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TESI DI LAUREA

COSMOLOGICAL PERTURBATIONS DURING INFLATION AND THE CONSISTENCY RELATION: AN EFFECTIVE FIELD THEORY APPROACH

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Chapter 1

Introduction

The birth of modern cosmology dates back to the second decade of the 20th century, when Einstein formulated his theory of General Relativity. Einstein equations, together with the Cosmological Principle, which states that the Universe is both homogeneous and isotropic, allowed the formulation of the Standard Hot Big Bang Model. It was Friedmann in 1922 who derived mathematically the equations describing the evolution of a homogeneous and isotropic Universe, which are a a set of two equations

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G\rho - \frac{k}{a^2}, \tag{1.0.1}$$

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p), \tag{1.0.2}$$

where $a(t)$ is the scale factor, ρ and p are the density and the pressure of the fluid constituting the Universe and k is the curvature parameter which can be $+1, 0, -1$ depending on whether the shape of Universe is a closed 3-sphere, flat or an open 3-hyperboloid. Usually we refer to this kind of cosmological model which evolution is given by the Friedmann equations and which metric is the Robertson-Walker metric as Friedmann-Robertson-Walker (FRW) Universe. One of the most interesting point of this model, which was initially difficult to acknowledge, is that our Universe is evolving. The first proof of the expansion of our Universe came in 1929 when Hubble observed that the Galaxies were moving away from the Earth, then in 1965 the discovery of the Cosmic Microwave Background (CMB) radiation was immediately considered as another good evidence of an evolving Universe which was hotter and denser in the past. However, even though the Hot Big Bang model had achieved great successes like predicting light-element abundances produced during cosmological nucleosynthesis and explaining how the CMB cooled, some problems remained unsolved: the cosmological horizon, the question of why the Universe is so close to being flat and the monopoles problem. The general features of these problems are:

- Horizon problem: CMB photons, which are propagating freely since they decoupled from matter at the moment of last scattering, appear to be in thermal equilibrium at almost the same temperature ($\Delta T/T \sim 10^{-5}$). The most natural explanation for this is that the Universe has indeed reached a state of thermal equilibrium through interactions between the different regions before the last scattering, this means that the cosmological scales we can now see must have been casually connected before the decoupling of radiation from matter. But this is not possible in the Standard Hot Big Bang model, in fact there was no possibility for the regions that became casually connected recently to interact before the last scattering because of the finite speed of light.

- Flatness problem: this problem regards the value of the density of the Universe. We define the ratio between the density of our Universe ρ and the density of the Universe if it would be flat $\rho_c = 3H^2/(8\pi G)$ as $\Omega = \rho/\rho_c$. The first Friedmann equation (1.0.1) can be written in the form
$$|\Omega - 1| = \frac{|k|}{a^2H^2}. \tag{1.0.3}$$
In the Standard Hot Big Bang model we expect that a^2H^2 decreases, hence Ω moves away from one, for example
$$\text{Matter domination } |\Omega - 1| \propto t^{2/3},$$
$$\text{Radiation domination } |\Omega - 1| \propto t.$$
So this means that if the Universe is flat then it stays flat forever otherwise the discrepancy between our Universe and the flat Universe would increase in time, in other words $\Omega = 1$ is an unstable critical point. From the observations we know that today $|\Omega(t_{now}) - 1| < 10^{-2}$ [1], so we can say that our Universe is very close to be a flat one. This is quite surprisingly because it means that in the past Ω must have been much closer to one, moreover there are not known reason for which the Universe density should be exactly ρ_c. On the other side, if in the primordial Universe there was a tiny departure of Ω from 1, it would have been magnified during billions of years of expansion to create a current density very far from the critical one.

- Monopoles problem: modern particle theories predict a large variety of "unwanted relics", which would violate observations. These are very massive particles which can be produced in the primordial Universe like magnetic monopoles, domain walls, supersymmetric particles (gravitino). We expect a huge contribution to the density of the Universe from these particles for two reasons: they are massive and they must be produced in great quantities (Kibble mechanism) [2]. So we would expect from them to become the dominant material in the Universe. However our observations don't show any contribution to the density of the Universe from these particles so during the history of Universe something must have happened that in some way erased the contribution of cosmic relics.

These problems were the signal that new physics was needed and this led Guth to the formulation in 1980 of a new theory that was able to overcome these problems: cosmological inflation [3, 4, 5]. It consists of a period of accelerated expansion in the very early Universe, $10^{-34}s$ after the Big Bang. Mathematically this request of accelerated expansion translate into the following condition on the scale factor $\ddot{a} > 0$. Since the result of the Standard Hot Big Bang model were undeniable, inflation wasn't proposed as an alternative model to describe our Universe but just as an epoch which takes place in the very early Universe, then comes to an end and it is followed by the conventional behaviour. The inflationary solutions to the problems outlined above are

- Horizon problem: inflation ensures that the portion of the Universe which was casually connected in the past was bigger than it is now. This allows an homogenization of the property of the Universe also on large scales. In other words the region of the Universe we can see after (even long after) inflation is much smaller than the region which would have been visible before inflation took place.

- Flatness problem: during the period of accelerated expansion the density parameter Ω is brought back to one since $\Omega - 1$ decreases exponentially. So, if Ω is close enough to it at the end of inflation, it will stay very close to it right to the present, despite being repelled from one as soon as inflation ends and starts the FRW Universe predicted by the Standard Hot Big Bang model.

- Monopoles problem: the accelerated expansion epoch in which consists the inflationary model dilutes the density of unwanted relics. The result is that the contribution of these unobserved particles to the Universe density is negligible. Obviously this require that between the end of inflation and the beginning of the FRW Universe occurs the process of reheating which turns the energy density of the Universe into conventional matter without creating the unwanted relics.

The observations helped to constrain the duration of the inflationary epoch. From the request that the cosmological scales of the order of the ones which we observe now are casually connected and the value of Ω predicted by the theory corresponds to the one observed [1], we inferred that inflation must have lasted at least $60 \div 70$ e-folds.

The power of inflation not only resides in the fact that it was able to answer the question left unresolved by the Hot Big Bang model, but it also provided some predictions which we recently found to be totally compatible with the measurements of the *WMAP* [6] and *Planck* [7] satellites. Firstly the inflationary paradigm tells us how an homogeneous and isotropic FRW Universe arises and in second place it provides an extremely appealing explanation for the formation of structure on large scales and the inhomogeneities of the CMB ($\Delta T/T \sim 10^{-5}$) through the generation of primordial perturbations. Microscopic quantum fluctuations get stretched by inflationary expansion to macroscopic scales, larger than the horizon, so no causal physics can affect them. Thus after a perturbations exits the horizon remains frozen with constant amplitude until it re-enters the horizon at a later time, when inflation has ended. One of the most important success of the inflationary theory is its prediction of almost scale invariant power spectrum of primordial fluctuations [8, 9, 10].

Since Guth proposal of an inflationary epoch [3], the theory of inflation has been studied and developed with great efforts [11, 12, 13, 14, 15]. Now the usual way to treat inflation is through a scalar field called the inflaton which under specific conditions on its potential acts like an effective cosmological constant (slow-roll inflation). The primordial perturbations are generated by the fluctuations of this field around its vacuum state and they are then promoted to classical perturbations at the time of horizon exit. These scalar perturbations induce small perturbations in the local density, which grow because of gravitational collapse and ends up by building the large scale structures we observe today in the Universe. On the other side we have small perturbations of the metric that, in the same way of the inflaton perturbations, when stretched outside the horizon they become classical producing anisotropies in the CMB. The BICEP2 experiment [16, 17] claimed to have detected for the first time in 2014 these tensor perturbations or primordial gravitational waves.

Nowadays inflation is considered a central paradigm in cosmology but there are still many aspects which are unknown, for example the potential of the inflaton. To unravel these pending questions we can count on experimental data, which precision is increasing greatly. An example is the detection of primordial gravitational waves, cited above, by BICEP2 which, if confirmed by other experiments, will set bounds on the energy at which inflation took place ($\sim 10^{16}$ GeV) and hence put constraints on the potential of the inflaton. The CMB spectrum is the most useful observable, in its shape are encoded large amounts of information even on the very early Universe.

A key role in a further understanding of the physics of inflation is played by the ratio between the tensor modes and the scalar ones, which is usually called r. This quantity tells us in which proportion scalar and tensor perturbations were produced in the early Universe and it is of paramount importance in order to catalogue the possible inflationary models. Hence a better measure of r will help to understand the correct model [18]. Moreover for a single-field slow-roll inflation the tensor to scalar ratio is linked to the tensor spectral index n_t by $r = -8n_t$, which is called consistency relation. If the experiments confirm this relation it would be an indisputable proof of the fact that inflation has actually been driven by a single scalar field otherwise it would mean that we need to consider alternative scenarios in which maybe there are more fields [19, 20]. The most challenging part from the experimental point of view, is to increase the accuracy in the measure of n_t [18].

While on one side we are receiving new experimental data with growing accuracy, on the other side we need theoretical models which help us to interpret those data. For this reason, in the last years, many efforts were spent on building several models for inflation, in particular great attention was dedicated in the construction of an effective field theory (EFT) for single-field inflation. This approach is very useful because it allows to write a very general theory relying only on the symmetries of the system and for which the leading contribution can be encoded in a finite number of operators. Moved by these recent developments in this field, this Thesis was conceived with the aim of understanding the basis of this new EFT approach to inflation and with the purpose of evaluating how the consistency relation $r = -8n_t$ is modified in this scenario.

This Thesis is divided into five parts.

- In Chapter 2 we will study the theory of cosmological perturbations, we will see how the perturbations are defined in cosmology and their geometrical interpretations. We will also discuss the important issue of the gauge dependence of the perturbations. In conclusion we will analyse what the Einstein equations predict for the evolution of the perturbations.
- In Chapter 3 we will focus on the dynamics of inflation, we will introduce the inflaton field and study the slow-roll inflation. We will also calculate the power spectra for both scalar and tensor perturbations. The last step will be explicitly finding the important relation between the ratio of the two power spectrum r and the spectral index of tensor perturbation n_t: the so called consistency relation $r = -8n_t$.
- In Chapter 4 we will consider one of the most recent approach to the study of inflationary perturbations: the effective field theory approach [86, 91]. This approach consists in writing the most general action for the inflaton perturbations starting from the underlying symmetries. Once we will have write the theory in its most general form we will check that particular models already studied in the literature can be found by setting the parameters of the theory to particular values and we will compute the power spectra.
- In Chapter 5 we will see how the different terms of the effective action modify the consistency relation introduced for the slow-roll inflation. This calculation is intended to be something original since in the literature there are no example of explicit calculations of a "generalized consistency relation" which takes into account the different operators that appear in the effective action.
- In Chapter 6 we summarize the results found and we discuss them.

Finally, we set the notations used throughout this Thesis. We choose the metric with the following signature $(-, +, +, +)$. We will use the dot above a function to indicate the derivative with respect the cosmic time t while we will use the apostrophe to indicate the derivative with respect the conformal time τ which is defined in the following way:

$$\tau = \int \frac{dt}{a}. \tag{1.0.4}$$

So if we have a function f the derivatives are expressed as

$$\dot{f} = \frac{df}{dt}, \qquad f' = \frac{df}{d\tau}, \qquad f_{,i} = \partial_i f = \frac{df}{dx^i}. \tag{1.0.5}$$

With ∇_μ we call the covariant derivative. The Hubble rate H is defined as the ratio

$$H = \frac{\dot{a}}{a}, \tag{1.0.6}$$

and we introduce also the conformal Hubble parameter

$$\mathcal{H} = aH, \tag{1.0.7}$$

which will be helpful to express some results while dealing with cosmological perturbations. When dealing with the perturbations in Chapter 2, we will generally use a number between parenthesis to indicate the order of the perturbation. We use this notation in order to avoid misunderstandings between the component of the tensors and the order of the perturbation.

Chapter 2

Cosmological perturbations

The first question we are going to answer in this section is: why do we need to study perturbations in cosmology? The main point which led to develop this formalism is the ineffectiveness of a homogeneous model in describing the complexity of the actual distribution of matter and energy in our observed Universe where stars and galaxies create clusters and superclusters of galaxies across a wide range of scales. The Standard Hot Big-Bang Model successfully described many observational characteristics of our Universe: its expansion and consequent cooling, the abundances of light nuclei, the CMB freely propagating since the last scattering. Even though these results outlined the effectiveness of this model, newer observations strongly reinforced the need for a further step: the presence of non-baryonic matter (dark matter), the structure of the Universe on large scales, the presence of anisotropies in the CMB indicating that the early Universe was not completely smooth. To understand these facts it is necessary to go beyond the Standard Model of Hot Big-Bang. Nevertheless there are few exact solutions of General Relativity that incorporate spatially inhomogeneous and anisotropic matter and hence geometry. For this reason small perturbations are the right tools to describe anisotropies (we know they are of order $\Delta T/T \sim 10^{-5}$) and structures formation on large scales.

In order to answer the opening question, a better description of the real physical Universe forced to include in the theory the perturbation approach. We start from a spatially homogeneous and isotropic FRW model as a background solution with simple properties, within which we can study the increasing complexity of inhomogeneous perturbations order by order.

Throughout the study of cosmological perturbations we will encounter different types of perturbations, such as scalar, vector and tensor perturbation modes, which play different roles in the evolution of the early Universe. These perturbations were produced during inflation as quantum fluctuations of the field leading inflation and then they evolved; for example scalar perturbations of the metric coupled to the density of matter and radiation and they are responsible for the most of the inhomogeneities and anisotropies in the Universe. These primordial perturbations slowly increased in amplitude due to gravitational instability to constitute the structures we see today on large-scales in the Universe. In a non expanding background this would have led to an exponential instability, while in an expanding Universe the gravitational force is counteracted by the expansion, so there is a power-law growth of perturbations instead of an exponential one. Inflation also generated tensor fluctuations in the gravitational metric, the so-called gravity waves. These are not coupled to the density (for more details see [21]) and so are not responsible for the large-scale structure of the Universe, but they induce perturbations in the CMB.

In our analysis of perturbations we will encounter the so called gauge issue which is directly inherited from the theory of General Relativity. If we call $\mathcal{M}_0$ the background manifold with Robertson-Walker metric (FRW) and $\mathcal{M}_{phys}$ the manifold of the "real" physical Universe with little inhomogeneities and anisotropies, then a generic map ϕ is called a gauge if it links a point in the background to the corresponding physical one by adding a little perturbation:

$$\phi : \begin{array}{ll} \mathcal{M}_0 & \to \mathcal{M}_{phys} \\ \rho_0(t) & \to \rho_{phys}(\vec{x},t) = \rho_0(t) + \delta\rho(\vec{x},t) \end{array} \tag{2.0.1}$$

where $\rho_0(t)$ can be for example the background value of the matter density. A gauge transformation, let it call ψ, is a change in the correspondence between background and physical points, keeping the background coordinates fixed. So if ϕ_1 and ϕ_2 are two different gauge choices which associate two different points in $\mathcal{M}_{phys}$ to the same point in $\mathcal{M}_0$ then ψ can be represented by

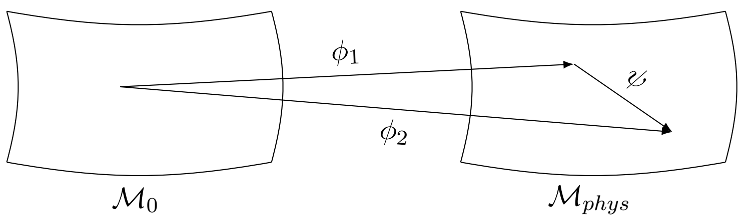

Figure 2.1: gauge transformation.

Obviously physics is invariant under gauge transformations and so we can choose every time the most suitable gauge to work with, to make easiest the calculations process. Because of the freedom in the choice of the gauge, not all the perturbed metrics correspond to perturbed space-times: it is possible to obtain an inhomogeneous form for the metric $g_{\mu\nu}(\vec{x},t)$ in a homogeneous and isotropic space-time by an inconvenient choice of coordinates. Hence it is important to be able to distinguish between physical (geometrical) inhomogeneities and mere coordinates artefacts. In this situation using gauge-independent variables, which are independent by the choice of gauge, is helpful because it gives an exact physical interpretation in the sense that these variables represent the same physical quantity in each gauge. For example also in electromagnetism we encounter the same problem and it is clearly easier to work with the electric and magnetic fields rather than the gauge-dependent scalar and vector potentials.

The pioneering work on perturbations in FRW cosmological model is the one of Lifshitz in [22] and summarized by Lifshitz and Khalatnikov in [23]. Then the subject was studied by many authors, the texts [24, 25, 26] treat cosmological perturbations in some details. The gauge-invariant approach was pioneered by Bardeen in [27, 28] and by Gerlach and Sengupta in [29]. Then this gauge-invariant approach to the problem was studied extensively in [30, 31, 32, 33], it has been applied to construct a self-consistent quantum theory of metric perturbations in [34, 35], to investigate eternal and stochastic inflation in [36, 37, 38], to follow the dynamics of inflationary Universe models in [39] and to analyse the stability of inflation in higher derivative theories of gravity in [40]. Den and Tomita have extended the gauge invariant formalism to anisotropic cosmologies [41, 42]. A gauge invariant formalism based on the $3+1$ Hamiltonian form of the General Relativity was developed by Durrer and Straumann in [43]. Most of the works done during all these years in the field of cosmological perturbations are reviewed in [44].

2.1 Defining perturbations

First of all we recall the assumptions we are going to make:

- our Universe can be described at zero order by a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) space-time;
- we consider a flat Universe.

So, according to our assumptions, the background space-time is described by a flat FRW metric which we can write as

$$ds^2 = a^2\big[-d\tau^2 + \delta_{ij}dx^i dx^j\big], \tag{2.1.1}$$

where $a = a(\tau)$ is the scale factor. We recall that τ is the conformal time which is linked to the cosmic time t by $t = \int a(\tau)d\tau$.

Another consequence of these assumptions is that we can decompose the physical quantities into a homogeneous background part, depending only on the cosmic time or alternatively the conformal time, and an inhomogeneous perturbations. If we consider a generic tensorial quantity, we can hence write it in agreement with our assumptions as

$$T(\tau,\vec{x}) = T_{(0)}(\tau) + \delta T(\tau,\vec{x}) \tag{2.1.2}$$

where $T_{(0)}$ is the background value, δT is the perturbation and τ is the conformal time. Moreover the perturbation part can be further expanded as a power series

$$\delta T(\tau,\vec{x}) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \delta T_{(n)}(\tau,\vec{x}) \tag{2.1.3}$$

where the subscript n denotes the order of the perturbations and ϵ is a small parameter[1]. Clearly this series contains an infinite number of terms but one has to take into account only a few depending on the situation: if the aim is to study linear perturbation theory then it is enough to consider only first order terms (the ones proportional to ϵ) and neglect the others, otherwise it would be necessary to consider also higher order perturbations. From now on we will omit the small parameter ϵ.

It is convenient to slice the space-time manifold into a one-parameter family of spatial hypersurfaces of constant time, which is the standard 3+1 split of space-time. This foliation was firstly introduced by Darmois in 1927 and popularized by Arnowitt, Deser and Misner [45] and for further details one can read the reference [46]. The foliation is given by spatial hypersurfaces of given conformal time and we call it time slicing while we refer to the identification of spatial coordinates on each hypersurface as the threading. As a consequence of this slicing, we can split our tensorial quantities into spatial and temporal parts as following.

2.1.1 Split of vectors

We can split any 4-vector into a temporal and a spatial part

$$V^\mu = (V^0, V^i). \tag{2.1.4}$$

Note that the temporal part V^0 is a scalar on spatial hypersurfaces. The spatial part can be further decomposed into a scalar part V and a vector part V^i_{vec},

$$V^i = \delta^{ij} V_{,j} + V^i_{vec}, \tag{2.1.5}$$

where $V_{,j} = \partial V/\partial x^j$ while the vector part satisfies $\partial V^i_{vec}/\partial x^i = 0$. The derivatives are defined with respect to the flat space metric of the background. The names "scalar" and "vector" parts were introduced by Bardeen

[1] The index labelling the perturbation order will be always written inside parenthesis and from now on it will appear indiscriminately as high or low index basing only on the presence of other indices.

in [27] and are due to the transformation behaviour under a change of coordinates of V and V^i_{vec} on spatial hypersurfaces [47]. The decomposition of a vector field into a curl-free and a divergence-free part in Euclidean space is known as Helmholtz theorem. Furthermore we have more constraints in our case because we are working on a FRW Universe which is in particular isotropic. As a consequence at zeroth order in perturbations there can't be spatial vector part otherwise there would be a preferred direction while there can be a non vanishing temporal part:

$$V^0_{(0)} \neq 0, \qquad V^i_{(0)} = 0. \tag{2.1.6}$$

Consequently we expect a non zero vector part only at first or higher order in perturbations.

2.1.2 Split of tensors

As for vectors, we can decompose a rank-2 tensor into a time part and spatial part but now there are also mixed time-space parts. We take for example the metric tensor $g_{\mu\nu}$ which by definition is symmetric and hence has only 10 independent components in 4 dimensions. First we split our metric tensor into a background part and a perturbed one using (2.1.2)

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu} = a^2(\tau)\eta_{\mu\nu} + \delta g_{\mu\nu}, \tag{2.1.7}$$

and then we split the perturbation into different parts labelled scalar, vector and tensor according to their transformation properties on spatial hypersurfaces. Thus we can write the perturbations for the metric tensor as:

$$\delta g_{00} = -2a^2\phi, \tag{2.1.8}$$
$$\delta g_{0i} = a^2 B_i, \tag{2.1.9}$$
$$\delta g_{ij} = 2a^2 C_{ij}. \tag{2.1.10}$$

As stated before, we can further decompose the $0i$ and the ij perturbations as:

$$B_i = B_{,i} - S_i, \tag{2.1.11}$$
$$C_{ij} = -\psi\delta_{ij} + E_{,ij} + F_{(i,j)} + \frac{1}{2}h_{ij}, \tag{2.1.12}$$

with $F_{(i,j)} = \frac{1}{2}\big(F_{i,j} + F_{j,i}\big)$. After all these decompositions we end up having four scalar perturbations ϕ, B, ψ, and E, two vector perturbations S_i and F_i and only one tensor perturbation h_{ij}. Each 3-vector, such as $B_{,i}$, constructed from a scalar is necessarily curl-free $B_{,[i,j]} = 0$. Instead vector perturbations are divergence-free. Finally there is h_{ij}, a tensor contribution which has the following properties:

$$\begin{cases} h_{ij} = h_{ji} & \text{simmetric}, \\ h_{ij,}{}^{j} = 0 & \text{transverse}, \\ h_i^i = 0 & \text{traceless}. \end{cases} \tag{2.1.13}$$

In terms of degrees of freedom, we have four of them coming from the four scalar functions, six from the two spatial vectors and nine from the tensor function. But there are also constraints to take into account: two for the divergence-free constraints on the vector functions and seven for the symmetric, traceless and transverse constraints on the tensor function. Subtracting the number of constraints from the number of degrees of freedom we are left with ten degrees of freedom[2] which are exactly the number of independent component in a symmetric 4-dimensional tensor like the metric $g_{\mu\nu}$. The reason for splitting the metric perturbations into scalars, vectors and tensors is that the governing equations decouple at linear order and hence we can solve each perturbation type separately. At higher order this is no longer true as outlined in [48, 49]. The choice of variables to describe the perturbed metric is not unique, already at first order there are different conventions in the literature for the split of the spatial part of the metric. Here we are following the notation of Mukhanov et al [50] so that the metric perturbation ψ can be identified directly with the intrinsic scalar curvature of spatial hypersurfaces at first order. Note that the metric perturbations written in (2.1.8-2.1.10) include all orders. If we write out the metric tensor up to second order in perturbations we have:

$$\begin{aligned} g_{00} &= -a^2\Big(1 + 2\phi_{(1)} + \phi_{(2)}\Big), \\ g_{0i} &= a^2\Big(B_{(1)i} + \frac{1}{2}B_{(2)i}\Big), \\ g_{ij} &= a^2\Big(\delta_{ij} + 2C_{(1)ij} + C_{(2)ij}\Big), \end{aligned} \tag{2.1.14}$$

where the first and second order perturbations $B_{(1)i}$ and $C_{(1)ij}$ and $B_{(2)i}$ and $C_{(2)ij}$ can be further split according to (2.1.11) and (2.1.12). The contravariant metric tensor follows from the constraint $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$, which up to the second order gives:

$$\begin{aligned} g^{00} &= -a^{-2}\Big(1 - 2\phi_{(1)} - \phi_{(2)} + 4\phi^2_{(1)} - B_{(1)k}B^{\ k}_{(1)}\Big) \\ g^{0i} &= a^{-2}\Big(B^{\ i}_{(1)} + \frac{1}{2}B^{\ i}_{(2)} - 2\phi_{(1)}B^{\ i}_{(1)} - 2B_{(1)k}C^{\ ki}_{(1)}\Big) \\ g^{ij} &= a^{-2}\Big(\delta^{ij} - 2C^{\ ij}_{(1)} - 2C^{\ ij}_{(2)} + 4C^{\ ik}_{(1)}C^{\ j}_{(1)k} - B^{\ i}_{(1)}B^{\ j}_{(1)}\Big). \end{aligned} \tag{2.1.15}$$

[2]Actually only six are physical degrees of freedom because, as we will see later, there is a freedom in the choice of the gauge and choosing a specific gauge we fix four degrees of freedom.

The detailed calculation of the contravariant metric tensor is carried out in Appendix 7.1. When lowering and raising spatial indices of perturbations we use the background spatial metric δ^{ij}.

2.2 Geometry of spatial hypersurfaces

In the perturbed metric given before we can define a vector field orthogonal to hypersurfaces of constant τ:

$$n_\mu = \alpha \frac{\partial \tau}{\partial x^\mu} \tag{2.2.1}$$

where α is a normalization constant. Let's check some of the properties of this vector field:

$$\frac{\partial \tau}{\partial x^0} = 1, \qquad \frac{\partial \tau}{\partial x^i} = 0, \tag{2.2.2}$$

$$n^\mu n_\mu = \alpha^2 \frac{\partial \tau}{\partial x^\mu} \frac{\partial \tau}{\partial x_\mu} = \alpha^2 \frac{\partial \tau}{\partial x^0} \left(-\frac{1}{a^2} \frac{\partial \tau}{\partial x^0} \right) = -\frac{\alpha^2}{a^2}, \tag{2.2.3}$$

which tell us that this vector field is time-like. To evaluate the components of the vector field up to the second order in perturbations we use the constraint

$$n^\mu n_\mu = \alpha^2 g^{\mu\nu} \frac{\partial \tau}{\partial x^\mu} \frac{\partial \tau}{\partial x^\nu} = \alpha^2 g^{00} = -1, \tag{2.2.4}$$

from which we get $\alpha^2 = -\left(g^{00}\right)^{-1}$. Now we simply use the expression for g^{00} at the second order in perturbations written in the first equation of (2.1.15) and we get

$$\alpha = \pm \left[a^{-2} \left(1 - \phi_{(1)} + 4\phi_{(1)}^2 - B_{(1)}^{\,i} B_{(1)\,i} - \phi_{(2)} \right) \right]^{-\frac{1}{2}}, \tag{2.2.5}$$

which we can formally rewrite as $\alpha = \pm\, a(1+x)^{-1/2}$ where x is small because contains inside all the perturbations terms; so expanding in Taylor series we get

$$\alpha = \pm\, a \left[1 + \phi_{(1)} - 2\phi_{(1)}^2 + \frac{1}{2} B_{(1)}^{\,i} B_{(1)\,i} + \frac{1}{2}\phi_{(2)} + \frac{3}{2}\phi_{(1)}^2 \right]. \tag{2.2.6}$$

This means that we can write our vector field as

$$n_\mu = -a \left(1 + \phi_{(1)} - \frac{1}{2}\phi_{(1)}^2 + \frac{1}{2} B_{(1)}^{\,i} B_{(1)\,i} + \frac{1}{2}\phi_{(2)}\,, \vec{0} \right), \tag{2.2.7}$$

where we choose the minus sign in front in order to have the temporal component negative. In the FRW background this vector field coincides with the 4-velocity of matter, while in the perturbed space-time need no longer to coincide with it at any perturbation order. The next step is the calculation of the contravariant vector field:

$$n^\mu = g^{\mu\nu}n_\nu \longrightarrow n^0 = g^{00}n_0 = \frac{1}{a}\left[1 - \phi_{(1)} + \frac{3}{2}\phi_{(1)}^2 - \frac{1}{2}\phi_{(2)} - \frac{1}{2}B_{(1)\,i}B_{(1)}^{\,i}\right],$$
$$n^i = g^{i0}n_0 = -\frac{1}{a}\left[B_{(1)}^{\,i} - \phi_{(1)}B_{(1)}^{\,i} + \frac{1}{2}B_{(2)}^{\,i} - 2B_{(1)\,k}C_{(1)}^{\,ki}\right]. \tag{2.2.8}$$

Observers moving along the hypersurface orthogonal vector field n^μ have a vanishing 3-velocity with respect to the spatial coordinates x^i when the shift vector B^i is zero. We will refer to these as orthogonal coordinate systems; in this case the threading is orthogonal to the slicing.

The covariant derivative of any time-like unit vector field n_μ can be decomposed uniquely as follows [51]:

$$n_{\mu;\nu} = \frac{1}{3}\theta\,\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu n_\nu, \tag{2.2.9}$$

where $\mathcal{P}_{\mu\nu}$ is the spatial projection tensor orthogonal to n^μ given by

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \tag{2.2.10}$$

θ is the overall expansion rate given by

$$\theta = n^\mu_{\ ;\mu}, \tag{2.2.11}$$

$\sigma_{\mu\nu}$ is the (traceless and symmetric) shear

$$\sigma_{\mu\nu} = \frac{1}{2}\mathcal{P}_\mu^{\ \alpha}\mathcal{P}_\nu^{\ \beta}(n_{\alpha;\beta} + n_{\beta;\alpha}) - \frac{1}{3}\theta\,\mathcal{P}_{\mu\nu}, \tag{2.2.12}$$

$\omega_{\mu\nu}$ is the (antisymmetric) vorticity

$$\omega_{\mu\nu} = \frac{1}{2}\mathcal{P}_\mu^{\ \alpha}\mathcal{P}_\nu^{\ \beta}(n_{\alpha;\beta} - n_{\beta;\alpha}), \tag{2.2.13}$$

and a_μ is the acceleration

$$a_\mu = n_{\mu;\nu}n^\nu. \tag{2.2.14}$$

On spatial hypersurfaces the expansion, shear, vorticity, acceleration coincide with their Newtonian counterparts in fluid dynamics [52, 53].

The projection tensor $\mathcal{P}_{\mu\nu}$ is the induced 3-metric on the spatial hypersurfaces, and the Lie derivative, which we denote by $\mathfrak{L}$, of $\mathcal{P}_{\mu\nu}$ along the vector field n^μ is the extrinsic curvature of the hypersurface embedded in the higher dimensional space-time [51, 54]. The extrinsic curvature of the spatial hypersurfaces defined by n_μ is thus given by

$$K_{\mu\nu} \equiv \frac{1}{2}\mathfrak{L}_n\mathcal{P}_{\mu\nu} = \mathcal{P}_\nu^{\ \lambda}n_{\mu;\lambda} = \frac{1}{3}\theta\,\mathcal{P}_{\mu\nu} + \sigma_{\mu\nu}. \tag{2.2.15}$$

At first order we can easily identify the metric perturbations with geometrical perturbations of the spatial hypersurfaces or the associated vector field, n_μ, as shown in [44].

2.3 Energy-momentum tensor for fluids

Einstein equations for General Relativity tells us that the geometry of the space-time and its energy content are strictly related. This implies that a small perturbation in the matter content of the Universe affects the metric tensor and hence the geometry of the space-time. The energy-momentum tensor for a perfect fluid with density ρ, isotropic pressure p and 4-velocity u^μ is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \tag{2.3.1}$$

2.3.1 Single fluids

The background value of a single fluid energy-momentum tensor is necessarily of the perfect fluid form, this means we can write it as

$$T^{(0)}_{\mu\nu} = (\rho_{(0)} + p_{(0)})u^{(0)}_\mu u^{(0)}_\nu + p_{(0)}g^{(0)}_{\mu\nu}, \tag{2.3.2}$$

with $\rho_{(0)} = \rho_{(0)}(\tau)$, $p_{(0)} = p_{(0)}(\tau)$ and $u^{(0)}_i = 0$, because the fluid in the background is at rest[3]. As regards the perturbation, we can identify two different contributions: one which keeps the energy-momentum tensor of the perfect fluid form and another one which adds an anisotropic contribution. Before writing these two contributions to $\delta T_{\mu\nu}$ we focus on the 4-velocity of matter which is defined by

$$u^\mu = \frac{dx^\mu}{d\tau}, \tag{2.3.3}$$

where τ is the proper time comoving with the fluid, and it is subject to the constraint

$$u^\mu u_\mu = -1. \tag{2.3.4}$$

The spatial components of the 4-velocity are

$$u^i = \frac{dx^i}{d\tau} = \frac{a}{a}\frac{dx^i}{d\tau} = \frac{1}{a}\frac{dr^i}{d\tau} = \frac{1}{a}v^i. \tag{2.3.5}$$

Here x^i are the comoving coordinates while r^i are the physical ones. On the background the velocity of the fluid vanishes (as a consequence of isotropy) so $v^i = v^i_{(1)} + \frac{1}{2}v^i_{(2)}$ contains only perturbations, hence

$$u^i = \frac{1}{a}\Big(v^i_{(1)} + \frac{1}{2}v^i_{(2)}\Big). \tag{2.3.6}$$

In order to get the temporal components of u^μ we need to use the constraint (2.3.4)

$$g_{\mu\nu}u^\mu u^\nu = g_{00}u^0u^0 + 2g_{0i}u^0u^i + g_{ij}u^iu^j = -1, \tag{2.3.7}$$

[3] If the background value of the velocity is different from zero it would means that there is a preferred direction in the background which is in contrast with our assumption of isotropy.

which gives the second order equation on the variable u^0

$$g_{00}u^0u^0 + 2g_{0i}u^0u^i + g_{ij}u^iu^j + 1 = 0, \tag{2.3.8}$$

which is solved by

$$u^0 = \frac{-g_{0i}u^i \pm \sqrt{(g_{0i}u^i)^2 - g_{00}(g_{ij}u^iu^j + 1)}}{g_{00}}. \tag{2.3.9}$$

Now we must write each term up to second order in perturbations and we find

$$u^0 = \frac{1}{a}\Big(1 - \phi_{(1)} - \frac{1}{2}\phi_{(2)} + \frac{3}{2}\phi^2_{(1)} + \frac{1}{2}v_{(1)i}v^i_{(1)} + B_{(1)i}v^i_{(1)}\Big). \tag{2.3.10}$$

Lowering the indices with the metric tensor we find

$$\begin{aligned} u_0 &= -a\Big(1 + \phi_{(1)} + \frac{1}{2}\phi_{(2)} - \frac{1}{2}\phi^2_{(1)} + \frac{1}{2}v_{(1)k}v^k_{(1)}\Big), \\ u_i &= a\Big(v_{(1)i} + B_{(1)i} + \frac{1}{2}(v_{(2)i} + B_{(2)i}) - \phi_{(1)}B_{(1)i} + 2C_{(1)ik}v^k_{(1)}\Big). \end{aligned} \tag{2.3.11}$$

As usually the spatial part of the velocity can be split into a scalar part and a vector part

$$v^i = \delta^{ij}v_{,j} + v^i_{vec}. \tag{2.3.12}$$

Note that v^i is the 3-velocity of matter defined considering the spatial coordinates x^i, and so it is not the velocity with respect to the hypersurface orthogonal vector field n^i, except in orthogonal coordinate systems for which $B^i = 0$.

At this point we can go back to writing the perturbations of the energy-momentum tensor. We pointed out that this perturbation can be written as the sum of two contributions; the first one, which preserves the perfect fluid form, can be written as

$$T_{\mu\nu} = T^{(0)}_{\mu\nu} + \delta T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \tag{2.3.13}$$

which differs from (2.3.2) by the fact that now $\rho = \rho_{(0)} + \delta\rho$, $p = p_{(0)} + \delta p$, $g_{\mu\nu} = g^{(0)}_{\mu\nu} + \delta g_{\mu\nu}$ and u_μ has the expression derived in (2.3.11). The other contribution can be written as an anisotropic stress tensor

$$\delta T_{\mu\nu} = \pi_{\mu\nu}. \tag{2.3.14}$$

This anisotropic stress tensor obviously vanishes on the background and it can be split into first and second order parts in the usual way

$$\pi_{\mu\nu} = \pi_{(1)\mu\nu} + \frac{1}{2}\pi_{(2)\mu\nu}, \tag{2.3.15}$$

and it is subject to the constraints

$$\pi_{\mu\nu}u^\nu = 0, \qquad \pi^\mu_{\ \mu} = 0. \tag{2.3.16}$$

The anisotropic stress vanishes for a perfect fluid or minimally coupled scalar fields, while when it is not null it contributes only to the perturbations because its value on the background is zero. The equations (2.3.16) constrain the stress tensor at each perturbation order:

$$\begin{aligned}
\text{order 0:}\quad & \pi_{(0)\mu\nu}u^\nu_{(0)} = 0 \longrightarrow \pi_{(0)\mu\nu} = 0,\\
\text{order 1:}\quad & \pi_{(1)\mu\nu}u^\nu_{(0)} + \pi_{(0)\mu\nu}u^\nu_{(1)} = \pi_{(1)\mu0}u^0_{(0)} = 0 \longrightarrow \pi_{(1)\mu0} = 0,\\
& \pi^i_{(1)i} = -\pi^0_{(1)0} = 0,\\
\text{order 2:}\quad & \pi_{(2)\mu\nu}u^\nu_{(0)} + \pi_{(1)\mu\nu}u^\nu_{(1)} + \pi_{(0)\mu\nu}u^\nu_{(2)} = \pi_{(2)\mu0}u^0_{(0)} + \pi_{(1)\mu i}u^i_{(1)} =\\
& = \pi_{(1)0i}v^i_{(1)} + \frac{1}{2}\pi_{(2)00} = 0 \longrightarrow \pi_{(2)00} = 0,\\
& \pi^i_{(2)i} = -\pi^0_{(2)0} = 0.
\end{aligned} \tag{2.3.17}$$

The second of equations (2.3.16) guarantees that the anisotropic stress tensor is traceless. In the same way as we did with the perturbation of the metric in Section 2.1, we can decompose the anisotropic stress tensor into a traceless scalar part Π, a vector part Π_i and a tensor part Π_{ij}, at each order according to [47]

$$\pi_{ij} = a^2\Big[\Pi_{,ij} - \frac{1}{3}\nabla^2\Pi\delta_{ij} + \frac{1}{2}\big(\Pi_{i,j} + \Pi_{j,i}\big) + \Pi_{ij}\Big]. \tag{2.3.18}$$

In conclusion the energy-momentum tensor for a single fluid can be written as [53, 55, 56]

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} + \pi_{\mu\nu}. \tag{2.3.19}$$

We follow [55] in defining the proper energy density as the eigenvalue of the energy-momentum tensor and the 4-velocity u^μ as the corresponding eigenvector

$$T^\mu_\nu u^\nu = -\rho u^\mu. \tag{2.3.20}$$

The components of the energy-momentum tensor on the background are

$$\begin{aligned}
T_{(0)\ 0}^{\ \ 0} &= -\rho_{(0)},\\
T_{(0)\ i}^{\ \ 0} &= 0,\\
T_{(0)\ j}^{\ \ i} &= \delta^i_{\ j}p_{(0)},
\end{aligned} \tag{2.3.21}$$

while at first order we have

$$\begin{aligned}
\delta T_{(1)\ 0}^{\ \ 0} &= -\delta\rho_{(1)},\\
\delta T_{(1)\ i}^{\ \ 0} &= \Big(\rho_{(0)} + p_{(0)}\Big)\Big(v_{(1)i} + B_{(1)i}\Big),\\
\delta T_{(1)\ j}^{\ \ i} &= \delta p_{(1)}\delta^i_{\ j} + a^{-2}\pi^i_{(1)j},
\end{aligned} \tag{2.3.22}$$

which we obtained from (2.3.19) considering first order perturbations at most. If we also take into account the second order perturbations we find:

$$
\begin{aligned}
\delta T^{\ 0}_{(2)0} &= -\delta\rho_{(2)} - 2\Big(\rho_{(0)} + p_{(0)}\Big)v_{(1)k}\Big(v^k_{(1)} + B^k_{(1)}\Big), \\
\delta T^{\ 0}_{(2)i} &= (\rho_{(0)} + p_{(0)})\Big[v_{(2)i} + B_{(2)i} + 4C_{(1)ik}v^k_{(1)} - 2\phi_{(1)}\Big(v_{(1)i} + 2B_{(1)i}\Big)\Big] + \\
&\qquad + 2\Big(\delta\rho_{(1)} + \delta p_{(1)}\Big)\Big(v_{(1)i} + B_{(1)i}\Big) + \frac{2}{a^2}\Big(B^k_{(1)} + v^k_{(1)}\Big)\pi_{(1)ik}, \\
\delta T^{\ i}_{(2)j} &= \delta p_{(2)}\delta^i_{\ j} + \frac{1}{a^2}\pi^{\ i}_{(2)j} - \frac{4}{a^2}C^{ik}_{(1)}\pi_{(1)jk} + 2\big(\rho_{(0)} + p_{(0)}\big)v^i_{(1)}\big(v_{(1)j} + B_{(1)j}\big).
\end{aligned}
\tag{2.3.23}
$$

Note that for simplicity of presentation we have not split perturbations into their constituent scalar, vector and tensor parts in the above expressions.

We will see that quantities like the density, pressure and 3-velocity are gauge-dependent and this implies that they change along with the choice of the gauge. On the contrary it is possible to show [47] that the anisotropic stress is gauge-invariant at first order but becomes gauge-dependent at second order.

2.3.2 Multiple fluids

The cosmological fluid consists of many components (photons, baryons, neutrinos, ...) so it is necessary to consider a energy-momentum tensor for multiple fluids. In this case the total energy-momentum tensor is the sum of the energy-momentum tensor of the individual fluids, labelled by the index α

$$T^{\mu\nu} = \sum_\alpha T^{\mu\nu}_{(\alpha)}. \tag{2.3.24}$$

The density and the pressure of the total fluid are related to the single components ones by

$$\rho = \sum_\alpha \rho_{(\alpha)}, \tag{2.3.25}$$

$$p = \sum_\alpha p_{(\alpha)}. \tag{2.3.26}$$

For each of the fluid we can define the local energy-momentum transfer 4-vector $Q^\nu_{(\alpha)}$ through the relation

$$\nabla_\mu T^{\mu\nu}_{(\alpha)} = Q^\nu_{(\alpha)}, \tag{2.3.27}$$

where ∇_μ is the covariant derivative. From the above relation we see that the local energy-momentum tensor, $T^{\mu\nu}_{(\alpha)}$, is locally conserved only for non-interacting fluids, for which $Q^\nu_{(\alpha)} = 0$. The fact that the energy-momentum tensor describing the matter content of the Universe must be covariantly conserved together with equation (2.3.27) implies that

$$\sum_\alpha Q^\nu_{(\alpha)} = 0. \tag{2.3.28}$$

Following [55, 57] we split the energy-momentum transfer 4-vector using the total fluid velocity u^μ as

$$Q^\mu_{(\alpha)} = Q_{(\alpha)} u^\mu + f^\mu_{(\alpha)}, \tag{2.3.29}$$

where $Q_{(\alpha)}$ is the energy transfer rate and $f^\mu_{(\alpha)}$ the momentum transfer rate, subject to the constraint

$$u_\mu f^\mu_{(\alpha)} = 0. \tag{2.3.30}$$

Writing this constraint at various perturbations order we find that

$$f^0_{(1)(\alpha)} = 0, \qquad f^0_{(2)(\alpha)} = 2 f^k_{(1)(\alpha)} \Big(v_{(1)k} + B_{(1)k} \Big). \tag{2.3.31}$$

We then find the temporal components of the energy transfer 4-vector to be

$$\begin{aligned}
Q^0_{(0)(\alpha)} &= \frac{1}{a} Q_{(0)\alpha}, \\
Q^0_{(1)(\alpha)} &= \frac{1}{a} \Big(\delta Q_{(1)\alpha} - \phi_{(1)} Q_{(0)\alpha} \Big), \\
Q^0_{(2)(\alpha)} &= \frac{1}{2a} \Bigg[\delta Q_{(2)\alpha} + Q_{(0)\alpha} \Big(3\phi^2_{(1)} - \phi_{(2)} \Big) - 2\phi_{(1)} \delta Q_{(1)\alpha} + \\
&\quad + \Big(v_{(1)k} + B_{(1)k} \Big) \Big(\frac{2}{a} f^k_{(1)(\alpha)} + Q_{(0)\alpha} v^k_{(1)} \Big) \Bigg],
\end{aligned} \tag{2.3.32}$$

where $Q_{(0)\alpha}$, $\delta Q_{(1)\alpha}$ and $\delta Q_{(2)\alpha}$ are the energy transfer to the α-fluid in the background, respectively at first and at second order. For the spatial components of the energy transfer 4-vector, the momentum part, we get at first and second order

$$\begin{aligned}
Q^i_{(1)(\alpha)} &= \frac{1}{a} Q_{(0)\alpha} v^i_{(1)} + \frac{1}{a^2} f^i_{(1)(\alpha)}, \\
Q^i_{(2)(\alpha)} &= \frac{1}{2a} \Bigg[\frac{1}{a} f^i_{(2)(\alpha)} + \delta Q_{(1)\alpha} v^i_{(1)} + Q_{(0)\alpha} \Big(v^i_{(2)} + 2\phi_{(1)} B^i_{(1)} - 4 C^i_{(1)k} v^k_{(1)} \Big) \Bigg],
\end{aligned} \tag{2.3.33}$$

where $f^i_{(1)(\alpha)}$ and $f^i_{(2)(\alpha)}$ are the spatial parts of the momentum transfer rate at first and second order.

Note that the homogeneous and isotropic FRW background excludes a zeroth order momentum transfer. The spatial momentum transfer vector of order n can be further decomposed into a scalar and a vector part

$$f^i_{(n)(\alpha)} = \delta^{ij} f_{(n)(\alpha)j} + \hat{f}^i_{(n)(\alpha)}. \tag{2.3.34}$$

2.4 Energy-momentum tensor for scalar fields

2.4.1 Single field

A minimally coupled scalar field is specified by the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi), \tag{2.4.1}$$

where the minus sign in front of the kinetic term is necessary for the scalar field in order to have a positive kinetic energy for our choice of the metric signature.

The energy-momentum tensor is defined as

$$T_{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}, \tag{2.4.2}$$

which for our scalar field φ becomes

$$T^\mu_\nu = g^{\mu\alpha} \partial_\alpha \varphi \partial_\nu \varphi - \delta^\mu_\nu \left(\frac{1}{2} g^{\beta\lambda} \partial_\beta \varphi \partial_\lambda \varphi + V(\varphi) \right). \tag{2.4.3}$$

Comparing (2.4.3) to the energy-momentum tensor of a perfect fluid (2.3.19) we can identify the non-linear 4-velocity, density, and pressure of the scalar field as in [58]

$$\begin{aligned} u_\mu &= \frac{\partial_\mu \varphi}{|g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi|}, \\ \rho &= -g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi + V, \\ p &= -g^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi - V. \end{aligned} \tag{2.4.4}$$

Note that the anisotropic stress $\pi_{\mu\nu}$ is identically zero for minimally coupled scalar fields.

Splitting the scalar field into a homogeneous background field and a perturbation

$$\varphi(\tau, x^i) = \varphi_{(0)}(\tau) + \delta\varphi(\tau, x^i), \tag{2.4.5}$$

and using the definitions above we find for the components of the energy-momentum tensor of a perturbed scalar field at linear order

$$\begin{aligned} T^0_{\ 0} &= -\frac{1}{2} a^{-2} \varphi'^{\,2}_{(0)} - V_{(0)} + a^{-2} \varphi'_{(0)} \Big(\phi_{(1)} \varphi'_{(0)} - \delta\phi'_{(1)} \Big) - \frac{\partial V}{\partial \varphi} \delta\varphi_{(1)}, \\ T^0_{\ i} &= -a^{-2} \Big(\varphi'_{(0)} \partial_i \delta\varphi_{(1)} \Big), \\ T^i_{\ j} &= \left[\frac{1}{2} a^{-2} \varphi'^{\,2}_{(0)} - V_{(0)} - \frac{\partial V}{\partial \varphi} \delta\varphi_{(1)} + a^{-2} \varphi'_{(0)} \Big(\delta\varphi'_{(1)} - \phi_{(1)} \varphi'_{(0)} \Big) \right] \delta^i{}_j, \end{aligned} \tag{2.4.6}$$

where $V_{(0)} = V(\varphi_{(0)})$ and the prime denotes the derivative with respect to the conformal time τ.

2.4.2 Multiple fields

For N minimally coupled scalar fields, labelled by the index I, the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{2}\sum_I (g^{\mu\nu}\partial_\mu\varphi_I\partial_\nu\varphi_I) - V(\varphi_1, \ldots, \varphi_N). \tag{2.4.7}$$

The energy-momentum tensor is

$$T_{\mu\nu} = \sum_I \left[\partial_\mu\varphi_I\partial_\nu\varphi_I - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi_I\partial_\beta\varphi_I\right] - g_{\mu\nu}V. \tag{2.4.8}$$

Similarly to the energy-momentum tensor for a single field, we can identify the non linear 4-velocity, density and pressure of each one of the scalar fields as in [57]

$$\begin{aligned} u_{(I)\mu} &= \frac{\partial_\mu\varphi_I}{|g^{\alpha\beta}\partial_\alpha\varphi_I\partial_\beta\varphi_I|}, \\ \rho_{(I)} &= -g^{\alpha\beta}\partial_\alpha\varphi_I\partial_\beta\varphi_I, \\ p_{(I)} &= g^{\alpha\beta}\partial_\alpha\varphi_I\partial_\beta\varphi_I. \end{aligned} \tag{2.4.9}$$

Again we can split the scalar fields φ_I into a background and perturbations

$$\varphi_I(\tau, x^i) = \varphi_{(0)I}(\tau) + \delta\varphi_{(1)I}(\tau, x^i) + \ldots \tag{2.4.10}$$

and similarly the potential

$$V(\varphi_I) = V(\varphi_{(0)I}) + \frac{\partial V}{\partial\varphi_I}(\varphi_{(0)I})\delta\varphi_{(1)I} + \ldots \tag{2.4.11}$$

2.5 Gauge transformations

A problem which arises in cosmological perturbation theory is the presence of spurious coordinate artefacts or gauge modes in the calculation. The gauge issue was resolved in a systematic way by Bardeen in [27]. The gauge issue arises in any approach to General Relativity that splits quantities into a background and a perturbation. In fact, although General Relativity is a covariant theory, i. e. manifestly independent by the coordinate choice, splitting variables into a background part and a perturbation is not a covariant procedure and therefore introduces a coordinate or gauge dependence. By construction this only affects the perturbations, the background quantities remain the same in the different coordinate systems.

We know from the study of General Relativity that solutions of the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi\, GT_{\mu\nu} \tag{2.5.1}$$

are invariant under diffeomorphism (the gauge transformation of General Relativity). Consequently, if $g_{\mu\nu}$ is a solution for a particular choice of $T_{\mu\nu}$, acting with a diffeomorphism we find $\tilde{g}_{\mu\nu}$ which is a solution for $\tilde{T}_{\mu\nu}$. The mathematical relation between $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ is

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\rho}{\partial \tilde{x}^\mu}\frac{\partial x^\sigma}{\partial \tilde{x}^\nu} g_{\rho\sigma}(x). \tag{2.5.2}$$

Now we consider an infinitesimal coordinate transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu - \xi^\mu, \tag{2.5.3}$$

described by four functions ξ^μ of space and time. For this infinitesimal transformation we can rewrite the left term of (2.5.2) as

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{g}_{\mu\nu}\big(x - \xi(x)\big) = \tilde{g}_{\mu\nu}(x) - \frac{\partial g_{\mu\nu}}{\partial \xi^\lambda}(x)\xi^\lambda + \mathcal{O}(\xi^2). \tag{2.5.4}$$

Using again (2.5.3) we can also rewrite the right part of (2.5.2):

$$\frac{\partial \tilde{x}^\mu}{\partial x^\rho} = \delta^\mu_{\ \rho} - \frac{\partial \xi^\mu}{\partial x^\rho} \longrightarrow \frac{\partial x^\rho}{\partial \tilde{x}^\mu} = \delta^\rho_{\ \mu} + \frac{\partial \xi^\rho}{\partial x^\mu} + \mathcal{O}(\xi^2), \tag{2.5.5}$$

hence

$$\begin{aligned}\tilde{g}_{\mu\nu}(\tilde{x}) &= \Big(\delta^\rho_{\ \mu} + \frac{\partial \xi^\rho}{\partial x^\mu}(x)\Big)\Big(\delta^\sigma_{\ \nu} - \frac{\partial \xi^\sigma}{\partial x^\nu}(x)\Big) g_{\rho\sigma}(x) \\ &= g_{\mu\nu}(x) + \frac{\partial \xi^\rho}{\partial x^\mu}(x) g_{\rho\nu}(x) + \frac{\partial \xi^\sigma}{\partial x^\nu}(x) g_{\mu\sigma}(x) + \mathcal{O}(\xi^2).\end{aligned} \tag{2.5.6}$$

Putting together the equations (2.5.4) and (2.5.6) we find

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \frac{\partial \xi^\rho}{\partial x^\mu}(x) g_{\rho\nu}(x) + \frac{\partial \xi^\sigma}{\partial x^\nu}(x) g_{\mu\sigma}(x) + \frac{\partial g_{\mu\nu}}{\partial \xi^\lambda}(x)\xi^\lambda + \mathcal{O}(\xi^2) \tag{2.5.7}$$

which is the expansion of the Lie derivative along the vector ξ^μ acting on $g_{\mu\nu}(x)$:

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) + \mathfrak{L}_\xi g_{\mu\nu}(x). \tag{2.5.8}$$

This tells us that the Lie dragging relates the metric tensor evaluated in the coordinate point x^μ with the transformed metric tensor under a diffeomorphism evaluated in the same coordinate point. Actually the relation (2.5.8) holds only at first order in ξ, however it can be generalized [59]: if we take the function T (which can be a scalar, vector or tensor) and taking into account also the terms $\mathcal{O}(\xi^2)$ we get:

$$\begin{aligned}\tilde{T}(x) &= e^{\mathfrak{L}_\xi} T(x) \\ &= T(x) + \mathfrak{L}_\xi T(x) + \frac{1}{2}\mathfrak{L}_\xi^2 T(x) + \dots\end{aligned} \tag{2.5.9}$$

Since background quantities are not affected by gauge transformations we can easily write the relation between perturbations in different gauges up to second order from (2.5.9)

$$\begin{aligned} \delta\tilde{T}(x) &= \tilde{T}(x) - T_{(0)}(x) = T(x) + \mathfrak{L}_\xi T(x) + \frac{1}{2}\mathfrak{L}_\xi^2 T(x) - T_{(0)}(x) \\ &= \delta T(x) + \mathfrak{L}_\xi T(x) + \frac{1}{2}\mathfrak{L}_\xi^2 T(x). \end{aligned} \tag{2.5.10}$$

There are two mathematically equivalent approaches to the problem: the passive and active methods

- **Active**: we study how perturbations change under mapping, where the map directly induces the transformation on the perturbed quantities. First we fix the coordinates on the background manifold $\mathcal{M}_0$, we call them for example x_b^μ where the b stands for background. Any diffeomorphism $\mathcal{D} : \mathcal{M}_0 \to \mathcal{M}_{phys}$ induces a system of coordinates on the physical manifold $\mathcal{M}_{phys}$ via $\mathcal{D} : x_b^\mu \to x^\mu$. For a given diffeomorphism $\mathcal{D}$ we define the perturbation δT of the generic function T (scalar, vector or tensor) defined on $\mathcal{M}_{phys}$ as

$$\delta T(p) = T(p) - T_{(0)}\big(\mathcal{D}^{-1}(p)\big), \tag{2.5.11}$$

where $T_{(0)}$ lives on the background. A second diffeomorphism $\tilde{\mathcal{D}}$ induces a new set of coordinates $\tilde{x}^\mu$ on $\mathcal{M}_{phys}$ via $\tilde{\mathcal{D}} : x_b^\mu \to \tilde{x}^\mu$ and a different $\widetilde{\delta T}$:

$$\widetilde{\delta T}(p) = \tilde{T}(p) - T_{(0)}\big(\tilde{\mathcal{D}}^{-1}(p)\big), \tag{2.5.12}$$

where $\tilde{T}$ is the value of T in the $\tilde{x}^\mu$ coordinates. In this approach, the gauge transformation $\delta T(p) \to \widetilde{\delta T}(p)$ is generated by the change of correspondence $\mathcal{D} \to \tilde{\mathcal{D}}$ between the manifolds $\mathcal{M}_0$ and $\mathcal{M}_{phys}$. We can associate to this change in the correspondence the change of coordinates $x^\mu \to \tilde{x}^\mu$ induced on $\mathcal{M}_{phys}$. We can think of the gauge transformation as a one to one correspondence between different points on the background. In fact $\mathcal{D}$ sends a background point b_1 to a point in the physical manifold, for example q: $\mathcal{D}(b_1) = q$. As regards $\tilde{\mathcal{D}}$, q won't be the image of b_1 but rather of another point in the background, for example b_2. So we can write

$$\mathcal{D}(b_1) = q = \tilde{\mathcal{D}}(b_2), \tag{2.5.13}$$

which can be rewritten as

$$b_1 = \mathcal{D}^{-1}\big(\tilde{\mathcal{D}}(b_2)\big) = \mathfrak{D}(b_2). \tag{2.5.14}$$

The map $\mathfrak{D}$ takes each point of $\mathcal{M}_0$ in a fixed coordinates system and sends it to another point in that coordinate system as shown in the figure below.

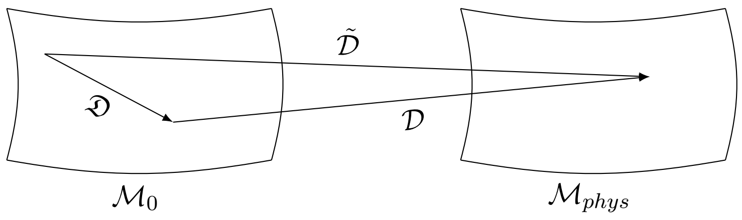

Figure 2.2: gauge transformation.

The starting point in this approach is the exponential map (2.5.9) that allows us to immediately write down how a function T transforms up to second order. The vector field generating the transformation, ξ^μ is up to second order

$$\xi^\mu = \xi^\mu_{(1)} + \frac{1}{2}\xi^\mu_{(2)}, \tag{2.5.15}$$

so the exponential map can be expanded up to second order as

$$e^{\mathfrak{L}_\xi} = 1 + \mathfrak{L}_{\xi_{(1)}} + \frac{1}{2}\mathfrak{L}^2_{\xi_{(1)}} + \frac{1}{2}\mathfrak{L}_{\xi_{(2)}}. \tag{2.5.16}$$

From equation (2.5.16) we get that tensorial quantities transform as

$$\begin{aligned}
\tilde{T}_{(0)} &= T_{(0)}, \\
\tilde{\delta T}_{(1)} &= \delta T_{(1)} + \mathfrak{L}_{\xi_{(1)}} T_{(0)}, \\
\tilde{\delta T}_{(2)} &= \delta T_{(2)} + \mathfrak{L}_{\xi_{(2)}} T_{(0)} + \mathfrak{L}^2_{\xi_{(1)}} T_{(0)} + 2\mathfrak{L}_{\xi_{(1)}} \delta T_{(1)}.
\end{aligned} \tag{2.5.17}$$

- **Passive**: we specify the relation between two coordinate systems directly and then calculate the change in the metric and matter variables when changing from one system to the other. First of all we choose some system of coordinates x^μ on the physical space-time manifold $\mathcal{M}_{phys}$. The background is defined by assigning to all functions T on $\mathcal{M}_{phys}$ a background value $T_{(0)}(x^\mu)$ which is a fixed function of the coordinates. Therefore in a second coordinate system $\tilde{x}^\mu$ the background function $T_{(0)}(\tilde{x}^\mu)$ will have exactly the same functional dependence on $\tilde{x}^\mu$. The perturbation δT in the system of coordinates x^μ is defined as

$$\delta T(p) = T\big(x^\mu(p)\big) - T_{(0)}\big(x^\mu(p)\big). \tag{2.5.18}$$

Similarly, in the second system of coordinates, the perturbation of T is

$$\tilde{\delta T}(p) = \tilde{T}\big(\tilde{x}^\mu(p)\big) - T_{(0)}\big(\tilde{x}^\mu(p)\big). \tag{2.5.19}$$

Here $\tilde{T}\big(\tilde{x}^\mu(p)\big)$ is the value of T in the new coordinate system at the same point p of $\mathcal{M}_{phys}$. The transformation $\delta T(p) \to \tilde{\delta T}(p)$ is called the gauge transformation associated with the change of variables $x^\mu \to$

$\tilde{x}^\mu$ on the manifold $\mathcal{M}_{phys}$. Applying the exponential map (2.5.9) to the functions x^μ, coordinates of the physical point q, we get the relation between the old coordinate system x^μ and the new one $\tilde{x}^\mu$ [49]

$$\begin{aligned}\tilde{x}^\mu(q) &= e^{\xi_\lambda \frac{\partial}{\partial x^\lambda}\big|_q} x^\mu(q) \\ &= x^\mu(q) - \xi^\mu_{(1)}(q) + \frac{1}{2}\Big(\big(\partial_\nu \xi^\mu_{(1)}(q)\big)\xi^\nu_{(1)}(q) - \xi^\mu_{(2)}(q)\Big). \end{aligned} \tag{2.5.20}$$

Now we consider a quantity, like the total density ρ that is a scalar under diffeomorphism which means it remains the same under a change of coordinate system

$$\tilde{\rho}(\tilde{x}^\mu) = \rho(x^\mu). \tag{2.5.21}$$

Because we are interested in the transformations of the perturbations we split the density as usual $\rho = \rho_{(0)} + \delta\rho$. Now expanding both sides of equation (2.5.21) up to first order in perturbations we find

$$\begin{aligned}\rho(x^\mu) &= \rho_{(0)}(x^0) + \delta\rho_{(1)}(x^\mu) \\ \tilde{\rho}(\tilde{x}^\mu) &= \rho_{(0)}(\tilde{x}^0) + \tilde{\delta\rho}_{(1)}(\tilde{x}^\mu)\end{aligned} \tag{2.5.22}$$

and using equation (2.5.20) to write $\tilde{x}^\mu$ in function of x^μ we get

$$\tilde{\rho}(\tilde{x}^\mu) = \rho_{(0)}(x^0) - \rho'_{(0)}(x^0)\xi^0_{(1)}(x^\mu) + \tilde{\delta\rho}_{(1)}(x^\mu). \tag{2.5.23}$$

Thus we obtain the transformation rule at first order

$$\tilde{\delta\rho}_{(1)} = \delta\rho_{(1)} + \rho'_{(0)}\xi^0_{(1)}. \tag{2.5.24}$$

Another important invariant is the line element ds^2 which allows us to deduce the transformation properties of the metric tensor:

$$ds^2 = \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.5.25}$$

Both approaches, the active and the passive one, are equivalent. From the point of view of physics, the active allows to understand how the amplitudes of the perturbations depend on the correspondence between background manifold $\mathcal{M}_0$ and physical manifold $\mathcal{M}_{phys}$. Instead the passive approach allows to connect the gauge transformation with the choice of the system of coordinates on $\mathcal{M}_{phys}$ in which the perturbations are described. However we decide to follow the active approach, instead for further developing of the passive approach see for example [31, 60].

2.5.1 Scalar perturbations

First of all we split the generating vector ξ^μ into a scalar temporal part $\alpha_{(1)}$ and a spatial scalar and vector part, $\beta_{(1)}$ and $\gamma^i_{(1)}$, according to

$$\xi^\mu_{(1)} = \Big(\alpha_{(1)}, \beta_{(1),}{}^{i} + \gamma^i_{(1)}\Big), \tag{2.5.26}$$

where the vector part is divergence free, $\partial_k \gamma^k_{(1)} = 0$. Now we consider a four scalar, like the energy density $\rho = \rho_{(0)} + \delta\rho_{(1)} + \frac{1}{2}\delta\rho_{(2)}$. We expect to find again the relation found in the passive approach (2.5.24), in fact both the approaches are equivalent. In the active approach the transformation for a quantity like $\delta\rho$ is given by the second equation of (2.5.17) where the Lie derivative is equal to [53]

$$\mathfrak{L}_\xi \rho = \xi^\lambda \partial_\lambda \rho. \tag{2.5.27}$$

The result is

$$\tilde{\delta\rho}_{(1)} = \delta\rho_{(1)} + \rho'_{(0)}\alpha_{(1)}, \tag{2.5.28}$$

which is exactly what we found in the passive approach. We see that the first order density perturbation is fully specified by prescribing the first order temporal gauge $\alpha_{(1)}$.

At second order we do the same: firstly we write the generating vector $\xi^\mu_{(2)}$ as

$$\xi^\mu_{(2)} = \left(\alpha_{(2)}, \beta_{(2),}{}^i + \gamma^i_{(2)}\right), \tag{2.5.29}$$

where the vector part is divergence free, $\partial_k \gamma^k_{(2)} = 0$. Then we take the third equation of (2.5.17) and using the expression for the Lie derivative above we get

$$\begin{aligned}\tilde{\delta\rho}_{(2)} = \delta\rho_{(2)} + \rho'_{(0)}\alpha_{(2)} + \alpha_{(1)}\left(\rho''_{(0)}\alpha_{(1)} + \rho'_{(0)}\alpha'_{(1)} + 2\delta\rho'_{(1)}\right) + \\ + \left(2\delta\rho_{(1)} + \rho'_{(0)}\alpha_{(1)}\right)_{,k}\left(\beta_{(1),}{}^k + \gamma^k_{(1)}\right).\end{aligned} \tag{2.5.30}$$

This time vector-like terms appear: $\gamma^k_{(1)}$ and the gradient $\beta_{(1),}{}^k$. At second order scalar perturbations are coupled to vectors. The gauge is specified only once $\alpha_{(1)}$, $\alpha_{(2)}$, $\beta_{(1)}$ and $\gamma^i_{(1)}$ are specified.

2.5.2 Vector perturbations

This time we use again the splitting defined before for the generator of gauge transformations ξ^μ and the second equation of (2.5.17), but in this case the Lie derivative is equal to [53]

$$\mathfrak{L}_\xi V_\mu = V_{\mu,\alpha}\xi^\alpha + V_\alpha \xi^\alpha_{\ ,\mu}. \tag{2.5.31}$$

Hence the vector perturbations transform at first order under a gauge transformation as

$$\delta\tilde{V}_{(1)\mu} = \delta V_{(1)\mu} + V'_{(0)\mu}\alpha_{(1)} + V_{(0)\lambda}\xi^\lambda_{(1),\mu}, \tag{2.5.32}$$

where we used the fact that on the background $V_{(0)\mu} = V_{(0)\mu}(\tau)$ and $V_{(0)i} = 0$. For the specific example of the 4-velocity, defined in (2.3.11), we find

$$\tilde{v}_{(1)i} + \tilde{B}_{(1)i} = v_{(1)i} + B_{(1)i} - \alpha_{(1),i}. \tag{2.5.33}$$

As we have done for the vector perturbation of the metric B_i in (2.1.11) we can decompose v_i in its scalar and vector part and hence, using the transformation relation for B_i which we will derive below in (2.5.42), we can divide the above equation (2.5.33) into

$$\begin{aligned}\tilde{v}_{(1)} &= v_{(1)} - \beta'_{(1)}, \\ \tilde{v}^i_{vec(1)} &= v^i_{vec(1)} - \gamma'^i_{(1)}.\end{aligned} \tag{2.5.34}$$

The next step is to find the relations for the vector perturbations in two different gauges at second order. This time we need the third equation of (2.5.17) and (2.5.29), the result is:

$$\begin{aligned}\delta\tilde{V}_{(2)\mu} = {} & \delta V_{(2)\mu} + V'_{(0)\mu}\alpha_{(2)} + V_{(0)0}\alpha_{(2),\mu} + V''_{(0)\mu}\alpha^2_{(1)} + V'_{(0)\mu}\alpha_{(1),\lambda}\xi^\lambda_{(1)} + \\ & + 2V'_{(0)0}\alpha_{(1)}\alpha_{(1),\mu} + V_{(0)0}\left(\xi^\lambda_{(1)}\alpha_{(1),\mu\lambda} + \alpha_{(1),\lambda}\xi^\lambda_{(1),\mu}\right) + \\ & + 2\left(\delta V_{(1)\mu,\lambda}\xi^\lambda_{(1)} + \delta V_{(1)\lambda}\xi^\lambda_{(1),\mu}\right).\end{aligned} \tag{2.5.35}$$

Focusing again on the 4-velocity and following a similar procedure as at first order, we find that the second order

$$\tilde{v}_{(2)i} = v_{(2)i} - \xi'_{(2)i} + \chi_i, \tag{2.5.36}$$

where χ_i contains the terms quadratic in the first order perturbations and it is given by

$$\begin{aligned}\chi_{ij} = {} & \xi'_{(1)i}\left(2\phi_{(1)} + \alpha'_{(1)} + 2\mathcal{H}\alpha_{(1)}\right) - \alpha_{(1)}\xi''_{(1)i} - \xi^k_{(1)}\xi'_{(1)i,k} + \xi'^k_{(1)}\xi_{(1)i,k} + \\ & - 2\alpha_{(1)}\left(v'_{(1)i} + \mathcal{H}v_{(1)i}\right) + 2v_{(1)i,k}\xi^k_{(1)} - 2v^k_{(1)}\xi_{(1)i,k},\end{aligned} \tag{2.5.37}$$

where $\mathcal{H} = aH = a'/a$ and we used the transformation relation of the metric perturbation $B_{(2)i}$ that we will write explicitly later.

2.5.3 Tensor perturbations

Now we can calculate how the first order metric perturbations change under a gauge transformation. δg_{00} is a 4-scalar so its transformation relation can be obtained dealing only with scalars. Once we know that the Lie derivative acts on a tensor like the metric as

$$\mathfrak{L}_\xi g_{\mu\nu} = g_{\mu\nu,\lambda}\xi^\lambda + g_{\mu\lambda}\xi^\lambda_{,\nu} + g_{\lambda\nu}\xi^\lambda_{,\mu}, \tag{2.5.38}$$

it is easy to verify using (2.5.26), (2.1.14) and the second of (2.5.17) that

$$\begin{aligned}\tilde{\delta g}^{(1)}_{00} &= \delta g^{(1)}_{00} + \delta g^{(0)}_{00,0}\xi^0_{(1)} + 2\delta g^{(0)}_{00}\xi^0_{(1),0} \\ &= \delta g^{(1)}_{00} - 2a^3H\alpha_{(1)} - 2a^2\alpha'_{(1)} \\ &= \delta g^{(1)}_{00} - 2a^2\mathcal{H}\alpha_{(1)} - 2a^2\alpha'_{(1)}.\end{aligned} \tag{2.5.39}$$

Actually $\delta g^{(1)}_{00} = -2a^2\phi_{(1)}$, thus from the equation above we can directly read how the scalar perturbation ϕ transforms under a gauge transformation:

$$\tilde{\phi}_{(1)} = \phi_{(1)} + \mathcal{H}\alpha_{(1)} + \alpha'_{(1)} \tag{2.5.40}$$

The next step is to find the transformation law for the $0i$ part of the metric. The change of the component δg_{0i} is slightly more involved, since this component contains scalar and vector perturbations. We therefore have to compute the overall transformation of this metric component using (2.5.38) and then split the result in the various components.

$$\begin{aligned}\tilde{\delta g}^{(1)}_{0i} &= \delta g^{(1)}_{0i} + \delta g^{(0)}_{00}\xi^0_{(1),i} + \delta g^{(0)}_{ki}\xi^k_{(1),0} \\ &= \delta g^{(1)}_{0i} + a^2\delta_{ij}\left(\beta'^{\,j}_{(1),} + \gamma'^{j}_{(1)}\right) - a^2\alpha_{(1),i},\end{aligned} \tag{2.5.41}$$

and hence, from $\delta g^{(1)}_{0i} = a^2 B_{(1)i}$, we get

$$\tilde{B}_{(1)i} = B_{(1)i} + \beta'_{(1),i} + \gamma'_{(1)i} - \alpha_{(1),i}. \tag{2.5.42}$$

But $B_{(1)i}$ consists of a vector divergence-free part $S_{(1)i}$ and a scalar part $B_{(1)}$, explicitly $B_{(1)i} = B_{(1),i} - S_{(1)i}$. The transformation law (2.5.42) only tells us how the whole vector $B_{(1)i}$ transforms and so to find the transformation rules for its components we firstly take the divergence of (2.5.42), so that it remains only the scalar part $B_{(1)}$

$$\nabla^2\tilde{B}_{(1)} = \nabla^2 B_{(1)} + \nabla^2\beta'_{(1)} - \nabla^2\alpha_{(1)}, \tag{2.5.43}$$

which gives

$$\tilde{B}_{(1)} = B_{(1)} + \beta'_{(1)} - \alpha_{(1)}. \tag{2.5.44}$$

Then the vector part transformation rule can be consequently found by subtracting the overall transformation rule (2.5.42) from the scalar part transformation (2.5.44):

$$\tilde{S}_{(1)i} = \tilde{B}_{(1),i} - \tilde{B}_{(1)i} = S_{(1)i} - \gamma'_{(1)i}. \tag{2.5.45}$$

The remaining terms of the metric are those with spatial indices only:

$$\begin{aligned}\tilde{\delta g}^{(1)}_{ij} &= \delta g^{(1)}_{ij} + \delta g^{(0)}_{ij,\lambda}\xi^\lambda_{(1)} + \delta g^{(0)}_{ik}\xi^k_{(1),j} + \delta g^{(0)}_{kj}\xi^k_{(1),i} \\ &= \delta g^{(1)}_{ij} + 2aa'\alpha_{(1)}\delta_{ij} + a^2\left(\xi_{(1)i,j} + \xi_{(1)j,i}\right) \\ &= \delta g^{(1)}_{ij} + 2a^2\mathcal{H}\alpha_{(1)}\delta_{ij} + a^2\left(\xi_{(1)i,j} + \xi_{(1)j,i}\right).\end{aligned} \tag{2.5.46}$$

The third equation of (2.1.14) tells us that $\delta g_{(1)ij} = 2a^2 C_{(1)ij}$ so the transformation relation becomes

$$2\tilde{C}_{(1)ij} = 2C_{(1)ij} + 2\mathcal{H}\alpha_{(1)}\delta_{ij} + \xi_{(1)i,j} + \xi_{(1)j,i}, \tag{2.5.47}$$

with $C_{(1)ij}$ given by (2.1.12). If we take the trace of (2.5.47) then we get

$$-3\tilde{\psi}_{(1)} + \nabla^2\tilde{E}_{(1)} = -3\psi_{(1)} + \nabla^2 E_{(1)} + 3\mathcal{H}\alpha_{(1)} + \nabla^2\beta_{(1)}. \qquad (2.5.48)$$

On the other hand the divergence is

$$\tilde{C}_{(1)ij}{}^{,j} = C_{(1)ij}{}^{,j} + \mathcal{H}\alpha_{(1),i} + \frac{1}{2}\nabla^2\xi_{(1)i} + \frac{1}{2}\nabla^2\beta_{(1),i}. \qquad (2.5.49)$$

Applying the double derivative $\partial^i\partial^j$ to (2.5.47) and then lowering the indices using the background comoving metric we get

$$-\nabla^2\tilde{\psi}_{(1)} + \nabla^2\nabla^2\tilde{E}_{(1)} = -\nabla^2\psi_{(1)} + \nabla^2\nabla^2 E_{(1)} + \mathcal{H}\nabla^2\alpha_{(1)} + \nabla^2\nabla^2\beta_{(1)}. \qquad (2.5.50)$$

Now obtaining $\tilde{\psi}_{(1)}$ from (2.5.48) and substituting its expression into (2.5.50) we find

$$\nabla^2\nabla^2\tilde{E}_{(1)} = \nabla^2\nabla^2 E_{(1)} + \nabla^2\nabla^2\beta_{(1)}, \qquad (2.5.51)$$

which gives us the transformation relation for $E_{(1)}$ under a gauge transformation

$$\tilde{E}_{(1)} = E_{(1)} + \beta_{(1)}. \qquad (2.5.52)$$

This relation can be inserted in (2.5.48) to find the transformation law for $\psi_{(1)}$

$$\tilde{\psi}_{(1)} = \psi_{(1)} - \mathcal{H}\alpha_{(1)}. \qquad (2.5.53)$$

In order to find the transformation law for the vector $F_{(1)i}$ we must consider (2.5.49) and using the transformation laws for $E_{(1)}$ and $\psi_{(1)}$ we have found before we get

$$\nabla^2\tilde{F}_{(1)i} = \nabla^2 F_{(1)i} + \nabla^2\xi_{(1)i} - \nabla^2\beta_{(1),i}, \qquad (2.5.54)$$

and using the expression for ξ written in (2.5.26) we conclude that

$$\tilde{F}_{(1)i} = F_{(1)i} + \gamma_{(1)i}. \qquad (2.5.55)$$

We need one more transformation law, the one for the tensor perturbation h_{ij}. In order to get it we have to insert the transformations law for $\psi_{(1)}$, $E_{(1)}$ and $F_{(1)i}$ into (2.5.47) and we find

$$\tilde{h}_{(1)ij} = h_{(1)ij}. \qquad (2.5.56)$$

Summarising, the first order transformation laws for the scalar perturbations of the metric are

$$\begin{aligned}
\tilde{\phi}_{(1)} &= \phi_{(1)} + \mathcal{H}\alpha_{(1)} + \alpha'_{(1)}, \\
\tilde{\psi}_{(1)} &= \psi_{(1)} - \mathcal{H}\alpha_{(1)}, \\
\tilde{B}_{(1)} &= B_{(1)} - \alpha_{(1)} + \beta'_{(1)}, \\
\tilde{E}_{(1)} &= E_{(1)} + \beta_{(1)},
\end{aligned} \qquad (2.5.57)$$

as regards the vectors we have found the following

$$\begin{aligned}\tilde{S}^i_{(1)} &= S^i_{(1)} - \gamma'^i_{(1)}, \\ \tilde{F}^i_{(1)} &= F^i_{(1)} + \gamma^i_{(1)},\end{aligned} \tag{2.5.58}$$

and we have found that the first order tensor perturbation is gauge-invariant

$$\tilde{h}_{(1)ij} = h_{(1)ij}. \tag{2.5.59}$$

The metric tensor transformation at second order is given by

$$\begin{aligned}\delta\tilde{g}^{(2)}_{\mu\nu} = {} & \delta g^{(2)}_{\mu\nu} + g^{(0)}_{\mu\nu,\lambda}\xi^\lambda_{(2)} + g^{(0)}_{\mu\lambda}\xi^\lambda_{(2)\ ,\nu} + g^{(0)}_{\nu\lambda}\xi^\lambda_{(2)\ ,\mu} + 2\Big[\delta g^{(1)}_{\mu\nu,\lambda}\xi^\lambda_{(1)} + \\ & + \delta g^{(1)}_{\mu\lambda}\xi^\lambda_{(1),\nu} + \delta g^{(1)}_{\nu\lambda}\xi^\lambda_{(1),\mu}\Big] + g^{(0)}_{\mu\nu,\lambda\alpha}\xi^\lambda_{(1)}\xi^\alpha_{(1)} + g^{(0)}_{\mu\nu,\lambda}\xi^\lambda_{(1),\alpha}\xi^\alpha_{(1)} + \\ & + 2\Big[g^{(0)}_{\mu\nu,\alpha}\xi^\alpha_{(1)}\xi^\lambda_{(1),\nu} + g^{(0)}_{\lambda\nu,\alpha}\xi^\alpha_{(1)}\xi^\lambda_{(1),\mu} + g^{(0)}_{\lambda\alpha}\xi^\lambda_{(1),\mu}\xi^\alpha_{(1),\nu}\Big] + \\ & + g^{(0)}_{\mu\lambda}\Big(\xi^\lambda_{(1),\nu\alpha}\xi^\alpha_{(1)} + \xi^\lambda_{(1),\alpha}\xi^\alpha_{(1),\nu}\Big) + g^{(0)}_{\nu\lambda}\Big(\xi^\lambda_{(1),\mu\alpha}\xi^\alpha_{(1)} + \xi^\lambda_{(1),\alpha}\xi^\alpha_{(1),\mu}\Big).\end{aligned} \tag{2.5.60}$$

The 00 component of this equation gives the transformation rule for the second order function $\phi_{(2)}$

$$\begin{aligned}\tilde{\phi}_{(2)} = {} & \phi_{(2)} + \mathcal{H}\alpha_{(2)} + \alpha'_{(2)} + \alpha_{(1)}\Big[\alpha''_{(1)} + 5\mathcal{H}\alpha'_{(1)} + \alpha_{(1)}\big(\mathcal{H}' + 2\mathcal{H}^2\big) + \\ & + 4\mathcal{H}\phi_{(1)} + 2\phi'_{(1)}\Big] + 2\alpha'_{(1)}\Big(\alpha'_{(1)} + 2\phi_{(1)}\Big) + \xi_{(1)k}\Big(\alpha'_{(1)} + \mathcal{H}\alpha_{(1)} + \\ & + 2\phi_{(1)}\Big)_{,}^{\ k} + \xi'_{(1)k}\Big[\alpha_{(1),}^{\ \ \ k} - 2B^k_{(1)} - \xi'^k_{(1)}\Big].\end{aligned} \tag{2.5.61}$$

Considering the $0i$ component we find the transformation rule for the vector perturbation $B_{(2)i}$, which is:

$$\tilde{B}_{(2)i} = B_{(2)i} + \xi'_{(2)i} - \alpha_{(2),i} + \chi_{Bi} \tag{2.5.62}$$

where

$$\begin{aligned}\chi_{Bi} = {} & 2\Big[\Big(2\mathcal{H}B_{(1)i} + B'_{(1)i}\Big)\alpha_{(1)} + B_{(1)i,k}\xi^k_{(1)} - 2\phi_{(1)}\alpha_{(1),i} + B_{(1)k}\xi^k_{(1),i} + \\ & + B_{(1)i}\alpha'_{(1)} + 2C_{(1)ik}\xi'^k_{(1)}\Big] + 4\mathcal{H}\alpha_{(1)}\Big(\xi'_{(1)i} - \alpha_{(1),i}\Big) + \\ & + \alpha'_{(1)}\Big(\xi'_{(1)i} - 3\alpha_{(1),i}\Big) + \alpha_{(1)}\Big(\xi''_{(1)i} - \alpha'_{(1),i}\Big) + \xi'^k_{(1)}\Big(\xi_{(1)i,k} + 2\xi_{(1)k,i}\Big) + \\ & + \xi^k_{(1)}\Big(\xi'_{(1)i,k} - \alpha_{(1),ik}\Big) - \alpha_{(1),k}\xi^k_{(1),i}.\end{aligned} \tag{2.5.63}$$

As in the first order perturbation case, the relation (2.5.62) tells us how the full vector B_i transforms at second order. If we want to find how the scalar part $B_{(2)}$ and the vector part $S_{(2)i}$ transform we have to proceed in an analogous way as we did for the first order perturbations. Firstly we take the divergence of (2.5.62) and we find the transformation law for the scalar part

$$\tilde{B}_{(2)} = B_{(2)} - \alpha_{(2)} + \beta'_{(2)} + \nabla^{-2}\chi_{Bk,}{}^{k} \tag{2.5.64}$$

where ∇^{-2} denotes the inverse of the Laplacian. As regards the vector part, we can find it by subtracting the scalar part from (2.5.62)

$$\tilde{S}_{(2)i} = S_{(2)i} - \gamma'^{i}_{(2)} - \chi_{Bi} + \nabla^{-2}\chi_{B}{}^{k}{}_{,ki}. \tag{2.5.65}$$

The last terms we deal with are the ones with spatial indices only for which equation (2.5.60) becomes

$$2\tilde{C}_{(2)ij} = 2C_{(2)ij} + 2\mathcal{H}\alpha_{(2)}\delta_{ij} + \xi_{(2)i,j} + \xi_{(2)j,i} + \chi_{ij}, \tag{2.5.66}$$

where we defined χ_{ij} to contain the terms quadratic in the first order perturbations as

$$\begin{aligned}
\chi_{ij} = {} & 2\Big[\Big(\mathcal{H}^2 + \frac{a''}{a}\Big)\alpha^2_{(1)} + \mathcal{H}\Big(\alpha_{(1)}\alpha'_{(1)} + \alpha_{(1),k}\xi^k_{(1)}\Big)\Big]\delta_{ij} + 2\Big(B_{(1)i}\alpha_{(1),j} + \\
& + B_{(1)j}\alpha_{(1),i}\Big) + 4\Big[\alpha_{(1)}\Big(C'_{(1)ij} + 2\mathcal{H}C_{(1)ij}\Big) + C_{(1)ij,k}\xi^k_{(1)} + \\
& + C_{(1)ik}\xi^k_{(1),j} + C_{(1)kj}\xi^k_{(1),i}\Big] + 4\mathcal{H}\alpha_{(1)}\Big(\xi_{(1)i,j} + \xi_{(1)j,i}\Big) + \\
& - 2\alpha_{(1),i}\alpha_{(1),j} + 2\xi_{(1)k,i}\xi^k_{(1),j} + \alpha_{(1)}\Big(\xi'_{(1)i,j} + \xi'_{(1)j,i}\Big) + \\
& + \Big(\xi_{(1)i,jk} + \xi_{(1)j,ik}\Big)\xi^k_{(1)} + \xi_{(1)i,k}\xi^k_{(1),j} + \xi_{(1)j,k}\xi^k_{(1),i} + \\
& + \xi'_{(1)i}\alpha_{(1),j} + \xi'_{(1)j}\alpha_{(1),i}.
\end{aligned} \tag{2.5.67}$$

Now we follow the same steps that led us to find the first order transformation relations above, so first of all we take the trace of (2.5.66)

$$-3\tilde{\psi}_{(2)} + \nabla^2\tilde{E}_{(2)} = -3\psi_{(2)} + \nabla^2 E_{(2)} + 3\mathcal{H}\alpha_{(2)} + \nabla^2\beta_{(2)} + \frac{1}{2}\chi^k_{\ k}, \tag{2.5.68}$$

then we take the divergence of (2.5.66)

$$2\tilde{C}_{(2)ij,}{}^{j} = 2C_{(2)ij,}{}^{j} + 2\mathcal{H}\alpha_{(1),i} + \nabla^2\xi_{(2)i} + \nabla^2\beta_{(2),i} + \chi_{ij,}{}^{k}, \tag{2.5.69}$$

and finally we also apply the double derivative $\partial^i\partial^j$ to (2.5.66)

$$-\nabla^2\tilde{\psi}_{(2)} + \nabla^2\nabla^2\tilde{E}_{(2)} = -\nabla^2\psi_{(2)} + \nabla^2\nabla^2 E_{(2)} + \mathcal{H}\nabla^2\alpha_{(2)} + \nabla^2\nabla^2\beta_{(2)} + \frac{1}{2}\chi^{ij}{}_{,ij}. \tag{2.5.70}$$

From (2.5.68) and (2.5.70) we get the second order scalar metric perturbations

$$\tilde{\psi}_{(2)} = \psi_{(2)} - \mathcal{H}\alpha_{(2)} - \frac{1}{4}\chi^k_{\ k} + \frac{1}{4}\nabla^{-2}\chi^{ij}{}_{,ij}, \tag{2.5.71}$$

and

$$\tilde{E}_{(2)} = E_{(2)} + \beta_{(2)} + \frac{3}{4}\nabla^{-2}\nabla^{-2}\chi^{ij}{}_{,ij} - \frac{1}{4}\nabla^{-2}\chi^{k}_{k}. \tag{2.5.72}$$

Substituting our results for $\tilde{\psi}_{(2)}$ and $\tilde{E}_{(2)}$ in (2.5.69) we obtain the second order vector metric perturbation

$$\tilde{F}_{(2)i} = F_{(2)i} + \gamma_{(2)i} + \nabla^{-2}\chi_{ik,}{}^{k} - \nabla^{-2}\nabla^{-2}\chi^{kl}{}_{,kli}. \tag{2.5.73}$$

We can now turn to the tensor perturbation at second order, in fact writing the expression of $\tilde{\psi}_{(2)}$, $\tilde{E}_{(2)}$ and $\tilde{F}_{(2)i}$ into equation (2.5.66) we get

$$\begin{aligned}\tilde{h}_{(2)ij} = h_{(2)ij} + \chi_{ij} + \frac{1}{2}\big(\nabla^{-2}\chi^{kl}{}_{,kl} - \chi^{k}_{k}\big)\delta_{ij} + \frac{1}{2}\nabla^{-2}\nabla^{-2}\chi^{kl}{}_{,klij} + \\ + \frac{1}{2}\nabla^{-2}\chi^{k}_{k,ij} - \nabla^{-2}\big(\chi_{ik,}{}^{k}{}_{j} + \chi_{jk,}{}^{k}{}_{i}\big).\end{aligned} \tag{2.5.74}$$

Although transformation rule for the second order tensor $h_{(2)ij}$ does not depend on the second order part of the gauge transformation $\xi^{\mu}_{(2)}$, it does depend on χ_{ij} and its derivatives which contains terms quadratic in first order perturbations. The tensor metric perturbations are no longer gauge-invariant at second and higher order.

2.6 Gauge-invariant variables

Previously we have seen how the perturbations are affected by gauge transformations, when we change the gauge also the perturbations change. Since we are free to work in the gauge coordinates best adapted to the problem at hand, we obtain apparently different results depending upon the arbitrary choice of the gauge. This problem can be overcome using quantities that are specified unambiguously, such that they have a gauge-invariant definition. Firstly we must highlight that gauge-invariance is not the same as gauge-independence: a quantity like the tensor metric perturbation, $h_{(1)ij}$, is gauge-independent at first order because the tensor part of the metric perturbation is the same in all gauges. Instead a gauge-invariant quantity is an appropriate combinations of gauge-dependent quantities whose dependence is compensated in the sense that the terms which appear after a gauge transformation because of the transformation of one component is erased by the terms due to the transformation of the other components. Bardeen, by studying the transformations of metric perturbations in [27], constructed two gauge-independent variables

$$\Phi \equiv \phi_{(1)} + \mathcal{H}\Big(B_{(1)} - E'_{(1)}\Big) + \Big(B_{(1)} - E'_{(1)}\Big)', \tag{2.6.1}$$

$$\Psi \equiv \psi_{(1)} - \mathcal{H}\Big(B_{(1)} - E'_{(1)}\Big). \tag{2.6.2}$$

Of course there are an infinite number of gauge-invariant variables, since any combination of gauge-invariant variables will also be gauge-invariant. At first order we can define scalar and vector type gauge-invariant quantities independently of each other and this can be understood by simply looking at the transformation relation at first order (2.5.57) and (2.5.58); in fact the first depends on the two scalar gauge functions $\alpha_{(1)}$ and $\beta_{(1)}$, while the seconds depends only on $\gamma_{(1)}$; instead at second and higher order things get more complicated.

If we want to work in a specific gauge, we have to specify at various order the value of the vector that generates the gauge transformations ξ^μ, which means we have to specify two scalar degrees of freedom α and β and one vector γ^i which is divergence-free. In the following we shall consider different particular choices of gauges which are often used in literature.

2.6.1 Longitudinal gauge

The longitudinal gauge or conformal Newtonian gauge is defined by the conditions (for the scalar perturbations)

$$E_l = 0, \tag{2.6.3}$$
$$B_l = 0, \tag{2.6.4}$$

which requires from equations (2.5.57) that at first order

$$\beta_{(1)l} = -E_{(1)}, \tag{2.6.5}$$
$$\alpha_{(1)l} = B_{(1)} - E'_{(1)}. \tag{2.6.6}$$

Using the first two equations of (2.5.57) we see that the remaining scalar metric perturbations, $\phi_{(1)}$ and $\psi_{(1)}$, are given by

$$\phi_{(1)l} = \phi_{(1)} + \mathcal{H}\Big(B_{(1)} - E'_{(1)}\Big) + \Big(B_{(1)} - E'_{(1)}\Big)' = \Psi, \tag{2.6.7}$$
$$\psi_{(1)l} = \psi_{(1)} - \mathcal{H}\Big(B_{(1)} - E'_{(1)}\Big) = \Phi. \tag{2.6.8}$$

We draw the important conclusion that in longitudinal gauge $\phi_{(1)}$ and $\psi_{(1)}$ coincide with the gauge-invariant variables Φ and Ψ. The fluid density perturbation and the scalar velocity are given from (2.5.28) and from (2.5.34)

$$\delta\rho_{(1)l} = \delta\rho_{(1)} + \rho'_{(0)}\Big(B_{(1)} - E'_{(1)}\Big), \tag{2.6.9}$$
$$v_{(1)l} = v_{(1)} + E'_{(1)}. \tag{2.6.10}$$

After imposing the gauge conditions, the metric tensor (accounting only the scalar perturbations) is diagonal:

$$ds^2 = a^2(\tau)\big[-(1+2\Phi)d\tau^2 + (1-2\Psi)\delta_{ij}dx^i dx^j\big]. \tag{2.6.11}$$

Moreover in many cases of physical interest (in the absence of anisotropic stress) one finds $\Phi = \Psi$ and there is only one variable required to describe all scalar metric perturbations which is a generalization of the Newtonian gravitational potential[4], which explains the choice of the name Newtonian gauge. This gauge is widely used, for example throughout reference [50]. It has also proven useful for calculations on small scales, since it gives evolution equations closest to the Newtonian ones [61].

The extension to include vector and tensor metric perturbations is called the Poisson gauge [49, 62]. The condition to impose on vector perturbations is

$$S_l^i = 0. \tag{2.6.12}$$

Recalling that we defined the splitting of the vector perturbations B_i of the metric as (2.1.11), the conditions (2.6.4) and (2.6.12) which fix the gauge to the Poisson gauge can be rewritten as

$$B_l^i = 0. \tag{2.6.13}$$

The condition (2.6.12) fixes the vector part of the spatial gauge transformation, in fact using the first of (2.5.58):

$$\gamma'^{i}_{(1)l} = S^i_{(1)} \tag{2.6.14}$$

hence

$$\gamma^i_{(1)l} = \int d\tau \, S^i_{(1)} + K^i_{(1)}(\vec{x}), \tag{2.6.15}$$

with $K^i_{(1)}(\vec{x})$ arbitrary constant 3-vector. The remaining vector metric perturbation hence is given by the second of (2.5.58)

$$F^i_{(1)l} = F^i_{(1)} + \int d\tau \, S^i_{(1)} + K^i_{(1)}(\vec{x}). \tag{2.6.16}$$

2.6.2 Spatially flat gauge

Another possible gauge choice is the spatially flat or uniform curvature gauge [63, 64]. Before showing the features of this gauge we introduce the curvature of spatial hypersurfaces and its perturbation. The curvature of spatial hypersurfaces is defined in the following way

$$\mathcal{R} = g^{ij} R_{ij}, \tag{2.6.17}$$

with

$$R_{ij} = R^k_{ikj} = \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ki} + \Gamma^k_{kl}\Gamma^l_{ji} - \Gamma^k_{jl}\Gamma^l_{ki} \tag{2.6.18}$$

[4]To see this it is necessary to write the Einstein equations at first order in perturbations in the longitudinal gauge.

which is the Riemann tensor. If we consider only scalar perturbations to the metric, in which case the spatial metric becomes

$$g_{ij} = a^2\Big[(1-2\psi)\delta_{ij} + 2E_{,ij}\Big], \tag{2.6.19}$$

it is easy to check that the background value of the Christoffel symbol is $\Gamma_{(0)ij}^{\ \ k} = 0$ hence the background value of the curvature of spatial hypersurfaces is

$$\mathcal{R}_{(0)} = 0. \tag{2.6.20}$$

Although the background value is zero, the first order perturbation of $\mathcal{R}$ is given by

$$\delta\mathcal{R}_{(1)} = \delta g^{ij}_{(1)} R^{(0)}_{ij} + g^{ij}_{(0)}\delta R^{(1)}_{ij} = -\frac{4}{a^2}\nabla^2\psi, \tag{2.6.21}$$

for this reason it is common to call ψ as "curvature perturbation".

In the spatially flat gauge one selects spatial hypersurfaces on which the induced 3-metric is left unperturbed by scalar or vector perturbations, which requires

$$\begin{aligned}\psi_{(1)f} &= 0,\\ E_{(1)f} &= 0,\\ F^i_{(1)f} &= 0.\end{aligned} \tag{2.6.22}$$

Using the second and the fourth transformation laws under a gauge transformations of (2.5.57) and the second of (2.5.58), this corresponds to a gauge transformation where

$$\alpha_{(1)f} = \frac{\psi_{(1)}}{\mathcal{H}}, \qquad \beta_{(1)f} = -E_{(1)}, \qquad \gamma^i_{(1)f} = -F^i_{(1)}. \tag{2.6.23}$$

The definitions of the remaining scalar metric degrees of freedom are then

$$\phi_{(1)f} = \phi_{(1)} + \psi_{(1)} + \left(\frac{\psi_{(1)}}{\mathcal{H}}\right)' \tag{2.6.24}$$

$$B_{(1)f} = B_{(1)} - \frac{\psi_{(1)}}{\mathcal{H}} - E'_{(1)}. \tag{2.6.25}$$

The definition of the remaining vector metric perturbation is the time derivative of the vector metric perturbation in the Poisson gauge

$$S^i_{(1)f} = S^i_{(1)} + F'^i_{(1)} = F'^i_{(1)l}. \tag{2.6.26}$$

It is possible to check using the transformations rules (2.5.57) and (2.5.58) that these three quantities $\phi_{(1)f}$, $B_{(1)f}$ and $S^i_{(1)f}$ are gauge-invariant variables. In this gauge also the perturbation of the density has a gauge-invariant definition, in fact from equation (2.5.28) we get

$$\delta\rho_{(1)f} = \delta\rho_{(1)} + \rho'_{(0)}\frac{\psi_{(1)}}{\mathcal{H}}. \tag{2.6.27}$$

The scalar part of the velocity is given by (2.5.34)

$$v_{(1)f} = v_{(1)} + E'_{(1)}. \tag{2.6.28}$$

Another example is the scalar field perturbation:

$$\delta\varphi_{(1)f} = \delta\varphi_{(1)} + \varphi'_{(0)}\frac{\psi_{(1)}}{\mathcal{H}}, \tag{2.6.29}$$

which is the gauge-invariant Sasaki-Mukhanov variable [65, 66], often denoted by Q.

2.6.3 Synchronous gauge

The synchronous gauge is defined by the conditions

$$\begin{aligned} \phi_s &= 0, \\ B^i_s &= 0, \end{aligned} \tag{2.6.30}$$

so that the proper time for observers at fixed spatial coordinates coincides with cosmic time in the FRW background. This gauge is very popular for numerical studies such as CMBFAST [67]. At first order the gauge conditions (2.6.30) fix the value of $\xi^\mu_{(1)}$ to

$$\alpha_{(1)s} = -\frac{1}{a}\int d\tau\,\Big(a\phi_{(1)} - J_{(1)}(\vec{x})\Big), \tag{2.6.31}$$

$$\beta_{(1)s} = \int d\tau\,\Big(\alpha_{(1)s} - B_{(1)}\Big) + K_{(1)}(\vec{x}), \tag{2.6.32}$$

$$\gamma^i_{(1)s} = \int d\tau\, S^i_{(1)} + K^i_{(1)}(\vec{x}). \tag{2.6.33}$$

Equations (2.6.31)-(2.6.33) do not determine the time slicing unambiguously and we are left with two arbitrary scalar functions of the spatial coordinates, $J_{(1)}$ and $K_{(1)}$. We are thus left with a residual gauge-freedom [44].

2.6.4 Comoving orthogonal gauge

The comoving orthogonal gauge is defined by choosing spatial coordinates such that the 3-velocity of the fluid vanishes, $v_i = 0$. Orthogonality of the constant-τ hypersurfaces to the 4-velocity, u^μ, then requires $v_i + B_i = 0$. This corresponds to

$$\alpha_{1com} = v_1 + B_1, \tag{2.6.34}$$

$$\beta_{1com} = \int d\tau\, v_1 + K(\vec{x}), \tag{2.6.35}$$

where $\check{A}(\vec{x})$ represents a residual gauge freedom, corresponding to a constant shift of the spatial coordinates. The scalar perturbations in the comoving orthogonal gauge can be written as

$$\phi_{(1)com} = \phi_{(1)} + \mathcal{H}\Big(v_{(1)} + B_{(1)}\Big) + \Big(v'_{(1)} + B'_{(1)}\Big), \tag{2.6.36}$$

$$\psi_{(1)com} = \psi_{(1)} - \mathcal{H}\Big(v_{(1)} + B_{(1)}\Big). \tag{2.6.37}$$

Defined in this way, these combinations are gauge-invariant under transformations of their component parts as one can easily verify. The density perturbation on the comoving orthogonal hypersurfaces is given by

$$\delta\rho_{(1)com} = \delta\rho_{(1)} + \rho'_{(0)}\Big(v_{(1)} + B_{(1)}\Big). \tag{2.6.38}$$

2.6.5 Total matter gauge

This gauge is also known as the velocity orthogonal isotropic gauge [44, 55, 68]. To fix the temporal and the spatial gauge we require

$$\begin{aligned} v_{(1)tm} + B_{(1)tm} &= 0, \\ E_{(1)tm} &= 0, \\ F_{(1)tm} &= 0. \end{aligned} \tag{2.6.39}$$

From these relations it follows that

$$\alpha_{(1)tm} = v_{(1)} + B_{(1)}, \qquad \beta_{(1)tm} = -E_{(1)}, \qquad \gamma^i_{(1)tm} = -F^i_{(1)}. \tag{2.6.40}$$

Hence using (2.5.57) we get the other metric perturbations in this gauge:

$$\phi_{(1)tm} = \phi_{(1)} + \mathcal{H}\Big(v_{(1)} + B_{(1)}\Big) + \Big(v'_{(1)} + B'_{(1)}\Big) = \phi_{(1)com}, \tag{2.6.41}$$

$$\psi_{(1)tm} = \psi_{(1)} - \mathcal{H}\Big(v_{(1)} + B_{(1)}\Big) = \psi_{(1)com}, \tag{2.6.42}$$

$$B_{(1)tm} = -v_{(1)} - E'_{(1)} = -v_{(1)l}. \tag{2.6.43}$$

As regards the matter quantities in this gauge we find

$$\delta\rho_{(1)tm} = \delta\rho_{(1)} + \rho'_{(0)}\Big(v_{(1)} + B_{(1)}\Big) = \delta\rho_{(1)com}, \tag{2.6.44}$$

$$v_{(1)tm} = v_{(1)} + E'_{(1)} = v_{(1)l}. \tag{2.6.45}$$

2.6.6 Uniform density gauge

We can use the matter to pick out a foliation of uniform density hypersurfaces on which to define perturbed quantities. Using equation (2.5.28) we see that $\widetilde{\delta\rho}_{(1)} = 0$ implies a temporal gauge transformation

$$\alpha_{(1)\delta\rho} = -\frac{\delta\rho_{(1)}}{\rho'_{(0)}}. \tag{2.6.46}$$

On these hypersurfaces one can define a gauge-invariant curvature perturbation as [54, 69]

$$-\zeta_{(1)} \equiv \psi_{(1)\delta\rho} = \psi_{(1)} + \mathcal{H}\frac{\delta\rho_{(1)}}{\rho'_{(0)}}. \tag{2.6.47}$$

There is still the freedom to chose the spatial gauge, in particular we can choose either $B_{\delta\rho}$, $E_{\delta\rho}$ or $v_{\delta\rho}$ to be zero and thus fix $\beta_{\delta\rho}$.

2.7 Dynamics

In General Relativity, the Einstein equations relate the local space-time curvature to the local energy-momentum tensor:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{2.7.1}$$

On the FRW background and using a metric tensor with the signature $(-,+,+,+)$, Einstein equations become Friedmann equations

$$\mathcal{H}^2 = \frac{8}{3}\pi G a^2 \rho, \tag{2.7.2}$$

$$\mathcal{H}' = -\frac{4}{3}\pi G a^2(\rho + 3p). \tag{2.7.3}$$

We also know that the energy-momentum tensor is covariantly conserved, $\nabla_\mu T^{\mu\nu} = 0$, and evaluating the temporal component of this relation on the background gives the continuity equation

$$\rho' = -3\mathcal{H}(\rho + p), \tag{2.7.4}$$

where ρ and p are the total energy density and the total pressure, which are related to the density and pressure of the component fluids by

$$\sum_\alpha \rho_\alpha = \rho, \qquad \sum_\alpha p_\alpha = p. \tag{2.7.5}$$

The continuity equation (2.3.27) for each individual fluid in the background is

$$\rho'_\alpha = -3\mathcal{H}(\rho_\alpha + p_\alpha) + aQ_\alpha, \tag{2.7.6}$$

where the energy transfer to the α-fluid is given by the component of the energy-momentum transfer vector

$$Q_\alpha \equiv -u_\mu Q^\mu_{(\alpha)}, \tag{2.7.7}$$

which obeys the constraint

$$\sum_\alpha Q_\alpha = 0. \tag{2.7.8}$$

Homogeneous scalar fields in the FRW metric obey the Klein-Gordon equation

$$\Box\varphi = \frac{\partial V}{\partial\varphi}. \tag{2.7.9}$$

This equation can be rewritten using the definition of the box operator in a curved space [68]

$$\Box\varphi \equiv g^{\mu\nu} D_\mu \partial_\nu \varphi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi\right), \tag{2.7.10}$$

that allows us to write (2.7.9) as

$$\varphi'' + 2\mathcal{H}\varphi' - a^2 \frac{\partial V}{\partial\varphi} = 0. \tag{2.7.11}$$

If we have more than a scalar field in our theory then we simply add an index labelling the various fields: φ_I with $I = 1, \ldots, N$

$$\varphi_I'' + 2\mathcal{H}\varphi_I' - a^2 \frac{\partial V}{\partial\varphi_I} = 0. \tag{2.7.12}$$

2.7.1 First order scalar perturbations

The perturbed Einstein equations at first order, in particular the trace of the ij component, yield the following evolution equation for the scalar metric perturbations

$$\psi''_{(1)} + 2\mathcal{H}\psi'_{(1)} + \mathcal{H}\phi'_{(1)} + \left(2\mathcal{H}' + \mathcal{H}^2\right)\phi_{(1)} = 4\pi G a^2 \Big(\delta p_{(1)} + \frac{2}{3}\nabla^2\Pi\Big), \tag{2.7.13}$$

where Π is scalar part of the anisotropic stress tensor defined in (2.3.18). The scalar metric perturbations in an arbitrary gauge are related to matter perturbations via the projection of the Einstein equations into components tangent to and orthogonal to the time-like 4-vector field n^μ. This relations can be written as [50, 31]

$$3\mathcal{H}\Big(\psi'_{(1)} + \mathcal{H}\phi_{(1)}\Big) - \nabla^2\Big(\psi_{(1)} + \mathcal{H}\sigma_{(1)}\Big) = -4\pi G a^2 \delta\rho_{(1)}, \tag{2.7.14}$$

$$\psi'_{(1)} + \mathcal{H}\phi_{(1)} = -4\pi G a^2 (\rho + p) V_{(1)}, \tag{2.7.15}$$

where the total covariant velocity perturbation is given by

$$V_{(1)} \equiv v_{(1)} + B_{(1)}, \tag{2.7.16}$$

and $v_{(1)}$ is the total scalar velocity potential defined in (2.3.12).

2.7.2 First order tensor perturbations

The spatial part of the Einstein equations yields a wave equation

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 8\pi G a^2 \Pi_{ij}. \tag{2.7.17}$$

We can decompose tensor perturbations into eigenmodes of the spatial Laplacian, $\nabla^2 e_{ij} = -(k^2/a^2)e_{ij}$, with comoving wavenumber k and scalar amplitude $h(t)$:

$$h_{ij} = h(t)e_{ij}^{(+,\times)}(\vec{x}), \tag{2.7.18}$$

with two possible polarization states, $+$ and $\times$. In the absence of any anisotropic stress (that corresponds for example to the situation in which we have only scalar fields and perfect fluids), the wave equations for the amplitude defined in (2.7.18) becomes

$$h'' + 2\mathcal{H}h' + k^2 h = 0, \tag{2.7.19}$$

which is the wave equation for a massless scalar field in the unperturbed FRW metric. Differently from the case of scalar perturbations, the tensor ones are not coupled to the density.

Chapter 3

Dynamics of inflation

In the previous Chapter we showed the importance of the primordial perturbations but there is still a question left: how could small primordial perturbations have been produced in a homogeneous and isotropic Universe? The answer is inflation, the same answer as for the other problems of the Standard Hot Big-Bang Model like the problem of horizon and flatness. Inflation was firstly introduced in 1981 by Alan Guth in [3] and then studied by Linde in [70, 71, 72] and also by Albrecht and Steinhardt in [4]; reviews on the argument are for example [73, 74, 75]. The inflationary epoch can be achieved by a "fluid" of negative pressure and we will show that a scalar field can act in this way under some conditions (slow-roll). However, until now, no evidence of a scalar field driving the inflationary epoch has been found so it is not to exclude the possibility that other kind of fields lead to inflation in the early Universe.

In this section we will explore the epoch of inflation to understand in which way the primordial perturbations have been produced. We have already seen previously that we can split our Universe in a FRW background and fluctuations and here we will do the same: we will have a uniform scalar field on top of its quantum fluctuations. At any given time the average fluctuations will be zero because there will be regions in which the fluctuations of the scalar field will be slightly larger than the average value and others in which it will be smaller. Nevertheless the average of the square of the fluctuations (variance) won't be zero as we will check. Our purpose will be to compute this variance and see how it evolves as inflation takes place.

Actually in the following pages we will not consider many aspects of inflation which are becoming more and more relevant in these days. Since the birth of inflation many models were developed to understand how this accelerated expansion took place and ended but up to now we are not able to say which model is the correct one. Various models have been ruled out and the precision of cosmological observables is increasing fast, in the next future we will be able to have tighter constraints and consequently to select the most suitable models. An important tool that is fundamental to verify the validity of the various models is the study of the non-Gaussianity. As stated above, here we won't enter in the details of this argument but a more information can be found in [75, 76].

3.1 The scalar field as the inflaton

First of all we show that we can describe the inflationary Universe using a scalar field $\varphi(x)$ under specific conditions which are commonly known as slow-roll conditions. During inflation the Universe expands in an accelerated way with an approximately constant Hubble parameter. So we can consider the Universe to be quasi-de Sitter, this means that at first approximation it has a privileged spatial slicing which can be realized by a time evolving scalar $\varphi(t)$. So we can split our field into a background part that respects this property and a perturbation which introduces the dependence on the spatial coordinates:

$$\varphi(x) = \varphi(t, \vec{x}) = \varphi_0(t) + \delta\varphi(t, \vec{x}). \tag{3.1.1}$$

The background value is given by the VEV taken by our field on the vacuum state $\varphi_0(t) = \langle 0|\varphi(t, \vec{x})|0\rangle$. We can write the Lagrangian for the scalar field φ as:

$$\mathcal{L}_\varphi = -\frac{1}{2} g^{\mu\nu}(\partial_\mu\varphi)(\partial_\nu\varphi) - V(\varphi), \tag{3.1.2}$$

where the minus sign in front of the kinetic term derives from the choice of the signature of metric we work with that is $(-, +, +, +)$. From this Lagrangian we get the following action for φ

$$\mathcal{S}_\varphi = \int d^4x \sqrt{-g}\mathcal{L}_\varphi, \tag{3.1.3}$$

which tells us that the energy-momentum tensor is

$$T^\varphi_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}}. \tag{3.1.4}$$

For the moment we focus on the background which means we consider only φ_0, so the energy-momentum tensor background components are given by:

$$\begin{aligned} T^{\varphi_0}_{\mu\nu} &= -\frac{2}{\sqrt{-g}} \frac{\delta S_{\varphi_0}}{\delta g^{\mu\nu}} = -2\frac{\partial \mathcal{L}\varphi_0}{\partial g^{\mu\nu}} - \frac{2}{\sqrt{-g}} \frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} \mathcal{L}_{\varphi_0} \\ &= (\partial_\mu\varphi_0)(\partial_\nu\varphi_0) - g_{\mu\nu}\left(\frac{1}{2} g^{\alpha\beta}(\partial_\alpha\varphi_0)(\partial_\beta\varphi_0) + V(\varphi_0)\right). \end{aligned} \tag{3.1.5}$$

From the Section 2.3 in the previous Chapter we know that in a homogeneous and isotropic Universe the energy-momentum tensor is the one of a perfect fluid:

$$T^{\mu}_{\nu} = Diag\Big(-\rho(t), p(t), p(t), p(t)\Big) \tag{3.1.6}$$

Using the cosmic time instead of the conformal time, the metric tensor has the form

$$g_{\mu\nu} = Diag\Big(-1, a^2(t), a^2(t), a^2(t)\Big), \tag{3.1.7}$$

we find that for the scalar field φ_0 the background components of the energy-momentum tensor are:

$$\begin{cases} T^0{}_0 & \longrightarrow \frac{1}{2}\dot{\varphi_0}^2 + V(\varphi_0) = \rho_{\varphi_0} \\ T^i{}_j & \longrightarrow \Big(\frac{1}{2}\dot{\varphi_0}^2 - V(\varphi_0)\Big)\delta^i_j = p_{\varphi_0}\delta^i_j. \end{cases} \tag{3.1.8}$$

We know from [73] that the inflationary solution to the Hot Big Bang model consists in the request that the comoving Hubble radius satisfies:

$$\dot{r}_H(t) = -\frac{\ddot{a}}{\dot{a}^2} < 0, \tag{3.1.9}$$

which is guaranteed by $\ddot{a} > 0$. This relation on $\ddot{a}$ can be rewritten in terms of the density and pressure of the cosmic fluid using the following Friedmann equation

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p), \tag{3.1.10}$$

since $a > 0$, the condition $\ddot{a} > 0$ implies:

$$-\frac{4}{3}\pi G(\rho + 3p) > 0 \Longrightarrow p < -\frac{1}{3}\rho. \tag{3.1.11}$$

If φ is the field driving the inflation we must require that its pressure and density satisfy (3.1.11) and looking at (3.1.8) it is clear that if

$$V(\varphi_0) \gg \frac{1}{2}\dot{\varphi_0}^2, \tag{3.1.12}$$

then $p_{\varphi_0} \simeq -\rho_{\varphi_0}$. In this way the condition (3.1.11) is satisfied and furthermore our scalar field acts like an effective cosmological constant (the state equation of a cosmological constant is $p_\Lambda = -\rho_\Lambda$).

The requirement on the potential $V(\varphi_0) \gg \frac{1}{2}\dot{\varphi_0}^2$ is important because it tells us that not all kind of potentials are able to produce inflation but only the ones that have a sufficiently smooth region in which the kinetic energy of the field is negligible with respect to its potential; in the region of the potential which is flat enough the scalar field moves very slowly (slow-roll). When this requirement is fulfilled the scalar field is called inflaton because it leads to inflation.

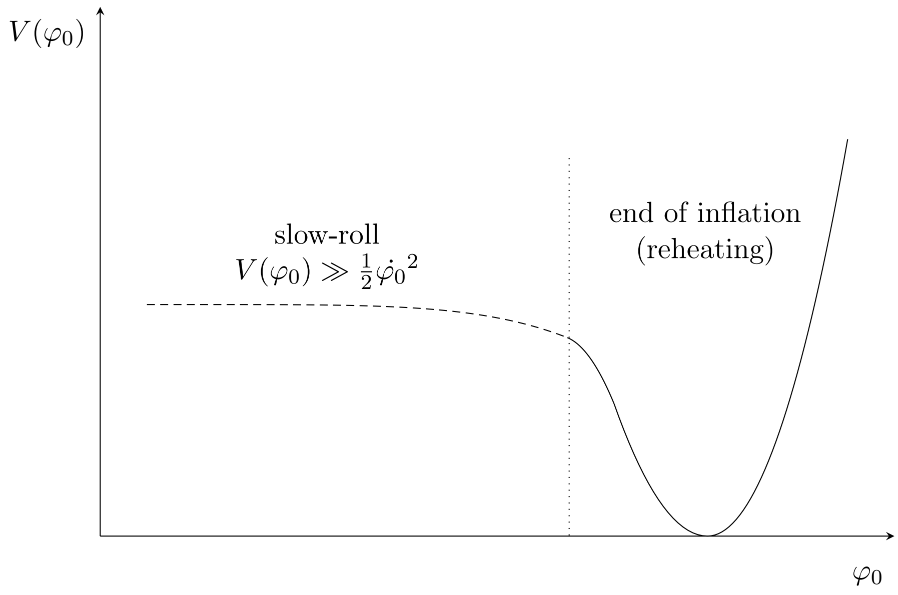

Figure 3.1: slow-roll potential.

In Figure 3.1 a typical potential for the inflaton is sketched; the dashed part is the slow-roll region in which the potential satisfies $V(\varphi_0) \gg \frac{1}{2}\dot{\varphi_0}^2$. Instead the solid line is the part of the potential associated to the end of inflation and the beginning of a later epoch in the Universe history: the reheating phase which is characterized by a potential well in which the field oscillates around the minimum. During the slow-roll period the Hubble rate is:

$$H^2 = \frac{8}{3}\pi G\rho_{\varphi_0} = \frac{8}{3}\pi G\Big(\frac{1}{2}\dot{\varphi}_0^2 + V(\varphi_0)\Big) \simeq \frac{8}{3}\pi G\, V(\varphi_0), \tag{3.1.13}$$

in fact, when enough inflation has occurred (when the number of e-folds is $N \gtrsim 60$), the densities of all other types of matter become negligible because of the accelerated expansion of the Universe and we can consider only the inflaton density.

3.1.1 Equation of motion for the inflaton

The equation of motion for the scalar field can be easily computed by variating the action (3.1.3) with respect to $\delta\varphi$

$$\begin{aligned}\frac{\delta \mathcal{S}_\varphi}{\delta\varphi} &= \frac{\delta}{\delta\varphi}\int d^4x\,\sqrt{-g}\left(-g^{\mu\nu}\partial_\mu\varphi\partial_\nu\delta\varphi - \frac{\partial V}{\partial\varphi}\delta\varphi\right)\\ &= \int d^4x\left[\partial_\mu\big(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\big) - \sqrt{-g}\frac{\partial V}{\partial\varphi}\right],\end{aligned} \tag{3.1.14}$$

and requiring this functional derivative is zero, which implies

$$\frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\big) = \frac{\partial V}{\partial\varphi}. \tag{3.1.15}$$

Using the box operator we already encountered in (2.7.10)

$$\Box\varphi \equiv g^{\mu\nu}D_\mu\partial_\nu\varphi = \frac{1}{\sqrt{-g}}\partial_\mu\big(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\big), \tag{3.1.16}$$

equation (3.1.15) becomes the Klein-Gordon equation:

$$\Box\varphi = \frac{\partial V}{\partial\varphi}. \tag{3.1.17}$$

Taking into account only the background part of the scalar field φ_0 and recalling that the background metric we choose to work with is (3.1.7), we find from (3.1.17)

$$\begin{aligned}\Box\varphi_0 &= \frac{1}{a^3}\partial_0(-a^3\dot{\varphi}_0) + \frac{1}{a^3}\partial_i\Big(\frac{a^3}{a^2}\partial_i\varphi_0\Big)\\ &= -\ddot{\varphi}_0 - 3H\dot{\varphi}_0 + \frac{\nabla^2\varphi_0}{a^2}\\ &= -\ddot{\varphi}_0 - 3H\dot{\varphi}_0.\end{aligned} \tag{3.1.18}$$

Obviously the spatial derivatives of the background scalar field φ_0 vanish because it depends only by the time coordinate. In conclusion, (3.1.17) becomes:

$$\ddot{\varphi}_0 + 3H\dot{\varphi}_0 = -\frac{\partial V}{\partial\varphi_0}, \tag{3.1.19}$$

which written in terms of the conformal time becomes exactly the equation found in the previous section for a scalar field on the FRW background (2.7.11).

Before analysing what happens for the quantum fluctuations of the scalar field, we focus on the background equation (3.1.19) and study it in more details. First of all, this equation is the most general equation for a scalar field in the sense that all scalar fields on FRW background satisfy this equation, not only the inflaton. Clearly the dynamics depends on the choice of the potential, in the particular case of the inflaton, the condition of slow-roll tell us that the potential must be flat enough. This means that if we study equation (3.1.19) in the slow-roll region we can consider the derivative of the potential to be small and constant:

$$\frac{\partial V}{\partial\varphi_0} \simeq F = const. \tag{3.1.20}$$

In this case, with a constant force F, the differential equation (3.1.19) has a solution with the first derivative of the scalar field constant in time and so its second derivative is zero; hence we can neglect the term $\ddot{\varphi}_0$ finding:

$$\dot{\varphi}_0 \simeq -\frac{1}{3H}\frac{\partial V}{\partial \varphi_0} = -\frac{V'}{3H}. \tag{3.1.21}$$

The previous equation (3.1.21) and the (3.1.12) are called slow-roll conditions:

$$\begin{cases} \frac{1}{2}\dot{\varphi}_0^2 & \ll V(\varphi_0) \\ \dot{\varphi}_0 & \simeq -\dfrac{V'}{3H}. \end{cases} \tag{3.1.22}$$

3.1.2 Slow-roll parameters

At this point we can introduce two parameters, ϵ and η, with which we can both rewrite the two slow-roll conditions (3.1.22) and have a more direct link to the observations because they are used to classify the various types of potential available for the inflaton. These two slow-roll parameters were firstly defined by Liddle and Lyth in [77] as:

- $\epsilon = -\dfrac{\dot{H}}{H^2}$
  During inflation the Hubble rate is $H^2 = \frac{8}{3}\pi G\big(V(\varphi_0) + \frac{1}{2}\dot{\varphi}_0^2\big)$ and deriving this equation we get $H\dot{H} = \frac{4}{3}\pi G\big(V'\dot{\varphi}_0 + \dot{\varphi}_0\ddot{\varphi}_0\big)$ which using (3.1.19) becomes $H\dot{H} = -4\pi GH\dot{\varphi}_0^2$. Now using the slow-roll conditions (3.1.22) we find $H^2 \simeq \frac{8}{3}\pi GV$ and $\dot{H} \simeq -4\pi \frac{G}{9H^2}\big(V'\big)^2$ and hence $-\frac{\dot{H}}{H^2} \simeq \frac{1}{16\pi G}\big(\frac{V'}{V}\big)^2$. So the value of this parameter establishes a relation between the potential and its first derivative. We can also rewrite the ratio $\dot{H}/H^2$ in function of the first derivative and φ_0 using the second equation of (3.1.22) and the result is $\epsilon = -\frac{\dot{H}}{H^2} \simeq \frac{3}{2}\frac{\dot{\varphi}_0^2}{V} \ll 1$. The parameter ϵ must be much smaller than one in order to have a kinetic term negligible with respect to the potential.

- $\eta = \dfrac{1}{3}\dfrac{V''}{H^2}$
  Using the expression for the Hubble parameter H written for ϵ we find that $\eta \simeq \frac{1}{8\pi G}\frac{V''}{V}$ during slow-roll. The fact that $|\eta| \ll 1$ is a consequence of the smallness of $\ddot{\varphi}_0$: deriving the second equation of (3.1.22) we get $3H\ddot{\varphi}_0 \simeq -V''\dot{\varphi}_0$ and so we can write $\ddot{\varphi}_0 \sim \frac{V''\dot{\varphi}_0}{H}$; using the equation of motion (3.1.19) and the fact that $\ddot{\varphi}_0$ is negligible we get $\ddot{\varphi}_0 \sim \frac{V''\dot{\varphi}_0}{H} \ll 3H\dot{\varphi}_0$ and hence $V'' \ll 3H^2$.

By their definition, we see that these parameters are directly connected to the potential, in fact they depend on the potential itself and also its derivatives. For this reason any constraint for ϵ or η automatically becomes a constraint for the potential. Moreover a small value of ϵ is, as already stated, due to the smallness of the kinetic term with respect to the potential; while the smallness of the η parameter is requested to have a flat enough potential during which the inflaton evolves slowly. As a result of this small value of η, the mass associated to the inflaton which is the second derivative of the potential must be much smaller than H^2.

Notice also that the condition for inflation to take place is $\ddot{a} > 0$ and we can rewrite in terms of the parameter ϵ:

$$\ddot{a} = \frac{d}{dt}\Big(\frac{a\dot{a}}{a}\Big) = \dot{a}H + a\dot{H} = a\big(\dot{H} + H^2\big) = aH^2\big(1 - \epsilon\big), \tag{3.1.23}$$

which means

$$\ddot{a} > 0 \Longleftrightarrow \epsilon < 1. \tag{3.1.24}$$

So usually when $\epsilon \gtrsim 1$ we can consider ended the epoch of inflation.

At the end of this short analysis on the slow-roll parameters we can see that they only restrict the form of the potential, but not the properties of dynamical solutions which instead are constrained by the second equation of (3.1.22). This implies that the scalar field evolves to approach an asymptotic attractor solution [78]. Although $\epsilon \ll 1$ and $|\eta| \ll 1$ are necessary conditions for the slow-rolls approximation to hold, they are not sufficient, since even if the potential is very flat it may be that the scalar field has a large velocity. A more elaborate version of the slow-roll approximation, based on the Hamilton-Jacobi formulation of inflation [79], exists and it is both sufficient and necessary [80].

3.2 Single harmonic oscillator

In order to compute the quantum fluctuations in the metric we have to quantize the fields. The way to do this, for both scalar and tensor perturbations, is to rewrite the problem so it looks like a harmonic oscillator. For this reason, before proceeding, we recall some basic facts about the quantization of this simple system.

The harmonic oscillator with unit mass and frequency ω is governed by the equation

$$\frac{d^2x}{dt^2} + w^2x = 0. \tag{3.2.1}$$

Upon quantization x becomes a quantum operator:

$$\hat{x} = v(\omega, t)\hat{a} + v^*(\omega, t)\hat{a}^\dagger, \tag{3.2.2}$$

where $v \propto e^{-i\omega t}$ is a solution of (3.2.1), $\hat{a}$ is the annihilation operator while $\hat{a}^\dagger$ is the creation one. These two operators satisfy the algebra relation

$$\big[\hat{a}, \hat{a}^\dagger\big] = 1, \tag{3.2.3}$$

while the other commutators vanish $\left[\hat{a}, \hat{a}\right] = 0 = \left[\hat{a}^\dagger, \hat{a}^\dagger\right]$. The average on the vacuum state is

$$\langle 0|\hat{x}|0\rangle = \langle 0|\left(v\hat{a} + v^*\hat{a}^\dagger\right)|0\rangle = 0 \tag{3.2.4}$$

because $\hat{a}|0\rangle = 0 = \langle 0|\hat{a}^\dagger$. While the quantum fluctuations on the ground state are non vanishing:

$$\begin{aligned}\langle 0|\hat{x}^\dagger\hat{x}|0\rangle &= \langle 0|\left(v^*\hat{a}^\dagger + v\hat{a}\right)\left(v\hat{a} + v^*\hat{a}^\dagger\right)|0\rangle \\ &= |v|^2\langle 0|\hat{a}\hat{a}^\dagger|0\rangle \\ &= |v|^2\langle 0|\left[\hat{a}, \hat{a}^\dagger\right] + \hat{a}^\dagger\hat{a}|0\rangle = |v|^2\langle 0|\left[\hat{a}, \hat{a}^\dagger\right]|0\rangle = |v|^2,\end{aligned} \tag{3.2.5}$$

where in the last step we used (3.2.3).

3.3 Quantum perturbations of the inflaton

Thanks to the previous summary on the properties of the quantum harmonic oscillator, now we can move towards the analogous but more laborious task of quantizing the scalar perturbations produced during inflation. We have previously seen that during inflation the Universe primarily consists of a uniform scalar field and a uniform background metric. Against this background, the fields fluctuate quantum mechanically. Now we have to write down the dynamical equation for the perturbations which can be obtained starting from the equation of motion of a scalar field (3.1.17). Using the expression we found in (3.1.18) we can write the Klein-Gordon equation as

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{\nabla^2\varphi}{a^2} = -\frac{\partial V}{\partial\varphi}. \tag{3.3.1}$$

Writing explicitly the value on the background and the perturbations $\varphi = \varphi_{(0)} + \delta\varphi$ and expanding the potential in Taylor series we get

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} - \frac{\nabla^2\delta\varphi}{a^2} = -\frac{\partial^2 V}{\partial\varphi^2}\left(\varphi_{(0)}\right)\delta\varphi. \tag{3.3.2}$$

In deriving the equation of motion for scalar perturbations we worked in the most general way never assuming for example slow-roll conditions are satisfied, this means that equation (3.3.2) holds for the every scalar field, in fact as already stated above $\varphi_{(0)}$ is the value on the background of a generic scalar field.

3.3.1 Quantum perturbations of the inflaton on large scales

As regards the inflaton, whose equation on the background is (3.1.19), we can achieve important information without doing any mathematically laborious calculation working on the regime in which $a^{-2}\nabla^2\delta\varphi$ is negligible in (3.3.2) and thus having

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} = -V''\delta\varphi. \tag{3.3.3}$$

To understand what it means that the spatial derivative term is negligible it is better to work in momentum space by operating a Fourier transform:

$$\frac{\nabla^2\delta\varphi(\vec{x},t)}{a^2} \quad \xrightarrow{\mathcal{F}} \quad \frac{k^2\tilde{\delta\varphi}(\vec{k},t)}{a^2}. \tag{3.3.4}$$

So neglecting the Laplacian means

$$\frac{k^2\tilde{\delta\varphi}(\vec{k},t)}{a^2} \ll 3H\dot{\tilde{\delta\varphi}} \sim 3H^2\tilde{\delta\varphi} \Longrightarrow k^2 \ll H^2a^2 = r_H^{-2}, \tag{3.3.5}$$

with r_H the comoving Hubble radius. Thus if we consider a scale $\lambda \propto k^{-1}$ which satisfies $\lambda \gg r_H$, we can neglect the term proportional to $\nabla^2\delta\varphi$ in equation (3.3.2) which becomes

$$\ddot{\delta\varphi} + 3H\dot{\delta\varphi} = -\frac{\partial^2 V}{\partial\varphi^2}\big(\varphi_{(0)}\big)\delta\varphi. \tag{3.3.6}$$

Now we take into account equation (3.1.19) and we derive it with respect to time, the result is

$$\frac{d^2}{dt^2}\big(\dot{\varphi}_0\big) + 3H\frac{d}{dt}\big(\dot{\varphi}_0\big) = -V''\big(\dot{\varphi}_0\big). \tag{3.3.7}$$

Hence equations (3.3.6) and (3.3.7) are formally equivalent to each other, except the fact that one is written for $\dot{\delta\varphi}$ while the other for φ_0. Moreover we can see by calculating the Wronskian that the solutions of (3.3.6) and (3.3.7) are not independent; it is well known that if the Wronskian matrix built with them has vanishing determinant, than they are dependent. In our case the Wronskian is not zero because

$$W\big(\dot{\varphi}_0, \delta\varphi\big) = \ddot{\varphi}_0\delta\varphi - \dot{\varphi}_0\dot{\delta\varphi} \tag{3.3.8}$$

but it has a peculiar behaviour, in fact its time derivative is

$$\dot{W}\big(\dot{\varphi}_0, \delta\varphi\big) = \dddot{\varphi}_0\delta\varphi - \dot{\varphi}_0\ddot{\delta\varphi} = -3HW\big(\dot{\varphi}_0, \delta\varphi\big), \tag{3.3.9}$$

using (3.3.3), (3.3.7). From this equation we get

$$W\big(\dot{\varphi}_0, \delta\varphi\big) = W_0\big(\dot{\varphi}_0, \delta\varphi\big)e^{-3Ht} \tag{3.3.10}$$

which implies that the Wronskian goes to zero on a time scale $t > H^{-1}$. This enables us to write

$$\delta\varphi(\vec{x},t) = -\delta t(\vec{x})\dot{\varphi}_0(t) \tag{3.3.11}$$

and consequently the scalar field becomes

$$\begin{aligned}\varphi(\vec{x},t) &= \varphi_0(t) + \delta\varphi(\vec{x},t)\\ &= \varphi_0(t) - \delta t(\vec{x})\dot{\varphi}_0(t)\\ &= \varphi_0(t - \delta t(\vec{x})),\end{aligned} \tag{3.3.12}$$

where we used the fact that the second line of the previous expression is the expansion in Taylor series of the third line. In conclusion we have found that on different regions of the Universe of typical size $\lambda \sim H^{-1}$ the scalar field driving inflation evolves in the same way (it takes the same values) but at slightly different times, due to its quantum fluctuations. This tells us that fluctuations of the inflaton field can be generated during an inflationary epoch on scales bigger than the horizon.

3.3.2 Quantum perturbations in momentum space

In the most general case we have to solve the equation (3.3.2) which is a second order differential equation. In order to make the equation easier to solve we move to momentum space by operating a 3d Fourier transform on the spatial coordinates, so we write

$$\delta\varphi(\vec{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\, e^{i\vec{k}\cdot\vec{x}}\tilde{\delta\varphi}(\vec{k},t), \tag{3.3.13}$$

where $\tilde{\delta\varphi}(\vec{k},t)$ is the Fourier transform of $\delta\varphi(\vec{x},t)$. Because the perturbation $\delta\varphi(\vec{x},t)$ is a real function, it follows directly from the properties of the Fourier transform that $\tilde{\delta\varphi}(\vec{k},t)^* = \tilde{\delta\varphi}(-\vec{k},t)$. Before showing which is the form this equation gets in momentum space, it is worthy to be more precise on what assumption we made in writing (3.3.13): clearly we used as Fourier modes the plane waves and this is due to our assumption that the spatial hypersurfaces of our FRW Universe are flat ($k = 0$), otherwise if the hypersurfaces were curved we would have used as modes the solutions of Helmoltz equation. From now on, to simplify the notation, we will omit the tilde on the top of the Fourier transform and we will write the dependence on the momentum by a subscript, so $\tilde{\delta\varphi}(\vec{k},t) \longrightarrow \delta\varphi_{\vec{k}}$. Now we can write equation (3.3.2) in momentum space as

$$\ddot{\delta\varphi}_{\vec{k}} + 3H\dot{\delta\varphi}_{\vec{k}} + \frac{k^2}{a^2}\delta\varphi_{\vec{k}} = -\frac{\partial^2 V}{\partial\varphi^2}\delta\varphi_{\vec{k}}. \tag{3.3.14}$$

The equation written above for the perturbations of a scalar field is not in the form of the harmonic oscillator equation (3.2.1), but we can massage it. The first thing we do is to rewrite the equation (3.3.14) using a different variable defined as $\check{\delta\varphi}(\vec{x},t) = a\,\delta\varphi(\vec{x},t)$. The equation we get for $\check{\delta\varphi}$ in momentum space is:

$$\frac{d^2}{dt^2}\left(\frac{\check{\delta\varphi}_{\vec{k}}}{a}\right)+3H\frac{d}{dt}\left(\frac{\check{\delta\varphi}_{\vec{k}}}{a}\right)+\frac{k^2}{a^2}\left(\frac{\check{\delta\varphi}_{\vec{k}}}{a}\right)= -\frac{\partial^2 V}{\partial\varphi^2}\left(\frac{\check{\delta\varphi}_{\vec{k}}}{a}\right), \tag{3.3.15}$$

which can be further simplified by working with the conformal time τ instead of the cosmic time t:

$$\frac{1}{a}\frac{d}{d\tau}\left[\frac{1}{a}\frac{d}{d\tau}\left(\frac{\delta\check{\varphi}_{\vec{k}}}{a}\right)\right]+3\left(\frac{1}{a^2}\frac{da}{d\tau}\right)\frac{1}{a}\frac{d}{d\tau}\left[\frac{\delta\check{\varphi}_{\vec{k}}}{a}\right]+\frac{k^2}{a^2}\frac{\delta\check{\varphi}_{\vec{k}}}{a}=-\frac{\partial^2 V}{\partial\varphi^2}\frac{\delta\check{\varphi}_{\vec{k}}}{a}$$

$$\delta\check{\varphi}''_{\vec{k}}-\frac{a''}{a}\delta\check{\varphi}_{\vec{k}}+k^2\delta\check{\varphi}_{\vec{k}}=-a^2\frac{\partial^2 V}{\partial\varphi^2}\delta\check{\varphi}_{\vec{k}}$$

$$\delta\check{\varphi}''_{\vec{k}}+\left(k^2-\frac{a''}{a}+a^2\frac{\partial^2 V}{\partial\varphi^2}\right)\delta\check{\varphi}_{\vec{k}}=0. \tag{3.3.16}$$

The equation for $\check{\varphi}$ is actually like the harmonic oscillator one so we can proceed with the usual quantization procedure. We write

$$\delta\check{\varphi}(\vec{x},t)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\left[u_k(t)\hat{a}_{\vec{k}}\,e^{i\vec{k}\cdot\vec{x}}+u^*_k(t)\hat{a}^\dagger_{\vec{k}}\,e^{-i\vec{k}\cdot\vec{x}}\right], \tag{3.3.17}$$

where $\hat{a}_{\vec{k}}$ and $\hat{a}^\dagger_{\vec{k}}$ are the annihilation and creation operators and they satisfy the algebra

$$\left[\hat{a}^\dagger_{\vec{k}},\hat{a}_{\vec{k}'}\right]=\delta^3(\vec{k}-\vec{k}'). \tag{3.3.18}$$

Instead the function $u_k(t)$ and $u^*_k(t)$ are functions of the cosmic time t and $k=|\vec{k}|$, so they inherit the isotropy and homogeneity of FRW Universe. In a spatially flat space-time the Klein-Gordon solution is

$$u_k(t)=\frac{e^{-i\omega_k t}}{\sqrt{2\omega_k}}\quad\text{with}\quad\omega_k=\sqrt{k^2+m^2}. \tag{3.3.19}$$

We expect to recover this result in the limit in which we can neglect the curvature of space-time, while in a more generic curved space the solution will be more complex.

The equation satisfied by the functions u_k is:

$$u''_k+\left(k^2-\frac{a''}{a}+a^2\frac{\partial^2 V}{\partial\varphi^2}\right)u_k=0. \tag{3.3.20}$$

Before explicitly solving the above equation for the amplitude u_k, we consider the case in which the scalar field is massless and the background is purely de Sitter ($H=const$), which means that equation (3.3.20) reduces to the simpler

$$u''_k+\left(k^2-\frac{a''}{a}\right)u_k=0, \tag{3.3.21}$$

with the scale factor a and the conformal time τ linked by:

$$\tau=\int\frac{dt}{a}=\int\frac{da}{aH^2}=-\frac{1}{aH}+\int\frac{da}{a}\frac{d}{da}\left[\frac{1}{H}\right]=-\frac{1}{aH}, \tag{3.3.22}$$

since H is a constant. Hence the ratio between a'' and a can be written as

$$\frac{a''}{a} = \left(-\frac{2}{\tau^3 H}\right)\left(-\frac{1}{\tau H}\right) = 2a^2H^2 = \frac{2}{\tau^2}. \tag{3.3.23}$$

Equation (3.3.21) can be consequently rewritten as

$$u_k'' + \left(k^2 - 2a^2H^2\right)u_k = 0. \tag{3.3.24}$$

We can study it in two different regimes: when the k^2 term is dominant hence for scale that are under the horizon or when the scales are above the horizon. The condition for the superhorizon or the subhorizon regime can be written in terms of the scale factor or the conformal time:

$$\text{Subhorizon:} \qquad k \gg aH, \qquad -k\tau \gg 1; \tag{3.3.25}$$

$$\text{Superhorizon:} \qquad k \ll aH, \qquad -k\tau \ll 1. \tag{3.3.26}$$

We can represent graphically this two regions:

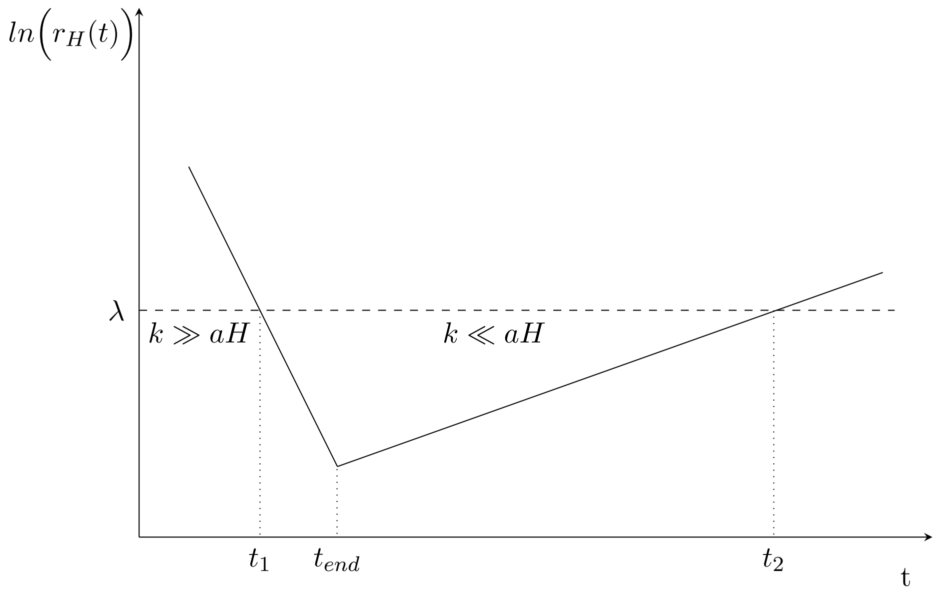

Figure 3.2: subhorizon and superhorizon regions.

In Figure 3.2 we consider a cosmological (comoving) scale λ which is initially smaller than the horizon ($k \gg aH$) then, during inflation, at the time t_1 it becomes bigger than the horizon ($k \ll aH$). While this scale λ is outside the horizon, inflation ends (t_{end}) and Universe starts to be dominated firstly by radiation and then by matter, so the comoving Hubble radius r_H grows and hence after enough time (t_2) the scale λ becomes observable, re-entering inside the horizon

In the subhorizon limit equations (3.3.24) becomes

$$u_k'' + k^2 u_k = 0, \tag{3.3.27}$$

which is solved by

$$u_k(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}. \tag{3.3.28}$$

This is a plane wave solution, which we should have expected since, for subhorizon scales, we can neglect the curvature of space obtaining the Klein-Gordon equation in a flat space-time. In the superhorizon limit the (3.3.24) equation becomes

$$u_k'' - \frac{2}{\tau^2} u_k = 0, \tag{3.3.29}$$

which is solved by

$$u_k(\tau) = A(k)\tau^2 + \frac{B(k)}{\tau} = B_+(k)a(\tau) + \frac{B_-(k)}{a^2(\tau)}. \tag{3.3.30}$$

This solution has two different modes, one increasing and one decreasing in time. Actually we are interested only in the increasing one since the other becomes negligible after enough time. So we can set to zero the decreasing mode B_- and write

$$u_k(\tau) \simeq B_+(k)a(\tau). \tag{3.3.31}$$

To determine the amplitude of this mode, B_+, we simply require that the two solutions in the subhorizon and superhorizon regime match at the time of horizon crossing which is defined as

$$a(t_k)H(t_k) = k. \tag{3.3.32}$$

From now on we will use the following notation for the Hubble rate evaluated at horizon crossing $H(t_k) = H_*$. If we now require that the moduli of the two solutions match at horizon crossing we get

$$|B_+(k)|\, a = \frac{1}{\sqrt{2k}}, \tag{3.3.33}$$

and using (3.3.32)

$$|B_+(k)| = \frac{H_*}{\sqrt{2k^3}}. \tag{3.3.34}$$

Actually this is not the solution for the physical perturbation of the scalar field. In fact at the beginning we changed the variable to $\breve{\delta\varphi}$ because it simplified the equations. Now that we have solved these equations we have to go back to the physical variable $\delta\varphi$ by dividing the result by the scale factor a. So the amplitude of the physical solution in the superhorizon regime is

$$|\delta\varphi_k| = \frac{u_k}{a} = \frac{|B_+(k)|\, a}{a} = |B_+(k)| = \frac{H}{\sqrt{2k^3}} \tag{3.3.35}$$

and we recall that in our approximation H is constant, hence $|\delta\varphi_k| = const$ when the scale is superhorizon. So we finally understand what happens in the two different regimes. When the scale is still under the horizon its amplitude oscillates as seen in (3.3.28) and, since the physical perturbation is given by u_k/a, the amplitude of its oscillations are damped. But, after the mode exits outside the horizon, the amplitude of the fluctuations becomes frozen to a constant. As a consequence of this mechanism, the mean value of the scalar perturbations on small, microscopic scales is zero, while on large scales we get a non zero value: quantum fluctuations of the field generate classical fluctuations when the scale becomes superhorizon.

Now we want to explicitly solve the equation for the u_k functions (3.3.20) because we will need its solution to find the power spectrum. Since the second derivative of the potential with respect to the field φ is the field mass, we call $M_\varphi^2 = \partial^2 V/\partial\varphi^2$. As regards the scale factor $a(\tau)$, in a quasi de Sitter Universe the expression in function of the conformal time τ becomes more complex than (3.3.22):

$$\begin{aligned}\tau &= \int \frac{dt}{a} = \int \frac{da}{aH^2} = -\frac{1}{aH} + \int \frac{da}{a}\frac{d}{da}\left[\frac{1}{H}\right] \\ &= -\frac{1}{aH} + \int \frac{da}{a}\frac{1}{\dot{a}}\frac{d}{dt}\left[\frac{1}{H}\right] = -\frac{1}{aH} + \epsilon \int \frac{da}{a^2 H},\end{aligned} \tag{3.3.36}$$

where ϵ is the slow-roll parameter defined in Section 3.1.2. Since in both sides of equation (3.3.36) there is the same integral we get

$$\tau = -\frac{1}{aH(1-\epsilon)}, \tag{3.3.37}$$

which can be rewritten as

$$a(\tau) = -\frac{1}{\tau H(1-\epsilon)}. \tag{3.3.38}$$

Now we can compute the first and the second derivative with respect the conformal time:

$$a' = \frac{1}{\tau^2 H(1-\epsilon)} + \frac{H'}{\tau H^2(1-\epsilon)} = \frac{1}{\tau^2 H(1-\epsilon)} + \frac{\epsilon}{\tau^2 H(1-\epsilon)^2}, \tag{3.3.39}$$

$$a'' = -\frac{2}{\tau^3 H(1-\epsilon)} - \frac{3\epsilon}{\tau^3 H(1-\epsilon)^2}, \tag{3.3.40}$$

where in (3.3.39) we used the fact that the explicit expression of ϵ in conformal is:

$$\epsilon = -\frac{\dot{H}}{H^2} \xrightarrow{t\to\tau} \epsilon = -\frac{1}{a(\tau)}\frac{H'}{H^2}. \tag{3.3.41}$$

Hence the ratio between the second derivative of a and the scale factor itself is

$$\frac{a''}{a} = \frac{1}{\tau^2}\left(2 + \frac{3\epsilon}{1-\epsilon}\right) \simeq \frac{2}{\tau^2}\left(1 + \frac{3}{2}\epsilon\right). \tag{3.3.42}$$

Now we have to express the term M^2a^2 in terms of the slow-roll parameters. From the definition of the slow-roll parameter η in Section 3.1.2 we see that it is linked to the mass of the scalar field by

$$\eta = \frac{1}{3}\frac{M^2}{H^2}. \tag{3.3.43}$$

From this it follows that

$$M^2a^2 \simeq \frac{1}{\tau^2}(3\eta - 6\epsilon), \tag{3.3.44}$$

so putting all the terms together we find

$$u_k'' + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\tau^2}\right)u_k = 0, \tag{3.3.45}$$

with $\nu = 3/2 + 3\epsilon - \eta$. This equation is actually the Bessel equation, in fact if we change the variable to $y_k(\tau) = u_k/\sqrt{-\tau}$, equation (3.3.45) becomes

$$\tau^2 y_k'' + \tau y_k' + (\tau^2k^2 - \nu^2)y_k = 0. \tag{3.3.46}$$

This equation is solved by a combination of the Bessel functions of the first kind J_ν and the Bessel functions of the second kind Y_ν multiplied by appropriate integration constants. Using the relation between u_k and y_k to express the former in function of the latter we are able to write down the solution of (3.3.45), which is

$$u_k(\tau) = \sqrt{-\tau}\Big[c_1(k)J_\nu(-k\tau) + c_2(k)Y_\nu(-k\tau)\Big]. \tag{3.3.47}$$

At this point we can introduce the Hankel functions which are given by a particular combination of the Bessel functions:

$$\begin{cases} H_\nu^{(1)} &= J_\nu + iY_\nu \\ H_\nu^{(2)} &= J_\nu - iY_\nu. \end{cases} \tag{3.3.48}$$

With these functions the solution can be rewritten as

$$u_k(\tau) = \sqrt{-\tau}\Big[C_1(k)H_\nu^{(1)}(-k\tau) + C_2(k)H_\nu^{(2)}(-k\tau)\Big]. \tag{3.3.49}$$

Since in our case ν is a real number the Hankel functions satisfy the relation

$$H_\nu^{(2)} = H_\nu^{(1)*}. \tag{3.3.50}$$

Moreover it is known that the asymptotic behaviour of the Hankel function of the first kind for a fixed value of the parameter ν is given by

$$H_\nu^{(1)}(x) = \begin{cases} \sqrt{\frac{2}{\pi x}}e^{i(x - \frac{\pi}{2}\nu - \frac{\pi}{4})} \sim \frac{1}{\sqrt{x}}e^{ix} & \text{for } x \to \infty, \\ \sqrt{\frac{2}{\pi}}e^{-i\frac{\pi}{2}}2^{\nu - \frac{3}{2}}\frac{\Gamma(\nu)}{\Gamma\left(\frac{3}{2}\right)}x^{-\nu} \sim x^{-\nu} & \text{for } x \to 0. \end{cases} \tag{3.3.51}$$

Therefore in the subhorizon limit the dominant term in equation (3.3.45) is k^2 so the equation and the solution are identical respectively to (3.3.27) and (3.3.28). Then, to recover the plane wave solution in the subhorizon limit, we have to impose that the integration constant are:

$$\begin{cases} C_1(k) &= \frac{\sqrt{\pi}}{2} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} \\ C_2(k) &= 0. \end{cases} \tag{3.3.52}$$

We can then write the general solution of (3.3.45) as

$$u_k(\tau) = \frac{\sqrt{\pi}}{2} e^{i(\nu+\frac{1}{2})\frac{\pi}{2}} \sqrt{-\tau} H_\nu^{(1)}(-k\tau). \tag{3.3.53}$$

In conclusion the amplitude on the superhorizon limit is given by

$$|u_k| = 2^{\nu-\frac{3}{2}-\frac{1}{2}} \frac{\Gamma(\nu)}{\Gamma\left(\frac{3}{2}\right)} k^{-\nu} \left(\frac{1}{aH(1-\epsilon)} \right)^{\frac{1}{2}-\nu}, \tag{3.3.54}$$

where we used the relation (3.3.37) between the conformal time and the scale factor. Now, as in the previous case where we neglected the mass term, we have to divide by the scale factor to find the amplitude of the physical perturbations and we find

$$|\delta\varphi_k| = \frac{|u_k|}{a} \simeq \frac{H}{\sqrt{2k^3}} \left(\frac{k}{aH} \right)^{-\epsilon}, \tag{3.3.55}$$

at the first order in the slow-roll parameters. Now the Hubble rate H is not constant because we are in a quasi de Sitter Universe, but it varies slowly $\dot{H} = -\epsilon H^2$ so at first sight we may say that the perturbations continue to evolve also when the scale becomes superhorizon. Nevertheless it is possible to show that in a quasi de Sitter Universe the Hubble rate is equal to

$$H \simeq H_* \left(\frac{k}{aH} \right)^{\epsilon}, \tag{3.3.56}$$

where $H_* = H(t_k)$ which is the value at the time of horizon crossing defined in (3.3.32). Thereby the amplitude of the perturbations becomes

$$|\delta\varphi_k| = \frac{H_*}{\sqrt{2k^3}}. \tag{3.3.57}$$

In conclusion, also in this more complete case, the amplitude of the scalar field perturbations is damped when the scale is in the subhorizon regime while it becomes constant once the scale λ has grown enough to become larger than the horizon.

3.4 Tensor perturbations

Now we consider the tensor perturbations to the metric and we want to understand how they evolve during inflation. Gravity waves (the tensor metric perturbations) are not coupled to the density, as we have seen in Section 2.7.2, and so they are not responsible for the large-scale structure of the Universe but they induce fluctuations in the CMB. These fluctuations turn out to be a unique signature of inflation and offer one of the best windows on the physics driving inflation. We remember from the previous section on the cosmological perturbations that these tensor modes are described by two degrees of freedom which we called h_a with $a = +, \times$. We also have seen that they both obey to the equation (2.7.19):

$$h_a'' + 2\mathcal{H}h_a' + k^2 h_a = 0. \tag{3.4.1}$$

We would like to massage this equation into the form of an harmonic oscillator, so we can easily quantize it. We define

$$\tilde{h}_a = \frac{a(\tau)}{\sqrt{32\pi G}} h_a, \tag{3.4.2}$$

where the $1/\sqrt{32\pi G}$ is introduced in order to have a canonically normalized kinetic term in the action [76]. The derivatives with respect to the conformal time τ can be written as:

$$\frac{h'}{\sqrt{32\pi G}} = \left(\frac{\tilde{h}}{a}\right)' = \frac{\tilde{h}'}{a} - \frac{a'}{a^2}\tilde{h}, \tag{3.4.3}$$

$$\frac{h''}{\sqrt{32\pi G}} = \frac{\tilde{h}''}{a} - 2\frac{a'}{a^2}\tilde{h}' - \frac{a''}{a^2}\tilde{h} + 2\frac{(a')^2}{a^3}\tilde{h}, \tag{3.4.4}$$

where we omitted the label of the polarization a for simplicity. We now can rewrite equation (3.4.2) using this new variable $\tilde{h}$:

$$\tilde{h}'' + \left(k^2 - \frac{a''}{a^2}\right)\tilde{h} = 0, \tag{3.4.5}$$

which is an equation like (3.2.1). We can now quantize the field $\tilde{h}$ in the usual way

$$\hat{\tilde{h}}(\tau, \vec{k}) = v(\tau, k)\hat{a}_{\vec{k}} + v^*(\tau, k)\hat{a}^\dagger_{\vec{k}}, \tag{3.4.6}$$

where the coefficients of the creation and annihilation operators satisfy the equation

$$v'' + \left(k^2 - \frac{a''}{a}\right)v = 0. \tag{3.4.7}$$

In conclusion we obtained the same equation as the one we obtained for the perturbations of the scalar field in (3.3.20), but this time the mass term is zero. During inflation we can write $a' = a^2 H \sim -a/\tau$ and hence $a''/a \sim 2/\tau^2$ so equation (3.4.7) becomes

$$v'' + \Big(k^2 - \frac{2}{\tau^2}\Big)v = 0. \tag{3.4.8}$$

If we consider this equation at very early times, before inflation has done most of its work and when the mode k is still inside the horizon, at that time $-1/\tau \ll k$ so the k^2 term dominates and (3.4.8) reduces to

$$v'' + k^2 v = 0, \tag{3.4.9}$$

which is solved by

$$v \propto \frac{e^{ik\tau}}{\sqrt{2k}}. \tag{3.4.10}$$

Hence, since the solution of the complete equation (3.4.8) must coincide with (3.4.10) in the limit in which the k term dominates, we find that

$$v = \frac{e^{-ik\tau}}{\sqrt{2k}}\Big(1 - \frac{i}{k\tau}\Big). \tag{3.4.11}$$

The behaviour of this solution is analogous to the one we found for the scalar perturbations, in fact when inflation has stretched the mode k to be larger than the horizon, so in the limit $-k\tau \to 0$, we find that the amplitude of $h \propto \tilde{h}/a$ becomes constant because

$$\lim_{-k\tau \to 0} v(k,\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\Big(-\frac{i}{k\tau}\Big), \tag{3.4.12}$$

so since $a = -1/\tau H$ we have

$$h \propto \frac{He^{-ik\tau}}{k^{\frac{3}{2}}}. \tag{3.4.13}$$

while when the mode is still inside the horizon, we have $h \propto 1/a$, so the amplitudes of the mode is reduced.

3.5 Power spectrum

Now that we have studied how perturbations evolve during inflation, we want to find an observable that can allow us to confront our theoretical results with the experimental data. Since the mean value of the fluctuations is expected to be zero, we need to consider different statistical variables such as the variance. Here we consider the perturbation of a generic function depending on time and spatial coordinates, $\delta f(t,\vec{x})$, whose Fourier transform is given by

$$\tilde{\delta f}(t,\vec{k}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3x e^{-i\vec{k}\cdot\vec{x}}\delta f(t,\vec{x}). \tag{3.5.1}$$

The Fourier transform of the two point function is

$$\langle \tilde{\delta f}(t, \vec{k}) \, \tilde{\delta f}^*(t, \vec{k'}) \rangle = \mathcal{P}(k) \delta^3(\vec{k} - \vec{k'}), \tag{3.5.2}$$

where the function $\mathcal{P}(k)$ is called power spectrum of the perturbation δf.

Usually, instead of working directly with the power spectrum $\mathcal{P}$, it is convenient to introduce a dimensionless function defined as

$$\triangle^2(k) = \frac{k^3}{2\pi^2} \mathcal{P}(k). \tag{3.5.3}$$

Now we want to evaluate the power spectrum for the two kinds of perturbations we have previously studied: scalar and tensor.

3.5.1 Power spectrum for scalar field perturbations

The two point function for scalar perturbations on the superhorizon region ($-k\tau \to 0$) is given by

$$\langle 0 | \delta\varphi_{\vec{k}} \delta\varphi^*_{\vec{k'}} | 0 \rangle = \left| \delta\varphi_{\vec{k}} \right|^2 \langle 0 | a(\vec{k}) a^\dagger(\vec{k'}) | 0 \rangle = \frac{H_*^2}{2k^3} \delta^3(\vec{k} - \vec{k'}), \tag{3.5.4}$$

where in the last step we used the commutation relations for the annihilation and creation operators written in (3.3.18) and the explicit expression for the perturbation amplitude of a scalar field in the superhorizon limit (3.3.57). Consequently we can write the power spectrum for the scalar field perturbations as

$$\mathcal{P}_{\delta\varphi}(k) = \frac{H_*^2}{2k^3}, \tag{3.5.5}$$

thus the dimensionless $\triangle^2_{\delta\varphi}$ is given by

$$\triangle^2_{\delta\varphi}(k) = \left(\frac{H_*}{2\pi} \right)^2. \tag{3.5.6}$$

Instead of working with the power spectrum of the scalar field perturbations $\delta\varphi$, it is preferable to work with the one of the curvature perturbation on uniform density hypersurfaces ζ which we defined previously in (2.6.47). The reason of doing this is that for models of single field inflation ζ is constant on superhorizon scales. Hence the power spectrum for the scalar curvature perturbation at horizon crossing becomes

$$\mathcal{P}_\zeta(k) = \frac{H_*^2}{\dot{\varphi}^2} \mathcal{P}_{\delta\varphi}(k) = \frac{H_*^4}{2k^3 \dot{\varphi}^2}, \tag{3.5.7}$$

and consequently

$$\triangle^2_\zeta(k) = \left(\frac{H_*^2}{2\pi\dot{\varphi}} \right)^2. \tag{3.5.8}$$

3.5.2 Power spectrum for tensor perturbations

The calculation of the power spectrum for the tensor perturbation follows straightforwardly from the one for the scalar perturbation, since we have seen previously that the solution of the equation for the tensor perturbation modes in the superhorizon limit (3.4.13) has the same form of the solution for the scalar perturbation equation and differs only in the normalization factor:

$$h_a = \sqrt{32\pi G}\frac{H_*}{\sqrt{2k^3}}e^{-ik\tau}. \tag{3.5.9}$$

Consequently we have that the power spectrum for tensor perturbation is

$$\mathcal{P}_{h_a}(k) = |h_a|^2 = 32\pi G\frac{H_*^2}{2k^3}, \tag{3.5.10}$$

consequently

$$\triangle^2_{h_a}(k) = \frac{8}{\pi M_{Pl}^2}H_*^2, \tag{3.5.11}$$

where we used the relation $G = 1/M_{Pl}^2$. This results is valid for each polarization state of the tensor modes, so considering both the polarization states we get

$$\triangle^2_{h}(k) = \frac{16}{\pi M_{Pl}^2}H_*^2. \tag{3.5.12}$$

3.6 Spectral indices and the consistency relation

We introduce the following quantities

$$n_s - 1 = \frac{d}{d(\ln k)}\Big[\ln\left(\triangle^2_\zeta(k)\right)\Big], \tag{3.6.1}$$

$$n_t = \frac{d}{d(\ln k)}\Big[\ln\left(\triangle^2_h(k)\right)\Big], \tag{3.6.2}$$

which are called spectral indices because they tell us in which way the quantity $\triangle^2(k)$ depends on k. In fact if we consider the case in which both n_s and n_t are constant (we will see that actually it is so since they can be written as a combination of the slow-roll parameters which vary slowly during inflation) we can integrate the two definitions finding

$$\triangle^2_\zeta(k) = A_\zeta\left(\frac{k}{k_0}\right)^{n_s-1}, \tag{3.6.3}$$

$$\triangle^2_h(k) = A_h\left(\frac{k}{k_0}\right)^{n_t}, \tag{3.6.4}$$

where A_ζ and A_h are two constants coming from the integration. Therefore, looking at the expressions above, we understand that, depending on the value of the spectral indexes, we have different dependences of the functions $\triangle^2(k)$ on k. In particular we see that the dependence on k vanishes for the scalars when $n_s = 1$ (Harrison-Zel'dovich spectrum) and for the tensors when $n_t = 0$.

Now we have to justify the assumption made of almost constant spectral indices by calculating their values in term of slow-roll parameters directly from their definition. First of all we can show that the derivative with respect to $\ln k$ can be rewritten as a derivative with respect to time. We start from the relation between k and H in a almost de Sitter Universe ($H \simeq const$)

$$k = aH = e^{Ht}H \tag{3.6.5}$$

and from this relation we can write

$$d\ln(k) = Hdt. \tag{3.6.6}$$

This result allows us to write:

$$\begin{aligned} n_s - 1 &\simeq \frac{d}{Hdt}\left[\ln\left(\triangle^2_\zeta(k)\right)\right] = \frac{d}{Hdt}\left[\ln\left(\frac{H^2}{2\pi\dot{\varphi}}\right)^2\right] = 4\frac{\dot{H}}{H^2} - 2\frac{\ddot{\varphi}}{H\dot{\varphi}} = \\ &= 2\eta - 6\epsilon, \end{aligned} \tag{3.6.7}$$

$$\begin{aligned} n_t &\simeq \frac{d}{Hdt}\left[\ln\left(\triangle^2_h(k)\right)\right] = \frac{d}{Hdt}\left[\ln\left(\frac{16}{\pi M_{Pl}^2}H^2\right)\right] = 2\frac{\dot{H}}{H^2} = \\ &= -2\epsilon, \end{aligned} \tag{3.6.8}$$

where we used the two expression for the $\triangle^2(k)$ functions (3.5.8) and (3.5.12), the expressions for the slow-roll parameters introduced in 3.1.2 and the relation $-\ddot{\varphi}/H\dot{\varphi} = \eta - \epsilon$ which is derived for example in [81].

One of the most important quantities to confront with observations is the ratio between the tensor and the scalar power spectrum. Before proceeding we rewrite the $\triangle^2_\zeta$ in terms of the slow-roll parameter ϵ:

$$\epsilon = -\frac{\dot{H}}{H^2} = 4\pi G\frac{\dot{\varphi}^2}{H^2} = \frac{4\pi}{M_{Pl}^2}\frac{\dot{\varphi}^2}{H^2}, \tag{3.6.9}$$

so

$$\triangle^2_\zeta(k) = \frac{H_*^2}{\pi M_{Pl}^2\epsilon}, \tag{3.6.10}$$

where we recall that $M_{Pl}^2 = 1/G$. Thereby using this expression for the scalar field perturbations and (3.5.12) for the tensor perturbations we find:

$$r = \frac{\triangle^2_h(k)}{\triangle^2_\zeta(k)} = 16\epsilon. \tag{3.6.11}$$

This result can also be expressed using n_t. Indeed, looking at the result we have found above for the spectral index for tensor modes in (3.6.8), it is clear that

$$r = -8n_t. \tag{3.6.12}$$

This relation is the so called "consistency relation". This relation is independent on the form of the potential and valid for all the single-field slow-roll inflation models with canonical kinetic terms [82]. So testing this relation provides a model-independent criteria to confirm or rule out the canonical single-field slow-roll inflation models. Actually this quantity, the ratio r, is what we can measure from the observations of the CMB power spectrum, so the gravitational wave tilt n_t is fixed once this ratio is known. Moreover its value not only gives us information about the validity of the inflation model but also it is directly linked to the value of the potential of the scalar field [76]

$$V^{1/4} \sim \left(\frac{r}{0,01}\right)^{1/4} 10^6 \text{ Gev}, \tag{3.6.13}$$

so its value gives also information about the energy scale at which inflation takes place.

Chapter 4

Effective field theory approach

In this Chapter we will study the theory of inflation with an effective field theory approach. This formalism was applied to the perturbations problem in cosmology only recently, in fact it made its first appearance in [83], but since then it has been developed and now is one of the most important tools we have to study the various inflationary models. Actually it was firstly used to study the coupling of the ghost condensate to gravity, but then it was applied to inflation in [84, 85] and more systematically developed in [86].

Many successful effective field theories have already been used in different areas of physics as in particle and nuclear physics and also condensed matter. The Standard Model of particle physics is itself an example of an effective field theory. In short, the basic idea of this approach is to write a general theory relying only upon the symmetries of the system for which it is clear at which energy scale each term contributes. In fact to build an effective field theory describing physics at a given energy scale E, one makes an expansion in powers of E/Λ where Λ is the scale involved in the process which is larger than E. The operators can be organized in terms of an increasing number of derivatives, or equivalently in powers of momentum; in the low-energy domain we are interested in, the terms with lower dimension will dominate. Hence, an effective field theory can be considered as a low energy approximation of a more general theory.

What we are going to do in this Chapter is to write the most general theory describing the fluctuations around a quasi de Sitter background using the underlying symmetries of the theory. The reason we need such an approach is that at the moment we don't have a thorough knowledge of the physics at Planckian energies, but we still want to study the dynamics of inflation at energies of the order of H, the Hubble constant. We will show that we can encode all the deviations from a standard slow-roll scenario in the size of higher order operators, similarly to what happens in the study of the Standard Model of particle physics. Then experiments will put bounds on the various operators, for example with measurements of the non-Gaussianity of perturbations and studying the deviation from the consistence relation for the gravitational wave tilt.

The advantages of such approach are that we have one unified theory, able to describe the whole physics of inflation. We can parametrize all the models in terms of some known operators so imposing that some operators get larger and other can be neglected we will be able to recover the various inflationary models studied in literature such as DBI inflation [87, 88] or Ghost Inflation [89].

As a first step we will focus on the derivation of the effective action for inflation, closely following [86]. Afterwards we will consider a few limits and result, highlighting which terms contribute to scalar perturbations and which to tensor perturbations in order to compute a generalized consistency relation.

4.1 Construction of the action

In this section we will understand how to write the effective action for inflation. Let us already introduce the final result:

$$\begin{aligned}\mathcal{S} = \int d^4x\,\sqrt{-g}\,\Bigg[&\frac{1}{2}M_{Pl}^2 R + M_{Pl}^2 \dot{H} g^{00} - M_{Pl}^2(3H^2 + \dot{H}) + \\ &+ \frac{1}{2!}M_2(t)^4(g^{00}+1)^2 + \frac{1}{3!}M_3(t)^4(g^{00}+1)^3 - \frac{\bar{M}_1(t)^3}{2}(g^{00}+1)\delta K^\mu_\mu + \\ &- \frac{\bar{M}_2(t)^2}{2}\left(\delta K^\mu_\mu\right)^2 - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \dots\Bigg], \end{aligned} \tag{4.1.1}$$

where $M_{1,2,3}$ and $\bar{M}_{1,2,3}$ are time-dependent mass scales, $\delta K_{\mu\nu} = K_{\mu\nu} - a^2 H h_{\mu\nu}$ is the perturbation of the extrinsic curvature of constant time surfaces and the dots stand for terms which are of higher order in the fluctuations or with more derivatives. Here we adopted the same convention of [86] for $M_{Pl}^2 = 1/(8\pi G)$. The first term in (4.1.1) is the well-known Einstein-Hilbert action.

From the previous parts, especially the one on the cosmological perturbations, it is clear that we can write the scalar field driving inflation as $\varphi(\vec{x},t) = \varphi_0(t) + \delta\varphi(\vec{x},t)$. Furthermore, from the study of gauge transformations, we know that the perturbation $\delta\varphi$ is a scalar only under spatial diffeomorphisms while it transforms non-linearly with respect to time diffeomorphisms:

$$t \to t + \xi^0(\vec{x},t) \qquad \delta\varphi \to \delta\varphi + \dot{\varphi}_0(t)\xi^0. \tag{4.1.2}$$

This suggests to choose a gauge in which the are not perturbations of the scalar field, in other words $\varphi(\vec{x},t) = \varphi_0(t)$, hence all degrees of freedom are in the metric. This is the so-called comoving or unitary gauge. Since our theory is invariant only under spatial diffeomorphisms, it means that there is a preferred slicing of space-time given by surfaces of constant value of $\tilde{t}(\vec{x})$, which corresponds to surfaces of constant value of the scalar field. Comoving gauge is the one in which the time coordinate t is chosen to coincide with $\tilde{t}$. As time diffeomorphisms have been fixed and are not a gauge symmetry anymore, the graviton now has three degrees of freedom (the scalar mode and the two tensor helicities): the scalar perturbation $\delta\varphi$ has been eaten by the graviton. The aim now is to write the most general Lagrangian in this gauge: we have to write down operators that are functions of the metric $g_{\mu\nu}$ and are invariant under the time dependent spatial diffeormophisms $x^i \to x^i + \xi^i(\vec{x}, t)$. Hence, besides of the usual terms with the Riemann tensor, which are invariant under all diffeomorphism, many extra terms are now allowed, because of the reduced symmetry of the system. They describe the additional degree of freedom eaten by the graviton. For example g^{00} is scalar under spatial diffeomorphisms so that it can appear freely in the Lagrangian. Moreover we have seen that there is a preferred slicing of the space-time, this allows us to use geometric objects describing this slicing like the extrinsic curvature $K_{\mu\nu}$. We are ready to review which terms can appear in the action.

1. Terms which are invariant under all kind of diffeomorphisms: these are just polynomials of the Riemann tensor $R_{\mu\nu\alpha\beta}$ and of its covariant derivatives contracted to give scalars.

2. We are free to use generic functions $f(\tilde{t})$, which in unitary gauge become generic functions of time $f(t)$, in front of any terms in the Lagrangian.

3. The gradient $\partial_\mu \tilde{t}$ becomes δ^0_μ in unitary gauge. Thus in every tensor we can always leave free an upper 0 index. For example we can use g^{00} and functions of it or the component of the Ricci tensor R^{00}.

4. It is useful to define a unit vector perpendicular to surfaces of constant $\tilde{t}$
$$n_\mu = \frac{\partial_\mu \tilde{t}}{\sqrt{-g^{\mu\nu}\partial_\mu \tilde{t}\partial_\nu \tilde{t}}}. \tag{4.1.3}$$
This allows to define the induced spatial metric on surfaces of constant $\tilde{t}$: $h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$. Every tensor can be projected on the surfaces using $h_{\mu\nu}$. In particular we can use in our action the Riemann tensor of the induced 3d metric ${}^{(3)}R_{\mu\nu\alpha\beta}$ and covariant derivatives with respect to the 3d metric.

5. Additional possibilities will come from the covariant derivative of $\partial_\mu \tilde{t}$ or equivalently covariant derivatives of n_μ: the derivative acting on the

normalization factor just gives terms like $\partial_\mu g^{00}$ which are covariant on their own and can be used in the unitary gauge Lagrangian. The covariant derivative of n_μ projected on the surfaces of constant $\tilde{t}$ gives the extrinsic curvature of these surfaces

$$K_{\mu\nu} \equiv h_\mu^\sigma \nabla_\sigma n_\nu. \tag{4.1.4}$$

The index ν is already projected on the surface because $n^\nu \nabla_\sigma n_\nu = \frac{1}{2}\nabla_\sigma(n^\nu n_\nu) = 0$. The covariant derivative of n_ν perpendicular to the surface can be rewritten as

$$\begin{aligned} n^\sigma \nabla_\sigma n_\nu &= n^\sigma \partial_\sigma n_\nu - n^\sigma \Gamma^\alpha_{\sigma\nu} n_\alpha \\ &= n^\sigma \partial_\sigma \left(\frac{\delta_\nu^0}{(-g^{00})^{\frac{1}{2}}} \right) - \frac{1}{(-g^{00})} \delta_\sigma^0 \delta_\alpha^0 \Gamma^\alpha_{\sigma\nu} \\ &= -\frac{1}{2}\frac{1}{(-g^{00})} n^\sigma n_\nu \partial_\sigma(-g^{00}) + \frac{1}{2}\frac{1}{(-g^{00})} \partial_\nu (g^{00}) \\ &= -\frac{1}{2}\frac{1}{(-g^{00})} (\delta^\sigma \nu + n^\sigma n_\nu) \partial_\sigma(-g^{00}) \\ &= -\frac{1}{2}\frac{1}{(-g^{00})} h_\nu^\sigma \partial_\sigma(-g^{00}), \end{aligned} \tag{4.1.5}$$

so it doesn't give rise to new terms. Therefore all covariant derivatives of n_μ can be written using the extrinsic curvature $K_{\mu\nu}$ (and its covariant derivatives) and derivatives of g^{00}.

6. Notice that using at the same time the Riemann tensor of the induced 3d metric and the extrinsic curvature is redundant as ${}^{(3)}R_{\mu\nu\alpha\beta}$ can be written using the Gauss-Codacci relation as shown in [51]

$${}^{(3)}R_{\alpha\beta\gamma\delta} = h_\alpha^\mu h_\beta^\nu h_\gamma^\rho h_\delta^\sigma R_{\mu\nu\rho\sigma} - K_{\alpha\gamma}K_{\beta\delta} + K_{\beta\gamma}K_{\alpha\delta}. \tag{4.1.6}$$

Thus one can forget about the 3d Riemann tensor altogether. We can also avoid to use the induced metric $h_{\mu\nu}$ explicitly: written in terms of the 4d metric $g_{\mu\nu}$ and n_μ one gets only terms already discussed above. Finally also the use of covariant derivatives with respect to the induced 3d metric can be avoided: the 3d covariant derivative of a projected tensor can be obtained as the projection of the 4d covariant derivative as shown in [51].

We conclude that the most generic action in unitary gauge is given by:

$$\mathcal{S} = \int d^4x \sqrt{-g}\, F(R_{\mu\nu\rho\sigma}, g^{00}, K_{\mu\nu}, \nabla_\mu, t), \tag{4.1.7}$$

where all the free indices inside the function F must be upper 0. We can write explicitly some operators included inside $F(R_{\mu\nu\rho\sigma}, g^{00}, K_{\mu\nu}, \nabla_\mu, t)$:

$$\begin{aligned}\mathcal{S} = \int d^4x \sqrt{-g} \Big[& \frac{1}{2} M_{Pl}^2 R - c(t) g^{00} - \Lambda(t) + \frac{1}{2!} M_2(t)^4 (g^{00}+1)^2 + \\ & + \frac{1}{3!} M_3(t)^4 (g^{00}+1)^3 - \frac{\bar{M}_1(t)^3}{2} (g^{00}+1) \delta K^\mu_\mu + \\ & - \frac{\bar{M}_2(t)^2}{2} \big(\delta K^\mu_\mu\big)^2 - \frac{\bar{M}_3(t)^2}{2} \delta K^\mu_\nu \delta K^\nu_\mu + \dots \Big]. \end{aligned} \tag{4.1.8}$$

Notice that only the first three terms in the action above contain linear perturbations around a chosen FRW solution, all the others are explicitly quadratic or higher. Therefore the coefficients $c(t)$ and $\Lambda(t)$ will be fixed by the requirement of having a given FRW evolution $H(t)$. Actually we would expect that there is an infinite number of operators which give a contribution at first order around the background solution, but in Appendix 7.2 we show that all the linear terms besides the ones in (4.1.8) can be integrated by parts to give a combination of the three linear terms we considered plus higher order terms. We conclude that the unperturbed history fixes $c(t)$ and $\Lambda(t)$, while the difference among different models will be encoded into higher order terms.

4.1.1 Fixing the background terms

Now we can fix the linear terms imposing that a given FRW evolution is a solution. As we discussed, the terms proportional to $c(t)$ and $\Lambda(t)$ are the only ones that give a energy-momentum tensor

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{S}_{matter}}{\delta g^{\mu\nu}}, \tag{4.1.9}$$

with

$$\mathcal{S}_{matter} = \int d^4x \sqrt{-g} \big(- c(t) g^{00} - \Lambda(t) \big). \tag{4.1.10}$$

Hence $T_{\mu\nu}$ does not vanish at zeroth order in the perturbations and therefore contributes to the right hand side of the Einstein equations. The components of the energy-momentum tensor are

$$
\begin{aligned}
T_{00} &= -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{00}}\Big[-c(t)g^{00}-\Lambda(t)\Big]-\frac{2}{\sqrt{-g}}\sqrt{-g}\frac{\delta}{\delta g^{00}}\Big[-c(t)g^{00}-\Lambda(t)\Big] \\
&= -g_{00}g^{00}c(t)-g_{00}\Lambda(t)+2c(t) \\
&= c(t)+\Lambda(t), \qquad (4.1.11) \\
T_{ij} &= -\frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}}{\delta g^{ij}}\Big[-c(t)g^{00}-\Lambda(t)\Big]-\frac{2}{\sqrt{-g}}\sqrt{-g}\frac{\delta}{\delta g^{ij}}\Big[-c(t)g^{00}-\Lambda(t)\Big] \\
&= -\frac{2}{\sqrt{-g}}\Big(-\frac{1}{2}g_{ij}\Big)\sqrt{-g}\Big(-c(t)g^{00}-\Lambda(t)\Big) \\
&= a^2\big[2c(t)-\Lambda(t)\big]\delta_{ij}, \qquad (4.1.12)
\end{aligned}
$$

where we used

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}. \qquad (4.1.13)$$

As regards the left side of the Einstein equations, in a flat FRW Universe we have

$$G_{00} = 3H^2, \qquad (4.1.14)$$

$$G_{ij} = -a^2\big[2\dot{H}+3H^2\big]\delta_{ij}, \qquad (4.1.15)$$

see Appendix 7.3 for full calculations. Hence, using the relation $8\pi G = M_{Pl}^{-2}$, we can write

$$3H^2 = \frac{1}{M_{Pl}^2}\big[c(t)+\Lambda(t)\big], \qquad (4.1.16)$$

$$2\dot{H}+3H^2 = \frac{1}{M_{Pl}^2}\big[\Lambda(t)-2c(t)\big], \qquad (4.1.17)$$

which can be rearranged to get the Friedmann equations:

$$H^2 = \frac{1}{3M_{Pl}^2}\big[c(t)+\Lambda(t)\big], \qquad (4.1.18)$$

$$\frac{\ddot{a}}{a} = \dot{H}+H^2 = -\frac{1}{3M_{Pl}^2}\big[2c(t)-\Lambda(t)\big]. \qquad (4.1.19)$$

Solving these two equations for $c(t)$ and $\Lambda(t)$ we obtain

$$c(t) = -M_{Pl}^2\dot{H}, \qquad (4.1.20)$$

$$\Lambda(t) = M_{Pl}^2\big[\dot{H}+3H^2\big]. \qquad (4.1.21)$$

Hence we can rewrite the action in unitary gauge (4.1.8) as

$$\begin{aligned}\mathcal{S} = \int d^4x\,\sqrt{-g}\,\Big[&\frac{1}{2}M_{Pl}^2 R + M_{Pl}^2 \dot{H} g^{00} - M_{Pl}^2(3H^2+\dot{H}) + \\ &+\frac{1}{2!}M_2(t)^4(g^{00}+1)^2 + \frac{1}{3!}M_3(t)^4(g^{00}+1)^3 - \frac{\bar{M}_1(t)^3}{2}(g^{00}+1)\delta K^\mu_\mu + \\ &-\frac{\bar{M}_2(t)^2}{2}\left(\delta K^\mu_\mu\right)^2 - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \dots\Big],\end{aligned} \tag{4.1.22}$$

where in the first line there is the contribution to the background and from the second line start operators of second order or higher in the perturbations. As already said, all the coefficients of the operators in the action above may have a generic time dependence. However we are interested in solutions where H and $\dot{H}$ do not vary significantly in one Hubble time. Therefore it is natural to assume that the same holds for all the other operators.

It is important to stress that this approach describe the most generic action not only for the scalar mode, but also for gravity. High energy effects will be encoded for example in operators containing the perturbations in the Riemann tensor $\delta R_{\mu\nu\rho\sigma}$. As these corrections are of higher order in derivatives, we will not explicitly talk about them.

We also recall that the action (4.1.22), which can be written in a more compact way as

$$\begin{aligned}\mathcal{S} = \int d^4x\,\sqrt{-g}\,\Big[&\frac{1}{2}M_{Pl}^2 R + M_{Pl}^2 \dot{H} g^{00} - M_{Pl}^2(3H^2+\dot{H}) + \\ &+ F^{(2)}\left(\delta g^{00}, \delta K_{\mu\nu}, \delta R_{\mu\nu\rho\sigma}; \nabla_\mu; t\right)\Big],\end{aligned} \tag{4.1.23}$$

with $F^{(2)}$ starting at second order in perturbations, encompasses all the possible models for inflation. For example a model with minimal kinetic term and a slow-roll potential $V(\varphi)$ can be written in unitary gauge as

$$\int d^4x\,\sqrt{-g}\,\Big[-\frac{1}{2}(\partial\varphi)^2 - V(\varphi)\Big] \longrightarrow \int d^4x\,\sqrt{-g}\,\Big[-\frac{\dot{\varphi}_0(t)^2}{2}g^{00} - V\big(\varphi_0(t)\big)\Big]. \tag{4.1.24}$$

As the Friedmann equations give

$$\dot{\varphi}_0(t)^2 = -2M_{Pl}^2\dot{H}, \qquad V\big(\varphi(t)\big) = M_{Pl}^2(3H^2+\dot{H}), \tag{4.1.25}$$

we see that the action (4.1.24) is of the form (4.1.22) with all but the first three terms set to zero: this tells us that the standard slow-roll inflation can be achieved starting from the most general effective action by setting to zero all the coefficients $M_{2,3,..}$ and $\bar{M}_{1,2,...}$. Clearly this cannot be true exactly as all the other terms will be generated by loop corrections: they encode all the possible effects of high energy physics on this simple slow-roll model of inflation.

4.2 The Stückelberg mechanism

In order to get the result (4.1.22) we had to pick up a specific gauge. In this so called unitary gauge, the scalar degree of freedom has been eaten by the metric. We will now use a trick that will restore the gauge invariance of the action and let the scalar mode explicitly appear, it is the so-called Stückelberg mechanism[1].The scalar mode will be reintroduced in the action after we perform a broken time diffeomorphism as the Goldstone boson which non-linearly realizes this symmetry. The Goldstone is associated to the spontaneously breaking of the time translations. In fact the expansion during inflation is quasi-de Sitter, because of the empirical evidence of a red tilt in the primordial power spectrum and because we need to exit the accelerating phase at some point, hence time translations, which are unbroken in de Sitter space, must be spontaneously broken [91]. The advantage of performing the Stückelberg mechanism is that the physics of the Goldstone decouples from the two graviton helicities at short distances, when the mixing can be neglected, in analogy with the equivalence theorem for the longitudinal components of a massive gauge boson [92].

First of all, in order to explain how this procedure works, we consider the particular case in which our action contains only the two following operators

$$\int d^4x \sqrt{-g}\left[A(t) + B(t)g^{00}(x)\right] \tag{4.2.1}$$

and we study its behavior under a broken time diffeomorphism

$$\begin{aligned} t &\to \tilde{t} = t + \xi^0(x), \\ x^i &\to \tilde{x}^i = x^i. \end{aligned} \tag{4.2.2}$$

g^{00} transforms as:

$$g^{00} \longrightarrow \tilde{g}^{00}\big(\tilde{x}(x)\big) = \frac{\partial \tilde{x}^0(x)}{\partial x^\mu}\frac{\partial \tilde{x}^0(x)}{\partial x^\nu} g^{\mu\nu}(x). \tag{4.2.3}$$

Then the action (4.2.1) written in terms of the transformed fields is given by

$$\int d^4x \sqrt{-\tilde{g}\big(\tilde{x}(x)\big)} \left|\frac{\partial \tilde{x}}{\partial x}\right| \Big[A(t) + B(t)\frac{\partial x^0}{\partial \tilde{x}^\mu}\frac{\partial x^0}{\partial \tilde{x}^\nu}\tilde{g}^{\mu\nu}\big(\tilde{x}(x)\big)\Big]. \tag{4.2.4}$$

Changing integration variables to $\tilde{x}$ we get

$$\begin{aligned} &\int d^4\tilde{x} \sqrt{-\tilde{g}(\tilde{x})}\Big[A\big(\tilde{t} - \xi^0\big(x(\tilde{x})\big)\big) + \\ &\qquad + B\big(\tilde{t} - \xi^0\big(x(\tilde{x})\big)\big)\frac{\partial\big(\tilde{t} - \xi^0\big(x(\tilde{x})\big)\big)}{\partial \tilde{x}^\mu}\frac{\partial\big(\tilde{t} - \xi^0\big(x(\tilde{x})\big)\big)}{\partial \tilde{x}^\nu}\tilde{g}^{\mu\nu}(\tilde{x})\Big]. \end{aligned} \tag{4.2.5}$$

[1]Originally the Stückelberg mechanism was introduced as an alternative way than Proca quantization to quantize massive vector fields, a thorough review on the the Stückelberg mechanism is [90]

Here comes the part in which the Goldstone boson appears: whenever ξ^0 appears in the action above, we make the substitution

$$\xi^0\big(x(\tilde{x})\big) \to -\tilde{\pi}(\tilde{x}). \tag{4.2.6}$$

In other words we promoted $\xi^0\big(x(\tilde{x})\big)$ to a field, the Goldstone field. This gives, dropping the tildes for simplicity:

$$\int d^4x \sqrt{-g}\Big[A\big(t+\pi(x)\big) + B\big(t+\pi(x)\big)\frac{\partial\big(t+\pi(x)\big)}{\partial x^\mu}\frac{\partial\big(t+\pi(x)\big)}{\partial x^\nu}g^{\mu\nu}(x)\Big]. \tag{4.2.7}$$

Requiring that the action above is invariant under broken time diffeomorphisms implies that the field π must satisfies the transformation relation

$$\pi(x) \longrightarrow \tilde{\pi}\big(\tilde{x}(x)\big) = \pi(x) - \xi^0(x), \tag{4.2.8}$$

with this definition π transforms as a scalar field plus an additional shift.

From this example we can derive the general rules which tell us how a generic tensor transforms when we apply the Stückelberg mechanism. The simplest case is the one of a scalar which does not transform under a change of coordinates

$$R(x^\mu) \longrightarrow \tilde{R}(\tilde{x}^\mu) = R(x^\mu). \tag{4.2.9}$$

Instead a time dependent function in the action transforms as:

$$f(t) \longrightarrow f(\tilde{t}) = f\big(t+\pi(x)\big) \simeq f(t) + \dot{f}(t)\pi(x) + \dots \tag{4.2.10}$$

Covariant and contravariant tensors are defined by the transformation rules under a diffeomorphism

$$\tilde{A}^{\mu\nu}(\tilde{x}) = \frac{\partial\tilde{x}^\mu}{\partial x^\sigma}\frac{\partial\tilde{x^\nu}}{\partial x^\rho}A^{\sigma\rho}(x), \qquad \tilde{A}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\sigma}{\partial\tilde{x}^\mu}\frac{\partial x^\rho}{\partial\tilde{x}^\nu}A_{\sigma\rho}(x), \tag{4.2.11}$$

so for the contravariant components of a tensor we have

$$\begin{aligned}\tilde{A}^{\mu\nu} &= \big(\delta^\mu{}_\sigma + \delta^\mu{}_0\partial_\sigma\pi\big)\big(\delta^\nu{}_\rho + \delta^\nu{}_0\partial_\rho\pi\big)A^{\sigma\rho} = \\ &= A^{\mu\nu} + \delta^\mu{}_0\partial_\sigma\pi A^{\sigma\nu} + \delta^\nu{}_0\partial_\rho\pi A^{\mu\rho} + \delta^\mu{}_0\delta^\nu{}_0\partial_\sigma\pi\partial_\rho\pi A^{\sigma\rho},\end{aligned} \tag{4.2.12}$$

while for the covariant components of a tensor

$$\begin{aligned}\tilde{A}_{\mu\nu} &= \big(\delta^\sigma{}_\mu - \delta^\sigma{}_0\partial_\mu\pi + \dots\big)\big(\delta^\rho{}_\nu - \delta^\rho{}_0\partial_\nu\pi + \dots\big)A_{\sigma\rho} = \\ &= A_{\mu\nu} - \partial_\nu\pi A_{\mu 0} - \partial_\mu\pi A_{0\nu} + \delta^\sigma{}_0\delta^\rho{}_0\partial_\mu\pi\partial_\nu\pi A_{\sigma\rho} + \dots\end{aligned} \tag{4.2.13}$$

Now we must apply this procedure to the whole action (4.1.22). Easily we can check that the components of the metric after the procedure are:

$$g^{00} \longrightarrow (1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})g^{0i}\partial_i\pi + g^{ij}\partial_i\pi\partial_j\pi, \tag{4.2.14}$$
$$g^{0i} \longrightarrow (1+\dot{\pi})g^{0i} + g^{ij}\partial_j\pi, \tag{4.2.15}$$
$$g^{ij} \longrightarrow g^{ij}. \tag{4.2.16}$$

These relations allow us to write below how terms of the effective action (4.1.22) change after the Stückelberg mechanism. It is evident, looking at the background part of the Lagrangian, that we need only the relation (4.2.14) since the only operator that appears is δg^{00}. As regards the perturbations the question becomes more complex because we have to take into account operators like $\delta K_{\mu\nu}$ and $\delta R_{\mu\nu\rho\sigma}$. The second one, the Riemann tensor, can be written in terms of only the metric $g_{\mu\nu}$, so the relations above fully determine how it changes after the Stückelberg mechanism. It remains to show that also the perturbation of the extrinsic curvature can be written only in terms of the metric. We already wrote its definition before in (4.1.4) but now we want to explicitly write it:

$$\begin{aligned} K_{\mu\nu} &= \delta_\mu{}^\sigma \nabla_\sigma n_\nu + n^\sigma n_\mu \nabla_\sigma n_\nu \\ &= \partial_\mu n_\nu - \Gamma^\epsilon_{\mu\nu} n_\epsilon + n^\sigma n_\mu \nabla_\sigma n_\nu. \end{aligned} \tag{4.2.17}$$

Using equation (4.1.5) we get

$$\begin{aligned} K_{\mu\nu} &= \partial_\mu n_\nu - \Gamma^\epsilon_{\mu\nu} n_\epsilon + \frac{1}{2} n_\mu \frac{1}{(-g^{00})} h_\nu{}^\sigma \partial_\sigma (g^{00}) \\ &= \frac{1}{2} \frac{\delta_\nu{}^0}{(-g^{00})^{\frac{3}{2}}} \partial_\mu (g^{00}) - \frac{1}{2(-g^{00})^{\frac{1}{2}}} g^{0\rho} \big(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \big) + \\ &\quad + \frac{1}{2} \frac{\delta_\mu{}^0}{(-g^{00})^{\frac{3}{2}}} \partial_\nu (g^{00}) + \frac{1}{2} \frac{\delta_\mu{}^0 \delta_\nu{}^0 g^{0\sigma}}{(-g^{00})^{\frac{5}{2}}} \partial_\sigma (g^{00}), \end{aligned} \tag{4.2.18}$$

hence also the extrinsic curvature can be written only in terms of the metric components.

As an example we write here how the three terms governing the background evolution, those written in the first line of (4.1.22), transform upon the Stückelberg mechanism:

$$\frac{1}{2} M_{Pl}^2 R \longrightarrow \frac{1}{2} M_{Pl}^2 R, \tag{4.2.19}$$

$$M_{Pl}^2 (3H^2 + \dot{H}) \longrightarrow M_{Pl}^2 \big(3H^2(t+\pi) + \dot{H}(t+\pi) \big), \tag{4.2.20}$$

$$\begin{aligned} M_{Pl}^2 (\dot{H} g^{00}) \longrightarrow M_{Pl}^2 \dot{H}(t+\pi) \big(& g^{00}(1+\dot{\pi})^2 + \\ & + 2(1+\dot{\pi}) \partial_i \pi g^{0i} + g^{ij} \partial_i \pi \partial_j \pi \big). \end{aligned} \tag{4.2.21}$$

4.3 Lagrangian for the Goldstone field

Now we want to write the expression of the Lagrangian for π, which can be achieved from (4.1.22) performing the Stückelberg mechanism. For the moment, for simplicity, we consider only the terms without the extrinsic curvature. So the action for the Goldstone field π, which represents the perturbation of the scalar field $\delta\varphi$ is

$$\begin{aligned}\mathcal{S} = \int d^4x\,\sqrt{-g}\,\Big[&\frac{1}{2}M_{Pl}^2 R - M_{Pl}^2\big(3H^2(t+\pi) + \dot{H}(t+\pi)\big) + \\ &+ M_{Pl}^2\dot{H}(t+\pi)\big((1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi g^{0i} + g^{ij}\partial_i\pi\partial_j\pi\big) + \\ &+ \frac{1}{2!}M_2(t+\pi)^4\big((1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi g^{0i} + g^{ij}\partial_i\pi\partial_j\pi\big)^2 + \\ &+ \frac{1}{3!}M_3(t+\pi)^4\big((1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi g^{0i} + g^{ij}\partial_i\pi\partial_j\pi\big)^3 + \ldots\Big].\end{aligned} \tag{4.3.1}$$

This action is rather complicated and now it is not clear what is the advantage of reintroducing the Goldstone π. But the simplification occurs because at sufficiently short distances, hence at sufficiently high energies, the physics of the Goldstone can be studied neglecting the metric fluctuations. The regime for which this is possible can be estimated just looking at the mixing terms in the Lagrangian above. In fact we see in equation (4.3.1) that quadratic terms which mix π and $g_{\mu\nu}$ contain fewer derivatives than the kinetic term of π, so they can be neglected above some high energy scale we have to determine. In general the answer will depend on which operators are present. We start with the simplest case in which only the tadpole terms are relevant ($M_2 = M_3 = \ldots = 0$), which corresponds to the standard slow-roll inflation case, as we have seen in Section 4.1.1. In this case, the action (4.3.1) reduces to

$$\begin{aligned}\mathcal{S}_{slow-roll} = \int d^4x\,\sqrt{-g}\,\Big[&\frac{1}{2}M_{Pl}^2 R - M_{Pl}^2\big(3H^2(t+\pi) + \dot{H}(t+\pi)\big) + \\ &+ M_{Pl}^2\dot{H}(t+\pi)\big((1+\dot{\pi})^2 g^{00} + 2(1+\dot{\pi})\partial_i\pi g^{0i} + g^{ij}\partial_i\pi\partial_j\pi\big)\Big].\end{aligned} \tag{4.3.2}$$

The leading term of the Goldstone and gravity mixing has the form

$$M_{Pl}^2\dot{H}\dot{\pi}\delta g^{00}. \tag{4.3.3}$$

After canonical normalization ($\pi_c \sim M_{Pl}\dot{H}^{\frac{1}{2}}\pi$, $\delta g_c^{00} \sim M_{Pl}\delta g^{00}$) the term becomes

$$\dot{H}^{\frac{1}{2}}\pi_c\delta g_c^{00}, \tag{4.3.4}$$

then the mixing terms can be neglected for energies above $E_{mix} \sim \dot{H}^{\frac{1}{2}} \sim \epsilon^{\frac{1}{2}}H$, where ϵ is the usual slow-roll parameter defined in Section 3.1.2. If we now consider the case in which more operators are present in (4.3.1), for example when $M_2 \neq 0$ gets large, then the resulting mixing term will be

$$M_2^4\dot{\pi}\delta g^{00}, \tag{4.3.5}$$

which, upon canonical normalization ($\pi_c \sim M_2^2\pi$), becomes negligible at energies larger than $E_{mix} \sim M_2^2/M_{Pl}$. We will see that models with a large value of M_2 have a sound speed smaller than unity, a feature often linked to high values for non-Gaussianities.

In the regime $E \gg E_{mix}$ the physics of the Goldstone boson decouples from the metric perturbations, which can be neglected. Therefore we have to consider a FRW background

$$ds^2 = dt^2 - a^2(t)d\vec{x}^2, \tag{4.3.6}$$

so the second line of (4.3.1) dramatically simplifies to

$$-1 - \dot{\pi}^2 - 2\dot{\pi} + \frac{(\partial_i\pi)^2}{a^2}. \tag{4.3.7}$$

But the term $-2M_{Pl}^2\dot{H}\dot{\pi}$ is no longer significant, since it is only at energies $E \sim E_{mix}$. We also neglect the terms $-3M_{Pl}^2H^2$ and $-M_{Pl}^2\dot{H}$ because we ignore the back-reaction of the perturbations on the metric. Thus the action (4.3.2) simplifies to

$$S_\pi = \int d^4x\,\sqrt{-g}\,\left[\frac{1}{2}M_{Pl}^2R - M_{Pl}^2\dot{H}\Big(\dot{\pi}^2 - \frac{(\partial_i\pi)^2}{a^2}\Big)\right]. \tag{4.3.8}$$

Now we consider also the terms multiplied by M_2 and M_3 in the action (4.3.1), if we keep terms up to the third order we get

$$\frac{1}{2!}M_2^4\bigg(-1 - \dot{\pi}^2 - 2\dot{\pi} + \frac{(\partial_i\pi)^2}{a^2} + 1\bigg)^2 \longrightarrow 2M_2^4\bigg(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\bigg), \tag{4.3.9}$$

$$\frac{1}{3!}M_3^4\bigg(-1 - \dot{\pi}^2 - 2\dot{\pi} + \frac{(\partial_i\pi)^2}{a^2} + 1\bigg)^3 \longrightarrow \frac{4}{3}M_3^4\dot{\pi}^3. \tag{4.3.10}$$

Thus the action (4.3.1) becomes

$$\begin{aligned} S_\pi = \int d^4x\,\sqrt{-g}\,\bigg[&\frac{1}{2}M_{Pl}^2R - M_{Pl}^2\dot{H}\Big(\dot{\pi}^2 - \frac{(\partial_i\pi)^2}{a^2}\Big) + \\ &+ 2M_2^4\Big(\dot{\pi}^2 + \dot{\pi}^3 - \dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\Big) - \frac{4}{3}M_3^4\dot{\pi}^3 + \ldots\bigg]. \end{aligned} \tag{4.3.11}$$

First of all we see that the dependence of H, $\dot{H}$ and the various M_i on the time t is disappeared. This is due to the fact that we are assuming that the time dependence of the coefficients in the comoving gauge Lagrangian is slow compared to the Hubble time, that is, suppressed by some generalized slow-roll parameters. This means that the additional π terms coming from the Taylor expansion of the coefficients are small.

Moreover, since we want to make predictions for the present cosmological observations, it seems that the decoupling limit is completely irrelevant for these extremely infrared scales. However, as for standard single field slow-roll inflation, one can prove that there exists a quantity, the usual ζ variable, which is constant out of the horizon at any order in perturbation theory [93, 94]. The intuitive reason for the existence of a conserved quantity is that after exiting the horizon different regions evolve exactly in the same way. The only difference is how much one has expanded with respect to another and it is this difference that remains constant.

Therefore the problem is reduced to calculating correlation functions just after horizon crossing. We are therefore interested in studying our Lagrangian with an IR energy cut-off of order H. If the decoupling scale E_{mix} is smaller than H, the Lagrangian for π (4.3.11) will give the correct predictions up to terms suppressed by E_{mix}/H.

In conclusion, with the Lagrangian (4.3.11) we are able to compute all the observables which are not dominated by the mixing with gravity. However the tilt of the spectrum can be calculated, at leading order, with this Lagrangian. Its value, in fact, can be deduced simply by the power spectrum at horizon crossing computed neglecting the mixing terms. It is important to stress that our approach does not lose its validity when the mixing with gravity is important; in this case we can't work in the decoupling limit, so the Goldstone action is not sufficient for predictions, but the action (4.1.22) contains all the information about the model and can be used to calculate all predictions even when the mixing with gravity is large.

4.4 The limit of slow-roll inflation

Now we want to retrieve the result found for the slow-roll inflation from starting from the action (4.1.22). As we have seen in Section 4.1.1 this corresponds to keep only the first three terms of (4.1.22), which are fixed once we know the background Hubble parameter $H(t)$, and setting to zero all the other operators of higher order: $M_2 = M_3 = \bar{M}_1 = \bar{M}_2 = \ldots = 0$. In this case, as discussed in the last section, predictions at the scale H can be made neglecting the mixing with gravity and concentrating on the Goldstone Lagrangian (4.3.11) which reduces to

$$S_\pi = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{Pl}^2 R - M_{Pl}^2 \dot{H} \left(\dot{\pi}^2 - \frac{(\partial_i \pi)^2}{a^2} \right) \right]. \tag{4.4.1}$$

We are interested in calculating, soon after the horizon crossing, the conserved quantity ζ. The relation between π and ζ is [86]

$$\zeta(t, \vec{x}) = -H\pi(t, \vec{x}), \tag{4.4.2}$$

For each mode k we are interested in the dynamics around horizon crossing $k \sim aH$. During this period the background can be approximated as de

Sitter up to slow-roll corrections. Therefore we expect to recover from the action (4.4.1) the power spectrum we have already found on Section 3.5.1. If we introduce the canonically normalized field π_c as

$$\pi_c = \sqrt{2\dot{H}} M_{Pl} \pi, \tag{4.4.3}$$

then it is easy to check that the equation of motions for π_c are:

$$\delta_{\pi_c} \mathcal{S}_{\pi_c} = 0 \Longleftrightarrow \ddot{\pi}_c + 3H\dot{\pi}_c - \frac{1}{a^2}\nabla^2 \pi_c = 0. \tag{4.4.4}$$

If we rewrite this equation in momentum space, using the conformal time τ instead of the cosmic time t and introducing the variable $\pi_c(\tau, \vec{k}) = u(\tau, \vec{k})/a(\tau)$ we find the same equation of (3.3.20) without the potential term. So we already know the 2 point function of π reads:

$$\langle \pi_c(\vec{k}_1)\pi_c(\vec{k}_2)\rangle = \delta(\vec{k}_1 - \vec{k}_2)\frac{H_*^2}{2k_1^3}, \tag{4.4.5}$$

where the $*$ means the value of the quantity is taken at horizon crossing. Hence the power spectrum of π is given by

$$\mathcal{P}_{\pi_c}(k) = \frac{H_*^2}{2k^3}. \tag{4.4.6}$$

This implies that the power spectrum of ζ is given by

$$\mathcal{P}_\zeta(k) = \frac{H_*^2}{2M_{Pl}^2|\dot{H}_*|}\mathcal{P}_{\pi_c}(k) = \frac{H_*^4}{4M_{Pl}^2|\dot{H}_*|}\frac{1}{k^3} = \frac{H_*^2}{4\epsilon M_{Pl}^2}\frac{1}{k^3}, \tag{4.4.7}$$

which is in agreement with the result found in the previous Chapter in (3.6.10), since $M_{Pl}^2 = 1/(8\pi G)$. This expression allows us to calculate the tilt of the spectrum at leading order in slow-roll:

$$\begin{aligned} n_s - 1 &= \frac{d}{d(\log k)}\left[\log\left(\triangle_\zeta^2(k)\right)\right] = \frac{d}{d(\log k)}\left[\log\left(\frac{k^3}{2\pi^2}\mathcal{P}_\zeta(k)\right)\right] \\ &\simeq \frac{1}{H_*}\frac{d}{dt}\log\left(\frac{H_*^4}{\dot{H}_*}\right) = 4\frac{\dot{H}_*}{H_*^2} - \frac{\ddot{H}_*}{H_*\dot{H}_*}. \end{aligned} \tag{4.4.8}$$

In the case of the field π the slow-roll parameters are not defined using the derivatives of the potential but in a different way. The first slow-roll parameter, ϵ is defined in the usual way as minus the ratio between the derivative of the Hubble rate and the square of the Hubble rate itself: $\epsilon = -\dot{H}/H^2$. While the parameter η is defined by the following expression: $\eta = \dot{\epsilon}/H\epsilon$. The definition of η can be written using the definition of ϵ as

$$\eta = \left(-\frac{\ddot{H}}{H^2} + 2\frac{\dot{H}^2}{H^3}\right)\left(-\frac{H}{\dot{H}}\right) = \frac{\ddot{H}}{\dot{H}H} - 2\frac{\dot{H}}{H^2}, \tag{4.4.9}$$

from which we get

$$\frac{\ddot{H}}{\dot{H}H} = \eta - 2\epsilon. \tag{4.4.10}$$

Consequently, using the slow-roll parameters we have just defined, we can express (4.4.8) as

$$n_s - 1 \simeq -4\epsilon - \eta + 2\epsilon = -\eta - 2\epsilon. \tag{4.4.11}$$

From the action (4.4.1) we were able to find the expected power spectrum of a scalar field; actually this result is not equal to the one found in Section 3.6 as it differs from equation (3.6.7). The difference is due to the fact that the slow-roll parameters we are using here in our EFT approach are more general than the ones defined for the scalar field in Section 3.1.2; the relation between the two set of slow-roll parameters (we call ϵ_V and η_V those defined previously) is [97]:

$$\epsilon = \epsilon_V, \tag{4.4.12}$$

$$\eta = -2\eta_V + 4\epsilon_V. \tag{4.4.13}$$

Rewriting (4.4.11) using ϵ_V and η_V we get exactly the result found in (3.6.7).

However not all observables can be calculated from this Lagrangian; the problem is that we neglected the mixing with the gravity since it is of higher order in the slow-roll expansion. This implies that in our Lagrangian there are not interaction terms between the field π and the gravity or self-interactions of the π. So if someone is interested in looking at the three point function he must take into account these subleading terms, otherwise it would turn out to be zero[2]. These subleading terms encodes the UV completion of our model. This is what usually happens in an effective field theory: we have some leading terms which describes the physics at a given energy and other higher dimensional operators which describe the higher energy scales. Another example of effective field theory is the Standard Model; anyway, in this case, there are many experimental data which allow us to put severe limits on the size of the higher dimensional operators. In the cosmological case the set of conceivable observations is much more limited. One example of a possible experimental limit on higher dimension operators is the consistency relation for the gravitational wave tilt.

Actually, in the slow-roll approximation, the only term of the Lagrangian (4.4.1) giving contributions to the tensor modes is the Einstein-Hilbert term, which gives the Einstein equations as equations of motion. So, in this approximation, we already know the power spectrum of primordial tensor perturbation, which in canonical normalization can be written as

$$\mathcal{P}_h(k) = \frac{4H_*^2}{M_{Pl}^2 k^3}. \tag{4.4.14}$$

[2]For the three point function of the curvature perturbation in single-field slow-roll see [98] and [99]

Then, as we have found previously in Section 3.6, $n_t = -2\epsilon$. So the ratio between the tensor power spectrum and the scalar one is given by

$$r = \frac{\triangle_h^2(k)}{\triangle_\zeta^2(k)} = 16\epsilon, \tag{4.4.15}$$

which is the same result we have found in Section 3.6. This prediction is valid if we assume $M_2 = 0$ which implies $c_s = 1$, in fact we will see that the scalar spectrum goes as c_s^{-1}, while predictions for gravitational waves are not changed by M_2. The experimental verification of the consistency relation would tell us that c_s cannot deviate substantially from 1 which implies

$$M_2^4 \lesssim M_{Pl}^2 |\dot{H}|. \tag{4.4.16}$$

Notice that the higher dimension operators will influence both scalar and tensor modes, but these corrections are much harder to test. Later we will see some examples.

4.5 Small speed of sound

The Goldstone action (4.3.11) shows that the spatial kinetic term $(\partial_i\pi)^2$ is completely fixed by the background evolution to be $M_{Pl}^2\dot{H}(\partial_i\pi)^2$. In particular only for $\dot{H} < 0$, it has the expected negative sign. However even if the operator $(\partial_i\pi)^2$ would have the wrong sign it won't be enough to conclude that the system is pathological: higher order terms such as $(\delta K^\mu{}_\mu)^2$ may become important in particular regimes. Reference [84] studies examples in which $\dot{H} > 0$ can be obtained without pathologies.

The coefficient of the kinetic term $\dot{\pi}^2$ is, on the contrary, not completely fixed by the background evolution, as it receives a contribution also from the quadratic operator $(g^{00} + 1)^2$. Looking at the action (4.3.11) we see that its coefficient is

$$\Big(- M_{Pl}^2 \dot{H} + 2M_2^4 \Big) \dot{\pi}^2. \tag{4.5.1}$$

This time to avoid instabilities we must have $-M_{Pl}^2\dot{H} + 2M_2^4 > 0$. The coefficient of the time kinetic term (4.5.1) is different from the coefficient of the spatial kinetic term, this implies that the speed of propagation of the π waves c_s is different from one and it is given by

$$c_s^{-2} = -\frac{\big(- M_{Pl}^2\dot{H} + 2M_2^4\big)}{M_{Pl}^2\dot{H}} = 1 - \frac{2M_2^4}{M_{Pl}^2\dot{H}}. \tag{4.5.2}$$

From this expression it is clear that for small value of M_2 we get back a field π propagating at the speed of light as in the case of slow-roll inflation; while if M_2 gets bigger the behaviour of c_s depends on the properties of $\dot{H}$, in particular if $\dot{H} < 0$, to avoid superluminal propagation we must have $M_2^4 > 0$.

We will restrict our study to the cases in which $c_s \leq 1$ because superluminal propagation would imply that the theory has no Lorentz invariant UV completion [95], see [96] for a phenomenological discussion of models with $c_s > 1$.

Using the expression for c_s in (4.5.2) the Goldstone action can be written at cubic order as

$$\begin{aligned} \mathcal{S}_\pi = \int d^4x \sqrt{-g} \Bigg[& \frac{1}{2} M_{Pl}^2 R - \frac{M_{Pl}^2 \dot{H}}{c_s^2} \left(\dot{\pi}^2 - c_s^2 \frac{(\partial_i \pi)^2}{a^2} \right) + \\ & + M_{Pl}^2 \dot{H} \left(1 - \frac{1}{c_s^2} \right) \left(\dot{\pi}^3 - \dot{\pi} \frac{(\partial_i \pi)^2}{a^2} \right) - \frac{4}{3} M_3^4 \dot{\pi}^3 + \dots \Bigg]. \end{aligned} \tag{4.5.3}$$

We remember that to write this action for the π we used the fact that above some energy scale we can neglect the mixing with the gravity and we found it to be $E \gg E_{mix} \simeq M_2^2/M_{Pl}$ in Section 4.3. This implies that predictions for cosmological observables, which are done at energies of order H, are captured at leading order by the Goldstone action (4.5.3) if $H \gg M_2^2/M_{Pl}$, or equivalently for $\epsilon/c_s^2 \ll 1$. It has been proven that this model allows large non-Gaussianities [86, 97, 100].

To solve for the equation of motion for π, we consider the term up to second order in the perturbations, that means only the first line of (4.5.3). To get the equation of motion we then have to follow the usual procedure of working in momentum space, with the conformal time τ and writing $\pi(\tau, \vec{k}) = u(\tau, \vec{k})/a(\tau)$ or we can simply notice that the first line of (4.5.3) is equal to the action we worked with in the slow-roll approximation once we rescale the momentum to $k_{res} = c_s k$ and take into account the additional factor c_s^{-2} in front of the time kinetic term. The result is

$$\langle \zeta(\vec{k}_1) \zeta(\vec{k}_2) \rangle = \delta(\vec{k}_1 - \vec{k}_2) \frac{1}{c_{s*}} \frac{H_*^4}{4 M_{Pl}^2 |\dot{H}_*|} \frac{1}{k^3} = \delta(\vec{k}_1 - \vec{k}_2) \frac{1}{c_{s*}} \frac{H_*^2}{4 \epsilon_* M_{Pl}^2} \frac{1}{k^3}. \tag{4.5.4}$$

The scalar spectral index is thus modified by the presence of the speed of sound

$$\begin{aligned} n_s - 1 &= \frac{d}{d(\log k)} \left[\log \left(\frac{H_*^4}{|\dot{H}_*| c_{s*}} \right) \right] \simeq \frac{1}{H_*} \frac{d}{dt} \left[\log \left(\frac{H_*^4}{|\dot{H}_*| c_{s*}} \right) \right] = \\ &= 4 \frac{\dot{H}_*}{H_*^2} - \frac{\ddot{H}_*}{\dot{H}_* H_*} - \frac{\dot{c}_{s*}}{c_{s*} H_*} = -\eta - 2\epsilon - \frac{\dot{c}_{s*}}{c_{s*} H_*}. \end{aligned} \tag{4.5.5}$$

We can think of the last term $\dot{c}_{s*}/(c_{s*} H_*)$ as another slow-roll parameter and we call it s, so

$$n_s - 1 = -\eta - 2\epsilon - s. \tag{4.5.6}$$

In the Lagrangian written above there are also cubic couplings for the Goldstone of the form $\dot{\pi}(\nabla \pi)^2$ and $\dot{\pi}^3$. These operators, at the contrary of the slow-roll case, give a non vanishing 3-points function which is linked to non-Gaussianities. The different terms give different size for the non-Gaussianities. The usual procedure to establish which is dominant is to compare the non-linear corrections with the quadratic terms around horizon crossing. We will see that at horizon crossing spatial derivatives are enhanced with respect to time derivatives since in momentum space $k \sim H/c_s$. This implies that in the limit $c_s \ll 1$ the cubic terms are led by the term $\dot{\pi}(\nabla\pi)^2$ and the quadratic ones are led by the $(\partial_i \pi)^2$. So the level of non-Gaussianities is given by the ratio:

$$\begin{aligned}\frac{\mathcal{L}_{\dot{\pi}(\nabla\pi)^2}}{\mathcal{L}_2} &\sim \frac{aM_{Pl}^2 \dot{H}\frac{c_s^2-1}{c_s^2} H\pi k^2\pi^2}{aM_{Pl}^2\dot{H}k^2\pi^2} \sim \frac{aM_{Pl}^2 \dot{H}\frac{c_s^2-1}{c_s^4} H^3\pi^3}{aM_{Pl}^2\dot{H}\frac{H^2}{c_s^2}\pi^2} \\ \text{for } c_s \ll 1 \quad &\sim \frac{aM_{Pl}^2\dot{H}\frac{H^3}{c_s^4}\pi^3}{aM_{Pl}^2\dot{H}\frac{H^2}{c_s^2}\pi^2} \sim \frac{H\pi}{c_s^2} \sim \frac{\zeta}{c_s^2}. \end{aligned} \tag{4.5.7}$$

Usually the magnitude of non-Gaussianities is given in terms of the parameters f_{NL} [101], which are parametrically of the form

$$\frac{\mathcal{L}_{\dot{\pi}(\nabla\pi)^2}}{\mathcal{L}_2} \sim f_{NL}\zeta. \tag{4.5.8}$$

So in our case the leading contribution gives

$$f_{NL,\dot{\pi}(\nabla\pi)^2} \sim \frac{1}{c_s^2}. \tag{4.5.9}$$

In the Goldstone Lagrangian (4.5.3) there is an additional independent operator, $M_3^4\dot{\pi}^3$, contributing to the 3-points function. We thus have two contributions of the form $\dot{\pi}^3$ which give

$$\begin{aligned}\frac{\mathcal{L}_{\dot{\pi}^3}}{\mathcal{L}_2} &\sim \frac{a^3\left(M_{Pl}^2\dot{H}c_s^{-2}(c_s^2-1) - \frac{4}{3}M_3^4\right)\dot{\pi}^3}{aM_{Pl}^2\dot{H}k^2\pi^2} \\ &\sim \frac{a^3\left(M_{Pl}^2\dot{H}c_s^{-2} - \frac{4}{3}M_3^4\right)H^3\pi^3}{aM_{Pl}^2\dot{H}\frac{H^2}{c_s^2}\pi^2} \sim \left(1 - \frac{4}{3}\frac{M_3^4 c_s^2}{M_{Pl}^2|\dot{H}|}\right)\zeta. \end{aligned} \tag{4.5.10}$$

This gives a parameter

$$f_{NL,\dot{\pi}^3} \sim 1 - -\frac{4}{3}\frac{M_3^4 c_s^2}{M_{Pl}^2|\dot{H}|}. \tag{4.5.11}$$

4.6 De Sitter limit and the ghost condensate

Now we take into account the higher derivative operators in the comoving gauge Lagrangian (4.1.22). This means that we consider only the contributions coming from the extrinsic curvature:

$$\int d^4x\,\sqrt{-g}\left(-\frac{\bar{M}_2(t)^2}{2}\delta K^\mu{}_\mu{}^2 - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu{}_\nu \delta K^\nu{}_\mu\right). \tag{4.6.1}$$

From this expression is still not clear what contributions these operators give to the π Lagrangian. The way to make it clear is to perform the Stückelberg mechanism. First of all it is worth noting that under a change of coordinates the extrinsic curvature does not transform covariantly. It changes as a geometrical quantity because the corresponding surface that it is referring to changes. For example we can see what happens to its spatial components. From the definition of the extrinsic curvature (4.2.18), we can write

$$K_{ij} = \frac{1}{2}\sqrt{-g^{00}}(\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}). \tag{4.6.2}$$

The transformations of the covariant metric under the Stückelberg mechanism can be found at linear order using (4.2.13), which gives:

$$g_{00} \longrightarrow g_{00} - 2\dot{\pi} g_{00}, \tag{4.6.3}$$
$$g_{0i} \longrightarrow g_{0i} - \partial_i \pi g_{00} - \dot{\pi} g_{0i}, \tag{4.6.4}$$
$$g_{ij} \longrightarrow g_{ij} - \partial_i \pi g_{0j} - \partial_j \pi g_{0i}. \tag{4.6.5}$$

At this point it is straightforward to see that

$$\begin{aligned} K_{ij} \longrightarrow \;& \frac{1}{2}\sqrt{-g^{00}}(1+\dot{\pi})\big[\partial_i(g_{0j} - \partial_j\pi) + \partial_j(g_{0i} - \partial_i\pi) - (1-\dot{\pi})\partial_0 g_{ij}\big] = \\ &= \frac{1}{2}\sqrt{-g^{00}}\big[\partial_i g_{0j} + \partial_j g_{0i} - \partial_0 g_{ij}\big] - \partial_i\partial_j\pi \\ &= K_{ij} - \partial_i\partial_j\pi, \end{aligned} \tag{4.6.6}$$

where K_{ij} in the last line is the extrinsic curvature orthogonal to the constant τ hypersurface of the new coordinates.

Since the operators in (4.6.1) are the perturbations of the extrinsic curvature $\delta K_{\mu\nu} = K_{\mu\nu} - K^{(0)}_{\mu\nu}$ and not the extrinsic curvature itself, we explicitly write the background value

$$K^{(0)}_{\mu\nu} = H h_{\mu\nu}. \tag{4.6.7}$$

Then acting with the Stückelberg mechanism on (4.6.1) we get

$$\int d^4x \sqrt{-g}\left[-\frac{\bar{M}^2}{2}\frac{1}{a^4}(\partial_i^2\pi)^2\right], \tag{4.6.8}$$

with $\bar{M}^2 = \bar{M}_2^2 + \bar{M}_3^2$. This spatial kinetic term will make the Goldstone propagate even in the limit $c_s \to 0$. It is therefore interesting to consider our general Lagrangian in the limit $\dot{H} = 0$, when the gravitational background is exactly de Sitter space which implies $c_s = 0$. In this case the Lagrangian (4.3.11) reduces to

$$\mathcal{S}_\pi = \int d^4x \sqrt{-g}\left[2M_2^4\dot{\pi}^2 - \frac{\bar{M}^2}{2}\frac{1}{a^4}(\partial_i^2\pi)^2\right], \tag{4.6.9}$$

where we have considered only quadratic terms in π. First of all we see that the kinetic term the action above has the "wrong" plus sign, this means π in this case is a ghost.

As H is now time independent, it is possible to impose an additional symmetry to the theory: the time independence of all the coefficients in (4.1.22). This has an important implication in the form of the action for π. In fact, looking back at the procedure done in Section 4.2 to introduce the Goldstone we can see that from the time dependent functions we got terms proportional to π without derivatives. If we now forbid the presence in the action of such time dependent functions, we are forbidding the presence of π without any kind of derivatives in the Goldstone action, which becomes invariant under shift of π:

$$\pi(t, \vec{x}) \to \pi(t, \vec{x}) + const. \tag{4.6.10}$$

This is called limit of Ghost Condensation, where the Goldstone has a dispersion relation $\omega \propto k^2$. For an in depth analysis of this particular inflation model read [83, 89, 102]. the name is due to the fact that in this model we have a de Sitter phase which arises from a derivatively coupled ghost scalar field which condenses in the background where it has a non-zero velocity [89].

As in previous models, we have to find the energy regime for which the mixing of the Goldstone with gravity can be neglected. After canonical normalization ($\pi_c \sim M_2^2 \pi$) we see that the mixing terms coming from (4.6.1) after having performed the Stückelberg mechanism are multiplied by a coefficient of the order

$$E_{mix} \simeq \frac{\bar{M}^2 k^3}{M_{Pl} M_2^2}. \tag{4.6.11}$$

Since the dispersion relation of the Goldstone is of the form

$$\omega^2 = (\bar{M}^2 / M_2^4) k^4, \tag{4.6.12}$$

we see that the energy E_{mix} under which the mixing is relevant is

$$E_{mix} \simeq \frac{\bar{M} M_2^2}{M_{Pl}^2}. \tag{4.6.13}$$

As we are interested in the inflaton, we concentrate on the case $H \gg E_{mix}$, when the mixing can be neglected.

To compute the power spectrum we firstly canonically normalize the field π writing

$$\pi_c = 2 M_2^2 \pi, \tag{4.6.14}$$

then we follow the usual procedure of deriving the equation of motion for π and then massage them in order to find a differential equation for the variable $u_{\vec{k}}(\tau) = \pi_{\vec{k}}(\tau)a(\tau)$. The resulting power spectrum for the scalar curvature perturbations is the following:

$$\mathcal{P}_\zeta(k) = \frac{4\pi^2}{\Gamma^2(1/4)}\frac{H}{M_2}\left(\frac{2H}{\bar{M}}\right)^{3/2}\frac{1}{k^3}. \tag{4.6.15}$$

An implication of the $\omega \propto k^2$ dispersion relation is that the way operators scale with energy does not coincide with its mass dimension as in the Lorentz invariant case [103]. A rescaling of the energy by a factor s, $E \to sE$, must go together with a momentum transformation $k \to s^{\frac{1}{2}}k$. Correspondingly the time rescales as $t \to s^{-1}t$ and the spatial coordinates as $x \to s^{-\frac{1}{2}}x$. If we want that the quadratic term of the action for π (4.6.9) to be invariant under this rescaling, we have to impose that π rescales as $\pi \to s^{\frac{1}{4}}\pi$. With this rule it is easy to check that all the allowed Goldstone operators, besides the kinetic term (4.6.9) are irrelevant: since they have positive scaling dimension they become less and less relevant going down in energy.

4.7 De Sitter limit without the ghost condensate

In the previous section we have neglected the operator

$$\int d^4x\,\sqrt{-g}\left(-\frac{\bar{M}_1(t)^3}{2}(g^{00}+1)\delta K^\mu{}_\mu\right). \tag{4.7.1}$$

Now we want to study the effect of this operator on the quadratic π action. We will see that, if the coefficient of this operator is sufficiently large, we obtain a new de Sitter limit, where the dispersion relation is of the form $\omega^2 \propto K^2$.

For simplicity we can take $\bar{M}_1$ to be time independent. Reintroducing the Goldstone by operating the Stückelberg mechanism, we get two different terms: the first is of the form $\bar{M}_1^3\dot{\pi}\nabla^2\pi/a^2$ and the second proportional to $H\dot{\pi}^2$. Here we will assume that the $H\dot{\pi}^2$ term is small compared to $M_2^4\dot{\pi}^2$. Integrating by parts the term proportional to $\dot{\pi}\nabla^2\pi$ we get a standard 2-derivative spatial kinetic term

$$\int d^4x\,\sqrt{-g}\,\frac{\bar{M}_1^3 H}{2}\left(\frac{\partial_i\pi}{a}\right)^2. \tag{4.7.2}$$

In the exact de Sitter limit ($\dot{H}=0$) and taking $M_2 \sim \bar{M}_1 \sim M$, this operator gives a dispersion relation of the form $\omega^2 = c_s^2 k^2$ with a small speed of sound

$$c_s^2 = \frac{H}{M} \ll 1. \tag{4.7.3}$$

This is true only if the operators we studied in the previous section, $\delta K^\mu{}_\mu{}^2$ and $\delta K^\mu{}_\nu\delta K^\nu{}_\mu$, are subdominant respect $(g^{00}+1)\delta K^\mu{}_\mu$. If we assume that they are characterized by the same mass scale, $\bar{M}_2 \sim \bar{M}_3 \sim M$, the dispersion relation will get two contributions

$$\omega^2 \sim \frac{H}{M}k^2 + \frac{k^4}{M^2}. \tag{4.7.4}$$

From the results found above it is clear that if the k^4 contribution is somewhat suppressed, it becomes irrelevant at freezing and therefore for inflationary predictions. In this limit we have a new kind of Ghost Inflation with an exactly de Sitter background, but with a $\omega^2 \propto k^2$ dispersion relationship. In this scenario we would find a two point function like [86]

$$\langle\zeta(\vec{k}_1)\zeta(\vec{k}_2)\rangle \sim \delta(\vec{k}_1 - \vec{k}_2)\left(\frac{H}{M}\right)^{5/2}\frac{1}{k_1^3}. \tag{4.7.5}$$

Now that we have found two different de Sitter limits, one dominated at freezing by $(g^{00}+1)\delta K^{\mu}{}_{\mu}$ and the other by $\delta K^{\mu}{}_{\mu}{}^{2}$ and $\delta K^{\mu}{}_{\nu}\delta K^{\nu}{}_{\mu}$, one may wonder if there are other possibilities. One could imagine that both these spatial kinetic terms are suppressed for some reason and the leading operators come at higher order. In this case one would end up with a dispersion relation of the form

$$\omega^2 \sim k^{2n} \qquad n \geq 3. \tag{4.7.6}$$

However this is not the case, because the theory would not make sense as an effective field theory. In fact, as shown in [86], the scaling dimension of the operator π would be $\pi \to s^{-\frac{1}{2}+\frac{3}{2n}}\pi$ and this implies that for $n \geq 3$ the operators that would modify the kinetic term will be strong at low energies. Clearly this would mean that the effective field theory does not make sense.

4.8 The most general action for π

In the previous Sections we outlined the procedure to write the effective action for the scalar perturbations and we specialized our study to special limits, such as slow-roll inflation, small speed of sound and de Sitter inflation. Now we want to write the most general second-order action for the scalar perturbations in comoving (unitary) gauge following [104]. First of all we write the expression of this action and then we specify how we got it starting from (4.1.22) and we spend some words on the various terms. So the the second-order effective action at leading order in slow-roll is

$$\begin{aligned}\mathcal{S}_2 = \int d^4x\,\sqrt{-g}\,\Big(M_{Pl}^2\dot{H}(\partial_\mu\pi)^2 + 2M_2^4\dot{\pi}^2 + \bar{M}_1^3 H\frac{(\partial_i\pi)^2}{2a^2} + \\ - \frac{\bar{M}_2^2}{2}\frac{1}{a^4}(\partial_i^2\pi)^2 - \frac{\bar{M}_3^2}{2}\frac{1}{a^4}(\partial_{ij}\pi)^2\Big)\end{aligned} \tag{4.8.1}$$

- The procedure followed in the previous Sections 4.1 and 4.3, which led us to writing the action above, was introduced in [86] and allows for a very general expression for inflation driven by a single scalar degree of freedom.

- In general the M_i coefficients are time dependent. Anyway, since we are interested in leading-order calculations, we can neglect this time dependence and consider the coefficients constant, as we have done previously.

- The action is written in the decoupling regime: for a sufficiently high energy range the dynamics of the scalar degree of freedom which drives inflation is decoupled from gravity. Here we can safely work with this action assuming $E > \epsilon^{\frac{1}{2}} H$ and $E > M_2^2/M_{Pl}$.

- If we turn off the coefficient $M_2 = 0 = \bar{M}_{1,2,3}$ we re-obtain the slow-roll inflation of Section 4.4. Switching on the M_2 term amounts to allowing $c_s^2 < 1$ as in Section 4.5. Working in the de Sitter limit and turning on $\bar{M}_{2,3}$ we rediscover Ghost Inflation. The list of correspondences can be continued with K-inflation [105, 106] theories and others proving that the effective field theory approach naturally provides a more unifying perspective on inflationary models.

- Actually this action is written with large non-Gaussianities in mind. This can be achieved requiring a small speed of sound: this assumption translates into bounds on the values of the coefficients driving quadratic operators in the Lagrangian. A speed of sound different than unity necessarily generates a different weight in Fourier space between time-like and space-like derivatives acting on the scalar. Consequently, this different weight for the derivatives gives us a meaningful criterion to establish which are the leading terms in writing the various operators. Consider for example the operators of (4.1.22) multiplied by the $\bar{M}_1^3$ coefficient
$$-\bar{M}_1^3\left(6H\dot{\pi}^2 - H\frac{(\partial_i\pi)^2}{2a^2}\right). \tag{4.8.2}$$
There are two terms: one with two temporal derivatives and one with two spatial derivatives. On the horizon crossing region we can assume the following estimates to hold: $\dot{\pi} \sim H\pi$, $\nabla\pi \sim (H/c_s)\pi$. Then, for $c_s \ll 1$ space-like derivatives are enhanced with respect time derivatives. So when we compare terms of the same perturbation order within the same M coefficient, we just count the number of space and time derivatives: the term with the highest number of spatial derivatives is the leading one. In conclusion, between the operators considered above

in (4.8.2), the leading one is

$$\bar{M}_1^3 H \frac{(\partial_i \pi)^2}{2a^2}. \tag{4.8.3}$$

- If we decide to include also the subleading contributions in the effective action, like for example the term with time derivative $\dot{\pi}^2$, the functional expression of the solution won't change. This happens because the types of operator $\dot{\pi}^2$, $(\partial_i\pi)^2$, $(\partial_i^2\pi)^2$ are already saturated in (4.8.1). So including also these subleading terms will end up only in a redefinition of some coefficients.

Now we can proceed with the calculation of the equation of motion. The procedure is always the same; varying the action and working in momentum space by using the Fourier transform, we get [104, 107]:

$$\ddot{\pi}_k + 3H\dot{\pi}_k + \frac{k^2}{a^2}\frac{M_{Pl}^2\dot{H} + \bar{M}_1^3}{M_{Pl}^2\dot{H} - 2M_2^4}\pi_k + \frac{k^4}{a^4}\frac{\bar{M}_2^2 + \bar{M}_3^2}{M_{Pl}^2\dot{H} - 2M_2^4}\pi_k = 0. \tag{4.8.4}$$

Now we make the substitution $\pi(\tau, \vec{k}) = u(\tau, \vec{k})/a(\tau)$ and using conformal time instead of t we find

$$u_k'' - \frac{2}{\tau^2}u_k + \alpha_0 k^2 u_k + \beta_0 k^4 \tau^2 u_k = 0, \tag{4.8.5}$$

with

$$\alpha_0 = \frac{M_{Pl}^2\dot{H} + \bar{M}_1^3 H}{M_{Pl}^2\dot{H} - 2M_2^4}, \qquad \beta_0 = \frac{(\bar{M}_2^2 + \bar{M}_3^2)H^2}{2M_2^4 - M_{Pl}^2\dot{H}}. \tag{4.8.6}$$

Before explicitly solving this equation we can make some guesses on the behaviour of the wavefunction. In the regime in which $2/\tau^2$ is negligible, we expect the typical oscillatory behaviour since both $\alpha_0 k^2$ and $\beta_0 k^4 \tau^2$ cause wavelike behaviour. On the other hand, in the $\tau \to 0$ limit, $-2/\tau^2$ will be the leading term and we expect to recover the usual frozen modes. Now we can justify the statement that in comparing terms at the same order in perturbations, the one with the most spatial derivatives are dominating in the $c_s \ll 1$ limit. We introduce the notion of effective horizon, placing it where the oscillatory behaviours stops being dominant

$$\alpha_0 k^2 + \beta_0 K^4 \tau_*^2 = \frac{2}{\tau_*^2} \to \tau_* = -\frac{2}{k\sqrt{\alpha_0 + \sqrt{\alpha_0^2 + 8\beta_0}}}. \tag{4.8.7}$$

For $\beta_0 = 0$ and $\alpha_0 \sim 1$ we recover the already known relation at horizon $-k^2\tau_*^2 \sim 1$. At this point we can relate the spatial derivatives $\nabla\pi \sim k\pi$ with the time ones $\dot{\pi} \sim H\pi$ at the horizon using (4.8.7) and the relation between the conformal time and the Hubble rate $\tau = -1/(aH)$:

$$k = \frac{\sqrt{2}H}{\sqrt{\alpha_0 + \sqrt{\alpha_0^2 + 8\beta_0}}}. \tag{4.8.8}$$

If we now restrict the parameters space to the $\alpha_0 \ll 1$ and $\beta_0 \ll 1$ region, we find

$$k \gg H. \tag{4.8.9}$$

Since the main contributions to correlators comes from the horizon-crossing region, this shows that we can identify the leading terms in the action according to the procedure followed.

Now we are ready to solve the equation of motion (4.8.5). The general wavefunction is

$$\begin{aligned} u_{\vec{k}}(\tau) = &\frac{ie^{\frac{i}{2}\sqrt{\beta_0}k^2\tau^2}}{2^{\frac{1}{4}}\tau}\mathcal{G}\left[-\frac{1}{4}-\frac{i\alpha_0}{4\sqrt{\beta_0}},-\frac{1}{2},-i\sqrt{\beta_0}k^2\tau^2\right]C_1(k)+ \\ &+\frac{ie^{\frac{i}{2}\sqrt{\beta_0}k^2\tau^2}}{2^{\frac{1}{4}}\tau}\mathcal{L}\left[\frac{1}{4}+\frac{i\alpha_0}{4\sqrt{\beta_0}},-\frac{3}{2},-i\sqrt{\beta_0}k^2\tau^2\right]C_2(k), \end{aligned} \tag{4.8.10}$$

where $\mathcal{G}$ stands for the confluent hypergeometric function and $\mathcal{L}$ is the generalized Laguerre polynomial. $C_1(k)$ and $C_2(k)$ are two integration constants and their value can be found in the way outlined in [104], that is requiring to re-obtain for $\beta_0 \to 0$ the wavefunction of DBI inflation (fixes $C_2(k) = 0$) and for $\alpha_0 = 0$ the wavefunction of Ghost Inflation (fixes $C_1(k)$). The final result we get is

$$\pi_k(\tau) = \frac{He^{\frac{i}{2}\sqrt{\beta_0}k^2\tau^2}k^{-3/2}\Gamma\left(\frac{5}{4}-\frac{i\alpha_0}{4\sqrt{\beta_0}}\right)\mathcal{G}\left(-\frac{1}{4}-\frac{i\alpha_0}{4\sqrt{\beta_0}},\frac{1}{2},0\right)}{2i\sqrt{M_{Pl}^2\epsilon H^2+2M_2^4}\gamma_0^{3/4}\Gamma\left(\frac{5}{4}+\frac{\alpha_0}{4\alpha_0-4i\sqrt{\beta_0}}\right)}, \tag{4.8.11}$$

where $\gamma_0 = \alpha_0 + \sqrt{\beta_0}$ and $\Gamma(x)$ is the Euler gamma function. Using this expression we can now evaluate the power spectrum in the superhorizon limit. If we compute the hypergeometric confluent function in the $\tau \to 0$ limit we find

$$\mathcal{G}\left(-\frac{1}{4}-\frac{i\alpha_0}{4\sqrt{\beta_0}},\frac{1}{2},0\right) = \frac{\sqrt{\pi}}{2\Gamma\left(\frac{5}{4}-\frac{i\alpha_0}{4\sqrt{\beta_0}}\right)}, \tag{4.8.12}$$

consequently we find

$$\mathcal{P}_\pi = |\pi_k(\tau\to 0)|^2 = \frac{\pi H^2}{16(M_{Pl}^2\epsilon H^2+2M_2^4)\gamma_0^{3/2}}\frac{1}{\left|\Gamma\left(\frac{5}{4}+\frac{\alpha_0}{4\alpha_0-4i\sqrt{\beta_0}}\right)\right|^2}\frac{1}{k^3}. \tag{4.8.13}$$

Clearly there is no time dependence in the result above, this means that the modes freeze outside the effective horizon. Using the relation between π and the gauge invariant quantity $\zeta = -H\pi$ we can write the expression of the power spectrum for the scalar perturbations

$$\mathcal{P}_\zeta = \frac{\pi H^4 k^{-3}}{16(M_{Pl}^2\epsilon H^2+2M_2^4)\gamma_0^{3/2}}\frac{1}{\left|\Gamma\left(\frac{5}{4}+\frac{\alpha_0}{4\alpha_0-4i\sqrt{\beta_0}}\right)\right|^2}. \tag{4.8.14}$$

4.9 Tensor contributions in the effective field theory

Since in the effective action there are also two degrees of freedom describing the tensor perturbations $h_{ij}^{(a)}$, we now want to investigate how the operators appearing in (4.1.22) are related to the second order tensor perturbations. Here we will use a different notation, we will call the tensor modes γ_{ij} instead of h_{ij} in order to be faithful with the notation used in the literature. Before proceeding in our analysis we recall that the induced metric on the constant t surfaces is defined as $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$ and this is linked to the scalar curvature perturbation ζ and the tensor perturbations γ_{ij} by [108]

$$h_{ij} = a^2 e^{2\zeta} (e^\gamma)_{ij}. \tag{4.9.1}$$

The tensor perturbations are traceless $\gamma_{ii} = 0$ and transverse $\partial_i \gamma_{ij} = 0$ so γ_{ij} describes two degrees of freedom. The operators of the effective Lagrangian (4.1.22) studied so far are

$$\begin{aligned} \mathcal{S} = \mathcal{S}_0 + \int d^4x\, \sqrt{-g} \Bigg[& \frac{M_2(t)^4}{2} (\delta g^{00})^2 - \frac{\bar{M}_1(t)^3}{2} \delta g^{00} \delta K^\mu{}_\mu + \\ & - \frac{\bar{M}_2(t)^2}{2} \delta K^\mu{}_\mu \delta K^\mu{}_\mu - \frac{\bar{M}_3(t)^2}{2} \delta K^\mu{}_\nu \delta K^\nu{}_\mu \Bigg], \end{aligned} \tag{4.9.2}$$

with

$$\mathcal{S}_0 = \int d^4x\, \sqrt{-g} \left[\frac{1}{2} M_{Pl}^2 R + M_{Pl}^2 \dot{H} g^{00} - M_{Pl}^2 \big(3H^2 + \dot{H} \big) \right]. \tag{4.9.3}$$

The four operators displayed in (4.9.2) are the only ones that contribute to the dispersion relation of primordial perturbations in the decoupling limit, as we have seen in the previous Sections.

Until now we studied only the perturbations of the scalar field π, now we focus on the tensor ones. Except for the Hilbert-Einstein action, in (4.9.2), there is another contribution to the second order tensor perturbations coming from $\delta K^\mu{}_\nu \delta K^\nu{}_\mu$. Using the expression for K_{ij} written in (4.6.2) we see that this term gives also a contribution like

$$\delta K_{ij} \supset \frac{1}{2} a^2 \dot{\gamma}_{ij}. \tag{4.9.4}$$

Thereby we expect to find a modification of the kinetic term for tensor of the form

$$\bar{M}_3^2 \delta K^\mu{}_\nu \delta K^\nu{}_\mu \supset -\frac{1}{4} \bar{M}_3^2 (\partial_0 e^{-\gamma} \partial_0 e^\gamma)_{ii} = -\frac{1}{4} \bar{M}_3^2 (\dot{\gamma}_{ij})^2. \tag{4.9.5}$$

Including this term in the tensor action we find

$$\mathcal{S}_\gamma = \int d^4x\, \sqrt{-g}\, \frac{M_{Pl}^2}{8} c_\gamma^{-2} \left[(\dot{\gamma}_{ij})^2 - c_\gamma^2 \frac{(\partial_k \gamma_{ij})^2}{a^2} \right], \tag{4.9.6}$$

where the tensor sound of speed is given by

$$c_\gamma^2 = \frac{M_{Pl}^2}{M_{Pl}^2 - \bar{M}_3^2}. \tag{4.9.7}$$

We can decompose γ_{ij} into the two helicity modes

$$\gamma_{ij} = \int \frac{d^3k}{(2\pi)^3} \sum_{s=\pm} \epsilon_{ij}^s(\vec{k})\gamma_{\vec{k}}^s(t)e^{i\vec{k}\vec{x}}, \tag{4.9.8}$$

where $s = \pm$ is the helicity index. These two helicity modes are quantized as

$$\gamma_{\vec{k}}^s(t) = a_{s,\vec{k}}v_k(t) + a_{s,-\vec{k}}^\dagger v_k^*(t). \tag{4.9.9}$$

The vacuum state is thus given by

$$v_k = \frac{H}{M_{Pl}} \frac{c_\gamma}{(c_\gamma k)^{3/2}}(1 + ic_\gamma k\tau)e^{-ic_\gamma k\tau}, \tag{4.9.10}$$

where τ is the conformal time. Then the power spectrum for each mode is given by

$$\mathcal{P}_{\gamma^s}(k) = c_\gamma^{-1}\frac{2H^2}{M_{Pl}^2 k^3}. \tag{4.9.11}$$

Since there are two modes the total power spectrum is given by

$$\mathcal{P}_\gamma(k) = c_\gamma^{-1}\frac{4H^2}{M_{Pl}^2 k^3}. \tag{4.9.12}$$

Chapter 5

Modification to the consistency relation

Finally, after having studied the effective field theory for the cosmological perturbations we can study how the various terms modify the consistency relation of standard slow-roll inflation $r = -8n_t$. We firstly encountered this relation in Section 3.6 and it constrains the value of the spectral index for the tensors n_t to the value of the tensor to scalar ratio r. This relation is fundamental in modern cosmology since, if confirmed by observational data, it would prove that slow-roll single field models of inflation are the responsible for the primordial fluctuations. It is also of paramount importance to check if there are deviations from this standard relation because they would mean that there are deviations from the simplest model. In fact inflationary models can be catalogued by the values of n_s and n_t or equivalently by n_s and r.

In this last part of the Thesis, we will work using the result found previously while studying the effective field theory approach to inflation; we will derive the consistency relation in a general case and then try to evaluate it in significant limits. The ingredients we will need to compute this relation are the power spectrum of scalar and tensor perturbations and the spectral index of the tensors which we haven't calculated yet. We want to put in evidence that it is the first time this calculation is done, so our aim is to provide a general result that can be specialized in different situations.

5.1 Generalization of the consistency relation

In Section 4.8 we have found that the power spectrum for a very general class of inflationary models is given by

$$\mathcal{P}_\zeta = \frac{\pi H^4 k^{-3}}{16(M_{Pl}^2 \epsilon H^2 + 2M_2^4)\gamma_0^{3/2}} \frac{1}{\left|\Gamma\left(\frac{5}{4} + \frac{\alpha_0}{4\alpha_0 - 4i\sqrt{\beta_0}}\right)\right|^2}. \tag{5.1.1}$$

While in Section 4.9 we have seen that the same operators which gives rise to the above power spectrum for the scalars, modify the tensors speed of sound and hence their power spectrum which becomes

$$\mathcal{P}_\gamma = c_\gamma^{-1} \frac{4H^2}{M_{Pl}^2} \frac{1}{k^3}, \tag{5.1.2}$$

with $c_\gamma = M_{Pl}^2/(M_{Pl}^2 - M_3^2)$. From this power spectrum we need to compute the tensor spectral index, which is given by

$$n_t = \frac{d}{d(\ln k)} \Big[\ln\left(\triangle_\gamma^2(k)\right) \Big] \simeq \frac{1}{H} \frac{1}{\mathcal{P}_\gamma} \frac{d\mathcal{P}_\gamma}{dt} = 2\frac{\dot{H}}{H^2} - \frac{\dot{c_\gamma}}{H c_\gamma}. \tag{5.1.3}$$

In the previous expression we recognise the slow-roll parameter ϵ, while the last term is a new slow-roll parameter that we call s_γ. Hence the relation between the spectral index for the tensors and the slow-roll parameters is

$$n_t = -2\epsilon - s_\gamma. \tag{5.1.4}$$

Now we can move towards the ratio r, which is given by

$$r = \frac{\triangle_\gamma^2(k)}{\triangle_\zeta^2(k)} = \frac{64H^2}{\pi M_{Pl}^2} c_\gamma^{-1} \frac{(\alpha_0 + \sqrt{\beta_0})^{3/2}\left(M_{Pl}^2 \epsilon H^2 + 2M_2^4\right)}{H^4} \times \\ \times \left|\Gamma\left(\frac{5}{4} + \frac{\alpha_0}{4\alpha_0 - 4i\sqrt{\beta_0}}\right)\right|. \tag{5.1.5}$$

From now on we will omit the argument of the Euler gamma function and we will call it simply

$$\Gamma\left(\frac{5}{4} + \frac{\alpha_0}{4\alpha_0 - 4i\sqrt{\beta_0}}\right) = \Gamma. \tag{5.1.6}$$

At this point we write explicitly the above product in (5.1.5) finding

$$r = \frac{64}{\pi}\frac{1}{c_\gamma}(\alpha_0 + \sqrt{\beta_0})^{3/2} \epsilon \, |\Gamma|^2 + \frac{64}{\pi}\frac{1}{c_\gamma}(\alpha_0 + \sqrt{\beta_0})^{3/2} \frac{1}{M_{Pl}^2 H^2}\left(2M_2^4\right) |\Gamma|^2 . \tag{5.1.7}$$

Using the definition of the scalar speed of sound

$$c_s^2 = \frac{M_{Pl}^2 \dot{H}}{M_{Pl}^2 \dot{H} - 2M_2^4}, \tag{5.1.8}$$

which we already encountered in (4.5.2), we can write the relation between M_2 and M_{Pl} as:

$$M_2^4 = \frac{M_{Pl}^2 \dot{H}}{2}\left(1 - \frac{1}{c_s^2}\right). \tag{5.1.9}$$

This relation allow us to rewrite the second term in (5.1.7) in terms of $\epsilon = -\dot{H}/H^2$ and c_s^2 finding

$$\begin{aligned} r &= \frac{64}{\pi}\frac{1}{c_\gamma}(\alpha_0 + \sqrt{\beta_0})^{3/2}\epsilon\,|\Gamma|^2 - \frac{64}{\pi}\frac{1}{c_\gamma}(\alpha_0 + \sqrt{\beta_0})^{3/2}\epsilon\left(1 - \frac{1}{c_s^2}\right)|\Gamma|^2 \\ &= \frac{64}{\pi}\frac{\epsilon}{c_\gamma c_s^2}(\alpha_0 + \sqrt{\beta_0})^{3/2}\,|\Gamma|^2\,. \end{aligned} \tag{5.1.10}$$

Finally we need to find a way to write r as a function of the n_t we have found in (5.1.4). This can be done by writing the slow-roll parameter ϵ in fiction of the index

$$\epsilon = -\frac{1}{2}n_t - \frac{1}{2}s_\gamma, \tag{5.1.11}$$

and then inserting this expression of ϵ into (5.1.10). The result we find is

$$r = -\frac{32}{\pi}\frac{n_t}{c_\gamma c_s^2}(\alpha_0 + \sqrt{\beta_0})^{3/2}\,|\Gamma|^2 - -\frac{32}{\pi}\frac{s_\gamma}{c_\gamma c_s^2}(\alpha_0 + \sqrt{\beta_0})^{3/2}\,|\Gamma|^2 \tag{5.1.12}$$

5.2 The consistency relation in different limits

The consistency relation we have found in (5.1.12) is the most general possible. Now, the first thing we can do, is to check our result is consistent with results that are already known in the literature. In particular we expect that in the slow-roll limit equation (5.1.12) reduces to the well known $r = -8n_t$. In the previous Chapter, where we analysed the effective field theory approach, we showed that the slow-roll inflation is described by the action (4.4.1). In order to retrieve that action we turned off all the M_i coefficients. Since in equation (5.1.12) the dependence on the M_i coefficients is encoded inside the two coefficients α_0 and β_0, which were defined in (4.8.6), the slow-roll limit is achieved through:

$$\alpha_0 = 1, \qquad \beta_0 = 0. \tag{5.2.1}$$

In this case the scalar speed of sound is $c_s^2 = 1$. As regards the tensor modes, going in the slow-roll limit means consider only the Hilbert-Einstein action and this implies:

$$c_\gamma^2 = 1, \qquad s_\gamma = 0. \tag{5.2.2}$$

Hence, using the values for α_0 and β_0 written above, we find

$$(\alpha_0 + \sqrt{\beta_0})^{3/2} = 1, \tag{5.2.3}$$

$$\left|\Gamma\Big(\frac{5}{4} + \frac{\alpha_0}{4\alpha_0 - 4i\sqrt{\beta_0}}\Big)\right|^2 = \left|\Gamma\Big(\frac{5}{4} + \frac{1}{4}\Big)\right|^2 = \left|\Gamma\Big(\frac{3}{2}\Big)\right|^2 = \frac{\pi}{4}. \tag{5.2.4}$$

Inserting these results in (5.1.12), we get

$$r = -\frac{32}{\pi} n_t \frac{\pi}{4} = -8n_t, \tag{5.2.5}$$

which is the result we were looking for.

We can now see what happens if we consider small speed of sound $c_s^2 \ll 1$. This corresponds to consider also the M_2 term in the scalar action (4.1.22) and assuming that the $\bar{M}_i$ are negligible. This case corresponds to the α_0 and β_0 values

$$\alpha_0 = \frac{M_{Pl}^2 \dot{H}}{M_{Pl}^2 \dot{H} - 2M_2^4}, \qquad \beta_0 = 0. \tag{5.2.6}$$

In this model the modification of the scalar speed of sound is linked to the M_2 coefficient, in fact, as we defined in (4.5.2), we have

$$c_s^2 = \frac{M_{Pl}^2 \dot{H}}{M_{Pl}^2 \dot{H} - 2M_2^4}. \tag{5.2.7}$$

Thus it is clear that in this case we have $\alpha_0 = c_s^2$. With this particular values of the two parameters α_0 and β_0 we find that the Euler gamma function becomes

$$\left|\Gamma\Big(\frac{5}{4} + \frac{\alpha_0}{4\alpha_0 - 4i\sqrt{\beta_0}}\Big)\right|^2 = \left|\Gamma\Big(\frac{5}{4} + \frac{\alpha_0}{4\alpha_0}\Big)\right|^2 = \left|\Gamma\Big(\frac{3}{2}\Big)\right|^2 = \frac{\pi}{4}. \tag{5.2.8}$$

As regards the tensors, their power spectrum is not modified by the M_2 coefficient because, as we have seen in Section 4.9, the only modifications come from the $\bar{M}_i$ coefficients. This implies that for tensors we have

$$c_\gamma^2 = 1, \qquad s_\gamma = 0. \tag{5.2.9}$$

In conclusion, this time the consistency relation (5.1.12) becomes

$$r = -8n_t c_s. \tag{5.2.10}$$

So in this case the result depends also by the speed of sound of scalars, hence by the coefficient M_2.

5.3 Further discussion on the tensor modes

We studied how the tensor modes are affected by the various terms in the effective action in Section 4.9. The final result was a power spectrum dependent on the tensor speed of sound c_γ as we showed in (4.9.12). The direct consequence of this is that also the generalized consistency relation found before in (5.1.12) is dependent on c_γ. Actually there is an ongoing discussion about how to properly treat tensor modes in some inflationary models which go beyond the standard ones. In particular it seems possible to define particular kind of transformations called disformal transformations [109] that allow us to set to unity the speed of propagation of gravitational waves during inflation. This is shown in [110, 111, 112] and the procedure consists first in a disformal transformation

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + (1 - c_\gamma^2) n_\mu n_\nu, \tag{5.3.1}$$

which rescales the extrinsic curvature terms in the following way

$$K_{ij} \longrightarrow \frac{K_{ij}}{c_\gamma}. \tag{5.3.2}$$

This transformation is followed by a conformal one that is necessary to recast the normalization coefficient of the Hilbert-Einstein term in a standard way. This conformal transformation is given by

$$g_{\mu\nu} \longrightarrow c_\gamma^{-1} g_{\mu\nu}. \tag{5.3.3}$$

So it appears that the predictions for the primordial tensor power spectrum cannot be modified at leading order in derivatives.

Here we won't enter in the details of this discussion, we are just interested to see what happens to the generalized consistency relation (5.1.12) if we require that tensor speed of sound is fixed to unity $c_\gamma = 1$. The modifications to the single field slow-roll inflation consistency relation $r = -8n_t$ becomes

$$r = -\frac{32}{\pi}\frac{n_t}{c_s^2}(\alpha_0 + \sqrt{\beta_0})^{3/2}\,|\Gamma|^2\,. \tag{5.3.4}$$

This means that the only contribution of tensor modes comes from their spectral index n_t.

Chapter 6

Conclusions

In this Thesis we studied the theory of perturbations during inflation, starting from the definitions of cosmological perturbations up to the construction of an effective field theory for perturbations during inflation. The main aim of this Thesis was to get in touch with this effective approach to the problem of perturbations, which is a relatively new and still developing theory. We learnt that the effective action can be written in the form

$$\begin{aligned}\mathcal{S} = \int d^4x\,\sqrt{-g}\,\Big[&\frac{1}{2}M_{Pl}^2 R + M_{Pl}^2 \dot{H} g^{00} - M_{Pl}^2(3H^2+\dot{H}) + \\ &+ \frac{1}{2!}M_2(t)^4(g^{00}+1)^2 + \frac{1}{3!}M_3(t)^4(g^{00}+1)^3 - \frac{\bar{M}_1(t)^3}{2}(g^{00}+1)\delta K^\mu_\mu + \\ &- \frac{\bar{M}_2(t)^2}{2}\left(\delta K^\mu_\mu\right)^2 - \frac{\bar{M}_3(t)^2}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \dots\Big],\end{aligned} \tag{6.0.1}$$

and we studied the effect of the various operators contained in it. Turning on or off different operators we were able to describe different models, from the slow-roll scenario to all possible deviations from it, which are described by higher order operators. From the action above we were able to calculate the power spectrum which allows us to discriminate between the various model comparing the theoretical results with the experimental ones.

Most importantly, we studied the effects of the various operators on the consistency relation, finding this result:

$$r = -\frac{32}{\pi}\frac{n_t}{c_\gamma c_s^2}(\alpha_0 + \sqrt{\beta_0})^{3/2}\,|\Gamma|^2 - -\frac{32}{\pi}\frac{s_\gamma}{c_\gamma c_s^2}(\alpha_0 + \sqrt{\beta_0})^{3/2}\,|\Gamma|^2\,. \tag{6.0.2}$$

It is the first time that someone attempts to write a generalization of the consistency relation through the effective field theory approach. We showed that setting the values of the parameters α_0 and β_0 conveniently we were able to find again the slow-roll consistency relation $r = -8n_t$. We see that in (6.0.2) there are contributions from both the scalar perturbations and the tensor ones but, as already stated, it seems that the tensor always propagate at the speed of light which corresponds to setting $c_\gamma = 1$, which means that the contribution to r coming from tensor modes propagating at a different speed than the speed of light vanishes.

Anyway many aspects of the theory need to be analysed deeply; a first step would be to evaluate the theoretical prediction for the other cosmological correlation functions than the power spectrum. In fact the experiments are now reaching enough precision to measure deviations from Gaussianity, which are given by the three point correlation function. This will allow us to get a better understanding of inflation by setting more constraints to our models. Moreover the effective theory of tensor perturbations is still not yet very clear, the issue of the propagation speed of tensor must be solved in order to identify the right model of inflation.

Finally, it is easy to think about possible extension of this formalism. It should be straightforward to introduce new additional fields and study multi-field inflationary models. The interesting point would be to see how the predictions for the observables made inside these alternative models for inflation change from the single field case. Another interesting perspective is to apply the effective field theory approach to study the perturbations in different energy regimes or, for example, to study the fluctuations in fluids like in radiation or matter dominance [113].

Chapter 7

Appendix

7.1 Calculation of contraviariant metric tensor

The starting points to calculate the perturbed contravariant metric are the covariant metric which up to second order in perturbations is

$$\begin{aligned} g_{00} &= -a^2\Big(1 + 2\phi_{(1)} + \phi_{(2)}\Big) \\ g_{0i} &= a^2\Big(B_{(1)i} + \frac{1}{2}B_{(2)i}\Big) \\ g_{ij} &= a^2\Big(\delta_{ij} + 2C_{(1)ij} + C_{(2)ij}\Big), \end{aligned} \tag{7.1.1}$$

and the relation valid at every perturbation order

$$g^{\mu\nu}g_{\nu\sigma} = \delta^{\mu}_{\sigma}. \tag{7.1.2}$$

Firstly we write the metric tensor in the following way

$$\begin{aligned} g_{\mu\nu} &= g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu} + \dots, \\ g^{\mu\nu} &= g^{\mu\nu}_{(0)} + g^{\mu\nu}_{(1)} + g^{\mu\nu}_{(2)} + \dots, \end{aligned} \tag{7.1.3}$$

where the numbers inside parenthesis refers to the order of the perturbation. Hence, using the explicit expressions for the covariant perturbed metric written above, we find

$$\begin{aligned} g^{(0)}_{00} &= -a^2, \\ g^{(1)}_{00} &= -2a^2\phi_{(1)}, \\ g^{(2)}_{00} &= -a^2\phi^{(2)}. \end{aligned} \tag{7.1.4}$$

It is straightforward to compute the contributions to the background and the various perturbation orders of the other components of the metric tensor g_{0i} and g_{ij}. The next step is to write (7.1.2) on the background

$$g^{(0)}_{\mu\nu} g^{\nu\sigma}_{(0)} = \delta_\mu^{\ \sigma}, \tag{7.1.5}$$

which gives

$$\begin{aligned} g^{(0)}_{0\nu} g^{\nu 0}_{(0)} &= -a^2 g^{00}_{(0)} = 1 &\longrightarrow&& g^{00}_{(0)} &= -a^{-2}, \\ g^{(0)}_{i\nu} g^{\nu j}_{(0)} &= a^2 \delta_{ik} g^{kj}_{(0)} = \delta_i^j &\longrightarrow&& g^{ij}_{(0)} &= a^{-2}\delta^{ij}. \end{aligned} \tag{7.1.6}$$

Obviously the third equation for the $0i$ component is $g^{0i}_{(0)} = 0$, which is a consequence of the assumptions we made to characterize the background, in particular isotropy. Now we can proceed to write (7.1.2) at the first order in perturbations

$$\begin{aligned} \left(g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu}\right)\left(g^{\nu\sigma}_{(0)} + g^{\nu\sigma}_{(1)}\right) &= \delta_\mu^{\ \sigma} \\ g^{(0)}_{\mu\nu} g^{\nu\sigma}_{(0)} + g^{(0)}_{\mu\nu} g^{\nu\sigma}_{(1)} + g^{(1)}_{\mu\nu} g^{\nu\sigma}_{(0)} &= \delta_\mu^{\ \sigma} \\ g^{(0)}_{\mu\nu} g^{\nu\sigma}_{(1)} + g^{(1)}_{\mu\nu} g^{\nu\sigma}_{(0)} &= 0, \end{aligned} \tag{7.1.7}$$

where we used the constraint on the background (7.1.5) to write the last line. Actually what we are interested in is an explicit relation for $g^{\mu\nu}_{(1)}$ so we multiply the last line of (7.1.7) by $g^{\beta\mu}_{(0)}$ and we get

$$g^{\mu\nu}_{(1)} = -g^{\mu\alpha}_{(0)} g^{(1)}_{\alpha\beta} g^{\beta\nu}_{(0)}, \tag{7.1.8}$$

which enable us to calculate each component of the first order perturbed contravariant metric tensor

$$\begin{aligned} g^{00}_{(1)} &= -g^{00}_{(0)} g^{(1)}_{00} g^{00}_{(0)} = \frac{2\phi_{(1)}}{a^2}, \\ g^{0i}_{(1)} &= -g^{00}_{(0)} g^{(1)}_{0j} g^{ji}_{(0)} = \frac{B_{(1)}^{\ i}}{a^2}, \\ g^{ij}_{(1)} &= -g^{ik}_{(0)} g^{(1)}_{kl} g^{lj}_{(0)} = -\frac{2C_{(1)}^{\ ij}}{a^2}. \end{aligned} \tag{7.1.9}$$

Finally we are ready to consider (7.1.2) at the second order in perturbations

$$\left(g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu} + g^{(2)}_{\mu\nu}\right)\left(g^{\nu\sigma}_{(0)} + g^{\nu\sigma}_{(1)} + g^{\nu\sigma}_{(2)}\right) = \delta_\mu^{\ \sigma}, \tag{7.1.10}$$

which using (7.1.5) and (7.1.7) gives

$$g^{(0)}_{\mu\nu} g^{\nu\sigma}_{(2)} + g^{(1)}_{\mu\nu} g^{\nu\sigma}_{(1)} + g^{(2)}_{\mu\nu} g^{\nu\sigma}_{(0)} = 0. \tag{7.1.11}$$

To achieve an expression for $g^{\mu\nu}_{(2)}$ we have to write $g^{\mu\nu}_{(1)}$ in terms of $g^{(1)}_{\mu\nu}$ using (7.1.8) and then multiply all the terms by $g^{\beta\mu}_{(0)}$

$$g^{\mu\nu}_{(2)} = g^{\mu\alpha}_{(0)} g^{(1)}_{\alpha\beta} g^{\beta\delta}_{(0)} g^{(1)}_{\delta\lambda} g^{\lambda\nu}_{(0)} - g^{\mu\alpha}_{(0)} g^{(2)}_{\alpha\beta} g^{\beta\nu}_{(0)}. \tag{7.1.12}$$

The various components are:

$$\begin{aligned}
g^{00}_{(2)} &= g^{00}_{(0)} g^{(1)}_{00}\, g^{00}_{(0)} g^{(1)}_{00}\, g^{00}_{(0)} + g^{00}_{(0)} g^{(1)}_{0i}\, g^{ij}_{(0)} g^{(1)}_{j0}\, g^{00}_{(0)} - g^{00}_{(0)} g^{(2)}_{00}\, g^{00}_{(0)} \\
&= a^{-2}\Big(- 4\phi^2_{(1)} + \delta^{ij} B_{(1)i} B_{(1)j} + \phi_{(2)}\Big), \\
g^{0i}_{(2)} &= g^{00}_{(0)} g^{(1)}_{00}\, g^{00}_{(0)} g^{(1)}_{0j}\, g^{ji}_{(0)} + g^{00}_{(0)} g^{(1)}_{0j}\, g^{jk}_{(0)} g^{(1)}_{kl}\, g^{li}_{(0)} - g^{00}_{(0)} g^{(2)}_{0j}\, g^{ji}_{(0)} \\
&= a^{-2}\Big(- 2\phi_{(1)} B^i_{(1)} - 2B_{(1)k} C^{ki}_{(1)} + \frac{1}{2} B^i_{(2)}\Big), \\
g^{ij}_{(2)} &= g^{ik}_{(0)} g^{(1)}_{k0}\, g^{00}_{(0)} g^{(1)}_{0l}\, g^{lj}_{(0)} + g^{ik}_{(0)} g^{(1)}_{kl}\, g^{lm}_{(0)} g^{(1)}_{mn} g^{nj}_{(0)} - g^{ik}_{(0)} g^{(2)}_{kl}\, g^{lj}_{(0)} \\
&= a^{-2}\Big(- B^i_{(1)} B^j_{(1)} + 4C^{ik}_{(1)} C^j_{(1)k} - C^{ij}_{(2)}\Big).
\end{aligned} \tag{7.1.13}$$

7.2 Expanding around a FRW solution

We want to prove that the most generic theory with broken time diffeomorphisms around a given FRW background (with $k = -1, 0, 1$ depending of the spatial curvature) can be written as

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\left[\frac{1}{2}M^2_{Pl}R + M^2_{Pl}\Big(\dot{H} - \frac{k}{a^2}\Big)g^{00} - M^2_{Pl}\Big(3H^2 + \dot{H} + 2\frac{k}{a^2}\Big) + \dots\right], \tag{7.2.1}$$

where the dots stand for terms which are invariant under spatial diffeomorphisms and of quadratic or higher order in the fluctuations around the given FRW background. We know that the displayed terms give rise to the wanted FRW evolution so that, if we do not want to move away from it, the additional operators must start quadratic around this solution. Every additional invariant term is quadratic or of higher order in perturbations, without cancellation of linear contributions among various operators. These terms, as shown in Section 4.1, will be written as polynomials (quadratic and higher) of linear operators like $\delta g^{00} = g^{00} + 1$, $\delta K_{\mu\nu} = K_{\mu\nu} - K^{(0)}_{\mu\nu}$, $\delta R_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - R^{(0)}_{\mu\nu\rho\sigma}$. These terms start linear in the perturbations as we have explicitly removed their value evaluated on the given FRW background. Because of the symmetries of a FRW metric, every tensor evaluated on the background ($K^{(0)}_{\mu\nu}$, $R^{(0)}_{\mu\nu\rho\sigma}$, $(\nabla_\alpha R_{\mu\nu\rho\sigma})^{(0)}$, ...) can be written just in terms of $g_{\mu\nu}$, n_μ and functions of time; for example [86]

$$K^{(0)}_{\mu\nu} = a^2 H h_{\mu\nu}, \tag{7.2.2}$$

$$R^{(0)}_{\mu\nu\rho\sigma} = 2\big(H + k\big) h_{\mu[\rho} h_{\sigma]\nu} + \Big[\big(\dot{H} + H^2\big) a^2 h_{\mu\sigma} \delta^0_\nu \delta^0_\rho + perm\Big]. \tag{7.2.3}$$

Let us now see how the Lagrangian can always be cast in the form (7.2.1). If we take an operator composed by the contraction of two tensors T and G, we can write:

$$TG = (T^{(0)} + \delta T)(G^{(0)} + \delta G) = \delta T \delta G + T^{(0)} G + T G^{(0)} - T^{(0)} G^{(0)}. \quad (7.2.4)$$

- The first term of the sum starts explicitly quadratic in the perturbation as we want.
- As regards the last term of (7.2.4), we recall that the unperturbed tensors $T^{(0)}$ and $G^{(0)}$ can be written as functions of $g_{\mu\nu}$, n_μ and t due to the symmetries of FRW background. Therefore $T^{(0)}G^{(0)}$ is just a polynomial of g^{00} with time dependent coefficients, it contains the terms $\sqrt{-g}\, g^{00}$ and $\sqrt{-g}$ plus operators which start explicitly quadratic in the perturbations.
- We are left with tensors of the form $T^{(0)}G$. By construction G will be linear either in $K_{\mu\nu}$ or $R_{\mu\nu\rho\sigma}$ with covariant derivatives acting on them. Covariant derivatives can be dealt with successive integrations by parts, letting them act on $T^{(0)}$ and the time dependent coefficient of the operator. In doing so we can generate extrinsic curvature terms. In this case we can reiterate equation (7.2.4) until no covariant derivatives are left. There can be also powers of g^{00} from $T^{(0)}$, we can deal with them by writing $g^{00} = -1 + \delta g^{00}$ and thus generating additional contributions to the g^{00} operator plus terms which are explicitly quadratic or higher in the perturbations. We are thus left with the only possible scalar linear terms with no covariant derivatives: K^μ_μ and R^{00}. Both terms can be rewritten in a more useful form:

$$\begin{aligned}\int d^4x \sqrt{-g}\, f(t) K^\mu_\mu &= \int d^4x \sqrt{-g}\, f(t) \nabla_\mu n^\mu = - \int d^4x \sqrt{-g}\, n^\mu \partial_\mu f(t) \\ &= \int d^4x \sqrt{-g}\, \sqrt{-g^{00}}\, \dot{f}(t). \end{aligned} \quad (7.2.5)$$

While we can deal with R^{00} using the following relationship [51]:

$$\begin{aligned}(-g^{00})^{-1} R^{00} &= R_{\mu\nu} n^\mu n^\nu \\ &= K^2 - K_{\mu\nu} K^{\mu\nu} - \nabla_\mu (n^\mu \nabla_\nu n^\nu) + \nabla_\nu (n^\mu \nabla_\mu n^\nu). \end{aligned} \quad (7.2.6)$$

The last two terms can again be integrated by parts:

$$\int d^4x \sqrt{-g}\, f(t) \nabla_\mu (n^\mu \nabla_\nu n^\nu) = - \int d^4x \sqrt{-g}\, \partial_\mu f(t) n^\mu K^\nu_\nu, \quad (7.2.7)$$

$$\int d^4x \sqrt{-g}\, f(t) \nabla_\nu (n^\mu \nabla_\mu n^\nu) = - \int d^4x \sqrt{-g}\, \partial_\nu f(t) n^\mu \nabla_\mu n^\nu = 0, \quad (7.2.8)$$

where in the last step we have used that $\partial_\nu f(t) \propto n_\nu$. This shows that K^μ_μ and R^{00} can be written in terms of the linear operators of (7.2.1) plus invariant terms that start quadratically in the fluctuations.

In conclusion, we have shown that the most general Lagrangian of a theory with broken time diffeomorphisms around a given FRW background can be written in the form:

$$\mathcal{S} = \int d^4x \sqrt{-g} \left[\frac{1}{2} M_{Pl}^2 R + M_{Pl}^2 \left(\dot{H} - \frac{k}{a^2} \right) g^{00} - M_{Pl}^2 \left(3H^2 + \dot{H} + 2\frac{k}{a^2} \right) + \right. \\ \left. + F^{(2)} \left(\delta g^{00}, \delta K_{\mu\nu}, \delta R_{\mu\nu\rho\sigma}; \nabla_\mu; t \right) \right], \tag{7.2.9}$$

where $F^{(2)}$ starts quadratic in the arguments $g^{00} + 1$, $\delta K_{\mu\nu}$ and $\delta R_{\mu\nu\rho\sigma}$.

7.3 Einstein equation

In this appendix we want to write explicitly the Einstein equations in a flat FRW Universe. First of all we compute the Christoffel symbols which are indispensable to compute the Riemann tensor. We know that the square of the line element in a flat FRW Universe can be written as

$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2. \tag{7.3.1}$$

So it is clear that the non vanishing components of the metric $g_{\mu\nu}$ are

$$\begin{aligned} g_{00} =& -1, \\ g_{ij} =& a^2(t)\delta_{ij}. \end{aligned} \tag{7.3.2}$$

Using the relation $g_{\mu\nu} g^{\nu\alpha} = \delta^\alpha_\mu$ we are able to find the contravariant metric

$$\begin{aligned} g^{00} =& -1, \\ g^{ij} =& \frac{1}{a^2(t)} \delta^{ij}. \end{aligned} \tag{7.3.3}$$

We recall that the Christoffel symbols are defined as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\epsilon} \left(\partial_\alpha g_{\epsilon\beta} + \partial_\beta g_{\epsilon\alpha} - \partial_\epsilon g_{\alpha\beta} \right), \tag{7.3.4}$$

hence their various components are

$$\begin{aligned} &\Gamma^0_{00} = 0 = \Gamma^i_{00} = \Gamma^0_{i0} = \Gamma^i_{jk}, \\ &\Gamma^0_{ij} = a\dot{a}\delta_{ij}, \\ &\Gamma^i_{j0} = H\delta^i_j. \end{aligned} \tag{7.3.5}$$

At this point we can move towards the Ricci tensor, which is defined as

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} = \partial_\alpha \Gamma^{\alpha}_{\nu\mu} - \partial_\nu \Gamma^{\alpha}_{\alpha\mu} + \Gamma^{\alpha}_{\alpha\varphi}\Gamma^{\varphi}_{\nu\mu} - \Gamma^{\alpha}_{\nu\varphi}\Gamma^{\varphi}_{\alpha\mu}. \tag{7.3.6}$$

Using the expressions for the components of Γ written above in (7.3.5) we find

$$\begin{aligned} R_{00} &= -3\dot{H} - 3H^2 = -3\frac{\ddot{a}}{a}, \\ R_{ij} &= a^2\big(\dot{H} + 3H^2\big)\delta_{ij} = 2\dot{a}^2\delta_{ij} + a\ddot{a}\delta_{ij}. \end{aligned} \tag{7.3.7}$$

The Ricci scalar is then given by

$$R = 6\dot{H} + 12H^2. \tag{7.3.8}$$

At this point everything is set up in order to write the Einstein equation, actually we are interested only in the 00 component and the ij components, so

$$\begin{aligned} G_{00} &= 3H^2, \\ G_{ij} &= -a^2\big[2\dot{H} + 3H^2\big]\delta_{ij}. \end{aligned} \tag{7.3.9}$$

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