore $R_s(V) \in \mathrm{Rep}_{\mathbf{k}}(Q_P, I_P)$.

Now let $f = (f_1, \dots, f_n)\colon V \to W$ be a morphism of representations of P. We simply define

$$R_s(f) \coloneqq (f_1, \dots, f_{n-1}) = f_{|P'}$$

that is a morphism between $R_s(V)$ and $R_s(W)$ because f is a morphism itself, and so we have all the needed commutative squares.

Lemma 4.27. *R_s is an additive functor.*

Proof. R_s is well defined on representations and their morphism. Moreover by the definitions for any $1_V\colon V \to V$, $f,h\colon V \to W$, $g\colon W \to Z$ we have that

$$R_s(1_V) = (1_V)_{|P'} = 1_{R_s(V)}$$

$$R_s(g \circ f) = (g \circ f)_{|P'} = g_{|P'} \circ f_{|P'} = R_s(g) \circ R_s(f)$$

and

$$R_s(h + f) = (h + f)_{|P'} = h_{|P'} + f_{|P'} = R_s(h) + R_s(f)$$

□

Definition 4.28. With the previous setting, we call the functor R_s the *reduction functor* of the subtraction given by $s\colon P' \to P$.

Now we look for the right adjoint of R_s. We define the following subposet of P'

$$Y(i) \coloneqq \{x \in P' \mid i \leq x\}$$

and we notice that

$$Y(i) \supseteq Y(j) \quad \text{if} \quad i \leq j \tag{4.2}$$

Indeed if $y \in Y(j)$ then $j \leq y$ but $i \leq j$ so $i \leq j \leq y$. therefore $y \in Y(i)$ for any $y \in Y(j)$ and we conclude.

We then define

$$E_s \colon \operatorname{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'}) \to \operatorname{Rep}_{\mathbf{k}}(Q_P, I_P)$$

in the following way: let $V = (V_1, \dots, V_{n-1}, \varphi_{i,j}) \in \operatorname{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})$ and set

$$E_s(V)_i := \lim V_{|Y(i)}$$

where $V_{|Y(i)}$ denotes the restriction of the representation V on the vertices in $Y(i)$ (i.e. $(V_{|Y(i)})_y = V_y$ for $y \in Y(i)$ and 0 otherwise).

We check that $E_s(V)_i = \lim V_{|Y(i)} = V_i$ for any $i \neq n$: clearly $i \in Y(i)$ so V_i is in vertex i in $V_{|Y(i)}$. We have

$$\begin{tikzcd}
& V_i \arrow[dl, "1"'] \arrow[dr, "\varphi_{i,j}"] & & & & V_i \arrow[dl, "\varphi_{i,h}"'] \arrow[dr, "\phi_{i,k}"] & \\
V_i \arrow[rr, "\varphi_{i,j}"'] & & V_j & & V_h \arrow[rr, dotted, "(\varphi_{h,k})"'] & & V_k
\end{tikzcd}$$

for any $j, h, k \in Y(i)$, with $\phi_{a,b}$ defined as in 4.3.1 (the dotted lines means that there may be no map if the elements h, k are not comparable). Now if we take a vector space L with maps $\psi_y \colon L \to V_y$ such that $\phi_{a,b}\psi_a = \psi_b$ for any $y, a, b \in Y(i)$ with $a \leq b$, we have the unique map $\psi_i \colon L \to V_i$ that makes the following diagram commutes

$$\begin{tikzcd}
& L \arrow[d, "\psi_i"'] \arrow[ddl, bend right, "\psi_h"'] \arrow[ddr, bend left, "\psi_k"] & \\
& V_i \arrow[dl, "\phi_{i,h}"'] \arrow[dr, "\phi_{i,k}"] & \\
V_h \arrow[rr, dotted, "(\phi_{h,k})"'] & & V_k
\end{tikzcd}$$

thanks to the the commutativity of

$$\begin{tikzcd}
& L \arrow[d, "\psi_i"'] \arrow[ddl, bend right, "\psi_i"'] \arrow[ddr, bend left, "\psi_j"] & \\
& V_i \arrow[dl, "1"'] \arrow[dr, "\phi_{i,j}"] & \\
V_i \arrow[rr, "\phi_{i,j}"'] & & V_j
\end{tikzcd}$$

Thus we conclude that for $i \neq n$ we have $E_s(V)_i = \lim V_{|Y(i)} = V_i$.

For $E_s(V)_n = \lim V_{|Y(n)}$ we denominate the maps of the limit as

$$\begin{tikzcd}
& E_s(V)_n \arrow[dl, "a_{n,j}"'] \arrow[dr, "a_{n,h}"] & \\
V_j \arrow[rr, dotted, "(\phi_{j,h})"'] & & V_h
\end{tikzcd}$$

for any $h, k \in Y(n)$ with

$$\phi_{j,h} a_{n,j} = a_{n,h} \quad \text{if} \quad j \leq h \tag{4.3}$$

They are unique: indeed, by (4.2) we have that $Y(n) \supseteq Y(j)$ for any $n \leq j$ (with our hyphothesis $j \in Y(n)$ since we are working removing just n so $j \in P'$ obviously) and so $V_{|Y(j)} \subseteq V_{|Y(n)}$. Therefore we have the commutative diagram

$$\begin{tikzcd}
& E_s(V)_n \arrow[d, dashed, "\exists!\psi"'] \arrow[ddl, bend right, "a_{n,h}"'] \arrow[ddr, bend left, "a_{n,k}"] & \\
& V_j \arrow[dl, "\phi_{j,h}"'] \arrow[dr, "\phi_{j,k}"] & \\
V_h \arrow[rr, dotted, "(\phi_{h,k})"'] & & V_k
\end{tikzcd}$$

for $j, h, k \in Y(j) \subseteq Y(n)$. Since $\lim V_{|Y(j)} = V_j$, by universal property of the limit we have a unique map $\psi \colon E_s(V)_n \to V_j$ such that $a_{n,h} = \phi_{j,h}\psi$. In particular we have

$$\begin{tikzcd}
& E_s(V)_n \arrow[d, dashed, "\exists!\psi"'] \arrow[ddl, bend right, "a_{n,j}"'] \arrow[ddr, bend left, "a_{n,h}"] & \\
& V_j \arrow[dl, "1"'] \arrow[dr, "\phi_{j,h}"] & \\
V_j \arrow[rr, "\phi_{j,h})"'] & & V_h
\end{tikzcd}$$

so $\psi = a_{n,j}$ and we conclude. Moreover we see that for $i \neq n$ we have (for $j \in Y(i)$)

$$a_{i,j} = \phi_{i,j} \tag{4.4}$$

since $E_s(V)_i = \lim V_{|Y(i)} = V_i$.

We define E_s on the **k**-linear maps. Let $i \leq_P j$ neighbours:

- if $i, j \neq n$ we define $E_s(\varphi_{i,j}) \coloneqq \varphi_{i,j}$ since we have

  $$E_s(V)_i = \lim V_{|Y(i)} = V_i \xrightarrow{\varphi_{i,j}} V_j = \lim V_{|Y(j)} = E_s(V)_j$$

  (as done above it is the unique map between limits given by the universal property);

- if $i = n$ we just define

  $$E_s(V)_n = \lim V_{|Y(n)} \xrightarrow{a_{n,j}} V_j = \lim V_{|Y(j)} = E_s(V)_j$$

- if $j = n$ we define

$$E_s(V)_i = \lim V_{|Y(i)} = V_i \xrightarrow{\widetilde{\varphi}_{i,n}} E_s(V)_n = \lim V_{|Y(n)}$$

  with $\widetilde{\varphi}_{i,n}$ the unique map between limits given by the universal property (similar procedure as before). Notice that we have

$$a_{n,j}\widetilde{\varphi}_{i,n} = \phi_{i,j} \tag{4.5}$$

  for any $j \in Y(n) \subseteq Y(i)$.

The relations in I_P are satisfied since we defined the maps through the universal property of limits, for example

$$\begin{tikzcd}
 & j & & & j & & \\
P' & k \arrow[u] \arrow[rr, "s"] & & k \arrow[ur] & & n \arrow[ul] & P \\
 & i \arrow[u] & & & i \arrow[ul] \arrow[ur] \arrow[uu, dash, dashed] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & V_j & & & V_j & & \\
V & V_k \arrow[u, "\varphi_{k,j}"] \arrow[rr, "E_s"] & & V_k \arrow[ur, "\varphi_{k,j}"] & & E_s(V)_n \arrow[ul, "a_{n,j}"'] & E_s(V) \\
 & V_i \arrow[u, "\varphi_{i,k}"] & & & V_i \arrow[ul, "\varphi_{i,k}"] \arrow[ur, "\widetilde{\varphi}_{i,n}"'] \arrow[uu, dash, dashed] & &
\end{tikzcd}$$

and we have $a_{n,j}\widetilde{\varphi}_{i,n} = \phi_{i,j} = \varphi_{k,j}\varphi_{i,k}$ (by definition of $\phi_{i,j}$). This can be easily generalised. Thus we conclude that $E_s(V) \in \mathrm{Rep}_{\mathbf{k}}(Q_P, I_P)$.

Now let $f = (f_1, \dots, f_{n-1})\colon V \to W$ be a morphism of representations in $(V_i, \varphi_{i,j}), (W_i, \psi_{i,j}) \in \mathrm{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})$ and we define $E_s(f)_i := f_i$ if $i \neq n$. For $i = n$ we define $E_s(f)_n := \widetilde{f}_n$ as the unique map $E_s(V)_n = \lim V_{|Y(n)} \xrightarrow{\widetilde{f}_n} \lim W_{|Y(n)} = E_s(W)_n$

given by the universal property of $E_s(W)_n$ that makes the following diagram commute

$$\begin{tikzcd}
E_s(V)_n \arrow[d, "a_{n,j}"] \arrow[dr, "a_{n,k}"] \arrow[ddd, bend right=50, "\exists! \widetilde{f}_n"'] & \\
V_j \arrow[r, dotted, "(\phi_{j,k})"'] \arrow[d, "f_j"'] & V_k \arrow[d, "f_k"] \\
W_j \arrow[r, dotted, "(\Psi_{j,k})"] & W_k \\
E_s(W)_n \arrow[u, "b_{n,j}"'] \arrow[ur, "b_{n,k}"'] &
\end{tikzcd}$$

with $j, k \in Y(n)$, $a_{n,j}$ the maps of the limit $E_s(V)_n$, $b_{n,j}$ the maps of the limit $E_s(W)_n$ and $\Psi_{j,k}$ defined in W as $\phi_{j,k}$ is defined in V. Indeed we have that (if $j \leq k$)

$$\Psi_{j,k} f_j a_{n,j} = f_k \phi_{j,k} a_{n,j} = f_k a_{n,k}$$

thanks to (4.3) and the fact that f is a morphism. Then we have that, for any $j \in Y(n)$

$$b_{n,j} \widetilde{f_n} = f_j a_{n,j} \tag{4.6}$$

Thanks to (4.4) it is easy to check that $f_i = \widetilde{f_i}$ for $i \neq n$ since $E_s(V)_i = \lim V_{|Y(i)} = V_i$. Therefore $E_s(f)$ is uniquely determined.

We have to check that it is a morphism of representations. If $i \leq_P j$ are neighbours with $i, j \neq n$ we conclude since f is a morphism. If $n \leq_P j$ are neighbours we would need to verify the commutativity of

$$\begin{tikzcd}
E_s(V)_n \arrow[r, "a_{n,j}"] \arrow[d, "\widetilde{f}_n"'] & V_j \arrow[d, "f_j"] \\
E_s(W)_n \arrow[r, "b_{n,j}"'] & W_j
\end{tikzcd}$$

but it holds thanks to (4.6).

We need to verify the case $i \leq_P n$ neighbours, i.e. the commutativity of

$$\begin{tikzcd}
V_i \arrow[r, "\widetilde{\varphi}_{i,n}"] \arrow[d, "f_i"'] & E_s(V)_n \arrow[d, "\widetilde{f}_n"] \\
W_i \arrow[r, "\widetilde{\psi}_{i,n}"'] & E_s(W)_n
\end{tikzcd}$$

(with $\widetilde{\psi}_{i,n}$ defined in W as $\widetilde{\varphi}_{i,n}$ is defined in V). We consider the diagram

$$\begin{tikzcd}
 & V_i \arrow[ddl, bend right, "f_j a_{n,j} \widetilde{\varphi}_{i,n}"'] \arrow[ddr, bend left, "f_k a_{n,k} \widetilde{\varphi}_{i,n}"] & \\
 & E_s(W)_n \arrow[dl, "b_{n,j}"'] \arrow[dr, "b_{n,k}"] & \\
W_j \arrow[rr, dotted, "(\Psi_{j,k})"'] & & W_k
\end{tikzcd}$$

for $j, k \in Y(n)$. If $j \leq k$ we have (similarly as before)

$$\Psi_{j,k} f_j a_{n,j} \widetilde{\varphi}_{i,n} = f_k \phi_{j,k} a_{n,j} \widetilde{\varphi}_{i,n} = f_k a_{n,k} \widetilde{\varphi}_{i,n}$$

Thus, by universal property of $E_s(W)_n$ there exists a unique map α

$$\begin{tikzcd}
 & V_i \arrow[ddl, bend right, "f_j a_{n,j} \widetilde{\varphi}_{i,n}"'] \arrow[ddr, bend left, "f_k a_{n,k} \widetilde{\varphi}_{i,n}"] \arrow[d, "\exists! \alpha"'] & \\
 & E_s(W)_n \arrow[dl, "b_{n,j}"'] \arrow[dr, "b_{n,k}"] & \\
W_j \arrow[rr, dotted, "(\Psi_{j,k})"'] & & W_k
\end{tikzcd}$$

such that $b_{n,j}\alpha = f_j a_{n,j} \widetilde{\varphi}_{i,n}$ for any $j \in Y(n)$. We check that $\widetilde{f}_n \widetilde{\varphi}_{i,n}$ and $\widetilde{\psi}_{i,n} f_i$ have the same behavior as α, so by uniqueness they must coincide, proving the commutativity we were looking for. For $\widetilde{f}_n \widetilde{\varphi}_{i,n}$ we have

$$b_{n,j} \widetilde{f}_n \widetilde{\varphi}_{i,n} \overset{(4.6)}{=} f_j a_{n,j} \widetilde{\varphi}_{i,n}$$

and for $\widetilde{\psi}_{i,n} f_i$ we have

$$b_{n,j} \widetilde{\psi}_{i,n} f_i \overset{(4.5)}{=} \Psi_{i,j} f_i = f_j \phi_{i,j} \overset{(4.5)}{=} f_j a_{n,j} \widetilde{\varphi}_{i,n}$$

By uniqueness we must have $\alpha = \widetilde{f}_n \widetilde{\varphi}_{i,n} = \widetilde{\psi}_{i,n} f_i$, so the diagram commutes. Therefore $E_s(f)$ is a morphism between $E_s(V)$ and $E_s(W)$.

Lemma 4.29. *E_s is an additive functor.*

Proof. E_s is well defined on representations and their morphism. Moreover, for any $1_V \colon V \to V$, $f, h \colon V \to W$, $g \colon W \to Z$ we have that

$$E_s(1_V) = 1_{E_s(V)} \qquad E_s(g \circ f) = E_s(g) \circ E_s(f) \qquad E_s(h + f) = E_s(h) + E_s(f)$$

by the uniqueness of the maps between the n-th vertices given by the universal property of limits. ☐

Definition 4.30. With the previous setting, we call the functor E_s the *extension functor* of the subtraction given by $s \colon P' \to P$.

Theorem 4.31. *We have that* $R_s E_s = 1_{\mathrm{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})}$ *and* $R_s \dashv E_s$ *(i.e.* E_s *is the right adjoint of* R_s*).*

Proof. Let $(V_i, \varphi_{i,j}) \in \mathrm{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})$ then we have that $R_s E_s(V)_i = V_i$ for any $i \in P'$ by definition of the two functors. We also have that for any $i \leq_{P'} j$ neighbours by definition $R_s E_s(\varphi_{i,j})_i = R_s(\varphi_{i,j}) = \varphi_{i,j}$ since $i, j \in P'$ means that $i, j \neq n$. Therefore we conclude that $R_s E_s = 1_{\mathrm{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})}$.

Now we want to prove that E_s is the right adjoint of the restriction functor R_s. By the previous point, we define the co-unit of the adjunction as the identity of functors (that is obviously a natural transformation)

$$\varepsilon \colon R_s E_s \xrightarrow{1} 1_{\mathrm{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})}$$

For the unit

$$\eta \colon 1_{\mathrm{Rep}_{\mathbf{k}}(Q_P, I_P)} \to E_s R_s$$

we consider $(V_i, \varphi_{i,j}) \in \mathrm{Rep}_{\mathbf{k}}(Q_P, I_P)$ and we set $\eta_V \colon V \to E_s R_s(V)$ as

$$(\eta_V)_i \colon V_i \xrightarrow{1_{V_i}} E_s R_s(V)_i = V_i$$

if $i \neq n$ while for $i = n$ we define

$$(\eta_V)_n \colon V_n \to E_s R_s(V)_n$$

as the unique map given by the universal property of $E_s R_s(V)_n$: indeed we have

$$\begin{tikzcd}
& V_n \arrow[d, "\exists!(\eta_V)_n"] \arrow[ddl, bend right, "\phi_{n,j}"'] \arrow[ddr, bend left, "\phi_{n,k}"] & \\
& E_s R_s(V)_n \arrow[dl, "a_{n,j}"'] \arrow[dr, "a_{n,k}"] & \\
V_j \arrow[rr, dotted, "(\phi_{j,k})"'] & & V_k
\end{tikzcd}$$

with $j, k \in Y(n)$ and if $j \leq k$ we clearly have $\phi_{j,k}\phi_{n,j} = \phi_{n,k}$, so we define $(\eta_V)_n$. We get

$$a_{n,j}(\eta_V)_n = \phi_{n,j} \quad \text{for any} \quad j \in Y(n) \tag{4.7}$$

We check that η is a natural transformation: let $f\colon (V_i, \varphi_{i,j}) \to (W_i, \psi_{i,j})$ be a morphism of representations. For $i \neq n$ we have

$$\begin{tikzcd}
V_i \arrow[r, "(\eta_V)_i = 1_{V_i}"] \arrow[d, "f_i"'] & V_i = E_sR_s(V)_i \arrow[d, "f_i"] \\
W_i \arrow[r, "(\eta_W)_i = 1_{W_i}"'] & W_i = E_sR_s(W)_i
\end{tikzcd}$$

that is clearly commutative. For $i = n$ we have

$$\begin{tikzcd}
V_n \arrow[r, "(\eta_V)_n"] \arrow[d, "f_n"'] & E_sR_s(V)_n \arrow[d, "\widetilde{f}_n"] \\
W_i \arrow[r, "(\eta_W)_n"'] & E_sR_s(W)_n
\end{tikzcd}$$

with $\widetilde{f}_n$ defined as before. We consider the diagram

$$\begin{tikzcd}
& V_n \arrow[d, "\exists!\alpha"] \arrow[ddl, bend right, "\Psi_{n,j}f_n"'] \arrow[ddr, bend left, "\Psi_{n,k}f_n"] & \\
& E_sR_s(W)_n \arrow[dl, "b_{n,j}"'] \arrow[dr, "b_{n,k}"] & \\
W_j \arrow[rr, dotted, "(\Psi_{j,k})"'] & & W_k
\end{tikzcd}$$

with $j, k \in Y(n)$ and if $j \leq k$ we clearly have $\Psi_{j,k}\Psi_{n,j}f_n = \Psi_{n,k}f_n$. Therefore by universal property of $E_sR_s(W)_n$ there exists a unique map α such that $b_{n,j}\alpha = \Psi_{n,j}f_n$ for any $j \in Y(n)$. We check that $\widetilde{f}_n(\eta_V)_n$ and $(\eta_W)_n f_n$ have the same behavior as α, so by uniqueness they must coincide, proving the commutativity we were looking for. For $\widetilde{f}_n(\eta_V)_n$ we have

$$b_{n,j}\widetilde{f}_n(\eta_V)_n \overset{(4.6)}{=} f_j a_{n,j}(\eta_V)_n \overset{(4.7)}{=} f_j \phi_{n,j} = \Psi_{n,j} f_n$$

and for $(\eta_W)_n f_n$ we have

$$b_{n,j}(\eta_W)_n f_n \overset{(4.7)}{=} \Psi_{n,j} f_n$$

By uniqueness we must have $\alpha = \widetilde{f}_n(\eta_V)_n = (\eta_W)_n f_n$, so the diagram commutes and η is a natural transformation.

To check the adjunction we must have the triangle identities

(1) (2)

$$\begin{tikzcd} R_s(V) \arrow[rr, "R_s(\eta_V)"] \arrow[dr, "1_{R_s(V)}"'] & & R_sE_sR_s(V) \arrow[dl, "\varepsilon_{R_s(V)}"] \\ & R_s(V) & \end{tikzcd} \qquad \begin{tikzcd} E_s(W) \arrow[rr, "\eta_{E_s(W)}"] \arrow[dr, "1_{E_s(W)}"'] & & E_sR_sE_s(W) \arrow[dl, "E_s(\varepsilon_W)"] \\ & E_s(W) & \end{tikzcd}$$

for any representations $V = (V_i, \varphi_{i,j}) \in \mathrm{Rep}_{\mathbf{k}}(Q_P, I_P)$ and $W = (W_i, \psi_{i,j}) \in \mathrm{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})$. We have the following:

- $R_sE_s = 1_{\mathrm{Rep}_{\mathbf{k}}(Q_{P'},I_{P'})}$, so $R_sE_sR_s(V) = R_s(V)$ in (1) and $E_sR_sE_s(W) = E_s(W)$ in (2);
- in (1), $R_s(\eta_V) = (\eta_V)_{P'} = 1_{R_s(V)}$ and $\varepsilon_{R_s(V)} = 1_{R_s(V)}$ by definition of the unit and co-unit;
- in (2), $E_s(\varepsilon_W) = 1_{E_s(W)}$ since $\varepsilon_W = 1_W$. We have that $\eta_{E_s(W)}$ is also equal to $1_{E_s(W)}$ since the i-th component for $i \neq n$ is the identity by definition of η, while the n-th component is the unique maps $(\eta_{E_s(W)})_n \colon E_s(W)_n \to E_s(W)_n$ defined as before that can be easily checked to be the identity on $E_s(W)_n$ by uniqueness.

Therefore the two triangle identities become

(1) (2)

$$\begin{tikzcd} R_s(V) \arrow[rr, "1_{R_s(V)}"] \arrow[dr, "1_{R_s(V)}"'] & & R_s(V) \arrow[dl, "1_{R_s(V)}"] \\ & R_s(V) & \end{tikzcd} \qquad \begin{tikzcd} E_s(W) \arrow[rr, "1_{E_s(W)}"] \arrow[dr, "1_{E_s(W)}"'] & & E_s(W) \arrow[dl, "1_{E_s(W)}"] \\ & E_s(W) & \end{tikzcd}$$

and they clearly commute. Therefore we conclude that $R_s \dashv E_s$. □

Corollary 4.32. *The extension functor E_s is fully faithful. In particular it preserves indecomposables and bricks.*

Proof. By Theorem 4.31 we have that $R_s \dashv E_s$, so by definition of adjunction, for any representations $X \in \mathrm{Rep}_{\mathbf{k}}(Q_P, I_P)$ and $Y \in \mathrm{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})$ we have that

$$\begin{array}{rcl} \mathrm{Hom}_{\mathrm{Rep}_{\mathbf{k}}(Q_{P'},I_{P'})}(R_s(X), Y) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathrm{Rep}_{\mathbf{k}}(Q_P,I_P)}(X, E_s(Y)) \\ f & \mapsto & E_s(f)\eta_X \\ \varepsilon_Y R_s(g) = R_s(g) & \leftarrow\!\shortmid & g \end{array}$$

Taking $X = E_s(V)$ and $Y = W$ for some representations $V, W \in \mathrm{Rep}_{\mathbf{k}}(Q_{P'}, I_{P'})$ and having $R_s E_s = 1_{\mathrm{Rep}_{\mathbf{k}}(Q_{P'},I_{P'})}$ we conclude that

$$\mathrm{Hom}(V, W) = \mathrm{Hom}(R_s E_s(V), W) \cong \mathrm{Hom}(E_s(V), E_s(W))$$

so the extension functor E_s is fully faithful. Following the same reasoning as in Theorem 4.26 we conclude that it also preserves indecomposables and bricks. □

4.4. Generic bricks of representation-infinite posets

We will now prove that, given any representation-infinite poset P, we can compute a generic brick in $\mathrm{Mod}(\mathcal{I}_{\mathbf{k}}(P))$.

Reduction of P. Let P be any representation-infinite poset. By Lemma 3.1 it can be reduced to a minimally representation-infinite poset C (that is a critical poset). As done before, we may assume that any step of the reduction process removes only one vertex. Let $\mathrm{card}(P) = n$ and $\mathrm{card}(C) = n - m$, so the reduction process involved m points of P through a sequence of subtractions and contractions. We obtain a list of posets P_i such that $\mathrm{card}(P_i) = n - i$ for $0 \leq i \leq m$ with $P = P_0$ and $C = P_m$ and we denote by r_j the reduction applied at the j-th step $(1 \leq j \leq m)$, that is

$$r_j = c\colon P_{j-1} \to P_j$$

if we applied an elementary contraction and

$$r_j = s\colon P_j \to P_{j-1}$$

if we applied a subtraction of a vertex. From this list of posets and reduction maps we can define a composition of the corresponding extension functors as

$$\mathrm{Rep}_{\mathbf{k}}(Q_C, I_C) \xrightarrow{E_{r_m}} \mathrm{Rep}_{\mathbf{k}}(Q_{P_{m-1}}, I_{P_{m-1}}) \xrightarrow{E_{r_{m-1}}} \cdots \xrightarrow{E_{r_2}} \mathrm{Rep}_{\mathbf{k}}(Q_{P_1}, I_{P_1}) \xrightarrow{E_{r_1}} \mathrm{Rep}_{\mathbf{k}}(Q_P, I_P)$$

that we denote by $\widetilde{E} := E_{r_1} E_{r_2} \cdots E_{r_{m-1}} E_{r_m}$. Let G_C be the generic brick of the critical poset C given by Theorem 4.22 and Remark 4.21. By the construction of G_C we have that $\dim_{\mathbf{k}}((G_C)_i) = \infty$ and $\dim_{\mathbf{k}(x)}((G_C)_i) = (y_C)_i$ for all $i \in (Q_C)_0$ since $(G_C)_i = \mathbf{k}(x)^{(y_C)_i}$ with y_C the minimal sincere positive radical vector of the quadratic form q_C (see A).

Then we have the following result:

Theorem 4.33. *$\widetilde{E}(G_C) = E_{r_1}E_{r_2}\cdots E_{r_{m-1}}E_{r_m}(G_C)$ is a generic brick for $\mathcal{I}_{\mathbf{k}}(P)$ with $(\widetilde{E}(G_C))_i = \mathbf{k}(x)^{h_i}$ for all $i \in (Q_P)_0$ for some $h_i > 0$ and* $\mathrm{End}(\widetilde{E}(G_C)) \cong \mathbf{k}(x)$.

Proof. We prove the statement by induction on the number m of reduction steps. Let $m = 1$, then we just removed a vertex through a contraction or a subtraction and, up to re-indexing, let $P = \{1, \dots, n-1, n\}$ and $C = \{1, \dots, n-1\}$.

(i) If we applied a contraction $c\colon P \to C$, contracting $n-1$ and n, we have $\widetilde{E} = E_c$. Now $(E_c(G_C))_i = (G_C)_i$ for the vertices $1, \dots, n-1$ and $(E_c(G_C))_{n-1} = (G_C)_{n-1}$ in vertex n by definition. So we get that

$$\dim_{\mathbf{k}}(E_c(G_C)) = \dim_{\mathbf{k}}(G_C) + \dim_{\mathbf{k}}((G_C)_{n-1})$$

and by Theorem 1.22 we have

$$\infty = \ell_{\mathcal{I}_{\mathbf{k}}(C)}(G_C) = \dim_{\mathbf{k}}(G_C)$$

since G_C is a generic brick. Therefore we conclude that

$$\ell_{\mathcal{I}_{\mathbf{k}}(P)}(E_c(G_C)) = \dim_{\mathbf{k}}(E_c(G_C)) = \dim_{\mathbf{k}}(G_C) + \dim_{\mathbf{k}}((G_C)_{n-1}) = \infty$$

Now E_c is fully faithful (Theorem 4.26) so $\mathrm{End}(E_c(G_C)) \cong \mathrm{End}(G_C) \cong \mathbf{k}(x)$. Applying the same idea as before, and using Lemma 1.21, we also get

$$\begin{aligned}\ell_{\mathbf{k}(x)^{\mathrm{op}}}(E_c(G_C)) = \dim_{\mathbf{k}(x)}(E_c(G_C)) &= \dim_{\mathbf{k}(x)}(G_C) + \dim_{\mathbf{k}(x)}((G_C)_{n-1}) \\ &= \ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_C) + (y_C)_{n-1} < \infty\end{aligned}$$

Therefore $E_c(G_C)$ has infinite length but finite endolength, so it is a generic module and by Remark 4.7 it is a generic brick. By definition we also conclude that $(\widetilde{E}(G_C))_i = (E_c(G_C))_i = \mathbf{k}(x)^{h_i}$ for all $i \in (Q_P)_0$ for some $h_i > 0$.

(ii) If we applied a subtraction $s\colon C \to P$, contracting $n-1$ and n, we have $\widetilde{E} = E_s$. Similarly to (i), by definition of E_s we have

$$\dim_{\mathbf{k}}(E_s(G_C)) = \dim_{\mathbf{k}}(G_C) + \dim_{\mathbf{k}}(\lim(G_C)_{|Y(n)})$$

$$\dim_{\mathbf{k}(x)}(E_s(G_C)) = \dim_{\mathbf{k}(x)}(G_C) + \dim_{\mathbf{k}(x)}(\lim(G_C)_{|Y(n)})$$

and $\lim(G_C)_{|Y(n)} \leq \prod_{j\in Y(n)}(G_C)_j = \prod_{j\in Y(n)} \mathbf{k}(x)^{(y_C)_j}$ (finite product). Moreover E_s is fully faithful by Corollary 4.32. Therefore, with a similar reasoning as in (i), we conclude that $\ell_{\mathcal{I}_{\mathbf{k}}(P)}(E_c(G_C)) = \infty$, $\mathrm{End}(E_s(G_C)) \cong \mathrm{End}(G_C) \cong \mathbf{k}(x)$ and $\ell_{\mathbf{k}(x)^{\mathrm{op}}}(E_c(G_C)) < \infty$. Thus, $E_s(G_C)$ has infinite length but finite endolength, so it is a generic module and by Remark 4.7 it is a generic brick. By definition we also conclude that $(\widetilde{E}(G_C))_i = (E_c(G_C))_i = \mathbf{k}(x)^{h_i}$ for all $i \in (Q_P)_0$ for some $h_i > 0$.

Now suppose the result holds for $m-1$, so $\widetilde{G} = E_{r_2}\cdots E_{r_{m-1}}E_{r_m}(G_C)$ is a generic brick for $\mathcal{I}_{\mathbf{k}}(P_1)$ with $(\widetilde{G})_i = \mathbf{k}(x)^{h_i}$ for all $i \in (Q_P)_0$ for some $h_i > 0$ and

$\mathrm{End}(\widetilde{G}) \cong \mathbf{k}(x)$. We have that r_1 can either be a contraction or a subtraction but in any case, since $\widetilde{G}$ has a similar behavior as G_C, we can apply the same reasoning used in (i) and (ii) to conclude that $\widetilde{E}(G_C) = E_{r_1}(\widetilde{G})$ is a generic brick for $\mathcal{I}_{\mathbf{k}}(P)$ with $(\widetilde{E}(G_C))_i = \mathbf{k}(x)^{h_i}$ for all $i \in (Q_P)_0$ for some $h_i > 0$ and $\mathrm{End}(\widetilde{E}(G_C)) \cong \mathbf{k}(x)$. So we proved the result. □

We can summarise our discussion with the following theorem:

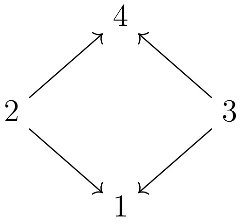
Theorem 4.34. *Any representation-infinite poset P admits a generic brick over $\mathcal{I}_{\mathbf{k}}(P)$ that can be explicitly computed.*

A. Appendix: proof of Theorem 4.22

In this Appendix we will give a complete proof of the existence and construction of generic bricks of critical posets, as stated in Theorem 4.22. The general idea of the proof is given at the beginning of Section 4.2. We recall that we will only study one particular orientation of the arrows in every class of critical poset. Indeed, as stated in Remark 4.21, proving the result for one orientation implies that it also holds for all the possible orientations of the undirected edges and all the opposites of these.

A.1. Generic brick of $\widetilde{\mathbb{A}}_3$

Consider the critical poset of type $P = \widetilde{\mathbb{A}}_3$ (see Table 2.1) with the following fixed orientation and labelling:



Its Tits form $q_{\widetilde{\mathbb{A}}_3}$ is

$$q_{\widetilde{\mathbb{A}}_3}(x) = \sum_{i=1}^{4} x_i^2 - x_2x_1 - x_3x_1 - x_2x_4 - x_3x_4$$

and the associated matrix $B_{\widetilde{\mathbb{A}}_3}$ is given by

$$B_{\widetilde{\mathbb{A}}_3} = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\widetilde{\mathbb{A}}_3}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\widetilde{\mathbb{A}}_3} = (1,1,1,1)^t$. Therefore

$$\mathrm{Rad}(q_{\widetilde{\mathbb{A}}_3}) = \ker(B_{\widetilde{\mathbb{A}}_3}) = \langle y_{\widetilde{\mathbb{A}}_3} \rangle$$

We define the representations $B(\lambda)_{\widetilde{\mathbb{A}}_3}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\widetilde{\mathbb{A}}_3} \coloneqq \begin{tikzcd} & \mathbf{k} & \\ \mathbf{k} \arrow[ur, "\lambda"] \arrow[dr, "1"'] & & \mathbf{k} \arrow[ul, "1"'] \arrow[dl, "1"] \\ & \mathbf{k} & \end{tikzcd}$$

We have that $\underline{\dim}(B(\lambda)_{\widetilde{\mathbb{A}}_3}) = y_{\widetilde{\mathbb{A}}_3}$ and we prove the following:

Theorem A.1. *The representations $B(\lambda)_{\widetilde{\mathbb{A}}_3}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{A}}_3\right)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\widetilde{\mathbb{A}}_3} \to B(\lambda')_{\widetilde{\mathbb{A}}_3}$ be a morphism of representations.

$$\begin{tikzcd} & \mathbf{k} \arrow[ddd, "f_4"] & & \\ \mathbf{k} \arrow[ur, "\lambda"] \arrow[drr, "1"'] \arrow[ddd, "f_2"'] & & & \mathbf{k} \arrow[ull, "1"'] \arrow[dl, "1"] \arrow[ddd, "f_3"] \\ & & \mathbf{k} \arrow[ddd, "f_1"'] & \\ & \mathbf{k} & & \\ \mathbf{k} \arrow[ur, "\lambda'"] \arrow[drr, "1"'] & & & \mathbf{k} \arrow[ull, "1"'] \arrow[dl, "1"] \\ & & \mathbf{k} & \end{tikzcd}$$

We have that f_1, f_2, f_3 and f_4 are just scalar multiplications of elements of $\mathbf{k}$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_1 = f_2$$

(2) The square involving f_1 and f_3 gives us

$$f_1 = f_3$$

(3) The square involving f_3 and f_4 gives us

$$f_3 = f_4$$

(4) The square involving f_2 and f_4 gives us $f_4(\lambda) = \lambda' f_2$ that means

$$\lambda f_4 = \lambda' f_2$$

because the f_i are $\mathbf{k}$-linear.

Now if $\lambda = \lambda'$ we conclude that any morphism $f\colon B(\lambda)_{\widetilde{\mathbb{A}}_3} \to B(\lambda)_{\widetilde{\mathbb{A}}_3}$ depends only on $f_1 \in \mathbf{k}$ since we have that

$$f_1 = f_2 = f_3 = f_4 \tag{A.1}$$

therefore $\operatorname{End}(B(\lambda)_{\widetilde{\mathbb{A}}_3}) \cong \mathbf{k}$, so $B(\lambda)_{\widetilde{\mathbb{A}}_3}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_4 = \lambda' f_2$, but by (A.1) we have $\lambda f_1 = \lambda' f_1$. If $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\widetilde{\mathbb{A}}_3}, B(\lambda')_{\widetilde{\mathbb{A}}_3}) = 0$$

Therefore $B(\lambda)_{\widetilde{\mathbb{A}}_3}$ and $B(\lambda')_{\widetilde{\mathbb{A}}_3}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\widetilde{\mathbb{A}}_3}\}_{\lambda\in\mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{A}}_3\right)$ is brick-continuous. $\square$

Construction of $G_{\widetilde{\mathbb{A}}_3}$ We have the brick-continuous family $\{B(\lambda)_{\widetilde{\mathbb{A}}_3}\}_{\lambda\in\mathbf{k}}$ and we define the representation $G_{\widetilde{\mathbb{A}}_3}$ substituting $\mathbf{k}$ with $\mathbf{k}(x)$ and λ with x:

$$\begin{tikzcd}
 & \mathbf{k}(x) & \\
\mathbf{k}(x) \arrow[ur, "x"] \arrow[dr, "1"'] & & \mathbf{k}(x) \arrow[ul, "1"'] \arrow[dl, "1"] \\
 & \mathbf{k}(x) &
\end{tikzcd}$$

Theorem A.2. *$G_{\widetilde{\mathbb{A}}_3}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{A}}_3\right)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\widetilde{\mathbb{A}}_3)}(G_{\widetilde{\mathbb{A}}_3}) = \dim_{\mathbf{k}}(G_{\widetilde{\mathbb{A}}_3}) = \sum_{i\in(Q_{\widetilde{\mathbb{A}}_3})_0} \dim_{\mathbf{k}}((G_{\widetilde{\mathbb{A}}_3})_i) = \infty$$

So $G_{\widetilde{\mathbb{A}}_3}$ have infinite length. Moreover, following the same procedure as in Theorem A.1, we see that all the components of a endomorphism $f\colon G_{\widetilde{\mathbb{A}}_3} \to G_{\widetilde{\mathbb{A}}_3}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\mathrm{End}(G_{\widetilde{\mathbb{A}}_3}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\widetilde{\mathbb{A}}_3}) = \dim_{\mathbf{k}(x)}(G_{\widetilde{\mathbb{A}}_3}) = \sum_{i\in(Q_{\widetilde{\mathbb{A}}_3})_0} \dim_{\mathbf{k}(x)}((G_{\widetilde{\mathbb{A}}_3})_i) < \infty$$

So $G_{\widetilde{\mathbb{A}}_3}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\widetilde{\mathbb{A}}_3}$ is a generic brick. □

A.2. Generic brick of $\widetilde{\mathbb{D}}_4$

Consider the critical poset of type $P = \widetilde{\mathbb{D}}_4$ (see Table 2.1) with the following fixed orientation and labelling:

$$\begin{tikzcd}
 & 1 \arrow[d] & \\
4 & 3 \arrow[l] \arrow[d] & 2 \arrow[l] \\
 & 5 &
\end{tikzcd}$$

Its Tits form $q_{\widetilde{\mathbb{D}}_4}$ is

$$q_{\widetilde{\mathbb{D}}_4}(x) = \sum_{i=1}^{5} x_i^2 - x_1x_3 - x_2x_3 - x_3x_4 - x_3x_5$$

and the associated matrix $B_{\widetilde{\mathbb{D}}_4}$ is given by

$$B_{\widetilde{\mathbb{D}}_4} = \frac{1}{2}\begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\widetilde{\mathbb{D}}_4}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\widetilde{\mathbb{D}}_4} = (1, 1, 2, 1, 1)^t$. Therefore

$$\operatorname{Rad}(q_{\widetilde{\mathbb{D}}_4}) = \ker(B_{\widetilde{\mathbb{D}}_4}) = \langle y_{\widetilde{\mathbb{D}}_4} \rangle$$

We define the representations $B(\lambda)_{\widetilde{\mathbb{D}}_4}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\widetilde{\mathbb{D}}_4} \coloneqq \begin{tikzcd} & \mathbf{k} \arrow[d, "\binom{1}{0}"'] & \\ \mathbf{k} & \mathbf{k}^2 \arrow[l, "(1\ 1)"'] \arrow[d, "(\lambda\ 1)"] & \mathbf{k} \arrow[l, "\binom{0}{1}"'] \\ & \mathbf{k} & \end{tikzcd}$$

We have that $\underline{\dim}(B(\lambda)_{\widetilde{\mathbb{D}}_4}) = y_{\widetilde{\mathbb{D}}_4}$ and we prove the following:

Theorem A.3. *The representations $B(\lambda)_{\widetilde{\mathbb{D}}_4}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{D}}_4\right)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\widetilde{\mathbb{D}}_4} \to B(\lambda')_{\widetilde{\mathbb{D}}_4}$ be a morphism of representations.

$$\begin{tikzcd} \mathbf{k} \arrow[ddd, "f_4"'] & & & \mathbf{k} \arrow[dl, "\binom{1}{0}"'] \arrow[ddd, "f_1"] & \\ & & \mathbf{k}^2 \arrow[ull, "(1\ 1)"'] \arrow[dl, "(\lambda\ 1)"'] \arrow[ddd, "f_3"'] & & \\ & \mathbf{k} \arrow[ddd, "f_5"'] & & & \mathbf{k} \arrow[ull, "\binom{0}{1}"'] \arrow[ddd, "f_2"] \\ \mathbf{k} & & & \mathbf{k} \arrow[dl, "\binom{1}{0}"] & \\ & & \mathbf{k}^2 \arrow[ull, "(1\ 1)"'] \arrow[dl, "(\lambda'\ 1)"] & & \\ & \mathbf{k} & & & \mathbf{k} \arrow[ull, "\binom{0}{1}"'] \end{tikzcd}$$

We have that f_1, f_2, f_4 and f_5 are just scalar multiplications of elements of $\mathbf{k}$, while $f_3 \in \mathbb{M}_2(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_3 gives us

$$f_3(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us

$$f_3(e_2) = \begin{pmatrix} 0 \\ f_2 \end{pmatrix}$$

(3) The square involving f_3 and f_4 gives us

$$\begin{pmatrix} 1 & 1 \end{pmatrix} f_3(e_1) = f_4 \quad \begin{pmatrix} 1 & 1 \end{pmatrix} f_3(e_2) = f_4$$

that means $f_1 = f_4$ and $f_2 = f_4$ respectively.

(4) The square involving f_3 and f_5 gives us

$$\begin{pmatrix} \lambda' & 1 \end{pmatrix} f_3(e_1) = f_5(\lambda) \quad \begin{pmatrix} \lambda' & 1 \end{pmatrix} f_3(e_2) = f_5$$

that means $\lambda' f_1 = \lambda f_5$ (by $\mathbf{k}$-linearity) and $f_2 = f_5$ respectively.

Therefore we have that

$$f_1 = f_2 = f_4 = f_5 \quad \text{and} \quad f_3 = f_1 \mathbb{1}_2 \tag{A.2}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f\colon B(\lambda)_{\widetilde{\mathbb{D}}_4} \to B(\lambda)_{\widetilde{\mathbb{D}}_4}$ depends only on $f_1 \in \mathbf{k}$ by (A.2), therefore $\operatorname{End}(B(\lambda)_{\widetilde{\mathbb{D}}_4}) \cong \mathbf{k}$, so $B(\lambda)_{\widetilde{\mathbb{D}}_4}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda' f_1 = \lambda f_5$, but by (A.2) we can write $\lambda f_1 = \lambda' f_1$. If $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\widetilde{\mathbb{D}}_4}, B(\lambda')_{\widetilde{\mathbb{D}}_4}) = 0$$

Therefore $B(\lambda)_{\widetilde{\mathbb{D}}_4}$ and $B(\lambda')_{\widetilde{\mathbb{D}}_4}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\widetilde{\mathbb{D}}_4}\}_{\lambda \in \mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{D}}_4\right)$ is brick-continuous. $\square$

Construction of $G_{\widetilde{\mathbb{D}}_4}$ We have the brick-continuous family $\{B(\lambda)_{\widetilde{\mathbb{D}}_4}\}_{\lambda \in \mathbf{k}}$ and we define the representation $G_{\widetilde{\mathbb{D}}_4}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & \mathbf{k}(x) \arrow[d, "\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)"'] & \\
\mathbf{k}(x) & \mathbf{k}(x)^2 \arrow[l, "(1\ 1)"] \arrow[d, "(x\ 1)"] & \mathbf{k}(x) \arrow[l, "\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)"'] \\
 & \mathbf{k}(x) &
\end{tikzcd}$$

Theorem A.4. *$G_{\widetilde{\mathbb{D}}_4}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{D}}_4\right)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{D}}_4\right)}(G_{\widetilde{\mathbb{D}}_4}) = \dim_{\mathbf{k}}(G_{\widetilde{\mathbb{D}}_4}) = \sum_{i\in(Q_{\widetilde{\mathbb{D}}_4})_0} \dim_{\mathbf{k}}((G_{\widetilde{\mathbb{D}}_4})_i) = \infty$$

So $G_{\widetilde{\mathbb{D}}_4}$ have infinite length. Moreover, following the same procedure as in Theorem A.3, we see that all the components of a endomorphism $f\colon G_{\widetilde{\mathbb{D}}_4} \to G_{\widetilde{\mathbb{D}}_4}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\operatorname{End}(G_{\widetilde{\mathbb{D}}_4}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\widetilde{\mathbb{D}}_4}) = \dim_{\mathbf{k}(x)}(G_{\widetilde{\mathbb{D}}_4}) = \sum_{i\in(Q_{\widetilde{\mathbb{D}}_4})_0} \dim_{\mathbf{k}(x)}((G_{\widetilde{\mathbb{D}}_4})_i) < \infty$$

So $G_{\widetilde{\mathbb{D}}_4}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\widetilde{\mathbb{D}}_4}$ is a generic brick. □

A.3. Generic brick of $\widetilde{\mathbb{E}}_6$

Consider the critical poset of type $P = \widetilde{\mathbb{E}}_6$ (see Table 2.1) with the following fixed orientation and labelling:

$$\begin{tikzcd}
 & & 5 \arrow[d] & & \\
 & & 4 \arrow[d] & & \\
1 \arrow[r] & 2 \arrow[r] & 3 & 6 \arrow[l] & 7 \arrow[l]
\end{tikzcd}$$

Its Tits form $q_{\widetilde{\mathbb{E}}_6}$ is

$$q_{\widetilde{\mathbb{E}}_6}(x) = \sum_{i=1}^{7} x_i^2 - x_1x_2 - x_2x_3 - x_4x_3 - x_5x_4 - x_6x_3 - x_7x_6$$

and the associated matrix $B_{\widetilde{\mathbb{E}}_6}$ is given by

$$B_{\widetilde{\mathbb{E}}_6} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\widetilde{\mathbb{E}}_6}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\widetilde{\mathbb{E}}_6} = (1, 2, 3, 2, 1, 2, 1)^t$. Therefore

$$\mathrm{Rad}(q_{\widetilde{\mathbb{E}}_6}) = \ker(B_{\widetilde{\mathbb{E}}_6}) = \langle y_{\widetilde{\mathbb{E}}_6} \rangle$$

We define the representations $B(\lambda)_{\widetilde{\mathbb{E}}_6}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\widetilde{\mathbb{E}}_6} := \begin{tikzcd} & & \mathbf{k} \arrow[d, "\varphi_{54}"] & & \\ & & \mathbf{k}^2 \arrow[d, "\varphi_{43}(\lambda)"] & & \\ \mathbf{k} \arrow[r, "\varphi_{12}"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"'] & \mathbf{k}^3 & \mathbf{k}^2 \arrow[l, "\varphi_{63}"] & \mathbf{k} \arrow[l, "\varphi_{76}"] \end{tikzcd}$$

where the $\mathbf{k}$-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{54} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{76} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{63} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \varphi_{43}(\lambda) = \begin{pmatrix} \lambda & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

We have that $\underline{\dim}(B(\lambda)_{\widetilde{\mathbb{E}}_6}) = y_{\widetilde{\mathbb{E}}_6}$ and we prove the following:

Theorem A.5. *The representations* $B(\lambda)_{\widetilde{\mathbb{E}}_6}$ *with* $\lambda \in \mathbf{k}$ *(with* $\mathbf{k}$ *algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence* $\mathbf{k}$*-algebra* $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{E}}_6\right)$ *is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\widetilde{\mathbb{E}}_6} \to B(\lambda')_{\widetilde{\mathbb{E}}_6}$ be a morphism of representations.

$$\begin{tikzcd}
 & & & & \mathbf{k} \arrow[dl, "\varphi_{54}"'] \arrow[ddd, "f_5"] & & \\
 & & & \mathbf{k}^2 \arrow[dl, "\varphi_{43}(\lambda)"'] \arrow[ddd, "f_4"'] & & & \\
\mathbf{k} \arrow[r, "\varphi_{12}"] \arrow[ddd, "f_1"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] \arrow[ddd, "f_2"'] & \mathbf{k}^3 \arrow[ddd, "f_3"'] & & & \mathbf{k}^2 \arrow[lll, "\varphi_{63}"'] \arrow[ddd, "f_6"] & \mathbf{k} \arrow[l, "\varphi_{76}"'] \arrow[ddd, "f_7"] \\
 & & & & \mathbf{k} \arrow[dl, "\varphi_{54}"] & & \\
 & & & \mathbf{k}^2 \arrow[dl, "\varphi_{43}(\lambda')"] & & & \\
\mathbf{k} \arrow[r, "\varphi_{12}"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"'] & \mathbf{k}^3 & & & \mathbf{k}^2 \arrow[lll, "\varphi_{63}"] & \mathbf{k} \arrow[l, "\varphi_{76}"]
\end{tikzcd}$$

We have that f_1, f_5 and f_7 are just scalar multiplications of elements of $\mathbf{k}$, while $f_2, f_4, f_6 \in \mathbb{M}_2(\mathbf{k})$ and $f_3 \in \mathbb{M}_3(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_4 and f_5 gives us

$$f_4(e_1) = \begin{pmatrix} f_5 \\ 0 \end{pmatrix}$$

(3) The square involving f_6 and f_7 gives us

$$f_6(e_2) = \begin{pmatrix} 0 \\ f_7 \end{pmatrix}$$

(4) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(5) The square involving f_3 and f_6 gives us that $\varphi_{63}f_6(e_i) = f_3(\varphi_{63}(e_i))$ that means

$$f_3(e_2) = \begin{pmatrix} 0 \\ f_6(e_1) \end{pmatrix} \qquad f_3(e_3) = \begin{pmatrix} 0 \\ f_6(e_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_7 \end{pmatrix}$$

But then $f_3(e_2) = \begin{pmatrix} 0 \\ f_6(e_1) \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$ means that $f_6(e_1) = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and $f_2(e_2) = \begin{pmatrix} 0 \\ a \end{pmatrix}$ for some $a \in \mathbf{k}$ (and so $f_3(e_2) = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}$).

(6) The square involving f_3 and f_4 gives us that $\varphi_{43}(\lambda')f_4(e_i) = f_3(\varphi_{43}(\lambda)(e_i))$ that means, for $i = 2$

$$\begin{pmatrix} \lambda' f_4(e_2)_1 + f_4(e_2)_2 \\ f_4(e_2)_1 + f_4(e_2)_2 \\ f_4(e_2)_1 \end{pmatrix} = \varphi_{43}(\lambda')f_4(e_2) = f_3(\varphi_{43}(\lambda)(e_2)) = f_3(e_1 + e_2) = \begin{pmatrix} f_1 \\ a \\ 0 \end{pmatrix}$$

So $f_4(e_2) = \begin{pmatrix} 0 \\ a \end{pmatrix}$ with $a = f_1$. Instead for $i = 1$ we have

$$\begin{pmatrix} \lambda' f_5 \\ f_5 \\ f_5 \end{pmatrix} = \varphi_{43}(\lambda')f_4(e_1) = f_3(\varphi_{43}(\lambda)(e_1)) = f_3(\lambda e_1 + e_2 + e_3) = \begin{pmatrix} \lambda f_1 \\ a \\ f_7 \end{pmatrix}$$

that means $a = f_5 = f_7$ and $\lambda f_1 = \lambda' f_5$.

Therefore we have that

$$f_1 = f_5 = f_7 \qquad f_2 = f_4 = f_6 = f_1 \mathbb{1}_2 \qquad f_3 = f_1 \mathbb{1}_3 \tag{A.3}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f\colon B(\lambda)_{\widetilde{\mathbb{E}}_6} \to B(\lambda)_{\widetilde{\mathbb{E}}_6}$ depends only on $f_1 \in \mathbf{k}$ by (A.3), therefore $\operatorname{End}(B(\lambda)_{\widetilde{\mathbb{E}}_6}) \cong \mathbf{k}$, so $B(\lambda)_{\widetilde{\mathbb{E}}_6}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_5$, but by (A.3) we can write $\lambda f_1 = \lambda' f_1$. If $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\widetilde{\mathbb{E}}_6}, B(\lambda')_{\widetilde{\mathbb{E}}_6}) = 0$$

Therefore $B(\lambda)_{\widetilde{\mathbb{E}}_6}$ and $B(\lambda')_{\widetilde{\mathbb{E}}_6}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\widetilde{\mathbb{E}}_6}\}_{\lambda \in \mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{E}}_6\right)$ is brick-continuous. □

Construction of $G_{\widetilde{\mathbb{E}}_6}$ We have the brick-continuous family $\{B(\lambda)_{\widetilde{\mathbb{E}}_6}\}_{\lambda\in\mathbf{k}}$ and we define the representation $G_{\widetilde{\mathbb{E}}_6}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
& & \mathbf{k}(x) \arrow[d, "\phi_{54}"] & & \\
& & \mathbf{k}(x)^2 \arrow[d, red, "\phi_{43,x}"] & & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"'] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"'] & \mathbf{k}(x)^3 & \mathbf{k}(x)^2 \arrow[l, "\phi_{63}"] & \mathbf{k}(x) \arrow[l, "\phi_{76}"]
\end{tikzcd}$$

with $\phi_{43,x} \coloneqq \varphi_{43}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.6. *$G_{\widetilde{\mathbb{E}}_6}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{E}}_6\right)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\widetilde{\mathbb{E}}_6)}(G_{\widetilde{\mathbb{E}}_6}) = \dim_{\mathbf{k}}(G_{\widetilde{\mathbb{E}}_6}) = \sum_{i\in(Q_{\widetilde{\mathbb{E}}_6})_0} \dim_{\mathbf{k}}((G_{\widetilde{\mathbb{E}}_6})_i) = \infty$$

So $G_{\widetilde{\mathbb{E}}_6}$ have infinite length. Moreover, following the same procedure as in Theorem A.5, we see that all the components of a endomorphism $f\colon G_{\widetilde{\mathbb{E}}_6} \to G_{\widetilde{\mathbb{E}}_6}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\mathrm{End}(G_{\widetilde{\mathbb{E}}_6}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\widetilde{\mathbb{E}}_6}) = \dim_{\mathbf{k}(x)}(G_{\widetilde{\mathbb{E}}_6}) = \sum_{i\in(Q_{\widetilde{\mathbb{E}}_6})_0} \dim_{\mathbf{k}(x)}((G_{\widetilde{\mathbb{E}}_6})_i) < \infty$$

So $G_{\widetilde{\mathbb{E}}_6}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\widetilde{\mathbb{E}}_6}$ is a generic brick. □

A.4. Generic brick of $\widetilde{\mathbb{E}}_7$

Consider the critical poset of type $P = \widetilde{\mathbb{E}}_7$ (see Table 2.1) with the following fixed orientation and labelling:

$$\begin{tikzcd}
& & & 5 \arrow[d] & & & \\
1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 & 6 \arrow[l] & 7 \arrow[l] & 8 \arrow[l]
\end{tikzcd}$$

Its Tits form $q_{\widetilde{\mathbb{E}}_7}$ is

$$q_{\widetilde{\mathbb{E}}_7}(x) = \sum_{i=1}^{8} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_5x_4 - x_6x_4 - x_7x_6 - x_8x_6$$

and the associated matrix $B_{\widetilde{\mathbb{E}}_7}$ is given by

$$B_{\widetilde{\mathbb{E}}_7} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\widetilde{\mathbb{E}}_7}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\widetilde{\mathbb{E}}_7} = (1, 2, 3, 4, 2, 3, 2, 1)^t$. Therefore

$$\mathrm{Rad}(q_{\widetilde{\mathbb{E}}_7}) = \ker(B_{\widetilde{\mathbb{E}}_7}) = \langle y_{\widetilde{\mathbb{E}}_7} \rangle$$

We define the representations $B(\lambda)_{\widetilde{\mathbb{E}}_7}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\widetilde{\mathbb{E}}_7} :=$$

$$\begin{tikzcd}
& & & \mathbf{k}^2 \arrow[d, "\varphi_{54}(\lambda)"] & & & \\
\mathbf{k} \arrow[r, "\varphi_{12}"] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"] & \mathbf{k}^4 & \mathbf{k}^3 \arrow[l, "\varphi_{64}"'] & \mathbf{k}^2 \arrow[l, "\varphi_{76}"'] & \mathbf{k} \arrow[l, "\varphi_{87}"']
\end{tikzcd}$$

where the **k**-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_{87} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \varphi_{76} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \varphi_{64} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \varphi_{54}(\lambda) = \begin{pmatrix} 1 & \lambda \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We have that $\underline{\dim}(B(\lambda)_{\widetilde{\mathbb{E}}_7}) = y_{\widetilde{\mathbb{E}}_7}$ and we prove the following:

Theorem A.7. *The representations $B(\lambda)_{\widetilde{\mathbb{E}}_7}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{E}}_7\right)$ is brick-continuous.*

Proof. Let $f \colon B(\lambda)_{\widetilde{\mathbb{E}}_7} \to B(\lambda')_{\widetilde{\mathbb{E}}_7}$ be a morphism of representations.

$$\begin{tikzcd}
 & & & & \mathbf{k}^2 \arrow[dl, "\varphi_{54}(\lambda)"'] \arrow[dd, "f_5"'] & & & \\
\mathbf{k} \arrow[r, "\varphi_{12}"] \arrow[dd, "f_1"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] \arrow[dd, "f_2"'] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"] \arrow[dd, "f_3"'] & \mathbf{k}^4 \arrow[dd, "f_4"'] & & \mathbf{k}^3 \arrow[ll, "\varphi_{64}"'] \arrow[dd, "f_6"] & \mathbf{k}^2 \arrow[l, "\varphi_{76}"'] \arrow[dd, "f_7"] & \mathbf{k} \arrow[l, "\varphi_{87}"'] \arrow[dd, "f_8"] \\
 & & & & \mathbf{k}^2 \arrow[dl, "\varphi_{54}(\lambda')"] & & & \\
\mathbf{k} \arrow[r, "\varphi_{12}"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"'] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"'] & \mathbf{k}^4 & & \mathbf{k}^3 \arrow[ll, "\varphi_{64}"] & \mathbf{k}^2 \arrow[l, "\varphi_{76}"] & \mathbf{k} \arrow[l, "\varphi_{87}"]
\end{tikzcd}$$

We have that f_1 and f_8 are just scalar multiplications of elements of $\mathbf{k}$, while $f_2, f_5, f_6 \in \mathbb{M}_2(\mathbf{k})$ and $f_3, f_6 \in \mathbb{M}_3(\mathbf{k})$ and $f_4 \in \mathbb{M}_4(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(3) The square involving f_3 and f_4 gives us that $\varphi_{34} f_3(e_i) = f_4(\varphi_{34}(e_i))$ that means

$$f_4(e_1) = \begin{pmatrix} f_3(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_2) = \begin{pmatrix} f_3(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \end{pmatrix}$$

(4) The square involving f_7 and f_8 gives us

$$f_7(e_2) = \begin{pmatrix} 0 \\ f_8 \end{pmatrix}$$

(5) The square involving f_6 and f_7 gives us that $\varphi_{76} f_7(e_i) = f_6(\varphi_{76}(e_i))$ that means

$$f_6(e_2) = \begin{pmatrix} 0 \\ f_7(e_1) \end{pmatrix} \qquad f_6(e_3) = \begin{pmatrix} 0 \\ f_7(e_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_8 \end{pmatrix}$$

(6) The square involving f_4 and f_6 gives us that $\varphi_{64} f_6(e_i) = f_4(\varphi_{64}(e_i))$ that means

$$f_4(e_2) = \begin{pmatrix} 0 \\ f_6(e_1) \end{pmatrix} \quad f_4(e_3) = \begin{pmatrix} 0 \\ f_6(e_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_7(e_1) \end{pmatrix} \quad f_4(e_4) = \begin{pmatrix} 0 \\ f_6(e_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_8 \end{pmatrix}$$

But now, comparing with (3), we have that

$$f_4(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ f_6(e_1) \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix}$$

and also

$$f_4(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_7(e_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \\ 0 \end{pmatrix}$$

for some $a, b \in \mathbf{k}$.
(7) The square involving f_4 and f_5 gives us that $\varphi_{54}(\lambda') f_5(e_i) = f_4(\varphi_{54}(\lambda)(e_i))$ that means for $i = 1$

$$\begin{pmatrix} f_1 \\ a \\ b \\ 0 \end{pmatrix} = f_4(e_1 + e_2 + e_3) = f_4(\varphi_{54}(\lambda)(e_1)) = \varphi_{54}(\lambda') f_5(e_1) = \begin{pmatrix} f_5(e_1)_1 + \lambda' f_5(e_1)_2 \\ f_5(e_1)_1 \\ f_5(e_1)_1 + f_5(e_1)_2 \\ f_5(e_1)_2 \end{pmatrix}$$

so we have $f_5(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$ with $f_1 = a = b$. Instead for $i = 2$ we get

$$\begin{pmatrix} \lambda f_1 \\ 0 \\ f_1 \\ f_8 \end{pmatrix} = f_4(\lambda e_1 + e_3 + e_4) = f_4(\varphi_{54}(\lambda)(e_2)) = \varphi_{54}(\lambda') f_5(e_2) = \begin{pmatrix} f_5(e_2)_1 + \lambda' f_5(e_2)_2 \\ f_5(e_2)_1 \\ f_5(e_2)_1 + f_5(e_2)_2 \\ f_5(e_2)_2 \end{pmatrix}$$

so we have $\lambda f_1 = \lambda' f_8$ and $f_5(e_2) = \begin{pmatrix} 0 \\ f_1 \end{pmatrix}$ with $f_1 = f_8$

Therefore we get

$$f_1 = f_8 \qquad f_2 = f_7 = f_1 \mathbb{1}_2 \qquad f_3 = f_5 = f_6 = f_1 \mathbb{1}_3 \qquad f_4 = f_1 \mathbb{1}_4 \tag{A.4}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f \colon B(\lambda)_{\widetilde{\mathbb{E}}_7} \to B(\lambda)_{\widetilde{\mathbb{E}}_7}$ depends only on $f_1 \in \mathbf{k}$ by (A.4), therefore $\operatorname{End}(B(\lambda)_{\widetilde{\mathbb{E}}_7}) \cong \mathbf{k}$, so $B(\lambda)_{\widetilde{\mathbb{E}}_7}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_8$, but by (A.4) we can write $\lambda f_1 = \lambda' f_1$. If $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\widetilde{\mathbb{E}}_7}, B(\lambda')_{\widetilde{\mathbb{E}}_7}) = 0$$

Therefore $B(\lambda)_{\widetilde{\mathbb{E}}_7}$ and $B(\lambda')_{\widetilde{\mathbb{E}}_7}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\widetilde{\mathbb{E}}_7}\}_{\lambda \in \mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{E}}_7\right)$ is brick-continuous. □

Construction of $G_{\widetilde{\mathbb{E}}_7}$ We have the brick-continuous family $\{B(\lambda)_{\widetilde{\mathbb{E}}_7}\}_{\lambda \in \mathbf{k}}$ and we define the representation $G_{\widetilde{\mathbb{E}}_7}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & & & \mathbf{k}(x)^2 \arrow[d, "\phi_{54,x}"] & & & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"] & \mathbf{k}(x)^3 \arrow[r, "\phi_{34}"] & \mathbf{k}(x)^4 & \mathbf{k}(x)^3 \arrow[l, "\phi_{64}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{76}"'] & \mathbf{k}(x) \arrow[l, "\phi_{87}"']
\end{tikzcd}$$

with $\phi_{54,x} \coloneqq \varphi_{54}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.8. *$G_{\widetilde{\mathbb{E}}_7}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{E}}_7\right)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\widetilde{\mathbb{E}}_7)}(G_{\widetilde{\mathbb{E}}_7}) = \dim_{\mathbf{k}}(G_{\widetilde{\mathbb{E}}_7}) = \sum_{i \in (Q_{\widetilde{\mathbb{E}}_7})_0} \dim_{\mathbf{k}}((G_{\widetilde{\mathbb{E}}_7})_i) = \infty$$

So $G_{\widetilde{\mathbb{E}}_7}$ have infinite length. Moreover, following the same procedure as in Theorem A.7, we see that all the components of a endomorphism $f \colon G_{\widetilde{\mathbb{E}}_7} \to G_{\widetilde{\mathbb{E}}_7}$ depends only on the map $f_1 \colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $x f_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = x f_1(x^{-1}) \implies x^{-1} f_1(1) = f_1(x^{-1})$$

Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\mathrm{End}(G_{\widetilde{\mathbb{E}}_7}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\widetilde{\mathbb{E}}_7}) = \dim_{\mathbf{k}(x)}(G_{\widetilde{\mathbb{E}}_7}) = \sum_{i \in (Q_{\widetilde{\mathbb{E}}_7})_0} \dim_{\mathbf{k}(x)}((G_{\widetilde{\mathbb{E}}_7})_i) < \infty$$

So $G_{\widetilde{\mathbb{E}}_7}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\widetilde{\mathbb{E}}_7}$ is a generic brick. □

A.5. Generic brick of $\widetilde{\mathbb{E}}_8$

Consider the critical poset of type $P = \widetilde{\mathbb{E}}_8$ (see Table 2.1) with the following fixed orientation and labelling:

$$\begin{tikzcd}
 & & 9 \arrow[d] & & & & & \\
7 \arrow[r] & 8 \arrow[r] & 6 & 5 \arrow[l] & 4 \arrow[l] & 3 \arrow[l] & 2 \arrow[l] & 1 \arrow[l]
\end{tikzcd}$$

Its Tits form $q_{\widetilde{\mathbb{E}}_8}$ is

$$q_{\widetilde{\mathbb{E}}_8}(x) = \sum_{i=1}^{9} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_5x_6 - x_7x_8 - x_8x_6 - x_9x_6$$

and the associated matrix $B_{\widetilde{\mathbb{E}}_8}$ is given by

$$B_{\widetilde{\mathbb{E}}_8} = \frac{1}{2}\begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\widetilde{\mathbb{E}}_8}$ we obtain

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{4}{3} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{5}{3} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{2}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{4}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\widetilde{\mathbb{E}}_8} = (1,2,3,4,5,6,2,4,3)^t$ (we want $y_{\widetilde{\mathbb{E}}_8} \in \mathbb{Z}^n$ because we will use it as a dimension vector). Therefore

$$\mathrm{Rad}(q_{\widetilde{\mathbb{E}}_8}) = \ker(B_{\widetilde{\mathbb{E}}_8}) = \langle y_{\widetilde{\mathbb{E}}_8} \rangle$$

We define the representations $B(\lambda)_{\widetilde{\mathbb{E}}_8}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\widetilde{\mathbb{E}}_8} := \begin{tikzcd} & & \mathbf{k}^3 \arrow[d, "\varphi_{96}(\lambda)"] & & & & & \\ \mathbf{k}^2 \arrow[r, "\varphi_{78}"] & \mathbf{k}^4 \arrow[r, "\varphi_{86}"] & \mathbf{k}^6 & \mathbf{k}^5 \arrow[l, "\varphi_{56}"'] & \mathbf{k}^4 \arrow[l, "\varphi_{45}"'] & \mathbf{k}^3 \arrow[l, "\varphi_{34}"'] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"'] & \mathbf{k} \arrow[l, "\varphi_{12}"'] \end{tikzcd}$$

where the $\mathbf{k}$-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_{45} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \varphi_{56} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \varphi_{78} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_{86} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \varphi_{96}(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

We have that $\underline{\dim}(B(\lambda)_{\widetilde{\mathbb{E}}_8}) = y_{\widetilde{\mathbb{E}}_8}$ and we prove the following:

Theorem A.9. *The representations $B(\lambda)_{\widetilde{\mathbb{E}}_8}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{E}}_8\right)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\widetilde{\mathbb{E}}_8} \to B(\lambda')_{\widetilde{\mathbb{E}}_8}$ be a morphism of representations.

$$\begin{tikzcd}
 & & & \mathbf{k}^3 \arrow[dl, "\varphi_{96}(\lambda)"'] \arrow[dd, "f_9"'] & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{78}"] \arrow[dd, "f_7"'] & \mathbf{k}^4 \arrow[r, "\varphi_{86}"] \arrow[dd, "f_8"'] & \mathbf{k}^6 \arrow[dd, "f_6"'] & & \mathbf{k}^5 \arrow[ll, "\varphi_{56}"'] \arrow[dd, "f_5"] & \mathbf{k}^4 \arrow[l, "\varphi_{45}"'] \arrow[dd, "f_4"] & \mathbf{k}^3 \arrow[l, "\varphi_{34}"'] \arrow[dd, "f_3"] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"'] \arrow[dd, "f_2"] & \mathbf{k} \arrow[l, "\varphi_{12}"'] \arrow[dd, "f_1"] \\
 & & & \mathbf{k}^3 \arrow[dl, "\varphi_{96}(\lambda')"] & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{78}"'] & \mathbf{k}^4 \arrow[r, "\varphi_{86}"'] & \mathbf{k}^6 & & \mathbf{k}^5 \arrow[ll, "\varphi_{56}"] & \mathbf{k}^4 \arrow[l, "\varphi_{45}"] & \mathbf{k}^3 \arrow[l, "\varphi_{34}"] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"] & \mathbf{k} \arrow[l, "\varphi_{12}"]
\end{tikzcd}$$

We have that f_1 is just a scalar multiplication of an element of $\mathbf{k}$, while $f_2, f_7 \in \mathbb{M}_2(\mathbf{k})$, $f_3, f_9 \in \mathbb{M}_3(\mathbf{k})$, $f_4, f_8 \in \mathbb{M}_4(\mathbf{k})$, $f_5 \in \mathbb{M}_5(\mathbf{k})$ and $f_6 \in \mathbb{M}_6(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(3) The square involving f_3 and f_4 gives us that $\varphi_{34} f_3(e_i) = f_4(\varphi_{34}(e_i))$ that means

$$f_4(e_1) = \begin{pmatrix} f_3(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_2) = \begin{pmatrix} f_3(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \end{pmatrix}$$

(4) The square involving f_4 and f_5 gives us that $\varphi_{45} f_4(e_i) = f_5(\varphi_{45}(e_i))$ that means

$$f_5(e_1) = \begin{pmatrix} f_4(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_2) = \begin{pmatrix} f_4(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f_5(e_3) = \begin{pmatrix} f_4(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_4) = \begin{pmatrix} f_4(e_4) \\ 0 \end{pmatrix}$$

(5) The square involving f_5 and f_6 gives us that $\varphi_{56} f_5(e_i) = f_6(\varphi_{56}(e_i))$ that means

$$f_6(e_1) = \begin{pmatrix} f_5(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_6(e_2) = \begin{pmatrix} f_5(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f_6(e_3) = \begin{pmatrix} f_5(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_6(e_4) = \begin{pmatrix} f_5(e_4) \\ 0 \end{pmatrix} = \begin{pmatrix} f_4(e_4) \\ 0 \\ 0 \end{pmatrix} \quad f_6(e_5) = \begin{pmatrix} f_5(e_5) \\ 0 \end{pmatrix}$$

(6) The square involving f_7 and f_8 gives us that $\varphi_{78} f_7(e_i) = f_8(\varphi_{78}(e_i))$ that means

$$f_8(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_1) \end{pmatrix} \quad f_8(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_2) \end{pmatrix}$$

(7) The square involving f_8 and f_6 gives us that $\varphi_{86} f_8(e_i) = f_6(\varphi_{86}(e_i))$ that means

$$f_6(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_8(e_1) \end{pmatrix} \quad f_6(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_8(e_2) \end{pmatrix}$$

$$f_6(e_5) = \begin{pmatrix} 0 \\ 0 \\ f_8(e_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_7(e_1) \end{pmatrix} \quad f_6(e_6) = \begin{pmatrix} 0 \\ 0 \\ f_8(e_4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_7(e_2) \end{pmatrix}$$

Comparing with (5) we get that

$$f_6(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_8(e_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_6(e_4) = \begin{pmatrix} f_4(e_4) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_8(e_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \\ c \\ 0 \\ 0 \end{pmatrix}$$

$$f_6(e_5) = \begin{pmatrix} f_5(e_5) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_7(e_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ d \\ 0 \end{pmatrix}$$

for some $a, b, c, d \in \mathbf{k}$.
(8) The square involving f_6 and f_9 gives us that $\varphi_{96}(\lambda')f_9(e_i) = f_6(\varphi_{96}(\lambda)(e_i))$ that means for $i = 3$

$$\begin{pmatrix} f_2(e_2)_1 \\ f_2(e_2)_2 \\ b \\ c \\ 0 \\ 0 \end{pmatrix} = f_6(e_2 + e_4) = f_6(\varphi_{96}(\lambda)(e_3)) = \varphi_{96}(\lambda')f_9(e_3) = \begin{pmatrix} \lambda' f_9(e_3)_1 + f_9(e_3)_2 \\ f_9(e_3)_3 \\ f_9(e_3)_1 + f_9(e_3)_2 \\ f_9(e_3)_1 + f_9(e_3)_3 \\ f_9(e_3)_1 + f_9(e_3)_2 \\ f_9(e_3)_2 \end{pmatrix}$$

and we get $f_9(e_3) = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$, $f_2(e_2) = \begin{pmatrix} 0 \\ c \end{pmatrix}$ and $b = 0$.

With $i = 2$ we get

$$\begin{pmatrix} f_1 \\ 0 \\ a \\ 0 \\ d + f_7(e_2)_1 \\ f_7(e_2)_2 \end{pmatrix} = f_6(e_1 + e_3 + e_5 + e_6) = f_6(\varphi_{96}(\lambda)(e_2)) = \varphi_{96}(\lambda')f_9(e_2) = \begin{pmatrix} \lambda' f_9(e_2)_1 + f_9(e_2)_2 \\ f_9(e_2)_3 \\ f_9(e_2)_1 + f_9(e_2)_2 \\ f_9(e_2)_1 + f_9(e_2)_3 \\ f_9(e_2)_1 + f_9(e_2)_2 \\ f_9(e_2)_2 \end{pmatrix}$$

and we get $f_9(e_2) = \begin{pmatrix} 0 \\ f_1 \\ 0 \end{pmatrix}$ and $f_7(e_2) = \begin{pmatrix} f_7(e_2)_1 \\ f_1 \end{pmatrix}$ with $a = f_1$ and $d + f_7(e_2)_1 = f_1$.

Finally, with $i = 1$ we get

$$\begin{pmatrix} \lambda f_1 \\ 0 \\ f_1 \\ c \\ d \\ 0 \end{pmatrix} = f_6(\lambda e_1 + e_3 + e_4 + e_5) = f_6(\varphi_{96}(\lambda)(e_1)) = \varphi_{96}(\lambda')f_9(e_1) = \begin{pmatrix} \lambda' f_9(e_1)_1 + f_9(e_1)_2 \\ f_9(e_1)_3 \\ f_9(e_1)_1 + f_9(e_1)_2 \\ f_9(e_1)_1 + f_9(e_1)_3 \\ f_9(e_1)_1 + f_9(e_1)_2 \\ f_9(e_1)_2 \end{pmatrix}$$

and we get $f_9(e_1) = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix}$ with $c = d = f_1$, $f_7(e_2)_1 = 0$ (since we had $d + f_7(e_2)_1 = f_1$) and $\lambda f_1 = \lambda' f_1$.

Therefore we get

$$f_2 = f_7 = f_1 \mathbb{1}_2 \quad f_3 = f_9 = f_1 \mathbb{1}_3 \quad f_4 = f_8 = f_1 \mathbb{1}_4 \quad f_5 = f_1 \mathbb{1}_5 \quad f_6 = f_1 \mathbb{1}_6 \tag{A.5}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f \colon B(\lambda)_{\widetilde{\mathbb{E}}_8} \to B(\lambda)_{\widetilde{\mathbb{E}}_8}$ depends only on $f_1 \in \mathbf{k}$ by (A.5), therefore $\operatorname{End}(B(\lambda)_{\widetilde{\mathbb{E}}_8}) \cong \mathbf{k}$, so $B(\lambda)_{\widetilde{\mathbb{E}}_8}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_1$ and if $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\widetilde{\mathbb{E}}_8}, B(\lambda')_{\widetilde{\mathbb{E}}_8}) = 0$$

Therefore $B(\lambda)_{\widetilde{\mathbb{E}}_8}$ and $B(\lambda')_{\widetilde{\mathbb{E}}_8}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\widetilde{\mathbb{E}}_8}\}_{\lambda \in \mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{E}}_8\right)$ is brick-continuous. □

Construction of $G_{\widetilde{\mathbb{E}}_8}$ We have the brick-continuous family $\{B(\lambda)_{\widetilde{\mathbb{E}}_8}\}_{\lambda \in \mathbf{k}}$ and we define the representation $G_{\widetilde{\mathbb{E}}_8}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & & \mathbf{k}(x)^3 \arrow[d, "\phi_{96,x}"] & & & & & \\
\mathbf{k}(x)^2 \arrow[r, "\phi_{78}"] & \mathbf{k}(x)^4 \arrow[r, "\phi_{86}"] & \mathbf{k}(x)^6 & \mathbf{k}(x)^5 \arrow[l, "\phi_{56}"'] & \mathbf{k}(x)^4 \arrow[l, "\phi_{45}"'] & \mathbf{k}(x)^3 \arrow[l, "\phi_{34}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{23}"'] & \mathbf{k}(x) \arrow[l, "\phi_{12}"']
\end{tikzcd}$$

with $\phi_{96,x} := \varphi_{96}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.10. *$G_{\widetilde{\mathbb{E}}_8}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}\left(\widetilde{\mathbb{E}}_8\right)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\widetilde{\mathbb{E}}_8)}(G_{\widetilde{\mathbb{E}}_8}) = \dim_{\mathbf{k}}(G_{\widetilde{\mathbb{E}}_8}) = \sum_{i \in (Q_{\widetilde{\mathbb{E}}_8})_0} \dim_{\mathbf{k}}((G_{\widetilde{\mathbb{E}}_8})_i) = \infty$$

So $G_{\widetilde{\mathbb{E}}_8}$ have infinite length. Moreover, following the same procedure as in Theorem A.9, we see that all the components of a endomorphism $f \colon G_{\widetilde{\mathbb{E}}_8} \to G_{\widetilde{\mathbb{E}}_8}$ depends only on the map $f_1 \colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $x f_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = x f_1(x^{-1}) \implies x^{-1} f_1(1) = f_1(x^{-1})$$

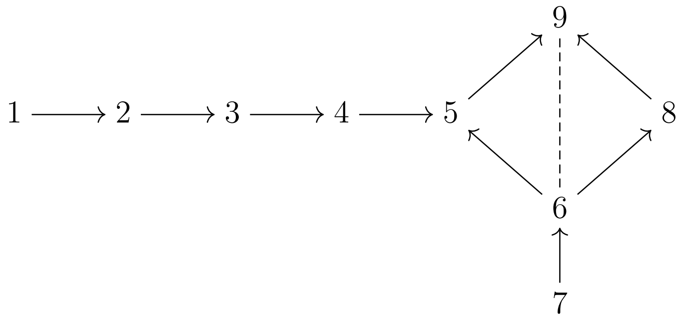
Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\operatorname{End}(G_{\widetilde{\mathbb{E}}_8}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\widetilde{\mathbb{E}}_8}) = \dim_{\mathbf{k}(x)}(G_{\widetilde{\mathbb{E}}_8}) = \sum_{i \in (Q_{\widetilde{\mathbb{E}}_8})_0} \dim_{\mathbf{k}(x)}((G_{\widetilde{\mathbb{E}}_8})_i) < \infty$$

So $G_{\widetilde{\mathbb{E}}_8}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\widetilde{\mathbb{E}}_8}$ is a generic brick. □

A.6. Generic brick of $\mathbb{R}_1$

Consider the critical poset of type $P = \mathbb{R}_1$ (see Table 2.1) with the following fixed orientation and labelling (we work with its dual):



Its Tits form $q_{\mathbb{R}_1}$ is

$$q_{\mathbb{R}_1}(x) = \sum_{i=1}^{9} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_6x_5 - x_5x_9 - x_6x_8 - x_8x_9 - x_7x_6 + x_6x_9$$

because $I_{\mathbb{R}_1} = \langle \alpha_{59}\alpha_{65} - \alpha_{89}\alpha_{68} \rangle$ so $r_{69} = 1$. The associated matrix $B_{\mathbb{R}_1}$ is given by

$$B_{\mathbb{R}_1} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\mathbb{R}_1}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\mathbb{R}_1} = (1,2,3,4,5,4,2,3,2)^t$ (we want $y_{\mathbb{R}_1} \in \mathbb{Z}^n$ because we will use it as a dimension vector). Therefore

$$\mathrm{Rad}(q_{\mathbb{R}_1}) = \ker(B_{\mathbb{R}_1}) = \langle y_{\mathbb{R}_1} \rangle$$

We define the representations $B(\lambda)_{\mathbb{R}_1}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$\begin{tikzcd}
 & & & & & & \mathbf{k}^2 \arrow[dd, dash, dashed] & \\
B(\lambda)_{\mathbb{R}_1} := & \mathbf{k} \arrow[r, "\varphi_{12}"] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"] & \mathbf{k}^4 \arrow[r, "\varphi_{45}"] & \mathbf{k}^5 \arrow[ur, "\varphi_{59}(\lambda)"] & & \mathbf{k}^3 \arrow[ul, "\varphi_{89}"'] \\
 & & & & & & \mathbf{k}^4 \arrow[ul, "\varphi_{65}"] \arrow[ur, "\varphi_{68}"'] & \\
 & & & & & & \mathbf{k}^2 \arrow[u, "\varphi_{76}"] &
\end{tikzcd}$$

where the $\mathbf{k}$-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_{45} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \varphi_{65} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \varphi_{76} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_{68} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \varphi_{89} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \varphi_{59}(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The relations are preserved since

$$\varphi_{59}(\lambda)\varphi_{65} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} = \varphi_{89}\varphi_{68}$$

We have that $\underline{\dim}(B(\lambda)_{\mathbb{R}_1}) = y_{\mathbb{R}_1}$ and we prove the following:

Theorem A.11. *The representations $B(\lambda)_{\mathbb{R}_1}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_1)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\mathbb{R}_1} \to B(\lambda')_{\mathbb{R}_1}$ be a morphism of representations.

$$\begin{tikzcd}
 & & & & & \mathbf{k}^2 \arrow[ddd, "f_9"] \arrow[ddr, dash, dashed] & & & \\
\mathbf{k} \arrow[r, "\varphi_{12}"] \arrow[ddd, "f_1"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] \arrow[ddd, "f_2"'] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"] \arrow[ddd, "f_3"'] & \mathbf{k}^4 \arrow[r, "\varphi_{45}"] \arrow[ddd, "f_4"'] & \mathbf{k}^5 \arrow[ur, "\varphi_{59}(\lambda)"] \arrow[ddd, "f_5"'] & & & \mathbf{k}^3 \arrow[ull, "\varphi_{89}"'] \arrow[ddd, "f_8"'] & \\
 & & & & & & \mathbf{k}^4 \arrow[ull, "\varphi_{65}"] \arrow[ur, "\varphi_{68}"] \arrow[ddd, "f_6"'] & & \mathbf{k}^2 \arrow[ll, "\varphi_{76}"] \arrow[ddd, "f_7"] \\
 & & & & & \mathbf{k}^2 \arrow[ddr, dash, dashed] & & & \\
\mathbf{k} \arrow[r, "\varphi_{12}"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"'] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"'] & \mathbf{k}^4 \arrow[r, "\varphi_{45}"'] & \mathbf{k}^5 \arrow[ur, "\varphi_{59}(\lambda')"'] & & & \mathbf{k}^3 \arrow[ull, "\varphi_{89}"'] & \\
 & & & & & & \mathbf{k}^4 \arrow[ull, "\varphi_{65}"] \arrow[ur, "\varphi_{68}"] & & \mathbf{k}^2 \arrow[ll, "\varphi_{76}"]
\end{tikzcd}$$

We have that f_1 is just a scalar multiplication of an element of $\mathbf{k}$, while $f_2, f_7, f_9 \in \mathbb{M}_2(\mathbf{k})$, $f_3, f_8 \in \mathbb{M}_3(\mathbf{k})$, $f_4, f_6 \in \mathbb{M}_4(\mathbf{k})$ and $f_5 \in \mathbb{M}_5(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(3) The square involving f_3 and f_4 gives us that $\varphi_{34} f_3(e_i) = f_4(\varphi_{34}(e_i))$ that means

$$f_4(e_1) = \begin{pmatrix} f_3(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_2) = \begin{pmatrix} f_3(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \end{pmatrix}$$

(4) The square involving f_4 and f_5 gives us that $\varphi_{45} f_4(e_i) = f_5(\varphi_{45}(e_i))$ that means

$$f_5(e_1) = \begin{pmatrix} f_4(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_2) = \begin{pmatrix} f_4(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f_5(e_3) = \begin{pmatrix} f_4(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_4) = \begin{pmatrix} f_4(e_4) \\ 0 \end{pmatrix}$$

(5) The square involving f_6 and f_7 gives us that $\varphi_{76} f_7(e_i) = f_6(\varphi_{76}(e_i))$ that means

$$f_6(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_1) \end{pmatrix} \quad f_6(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_2) \end{pmatrix}$$

(6) The square involving f_5 and f_6 gives us that $\varphi_{65} f_6(e_i) = f_5(\varphi_{65}(e_i))$ that means

$$f_5(e_2) = \begin{pmatrix} 0 \\ f_6(e_1) \end{pmatrix} \quad f_5(e_3) = \begin{pmatrix} 0 \\ f_6(e_2) \end{pmatrix}$$

$$f_5(e_4) = \begin{pmatrix} 0 \\ f_6(e_3) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_7(e_1) \end{pmatrix} \quad f_5(e_5) = \begin{pmatrix} 0 \\ f_6(e_4) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_7(e_2) \end{pmatrix}$$

Comparing with (4) we get that

$$f_5(e_2) = \begin{pmatrix} 0 \\ f_6(e_1) \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_3) = \begin{pmatrix} 0 \\ f_6(e_2) \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ c \\ 0 \\ 0 \end{pmatrix}$$

$$f_5(e_4) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_7(e_1) \end{pmatrix} = \begin{pmatrix} f_4(e_4) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \\ 0 \end{pmatrix} \quad f_5(e_5) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ f_7(e_2) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e \\ f \end{pmatrix}$$

for some $a, b, c, d, e, f \in \mathbf{k}$
(7) The square involving f_6 and f_8 gives us that $\varphi_{68} f_6(e_i) = f_8(\varphi_{68}(e_i))$ that means

$$f_8(e_1) = \varphi_{68} f_6(e_1) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \quad f_8(e_2) = \varphi_{68} f_6(e_2) = \begin{pmatrix} b \\ c \\ 0 \end{pmatrix}$$

$$f_8(e_3) = \varphi_{68} f_6(e_3) = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix} \quad f_8(e_2) = \varphi_{68} f_6(e_4) = \begin{pmatrix} 0 \\ f \\ e \end{pmatrix} = \begin{pmatrix} b \\ c \\ 0 \end{pmatrix}$$

so we conclude that $b = e = 0$ and $c = f$.
(8) The square involving f_8 and f_9 gives us that $\varphi_{89} f_8(e_i) = f_9(\varphi_{89}(e_i))$ that means

$$f_9(e_1) = \varphi_{89} f_8(e_1) = \begin{pmatrix} a \\ 0 \end{pmatrix} \qquad f_9(e_2) = \varphi_{89} f_8(e_2) = \begin{pmatrix} 0 \\ c \end{pmatrix}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} = f_9(e_1 + e_2) = \varphi_{89} f_8(e_3) = \begin{pmatrix} d \\ d \end{pmatrix}$$

so we conclude that $a = d = c\,(= f)$.
(9) The square involving f_5 and f_9 gives us that $\varphi_{59}(\lambda') f_5(e_i) = f_9(\varphi_{59}(\lambda)(e_i))$. Since we have relations $i = 2, 3, 4, 5$ lead to known facts, while $i = 1$ gives us

$$\begin{pmatrix} \lambda a \\ a \end{pmatrix} = f_9(\lambda e_1 + e_2) = \varphi_{89}(\lambda') f_5(e_1) = \begin{pmatrix} \lambda' f_1 \\ f_1 \end{pmatrix}$$

and we get $a = f_1$ and $\lambda f_1 = \lambda' f_1$.

Therefore we get

$$f_2 = f_7 = f_9 = f_1 \mathbb{1}_2 \quad f_3 = f_8 = f_1 \mathbb{1}_3 \quad f_4 = f_6 = f_1 \mathbb{1}_4 \quad f_5 = f_1 \mathbb{1}_5 \tag{A.6}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f\colon B(\lambda)_{\mathbb{R}_1} \to B(\lambda)_{\mathbb{R}_1}$ depends only on $f_1 \in \mathbf{k}$ by (A.6), therefore $\operatorname{End}(B(\lambda)_{\mathbb{R}_1}) \cong \mathbf{k}$, so $B(\lambda)_{\mathbb{R}_1}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_1$ and if $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\mathbb{R}_1}, B(\lambda')_{\mathbb{R}_1}) = 0$$

Therefore $B(\lambda)_{\mathbb{R}_1}$ and $B(\lambda')_{\mathbb{R}_1}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\mathbb{R}_1}\}_{\lambda \in \mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_1)$ is brick-continuous. $\square$

Construction of $G_{\mathbb{R}_1}$ We have the brick-continuous family $\{B(\lambda)_{\mathbb{R}_1}\}_{\lambda \in \mathbf{k}}$ and we define the representation $G_{\mathbb{R}_1}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & & & & & \mathbf{k}(x)^2 \arrow[dd, dash, dashed] & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"] & \mathbf{k}(x)^3 \arrow[r, "\phi_{34}"] & \mathbf{k}(x)^4 \arrow[r, "\phi_{45}"] & \mathbf{k}(x)^5 \arrow[ur, red, "\phi_{59,x}"] & & \mathbf{k}(x)^3 \arrow[ul, "\phi_{89}"'] \\
 & & & & & \mathbf{k}(x)^4 \arrow[ul, "\phi_{65}"] \arrow[ur, "\phi_{68}"'] & \\
 & & & & & \mathbf{k}(x)^2 \arrow[u, "\phi_{76}"] &
\end{tikzcd}$$

with $\phi_{59,x} := \varphi_{59}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.12. *$G_{\mathbb{R}_1}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_1)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\mathbb{R}_1)}(G_{\mathbb{R}_1}) = \dim_{\mathbf{k}}(G_{\mathbb{R}_1}) = \sum_{i \in (Q_{\mathbb{R}_1})_0} \dim_{\mathbf{k}}((G_{\mathbb{R}_1})_i) = \infty$$

So $G_{\mathbb{R}_1}$ have infinite length. Moreover, following the same procedure as in Theorem A.11, we see that all the components of a endomorphism $f\colon G_{\mathbb{R}_1} \to G_{\mathbb{R}_1}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

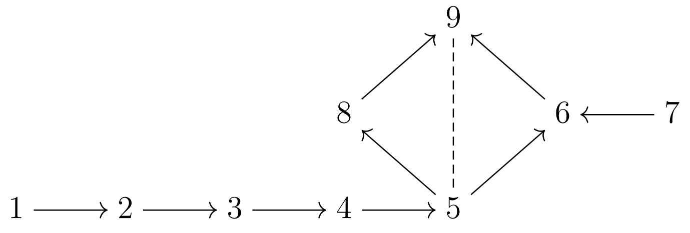
Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\operatorname{End}(G_{\mathbb{R}_1}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\mathbb{R}_1}) = \dim_{\mathbf{k}(x)}(G_{\mathbb{R}_1}) = \sum_{i \in (Q_{\mathbb{R}_1})_0} \dim_{\mathbf{k}(x)}((G_{\mathbb{R}_1})_i) < \infty$$

So $G_{\mathbb{R}_1}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\mathbb{R}_1}$ is a generic brick. □

A.7. Generic brick of $\mathbb{R}_2$

Consider the critical poset of type $P = \mathbb{R}_2$ (see Table 2.1) with the following fixed orientation and labelling (we work with its dual):



Its Tits form $q_{\mathbb{R}_2}$ is

$$q_{\mathbb{R}_2}(x) = \sum_{i=1}^{9} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_5x_8 - x_5x_6 - x_6x_9 - x_8x_9 - x_7x_6 + x_5x_9$$

because $I_{\mathbb{R}_2} = \langle \alpha_{89}\alpha_{58} - \alpha_{69}\alpha_{56} \rangle$ so $r_{59} = 1$. The associated matrix $B_{\mathbb{R}_2}$ is given by

$$B_{\mathbb{R}_2} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\mathbb{R}_2}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\mathbb{R}_2} = (1, 2, 3, 4, 5, 4, 2, 3, 1)^t$. Therefore

$$\mathrm{Rad}(q_{\mathbb{R}_2}) = \ker(B_{\mathbb{R}_2}) = \langle y_{\mathbb{R}_2} \rangle$$

We define the representations $B(\lambda)_{\mathbb{R}_2}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\mathbb{R}_2} := \begin{tikzcd} & & & & \mathbf{k} & & \\ & & & \mathbf{k}^3 \arrow[ur, "\varphi_{89}(\lambda)"] & & \mathbf{k}^4 \arrow[ul, "\varphi_{69}(\lambda)"'] & \mathbf{k}^2 \arrow[l, "\varphi_{76}"'] \\ \mathbf{k} \arrow[r, "\varphi_{12}"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"'] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"'] & \mathbf{k}^4 \arrow[r, "\varphi_{45}"'] & \mathbf{k}^5 \arrow[ul, "\varphi_{58}"] \arrow[ur, "\varphi_{56}"'] \arrow[uu, dash, dashed] & & \end{tikzcd}$$

where the $\mathbf{k}$-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_{45} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \varphi_{56} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \varphi_{76} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_{58} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \varphi_{89}(\lambda) = \begin{pmatrix} 1 & \lambda & (-1-\lambda) \end{pmatrix} \quad \varphi_{69}(\lambda) = \begin{pmatrix} 1 & \lambda & (-1-\lambda) & \lambda \end{pmatrix}$$

The relations are preserved since

$$\varphi_{89}(\lambda)\varphi_{58} = \begin{pmatrix} 1 & \lambda & (-1-\lambda) & \lambda & 0 \end{pmatrix} = \varphi_{69}(\lambda)\varphi_{56}$$

We have that $\underline{\dim}(B(\lambda)_{\mathbb{R}_2}) = y_{\mathbb{R}_2}$ and we prove the following:

Theorem A.13. *The representations $B(\lambda)_{\mathbb{R}_2}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_2)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\mathbb{R}_2} \to B(\lambda')_{\mathbb{R}_2}$ be a morphism of representations.

$$\begin{tikzcd}
 & & & & & & \mathbf{k} \arrow[ddd, red, "f_9"] & & \\
 & & & & \mathbf{k}^3 \arrow[urr, "\varphi_{89}(\lambda)"] \arrow[ddd, red, "f_8"'] & & & \mathbf{k}^4 \arrow[ul, "\varphi_{69}(\lambda)"'] \arrow[ddd, red, "f_6"] & \mathbf{k}^2 \arrow[l, "\varphi_{76}"'] \arrow[ddd, red, "f_7"] \\
\mathbf{k} \arrow[r, "\varphi_{12}"] \arrow[ddd, red, "f_1"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] \arrow[ddd, red, "f_2"'] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"] \arrow[ddd, red, "f_3"'] & \mathbf{k}^4 \arrow[rr, "\varphi_{45}"] \arrow[ddd, red, "f_4"'] & & \mathbf{k}^5 \arrow[ul, "\varphi_{58}"'] \arrow[urr, "\varphi_{56}"'] \arrow[uur, dash, dashed] \arrow[ddd, red, "f_5"] & & & \\
 & & & & & & \mathbf{k} & & \\
 & & & & \mathbf{k}^3 \arrow[urr, "\varphi_{89}(\lambda')"] & & & \mathbf{k}^4 \arrow[ul, "\varphi_{69}(\lambda')"'] & \mathbf{k}^2 \arrow[l, "\varphi_{76}"'] \\
\mathbf{k} \arrow[r, "\varphi_{12}"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"'] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"'] & \mathbf{k}^4 \arrow[rr, "\varphi_{45}"'] & & \mathbf{k}^5 \arrow[ul, "\varphi_{58}"] \arrow[urr, "\varphi_{56}"'] \arrow[uur, dash, dashed] & & &
\end{tikzcd}$$

We have that f_1, f_9 are just scalar multiplications of elements of $\mathbf{k}$, while $f_2, f_7 \in \mathbb{M}_2(\mathbf{k})$, $f_3, f_8 \in \mathbb{M}_3(\mathbf{k})$, $f_4, f_6 \in \mathbb{M}_4(\mathbf{k})$ and $f_5 \in \mathbb{M}_5(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(3) The square involving f_3 and f_4 gives us that $\varphi_{34} f_3(e_i) = f_4(\varphi_{34}(e_i))$ that means

$$f_4(e_1) = \begin{pmatrix} f_3(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_2) = \begin{pmatrix} f_3(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \end{pmatrix}$$

(4) The square involving f_4 and f_5 gives us that $\varphi_{45} f_4(e_i) = f_5(\varphi_{45}(e_i))$ that means

$$f_5(e_1) = \begin{pmatrix} f_4(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_2) = \begin{pmatrix} f_4(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f_5(e_3) = \begin{pmatrix} f_4(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_4) = \begin{pmatrix} f_4(e_4) \\ 0 \end{pmatrix}$$

(5) The square involving f_6 and f_7 gives us that $\varphi_{76} f_7(e_i) = f_6(\varphi_{76}(e_i))$ that means

$$f_6(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_1) \end{pmatrix} \quad f_6(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_2) \end{pmatrix}$$

(6) The square involving f_5 and f_6 gives us that $\varphi_{65} f_5(e_i) = f_6(\varphi_{65}(e_i))$ that means

$$f_6(e_1) = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_6(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}$$

$$f_6(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_7(e_1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \\ 0 \end{pmatrix} \quad f_6(e_4) = f_4(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d \\ e \end{pmatrix}$$

(we used point (5)) and for $i = 5$ we get $f_5(e_5) = \begin{pmatrix}0\\0\\0\\0\\f\end{pmatrix}$ for some $a, b, c, d, e, f \in \mathbf{k}$.

(7) The square involving f_5 and f_8 gives us that $\varphi_{58} f_5(e_i) = f_8(\varphi_{58}(e_i))$ that means

$$f_8(e_1) = \varphi_{58} f_5(e_1) = \begin{pmatrix}f_1\\0\\0\end{pmatrix} \quad f_8(e_2) = \varphi_{58} f_5(e_2) = \begin{pmatrix}a\\b\\0\end{pmatrix}$$

$$f_8(e_3) = \varphi_{58} f_5(e_3) = \begin{pmatrix}0\\0\\c\end{pmatrix} \quad \begin{pmatrix}a\\b\\0\end{pmatrix} = f_8(e_2) = \varphi_{58} f_5(e_4) = \begin{pmatrix}0\\e\\d\end{pmatrix}$$

so $a = d = 0$ and $b = e$ and also

$$\begin{pmatrix}f_1\\b\\c\end{pmatrix} = f_8(e_1 + e_2 + e_3) = \varphi_{58} f_5(e_5) = \begin{pmatrix}f\\f\\f\end{pmatrix}$$

so we conclude that $f_1 = b = c = e = f$.

(8) The square involving f_8 and f_9 gives us that $\varphi_{89}(\lambda') f_8(e_i) = f_9(\varphi_{89}(\lambda)(e_i))$ that means for $i = 1$

$$f_9 = \varphi_{89}(\lambda) f_8(e_1) = f_1$$

and for $i = 2$

$$\lambda' f_9 = \varphi_{89}(\lambda) f_8(e_1) = \lambda f_1$$

so we conclude that $f_1 = f_9$ and $\lambda f_1 = \lambda' f_1$.

(9) The square involving f_6 and f_9 gives us the same informations of (8).

Therefore we get

$$f_1 = f_9 \quad f_2 = f_7 = f_1 \mathbb{1}_2 \quad f_3 = f_8 = f_1 \mathbb{1}_3 \quad f_4 = f_6 = f_1 \mathbb{1}_4 \quad f_5 = f_1 \mathbb{1}_5 \tag{A.7}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f\colon B(\lambda)_{\mathbb{R}_2} \to B(\lambda)_{\mathbb{R}_2}$ depends only on $f_1 \in \mathbf{k}$ by (A.7), therefore $\mathrm{End}(B(\lambda)_{\mathbb{R}_2}) \cong \mathbf{k}$, so $B(\lambda)_{\mathbb{R}_2}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_1$ and if $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\mathrm{Hom}(B(\lambda)_{\mathbb{R}_2}, B(\lambda')_{\mathbb{R}_2}) = 0$$

Therefore $B(\lambda)_{\mathbb{R}_2}$ and $B(\lambda')_{\mathbb{R}_2}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\mathbb{R}_2}\}_{\lambda \in \mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_2)$ is brick-continuous. □

Construction of $G_{\mathbb{R}_2}$ We have the brick-continuous family $\{B(\lambda)_{\mathbb{R}_2}\}_{\lambda\in\mathbf{k}}$ and we define the representation $G_{\mathbb{R}_2}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & & & & \mathbf{k}(x) & & \\
 & & & \mathbf{k}(x)^3 \arrow[ur, "\phi_{89,x}"] & & \mathbf{k}(x)^4 \arrow[ul, "\phi_{69,x}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{76}"'] \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"'] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"'] & \mathbf{k}(x)^3 \arrow[r, "\phi_{34}"'] & \mathbf{k}(x)^4 \arrow[r, "\phi_{45}"'] & \mathbf{k}(x)^5 \arrow[ul, "\phi_{58}"'] \arrow[ur, "\phi_{56}"'] \arrow[uu, dash, dashed] & &
\end{tikzcd}$$

with $\phi_{69,x} \coloneqq \varphi_{69}(x)$, $\phi_{89,x} \coloneqq \varphi_{89}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.14. *$G_{\mathbb{R}_2}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_2)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\mathbb{R}_2)}(G_{\mathbb{R}_2}) = \dim_{\mathbf{k}}(G_{\mathbb{R}_2}) = \sum_{i\in(Q_{\mathbb{R}_2})_0} \dim_{\mathbf{k}}((G_{\mathbb{R}_2})_i) = \infty$$

So $G_{\mathbb{R}_2}$ have infinite length. Moreover, following the same procedure as in Theorem A.13, we see that all the components of a endomorphism $f\colon G_{\mathbb{R}_2} \to G_{\mathbb{R}_2}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

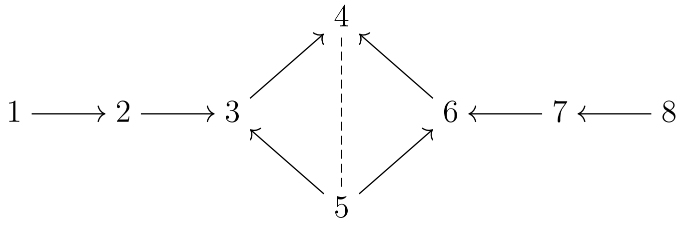
Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\operatorname{End}(G_{\mathbb{R}_2}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\mathbb{R}_2}) = \dim_{\mathbf{k}(x)}(G_{\mathbb{R}_2}) = \sum_{i\in(Q_{\mathbb{R}_2})_0} \dim_{\mathbf{k}(x)}((G_{\mathbb{R}_2})_i) < \infty$$

So $G_{\mathbb{R}_2}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\mathbb{R}_2}$ is a generic brick. $\square$

A.8. Generic brick of $\mathbb{R}_3$

Consider the critical poset of type $P = \mathbb{R}_3$ (see Table 2.1) with the following fixed orientation and labelling:



Its Tits form $q_{\mathbb{R}_3}$ is

$$q_{\mathbb{R}_3}(x) = \sum_{i=1}^{8} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_5x_3 - x_5x_6 - x_6x_4 - x_7x_6 - x_8x_7 + x_5x_4$$

because $I_{\mathbb{R}_3} = \langle \alpha_{34}\alpha_{53} - \alpha_{64}\alpha_{56} \rangle$ so $r_{54} = 1$. The associated matrix $B_{\mathbb{R}_3}$ is given by

$$B_{\mathbb{R}_3} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\mathbb{R}_3}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\mathbb{R}_3} = (1, 2, 3, 2, 2, 3, 2, 1)^t$. Therefore

$$\mathrm{Rad}(q_{\mathbb{R}_3}) = \ker(B_{\mathbb{R}_3}) = \langle y_{\mathbb{R}_3} \rangle$$

We define the representations $B(\lambda)_{\mathbb{R}_3}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\mathbb{R}_3} := \begin{tikzcd} & & & \mathbf{k}^2 \arrow[dd, dash, dashed] & & & \\ \mathbf{k} \arrow[r, "\varphi_{12}"] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] & \mathbf{k}^3 \arrow[ur, "\varphi_{34}(\lambda)"] & & \mathbf{k}^3 \arrow[ul, "\varphi_{64}"'] & \mathbf{k}^2 \arrow[l, "\varphi_{76}"'] & \mathbf{k} \arrow[l, "\varphi_{87}"'] \\ & & & \mathbf{k}^2 \arrow[ul, "\varphi_{53}"] \arrow[ur, "\varphi_{56}"'] & & & \end{tikzcd}$$

where the $\mathbf{k}$-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{53} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \varphi_{34}(\lambda) = \begin{pmatrix} \lambda & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\varphi_{87} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \varphi_{76} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \varphi_{56} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{64} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The relations are preserved since

$$\varphi_{34}(\lambda)\varphi_{53} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \varphi_{64}\varphi_{56}$$

We have that $\underline{\dim}(B(\lambda)_{\mathbb{R}_3}) = y_{\mathbb{R}_3}$ and we prove the following:

Theorem A.15. *The representations $B(\lambda)_{\mathbb{R}_3}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_3)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\mathbb{R}_3} \to B(\lambda')_{\mathbb{R}_3}$ be a morphism of representations.

$$\begin{tikzcd}
 & & & & \mathbf{k}^2 \arrow[ddd, "f_4"'] \arrow[dl, dashed, no head] & & & \\
\mathbf{k} \arrow[r, "\varphi_{12}"] \arrow[ddd, "f_1"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] \arrow[ddd, "f_2"'] & \mathbf{k}^3 \arrow[urr, "\varphi_{34}(\lambda)"] \arrow[ddd, "f_3"'] & & & \mathbf{k}^3 \arrow[ul, "\varphi_{64}"'] \arrow[ddd, "f_6"] & \mathbf{k}^2 \arrow[l, "\varphi_{76}"'] \arrow[ddd, "f_7"] & \mathbf{k} \arrow[l, "\varphi_{87}"'] \arrow[ddd, "f_8"] \\
 & & & \mathbf{k}^2 \arrow[ul, "\varphi_{53}"] \arrow[urr, "\varphi_{56}"'] \arrow[ddd, "f_5"] & & & & \\
 & & & & \mathbf{k}^2 \arrow[dl, dashed, no head] & & & \\
\mathbf{k} \arrow[r, "\varphi_{12}"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"'] & \mathbf{k}^3 \arrow[urr, "\varphi_{34}(\lambda')"] & & & \mathbf{k}^3 \arrow[ul, "\varphi_{64}"'] & \mathbf{k}^2 \arrow[l, "\varphi_{76}"] & \mathbf{k} \arrow[l, "\varphi_{87}"] \\
 & & & \mathbf{k}^2 \arrow[ul, "\varphi_{53}"] \arrow[urr, "\varphi_{56}"'] & & & &
\end{tikzcd}$$

We have that f_1, f_8 are just scalar multiplications of elements of $\mathbf{k}$, while $f_2, f_4, f_5, f_7 \in \mathbb{M}_2(\mathbf{k})$ and $f_3, f_6 \in \mathbb{M}_3(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(3) The square involving f_7 and f_8 gives us

$$f_7(e_2) = \begin{pmatrix} 0 \\ f_8 \end{pmatrix}$$

(4) The square involving f_6 and f_7 gives us that $\varphi_{76} f_7(e_i) = f_6(\varphi_{76}(e_i))$ that means

$$f_6(e_2) = \begin{pmatrix} 0 \\ f_7(e_1) \end{pmatrix} \qquad f_6(e_3) = \begin{pmatrix} 0 \\ f_7(e_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_8 \end{pmatrix}$$

(5) The square involving f_3 and f_5 gives us that $\varphi_{53} f_5(e_i) = f_3(\varphi_{53}(e_i))$ that means

$$f_3(e_2) = \begin{pmatrix} 0 \\ f_5(e_1) \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} \qquad f_3(e_3) = \begin{pmatrix} 0 \\ f_5(e_2) \end{pmatrix}$$

for some $a \in \mathbf{k}$ (we used (3)).
(6) The square involving f_5 and f_6 gives us that $\varphi_{56} f_5(e_i) = f_6(\varphi_{56}(e_i))$ that means

$$f_6(e_1) = \begin{pmatrix} f_5(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \qquad f_6(e_2) = \begin{pmatrix} f_5(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ f_7(e_1) \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$$

for some $b \in \mathbf{k}$ (we used (4)).
(7) The square involving f_4 and f_6 gives us that $\varphi_{64} f_6(e_i) = f_4(\varphi_{64}(e_i))$ that means

$$f_4(e_1) = \varphi_{64} f_6(e_1) = \begin{pmatrix} a \\ 0 \end{pmatrix} \qquad f_4(e_2) = \varphi_{64} f_6(e_3) = \begin{pmatrix} 0 \\ f_8 \end{pmatrix}$$

$$\begin{pmatrix} a \\ f_8 \end{pmatrix} = f_4(e_1 + e_2) = \varphi_{64} f_6(e_2) = \begin{pmatrix} b \\ b \end{pmatrix}$$

so we conclude that $a = b = f_8$.
(8) The square involving f_3 and f_4 gives us that $\varphi_{34} f_3(e_i) = f_4(\varphi_{34}(e_i))$. For $i = 1$ we have that

$$\begin{pmatrix} \lambda f_8 \\ f_8 \end{pmatrix} = f_4(\lambda e_1 + e_2) = \varphi_{34} f_3(e_1) = \begin{pmatrix} \lambda' f_1 \\ f_1 \end{pmatrix}$$

so we conclude that $f_1 = f_8$ and $\lambda f_1 = \lambda' f_1$.

Therefore we get

$$f_1 = f_8 \qquad f_2 = f_4 = f_5 f_7 = f_1 \mathbb{1}_2 \qquad f_3 = f_6 = f_1 \mathbb{1}_3 \tag{A.8}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f \colon B(\lambda)_{\mathbb{R}_3} \to B(\lambda)_{\mathbb{R}_3}$ depends only on $f_1 \in \mathbf{k}$ by (A.8), therefore $\mathrm{End}(B(\lambda)_{\mathbb{R}_3}) \cong \mathbf{k}$, so $B(\lambda)_{\mathbb{R}_3}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_1$ and if $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\mathbb{R}_3}, B(\lambda')_{\mathbb{R}_3}) = 0$$

Therefore $B(\lambda)_{\mathbb{R}_3}$ and $B(\lambda')_{\mathbb{R}_3}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\mathbb{R}_3}\}_{\lambda\in\mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_3)$ is brick-continuous. $\square$

Construction of $G_{\mathbb{R}_3}$ We have the brick-continuous family $\{B(\lambda)_{\mathbb{R}_3}\}_{\lambda\in\mathbf{k}}$ and we define the representation $G_{\mathbb{R}_3}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & & & \mathbf{k}(x)^2 \arrow[dd, dash, dashed] & & & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"] & \mathbf{k}(x)^3 \arrow[ur, red, "\phi_{34,x}"] & & \mathbf{k}(x)^3 \arrow[ul, "\phi_{64}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{76}"'] & \mathbf{k}(x) \arrow[l, "\phi_{87}"'] \\
 & & & \mathbf{k}(x)^2 \arrow[ul, "\phi_{53}"] \arrow[ur, "\phi_{56}"'] & & &
\end{tikzcd}$$

with $\phi_{34,x} := \varphi_{34}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.16. *$G_{\mathbb{R}_3}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_3)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\mathbb{R}_3)}(G_{\mathbb{R}_3}) = \dim_{\mathbf{k}}(G_{\mathbb{R}_3}) = \sum_{i\in(Q_{\mathbb{R}_3})_0} \dim_{\mathbf{k}}((G_{\mathbb{R}_3})_i) = \infty$$

So $G_{\mathbb{R}_3}$ have infinite length. Moreover, following the same procedure as in Theorem A.15, we see that all the components of a endomorphism $f\colon G_{\mathbb{R}_3} \to G_{\mathbb{R}_3}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\operatorname{End}(G_{\mathbb{R}_3}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\mathbb{R}_3}) = \dim_{\mathbf{k}(x)}(G_{\mathbb{R}_3}) = \sum_{i\in(Q_{\mathbb{R}_3})_0} \dim_{\mathbf{k}(x)}((G_{\mathbb{R}_3})_i) < \infty$$

So $G_{\mathbb{R}_3}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\mathbb{R}_3}$ is a generic brick. $\square$

A.9. Generic brick of $\mathbb{R}_4$

Consider the critical poset of type $P = \mathbb{R}_4$ (see Table 2.1) with the following fixed orientation and labelling:

$$\begin{tikzcd}
 & & 9 & & & & & \\
8 \arrow[r] & 7 \arrow[ur] & & 5 \arrow[ul] & 4 \arrow[l] & 3 \arrow[l] & 2 \arrow[l] & 1 \arrow[l] \\
 & & 6 \arrow[ul] \arrow[ur] \arrow[uu, dash, dashed] & & & & &
\end{tikzcd}$$

Its Tits form $q_{\mathbb{R}_4}$ is

$$q_{\mathbb{R}_4}(x) = \sum_{i=1}^{9} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_5x_9 - x_6x_5 - x_6x_7 - x_7x_9 - x_8x_7 + x_6x_9$$

because $I_{\mathbb{R}_4} = \langle \alpha_{79}\alpha_{67} - \alpha_{59}\alpha_{65} \rangle$ so $r_{69} = 1$. The associated matrix $B_{\mathbb{R}_4}$ is given by

$$B_{\mathbb{R}_4} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\mathbb{R}_4}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{4}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\mathbb{R}_4} = (1, 2, 3, 4, 5, 3, 4, 2, 3)^t$ (we want $y_{\mathbb{R}_4} \in \mathbb{Z}^n$ because we will use it as a dimension vector). Therefore

$$\operatorname{Rad}(q_{\mathbb{R}_4}) = \ker(B_{\mathbb{R}_4}) = \langle y_{\mathbb{R}_4} \rangle$$

We define the representations $B(\lambda)_{\mathbb{R}_4}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\mathbb{R}_4} := \begin{tikzcd}
 & & \mathbf{k}^3 \arrow[dd, dash, dashed] & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{87}"] & \mathbf{k}^4 \arrow[ur, "\varphi_{79}(\lambda)"] & & \mathbf{k}^5 \arrow[ul, "\varphi_{59}"'] & \mathbf{k}^4 \arrow[l, "\varphi_{45}"'] & \mathbf{k}^3 \arrow[l, "\varphi_{34}"'] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"'] & \mathbf{k} \arrow[l, "\varphi_{12}"'] \\
 & & \mathbf{k}^3 \arrow[ul, "\varphi_{67}"] \arrow[ur, "\varphi_{65}"'] & & & & &
\end{tikzcd}$$

where the $\mathbf{k}$-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \varphi_{45} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_{65} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \varphi_{67} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \varphi_{87} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\varphi_{59} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \varphi_{79}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

The relations are preserved since

$$\varphi_{79}(\lambda)\varphi_{67} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \varphi_{59}\varphi_{65}$$

We have that $\underline{\dim}(B(\lambda)_{\mathbb{R}_4}) = y_{\mathbb{R}_4}$ and we prove the following:

Theorem A.17. *The representations $B(\lambda)_{\mathbb{R}_4}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_4)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\mathbb{R}_4} \to B(\lambda')_{\mathbb{R}_4}$ be a morphism of representations.

$$\begin{tikzcd}
 & & \mathbf{k}^3 \arrow[ddd, red, "f_9"] \arrow[dr, dash, dashed] & & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{87}"] \arrow[ddd, red, "f_8"'] & \mathbf{k}^4 \arrow[ur, "\varphi_{79}(\lambda)"] \arrow[ddd, red, "f_7"'] & & & \mathbf{k}^5 \arrow[ull, "\varphi_{59}"'] \arrow[ddd, red, "f_5"] & \mathbf{k}^4 \arrow[l, "\varphi_{45}"'] \arrow[ddd, red, "f_4"] & \mathbf{k}^3 \arrow[l, "\varphi_{34}"'] \arrow[ddd, red, "f_3"] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"'] \arrow[ddd, red, "f_2"] & \mathbf{k} \arrow[l, "\varphi_{12}"'] \arrow[ddd, red, "f_1"] \\
 & & & \mathbf{k}^3 \arrow[ull, "\varphi_{67}"] \arrow[ur, "\varphi_{65}"'] \arrow[ddd, red, "f_6"] & & & & & \\
 & & \mathbf{k}^3 \arrow[ddr, dash, dashed] & & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{87}"'] & \mathbf{k}^4 \arrow[ur, "\varphi_{79}(\lambda')"'] & & & \mathbf{k}^5 \arrow[ull, "\varphi_{59}"'] & \mathbf{k}^4 \arrow[l, "\varphi_{45}"] & \mathbf{k}^3 \arrow[l, "\varphi_{34}"] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"] & \mathbf{k} \arrow[l, "\varphi_{12}"] \\
 & & & \mathbf{k}^3 \arrow[ull, "\varphi_{67}"] \arrow[ur, "\varphi_{65}"'] & & & & &
\end{tikzcd}$$

We have that f_1 is just a scalar multiplication of an element of $\mathbf{k}$, while $f_2, f_8 \in \mathbb{M}_2(\mathbf{k})$, $f_3, f_6, f_9 \in \mathbb{M}_3(\mathbf{k})$, $f_4, f_7 \in \mathbb{M}_4(\mathbf{k})$ and $f_5 \in \mathbb{M}_5(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(3) The square involving f_3 and f_4 gives us that $\varphi_{34} f_3(e_i) = f_4(\varphi_{34}(e_i))$ that means

$$f_4(e_1) = \begin{pmatrix} f_3(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_2) = \begin{pmatrix} f_3(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \end{pmatrix}$$

(4) The square involving f_4 and f_5 gives us that $\varphi_{45} f_4(e_i) = f_5(\varphi_{45}(e_i))$ that means

$$f_5(e_1) = \begin{pmatrix} f_4(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_2) = \begin{pmatrix} f_4(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f_5(e_3) = \begin{pmatrix} f_4(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_4) = \begin{pmatrix} f_4(e_4) \\ 0 \end{pmatrix}$$

(5) The square involving f_5 and f_6 gives us that $\varphi_{65} f_6(e_i) = f_5(\varphi_{65}(e_i))$ that means

$$f_5(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_6(e_1) \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \end{pmatrix}$$

$$f_5(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_6(e_2) \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b \\ c \\ 0 \end{pmatrix} \quad f_5(e_5) = \begin{pmatrix} 0 \\ 0 \\ f_6(e_3) \end{pmatrix}$$

for some $a, b, c \in \mathbf{k}$ (we used (4)).

(6) The square involving f_6 and f_7 gives us that $\varphi_{67} f_6(e_i) = f_7(\varphi_{67}(e_i))$ that means

$$f_7(e_1) = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_7(e_2) = \begin{pmatrix} b \\ c \\ 0 \\ 0 \end{pmatrix} \quad f_7(e_3) = \begin{pmatrix} f_6(e_3) \\ 0 \end{pmatrix}$$

(7) The square involving f_7 and f_8 gives us that $\varphi_{87} f_8(e_i) = f_7(\varphi_{87}(e_i))$ that means

$$f_8(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_8(e_1) \end{pmatrix} = \begin{pmatrix} f_6(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d \\ 0 \end{pmatrix} \quad f_7(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_8(e_2) \end{pmatrix}$$

for some $d \in \mathbf{k}$ (we used (6)).

(8) The square involving f_5 and f_9 gives us that $\varphi_{59} f_5(e_i) = f_9(\varphi_{59}(e_i))$ that means

$$f_9(e_1) = \varphi_{59} f_5(e_3) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \quad f_9(e_2) = \varphi_{59} f_5(e_4) = \begin{pmatrix} b \\ c \\ 0 \end{pmatrix} \quad f_9(e_3) = \varphi_{59} f_5(e_5) = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}$$

Thus, for $i = 1$

$$\begin{pmatrix} b \\ c \\ d \end{pmatrix} = f_9(e_2 + e_3) = \varphi_{59} f_5(e_1) = \begin{pmatrix} f_1 \\ f_1 \\ 0 \end{pmatrix}$$

therefore $b = 0$ and $f_1 = c = d$. For $i = 2$

$$\begin{pmatrix} a \\ 0 \\ f_1 \end{pmatrix} = f_9(e_1 + e_3) = \varphi_{59} f_5(e_2) = \begin{pmatrix} f_2(e_2)_2 \\ f_2(e_2)_1 \\ f_2(e_2)_1 + f_2(e_2)_2 \end{pmatrix}$$

so $f_1 = a = d$ and $f_2(e_2) = \begin{pmatrix} 0 \\ f_1 \end{pmatrix}$.

(8) The square involving f_7 and f_9 gives us that $\varphi_{79}(\lambda')f_7(e_i) = f_9(\varphi_{79}(\lambda)(e_i))$. For $i = 4$ we get

$$\begin{pmatrix} \lambda f_1 \\ f_1 \\ 0 \end{pmatrix} = f_9(\lambda e_1 + e_2) = \varphi_{79}(\lambda')f_7(e_4) = \begin{pmatrix} \lambda' f_8(e_2)_2 \\ f_8(e_2)_2 \\ f_8(e_2)_1 \end{pmatrix}$$

so $f_8(e_2) = \begin{pmatrix} 0 \\ f_1 \end{pmatrix}$ and $\lambda f_1 = \lambda' f_1$.

Therefore we get

$$f_2 = f_8 = f_1 \mathbb{1}_2 \quad f_3 = f_6 = f_9 = f_1 \mathbb{1}_3 \quad f_4 = f_7 = f_1 \mathbb{1}_4 \quad f_5 = f_1 \mathbb{1}_5 \tag{A.9}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f \colon B(\lambda)_{\mathbb{R}_4} \to B(\lambda)_{\mathbb{R}_4}$ depends only on $f_1 \in \mathbf{k}$ by (A.9), therefore $\operatorname{End}(B(\lambda)_{\mathbb{R}_4}) \cong \mathbf{k}$, so $B(\lambda)_{\mathbb{R}_4}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_1$ and if $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\mathbb{R}_4}, B(\lambda')_{\mathbb{R}_4}) = 0$$

Therefore $B(\lambda)_{\mathbb{R}_4}$ and $B(\lambda')_{\mathbb{R}_4}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\mathbb{R}_4}\}_{\lambda \in \mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_4)$ is brick-continuous. □

Construction of $G_{\mathbb{R}_4}$ We have the brick-continuous family $\{B(\lambda)_{\mathbb{R}_4}\}_{\lambda \in \mathbf{k}}$ and we define the representation $G_{\mathbb{R}_4}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & & \mathbf{k}(x)^3 & & & & & \\
\mathbf{k}(x)^2 \arrow[r, "\phi_{87}"] & \mathbf{k}(x)^4 \arrow[ur, red, "\phi_{79,x}"] & & \mathbf{k}(x)^5 \arrow[ul, "\phi_{59}"'] & \mathbf{k}(x)^4 \arrow[l, "\phi_{45}"'] & \mathbf{k}(x)^3 \arrow[l, "\phi_{34}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{23}"'] & \mathbf{k}(x) \arrow[l, "\phi_{12}"'] \\
 & & \mathbf{k}(x)^3 \arrow[ul, "\phi_{67}"] \arrow[ur, "\phi_{65}"'] \arrow[uu, dash, dashed] & & & & &
\end{tikzcd}$$

with $\phi_{79,x} := \varphi_{79}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.18. *$G_{\mathbb{R}_4}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_4)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\mathbb{R}_4)}(G_{\mathbb{R}_4}) = \dim_{\mathbf{k}}(G_{\mathbb{R}_4}) = \sum_{i \in (Q_{\mathbb{R}_4})_0} \dim_{\mathbf{k}}((G_{\mathbb{R}_4})_i) = \infty$$

So $G_{\mathbb{R}_4}$ have infinite length. Moreover, following the same procedure as in Theorem A.17, we see that all the components of a endomorphism $f\colon G_{\mathbb{R}_4} \to G_{\mathbb{R}_4}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

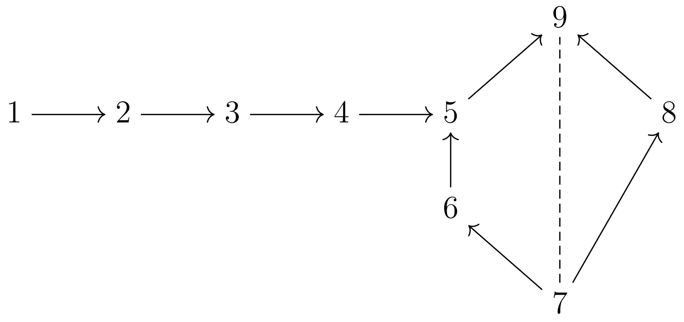
Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\mathrm{End}(G_{\mathbb{R}_4}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\mathbb{R}_4}) = \dim_{\mathbf{k}(x)}(G_{\mathbb{R}_4}) = \sum_{i \in (Q_{\mathbb{R}_4})_0} \dim_{\mathbf{k}(x)}((G_{\mathbb{R}_4})_i) < \infty$$

So $G_{\mathbb{R}_4}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\mathbb{R}_4}$ is a generic brick. □

A.10. Generic brick of $\mathbb{R}_5$

Consider the critical poset of type $P = \mathbb{R}_5$ (see Table 2.1) with the following fixed orientation and labelling:



Its Tits form $q_{\mathbb{R}_5}$ is

$$q_{\mathbb{R}_5}(x) = \sum_{i=1}^{9} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_4x_5 - x_5x_9 - x_6x_5 - x_7x_6 - x_7x_8 - x_8x_9 + x_7x_9$$

because $I_{\mathbb{R}_5} = \langle \alpha_{59}\alpha_{65}\alpha_{76} - \alpha_{89}\alpha_{78} \rangle$ so $r_{79} = 1$. The associated matrix $B_{\mathbb{R}_5}$ is given by

$$B_{\mathbb{R}_5} = \frac{1}{2} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\mathbb{R}_5}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{4}{3} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\mathbb{R}_5} = (1, 2, 3, 4, 5, 3, 1, 2, 3)^t$ (we want $y_{\mathbb{R}_5} \in \mathbb{Z}^n$ because we will use it as a dimension vector). Therefore

$$\mathrm{Rad}(q_{\mathbb{R}_5}) = \ker(B_{\mathbb{R}_5}) = \langle y_{\mathbb{R}_5} \rangle$$

We define the representations $B(\lambda)_{\mathbb{R}_5}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\mathbb{R}_5} := \begin{tikzcd} & & & & & & \mathbf{k}^3 & \\ \mathbf{k} \arrow[r, "\varphi_{12}"] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"] & \mathbf{k}^4 \arrow[r, "\varphi_{45}"] & \mathbf{k}^5 \arrow[urr, "\varphi_{59}(\lambda)"] & & & \mathbf{k}^2 \arrow[ul, "\varphi_{89}(\lambda)"'] \\ & & & & \mathbf{k}^3 \arrow[u, "\varphi_{65}"] & & & \\ & & & & & & \mathbf{k} \arrow[ull, "\varphi_{76}"] \arrow[uur, "\varphi_{78}"'] \arrow[uuu, dash, dashed] & \end{tikzcd}$$

where the **k**-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \varphi_{45} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_{65} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \varphi_{76} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \varphi_{78} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\varphi_{89}(\lambda) = \begin{pmatrix} 1 & \lambda \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \varphi_{59}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 & \lambda \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

The relations are preserved since

$$\varphi_{59}(\lambda)\varphi_{65}\varphi_{76} = \begin{pmatrix} \lambda \\ 1 \\ 1 \end{pmatrix} = \varphi_{89}(\lambda)\varphi_{78}$$

We have that $\underline{\dim}(B(\lambda)_{\mathbb{R}_5}) = y_{\mathbb{R}_5}$ and we prove the following:

Theorem A.19. *The representations $B(\lambda)_{\mathbb{R}_5}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_5)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\mathbb{R}_5} \to B(\lambda')_{\mathbb{R}_5}$ be a morphism of representations.

$$\begin{tikzcd}
 & & & & & & & \mathbf{k}^3 \arrow[dddd, red, "f_9"] & \\
\mathbf{k} \arrow[r, "\varphi_{12}"] \arrow[dddd, red, "f_1"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"] \arrow[dddd, red, "f_2"'] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"] \arrow[dddd, red, "f_3"'] & \mathbf{k}^4 \arrow[r, "\varphi_{45}"] \arrow[dddd, red, "f_4"'] & \mathbf{k}^5 \arrow[urrr, "\varphi_{59}(\lambda)"] \arrow[dddd, red, "f_5"'] & & & & \mathbf{k}^2 \arrow[ul, "\varphi_{89}(\lambda)"'] \arrow[dddd, red, "f_8"] \\
 & & & & & \mathbf{k}^3 \arrow[ul, "\varphi_{65}"] \arrow[dddd, red, "f_6"'] & & & \\
 & & & & & & \mathbf{k} \arrow[ul, "\varphi_{76}"] \arrow[uurr, "\varphi_{78}"'] \arrow[uuur, dash, dashed] \arrow[dddd, red, "f_7"] & & \\
 & & & & & & & \mathbf{k}^3 & \\
\mathbf{k} \arrow[r, "\varphi_{12}"'] & \mathbf{k}^2 \arrow[r, "\varphi_{23}"'] & \mathbf{k}^3 \arrow[r, "\varphi_{34}"'] & \mathbf{k}^4 \arrow[r, "\varphi_{45}"'] & \mathbf{k}^5 \arrow[urrr, "\varphi_{59}(\lambda')"] & & & & \mathbf{k}^2 \arrow[ul, "\varphi_{89}(\lambda')"'] \\
 & & & & & \mathbf{k}^3 \arrow[ul, "\varphi_{65}"] & & & \\
 & & & & & & \mathbf{k} \arrow[ul, "\varphi_{76}"] \arrow[uurr, "\varphi_{78}"'] \arrow[uuur, dash, dashed] & &
\end{tikzcd}$$

We have that f_1, f_7 are just scalar multiplications of elements of $\mathbf{k}$, while $f_2, f_8 \in \mathbb{M}_2(\mathbf{k})$, $f_3, f_6, f_9 \in \mathbb{M}_3(\mathbf{k})$, $f_4 \in \mathbb{M}_4(\mathbf{k})$ and $f_5 \in \mathbb{M}_5(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(3) The square involving f_3 and f_4 gives us that $\varphi_{34} f_3(e_i) = f_4(\varphi_{34}(e_i))$ that means

$$f_4(e_1) = \begin{pmatrix} f_3(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_2) = \begin{pmatrix} f_3(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \end{pmatrix}$$

(4) The square involving f_4 and f_5 gives us that $\varphi_{45}f_4(e_i) = f_5(\varphi_{45}(e_i))$ that means

$$f_5(e_1) = \begin{pmatrix} f_4(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_2) = \begin{pmatrix} f_4(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \\ 0 \end{pmatrix} := \begin{pmatrix} a \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$f_5(e_3) = \begin{pmatrix} f_4(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_4) = \begin{pmatrix} f_4(e_4) \\ 0 \end{pmatrix}$$

for some $a, b \in \mathbf{k}$.

(5) The square involving f_6 and f_7 gives us

$$f_6(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_7 \end{pmatrix}$$

(6) The square involving f_5 and f_6 gives us that $\varphi_{65}f_6(e_i) = f_5(\varphi_{65}(e_i))$ that means

$$f_5(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_6(e_1) \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ c \\ 0 \\ 0 \end{pmatrix}$$

$$f_5(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_6(e_2) \end{pmatrix} = \begin{pmatrix} f_4(e_4) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d \\ e \\ 0 \end{pmatrix} \quad f_5(e_5) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ f_7 \end{pmatrix}$$

for some $c, d, e \in \mathbf{k}$ (we used (4)).

(7) The square involving f_5 and f_9 gives us that $\varphi_{59}(\lambda')f_5(e_i) = f_9(\varphi_{59}(\lambda)(e_i))$ that means

$$f_9(e_1) = \varphi_{59}(\lambda')f_5(e_1) = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \quad f_9(e_2) = \varphi_{59}(\lambda')f_5(e_2) = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

$$f_9(e_3) = \varphi_{59}(\lambda')f_5(e_3) = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$$

Then for $i = 4$

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = f_9(e_2) = \varphi_{59}(\lambda')f_5(e_4) = \begin{pmatrix} 0 \\ e \\ d \end{pmatrix}$$

so $a = d = 0$ and $b = e$. For $i = 5$

$$\begin{pmatrix} \lambda f_1 \\ b \\ c \end{pmatrix} = f_9(\lambda e_1 + e_2 + e_3) = \varphi_{59}(\lambda')f_5(e_5) = \begin{pmatrix} \lambda' f_7 \\ f_7 \\ f_7 \end{pmatrix}$$

so $b = c = f_7$ and $\lambda f_1 = \lambda' f_7$.
(8) The square involving f_7 and f_8 gives us

$$f_8(e_2) = \begin{pmatrix} 0 \\ f_7 \end{pmatrix}$$

(9) The square involving f_8 and f_9 gives us that $\varphi_{89}(\lambda')f_8(e_i) = f_9(\varphi_{89}(\lambda)(e_i))$. For $i = 1$ we have

$$\begin{pmatrix} f_1 \\ f_7 \\ 0 \end{pmatrix} = f_9(e_1 + e_2) = \varphi_{89}(\lambda')f_8(e_1) = \begin{pmatrix} f_8(e_1)_1 + \lambda' f_8(e_1)_2 \\ f_8(e_1)_1 + f_8(e_1)_2 \\ f_8(e_1)_2 \end{pmatrix}$$

so we have $f_8(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$ with $f_1 = f_7$, therefore from (7) we get $\lambda f_1 = \lambda' f_1$.

Therefore we get

$$f_1 = f_7 \quad f_2 = f_8 = f_1 \mathbb{1}_2 \quad f_3 = f_6 = f_9 = f_1 \mathbb{1}_3 \quad f_4 = f_1 \mathbb{1}_4 \quad f_5 = f_1 \mathbb{1}_5 \tag{A.10}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f \colon B(\lambda)_{\mathbb{R}_5} \to B(\lambda)_{\mathbb{R}_5}$ depends only on $f_1 \in \mathbf{k}$ by (A.10), therefore $\operatorname{End}(B(\lambda)_{\mathbb{R}_5}) \cong \mathbf{k}$, so $B(\lambda)_{\mathbb{R}_5}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_1$ and if $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\mathbb{R}_5}, B(\lambda')_{\mathbb{R}_5}) = 0$$

Therefore $B(\lambda)_{\mathbb{R}_5}$ and $B(\lambda')_{\mathbb{R}_5}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\mathbb{R}_5}\}_{\lambda \in \mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_5)$ is brick-continuous. □

Construction of $G_{\mathbb{R}_5}$ We have the brick-continuous family $\{B(\lambda)_{\mathbb{R}_5}\}_{\lambda\in\mathbf{k}}$ and we define the representation $G_{\mathbb{R}_5}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & & & & & \mathbf{k}(x)^3 \arrow[ddd, dash, dashed] & \\
\mathbf{k}(x) \arrow[r, "\phi_{12}"] & \mathbf{k}(x)^2 \arrow[r, "\phi_{23}"] & \mathbf{k}(x)^3 \arrow[r, "\phi_{34}"] & \mathbf{k}(x)^4 \arrow[r, "\phi_{45}"] & \mathbf{k}(x)^5 \arrow[ur, red, "\phi_{59,x}"] & & \mathbf{k}(x)^2 \arrow[ul, red, "\phi_{89,x}"'] \\
 & & & & \mathbf{k}(x)^3 \arrow[u, "\phi_{65}"] & & \\
 & & & & & \mathbf{k}(x) \arrow[ul, "\phi_{76}"] \arrow[uur, "\phi_{78}"'] &
\end{tikzcd}$$

with $\phi_{59,x} \coloneqq \varphi_{59}(x)$, $\phi_{89,x} \coloneqq \varphi_{89}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.20. *$G_{\mathbb{R}_5}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_5)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\mathbb{R}_5)}(G_{\mathbb{R}_5}) = \dim_{\mathbf{k}}(G_{\mathbb{R}_5}) = \sum_{i\in(Q_{\mathbb{R}_5})_0} \dim_{\mathbf{k}}((G_{\mathbb{R}_5})_i) = \infty$$

So $G_{\mathbb{R}_5}$ have infinite length. Moreover, following the same procedure as in Theorem A.19, we see that all the components of a endomorphism $f\colon G_{\mathbb{R}_5} \to G_{\mathbb{R}_5}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

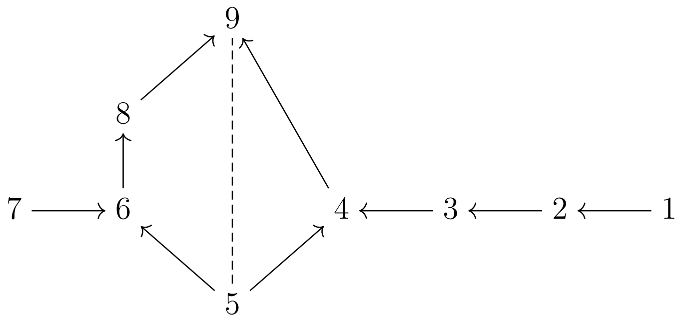
Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\mathrm{End}(G_{\mathbb{R}_5}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\mathbb{R}_5}) = \dim_{\mathbf{k}(x)}(G_{\mathbb{R}_5}) = \sum_{i\in(Q_{\mathbb{R}_5})_0} \dim_{\mathbf{k}(x)}((G_{\mathbb{R}_5})_i) < \infty$$

So $G_{\mathbb{R}_5}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\mathbb{R}_5}$ is a generic brick. □

A.11. Generic brick of $\mathbb{R}_6$

Consider the critical poset of type $P = \mathbb{R}_6$ (see Table 2.1) with the following fixed orientation and labelling (we work with its dual):



Its Tits form $q_{\mathbb{R}_6}$ is

$$q_{\mathbb{R}_6}(x) = \sum_{i=1}^{9} x_i^2 - x_1x_2 - x_2x_3 - x_3x_4 - x_5x_4 - x_5x_6 - x_7x_6 - x_6x_8 - x_8x_9 - x_4x_9 + x_5x_9$$

because $I_{\mathbb{R}_6} = \langle \alpha_{89}\alpha_{68}\alpha_{56} - \alpha_{49}\alpha_{54} \rangle$ so $r_{59} = 1$. The associated matrix $B_{\mathbb{R}_6}$ is given by

$$B_{\mathbb{R}_6} = \frac{1}{2} \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\mathbb{R}_6}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\mathbb{R}_6} = (1, 2, 3, 4, 3, 4, 2, 3, 2)^t$ (we want $y_{\mathbb{R}_6} \in \mathbb{Z}^n$ because we will use it as a dimension vector). Therefore

$$\mathrm{Rad}(q_{\mathbb{R}_6}) = \ker(B_{\mathbb{R}_6}) = \langle y_{\mathbb{R}_6} \rangle$$

We define the representations $B(\lambda)_{\mathbb{R}_6}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\mathbb{R}_6} := \begin{tikzcd}
 & & \mathbf{k}^2 \arrow[ddd, dash, dashed] & & & & \\
 & \mathbf{k}^3 \arrow[ur, "\varphi_{89}"] & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{76}"] & \mathbf{k}^4 \arrow[u, "\varphi_{68}"] & & \mathbf{k}^4 \arrow[uul, "\varphi_{49}(\lambda)"'] & \mathbf{k}^3 \arrow[l, "\varphi_{34}"'] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"'] & \mathbf{k} \arrow[l, "\varphi_{12}"'] \\
 & & \mathbf{k}^3 \arrow[ul, "\varphi_{56}"] \arrow[ur, "\varphi_{54}"'] & & & &
\end{tikzcd}$$

where the $\mathbf{k}$-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{34} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \varphi_{54} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\varphi_{56} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \varphi_{76} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \varphi_{68} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\varphi_{89} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \varphi_{49}(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

The relations are preserved since

$$\varphi_{89}\varphi_{68}\varphi_{56} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \varphi_{49}(\lambda)\varphi_{54}$$

We have that $\underline{\dim}(B(\lambda)_{\mathbb{R}_6}) = y_{\mathbb{R}_6}$ and we prove the following:

Theorem A.21. *The representations $B(\lambda)_{\mathbb{R}_6}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_6)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\mathbb{R}_6} \to B(\lambda')_{\mathbb{R}_6}$ be a morphism of representations.

$$\begin{tikzcd}
 & & & \mathbf{k}^2 \arrow[dddd, red, "f_9"] \arrow[dddr, dash, dashed] & & & & & \\
 & & \mathbf{k}^3 \arrow[ur, "\varphi_{89}"] \arrow[dddd, red, "f_8"'] & & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{76}"] \arrow[dddd, red, "f_7"'] & \mathbf{k}^4 \arrow[ur, "\varphi_{68}"] \arrow[dddd, red, "f_6"'] & & & & \mathbf{k}^4 \arrow[uull, "\varphi_{49}(\lambda)"'] \arrow[dddd, red, "f_4"] & \mathbf{k}^3 \arrow[l, "\varphi_{34}"'] \arrow[dddd, red, "f_3"] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"'] \arrow[dddd, red, "f_2"] & \mathbf{k} \arrow[l, "\varphi_{12}"'] \arrow[dddd, red, "f_1"] \\
 & & & & \mathbf{k}^3 \arrow[ulll, "\varphi_{56}"'] \arrow[ur, "\varphi_{54}"] \arrow[dddd, red, "f_5"] & & & & \\
 & & & \mathbf{k}^2 \arrow[dddr, dash, dashed] & & & & & \\
 & & \mathbf{k}^3 \arrow[ur, "\varphi_{89}"'] & & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{76}"'] & \mathbf{k}^4 \arrow[ur, "\varphi_{68}"'] & & & & \mathbf{k}^4 \arrow[uull, "\varphi_{49}(\lambda')"'] & \mathbf{k}^3 \arrow[l, "\varphi_{34}"] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"] & \mathbf{k} \arrow[l, "\varphi_{12}"] \\
 & & & & \mathbf{k}^3 \arrow[ulll, "\varphi_{56}"] \arrow[ur, "\varphi_{54}"'] & & & &
\end{tikzcd}$$

We have that f_1 is just a scalar multiplication of an element of $\mathbf{k}$, while $f_2, f_7, f_9 \in \mathbb{M}_2(\mathbf{k})$, $f_3, f_5, f_8 \in \mathbb{M}_3(\mathbf{k})$ and $f_4, f_6 \in \mathbb{M}_4(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(3) The square involving f_3 and f_4 gives us that $\varphi_{34} f_3(e_i) = f_4(\varphi_{34}(e_i))$ that means

$$f_4(e_1) = \begin{pmatrix} f_3(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_2) = \begin{pmatrix} f_3(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} \quad f_4(e_3) = \begin{pmatrix} f_3(e_3) \\ 0 \end{pmatrix}$$

(4) The square involving f_4 and f_5 gives us that $\varphi_{54}f_5(e_i) = f_4(\varphi_{54}(e_i))$ that means

$$f_4(e_2) = \begin{pmatrix} 0 \\ f_5(e_1) \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \\ 0 \end{pmatrix}$$

$$f_4(e_3) = \begin{pmatrix} 0 \\ f_5(e_2) \end{pmatrix} = \begin{pmatrix} f_3(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ c \\ 0 \end{pmatrix} \quad f_4(e_4) = \begin{pmatrix} 0 \\ f_5(e_3) \end{pmatrix}$$

for some $a, b, c \in \mathbf{k}$ (we used (3)).
(5) The square involving f_5 and f_6 gives us that $\varphi_{56}f_5(e_i) = f_6(\varphi_{56}(e_i))$ that means

$$f_6(e_1) = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_6(e_2) = \begin{pmatrix} b \\ c \\ 0 \\ 0 \end{pmatrix} \quad f_6(e_3) = \begin{pmatrix} f_5(e_3) \\ 0 \end{pmatrix}$$

(6) The square involving f_6 and f_7 gives us that $\varphi_{76}f_7(e_i) = f_6(\varphi_{76}(e_i))$ that means

$$f_6(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_1) \end{pmatrix} = \begin{pmatrix} f_5(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d \\ 0 \end{pmatrix} \quad f_6(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ e \\ f \end{pmatrix}$$

for some $d, e, f \in \mathbf{k}$ (we used (5)).
(7) The square involving f_6 and f_8 gives us that $\varphi_{68}f_6(e_i) = f_8(\varphi_{68}(e_i))$ that means

$$f_8(e_1) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \quad f_8(e_2) = \begin{pmatrix} b \\ c \\ 0 \end{pmatrix} \quad f_8(e_3) = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}$$

For $i = 4$

$$\begin{pmatrix} b \\ c \\ 0 \end{pmatrix} = f_8(e_2) = \varphi_{68}f_6(e_4) = \begin{pmatrix} 0 \\ f \\ e \end{pmatrix}$$

so $b = e = 0$ and $f = c$.
(8) The square involving f_8 and f_9 gives us that $\varphi_{89}f_8(e_i) = f_9(\varphi_{89}(e_i))$ that means

$$f_9(e_1) = \varphi_{89}f_8(e_1) = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad f_9(e_2) = \varphi_{89}f_8(e_2) = \begin{pmatrix} 0 \\ c \end{pmatrix}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} = f_9(e_1+e_2) = \varphi_{89} f_8(e_3) = \begin{pmatrix} d \\ d \end{pmatrix}$$

So $a = d = c\,(= f)$.
(9) The square involving f_4 and f_9 gives us that $\varphi_{49}(\lambda')f_4(e_i) = f_9(\varphi_{49}(\lambda)(e_i))$. For $i = 1$ we get

$$\begin{pmatrix} \lambda a \\ a \end{pmatrix} = f_9(\lambda e_1+e_2) = \varphi_{49}(\lambda')f_4(e_1) = \begin{pmatrix} \lambda' f_1 \\ f_1 \end{pmatrix}$$

that means $a = f_1$ and $\lambda f_1 = \lambda' f_1$.

Therefore we get

$$f_2 = f_7 = f_9 = f_1\mathbb{1}_2 \quad f_3 = f_5 = f_8 = f_1\mathbb{1}_3 \quad f_4 = f_6 = f_1\mathbb{1}_4 \tag{A.11}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f\colon B(\lambda)_{\mathbb{R}_6} \to B(\lambda)_{\mathbb{R}_6}$ depends only on $f_1 \in \mathbf{k}$ by (A.11), therefore $\operatorname{End}(B(\lambda)_{\mathbb{R}_6}) \cong \mathbf{k}$, so $B(\lambda)_{\mathbb{R}_6}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_1$ and if $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\mathbb{R}_6}, B(\lambda')_{\mathbb{R}_6}) = 0$$

Therefore $B(\lambda)_{\mathbb{R}_6}$ and $B(\lambda')_{\mathbb{R}_6}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\mathbb{R}_6}\}_{\lambda\in\mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_6)$ is brick-continuous. □

Construction of $G_{\mathbb{R}_6}$ We have the brick-continuous family $\{B(\lambda)_{\mathbb{R}_6}\}_{\lambda\in\mathbf{k}}$ and we define the representation $G_{\mathbb{R}_6}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & & \mathbf{k}(x)^2 \arrow[ddd, dash, dashed] & & & & \\
 & \mathbf{k}(x)^3 \arrow[ur, "\phi_{89}"] & & & & & \\
\mathbf{k}(x)^2 \arrow[r, "\phi_{76}"] & \mathbf{k}(x)^4 \arrow[u, "\phi_{68}"] & & \mathbf{k}(x)^4 \arrow[uul, red, "\phi_{49,x}"'] & \mathbf{k}(x)^3 \arrow[l, "\phi_{34}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{23}"'] & \mathbf{k}(x) \arrow[l, "\phi_{12}"'] \\
 & & \mathbf{k}(x)^3 \arrow[ul, "\phi_{56}"] \arrow[ur, "\phi_{54}"'] & & & &
\end{tikzcd}$$

with $\phi_{49,x} := \varphi_{49}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.22. *$G_{\mathbb{R}_6}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_6)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\mathbb{R}_6)}(G_{\mathbb{R}_6}) = \dim_{\mathbf{k}}(G_{\mathbb{R}_6}) = \sum_{i \in (Q_{\mathbb{R}_6})_0} \dim_{\mathbf{k}}((G_{\mathbb{R}_6})_i) = \infty$$

So $G_{\mathbb{R}_6}$ have infinite length. Moreover, following the same procedure as in Theorem A.21, we see that all the components of a endomorphism $f\colon G_{\mathbb{R}_6} \to G_{\mathbb{R}_6}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

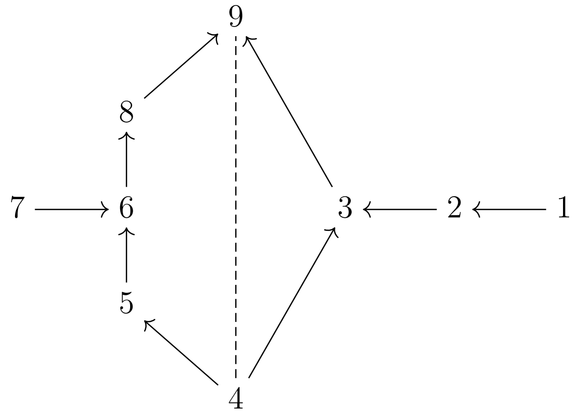
Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\mathrm{End}(G_{\mathbb{R}_6}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\mathbb{R}_6}) = \dim_{\mathbf{k}(x)}(G_{\mathbb{R}_6}) = \sum_{i \in (Q_{\mathbb{R}_6})_0} \dim_{\mathbf{k}(x)}((G_{\mathbb{R}_6})_i) < \infty$$

So $G_{\mathbb{R}_6}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\mathbb{R}_6}$ is a generic brick. □

A.12. Generic brick of $\mathbb{R}_7$

Consider the critical poset of type $P = \mathbb{R}_7$ (see Table 2.1) with the following fixed orientation and labelling:



Its Tits form $q_{\mathbb{R}_7}$ is

$$q_{\mathbb{R}_7}(x) = \sum_{i=1}^{9} x_i^2 - x_1x_2 - x_2x_3 - x_4x_3 - x_3x_9 - x_4x_5 - x_5x_6 - x_7x_6 - x_6x_8 - x_8x_9 + x_4x_9$$

because $I_{\mathbb{R}_7} = \langle \alpha_{89}\alpha_{68}\alpha_{56}\alpha_{45} - \alpha_{39}\alpha_{43} \rangle$ so $r_{59} = 1$. The associated matrix $B_{\mathbb{R}_7}$ is given by

$$B_{\mathbb{R}_7} = \frac{1}{2}\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Applying Gauss-Jordan elimination method to $B_{\mathbb{R}_7}$ we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that has kernel of dimension 1 generated by $y_{\mathbb{R}_7} = (1, 2, 3, 2, 3, 4, 2, 3, 2)^t$ (we want $y_{\mathbb{R}_7} \in \mathbb{Z}^n$ because we will use it as a dimension vector). Therefore

$$\operatorname{Rad}(q_{\mathbb{R}_7}) = \ker(B_{\mathbb{R}_7}) = \langle y_{\mathbb{R}_7} \rangle$$

We define the representations $B(\lambda)_{\mathbb{R}_7}$ (with $\lambda \in \mathbf{k}$) in the following way:

$$B(\lambda)_{\mathbb{R}_7} := \begin{tikzcd} & & \mathbf{k}^2 \arrow[dddd, dashed, no head] & & & \\ & \mathbf{k}^3 \arrow[ur, "\varphi_{69}"] & & & & \\ \mathbf{k}^2 \arrow[r, "\varphi_{76}"] & \mathbf{k}^4 \arrow[u, "\varphi_{68}"] & & \mathbf{k}^3 \arrow[uul, "\varphi_{39}(\lambda)"'] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"'] & \mathbf{k} \arrow[l, "\varphi_{12}"'] \\ & \mathbf{k}^3 \arrow[u, "\varphi_{56}"] & & & & \\ & & \mathbf{k}^2 \arrow[ul, "\varphi_{45}"] \arrow[uur, "\varphi_{43}"'] & & & \end{tikzcd}$$

where the **k**-linear maps are given by

$$\varphi_{12} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \varphi_{23} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{43} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \varphi_{45} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \varphi_{56} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\varphi_{76} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \varphi_{68} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \varphi_{89} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \varphi_{49}(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The relations are preserved since

$$\varphi_{89}\varphi_{68}\varphi_{56}\varphi_{45} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \varphi_{39}(\lambda)\varphi_{43}$$

We have that $\underline{\dim}(B(\lambda)_{\mathbb{R}_7}) = y_{\mathbb{R}_7}$ and we prove the following:

Theorem A.23. *The representations $B(\lambda)_{\mathbb{R}_7}$ with $\lambda \in \mathbf{k}$ (with $\mathbf{k}$ algebraically closed field) form an infinite family of non-isomorphic bricks with the same dimension. Therefore, the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_7)$ is brick-continuous.*

Proof. Let $f\colon B(\lambda)_{\mathbb{R}_7} \to B(\lambda')_{\mathbb{R}_7}$ be a morphism of representations.

$$\begin{tikzcd}
 & & & & \mathbf{k}^2 \arrow[ddddd, red, "f_9"'] \arrow[ddddr, dashed, no head] & & & & \\
 & & \mathbf{k}^3 \arrow[rru, "\varphi_{69}"] \arrow[ddddd, red, "f_8"'] & & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{76}"] \arrow[ddddd, red, "f_7"'] & \mathbf{k}^4 \arrow[ru, "\varphi_{68}"] \arrow[ddddd, red, "f_6"'] & & & & & \mathbf{k}^3 \arrow[lluu, "\varphi_{39}(\lambda)"'] \arrow[ddddd, red, "f_3"] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"'] \arrow[ddddd, red, "f_2"] & \mathbf{k} \arrow[l, "\varphi_{12}"'] \arrow[ddddd, red, "f_1"] \\
 & & & \mathbf{k}^3 \arrow[ull, "\varphi_{56}"'] \arrow[ddddd, red, "f_5"] & & & & & \\
 & & & & & \mathbf{k}^2 \arrow[ull, "\varphi_{45}"'] \arrow[uur, "\varphi_{43}"] \arrow[ddddd, red, "f_4"'] & & & \\
 & & & & \mathbf{k}^2 \arrow[ddddr, dashed, no head] & & & & \\
 & & \mathbf{k}^3 \arrow[rru, "\varphi_{69}"] & & & & & & \\
\mathbf{k}^2 \arrow[r, "\varphi_{76}"] & \mathbf{k}^4 \arrow[ru, "\varphi_{68}"] & & & & & \mathbf{k}^3 \arrow[lluu, "\varphi_{39}(\lambda')"'] & \mathbf{k}^2 \arrow[l, "\varphi_{23}"'] & \mathbf{k} \arrow[l, "\varphi_{12}"'] \\
 & & & \mathbf{k}^3 \arrow[ull, "\varphi_{56}"] & & & & & \\
 & & & & & \mathbf{k}^2 \arrow[ull, "\varphi_{45}"] \arrow[uur, "\varphi_{43}"'] & & &
\end{tikzcd}$$

We have that f_1 is just a scalar multiplication of an element of $\mathbf{k}$, while $f_2, f_4, f_7, f_9 \in \mathbb{M}_2(\mathbf{k})$, $f_3, f_5, f_8 \in \mathbb{M}_3(\mathbf{k})$ and $f_6 \in \mathbb{M}_4(\mathbf{k})$. In order to understand the property that f must have, we study the commutativity of the squares in the previous diagram (given by the fact that f is a morphism of representations).

(1) The square involving f_1 and f_2 gives us

$$f_2(e_1) = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}$$

(2) The square involving f_2 and f_3 gives us that $\varphi_{23} f_2(e_i) = f_3(\varphi_{23}(e_i))$ that means

$$f_3(e_1) = \begin{pmatrix} f_2(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ 0 \\ 0 \end{pmatrix} \qquad f_3(e_2) = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix}$$

(3) The square involving f_3 and f_4 gives us that $\varphi_{43} f_4(e_i) = f_3(\varphi_{43}(e_i))$ that means

$$f_3(e_2) = \begin{pmatrix} 0 \\ f_4(e_1) \end{pmatrix} = \begin{pmatrix} f_2(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} \quad f_3(e_3) = \begin{pmatrix} 0 \\ f_4(e_2) \end{pmatrix}$$

for some $a \in \mathbf{k}$ (we used (2)).

(4) The square involving f_4 and f_5 gives us that $\varphi_{45} f_4(e_i) = f_5(\varphi_{45}(e_i))$ that means

$$f_5(e_1) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \quad f_5(e_2) = \begin{pmatrix} f_4(e_2) \\ 0 \end{pmatrix}$$

(5) The square involving f_5 and f_6 gives us that $\varphi_{56} f_5(e_i) = f_6(\varphi_{56}(e_i))$ that means

$$f_6(e_1) = \begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad f_6(e_2) = \begin{pmatrix} f_4(e_2) \\ 0 \\ 0 \end{pmatrix} := \begin{pmatrix} b \\ c \\ 0 \\ 0 \end{pmatrix} \quad f_6(e_3) = \begin{pmatrix} f_5(e_3) \\ 0 \end{pmatrix}$$

for some $b, c \in \mathbf{k}$.

(6) The square involving f_6 and f_7 gives us that $\varphi_{76} f_7(e_i) = f_6(\varphi_{76}(e_i))$ that means

$$f_6(e_3) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_1) \end{pmatrix} = \begin{pmatrix} f_5(e_3) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ d \\ 0 \end{pmatrix} \quad f_6(e_4) = \begin{pmatrix} 0 \\ 0 \\ f_7(e_2) \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ e \\ f \end{pmatrix}$$

for some $d, e, f \in \mathbf{k}$ (we used (5)).

(7) The square involving f_6 and f_8 gives us that $\varphi_{68} f_6(e_i) = f_8(\varphi_{68}(e_i))$ that means

$$f_8(e_1) = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \quad f_8(e_2) = \begin{pmatrix} b \\ c \\ 0 \end{pmatrix} \quad f_8(e_3) = \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}$$

For $i = 4$

$$\begin{pmatrix} b \\ c \\ 0 \end{pmatrix} = f_8(e_2) = \varphi_{68} f_6(e_4) = \begin{pmatrix} 0 \\ f \\ e \end{pmatrix}$$

so $b = e = 0$ and $f = c$.
(8) The square involving f_8 and f_9 gives us that $\varphi_{89} f_8(e_i) = f_9(\varphi_{89}(e_i))$ that means

$$f_9(e_1) = \begin{pmatrix} a \\ 0 \end{pmatrix} \quad f_9(e_2) = \begin{pmatrix} 0 \\ c \end{pmatrix}$$

$$\begin{pmatrix} a \\ c \end{pmatrix} = f_9(e_1 + e_2) = \varphi_{89} f_8(e_3) = \begin{pmatrix} d \\ d \end{pmatrix}$$

So $a = d = c \, (= f)$.
(9) The square involving f_3 and f_9 gives us that $\varphi_{39}(\lambda') f_3(e_i) = f_9(\varphi_{39}(\lambda)(e_i))$. For $i = 1$ we get

$$\begin{pmatrix} \lambda a \\ a \end{pmatrix} = f_9(\lambda e_1 + e_2) = \varphi_{39}(\lambda') f_3(e_1) = \begin{pmatrix} \lambda' f_1 \\ f_1 \end{pmatrix}$$

that means $a = f_1$ and $\lambda f_1 = \lambda' f_1$.

Therefore we get

$$f_2 = f_4 = f_7 = f_9 = f_1 \mathbb{1}_2 \quad f_3 = f_5 = f_8 = f_1 \mathbb{1}_3 \quad f_6 = f_1 \mathbb{1}_4 \tag{A.12}$$

Now if $\lambda = \lambda'$ we conclude that any morphism $f\colon B(\lambda)_{\mathbb{R}_7} \to B(\lambda)_{\mathbb{R}_7}$ depends only on $f_1 \in \mathbf{k}$ by (A.12), therefore $\operatorname{End}(B(\lambda)_{\mathbb{R}_7}) \cong \mathbf{k}$, so $B(\lambda)_{\mathbb{R}_7}$ is a brick for any $\lambda \in \mathbf{k}$.

If $\lambda \neq \lambda'$ we have that $\lambda f_1 = \lambda' f_1$ and if $f_1 \neq 0$ it would mean that $\lambda = \lambda'$, contradiction. Thus, for any $\lambda \neq \lambda'$ we must have

$$\operatorname{Hom}(B(\lambda)_{\mathbb{R}_7}, B(\lambda')_{\mathbb{R}_7}) = 0$$

Therefore $B(\lambda)_{\mathbb{R}_7}$ and $B(\lambda')_{\mathbb{R}_7}$ cannot be isomorphic. We conclude that $\{B(\lambda)_{\mathbb{R}_7}\}_{\lambda \in \mathbf{k}}$ is an infinite family of non-isomorphic bricks with the same dimension and consequently the incidence $\mathbf{k}$-algebra $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_7)$ is brick-continuous. □

Construction of $G_{\mathbb{R}_7}$ We have the brick-continuous family $\{B(\lambda)_{\mathbb{R}_7}\}_{\lambda \in \mathbf{k}}$ and we define the representation $G_{\mathbb{R}_7}$ substituting $\mathbf{k}^n$ with $\mathbf{k}(x)^n$ (for any n) and λ with x:

$$\begin{tikzcd}
 & & \mathbf{k}(x)^2 & & & \\
 & \mathbf{k}(x)^3 \arrow[ur, "\phi_{69}"] & & & & \\
\mathbf{k}(x)^2 \arrow[r, "\phi_{76}"] & \mathbf{k}(x)^4 \arrow[u, "\phi_{68}"] & & \mathbf{k}(x)^3 \arrow[uul, red, "\phi_{39,x}"'] & \mathbf{k}(x)^2 \arrow[l, "\phi_{23}"'] & \mathbf{k}(x) \arrow[l, "\phi_{12}"'] \\
 & \mathbf{k}(x)^3 \arrow[u, "\phi_{56}"] & & & & \\
 & & \mathbf{k}(x)^2 \arrow[ul, "\phi_{45}"] \arrow[uur, "\phi_{43}"'] \arrow[uuuu, dash, dashed] & & &
\end{tikzcd}$$

with $\phi_{39,x} \coloneqq \varphi_{39}(x)$ and $\phi_{ij} = \varphi_{ij}$ for the other maps.

Theorem A.24. *$G_{\mathbb{R}_7}$ is a generic brick over $\mathcal{I}_{\mathbf{k}}(\mathbb{R}_7)$.*

Proof. By Theorem 1.22 we have that

$$\ell_{\mathcal{I}_{\mathbf{k}}(\mathbb{R}_7)}(G_{\mathbb{R}_7}) = \dim_{\mathbf{k}}(G_{\mathbb{R}_7}) = \sum_{i \in (Q_{\mathbb{R}_7})_0} \dim_{\mathbf{k}}((G_{\mathbb{R}_7})_i) = \infty$$

So $G_{\mathbb{R}_7}$ have infinite length. Moreover, following the same procedure as in Theorem A.23, we see that all the components of a endomorphism $f\colon G_{\mathbb{R}_7} \to G_{\mathbb{R}_7}$ depends only on the map $f_1\colon \mathbf{k}(x) \to \mathbf{k}(x)$ that is $\mathbf{k}$-linear by definition and satisfies $xf_1(1) = f_1(x)$ (as done in the previous proof, substituting λ with x). We also have that

$$f_1(1) = f_1(xx^{-1}) = xf_1(x^{-1}) \implies x^{-1}f_1(1) = f_1(x^{-1})$$

Therefore f_1 is completely determined by the element $f_1(1) \in \mathbf{k}(x)$, thus we have that $\operatorname{End}(G_{\mathbb{R}_7}) \cong \mathbf{k}(x)$ and by Lemma 1.21

$$\ell_{\mathbf{k}(x)^{\mathrm{op}}}(G_{\mathbb{R}_7}) = \dim_{\mathbf{k}(x)}(G_{\mathbb{R}_7}) = \sum_{i \in (Q_{\mathbb{R}_7})_0} \dim_{\mathbf{k}(x)}((G_{\mathbb{R}_7})_i) < \infty$$

So $G_{\mathbb{R}_7}$ has finite endolength too. By Remark 4.7 we conclude that $G_{\mathbb{R}_7}$ is a generic brick. $\square$

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